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The Dynamic Portfolio Management under Incomplete Information

Revised Version

by

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Abstract

In this paper, we propose dynamic portfolio management schemes for deriving the optimal portfolio under incomplete information which assures the log-utility investors of maximizing their expected utility, at any *finite* terminal-time. One of the above schemes is the sample path-wise optimal portfolio (SPOP), which is consistent with the *back-test* framework used in the actual investment. It is proven that, at any *finite* terminal-time, this SPOP is asymptotically optimal among all the portfolios which is predictable under investors' incomplete information. The optimality is guaranteed by the continuous Bayesian updating scheme with the prior distribution for unknown drift parameters being endowed with asymptotically infinite differential entropy. Another scheme is the universal portfolio (UP). Although more relaxed constraints for portfolio weights are required, the UP is again optimal among all the portfolios which is predictable for the incomplete information. Also, as we extend the investor's utility class to the general power-utility, we show the above schemes guarantee its convergence to the optimal portfolio which is predictable for complete information. After proposing algorithms for the two schemes, we provide an empirical analysis which verifies the above schemes, and compensates for what the schemes lacks.

Contents

Chapter 1

Introduction

Modern portfolio theory has been developed with the models which maximize expected value of von Neumann-Morgenstern utility. Among these utilities, the log-utility and power-utility we are to treat in this paper can be used for reinvesting investors' wealth, and financial literature has shown its important properties. One of those properties is that the degree of Arrow-Pratt's relative risk aversion is constant, and the optimal portfolio policy for power-utility investors is constant through their i.i.d. investment horizons, and it is proven that this property holds if and only if the utility belongs to this class[3, 40]. Concerning the log-utility which is included in the power-utility, maximizing the terminal expected log-utility coincides with the growth maximization criterion, which is also known as the *Kelly criterion*, and several authors showed its favorable properties[32, 4, 44, and 1]. On the other hand, several authors paid attention to the problem of how to apply these portfolio theories in the practical market where only the incomplete information can be obtained. In the past, there are two approaches to treat this problem. In the first approach, the continuous estimation problem for the asset price process is considered[12, 10, 18, 11, 29, 17, 35, 36, 37, and 30]. In the analysis of [12, 10, 18, 11, 17, and 37], the same continuous Bayesian updating scheme of Liptser-Shiryayev is employed[38]. Then by using different techniques, the optimal portfolio under incomplete information is derived. The second approach for deriving optimal portfolios is different from the first one in that the estimation of asset distribution and the portfolio optimization are jointly executed. Recently, several authors propose this scheme, called *universal portfolios*. Given the information of observed asset prices alone, the universal portfolio converges to the portfolio which maximize the expected log-utility. In this area, Cover pioneered the universal portfolio and got an asymptotic result on general discrete time series[7], and Cover-Ordentlich extended that result with side information taken into consideration[8]. Also Jamshidian provided the same results in a continuous time framework[25]. Shirakawa-Ishijima extended the result into the finite time framework and showed that it provides better expected log-utility compared to the portfolio using estimated mean and covariance from the past[42].

Our motivation is quite similar to these two approaches, and our treatment resembles the latter approach. But our approach to treat this problem is different in that we are to build a theoretical framework directly applicable to actual investments. Practically, when investors decide their portfolio positions for the next investment period $t+1$, they carry out so-called *back-test* from past observations of asset returns. That is, any joint observation of asset returns, in each period from $t-L+1$ to t, is presumed to be uniformly distributed, with probability $1/L$ of occurring in period $t+1$[19, 20, 21, and 22]. Then investors adopt the portfolio which maximize the expected value of their utility using the above probability distribution. Our question is whether this portfolio is really optimal among all the portfolios which are predictable for the available information set. The objective of this paper is to give the answer for this question. That is, as we restrict the power-utility class only to the log-utility, we show the optimality which assures log-utility investors of maximizing their expected utility, at any *finite* terminal-time. Another objective is to show these schemes almost surely learn the ideal portfolio asymptotically which maximize the terminal expected value of investors' utility under *complete information*. That is, we provide the schemes which can derive the optimal portfolios within the shorter incomplete information set, and also assure their convergence to the ideal optimal portfolio within the longest complete information set, without contradiction.

After proposing the algorithms for the above schemes, we empirically verify the schemes and make up for what they lacks in theoretical aspects, at the practical US stock market. Concerning the preceding empirical works on the asset management for power-utility investors, several authors published on this theme [19, 20, 21, and 22]. In these papers, the portfolio composition and the ex post geometric mean and variance are examined by varying risk attitude, i.e. the Arrow-Pratt relative risk aversion. But these empirical analyses were not based on a scheme for how to apply the *Expected Utility Maximization Theorem* of von Neumann-Morgenstern in the practical market, where asset prices are observed only once. In this paper, using the several schemes described above, we verify if these schemes can provide the maximal ex post power-utility in the settings of the practical US stock market.

This paper is organized as follows. In chapter 2, the continuous model of asset prices is defined. After defining the complete information given to power-utility investors, we derive the optimal constant portfolio (the ExPow portfolio) applying the Expected Utility Maximization Theorem of von Neumann-Morgenstern.

In chapter 3, we restrict the power-utility class only to the log-utility, or equivalently restrict the degree of Arrow-Pratt's relative risk aversion α only to one. After defining the incomplete information given to investors, we address the expected utility maximization problem whose expectation is taken under the sequence of probability measure $\mathcal{P}^{(k)}$. Then we propose the portfolio, called the *SPOP*. The SPOP maximize the sample path-wise value of the constant portfolio. The asymptotic optimality of the SPOP at any *finite* terminal-time, among all the portfolio which is predictable for the incomplete information, is proved by introducing the optimal portfolio based on the continuous Bayesian updating formula (referred to as the *CBOP*) of Liptser-Shiryayev[38].

In chapter 4, we propose another scheme called the *universal portfolio (UP)*. As the SPOP, the UP is also optimal among the portfolios which is predictable for the incomplete information. But more relaxed constraints for portfolio weights are required to show its optimality. Moreover, we evaluate the gap in the expected log-utility base, at any finite terminal-time, among both the SPOP and UP with incomplete information, and the ideal ExLog portfolio with complete information. We also establish the advantage of both the SPOP and UP compared to the portfolio which uses the unbiased estimators for the drift and diffusion parameters in asset price processes. Furthermore, the peformance of unbiased estimator approach converges to that of both the SPOP and UP, as the number of observed asset prices increases.

In chapter 5, as we extend the utility class from the log-utility to the general power-utility, we propose two schemes which almost surely learn the ExPow portfolio asymptotically. One is the *α-scaled sample path-wise optimal portfolio (αSPOP)* which is the generalization of the SPOP. The other is the *α-scaled universal portfolio (αUP)* which is the generalization of the UP. It shown that both the αSPOP and αUP with incomplete information almost surely converge or learn the ExPow portfolio which is predictable for complete information. In chapter 6, we propose algorithms to search for the αSPOP, and for the αUP.

In chapter 7, we provide an empirical analysis which verify our theory in the continuous-time framework, and make up for what the theory lacks. The target of this analysis is the entire NYSE/AMEX stock market. Our analysis is rather new in that we take the exact transaction costs into account, whenever we evaluate the ex post power-utility. And in chapter 8, the conclusion and the direction of our future research are stated.

Chapter 2

The Optimal Portfolio under Complete Information

Under the complete information given to the power-utility investors, we derive the *ideal* optimal portfolio, among all the portfolios which are predictable for the complete information. That is, with complete information about how asset prices are stochastically generated, and with the assumption of optimal portfolio constancy, we derive the optimal constant portfolio for the power-utility investor. Then we verify the optimality of the constant portfolio.

2.1 The Model of Asset Prices and Investor's Objective

We consider a security market with m assets which prices S_{it} $(i = 1, \ldots, m)$ follow geometric Brownian motion as follows:

$$\begin{aligned} \frac{dS_{it}}{S_{it}} &= \mu_i dt + \sum_{j=1}^{m} \sigma_{ij} dW_{jt} \quad (i = 1, \ldots, m) \ , \\ \text{or} \quad (\mathrm{diag}(\boldsymbol{S}_t))^{-1} d\boldsymbol{S}_t &= \boldsymbol{\mu} dt + \boldsymbol{\Sigma} d\mathbf{W}_t \ , \end{aligned} \tag{2.1}$$

where $\mathrm{diag}(\boldsymbol{S}_t)$ is a diagonal matrix whose element is $\boldsymbol{S}_t$. Here $\mathbf{W}_t = (W_{1t}, \ldots, W_{mt})'$ denotes an m-dimensional standard Brownian motion and the filtration $\mathcal{F}_t$ is generated by $\sigma(\mathbf{W}_u; 0 \leq u \leq t)$ and $\mathcal{F}_0$. We assume that $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_m)'$ is an $\mathcal{F}_0$-measurable random vector following multivariate normal distribution and $\boldsymbol{\Sigma} = (\sigma_{ij})_{1\leq i,j\leq m}$ is a constant diffusion parameter. It is supposed that $\{\mathbf{W}_t;\ t \geq 0\}$ and $\boldsymbol{\mu}$ are independent and investors have the following class of information:

Information 1

Complete information: With the $\mathcal{F}_0$-measurable drift parameter $\boldsymbol{\mu}$ being known, asset prices follow the stochastic differential equation (s.d.e) (2.1).

We assume the investors' utility is expressed as a power-utility function, such that

$$u(x) \stackrel{\triangle}{=} \frac{1}{1-\alpha} x^{1-\alpha} \quad (\alpha \geq 0) \ , \tag{2.2}$$

where α is the degree of Arrow-Pratt's relative risk aversion, i.e. $\alpha \stackrel{\triangle}{=} -xu''(x)/u'(x)$. When $\alpha = 1$, the investors' utility is logarithmic one.

Then the investors having the power-utility continuously select their optimal portfolios within all the $\mathcal{F}_t$-predictable portfolios. And the portfolio selection is made within the following simplex $\mathbf{D}$:[1]

$$\mathbf{D} \stackrel{\triangle}{=} \{\boldsymbol{b} \in \mathbf{R}^m \ \mid \boldsymbol{b}'\mathbf{1} = 1,\ \boldsymbol{b} \geq \mathbf{0}\} \ . \tag{2.3}$$

The instantaneous return of the portfolio value process is given by

$$\frac{dV_t(\boldsymbol{b}_\bullet)}{V_t} \ = \ \boldsymbol{b}_t'(\mathrm{diag}(\boldsymbol{S}_t))^{-1} d\boldsymbol{S}_t.$$

[1] Though we can incorporate the general convex constraints, $\mathbf{A}\boldsymbol{b} \leq \boldsymbol{c}$, as the study in [23], we assume investors are allowed to select their portfolios in the simplex (2.3). The reason for this is that we do not consider them in the following empirical analysis (chapter 7), and we make our theory consistent with it. This reason applies to the treatment of the simplex $\mathbf{D}'$ in chapter 4.

Provided that the investor's initial wealth is $V_0 = 1$, we can easily check that the portfolio value at time T is given by

$$V_T(\boldsymbol{b}_\bullet) \;=\; \exp\left[-\frac{1}{2}\int_0^T \boldsymbol{b}_u'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b}_u du + \int_0^T \boldsymbol{b}_u'(\mathrm{diag}(\boldsymbol{S}_u))^{-1}d\boldsymbol{S}_u\right] . \tag{2.4}$$

Under these conditions, with terminal-time T provided, the objective problem for the power-utility investors is to maximize the terminal value of their expected utility as follows:

$$\mathbf{P}_0 \left| \begin{array}{ll} \underset{\mathbf{b}_\bullet}{\text{maximize}} & E[u(\ V_T(\boldsymbol{b}_\bullet)\ )] \\ \text{subject to} & \boldsymbol{b}_t \in \mathbf{D}\ , \\ & \boldsymbol{b}_t \text{ is } \mathcal{F}_t\text{−predictable process }, \end{array}\right.$$

where $\mathbf{1}$ is a vector of ones.

2.2 The Optimal Constant Portfolio $\boldsymbol{b}^*$

The outline of the following analysis is as follows. First, we assume the portfolio maximizing the expected power-utility among all the $\mathcal{F}_t$-predictable portfolios is constant through time and drive the optimal constant portfolio. And then we verify its optimality among all the admissible portfolios. If the portfolio at t is constant, i.e. $\boldsymbol{b}_t = \boldsymbol{b} =$ const., the portfolio value at time T is reduced to be simply

$$V_T(\boldsymbol{b}) \;=\; \exp\left[\left(\boldsymbol{b}'\boldsymbol{\mu} - \frac{1}{2}\boldsymbol{b}'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b}\right)T + \boldsymbol{b}'\boldsymbol{\Sigma}\boldsymbol{W}_t\right].$$

And its expectation at terminal-time T, under Information 1, is

$$\begin{aligned} E[u(V_T(\boldsymbol{b}))] \;&=\; \frac{1}{1-\alpha}E\left[\exp\left[(1-\alpha)\left\{\left(\boldsymbol{b}'\boldsymbol{\mu} - \frac{1}{2}\boldsymbol{b}'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b}\right)T + \boldsymbol{b}'\boldsymbol{\Sigma}\boldsymbol{W}_T\right\}\right]\right] \\ &=\; \frac{1}{1-\alpha}\exp\left[(1-\alpha)\boldsymbol{b}'\boldsymbol{\mu}T - \frac{1}{2}(1-\alpha)\alpha\boldsymbol{b}'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b}T\right]. \end{aligned} \tag{2.5}$$

Then the terminal expected power-utility maximization problem for the investor holding constant portfolio is then stated as

$$\mathbf{P}_1 \left| \begin{array}{ll} \underset{\mathbf{b}}{\text{maximize}} & E[u(V_T(\boldsymbol{b}))] \\ \text{subject to} & \boldsymbol{b} \in \mathbf{D}\ . \end{array}\right.$$

Utilizing Eq. (2.5), this problem is equivalent to

$$\mathbf{P}_1' \left| \begin{array}{ll} \underset{\{\mathbf{b}\}}{\text{minimize}} & \frac{1}{2}\alpha \boldsymbol{b}'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b} - \boldsymbol{b}'\boldsymbol{\mu} \\ \text{subject to} & \boldsymbol{b} \in \mathbf{D} \ . \end{array} \right.$$

From the Karush-Kuhn-Tucker (KKT) condition[31, 34], the optimal constant portfolio for the problem $\mathbf{P}_1'$ is $(\boldsymbol{b}^*, \eta^*, \boldsymbol{\nu}^*)$ satisfying the following equations.

$$\begin{aligned} &\alpha \boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b}^* + \eta^* \mathbf{1} - \boldsymbol{\nu}^* = \boldsymbol{\mu}, \\ &\boldsymbol{b}^{*'}\mathbf{1} = 1, \quad \boldsymbol{\nu}^{*'}\boldsymbol{b}^* = 0, \quad \boldsymbol{\nu}^* \geq \mathbf{0}, \end{aligned} \tag{2.6}$$

where η^* and $\boldsymbol{\nu}^*$ are Lagrange multipliers. Then the optimality of the constant portfolio $\boldsymbol{b}^*$, among all the $\mathcal{F}_t$-predictable portfolios, is guaranteed by the following theorem.

Theorem 1 (Optimality of constant portfolio $\boldsymbol{b}^*$)

Under Information 1, the expected power-utility is maximized by constant portfolio $\boldsymbol{b}^*$ at any finite terminal-time T. That is

$$(\ \{\forall \mathcal{F}_t\text{−predictable } \boldsymbol{b}_t \in \mathbf{D} \ ; \ 0 \leq t \leq T \ \} \ ) \ (\ E[u(\ V_T(\boldsymbol{b}_\bullet) \ )] \leq E[u(\ V_T(\boldsymbol{b}^*) \ )] \ ) \ .$$

Proof. Since $u(x)$ is strictly concave, we have

$$E[u(V_T(\boldsymbol{b}_\bullet))] \ \leq \ E[u(V_T(\boldsymbol{b}^*_\bullet))] + \left\{ E[V_T(\boldsymbol{b}^*_\bullet)^{-\alpha} \cdot V_T(\boldsymbol{b}_\bullet)] - E[V_T(\boldsymbol{b}^*_\bullet)^{1-\alpha}] \right\}$$

Then, it is necessary and sufficient to prove $E[V_T(\boldsymbol{b}^*_\bullet)^{-\alpha} \cdot V_T(\boldsymbol{b}_\bullet)] - E[V_T(\boldsymbol{b}^*_\bullet)^{1-\alpha}] \leq 0$. We can easily see that

$$\begin{aligned} V_T(\boldsymbol{b}^*_\bullet)^{1-\alpha} &= \exp\left[(1-\alpha) \left\{ \int_0^T \left(\boldsymbol{b}_t^{*'}\boldsymbol{\mu} - \frac{1}{2}\boldsymbol{b}_t^{*'}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b}_t^* \right) dt + \int_0^T \boldsymbol{b}_t^{*'}\boldsymbol{\Sigma} d\boldsymbol{W}_t \right\} \right] \\ &= A \cdot B \ , \end{aligned}$$

where

$$\begin{aligned} A &= \exp\left[(1-\alpha) \int_0^T \boldsymbol{b}_t^{*'}\boldsymbol{\mu} dt - \frac{1}{2}\alpha(1-\alpha) \int_0^T \boldsymbol{b}_t^{*'}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b}_t^* dt \right] \ , \\ B &= \exp\left[-\frac{1}{2}(1-\alpha)^2 \int_0^T \boldsymbol{b}_t^{*'}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b}_t^* dt + (1-\alpha) \int_0^T \boldsymbol{b}_t^* \boldsymbol{\Sigma} d\mathbf{W}_t \right] . \end{aligned}$$

Also,

$$
\begin{aligned}
& V_T(\boldsymbol{b}^*_\bullet)^{-\alpha} \cdot V_T(\boldsymbol{b}_\bullet) \\
& = \exp\left[\int_0^T \left\{(\boldsymbol{b}_t - \alpha \boldsymbol{b}^*_t)' \boldsymbol{\mu} - \frac{1}{2}\left(\boldsymbol{b}_t' \boldsymbol{\Sigma}\boldsymbol{\Sigma}' \boldsymbol{b}_t - \alpha \boldsymbol{b}^{*'}_t \boldsymbol{\Sigma}\boldsymbol{\Sigma}' \boldsymbol{b}^*_t\right)\right\} dt + \int_0^T (\boldsymbol{b}_t - \alpha \boldsymbol{b}^*_t)' \boldsymbol{\Sigma} d\mathbf{W}_t\right] \\
& = A \cdot C \exp\left[\int_0^T (\boldsymbol{b}_t - \boldsymbol{b}^*_t)' (\boldsymbol{\mu} - \alpha \boldsymbol{\Sigma}\boldsymbol{\Sigma}' \boldsymbol{b}^*_t)\, dt\right], \qquad (2.7)
\end{aligned}
$$

where

$$
C \;=\; \exp\left[-\frac{1}{2}\int_0^T (\boldsymbol{b}_t - \alpha \boldsymbol{b}^*_t)' \boldsymbol{\Sigma}\boldsymbol{\Sigma}' (\boldsymbol{b}_t - \alpha \boldsymbol{b}^*_t)\, dt + \int_0^T (\boldsymbol{b}_t - \alpha \boldsymbol{b}^*_t)' \boldsymbol{\Sigma} d\mathbf{W}_t\right].
$$

The last term of (2.7) is

$$
\begin{aligned}
\exp\left[\int_0^T (\boldsymbol{b}_t - \boldsymbol{b}^*_t)' (\boldsymbol{\mu} - \alpha \boldsymbol{\Sigma}\boldsymbol{\Sigma}' \boldsymbol{b}^*_t)\, dt\right] &= \exp\left[\int_0^T (\boldsymbol{b}_t - \boldsymbol{b}^*_t)' \{\boldsymbol{\mu} - (\boldsymbol{\mu} - \eta^*_t \mathbf{1} + \boldsymbol{\nu}^*_t)\}\, dt\right] \\
&= \exp\left[-\int_0^T \boldsymbol{b}_t' \boldsymbol{\nu}^*_t dt\right] \\
&\le 1\,. \qquad (2.8)
\end{aligned}
$$

The second equality follows from the constraint $\boldsymbol{b}_t'\mathbf{1} = \boldsymbol{b}^{*'}_t\mathbf{1} = 1$, and the KKT condition $\boldsymbol{b}^{*'}_t \boldsymbol{\nu}^*_t = 0$ in Eq. (2.6). And the inequality follows from $\boldsymbol{b}_t \ge \mathbf{0}$ and $\boldsymbol{\nu}^*_t \ge \mathbf{0}$. Then we can rewrite (2.7) as

$$
V_T(\boldsymbol{b}^*_\bullet)^{-\alpha} \cdot V_T(\boldsymbol{b}_\bullet) \le A \cdot C.
$$

Since $\boldsymbol{b}^*_t$ is uniformly bounded, we can show $E[B] = 1$. From this together with $A \ge 0$ and $E[C] \le 1$, we have

$$
\begin{aligned}
E[V_T(\boldsymbol{b}^*_\bullet)^{-\alpha} V_T(\boldsymbol{b}_\bullet)] - E[V_T(\boldsymbol{b}^*_\bullet)^{1-\alpha}] &\le AE[C] - AE[B] \\
&\le A - A = 0\,.
\end{aligned}
$$

□

This theorem asserts that the constant portfolio $\boldsymbol{b}^*$ is the solution of $\mathbf{P}_0$. In other words, $\boldsymbol{b}^*$ is the portfolio maximizing the expected power-utility at any finite terminal-time, among all the $\mathcal{F}_t$-predictable portfolios. Hereafter we call $\boldsymbol{b}^*$ the *ExPow portfolio*, or the *ExLog portfolio* especially for the $\alpha = 1$(log) investors.

Chapter 3

The Sample Path-Wise Optimal Portfolio

In this chapter, as we restrict the degree of Arrow-Pratt's relative risk aversion α only to one, i.e. the log-utility investor, we propose a scheme of how to derive the optimal portfolio, among all the portfolios which are predictable for the incomplete information. The optimality is proved by utilizing the continuous Bayesian updating formula of [38]. On the other hand, the above optimal portfolio almost surely learns the ExPow portfolio in the long run, but we leave this asymptotic analysis in Chapter 5.

3.1 Addressing the Problem under the Incomplete Information

As in the previous chapter, we consider the security market with m assets. We suppose, however, that investors are allowed to observe a realization of asset price process (2.1), only once. In other words, they are provided with the following class of information which is reasonable for the practical market environment.

Information 2

Incomplete information: Investors have no prior distribution information for the drift parameter $\boldsymbol{\mu}$*. That is, the differential entropy of the prior distribution for the drift parameter is asymptotically infinite. And they are only provided with the information* $\mathcal{G}_t \subset \mathcal{F}_t$ *generated by a realized asset price process of Eq. (2.1) as follows:*

$$\mathcal{G}_t \overset{\triangle}{=} \sigma\left(\boldsymbol{S}_u; 0 \leq u \leq t\right) \ . \tag{3.1}$$

Remark 1

The information $\mathcal{G}_t$ *is enough to derive* $\boldsymbol{\Sigma\Sigma}'$ *exactly, since the Doob-Meyer decomposition of the quadratic process* $(d\boldsymbol{S}_t/\boldsymbol{S}_t)(d\boldsymbol{S}_t/\boldsymbol{S}_t)'$ *yields the finite process* $\boldsymbol{\Sigma\Sigma}'t$*.*

Remark 2

It is well known that if $\boldsymbol{\mu}$ *follows the multivariate normal distribution* $N(\boldsymbol{m}, \boldsymbol{\Gamma})$*, its differential entropy is given by* $1/2\log(2\pi e)^m|\boldsymbol{\Gamma}|$ *[Th.9.4.1, 6]. Since we aim to treat the prior distribution for* $\boldsymbol{\mu}$ *with infinite differential entropy being endowed, we consider the sequence of the probability measure* $\mathcal{P}^{(k)}$ *on* $(\mathcal{F}_0, \Omega)$*, such that each prior distribution for* $\boldsymbol{\mu}$ *is* $N\left(\boldsymbol{m}_0^{(k)}, \boldsymbol{\Gamma}_0^{(k)}\right) \overset{\triangle}{=} N\left(\boldsymbol{m}_0, k\boldsymbol{\Gamma}_0\right)$*. Then we can guarantee that the differential entropy of the prior distribution for* $\boldsymbol{\mu}$ *is asymptotically infinite, since* $1/2\log(2\pi e)^m|\boldsymbol{\Gamma}_0^{(k)}| = k/2\log(2\pi e)^m|\boldsymbol{\Gamma}_0| \to \infty$ *as* $k \to \infty$*.*

We assume the investors' utility is expressed as the log-utility function

$$u(x) \overset{\triangle}{=} \log x \ (x > 0) \ . \tag{3.2}$$

Then the investors having the log-utility continuously select the optimal portfolios within all the $\mathcal{G}_t$-predictable portfolios. And the portfolio selection is made within the simplex $\mathbf{D}$ defined by (2.3). The instantaneous return of the portfolio value process is given by

$$\frac{dV_t(\boldsymbol{b}_\bullet)}{V_t} = \boldsymbol{b}_t'(\mathrm{diag}(\boldsymbol{S}_t))^{-1}d\boldsymbol{S}_t.$$

Without loss of generality, we assume that the investor's initial wealth is $V_0 = 1$. Then we can easily check that the portfolio value at terminal-time T is given by

$$V_T(\boldsymbol{b}_\bullet) \;=\; \exp\left[-\frac{1}{2}\int_0^T \boldsymbol{b}_u'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b}_u du + \int_0^T \boldsymbol{b}_u'(\mathrm{diag}(\boldsymbol{S}_u))^{-1}d\boldsymbol{S}_u\right] . \tag{3.3}$$

The problem we are to treat in this chapter emerges, because, under weak Information 2, we *cannot* observe the realization of $\boldsymbol{\mu}$. Then the terminal expected log-utility maximization problem we treat is:

$$\mathbf{P}_2 \left| \begin{array}{ll} \underset{\mathbf{b}_\bullet}{\text{maximize}} & \lim_{k\to\infty} E^{(k)}[u(\ V_T(\boldsymbol{b}_\bullet)\ )] \\ \text{subject to} & \boldsymbol{b}_t \in \mathbf{D}\ , \\ & \boldsymbol{b}_t \text{ is } \mathcal{G}_t\text{−predictable process}\ , \end{array} \right.$$

where $E^{(k)}[\cdot]$ is the expectation under the probability measure $\mathcal{P}^{(k)}$, and $\mathbf{1}$ is a vector of ones. In the following discussion, we propose a scheme of how to derive the optimal portfolio for $\mathbf{P}_2$ within the $\mathcal{G}_t$-predictable portfolios.

3.2 Deriving the Sample Path-Wise Optimal Portfolio $\boldsymbol{b}_t^{**}$

Under the limited Information 2 given to the log-utility investor, we propose a method to derive the optimal portfolio among all the $\mathcal{G}_t$-predictable portfolios. Under Information 2, the log-utility investors cannot use the filtration $\mathcal{F}_t$. Then they can only maximize their log-utility along this sample path-wise information $\mathcal{G}_t$, using $\mathcal{G}_t$-predictable portfolios $\boldsymbol{b}_\bullet$.

First we sketch the framework of *back-test* which is frequently employed in the actual investment. Divide $[0, t]$ into N, then the time interval is $\Delta \overset{\triangle}{=} t/N$ and the price observations are $\{\Delta \boldsymbol{S}_i\ ;\ i = 1, \ldots, N\}$. The back-test framework assumes that the next increment $\Delta \boldsymbol{S}$ during $[t, t + \Delta]$ is uniformly distributed as the historical data, i.e.,

$$(\mathrm{diag}\,(\boldsymbol{S}))^{-1} \cdot \Delta\boldsymbol{S} = (\mathrm{diag}\,(\boldsymbol{S}_i))^{-1} \cdot \Delta\boldsymbol{S}_i\ ,\ w.p.\ \frac{1}{N}\ ,\ 1 \le i \le N\ .$$

Then we can easily see that

$$E\left[\log\left(\frac{V_{t+\Delta}(\boldsymbol{b}_\bullet)}{V_t(\boldsymbol{b}_\bullet)}\right)\middle|\mathcal{G}_t\right] \cong \frac{1}{N}\sum_{i=1}^{N}\log\left(1+\boldsymbol{b}_t'\left(\mathrm{diag}\left(\boldsymbol{S}_i\right)\right)^{-1}\cdot\Delta\boldsymbol{S}_i\right)\ . \tag{3.4}$$

If we maximize the left-hand side of (3.4) continuously, we can optimize $\mathbf{P}_2$ for any realization of the drift parameter $\boldsymbol{\mu}$. From the Ito's lemma, if we take the limit $N \to \infty$,

$$\frac{1}{N}\sum_{i=1}^{N}\log\left(1+\boldsymbol{b}_t'\left(\mathrm{diag}\left(\boldsymbol{S}_i\right)\right)^{-1}\cdot\Delta\boldsymbol{S}_i\right) \to -\frac{1}{2}\boldsymbol{b}_t'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b}_t t + \int_0^t \boldsymbol{b}_t'\left(\mathrm{diag}\left(\boldsymbol{S}_u\right)\right)^{-1}\cdot d\boldsymbol{S}_u\ .$$

Then the back-test framework of the portfolio optimization results in the following parametric problem with respect to t:

$$\mathbf{P}_2(t)\left|\begin{array}{ll}\underset{\mathbf{b}}{\text{maximize}} & V_t(\boldsymbol{b}) = \exp\left[\left(\boldsymbol{b}'\boldsymbol{\mu}-\frac{1}{2}\boldsymbol{b}'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b}\right)t+\boldsymbol{b}'\boldsymbol{\Sigma}\mathbf{W}_t\right] \\ \text{subject to} & \boldsymbol{b}\in\mathbf{D}\ .\end{array}\right.$$

Hereafter, we call the optimal solution of $\mathbf{P}_2(t)$, based on the back-test framework, the *sample path-wise optimal portfolio (SPOP)*. $\mathbf{P}_2(t)$ is equivalent to the following problem.

$$\mathbf{P}_2'(t)\left|\begin{array}{ll}\underset{\mathbf{b}}{\text{minimize}} & \frac{1}{2}\boldsymbol{b}'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b}-\boldsymbol{b}'\tilde{\boldsymbol{\mu}}_t \\ \text{subject to} & \boldsymbol{b}\in\mathbf{D}\ ,\end{array}\right.$$

where $\tilde{\boldsymbol{\mu}}_t \stackrel{\triangle}{=} \boldsymbol{\mu}+1/t\boldsymbol{\Sigma}\mathbf{W}_t$. Since

$$\tilde{\boldsymbol{\mu}}_t = \frac{1}{t}\int_0^t(\mathrm{diag}(\boldsymbol{S}_u))^{-1}d\boldsymbol{S}_u\ , \tag{3.5}$$

this problem is well-defined under Information 2. Also note that we admit the SPOP is $\mathcal{G}_t$-predictable, since it seeks the constant portfolio which maximizes the sample path-wise portfolio value at time t, according to one sample path $\{\boldsymbol{S}_u; 0 \le u \le t\}$. Then, at time t (> 0), the KKT condition for the problem $\mathbf{P}_2'(t)$ is $(\boldsymbol{b}_t^{**}, \eta_t^{**}, \boldsymbol{\nu}_t^{**})$ satisfying

$$\left\{\begin{array}{l}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b}_t^{**}+\eta_t^{**}\mathbf{1}-\boldsymbol{\nu}_t^{**}=\tilde{\boldsymbol{\mu}}_t\ , \\ \boldsymbol{b}_t^{**'}\mathbf{1}=1\ , \\ \boldsymbol{\nu}_t^{**'}\boldsymbol{b}_t^{**}=0\ , \boldsymbol{\nu}_t^{**}\ge\mathbf{0}\ ,\end{array}\right. \tag{3.6}$$

where η_t^{**} and $\boldsymbol{\nu}_t^{**}$ are Lagrange multipliers at t. Hereafter, $\boldsymbol{b}_t^{**}$ denotes the SPOP.

3.3 The Asymptotic Optimality of the SPOP via Continuous Bayesian Updating Schemes

In this section, we shall prove the asymptotic optimality of the SPOP among all the $\mathcal{G}_t$-predictable portfolios, through the continuous Bayesian updating scheme. The portfolio selection problem under incomplete information, which is quite similar to our model, is considered by several authors [12, 10, 18, 11, 17, and 37]. Let us consider the portfolio learning scheme of Gennotte[18], based on the continuous Bayesian updating formula of Liptser-Shiryayev[38].

We suppose the investors only have Information 2. Then as we described in Remark 1, they know the diffusion parameter $\boldsymbol{\Sigma\Sigma}'$ exactly, but do not know the $\mathcal{F}_0$-measurable drift parameter $\boldsymbol{\mu}$. Hereafter, as noted in Remark 2, we consider the probability measure $\mathcal{P}^{(k)}$ on $(\mathcal{F}_0, \Omega)$, where the prior distribution for $\boldsymbol{\mu}$ follows $N\left(\boldsymbol{m}_0^{(k)}, \boldsymbol{\Gamma}_0^{(k)}\right) \triangleq N\left(\boldsymbol{m}_0, k\boldsymbol{\Gamma}_0\right)$. Then we can study the infinite differential entropy behavior by taking the limit $k \to \infty$. Utilizing the information $\mathcal{G}_t$, the investors estimate the $\mathcal{F}_0$-measurable $\boldsymbol{\mu}$ as follows:

$$\begin{aligned} \boldsymbol{m}_t^{(k)} &\triangleq E^{(k)}\left[\boldsymbol{\mu} \mid \mathcal{G}_t\right] , \\ \boldsymbol{\Gamma}_t^{(k)} &\triangleq E^{(k)}\left[\left(\boldsymbol{\mu} - \boldsymbol{m}_t^{(k)}\right)\left(\boldsymbol{\mu} - \boldsymbol{m}_t^{(k)}\right)' \,\middle|\, \mathcal{G}_t\right] , \end{aligned}$$

where $\boldsymbol{m}_t^{(k)}$ is the estimation for $\boldsymbol{\mu}$, and $\boldsymbol{\Gamma}_t^{(k)}$ is its estimation error, under $\mathcal{P}^{(k)}$. Using infinitesimal observations $d\boldsymbol{S}_t$, we can improve the estimation by the continuous Bayesian updating formula of [38], to obtain the improvement $d\boldsymbol{m}_t^{(k)}$. By assumption, we have $E^{(k)}[|\boldsymbol{\mu}|^4] = E^{(k)}\left[|\boldsymbol{\mu}|^4 \middle| \mathcal{G}_0\right] < \infty$. According to Theorem 12.8 in [38], the estimation of $\boldsymbol{\mu}$ conditioned on $\mathcal{G}_t$ is given by

$$\begin{aligned} \boldsymbol{m}_t^{(k)} &= \left[\mathbf{I} + \boldsymbol{\Gamma}_0^{(k)}\left(\boldsymbol{\Sigma\Sigma}'\right)^{-1} t\right]^{-1} \cdot \left[\boldsymbol{m}_0^{(k)} + \boldsymbol{\Gamma}_0^{(k)}\left(\boldsymbol{\Sigma\Sigma}'\right)^{-1} \int_0^t \left(\mathrm{diag}(\boldsymbol{S}_u)\right)^{-1} d\boldsymbol{S}_u\right] \qquad (3.7) \\ &= \left[\frac{1}{k}\boldsymbol{\Gamma}_0^{-1} + \left(\boldsymbol{\Sigma\Sigma}'\right)^{-1} t\right]^{-1} \cdot \left[\frac{1}{k}\boldsymbol{\Gamma}_0^{-1}\boldsymbol{m}_0 + \left(\boldsymbol{\Sigma\Sigma}'\right)^{-1} \int_0^t \left(\mathrm{diag}(\boldsymbol{S}_u)\right)^{-1} d\boldsymbol{S}_u\right] \end{aligned}$$

where $\mathbf{I}$ is an identity matrix. Since we aim to analyze the infinite differential entropy case, we should take the limit $k \to \infty$. Then the estimator $\boldsymbol{m}_t^{(k)}$ converges to

$$\boldsymbol{m}_t \triangleq \lim_{k\to\infty} \boldsymbol{m}_t^{(k)} = \frac{1}{t}\int_0^t (\mathrm{diag}(\boldsymbol{S}_u))^{-1}\, d\boldsymbol{S}_u = \tilde{\boldsymbol{\mu}}_t \ , \tag{3.8}$$

where $\tilde{\boldsymbol{\mu}}_t$ is given by Eq. (3.5). Next transform Eq. (2.1) into entirely observable s.d.e. :

$$(\mathrm{diag}(\boldsymbol{S}_t))^{-1}\, d\boldsymbol{S}_t = \boldsymbol{m}_t^{(k)} dt + \boldsymbol{\Sigma} d\tilde{\mathbf{W}}_t^{(k)} \ , \tag{3.9}$$

where

$$\tilde{\mathbf{W}}_t^{(k)} \triangleq \boldsymbol{\Sigma}^{-1}\left[\int_0^t (\mathrm{diag}(\boldsymbol{S}_u))^{-1}\, d\boldsymbol{S}_u - \int_0^t \boldsymbol{m}_u^{(k)} du\right] = \boldsymbol{\Sigma}^{-1}\int_0^t (\boldsymbol{\mu} - \boldsymbol{m}_u^{(k)}) du + \mathbf{W}_t \ .$$

Then $\tilde{\mathbf{W}}_t^{(k)}$ is $\mathcal{G}_t$-measurable, and

$$E^{(k)}\left[\tilde{\mathbf{W}}_t^{(k)} - \tilde{\mathbf{W}}_s^{(k)} \Big| \mathcal{G}_s\right] = E^{(k)}\left[\boldsymbol{\Sigma}^{-1}\int_s^t (\boldsymbol{\mu} - \boldsymbol{m}_u^{(k)}) du \Big| \mathcal{G}_s\right] = 0 \ .$$

Furthermore, $\left\langle \tilde{\mathbf{W}}^{(k)} \right\rangle_t = \langle \mathbf{W} \rangle_t = t$. Hence $\tilde{\mathbf{W}}_t^{(k)}$ is $\mathcal{G}_t$ standard Brownian motion owing to Lévy's theorem under the probability measure $\mathcal{P}^{(k)}$. Then the portfolio value at terminal-time T, using the $\mathcal{G}_t$-predictable drift $\boldsymbol{m}_t^{(k)}$ and portfolios $\boldsymbol{b}_\bullet$, is given by

$$V_T(\boldsymbol{b}_\bullet) \ = \ \exp\left[-\frac{1}{2}\int_0^T \boldsymbol{b}_u' \boldsymbol{\Sigma}\boldsymbol{\Sigma}' \boldsymbol{b}_u du + \int_0^T \boldsymbol{b}_u' \left(\boldsymbol{m}_u^{(k)} du + \boldsymbol{\Sigma} d\tilde{\mathbf{W}}_u^{(k)}\right)\right], \ k \geq 1 \ . \tag{3.10}$$

Let $\boldsymbol{b}_t^{\dagger(k)}$ be the optimal portfolio under $\mathcal{P}^{(k)}$ for the problem $\mathbf{P}_2$, which is based on the continuous Bayesian formula (referred to as the CBOP) under the finite uniform distribution. Then $\boldsymbol{b}_t^{\dagger(k)}$ satisfies the KKT condition (3.6) in which $\boldsymbol{m}_t^{(k)}$ is substituted for $\tilde{\boldsymbol{\mu}}_t$. That is $(\boldsymbol{b}_t^{\dagger(k)}, \eta_t^{\dagger(k)}, \boldsymbol{\nu}_t^{\dagger(k)})$ satisfying

$$\begin{cases} \boldsymbol{\Sigma}\boldsymbol{\Sigma}' \boldsymbol{b}_t^{\dagger(k)} + \eta_t^{\dagger(k)} \mathbf{1} - \boldsymbol{\nu}_t^{\dagger(k)} = \boldsymbol{m}_t^{(k)} \ , \\ \boldsymbol{b}_t^{\dagger(k)\prime} \mathbf{1} = 1 \ , \\ \boldsymbol{\nu}_t^{\dagger(k)\prime} \boldsymbol{b}_t^{\dagger(k)} = 0, \ \boldsymbol{\nu}_t^{\dagger(k)} \geq \mathbf{0} \ , \end{cases} \tag{3.11}$$

where $\eta_t^{\dagger(k)}$ and $\boldsymbol{\nu}_t^{\dagger(k)}$ are Lagrange multipliers associated with $\boldsymbol{m}_t^{(k)}$. From (3.8), $\boldsymbol{m}_t^{(k)} \to \boldsymbol{m}_t$ in the limit. Then $(\boldsymbol{b}_t^{\dagger(k)}, \eta_t^{\dagger(k)}, \boldsymbol{\nu}_t^{\dagger(k)})$ converges to the solution of (3.11) in which $\boldsymbol{m}_t$ is substituted for $\boldsymbol{m}_t^{(k)}$. That is $(\boldsymbol{b}_t^\dagger, \eta_t^\dagger, \boldsymbol{\nu}_t^\dagger)$ satisfying

$$\begin{cases} \boldsymbol{\Sigma\Sigma}'\boldsymbol{b}_t^\dagger + \eta_t^\dagger \mathbf{1} - \boldsymbol{\nu}_t^\dagger = \boldsymbol{m}_t \ , \\ \boldsymbol{b}_t^{\dagger'}\mathbf{1} = 1 \ , \\ \boldsymbol{\nu}_t^{\dagger'}\boldsymbol{b}_t^\dagger = 0, \ \boldsymbol{\nu}_t^\dagger \geq \mathbf{0} \ , \end{cases} \tag{3.12}$$

where $\eta_t^\dagger$, and $\boldsymbol{\nu}_t^\dagger$ are Lagrange multipliers associated with $\boldsymbol{m}_t$. And $\boldsymbol{b}_t^\dagger$ is the asymptotic form of the optimal portfolio based on the continuous Bayesian formula (referred to as the ACBOP).

The problem $\mathbf{P}_2$ without convex constraints for the portfolio weights is explicitly solved by Lakner[36, 37]. In Lakner's approach, the filtering techniques are introduced to treat $\mathbf{P}_2$, i.e. the optional projection of the Radon-Nikodym derivative to $\mathcal{G}_t$ is considered, and an explicit solution of $\mathbf{P}_2$ is derived. Also in the complete market with complete information $\mathcal{F}_t$, explicit solutions without convex constraints are obtained in [28] and with convex constraints are obtained in [9]. And recently, under incomplete information, explicit solutions are also obtained in [30], using both martingale and partial differential equation methodologies. Here we emphasize that the optimal solution of $\mathbf{P}_2$ can be also obtained by the SPOP. The advantage of the SPOP is that the optimal portfolio under incomplete information can be obtained by just maximizing the sample path-wise value, and then it can be directly applicable in actual investments. In the following theorem, among all the $\mathcal{G}_t$-predictable portfolios, we prove the optimality of both the CBOP $\boldsymbol{b}_t^{\dagger(k)}$, and ACBOP $\boldsymbol{b}_t^\dagger$ which is the asymptotic form of CBOP.

Theorem 2 (Optimality of $\boldsymbol{b}_t^\dagger$)

Under Information 2 and with the observable s.d.e. (3.9), the asymptotic optimal portfolio is given by the ACBOP, $\boldsymbol{b}_\bullet^\dagger$, for any finite terminal-time T. That is

$$(\ \{\forall \mathcal{G}_t\text{−predictable } \boldsymbol{b}_t \in \mathbf{D} \ ; \ 0 \leq t \leq T \ \} \ ) \Big(\ E^{(k)}[u(\ V_T(\boldsymbol{b}_\bullet) \ )] \leq E^{(k)}[u(\ V_T(\boldsymbol{b}_\bullet^{\dagger(k)}) \ )] \ \Big) \ ,$$

and

$$\lim_{k\to\infty} E^{(k)}[u(\ V_T(\boldsymbol{b}_\bullet^\dagger) \ )] = \lim_{k\to\infty} E^{(k)}[u(\ V_T(\boldsymbol{b}_\bullet^{\dagger(k)}) \ )] \ .$$

Proof. Since $\log(\cdot)$ is strictly concave, we have

$$E^{(k)}[\log(V_T(\boldsymbol{b}_\bullet))] \leq E^{(k)}[\log(V_T(\boldsymbol{b}_\bullet^{\dagger(k)}))] + E^{(k)}\left[\frac{V_T(\boldsymbol{b}_\bullet)}{V_T(\boldsymbol{b}_\bullet^{\dagger(k)})}\right] - 1$$

Then, it is necessary and sufficient to prove $E^{(k)}\left[\frac{V_T(\boldsymbol{b}_\bullet)}{V_T(\boldsymbol{b}_\bullet^{\dagger(k)})}\right] \leq 1$. We can easily see that

$$\begin{aligned}
\frac{V_T(\boldsymbol{b}_\bullet)}{V_T(\boldsymbol{b}_\bullet^{\dagger(k)})} &= \exp\left[\int_0^T \left\{\left(\boldsymbol{b}_t - \boldsymbol{b}_t^{\dagger(k)}\right)' \boldsymbol{m}_t^{(k)} - \frac{1}{2}\left(\boldsymbol{b}_t'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b}_t - \boldsymbol{b}_t^{\dagger(k)'}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b}_t^{\dagger(k)}\right)\right\} dt \right. \\
&\quad \left. + \int_0^T \left(\boldsymbol{b}_t - \boldsymbol{b}_t^{\dagger(k)}\right)' \boldsymbol{\Sigma} d\tilde{\mathbf{W}}_t^{(k)}\right] \\
&= A \cdot \exp\left[\int_0^T \left(\boldsymbol{b}_t - \boldsymbol{b}_t^{\dagger(k)}\right)' \left(\boldsymbol{m}_t^{(k)} - \boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b}_t^{\dagger(k)}\right) dt\right], \qquad (3.13)
\end{aligned}$$

where

$$A \triangleq \exp\left[-\frac{1}{2}\int_0^T \left(\boldsymbol{b}_t - \boldsymbol{b}_t^{\dagger(k)}\right)' \boldsymbol{\Sigma}\boldsymbol{\Sigma}' \left(\boldsymbol{b}_t - \boldsymbol{b}_t^{\dagger(k)}\right) dt + \int_0^T \left(\boldsymbol{b}_t - \boldsymbol{b}_t^{\dagger(k)}\right)' \boldsymbol{\Sigma} d\tilde{\mathbf{W}}_t^{(k)}\right] .$$

The last term of (3.13) is

$$\begin{aligned}
&\exp\left[\int_0^T \left(\boldsymbol{b}_t - \boldsymbol{b}_t^{\dagger(k)}\right)' \left(\boldsymbol{m}_t^{(k)} - \boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b}_t^{\dagger(k)}\right) dt\right] \\
&= \exp\left[\int_0^T \left(\boldsymbol{b}_t - \boldsymbol{b}_t^{\dagger(k)}\right)' \left\{\boldsymbol{m}_t^{(k)} - \left(\boldsymbol{m}_t^{(k)} - \eta_t^{\dagger(k)}\mathbf{1} + \boldsymbol{\nu}_t^{\dagger(k)}\right)\right\} dt\right] \\
&= \exp\left[-\int_0^T \boldsymbol{b}_t'\boldsymbol{\nu}_t^{\dagger(k)} dt\right] \\
&\leq 1 . \qquad (3.14)
\end{aligned}$$

The second equality follows from the constraint $\boldsymbol{b}_t'\mathbf{1} = \boldsymbol{b}_t^{\dagger(k)'}\mathbf{1} = 1$, and the KKT condition $\boldsymbol{b}_t^{\dagger(k)'}\boldsymbol{\nu}_t^{\dagger(k)} = 0$ in Eq. (3.11). And the inequality follows from the constraint $\boldsymbol{b}_t \geq \mathbf{0}$, and the KKT condition $\boldsymbol{\nu}_t^{\dagger(k)} \geq \mathbf{0}$. Then we can rewrite (3.13) as

$$\frac{V_T(\boldsymbol{b}_\bullet)}{V_T(\boldsymbol{b}_\bullet^{\dagger(k)})} \leq A.$$

Since $\boldsymbol{b}_t - \boldsymbol{b}_t^{\dagger(k)}$ is uniformly bounded, we can show $E^{(k)}[A] = 1$. Then we have

$$E^{(k)}\left[\frac{V_T(\boldsymbol{b}_\bullet)}{V_T(\boldsymbol{b}_\bullet^{\dagger(k)})}\right] \leq 1 .$$

Since $\lim_{k\to\infty} \boldsymbol{b}_t^{\dagger(k)} = \boldsymbol{b}_t^{\dagger}$ for all t, we can easily see that

$$\lim_{k\to\infty} \left| E^{(k)}[u(\ V_T(\boldsymbol{b}_\bullet^{\dagger})\ )] - E^{(k)}[u(\ V_T(\boldsymbol{b}_\bullet^{\dagger(k)})\ )] \right| = 0\ .$$

$\square$

This result asserts that both the CBOP and ACBOP maximize the expected log-utility under $\mathcal{P}^{(k)}$. Furthermore, recalling $\boldsymbol{m}_t = \tilde{\boldsymbol{\mu}}_t$ from (3.5), (3.8), and also $\boldsymbol{b}_t^{**} = \boldsymbol{b}_t^{\dagger}$ from (3.6), (3.12), we obtain that $\boldsymbol{b}_t^{\dagger(k)}$ coincides with $\boldsymbol{b}_t^{**}$ asymptotically, if we choose the prior distribution for $\boldsymbol{\mu}$ which has infinite differential entropy. Since the prior distribution with infinite differential entropy means no prior information, we can conclude that the SPOP is asymptotically optimal solution for $\mathbf{P}_2$.

Chapter 4

The Universal Portfolio

In this chapter, as we restrict the investors' utility class only to the log-utility, we introduce another scheme, called *universal portfolios*, which is optimal among all the $\mathcal{G}_t$-predictable portfolios. To prove its optimality, however, we require more relaxed constraints for portfolio weights, than the simplex $\mathbf{D}$ defined in (2.3). Also, the universal portfolio almost surely learns the ExPow portfolio over the long run, but we leave this asymptotic analysis to Chapter 5.

4.1 The Optimality of the Universal Portfolio

As in the foregoing discussion, we consider the market with m assets. And again, we suppose investors are allowed to observe a realization of asset price process Eq. (2.1), only once. That is, they are provided with Information 2. Then the investors, having the log-utility of (3.2), continuously select the optimal portfolios within all the $\mathcal{G}_t$-predictable portfolios. However, the portfolio selection is made within more relaxed constraints for portfolio weights than the simplex $\mathbf{D}$ of (2.3). In this chapter, the simplex is defined, without the non-negativity requirement for portfolio weights, as follows: [1]

$$\mathbf{D}' \stackrel{\triangle}{=} \{\boldsymbol{b} \mid \boldsymbol{b} \in \mathbf{R}^m \} \ . \tag{4.1}$$

Under the settings above, we introduce the *universal portfolio*, which is optimal among all the $\mathcal{G}_t$-predictable portfolios. Hereafter, we abbreviate the universal portfolio to the *UP*.

Define the UP at time t as

$$\boldsymbol{b}_t^{\sharp} \ \stackrel{\triangle}{=} \ \int_{\boldsymbol{b} \in \mathbf{D}'} \boldsymbol{b} f_t(\boldsymbol{b}) d\boldsymbol{b} \qquad (t > 0) \ , \tag{4.2}$$

where

$$f_t(\boldsymbol{b}) \stackrel{\triangle}{=} \frac{V_t(\boldsymbol{b})}{\int_{\boldsymbol{b} \in \mathbf{D}'} V_t(\boldsymbol{b}) d\boldsymbol{b}} \tag{4.3}$$

is the weighting density function of constant portfolios, and where $V_t(\boldsymbol{b})$ is given in the objective function of the problem $\mathbf{P}_2(t)$. The following theorem asserts that the UP can be used as one of the schemes which can derive the optimal portfolio among all the $\mathcal{G}_t$-predictable portfolios.

Theorem 3

Under incomplete Information 2 and $\mathbf{D}'$*, the UP coincides with the SPOP.*

$$\boldsymbol{b}_t^{\sharp} \equiv \boldsymbol{b}_t^{**} \ (t > 0) \ .$$

Hence the ACBOP, $\boldsymbol{b}_{\bullet}^{\dagger}$*, is given by the UP. That is,*

$$(\ \{\forall \mathcal{G}_t\text{−predictable } \boldsymbol{b}_t \in \mathbf{D}' \ ; \ 0 \le t \le T \ \} \ ) \Big(\ E[u(\ V_T(\boldsymbol{b}_{\bullet}) \ )] \le E[u(\ V_T(\boldsymbol{b}_{\bullet}^{\sharp}) \ )] \ \Big) \ .$$

Proof. From (4.3) and $V_t(\boldsymbol{b})$ which is given as the objective function of the problem $\mathbf{P}_2(t)$, the weighting density function is

$$f_t(\boldsymbol{b})$$

[1] Dropping off the budget constraint $\boldsymbol{b}_t'\mathbf{1} = 1$ in $\mathbf{D}$ of (2.3) seems a singular leap of discussion. But note that one can incorporate this constraint by introducing a safety asset which does not include stochastic terms and can be lent or borrowed in the market. The reason for this leap of discussion is that we do not treat a safety asset in the following empirical analysis, and we make our theory consistent with it.

$$
\begin{aligned}
&= CV_t(\boldsymbol{b}) \\
&= C \exp\left[\left(\boldsymbol{b}'\boldsymbol{\mu} - \frac{1}{2}\boldsymbol{b}'\boldsymbol{\Sigma\Sigma}'\boldsymbol{b}\right)t + \boldsymbol{b}'\boldsymbol{\Sigma W}_t\right] \\
&= C' \exp\left[-\frac{1}{2}\left\{\boldsymbol{b} - (\boldsymbol{\Sigma\Sigma}')^{-1}\tilde{\boldsymbol{\mu}}_t\right\}'(\boldsymbol{\Sigma\Sigma}'t)\left\{\boldsymbol{b} - (\boldsymbol{\Sigma\Sigma}')^{-1}\tilde{\boldsymbol{\mu}}_t\right\}\right] ,
\end{aligned}
$$

where $\tilde{\boldsymbol{\mu}}_t \triangleq \boldsymbol{\mu} + \frac{1}{t}\boldsymbol{\Sigma W}_t$ of (3.5). And where C and C' are constants. This means that $\boldsymbol{b}$ follows a multivariate normal distribution :

$$
N\left((\boldsymbol{\Sigma\Sigma}')^{-1}\tilde{\boldsymbol{\mu}}_t,\ \frac{1}{t}(\boldsymbol{\Sigma\Sigma}')^{-1}\right).
$$

Hence the universal portfolio $\boldsymbol{b}_t^\sharp$ is given by the mean, that is

$$
\boldsymbol{b}_t^\sharp = \int_{\mathbf{D}'} \boldsymbol{b} f_t(\boldsymbol{b}) d\boldsymbol{b} = (\boldsymbol{\Sigma\Sigma}')^{-1}\tilde{\boldsymbol{\mu}}_t .
$$

Whilst, the KKT condition of the SPOP under $\mathbf{D}'$ is given by (3.6), with $\boldsymbol{\nu}_t^{**}$ and η_t^{**} being dropped off. Hence the SPOP is given by the UP:

$$
\boldsymbol{b}_t^{**} = \boldsymbol{b}_t^\sharp .
$$

Moreover, Theorem 2 still holds under $\mathbf{D}'$. Then we have

$$
\boldsymbol{b}_t^\dagger = \boldsymbol{b}_t^{**} = \boldsymbol{b}_t^\sharp .
$$

This completes the proof. □

Remark 3

For the notational convenience, $\boldsymbol{b}_t^{**}$ *stands for both the SPOP and UP, in the rest of this chapter. And* $\log V_T^{**}$ *represents the logarithm of the portfolio value given by both portfolios, at* T.

4.2 Evaluating the Expected Log-Utility Gap among the SPOP, UP, and ExLog Portfolio

In this section, we evaluate the expected log-utility gap among the SPOP, the UP, and the ExLog portfolio, at finite terminal-time T. To do this, we need a certain number of periods in which all the investors have the same constant portfolio $\boldsymbol{b}^L$, regardless of their portfolio criteria. We call such a period *the learning period* and it is denoted by $[0, L]$, where $0 < L \leq T$. With learning periods and by (2.6) with η^* and $\boldsymbol{\nu}^*$ being eliminated, the ExLog ($\alpha = 1$) portfolio $\{\boldsymbol{b}^* \ ; \ 0 \leq t\}$ is given by:

$$\boldsymbol{b}_t^* = \begin{cases} \boldsymbol{b}^L, & 0 \leq t \leq L, \\ (\boldsymbol{\Sigma\Sigma}')^{-1}\boldsymbol{\mu}, & L \leq t, \end{cases} .$$

And at $T \ (\geq L)$, the expected value of the log-utility using the ExLog portfolio is given by:

$$E[\log V_T^*] = \left(\boldsymbol{b}^{L\prime}\boldsymbol{\mu} - \frac{1}{2}\boldsymbol{b}^{L\prime}\boldsymbol{\Sigma\Sigma}'\boldsymbol{b}^L\right) L + \frac{1}{2}\boldsymbol{\mu}'(\boldsymbol{\Sigma\Sigma}')^{-1}\boldsymbol{\mu}(T - L) \ . \tag{4.4}$$

Also by Theorem 3, the SPOP $\{\boldsymbol{b}^{**} \ ; \ 0 \leq t\}$ is given by:

$$\boldsymbol{b}_t^{**} = \begin{cases} \boldsymbol{b}^L, & 0 \leq t \leq L, \\ (\boldsymbol{\Sigma\Sigma}')^{-1}\tilde{\boldsymbol{\mu}}_t, & L \leq t, \end{cases} .$$

Then we have the following theorem concerning the expected log gap.

Theorem 4

For $T \geq L$, the expected log-utility using the SPOP or the UP is given by:

$$E[\log V_T^{**}] = \left(\boldsymbol{b}^{L\prime}\boldsymbol{\mu} - \frac{1}{2}\boldsymbol{b}^{L\prime}\boldsymbol{\Sigma\Sigma}'\boldsymbol{b}^L\right) L + \frac{1}{2}\boldsymbol{\mu}'(\boldsymbol{\Sigma\Sigma}')^{-1}\boldsymbol{\mu}(T - L) - \frac{m}{2}(\log T - \log L). \tag{4.5}$$

Hence the expected log-utility gap between the $\mathcal{F}_t$-predictable ExPow portfolio, and the $\mathcal{G}_t$-predictable SPOP or the UP is given by :

$$E[\log V_T^*] - E[\log V_T^{**}] = \frac{m}{2}(\log T - \log L), \quad T \geq L. \tag{4.6}$$

Proof. We can easily see that

$$
\begin{aligned}
E[\boldsymbol{b}_u^{**}] &= (\boldsymbol{\Sigma\Sigma'})^{-1}\boldsymbol{\mu}, \\
E[\boldsymbol{b}_u^{**'}\boldsymbol{\Sigma\Sigma'}\boldsymbol{b}_u^{**}] &= E\left[\left(\boldsymbol{\mu'} + \frac{1}{u}\boldsymbol{W}'_u\boldsymbol{\Sigma'}\right)(\boldsymbol{\Sigma\Sigma'})^{-1}\boldsymbol{\Sigma\Sigma'}(\boldsymbol{\Sigma\Sigma'})^{-1}\left(\boldsymbol{\mu} + \frac{1}{u}\boldsymbol{\Sigma W}_u\right)\right] \\
&= \boldsymbol{\mu'}(\boldsymbol{\Sigma\Sigma'})^{-1}\boldsymbol{\mu} + \frac{m}{u}.
\end{aligned}
$$

Then by utilizing the above, we obtain

$$
\begin{aligned}
E[\log V_T^{**}] &= E\left[\left(\boldsymbol{b}^{L'}\boldsymbol{\mu} - \frac{1}{2}\boldsymbol{b}^{L'}\boldsymbol{\Sigma\Sigma'}\boldsymbol{b}^{L}\right)L + \boldsymbol{b}^{L'}\boldsymbol{\Sigma}\mathbf{W}_L \right. \\
&+ \left. \int_L^T \left(\boldsymbol{b}_u^{**'}\boldsymbol{\mu} - \frac{1}{2}\boldsymbol{b}_u^{**'}\boldsymbol{\Sigma\Sigma'}\boldsymbol{b}_u^{**}\right)du + \int_L^T \boldsymbol{b}_u^{**'}\boldsymbol{\Sigma}d\mathbf{W}_u\right] \\
&= E[\log V_T^{*}] - \frac{m}{2}(\log T - \log L)\ .
\end{aligned}
$$

□

Remark 4

If $L \to 0$, then $E[\log V_t^{**}] \to -\infty$. This means that we need the strict positive learning periods to get the finite expected log-utility with $\boldsymbol{b}_\bullet^{**}$.

4.3 The Unbiased Estimator Approach

As in the previous chapter, by restricting the degree of Arrow-Pratt's relative risk aversion only to $\alpha = 1$, we show the superiority of both the SPOP and UP from another aspect. That is, we study the standard unbiased estimator for the optimal portfolio process and compare the performance with both the SPOP and UP. We shall discretize the time interval $[0, T]$ into N units $dt = \frac{T}{N}$ and approximate the return process by :

$$
\frac{\Delta S_{i,n}^{(N)}}{S_{i,n}^{(N)}} = \mu_i \frac{T}{N} + \sum_{1 \le j \le m} \sigma_{i,j}\left(W_{j,\frac{T}{N}n} - W_{j,\frac{T}{N}(n-1)}\right). \tag{4.7}
$$

Then the discretized return process $\frac{\Delta S_{i,n}^{(N)}}{S_{i,n}^{(N)}}$ follows the normal distribution $N(\boldsymbol{\mu}\frac{T}{N},\ \boldsymbol{\Sigma\Sigma'}\frac{T}{N})$. It is easy to see that the discretized price process $\{\boldsymbol{S}_{[\frac{tN}{T}]}^{(N)}; 0 \le t \le T\}$ converges to the continuous price process of Eq. (2.1). Thus, under Incomplete Information 2, we can construct the unbiased estimator for the mean vector $\boldsymbol{\mu}\frac{T}{N}$ and the variance-covariance matrix $\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\frac{T}{N}$ by

$$\begin{aligned}\hat{\mu}_{i,n}^{(N)} &= \frac{1}{n}\sum_{l=1}^{n}\frac{\Delta S_{i,l}^{(N)}}{S_{i,l}^{(N)}}, \quad 1 \le i \le m, \\ \hat{\sigma}_{i,j,n}^{(N)} &= \frac{1}{n-1}\sum_{l=1}^{n}\left(\frac{\Delta S_{i,l}^{(N)}}{S_{i,l}^{(N)}} - \hat{\mu}_{i,l}^{(N)}\right)\left(\frac{\Delta S_{j,l}^{(N)}}{S_{j,l}^{(N)}} - \hat{\mu}_{j,l}^{(N)}\right), \quad 1 \le i,j \le m.\end{aligned}$$

The following properties are well-known for these estimators (e.g., see [2, 26, and 43]).

Lemma 1

$\hat{\boldsymbol{\mu}}_n^{(N)}$ *and* $\hat{\boldsymbol{\Sigma}}_n^{(N)}$ *are mutually independent, and each follows*

$$\begin{aligned}\hat{\boldsymbol{\mu}}_n^{(N)} &\sim N\left(\frac{T}{N}\boldsymbol{\mu}, \frac{T}{nN}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right), \\ \hat{\boldsymbol{\Sigma}}_n^{(N)} &= \frac{T}{(n-1)N}\boldsymbol{W}_{m,n-1}^{(N)}, \quad \boldsymbol{W}_{m,n}^{(N)} \sim \boldsymbol{W}_m\left(n, \boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right),\end{aligned}$$

where $W_m(n, \boldsymbol{A})$ *denotes the Wishart distribution with dimension* m*, degree of freedom* n *and variance-covariance matrix* $\boldsymbol{A}$*.* □

Lemma 2

Let $\boldsymbol{X} \sim W_m(n, \boldsymbol{A})$*. Then*

$$\begin{aligned}E[X] &= n\boldsymbol{A}, \\ E[X^{-1}] &= \frac{1}{n-m-1}\boldsymbol{A}^{-1}.\end{aligned}$$

□

Lemma 3

Let $\boldsymbol{X} \sim W_m(n, \boldsymbol{A})$ *and* $\boldsymbol{C}$ *is* $m \times m$ *nonsingular matrix. Then* $\boldsymbol{Y} = \boldsymbol{C}'\boldsymbol{X}\boldsymbol{C} \sim W_m(n, \boldsymbol{C}'\boldsymbol{A}\boldsymbol{C})$*.*

□

Before we proceed to derive the unbiased optimal portfolio estimator, we quote the some results without proof which are easy to prove.

Lemma 4

Let $\boldsymbol{b}$ be n dimensional arbitrary given vector and $\boldsymbol{A}$ be a square matrix. Then the function $f(\boldsymbol{A}) = \boldsymbol{b}'\boldsymbol{A}'\boldsymbol{A}\boldsymbol{b}$ is convex with respect to $\boldsymbol{A}$. □

Lemma 5

Let $\boldsymbol{A}$ be a symmetric matrix and $\boldsymbol{X}$ be a random vector such that

$$\begin{cases} \boldsymbol{\mu} &= E[\boldsymbol{X}], \\ \boldsymbol{\Sigma} &= E[(\boldsymbol{X}-\boldsymbol{\mu})'(\boldsymbol{X}-\boldsymbol{\mu})]. \end{cases}$$

Then the mean of quadratic form $Y = \boldsymbol{X}'\boldsymbol{A}\boldsymbol{X}$ is given by

$$E[Y] = \boldsymbol{\mu}'\boldsymbol{A}\boldsymbol{\mu} + trace[\boldsymbol{A}\boldsymbol{\Sigma}].$$

□

From Lemma 1 through 5, we can evaluate the mean and the quadratic form of the estimated optimal control process : $(\hat{\boldsymbol{\Sigma}}_n^{(N)})^{-1}\hat{\boldsymbol{\mu}}_n^{(N)}$.

Theorem 5

$$1) \quad E\left[\left(\hat{\boldsymbol{\Sigma}}_n^{(N)}\right)^{-1}\hat{\boldsymbol{\mu}}_n^{(N)}\right] = \frac{n-1}{n-m-2}\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\mu}, \tag{4.8}$$

$$\begin{aligned} 2) \quad & E\left[\hat{\boldsymbol{\mu}}_n^{(N)\prime}\left(\hat{\boldsymbol{\Sigma}}_n^{(N)}\right)^{-1}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\left(\hat{\boldsymbol{\Sigma}}_n^{(N)}\right)^{-1}\hat{\boldsymbol{\mu}}_n^{(N)}\right] \\ & \geq \left(\frac{n-1}{n-m-2}\right)^2\left[\boldsymbol{\mu}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\mu} + \frac{mN}{nT}\right]. \end{aligned} \tag{4.9}$$

Expecially if $m = 1$, then

$$E\left[\left(\frac{\hat{\mu}_{1,n}^{(N)}\sigma_{11}}{\hat{\sigma}_{11,n}^{2(N)}}\right)^2\right] = \frac{(n-1)^2}{(n-3)(n-5)}\left(\frac{\mu_1^2}{\sigma_{11}^2} + \frac{N}{nT}\right). \tag{4.10}$$

Proof. 1) From Lemma 2,

$$E[(\hat{\boldsymbol{\Sigma}}_n^{(N)})^{-1}\hat{\boldsymbol{\mu}}_n^{(N)}] = E[(\hat{\boldsymbol{\Sigma}}_n^{(N)})^{-1}]E[\hat{\boldsymbol{\mu}}_n^{(N)}]$$

$$
\begin{aligned}
&= \frac{(n-1)T}{N} E[(\boldsymbol{W}_{m,n-1}^{(N)})^{-1}]\boldsymbol{\mu} \\
&= \frac{n-1}{n-m-2} (\boldsymbol{\Sigma}\boldsymbol{\Sigma}')^{-1}\boldsymbol{\mu}.
\end{aligned}
$$

2) Let

$$
\begin{aligned}
\boldsymbol{Z} &= \frac{T}{(n-1)N} \boldsymbol{\Sigma}'(\hat{\boldsymbol{\Sigma}}_n^{(N)})^{-1}\boldsymbol{\Sigma}, \\
M &= \hat{\boldsymbol{\mu}}_n^{(N)\prime}(\hat{\boldsymbol{\Sigma}}_n^{(N)})^{-1}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'(\hat{\boldsymbol{\Sigma}}_n^{(N)})^{-1}\hat{\boldsymbol{\mu}}_n^{(N)} \\
&= \frac{((n-1)N)^2}{T^2} \|\boldsymbol{Z}\boldsymbol{\Sigma}^{-1}\hat{\boldsymbol{\mu}}_n^{(N)}\|^2.
\end{aligned}
$$

Then from Lemma 1,

$$
\begin{aligned}
E[M] &= \frac{((n-1)N)^2}{T^2} E[\|\boldsymbol{Z}\boldsymbol{\Sigma}^{-1}\hat{\boldsymbol{\mu}}_n^{(N)}\|^2] \\
&= \frac{((n-1)N)^2}{T^2} E[E[\|\boldsymbol{Z}\boldsymbol{\Sigma}^{-1}\hat{\boldsymbol{\mu}}_n^{(N)}\|^2 \mid \hat{\boldsymbol{\mu}}_n^{(N)}]] \\
&\geq \frac{((n-1)N)^2}{T^2} E[\|E[\boldsymbol{Z}\boldsymbol{\Sigma}^{-1}\hat{\boldsymbol{\mu}}_n^{(N)}|\hat{\boldsymbol{\mu}}_n^{(N)}]\|^2] \\
&= \frac{((n-1)N)^2}{T^2} E[\|E[\boldsymbol{Z}]\boldsymbol{\Sigma}^{-1}\hat{\boldsymbol{\mu}}_n^{(N)}\|^2].
\end{aligned}
$$

Here the inequality follows from Lemma 4 and the Jensen's inequality. Furthermore from Lemma 3, $\boldsymbol{Z}$ follows the m dimensional Inverted Wishart distribution $\boldsymbol{W}_m^{-1}(n-1, \boldsymbol{I})$. Hence from Lemma 2,

$$
E[\boldsymbol{Z}] = \frac{1}{n-m-2}\boldsymbol{I}.
$$

This together with Lemma 5 yields

$$
\begin{aligned}
E[M] &\geq \frac{((n-1)N)^2}{((n-m-2)T)^2} E[\hat{\boldsymbol{\mu}}_n^{(N)\prime}(\boldsymbol{\Sigma}\boldsymbol{\Sigma}')^{-1}\hat{\boldsymbol{\mu}}_n^{(N)}] \\
&= \frac{((n-1)N)^2}{((n-m-2)T)^2} \left\{ \frac{T^2}{N^2}\boldsymbol{\mu}'(\boldsymbol{\Sigma}\boldsymbol{\Sigma}')^{-1}\boldsymbol{\mu} + \frac{T}{nN} trace(\boldsymbol{I}) \right\} \\
&= \frac{(n-1)^2}{(n-m-2)^2} \left\{ \boldsymbol{\mu}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\mu} + \frac{mN}{nT} \right\}.
\end{aligned}
$$

Especially when m=1, $\boldsymbol{Z}^{-1}$ follows $W_1(n-1,1)$, that is Chi-square distribution with $n-1$ degree of freedom. Therefore

$$E[\boldsymbol{Z}^2] = E\left[\frac{1}{\chi^4}\right] = \frac{1}{(n-3)(n-5)}.$$

Thus

$$\begin{aligned} E[M] &= \frac{((n-1)N)^2}{T^2} E[\|\boldsymbol{Z}\boldsymbol{\Sigma}^{-1}\hat{\boldsymbol{\mu}}_n^{(N)}\|^2 | \hat{\boldsymbol{\mu}}_n^{(N)}] \\ &= \frac{((n-1)N)^2}{T^2} E[\boldsymbol{Z}^2] E\left[\left(\frac{\hat{\boldsymbol{\mu}}_n^{(N)}}{\sigma_{11}}\right)^2\right] \\ &= \frac{(n-1)^2}{(n-3)(n-5)}\left(\frac{\mu_1^2}{\sigma_{11}^2} + \frac{N}{nT}\right). \end{aligned}$$

□

Using Theorem 5, we can construct the unbiased optimal portfolio estimator by

$$\hat{\boldsymbol{b}}_n^{(N)} = \begin{cases} \boldsymbol{b}^L, & 0 \le n < n_L, \\ \frac{n-m-2}{n-1}(\hat{\boldsymbol{\Sigma}}_n^{(N)})^{-1}\hat{\boldsymbol{\mu}}_n^{(N)}, & n_L \le n \le N-1. \end{cases}$$

Then the log-utility under this unbiased estimator results in

$$\log \hat{V}_T^{(N)} = \left[\begin{array}{l} \left(\boldsymbol{b}^{L'}\boldsymbol{\mu} - \frac{1}{2}\boldsymbol{b}^{L'}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b}^L\right)L \\ +\sum_{n_L \le n \le N-1}\left\{\begin{array}{l} \left(\hat{\boldsymbol{b}}_n^{(N)'}\boldsymbol{\mu} - \frac{1}{2}\hat{\boldsymbol{b}}_n^{(N)'}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\hat{\boldsymbol{b}}_n^{(N)}\right)\frac{T}{N} \\ +\hat{\boldsymbol{b}}_n^{(N)'}\left(\boldsymbol{W}_{\frac{T}{N}(n+1)} - \boldsymbol{W}_{\frac{T}{N}n}\right) \end{array}\right\} \end{array}\right], \tag{4.11}$$

where $n_L = \min\{n \in \boldsymbol{Z}; n \ge \frac{LN}{T}\}$. Now we can show the superiority of both the SPOP and UP, compared to the unbiased estimator approach.

Theorem 6

The expected log-utility for the portfolio value process attained by unbiased estimator is bounded from above by the SPOP or the UP. That is,

$$\begin{aligned} & E\left[\log \hat{V}_T^{(N)}\right] \\ \le\ & \left(\boldsymbol{b}^{L'}\boldsymbol{\mu} - \frac{1}{2}\boldsymbol{b}^{L'}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b}^L\right)L + \frac{1}{2}\boldsymbol{\mu}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\mu}(T-L) - \frac{m}{2}(\log T - \log L) \quad (4.12) \\ =\ & E\left[\log V_T^{**}\right]. \end{aligned}$$

Hence the expected log-utility gap between the ExPow portfolio and the unbiased estimator portfolio is bounded from above by

$$E[\log \hat{V}_T^{(N)}] - E[\log V_T^*] \leq -\frac{m}{2}(\log T - \log L) = E[\log V_T^{**}] - E[\log V_T^*]. \tag{4.13}$$

Proof. Note that

$$E[\hat{\boldsymbol{b}}_n^{(N)'}(\boldsymbol{W}_{\frac{T}{N}(n+1)} - \boldsymbol{W}_{\frac{T}{N}n})|\mathcal{F}_{\frac{T}{N}n}] = 0.$$

Then from (4.11),

$$E\left[\log \hat{V}_T^{(N)}\right] = \left(\boldsymbol{b}^{L'}\boldsymbol{\mu} - \frac{1}{2}\boldsymbol{b}^{L'}\boldsymbol{\Sigma\Sigma}'\boldsymbol{b}^L\right)L + \sum_{n=n_L}^{N-1}\left\{E\left[\hat{\boldsymbol{b}}_n^{(N)}\right]'\boldsymbol{\mu} - \frac{1}{2}E\left[\hat{\boldsymbol{b}}_n^{(N)'}\boldsymbol{\Sigma\Sigma}'\hat{\boldsymbol{b}}_n^{(N)}\right]\right\}\frac{T}{N}.$$

Substituting (4.8) and (4.9) into the above equation, we get

$$\begin{aligned}
&E\left[\log \hat{V}_T^{(N)}\right] \\
=\ & \left(\boldsymbol{b}^{L'}\boldsymbol{\mu} - \frac{1}{2}\boldsymbol{b}^{L'}\boldsymbol{\Sigma\Sigma}'\boldsymbol{b}^L\right)L + \sum_{n=n_L}^{N-1}\left\{\frac{n-m-2}{n-1}E\left[\left(\hat{\boldsymbol{\Sigma}}_n^{(N)}\right)^{-1}\hat{\boldsymbol{\mu}}_n^{(N)}\right]'\boldsymbol{\mu}\right. \\
&\qquad \left. - \frac{1}{2}\left(\frac{n-m-2}{n-1}\right)^2 E\left[\hat{\boldsymbol{\mu}}_n^{(N)'}\left(\hat{\boldsymbol{\Sigma}}_n^{(N)}\right)^{-1}\boldsymbol{\Sigma\Sigma}'\left(\hat{\boldsymbol{\Sigma}}_n^{(N)}\right)^{-1}\hat{\boldsymbol{\mu}}_n^{(N)}\right]\right\}\frac{T}{N} \\
\leq\ & \left(\boldsymbol{b}^{L'}\boldsymbol{\mu} - \frac{1}{2}\boldsymbol{b}^{L'}\boldsymbol{\Sigma\Sigma}'\boldsymbol{b}^L\right)L + \boldsymbol{\mu}'\left(\boldsymbol{\Sigma\Sigma}'\right)^{-1}\boldsymbol{\mu}\sum_{n=n_L}^{N-1}\frac{T}{N} - \frac{1}{2}\sum_{n=n_L}^{N-1}\left\{\boldsymbol{\mu}'\left(\boldsymbol{\Sigma\Sigma}'\right)^{-1}\boldsymbol{\mu}\frac{T}{N} + \frac{m}{l}\right\} \\
\leq\ & \left(\boldsymbol{b}^{L'}\boldsymbol{\mu} - \frac{1}{2}\boldsymbol{b}^{L'}\boldsymbol{\Sigma\Sigma}'\boldsymbol{b}^L\right)L + \boldsymbol{\mu}'\left(\boldsymbol{\Sigma\Sigma}'\right)^{-1}\boldsymbol{\mu}\int_L^T d\ell - \frac{1}{2}\int_L^T\left\{\boldsymbol{\mu}'\left(\boldsymbol{\Sigma\Sigma}'\right)^{-1}\boldsymbol{\mu} + \frac{m}{\ell}\right\}d\ell \\
=\ & \left(\boldsymbol{b}^{L'}\boldsymbol{\mu} - \frac{1}{2}\boldsymbol{b}^{L'}\boldsymbol{\Sigma\Sigma}'\boldsymbol{b}^L\right)L + \frac{1}{2}\boldsymbol{\mu}'\left(\boldsymbol{\Sigma\Sigma}'\right)^{-1}\boldsymbol{\mu}(T-L) - \frac{m}{2}(\log T - \log L).
\end{aligned}$$

□

Finally, we establish the asymptotic attainability of the SPOP or the UP, as the number of observation increases for the two securities market model, i.e., $m = 1$.

Theorem 7

Let us assume $m = 1$. Then the expected log-utility for the portfolio value process attained by the unbiased estimator converges to that of the SPOP or the UP. That is,

$$\lim_{N\to\infty} E\left[\log \hat{V}_T^{(N)}\right] = E[\log V_T^{**}]. \tag{4.14}$$

Proof. From (4.8), (4.10) and (4.11),

$$
\begin{aligned}
&E\left[\log \hat{V}_T^{(N)}\right] \\
&= \left(b_1^L\mu_1 - \frac{1}{2}(\sigma_{11}b_1^L)^2\right)L + \sum_{n=n_L}^{N-1}\left\{\frac{n-3}{n-1}E\left[\frac{\hat{\mu}_1^{(N)}}{\hat{\sigma}_{11}^{(N)}}\right]\mu_1 - \frac{1}{2}\left(\frac{n-3}{n-1}\right)^2 E\left[\left(\frac{\sigma_{11}\hat{\mu}_1^{(N)}}{\hat{\sigma}_{11}^{(N)}}\right)^2\right]\right\}\frac{T}{N} \\
&= \left(b_1^L\mu_1 - \frac{1}{2}(\sigma_{11}b_1^L)^2\right)L + \frac{\mu_1^2}{\sigma_{11}^2}\left(T - \frac{n_L T}{N}\right) - \frac{1}{2}\sum_{n=n_L}^{N-1}\frac{n-3}{n-5}\left(\frac{\mu_1^2}{\sigma_{11}^2} + \frac{N}{nT}\right)\frac{T}{N}.
\end{aligned}
$$

Since

$$
\lim_{N\to\infty}\frac{n_L}{N} = \frac{L}{T},
$$

and

$$
\begin{aligned}
&\lim_{N\to\infty}\sum_{n=n_L}^{N-1}\frac{n-3}{n-5}\left(\frac{\mu_1^2}{\sigma_{11}^2} + \frac{N}{nT}\right)\frac{T}{N} \\
&= \int_L^T\left(\frac{\mu_1^2}{\sigma_{11}^2} + \frac{1}{\ell}\right)d\ell \\
&= \frac{\mu_1^2}{\sigma_{11}^2}(T-L) + \log T - \log L,
\end{aligned}
$$

we get

$$
\begin{aligned}
&E\left[\log \hat{V}_T^{(N)}\right] \\
&= \left(b_1^L\mu_1 - \frac{1}{2}(\sigma_{11}b_1^L)^2\right)L + \frac{\mu_1^2}{2\sigma_{11}^2}(T-L) - \frac{1}{2}(\log T - \log L) \\
&= E[\log V_T^{**}].
\end{aligned}
$$

□

Chapter 5

The Asymptotic Convergence to the ExPow Portfolio

In this chapter, as we extend the utility class from the log-utility to the general power-utility, we propose two schemes under the incomplete Information 2. One is the *α-scaled sample path-wise optimal portfolio* (αSPOP) which is the generalization of the SPOP. The other is the *α-scaled universal portfolio* (αUP) which is the generalization of the UP. It is shown that both the αSPOP and αUP, which are $\mathcal{G}_t$-predictable portfolios, converge to the $\mathcal{F}_t$-predictable ExPow portfolio, asymptotically. The outline in this chapter is as follows. First, under the complete Information 1, we represent the $\mathcal{F}_t$-predictable ExPow portfolio as the basic solution of the problem $\mathbf{P}'_1$. Second, we introduce the αSPOP under the incomplete Information 2. And we represent the αSPOP in the basic solution form. Then, by showing the optimal base of the αSPOP converges to that of the ExPow portfolio, we prove the αSPOP converges to the ExPow portfolio. Third, also under the incomplete Information 2, we propose the αUP. By showing the weighting density function converges to the Dirac's delta function, we prove the αUP converges to the ExPow portfolio.

The consistent settings through this chapter is as follows. As in the foregoing analysis, we consider the market with m assets. And the investors, having the power-utility of (2.2), select their portfolios within the simplex $\mathbf{D}$ of (2.3), not within $\mathbf{D}'$ of (4.1).

5.1 The Basic ExPow Portfolio

In this section, we suppose the power-utility investors are provided with the complete Information 1. Then the KKT condition for the problem $\mathbf{P}_1'$, which is treated in chapter 2, is given by (2.6). From (2.6), we consider the basic solution for the problem $\mathbf{P}_1'$. Define the set for the basic index for the problem $\mathbf{P}_1'$ as $\mathbf{B} = \{i_1, \ldots, i_B\} \in \mathbf{I} = \{1, \ldots, m\}$, and $\mathbf{B} \neq \phi$. Also, $\mathbf{N} = \mathbf{I} \setminus \mathbf{B}$. Using this definition, the basic solution of the problem is stated as

$$\left(\boldsymbol{b} = \begin{pmatrix} \boldsymbol{b}_{\mathbf{B}} \\ \boldsymbol{b}_{\mathbf{N}} \end{pmatrix}, \quad \eta_{\mathbf{B}}, \quad \boldsymbol{\nu} = \begin{pmatrix} \boldsymbol{\nu}_{\mathbf{B}} \\ \boldsymbol{\nu}_{\mathbf{N}} \end{pmatrix} \right).$$

So, the active basic solution for $\mathbf{P}_1'$ is

$$\mathbf{U}_{\mathbf{B}} \begin{pmatrix} \boldsymbol{b}_{\mathbf{B}} \\ \eta_{\mathbf{N}} \end{pmatrix} = \boldsymbol{v}_{\mathbf{B}},$$

where $\mathbf{U}_{\mathbf{B}} = \begin{bmatrix} \mathbf{A}_{\mathbf{B}} & \mathbf{1}_{\mathbf{B}} \\ \mathbf{1}_{\mathbf{B}}' & \mathbf{0} \end{bmatrix}$ and where $\mathbf{A}_{\mathbf{B}} = \alpha \left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}' \right)_{\mathbf{B}}$, and $\boldsymbol{v}_{\mathbf{B}} = \begin{bmatrix} \boldsymbol{\mu}_{\mathbf{B}} \\ 1 \end{bmatrix}$.
For the assurance of a unique solution, we need the following lemma.

Lemma 6 (Non-singularity)

If $\boldsymbol{\Sigma}$ is non-singular, then $U_{\mathbf{B}}$ is non-singular.

Proof. Let $\mathbf{A} = \boldsymbol{\Sigma}\boldsymbol{\Sigma}'$. Since $\boldsymbol{\Sigma}$ is non-singular and strictly positive definite, so $\mathbf{A}$ is. Let the eigen values and the eigen vectors of $\mathbf{A}$ be λ_i and ξ_i respectively, and for $i = 1, \ldots, m$, $\lambda_i > 0$, $\xi_i \xi_j = 1$ (if $i = j$), and $\xi_i \xi_j = 0$ (if $i \neq j$). Then we apply the spectral decomposition to $\mathbf{A}$, $\mathbf{A} = \sum_{i=1}^m \lambda_i \xi_i \xi_i'$. So we have $\mathbf{A}_{\mathbf{B}} = \sum_{i=1}^m \lambda_i \xi_{\mathbf{B},i} \xi_{\mathbf{B},i}'$, and $\mathbf{A}_{\mathbf{B}}$ is strictly positive definite and hence non-singular.
Suppose $(\exists \boldsymbol{x}, v)$ s.t. $\mathbf{U}_{\mathbf{B}} \begin{pmatrix} \boldsymbol{x} \\ v \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 0 \end{pmatrix}$. Then, $\mathbf{A}_{\mathbf{B}} \boldsymbol{x} = -v \mathbf{1}_{\mathbf{B}}$, and $\mathbf{1}_{\mathbf{B}}' \boldsymbol{x} = 0$. Since

$\mathbf{A_B}$ is non-singular, $\boldsymbol{x} = -\upsilon \mathbf{A_B}^{-1}\mathbf{1_B}$, then $\mathbf{1}'_{\mathbf{B}}\boldsymbol{x} = -\upsilon \mathbf{1}'_{\mathbf{B}}\mathbf{A_B}^{-1}\mathbf{1_B} = 0$. Furthermore, since $\mathbf{A_B}$ is positive definite, so $\mathbf{A_B}^{-1}$ is. Then $\mathbf{1}'_{\mathbf{B}}\mathbf{A_B}^{-1}\mathbf{1_B} > 0$, meaning $\upsilon = 0$ and $\boldsymbol{x} = -\upsilon \mathbf{A_B}^{-1}\mathbf{1_B} = 0$. Hence $\mathbf{U_B}$ is non-singular for $\mathbf{B} \subset \mathbf{I}$, $\mathbf{B} \neq \phi$. □

From Lemma 6, there always exists a unique solution for $\mathbf{P}'_1$. The basic solution for $\mathbf{P}'_1$ is

$$\boldsymbol{b}_{\mathbf{B}^*} = \mathbf{A}^{-1}_{\mathbf{B}^*}\left(\boldsymbol{\mu}_{\mathbf{B}^*} - \eta_{\mathbf{B}^*}\mathbf{1}_{\mathbf{B}^*}\right),$$

where $\mathbf{B}^*$ is the optimal base for the problem $\mathbf{P}'_1$. Since

$$\mathbf{1}'_{\mathbf{B}^*}\boldsymbol{b}_{\mathbf{B}^*} = \mathbf{1}'_{\mathbf{B}^*}\mathbf{A}^{-1}_{\mathbf{B}^*}\boldsymbol{\mu}_{\mathbf{B}^*} - \eta_{\mathbf{B}^*}\mathbf{1}'_{\mathbf{B}^*}\mathbf{A}^{-1}_{\mathbf{B}^*}\mathbf{1}_{\mathbf{B}^*} = 1\ ,$$

then

$$\eta_{\mathbf{B}^*} = \frac{\mathbf{1}'_{\mathbf{B}^*}\mathbf{A}^{-1}_{\mathbf{B}^*}\boldsymbol{\mu}_{\mathbf{B}^*} - 1}{\mathbf{1}'_{\mathbf{B}^*}\mathbf{A}^{-1}_{\mathbf{B}^*}\mathbf{1}_{\mathbf{B}^*}} = \frac{\mathbf{1}'_{\mathbf{B}^*}(\boldsymbol{\Sigma}\boldsymbol{\Sigma}')^{-1}_{\mathbf{B}^*}\boldsymbol{\mu}_{\mathbf{B}^*} - \alpha}{\mathbf{1}'_{\mathbf{B}^*}(\boldsymbol{\Sigma}\boldsymbol{\Sigma}')^{-1}_{\mathbf{B}^*}\mathbf{1}_{\mathbf{B}^*}}.$$

Thus, the basic portfolio is

$$\begin{aligned} \boldsymbol{b}_{\mathbf{B}^*} &= \mathbf{A}^{-1}_{\mathbf{B}^*}\boldsymbol{\mu}_{\mathbf{B}^*} - \eta_{\mathbf{B}^*}\mathbf{A}^{-1}_{\mathbf{B}^*}\mathbf{1}_{\mathbf{B}^*} \\ &= \frac{1}{\alpha}(\boldsymbol{\Sigma}\boldsymbol{\Sigma}')^{-1}_{\mathbf{B}^*}\boldsymbol{\mu}_{\mathbf{B}^*} - \frac{\eta_{\mathbf{B}^*}}{\alpha}(\boldsymbol{\Sigma}\boldsymbol{\Sigma}')^{-1}_{\mathbf{B}^*}\mathbf{1}_{\mathbf{B}^*}, \end{aligned} \tag{5.1}$$

$$\boldsymbol{\nu}_{\mathbf{N}^*} = \alpha(\boldsymbol{\Sigma}\boldsymbol{\Sigma}')_{\mathbf{N}^*}\boldsymbol{b}_{\mathbf{B}^*} + \eta_{\mathbf{B}^*}\mathbf{1}_{\mathbf{N}^*} - \boldsymbol{\mu}_{\mathbf{N}^*},$$

where $\mathbf{N}^* = \mathbf{I} \setminus \mathbf{B}^*$. Summarizing above results, if $\nu_{\mathbf{N}^*} \geq 0$ holds, the problem $\mathbf{P}'_1$ has the optimal solution as follows.

$$\alpha(\boldsymbol{\Sigma}\boldsymbol{\Sigma}')\begin{pmatrix} \boldsymbol{b}_{\mathbf{B}^*} \\ \mathbf{0} \end{pmatrix} + \eta_{\mathbf{B}^*}\mathbf{1} - \begin{pmatrix} \mathbf{0} \\ \boldsymbol{\nu}_{\mathbf{N}^*} \end{pmatrix} = \boldsymbol{\mu},$$

$$\mathbf{1}'\begin{pmatrix} \boldsymbol{b}_{\mathbf{B}^*} \\ \mathbf{0} \end{pmatrix} = 1, \quad (\mathbf{0}', \boldsymbol{\nu}'_{\mathbf{N}^*})\begin{pmatrix} \boldsymbol{b}_{\mathbf{B}^*} \\ \mathbf{0} \end{pmatrix} = 0, \quad \begin{pmatrix} \mathbf{0} \\ \boldsymbol{\nu}_{\mathbf{N}^*} \end{pmatrix} \geq \mathbf{0}.$$

And the optimal portfolio for the problem $\mathbf{P}'_1$ is

$$\boldsymbol{b}^* = \begin{pmatrix} \boldsymbol{b}_{\mathbf{B}^*} \\ \mathbf{0} \end{pmatrix}, \tag{5.2}$$

where $\boldsymbol{b}_{\mathbf{B}^*}$ is given in Eq. (5.1). And the existence of the optimal base $\mathbf{B}^*$ for $\mathbf{P}'_1$ is assumed in the following discussion.

5.2 The α-scaled Sample Path-Wise Optimal Portfolio and its Convergence

In this section, we suppose the power-utility investors are provided with the incomplete Information 2. And we introduce one scheme which can learn the $\mathcal{F}_t$-predictable ExPow portfolio in the long run. By analogy with the discussion in section 3.2, we first assume the power-utility investors use constant portfolios $\boldsymbol{b}_t = \boldsymbol{b}$ as the $\mathcal{G}_t$-predictable portfolios and derive the portfolio which maximize the sample path-wise value $V_t(\boldsymbol{b})$. Moreover, as we prove later, to assure this sample path-wise portfolio converges to the $\mathcal{F}_t$-predictable ExPow portfolio for all the power-utility investors, the constant portfolio $\boldsymbol{b}$ must be scaled by the relative risk aversion coefficient α. We call such a portfolio the *α-scaled sample path-wise optimal portfolio* (hereafter, referred to as the *αSPOP*, especially when $\alpha = 1$, just the *SPOP*). Then the optimal αSPOP is obtained by the following problem.

$$\mathbf{P}_3(t) \left| \begin{array}{ll} \underset{\mathbf{b}}{\text{maximize}} & V_t(\alpha\boldsymbol{b}) = \exp\left[\left(\alpha\boldsymbol{b}'\boldsymbol{\mu} - \frac{1}{2}\alpha^2\boldsymbol{b}'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b}\right)t + \alpha\boldsymbol{b}'\boldsymbol{\Sigma}\mathbf{W}_t\right] \\ \text{subject to} & \boldsymbol{b} \in \mathbf{D}\ . \end{array} \right.$$

This is equivalent to the following problem.

$$\mathbf{P}_3'(t) \left| \begin{array}{ll} \underset{\mathbf{b}}{\text{minimize}} & \frac{\alpha}{2}\boldsymbol{b}'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b} - \boldsymbol{b}'\tilde{\boldsymbol{\mu}}_t \\ \text{subject to} & \boldsymbol{b} \in \mathbf{D}\ , \end{array} \right.$$

where $\tilde{\boldsymbol{\mu}}_t \stackrel{\triangle}{=} \boldsymbol{\mu} + 1/t\boldsymbol{\Sigma}\mathbf{W}_t$ of (3.5). Note that

$$\tilde{\boldsymbol{\mu}}_t = \boldsymbol{\mu} + \frac{1}{t}\boldsymbol{\Sigma}\mathbf{W}_t = \frac{1}{t}\int_0^t (\text{diag}(\boldsymbol{S}_u))^{-1} d\boldsymbol{S}_u\ .$$

Hence this problem is well-defined under Information 2. Also note that we admit the optimal αSPOP is $\mathcal{G}_t$-predictable, since it seeks the constant portfolio which maximizes the sample path-wise portfolio value at time t, according to one sample path $\{\mathbf{S}_u; 0 \le u \le t\}$. Then, at time t (>0), the KKT condition for the problem $\mathbf{P}_3'(t)$ is $(\boldsymbol{b}_t^{**}, \eta_t^{**}, \boldsymbol{\nu}_t^{**})$ satisfying

$$\begin{cases} \alpha \boldsymbol{\Sigma}\boldsymbol{\Sigma}' \boldsymbol{b}_t^{**} + \eta_t^{**}\mathbf{1} - \boldsymbol{\nu}_t^{**} = \tilde{\boldsymbol{\mu}}_t \ , \\ \boldsymbol{b}_t^{**'}\mathbf{1} = 1 \ , \\ \boldsymbol{\nu}_t^{**'}\boldsymbol{b}_t^{**} = 0 \ , \\ \boldsymbol{\nu}_t^{**} \geq \mathbf{0} \ , \end{cases} \tag{5.3}$$

where η_t^{**} and $\boldsymbol{\nu}_t^{**}$ are Lagrange multipliers at t. Hereafter, $\boldsymbol{b}_t^{**}$ represent the αSPOP including the SPOP. Then, for the base

$$\mathbf{B}_t = \{i_{1,t}, \ldots, i_{m',t}\} \subset \mathbf{I} = \{1, \ldots, m\} \, , \quad \mathbf{B}_t \neq \phi \ ,$$

and $\mathbf{N}_t = \mathbf{I} \setminus \mathbf{B}_t$, the basic solution of the problem is stated as

$$\left(\boldsymbol{b}_t = \begin{pmatrix} \boldsymbol{b}_{\mathbf{B}_t} \\ \boldsymbol{b}_{\mathbf{N}_t} \end{pmatrix}, \quad \eta_{\mathbf{B}_t}, \quad \boldsymbol{\nu}_t = \begin{pmatrix} \boldsymbol{\nu}_{\mathbf{B}_t} \\ \boldsymbol{\nu}_{\mathbf{N}_t} \end{pmatrix} \right) \ .$$

And the active basic solution above in matrix form is described as follows.

$$\mathbf{U}_{\mathbf{B}} \begin{pmatrix} \boldsymbol{b}_{\mathbf{B}_t} \\ \eta_{\mathbf{B}_t} \end{pmatrix} = \boldsymbol{v}_{\mathbf{B}_t},$$

where $\mathbf{U}_{\mathbf{B}} = \begin{bmatrix} \alpha \left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)_{\mathbf{B}} & \mathbf{1}_{\mathbf{B}} \\ \mathbf{1}_{\mathbf{B}}' & \mathbf{0} \end{bmatrix}$, and $\boldsymbol{v}_{\mathbf{B}_t} = \begin{bmatrix} \boldsymbol{\mu}_{\mathbf{B}} + \frac{1}{t}(\boldsymbol{\Sigma}\mathbf{W}_t)_{\mathbf{B}} \\ 1 \end{bmatrix}$.

Using Lemma 6, we have the basic solution of the problem $\mathbf{P}_3'(t)$ as follows.

$$\begin{aligned} \eta_{\mathbf{B}_t^{**}} &= \frac{\mathbf{1}_{\mathbf{B}^{**}}'(\boldsymbol{\Sigma}\boldsymbol{\Sigma}')_{\mathbf{B}^{**}}^{-1}\left\{\boldsymbol{\mu}_{\mathbf{B}^{**}} + \frac{1}{t}(\boldsymbol{\Sigma}\mathbf{W}_t)_{\mathbf{B}_t^{**}}\right\} - \alpha}{\mathbf{1}_{\mathbf{B}^{**}}'(\boldsymbol{\Sigma}\boldsymbol{\Sigma}')_{\mathbf{B}^{**}}^{-1}\mathbf{1}_{\mathbf{B}^{**}}} = \eta_{\mathbf{B}^*} + \Delta\eta_{\mathbf{B}_t^{**}}, \\ \boldsymbol{b}_{\mathbf{B}_t^{**}} &= \frac{1}{\alpha}(\boldsymbol{\Sigma}\boldsymbol{\Sigma}')_{\mathbf{B}^{**}}^{-1}\left\{\boldsymbol{\mu}_{\mathbf{B}^{**}} + \frac{1}{t}(\boldsymbol{\Sigma}\mathbf{W}_t)_{\mathbf{B}_t^{**}}\right\} - \frac{\eta_{\mathbf{B}_t^{**}}}{\alpha}(\boldsymbol{\Sigma}\boldsymbol{\Sigma}')_{\mathbf{B}^{**}}^{-1}\mathbf{1}_{\mathbf{B}^{**}} = \boldsymbol{b}_{\mathbf{B}^*} + \Delta\boldsymbol{b}_{\mathbf{B}_t^{**}}, \\ \boldsymbol{\nu}_{\mathbf{N}_t^{**}} &= \alpha(\boldsymbol{\Sigma}\boldsymbol{\Sigma}')_{\mathbf{N}^{**}}\boldsymbol{b}_{\mathbf{B}_t^{**}} + \eta_{\mathbf{B}_t^{**}}\mathbf{1}_{\mathbf{N}^{**}} - \left\{\boldsymbol{\mu}_{\mathbf{N}^{**}} + \frac{1}{t}(\boldsymbol{\Sigma}\mathbf{W}_t)_{\mathbf{B}_t^{**}}\right\} = \boldsymbol{\nu}_{\mathbf{N}^*} + \Delta\boldsymbol{\nu}_{\mathbf{N}_t^{**}}, \end{aligned}$$

where $\mathbf{B}_t^{**}$ is the optimal base, $\mathbf{N}_t^{**} = \mathbf{I} \setminus \mathbf{B}_t^{**}$, and

$$\begin{aligned} \Delta\eta_{\mathbf{B}_t^{**}} &= \frac{\mathbf{1}_{\mathbf{B}^{**}}'(\boldsymbol{\Sigma}\boldsymbol{\Sigma}')^{-1}(\boldsymbol{\Sigma}\frac{1}{t}\mathbf{W}_t)_{\mathbf{B}_t^{**}}}{\mathbf{1}_{\mathbf{B}^{**}}'(\boldsymbol{\Sigma}\boldsymbol{\Sigma}')^{-1}\mathbf{1}_{\mathbf{B}^{**}}}, \\ \Delta\boldsymbol{b}_{\mathbf{B}_t^{**}} &= \frac{1}{\alpha}(\boldsymbol{\Sigma}\boldsymbol{\Sigma}')_{\mathbf{B}^{**}}^{-1}(\boldsymbol{\Sigma}\frac{1}{t}\mathbf{W}_t)_{\mathbf{B}_t^{**}} - \frac{\Delta\eta_{\mathbf{B}_t^{**}}}{\alpha}(\boldsymbol{\Sigma}\boldsymbol{\Sigma}')_{\mathbf{B}^{**}}^{-1}\mathbf{1}_{\mathbf{B}^{**}}, \\ \Delta\boldsymbol{\nu}_{\mathbf{N}_t^{**}} &= \alpha(\boldsymbol{\Sigma}\boldsymbol{\Sigma}')_{\mathbf{N}^{**}}^{-1}\Delta\boldsymbol{b}_{\mathbf{B}_t^{**}} + \Delta\eta_{\mathbf{B}_t^{**}}\mathbf{1}_{\mathbf{N}^{**}} - (\boldsymbol{\Sigma}\frac{1}{t}\mathbf{W}_t)_{\mathbf{B}_t^{**}}. \end{aligned}$$

If t is large enough, the optimal base of the αSPOP congerges to that of the ExPow portfolio. As $t \to \infty$, $\Delta \frac{1}{t} \mathbf{W}_t \to 0$ a.s. ([27], p.104). Hence

$$\Delta \eta_{\mathbf{B}_t^{**}} \to 0, \quad \Delta \boldsymbol{b}_{\mathbf{B}_t^{**}} \to 0, \quad \Delta \boldsymbol{\nu}_{\mathbf{B}_t^{**}} \to 0.$$

Then,

$$\eta_{\mathbf{B}_t^{**}} \to \eta_{\mathbf{B}^*}, \quad \boldsymbol{b}_{\mathbf{B}_t^{**}} \to \boldsymbol{b}_{\mathbf{B}^*}, \quad \boldsymbol{\nu}_{\mathbf{B}_t^{**}} \to \boldsymbol{\nu}_{\mathbf{B}^*}.$$

Summarizing above results, if $\nu_{\mathbf{N}^{**}} \geq 0$, the problem $\mathbf{P}_3'(t)$ has the basic solution as follows:

$$\alpha(\boldsymbol{\Sigma}\boldsymbol{\Sigma}') \begin{pmatrix} \boldsymbol{b}_{\mathbf{B}_t^{**}} \\ \mathbf{0} \end{pmatrix} + \eta_{\mathbf{B}_t^{**}} \mathbf{1} - \begin{pmatrix} \mathbf{0} \\ \boldsymbol{\nu}_{\mathbf{N}_t^{**}} \end{pmatrix} = \boldsymbol{\mu} + \frac{1}{t} \boldsymbol{\Sigma} \mathbf{W}_t,$$

$$\mathbf{1}' \begin{pmatrix} \boldsymbol{b}_{\mathbf{B}_t^{**}} \\ \mathbf{0} \end{pmatrix} = 1, \quad \left(\mathbf{0}', \boldsymbol{\nu}_{\mathbf{N}_t^{**}}'\right) \begin{pmatrix} \boldsymbol{b}_{\mathbf{B}_t^{**}} \\ \mathbf{0} \end{pmatrix} = 0, \quad \begin{pmatrix} \mathbf{0} \\ \boldsymbol{\nu}_{\mathbf{N},t^{**}} \end{pmatrix} \geq \mathbf{0}.$$

Hence $\boldsymbol{b}_t^{**} = \begin{pmatrix} \boldsymbol{b}_{\mathbf{B}_t^{**}} \\ \mathbf{0} \end{pmatrix}$ is the basic αSPOP. And the existence of the optimal base $\mathbf{B}_t^{**}$ for $\mathbf{P}_3'(t)$ is assumed in the following discussion.

The following theorem assures that the αSPOP converges to the ExPow portfolio. That is, under Information 2, it is a good portfolio which can learn the ExPow portfolio asymptotically.

Theorem 8 (Convergence of the αSPOP $\boldsymbol{b}_\bullet^{**}$)

Under Information 2, the αSPOP $\boldsymbol{b}_\bullet^{**}$ converges to the ExPow portfolio $\boldsymbol{b}^*$. That is

$$\lim_{t\to\infty} \boldsymbol{b}_t^{**} = \boldsymbol{b}^*, \qquad w.p. \quad 1.$$

Proof. Let $\mathbf{B}_t^{**}$ be the optimal basic set for the problem $\mathbf{P}_3'(t)$.

From the convergence $\boldsymbol{\nu}_{\mathbf{N}_t^{**}} \to \boldsymbol{\nu}_{\mathbf{N}^*}$ and the assumption $(\exists \mathbf{B}^*)(s.t. \quad \boldsymbol{\nu}_{\mathbf{N}^*} \geq \mathbf{0}, \quad \mathbf{N}^* = \mathbf{I} \backslash \mathbf{B}^*)$, $\lim_{t\to\infty} \mathbf{B}_t^* = \mathbf{B}^*$. Hence,

$$\begin{aligned} \lim_{t\to\infty} \boldsymbol{b}_{\mathbf{B}_t,t}^{**} &= \lim_{t\to\infty} \boldsymbol{b}_{\mathbf{B}_t^{**},t} = \lim_{t\to\infty} \boldsymbol{b}_{\mathbf{B}^*,t} = \boldsymbol{b}_{\mathbf{B}^*} = \boldsymbol{b}_{\mathbf{B}}^* \\ \lim_{t\to\infty} \boldsymbol{b}_{\mathbf{N}_t,t}^{**} &= \lim_{t\to\infty} \boldsymbol{b}_{\mathbf{N}_t^{**},t} = \lim_{t\to\infty} \boldsymbol{b}_{\mathbf{N}^*,t} = \boldsymbol{b}_{\mathbf{N}^*} = \boldsymbol{b}_{\mathbf{N}}^*. \end{aligned}$$

Then,

$$\lim_{t\to\infty} \boldsymbol{b}_t^{**} = \boldsymbol{b}^* \ .$$

□

Moreover, the growth rate of $\boldsymbol{b}_\bullet^{**}$ converges to the ideal optimal one.

Theorem 9 (Convergence of the growth rate for $\boldsymbol{b}_\bullet^{**}$)

Under Information 2, the αSPOP $\boldsymbol{b}_\bullet^{**}$ has the same growth rate as the ideal ExPow portfolio, asymptotically. That is

$$\lim_{T\to\infty} \frac{1}{T} \log V_T(\boldsymbol{b}_\bullet^{**}) = \lim_{T\to\infty} \frac{1}{T} \log V_T(\boldsymbol{b}^*) = \boldsymbol{b}^{*'}\boldsymbol{\mu} - \frac{1}{2}\boldsymbol{b}^{*'}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b}^* \ .$$

Proof. First,

$$\lim_{T\to\infty} \frac{1}{T} \log V_T(\boldsymbol{b}_\bullet^{**}) = \lim_{T\to\infty} \left\{ \left(\frac{1}{T}\int_0^T \boldsymbol{b}_t^{**} dt \right)' \boldsymbol{\mu} - \frac{1}{2}\left(\frac{1}{T}\int_0^T \boldsymbol{b}_t^{**'}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b}_t^{**} dt \right) + \frac{1}{T}\int_0^T \boldsymbol{b}_t^{**'}\boldsymbol{\Sigma} d\mathbf{W}_t \right\} \ .$$

Using the result $\lim_{T\to\infty} \boldsymbol{b}_t^{**} = \boldsymbol{b}^*$ of Theorem 8,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \boldsymbol{b}_t^{**} dt = \boldsymbol{b}^* \ ,$$
$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \boldsymbol{b}_t^{**'}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b}_t^{**} dt = \boldsymbol{b}^{*'}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b}^* \ .$$

And taking the time change into consideration, we get $\int_0^T \boldsymbol{b}_t^{**'}\boldsymbol{\Sigma} d\mathbf{W}_t = \widehat{\mathbf{W}}_{\tau_T} \quad a.s.,$ where $\widehat{\mathbf{W}}_t$ is standard Brownian motion, and $\tau_T = \int_0^T \boldsymbol{b}_t^{**'}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b}_t^{**} dt$. Also from $\lim_{T\to\infty} \boldsymbol{b}_t^{**} = \boldsymbol{b}^* \neq 0$, $\lim_{T\to\infty} \tau_T = +\infty \quad a.s.$. Then

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \boldsymbol{b}_t^{**'}\boldsymbol{\Sigma} d\mathbf{W}_t = \lim_{T\to\infty} \frac{\tau_T}{T}\cdot\frac{1}{\tau_T}\widehat{\mathbf{W}}_{\tau_T} = \lim_{T\to\infty} \frac{\tau_T}{T} \lim_{T'\to\infty} \frac{1}{T'}\widehat{\mathbf{W}}_{T'} = \boldsymbol{b}^{*'}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b}^* 0 = 0 \ .$$

Hence,

$$\lim_{T\to\infty} \frac{1}{T} \log V_T(\boldsymbol{b}_\bullet^{**}) = \boldsymbol{b}^{*'}\boldsymbol{\mu} - \frac{1}{2}\boldsymbol{b}^{*'}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b}^* \ .$$

On the other hand, the second term of this theorem statement is

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{T}\log V_T(\boldsymbol{b}^*) &= \lim_{T\to\infty}\frac{1}{T}\left\{\left(\boldsymbol{b}^{*'}\boldsymbol{\mu}-\frac{1}{2}\boldsymbol{b}^{*'}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b}^*\right)T+\boldsymbol{b}^*\boldsymbol{\Sigma}\mathbf{W}_T\right\}\\ &= \boldsymbol{b}^{*'}\boldsymbol{\mu}-\frac{1}{2}\boldsymbol{b}^{*'}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b}^*+\lim_{T\to\infty}\boldsymbol{b}^{*'}\frac{1}{T}\boldsymbol{\Sigma}\mathbf{W}_T\\ &= \boldsymbol{b}^{*'}\boldsymbol{\mu}-\frac{1}{2}\boldsymbol{b}^{*'}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b}^*\ .\end{aligned}$$

This completes the proof. □

Remark 5

From Theorem 9 and Corollary 1, both the αSPOP and αUP (which is treated in the next section) converge to the ideal optimal one, i.e. the ExPow portfolio. But it is difficult to evaluate the gap in the expected power-utility base among those three portfolios. So instead of comparing in the utility base, we evaluated the asymptotic growth rate among the three.

5.3 The α-scaled Universal Portfolio and its Convergence

In this section, we suppose the power-utility investors are provided with the incomplete Information 2 as in the previous section. And we propose another scheme which can learn the $\mathcal{F}_t$-predictable ExPow portfolio in the long run. It is the *α-scaled universal portfolio* (*αUP*, and especially when $\alpha = 1$, just the *UP*).

Define the αUP at time t (>0) as

$$\boldsymbol{b}_t^\sharp \triangleq \int_{\boldsymbol{b}\in\mathbf{D}} \boldsymbol{b} f_{\alpha,t}(\boldsymbol{b})d\boldsymbol{b}\ , \tag{5.4}$$

where

$$f_{\alpha,t}(\boldsymbol{b}) \triangleq \frac{V_t(\alpha\boldsymbol{b})}{\int_{\boldsymbol{b}\in\mathbf{D}} V_t(\alpha\boldsymbol{b})d\boldsymbol{b}} \tag{5.5}$$

is the weighting density function of constant portfolio, and where $V_t(\alpha \boldsymbol{b})$ is given in the objective function of the problem $\mathbf{P}_3(t)$. The original universal portfolio proposed by Cover is defined as $\alpha = 1$ in the weighting density function[7]. The assumptions required to prove the theorems in this section is completely the same as the previous section. Under the above settings, the following theorem asserts that the αUP provides one of the most superior scheme to learn the ExPow portfolio asymptotically.

Theorem 10 (Convergence of the αUP $\boldsymbol{b}^{\sharp}_{\bullet}$)

Under Information 2, the αUP converges to the ideal ExPow portfolio over the long run. That is

$$\lim_{t\to\infty} \boldsymbol{b}^{\sharp}_t = \boldsymbol{b}^*, \quad a.s. \; .$$

Proof. Let $\boldsymbol{b}$, $\tilde{\boldsymbol{b}} \in \mathbf{D}$ and $\boldsymbol{b} \neq \tilde{\boldsymbol{b}}$. Then,

$$\begin{aligned}
\lim_{t\to\infty} \left(\frac{V_t(\alpha \boldsymbol{b})}{V_t(\alpha \tilde{\boldsymbol{b}})} \right) &= \lim_{t\to\infty} \exp \alpha \left[\boldsymbol{b}'\boldsymbol{\mu} - \frac{1}{2}\alpha \boldsymbol{b}'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b} - \left(\tilde{\boldsymbol{b}}'\boldsymbol{\mu} - \frac{1}{2}\alpha \tilde{\boldsymbol{b}}'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\tilde{\boldsymbol{b}} \right) + \left(\boldsymbol{b} - \tilde{\boldsymbol{b}} \right)' \boldsymbol{\Sigma} \frac{1}{t} \mathbf{W}_t \right] t \\
&= \lim_{t\to\infty} \exp \alpha \left[\boldsymbol{b}'\boldsymbol{\mu} - \frac{1}{2}\alpha \boldsymbol{b}'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b} - \left(\tilde{\boldsymbol{b}}'\boldsymbol{\mu} - \frac{1}{2}\alpha \tilde{\boldsymbol{b}}'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\tilde{\boldsymbol{b}} \right) \right] t \\
&= \begin{cases} +\infty, & \text{if} \quad \boldsymbol{b}'\boldsymbol{\mu} - \frac{1}{2}\alpha \boldsymbol{b}'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b} > \tilde{\boldsymbol{b}}'\boldsymbol{\mu} - \frac{1}{2}\alpha \tilde{\boldsymbol{b}}'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\tilde{\boldsymbol{b}} \\ 0, & \text{if} \quad \boldsymbol{b}'\boldsymbol{\mu} - \frac{1}{2}\alpha \boldsymbol{b}'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b} < \tilde{\boldsymbol{b}}'\boldsymbol{\mu} - \frac{1}{2}\alpha \tilde{\boldsymbol{b}}'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\tilde{\boldsymbol{b}} \; . \end{cases}
\end{aligned}$$

Hence, the weighting density function of constant portfolio is

$$\begin{aligned}
f_{\alpha,t}(\tilde{\boldsymbol{b}}) &= \lim_{t\to\infty} \frac{1}{\int_{\boldsymbol{b}\in\mathbf{D}} \left(\frac{V_t(\alpha \boldsymbol{b})}{V_t(\alpha \tilde{\boldsymbol{b}})} \right) d\boldsymbol{b}} = \lim_{t\to\infty} \frac{1}{\int_{\boldsymbol{b}\in\mathbf{D}(\tilde{\boldsymbol{b}})} \left(\frac{V_t(\alpha \boldsymbol{b})}{V_t(\alpha \tilde{\boldsymbol{b}})} \right) d\boldsymbol{b}} \\
&= \begin{cases} +\infty, & \text{if} \quad L(\mathbf{D}(\tilde{\boldsymbol{b}})) = 0 \\ 0, & \text{if} \quad L(\mathbf{D}(\tilde{\boldsymbol{b}})) > 0 \; , \end{cases}
\end{aligned}$$

where $\mathbf{D}(\tilde{\boldsymbol{b}}) = \left\{ \boldsymbol{b} \in \mathbf{D} \,|\, \text{if} \quad \boldsymbol{b}'\boldsymbol{\mu} - \frac{1}{2}\alpha \boldsymbol{b}'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{b} > \tilde{\boldsymbol{b}}'\boldsymbol{\mu} - \frac{1}{2}\alpha \tilde{\boldsymbol{b}}'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\tilde{\boldsymbol{b}} \right\}$, and $L(D)$ is the Lebesgue measure for the area $\mathbf{D}$. Noticing $L(\mathbf{D}(\tilde{\boldsymbol{b}})) = 0$ if $\tilde{\boldsymbol{b}} = \boldsymbol{b}^*$ and $L(\mathbf{D}(\tilde{\boldsymbol{b}})) > 0$ if $\tilde{\boldsymbol{b}} \neq \boldsymbol{b}^*$. Then,

$$\begin{aligned}
\lim_{t\to\infty} f_{\alpha,t}(\tilde{\boldsymbol{b}}) &= \begin{cases} +\infty, & \text{if } \tilde{\boldsymbol{b}} = \boldsymbol{b}^* \\ 0, & \text{if } \tilde{\boldsymbol{b}} \neq \boldsymbol{b}^* \end{cases} \\
&= \delta(\boldsymbol{b}, \boldsymbol{b}^*) \; .
\end{aligned}$$

where $\delta(\bullet)$ is the Dirac delta function. So we have

$$\lim_{t\to\infty} \boldsymbol{b}_t^\sharp = \lim_{t\to\infty} \int_{\boldsymbol{b}\in\mathbf{D}} \boldsymbol{b} f_{\alpha,t}(\boldsymbol{b}) d\boldsymbol{b} = \boldsymbol{b}^* \ .$$

□

Also the universal portfolio $\boldsymbol{b}_\bullet^\sharp$ has the following property.

Corollary 1 (Convergence of the growth rate for $\boldsymbol{b}_\bullet^\sharp$)

Under Information 2, the αUP has the same growth rate as the ideal ExPow portfolio, asymptotically. That is

$$\lim_{T\to\infty} \frac{1}{T} \log V_T(\boldsymbol{b}_\bullet^\sharp) = \lim_{T\to\infty} \frac{1}{T} \log V_T(\boldsymbol{b}^*) = \boldsymbol{b}^{*'} \boldsymbol{\mu} - \frac{1}{2} \boldsymbol{b}^{*'} \boldsymbol{\Sigma}\boldsymbol{\Sigma}' \boldsymbol{b}^* \quad a.s.$$

Proof. By similar argument to Theorem 9. □

Chapter 6

Algorithms for the αSPOP and the αUP

The continuous model enables us to derive the asymptotically optimal portfolio among all the $\mathcal{G}_t$-predictable portfolios. It is the SPOP within the simplex $\mathbf{D}$, and it is the SPOP or UP within $\mathbf{D}'$. Moreover, the $\mathcal{G}_t$-predictable αSPOP and αUP can learn the $\mathcal{F}_t$-predictable ExPow portfolio in the long run. Our next interest is focused on the algorithms for searching these portfolios in discrete time framework. The difficulty here is that since asset prices in practical markets are observed only once in a discrete time interval, the portfolio maximizing the expected concave utility cannot be directly obtained from these observations, and we cannot make assumptions on asset prices other than as to their non-negativity. To address these limitations, we utilize the results in a continuous model and we derive the algorithms for the αSPOP and the αUP.

We consider the market with m assets, and let the price-relative vector of assets be $\boldsymbol{X}_t = (X_{1,t}, \ldots, X_{m,t})'$. It is assumed that the prices of each security $S_{i,t}$ are observed at the market in discrete time $t = 0, 1, \ldots$, and so its price-relatives $X_{i,t} \triangleq \dfrac{S_{i,t}}{S_{i,t-1}}$. And the power-utility investors select their portfolios within the simplex $\mathbf{D}$ of (2.3).

6.1 An Algorithm for the αSPOP at Discrete Time Intervals

According to the continuous theory discussed in chapter 3, if investors' utility belongs to the log-utility class, their optimal portfolios are the SPOP at any finite terminal-time. And by the discussion in chapter 5, the $\mathcal{G}_t$-predictable αSPOP assures the general power-utility investor of learning the $\mathcal{F}_t$-predictable ExPow portfolio asymptotically. Hence, at any time $t > 0$, the power-utility investors' optimal policy is stated as

$$\mathbf{P}_4 \left| \begin{array}{ll} \underset{\mathbf{b}}{\text{maximize}} & V_t(\alpha \boldsymbol{b}) = \prod_{u=1}^{t} \left(1 + \alpha \boldsymbol{b}'(\mathbf{X}_u - \mathbf{1})\right) = \prod_{u=1}^{t} \boldsymbol{b}'\boldsymbol{Y}_u \\ \text{subject to} & \boldsymbol{b} \in \mathbf{D}\ , \end{array} \right.$$

where $\boldsymbol{Y}_u \stackrel{\triangle}{=} \alpha \boldsymbol{X}_u + (1-\alpha)\mathbf{1}$. And this is equivalent to the following problem:

$$\mathbf{P}_4' \left| \begin{array}{ll} \underset{\mathbf{b}}{\text{maximize}} & g(\boldsymbol{b}) = \frac{1}{t} \sum_{u=1}^{t} \log \boldsymbol{b}'\boldsymbol{Y}_u \\ \text{subject to} & \boldsymbol{b} \in \mathbf{D}\ . \end{array} \right.$$

If we assume $\boldsymbol{Y}_u > \mathbf{0}$, which is generally held, we can utilize the algorithm for the expected log-utility maximization proposed by Cover [5]. And note that if the investor's relative risk aversion α is particularly high, no feasible solution may be gotten, for there is no assurance of $\boldsymbol{Y}_u > \mathbf{0}$.

Hereafter we abbreviate the operation $\frac{1}{t}\sum_{u=1}^{t}$ to E. The solution of $\mathbf{P}_4'$ is given by the following iterative algorithm.

(Algorithm for the αSPOP)

The iteration $\boldsymbol{b}^{(n)}$ is defined as :

$$\left| \begin{array}{ll} b_i^{(n+1)} = b_i^{(n)} E\left[\frac{Y_i}{\boldsymbol{b}^{(n)'}\mathbf{Y}}\right]\ , & \\ b_i^{(0)} = \frac{1}{m}\ , & i = 1, \ldots, m\ . \end{array} \right. \tag{6.1}$$

Remark 6

The sequence of the algorithm is always satisfying the constraints. For $b_i^{(n+1)}$ is positive, and $\boldsymbol{b}^{(n+1)'}\mathbf{1} = \sum_{i=1}^{m} b_i^{(n)} E\left[\frac{Y_i}{\boldsymbol{b}^{(n)'}\mathbf{Y}}\right] = 1$.

The KKT condition for $\mathbf{P}_4'$ is

$$\begin{aligned} E\left[\frac{Y_i}{\boldsymbol{b}^{*'}\mathbf{Y}}\right] &= 1 \quad \text{if} \quad b_i^* > 0 \ , \\ E\left[\frac{Y_i}{\boldsymbol{b}^{*'}\mathbf{Y}}\right] &\le 1 \quad \text{if} \quad b_i^* = 0 \ . \end{aligned} \tag{6.2}$$

Here we have the following two theorems for the iterative algorithm which are similar to the argument shown by Cover[5].

Theorem 11 (Monotonicity)

The algorithm improves the objective function of $\mathbf{P}_4'$ *monotonically. That is*

$g(\boldsymbol{b}^{(n+1)}) \ge g(\boldsymbol{b}^{(n)})$.

Proof.

$$\begin{aligned} g(\boldsymbol{b}^{(n+1)}) - g(\boldsymbol{b}^{(n)}) &= E\left[\log \boldsymbol{b}^{(n+1)'}\mathbf{Y}\right] - E\left[\log \boldsymbol{b}^{(n)'}\mathbf{Y}\right] \\ &= E\left[\log \frac{\boldsymbol{b}^{(n+1)'}\mathbf{Y}}{\boldsymbol{b}^{(n)'}\mathbf{Y}}\right] = E\left[\log \sum_{i=1}^m b_i^{(n+1)} \frac{Y_i}{\boldsymbol{b}^{(n)'}\mathbf{Y}}\right] \\ &= E\left[\log \sum_{i=1}^m \left(\frac{b_i^{(n)} Y_i}{\boldsymbol{b}^{(n)'}\mathbf{Y}} E\left[\frac{Y_i}{\boldsymbol{b}^{(n)'}\mathbf{Y}}\right]\right)\right] \\ &\ge E\left[\sum_{i=1}^m \frac{b_i^{(n)} Y_i}{\boldsymbol{b}^{(n)'}\mathbf{Y}} \log E\left[\frac{Y_i}{\boldsymbol{b}^{(n)'}\mathbf{Y}}\right]\right] \qquad (6.3) \\ &= \sum_{i=1}^m b_i^{(n)} E\left[\frac{Y_i}{\boldsymbol{b}^{(n)'}\mathbf{Y}}\right] \log\left(\frac{b_i^{(n+1)}}{b_i^{(n)}}\right) \\ &= \sum_{i=1}^m b_i^{(n+1)} \log\left(\frac{b_i^{(n+1)}}{b_i^{(n)}}\right) = D\left(\boldsymbol{b}^{(n+1)} \| \boldsymbol{b}^{(n)}\right) \\ &\ge 0 \ . \qquad (6.4) \end{aligned}$$

(6.3) is from the Jensen's inequality, and $D\left(\boldsymbol{b}^{(n+1)} \| \boldsymbol{b}^{(n)}\right)$ is the Kullback-Leibler information number. And (6.4) is from

$$\begin{aligned} D\left(\boldsymbol{b}^{(n+1)} \| \boldsymbol{b}^{(n)}\right) &= \sum_{i=1}^m b_i^{(n+1)} \log\left(\frac{b_i^{(n+1)}}{b_i^{(n)}}\right) = -\sum_{i=1}^m b_i^{(n+1)} \log\left(\frac{b_i^{(n)}}{b_i^{(n+1)}}\right) \\ &\ge -\log\left(\sum_{i=1}^m b_i^{(n+1)} \frac{b_i^{(n)}}{b_i^{(n+1)}}\right) = 0 \ . \end{aligned}$$

This completes the proof. □

Theorem 12 (Convergence)

The algorithm converges to the optimal solution. That is

$$\lim_{n\to\infty} \boldsymbol{b}^{(n)} = \boldsymbol{b}^* \ .$$

Proof. Let $\Delta(\boldsymbol{b}^{(n)}) = g(\boldsymbol{b}^{(n+1)}) - g(\boldsymbol{b}^{(n)})$, and $\Delta(\boldsymbol{b}^{(n)}) \geq 0$ from Theorem 11. From the Bolzano-Weierstrass theorem and $g(\boldsymbol{b})$ is strictly concave, the algorithm sequence $\left\{\boldsymbol{b}^{(n)}\right\}$ has only one accumulate point $\tilde{\boldsymbol{b}}$ in the area $\mathbf{D} = \{\boldsymbol{b}|\boldsymbol{b}'\mathbf{1} = 1, \boldsymbol{b} \geq \mathbf{0}\}$. So $\Delta(\tilde{\boldsymbol{b}}) \to 0$.

$\Delta(\tilde{\boldsymbol{b}}) = 0$ is reached only when

$$\tilde{b}_i^{(N+1)} = \tilde{b}_i^{(N)} = \tilde{b}_i^{(N)} E\left[\frac{Y_i}{\tilde{\boldsymbol{b}}^{(N)'}\mathbf{Y}}\right]$$
$$\tilde{b}_i^{(N)}\left(E\left[\frac{Y_i}{\tilde{\boldsymbol{b}}^{(N)'}\mathbf{Y}}\right] - 1\right) = 0.$$

(i)Case $E\left[\frac{Y_i}{\tilde{\boldsymbol{b}}^{(N)'}\mathbf{Y}}\right] = 1$. Clearly, $\tilde{b}_i$ satisfies the KKT condition (6.2).

(ii)Case $\tilde{b}_i^{(N)} = 0$. Assume that $E\left[\frac{Y_i}{\tilde{\boldsymbol{b}}^{(N)'}\mathbf{Y}}\right] \geq 1$. From this and the definition of the algorithm, $\tilde{b}_i^{(N)} = \frac{1}{m}\lim_{N\to\infty}\prod_{k=0}^{N-1} E\left[\frac{Y_i}{\boldsymbol{b}^{(k)'}\mathbf{Y}}\right] \neq 0$. This is contradiction. Then $E\left[\frac{Y_i}{\tilde{\boldsymbol{b}}^{(N)'}\mathbf{Y}}\right] < 1$, and this satisfies the KKT condition.

From (i) and (ii), the accumulate point $\tilde{\boldsymbol{b}}$ satisties the KKT condition, and $\tilde{\boldsymbol{b}} = \boldsymbol{b}^*$. □

6.2 An Algorithm for the αUP

Next, an algorithm for the αUP is dealt with. Discrete sampling expression of the αUP $\boldsymbol{b}_t^\sharp$ at time $t > 0$ is stated as

$$b_{i,t}^\sharp \triangleq \sum_{k=1}^{N} b_i^{(k)} f_{\alpha,t}(\boldsymbol{b}) = \sum_{k=1}^{N} b_i^{(k)} \frac{\prod_{u=1}^{t}\boldsymbol{b}^{(k)'}\boldsymbol{Y}_u}{\sum_{k=1}^{N}\left(\prod_{u=1}^{t}\boldsymbol{b}^{(k)'}\boldsymbol{Y}_u\right)} \quad (i = 1, \ldots, m) \ , \tag{6.5}$$

where N is the number of samplings for constant portfolios, $\boldsymbol{b}^{(k)}$ is the k-th sampled constant portfolio, and $\boldsymbol{Y}_u$ is the scaled price-relative vector introduded in the previous subsection.

The algorithm of the αUP only depends on how to sample constant portfolio $\boldsymbol{b}$ efficiently. If we are to sample the constant portfolio $\boldsymbol{b}$ uniformly with precision n, the number of sampling cases depends upon how you allocate n indiscriminate balls with the size $\frac{1}{n}$ (portfolio weights) into m discriminated box (assets), with the redundancy allowed. Hence this is

$$\frac{(m+n-1)!}{n!(m-1)!} \quad \text{cases} . \tag{6.6}$$

If the investor is faced with a middle sized portfolio problem, for example, one constituted of 100 securities, the number of cases is about 2.455×10^{22}, with precision $n = 20$. But here we propose the most rough $(m+1)$-sampling, i.e. $\boldsymbol{b} = \left\{\boldsymbol{b}^{(1)}, \ldots, \boldsymbol{b}^{(k)}, \ldots, \boldsymbol{b}^{(m+1)}\right\} = \left\{(1, 0, \ldots, 0)', (0, 1, 0, \ldots, 0)', \ldots, (0, \ldots, 0, 1)', (\frac{1}{m}, \ldots, \frac{1}{m})'\right\}$. Using above constant portfolio sampling, an algorithm for the αUP is given as follows.

(Algorithm for the αUP)

The αUP $\boldsymbol{b}_t^\sharp$ at time t is :

(At $t = 0$) Define $b_{i,0}^\sharp = \frac{1}{m}$ $(i = 1, \ldots, m)$.

(At $t > 0$) Calculate $b_{i,t}^\sharp$ $(i = 1, \ldots, m)$ of (6.5) using sampling $\boldsymbol{b} = \left\{\boldsymbol{b}^{(1)}, \ldots, \boldsymbol{b}^{(m+1)}\right\}$.

Chapter 7

The Empirical Analysis

In this chapter, we provide an empirical analysis which verify our theory in the continuous-time framework, and compensate for what the theory lacks. The target of this analysis is the entire NYSE/AMEX stock market.

We previously constructed two schemes for the power-utility investors, called the αSPOP and the αUP. For the general power-utility investors, they are proved to learn the $\mathcal{F}_t$-predictable ExPow portfolio asymptotically. Also by restricting the investors' utility class to the log-utility, the two schemes are proved to be asymptotically optimal among the $\mathcal{G}_t$-predictable portfolios. However, we couldn't prove theoretically that the two schemes are optimal for the general power-utility investors under $\mathcal{G}_t$, at any finite terminal-time. Furthermore, the favorable properties of both the αSPOP and αUP, which are guaranteed in the continuous-time framework, may not hold in the general discrete-time framework as in the practical market environment. Besides the validity of the assumptions required for our theory, several parameters are required when we try to apply our theory to the actual investment. These parameters, such as the length of learning (estimating) periods, the number of securities, and even the degree of Arrow-Pratt's relative risk aversion, cannot be determined theoretically.

Hence our empirical analysis is executed using a sensitivity analysis-like method. That is, by varying one specified parameter with all the other parameters fixed, we examine the sensitivity of the object, the ex post power-utility, for each scheme. Then we are able to evaluate simultaneously which scheme provides the best posterior power-utility, which scheme is robust in parameters, and which information set $\mathcal{G}_t$ provide the best posterior power-utility.

Also, we regard the several properties, which the ExPow portfolio is supposed to have, as portfolio performance measures. Then by observing these performance measures, we examine whether two schemes can learn the ExPow portfolio over the long run.

And these analysis is carried out with transaction costs taken into account as these are present in the practical market environment. Our treatment for transaction costs is rather new in that costs are incurred for each asset's rebalance level.

7.1 The Model

First we define the objective of the power-utility investors. We consider the market with m assets. Their prices $S_{i,t}$ $(i = 1, \ldots, m)$ are observed in the market, in discrete time $t = 0, 1, \ldots$. And we let the price-relative vector of assets be $\boldsymbol{X}_t = (X_{1,t}, \ldots, X_{m,t})'$, where $X_{i,t} \triangleq \dfrac{S_{i,t}}{S_{i,t-1}}$, and superscript $'$ stands for transpose. Suppose that the investors rebalance their portfolios in discrete time intervals, within the simplex $\mathbf{D}$ of (2.3). If the log-utility investors continuously observe asset prices as a realization of the s.d.e. (2.1), the optimal portfolio among all the $\mathcal{G}_t$-predictable portfolios is given by the SPOP. This is shown in chapter 3. Moreover, by utilizing the αSPOP, the general power-utility investors with $\mathcal{G}_t$ can attain the $\mathcal{F}_t$-predictable ExPow portfolio asymptotically. This is proved in chapter 5. The αSPOP states that the portfolio providing such favorable properties for the power-utility investors is given by some constant portfolio using the information $\mathcal{G}_t$. Also, to assure all the power-utility investors of holding such properties, we must scale the constant portfolio by the relative risk aversion α. Since asset prices can be observed only once in discrete-time intervals, investors need some number of periods in which to learn, letting L. With $V_0 = 1$ provided, the αSPOP at time $t \geq L$, in the discrete-time framework, is the solution of the following problem.

$$\mathbf{P}_5 \left| \begin{array}{ll} \underset{\mathbf{b}}{\text{maximize}} & V_t(\alpha \boldsymbol{b}) = \prod_{u=t-L+1}^{t} \left(1 + \alpha \boldsymbol{b}'(\mathbf{X}_u - \mathbf{1})\right) = \prod_{u=t-L+1}^{t} \left(\boldsymbol{b}'\boldsymbol{Y}\right) \\ \text{subject to} & \boldsymbol{b} \in \mathbf{D} \ , \end{array} \right.$$

where $\boldsymbol{Y}_u \stackrel{\triangle}{=} \alpha \boldsymbol{X}_u + (1-\alpha)\mathbf{1}$. Note that this discrete-time version of the αSPOP may not be optimal among all the $\mathcal{G}_t$-predictable portfolio, even for the log-utility investors, since asset prices are not observed continuously as a realization of Eq. (2.1). Also note that the operation product for portfolio value V_t should start from $u = 1$ to be accommodated by the theory. However, the reason why $u = t - L + 1$ is used instead of $u = 1$ is from the same reason which will be described at the third item *Information*, in subsection 7.2.1. Also note that we admit the αSPOP is $\mathcal{G}_t$-predictable, since it seeks the constant portfolio which maximize the sample path-wise portfolio value at time t, according to one sample path $\{\boldsymbol{S}_u \ ; \ u = 0, \ldots, t \ \}$. The important point is that, as in the continuous-time framework, we may expect the αSPOP given by the problem $\mathbf{P}_5$ assures the log-utility investors of the optimality under $\mathcal{G}_t$, and assures the general power-utility investors of attaining the ExPow portfolio over the long run. Problem $\mathbf{P}_5$ is equivalent to the following:

$$\mathbf{P}'_5 \left| \begin{array}{ll} \underset{\mathbf{b}}{\text{maximize}} & g(\boldsymbol{b}) = \frac{1}{L} \sum_{u=t-L+1}^{t} \log \boldsymbol{b}'\boldsymbol{Y}_u \\ \text{subject to} & \boldsymbol{Y}_u = \alpha \boldsymbol{X}_u + (1-\alpha)\mathbf{1} \quad (\ u = t-L+1, \ldots, t \ ) \ , \\ & \boldsymbol{b} \in \mathbf{D} \ . \end{array} \right.$$

Another scheme we can expect, in the discrete-time framework, to assure the power-utility investors of having the portfolio with the properties described above, is the αUP. With learning periods L and $V_0 = 1$ provided, the αUP at time $t \geq L$ is defined as

$$b^{\sharp}_{i,t} \stackrel{\triangle}{=} \sum_{k=1}^{N} b_i^{(k)} f_{\alpha,t}(\boldsymbol{b}) = \sum_{k=1}^{N} b_i^{(k)} \frac{\prod_{u=t-L+1}^{t} \boldsymbol{b}^{(k)'} \boldsymbol{Y}_u}{\sum_{k=1}^{N} \left(\prod_{u=t-L+1}^{t} \boldsymbol{b}^{(k)'} \boldsymbol{Y}_u \right)} \quad (i = 1, \ldots, m) \ , \tag{7.1}$$

where N is the number of samplings for constant portfolios, $\boldsymbol{b}^{(k)}$ is the k-th sampled constant portfolio, and $\boldsymbol{Y}_u$ is the scaled price-relative vector.

The above two schemes are valid in the continuous-time framework. So we empirically verify these schemes are valid again in the setting of the practical stock market.

7.2 Method of Empirical Analysis

In this section, we describe the investment rules which are employed in our analysis, the parameters which are necessary to answer our questions by a sensitivity analysis-like method, and the portfolio performance measures which can judge if our schemes has the favorable properties as in the continuous-time framework.

7.2.1 Investment Rules

It is supposed that investors are specified by the relative risk aversion coefficient α. And we suppose 20 types, ranging from near the risk-neutral investor $\alpha = 0.1$, to the log investor $\alpha = 1$ who wants to maximize the growth rate, and on to the risk-averse investor $\alpha = 10$.

These investors reinvest their wealth only in the NYSE and AMEX stock markets according to the following rules.

1. Rebalancing monthly: Investors decide their optimal portfolios at the beginning of each month.

2. Three portfolios: Investors have three schemes to determine their portfolios which are expected to maximize their power-utility. These are the αSPOP, the αUP and the equally weighted portfolio (EP) as benchmark.

3. Information: Investors only have the information of rates of return plus one. These stock returns are obtained from the CRSP CD-ROM, so every stock price and dividend

are adjusted at each period. Since the information can be obtained in discrete-time intervals, investors require certain positive periods in which to derive or learn the portfolios by their criteria. We call such periods the *learning periods* and denoted by L. Hence at decision point t, the price-relatives' information from L months past to the last month of t, i.e., $\{\boldsymbol{X}_{t-L+1}, \ldots, \boldsymbol{X}_t\}$ is available for investors. If the power-utility investors are to learn the ExPow portfolio, possibly longest learning periods are the most relevant ones. But investors do not necessarily use very long learning periods, when they are to derive the $\mathcal{G}_t$-predictable portfolio. In such cases, investors may be able to extract the worthwhile information from the shorter $\mathcal{G}_t$. For our purposes, the entire universe of the market is composed of 749 stocks, whose rates of return have been observed in full from January 1971 up to December 1995, i.e. 300 months.

4. Initial portfolio: Investors have the same EP at the beginning of the performance measurement horizon, regardless of their decision criteria.

5. Transaction costs, divisibility, and budget constraints: Investors incur costs for every rebalance. If their optimal portfolios change their weights drastically, their utility may decrease drastically. And we assume investors' wealth is infinitely divisible. Also investments are made within the investors' own wealth, with short-selling not allowed, i.e. with the constraints $\boldsymbol{b}_t'\mathbf{1} = 1, \quad \boldsymbol{b}_t \geq \mathbf{0}$.

7.2.2 Parameters

When investors reinvest their wealth in the stock market, a few more elements may affect the ex post mean of their power utility besides the choice of portfolios. Those elements are viewed as parameters and listed below.

1. Learning periods L : As we described earlier, investors can use the possibly longest L to learn the ExPow portfolio. While they have alternative choices to use shorter L

to extract the worthwhile information which may be involved in shorter $\mathcal{G}_t$. We set L as $L = 3, 12, 36, 60, 120$ months, and as all the available learning months at portfolio decision point t.

2. Number of securities m : From the mathematical viewpoint, the larger the investment universe gets, the higher investors can expect their objectives to increase. But asset choice may have an affect. So we vary the data-set in several cases. Ordering from small to large, we prepare the 27 securities composing the DJIA, the 74 securities composing the S & P 100, the 274 securities composing the S & P 500, and the 749 securities of the entire market universe [1].

3. Transaction costs γ : Transaction costs are not taken into account in our analysis in the continuous-time framework. But in the practical market, costs are incurred every time investors rebalance. If investors allocate their wealth V_{t-1} via the portfolio $\boldsymbol{b}_{t-1}$ at the beginning of the period t, the wealth becomes $V_t = V_{t-1}\boldsymbol{b}'_{t-1}\boldsymbol{X}_t$, without costs. But with costs γ, the wealth becomes

$$V_t = V_{t-1}\frac{\sum_{i=1}^{m}\frac{b_{i,t-1}X_{i,t}}{1-\gamma\zeta_{i,t}}}{\sum_{j=1}^{m}\frac{b_{j,t}}{1-\gamma\zeta_{j,t}}}\ . \tag{7.2}$$

where $\zeta_{i,t} \overset{\triangle}{=} 1\ \left(\text{ if } b_{i,t} \geq \kappa_t b^-_{i,t}\right),\ \zeta_{i,t} \overset{\triangle}{=} -1\ \left(\text{ if } b_{i,t} < \kappa_t b^-_{i,t}\right)$, $b_{i,t}$ is the portfolio at the beginning of the period $t+1$, and $b^-_{i,t}$ is the weight (ratio) of the i-th asset value relative to the value of portfolio at the end of the period t, i.e. $b^-_{i,t} = \frac{b_{i,t-1}X_{i,t}}{\boldsymbol{b}'_{t-1}\boldsymbol{X}_t}$. And the adjustment coefficient κ_t is the fixed point obtained from the following equation.

$$\kappa_t = \frac{\sum_{i=1}^{m} b_{i,t-1}X_{i,t}\cdot\sum_{j=1}^{m}\frac{b_{j,t}}{1-\gamma\zeta_{j,t}}}{\sum_{l=1}^{m}\frac{b_{l,t-1}X_{l,t}}{1-\gamma\zeta_{l,t}}}\ . \tag{7.3}$$

The derivation of above formulae (7.2) (7.3) is given in section 7.5. We set costs as $\gamma = 0,\ 1,\ 2\ (\%)$ in the analysis.

[1] Note that the list of securities which comprise each index were obtained from 1995, so we have had to eliminate some securities which have lost data at any point during the learning or the measurement periods.

7.2.3 Portfolio Performance Measures

Under above rules and parameters, we measure the performance of three portfolios. We set the measurement horizon from January 1993 to December 1995. At January 1993, the initial point in the measurement horizon, every investor has the EP. Then at February 1993, which is the first decision point for investors, maximally 265 returns for each of the 749 stocks are available for investors. And the measures we use in the analysis are as follows.

1. Ex post mean of investors' power-utility: Except the log-utility, the gap in expected power-utility base is not shown in our theory (see Theorem 4 and Remark 5), among the ExPow portfolio, the αSPOP, and the αUP, then it is of interest to juxtapose these numerically and empirically.

2. Ex post logarithm mean of portfolio wealth-relatives: Since it is proved, by Theorem 8, 10, 9, and Corollary 1, that the αSPOP and the αUP converge to the ExPow portfolio and so their asymptotic growth rates, i.e. $\lim_{T\to\infty}\frac{1}{T}log(V_T)$ are the same, we investigate the log mean empirically at finite measurement periods of the portfolio performance. So we compare these for each investor identified by the relative risk aversion coefficient α.

3. Turnover: This is the measure revealing how much the portfolio changes its weights. This is defined as the ratio of the amounts of rebalances being summed up for all securities in absolute value relative to the entire portfolio value before rebalancing. Derivation and details are placed in section 7.5.

4. The Kullback-Leibler distance: The Kullback-Leibler distance (KL distance) measures how divergent any portfolio is from the EP in our analysis. This is an information number and satisfies the axioms of distance. The KL distance from the EP $\boldsymbol{b}^m =$

$\left(\frac{1}{m}, \ldots, \frac{1}{m}\right)'$ to any portfolio $\boldsymbol{b}$ is defined as follows.

$$D(\boldsymbol{b}\|\boldsymbol{b}^m) = \sum_{i=1}^{m} b_i \log \frac{b_i}{b_i^m} = \sum_{i=1}^{m} b_i(\log m + \log b_i) = \log m - H(\boldsymbol{b})$$

where $H(\boldsymbol{b}) = \sum_{i=1}^{m} b_i(-\log b_i)$ is entropy, and so, is non-negative. One reason for using this distance is that this number is bounded by $\log m$. I.e., $0 \leq D(\boldsymbol{b}\|\boldsymbol{b}^m) \leq \log m$. And note that $\boldsymbol{b} = \boldsymbol{b}^m$ reaches the lower bound. And the upper bound is given by the portfolio investing in only one stock.

Now we proceed to the results.

7.3 Results in the U.S. Stock Market

7.3.1 Varying Learning Periods L

First, fix the relative risk aversion, the number of stocks, and the transaction costs to be $\alpha = 1.0$ (the ExLog investor), $m = 74$ (composing the S & P 100), and $\gamma = 1\%$, respectively. The results for varying the learning periods are given in Table 7.1, and Figure 7.1. The result here represents the main finding and the implication of this empirical analysis, and subsequent results are given to support these.

Generally, both the SPOP and the UP cannot exceed the EP in the ex post mean of the power-utility, with any leaning periods. The shape of ex post utility measured against the learning periods L is U-shape, for both the SPOP and the UP. But a distinct difference between the two is the robustness in learning periods. As we can see in the turnover in Table 7.1, the SPOP is not able to learn long-living portfolios consistently, except $L = ALL$. Then it changes its weights frequently. To make matters worse, the ex post utility of the SPOP varies among learning periods. From the KL distance given in Table 7.1, the UP is to learn the portfolio near the EP, regardless of learning periods. So from a viewpoint of learning periods, the UP is a better learning scheme than the SPOP.

To consider the U-shape of the ex post utility, let us pay attention to the shortest and the longest learning periods. Slight as it is, the excess ex post utility compared to the EP can be seen for both the SPOP and the UP with the longest learning periods ($L = ALL$). With the shortest learning periods ($L = 3$), the ex post utility is at the same level. These two similar results imply quite different meanings.

With $L = ALL$, the KL distance, which is bounded by the natural logarithm of the number of stocks m and about 4.3 in this case, shows both the SPOP and the UP are near to the EP. This means that after long enough learning periods, both the SPOP and the UP get nearer to the portfolio which is close to the EP. And both the SPOP and the UP with longest learning periods keep their turnover low compared to the other learning periods (Table 7.1). This is because these portfolios don't have to change their weights drastically, for the chosen portfolios will be good ones as they were in the past. Then, we can say that both the SPOP and the UP with enough learning are roughly constant as the ideal ExPow portfolio is supposed to be. This finding implies that the two portfolios, after long enough learning periods, converge to the portfolios which duplicate the ExPow portfolio. Also, these duplicated portfolios are near the EP in the KL distance sense and achieve almost the same highest ex post utility as the EP does. Furthermore, the results in the continuous-time framework, given in chapter 3 and 4, asserts that both the SPOP and the UP are the optimal $\mathcal{G}_t$-predictable portfolios for the log-utility investors, under incomplete Information 2. Then, for the $\alpha = 1$ (log-utility) investor in this case, these observations imply that the EP can be roughly regarded as the market portfolio for the log-utility investors. That's why most fund managers rarely outperform the EP in the practical market where transaction costs exist, recalling that the log-utility investor aims for growth maximization.

Then turning to the result with $L = 3$, the SPOP is quite different from the EP in the KL distance sense. This is because the concave optimization is executed with few states of returns relative to the number of stocks, and its optimal portfolio is composed of only a few stocks having positive weights [2]. But the UP is quite close to the EP. This is from the modestly adaptive property of the UP. Since the UP starting from the EP is intended to become the "winning horse", it requires longer price-relative sequences. So with shorter learning periods such as three months, the additional information from the EP is not incorporated into the UP. Since the EP is near the ExPow portfolio according to the result with $L = ALL$, the consequence that both the EP and the UP achieve the same utility is acceptable. While the SPOP is quite different from the EP and UP, it does yield the same performance. This is because extracting a portfolio in maximal growth from some information batch and changing the portfolio weights from time to time to get prominent utility falls victim to transaction costs. This is exhibited in the turnover in Table 7.1. In the SPOP, the turnover is relatively high compared to the other portfolios, except the SPOP with the longest learning periods. Hence the SPOP with shorter learning periods has a potential to provide a prominent utility compared to the duplicated ExPow portfolios, which are equivalent to the SPOP and the UP with all the available information, which, in turn, are near the EP. But that potential heavily depends on the batch of price-relative vectors, judged from Figure 7.1 and the first column of Table 7.1. The SPOP with shorter learning periods has a possibility to achieve the same or higher utility only when the investor composes the SPOP using a batch of price-relative vectors which affect the next holding period's realization sufficiently enough to reverse the disadvantage in transaction costs.

According to the above results with shortest and longest learning periods, we can say that the general character of the U.S. stock market is such that extremely short or long sequences of returns in the past reflect worthwhile information for the log-utility investor and affect the ex post portfolio performance. That is, short term investors who are investing in the portfolios as if they were concentrating on a series of short term play-offs, and longer

[2]Strictly speaking from the mathematical programming, the number is bounded by the number of states plus the number of the budget constraint. So the number of stocks having positive weights in our analysis is $(L + 1)$, at most.

term investors who are equally allocating their wealth to stocks which have been blue chips for a long time, both get rational utility provided by the market EP. These findings can be linked with the concept, which is used in explaining stock price behavior, called "mean reversion"[15, 41]. There seems to exist permanent and worthwhile information in stock prices for the log-utility investor, but some fads or swaying information (temporary information) may be added and overwhelm it. Investors can directly observe prices only, and are unable to directly observe this mixture of information that make up these prices. Thus, they are unable to directly pick up worthwhile information. Reviewing our results given this point of view, longest learning periods enable the log-utility investors to pick up the permanent components. Extra ex post utility with shortest L is provided by the transitory components.

Table 7.1: Portfolio performance measured against the learning months L. All the other parameters are set as $\alpha = 1.0$, $m = 74$, and $\gamma = 1\%$. First column title "$\overline{u(x; \alpha = 1.0)}$" shows the ex post mean of the power-utility in percentage, the second shows the turnover in percentage for each portfolio, and the third column title "KL Distance" shows the Kullback-Leibler distance from the EP, which is bounded by 4.3.

L	$\overline{\mathbf{u}(\mathbf{x}; \alpha = \mathbf{1.0})}$(%)			**TurnOver** (%)			**KL Distance**		
	smpl.	univ.	Eq.	smpl.	univ.	Eq.	smpl.	univ.	Eq.
3	1.457124	1.487579	1.484813	98.579833	4.569605	4.585246	3.298277	0.0056	0
12	-0.659788	1.473479	1.484813	65.862496	5.047974	4.585246	2.624977	0.019664	0
36	0.729961	1.39445	1.484813	38.581654	6.132685	4.585246	2.24419	0.072564	0
60	0.491328	1.340188	1.484813	26.683572	5.370208	4.585246	1.309914	0.082795	0
120	1.139812	1.352259	1.484813	17.223358	6.128498	4.585246	0.843155	0.178007	0
ALL	1.50841	1.520991	1.484813	4.330022	2.016586	4.585246	0.362633	0.355289	0

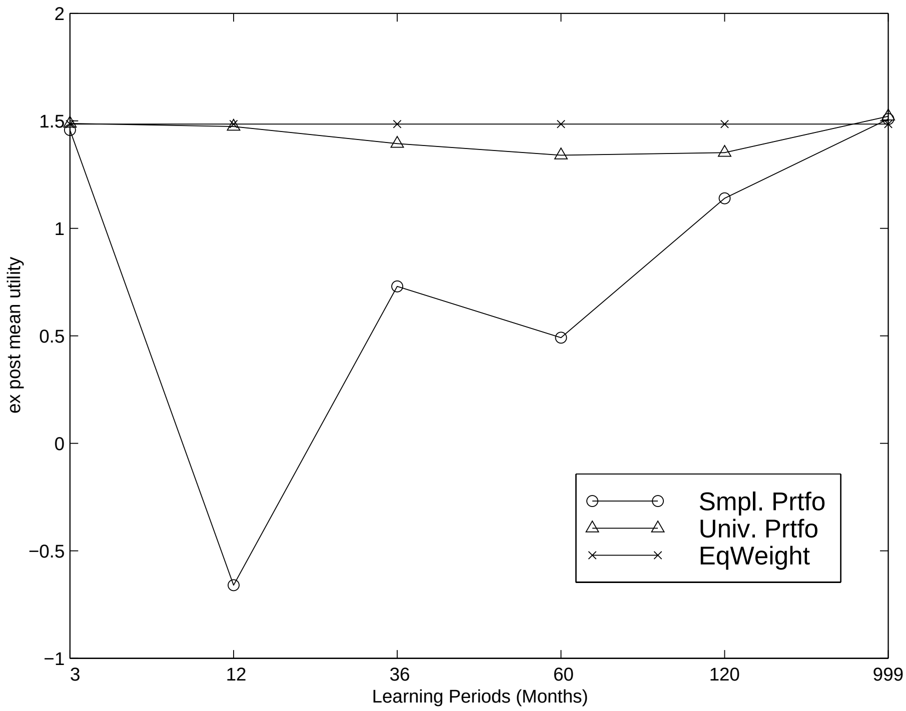

Figure 7.1: Ex post mean of the power-utility measured against the learning months L. This is a visualization of the first column at Table 7.1. The legend “Smpl. Prtfo” shows the αSPOP, “Univ. Prtfo” shows the αUP, and “EqWeight” shows the EP. These legends are also used in the subsequent Figures 7.2, 7.3, 7.4, 7.5, 7.6 and 7.7.

7.3.2 Varying the Relative Risk Aversion α

The results for varying the relative risk aversion α, with all the other parameters fixed except learning periods L, are given in this subsection. As we remarked in the previous subsection, all the available learning periods $L = ALL$ and the shortest learning periods $L = 3$ have special meaning in the U.S. stock market. Then we will juxtapose these results simultaneously and the subsequent results will be shown in this way. The result with $L = ALL$ is given in Tables 7.2, 7.3, and Figure 7.2. The result with $L = 3$ is in Tables 7.4, 7.5, and Figure 7.3.

Almost the same result as the previous subsection is obtained for a wide variety of α. When we visualize the ex post utility in Figures 7.2 and 7.3, these seem to provide the same results. That is the ex post utility for both the αSPOP and the αUP is bounded by the same ex post utility as the EP, and this holds at any degree of α. Moreover, for any fixed α, the ex post utility with $L = ALL$ and one with $L = 3$ are at the same level. But as described in the results obtained by varying L, these two results imply quite different meanings.

Let us begin with $L = ALL$. From the KL distance in the second column in Table 7.3, if relative risk aversion α is less than or equal to unit, both the αSPOP and the αUP with enough learning stay near the EP. But as the investor gets more risk averse than unit, the αSPOP and the αUP get more divergent from the EP. The difference is that the former is not monotonic divergence, but the latter is monotonic divergence. It is conjectured that as we ruled out safer assets from investors' choices, the high α investors search for and choose more modest stocks from time to time. As a result, their optimal portfolios are composed of a few stocks, far from the EP. The relatively high turnover in the first column of Table 7.3 support this. And the different tendencies in divergence for two portfolios are a result of the different learning algorithms and different techniques to eliminate α-scaled price-relatives in negative value that are used.

These tendencies are also seen to support this conjecture in the log mean (the second column in Table 7.2). Since we proved theoretically that both the αSPOP and the αUP converge to the ExPow portfolio, and so the expected growth rate, it is expected that this property holds empirically. From the result, when α is equal to or less than 2, the difference of log mean among the three portfolios is within 10 %. But when α is more than 2, the differences increase to about 35 %.

Another point of view for the ex post log mean is whether the phenomenon, to coin a phrase, "ExLog-reversion", really occurs. That is, the ex post log mean is really maximized when the investor has α of one or the log-utility investor. Since we observed that batches of stock price processes are not uniform concerning whether they contain worthwhile information for the log-utility investors, we are apt to conjecture that the stock market involves undulation, and so a winning horse often turns into a losing one. Then we attempt to think that it may occur that some α other than $\alpha = 1$ provides the best log mean in the practical market. Such ExLog-reversion, however, was not obtained in our analysis. Within the 0.15 % error allowed for the αUP, the best ex post log mean is obtained by $\alpha = 1$ investor with all the available learning periods. And this holds for both the αSPOP and the αUP. I.e., we can conclude the log-utility investor captures the best log mean.

Then $L = 3$ is dealt with. Surprising enough, though the KL distance in the second column at Table 7.5 is quite different, the three portfolios are at the same ex post utility level for any α. It is understandable that since the αUP with shortest learning doesn't incorporate additional information from the EP, it performs as well as the EP. The reason why the αSPOP is also good is conjectured as follows. As we described earlier, the permanent and worthwhile information for the power-utility investor, as well as the log-utility investor, can be extracted with sufficiently long learning periods. And with all the other shorter learning periods, they fail to pick up the valuable information due to transitory components in stock prices. But the shortest learning periods, such as three months, have special meaning in the U.S. market. The U.S. market is likely to respond to "news" rapidly, and decide its position in haste. So, transitory as it is, the valuable information and the possibility to catch a favorable short term move in the market by taking advantage of short term information seems to exist.

For the end of this subsection, we comment on the EP. From the above results with $L = ALL$ and $L = 3$, the EP cannot always be regarded as portfolios duplicating ExPow portfolios, but it surely serves as an index or benchmark for any power-utility investor, under the existence of transaction costs.

Table 7.2: Portfolio performance measured against the relative risk aversion α, with all the learning months, part 1.

	$\overline{\mathbf{u}(\mathbf{x};\alpha)}$			$\overline{\mathbf{log}(\mathbf{x})}$ (%)		
α	smpl.	univ.	Eq.	smpl.	univ.	Eq.
0.1	1.126326	1.126428	1.126326	1.484853	1.495246	1.484813
0.2	1.265174	1.265364	1.265173	1.48489	1.50419	1.484813
0.3	1.443705	1.443967	1.443704	1.484925	1.511457	1.484813
0.4	1.68176	1.682075	1.681758	1.484956	1.516901	1.484813
0.5	2.015053	2.015404	2.015051	1.484985	1.520516	1.484813
0.6	2.514912	2.515383	2.51501	1.475037	1.522511	1.484813
0.7	3.348343	3.348683	3.348303	1.489066	1.52318	1.484813
0.8	5.015077	5.015307	5.014929	1.49976	1.522874	1.484813
0.9	10.015108	10.01526	10.014889	1.506762	1.522079	1.484813
1	0.015084	0.01521	0.014848	1.50841	1.520991	1.484813
1.1	-9.98611	-9.984906	-9.985192	1.394915	1.513298	1.484813
1.2	-4.985947	-4.985033	-4.985232	1.417199	1.504443	1.484813
1.3	-3.319126	-3.318495	-3.318606	1.438279	1.495508	1.484813
1.4	-2.48567	-2.485295	-2.485313	1.456114	1.486063	1.484813
1.5	-1.985573	-1.985437	-1.985353	1.471193	1.47573	1.484813
2	-0.987057	-0.986421	-0.985552	1.345792	1.396693	1.484813
2.5	-0.653051	-0.654594	-0.652416	1.43868	1.264007	1.484813
5	-0.241311	-0.240647	-0.236709	1.005266	1.161833	1.484813
7.5	-0.145522	-0.148104	-0.141476	1.060512	0.9645	1.484813
10	-0.101741	-0.104782	-0.099626	1.300862	1.156834	1.484813

Table 7.3: Portfolio performance measured against the relative risk aversion α, with all the available learning months, part 2.

	TurnOver (%)			**KL Distance**		
α	smpl.	univ.	Eq.	smpl.	univ.	Eq.
0.1	4.583858	4.295813	4.585246	0	0.005316	0
0.2	4.582467	4.052833	4.585246	0	0.020612	0
0.3	4.581073	3.815362	4.585246	0.000001	0.04458	0
0.4	4.579676	3.568026	4.585246	0.000001	0.075693	0
0.5	4.578275	3.310423	4.585246	0.000002	0.112493	0
0.6	5.957042	3.043942	4.585246	0.008831	0.153833	0
0.7	5.240757	2.776918	4.585246	0.07005	0.198998	0
0.8	4.7758	2.518529	4.585246	0.163392	0.247677	0
0.9	4.468189	2.261662	4.585246	0.265749	0.299799	0
1	4.330022	2.016586	4.585246	0.362633	0.355289	0
1.1	11.999	2.422605	4.585246	3.052692	0.413813	0
1.2	11.974684	2.932496	4.585246	3.00145	0.474615	0
1.3	11.890004	3.414844	4.585246	2.964329	0.536499	0
1.4	11.755959	3.87537	4.585246	2.937174	0.597961	0
1.5	11.640188	4.308086	4.585246	2.916774	0.657436	0
2	21.842174	6.173865	4.585246	2.724796	0.893774	0
2.5	12.173995	7.667517	4.585246	2.44517	1.075259	0
5	16.389422	6.646638	4.585246	1.853065	3.963973	0
7.5	11.173294	8.225325	4.585246	2.035929	4.156608	0
10	12.828976	6.10599	4.585246	2.47477	4.255995	0

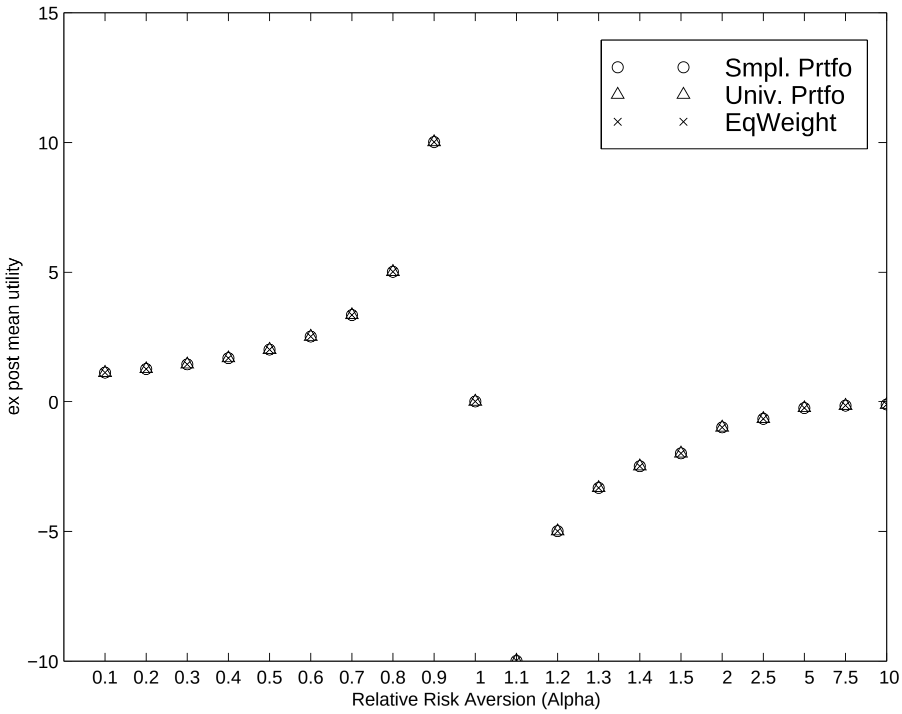

Figure 7.2: Ex post mean of the power-utility measured against the relative risk aversion α, with all the available learning months.

Table 7.4: Portfolio performance measured against the relative risk aversion α, with three learning months, part 1. The results for αSPOPs of $\alpha = 7.5$ and 10 are not listed, since no feasible solutions are obtained.

	$\overline{\mathbf{u}(\mathbf{x};\alpha)}$			$\overline{\mathbf{log}(\mathbf{x})}$ (%)		
α	smpl.	univ.	Eq.	smpl.	univ.	Eq.
0.1	1.125977	1.126372	1.126326	1.441137	1.489451	1.484813
0.2	1.2653	1.265258	1.265173	1.460891	1.493361	1.484813
0.3	1.443842	1.443819	1.443704	1.44927	1.496387	1.484813
0.4	1.681865	1.681894	1.681758	1.442701	1.498483	1.484813
0.5	2.015115	2.015194	2.015051	1.440835	1.499296	1.484813
0.6	2.515055	2.515149	2.51501	1.444914	1.498839	1.484813
0.7	3.348289	3.348426	3.348303	1.447756	1.497274	1.484813
0.8	5.014827	5.015027	5.014929	1.449586	1.494789	1.484813
0.9	10.014725	10.014955	10.014889	1.455346	1.491527	1.484813
1	0.014571	0.014876	0.014848	1.457124	1.487579	1.484813
1.1	-9.985837	-9.985207	-9.985192	1.436656	1.483212	1.484813
1.2	-4.985991	-4.985293	-4.985232	1.441911	1.47854	1.484813
1.3	-3.319504	-3.318713	-3.318606	1.444988	1.47367	1.484813
1.4	-2.486371	-2.485468	-2.485313	1.446278	1.468663	1.484813
1.5	-1.986549	-1.985557	-1.985353	1.449872	1.463501	1.484813
2	-0.987623	-0.986006	-0.985552	1.450348	1.436754	1.484813
2.5	-0.65534	-0.65312	-0.652416	1.453682	1.409322	1.484813
5	-0.244256	-0.23858	-0.236709	1.410812	1.272905	1.484813
7.5	-	-0.144305	-0.141476	-	1.151672	1.484813
10	-	-0.103428	-0.099626	-	1.033669	1.484813

Table 7.5: Portfolio performance measured against the relative risk aversion α, with three learning months, part 2. The results for αSPOPs of $\alpha = 7.5$ and 10 are not listed, since no feasible solutions are obtained.

	TurnOver (%)			**KL Distance**		
α	smpl.	univ.	Eq.	smpl.	univ.	Eq.
0.1	65.441217	4.142574	4.585246	0.89973	0.000057	0
0.2	84.604299	3.773677	4.585246	1.918163	0.000228	0
0.3	90.252877	3.494295	4.585246	2.397556	0.000512	0
0.4	93.089367	3.309223	4.585246	2.679378	0.000907	0
0.5	94.800474	3.253532	4.585246	2.865959	0.001414	0
0.6	95.948008	3.325854	4.585246	2.999513	0.002032	0
0.7	96.796242	3.510219	4.585246	3.099743	0.00276	0
0.8	97.52226	3.787685	4.585246	3.180501	0.003597	0
0.9	98.082771	4.143887	4.585246	3.247029	0.004544	0
1	98.579833	4.569605	4.585246	3.298277	0.0056	0
1.1	100.458701	5.038305	4.585246	3.553638	0.006764	0
1.2	100.588142	5.538401	4.585246	3.577047	0.008036	0
1.3	100.752466	6.05933	4.585246	3.596629	0.009416	0
1.4	100.944397	6.594983	4.585246	3.615464	0.010904	0
1.5	101.120767	7.146903	4.585246	3.631083	0.012501	0
2	101.966795	10.013566	4.585246	3.694865	0.022121	0
2.5	102.558725	12.917971	4.585246	3.734717	0.034297	0
5	104.013557	26.665728	4.585246	3.818912	0.130009	0
7.5	-	38.886159	4.585246	-	0.277736	0
10	-	49.237352	4.585246	-	0.448784	0

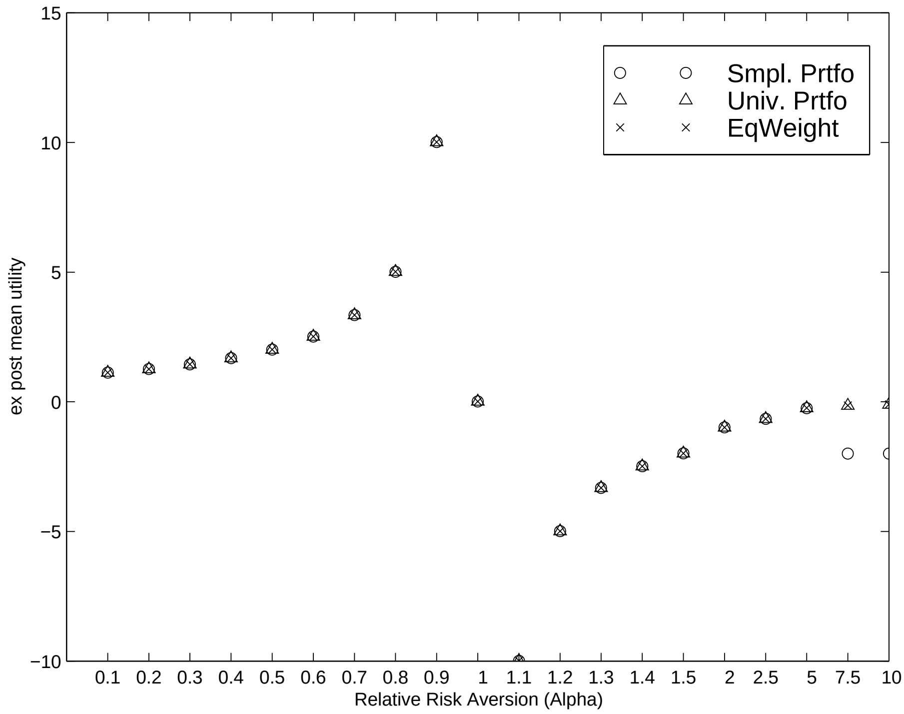

Figure 7.3: Ex post mean of the power-utility measured against the relative risk aversion α, with three learning months.

7.3.3 Varying Number of Stocks m

The results with $L = ALL$ and $L = 3$ are given in Table 7.6, Figure 7.4, and Table 7.7, Figure 7.5, respectively.

With $L = ALL$, from Table 7.6 and Figure 7.4: as the asset universe gets smaller, all three portfolios increase their ex post utility, except when $m = 295$. This trend shows that the effect of asset choice exists. That is, the EP composed of selected stocks in an elitism course outperforms the EP of the entire market. In every asset universe except $m = 74$, the EP performs best. And as the universe gets larger, the KL distance (third column in Table 7.6) of the SPOP and the UP gets larger; and the ex post utility becomes worse (Figure 7.4). This trend is most striking in the SPOP, which is conjectured as follows. As we pointed out in the result by varying learning periods, when the concave optimization is executed with few states of returns relative to the number of stocks, its optimal portfolio is composed of only a few stocks having positive weights. So even if investors use the full 299 months of learning periods, the SPOP for the next (300th) holding period is at most composed of 300 stocks having positive weights. Then the larger the universe gets, the more divergent the SPOP becomes from the EP. This reasoning can also be applied to the UP.

Then proceed to $L = 3$. From Figure 7.5, different results are obtained for the SPOP. As asset universes get larger, the ex post utility of the SPOP portfolio increases, except when $m = 295$. When $m = 749$, for the first time, a portfolio clearly outperforms the index EP in ex post utility. Turnover above 100 % (Table 7.7) tell us too much transaction cost is being incurred. As we stated earlier, it can be said that transitory but worthwhile information exists sufficient to reverse the disadvantage in transaction costs. And to extract these most successfully, the SPOP with the entire asset universe and three months learning should be used, because the SPOP selects its optimal portfolio in the maximal growth. From the mathematical programming viewpoint, the larger the universe gets, the more likely a bigger objective can be found. But note that this SPOP is no longer a good duplication of the

ExLog portfolio. Rather, this SPOP should be regarded as a transitory one which will become stationary if given sufficient learning periods. Or as a counterexample, the U.S. stock market is not efficient in the short term even if the market requires proper costs.

Table 7.6: Portfolio performance measured against the number of stocks m, with all the available learning months. The figures in parentheses in the third column show the maximal KL distance from EP.

	$\overline{\mathbf{u}}(\mathbf{x};\alpha=\mathbf{1.0})$(%)			**TurnOver** (%)			**KL Distance**		
	smpl.	univ.	Eq.	smpl.	univ.	Eq.	smpl.	univ.	Eq.
27	1.625279	1.607867	1.647735	3.298554	1.972046	3.838582	0.272942	0.330106	0 (3.295837)
74	1.50841	1.520991	1.484813	4.330022	2.016586	4.585246	0.362633	0.355289	0 (4.304065)
295	0.628351	0.936103	1.15648	5.134374	2.176637	4.593715	1.758318	0.528562	0 (5.686975)
749	0.652023	0.950024	1.165783	6.254216	2.249831	5.65667	2.162602	0.578509	0 (6.618739)

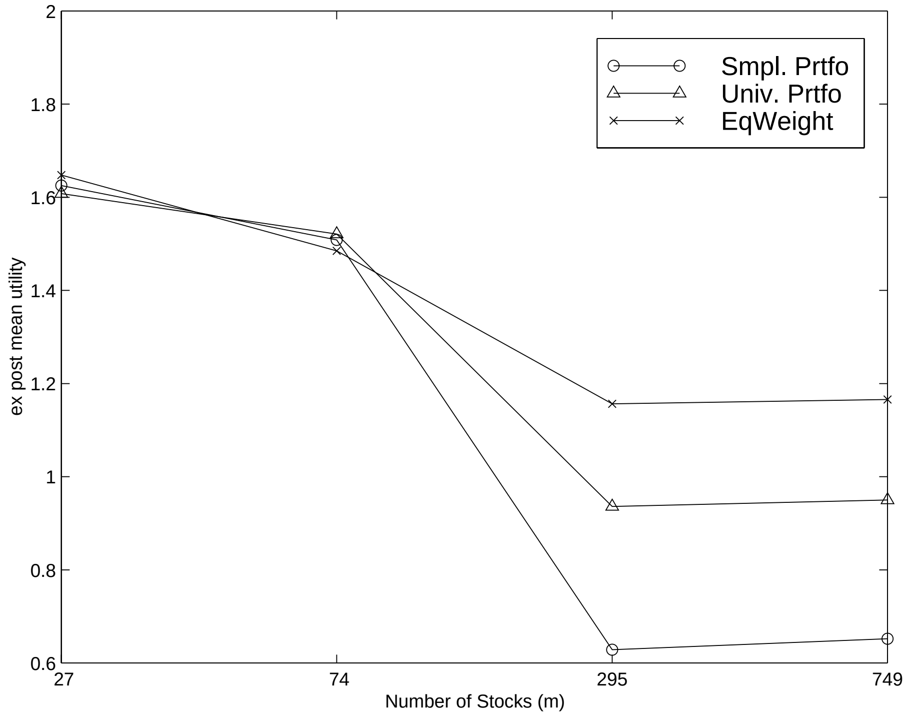

Figure 7.4: Ex post mean of the power-utility measured against the number of stocks m, with all the available learning months.

Table 7.7: Portfolio performance measured against the number of stocks m, with three learning months. The figures in parentheses in the third column show the maximal KL distance from EP.

	$\overline{\mathbf{u}(\mathbf{x};\alpha = \mathbf{1.0})}(\%)$			**TurnOver** (%)			**KL Distance**		
	smpl.	univ.	Eq.	smpl.	univ.	Eq.	smpl.	univ.	Eq.
27	0.202003	1.642577	1.647735	98.949365	3.748113	3.838582	2.127756	0.003615	0 (3.295837)
74	1.457124	1.487579	1.484813	98.579833	4.569605	4.585246	3.298277	0.0056	0 (4.304065)
295	-0.855513	1.133832	1.15648	115.947728	4.591277	4.593715	4.738275	0.00532	0 (5.686975)
749	3.09197	1.171303	1.165783	108.998712	5.740708	5.65667	5.875138	0.010179	0 (6.618739)

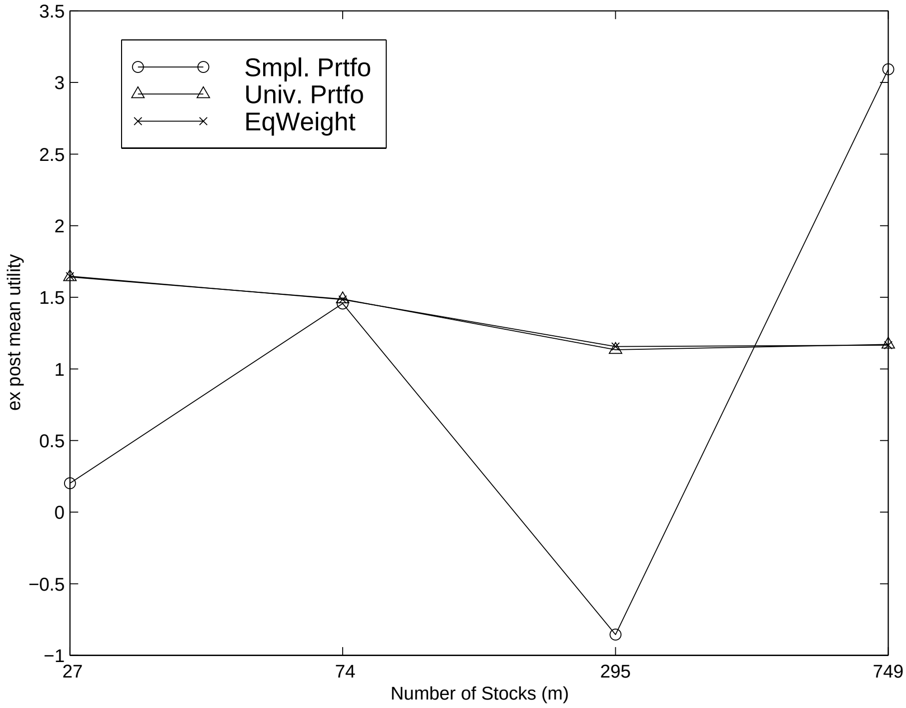

Figure 7.5: Ex post mean of the power-utility measured against the number of stocks m, with three learning months.

7.3.4 Varying Transaction Costs γ

The results with $L = ALL$ and $L = 3$ are given in Table 7.8, Figure 7.6, and Table 7.9, Figure 7.7, respectively.

With $L = ALL$, the order in ex post utility among the three portfolios basically does not change, according to transaction costs. The differences among portfolios are within 5 %. This is because all the portfolios with $L = ALL$ are in the duplicated ExLog portfolios and are almost constant. These then, will not fall victim to transaction costs. Among these, the UP is most robust in costs. The adaptive property of the UP to newly added information has the advantages of not only attaining the ExLog portfolio but also reducing the redundancy of a perfectly constant portfolio in the presence of costs. By adapting to the winning portfolio as it was in the past, the UP with $L = ALL$ will do well in the next holding period, does not have to change its weights, and keeps the turnover low.

With $L = 3$, the SPOP is quite different from the index EP and changes its ex post utility badly as cost requirements increase. As pointed out earlier, given no cost requirement, there seems to exist a favorable portfolio to get prominent utility. But such a portfolio is likely to change its weights excessively due to short-term information and be defeated by costs. From Figure 7.7 and Figure 7.1, when market impact is put aside, it is rational for the market to require 1 % cost to use shorter-term information which is highly correlated to the next holding period.

Table 7.8: Portfolio performance measured against the transaction costs γ, with all the available learning months. Note that in this table and Table 7.9, the KL distance for both the SPOP and UP is the same at any cost level. This is because transaction costs are not taken into account in searching for the optimal solutions in either portfolio. Just using the optimal portfolio sequence, each portfolio value process with transaction costs defined by (7.2) is calculated to provide the results.

	$\overline{\mathbf{u}(\mathbf{x};\alpha=\mathbf{1.0})}(\%)$			**TurnOver** (%)			**KL Distance**		
$\gamma(\%)$	smpl.	univ.	Eq.	smpl.	univ.	Eq.	smpl.	univ.	Eq.
0	1.552051	1.54123	1.531048	0	0	0	0.362633	0.355289	0
1	1.50841	1.520991	1.484813	4.330022	2.016586	4.585246	0.362633	0.355289	0
2	1.464772	1.500813	1.438561	4.296127	2.006244	4.548191	0.362633	0.355289	0

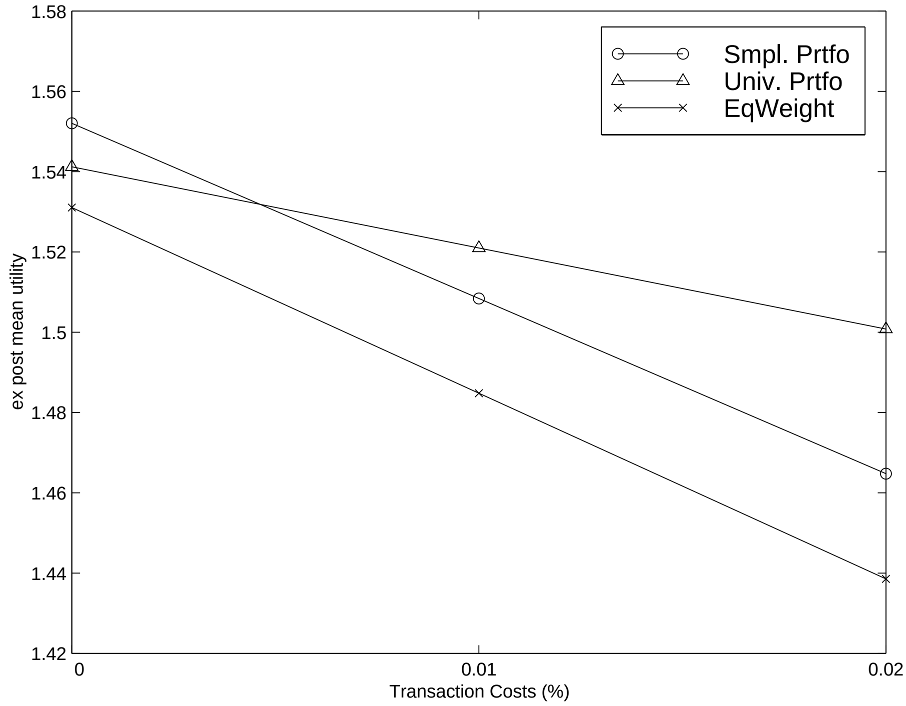

Figure 7.6: Ex post mean of the power-utility measured against the transaction cost γ, with all the available learning months.

Table 7.9: Portfolio performance measured against the transaction costs γ, with three learning months.

$\gamma(\%)$	$\overline{\mathbf{u}(\mathbf{x};\alpha=\mathbf{1.0})}(\%)$			**TurnOver** (%)			**KL Distance**		
	smpl.	univ.	Eq.	smpl.	univ.	Eq.	smpl.	univ.	Eq.
0	2.674708	1.533655	1.531048	0	0	0	3.298277	0.0056	0
1	1.457124	1.487579	1.484813	98.579833	4.569605	4.585246	3.298277	0.0056	0
2	0.239395	1.441493	1.438561	81.090473	4.532501	4.548191	3.298277	0.0056	0

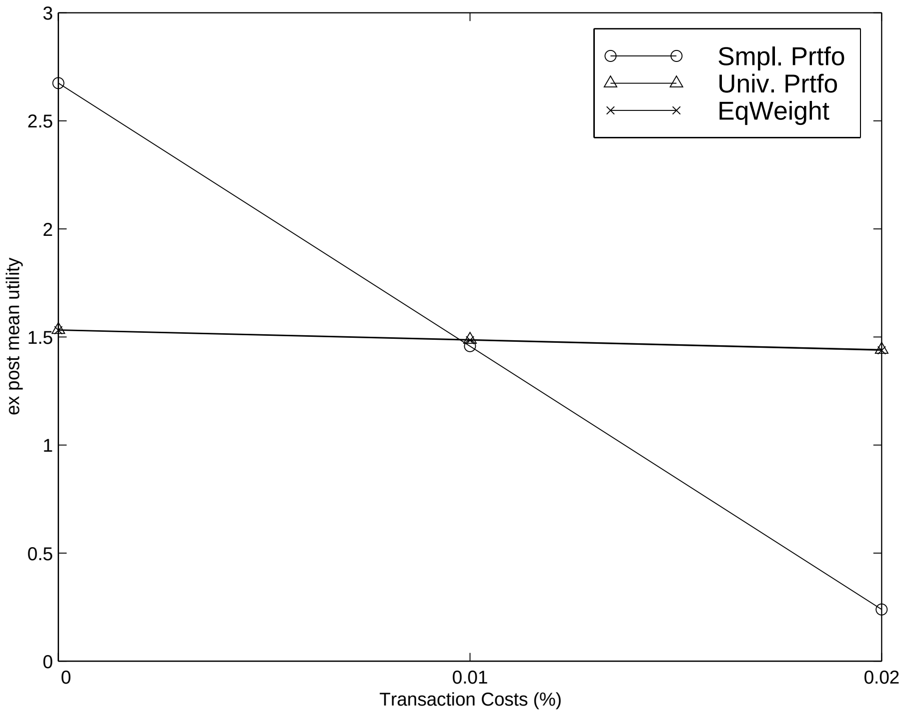

Figure 7.7: Ex post mean of the power-utility measured against the transaction cost γ, with three learning months.

7.4 Summary

The results of our continuous-time analysis hold to some extent in the practical stock market. That is, after long enough learning, both the αSPOP and the αUP converge to the portfolio duplicating the ideal ExPow portfolio if the α is less than or equal to one. The reason for this conclusion is stated as follows.

1. With all the available learning periods, the ex post utility is obtained within the highest range.

2. Two portfolios steadily learned the portfolios at the same distance from the EP in the KL distance sense.

3. Two portfolios learned to be constant (myopic), as the ExPow portfolio is supposed to be. This can be seen in their turnover which is kept as low as the EP.

4. Our theory given in the previous chapters asserts that both the SPOP and the UP are asymptotically optimal for the log-utility investors under incomplete Information 2.

However, this conclusion should not be interpreted to mean power-utility investors could use any learning portfolio to achieve the ExPow portfolio. Concerning the learning periods, the αUP is very robust. Moreover, regarding transaction costs, the αUP is rarely defeated by them. Even if compared to the EP of a perfectly constant one, the αUP is superior. This is because the αUP is endowed by its definition with a modestly adaptive property to newly added price-relatives. In this sense, the αUP has the possibility to learn the ExPow portfolio most successfully.

For other α, however, it cannot be said that both the αSPOP and the αUP have learned the portfolio duplicating the ExPow portfolio due to lack of evidence. For the two portfolios are located in quite distinct points and these change their weights rather frequently. The reason for this is there are no candidates to be selected steadily in the U.S. stock market for highly risk averse investors. This is quite a natural consequence and so future empirical analysis including safer assets needs to be conducted.

Besides the above results verifying our theory in the continuous-time framework, we have shown some interesting findings regarding the U.S. stock market via this empirical analysis. First, extremely short or long sequences of stock prices in the past include worthwhile information for the power-utility investor and affect the ex post portfolio performance. This implies that the permanent components essential for the power-utility investors surely exist, and allows the investors to learn the ExPow portfolio. And the transitory components that affect the holding period performance also exist. These are most successfully extracted by using the αSPOP for the larger universe, and have the possibility to provide the prominent utility. The αSPOP using these components, however, has to change its weights frequently, and is defeated by transaction costs. Thus, it cannot duplicate the ExPow portfolio. So batches of stock price sequences in the U.S. stock market are not uniform concerning whether they contain worthwhile information for the power-utility investors. But judging from these findings, one should not fall into the dogma that there exists ExLog-reversion. For it is not observed that any α investor gets a higher ex post log mean than the $\alpha = 1$ investor.

Second, for any power-utility investor, the ex post utility obtained by both the αSPOP and αUP rarely exceed the one obtained by the EP, under the existence of transaction costs. This result is quite surprising since these three portfolios are quite distinct ones, judging from the KL distance. And limiting α to near unit, the ExPow portfolio is learned to be near the EP in the KL sense. Taking the above findings into account, the EP can be regarded as the index or benchmark portfolio for any power-utility investor. Moreover, especially when investors have the utility near $\alpha = 1$ (log-utility), the EP can be roughly regarded as the market ExLog portfolio. Recalling that the $\alpha = 1$ investor is the log-utility investor and aims for growth maximization, it becomes clear why fund managers can rarely outperform the EP in the market.

Third, we can say that the U.S. stock market is efficient in the weak form for the power-utility investors, given costs of 1 % [14, 16]. That is, stock returns reflect all the available information which is wothwhile for them, and so any portfolio cannot provide extra ex post utility beyond the index EP. Or viewing these from another aspect, the transaction costs of 1 % are the rational requirement for the stock market to prevent any portfolio from extracting transitory information which has close correlation to get the prominent ex post utility.

7.5 Appendix for Chapter 7: Derivation of Portfolio Value after Rebalancing

Here the portfolio value after rebalancing is derived. Let the portfolio and its value at $t-1$ be $\boldsymbol{b}_{t-1}$ and V_{t-1} respectively. The investment on period t is carried out using this portfolio. If the realized price-relative of assets is $\mathbf{X}_t$, then the ratio of i-th asset value to the portfolio value is $b^-_{i,t} \triangleq \frac{b_{i,t-1}X_{i,t}}{\boldsymbol{b}'_{t-1}\boldsymbol{X}_t}$. If we assume $\Delta_{i,t}$ to be the rebalance amount of asset i, then the i-th asset value after rebalancing at time t is

$$V_{i,t} = V_{t-1}b_{i,t-1}X_{i,t} + \Delta_{i,t}\left(1 - \gamma\zeta_{i,t}\right), \tag{7.4}$$

where $\zeta_{i,t} \triangleq 1 \left(\text{ if } b_{i,t} \geq \kappa_t b^-_{i,t}\right)$, $\zeta_{i,t} \triangleq -1 \left(\text{ if } b_{i,t} < \kappa_t b^-_{i,t}\right)$, and κ_t is the rebalance adjustment coefficient. And the entire value of the portfolio after rebalancing is

$$V_t = V_{t-1}\sum_{i=1}^{m} b_{i,t-1}X_{i,t} - \gamma\sum_{i=1}^{m}\Delta_{i,t}\zeta_{i,t} \ . \tag{7.5}$$

At time t, the ratio of i-th asset value to the entire portfolio value should be $b_{i,t}$, so we obtain

$$b_{i,t} = \frac{V_{i,t}}{V_t} = \frac{V_{t-1}b_{i,t-1}X_{i,t} + \Delta_{i,t}\left(1-\gamma\zeta_{i,t}\right)}{V_{t-1}\sum_{i=1}^{m} b_{i,t-1}X_{i,t} - \gamma\sum_{i=1}^{m}\Delta_{i,t}\zeta_{i,t}} \ .$$

Expanding this equation, we get ρ not depending on i as follows.

$$\rho \ = \ \sum_{j=1}^{m} b_{j,t-1}X_{j,t} - \gamma\sum_{k=1}^{m}\widetilde{\Delta}_{k,t}\zeta_{k,t}$$

$$= \frac{b_{i,t-1}}{b_{i,t}} X_{i,t} + \widehat{\Delta}_{i,t} \left(1 - \gamma \zeta_{i,t}\right) \ , \tag{7.6}$$

where $\widetilde{\Delta}_{i,t} = \frac{\Delta_{i,t}}{V_{t-1}}$, $\widehat{\Delta}_{i,t} = \frac{\widetilde{\Delta}_{i,t}}{b_{i,t}}$. And we solve the equation of ρ about $\widehat{\Delta}_{i,t}$, $\widetilde{\Delta}_{i,t}$ and $\Delta_{i,t}$, then

$$\begin{aligned}
\widehat{\Delta}_{i,t} &= \frac{\rho - \frac{b_{i,t-1}}{b_{i,t}} X_{i,t}}{1 - \gamma \zeta_{i,t}} \ , \\
\widetilde{\Delta}_{i,t} &= b_{i,t} \widehat{\Delta}_{i,t} = \frac{b_{i,t}\rho - b_{i,t-1} X_{i,t}}{1 - \gamma \zeta_{i,t}} \ , \\
\Delta_{i,t} &= V_{t-1} \widetilde{\Delta}_{i,t} = V_{t-1} \frac{b_{i,t}\rho - b_{i,t-1} X_{i,t}}{1 - \gamma \zeta_{i,t}} \ .
\end{aligned} \tag{7.7}$$

Assigning $\widehat{\Delta}_{i,t}$ into (7.6), then

$$\begin{aligned}
\rho &= \sum_{j=1}^{m} b_{j,t-1} X_{j,t} - \gamma \sum_{k=1}^{m} \frac{\left(b_{k,t}\rho - b_{k,t-1} X_{k,t}\right) \zeta_{k,t}}{1 - \gamma \zeta_{k,t}} \\
&= \sum_{j=1}^{m} b_{j,t-1} X_{j,t} + \sum_{k=1}^{m} \left(1 - \frac{1}{1 - \gamma \zeta_{k,t}}\right) \cdot \left(b_{k,t}\rho - b_{k,t-1} X_{k,t}\right) \ .
\end{aligned}$$

From above, ρ is

$$\rho = \frac{\sum_{i=1}^{m} \frac{b_{i,t-1} X_{i,t}}{1 - \gamma \zeta_{i,t}}}{\sum_{j=1}^{m} \frac{b_{j,t}}{1 - \gamma \zeta_{j,t}}} \ .$$

Whilst, assigning $\Delta_{i,t}$ into (7.5), the portfolio value after rebalancing at t becomes

$$\begin{aligned}
V_t &= V_{t-1} \left(\sum_{i=1}^{m} b_{i,t-1} X_{i,t} - \gamma \sum_{i=1}^{m} \frac{\left(b_{i,t}\rho - b_{i,t-1} X_{i,t}\right) \zeta_{i,t}}{1 - \gamma \zeta_{i,t}} \right) \\
&= V_{t-1} \left\{ \sum_{i=1}^{m} b_{i,t-1} X_{i,t} + \sum_{i=1}^{m} \left(1 - \frac{1}{1 - \gamma \zeta_{i,t}}\right) \left(b_{i,t}\rho - b_{i,t-1} X_{i,t}\right) \right\} \\
&= V_{t-1} \cdot \rho = V_{t-1} \cdot \frac{\sum_{i=1}^{m} \frac{b_{i,t-1} X_{i,t}}{1 - \gamma \zeta_{i,t}}}{\sum_{j=1}^{m} \frac{b_{j,t}}{1 - \gamma \zeta_{j,t}}} \ .
\end{aligned}$$

Next, taking care of the sign of the rebalance amount for i-th asset, the rebalance adjustment coefficient κ_t defined at (7.4) is determined as

$$\Delta_{i,t} = V_{t-1} \frac{b_{i,t}\rho - b_{i,t-1} X_{i,t}}{1 - \gamma \zeta_{i,t}} \begin{matrix} > \\ < \end{matrix} 0$$

$$\iff \quad b^-_{i,t} = \frac{b_{i,t-1}X_{i,t}}{\sum_{j=1}^m b_{j,t-1}X_{j,t}} \begin{array}{c} > \\ < \end{array} \frac{b_{i,t}\rho}{\sum_{j=1}^m b_{j,t-1}X_{j,t}}$$

$$\iff \quad \kappa_t = \frac{\sum_{j=1}^m b_{j,t-1}X_{j,t}}{\rho} \begin{array}{c} > \\ < \end{array} \frac{b_{i,t}}{b^-_{i,t}} \; .$$

Then the rebalance adjustment coefficient κ_t is

$$\kappa_t = \frac{\sum_{i=1}^m \frac{b_{i,t}}{1-\gamma\zeta_{i,t}} \cdot \sum_{j=1}^m b_{j,t-1}X_{j,t}}{\sum_{k=1}^m \frac{b_{k,t-1}X_{k,t}}{1-\gamma\zeta_{k,t}}} \; .$$

Finally, since turn-over is defined as the ratio of the amount rebalanced being summed up for all securities in absolute value relative to the entire portfolio value before rebalancing, then

$$\begin{aligned} \text{Turnover}(\%) &= 100 \cdot \frac{\sum_{i=1}^m |\Delta_{i,t}|}{V_{t-1}\sum_{j=1}^m b_{j,t-1}X_{j,t}} \\ &= 100 \cdot \frac{\sum_{i=1}^m \left| \frac{b_{i,t}\rho - b_{i,t-1}X_{i,t}}{1-\gamma\zeta_{i,t}} \right|}{\sum_{j=1}^m b_{j,t-1}X_{j,t}} \quad (\text{ From } (7.7)) \; . \end{aligned}$$

Chapter 8

Conclusion and the Direction of Future Research

In this paper, we comprehensively treat the dynamic portfolio management for the log-utility and power-utility investors, under incomplete information. As we restrict the power-utility class only to the log-utility, we have shown that the optimal portfolio can be obtained at any finite terminal-time by utilizing the SPOP and the UP. Concerning the SPOP, we showed that the SPOP means the back-test framework of continuous portfolio selection under incomplete information. And we verified the asymptotic optimality of the SPOP by proposing the CBOP and its asymptotic form, the ACBOP, under the prior distribution for $\boldsymbol{\mu}$ being endowed with infinite differential entropy. Our question of how to make the optimal portfolio among all the portfolios which are predictable for the incomplete information emerges quite naturally when we try to put the Expected Utility Maximization Theorem of von Neumann-Morgenstern into application. And we resolve this question to some extent by proposing the SPOP and UP for the log-utility investors. Moreover, as we extend the utility class to the general power-utility, we can show that both the αSPOP and αUP almost surely learns the ExPow portfolio asymptotically, either in theory or by use of algorithm.

Furthermore, the results of our analysis in the continuous-time framework hold to some extent in the practical stock market. That is, after long enough learning, the αSPOP and αUP converge to the portfolio duplicating the ideal ExPow portfolio. Or in some cases, we showed that both the αSPOP and αUP with shorter incomplete information can provide the same ex post utility as that with long enough information.

However, we leave several open problems to our future research. First, we didn't treat the information other than asset price process. Since the practical economic market has several states, for a simple example, the market is in good times or is in hard times. Investors may actively change their investment policy according to the market states. Then our interest emerges naturally. Admitting and incorporating the several economic states which may precede or affect the generations of asset price process, our treatment for the dynamic portfolio management is no longer optimal. Second, the algorithm for the αUP is not enough established. In our future research, we solve the numerous calculation requirements for that algorithm with the help of theories for the αUP. Third, our empirical analysis is rather poor to derive our general results. That is, since the target of our analysis is only the entire US stock market, then we have to support our results hold generally, by applying our schemes to another markets.

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