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A Formal Comparison Theory of Coalition Influence for Group Decision and Negotiation

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Abstract

This thesis deals with formal methods to compare coalition influence for group decision and negotiation. The proposed methods to compare coalition influence are binary relations which compare a pair of coalitions. Games in characteristic function form is often used as a model of coalition formation and negotiation. We assume that all players can communicate each other with complete information. Social welfare function and social choice function are used to describe the situation of group decision in this thesis. The proposed methods compare coalition influence formally in these models. Blockability relation, viability relation and profitability relation for games in characteristic function form are proposed. These relations compare coalition influence from each basis. Examples how these proposed methods work are provided and properties that the proposed methods satisfy in the frameworks are given. The proposed method to compare coalition influence for social welfare function compares two coalitions from the point of view how the opinions which the members of the coalition have are close to the result determined by the decision rule. Proposed methods in this thesis are defined on social welfare function, social choice function or the games in characteristic function form. Coalition influence and allocation result are impacted each other. A method to compare a pair of payoff configuration whose components are payoff to players and coalition structure is provided. Some mathematical properties of the proposed method for payoff configuration are verified by the given theorems. This thesis also provides methods to evaluate coalition influence for group decision and negotiation. Some properties which proposed functions satisfy are verified. We propose blockability value, viability value and profitability value, which are derived from relations for games in characteristic function respectively. It is shown that proposed functions satisfy some ideal properties, called axioms. Moreover, it is given a proposition which shows that blockability value and viability value have complementary relationship. Coalition values derived from existing values for players, which are Shapley value and Banzhaf value, are defined. Propositions which show that the defined coalition values satisfy the proposed axioms are given. Properties of the proposed coalition values are compared through numerical examples.

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Chapter 1

Introduction

1.1 Background

Almost all of social activities are derived from group decision or negotiation in the world. Players who join the group decision or negotiation take actions to increase profit which the players can get in the situation. One of actions which make players' profit increase is forming coalitions by players. The result of the coalition formation will be determined based on the power in the situation. Meanwhile, the rule which determines the result of group decision or resource allocation also affects the coalition influence in the situation. Change of the decision rule fills the gap of powers of players or coalitions in the situation. Such situations can be described as games or collective choice which are mathematical models in cooperative game theory. A group of players all of the members of which agree to take cooperative actions each other is called coalition.

Comparison of coalition influence will help to know properties of group decision rules in such situations. The result of coalition influence comparison will guide a policy which tactics in alternatives each player should take in the situation. Also difference of power between individual players and coalitions can be clarified by evaluation of coalition influence. Providing mathematical method to compare coalition influence will help to analyze group decision and negotiation with a lot of players such as countries and businesses. Studying about impact of decision rule change to coalition influence will also make a contribution to know which coalitions will form.

1.2 Purpose of this Thesis

The purpose of this study is to develop a theory of comparison and evaluation of coalition influence under the background provided above. This thesis proposes formal methods to compare coalition influence for group decision and negotiation. Properties and numerical examples which show how the proposed methods work are provided. Binary relations and functions are employed to compare and evaluate coalition influence respectively in this thesis. A binary relation compares coalitions through pairwise comparison, and a function assigns a real number to each coalition. The formal methods provided in this thesis will allow numerical experiments for the situations of group decision and negotiation.

1.3 Previous Works of this Study

In the framework of simple games, which constitute a special class of games in characteristic function form, there are such methods to compare coalition influence as the desirability relation [7, 48], the blockability relation [18], and the viability relation [22]. The desirability relation compares coalitions with respect to how much the coalitions are close to have enough power to completely control the decision of the situation. The blockability relation compares coalitions with respect to how much they can make other coalitions not have such power. The viability relation compares coalitions with respect to how robust they are over deviation of members. These relations are mathematically defined using the concepts of winning and losing. Because being winning and losing coalitions can be expressed by payoffs 1 and 0, respectively, analogous relations can be defined on games in characteristic function form. The blockability relation and the viability relation are extended to those for games in characteristic function form, respectively.

There are existing values to evaluate players' influence in games in characteristic function form. Shapley [46] proposed a function which assigns a real number to each player, and the real number is interpreted as the expected value of marginal contribution of the player in the case that the players form the grand coalition with a random sequence. Banzhaf value [36] which is another existing function which assigns a real number to each player, and the value is interpreted as the expected value of marginal contribution of the player in the case that the players form the grand coalition when every coalition has the same probability to be formed. Both of Shapley value and Banzhaf value are extended to coalition values which evaluate coalition influence for games in characteristic function form in this thesis.

1.4 Models and Methodologies

This thesis confirms that the binary relation and viability relation for games in characteristic function form are exactly extensions of blockability relation and viability relation for simple games, respectively. Profitability relation is also defined in this thesis.

Profitability relation compares coalitions with a pair from the viewpoint how much the coalition can generate profit by forming coalition with other coalitions. Propositions which shows that the proposed relations satisfies completeness is provided. Relationship between the new relations is verified. These methods are binary relations, which require pairwise comparison of coalitions. Then, in order to know the results of the comparison of the influences of all coalitions, therefore, one needs much computational complexity. So, this thesis proposes new values which show coalition influence based on blockability relation, viability relation and profitability relation to compare coalition influence easily. Each of the values indicates a coalition's influence by a real number, and the bigger the number is, the more influence the coalition has. Two axioms, which are null coalition axiom and symmetry axiom, are introduced. Propositions which shows that the proposed values satisfy these axioms are provided.

There is currently no method to compare coalition influence in the framework of social welfare functions and social choice functions. A new method to compare coalition influence in the framework of social welfare function is provided. The provided method compares coalitions through pairwise comparison from the viewpoint how the coalition's opinion is close to the decision rule. A proposition which expresses that the provided method satisfies monotonicity, which requires that a bigger coalition has more power with respect to the group decision. Blockability relation, viability relation and profitability relation are extended to social choice function. Properties of the binary relations for social choice function are verified.

Acyclicity of relations is examined within the framework of games in characteristic function form. Acyclicity is a weaker concept than transitivity as a property of relation. Acyclicity is one of the important properties of relations, because one can determine the maximal elements with this property. In this paper a proposition which shows that the newly proposed relation on the set of coalitions satisfies acyclicity.

One of the important issues in the field of cooperative game theory is to identify the payoff allocation for players. The payoff allocations in the core [9], the bargaining set [2], the kernel [6], or the nucleolus [44] of a cooperative game have stability in the sense of each definition. Identifying an appropriate payoff allocation for players often requires the consideration on the influence of coalitions in the game [24] upon the payoff allocation. Therefore, this thesis deals with payoff configurations, each of which is defined as a pair of a payoff allocation for players and a coalition structure, and develop a new binary relation for the comparison of the payoff configurations.

1.5 Structure of this Thesis

The structure of this thesis is as follows: The next chapter introduces basic concepts and models which are employed through this thesis. The models which are dealt with in this thesis are binary relations, social welfare functions, social choice functions, transferable utility games and non-transferable utility games.

Chapter 3 proposes methods to compare coalition influence for the models introduced in Chapter 2. Existing comparison methods for simple games are introduced at first in this chapter. For games in characteristic function form, binary relation, viability relation and profitability relation are proposed. Some examples which shows how the proposed relations for games in characteristic function form work are given. Properties of the proposed methods and interrelationship between the proposed relations are verified. Next, methods for comparison of coalition influence for social welfare functions and social choice functions are defined. Some examples which show how the proposed relations for social welfare functions and social choice functions work are provided. Some propositions which show which properties the proposed methods satisfy are given. A method which compares bargaining results for non-transferable utility games is also proposed. Properties of the proposed methods for non-transferable utility games are given. Lastly, conclusions of Chapter 3 is provided.

Chapter 4 deals with evaluation of coalition influence. A function which assigns real number to each coalition is used to evaluate coalition influence. Three coalition values, blockability value, viability value and profitability value, are proposed in this Chapter. Blockability value is derived from blockability relation defined in Chapter 2. Viability value is derived from viability relation defined in Chapter 2. Profitability value is derived from profitability relation defined in Chapter 2. Axioms which are properties that a coalition value should satisfy are defined. Propositions that shows which axioms the proposed coalition values satisfy are given. Some numerical examples of the proposed coalition values are provided. Lastly, conclusions of Chapter 4 is discussed.

Chapter 5 contains a summary of this thesis and future research topics of this study. The summary mentions about this thesis's contributions to the background of this study. The future research topics of this study is discussed lastly.

Chapter 2

Basic Definitions

This chapter introduces the notation and the frameworks to describe the situations of group decision and negotiation. Binary relations are employed to compare coalition influence in this thesis. It is discussed which properties that the proposed binary relations in this thesis satisfy are. Social welfare functions, social choice functions, transferable utility games, and non-transferable utility games, which are used as models to describe the situations of group decision and negotiation throughout this thesis are also introduced in this chapter. A social welfare function is a method of associating with every individual ordering a social preference relation. A social choice function chooses a single alternative as a decision by the society. These two models were discussed in Arrow [1], Fishburn [8], Sen [43], Mas-Collel and Sonnenschein [30], and so on. Transferable utility games describe coalition formation with complete information. Players are allowed to communicate with each other. Each value of a utility function can be transferred between players in a transferable utility game. Some models for negotiation were proposed by Nash [35], Harsanyi [14], Selten [45], Aumann and Mashler [2], Rubinstien [42], and so on. In contrast, it is not assumed that each value of a utility function can be transferred between players in a non-transferable utility game. Properties of players or coalitions which join the group decision and negotiation are also introduced.

2.1 Framework of Collective Choice

2.1.1 Binary Relations

The following properties of binary relations are discussed in this thesis. Let A be a set, and let R be a binary relation on A. For $x, y \in A$, $x \not R y$ denotes that xRy does not hold.

Definition 2.1.1 (Completeness)**.** R is said to be *complete* if xRy or yRx holds for all $x, y \in A$. □

A complete relation R always can determine a result of comparison between any two elements in A.

Definition 2.1.2 (Transitivity)**.** R is said to be *transitive* if it is satisfied that if xRy and yRz, then xRz for all $x, y, z \in A$. □

Definition 2.1.3 (Negative Transitivity)**.** R is said to be *negatively transitive* if it is satisfied that if $x \not R y$ and $y \not R z$, then $x \not R z$ for all $x, y, z \in A$. □

Definition 2.1.4 (Antisymmetry)**.** R is said to be *antisymmetric* if xRy and yRx imply that $y = x$ for all $x, y \in A$. □

In an antisymmetric relation R, two elements in A are the same if they have the relation with each other.

Definition 2.1.5 (Irreflexivity)**.** R is said to be *irreflexive* if $x \not R x$ for all $x \in A$. □

An irreflexive relation R expresses a relation that each element in A does not have relation with itself.

Definition 2.1.6 (Asymmetry)**.** R is said to be *asymmetric* if for all $x, y \in A$, xRy implies $y \not R x$. □

In an asymmetric relation R, there is only unilateral relation.

Definition 2.1.7 (Acyclicity)**.** R is said to be *acyclic* if it satisfies the following condition: If $x_1, \ldots, x_k \in A$, $k \geq 2$, and $x_i R x_{i+1}$ for $i = 1, \ldots, k-1$, then $x_k R x_1$ does not hold. □

Definition 2.1.8 (Linear order)**.** R is said to be *linear order* if it is complete, transitive, and antismmetric. □

$L = L(A)$ denotes the set of all linear orders on A.

2.1.2 Social Welfare Functions

Let N be a finite set with n members. N is called a *society*, a member of N is called a *voter* or a *player*. A non-empty subset of N is called a *coalition*. For a coalition S, $|S|$ denotes the number of members of S. For coalition S, L^S denotes the set of all combination of linear orders on A of the members of S.

Definition 2.1.9 (Social Welfare Function (SWF))**.** Consider a pair (N, A). A *social welfare function* (SWF) is a function F from L^N to L. □

An SWF F determines a preference relation over the alternatives. The next example is dealt with throughout this thesis. Let R be a binary relations A. For $a_1, a_2, a_3 \in A$, $a_1 R a_2 R a_3$ denotes that $a_1 R a_2$ and $a_2 R a_3$ hold.

Example 2.1.1. Consider a pair (N, A) such that $N = \{1, 2, 3\}$ and $A = \{a_1, a_2, a_3\}$. In this case, $L = \{R_1, R_2, R_3, R_4, R_5, R_6\}$, where $a_1 R_1 a_2 R_1 a_3$, $a_1 R_2 a_3 R_2 a_2$, $a_2 R_3 a_1 R_3 a_3$, $a_2 R_4 a_3 R_4 a_1$, $a_3 R_5 a_1 R_5 a_2$, and $a_3 R_6 a_2 R_6 a_1$ hold. Then, we get $|L^N| = 6^3 = 216$. Consider a function $F : L^N \to L$, which determines a relation on A as follows: If $P_2 = P_3$, then $F(P_1, P_2, P_3) = P_2$. Otherwise $F(P_1, P_2, P_3) = P_1$. F gives a relation on A for each element in L^N, hence F is SWF. □

An interesting property related to an SWF is introduced.

Definition 2.1.10 (Dictatorship)**.** Consider a pair (N, A) and an SWF $F : L^N \to L$. An SWF is called dictatorial if there exists $i \in N$ such that $F(P_1, \ldots, P_n) = P_i$ for all $(P_1, \ldots, P_n) \in L^N$. The player i is called a *dictator*. □

If an SWF is dictatorial, the dictator's opinion is always selected by the SWF. Any changes of the other players' opinion do not affect the result of the SWF.

Example 2.1.2. Consider a pair (N, A) such that $N = \{1, 2, 3\}$ and $A = \{a_1, a_2, a_3\}$.. Let $F : L^N \to L$ be $F(P_1, \ldots, P_n) = P_1$ for all $(P_1, \ldots, P_n) \in L^N$. In this case, F is dictatorial. Player 1 is a dictator. □

Consider a non-empty set X. A real valued function $d : X \times X \to \mathbb{R}$ called a *distance function* on X if the following conditions are satisfied: (i) $d(x, y) \geq 0$ for all $x, y \in X$. (ii) $d(x, y) = 0$ is equivalent to $x = y$ for all $x, y \in X$. (iii) $d(x, y) = d(y, x)$ for all $x, y \in X$. (iv) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Inohara [16] [17] introduced the following distance function on L.

Example 2.1.3 (Distance between preferences)**.** A linear order $P \in L$ can be expressed as a sequence of $(a_m, a_{m-1}, \ldots, a_1)$ which means $a_m\ P\ a_{m-1}$, $a_{m-1}\ P\ a_{m-2}$, $\cdots$, $a_2 P a_1$. For a positive integer k, we value an alternating operation n-th alternative a_n and $n+1$-th alternative a_{n+1} as k^{n-1}. For all $P, P' \in L$, the distance is defined as 0 if $P = P'$. If $P \neq P'$, the distance is defined as the sum of the values which are required to the minimum alternating operations to match P and P'. $d^k(P, P')$ denotes the defined distance. □

The bigger the integer k is, the more the patience is required to alternate the alternatives. It is easy to verify that a pair d^k satisfies the conditions of distance function. A proof can be found in [16]

Example 2.1.4. Consider a pair (N, A) in Example 2.1.1. Let d^3 be a distance function on L in Example 2.1.3. For liner orders R_1 and R_2, $d^3(R_1, R_2) = 1$. For R_1 and R_6, $d^3(R_1, R_6) = 5$. It implies that R_6 is farther away than R_2 from R_1 with respect to d^3. □

This example shows that we can provide a numerical evaluation of the distance between two linear orders.

2.1.3 Social Choice Functions

Social choice function expresses a collective decision rule which determines an alternative from preferences of the members in a society over alternatives.

Let $N = \{1, 2, \ldots, n\}$ be a set of n players. Each non-empty subset of N is called a *coalition*, and a coalition $S = \{i_1, i_2, \ldots, i_m\}$ is often denoted by $i_1 i_2 \cdots i_m$ for monotonicity.

Definition 2.1.11 (Social choice function)**.** A *social choice function* (SCF) is a function from L^N to A. □

A social choice function determines an alternative from all preferences of the players.

Example 2.1.5. Consider a pair (N, A) such that $N = \{1, 2, 3\}$, $A = \{x, y, z\}$ and $L = \{R^1, R^2, R^3, R^4, R^5, R^6\}$. Let a function F be $F(R^i, R^j, R^k) = x$ for all $i, j, k \in \{1, 2, 3, 4, 5, 6\}$. In this case, the function F is a social choice function. □

The following properties of an SCF are interesting.

Definition 2.1.12 (Anonymity)**.** An SCF is said to be *anonymous* if for all permutations π of N and for all members $R^N = (R^1, \ldots, R^n)$ of L^N, $F(R^N) = F(R^{\pi(1)}, \ldots, R^{\pi(n)})$. □

The property of anonymity of an SCF means that there is no effect on the value of the SCF of the reshuffle of indices of players.

Example 2.1.6. Consider a pair (N, A) such that $N = \{1, 2, 3\}$, $A = \{x, y, z\}$ and $L = \{R^1, R^2, R^3, R^4, R^5, R^6\}$. Let a function F be $F(R^i, R^j, R^k) = x$ for all $i, j, k \in \{1, 2, 3, 4, 5, 6\}$. In this case, the function F is anonymous because of for all permutations π of N and for all members $R^N = (R^1, R^2, R^3)$ of L^N, $F(R^N) = F(R^{\pi(1)}, R^{\pi(2)}, R^{\pi(3)}) = x$. □

Definition 2.1.13 (Monotonicity)**.** A SCF is said to be *monotonic* if it is satisfied that if the following conditions hold then $F(R_1^N) = x$ holds.
i) $F(R^N) = x$,
ii) $R_1^N \in L^N$,
iii) For all $a, b \in A \setminus \{x\}$ and all $i \in N$, aR^ib if and only if aR_1^ib and xR^ia implies xR_1^ia.
□

Definition 2.1.14 (Paretian)**.** A SCF is said to be *paretian* if if is satisfied that if the following conditions hold then $F(R^N) \neq y$.
i) $R^N \in L^N$,

ii) $x, y \in A$, $x \neq y$,
iii) xR^iy for all $i \in N, x, y \in A$.

□

A paretian SCF does not select an alternative y if there is an alternative x which is better than the alternative y for all players.

Definition 2.1.15 (Winning coalition with respect to SCF)**.** Consider a SCF F. A winning coalition S with respect to F is defined as: if the following conditions hold, then $F(R^N) = x$.
i) $R^N \in L^N$,
ii) $x \in A$,
iii) xR^iy for all $i \in S$ and all $y \in A$. □

If all members in coalition S prefer to x than any other alternatives, the coalition S can get x decision of the society. A coalition which is not winning coalition with respect to a SCF F and empty set are called losing coalition with respect to F.

Example 2.1.7. Consider a pair (N, A) such that $N = \{1, 2, 3\}$, $A = \{x, y, z\}$ and $L = \{R^1, R^2, R^3, R^4, R^5, R^6\}$. Let F be a function such that if aR^mb holds for all $m \in 12$ and all $a, b \in A$, then $F(R^i, R^j, R^k) = a$. Otherwise, let F be a function such that $F(R^i, R^j, R^k) = z$ for all $i, j, k \in \{1, 2, 3, 4, 5, 6\}$. In this case, coalition 12 is a winning coalition with respect to F. □

There is a concept for describing that coalition prevents an alternative from being selected by preference change.

Definition 2.1.16 (Prevention of Collective Choice)**.** Let F be a SCF, let $x \in A$ and let S be a coalition. S is said to *prevent* x if there exists $Q^S \in L^S$ such that for all $R^{N \setminus S} \in L^{N \setminus S}$, $F(Q^S, R^{N \setminus S}) \neq x$. □

Example 2.1.8. Consider a pair (N, A) such that $N = \{1, 2, 3\}$, $A = \{x, y, z\}$ and $L = \{R^1, R^2, R^3, R^4, R^5, R^6\}$. Let F be a function such that if aR^mb holds for all $m \in 12$ and all $a, b \in A$, then $F(R^i, R^j, R^k) = a$. Otherwise, let F be a function such that $F(R^i, R^j, R^k) = z$ for all $i, j, k \in \{1, 2, 3, 4, 5, 6\}$. In this case, coalition 12 prevents x because there exists $Q^{12} \in L^{12}$ such that for all $V^3 \in L^3$, $F(Q^{12}, V^3) \neq x$. □

Definition 2.1.17 (Social choice correspondence)**.** A *social choice correspondence* (SCC) is a function $H : L^N \to 2^A$ for all $R^N \in L^N$, $H(R^N) \neq \emptyset$. □

A social choice correspondence determines a non-empty set of alternatives from a list of preferences of the players.

Definition 2.1.18 (Winning coalition with respect to SCC)**.** Let $S \in 2^N$, $S \neq \emptyset$, and let $B \in 2^A$. S is said to be *winning coalition* for B is defined as: if the following conditions hold, then $H(R^N) \subset B$.
i) $R^N \in L^N$,
ii) xR^iy for all $x \in B$, $y \notin B$, and $i \in S$. □

$\mathbb{W}_B^H$ denotes that the set of all winning coalitions for B with respect to H.

Definition 2.1.19 (α-effective)**.** Let $S \in 2^N$, $S \neq \emptyset$, and let $B \in 2^A$. S is *α-effective for B* is defined as: there exists $R^S \in L^S$ such that $H(R^S, Q^{N\setminus S}) \subset B$ holds for all $Q^{N\setminus S} \in L^{N\setminus S}$. □

Definition 2.1.20 (α-effectivity function)**.** Let $S \in 2^N$, $S \neq \emptyset$, and let $B \in 2^A$. The *α-effectivity function associated with H* is a function $E_\alpha^H : 2^N \to P(2^A)$ such that $E_\alpha^H(S) = \{B | B \in 2^A$ and S is α-effective for $B\}$, for $S \in s^N$ such that $S \neq \emptyset$. We define $E_\alpha^H(\emptyset) = \emptyset$. □

An α-effectivity function selects a family of sets for which S is α-effective, of alternatives.

Definition 2.1.21 (β-effective)**.** Let $S \in 2^N$, $S \neq \emptyset$, and let $B \in 2^A$. S is *β-effective for B* is defined as: for every $Q^{N\setminus S} \in L^{N\setminus S}$, there exists $R^S \in L^S$ such that $H(R^S, Q^{N\setminus S}) \subset B$. □

Definition 2.1.22 (β-effectivity function)**.** Let $S \in 2^N$, $S \neq \emptyset$, and let $B \in 2^A$. The *β-effectivity function associated with H* is a function $E_\beta^H : 2^N \to P(2^A)$ such that $E_\beta^H(S) = \{B | B \in 2^A$ and S is β-effective for $B\}$, for $S \in s^N$ such that $S \neq \emptyset$. We define $E_\beta^H(\emptyset) = \emptyset$. □

An β-effectivity function selects a family of sets for which S is β-effective, of alternatives.

2.2 Framework of Games in Characteristic Function Form

2.2.1 Transferable Utility Games

A framework of transferable utility games is introduced in this subsection.

Let $N = \{1, 2, \ldots, n\}$ be a set of n players. Each subset of N is called a *coalition*, and a coalition $S = \{i_1, i_2, \ldots, i_m\}$ is often denoted by $i_1 i_2 \cdots i_m$ for simplicity. A characteristic function $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$ assigns a real number to each coalition, where 2^N and $\mathbb{R}$ denote the power set of N and the set of all real numbers, respectively. For each coalition, $v(S)$ denotes the payoff which the coalition S can obtain through cooperation.

Definition 2.2.1 (Games in characteristic function form)**.** A pair (N, v) is said to be a game in characteristic function form with transferable utility, simply called a game in this thesis. □

An example of games is given.

Example 2.2.1. Consider a pair (N, v) such that $N = \{1, 2, 3, 4\}$ and a characteristic function v that $v(\{i\}) = 0$ for all $i \in N$; $v(14) = v(24) = v(34) = 0$; $v(12) = v(13) = v(124) = v(134) = 36$; $v(23) = v(234) = 24$; $v(123) = v(1234) = 42$. Then, (N, v) is a game. □

Some properties of games are introduced.

Definition 2.2.2 (Constant-sum)**.** Consider a game (N, v). (N, v) is said to be a constant-sum game if the following formula holds for all coalition S:

$$v(N) = v(S) + v(N \setminus S).$$

□

A constant-sum game expresses a competitive situation that total payoff is constant.

Definition 2.2.3 (Super-additivity)**.** Consider a game (N, v). (N, v) is said to be a super-additive game if the following formula holds for all coalition S and T such that $S \cap T = \emptyset$:

$$v(S \cup T) \geq v(S) + v(T).$$

□

Definition 2.2.4 (Monotonicity)**.** Consider a game (N, v). (N, v) is said to be a monotonic game if the following formula holds for all coalitions S and T such that $S \supset T$:

$$v(S) \geq v(T).$$

□

Definition 2.2.5 (Inessential)**.** Consider a game (N, v). (N, v) is said to be an inessential game if the following formula holds:

$$v(N) = \sum_{i=1}^{n} v(i).$$

□

A game (N, v) is said to be an essential game if (N, v) is not an inessential game.

Simple games constitute a class of games in characteristic function form. A voting rule is described by a simple game.

Definition 2.2.6 (Simple games)**.** A game (N, v) which satisfies the following conditions is called a simple game:
i) $v(S) \in \{0, 1\}$ for all $S \subset N$,
ii) $v(N) = 1$ and
iii) for $S, T \subset N$ if $S \subset T$ then $v(S) \leq v(T)$. □

In a simple game, a coalition S such that $v(S) = 1$ is said to be a winning coalition. A coalition S such that $v(S) = 0$ is said to be a losing coalition.

Example 2.2.2. Consider the pair (N, v) such that $N = \{1, 2, 3, 4\}$, $v(S) = 1$ if S is 12, 123, 124, 234, or 1234, and $v(S) = 0$, otherwise. Then, (N, v) is a simple game. In this case, coalitions 12, 123, 124, 234 and 1234 are winning coalitions. □

Some types of players in a game are introduced as follows:

Definition 2.2.7 (Null players [39])**.** Consider a game (N, v). For $i \in N$, player i is said to be a *null player*, if and only if $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N \setminus \{i\}$. □

Because a null player brings no contribution toward other coalitions, other coalitions do not have any positive incentive to form coalitions with a null player. In many cases a bigger coalition gains a bigger payoff. A null player, however, does not generate any additional payoff even if he/she joins whatever another coalition.

Example 2.2.3. Consider a game (N, v) in Example 2.2.1. Then, player 4 is a null player. In fact, $v(4) = 0 = v(\emptyset)$, $v(14) = v(24) = v(34) = 0 = v(1) = v(2) = v(3)$, $v(124) = 36 = v(12)$, $v(134) = 36 = v(13)$, $v(234) = 2 = v(23)$ and $v(1234) = 42 = v(123)$, so that $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N \setminus \{i\}$. □

Definition 2.2.8 (Symmetric players [39])**.** Consider a game (N, v). For $i, j \in N$, player i and player j are said to be *symmetric players*, if and only if $v(T \cup \{i\}) = v(T \cup \{j\})$ for all $T \subseteq N \setminus \{i, j\}$. □

Symmetric players i and j have the same contribution when one of them joins a coalition which contains neither i nor j.

Example 2.2.4. Consider a game (N, v) in Example 2.2.1. Then, player 2 and player 3 are symmetric players. In fact, $v(12) = v(13) = 36$, $v(24) = v(34) = 0$ and $v(124) = v(134) = 36$. □

A *coalition structure* $\mathcal{P}$ of N is defined as a partition of N, which is defined as a family $\{P_1, \ldots, P_m\}$ of pairwise disjoint (that is, $P_j \cap P_{j'} = \emptyset$ if $j \neq j'$) non-empty coalitions P_j $(j = 1, \ldots, m)$ whose union $\cup_{j=1}^{m} P_j$ is N. A coalition structure represents the breaking up of N.

A pair $(x; \mathcal{P})$ which consists of an n-vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and a coalition structure $\mathcal{P} = \{P_1, \ldots, P_m\}$ of N satisfying $\sum_{i \in P_j} x_i = v(P_j)$ for $j = 1, \ldots, m$ is called a payoff configuration. If a payoff configuration $(x; \mathcal{P})$ satisfies $x_i \geq v(\{i\})$ for all $i \in N$, $(x; \mathcal{P})$ is said to be *individually rational*. An individually rational payoff configuration is often abbreviated by an i.r.p.c..

The following definitions of objections, counter-objections, a relation $\succ$ on players, and acyclicity of relations are based on [5, 37].

Definition 2.2.9 (Objections)**.** Consider a game (N, v), and let $(x; \mathcal{P})$ be an i.r.p.c. for (N, v). Let, moreover, h and k be two distinct players in coalition $T \in \mathcal{P}$. An *objection* of k against h in $(x; \mathcal{P})$ is such an i.r.p.c. $(y; \mathcal{P}')$ for (N, v) that there exists $T' \in \mathcal{P}'$ such that $k \in T'$, $h \notin T'$, and $y_i > x_i$ for all $i \in T'$. □

An objection of k against h expresses the situation that player k is insisting that player h does not have to be a member of k's coalition, because k can form another coalition T', in which h is not contained, such that the payoff y_i of each member i of the new coalition T' will be more than x_i.

Definition 2.2.10 (Counter-objections)**.** Consider a game (N, v), and let $(x; \mathcal{P})$ be an i.r.p.c. for (N, v). Let, moreover, h and k be two distinct players in coalition $T \in \mathcal{P}$. Suppose an objection $(y; \mathcal{P}')$ of k against h, where $T' \in \mathcal{P}'$ satisfies that $k \in T'$, $h \notin T'$, and $y_i > x_i$ for all $i \in T'$. Then, a *counter-objection* of h against k with respect to the objection $(y; \mathcal{P}')$ is such an i.r.p.c. $(z; \mathcal{P}'')$ that there exists $T'' \in \mathcal{P}''$ such that $h \in T''$, $k \notin T''$, $z_i \geq x_i$ for all $i \in T''$, and $z_i \geq y_i$ for all $i \in T' \cap T''$. □

A counter-objection of h with respect to the objection of k to form the coalition $T' \in \mathcal{P}'$, in which h is not contained, weakens the power of the objection, because h can form the coalition $T'' \in \mathcal{P}''$, in which k is not contained and each member obtains equal or more payoff than in the case he/she participates in the original coalition $T \in \mathcal{P}$ or in the coalition T' proposed in the objection of k.

The next gives an example of objections and counter-objections.

Example 2.2.5. Consider a game (N, v) such that $N = \{1, 2, 3\}$, $v(1) = v(2) = v(3) = 0$, $v(12) = v(13) = v(123) = 100$, and $v(23) = 50$. Then, consider the i.r.p.c. $(x; \mathcal{P}) = ((75, 25, 0); \{12, 3\})$. In this case, player 2 has an objection $(y; \mathcal{P}') = ((0, 26, 24); \{1, 23\})$ against player 1, and player 1 has a counter-objection $(z; \mathcal{P}'') = ((76, 0, 24); \{13, 2\})$ with respect to the objection $(y; \mathcal{P}')$ of player 2. □

A relation on the set N of all players can be defined based on the concepts of objections and counter-objections.

Definition 2.2.11 (Relation $\succ$ on players in $(x; \mathcal{P})$)**.** Consider a game (N, v), and let $(x; \mathcal{P})$ be an i.r.p.c. for (N, v). Suppose two players h and k in N. Then, player k is said to be *stronger* than player h (or, equivalently, player h is *weaker* than player k) in $(x; \mathcal{P})$, if and only if k has an objection against h, but h does not have any counter-objections with respect to the objection, denoted by $k \succ h$. k is said to be equal to h, denoted by $k \sim h$, if and only if neither $k \succ h$ nor $h \succ k$ hold. □

We see, Definition 2.2.9 of objections, that if $k \succ h$, then k and h are elements of the same coalition in $\mathcal{P}$. In other words, one has neither $k \succ h$ nor $h \succ k$, if k and h belong to different coalitions in $\mathcal{P}$. That is, the relation $\succ$ is, in general, a partial relation.

The next gives a numerical example of the relation $\succ$ on the set N of all players.

Example 2.2.6. Consider the game (N, v) in Example 2.2.5, and suppose the i.r.p.c. $(x; \mathcal{P}) = ((80, 20, 0); \{12, 3\})$. The i.r.p.c. $(y; \mathcal{P}') = ((0, 21, 29); \{1, 23\})$ is an objection of player 2 against player 1. Player 1, however, does not have any counter-objection $(z; \mathcal{P}'')$ with respect to this objection $(y; \mathcal{P}')$, because player 1 cannot obtain

80 if he/she offers 29 or more to player 3. Hence, we have that $2 \succ 1$ in $(x;\mathcal{P}) = ((80,20,0);\{12,3\})$. □

We have, in Example 2.2.5, that $2 \succ 1$, but we never have that $1 \succ 2$. This fact is guaranteed by the acyclicity of the relation $\succ$. Acyclicity of relations is defined as follows:

Definition 2.2.12 (Acyclicity of relations)**.** Consider a game (N,v) and the relation $\succ$ on the set N of players in $(x;\mathcal{P})$. The relation $\succ$ is said to be acyclic, if and only if there do not exist such players $1,2,\ldots,t$ that $1 \succ 2 \succ \cdots \succ t \succ 1$. □

Under the acyclicity of a relation on the set N of all players, one can find the maximal players from N with respect to the relation. The next lemma verifies that the relation $\succ$ defined in Definition 2.2.11 is acyclic.

Lemma 2.2.1. *Let $(x;\mathcal{P})$ be an i.r.p.c. for a game (N,v), then the relation $\succ$ on the set N of all players is acyclic.* □

Proof See [5]. ■

This lemma implies, in particular, that the relation $\succ$ is asymmetric, that is, for i and j in N, if $i \succ j$, then $j \succ i$ is not true.

As defined in Definition 2.2.11, for h and k in N, $k \sim h$ denotes that neither $k \succ h$ nor $h \succ k$ hold. Using this relation $\sim$ on N, Aumann and Maschler [2] defines the concept of M-stability of i.r.p.c.s for a game (N,v).

Definition 2.2.13 (M-stability of i.r.p.c.s [2])**.** Consider a game (N,v). An i.r.p.c $(x,\mathcal{P})$ for (N,v) is said to be M-stable, if and only if for all i and j in N, $i \sim j$ holds. □

Then, for a game (N,v), the set of all M-stable i.r.p.c.s $(x,\mathcal{P})$ for (N,v) is called the bargaining set of (N,v).

One of important problems of cooperative game theory is how the total payoff is allocated to members. Consider a game (N,v) and x_i which is player i's payoff, then $x = (x_1, x_2, \ldots, x_n)$ is said to be *payoff vector* in game (N,v).

Definition 2.2.14 (Imputations)**.** Consider a game (N, v). A payoff vector $x = (x_1, x_2, \ldots, x_n)$ is said to be an *imputation* if the following conditions are satisfied:
(1) $x_i \geq v(i)$, $i = 1, \ldots, n$,
(2) $\sum_{i=1}^{n} x_i = v(N)$. □

There is a concept of stability for imputations, which is called core [9].

Definition 2.2.15 (Domination)**.** Consider a game (N, v). For imputations $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ and coalition S, x dominates y via S if and only if the following conditions are satisfied:
(1) $v(S) \geq \sum_{i \in S} x_i$,
(2) $x_i > y_i$ for all $i \in S$. □

$x \ dom_s \ y$ denotes that imputation x dominates imputation y via coalition S. Core is the set of all imputations which are not dominated by any other imputations.

Definition 2.2.16 (Balanced game)**.** Consider a game (N, v). (N, v) is said to be a balanced game if the following formula holds for all $C = S_1, \ldots, S_m$ and all $\gamma = (\gamma_1, \ldots, \gamma_m)$:

$$\sum_{j=1}^{m} \gamma_j v(S_j) \leq v(N).$$

□

Next theorem mentions about relationship between core and balanced game.

Theorem 2.2.1. *Consider a game (N, v). Core of (N, v) is not empty if and only if (N, v) is balanced game.* □

Definition 2.2.17 (Market game with transferable utility)**.** Consider a situation that n players exchange $m+1$ kinds of goods. Let $w_i = (w_i^1, \ldots, w_i^m, w_i^{m+1})$ be initial goods vector for player i. The $m+1$-th good is called money which allows side payments arbitrarily. Player i's utility function $U_i(x)$ can be expressed as follows:

$$U_i(x^1, \ldots, x^m, x^{m+1}) = u_i(x^1, \ldots, x^m) + x^{m+1},$$

where $x = (x^1, \ldots, x^m, x^{m+1})$.

Then, $(N, v, \{w_i\}_{i \in N}, \{u_i\}_{i \in N})$ is said to be market game with transferable utility if v satisfies the following formula.

$$v(S) = \max \sum_{i \in S} u_i(x_i^1, \ldots, x_i^m)$$

$$\text{s.t. } \sum_{i \in S} x_i^j \leq \sum_{i \in S} w_i^j \ \ j = 1, \ldots, m$$

□

Theorem 2.2.2. *Market game with transferable utility is balanced.* □

This theorem shows that market game with transferable utility has non-empty core.

Definition 2.2.18 (Competitive equilibrium)**.** Consider a market game with transferable utility $(N, v, \{w_i\}_{i \in N}, \{u_i\}_{i \in N})$. For price vector $p^* = (p_1^*, \ldots, p_m^*)$ and player i's goods vector $x_i^* \in R_+^m$, a pair $(p^*, x_1^*, \ldots, x_n^*)$ is said to be competitive equilibrium if the following conditions are satisfied.

(1) $u_i(x_i^*) - p^* \cdot (x_i^* - w_i) \geq u_i(x_i) - p^* \cdot (x_i - w_i)$ holds for all player i and all goods vector $x_i \in R_+^m$,

(2) $\sum_{i \in N} x_i^* = \sum_{i \in N} w_i$. □

2.2.2 Non-Transferable Utility Games

In this subsection, a framework of non-transferable utility game is introduced, simply called an NTU-game in this thesis.

Let N be a finite set of players. A non-empty subset of N is called a coalition. For a coalition S, $|S|$ is the number of elements in N. Coalition structure $\mathcal{U} = \{U_1, U_2, \ldots, U_m\}$ is a partition of N. A partition of N satisfies the following three conditions: i) each element of $\mathcal{U}$ is a non-empty subset of N; ii) U_j and U_k in $\mathcal{U}$ such that $j \neq k$ have a relation of $U_j \cap U_k = \emptyset$; iii) $\cup_{i=1}^m U_i = N$. A coalition structure is an m-tuple of coalitions in N such that each player surely belongs to just one of the coalitions.

For a coalition K, $\mathbb{R}^K$ denotes the $|K|$-dimensional real number space. x^S is projection of x on $\mathbb{R}^S$ for $x \in \mathbb{R}^N$ and coalition S. For $x, y \in \mathbb{R}^N$ and a coalition S, $x \gg^S y$ if and only if $x_i > y_i$ for all $i \in S$; $x >^S y$ if and only if $x_j > y_j$ for some $j \in S$ and $x_i \geq y_i$ for all $i \in S$; $x \geq^S y$ if and only if $x_i \geq y_i$ for all $i \in S$. For a coalition S, $\mathbb{R}_+^S = \{x \in \mathbb{R}^S \mid x_i \geq 0 \text{ for all } i \in S\}$ and $\mathbb{R}_{++} = \{x \in \mathbb{R} | x > 0\}$. More, for a coalition S and $\lambda \in \mathbb{R}_{++}$, $\lambda \cdot x^S$ is defined as $(\lambda x_i)^{i \in S}$. For $W \subset \mathbb{R}^S$, W is said to be comprehensive if and only if $w \in W$ and $w \geq z$ imply $z \in W$. For $x \in \mathbb{R}^N$, $x + W$ is defined as the set $\{x + w \mid w \in W\}$. For a set X, ∂X represents the set of all boundary of X with respect to the usual topology. A subset S of X is said to be *bounded* if S is contained in a ball of finite radius.

Definition 2.2.19 (Characteristic function form games with non-transferable utility)**.** A characteristic function form game with non-transferable utility, called an NTU-game, is a pair (N, V) where V is a function which associates with each coalition $S \subset N$ a subset $V(S)$ of $\mathbb{R}^S$ such that
(i) $V(S) \neq \emptyset$ if $S \neq \emptyset$, and $V(\emptyset) = \emptyset$,
(ii) $V(S)$ is comprehensive and closed,
(iii) $V(S) \cap (x^S + \mathbb{R}^S_+)$ is bounded for every $x^S \in \mathbb{R}^S$. □

Let (N, V) be an NTU-game. For every $i \in N$, let $v^i = \max\ \{x_i | x_i \in V(\{i\})\}$. Then $V(\{i\}) = \{x_i \in \mathbb{R}^{\{i\}} | x_i \leq v^i\}$.

$x \in \mathbb{R}^N$ is said to be individually rational if and only if $x_i \geq v^i$ for all $i \in N$. For $x \in \mathbb{R}^N$ and a coalition S, x^S is said to be weakly efficient for S if and only if there is no $y \in V(S)$ which satisfies $y \gg^S x$.

Definition 2.2.20 (Payoff configuration)**.** Let (N, V) be an NTU-game. Payoff configuration $(x;\mathcal{U})$ is a pair of $x \in R^N$ and coalition structure $\mathcal{U}$ which satisfies $x^U \in V(U)$ for all $U \in \mathcal{U}$. □

A payoff configuration $(x,\mathcal{U})$ for (N, V) where x is individually rational is said to be an *individually rational payoff configuration*,which is often abbreviated by an i.r.p.c.. A payoff configuration $(x;\mathcal{U})$ for (N, V) which x^U is a weakly efficient for all $U \in \mathcal{U}$ is said to be *weakly efficient payoff configuration*, which is often abbreviated by an w.e.p.c..

Definition 2.2.21 (Objection)**.** Let (N, V) and $(x;\mathcal{U})$ be an NTU-game and a w.e.p.c. respectively. For $U \in \mathcal{U}$ and $k, l \in U$, an *objection* of k against l in $(x;\mathcal{U})$ is such a w.e.p.c. $(y;\mathcal{U}')$ for (N, V) that there exists $U' \in \mathcal{U}'$ such that $k \in U'$, $l \notin U'$, and $y^{U'} \gg x^{U'}$. □

An objection of k against l expresses the situation that k maintains that l does not have to be a member of k's coalition, because k can form another coalition C, in which l is not included, such that the payoff y_i of each member i of the new coalition C will gets more than x_i.

Definition 2.2.22 (Counter-objection)**.** Consider an NTU-game (N, V), and let $(x;\mathcal{U})$ be an w.e.p.c. for (N, V). Let, moreover, k and l be two distinct players in coalition $U \in \mathcal{U}$. Suppose an objection $(y;\mathcal{U}')$ of k against l, where $U' \in \mathcal{U}'$ satisfies that $k \in U'$,

$l \notin U'$, and $y^{U'} \gg x^{U'}$. Then, a *counter-objection* of l against k with respect to the objection $(y;\mathcal{U}')$ is such an w.e.p.c. $(z;\mathcal{U}'')$ that there exists $U'' \in \mathcal{U}''$ such that $l \in U''$, $k \notin U''$, $z^{U''} \gg x^{U''}$, and $z^{U' \cap U''} \geq y^{U' \cap U''}$. □

A counter-objection of l with respect to the objection of k to form the coalition C, in which h is not included, weakens the power of the objection, because h can form the coalition D, in which k is not contained and each member obtains equal or more payoff than in the case he/she participates the original coalition $U \in \mathcal{U}$ or the coalition C proposed in the objection of k.

Definition 2.2.23 (Justified objection)**.** Let (N, V) and $(x;\mathcal{U})$ be an NTU-game and a w.e.p.c.. For $U \in \mathcal{U}$ and $k, l \in U$, an objection $(y;\mathcal{U}')$ of k against l in $(x;\mathcal{U})$ is said to be a justified objection of k against l in $(x;\mathcal{U})$ if and only if there is no counter-objection of l against k with respect to $(y;\mathcal{U}')$. □

The definition of justified objection means that if there exists some justified objection from k to l for all $k, l \in N$ at some payoff configuration $(x;\mathcal{U})$, $(x;\mathcal{U})$ is not stable by difference of negotiation power which each player has.

2.3 Summary of Chapter 2

This chapter introduced existing mathematical models which describe group decision and negotiation. These models are called social welfare function (Definition 2.1.9), social choice function (Definition 2.1.11), game (Definition 2.2.1) and NTU-game (Definition 2.2.19), respectively. Numerical examples which show how each model works were provided in this chapter. Properties of coalitions which are α-effective and β-effective were given. Properties of players which are symmetric players and null players and concepts of negotiation which are objections, counter-objections and justified objections were also introduced.

Comparison of coalition influence on the introduced models is studied in the next chapter. Next chapter uses binary relations to compare coalition influence in the models, the introduced properties of binary relations are discussed through the provided propositions.

Chapter 3

Comparison of Coalition Influence

This chapter proposes relations to compare coalition influence for frameworks of group decision and negotiation. In the framework of simple games, which constitute a special class of games in characteristic function form, there are such methods to compare coalition influence as the desirability relation [7, 48], the blockability relation [18], and the viability relation [22]. The blockability relation and viablity relation for simple games are extended to games in characteristic function form. Examples how the proposed relations work are provided. It is verified that some properties are satisfied by the proposed relations. There is no existing methods to compare coalition influence in the framework of collective choice. New methods to comapre coalition influece for social welfare functions and social choice functions are also defined in this chapter. Exmaples which show how the defined methods work are provided. Lastly, models which expresses the situation that players negotiate each other are discussed. Some models for negotiation were proposed by Nash [35], Harsanyi [14], Selten [45] and Rubistein [42].

Comparison of coalition influence for coalition formation is defined in the first section. Next, method to compare coalition bargaining power is given. The content of this chapter is due to [23], [24], [25], [26], [27] and [29].

3.1 Comparison of Coalition Influence for Coalition Formation

This section deals with comparison of coalition influence for games in characteristic function form that is a model which describes coalition formation situation.

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3.1.1 Existing Comparison Methods for Simple Games

This section introduces existing methods to compare coalition influence for simple games. The blockability relation for simple games is defined as follows.

Definition 3.1.1 (Blockability relations for simple games [18])**.** Consider a simple game (N, v). For coalitions S and S', $S \succeq^b S'$ is defined as: for all winning coalition T, if $T \setminus S'$ is a losing coalition, then $T \setminus S$ is also a losing coalition. $\succeq^b$ is called the *blockability relation* for (N, v). □

$S \succeq^b S'$ expresses that if coalition S' can make winning coalition T losing by deviation then coalition S can also make T losing by that.

The next lemma is convenient to specify the blockability relation $\succeq^b$ between two coalitions.

Lemma 3.1.1 ([18])**.** *Consider a simple game (N, v) and the blockability relation $\succeq^b$ for (N, v). Then, it is satisfied that for all coalitions S and S', $S \succeq^b S'$ is equivalent to $B(S) \supset B(S')$, where for $S \subset N, B(S) = \{T \mid v(T) = 1$ and $v(T \setminus S) = 0\}$.* □

The next example shows how blockability relation and this Lemma does work.

Example 3.1.1. Consider the simple game in Example 2.2.2. Then, we have $B(12) = \{12, 123, 124, 234, 1234\}$ and $B(34) = \{234\}$, because, for example, $234 \in B(34)$ since $v(234) = 1$ and $v(234 \setminus 34) = v(2) = 0$. By Lemma 3.1.1, $12 \succeq^b 34$ holds, because $B(12) \supset B(34)$. That is, all winning coalitions become losing by the deviation of 12, while winning coalitions other than 234 do not become losing by the deviation of 34. □

The definition of viability relation for simple games can be given as follows:

Definition 3.1.2 (Viability relations for simple games [22])**.** Consider a simple game $(N,\ v)$. For coalitions S and S', $S \succeq^v S'$ is defined as: for all coalition $T \in 2^N$, if $S' \setminus T$ is a winning coalition, then $S \setminus T$ is also a winning coalition. $\succeq^v$ is called the *viability relation* for (N, v). □

This relation says that if coalition S' does not become losing by the deviation of T, then S does not become losing coalition by that, ether.

The next lemma is useful for specifying the viability relation $\succeq^v$ for simple games.

Lemma 3.1.2 ([22])**.** *Consider a simple game* (N, v) *and the blockability relation* $\succeq^v$ *for* (N, v)*. Then, it is satisfied that for all coalitions* S *and* S', $S \succeq^v S'$ *is equivalent to* $V(S) \supset V(S')$*, where for* $S \subset N, V(S) = \{T \mid v(S \setminus T) = 1\}$*.* □

The next example shows how viability relation and this Lemma does work.

Example 3.1.2. Consider the simple game in Example 2.2.2. Then, we have $V(1234) = \{1, 3, 4, 34\}$ and $V(124) = \{3, 4, 34\}$, because, for example, $34 \in V(124)$ since $v(124 \setminus 34) = v(12) = 1$. By Lemma 2, $1234 \succeq^v 124$ holds because $V(1234) \supset V(124)$. □

3.1.2 Blockability Relations for Games in Characteristic Function Form

In this section, the relations for simple games are extended to those for games in characteristic function form, and their some properties are verified. Some propositions imply that these relations for games in characteristic function form are indeed extensions of the corresponding relations for simple games.

Definition 3.1.3 (Blockability relations for games in characteristic function form)**.** Consider a game (N, v). For a coalition T, let $B^*(T)$ be $\sum_{U \subset N} v(U \setminus T)$. For coalitions S and S', $S \succeq^B S'$ is defined as $B^*(S) \leq B^*(S')$. $\succeq^B$ is called the *blockability relation* for (N, v). □

$S \succeq^B S'$ expresses that coalition S can decrease the value of the characteristic function v by deviating from U more than coalition S' can do.

The next example shows how Definition 3.1.3 works.

Example 3.1.3. Consider the simple game in Example 2.2.2. For coalitions 12 and 34, we have that

$$B^*(12) = \textstyle\sum_{U \subset N} v(U \setminus 12) = 4 \cdot [v(\emptyset) + v(3) + v(4) + v(34)] = 0, \text{ and}$$
$$B^*(34) = \textstyle\sum_{U' \subset N} v(U' \setminus 34) = 4 \cdot [v(\emptyset) + v(1) + v(2) + v(12)] = 4.$$

By the definition of $\succeq^B$, it holds that $12 \succeq^B 34$. □

Since simple games constitute a special class of games in characteristic function form, the blockability relation $\succeq^B$ for games in characteristic function form can be applied to every simple game. The next proposition shows that the blockability relation $\succeq^B$ which is applied to a simple game is implied by $\succeq^b$.

Proposition 3.1.1. *For a simple game (N, v) and coalitions $S_1, S_2 \subset N$, we have that if $S_1 \succeq^b S_2$ then $S_1 \succeq^B S_2$.* □

Proof Assume that $S_1 \succeq^b S_2$. Then, by Lemma 3.1.1, we have $B(S_1) \supset B(S_2)$, which implies $|B(S_1)| \geq |B(S_2)|$. Elements of $B(S)$ are winning coalitions which become losing by the deviation of S. Thus, we see that $B^*(S_1) \leq B^*(S_2)$, which means $S_1 \succeq^B S_2$. ■

The next proposition gives some properties which blockability relation for games in characteristic function form satisfies.

Proposition 3.1.2. The blockability relation $\succeq^B$ for a game (N, v) is transitive and complete. □

Proof
(Transitivity) If $S_1 \succeq^B S_2$ and $S_2 \succeq^B S_3$ for $S_1, S_2, S_3 \subset N$, then $B^*(S_1) \leq B^*(S_2)$ and $B^*(S_2) \leq B^*(S_3)$ hold. This implies that $B^*(S_1) \leq B^*(S_3)$, which means $S_1 \succeq^B S_3$.
(Completeness) For $S_1, S_2 \subset N$, $B^*(S_1)$ and $B^*(S_2)$ are real numbers. Hence, we have $B^*(S_1) \leq B^*(S_2)$ or $B^*(S_2) \leq B^*(S_1)$. This implies that $S_1 \succeq^B S_2$ or $S_2 \succeq^B S_1$. ■

3.1.3 Viability Relations for Games in Characteristic Function Form

The viability relation for games in characteristic function form is defined as follows:

Definition 3.1.4 (Viability relations for games in characteristic function form). Consider a game (N, v). For a coalition T, let $V^*(T)$ be $\sum_{U \subset N} v(T \setminus U)$. For coalitions S and S', $S \succeq^V S'$ is defined as $V^*(S) \geq V^*(S')$. $\succeq^V$ is called the *viability relation* for (N, v). □

$S \succeq^V S'$ expresses that coalition S can defend the value of the characteristic function from the deviation of U more than coalition S' can do.

The next example shows how Definition 3.1.4 works.

Example 3.1.4. Consider the simple game in Example 2.2.2. For coalitions 124 and 234, we have

$$V^*(124) = \sum_{U \subset N} v(124 \setminus U) = 2 \cdot \sum_{U' \subset 124} v(U') = 4,$$
$$V^*(234) = \sum_{V \subset N} v(234 \setminus V) = 2 \cdot \sum_{V' \subset 234} v(V') = 2.$$

By the definition of $\succeq^V$, it holds that $124 \succeq^V 234$. □

The next proposition shows that the viability relation $\succeq^V$ which is applied to a simple game is implied by $\succeq^v$.

Proposition 3.1.3. For a simple game (N, v) and coalitions $S_1, S_2 \subset N$, we have that if $S_1 \succeq^v S_2$, then $S_1 \succeq^V S_2$. □

Proof Assume that $S_1 \succeq^v S_2$. By Lemma 3.1.2, we have $V(S_1) \supset V(S_2)$, which implies $|V(S_1)| \geq |V(S_2)|$. Elements of $V(S)$ are coalitions which cannot make S losing by deviation. Thus, we have that $V^*(S_1) \geq V^*(S_2)$, which means $S_1 \succeq^V S_2$. ■

The next proposition gives some properties which viability relation for games in characteristic function form satisfies.

Proposition 3.1.4. The viability relation $\succeq^V$ for a game (N, v) is transitive and complete. □

Proof

(Transitivity) If $S_1 \succeq^V S_2$ and $S_2 \succeq^V S_3$ for $S_1, S_2, S_3 \subset N$, then $V^*(S_1) \geq V^*(S_2)$ and $V^*(S_2) \geq V^*(S_3)$ hold. This implies that $V^*(S_1) \geq V^*(S_3)$, which means $S_1 \succeq^V S_3$.
(Completeness) For $S_1, S_2 \subset N$, $V^*(S_1)$ and $V^*(S_2)$ are real numbers. Hence, we have $V^*(S_1) \geq V^*(S_2)$ or $V^*(S_2) \geq V^*(S_1)$. This implies that $S_1 \succeq^V S_2$ or $S_2 \succeq^V S_1$. ■

3.1.4 Profitability Relations for Games in Characteristic Function Form

In this section, a binary relation which compares two coalitions how much the coalitions can bring profit to other coalitions in a game is introduced.

Definition 3.1.5 (Profitability relations)**.** Consider a game (N, v). For a coalition T, let $P^*(T)$ be $\sum_{U \subseteq N} v(U \cup T)$. For coalitions S and S', $S \succeq^P S'$ is defined as $P^*(S) \geq P^*(S')$. $\succeq^P$ is called the *profitability relation* for (N, v). □

$S \succeq^P S'$ expresses that coalition S can increase the value of the characteristic function v by merging with other coalitions equally to or more than coalition S' can do.

Example 3.1.5. Consider a game (N, v) in Example 2.2.1. For coalitions 12 and 34, we have

$$\begin{aligned} P^*(12) &= \sum_{U \subseteq N} v(12 \cup U) \\ &= 4 \cdot [v(12) + v(123) + v(124) + v(1234)] \\ &= 480, \end{aligned}$$

$$\begin{aligned} P^*(34) &= \sum_{U \subseteq N} v(34 \cup U) \\ &= 4 \cdot [v(34) + v(134) + v(234) + v(1234)] \\ &= 360. \end{aligned}$$

By the definition of profitability relations, we have $12 \succeq^P 34$. □

If profitability relation $\succeq^P$ for a game (N, v) is transitive and complete, a function which assigns a real number that expresses profitability of the coalition to every coalition can be generated because there exists maximal number and minimal number on all numbers assigned by the function.

Proposition 3.1.5. Profitability relation $\succeq^P$ for a game (N, v) is transitive and complete. □

Proof (Transitivity) If $S_1 \succeq^P S_2$ and $S_2 \succeq^P S_3$ for $S_1, S_2, S_3 \subset N$, then $P^*(S_1) \geq P^*(S_2)$ and $P^*(S_2) \geq P^*(S_3)$ hold. This implies that $P^*(S_1) \geq P^*(S_3)$, which means $S_1 \succeq^P S_3$.
(Completeness) For $S_1, S_2 \subset N$, $P^*(S_1)$ and $P^*(S_2)$ are real numbers. Hence, we have $P^*(S_1) \leq P^*(S_2)$ or $P^*(S_2) \leq P^*(S_1)$. This implies that $S_1 \succeq^P S_2$ or $S_2 \succeq^P S_1$. ■

Proposition 3.1.6. Consider a game (N, v). If (N, v) is constant-sum game, then it holds that $S \succeq^B S'$ if and only if $S \succeq^P S'$ for all coalitions $S, S' \subseteq N$. □

Proof Assume that $S_1 \succeq^B S_2$. Then we have $B^*(S_1) \leq B^*(S_2)$, which means $\sum_{T \subset N} v(T \setminus S_1) \leq \sum_{T \subset N} v(T \setminus S_2)$. Since $T \subset N$ can be expressed by $N \setminus U$, if one takes $N \setminus T$ as U, $\sum_{T \subset N} v(T \setminus S_1) \leq \sum_{T \subset N} v(T \setminus S_2)$ can be rewritten as $\sum_{U \subset N} v((N \setminus U) \setminus S_1) \leq \sum_{U \subset N} v((N \setminus U) \setminus S_2)$. For sets X, Y, and $Z \subset N$, we generally have that $(X \setminus Y) \setminus Z = X \setminus (Y \cup Z)$. Therefore, $\sum_{U \subset N} v((N \setminus U) \setminus S_1) \leq \sum_{U \subset N} v((N \setminus U) \setminus S_2)$ can be rewritten $\sum_{U \subset N} v(N \setminus (S_1 \cup U)) \leq \sum_{U \subset N} v(N \setminus (S_2 \cup U))$. $\sum_{U \subset N} v(N \setminus (S_1 \cup U)) \leq \sum_{U \subset N} v(N \setminus (S_2 \cup U))$ can be written $\sum_{U \subset N}[v(N) - v(S_1 \cup U)] \leq \sum_{U \subset N}[v(N) - v(S_2 \cup U)]$ because v is constant-sum. Calculated both sides, it results $\sum_{U \subset N} v(S_1 \cup U)] \geq \sum_{U \subset N} v(S_2 \cup U)$. It means that $P^*(S_1) \geq P^*(S_2)$. By the definition of the profitability relation, we have $S_1 \succeq^P S_2$. ■

This proposition implies that decreasing profit and increasing profit are the same in the situation that all participators want for bigger profit against limited resources .

3.1.5 Interrelationships of New Relations

This section shows a complementary interrelationship between the blockability relation and the viability relation for games in characteristic function form.

Proposition 3.1.7. *Consider a game (N, v). Let $\succeq^B$ and $\succeq^V$ be the blockability relation and the viability relation for (N, v), respectively. For $S_1, S_2 \subset N$, we have that $S_1 \succeq^B S_2$ if and only if $N \setminus S_2 \succeq^V N \setminus S_1$.* □

Proof Assume that $S_1 \succeq^B S_2$. Then we have $B^*(S_1) \leq B^*(S_2)$, which means $\sum_{T \subset N} v(T \setminus S_1) \leq \sum_{T \subset N} v(T \setminus S_2)$. Since $T \subset N$ can be expressed by $N \setminus U$, if one takes $N \setminus T$ as U, $\sum_{T \subset N} v(T \setminus S_1) \leq \sum_{T \subset N} v(T \setminus S_2)$ can be rewritten as $\sum_{U \subset N} v((N \setminus U) \setminus S_1) \leq \sum_{U \subset N} v((N \setminus U) \setminus S_2)$. For sets X, Y, and $Z \subset N$, we generally have that $(X \setminus Y) \setminus Z = (X \setminus Z) \setminus Y$. Therefore, $\sum_{U \subset N} v((N \setminus U) \setminus S_1) \leq \sum_{U \subset N} v((N \setminus U) \setminus S_2)$ can be rewritten $\sum_{U \subset N} v((N \setminus S_1) \setminus U) \leq \sum_{U \subset N} v((N \setminus S_2) \setminus U)$. By the definition of the viability relation, we have $N \setminus S_2 \succeq^V N \setminus S_1$. ■

This proposition shows that for every game in characteristic function form the blockability relation and the viability relation have a complementary interrelationship.

3.2 Comparison of Coalition Influence for Group Decision

This section deals with comparison of coalition influence for social welfare function or social choice function, which are models of group decision situations.

3.2.1 A Method to Compare Coalition Influence with Preference Distance

This section introduces a method to compare coalition influence for an SWF (Definition 2.1.9). The introduced method compares a pair of coalitions with respect to the distance between preferences to the value of the SWF. The concept of the method means that the coalition would lose power in the group if the coalition had opinions which are different from the result of the SWF, because an SWF is a rule of the group decision. Such properties of coalitions as symmetric coalitions and null coalitions are also introduced. A relationship between a property of coalitions and coalition influence is given in this section.

To prepare the method, a definition of the distance between a player's preference and the value of an SWF is provided.

Definition 3.2.1 (Distance between a player and SWF)**.** Consider a pair (N, A) and an SWF $F : L^N \to L$. Let d be a distance function on L. For a player $i \in N$, i's preference distance to SWF F is defined as follows:

$$D^i(F) = \sum_{P \in L^N} d(P_i, F(P)),$$

where P_i is an ith component of $P = (P_1, P_2, \cdots, P_i, \cdots, P_n)$. □

What the player has low number of preference distance to SWF means that the preference of the player is close to the group decision.

Example 3.2.1. Consider a pair (N, A) and an SWF $F : L^N \to L$ in Example 2.1.1. Let d^2 be a distance function on L in Example 2.1.3. For instance, we get each distance between R_1 and the other elements of L.

$$d^2(R_1, R_1) = 0.$$

$$d^2(R_1, R_2) = 1.$$

$$d^2(R_1, R_3) = 2.$$

$$d^2(R_1, R_4) = 3.$$

$$d^2(R_1, R_5) = 3.$$

$$d^2(R_1, R_6) = 4.$$

In this case, distance between each player's preference and the SWF is calculated as follows:

$$D^1(F) = \sum_{j=1}^{6}\sum_{k=1}^{6} d^2(R_j, R_k) = 13 \times 6 = 78.$$

$$D^2(F) = \sum_{P \in L^N} d^2(P_2, F(P)) = 13 \times 5 \times 6 = 390.$$

$$D^3(F) = \sum_{P \in L^N} d^2(P_3, F(P)) = 13 \times 5 \times 6 = 390.$$

□

It is clear that preference distance to SWF for dictator is zero because dictator's preference is always accepted by SWF.

Next, coalition preference distance to the SWF is discussed. Coalition preference distance can be defined as maximum preference distance to the SWF for the member of the coalition, average of preference distance for the member of the coalition, median point of preference distance for the member of the coalition or weighted average of preference distance for the member of the coalition. In this thesis, coalition preference distance to an SWF is sum of the minimum preference distances to the SWF for the members of the coalition to preserve monotonicity with regard to coalition sizes.

Definition 3.2.2 (Coalition preference distance to SWF)**.** Consider a pair (N, A) and an SWF $F : L^N \to L$. Let d be a distance function on L. For a coalition S, S's preference distance to the SWF F is defined as follows:

$$D^S(F) = \sum_{P \in L^N} \min_{i \in S} \{d(P_i, F(P))\},$$

where P_i is an ith component of $P = (P_1, P_2, \cdots, P_i, \cdots, P_n)$. □

This definition expresses that the coalition preference distance to an SWF F gets lower number if the coalition has some member whose preference is close to the result of the SWF F. If an SWF F is dictatorial, preference distance to SWF F of every coalition which has the dictator as a member of the coalition is zero.

Example 3.2.2. Consider a pair (N, A) and an SWF $F : L^N \to L$ in Example 2.1.1. Let d^2 be a distance function on L in Example 2.1.3. In this case, coalition preference distances to F are calculated as follows:

$$D^1(F) = \sum_{P \in L^N} d^2(P_1, F(P)) = 78.$$

$$D^2(F) = \sum_{P \in L^N} d^2(P_2, F(P)) = 390.$$

$$D^3(F) = \sum_{P \in L^N} d^2(P_3, F(P)) = 390.$$

$$D^{12}(F) = \sum_{P \in L^N} \min_{i \in 12} \{d^2(P_i, F(P))\} = 0.$$

$$D^{13}(F) = \sum_{P \in L^N} \min_{i \in 13} \{d^2(P_i, F(P))\} = 0.$$

$$D^{23}(F) = \sum_{P \in L^N} \min_{j \in 23} \{d^2(P_j, F(P))\} = (8 + 7 + 6 + 7 + 10) \times 6 = 228.$$

$$D^{123}(F) = \sum_{P \in L^N} \min_{i \in 123} \{d^2(P_i, F(P))\} = 0.$$

□

This example shows that coalition preference distance to SWF provides a real number to every coalition.

We compare a pair of coalition by coalition preference distance to SWF.

Definition 3.2.3 (Relation on coalitions for SWF)**.** Consider a pair (N, A) and an SWF $F : L^N \to L$. Let d be a distance function on L. For coalition S and S', $S \succ^F S'$ is defined as $D^S(F) < D^{S'}(F)$. For coalition S and S', $S \succeq^F S'$ is defined as $D^S(F) \leq D^{S'}(F)$. For coalition S and S', $S \sim^F S'$ denotes $S \succeq^F S'$ and $S' \succeq^F S$. For coalition S and S', $S \not\succeq^F S'$ is defined as $D^S(F) > D^{S'}(F)$. □

This definition expresses that coalition which has smaller coalition distance to SWF F has more power to the decision.

The next example shows how the proposed relation works in the framework of SWF.

Example 3.2.3. Consider a pair (N, A) and an SWF $F : L^N \to L$ in Example 2.1.1. Let d^2 be a distance function on L in Example 2.1.3. From the definition of $\succeq^F$, $123 \sim^F 12 \sim^F 13 \succ^F 1 \succ^F 23 \succ^F 2 \sim^F 3$ holds. □

The proposed method for comparison of coalition influence satisfies some properties.

Proposition 3.2.1. Consider a pair (N, A) and an SWF $F : L^N \to L$. Let d be a distance function on L. $\succeq^F$ is reflexive, complete, transitive and negatively transitive. ■

Proof (Reflexivity) For coalition S, $D^S(F)$ is a real number. $D^S(F) \leq D^S(F)$ holds, hence we get $S \succeq^F S$.
(Completeness) For coalition S, $D^S(F)$ is a real number. $\leq$ is complete on $\mathbb{R}$. Hence, $\succeq^F$ is complete on 2^N.
(Transitivity) For coalition S, S' and S'', assume that $S \succeq^F S'$ and $S' \succeq^F S''$ hold. It implies that $D^S(F) \leq D^{S'}(F)$ and $D^{S'}(F) \leq D^{S''}(F)$ hold. Then, we get $D^S(F) \leq D^{S''}(F)$. Hence, $S \succeq^F S''$ holds.
(Negatively transitivity) For coalition S, S' and S'', assume that $S \not\succeq^F S'$ and $S' \not\succeq^F S''$ hold. It implies that $D^S(F) > D^{S'}(F)$ and $D^{S'}(F) > D^{S''}(F)$ hold. Then, we get $D^S(F) > D^{S''}(F)$. Hence, $S \not\succeq^F S''$ holds. ■

This proposition means that the proposed method can assign power index to each coalition for SWF.

Proposition 3.2.2. Consider a pair (N, A) and an SWF $F : L^N \to L$. Let d be a distance function on L. For coalition S and S', if $S' \subseteq S$ then $S \succeq^F S'$ holds. □

Proof For coalitions S and S' such that $S' \subseteq S$, assume that $S \not\succeq^F S'$ holds. If $S = S'$, then $D^S(F) = D^{S'}(F)$ holds. This is contradictory to $S \not\succeq^F S'$. If $S \neq S'$, there exists players in $S \setminus S'$. If there exists players $i \in S \setminus S'$ and $j \in S'$ such that $D^i(F) < D^j(F)$, $D^S(F) < D^{S'}(F)$ holds. It means $S \succeq^F S'$ by the Definition 3.2.3. This is contradictory to $S \not\succeq^F S'$. If there is no players $i \in S \setminus S'$ and $j \in S'$ such that $D^i(F) < D^j(F)$, $D^S(F) = D^{S'}(F)$ holds. It means $S \succeq^F S'$ by the Definition 3.2.3. This is contradictory to $S \not\succeq^F S'$. ■

This proposition shows that a coalition has greater or equal influence on the group decision than the influence which a subgroup of the coalition has.

Definition 3.2.4 (Symmetric players for SWFs)**.** Consider a pair (N, A) and an SWF $F : L^N \to L$. Let d be a distance function on L. Players i and j are called *symmetric players* for F if $D^i(F) = D^j(F)$ holds. □

Symmetric players for an SWF have the same influence in the group decision.

Example 3.2.4. Consider a pair (N, A) and an SWF $F : L^N \to L$ in Example 2.1.1. Let d^2 be a distance function on L in Example 2.1.3. In this case, player 2 and 3 are symmetric players for F because $D^2(F) = D^3(F)$ holds. □

Player 2 and 3 have the same influence in regard to SWF.

Definition 3.2.5 (Symmetric coalitions for SWFs)**.** Consider a pair (N, A) and an SWF $F : L^N \to L$. Let d be a distance function on L. Coalitions S and S' are called symmetric coalitions for F if there exists a bijection $h : S \to S'$ such that $j \in S$ and $h(j) \in S'$ are symmetric players for F. □

Symmetric coalitions for an SWF have the same coalition influence on the group decision.

Example 3.2.5. Consider a pair (N, A) and an SWF $F : L^N \to L$ in Example 2.1.1. Let d^2 be a distance function on L in Example 2.1.3. In this case, coalition 12 and 13 are symmetric coalitions for F because there exists a bijection $h : 12 \to 13$ such that $h(1) = h(1)$ and $h(2) = h(3)$ hold. □

Coalition 12 and 13 have the same influence on the group decision.

Proposition 3.2.3. Consider a piar (N, A) and an SWF $F : L^N \to L$. Let d be a distance function on L. For all coalitions S, S', if S and S' are symmetric coalitions then $S \sim^F S'$ holds, where $\sim^F$ is the relation defined in Definition 3.2.3. ■

Proof Assume that coalition S and S' are symmetric coalitions for F for coalition S and S'. By the Definition 3.2.5, there exists a bijection $h : S \to S'$ such that $j \in S$ and $h(j) \in S'$ are symmetric players for F. It implies that $|S| = |S'|$ holds. For all $j \in S$, there exists $h(j) \in S'$ such that $D^i(F) = D^{h(j)}(F)$. By the Definition 3.2.2, $D^S(F) = D^{S'}(F)$ holds. Hence, $S \succeq^F S'$ and $S' \succeq^F S$ holds, which means that $S \sim^F S'$ holds by the Definition 3.2.3. ■

This proposition shows that the proposed method evaluates symmetric coalitions as indifferent from the point of view how the opinions of the coalitions are different from the group decision.

Example 3.2.6. Consider a pair (N, A) and an SWF $F : L^N \to L$ in Example 2.1.1. Let d^2 be a distance function on L in Example 2.1.3. From the Example 3.2.5, coalitions 12 and 13 are symmetric. From Example 3.2.3, $12 \sim^F 13$ holds. □

A case that Proposition 3.2.3 supports is shown in the next example.

Example 3.2.7. Consider a pair (N, A) and an SWF $F : L^N \to L$ in Example 2.1.1. Let d^2 be a distance function on L in Example 2.1.3. From Example 3.2.2, coalitions $123 \sim^F 12$ holds. But, coalition 123 and 12 are not symmetric coalition by Definition 3.2.5. □

This example shows that coalitions which are indifferent based on $\succeq^F$ for an SWF are not always symmetric for the SWF.

Definition 3.2.6 (Null player for SWF)**.** Consider a pair (N, A) and an SWF $F : L^N \to L$. Let d be a distance function on L. A player i is called a *null player* for F if for all coalition S, $D^F(S) = D^F(S \cup \{i\})$ holds. □

A null player for an SWF does not have any influence in the group decision.

Example 3.2.8. Consider a pair (N, A) in Example 2.1.1. Let d^2 be a distance function on L in Example 2.1.3. Consider a function $F : L^N \to L$ defined as follows: For all $(P_1, P_2, P_3) \in L$, $F(P_1, P_2, P_3) = \arg\max_{P \in L} d^2(P_1, P)$. Player 1 is a null player for F because $D^F(S) = D^F(S \cup \{1\})$ holds for all coalition S. □

We extend the concept of null player to null coalition by the next definition.

Definition 3.2.7 (Null coalition for SWF)**.** Consider a pair (N, A) and an SWF $F : L^N \to L$. Let d be a distance function on L. A coalition T is called a *null coalition* for F if for all coalition S, $D^F(S) = D^F(S \cup T)$ holds. □

A null coalition for an SWF does not have any influence on the group decision.

Consider a pair (N, A) and an SWF $F : L^N \to L$. Let d be a distance function on L. It is clear that for a null coalition S and all coalition T, $T \succeq^F S$ holds.

Proposition 3.2.4. Consider a pair (N, A) and an SWF $F : L^N \to L$. Let d be a distance function on L. For a coalition S, S is a null coalition for F if and only if i is a null player for F for all $i \in S$. ■

Proof Assume that S is a null coalition for F. By the Definition 3.2.7, $D^F(S) = D^F(T \cup S)$ for all coalition T. The coalition is described as $S = \{s_1, s_2, \ldots, s_m\}$. Then, $D^F(T) = D^F(T \cup S) = D^F(T \cup \{s_1\} \cup \{s_2\} \cup \ldots \cup \{s_m\})$ holds for all coalition T. If there exists $j \in S$ such that j is not a null player, this equation does not hold, this is contradiction.
For coalition S, assume that i is a null player for F for all $i \in S$. The coalition is described as $S = \{s_1, s_2, \ldots, s_m\}$. Then, $D^F(T) = D^F(T \cup \{s_1\} \cup \{s_2\} \cup \ldots \cup \{s_m\}) = D^F(T \cup S)$ holds for all coalition T. It indicates that the coalition S is a null coalition for F. ■

This proposition shows that a null coalition for an SWF always contains only null players for the SWF.

3.2.2 A Method to Compare Coalition Influence for Social Choice Functions

This section, the definition of blockability relation for social choice functions is defined.

Definition 3.2.8 (Blockability relations for SCF)**.** Consider an SCF F. For coalitions S and S', $S \succ^{b(F)} S'$ is defined as: for all winning coalition T with respect to F, if $T \setminus S'$ is a losing coalition with respect to F, then $T \setminus S$ is also a losing coalition with respect to F. $\succ^{b(F)}$ is called the *blockability relation* for F. □

$S \succ^{b(F)} S'$ expresses that if coalition S' can make winning coalition T losing by deviation then coalition S can also make T losing by that with respect to the SCF F.

For coalitions S and S', $S \sim^{b(F)} S'$ means that both $S \succ^{b(F)} S'$ and $S' \succ^{b(F)} S$ hold. For coalitions S and S', $S \succeq^{b(F)} S'$ denotes that $S \succ^{b(F)} S'$ and not $S' \succ^{b(F)} S$'. For coalition S, let $B^F(S)$ be the set of winning coalitions which become losing coalitions by deviation of S with respect to SCF F.

Example 3.2.9. Consider a 3-tuple (N, A, R) such that $N = \{1, 2, 3\}$, $A = \{x, y, z\}$ and $R = \{R^1, R^2, R^3, R^4, R^5, R^6\}$. Let a function F be $F(R^i, R^j, R^k) = x$ if $x, w \in A$, $x \neq w$ and $xR^m w$ for all $m \in N$. Let a function F be $F(R^i, R^j, R^k) = y$ if $y, w \in A$, $y \neq w$ and $yR^l w$ for all $l \in \{1, 2\}$. Otherwise, let F be $F(R^i, R^j, R^k) = z$ for all $i, j, k \in \{1, 2, 3, 4, 5, 6\}$. In this case, $12 \succ^{b(F)} 3$ holds because coalition 3 can make 123 losing coalition by deviation and coalition 12 can also make 123 losing coalition by the same action. □

Proposition 3.2.5. Consider an SCF F and coalitions S and S'. It holds that $S \succeq^{b(F)} S'$ is equivalent to $B^F(S) \supseteq B^F(S')$. □

Proof For coalition S and S', assume $S \succeq^{b(F)} S'$. By the definition 3.2.8, if $T \setminus S'$ is a losing coalition with respect to F, then $T \setminus S$ is also a losing coalition with respect to F for all coalition T. It implies that all coalition $U \in B^F(S')$ is included in $B^F(S)$, which means $B^F(S) \supseteq B^F(S')$.

For coalition S and S', assume $B^F(S) \supseteq B^F(S')$. For all coalition T, the assumption says that if $T \setminus S'$ is a losing coalition with respect to F, then $T \setminus S$ is also a losing coalition with respect to F. By Definition 3.2.8, $S \succeq^{b(F)} S'$ holds. ■

By this proposition, the inclusion relation on the sets of the winning coalition which become losing coalition by the deviation of coalition becomes congruent with the comparison result by blockability relations for SCF.

Example 3.2.10. In Example 3.2.9, we had $12 \succ^{b(F)} 3$. In this case, $B^F(12) = \{12, 123\}$ and $B^F(3) = \{123\}$ hold. Then, we get $B^F(12) \supseteq B^F(3)$. □

This example provides a case that Proposition 3.2.5 supports.

Lemma 3.2.1. Consider an SCF F. If coalition S is a winning coalition, then S' such that $S \subseteq S'$ is also a winning coalition, □

Proof If $S = S'$, it is clear that S' is also a winning coalition.
Assume that $S \subset S'$ holds. $S' \backslash S$ can be written as $\{s^1, s^2, \ldots, s^m\}$. $S \cup \{s^1\}$ is also a winning coalition because $x \in A$ and xR^iy for all $i \in S$ and all $y \in A$, then $F(R^N) = x$ for all $R^N \in L^N$. $S \cup \{s^1\} \cup \{s^2\}$ is also a winning coalition due to same reason. By m times same operations, $S \cup \{s^1, s^2, \ldots, s^m\} = S'$ is also a winning coalition. ■

This lemma shows that every coalition which contains a winning coalition in terms of an SCF is also a winning coalition with respect to the same SCF.

Proposition 3.2.6. Consider an SCF F and coalitions S and S'. If $S \supseteq S'$, then $S \succeq^{b(F)} S'$. □

Proof Assume that $S \succ^{b(F)} S'$ does not hold for coalition S and S' such that $S \supseteq S'$. By Proposition 3.2.5, there is a winning coalition $T \in B^F(S') \backslash B^F(S)$. By Lemma 3.2.1, the winning coalition T is blocked by S because of $S \supseteq S'$ which is contradiction. ■

Bigger coalition has larger or equal influence from the point of view of the blockability relation for an SCF.

Proposition 3.2.7. Consider an SCF F and coalitions S and S'. Blockability relations $\succeq^{b(F)}$ for the SCF satisfies transitivity. □

Proof By Proposition 3.2.5, $S \succeq^{b(F)} S'$ is equivalent to $B^F(S) \supseteq B^F(S')$ for all coalitions S and S'. The binary relation $\supseteq$ satisfies transitivity on 2^N. Hence, the relation $\succeq^{b(F)}$ also satisfies transitivity. ■

This proposition shows that blockability relations $\succ^{b(F)}$ for SCF determines a maximal element on coalitions.

Example 3.2.11. Consider Example 3.2.9. It was seen that $12 \succ^{b(F)} 3$ in Example 3.2.9. It also holds that $123 \succ^{b(F)} 12$ because of $B^F(123) = \{12, 123\} \supseteq B^F(12)$. Then, $B^F(123) \supseteq B^F(3)$ holds which means $123 \succ^{b(F)} 3$ by the Definition 3.2.8. □

A case of the proposition 3.2.7 is shown in Example 3.2.11.

For coalition S and permutation π of N, we define $\pi(S)$ as the set $\{\pi(i)|i \in S\}$.

Example 3.2.12. Consider Example 3.2.9. Give a permutation π of N such that $\pi(1) = 1$, $\pi(2) = 3$ and $\pi(3) = 2$. In this case, $\pi(12) \succ^{b(F)} \pi(3)$ holds because of $13 \succ^{b(F)} 2$. □

This example shows that the permutation of N does not affect to the coalition influence for the coalition 12 and 3 with respect to F.

Definition 3.2.9 (α-effective relation)**.** Consider an SCF F. For coalitions S and S', $S \succeq^{\alpha(F)} S'$ is defined as:

$$E^F_\alpha(S) \supseteq E^F_\alpha(S'),$$

for all $B \in 2^A$.

$S \sim^{\alpha(F)} S'$ is denoted that both $S \succeq^{\alpha(F)} S'$ and $S' \succeq^{\alpha(F)} S$ hold. □

Definition 3.2.10 (β-effective relation)**.** Consider an SCF F. For coalitions S and S', $S \succeq^{\beta(F)} S'$ is defined as: for all

$$E^F_\beta(S) \supseteq E^F_\beta(S'),$$

for all $B \in 2^A$.

$S \sim^{\beta(F)} S'$ expresses that both $S \succeq^{\beta(F)} S'$ and $S' \succeq^{\beta(F)} S$ hold. □

The next example gives the differences among blockability relation for SCF, α-effective relation, and β-effective relation.

Example 3.2.13. Consider Example 3.2.9. It was seen that $12 \succ^{b(F)} 3$ in Example 3.2.9. $12 \succ^{\alpha(F)} 3$ holds because of $E^H_\alpha(12) = \{x, y, z\}$ and $E^H_\alpha(3) = \emptyset$. $12 \sim^{\beta(F)} 3$ holds because of $E^H_\beta(12) = \{x, y, z\}$ and $E^H_\beta(3) = \{x, y, z\}$. □

This example shows that blockability relation for SCF and β-effective relation are different.

Preference distance between coalition and SWF was proposed in the last section. As similiar to the proposed preference distance function for SWF, a function which evaluates how different alternatives selected by SCC and player's preference is proposed.

Definition 3.2.11 (Alternative-preference measurement)**.** Consider a pair (N, A). An alternative-preference measurement is a function $e : A \times L \to \mathbb{R}_+$. □

Alternative-preference measurement assigns a non-negative real number to a pair of alternative and preference on the set of alternatives.

Example 3.2.14. Consider a pair (N, A) such that $N = \{1, 2, \ldots, n\}$ and $A = \{a_1, a_2, \ldots, a_m\}$. Any linear preference on A can be expressed by a sequence $(b_1, \ldots, b_j, \ldots, b_m)$, where $b_1, \ldots, b_j, \ldots b_m \in A$. For any alternative a_k, consider a function $e\ (a_k\ ; b_1, \ldots, b_j, \ldots, b_m) = j - 1$ such that $a_k = b_j$ holds. Then, the function e is an alternative-preference measurement. □

Definition 3.2.12 (Player's alternative-preference measurement for SCC)**.** Consider an SCC H and an alternative-preference measurement e. Player i's alternative-preference measurement for H is defined as follows:

$$E^i(H) = \sum_{P \in L^N} \sum_{x \in H(P)} e(x, P^i),$$

where P^i is i-th component of P. □

Example 3.2.15. Consider SWF F in Example 2.1.1. For all $(P^1, P^2, P^3) \in L^N$, define $H(P^1, P^2, P^3) = \{b\}$ such that $bF(P^1, P^2, P^3)c$ for all $c \in A$. Let e be an alternative-preference measurement in Example 3.2.14. The values of the e are below:

$$e(a_1, R_1) = 0, e(a_1, R_2) = 0, e(a_1, R_3) = 1, e(a_1, R_4) = 2,$$

$$e(a_1, R_5) = 1, e(a_1, R_6) = 2, e(a_2, R_1) = 1, e(a_2, R_2) = 2,$$

$$e(a_2, R_3) = 0, e(a_2, R_4) = 0, e(a_2, R_5) = 2, e(a_2, R_6) = 1,$$

$$e(a_3, R_1) = 2, e(a_3, R_2) = 1, e(a_3, R_3) = 2, e(a_3, R_4) = 1,$$

$$e(a_3, R_5) = 0, e(a_3, R_6) = 0.$$

In this case, players alternative-preference measurement are calculated as follows:

$$E^1(H) = \sum_{P \in L^N} \sum_{x \in H(P)} e(x, P^1) = 5 \times 6 = 30,$$

$$E^2(H) = \sum_{P \in L^N} \sum_{x \in H(P)} e(x, P^2) = 5 \times 5 \times 6 = 150,$$

$$E^3(H) = \sum_{P \in L^N} \sum_{x \in H(P)} e(x, P^3) = 5 \times 5 \times 6 = 150.$$

□

We see that $E^1(H) < E^2(H) = E^3(H)$ holds in this example. The provided SCC H and alternative-preference measurement e preserve the magnitude relation $D^1(F) < D^2(F) = D^3(F)$ in Example 3.2.1.

One of future research is to find the transformation from SWF to SCC and alternative-preference measurement which magnitude relation of D^i and E^i is preserved.

3.3 Comparison of Coalition Influence for Negotiation

This section deals with comparison of bargaining power of coalitions by using the concepts of objection and counter-objection.

3.3.1 Coalition Bargaining Power

This section proposes a definition of a relation on the set of all coalitions in a game. An example demonstrates how the newly proposed relation works, and a theorem shows that the proposed relation is acyclic.

Definition 3.3.1 (Relation $\gg$ on coalitions in $(x;\mathcal{P})$)**.** Consider a game (N,v), and let $(x;\mathcal{P})$ be an i.r.p.c. for (N,v). Suppose two coalitions S^1 and S^2 in N. Then, coalition S^1 is said to be *stronger* than coalition S^2 (or, equivalently, coalition S^2 is *weaker* than coalition S^1) in $(x;\mathcal{P})$, denoted by $S^1 \gg S^2$, if and only if

1. for each $i \in S^1$, there exists $j \in S^2$ such that $i \succ j$, and

2. for each $i \in S^1$ and each $j \in S^2$, it is not satisfied that $j \succ i$.

Then, S^1 is said to be equal to S^2, denoted by $S^1 \sim S^2$, if and only if neither $S^1 \gg S^2$ nor $S^2 \gg S^1$ hold. □

Note that in Definition 3.3.1, S^1 and S^2 can be arbitrary non-empty subsets of N, and in particular, it is *not* assumed that S^1 or S^2 are coalitions in the coalition structure $\mathcal{P}$. We see, from Definition 3.3.1 and the comments just after Definition 2.2.11, that if $S^1 \gg S^2$ in $(x;\mathcal{P})$ and $S^1 \cap T \neq \emptyset$ for some $T \in \mathcal{P}$, then $S^2 \cap T \neq \emptyset$.

The following two numerical examples show how the newly proposed relation $\gg$ on the set of all coalitions works.

Example 3.3.1 demonstrates that if a player $i \in N$ is identified with a one-player coalition $\{i\}$ in N, then the newly proposed relation $\gg$ on coalitions reserves the relation $\succ$ on players.

Example 3.3.1. In Example 2.2.6, we see that $2 \succ 1$ in $(x;\mathcal{P}) = ((80,20,0);\{12,3\})$ in the game (N,v) given in Example 2.2.5. We also see, by Lemma 2.2.1, that $1 \succ 2$ does not hold in $(x;\mathcal{P}) = ((80,20,0);\{12,3\})$. Therefore, it holds $\{2\} \gg \{1\}$ in $(x;\mathcal{P}) = ((80,20,0);\{12,3\})$. □

Example 3.3.2 demonstrates how the newly proposed relation $\gg$ on the set of all coalitions works for comparing coalitions with two or more members.

Example 3.3.2. Consider the game (N,v) such that $N = \{1,2,3,4\}$, $v(1) = v(2) = v(3) = v(4) = 0$, $v(12) = v(13) = v(123) = v(134) = v(124) = 80$, $v(14) = v(23) = v(24) = v(34) = 65$, $v(234) = 75$, and $v(1234) = 120$. Let us compare two coalitions, 12 and 34, in the i.r.p.c. $(x;\mathcal{P}) = ((30,30,30,30);\ \{1234\})$.

The i.r.p.c. $(y;\mathcal{P}') = ((40,40,0,0);\{12,3,4\})$ is an objection of player 1 against player 3, and player 3 does not have any counter-objections $(z;\mathcal{P}'')$ against 1 with respect to this objection. Thus, we have $1 \succ 3$, and thus, $3 \succ 1$ is not satisfied by the asymmetry of $\succ$. Similarly, the i.r.p.c. $(y;\mathcal{P}') = ((40,40,0,0);\{12,3,4\})$ is an objection of player 2 against player 4, and player 4 does not have any counter-objections $(z;\mathcal{P}'')$ against 2 with respect to this objection. So, we have $2 \succ 4$, and thus, $4 \succ 2$ is not satisfied by the asymmetry of $\succ$.

Player 1 has a counter-objection against player 4 with respect to each objection $(y;\mathcal{P}')$ of player 4 against player 1, that is, $4 \succ 1$ is not satisfied. Similarly, player 2 has a counter-objection $(z;\mathcal{P}'')$ against player 3 with respect to each objection $(y;\mathcal{P}')$ of player 3 against player 2, that is, $3 \succ 2$ is not satisfied.

Therefore, since we have $1 \succ 3$, "not $3 \succ 1$," $2 \succ 4$, "not $4 \succ 2$," "not $4 \succ 1$," and "not $3 \succ 2$," we have $12 \gg 34$ in $(x;\mathcal{P}) = ((30,30,30,30);\{1234\})$. □

The next theorem verifies that the relation $\gg$ defined in Definition 3.3.1 is acyclic.

Theorem 3.3.1. *Let $(x;\mathcal{P})$ be an i.r.p.c. for a game (N,v). Then, the relation $\gg$ on the set of all coalitions is acyclic.* □

Proof Assume that coalitions $S^1, S^2, \dots, S^t$ in N satisfies that $S^1 \gg S^2 \gg \cdots \gg S^t \gg S^1$. Then, for each u $(u = 1,2,\dots,t-1)$ and each $k^u \in S^u$, there exists a player $k^{u+1} \in S^{u+1}$ such that $k^u \succ k^{u+1}$. This implies that there exists a sequence of players $k^1, k^2, \dots, k^t, k^{t+1}, \dots$ such that

$$k^1 \succ k^2 \succ \cdots \succ k^t \succ k^{t+1} \succ \cdots \succ k^{2t} \succ k^{2t+1} \succ \cdots \succ k^{3t} \succ \cdots,$$

where $k^1, k^{t+1}, k^{2t+1}, \dots \in S^1$, $k^2, k^{t+2}, k^{2t+2}, \dots \in S^2$, ..., and $k^t, k^{2t}, k^{3t}, \dots \in S^t$.

Since the set N of all player is finite, one can find v and w such that $w > v$ and $k^v = k^w$, that is, the sub-sequence

$$k^v \succ k^{v+1} \succ \cdots \succ k^{w-1} \succ k^w = k^v$$

of the above sequence is cyclic, but this contradicts Lemma 2.2.1. Hence, the relation $\gg$ on the set of all coalitions is acyclic. ■

By Theorem 3.3.1, it is verified that one can find the maximal coalitions from all coalitions with respect to the newly proposed relation $\gg$ on the set of all coalitions.

More, as in the case of the relation $\succ$ on the set of all players, the relation $\gg$ on the set of all coalitions is asymmetric, that is, for coalitions S^1 and S^2 in N, if $S^1 \gg S^2$, then $S^2 \succ S^1$ is not true.

The next example shows that the relation $\gg$ on the set of all coalitions is *not* necessarily transitive.

Example 3.3.3 ([5]). Consider the game (N, v) such that $N = \{1, 2, 3, 4, 5\}$, $v(1) = v(2) = v(3) = 0$, $v(12) = v(13) = v(123) = 30$, $v(14) = 40$, $v(35) = 20$, $v(245) = 30$, and for $B \subset N$, $v(B) = 0$, otherwise. We see $\{1\} \gg \{2\}$, $\{2\} \gg \{3\}$ and $\{1\} \sim \{3\}$ in $(x; \mathcal{P}) = ((10, 10, 10, 0, 0); \{123, 4, 5\})$. □

Theorem 3.3.2 verifies that the set of all i.r.p.c.s under which all coalitions have the equal bargaining power coincides with the bargaining set.

Theorem 3.3.2. *Let (N, v) be a game. Then, for each i.r.p.c. $(x; \mathcal{P})$ for (N, v), we have that*

$(x; \mathcal{P})$ is M-stable if and only if $S^1 \sim S^2$ in $(x; \mathcal{P})$ for all coalitions S^1 and S^2 in N.

■

Proof Assume that $(x; \mathcal{P})$ is M-stable. Then, for all players i_1 and i_2 in N, we have $i_1 \sim i_2$ by Definition 2.2.13. Therefore, we have that for all coalitions S^1 and S^2, $S^1 \sim S^2$ by Definition 3.3.1.

If $S^1 \sim S^2$ for all coalitions S^1 and S^2, then considering all one-player coalitions $\{i_1\}$ and $\{i_2\}$ in N, we have $i_1 \sim i_2$ for all players i_1 and i_2 in N, which means that $(x; \mathcal{P})$ is M-stable. ■

3.3.2 Influence of Bargaining Results

In this section, a relation on payoff configurations for comparison of coalition allocations in a game is proposed.

For a w.e.p.c. $(x;\mathcal{U})$ at an NTU-game (N, V), $\mathcal{J}(x;\mathcal{U})$ denotes the set of all justified objections of k against l in $(x;\mathcal{U})$ for some k, l in U for some $U \in \mathcal{U}$. The $\mathcal{J}$ can be regarded as a operation which assigns a set to a payoff configuration.

Definition 3.3.2 (Relation on payoff configurations). Consider w.e.p.c.s $(x;\mathcal{U})$ and $(y;\mathcal{U}')$ at an NTU-game (N, V). Then, a relation on payoff configurations at NTU-game (N, V) is defined as, for $(x;\mathcal{U})$ and $(y;\mathcal{U}')$, $(x;\mathcal{U}) >^J (y;\mathcal{U}')$ if and only if $\mathcal{J}(x;\mathcal{U})$ is a proper subset of $\mathcal{J}(y;\mathcal{U}')$. Neither $(x;\mathcal{U}) >^J (y;\mathcal{U}')$ nor $(y;\mathcal{U}') >^J (x;\mathcal{U})$ is denoted by $(y;\mathcal{U}') \sim (x;\mathcal{U})$ □

This relation compares a pair of payoff configurations from the viewpoint which payoff configurations have less justified objections. Next, a numerical example of proposed method is given.

Example 3.3.4. Assume that $N = \{1, 2, 3\}$ and V satisfies the following conditions:
$V(\{i\}) = \{(x_i) \mid x_i \leq 0\}$ for all $i \in N$,
$V(\{1, 2\}) = \{(x_1, x_2) \mid x_1 + x_2 \leq 10\}$,
$V(\{1, 3\}) = \{(x_1, x_3) \mid x_1 + x_3 \leq 16\}$,
$V(\{2, 3\}) = \{(x_2, x_3) \mid x_2 + x_3 \leq 10\}$, and
$V(\{1, 2, 3\}) = \{(x_1, x_2, x_3) | x_1 + x_2 + x_3 \leq 18\}$.
In this case, we get
$\mathcal{J}((6, 4, 0); \{\{1, 2\}, \{3\}\}) = \{(x; \{\{1, 3\}, \{2\}\}) \mid x^{\{1,3\}} \in \partial V(\{1, 3\}), x_1 > 6$ and $x_3 > 6\}$,
$\mathcal{J}((0, 5, 5); \{\{2, 3\}, \{1\}\}) = \{(x; \{\{1, 3\}, \{2\}\}) \mid x^{\{1,3\}} \in \partial V(\{1, 3\}), x_1 > 5$ and $x_3 > 5\}$.
Then $\mathcal{J}((6, 4, 0); \{\{1, 2\}, \{3\}\})$ is proper subset of $\mathcal{J}((0, 5, 5); \{\{2, 3\}, \{1\}\})$, hence it holds that $((6, 4, 0); \{\{1, 2\}, \{3\}\}) >^J ((0, 5, 5); \{\{2, 3\}, \{1\}\})$. □

This example means that $((0, 5, 5); \{\{2, 3\}, \{1\}\})$ is harder to be achieved than $((6, 4, 0); \{\{1, 2\}, \{3\}\})$.

Next example shows that there exists a case that $\sim$ holds.

Example 3.3.5. Consider Example 3.3.4. It holds that
$\mathcal{J}((5, 5, 0); \{\{1, 2\}, \{3\}\}) = \{(x; \{\{1, 3\}, \{2\}\}) \mid x^{\{1,3\}} \in \partial V(\{1, 3\}), x_1 > 5$ and $x_3 > 5\}$,
$\mathcal{J}((0, 5, 5); \{\{2, 3\}, \{1\}\}) = \{(x; \{\{1, 3\}, \{2\}\}) \mid x^{\{1,3\}} \in \partial V(\{1, 3\}), x_1 > 5$ and $x_3 > 5\}$.
Then, $\mathcal{J}((5, 5, 0); \{\{1, 2\}, \{3\}\})$ is not a proper subset of $\mathcal{J}((0, 5, 5); \{\{2, 3\}, \{1\}\})$. Similarly, $\mathcal{J}((0, 5, 5); \{\{2, 3\}, \{1\}\})$ is not a proper subset of $\mathcal{J}((5, 5, 0); \{\{1, 2\}, \{3\}\})$. These mean that $((5, 0, 5); \{\{1, 2\}, \{3\}\}) \sim ((0, 5, 5); \{\{2, 3\}, \{1\}\})$ by Definition 3.3.2. ■

To clarify which properties are satisfied by the proposed relation, some concepts for NTU-games are introduced.

Definition 3.3.3 (λ-scale NTU-games)**.** For an NTU-game (N, V), $(N, \lambda V)$ is defined as an NTU-game, where $\lambda V(S) = \{\lambda \cdot x^S | x^S \in V(S)\}$ for coalition S and $\lambda \in \mathbb{R}_{++}$.

□

$(N, \lambda V)$ is a situation that each payoff of coalitions on (N, V) is multiplied λ times. There is no change about the structure of (N, V) by this transformation because this transformation is positive linear transformation.

Definition 3.3.4 (Independence of common utility scale)**.** An NTU-game (N, V), w.e.p.c. $(x;\mathcal{U})$ and $(y;\mathcal{U}')$ are given. A relation R on payoff configurations is said to be independent of common utility scale, if and only if for every $\lambda \in \mathbb{R}_{++}$, $(x;\mathcal{U})$ R $(y;\mathcal{U}')$ at NTU-game (N, V) if and only if $(\lambda \cdot x;\mathcal{U})$ R $(\lambda \cdot y;\mathcal{U}')$ at NTU-game $(N, \lambda V)$ □

Independence of common utility scale is expressing that an order by some relation is preserved even if the payoffs of coalitions multiplied by a constant. The following lemma is useful to give a proof of main theorems.

Lemma 3.3.1. For an NTU-game (N, V), consider a w.e.p.c. $(x;\mathcal{U})$ at (N, V). Then, it holds that $\{(\lambda \cdot y;\mathcal{V}) | (y;\mathcal{V}) \in \mathcal{J}(x;\mathcal{U})\}$ corresponds with $\mathcal{J}(\lambda \cdot x;\mathcal{U})$ at NTU-game $(N, \lambda V)$ for every $\lambda \in \mathbb{R}_{++}$ by the definition of $(N, \lambda V)$. ■

Proof Assume that $(x;\mathcal{U})$ is a w.e.p.c. at an NTU-game (N, V) and $\lambda \in \mathbb{R}_{++}$. Then, $(\lambda \cdot x;\mathcal{U})$ is a w.e.p.c. at an NTU-game $(N, \lambda V)$. It is clear that if $\mathcal{J}(x;\mathcal{U})$ is empty, also $\mathcal{J}(\lambda \cdot x;\mathcal{U})$ is empty at each NTU-game. In this case, they are matched obviously. For all $(y;\mathcal{V}) \in \mathcal{J}(x;\mathcal{U}) \neq \emptyset$ at (N, V) and $(z;\mathcal{W}) \in \mathcal{J}(\lambda \cdot x;\mathcal{U}) \neq \emptyset$ at $(N, \lambda V)$, there exists identity from $\mathcal{V}$ to $\mathcal{W}$ because $(N, \lambda V)$ is positive linear transformation of (N, V). By the same reason, it holds that if $\mathcal{V} = \mathcal{W}$ then there exists bijection $\lambda \cdot y = z$. Hence, $|\mathcal{J}(x;\mathcal{U})| = |\mathcal{J}(\lambda \cdot x;\mathcal{U})|$ holds. It is shown that $\mathcal{J}(\lambda \cdot x;\mathcal{U}) = \{(\lambda \cdot y;\mathcal{V}) | (y;\mathcal{V}) \in \mathcal{J}(x;\mathcal{U})\}$. ■

Definition 3.3.5 (w-parallel shift NTU-games)**.** For an NTU-game (N, V), $(N, V + w)$ is defined as an NTU-game which has a characteristic function that $V + w(S) = \{(x_i + w)^{i \in S} | x^S \in V(S)\}$ for all coalition S and $w \in \mathbb{R}$. □

$(N, V + w)$ is a situation that each coalition at (N, V) can gather each payoff of coalition plus w. There is no change about the structure of (N, V) by this transformation because this transformation is positive linear transformation.

Definition 3.3.6 (Independence of parallel shift utilities)**.** An NTU-game (N, V), w.e.p.c. $(x;\mathcal{U})$ and $(y;\mathcal{U}')$ are given. A relation R on payoff configurations satisfies independence of parallel shift utilities, if and only if for every $w \in \mathbb{R}$, $(x;\mathcal{U})\ R\ (y;\mathcal{U}')$ at NTU-game (N, V) if and only if $((x_i + w)^{i\in N};\mathcal{U})\ R\ ((y_i + w)^{i\in N};\mathcal{U}')$ at NTU-game $(N, V + w)$. □

Independence of parallel shift utilities is representing that a order by some relation is preserved in spite of the addition of a constant to each payoff of coalitions.

Lemma 3.3.2. For an NTU-game (N, V), consider a w.e.p.c. $(x;\mathcal{U})$ at (N, V). Then, it holds that $\{y + w | y \in \mathcal{J}(x;\mathcal{U})\}$ corresponds with $\mathcal{J}(x + w;\mathcal{U})$ at NTU-game $(N, V + w)$ for every $w \in \mathbb{R}$. □

Proof Assume that $(x;\mathcal{U})$ is a w.e.p.c. at an NTU-game (N, V) and $w \in \mathbb{R}$. Then, $(x+w;\mathcal{U})$ is a w.e.p.c. at an NTU-game $(N, V+w)$. For all $(y;\mathcal{V}) \in \mathcal{J}(x;\mathcal{U})$ at (N, V) and $(z;\mathcal{W}) \in \mathcal{J}(x + w;\mathcal{U})$ at $(N, V + w)$, there exists identity from $\mathcal{V}$ to $\mathcal{W}$ because $(N, V + w)$ is positive linear transformation of (N, V). By the same reason, it holds that if $\mathcal{V} = \mathcal{W}$ then there exists bijection $y+w = z$. Hence, $|\mathcal{J}(x;\mathcal{U})| = |\mathcal{J}(x+w;\mathcal{U})|$ holds. It is shown that $\mathcal{J}(x + w;\mathcal{U}) = \{(y + w;\mathcal{V})|(y;\mathcal{V}) \in \mathcal{J}(x;\mathcal{U})\}$. ■

Definition 3.3.7 (Monotonicity)**.** For an NTU-game (N, V) and any payoff configurations $(x;\mathcal{U})$ and $(y, \mathcal{V})$, relation R satisfies that if $x \gg^N y$, then $(x;\mathcal{U})R(y, \mathcal{V})$. It is said that R satisfies *monotonicity.* □

The rest of this section treats the properties which the newly proposed relation satisfies. And, it will be shown that $>^J$ does not satisfy monotonicity.

Theorem 3.3.3. *Relation $>^J$ is strict partial order. That is, $>^J$ satisfies irreflexivity, asymmetry and transitivity.* □

Proof Irreflexivity: For any w.e.p.c. $(x;\mathcal{U})$ at an NTU-game (N, V), $\mathcal{J}(x;\mathcal{U})$ is not a proper subset of $\mathcal{J}(x;\mathcal{U})$, hence it does not hold that $(x;\mathcal{U}) >^J (x;\mathcal{U})$.
Asymmetry: Let $(x;\mathcal{U})$ and $(y;\mathcal{U}')$ be w.e.p.c.s at an NTU-game (N, V). If $(x;\mathcal{U}) >^J (y;\mathcal{U}')$, then $\mathcal{J}(x;\mathcal{U})$ is a proper subset of $\mathcal{J}(y;\mathcal{U}')$. $\mathcal{J}(y;\mathcal{U}')$ is not a proper subset of $\mathcal{J}(x;\mathcal{U})$, hence it does not hold that $(y;\mathcal{U}') >^J (x;\mathcal{U})$.
Transitivity: Let $(x;\mathcal{U})$, $(y;\mathcal{U}')$ and $(z;\mathcal{U}'')$ be w.e.p.c.s at an NTU-game (N, V).

If $(x;\mathcal{U}) >^J (y;\mathcal{U}')$ and $(y;\mathcal{U}') >^J (z;\mathcal{U}'')$, then $\mathcal{J}(x;\mathcal{U}) \subset \mathcal{J}(y;\mathcal{U}')$ and $\mathcal{J}(y;\mathcal{U}') \subset \mathcal{J}(z;\mathcal{U}'')$ hold. Hence $\mathcal{J}(x;\mathcal{U}) \subset \mathcal{J}(z;\mathcal{U}'')$ holds. By definition, it holds that $(x;\mathcal{U}) >^J (z;\mathcal{U}'')$. ■

This theorem implies that one can assign a real number to each payoff configuration, so that the bigger the payoff configuration is in the sense of $>^J$, the bigger the real number assigned to the payoff configuration is.

The next two theorems shows that $>^J$ is independent from linear transformation of NTU-game.

Theorem 3.3.4. *Relation $>^J$ satisfies independence of common utility scale.* ■

Proof Assume $(x;\mathcal{U}) >^J (y;\mathcal{U}')$ at NTU-game (N, V). Then it holds that $\mathcal{J}(x;\mathcal{U})$ is a proper subset of $\mathcal{J}(y;\mathcal{U}')$. For $(\lambda \cdot x;\mathcal{U})$ and $(\lambda \cdot y;\mathcal{U}')$ at NTU-game $(N, \lambda V)$, each set of justified objection is described as $\mathcal{J}(\lambda \cdot x;\mathcal{U}) = \{(\lambda \cdot z;\mathcal{V})|(z;\mathcal{V}) \in \mathcal{J}(x;\mathcal{U})\}$, $\mathcal{J}(\lambda \cdot y;\mathcal{U}') = \{(\lambda \cdot w;\mathcal{W})|(w;\mathcal{W}) \in \mathcal{J}(y;\mathcal{U}')\}$ by Lemma 3.3.1. It means that $\mathcal{J}(\lambda \cdot x;\mathcal{U})$ is a proper subset of $\mathcal{J}(\lambda \cdot y;\mathcal{U}')$. Hence, $(\lambda \cdot x;\mathcal{U}) >^J (\lambda \cdot y;\mathcal{U}')$ holds.

Assume $(\lambda \cdot x;\mathcal{U}) >^J (\lambda \cdot y;\mathcal{U}')$ at NTU-game $(N, \lambda V)$. Then it holds that $\mathcal{J}(\lambda \cdot x;\mathcal{U})$ is a proper subset of $\mathcal{J}(\lambda \cdot y;\mathcal{U}')$. For $(x;\mathcal{U})$ and $(y;\mathcal{U}')$ at NTU-game (N, V), each set of justified objection is described as $\mathcal{J}(x;\mathcal{U}) = \{(1/\lambda \cdot z;\mathcal{V})|(z;\mathcal{V}) \in \mathcal{J}(\lambda \cdot x;\mathcal{U})\}$, $\mathcal{J}(y;\mathcal{U}') = \{(1/\lambda \cdot w;\mathcal{W})|(w;\mathcal{W}) \in \mathcal{J}(\lambda \cdot y;\mathcal{U}')\}$ by Lemma 3.3.1. It means that $\mathcal{J}(x;\mathcal{U})$ is a proper subset of $\mathcal{J}(y;\mathcal{U}')$. Hence, $(x;\mathcal{U}) >^J (y;\mathcal{U}')$ at NTU-game (N, V) holds.

It is shown that $>^J$ satisfies independence of common utility scale. ■

Theorem 3.3.5. *Relation $>^J$ satisfies independence of parallel shift utilities.* ■

Proof Assume $(x;\mathcal{U}) >^J (y;\mathcal{U}')$ at NTU-game (N, V). Then it holds that $\mathcal{J}(x;\mathcal{U})$ is a proper subset of $\mathcal{J}(y;\mathcal{U}')$. For $(x+w;\mathcal{U})$ and $(y+w;\mathcal{U}')$ at NTU-game $(N, V+w)$ where $w \in \mathbb{R}$, each set of justified objection is described as $\mathcal{J}(x+w;\mathcal{U}) = \{(z+w;\mathcal{V})|(z;\mathcal{V}) \in \mathcal{J}(x;\mathcal{U})\}$, $\mathcal{J}(y+w;\mathcal{U}') = \{(v+w;\mathcal{W})|(v,\mathcal{W}) \in \mathcal{J}(y;\mathcal{U}')\}$ by Lemma 3.3.2. It means that $\mathcal{J}(x+w;\mathcal{U})$ is a proper subset of $\mathcal{J}(y+w;\mathcal{U}')$. Hence, $(x+w;\mathcal{U}) >^J (y+w;\mathcal{U}')$ holds.

Assume $(x+w;\mathcal{U}) >^J (y+w;\mathcal{U}')$ at NTU-game $(N, V+w)$ for some $w \in \mathbb{R}$. Then it holds that $\mathcal{J}(x+w;\mathcal{U})$ is a proper subset of $\mathcal{J}(y+w;\mathcal{U}')$. For $(x;\mathcal{U})$ and $(y;\mathcal{U}')$

at NTU-game (N, V), each set of justified objection is described as $\mathcal{J}(x;\mathcal{U}) = \{(z - w;\mathcal{V})|(z;\mathcal{V}) \in \mathcal{J}(x+w;\mathcal{U})\}$, $\mathcal{J}(y;\mathcal{U}') = \{(v-w;\mathcal{W})|(v;\mathcal{W}) \in \mathcal{J}(y+w;\mathcal{U}')\}$ by Lemma 3.3.1. It means that $\mathcal{J}(x;\mathcal{U})$ is a proper subset of $\mathcal{J}(y;\mathcal{U}')$. Hence, $(x;\mathcal{U}) >^J (y;\mathcal{U}')$ at NTU-game (N, V) holds.

It is shown that $>^J$ satisfies independence of individual zero of utilities.

■

Next example shows that relation $>^J$ does not satisfy monotonicity.

Example 3.3.6. Consider Example 3.3.4. The following sets are decided.
$\mathcal{J}^V((4,7,7);\{\{1,2,3\}\}) = \{(x;\{\{1,3\},\{2\}\}) \mid x^{\{1,3\}} \in \partial V(\{1,3\}), x_1 > 4, x_2 = 0$ and $x_3 > 7\}$,
$\mathcal{J}^V((0,6,4);\{\{2,3\},\{1\}\}) = \{(x;\{\{1,3\},\{2\}\}) \mid x^{\{1,3\}} \in \partial V(\{1,3\}), x_1 > 5, x_2 = 0$ and $x_3 > 5\}$.
Then, $(4,7,7) \gg^N (0,6,4)$ holds, but $((4,7,7);\{\{1,2,3\}\}) >^J ((0,6,4);\{\{2,3\},\{1\}\})$ does not holds because $\mathcal{J}^V((4,7,7);\ \{\{1,2,3\}\})$ is not a proper subset of $\mathcal{J}^V((0,6,4);\ \{\{2,3\},\{1\}\})$. □

This example says that increasing all individual payoff does not always get stable in the sense of $>^J$.

3.4 Summary of Chapter 3

This chapter provided methods to compare coalition influence on the models of group decision and negotiation. Blockability relation (Definition 3.1.3), viability relation (Definition 3.1.4) and profitability relation (Definition 3.1.5) for games in characteristic function form were defined in the first section of this chapter. Blockability relation compares a pair of coalitions from the viewpoint how the coalition can make coalitions payoff decreased by the deviation from the coalitions. Blockability relation satisfies Viability relation compares a pair of coalitions from the viewpoint how the coalition can protect the coalition's payoff by the deviation performed by members of the coalition. It was confirmed that profitability relation compares a pair of coalitions from the viewpoint how the coalition can make coalitions payoff increased by the forming the coalitions. It was verified that profitability relation satisfies transitivity and completeness. Examples which shows how the provided relations work were also devoted. It was confirmed that the provided relations satisfy transitivity and completeness which allows to assign a real number to each coalition.

A method to compare coalition influence from viewpoint how the opinion of the coalition matches the group decision was defined in the Definition 3.2.3. A proposition which shows that the proposed method satisfies reflexivity, completeness, transitivity and negatively transitivity was given.

It was studied how change of the group decision rule affects to coalition influence through dealing with blockability relation for SCF (Definition 3.2.8).

This chapter also provided a method to compare bargaining power of coalition for games in characteristic function form. It was verified that the provided method satisfies acyclicity which enables to determine a maximal element. A theorem which shows that bargaining set is equivalent to the set such that all coalitions are indifferent based on the proposed method to compare the bargaining power of coalition.

For non-transferable utility games, a comparison method for payoff configurations was given (Definition 3.3.2). The given method compares a pair of payoff configurations based on the set of justified objections against each payoff configuration . Theorems which shows that the given method satisfies independence of common utility scale and independence of parallel shift utilities were devoted. An example which is a case that the given method does not satisfy monotonicity was provided.

The methods which were proposed in Chapter 3 enable us to know coalition influence in group decision and negotiation. The result calculated by the proposed methods provides a prediction of coalitions' action in terms of coalition influence. The provided predictions will be useful for us to make a decision about coalitions' action. On the otherhand, the method to compare coalition influence based on preference distance throws up which coaltion's opinion is matched with the group decision rule. In other words, the method clarifies how much power of control the coalitions have in the group decision. This point will contribute to know what coalitions will form in the group decision.

Chapter 4 proposes coalition values which assign a real number to each coalition. Some proposed coalition values are based on concepts of the methods proposed in Chapter 3.

Chapter 4

Evaluation of Coalition Influence

The more number of players join a game, the larger computational effort is required to know all coalition influence determined by binary relations which carry out pairwise comparison of coalitions. Methods to evaluate coalition influence with numerical value will help to figure out the comparison result of coalition influence with lower computational complexity.

This chapter deals with evaluation of coalition influence. Some methods which assign a number which expresses coalition influence to each coalition are proposed. Properties of the proposed methods are provided and discussed. This study will enable us to carry out numerical experiment to know which coalition will be formed in group decision making.

Shapley [46] proposed a function which assigns a real number to each player, and the real number is interpreted as the expected value of marginal contribution of the player in the case that the players form the grand coalition with a random sequence.

Banzhaf [36] value which is another existing function which assigns a real number to each player, and the value is interpreted as the expected value of marginal contribution of the player in the case that the players form the grand coalition when every coalition has the same probability to be formed.

These existing values for players are extended to those for coalitions in the framework of games in characteristic function form. Properties and examples of the extended values for coalitions are provided to know how the extended values work for our objects.

This chapter is due to [28].

4.1 Existing Values for Players

This section introduces some existing values for players in games in characteristic function form.

Definition 4.1.1 (value for players)**.** Consider a game (N, v). A value for players is a function $\phi : 2^N \to \mathbb{R}$. □

A value for players in a game assigns a real number to each player.

Definition 4.1.2 (Shapley value [46])**.** Consider a game (N, v). The Shapley value of player $i \in N$ is defined as follows:

$$\phi_i(N, v) = \sum_{T \subset N \setminus \{i\}} \frac{t!(n-t-1)!}{n!} [v(T \cup \{i\}) - v(T)],$$

where t is the number of elements of the set T.

The Shapley value of player i is interpreted as the expected value of i's marginal contribution against the coalitions when n players form the grand coalition N with a random order.

Next, the definition of a value for players proposed by Owen [36] is introduced.

Definition 4.1.3 (Banzhaf value [36])**.** Consider a game (N, v). Banzhaf value of player $i \in N$ is defined as follows:

$$\beta_i(N, v) = \frac{1}{2^{n-1}} \sum_{T \subset N \setminus \{i\}} [v(T \cup \{i\}) - v(T)]$$

The Banzhaf value of player i is interpreted as the expected value of i's marginal contribution against the coalitions when the possibilities that coalitions which i join are formed are the same .

Next, properties defined on players are introduced.

Definition 4.1.4 (Symmetry, null players, dummy players)**.** Consider a game (N, v).
(1) Player i and j are said to be *symmetric players* if

$$v(S \cup \{i\}) = v(S \cup \{j\})$$

holds for all coalition $S \subseteq N \setminus \{i, j\}$.
(2) Player i is said to be a *null player* if

$$v(T \cup \{i\}) = v(T)$$

holds for all coalition T.
(3) Player i is said to be a *dummy player* if

$$v(T \cup \{i\}) = v(T) + v(\{i\})$$

holds for all $T \subset N \setminus \{i\}$. □

Symmetric players i and j have the same influence to characteristic value of coalitions. Null players cannot bring any marginal contribution to all coalitions. There is no positive incentive to form coalition with dummy players.

Axioms, which are properties that values for players should satisfy, are introduced.

Axiom 4.1.1 (Efficiency)**.** Consider a game (N, v) and a value for players ϕ. ϕ satisfies *efficiency* if and only if it holds that

$$\sum_{i \in N} \phi_i(N, v) = v(N).$$

□

Efficiency means that all of $v(N)$ are allocated to the players.

Axiom 4.1.2 (Null players)**.** Consider a game (N, v) and a value for players ϕ. ϕ satisfies *null players* if and only if it holds that

$$\phi_i(N, v) = 0$$

for all null players i. □

This axiom contains the meaning that null players should be assigned zero value.

Axiom 4.1.3 (Symmetry)**.** Consider a game (N, v) and a value for players ϕ. ϕ satisfies *symmetry* if and only if it holds that

$$\phi_i(N, v) = \phi_j(N, v)$$

for all symmetric players i and j. □

Symmetry axiom expresses that symmetric players should get the same value.

To introduce additivity axiom, we provides a definition of addition of games.

Definition 4.1.5 (Game addition)**.** Consider two game (N, v_1) and (N, v_2). A game (N, v) is said to be addition of (N, v_1) and (N, v_2) if the following condition holds: For all coalition S,

$$v(S) = v_1(S) + v_2(S).$$

$v = v_1 + v_2$ denotes that a game (N, v) is addition of (N, v_1) and (N, v_2). □

Game addition is used in the following axiom.

Axiom 4.1.4 (Additivity)**.** Consider two different games (N, v_1) and (N, v_2). Let v be $v_1 + v_2$. ϕ satisfies *additivity* if and only if it holds that

$$\phi_i(N, v) = \phi_i(N, v_1) + \phi_i(N, v_2),$$

where ϕ is a value for players in (N, v), (N, v_1) and (N, v_2).

Additivity axiom means that the value preserve the results with the respect to the addition of the games.

Theorem 4.1.1 ([46])**.** *Shapley value is unique value which satisfies axioms of efficiency, null players, symmetry and additivity.* ■

Proof See [46].

4.2 Coalition Values Derived from Comparison of Coalition Influence

This section proposes coalition values which assign a real number to each coalition for games in characteristic function form. The proposed coalition values are derived from binary relations, which are blockability relation, viability relation, and profitability relation for games in characteristic function form, provided in Chapter 3.

4.2.1 Blockability Value

In this section, coalition values which indicate coalition influence are introduced. A coalition value is a function which assigns a real number to every coalition in a game.

Definition 4.2.1 (Blockability value)**.** Consider a game (N, v). *Blockability value* of coalition $S \subseteq N$ is defined as follows:

$$\hat{B}_S(N, v) = \frac{\sum_{T \subseteq N} v(T) - B^*(S)}{\sum_{T \subseteq N} v(T) - B^*(N)} \cdot v(N).$$

□

Blockability value of a coalition indicates the influence value of the coalition, and lesser $B^*(S)$ makes more $\hat{B}_S(N, v)$. Therefore, it is consistent with the concept of blockability relation for games in characteristic function form.

The next example shows how blockability value works.

Example 4.2.1. Consider a game (N, v) in Example 2.2.1. For coalitions 12 and 34, we have

$$\hat{B}_{12}(N, v) = \frac{\sum_{T \subseteq 1234} v(T) - B^*(12)}{\sum_{T \subseteq 1234} v(T) - B^*(1234)} \cdot v(1234) = \frac{276 - 0}{276 - 0} \cdot 42 = 42, \text{and}$$

$$\hat{B}_{34}(N, v) = \frac{\sum_{T \subseteq 1234} v(T) - B^*(34)}{\sum_{T \subseteq 1234} v(T) - B^*(1234)} \cdot v(1234) = \frac{276 - 144}{276 - 0} \cdot 42 \fallingdotseq 20.$$

□

Thus, we have $\hat{B}_{12}(N, v) > \hat{B}_{34}(N, v)$, which is consistent with the blockability relation of the coalitions, that is, $12 \succeq^B 34$. The next proposition shows that this is a general property between blockability relation and blockability value.

Proposition 4.2.1. *Consider a game (N, v). For all coalitions S^1 and S^2, $S^1 \succeq^B S^2$ is equivalent to $\hat{B}_{S^1}(N, v) \geq \hat{B}_{S^2}(N, v)$.* □

Proof Consider a game (N, v) and assume that $S^1 \succeq^B S^2$ for coalitions S^1 and S^2. By Definition 3.1.3, $B^*(S^1) \leq B^*(S^2)$ holds. Hence, we have the following inequality:

$$\frac{\sum_{T \subseteq N} v(T) - B^*(S^1)}{\sum_{T \subseteq N} v(T) - B^*(N)} \cdot v(N) \geq \frac{\sum_{T \subseteq N} v(T) - B^*(S^2)}{\sum_{T \subseteq N} v(T) - B^*(N)} \cdot v(N).$$

By Definition 4.2.1, $\hat{B}_{S^1}(N, v) \geq \hat{B}_{S^2}(N, v)$ holds.

Next, assume that $\hat{B}_{S^1}(N, v) \geq \hat{B}_{S^2}(N, v)$ for coalition S^1 and S^2. By Definition 4.2.1, we have

$$\frac{\sum_{T \subseteq N} v(T) - B^*(S^1)}{\sum_{T \subseteq N} v(T) - B^*(N)} \cdot v(N) \geq \frac{\sum_{T \subseteq N} v(T) - B^*(S^2)}{\sum_{T \subseteq N} v(T) - B^*(N)} \cdot v(N).$$

Hence, it holds that $B^*(S^1) \leq B^*(S^2)$. By Definition 3.1.3, $S^1 \succeq^B S^2$ holds. ■

This proposition verifies that blockability value has consistency with blockability relation for games in characteristic function form.

4.2.2 Viability Value

The next introduced value is derived from viability relations for games in characteristic function form.

Definition 4.2.2 (Viability value)**.** Consider a game (N, v). *Viability value* of coalition $S \subseteq N$ is defined as follows:

$$\hat{V}_S(N, v) = \frac{V^*(S)}{V^*(N)} \cdot v(N).$$

□

Viability values of coalitions indicate influence value of the coalitions, and are consistent with the viability relation of the coalitions. In fact, from the definition, it is clear that more $V^*(S)$ makes more $\hat{V}_S(N, v)$.

Viability values of coalitions 12 and 34 in Example 2.2.1 can be calculated as in the next example.

Example 4.2.2. Consider a game (N, v) in Example 2.2.1. For coalitions 12 and 34, we have

$$\hat{V}_{12}(N, v) = \frac{V^*(12)}{V^*(1234)} \cdot v(1234) = \frac{144}{276} \cdot 42 \fallingdotseq 22, \text{and}$$

$$\hat{V}_{34}(N, v) = \frac{V^*(34)}{V^*(1234)} \cdot v(1234) = \frac{0}{276} \cdot 42 = 0.$$

□

The result is consistent with the comparison by viability relation in Example 3.1.4, that is, $12 \succeq^V 34$. This is also a general property, which is verified by the next proposition.

Proposition 4.2.2. Consider a game (N, v). For all coalition S^1 and S^2, $S^1 \succeq^V S^2$ is equivalent to $\hat{V}_{S^1}(N, v) \geq \hat{V}_{S^2}(N, v)$. □

Proof Consider a game (N, v) and assume that $S^1 \succeq^V S^2$ for coalition S^1 and S^2. By Definition 3.1.4, $V^*(S^1) \geq V^*(S^2)$ holds. Hence, we have the following inequality:

$$\frac{V^*(S^1)}{V^*(N)} \cdot v(N) \geq \frac{V^*(S^2)}{V^*(N)} \cdot v(N).$$

By Definition 4.2.2, $\hat{V}_{S^1}(N, v) \geq \hat{V}_{S^2}(N, v)$ holds.

Next, assume that $\hat{V}_{S^1}(N, v) \geq \hat{V}_{S^2}(N, v)$ for coalition S^1 and S^2. By Definition 4.2.2, we have

$$\frac{V^*(S^1)}{V^*(N)} \cdot v(N) \geq \frac{V^*(S^2)}{V^*(N)} \cdot v(N).$$

Hence, it holds that $V^*(S^1) \geq V^*(S^2)$. By Definition 3.1.4, $S^1 \succeq^V S^2$ holds. ■

This proposition verifies that viability value has consistency with viability relation for games in characteristic function form.

4.2.3 Profitability Value

In this section, a function which evaluates coalition influence by a real number based on profitability relations is introduced. We call this function as coalition value. An example which expresses how to calculate introduced value is provided.

Definition 4.2.3 (Profitability value)**.** Consider a game (N, v). *Profitability value* of coalition $S \subseteq N$ is defined as follows:

$$\hat{P}_S(N, v) = \frac{\sum_{T \subseteq N} P^*(S)}{\sum_{T \subseteq N} P^*(N)} \cdot v(N).$$

□

It is clear that profitability value always assigns $v(N)$ to the grand coalition for a game (N, v). The next example shows how the profitability value for games in characteristic function form works.

Example 4.2.3. Consider a game (N, v) in Example 2.2.1.

$$\hat{P}_{12}(N, v) = \frac{\sum_{T \subseteq N} P^*(12)}{\sum_{T \subseteq N} P^*(N)} \cdot v(N) = \frac{624}{672} \cdot 42 = 39, \text{and}$$

$$\hat{P}_{34}(N,v) = \frac{V^*(34)}{V^*(1234)} \cdot v(1234) = \frac{404}{672} \cdot 42 = 25.25.$$

□

It is confirmed that the coalition 12 has more influence than the coalition 34 has from the viewpoint of profitability value in Example 2.2.1.

Proposition 4.2.3. Consider a game (N,v). For all coalitions S^1 and S^2, $S^1 \succeq^P S^2$ is equivalent to $\hat{P}_{S^1}(N,v) \geq \hat{P}_{S^2}(N,v)$. □

Proof Consider a game (N,v) and assume that $S^1 \succeq^P S^2$ for coalitions S^1 and S^2. By Definition 3.1.5, $P^*(S^1) \geq P^*(S^2)$ holds. Hence, we have the following inequality:

$$\frac{P^*(S^1)}{P^*(N)} \cdot v(N) \geq \frac{P^*(S^2)}{P^*(N)} \cdot v(N).$$

By Definition 4.2.3, $\hat{P}_{S^1}(N,v) \geq \hat{P}_{S^2}(N,v)$ holds.

Next, assume that $\hat{P}_{S^1}(N,v) \geq \hat{P}_{S^2}(N,v)$ for coalitions S^1 and S^2. By Definition 4.2.3, we have

$$\frac{P^*(S^1)}{P^*(N)} \cdot v(N) \geq \frac{P^*(S^2)}{P^*(N)} \cdot v(N).$$

Hence, it holds that $P^*(S^1) \geq P^*(S^2)$. By Definition 3.1.5, $S^1 \succeq^P S^2$ holds. ■

This proposition shows that profitability relation for games in characteristic function form is exactly extended to profitability value.

4.2.4 Properties of Coalition Values Derived from Coalition Influence

The concepts of null players and symmetric players are extended to those of null coalitions and symmetric coalitions to define conditions, called axioms below, which should be satisfied by coalition values.

Definition 4.2.4 (Null coalitions)**.** Consider a game (N,v). For all coalition $S \subseteq N$, S is a *null coalition*, if and only if $v(T \cup S) = v(T)$ for all $T \subseteq N$. □

A null coalition always does not bring any additional contribution toward the other coalitions through cooperation.

Example 4.2.4. Consider a game (N, v) in Example 2.2.1. Then, coalition $\{4\}$ is a null coalition because player 4 is a null player in (N, v). □

The next lemma provides a type of players included in a null coalition.

Lemma 4.2.1. Consider a game (N, v). For all coalition S, S is a null coalition in (N, v), if and only if for all $i \in S$, player i is null player in (N, v). □

Proof Fix a null coalition S in game (N, v). Then, for all player $i \in S$ and all $T \subseteq N$, $v(T) = v(T \cup S) = v(T \cup \{i\} \cup S) = v(T \cup \{i\})$ holds by the definition of null coalitions S. It means that every player in null coalition S is null player in (N, v).

Assume that every player in coalition S is a null player in (N, v). For all $T \subseteq N$, $v(T \cup S) = v(T \amalg (S \setminus T))$ holds, where $\amalg$ is disjoint union. If $i \in T$ then $v(T \cup \{i\}) = v(T)$ holds. For each player $i \in S \setminus T$, $v(T \amalg \{i\}) = v(T)$ holds because i is a null player in (N, v). After repeating this operation $|S \setminus T|$ times, it results in $v(T \amalg (S \setminus T)) = v(T)$ which means $v(T \cup S) = v(T)$ for all $T \subseteq N$. ■

Definition 4.2.5 (Symmetric coalitions)**.** Consider a game (N, v). For all $S^1, S^2 \subseteq N$, S^1 and S^2 are said to be *symmetric coalitions*, if and only if there exists a bijection $f : S^1 \to S^2$ such that i and $f(i)$ are symmetric players in (N, v). □

This definitions means that if S^1 and S^2 are symmetric coalitions in a game, then $|S_1| = |S_2|$ holds. For all symmetric coalitions S^1 and S^2, the contribution that a member of S^1 has is matched with the contribution that someone of S^2 has.

Example 4.2.5. Consider a game (N, v) in Example 2.2.1. Then, coalition 24 and coalition 34 are symmetric coalitions because there exists bijection $f : \{2, 4\} \to \{3, 4\}$ such that $f(2) = 3$ and $f(4) = 4$, where player 2 and player 3 are symmetric players, and player 4 and player 4 are symmetric players, too. □

Lemma 4.2.2. Consider a game (N, v) and symmetric coalitions S^1 and S^2 in (N, v). Then, $v(T \amalg S^1) = v(T \amalg S^2)$ for all $T \subseteq N \setminus S^1 \setminus S^2$, where $\amalg$ is disjoint union. □

Proof A bijection $g : S^1 \to S^2$ such that for all $x \in S^1 \backslash S^2$ and $g(x) \in S^2 \backslash S^1$ are symmetric players, and for all $y \in S^1 \cap S^2$ and $g(y) \in S^2 \cap S^1$ are symmetric players, can be generated from the bijection $f : S^1 \to S^2$ because it is clear that symmetric relation on N is equivalence relation (See Lemma A.1.1 in Appendix).

For all coalition $U \subseteq N \setminus S^1 \setminus S^2$, it holds that

$$v(U \amalg (S^1 \cap S^2)) = v(U \amalg (S^1 \cap S^2)).$$

For player $i \in S^1 \setminus S^2$ and $g(i) \in S^2 \setminus S^1$, it holds that

$$v(U \amalg (S^1 \cap S^2) \amalg \{i\}) = v(U \amalg (S^1 \cap S^2) \amalg \{g(i)\}).$$

For player $i \neq j \in S^1 \setminus S^2$ and $g(j) \in S^2 \setminus S^1$, it holds that

$$v(U \amalg (S^1 \cap S^2) \amalg \{i\} \amalg \{j\}) = v(U \amalg (S^1 \cap S^2) \amalg \{i\} \amalg \{g(j)\}) = v(U \amalg (S^1 \cap S^2) \amalg \{g(i)\} \amalg \{g(j)\}).$$

Repeating this discussion and $|S^1| = |S^2|$, it results in $v(U \amalg S^1) = v(U \amalg S^2)$.

■

This lemma shows that the concept of symmetric coalitions is exactly an extension of the concept of symmetric players.

The next lemma is used in the proof of the following proposition .

Lemma 4.2.3. Consider a game (N, v) and symmetric coalitions S^1 and S^2 in (N, v). Then, $v(S^1 \setminus (S^1 \cap U)) = v(S^2 \setminus g(S^1 \cap U))$ holds for all $U \subseteq N$, where g is a function defined in Lemma A.1.1 in Appendix.

Proof By Lemma 4.2.2, $v(S^1) = v(S^2)$ holds. For all $U \subseteq N$, fix $i \in S^1 \cap U$. Then, it holds that

$$v(S^1 \setminus \{i\}) = v(S^2 \setminus \{g(i)\})$$

because $S^1 \setminus \{i\}$ and $S^2 \setminus \{g(i)\}$ are symmetric coalitions in (N, v). By the same operation with player j such that $j \neq i$ and $j \in S^1 \cap U$, it results in the following formula:

$$v(S^1 \setminus \{i\} \setminus \{j\}) = v(S^2 \setminus \{g(i)\} \setminus \{g(j)\}).$$

Repeating this step makes $v(S^1 \setminus (S^1 \cap U)) = v(S^2 \setminus g(S^1 \cap U))$ for all $U \subseteq N$. ■

The following two axioms should be satisfied by coalition values:

Axiom 4.2.1 (Null coalitions)**.** Consider a game (N, v) and a coalition value ϕ. Coalition value ϕ is said to satisfy *null coalition axiom*, if and only if for all null coalitions $S \subseteq N$, $\phi_S(N, v) = 0$ holds. □

Null coalitions axiom means that each coalition which has no contribution will get zero evaluation.

Axiom 4.2.2 (Symmetry)**.** Consider a game (N, V) and a coalition value ϕ. Coalition value ϕ is said to satisfy *symmetry axiom*, if and only if for all symmetric coalitions S^1 and S^2, $\phi_{S^1}(N, v) = \phi_{S^2}(N, v)$ holds. □

Symmetry axiom means that all coalitions which have the same contribution will get the same evaluation.

Axiom 4.2.3 (Super additivity)**.** Consider a game (N, V) and a coalition value ϕ. Coalition value ϕ is said to satisfy *super additivity axiom*, if and only if $\phi(S_1)+\phi(S_2) \leq \phi(S_1 \cup S_2)$ for all coalitions $S_1, S_2 \subseteq N$ such that $S_1 \cap S_2 = \emptyset$. □

Super additivity axiom implies that all two coalitions which have no common players have incentives to merge because the merged coalition can get equal to or more evaluation than the total of the evaluation of the coalitions.

The nextt two propositions show that blockability value and viability value satisfy null coalitions axiom and symmetry axiom.

Proposition 4.2.4. Blockability value satisfies null coalitions axiom and symmetry axiom. □

Proof (Null coalitions axiom) Consider a game (N, v) and a null coalition S. Then, $B^*(S) = \sum_{T \subseteq N} v(T)$ because $v(T \setminus S) = v((T \setminus S) \cup S) = v(T \cup S) = v(T)$ holds for all coalition $T \subseteq N$. Hence, $\hat{B}_S = 0$ holds.

(Symmetry axiom) Consider a game (N, v) and symmetric coalitions S^1 and S^2. $B^*(S^1) = B^*(S^2)$ should be shown. The function $g : S^1 \to S^2$ defined in Lemma A.1.1 in Appendix is available for symmetric coalition S^1 and S^2. For all coalition $U \subseteq N$, $U = (U \setminus S^1) \amalg (U \cap S^1)$ holds. By the same operation, we get

$$U \setminus S^1 = ((U \setminus S^1) \setminus S^2) \amalg ((U \setminus S^1) \cap S^2).$$

Let the following formula hold.

$$U' = ((U \setminus S^1) \setminus S^2) \amalg g^{-1}((U \setminus S^1) \cap S^2) \amalg g(U \cap S^1).$$

Then, the following formula holds by the definition of g in Lemma A.1.1 in Appendix, in particular, $g(U \cup S_1) \subseteq S_2$.

$$U' \setminus S^2 = ((U \setminus S^1) \setminus S^2) \amalg g^{-1}((U \setminus S^1) \cap S^2).$$

Because $((U \setminus S^1) \cap S^2)$ and $g^{-1}((U \setminus S^1) \cap S^2)$ are symmetric coalitions, it results in $v(U \setminus S^1) = v(U' \setminus S^2)$ by Lemma 4.2.2. Consider a function $\mathscr{F} : 2^N \to 2^N$ such that $\mathscr{F}(U) = U'$ for all $U \in 2^N$. One can see that $\mathscr{F}$ is an injection as follows: for U and $V \subseteq N$, assume that $U' = V'$, that is,

$$\begin{aligned}&((U \setminus S^1) \setminus S^2) \amalg g^{-1}((U \setminus S^1) \cap S^2) \amalg g(U \cap S^1)\\ &= ((V \setminus S^1) \setminus S^2) \amalg g^{-1}((V \setminus S^1) \cap S^2) \amalg g(V \cap S^1).\end{aligned}$$

Because the items in each side are mutually disjoint, it follows that $(U \setminus S^1) \setminus S^2 = (V \setminus S^1) \setminus S^2$, $g^{-1}((U \setminus S^1) \cap S^2) = g^{-1}((V \setminus S^1) \cap S^2)$, and $g(U \cap S^1) = g(V \cap S^1)$. From bijectiveness of g and g^{-1}, one sees that $(U \setminus S^1) \setminus S^2 = (V \setminus S^1) \setminus S^2$, $(U \setminus S^1) \cap S^2 = (V \setminus S^1) \cap S^2$, and $U \cap S^1 = V \cap S^1$. Therefore, it is satisfied that

$$\begin{aligned}U &= ((U \setminus S^1) \setminus S^2) \amalg ((U \setminus S^1) \cap S^2) \amalg (U \cap S^1)\\ &= ((V \setminus S^1) \setminus S^2) \amalg ((V \setminus S^1) \cap S^2) \amalg (V \cap S^1)\\ &= V,\end{aligned}$$

which implies that $\mathscr{F}$ is a bijection by the finiteness of 2^N.

Hence, it results in $\sum_{U \subseteq N} v(U \setminus S^1) = \sum_{U' \subseteq N} v(U' \setminus S^1)$ which means that $B^*(S^1) = B^*(S^2)$ holds. ■

Proposition 4.2.5. Viability value satisfies null coalitions axiom and symmetry axiom. □

Proof (Null coalitions axiom) Consider a game (N, v) and a null coalition S. By Lemma 4.2.1, every player $i \in S$ is a null player in (N, v). For all $T \subseteq N$, $v(S \setminus T) = v(\emptyset) = 0$ because $(S \setminus T) \subseteq S$ holds. Then, $V^*(S) = 0$ which results in $\hat{V}_S(N, v) = 0$.

(Symmetry axiom) Consider a game (N, v), and symmetric coalitions S^1 and S^2. $V^*(S^1) = V^*(S^2)$ should be shown. A function $g : S^1 \to S^2$ defined in Lemma A.1.1 in Appendix is available for symmetric coalition S^1 and S^2. For all coalition $U \subseteq N$, the following formula holds.

$$U = (U \setminus S^1) \amalg (U \cap S^1) = ((U \setminus S^1) \setminus S^2) \amalg ((U \setminus S^1) \cap S^2) \amalg (U \cap S^1).$$

And, $S^1 \setminus U = S^1 \setminus (S^1 \cap U)$ holds.

Let the following formula holds.

$$U' = ((U \setminus S^1) \setminus S^2) \amalg g^{-1}((U \setminus S^1) \cap S^2) \amalg g(U \cap S^1).$$

Then, $S^2 \setminus U' = S^2 \setminus g(S^1 \cap U)$ because $g^{-1}((U \setminus S^1) \cap S^2) \subseteq N \setminus S^2$, $g(U \cap S^1) \subseteq S^2$, and $(U \setminus S^1) \setminus S^2 \subseteq N \setminus S^2$ hold. By Lemma 4.2.2, $v(S^1 \setminus U) = v(S^2 \setminus U')$ holds. Consider the function $\mathscr{F} : 2^N \to 2^N$, such that $\mathscr{F}(U) = U'$ for all $U \in 2^N$, defined in the proof of Proposition 1. As seen in the proof, $\mathscr{F}$ is a bijection.

Hence, it results in $\sum_{U \subseteq N} v(S^1 \setminus U) = \sum_{U' \subseteq N} v(S^2 \setminus U')$ which means that $V^*(S^1) = V^*(S^2)$ holds. ■

It was clarified that blockability value and viability value assign zero to a coalition which does not bring any additional contributions to the other coalitions, and the same evaluation to two coalitions which bring the same contribution to the other coalitions.

Proposition 4.2.6. Profitability value satisfies null coalitions axiom and symmetry axiom in games in characteristic function form. Profitability value satisfies super additivity axiom in convex game. □

Proof (Null coalitions axiom) Consider a game (N, v) and a null coalition S. By Lemma 4.2.1, every player $i \in S$ is a null player in (N, v). Then, $v(T \cup S) = v(T)$ holds for all $T \subseteq N$. Hence, $P^*(S) - \sum_{T \subseteq N} v(T) = 0$ which results in $\hat{P}_S(N, v) = 0$.

(Symmetry axiom) Consider a game (N, v) and symmetric coalitions S^1, S^2. S^1 and S^2. $P^*(S^1) = P^*(S^2)$ should be shown. A function $g : S^1 \to S^2$ defined in Lemma A.1.1 in Appendix is available for symmetric coalition S^1 and S^2. For coalition $T \subseteq N \backslash S^1 \backslash S^2$, $v(T \amalg S^1) = v(T \amalg S^2)$ for all $T \subseteq N \setminus S^1 \setminus S^2$ hold by Lemma 4.2.2.

For coalition $T \not\subseteq N \backslash S^1 \backslash S^2$, T can be described as the following formula.

$$T = (T \backslash S^1 \backslash S^2) \amalg ((T \backslash S^1) \cap S^2) \amalg ((T \backslash S^2) \cap S^1) \amalg (T \cap S^2 \cap S^1).$$

Then, the following formula holds.

$$v(T \cup S^1) = v(((T \backslash S^1 \backslash S^2) \amalg ((T \backslash S^1) \cap S^2) \amalg ((T \backslash S^2) \cap S^1) \amalg (T \cap S^2 \cap S^1)) \cup S^1).$$

Because of $((T \backslash S^2) \cap S^1) \amalg (T \cap S^2 \cap S^1) \subseteq S^1$, $v(T \cup S^1) = v(((T \backslash S^1 \backslash S^2) \amalg ((T \backslash S^1) \cap S^2) \cup S^1)$ holds. It holds that $(T \backslash S^1 \backslash S^2) \cap S^1 =$ and $((T \backslash S^1) \cap S^2) \cap S^1 =$, we get the following:

$$v(T \cup S^1) = v(((T \backslash S^1 \backslash S^2) \amalg ((T \backslash S^1) \cap S^2) \amalg S^1).$$

Consider $\mathscr{F} : 2^N \to 2^N$ such that $\mathscr{F}$ assigns a coalition $T' = (T\backslash S^2) \cup g^{-1}(T \cap S^2)$ such that $v(T \cup S^1) = v(((T'\backslash S^1\backslash S^2) \amalg ((T'\backslash S^2) \cap S^1) \amalg S^2)$ to the coalition T where $g((T'\backslash S^2) \cap S^1) = (T\backslash S^1) \cap S^2$ due to Lemma 4.2.2. If $U \neq U'$ then $\mathscr{F}(U) \neq \mathscr{F}(U')$ holds. $|2^N| < \infty$, hence $\mathscr{F}$ is bijection. By symmetry of the set 2^N, $\sum_{T \not\subseteq N\backslash S^1\backslash S^2} v(T \cup S^1) = \sum_{T \not\subseteq N\backslash S^1\backslash S^2} v(T \cup S^2)$ holds. ■

This proposition gives that profitability value assigns zero to a coalition which does not bring any additional contributions to the other coalitions, and the same evaluation to two coalitions which bring the same contribution to the other coalitions. Profitability value also has a property that integration of any two coalitions brings more evaluation in a game.

4.3 Coalition Values Derived from Existing Values of Players

This section proposes other coalition values which are derived from existing value for players, which are Shapley value and Banzhaf value. Shapley value and Banzhaf value assign a real number to each player, and these two values are extended to coalition values in this section. Some properties of the extended coalition values are given.

4.3.1 Group Shapley Value

This section defines group Shapley value.

Definition 4.3.1 (Group Shapley value). Consider a game (N, v). For a coalition S, *Group Shapley value* of S is defined as

$$\hat{\phi}_S(N, v) = \sum_{T \subset N\backslash S} \frac{t!(n-t-s)!}{(n-s+1)!} \left[v(T \cup S) - v(T) \right],$$

where t, s and n are the number of sets T, S and N, respectively. □

Group Shapley value of coalition S is interpreted as expected value of marginal contribution in the case that coalition S and the other players form the grand coalition N with a random sequence.

Example 4.3.1. Consider a game (N, v) in Example 2.2.1.

$$\hat{\phi}_{12}(N,v) = \sum_{T \subset 34} \frac{t!(4-t-2)!}{(4-2+1)!} [v(T \cup 12) - v(T)]$$

$$= \frac{0!2!}{3!} [v(12) - v(\emptyset)] + \frac{1!1!}{3!} [v(123) - v(3)] + \frac{1!1!}{3!} [v(14) - v(4)] + \frac{2!0!}{3!} [v(1234) - v(34)]$$

$$= 7 + 12 + 6 + 14 = 39, \text{and}$$

$$\hat{\phi}_{34}(N,v) = \sum_{T \subset 34} \frac{t!(4-t-2)!}{(4-2+1)!} [v(T \cup 34) - v(T)]$$

$$= \frac{0!2!}{3!} [v(34) - v(\emptyset)] + \frac{1!1!}{3!} [v(134) - v(1)] + \frac{1!1!}{3!} [v(234) - v(2)] + \frac{2!0!}{3!} [v(1234) - v(12)]$$

$$= 0 + 12 + 4 + 2 = 18.$$

□

This example shows that group Shapley value assigns a larger number to the coalition 12 than to the coalition 34 in Example 2.2.1.

4.3.2 Group Banzhaf Value

This section defines group Banzhaf value.

Definition 4.3.2 (Group Banzhaf value)**.** Consider a game (N, v). For coalition S, *Group Banzhaf value* of S is defined as

$$\hat{\beta}_S(N,v) = \frac{1}{2^{n-s}} \sum_{T \subset (N \setminus S)} [v(T \cup S) - v(T)],$$

where s and n are the number of S and N, respectively. □

Group Banzhaf value of coalition S is interpreted as expected value of marginal contribution in the case that coalition S and the other players form the grando coalition N when every coalition has the same probability to be formed.

Example 4.3.2. Consider a game (N, v) in Example 2.2.1.

$$\hat{\beta}_{12}(N,v) = \sum_{T \subset 34} \frac{1}{2^{4-2}} [v(T \cup 12) - v(T)]$$

$$= \frac{1}{4}[v(12) - v(\emptyset)] + \frac{1}{4}[v(123) - v(3)] + \frac{1}{4}[v(14) - v(4)] + \frac{1}{4}[v(1234) - v(34)]$$

$$= 10.5 + 9 + 9 + 10.5 = 39, \text{and}$$

$$\hat{\beta}_{34}(N, v) = \sum_{T \subset 34} \frac{t!(4-t-2)!}{(4-2+1)!}[v(T \cup 34) - v(T)]$$

$$= \frac{1}{4}[v(34) - v(\emptyset)] + \frac{1}{4}[v(134) - v(1)] + \frac{1}{4}[v(234) - v(2)] + \frac{1}{4}[v(1234) - v(12)]$$

$$= 0 + 9 + 6 + 1.5 = 16.5.$$

□

This example shows that group Banzhaf value assigns a larger number to the coalition 12 than to the coalition 34 in Example 2.2.1. One can see that group Banzhaf value assigns a real number which is different from the one that group Shapley value does to the coalition 12 and 34, respectively.

4.3.3 Shapley Coalition Value

This section defines Shapley coalition value and provides an calculation example.

Definition 4.3.3 (Shapley coalition value)**.** Consider a game (N, v). For coalition S, *Shapley coalition value* of S is defined as

$$\phi_S(N, v) = \sum_{i \in S} \phi_i(N, v).$$

□

Shapley coalition value of coalition S is defined as the sum of Shapley value of the member of S.

Example 4.3.3. Consider a game (N, v) in Example 2.2.1.

$$\phi_{12}(N, v) = \sum_{T \subset 234} \frac{t!(4-t-1)!}{4!}[v(T \cup 1) - v(T)] + \sum_{T \subset 134} \frac{t!(4-t-1)!}{4!}[v(T \cup 2) - v(T)]$$

$$= \frac{1}{4}[v(1) + v(2) - 2 \cdot v(\emptyset) +] + \frac{1}{12}[2 \cdot v(12) - v(1) - v(2)]$$

$$+ \frac{1}{12}[v(13) + v(23) - 2 \cdot v(3)] + \frac{1}{12}[v(14) + v(24) - 2 \cdot v(4)]$$

$$+\frac{1}{12}\left[2\cdot v(123)-v(13)-v(23)\right]+\frac{1}{12}\left[2\cdot v(124)-v(14)-v(24)\right]$$
$$+\frac{1}{12}\left[v(134)+v(234)-2\cdot v(34)\right]+\frac{1}{4}\left[2\cdot v(1234)-v(134)-v(234)\right]$$
$$=18+23+4=45, \text{and}$$

$$\phi_{34}(N,v)=\sum_{T\subset 124}\frac{t!(4-t-1)!}{4!}\left[v(T\cup 3)-v(T)\right]+\sum_{T\subset 123}\frac{t!(4-t-1)!}{4!}\left[v(T\cup 4)-v(T)\right]$$
$$=\frac{1}{4}\left[v(3)+v(4)-2\cdot v(\emptyset)+\right]+\frac{1}{12}\left[2\cdot\left[v(34)-v(3)-v(4)\right]\right]$$
$$+\frac{1}{12}\left[v(13)+v(14)-2\cdot v(1)\right]+\frac{1}{12}\left[v(23)+v(24)-2\cdot v(2)\right]$$
$$+\frac{1}{12}\left[2\cdot v(134)-v(14)-v(13)\right]+\frac{1}{12}\left[2\cdot v(234)-v(24)-v(23)\right]$$
$$+\frac{1}{12}\left[v(123)+v(124)-2\cdot v(12)\right]+\frac{1}{4}\left[2\cdot v(1234)-v(124)-v(123)\right]$$
$$=1.5+15.75+1.5=18.75.$$

□

4.3.4 Banzhaf Coalition Value

This section defines Banzhaf coalition value.

Definition 4.3.4 (Banzhaf coalition value)**.** Consider a game (N,v). For coalition S, *Banzhaf coalition value* of S is defined as

$$\beta_S(N,v)=\sum_{i\in S}\beta_i(N,v).$$

□

Banzhaf coalition value of coalition S is interpreted as the sum of Banzhaf value of the member of S.

Example 4.3.4. Consider a game (N,v) in Example 2.2.1.

$$\beta_{12}(N,v)=\sum_{T\subset 234}\frac{1}{2^{4-1}}\left[v(T\cup 1)-v(T)\right]+\sum_{T\subset 134}\frac{1}{2^{4-1}}\left[v(T\cup 2)-v(T)\right]$$

$$=\frac{1}{8}\left[v(1)+v(2)-2\cdot v(\emptyset)+\right]+\frac{1}{8}\left[2\cdot v(12)-v(1)-v(2)\right]$$

$$+\frac{1}{8}[v(13)+v(23)-2\cdot v(3)]+\frac{1}{8}[v(14)+v(24)-2\cdot v(4)]$$

$$+\frac{1}{8}[2\cdot v(123)-v(13)-v(23)]+\frac{1}{8}[2\cdot v(124)-v(14)-v(24)]$$

$$+\frac{1}{8}[v(134)+v(234)-2\cdot v(34)]+\frac{1}{8}[2\cdot v(1234)-v(134)-v(234)]$$

$$=\frac{372}{8}=46.5, \text{and}$$

$$\beta_{34}(N,v)=\sum_{T\subset 124}\frac{1}{2^{4-1}}[v(T\cup 3)-v(T)]+\sum_{T\subset 123}\frac{1}{2^{4-1}}[v(T\cup 4)-v(T)]$$

$$=\frac{1}{8}[v(3)+v(4)-2\cdot v(\emptyset)+]+\frac{1}{8}[2\cdot v(34)-v(3)-v(4)]$$

$$+\frac{1}{8}[v(13)+v(14)-2\cdot v(1)]+\frac{1}{8}[v(23)+v(24)-2\cdot v(2)]$$

$$+\frac{1}{8}[2\cdot v(134)-v(14)-v(13)]+\frac{1}{8}[2\cdot v(234)-v(24)-v(23)]$$

$$+\frac{1}{8}[v(123)+v(124)-2\cdot v(12)]+\frac{1}{8}[2\cdot v(1234)-v(124)-v(123)]$$

$$=\frac{132}{8}=16.5.$$

□

4.3.5 Properties of Coalition Value Derived from Existing Values

This section targets to characterize the coalition values derived from existing values with the defined axioms.

Proposition 4.3.1. Group Shapley value satisfies null coalitions axiom and symmetry axiom. □

Proof (Null coalitions axiom) Consider a game (N,v) and let S be a null coalition in (N,v). $v(T)=v(T\cup S)$ holds for all coalition T. Hence, $\hat{\phi}_S(N,v)=0$ holds by the Definition 4.3.1.
(Symmetry axiom) Consider a game (N,v) and let S^1 and S^2 be symmetric coalitions in (N,v). For coalition $T\subseteq N\backslash S^1\backslash S^2$, $v(T\amalg S^1)=v(T\amalg S^2)$ for all $T\subseteq N\setminus S^1\setminus S^2$ hold by Lemma 4.2.2.

For coalition $T \not\subseteq N \backslash S^1 \backslash S^2$, T can be described as follows:

$$T = (T \backslash S^1 \backslash S^2) \amalg ((T \backslash S^1) \cap S^2) \amalg ((T \backslash S^2) \cap S^1),$$

because $T \subseteq N \backslash S^1$ or $T \subseteq N \backslash S^2$ holds. Then, the following formula holds.

$$v(T \cup S^1) = v(((T \backslash S^1 \backslash S^2) \amalg ((T \backslash S^1) \cap S^2) \amalg ((T \backslash S^2) \cap S^1) \cup S^1).$$

Because of $((T \backslash S^2) \cap S^1) \subseteq S^1$, $v(T \cup S^1) = v(((T \backslash S^1 \backslash S^2) \amalg ((T \backslash S^1) \cap S^2) \cup S^1)$ holds. It holds that $(T \backslash S^1 \backslash S^2) \cap S^1 = \emptyset$ and $((T \backslash S^1) \cap S^2) \cap S^1 = \emptyset$, then we get

$$v(T \cup S^1) = v(((T \backslash S^1 \backslash S^2) \amalg ((T \backslash S^1) \cap S^2) \amalg S^1).$$

Consider $\mathscr{F} : 2^N \to 2^N$ such that $\mathscr{F}(T) = (T \backslash S^2) \cup g^{-1}(T \cap S^2)$ hold. It hods that

$$v(T \cup S^1) = v(((\mathscr{F}(T) \backslash S^1 \backslash S^2) \amalg ((\mathscr{F}(T) \backslash S^2) \cap S^1) \amalg S^2),$$

where $g((\mathscr{F}(T) \backslash S^2) \cap S^1) = (T \backslash S^1) \cap S^2$ due to Lemma 4.2.2. If $U \neq U'$ then $\mathscr{F}(U) \neq \mathscr{F}(U')$ holds. $|2^N| < \infty$, hence $\mathscr{F}$ is bijection. By symmetry of the set 2^N, it holds that

$$\sum_{T \subseteq (N \backslash S^1) \text{s.t.} T \cap S^2 \neq \emptyset} v(T \cup S^1) = \sum_{T \subseteq (N \backslash S^2) \text{s.t.} T \cap S^1 \neq \emptyset} v(T \cup S^2).$$

Then, we get

$$\sum_{T \subseteq N \backslash S^1} v(T \cup S^1) - v(T) = \sum_{U \subseteq N \backslash S^2} v(U \cup S^2) - v(U).$$

For coalition T, $|T| = |\mathscr{F}(T)|$ holds because the function g is bijection, hence the following formula holds:

$$\sum_{T \subseteq N \backslash S^1} \frac{t!(n-t-s)!}{(n-s+1)!} \left[v(T \cup S^1) - v(T) \right]$$

$$= \sum_{U \subseteq N \backslash S^2} \frac{t!(n-t-s)!}{(n-s+1)!} \left[v(\mathscr{F}^{-1}(U) \cup S^2) - v(\mathscr{F}^{-1}(U)) \right],$$

which implies $\hat{\phi}_{S^1}(N, v) = \hat{\phi}_{S^2}(N, v)$. ■

This proposition shows that group Shapley value assigns zero to null coalitions and the same number to symmetric coalitions.

Proposition 4.3.2. Group Banzhaf value satisfies null coalitions axiom and symmetry axiom. □

Proof (Null coalitions axiom) Consider a game (N, v) and let S be a null coalition in (N, v). $v(T) = v(T \cup S)$ holds for all coalition T. Hence, $\hat{\beta}_S(N, v) = 0$ holds by the Definition 4.3.2.
(Symmetry axiom) Consider a game (N, v) and let S^1 and S^2 be symmetric coalitions in (N, v). For coalition $T \subseteq N \backslash S^1 \backslash S^2$, $v(T \amalg S^1) = v(T \amalg S^2)$ for all $T \subseteq N \setminus S^1 \setminus S^2$ hold by Lemma 4.2.2.

For coalition $T \not\subseteq N \backslash S^1 \backslash S^2$, T can be described as follows:

$$T = (T \backslash S^1 \backslash S^2) \amalg ((T \backslash S^1) \cap S^2) \amalg ((T \backslash S^2) \cap S^1),$$

because $T \subseteq N \backslash S^1$ or $T \subseteq N \backslash S^2$ holds. Then, it holds that

$$v(T \cup S^1) = v(((T \backslash S^1 \backslash S^2) \amalg ((T \backslash S^1) \cap S^2) \amalg ((T \backslash S^2) \cap S^1) \cup S^1).$$

Because of $((T \backslash S^2) \cap S^1) \subseteq S^1$, the following formula holds.

$$v(T \cup S^1) = v(((T \backslash S^1 \backslash S^2) \amalg ((T \backslash S^1) \cap S^2) \cup S^1).$$

It holds that $(T \backslash S^1 \backslash S^2) \cap S^1 = \emptyset$ and $((T \backslash S^1) \cap S^2) \cap S^1 = \emptyset$, we get

$$v(T \cup S^1) = v(((T \backslash S^1 \backslash S^2) \amalg ((T \backslash S^1) \cap S^2) \amalg S^1).$$

Consider $\mathscr{F} : 2^N \to 2^N$ such that $\mathscr{F}(T) = (T \backslash S^2) \cup g^{-1}(T \cap S^2)$ hold. It holds that

$$v(T \cup S^1) = v(((\mathscr{F}(T) \backslash S^1 \backslash S^2) \amalg ((\mathscr{F}(T) \backslash S^2) \cap S^1) \amalg S^2),$$

where $g((\mathscr{F}(T) \backslash S^2) \cap S^1) = (T \backslash S^1) \cap S^2$ due to Lemma 4.2.2. If $U \neq U'$ then $\mathscr{F}(U) \neq \mathscr{F}(U')$ holds. $|2^N| < \infty$, hence $\mathscr{F}$ is bijection. By symmetry of the set 2^N, it holds that

$$\sum_{T \subseteq (N \backslash S^1) \text{s.t.} T \cap S^2 \neq \emptyset} v(T \cup S^1) = \sum_{T \subseteq (N \backslash S^2) \text{s.t.} T \cap S^1 \neq \emptyset} v(T \cup S^2).$$

Hence, we get

$$\sum_{T \subseteq N \backslash S^1} v(T \cup S^1) - v(T) = \sum_{U \subseteq N \backslash S^2} v(U \cup S^2) - v(U),$$

which means $\hat{\beta}_{S^1}(N, v) = \hat{\beta}_{S^2}(N, v)$. ■

This proposition shows that group Banzhaf value assigns zero to null coalitions and the same number to symmetric coalitions.

Proposition 4.3.3. Shapley coalition value satisfies null coalitions axiom and symmetry axiom. ■

Proof (Null coalitions axiom) Shapley value satisfies the null players axiom, hence Shapley coalition value always assigns zero to null coalitions.
(Symmetry axiom) Shapley value satisfies symmetry axiom, hence Shapley coalition value satisfies symmetry axiom by the Definition 4.2.5. ■

This proposition shows that Shapley coalition value assigns zero to null coalitions and the same number to symmetric coalitions.

Proposition 4.3.4. Banzhaf coalition value satisfies null coalitions axiom and symmetry axiom. ■

Proof (Null coalitions axiom) Banzhaf value satisfies null players axiom, hence Banzhaf coalition value always assigns zero to null coalitions.
(Symmetry axiom) Banzhaf value satisfies symmetry axiom, hence Banzhaf coalition value satisfies symmetry axiom by the Definition 4.2.5. ■

This proposition shows that Banzhaf coalition value assigns zero to null coalitions and the same number to symmetric coalitions.

4.4 Coalition Values for Group Decision

This section proposes coalition values for group decision. The size of the group decision is determined by the number of players and the number of alternatives in the situation. The value of preference distance is depending on the size of group decision, thus it is difficult that to compare coalition influence in multiple group decisions. Indices which show coalition influence in multiple group decision will help to know what coalitions will form in the situation.

The proposed coalition values in this section express coalition influence based on preference distance with respect to social welfare function.

Definition 4.4.1 (Preference-distance coalition index)**.** Consider a pair (N, A) and an SWF $F : L^N \to L$. Let d be a distance function on L. For a coalition S, *preference-distance coalition index* of S with respect to SWF F is defined as follows:

$$\delta_S(F) = \left(\frac{|S|}{|N|}\right)^{\frac{D_S(F)}{\sum_{P \in L^N} \max_{i \in N}\{d(P_i, F(P))\}}},$$

where $D_S(F)$ is given in Definition 3.2.2. □

This definition means that a coalition with more members whose opinions are closer to SWF F has more power in the decision. This index can deal with coalition influence without the dependency of group decision size.

Example 4.4.1. Consider a pair (N, A) and an SWF $F : L^N \to L$ in Example 2.1.1. Let d^2 be a distance function on L in Example 2.1.3. In this case, preference-distance coalition index of each coalition is calculated as follows:

$$\delta_1(F) = \left(\frac{1}{3}\right)^{\frac{78}{468}} \fallingdotseq 0.833.$$

$$\delta_2(F) = \left(\frac{1}{3}\right)^{\frac{390}{468}} \fallingdotseq 0.400.$$

$$\delta_3(F) = \left(\frac{1}{3}\right)^{\frac{390}{468}} \fallingdotseq 0.400.$$

$$\delta_{12}(F) = \left(\frac{2}{3}\right)^{\frac{0}{468}} = 1.$$

$$\delta_{13}(F) = \left(\frac{2}{3}\right)^{\frac{0}{468}} = 1.$$

$$\delta_{23}(F) = \left(\frac{2}{3}\right)^{\frac{228}{468}} \fallingdotseq 0.821.$$

$$\delta_{123}(F) = \left(\frac{3}{3}\right)^{\frac{0}{468}} = 1.$$

□

It is confirmed how the preference-distance coalition index work in this example.

This coalition index assigns a real number to every coalition which can be regarded as a game in characteristic function form. Study from the perspective of cooperative game theory will be a future research.

4.5 Computational Examples of Coalition Values

This section provides computational examples of the defined coalition values.

Example 4.5.1. Consider a game (N, v) such that $N = \{1, 2, 3, 4\}$, $v(1) = 20$, $v(2) = 5$, $v(3) = 12$, $v(4) = 9$, $v(12) = 27$, $v(13) = 35$, $v(14) = 32$, $v(23) = 20$, $v(24) = 20$, $v(34) = 30$, $v(123) = 40$, $v(124) = 40$, $v(134) = 50$, $v(234) = 40$ and $v(1234) = 70$
In this case, each coalition value is calculated as follows:

S	$\hat{B}_S(N,v)$	$\hat{V}_S(N,v)$	$\hat{P}_S(N,v)$	$\hat{\phi}_S(N,v)$	$\hat{\beta}_S(N,v)$	$\phi_S(N,v)$	$\beta_S(N,v)$
$\emptyset$	0	0	0	0	0	0	0
1	27.2	24.9	18.6	23.2	22.3	23.2	22.3
2	11.5	6.2	7.7	10.3	9.3	10.3	9.3
3	22.4	14.9	15.0	19.0	18.0	19.0	18.0
4	20.5	11.2	13.8	17.5	16.5	17.5	16.5
12	38.3	32.4	27.0	32.2	31.5	33.5	31.5
13	48.8	41.7	34.5	41.0	40.3	42.2	40.3
14	47.0	38.0	33.2	39.5	38.8	40.7	38.8
23	32.0	23.0	24.0	27.8	27.3	29.3	27.3
24	28.3	21.2	24.0	26.3	25.8	27.8	25.8
34	37.6	31.7	32.4	35.2	34.5	36.5	34.5
123	58.8	49.5	44.9	50.5	50.5	52.5	49.5
124	55.1	47.6	44.9	49.0	49.0	51.0	48.0
134	63.8	58.5	53.3	57.5	57.5	59.7	56.8
234	45.1	42.3	44.9	45.0	45.0	46.8	43.8
1234	70.0	70.0	70.0	70.0	70.0	70.0	66.0

The following points regarding coalition values can be seen from this numerical example.

- Blockability value tends to assign a greater or equal real number to the coalition than the value assigned by viability value.
- Banzhaf coalition value does not always assign the characteristic function value $v(N)$ to the grand coalition.
- Group Shapley value is smaller than Shapley coalition value of all coalitions in this example.
- Group Banzhaf value is greater than Banzhaf coalition value of all coalitions in this example.

Next, coalition values proposed in this thesis are applied to the assignment game. The assignment game is a model for a two-sided market in which a product is exchanged for money. Each player can buy or sell exactly one unit.

Example 4.5.2 (The Assignment Game [47])**.** Consider two disjoint sets of players $S = \{1, 2, \ldots, m\}$ and $D = \{m+1, m+2, \ldots, 2m\}$. Assume that S is the set of seller and D is the set of buyer. The players of S have a value (price) for own unit which is expressed as the set $A = \{a_1, a_2, \ldots, a_m\}$. The players of D also have values (price) for their units which is expressed as $B = \{b_{a_1 m+1}, b_{a_2 m+1}, \ldots, b_{a_m m+1}, \ldots, b_{a_1 2m}, \ldots b_{m2m}\}$. If the buyer's price is greater than seller's price, the trade goes through. Otherwise, the players do not trade the unit. For all $a \in A$ and $binB$, the profit of members of S is calculated with the following formula.

$$p(a, b) = \begin{cases} b - a & (a < b) \\ 0 & (a \geq b) \end{cases}$$

The characteristic function is described as follows:

$$v(T) = \begin{cases} 0 & (S \cap T = \emptyset \text{ or } D \cap T = \emptyset) \\ max\left\{\sum_{i \in S \cap T, j \in D \cap T} p(a_i, b_{ij})\right\} & (\text{otherwise}) \end{cases}$$

□

It is known that the assignment game is super additive and balanced. Every core outcome is competitive and vice versa in the assignment game.

Example 4.5.3 ([47])**.** Consider a game in Example 4.5.2. Let S be a set $\{1, 2, 3\}$. Let D be a set $\{4, 5, 6\}$. Then, assume that sellers' prices for unit are as follows:

$$a_1 = 18, a_2 = 15, a_3 = 19.$$

Buyers' prices are as below:

$$b_{14} = 23, b_{15} = 26, b_{16} = 20, b_{24} = 22,$$

$$b_{25} = 24, b_{26} = 21, b_{34} = 21, b_{35} = 22, b_{36} = 17.$$

In this case, characteristic function value for coalitions are determines as follows:

$$v(\emptyset) = 0, v(1) = 0, v(2) = 0, v(12) = 0,$$

$$v(3) = 0, v(13) = 0, v(23) = 0, v(123) = 0,$$

$$v(4) = 0, v(14) = 5, v(24) = 7, v(124) = 7,$$

$$v(34) = 2, v(134) = 5, v(234) = 7, v(1234) = 7,$$

$$v(5) = 0, v(15) = 8, v(25) = 9, v(125) = 9,$$

$$v(35) = 3, v(135) = 8, v(235) = 9, v(1235) = 9,$$

$$v(45) = 0, v(145) = 8, v(245) = 9, v(1245) = 15,$$

$$v(345) = 3, v(1345) = 10, v(2345) = 11, v(12345) = 15,$$

$$v(6) = 0, v(16) = 2, v(26) = 6, v(126) = 6,$$

$$v(36) = 0, v(136) = 2, v(236) = 6, v(1236) = 6,$$

$$v(46) = 0, v(146) = 5, v(246) = 7, v(1246) = 11,$$

$$v(346) = 2, v(1346) = 5, v(2346) = 8, v(12346) = 11,$$

$$v(56) = 0, v(156) = 8, v(256) = 9, v(1256) = 14,$$

$$v(356) = 3, v(1356) = 8, v(2356) = 9, v(12356) = 14,$$

$$v(456) = 0, v(1456) = 8, v(2456) = 9, v(12456) = 15,$$

$$v(3456) = 3, v(13456) = 10, v(23456) = 11, v(123456) = 16.$$

Each coalition value for this game of characteristic function form is calculated as follows:

S	$\hat{B}_S(N,v)$	$\hat{V}_S(N,v)$	$\hat{P}_S(N,v)$	$\hat{\phi}_S(N,v)$	$\hat{\beta}_S(N,v)$	$\phi_S(N,v)$	$\beta_S(N,v)$
$\emptyset$	0	0	0	0	0	0	0
1	4.8	0.0	2.8	3.3	3.6	3.3	3.6
2	6.9	0.0	4.1	4.5	5.1	4.5	5.1
1 2	13.3	0.0	6.0	8.0	8.7	7.9	8.7
3	1.1	0.0	0.6	0.7	0.8	0.7	0.8
1 3	6.6	0.0	3.1	4.2	4.4	4.1	4.4
2 3	8.6	0.0	4.4	5.4	5.9	5.3	5.9
1 2 3	16.0	0.0	6.1	9.2	9.8	8.6	9.5
4	3.5	0.0	2.1	2.2	2.6	2.2	2.6
1 4	6.9	3.4	5.8	6.1	6.2	5.5	6.2
2 4	8.9	4.7	7.1	7.7	7.8	6.7	7.8
1 2 4	14.0	6.4	9.8	10.9	11.4	10.1	11.3
3 4	4.0	1.3	3.1	3.0	3.4	2.9	3.4
1 3 4	7.9	4.0	6.3	6.6	6.9	6.3	7.0
2 3 4	9.9	5.4	7.7	8.2	8.5	7.5	8.6
1 2 3 4	16.0	6.7	10.0	12.0	12.3	10.8	12.1
5	6.1	0.0	3.6	4.0	4.6	4.0	4.6
1 5	8.4	5.4	8.0	8.1	8.1	7.3	8.1
2 5	11.3	6.1	8.7	9.8	9.7	8.5	9.7
1 2 5	14.7	8.8	11.8	12.4	12.9	11.8	13.3
3 5	6.6	2.0	4.7	5.0	5.4	4.7	5.4
1 3 5	9.3	6.4	8.4	8.7	8.8	8.0	8.9
2 3 5	12.0	7.1	9.2	10.2	10.3	9.3	10.5
1 2 3 5	16.0	9.3	12.0	13.2	13.5	12.6	14.1
4 5	11.3	0.0	4.8	6.5	7.2	6.2	7.2
1 4 5	12.0	7.1	9.8	10.1	10.6	9.5	10.8
2 4 5	14.7	8.4	10.6	11.9	12.1	10.7	12.3
1 2 4 5	16.0	13.0	14.8	15.3	15.3	14.0	15.9

S	$\hat{B}_S(N,v)$	$\hat{V}_S(N,v)$	$\hat{P}_S(N,v)$	$\hat{\phi}_S(N,v)$	$\hat{\beta}_S(N,v)$	$\phi_S(N,v)$	$\beta_S(N,v)$
3 4 5	11.3	2.7	6.3	7.6	8.1	6.9	8.0
1 3 4 5	12.0	8.8	10.8	10.8	11.3	10.2	11.6
2 3 4 5	14.7	10.1	11.6	12.7	12.8	11.5	13.1
1 2 3 4 5	16.0	14.0	15.2	15.5	15.5	14.8	16.7
6	2.0	0.0	1.2	1.2	1.5	1.2	1.5
1 6	5.9	1.3	4.6	4.6	5.1	4.6	5.1
2 6	7.2	4.0	6.3	6.5	6.6	5.8	6.6
1 2 6	13.3	4.7	9.0	10.3	10.6	9.1	10.2
3 6	3.0	0.0	1.9	1.8	2.3	2.0	2.3
1 3 6	7.6	1.3	4.9	5.4	5.9	5.3	5.9
2 3 6	8.9	4.0	6.7	7.3	7.5	6.5	7.4
1 2 3 6	16.0	4.7	9.2	11.5	11.8	9.8	11.0
4 6	6.7	0.0	2.6	4.0	4.1	3.4	4.1
1 4 6	8.9	4.0	6.7	7.0	7.5	6.8	7.7
2 4 6	9.6	6.7	8.0	8.3	8.6	8.0	9.3
1 2 4 6	14.0	9.4	11.6	12.3	12.5	11.3	12.8
3 4 6	7.2	1.3	3.7	4.8	5.0	4.2	4.9
1 3 4 6	9.9	4.7	7.3	7.5	8.3	7.5	8.5
2 3 4 6	10.6	7.6	8.8	9.0	9.5	8.7	10.1
1 2 3 4 6	16.0	9.9	12.0	13.5	13.5	12.0	13.6
5 6	9.3	0.0	4.2	5.6	6.1	5.2	6.1
1 5 6	10.6	6.1	9.0	9.3	9.6	8.5	9.6
2 5 6	12.0	8.1	9.8	10.4	10.6	9.8	11.2
1 2 5 6	14.7	12.0	14.0	14.2	14.3	13.1	14.8
3 5 6	9.6	2.0	5.3	6.6	6.9	6.0	6.9
1 3 5 6	11.3	7.1	9.6	9.7	10.3	9.3	10.4
2 3 5 6	12.6	9.1	10.4	10.8	11.3	10.5	12.0
1 2 3 5 6	16.0	12.5	14.4	15.0	15.0	13.8	15.6
4 5 6	16.0	0.0	4.9	8.7	9.0	7.4	8.7
1 4 5 6	16.0	7.4	10.0	12.2	12.3	10.7	12.3
2 4 5 6	16.0	9.4	10.8	12.7	12.8	12.0	13.8
1 2 4 5 6	16.0	14.9	15.2	15.5	15.5	15.3	17.4
3 4 5 6	16.0	2.7	6.5	9.8	10.0	8.2	9.5
1 3 4 5 6	16.0	9.1	11.2	13.0	13.0	11.5	13.1
2 3 4 5 6	16.0	11.2	12.0	13.5	13.5	12.7	14.6
1 2 3 4 5 6	16.0	16.0	16.0	16.0	16.0	16.0	18.2

□

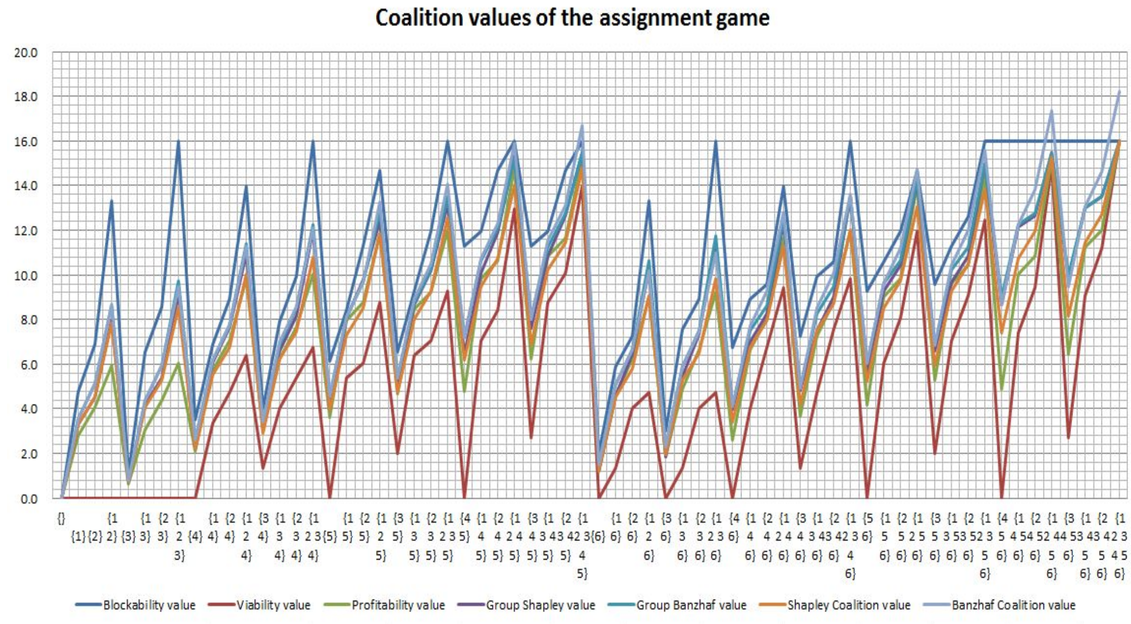

Figure 4.1:

Discussion from the application to the assignment game

Features of the proposed coalition values and their interrelationships from the provided application to the assignment game are discussed.

Blockability value

Blockability value assigns characteristic function value of grand coalition to the set of sellers/buyers. Blockability value assigns larger numbers to most coalitions than the other coalition values assign.

Viability value

Viability value assigns zero to the set of sellers/buyers. Viability value assigns smaller numbers to most coalitions than the other coalition values assign.

Profitability value

Profitability value of the coalitions that there is a matching between sellers and buyers gets the profitability value of the player 1 or zero when player 1 joins the coalition. Profitability value of every coalition is smaller than or equal to blockability value of the coalition. Profitability value of every coalition is larger than or equal to viability value of the coalition.

Group Shapley value

Group Shapley value of the set of sellers/buyers is greater than Shapley coalition value.

Group Banzhaf value

Group Banzhaf value of the set of sellers/buyers is greater than Banzhaf coalition value.

Shapley coalition value

Shapley coalition value is smaller than or equal to group Shapley value for all coalitions.

Banzhaf coalition value

Banzhaf coalition value of grand coalition is not matched to characteristic function value of grand coalition. The size of number of Banzhaf coalition value is depending on the matching of sellers and buyers in the coalition.

Example 4.5.4. Consider an inessential game (N, v) such that $N = \{1, 2, 3, 4, 5, 6\}$,

$$v(\emptyset) = 0, v(1) = 2, v(2) = 1, v(12) = 2,$$
$$v(3) = 1, v(13) = 3, v(23) = 2, v(123) = 4,$$
$$v(4) = 1, v(14) = 3, v(24) = 2, v(124) = 4,$$
$$v(34) = 2, v(134) = 4, v(234) = 3, v(1234) = 5,$$
$$v(5) = 1, v(15) = 3, v(25) = 2, v(125) = 4,$$
$$v(35) = 2, v(135) = 4, v(235) = 3, v(1235) = 5,$$
$$v(45) = 2, v(145) = 4, v(245) = 3, v(1245) = 5,$$
$$v(345) = 3, v(1345) = 5, v(2345) = 4, v(12345) = 6,$$
$$v(6) = 1, v(16) = 3, v(26) = 2, v(126) = 4,$$
$$v(36) = 2, v(136) = 4, v(236) = 2, v(1236) = 5,$$
$$v(46) = 2, v(146) = 4, v(246) = 3, v(1246) = 5,$$
$$v(346) = 3, v(1346) = 5, v(2346) = 4, v(12346) = 6,$$
$$v(56) = 2, v(156) = 4, v(256) = 3, v(1256) = 5,$$
$$v(356) = 3, v(1356) = 5, v(2356) = 4, v(12356) = 6,$$
$$v(456) = 3, v(1456) = 5, v(2456) = 4, v(12456) = 6,$$

$$v(3456) = 4, v(13456) = 6, v(23456) = 5, v(123456) = 7.$$

In this case, coalition values of the constant-sum game are follows:

S	$\hat{B}_S(N,v)$	$\hat{V}_S(N,v)$	$\hat{P}_S(N,v)$	$\hat{\phi}_S(N,v)$	$\hat{\beta}_S(N,v)$	$\phi_S(N,v)$	$\beta_S(N,v)$
∅	0	0	0	0	0	0	0
1	2.0	2.0	2.0	2.0	2.0	2.0	2.0
2	1.0	1.0	1.0	1.0	1.0	1.0	1.0
1 2	3.0	3.0	3.0	3.0	3.0	3.0	3.0
3	1.0	1.0	1.0	1.0	1.0	1.0	1.0
1 3	3.0	3.0	3.0	3.0	3.0	3.0	3.0
2 3	2.0	2.0	2.0	2.0	2.0	2.0	2.0
1 2 3	4.0	4.0	4.0	4.0	4.0	4.0	4.0
4	1.0	1.0	1.0	1.0	1.0	1.0	1.0
1 4	3.0	3.0	3.0	3.0	3.0	3.0	3.0
2 4	2.0	2.0	2.0	2.0	2.0	2.0	2.0
1 2 4	4.0	4.0	4.0	4.0	4.0	4.0	4.0
3 4	2.0	2.0	2.0	2.0	2.0	2.0	2.0
1 3 4	4.0	4.0	4.0	4.0	4.0	4.0	4.0
2 3 4	3.0	3.0	3.0	3.0	3.0	3.0	3.0
1 2 3 4	5.0	5.0	5.0	5.0	5.0	5.0	5.0
5	1.0	1.0	1.0	1.0	1.0	1.0	1.0
1 5	3.0	3.0	3.0	3.0	3.0	3.0	3.0
2 5	2.0	2.0	2.0	2.0	2.0	2.0	2.0
1 2 5	4.0	4.0	4.0	4.0	4.0	4.0	4.0
3 5	2.0	2.0	2.0	2.0	2.0	2.0	2.0
1 3 5	4.0	4.0	4.0	4.0	4.0	4.0	4.0
2 3 5	3.0	3.0	3.0	3.0	3.0	3.0	3.0
1 2 3 5	5.0	5.0	5.0	5.0	5.0	5.0	5.0
4 5	2.0	2.0	2.0	2.0	2.0	2.0	2.0
1 4 5	4.0	4.0	4.0	4.0	4.0	4.0	4.0
2 4 5	3.0	3.0	3.0	3.0	3.0	3.0	3.0
1 2 4 5	5.0	5.0	5.0	5.0	5.0	5.0	5.0

S	$\hat{B}_S(N,v)$	$\hat{V}_S(N,v)$	$\hat{P}_S(N,v)$	$\hat{\phi}_S(N,v)$	$\hat{\beta}_S(N,v)$	$\phi_S(N,v)$	$\beta_S(N,v)$
3 4 5	3.0	3.0	3.0	3.0	3.0	3.0	3.0
1 3 4 5	5.0	5.0	5.0	5.0	5.0	5.0	5.0
2 3 4 5	4.0	4.0	4.0	4.0	4.0	4.0	4.0
1 2 3 4 5	6.0	6.0	6.0	6.0	6.0	6.0	6.0
6	1.0	1.0	1.0	1.0	1.0	1.0	1.0
1 6	3.0	3.0	3.0	3.0	3.0	3.0	3.0
2 6	2.0	2.0	2.0	2.0	2.0	2.0	2.0
1 2 6	4.0	4.0	4.0	4.0	4.0	4.0	4.0
3 6	2.0	2.0	2.0	2.0	2.0	2.0	2.0
1 3 6	4.0	4.0	4.0	4.0	4.0	4.0	4.0
2 3 6	3.0	3.0	3.0	3.0	3.0	3.0	3.0
1 2 3 6	5.0	5.0	5.0	5.0	5.0	5.0	5.0
4 6	2.0	2.0	2.0	2.0	2.0	2.0	2.0
1 4 6	4.0	4.0	4.0	4.0	4.0	4.0	4.0
2 4 6	3.0	3.0	3.0	3.0	3.0	3.0	3.0
1 2 4 6	5.0	5.0	5.0	5.0	5.0	5.0	5.0
3 4 6	3.0	3.0	3.0	3.0	3.0	3.0	3.0
1 3 4 6	5.0	5.0	5.0	5.0	5.0	5.0	5.0
2 3 4 6	4.0	4.0	4.0	4.0	4.0	4.0	4.0
1 2 3 4 6	6.0	6.0	6.0	6.0	6.0	6.0	6.0
5 6	2.0	2.0	2.0	2.0	2.0	2.0	2.0
1 5 6	4.0	4.0	4.0	4.0	4.0	4.0	4.0
2 5 6	3.0	3.0	3.0	3.0	3.0	3.0	3.0
1 2 5 6	5.0	5.0	5.0	5.0	5.0	5.0	5.0
3 5 6	3.0	3.0	3.0	3.0	3.0	3.0	3.0
1 3 5 6	5.0	5.0	5.0	5.0	5.0	5.0	5.0
2 3 5 6	4.0	4.0	4.0	4.0	4.0	4.0	4.0
1 2 3 5 6	6.0	6.0	6.0	6.0	6.0	6.0	6.0
4 5 6	3.0	3.0	3.0	3.0	3.0	3.0	3.0
1 4 5 6	5.0	5.0	5.0	5.0	5.0	5.0	5.0
2 4 5 6	4.0	4.0	4.0	4.0	4.0	4.0	4.0
1 2 4 5 6	6.0	6.0	6.0	6.0	6.0	6.0	6.0
3 4 5 6	4.0	4.0	4.0	4.0	4.0	4.0	4.0
1 3 4 5 6	6.0	6.0	6.0	6.0	6.0	6.0	6.0
2 3 4 5 6	5.0	5.0	5.0	5.0	5.0	5.0	5.0
1 2 3 4 5 6	7.0	7.0	7.0	7.0	7.0	7.0	7.0

□

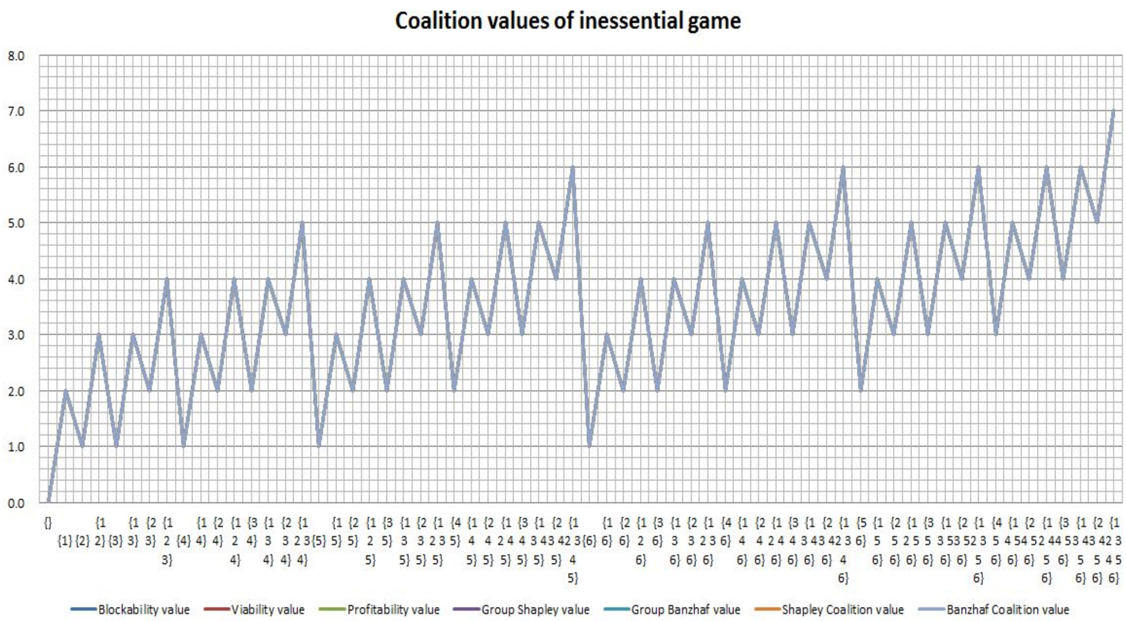

Figure 4.2:

As seen from the table, blockability value, viability value, profitability value, group Shapley value, group Banzhaf value, Shapley coalition value and Banzhaf coalition value of every coalition are the same number in the inessential game

Example 4.5.5. Consider a nonmonotonic game such that $N = \{1, 2, 3, 4, 5, 6\}$,

$$v(\emptyset) = 0, v(1) = 1, v(2) = 1, v(12) = 2,$$

$$v(3) = 1, v(13) = 2, v(23) = 2, v(123) = 3,$$

$$v(4) = 1, v(14) = 0, v(24) = 0, v(124) = 1,$$

$$v(34) = 0, v(134) = 1, v(234) = 1, v(1234) = 2,$$

$$v(5) = 1, v(15) = 0, v(25) = 0, v(125) = 1,$$

$$v(35) = 0, v(135) = 1, v(235) = 1, v(1235) = 2,$$

$$v(45) = 2, v(145) = 1, v(245) = 1, v(1245) = 0,$$

$$v(345) = 1, v(1345) = 0, v(2345) = 0, v(12345) = 1,$$

$$v(6) = 1, v(16) = 0, v(26) = 0, v(126) = 1,$$

$$v(36) = 2, v(136) = 1, v(236) = 1, v(1236) = 2,$$

$$v(46) = 2, v(146) = 1, v(246) = 1, v(1246) = 0,$$

$$v(346) = 1, v(1346) = 0, v(2346) = 0, v(12346) = 1,$$

$$v(56) = 2, v(156) = 1, v(256) = 1, v(1256) = 0,$$

$$v(356) = 1, v(1356) = 0, v(2356) = 0, v(12356) = 1,$$

$$v(456) = 3, v(1456) = 2, v(2456) = 2, v(12456) = 1,$$

$$v(3456) = 2, v(13456) = 1, v(23456) = 1, v(123456) = 7.$$

In this case, coalition values of the nonmonotonic game are follows:

S	$\hat{B}_S(N,v)$	$\hat{V}_S(N,v)$	$\hat{P}_S(N,v)$	$\hat{\phi}_S(N,v)$	$\hat{\beta}_S(N,v)$	$\phi_S(N,v)$	$\beta_S(N,v)$
$\emptyset$	0	0	0	0	0	0	0
1	0.7	3.3	0.1	1.2	0.2	1.2	0.2
2	0.7	3.3	0.1	1.2	0.2	1.2	0.2
1 2	-0.5	6.7	0.6	1.4	0.4	2.3	0.4
3	0.7	3.3	0.1	1.2	0.2	1.2	0.2
1 3	-0.5	6.7	0.6	1.4	0.4	2.3	0.4
2 3	-0.5	6.7	0.6	1.4	0.4	2.3	0.4
1 2 3	-3.0	10.0	1.6	1.8	0.9	3.5	0.7
4	0.7	3.3	0.1	1.2	0.2	1.2	0.2
1 4	2.0	3.3	0.2	1.4	0.4	2.3	0.4
2 4	2.0	3.3	0.2	1.4	0.4	2.3	0.4
1 2 4	2.0	5.0	0.7	1.8	0.9	3.5	0.7
3 4	2.0	3.3	0.2	1.4	0.4	2.3	0.4
1 3 4	2.0	5.0	0.7	1.8	0.9	3.5	0.7
2 3 4	2.0	5.0	0.7	1.8	0.9	3.5	0.7
1 2 3 4	0.3	7.5	2.0	2.3	1.8	4.7	0.9
5	0.7	3.3	0.1	1.2	0.2	1.2	0.2
1 5	2.0	3.3	0.2	1.4	0.4	2.3	0.4
2 5	2.0	3.3	0.2	1.4	0.4	2.3	0.4
1 2 5	2.0	5.0	0.7	1.8	0.9	3.5	0.7
3 5	2.0	3.3	0.2	1.4	0.4	2.3	0.4
1 3 5	2.0	5.0	0.7	1.8	0.9	3.5	0.7
2 3 5	2.0	5.0	0.7	1.8	0.9	3.5	0.7
1 2 3 5	0.3	7.5	2.0	2.3	1.8	4.7	0.9
4 5	-0.5	6.7	0.6	1.4	0.4	2.3	0.4
1 4 5	2.0	5.0	0.7	1.8	0.9	3.5	0.7
2 4 5	2.0	5.0	0.7	1.8	0.9	3.5	0.7
1 2 4 5	3.7	5.0	1.4	2.3	1.8	4.7	0.9

S	$\hat{B}_S(N,v)$	$\hat{V}_S(N,v)$	$\hat{P}_S(N,v)$	$\hat{\phi}_S(N,v)$	$\hat{\beta}_S(N,v)$	$\phi_S(N,v)$	$\beta_S(N,v)$
3 4 5	2.0	5.0	0.7	1.8	0.9	3.5	0.7
1 3 4 5	3.7	5.0	1.4	2.3	1.8	4.7	0.9
2 3 4 5	3.7	5.0	1.4	2.3	1.8	4.7	0.9
1 2 3 4 5	3.7	6.3	3.5	3.5	3.5	5.8	1.1
6	0.7	3.3	0.1	1.2	0.2	1.2	0.2
1 6	2.0	3.3	0.2	1.4	0.4	2.3	0.4
2 6	2.0	3.3	0.2	1.4	0.4	2.3	0.4
1 2 6	2.0	5.0	0.7	1.8	0.9	3.5	0.7
3 6	2.0	3.3	0.2	1.4	0.4	2.3	0.4
1 3 6	2.0	5.0	0.7	1.8	0.9	3.5	0.7
2 3 6	2.0	5.0	0.7	1.8	0.9	3.5	0.7
1 2 3 6	0.3	7.5	2.0	2.3	1.8	4.7	0.9
4 6	-0.5	6.7	0.6	1.4	0.4	2.3	0.4
1 4 6	2.0	5.0	0.7	1.8	0.9	3.5	0.7
2 4 6	2.0	5.0	0.7	1.8	0.9	3.5	0.7
1 2 4 6	3.7	5.0	1.4	2.3	1.8	4.7	0.9
3 4 6	2.0	5.0	0.7	1.8	0.9	3.5	0.7
1 3 4 6	3.7	5.0	1.4	2.3	1.8	4.7	0.9
2 3 4 6	3.7	5.0	1.4	2.3	1.8	4.7	0.9
1 2 3 4 6	3.7	6.3	3.5	3.5	3.5	5.8	1.1
5 6	-0.5	6.7	0.6	1.4	0.4	2.3	0.4
1 5 6	2.0	5.0	0.7	1.8	0.9	3.5	0.7
2 5 6	2.0	5.0	0.7	1.8	0.9	3.5	0.7
1 2 5 6	3.7	5.0	1.4	2.3	1.8	4.7	0.9
3 5 6	2.0	5.0	0.7	1.8	0.9	3.5	0.7
1 3 5 6	3.7	5.0	1.4	2.3	1.8	4.7	0.9
2 3 5 6	3.7	5.0	1.4	2.3	1.8	4.7	0.9
1 2 3 5 6	3.7	6.3	3.5	3.5	3.5	5.8	1.1
4 5 6	-3.0	10.0	1.6	1.8	0.9	3.5	0.7
1 4 5 6	0.3	7.5	2.0	2.3	1.8	4.7	0.9
2 4 5 6	0.3	7.5	2.0	2.3	1.8	4.7	0.9
1 2 4 5 6	3.7	6.3	3.5	3.5	3.5	5.8	1.1
3 4 5 6	0.3	7.5	2.0	2.3	1.8	4.7	0.9
1 3 4 5 6	3.7	6.3	3.5	3.5	3.5	5.8	1.1
2 3 4 5 6	3.7	6.3	3.5	3.5	3.5	5.8	1.1
1 2 3 4 5 6	7.0	7.0	7.0	7.0	7.0	7.0	1.3

□

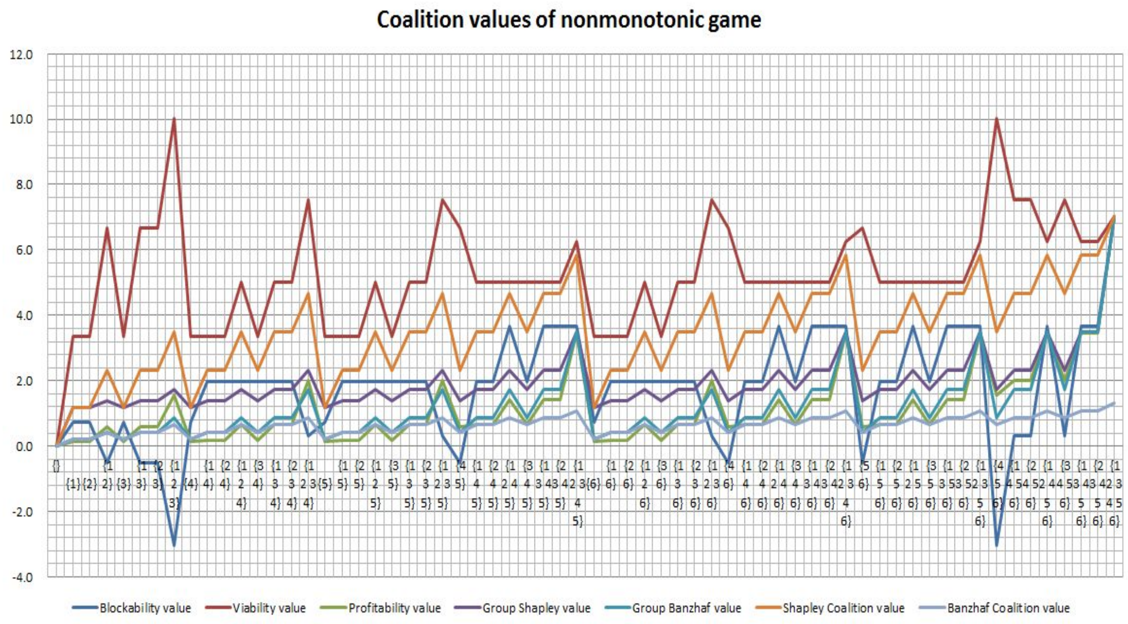

Figure 4.3:

There are in this game cases that players makes negative contrbution to the coalition, which causes that Blockability value assigns negative number to coalitions.

Properties are given for interrelationships between coalition values as below.

Proposition 4.5.1. Consider a game (N, v). For all coalition S, the following formula holds.

$$\hat{B}_S(N, v) + \hat{V}_{N \setminus S}(N, v) = v(N).$$

$\square$

Proof Consider a game (N, v). For all coalition S, the following formula holds for game (N, v) by Definition 4.2.1 and 4.2.2.

$$\hat{B}_S(N, v) + \hat{V}_{N \setminus S}(N, v) = \frac{\sum_{T \subset N} v(T) - B^*(S)}{\sum_{T \subset N} v(T) - B^*(N)} \; v(N) + \frac{V^*(N \setminus S)}{V^*(N)} \; v(N)$$

$$= \frac{\sum_{T \subset N} v(T) - \sum_{T \subset N} v(T \setminus S)}{\sum_{T \subset N} v(T)} v(N) + \frac{\sum_{T \subset N} v(N \setminus S \setminus T)}{\sum_{T \subset N} v(N \setminus T)} v(N).$$

Let U be $N \setminus T$, then the following formula holds.

$$\sum_{U \subset N} v(U \setminus S) = \sum_{T \subset N} v(N \setminus T \setminus S).$$

For all $S, T \subset N$, we get the below formulas.

$$v(N \setminus T \setminus S) = v(N \setminus S \setminus T),$$

$$\sum_{T \subset N} v(T \setminus S) = \sum_{T \subset N} v(N \setminus S \setminus T),$$

$$\sum_{T \subset N} v(T) = \sum_{T \subset N} v(N \setminus T).$$

Hence, the following formula holds.

$$\frac{\sum_{T \subset N} v(T) - \sum_{T \subset N} v(T \setminus S) + \sum_{T \subset N} v(T \setminus S)}{\sum_{T \subset N} v(T)} v(N) = v(N).$$

■

This propostion shows that there is a complementary relationship between blockability value and viability value.

Proposition 4.5.2. Consider a constant-sum game (N, v). For all coalition S, the following formula holds.

$$\hat{B}_S(N, v) = \hat{P}_S(N, v).$$

□

Proof If $S = N$, we get

$$\begin{aligned}
\hat{B}_N(N, v) &= \frac{\sum_{T \subset N} v(T) - B^*(N)}{\sum_{T \subset N} v(T) - B^*(N)} v(N) \\
&= v(N) \\
&= \frac{P^*(N) - \sum_{T \subset N} v(T)}{P^*(N) - \sum_{T \subset N} v(T)} v(N) \\
&= \hat{P}_N(N, v)
\end{aligned}$$

For all coalition S such that $S \neq N$, it holds that

$$\begin{aligned}
\hat{P}_S(N,v) &= \frac{P^*(S) - \sum_{T\subset N} v(T)}{P^*(N) - \sum_{T\subset N} v(T)} v(N) \\
&= \frac{\sum_{T\subset N} v(T\cup S) - \sum_{T\subset N} v(T)}{\sum_{T\subset N} v(T\cup N) - \sum_{T\subset N} v(T)} v(N) \\
&= \frac{\sum_{T\subset N}[v(N) - v(N\setminus(T\cup S))] - \sum_{T\subset N}[v(N) - v(N\setminus T)]}{\sum_{T\subset N} v(N) - \sum_{T\subset N} v(T)} v(N) \\
&= \frac{\sum_{T\subset N} v(N\setminus T) - \sum_{T\subset N} v(N\setminus(T\cup S))}{\sum_{T\subset N}[v(T) + v(N\setminus T)] - \sum_{T\subset N} v(T)} v(N) \\
&= \frac{\sum_{T\subset N} v(N\setminus T) - \sum_{T\subset N} v(N\setminus T\setminus S)}{\sum_{T\subset N} v(N\setminus T)} v(N) \\
&= \frac{\sum_{U\subset N} v(U) - \sum_{U\subset N} v(U\setminus S)}{\sum_{U\subset N} v(U)} v(N) \\
&= \frac{\sum_{U\subset N} v(U) - \sum_{U\subset N} v(U\setminus S)}{\sum_{U\subset N} v(U) - \sum_{U\subset N} v(U\setminus N)} v(N) \\
&= \frac{\sum_{U\subset N} v(U) - B^*(S)}{\sum_{U\subset N} v(U) - B^*(N)} v(N) \\
&= \hat{B}_S(N,v)
\end{aligned}$$

■

This proposition shows that blockability value matches to profitability value in constant-sum games. In other words, there is no difference betweeen blockability evaluation and profitability evaluation in the conflict of interest

Next table shows properties of each coalition value which were confirmed in this chapter.

Coalition value	Null coalition axiom	Symmetric coalition axiom	Imputation
$\hat{B}_S(N,v)$	Yes	Yes	No
$\hat{V}_S(N,v)$	Yes	Yes	No
$\hat{P}_S(N,v)$	Yes	Yes	No
$\hat{\phi}_S(N,v)$	Yes	Yes	No
$\hat{\beta}_S(N,v)$	Yes	Yes	No
$\phi_S(N,v)$	Yes	Yes	Yes
$\beta_S(N,v)$	Yes	Yes	No

Table 1.

To characterise these coalition values, other axioms will need to be proposed.

It was already mentioned that the assignment game was super additive and balanced. If the characteristic functino value of N was changed to remove monotonicity condition, these coalition values show different trend of value.

4.6 Summary of Chapter 4

This chapter defined methods to evaluate coalition influence for group decision or negotiation.

Blockability value (Definition 4.2.1) assigns a real number to each coalition for games in characteristic function form. Blockability values evaluates coalition influence based on blockability relation for games in characteristic function form. The comparison result by blockability relation and evaluation result by blockability value are matched in games. Blockability value satisfies null coalition axiom and symmetry axiom.

Viability value (Definition 4.2.2) assigns a real number to each coalition for games in characteristic function form. Viability values evaluates coalition influence based on blockability relation for games in characteristic function form. The comparison result by viability relation and evaluation result by viability value are matched in games. It was verified that viability value satisfies null coalition axiom and symmetry axiom.

Profitability value (Definition 4.2.3) assigns a real number to each coalition for games in characteristic function form. Profitability values evaluates coalition influence based on profitability relation for games in characteristic function form. The comparison result by profitability relation and evaluation result by profitability value are matched in games.

Propositions which shows an interrelationship between blockability value and viability value for games in characteristic function form.

Group Shapley value (Definition 4.3.1) assigns a real number to each coalition for games in characteristic function form. Group Shapley value evaluates the coalition as expected value of marginal contribution in case that the coalition forms the grand coalition with random sequence. Group Shapley value satisfies null coalitions axiom and symmetry axioms.

Group Banzhaf value (Definition 4.3.2) assigns a real number to each coalition for games in characteristic function form. Group Banzhaf value evaluates the coalition as expected value of marginal contribution in case that the coalition forms the grand coalition when every coalition has same probability to be formed. Group Banzhaf value satisfies null coalitions axiom and symmetry axioms.

Shapley coalition value (Definition 4.3.3) assigns a real number to each coalition for games in characteristic function form. Shapley coalition value is the sum of Shapley value of the coalition's members. Shapley coalition value satisfies null coalitions axiom and symmetry axioms.

Banzhaf coalition value (Definition 4.3.4) assigns a real number to each coalition for games in characteristic function form. Banzhaf coalition value is the sum of Banzhaf value of the coalition's members. Banzhaf coalition value satisfies null coalitions axiom and symmetry axioms.

Preference distance coalition index (Definition 4.4.1) assigns a real number to each coalition in the framework of social welfare function. An example which shows how preference distance coalition index works was given.

Chapter 5

Conclusion and Further Research

This chapter provides summary and further research of this thesis.

5.1 Conclusion of this Thesis

In this thesis methods to compare coalition influence for frameworks of social choice and games were discussed. Blockability relation, viability relation and profitability relation were proposed for games in characteristic function form. Examples which show how the proposed methods work were given. These relations compare coalition influence with a pair from each perspective and satisfy transitivity which enables us to assign an index to each coalition to show the coalition influence by a real number. It was verified that blockability relation and viability relation for games have a complementary relationship.

A comparison method for coalition influence based on preference distance with respect to social welfare function was proposed. The proposed method gave knowledge of the relationships between coalition influence and decision rules of the group decision. The methods to compare coalition influence for games in characteristic function form were extended to social choice function. We reviewed the relationship between the definition of winning coalitions and coalition influence through some examples.

A method to compare coalition bargaining power was provided on the basis of the concepts of objection and counter-objection. It was confirmed that the proposed method satisfies acyclicity which allow us to determine a maximal element. The provided theorem showed that the set which all coalitions are indifferent from the viewpoint of the proposed method is matched with the bargaining set.

In the framework of non-transferable utility games, comparison of bargaining results which are expressed by payoff configurations was discussed. This thesis provided a method to compare payoff configurations for NTU-game. Propositions which show some properties of the proposed method for payoff configurations were given.

Coalition values derived from proposed relation to compare coalition influence for games were provided. Blockability value, viability value and profitability value assign a real number to each coalition based on the concept of blockability relation, viability relation, and profitability relation for games in characteristic function form, respectively. The provided propositions confirmed that these coalition values surely express each relation by function.

Axioms which are properties that coalition values should satisfy were provided. Some propositions confirmed that the proposed coalition values satisfy the provided axioms.

Coalition values derived from existing values for players were given. Group Shapley value, group Banzhaf value, Shapley coalition value and Banzhaf coalition value were defined. Examples which show how these value work were provided. It was verified that these defined coalition values satisfy null coalitions axiom and symmetry axiom.

A coalition index derived from preference distance for social welfare function was provided. The provided coalition index assigns to each coalition a real number which expresses how the coalition's opinion matches the decision rule.

The following were made the contribution made through this study for solving the problem to the problem which coalition will form in the situation of group decision and negotiation.

- Developed analysis tools to compare coalition influence.
  The proposed methods in this thesis can detect the coalitions which have more influence in the situation of group decision and negotiation. It allows us to calculate a result of the coalition formation.

- Gave properties of the proposed methods for comparison of coalition influence.
  Some propositions which show which properties the proposed methods satisfy were given. How the proposed methods calculate coalition influence was explained in this paper.

- Provided knowledge for coalition formation strategy.
  If players changed the method to compare coalition influence, the forming coalition strategy might need to be changed because the comparison result calculated by each method is different. If we knew the forming coalition strategy, we might calculate the results of coalition formation in the situation.

- Got relationships between decision rules and coalition influence.
  Change of decision rules has an impact to coalition influence in almost cases, but there are some cases that coalition influence does not get any effect from change of decision rules.

- Allowed numerical experimentations of coalition formation.
  Computer simulation of coalition formation is available through the provided methods in this thesis. This thesis provided the coalition values, and demonstrated how the provided coalition values are calculated.

This thesis approached to group decision and negotiation with top-down model (social welfare function) and bottom-up model (games in characteristic function form). Using both models would be better to describe the real situations.

5.2 Comments for Further Research

There are further topics that we can discuss in comparison of coalition influence for group decision and negotiation.

- Other methods to compare coalition influence.
  Viewing coalition influence from other bases will help to find out new features of group decision and negotiation. Hybrid methods of the proposed methods in this thesis may be one of new methods to compare coalition influence.

- Other models to describe group decision and negotiation.
  The models to describe group decision and negotiation in this thesis are composed by simple parameters. The real situations would be more complicated as discussed in [13], so change of the models may develop this study. There are existing group decision models which have infinite alternatives [11], infinite players [10] or nonlinear preferences [15]. These models are possibilities which comparison methods proposed in this paper are extended to.

- Characterization of the proposed methods.
  Characterization of the proposed methods would be useful to know the meaning of comparison results calculated by the proposed methods.

- Interrelationships of comparison methods on between SWF and SCC.
  SCC is consdered as a class of SWF, therefore there may exist some interrelationships of comparison methods on between SWF and SCC.

- Computer simulations with using the proposed methods.
  Simulation of coalition formation with the large number of players may find other properties of the proposed methods. Repeated games also may reveal the intimate coalition influence.

- Comparison of methods to compare coalition influence.
  "Which is the best method to compare coalition influence?" is a natural question to be asked. It may be required that we need to define a method to compare the methods to compare coalition influence for group decision and negotiation.

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Appendix A

Lemma for Theorems in this Thesis

A.1 ERC bijections

Equivalence-relation-consistent (ERC) bijections are employed in the field of cooperative game theory, in particular, for the research on evaluation of coalitions' influence in a group decision making situation. In fact, the concept of coalition symmetry is defined with an ERC bijection. In this thesis, it is verified in Lemma A.1.1 that an ERC bijection can be decomposed into two ERC bijections, and the domain of the original ERC bijection is the disjoint union of the domains of the two decomposing ERP bijections.

Lemma A.1.1. Let N be a finite set, and R an equivalence (that is, reflexive, symmetric, and transitive) relation on N. Consider a bijection f from S_1 to S_2, where S_1 and S_2 are subsets of N and may intersect with each other, such that for all $x \in S_1$, $xRf(x)$. In this case, f is said to be an equivalence-relation-consistent (ERC) bijection from S_1 to S_2 in N with respect to R.

Then, there exists a bijection g from S_1 to S_2 such that (i) for all $x \in S_1$, $xRg(x)$, (ii) the restriction $g|_{S_1 \backslash S_2}$ of g on the set $S_1 \backslash S_2$ is a bijection from $S_1 \backslash S_2$ to $S_2 \backslash S_1$, and (iii) the restriction $g|_{S_1 \cap S_2}$ of g on the set $S_1 \cap S_2$ is a bijection on $S_1 \cap S_2$. That is, g is an ERC bijection from S_1 to S_2 in N with respect to R, $g|_{S_1 \backslash S_2}$ is an ERC bijection from $S_1 \backslash S_2$ to $S_2 \backslash S_1$ in N with respect to R, and $g|_{S_1 \cap S_2}$ is an ERC bijection from $S_1 \cap S_2$ to $S_1 \cap S_2$ in N with respect to R. □

Note that when S_1 and S_2 are disjoint with each other, this lemma is evidently true; in fact, the original bijection f itself satisfies the conditions (i), (ii), and (iii) because $S_1 \backslash S_2 = S_1$ and $S_1 \cap S_2 = \emptyset$. Thus, the case $S_1 \cap S_2 \neq \emptyset$ is essencial in this lemma. It should be also noted that S_1 is a disjoint union of $S_1 \backslash S_2$ and $S_1 \cap S_2$, and similarly, S_2 is a disjoint union of $S_2 \backslash S_1$ and $S_1 \cap S_2$.

Proof of Lemma A.1.1:

1. **Construction of g from f.**

   (a) Definition of four subsets B_1, B_2, B_3, and B_4 of S_1.

   Define four subsets B_1, B_2, B_3, and B_4 of S_1 as follows:

$$B_1 = \{x \in S_1 \backslash S_2 \mid f(x) \in S_2 \backslash S_1\};$$
$$B_2 = \{x \in S_1 \backslash S_2 \mid f(x) \in S_1 \cap S_2\};$$
$$B_3 = \{x \in S_1 \cap S_2 \mid f(x) \in S_2 \backslash S_1\};$$
$$B_4 = \{x \in S_1 \cap S_2 \mid f(x) \in S_1 \cap S_2\}.$$

   Claim 1. (i) B_1, B_2, B_3, and B_4 are mutually disjoint, and (ii) $B_1 \cup B_2 \cup B_3 \cup B_4 = S_1$.

   Proof of Claim 1: The set S_1 is the domain of the bijection f, and this set is the disjoint union of $S_1 \backslash S_2$ and $S_1 \cap S_2$. The set S_2 is the codomain of the bijection f, and this is the disjoint union of $S_1 \cap S_2$ and $S_2 \backslash S_1$. Hence, from the definitions of B_1, B_2, B_3, and B_4, they are mutually disjoint, that is, if $i \neq j$ then $B_i \cap B_j = \emptyset$.

   For each i, $B_i \subseteq S_1$, and hence, one has $B_1 \cup B_2 \cup B_3 \cup B_4 \subseteq S_1$. For each $x \in S_1$, either $x \in S_1 \backslash S_2$ or $x \in S_1 \cap S_2$ is true. For each cases, either $f(x) \in S_1 \cap S_2$ or $f(x) \in S_2 \backslash S_1$ is true. Therefore, one has $B_1 \cup B_2 \cup B_3 \cup B_4 \subseteq S_1$.
   □ (**End of proof** of Claim 1)

   Note that $\{B_1, B_2, B_3, B_4\}$ may not be a partition of S_1, because one of these sets can be empty.

   (b) Definition of four functions g_1, g_2, g_3, and g_4, whose domains are B_1, B_2, B_3, and B_4, respectively.

      i. Definition of $g_1 : B_1 \to S_2 \backslash S_1$.
      For each $x \in B_1$, $g_1(x)$ is defined as $f(x)$.

      Claim 2. g_1 is well-defined.

      Proof of Claim 2: For each $x \in B_1$, $x \in S_1 \backslash S_2$ and $f(x) \in S_2 \backslash S_1$, and hence an element of $S_2 \backslash S_1$ is uniquely determined by g_1 for each $x \in B_1$. □ (**End of proof** of Claim 2)

      Claim 3. g_1 is injective.

      Proof of Claim 3: Because f is injective, g_1 is also injective. □ (**End of proof** of Claim 3)

      ii. Definition of $g_2 : B_2 \to S_2 \backslash S_1$.
      For each $x \in B_2$, there exists $k \geq 2$ such that $f^k(x) \in S_2 \backslash S_1$. In fact, if $f^k(x) \in S_1 \cap S_2$ for all $k \geq 2$, then there exist i and j such that $1 \leq i < j$

and $f^i(x) = f^j(x)$ because of the finiteness of $S_1 \cap S_2$. Considering the minimum of such js, one has that $f^{i-1}(x) \neq f^{j-1}(x)$, where $f^0(x)$ is assumed to be x, which contradicts with the injectiveness of f.
Let k_x denote the minimum of such ks that satisfy $k \geq 2$ and $f^k(x) \in S_2 \backslash S_1$, and define $g_2(x)$ as $f^{k_x}(x)$ for each $x \in B_2$.

Claim 4. g_2 is well-defined.

Proof of Claim 4: For each $x \in B_2$, $x \in S_1 \backslash S_2$ and $f(x) \in S_1 \cap S_2$, and k_x is uniquely determined for each $x \in B_2$ as the minimum of such ks that satisfy $k \geq 2$ and $f^k(x) \in S_2 \backslash S_1$. Hence, an element of $S_2 \backslash S_1$ is uniquely determined by g_2 for each $x \in B_2$. □ (**End of proof** of Claim 4)

Claim 5. g_2 is injective.

Proof of Claim 5: Assume that $g_2(x) = g_2(x')$, that is, $f^{k_x}(x) = f^{k_{x'}}(x')$, for x and x' in B_2. There are two cases: (a) $k_x \neq k_{x'}$, and (b) $k_x = k_{x'}$. For the case (a), assume that $k_x < k_{x'}$. Then, it is satisfied that $f^{-k_x}(f^{k_x}(x)) = f^{-k_x}(f^{k_{x'}}(x'))$, which implies that $x = f^{k_{x'}-k_x}(x')$. This is a contradiction, however, because $x \in S_1 \backslash S_2$ and $f^{k_{x'}-k_x}(x') \in S_1 \cap S_2$ by the definition of $k_{x'}$. One has a similar result in the case of $k_x > k_{x'}$, thus it turns out that in the case of $f^{k_x}(x) = f^{k_{x'}}(x')$, $k_x \neq k_{x'}$ cannot to be true. For the case (b), $x = x'$ because f is injective. □ (**End of proof** of Claim 5)

iii. Definition of $g_3 : B_3 \to S_1 \cap S_2$.
For each $x \in B_3$, there exists $l \geq 1$ such that $f^{-l}(x) \in S_1 \backslash S_2$. In fact, if $f^{-l}(x) \in S_1 \cap S_2$ for all $l \geq 1$, then there exist i and j such that $1 \leq i < j$ and $f^{-i}(x) = f^{-j}(x)$ because of the finiteness of $S_1 \cap S_2$. Considering the minimum of such js, one has that $f^{-(i-1)}(x) \neq f^{-(j-1)}(x)$, where $f^0(x)$ is assumed to be x, which contradicts with the injectiveness of f^{-1}.
Let l_x denote the minimum of such ls that satisfy $l \geq 1$ and $f^{-l}(x) \in S_1 \backslash S_2$, and define $g_3(x)$ as $f^{-(l_x-1)}(x)$ for each $x \in g_3$.

Claim 6. g_3 is well-defined.

Proof of Claim 6: For each $x \in B_3$, $x \in S_1 \cap S_2$ and $f(x) \in S_2 \backslash S_1$, and l_x is uniquely determined for each $x \in B_3$ as the minimum of such ls that satisfy $l \geq 1$ and $f^{-l}(x) \in S_1 \backslash S_2$. Then, $f^{-(l_x-1)}(x) \in S_1 \cap S_2$. Hence, an element of $S_1 \cap S_2$ is uniquely determined by g_3 for each $x \in B_3$. □ (**End of proof** of Claim 6)

Claim 7. g_3 is injective.

Proof of Claim 7: Assume that $g_3(x) = g_3(x')$, that is, $f^{-(l_x-1)}(x) = f^{-(l_{x'}-1)}(x')$, for x and x' in B_3. There are two cases: (a) $l_x \neq l_{x'}$, and

(b) $l_x = l_{x'}$. For the case (a), assume that $l_x < l_{x'}$. Then, it is satisfied that $f^{l_x}(f^{-(l_x-1)}(x)) = f^{l_x}(f^{-(l_{x'}-1)}(x'))$, which implies that $f(x) = f^{-(l_{x'}-l_x-1)}(x')$. This is a contradiction, however, because $f(x) \in S_2 \backslash S_1$ and $f^{-(l_{x'}-l_x-1)}(x') \in S_1 \cap S_2$ by the definition of $l_{x'}$. One has a similar result in the case of $l_x > l_{x'}$, thus it turns out that in the case of $f^{-(l_x-1)}(x) = f^{-(l_{x'}-1)}(x')$, $k_x \neq k_{x'}$ cannot to be true. For the case (b), $x = x'$ because $x = f^{l_x-1}(f^{-(l_x-1)}(x)) = f^{l_{x'}-1}(f^{-(l_{x'}-1)}(x')) = x'$. □ (**End of proof** of Claim 7)

iv. Definition of $g_4 : B_4 \to S_1 \cap S_2$.
For each $x \in B_4$, $g_4(x)$ is defined as $f(x)$.

Claim 8. g_4 is well-defined.

Proof of Claim 8: For each $x \in B_4$, $x \in S_1 \cap S_2$ and $f(x) \in S_1 \cap S_2$, and hence an element of $S_1 \cap S_2$ is uniquely determined by g_4 for each $x \in B_4$. □ (**End of proof** of Claim 8)

Claim 9. g_4 is injective.

Proof of Claim 9: Because f is injective, g_4 is also injective. □ (**End of proof** of Claim 9)

(c) Definition of $g : S_1 \to S_2$.
The function g from S_1 to S_2 is defined by using g_1, g_2, g_3, and g_4 as follows:

for each $x \in S_1$,

$$g(x) = \begin{cases} g_1(x) & \text{if} \quad x \in B_1 \\ g_2(x) & \text{if} \quad x \in B_2 \\ g_3(x) & \text{if} \quad x \in B_3 \\ g_4(x) & \text{if} \quad x \in B_4. \end{cases}$$

2. **Examination of the conditions in the proposition.**

(a) Well-definiteness of g.

Claim 10. g is well-defined.

Proof of Claim 10: From Claim 1, one has (i) B_1, B_2, B_3, and B_4 are mutually disjoint, and (ii) $B_1 \cup B_2 \cup B_3 \cup B_4 = S_1$. From Claim 2, Claim 4, Claim 6, and Claim 8, g_1, g_2, g_3, and g_4 are well-defined on B_1, B_2, B_3, and B_4, respectively, and their codomains, that is, $S_2 \backslash S_1$, $S_2 \backslash S_1$, $S_1 \cap S_2$, and $S_1 \cap S_2$, respectively, are subsets of S_2. Hence, an element of S_2 is uniquely determined by g for each $x \in S_1$. □ (**End of proof** of Claim 10)

(b) Injectiveness of g.

Claim 11. g is injective.

Proof of Claim 11: One needs to verify that for each x and x' in S_1, if $g(x) = g(x')$ then $x = x'$.

From Claim 3, Claim 5, Claim 7, and Claim 9, g_1, g_2, g_3, and g_4 are injective with the domains B_1, B_2, B_3, and B_4, respectively. Therefore, the cases that both x and x' belong to the same B_m ($m = 1, 2, 3, 4$) are already verified. Moreover, Because g_1, g_2, g_3, and g_4 have the codomains $S_2 \backslash S_1$, $S_2 \backslash S_1$, $S_1 \cap S_2$, and $S_1 \cap S_2$, respectively, and $S_2 \backslash S_1$ and $S_1 \cap S_2$ are mutually disjoint. Therefore, it suffices to confirm the following two cases in which $g(x) = g(x')$ can hold: (a) $x \in B_1$ and $x' \in B_2$, and (b) $x \in B_3$ and $x' \in B_4$.

i. Case (a): if $x \in B_1$ and $x' \in B_2$ hold, one has $g(x) = g_1(x) = f(x)$ and $g(x') = g_2(x') = f^{k_{x'}}(x')$ ($k_{x'} \geq 2$) (see 1(b)i and 1(b)ii). If $g(x) = g(x')$, then $f(x) = f^{k_{x'}}(x')$, which implies that $x = f^{-1}(f(x)) = f^{-1}(f^{k_{x'}}(x')) = f^{k_{x'}-1}(x')$. This is, however, a contradiction, because $x \in S_1 \backslash S_2$ and $f^{k_{x'}-1}(x') \in S_1 \cap S_2$. Therefore, $g(x) = g(x')$ cannot to be true in this case.

ii. Case (b): if $x \in B_3$ and $x' \in B_4$ hold, one has $g(x) = g_3(x) = f^{-(l_x-1)}(x)$ and $g(x') = g_4(x') = f(x')$ ($l_x \geq 1$) (see 1(b)iii and 1(b)iv). If $g(x) = g(x')$, then $f^{-(l_x-1)}(x) = f(x')$, which implies that $f^{l_x}(x) = f^{-1}(f^{-(l_x-1)}(x)) = f^{-1}(f(x')) = x'$. This is, however, a contradiction, because $f^{l_x}(x) \in S_1 \backslash S_2$ and $x' \in S_1 \cap S_2$. Therefore, $g(x) = g(x')$ cannot to be true in this case.

Thus, g is injective. □ (**End of proof** of Claim 11)

(c) Surjectiveness of g.

Claim 12. g is surjective.

Proof of Claim 12: One has to see that for each $y \in S_2$, there exists $x \in S_1$ such that $g(x) = y$. Define four subsets C_1, C_2, C_3, and C_4 of S_2 as follows:

$$C_1 = \{y \in S_2 \backslash S_1 \mid f^{-1}(y) \in S_1 \backslash S_2\};$$
$$C_2 = \{y \in S_2 \backslash S_1 \mid f^{-1}(y) \in S_1 \cap S_2\};$$
$$C_3 = \{y \in S_1 \cap S_2 \mid f^{-1}(y) \in S_1 \cap S_2\};$$
$$C_4 = \{y \in S_1 \cap S_2 \mid f^{-1}(y) \in S_1 \backslash S_2\}.$$

C_1, C_2, C_3, and C_4 are mutually disjoint, because S_2 is a disjoint union of $S_2 \backslash S_1$ and $S_1 \cap S_2$, and S_1 is a disjoint union of $S_1 \backslash S_2$ and $S_1 \cap S_2$. For each $y \in S_2$, moreover, one has either $y \in S_2 \backslash S_1$ or $y \in S_1 \cap S_2$, and for each case, one has either $f^{-1}(y) \in S_1 \backslash S_2$ or $f^{-1}(y) \in S_1 \cap S_2$. Thus, $S_2 = C_1 \cup C_2 \cup C_3 \cup C_4$.

Let us consider four cases, that is, (a) $y \in C_1$; (b) $y \in C_2$; (c) $y \in C_3$; (d) $y \in C_4$, and examine whether there exists $x \in S_1$ such that $g(x) = y$ for each case.

i. Case (a): take $f^{-1}(y)$ as x. Then, one has $x = f^{-1}(y) \in S_1 \backslash S_2$, and $f(x) = f(f^{-1}(y)) = y \in S_2 \backslash S_1$, and hence, $x \in B_1$ (see 1a). Therefore, $g(x) = g_1(x) = f(x) = f(f^{-1}(y)) = y$ (see 1c and 1(b)i).

ii. Case (b): Because $f^{-1}(y) \in S_1 \cap S_2$ and $f(f^{-1}(y)) = y \in S_2 \backslash S_1$, one has $f^{-1}(y) \in B_3$ (see 1a). As seen in 1(b)iii, $l_{f^{-1}(y)}$ is determined as the minimum of such l that satisfies $f^{-l}(f^{-1}(y)) \in S_1 \backslash S_2$. Take $f^{-l_{f^{-1}(y)}}(f^{-1}(y))$ as x. Then, one has $x \in S_1 \backslash S_2$ and $f(x) \in S_1 \cap S_2$ from the way to determine $l_{f^{-1}(y)}$, so that $x \in B_2$ (see 1a). Therefore, $g(x) = g_2(x)$ (see 1c).
From 1(b)ii, k_x is determined as the minimum of such k that satisfies $f^k(x) \in S_2 \backslash S_1$, and it coincides with $l_{f^{-1}(y)} + 1$ from the way to determine $l_{f^{-1}(y)}$, x, and k_x. Therefore, $g_2(x) = f^{k_x}(x) = f^{k_x}(f^{-l_{f^{-1}(y)}}(f^{-1}(y))) = y$.

iii. Case (c): take $f^{-1}(y)$ as x. Then, one has $x = f^{-1}(y) \in S_1 \cap S_2$, and $f(x) = f(f^{-1}(y)) = y \in S_2 \cap S_1$, and hence, $x \in B_4$ (see 1a). Therefore, $g(x) = g_4(x) = f(x) = f(f^{-1}(y)) = y$ (see 1c and 1(b)iv).

iv. Case (d): Because $f^{-1}(y) \in S_1 \backslash S_2$ and $f(f^{-1}(y)) = y \in S_1 \cap S_2$, one has $f^{-1}(y) \in B_2$ (see 1a). As seen in 1(b)ii, $k_{f^{-1}(y)}$ is determined as the minimum of such k that satisfies $f^k(f^{-1}(y)) \in S_2 \backslash S_1$. Take $f^{(k_{f^{-1}(y)}-1)}(f^{-1}(y))$ as x. Then, one has $x \in S_1 \cap S_2$ and $f(x) \in S_2 \backslash S_1$ from the way to determine $k_{f^{-1}(y)}$, so that $x \in B_3$ (see 1a). Therefore, $g(x) = g_3(x)$ (see 1c).
From 1(b)iii, l_x is determined as the minimum of such l that satisfies $f^{-l}(x) \in S_1 \backslash S_2$, and it coincides with $k_{f^{-1}(y)} - 1$ from the way to determine $k_{f^{-1}(y)}$, x, and l_x. Therefore, $g_3(x) = f^{-(l_x-1)}(x) = f^{-(l_x-1)}(f^{(k_{f^{-1}(y)}-1)}(f^{-1}(y))) = y$.

□ (**End of proof** of Claim 12)

(d) Condition (i), (ii), and (iii).

Claim 13. g satisfies the conditions (i), (ii), and (iii).

Proof of Claim 13:

i. Condition (i) (for all $x \in S_1$, $xRg(x)$):

A. R is an equivalence relation on N, that is, it is reflexive, symmetric, and transitive. Therefore, for each $x \in S_1$, one has xRx.

B. Because $xRf(x)$ for each $x \in S_1$, it holds that for each $m \geq 1$, if $f^m(x) \in S_1 \cap S_2 \subseteq S_1$, then $f^m(x)Rf^{m+1}(x)$. Therefore, for each $m \geq 2$, if $f^p(x) \in S_1 \cap S_2 \subseteq S_1$ for each p such that $1 \leq p \leq m-1$, then one has $xRf(x)$, $f(x)Rf^2(x)$, …, $f^p(x)Rf^{p+1}(x)$, …, $f^{m-1}(x)Rf^m(x)$, which implies $xRf^m(x)$ from the transitivity of R.

C. For each $x \in S_1 \cap S_2 \subset S_1$, if $f^{-1}(x) \in S_1 \cap S_2 \subseteq S_1$, then one has $f^{-1}(x)Rx$, because $f(f^{-1}(x)) = x$ and $xRf(x)$ for each $x \in S_1$. Symmetry of R implies $xRf^{-1}(x)$. Moreover, for each $m \geq 1$, if $f^{-m}(x) \in S_1 \cap S_2 \subseteq S_1$, then $f^{-m}(x)Rf^{-(m-1)}(x)$ because $f(f^{-m}(x)) = f^{-(m-1)}(x)$ and $xRf(x)$ for each $x \in S_1$. This implies $f^{-(m-1)}(x)Rf^{-m}$ from the symmetry of R.
Thus, if for each p such that $1 \leq p \leq m-1$, $f^{-p}(x) \in S_1 \cap S_2 \subseteq S_1$, then one has $xRf^{-1}(x)$, $f^{-1}(x)Rf^{-2}(x)$, ..., $f^{-p}(x)Rf^{-(p+1)}(x)$, ... , $f^{-(m-1)}(x)Rf^{-m}(x)$, which implies $xRf^{-m}(x)$ from the transitivity of R.

From A, B, and C above, and the definitions of g_1, g_2, g_3, g_4, and g (see 1(b)i, 1(b)ii, 1(b)iii, 1(b)iv, and 1c, respectively), one has that g satisfies the condition (i).

ii. Condition (ii) (the restriction $g|_{S_1 \backslash S_2}$ of g on the set $S_1 \backslash S_2$ is a bijection from $S_1 \backslash S_2$ to $S_2 \backslash S_1$):
From Claim 11 and Claim 12, g is a bijection from S_1 to S_2, which implies that the restriction $g|_{S_1 \backslash S_2}$ of g on the set $S_1 \backslash S_2$ is a bijection from $S_1 \backslash S_2$ to $g(S_1 \backslash S_2)$. Thus, it suffices to see that $g(S_1 \backslash S_2) = S_2 \backslash S_1$.
$S_1 \backslash S_2 \subseteq S_1$ is a disjoint union of B_1 and B_2 (see 1a). Thus, one has that $g(S_1 \backslash S_2) = g(B_1) \cup g(B_2) = g_1(B_1) \cup g_2(B_2) \subseteq S_2 \backslash S_1$.
$S_2 \backslash S_1 \subseteq S_2$ is a disjoint union of C_1 and C_2 (see 12). From 2(c)i, it is satisfied that the inverse image of $y \in C_1$ by g is an element of $S_1 \backslash S_2$. Similarly, from 2(c)ii, one can see that the inverse image of $y \in C_2$ by g is an element of $S_1 \backslash S_2$. Thus, it holds that $S_2 \backslash S_1 \subseteq g(S_1 \backslash S_2)$.
These imply that $g(S_1 \backslash S_2) = S_2 \backslash S_1$.

iii. Condition (iii) (the restriction $g|_{S_1 \cap S_2}$ of g on the set $S_1 \cap S_2$ is a bijection on $S_1 \cap S_2$.):
Similarly to the proof for Condition (iii) above, it suffices to see that $g(S_2 \cap S_1) = S_2 \cap S_1$.
$S_1 \cap S_2 \subseteq S_1$ is a disjoint union of B_3 and B_4 (see 1a). Thus, one has that $g(S_1 \cap S_2) = g(B_3) \cup g(B_4) = g_3(B_3) \cup g_4(B_4) \subseteq S_2 \cap S_1$.
$S_1 \cap S_2 \subseteq S_2$ is a disjoint union of C_3 and C_4 (see 12). From 2(c)iii, it is satisfied that the inverse image of $y \in C_3$ by g is an element of $S_1 \cap S_2$. Similarly, from 2(c)iv, one can see that the inverse image of $y \in C_4$ by g is an element of $S_1 \cap S_2$. Thus, it holds that $S_2 \cap S_1 \subseteq g(S_1 \cap S_2)$.
These imply that $g(S_2 \cap S_1) = S_2 \cap S_1$.

□ (**End of proof** of Claim 13)

□ (**End of proof** of Lemma A.1.1)