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BPS States in Supersymmetric Theories

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Abstract

The purpose of this thesis is to explore various BPS states in supersymmetric theories. The BPS states are related to central charges. There are three types of central charges, $(0, 0)$, $(1, 0)$ and $(1/2, 1/2)$ central charges, in four dimensional supersymmetric theories. $\mathcal{N} = 2$ supersymmetric gauge theories have the BPS states related to the $(0, 0)$ central charge. Seiberg-Witten obtained exact results of the low energy physics of $\mathcal{N} = 2$ $SU(2)$ Yang-Mills theory. The BPS states play an important role in analyzing the model. On the other hand, $\mathcal{N} = 1$ SUSY theories cannot have the $(0, 0)$ central charges. However, $\mathcal{N} = 1$ supersymmetric theories have the other types of central charges. Domain walls are the BPS states which are related to only the $(1, 0)$ central charges. BPS strings are the BPS states which are related to only the $(1/2, 1/2)$ central charges. Domain wall junctions are the BPS states which are related to both of the central charges.

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Chapter 1

Introduction

Quantum chromodynamics (QCD) is asymptotically free. The coupling constant becomes strong at low energy scale. Generally, non-Abelian gauge theories which do not have so many flavors are asymptotically free. Therefore, nonperturbative effect is crucial at low energy or at long distance scale. A typical phenomenon of the nonperturbative effect is quark confinement. In order to understand these phenomena, the study of the nonperturbative effect is needed. On the other hand, Supersymmetric (SUSY) theories are supposed to be one of the most promising candidates for unified theory. It is also important to study the nonperturbative effect in SUSY theories. The nonperturbative effects play crucial roles in the dynamics of SUSY gauge theories, especially SUSY breaking. In fact, the perturbative non-renormalization theorem [1] shows that superpotentials receive no perturbative quantum corrections. Hence, only the nonperturbative effects break the supersymmetry as long as supersymmetry does not break spontaneously at tree-level.

In several years, good progress has been made on nonperturbative aspects of SUSY gauge theories. There are some techniques to analyze the nonperturbative aspect of various SUSY gauge theories — *holomorphy, duality, BPS state, brane approach*. In 1994, Seiberg and Witten [2][3] obtained exact results of the low energy physics of $SU(2)$ Yang-Mills theory which have two supersymmetries, so-called $\mathcal{N} = 2$ SUSY theory. The BPS states play an important role in analyzing the model with holomorphy and duality.

Originally, Prasad and Sommerfield [4], and Bogomol'nyi [5] discussed the non-SUSY $SU(2)$ (or $O(3)$) Yang-Mills theory coupled to a Higgs field at the classical level. They argued that the mass of a dyon with electric and magnetic charges (n_e, n_m) is

$$M \geq \sqrt{2}|Z_{\text{SW}}|, \tag{1.1}$$

$$Z_{\text{SW}} = v(n_e + i4\pi/g^2 n_m), \tag{1.2}$$

where v is the vacuum expectation value of the Higgs and g is the gauge coupling constant. The inequality is called Bogomol'nyi bound. If the inequality is saturated, the state is called BPS state:

$$M_{\text{BPS}} = \sqrt{2}|Z_{\text{SW}}|. \quad (1.3)$$

Witten and Olive [6] pointed out that the same inequality must hold in $\mathcal{N} = 2$ SUSY $SU(2)$ pure Yang-Mills theory. They interpreted Z_{SW} as a central charge of $\mathcal{N} = 2$ SUSY algebra. We call it $(0, 0)$ central charge. In SUSY theories context, the BPS states preserve some supersymmetries and, therefore, belong to special representations of the supersymmetry algebra known as “short” multiplets, which contain fewer states than standard “long” multiplets of the superalgebra. Because of the symmetries, some of the properties of the BPS states can be shown to be independent of the coupling constants, and thus certain properties can be computed exactly by weak coupling methods. In fact, the inequality (1.1) and the equation (1.3) must hold at strong coupling. The remaining question is how the central charge Z_{SW} is modified by quantum effect. By analogy with the classical result (1.2), Seiberg and Witten [2] assumed that at low energies the central charge Z_{SW} become

$$Z_{\text{SW}} = an_e + a_D n_m, \quad (1.4)$$

where a is the vacuum expectation value of the scalar component of $\mathcal{N} = 2$ SUSY vector multiplet in low energy effective theory, and a_D is a dual partner of a . The assumption plays crucial roles in Seiberg-Witten theory.

The Seiberg-Witten theory was generalized to various SUSY gauge theories [3][7]–[14]. Their techniques are very powerful for analyzing nonperturbative aspects of various SUSY gauge theories. The generalized Seiberg-Witten theories pass some consistency checks.

The BPS states were found in superstring theory, too. They are p -dimensional objects, so-called p -branes (for review, see Ref. [15]). In recent years, a deeper understanding of nonperturbative aspect of SUSY gauge theories in various dimensions has been gained by realizing them on the world-volumes of branes [16]–[55] (for review, see Ref. [56]). Using the brane approach, one can understand various aspects of SUSY gauge theories that one cannot understand using field-theoretic approaches. For instance, one cannot understand the low energy physics of $\mathcal{N} = 2$ $SU(N_c)$ SUSY Yang-Mills theory with an antisymmetric tensor matter, using field-theoretic approach. However, Landsteiner and Lopez [46] proposed the low energy physics of it, using the brane approach.

On the other hand, the exact low energy physics of four-dimensional $\mathcal{N} = 1$ SUSY theories cannot be obtained. $\mathcal{N} = 1$ SUSY theories cannot have

$(0, 0)$ central charges. However, $\mathcal{N} = 1$ SUSY theories have the other types of central charges — $(1, 0)$ and $(1/2, 1/2)$ central charges. There are BPS states related to the $(1, 0)$ and $(1/2, 1/2)$ central charges — Domain walls, string (vortex line) and domain wall junctions. Bogomol’nyi [5] discussed the domain wall and the string (vortex line) in non-SUSY theories at classical level. In SUSY theories, the BPS domain walls and the BPS strings preserve a half of supersymmetry. The BPS domain wall junctions preserve a quarter of supersymmetry.

A domain wall is defined as an object which interpolates between degenerate discrete minima of a potential and spread over two dimensions [57]–[64]. The BPS domain wall is related to the $(1, 0)$ central charges. Contrary to the classically Z_{SW} nonvanishing in Eq. (1.2), the $(1, 0)$ central charges in $\mathcal{N} = 1$ SUSY $SU(N_c)$ Yang-Mills theory vanish at tree-level. However, at the loop level quantum anomaly modifies the central charges. Therefore, the central charges become non-zero.

A BPS string is defined as a field configuration whose energy density is localized near one or some parallel axes [65]–[69]. The BPS string is related to the $(1/2, 1/2)$ central charges. In weak coupling, the only gauge model admitting the BPS string is SUSY quantum electrodynamics (QED) with a Fayet-Iliopoulos (FI) term. However, other gauge theories may have BPS string solution at strong coupling.

Very recently, another interesting possibility for BPS states has attracted much attention [70]–[73][69][74]–[83]. These states are junctions of BPS domain walls which are related to the $(1, 0)$ and $(1/2, 1/2)$ central charges.

The purpose of this thesis is to explore various BPS states in SUSY theories. In Chapter 2, we summarize the $(0, 0)$, $(1, 0)$ and $(1/2, 1/2)$ central charges, and the BPS states. Part of results in Chapter 2 is based on my work [76] in collaboration with K. Ito, M. Naganuma and N. Sakai. Chapter 3 briefly reviews Seiberg-Witten theory. The Seiberg-Witten theory is generalized to $\mathcal{N} = 2$ SUSY QCD in Chapter 4. In Chapter 4 we introduce a consistency check on the Seiberg-Witten theory. The method is called “integrating in” method [84]. Chapter 5 briefly reviews brane approach. In Chapter 6 we discuss $\mathcal{N} = 2$ SUSY Yang-Mills theory with a matter in the antisymmetric representation, using brane approach and “integrating in” method. The results in Chapter 6 are based on my work [85] in collaboration with S. Tomizawa, N. Sakai and T. Sakai. In Chapter 7, 8 and 9, we review the BPS domain wall, the BPS string and BPS domain wall junction, respectively. Part of results in Chapter 9 is based on my work [76] in collaboration with K. Ito, M. Naganuma and N. Sakai.

Chapter 2

BPS state

2.1 $(0, 0)$ central charges

Consider $\mathcal{N} \geq 2$ SUSY theories. Using the convention of Ref. [86], we denote the left-handed and right-handed supercharges of the four-dimensional field theory with extended supersymmetry as Q_α^A , $\bar{Q}_{\dot{\beta}B}$. The indices α and $\dot{\beta}$ run from 1 to 2 and denote two-component Weyl spinors. The indices A, B run from 1 to $\mathcal{N}$ (≥ 2) which is the number of supersymmetry. The supersymmetry algebra is

$$\begin{aligned}\{Q_\alpha^A, \bar{Q}_{\dot{\beta}B}\} &= 2\sigma_{\alpha\dot{\beta}}^\mu P_\mu \delta^A_B, \\ \{Q_\alpha^A, Q_\beta^B\} &= \epsilon_{\alpha\beta} Z^{AB}, \\ Z^{AB} &= -Z^{BA},\end{aligned}\tag{2.1}$$

where P_μ , $\mu = 0, 1, 2, 3$ are the energy-momentum four-vector of the system. The Z^{AB} are called $(0, 0)$ central charges. The unitary transformation can rotate Z^{AB} into a standard form:

$$\begin{aligned}Z &= \epsilon \otimes D, & (\mathcal{N} \text{ even}) \\ Z &= \begin{pmatrix} \epsilon \otimes D & 0 \\ 0 & 0 \end{pmatrix}, & (\mathcal{N} \text{ odd})\end{aligned}\tag{2.2}$$

where D is diagonal with nonnegative real eigenvalues and ϵ is the 2×2 antisymmetric matrix with $\epsilon^{12} = 1$.

For simplicity, we take $\mathcal{N} = 2$. In the rest frame, $P_\mu = (-H, 0, 0, 0)$, the SUSY algebra (2.1) becomes the following form:

$$\begin{aligned}\{Q_\alpha^A, (Q_\beta^B)^\dagger\} &= 2H\delta_\alpha^\beta \delta^A_B, \\ \{Q_\alpha^A, Q_\beta^B\} &= \epsilon_{\alpha\beta} \epsilon^{AB} |Z|, \\ \{(Q_\alpha^A)^\dagger, (Q_\beta^B)^\dagger\} &= \epsilon^{\alpha\beta} \epsilon_{AB} |Z|.\end{aligned}\tag{2.3}$$

We consider the following linear combinations of Q and $Q^\dagger$:

$$\begin{aligned} a_\alpha &= \frac{1}{\sqrt{2}}[Q_\alpha^1 + \epsilon_{\alpha\beta}(Q_\beta^2)^\dagger], \\ b_\alpha &= \frac{1}{\sqrt{2}}[Q_\alpha^1 - \epsilon_{\alpha\beta}(Q_\beta^2)^\dagger]. \end{aligned} \quad (2.4)$$

The operators a_α and b_β satisfy the following algebra:

$$\begin{aligned} \{a_\alpha, a_\beta\} &= \{b_\alpha, b_\beta\} = \{a_\alpha, b_\beta\} = 0, \\ \{a_\alpha, (a_\beta)^\dagger\} &= \delta_{\alpha\beta}(2H + |Z|), \\ \{b_\alpha, (b_\beta)^\dagger\} &= \delta_{\alpha\beta}(2H - |Z|). \end{aligned} \quad (2.5)$$

The expectation value of $\{b_\alpha, (b_\alpha)^\dagger\}$ (no sum over α) over any state is non-negative definite

$$\langle S | \{b_\alpha, (b_\alpha)^\dagger\} | S \rangle = \langle S | b_\alpha (b_\alpha)^\dagger | S \rangle + \langle S | (b_\alpha)^\dagger b_\alpha | S \rangle = 2\langle H \rangle - |\langle Z \rangle| \geq 0. \quad (2.6)$$

The bound is called Bogomol'nyi bound [5]. If the inequality is saturated, The state is called BPS state [4][5]. If $2\langle H \rangle > |\langle Z \rangle|$, then the algebra (2.5) has 4 creation and annihilation operators. The dimension of the representation of the algebra is $\sum_{n=0}^4 \binom{4}{n} = 16$. If $2\langle H \rangle = |\langle Z \rangle|$, then the state is BPS state and preserves the half of the supersymmetry

$$b_\alpha |\text{BPS}\rangle = (b_\alpha)^\dagger |\text{BPS}\rangle = 0. \quad (2.7)$$

The algebra (2.5) has 2 creation and annihilation operators. The dimension of the representation of the algebra is $\sum_{n=0}^2 \binom{2}{n} = 4$.

For $\mathcal{N} \geq 2$ case, the Bogomol'nyi bound is

$$2\langle H \rangle \geq \max_k \{D_k\}, \quad (2.8)$$

where D_k ($k = 1, \dots, [\mathcal{N}/2]$) are the eigenvalues of the diagonal matrix D . If $2\langle H \rangle = D_k$ ($k = 1, \dots, r$), then the BPS state preserves $r/\mathcal{N}$ of the whole supersymmetry.

In $\mathcal{N} \geq 2$ theories, there are particle-like BPS states, magnetic monopoles, dyons, and so on. We will discuss these particle-like BPS states in Chapter 3 and the subsequent chapters.

2.2 $(1, 0)$ and $(1/2, 1/2)$ central charges

Part of results in this section is based on my work [76] in collaboration with K. Ito, M. Naganuma and N. Sakai.

In $\mathcal{N} \geq 2$ theories, the $(0, 0)$ central charges Z^{AB} must be antisymmetric, *i.e.*, $Z^{AB} = -Z^{BA}$. Therefore, $\mathcal{N} = 1$ SUSY theories cannot have the $(0, 0)$ central charge. However, $\mathcal{N} = 1$ theories can have other types of central charges:

$$\{Q_\alpha, Q_\beta\} = 2i(\sigma^k \bar{\sigma}^0)_\alpha{}^\gamma \epsilon_{\gamma\beta} Z_k, \quad k = 1, 2, 3. \quad (2.9)$$

The new central charges Z_k are called $(1, 0)$ central charges, since the right hand side of Eq. (2.9) have two symmetric undotted spinor indices α and β , and transform under the $(1, 0)$ representation of the Lorentz group. The Coleman-Mandula [87] theorem and the four-dimensional full Lorentz symmetry forbid the $(1, 0)$ central charges Z_k to become non-zero [88]. However, the $(1, 0)$ central charges Z_k may become non-zero when the Lorentz symmetry is spontaneously broken.

Furthermore, $\mathcal{N} = 1$ SUSY theories have other central charges:

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2(\sigma_{\alpha\dot{\alpha}}^\mu P_\mu + \sigma_{\alpha\dot{\alpha}}^k Y_k), \quad k = 1, 2, 3. \quad (2.10)$$

The central charges Y_k are called $(1/2, 1/2)$ central charges, since $\sigma_{\alpha\dot{\alpha}}^k Y_k$ have an undotted spinor index α and a dotted spinor index $\dot{\alpha}$, and transform under the $(1/2, 1/2)$ representation of the Lorentz group. The central charges Z_k are complex and Y_k are real: $(Y_k)^* = Y_k$.

These central charges come from the total divergence and are non-vanishing when there are nontrivial differences in asymptotic behavior in different region of spatial infinity. Therefore these charges are topological in the sense that they are determined completely by the boundary conditions at infinity. For instance, we can compute the anticommutators (2.9) and (2.10) at tree-level in the general Wess-Zumino models with arbitrary number of chiral superfields Φ^i and arbitrary superpotential $\mathcal{W}$ to find the contributions from the bosonic component fields of the chiral superfields to the central charges

$$Z_k = 2 \int d^3x \partial_k \mathcal{W}^*(A^*), \quad (2.11)$$

$$Y_k = i\epsilon^{knm} \int d^3x K_{ij^*} \partial_n (A^{*j} \partial_m A^i), \quad \epsilon^{123} = 1, \quad (2.12)$$

where the scalar component of the i -th chiral superfield Φ^i is denoted as A^i and the Kähler metric $K_{ij^*} = \partial^2 K(A^*, A) / \partial A^i \partial A^{*j}$ is obtained from the Kähler potential K . We see that the central charges Z_k are completely determined by the difference of values of the superpotential $\mathcal{W}$ at spatial infinities where different discrete vacua are chosen for different directions. The central charges Y_k can be non-vanishing, if the field configuration at infinity is nontrivial in at least two-dimensions.

No particle-like BPS objects are known in $\mathcal{N} = 1$ theories. There are BPS objects in $\mathcal{N} = 1$ theories whose field configuration are extended in one- or two-dimensions. We will discuss these BPS states in Chapter 7 and the subsequent chapters.

To examine the lower bound for the energy due to the hermiticity of the supercharges, we consider two hermitian linear combinations of operators Q and $\bar{Q}$ with two arbitrary complex two-vectors β^α , β'^α and their complex conjugates $\bar{\beta}^{\dot{\alpha}} = (\beta^\alpha)^*$, $\bar{\beta}'^{\dot{\alpha}} = (\beta'^\alpha)^*$ as coefficients

$$K = \beta^\alpha Q_\alpha + \bar{\beta}^{\dot{\alpha}} \bar{Q}_{\dot{\alpha}}, \quad K' = \beta'^\alpha Q_\alpha + \bar{\beta}'^{\dot{\alpha}} \bar{Q}_{\dot{\alpha}}. \quad (2.13)$$

We treat β^α and β'^α as c-numbers rather than the Grassmann numbers. The anti-commutator between K and K' is

$$\frac{1}{2}\{K', K\} \equiv (\bar{\beta}'^{\dot{1}}, \bar{\beta}'^{\dot{2}}, \beta'^1, \beta'^2) \hat{K}^2 \begin{pmatrix} \beta^1 \\ \beta^2 \\ \bar{\beta}^{\dot{1}} \\ \bar{\beta}^{\dot{2}} \end{pmatrix}. \quad (2.14)$$

Since we are interested in field configurations at rest, we obtain $P^k = 0$, ($k = 1, 2, 3$) and the matrix $\hat{K}^2$ in terms of the central charges Z_k , Y_k and the Hamiltonian H explicitly

$$\hat{K}^2 = \begin{pmatrix} H + Y_3 & Y_1 + iY_2 & -Z_2^* + iZ_1^* & -iZ_3^* \\ Y_1 - iY_2 & H - Y_3 & -iZ_3^* & -Z_2^* - iZ_1^* \\ -Z_2 - iZ_1 & iZ_3 & H + Y_3 & Y_1 - iY_2 \\ iZ_3 & -Z_2 + iZ_1 & Y_1 + iY_2 & H - Y_3 \end{pmatrix}. \quad (2.15)$$

For simplicity, let us assume that field configuration is two-dimensional, for instance, depends on x_1 , x_2 only. Then we obtain $\langle Z_3 \rangle = \langle Y_1 \rangle = \langle Y_2 \rangle = 0$. We denote the eigenvalues and the eigenvectors of the hermitian matrix $\hat{K}^2$ as e_i and v_i ($i = 1, \dots, 4$).

$$\hat{K}^2 v_i = e_i v_i, \quad (2.16)$$

$$v_i^\dagger v_j = c_i \delta_{ij}, \quad c_i \in \mathbb{R}^+, \quad (2.17)$$

where c_i are norms of v_i . The eigenvalues and the eigenvectors are

$$\begin{aligned} e_1 &= H + Y_3 - |Z_2 + iZ_1|, & v_1 &= \left(\beta^1, 0, \frac{Z_2 + iZ_1}{|Z_2 + iZ_1|} \beta^1, 0 \right)^t \\ e_2 &= H + Y_3 + |Z_2 + iZ_1|, & v_2 &= \left(\beta^1, 0, -\frac{Z_2 + iZ_1}{|Z_2 + iZ_1|} \beta^1, 0 \right)^t \\ e_3 &= H - Y_3 - |Z_2 - iZ_1|, & v_3 &= \left(0, \beta^2, 0, \frac{Z_2 - iZ_1}{|Z_2 - iZ_1|} \beta^2 \right)^t \\ e_4 &= H - Y_3 + |Z_2 - iZ_1|, & v_4 &= \left(0, \beta^2, 0, -\frac{Z_2 - iZ_1}{|Z_2 - iZ_1|} \beta^2 \right)^t. \end{aligned} \quad (2.18)$$

Note that β^α of v_i are not the same as β^α of v_j ($i \neq j$). When $(\beta^1, \beta^2, \bar{\beta}^1, \bar{\beta}^2)^t = v_i$, we denote K as K_i . Hence,

$$\frac{1}{2}\{K_i, K_j\} = v_i^\dagger \hat{K}^2 v_j = e_i c_i \delta_{ij}. \quad (2.19)$$

Since K_i are hermitian, the expectation value of K_i^2 (no sum over i) over any state is non-negative definite

$$\langle S|K_i^2|S\rangle = \frac{1}{2}\langle S|\{K_i, K_i\}|S\rangle = c_i \langle e_i \rangle \geq 0. \quad (2.20)$$

The equality holds if and only if the linear combination of supercharges K_i is preserved by the state $|S\rangle$. Since $\langle e_2 \rangle \geq \langle e_1 \rangle$ and $\langle e_4 \rangle \geq \langle e_3 \rangle$ always, the Bogomol'nyi bound becomes

$$\langle H \rangle \geq \max\{H_I, H_{II}\}, \quad (2.21)$$

$$H_I \equiv |\langle -iZ_1 - Z_2 \rangle| - \langle Y_3 \rangle, \quad (2.22)$$

$$H_{II} \equiv |\langle iZ_1 - Z_2 \rangle| + \langle Y_3 \rangle. \quad (2.23)$$

When $\langle e_i \rangle > 0$ for all i , the algebra of K_i (2.19) is the Clifford algebra of $SO(4)$. The dimension of the representations is four. When one or two $\langle e_i \rangle$ are zero, the algebra of K_i (2.19) becomes the Clifford algebra of $SO(3)$ or $SO(2)$. The dimension of the representations is two.

When $\langle H \rangle = H_I$, the combination of supercharges K_1 is conserved

$$\left(Q_1 + \frac{\langle iZ_1 + Z_2 \rangle}{|\langle iZ_1 + Z_2 \rangle|} \bar{Q}_1 \right) |\text{BPS}\rangle = 0. \quad (2.24)$$

The condition of supercharge conservation (2.24) for $\langle H \rangle = H_I$ applied to chiral superfield $\Phi^i = (A^i, \psi^i, F^i)$ gives after eliminating the auxiliary field F^i

$$2iK_{ij}^* \frac{\langle iZ_1 + Z_2 \rangle}{|\langle iZ_1 + Z_2 \rangle|} \mathcal{D}_{\bar{z}} A^i = -\frac{\partial \mathcal{W}^*}{\partial A^{*j}}, \quad (2.25)$$

where complex coordinates $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$, gauge covariant derivatives $\mathcal{D}_\mu$, $\mathcal{D}_{\bar{z}} = \frac{1}{2}(\mathcal{D}_1 + i\mathcal{D}_2)$ and $\mathcal{D}_z = \frac{1}{2}(\mathcal{D}_1 - i\mathcal{D}_2)$ are introduced. The same BPS condition (2.24) applied to a vector superfield $V = (v_\mu, \lambda, D)$ in the Wess-Zumino gauge gives

$$v_{12} = -D, \quad v_{03} = 0, \quad v_{01} = v_{31}, \quad v_{23} = -v_{02}, \quad (2.26)$$

where $v_{\mu\nu} \equiv \frac{1}{ig}[\mathcal{D}_\mu, \mathcal{D}_\nu]$ and g is the gauge coupling constant. These Eqs. (2.25) and (2.26) are called BPS equations.

Similarly, when $\langle H \rangle = H_{\text{II}}$, the combination of supercharges K_3 is conserved

$$\left(Q_2 + \frac{\langle -iZ_1 + Z_2 \rangle}{|\langle -iZ_1 + Z_2 \rangle|} \bar{Q}_2 \right) |\text{BPS}\rangle = 0. \quad (2.27)$$

The condition of supercharge conservation (2.27) for $\langle H \rangle = H_{\text{II}}$ applied to chiral superfield gives after eliminating the auxiliary field

$$2iK_{ij^*} \frac{\langle iZ_1 - Z_2 \rangle}{|\langle iZ_1 - Z_2 \rangle|} \mathcal{D}_z A^i = -\frac{\partial \mathcal{W}^*}{\partial A^{*j}}. \quad (2.28)$$

The BPS condition (2.27) applied to a vector superfield in the Wess-Zumino gauge gives

$$v_{12} = D, \quad v_{03} = 0, \quad v_{01} = -v_{31}, \quad v_{23} = v_{02}. \quad (2.29)$$

We classify the BPS states into three types.

- $H_{\text{I}} \neq H_{\text{II}}$ and non-zero $\langle Z_k \rangle$,
- $H_{\text{I}} = H_{\text{II}}$ and non-zero $\langle Z_k \rangle$,
- $\langle Z_k \rangle = 0$ ($k = 1, 2$).

Domain wall junctions can satisfy that $H_{\text{I}} \neq H_{\text{II}}$ and non-zero $\langle Z_k \rangle$. Domain walls can satisfy that $H_{\text{I}} = H_{\text{II}}$ and non-zero $\langle Z_k \rangle$. String-like objects can satisfy $\langle Z_k \rangle = 0$ ($k = 1, 2$). BPS-saturated string-like objects are called BPS strings.

Domain wall junction First, consider the case of $H_{\text{I}} \neq H_{\text{II}}$ and non-zero $\langle Z_k \rangle$.

When $H_{\text{I}} > H_{\text{II}}$, the only one combination of supercharges K_1 (2.24) is conserved. These cases correspond to the 1/4 BPS state. The BPS states satisfy the BPS equations (2.25) and (2.26).

When $H_{\text{II}} > H_{\text{I}}$, the only one combination of supercharges K_3 (2.27) is conserved. The BPS states satisfy the BPS equations (2.28) and (2.29).

The typical BPS state which satisfies that $H_{\text{I}} \neq H_{\text{II}}$ and non-zero $\langle Z_k \rangle$ is domain wall junction. We will discuss the BPS domain wall junction in Chapter 9.

Domain wall Secondly, consider the case of $H_I = H_{II}$ and non-zero $\langle Z_k \rangle$. In this case, the two combinations of supercharges K_1 (2.24) and K_3 (2.27) are conserved. These cases correspond to the 1/2 BPS state. Therefore, the all BPS equations (2.25)–(2.29) are satisfied. From Eqs. (2.25) and (2.28), the scalar fields A^i depend on only one-dimensional spatial direction. For instance, when $\langle Z_2 \rangle = 0$, the scalar fields A^i depend on only x_1 :

$$K_{ij^*} \frac{\langle Z_1 \rangle}{|\langle Z_1 \rangle|} \frac{dA^i}{dx_1} = \frac{\partial \mathcal{W}^*}{\partial A^{*j}}. \quad (2.30)$$

From Eqs. (2.26) and (2.29),

$$v_{\mu\nu} = D = 0. \quad (2.31)$$

For the general Wess-Zumino model, the $(1/2, 1/2)$ central charges Y_3 is given by Eq. (2.12). Therefore, when the scalar fields A^i of the general Wess-Zumino model depend on only one-dimensional spatial direction, $\langle Y_3 \rangle$ must vanish. We believe that $\langle Y_3 \rangle$ vanish for any models when the fields depend on only one-dimensional spatial direction.

The typical BPS state which satisfies that $H_I = H_{II}$ and non-zero $\langle Z_k \rangle$ is domain wall. We will discuss the BPS domain wall in Chapter 7.

BPS string Lastly, consider the case of $\langle Z_1 \rangle = \langle Z_2 \rangle = 0$.

When $H_I > H_{II}$, the two combinations of supercharges K_1 and K_2 are conserved:

$$Q_1 |\text{BPS}\rangle = \bar{Q}_1 |\text{BPS}\rangle = 0. \quad (2.32)$$

When $H_{II} > H_I$, the two combinations of supercharges K_3 and K_4 are conserved:

$$Q_2 |\text{BPS}\rangle = \bar{Q}_2 |\text{BPS}\rangle = 0. \quad (2.33)$$

These cases correspond to the 1/2 BPS state.

For $\langle H \rangle = H_I$, the BPS equations are

$$\begin{aligned} \mathcal{D}_{\bar{z}} A^i &= \frac{\partial \mathcal{W}^*}{\partial A^{*i}} = 0, \\ v_{12} &= -D, \quad v_{03} = 0, \quad v_{01} = v_{31}, \quad v_{23} = -v_{02}. \end{aligned} \quad (2.34)$$

For $\langle H \rangle = H_{II}$, the BPS equations are

$$\begin{aligned} \mathcal{D}_z A^i &= \frac{\partial \mathcal{W}^*}{\partial A^{*i}} = 0, \\ v_{12} &= D, \quad v_{03} = 0, \quad v_{01} = -v_{31}, \quad v_{23} = v_{02}. \end{aligned} \quad (2.35)$$

The typical BPS state which satisfies that $\langle Z_1 \rangle = \langle Z_2 \rangle = 0$ is BPS string. We will discuss the BPS string in Chapter 8.

2.3 Branes in d -dimensional theories

In recent years, BPS states in d -dimensional theories have been considered. The BPS objects are p -dimensional objects, so-called p -branes (for review, see Ref. [15]), *i.e.*, 0-branes are particles, 1-branes are strings, 2-branes are membranes, $\dots$. Generically, the $\mathcal{N}$ supersymmetry algebra in d dimensions has the form:

$$\{\mathbf{Q}_\alpha^A, \mathbf{Q}_\beta^B\} = \sum_p Z_{\mu_1 \dots \mu_p}^{AB} \frac{1}{p!} (\Gamma^{\mu_1 \dots \mu_p} C)_{\alpha\beta}, \quad A, B = 1, \dots, \mathcal{N}, \quad (2.36)$$

where $\Gamma^{\mu_1 \dots \mu_p}$ are antisymmetrized Γ matrices, α and β are spinor indices. C is (real antisymmetric) charge conjugation matrix, and $Z_{\mu_1 \dots \mu_p}^{AB}$ are the central charges. The vector central charge Z_μ^{AB} contains the energy-momentum vector $P_\mu \delta^{AB}$. The tensor central charges $\Gamma^{\mu_1 \dots \mu_p}$ are related to (BPS-saturated) p -branes.

For $d = 4$ case, the supercharges $\mathbf{Q}_\alpha^A$, the Γ matrices and the charge conjugation matrix C are given by

$$\begin{aligned} \mathbf{Q}_\alpha^A &= \begin{pmatrix} Q_\alpha^A \\ \bar{Q}_{\dot{\alpha}A} \end{pmatrix}, \quad \Gamma_{\alpha\beta} = \begin{pmatrix} 0 & \sigma_{\alpha\dot{\beta}}^\mu \\ \bar{\sigma}^{\mu\dot{\alpha}\beta} & 0 \end{pmatrix}, \\ C &= i\Gamma^2\Gamma^0 = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix}. \end{aligned} \quad (2.37)$$

From Eqs. (2.37), the supersymmetry algebra (2.36) becomes [89]

$$\begin{aligned} \{Q_\alpha^A, \bar{Q}_{\dot{\beta}B}\} &= \sigma_{\alpha\dot{\beta}}^\mu P_\mu \delta_B^A + \sigma_{\alpha\dot{\beta}}^k Y_k \delta_B^A + \sigma_{\alpha\dot{\beta}}^\mu Z_{\mu B}^A, \\ \{Q_\alpha^A, Q_\beta^B\} &= \epsilon_{\alpha\beta} Z^{AB} + \sigma_{\alpha\beta}^{\mu\nu} \tilde{Z}_{\mu\nu}^{AB}, \\ Z^{AB} &= -Z^{BA}, \quad \tilde{Z}_{\mu\nu}^{AB} = \tilde{Z}_{\mu\nu}^{BA}, \\ Z_{\mu A}^A &= 0 \quad (\text{traceless}). \end{aligned} \quad (2.38)$$

Here, Z^{AB} are 0-brane charges, Y_k and $Z_{\mu B}^A$ are 1-brane charges and $\tilde{Z}_{\mu\nu}^{AB}$ are 2-brane charges.

In fact, Z^{AB} are the $(0,0)$ charges in Eq. (2.1). In Chapter 3, we will find that the $(0,0)$ charges are related to a monopole and a dyon in $\mathcal{N} = 2$ theories. Y_k are nothing but the $(1/2, 1/2)$ central charges in Eq. (2.10). The $(1/2, 1/2)$ central charges are related to BPS strings. The $(1,0)$ central charges Z_k , which are domain wall's charges, are identified with some components of the membrane charges $\tilde{Z}_{\mu\nu}$ in $\mathcal{N} = 1$ theories:

$$Z_k = \frac{i}{2} \tilde{Z}_{k0} + \frac{1}{4} \epsilon_{0ijk} \tilde{Z}_{ij}. \quad (2.39)$$

Chapter 3

Seiberg-Witten theory

In several years, good progress has been made on nonperturbative aspect of SUSY gauge theories. Seiberg and Witten obtained exact results of the low energy effective theories of $\mathcal{N} = 2$ $SU(2)$ pure Yang-Mills theory [2]. Their techniques are very powerful for analyzing nonperturbative aspect of various SUSY gauge theories. The BPS state plays important roles in the techniques. In this chapter, we will briefly review this Seiberg-Witten theory [2] .

3.1 Lagrangian

The $\mathcal{N} = 2$ vector multiplet contains gauge fields A_μ , two Weyl fermions λ , ψ and a scalar ϕ , all in the adjoint representation of gauge group. In terms of $\mathcal{N} = 1$ supersymmetry, these fields can be organized into a vector multiplet W_α (containing (A_μ, λ)) and a chiral multiplet Φ (containing (ϕ, ψ)). The Lagrangian is

$$\begin{aligned} 4\pi\mathcal{L} &= 2\text{Im} \left[\tau_{\text{cl}} \int d^2\theta d^2\bar{\theta} \text{tr} \Phi^\dagger e^{-2V} \Phi + \tau_{\text{cl}} \int d^2\theta \frac{1}{2} \text{tr} W W \right] \\ &= 4\pi \text{tr} \left\{ \frac{2}{g^2} \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \bar{\lambda} i \bar{\sigma}^\mu \mathcal{D}_\mu \lambda - \bar{\psi} i \bar{\sigma}^\mu \mathcal{D}_\mu \psi - \mathcal{D}_\mu \phi \mathcal{D}^\mu \phi^\dagger \right. \right. \\ &\quad \left. \left. + i\sqrt{2}\phi[\bar{\lambda}, \bar{\psi}] + i\sqrt{2}\phi^\dagger[\lambda, \psi] - \frac{1}{2}[\phi, \phi^\dagger]^2 \right) \right. \\ &\quad \left. - i \frac{\theta}{32\pi^2} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \right\}, \end{aligned} \tag{3.1}$$

where τ_{cl} is the gauge coupling constant ($\tau_{\text{cl}} = \theta/2\pi + i4\pi/g^2$). One of the SUSY transformations [86] in the Wess-Zumino gauge is

$$\delta_\xi \phi = \sqrt{2}\xi\psi,$$

$$\begin{aligned}
\delta_\xi \psi &= i\sqrt{2}\sigma^\mu \bar{\xi} \mathcal{D}_\mu \phi, \\
\delta_\xi A^\mu &= -i\bar{\lambda} \bar{\sigma}^\mu \xi + i\bar{\xi} \bar{\sigma}^\mu \lambda, \\
\delta_\xi \lambda &= \sigma^{\mu\nu} \xi F_{\mu\nu} - i\xi[\phi, \phi^\dagger].
\end{aligned} \tag{3.2}$$

The other SUSY transformation is the same except that λ is replaced with ψ each other. $\mathcal{N} = 2$ theories have a global $U(1)_R \times SU(2)_R$ symmetry:

$$SU(2)_R : \quad \theta_a \rightarrow U_a^b \theta_b \quad U_a^b \in SU(2), \tag{3.3}$$

$$U(1)_R : \quad \theta_a \rightarrow e^{-i\alpha} \theta_a. \tag{3.4}$$

Under this $SU(2)_R$, A_μ and ϕ are singlets, and λ and ψ form a doublet. Though the $SU(2)_R$ symmetry cannot be made manifest in terms of $\mathcal{N} = 1$ superfields, their transformations under an R-symmetry $U(1)_J \subset SU(2)_R$

$$U(1)_J : \quad \theta_1 \rightarrow e^{-i\alpha} \theta_1 \quad \theta_2 \rightarrow e^{i\alpha} \theta_2 \tag{3.5}$$

are manifest. The $U(1)_R$ and $U(1)_J$ act as

$$U(1)_R : \quad \begin{cases} W_\alpha(\theta) & \rightarrow e^{i\alpha} W_\alpha(e^{-i\alpha}\theta) \\ \Phi(\theta) & \rightarrow e^{2i\alpha} \Phi(e^{-i\alpha}\theta) \end{cases} \tag{3.6}$$

$$U(1)_J : \quad \begin{cases} W_\alpha(\theta) & \rightarrow e^{i\alpha} W_\alpha(e^{-i\alpha}\theta) \\ \Phi(\theta) & \rightarrow \Phi(e^{-i\alpha}\theta). \end{cases} \tag{3.7}$$

In the quantum theory, the $U(1)_R$ symmetry is generally broken by the anomaly. For the case of $SU(N_c)$ QCD with N_f flavors, $U(1)_R$ is broken to $\mathbb{Z}_{4N_c - 2N_f}$. In this $N_c = 2$ and $N_f = 0$ case, the global symmetry is $(SU(2)_R \times \mathbb{Z}_8)/\mathbb{Z}_2$ (the division by $\mathbb{Z}_2$ arises because the center of $SU(2)_R$ is contained in $\mathbb{Z}_8$).

3.2 Classical central charge

From the SUSY transformations (3.2), the supercurrents and the supercharges Q_α^1 and Q_α^2 can be computed. Witten and Olive [6] computed the anti-commutator of the two supercharges by using the canonical commutators of fields at the tree-level.

$$\begin{aligned}
\{Q_\alpha^1, Q_\beta^2\} &= \epsilon_{\alpha\beta} \epsilon^{12} Z \\
&= -2\sqrt{2}i Z_{\text{SW}}^* \epsilon_{\alpha\beta} \\
Z_{\text{SW}} &= a(n_e + \tau_{\text{cl}} n_m) \quad (\text{classical}),
\end{aligned} \tag{3.8}$$

where n_e and n_m are the electric and magnetic charges. The Bogomol'nyi bound condition is

$$M \geq \sqrt{2}|Z_{\text{SW}}|. \tag{3.9}$$

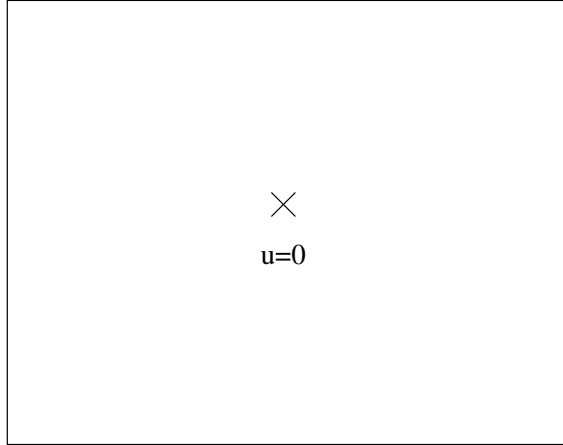


Figure 3.1: The classical moduli space of vacua of $\mathcal{N} = 2$ $SU(2)$ Yang-Mills theory. At singular point $u = 0$, the full $SU(2)$ gauge symmetry is restored.

Later, we will find that the central charges Z_{SW} is modified by the quantum effect.

3.3 Classical moduli space

The classical scalar potential is

$$V(\phi) = \frac{1}{g^2} \text{tr}([\phi, \phi^\dagger]^2). \quad (3.10)$$

Therefore, the classical vacua are found by setting $[\phi, \phi^\dagger]$ to zero. In the $SU(2)$ case, up to gauge transformation we can take $\phi = \frac{1}{2}a\sigma^3$, with a complex parameter a labeling the vacua. Since the Weyl group of $SU(2)$ acts by $a \leftrightarrow -a$, the gauge-invariant quantity parametrizing the space of vacua is

$$u = \frac{1}{2}a^2 = \text{tr}\phi^2. \quad (3.11)$$

For non-zero u , the gauge symmetry is broken to $U(1)$, and the gauge bosons, except for a component of Cartan subalgebra, become massive.

Classically, there is a singularity at $u = 0$, where the full $SU(2)$ gauge symmetry is restored and more field become massless. The classical moduli space of vacua is illustrated in Fig. 3.1.

Since $U(1)_R$ charge of ϕ and u is 2 and 4, respectively, the generator of $\mathbb{Z}_8$ acts as,

$$\phi \rightarrow e^{i2\pi\frac{2}{8}}\phi = i\phi, \quad (3.12)$$

$$u \rightarrow e^{i2\pi\frac{4}{8}}u = -u. \quad (3.13)$$

For non-zero u , the global $\mathbb{Z}_8$ symmetry is broken to $\mathbb{Z}_4$:

$$\begin{aligned} \phi &\rightarrow -\phi \simeq \phi, \\ u &\rightarrow u, \end{aligned} \quad (3.14)$$

where $-\phi$ are identified with ϕ by the Weyl group. The residual $\mathbb{Z}_4$ acts trivially in the u plane. The global symmetry group acts on the u plane as a spontaneously broken $\mathbb{Z}_2$, acting by $u \leftrightarrow -u$.

3.4 Low energy effective theory

Low energy effective theory is defined as a theory whose modes are integrated down to an infrared cutoff in momentum regions. Now, we are interested in a moduli space of vacua. Therefore we set the infrared cutoff under the energy scale at which Higgs mechanism occurs. Then, the low energy effective theory is written in terms of only massless fields.

In $\mathcal{N} = 2$ $SU(2)$ Yang-Mills theory case, since gauge symmetry $SU(2)$ is broken by Higgs mechanism, the gauge symmetry of its effective theory is $U(1)$. If an extra gauge symmetry arises or additional massless particles appear by quantum effect, then the low energy effective theory is not valid. These points, where the low energy effective theory is not valid, in the u plane is called “singular” points. Classically, the point $u = 0$ is a singular point.

Although the effective Lagrangian is very complicated, it can be expressed in terms of a single holomorphic function $\mathcal{F}(\mathcal{A})$ because of $\mathcal{N} = 2$ SUSY [90][91]:

$$\mathcal{L}_{\text{eff}} = \frac{1}{4\pi} \int d^2\theta_1 d^2\theta_2 \mathcal{F}(\mathcal{A}). \quad (3.15)$$

In $\mathcal{N} = 1$ superspace, the Lagrangian is

$$\mathcal{L}_{\text{eff}} = \frac{1}{4\pi} \text{Im} \left[\int d^2\theta d^2\bar{\theta} \frac{\partial \mathcal{F}(A)}{\partial A} \bar{A} + \int d^2\theta \frac{1}{2} \frac{\partial^2 \mathcal{F}(A)}{\partial A^2} W W \right], \quad (3.16)$$

where A is the $\mathcal{N} = 1$ chiral multiplet in the $\mathcal{N} = 2$ vector multiplet $\mathcal{A}$ whose scalar component is a . The purpose of this chapter is to obtain the holomorphic function $\mathcal{F}(A)$.

Classically, the $\mathcal{F}$ function is

$$\mathcal{F}_0 = \frac{1}{2} \tau_{\text{cl}} \mathcal{A}^2. \quad (3.17)$$

The quantum corrections were analyzed in Ref. [91]. The tree-level and one-loop contributions add up to

$$\mathcal{F}_{1\text{-loop}} = i \frac{1}{2\pi} \mathcal{A}^2 \ln \frac{\mathcal{A}^2}{\Lambda^2}, \quad (3.18)$$

where Λ is the dynamically generated scale. This logarithm is related to the one-loop β function of $\mathcal{N} = 2$ $SU(2)$ SUSY Yang-Mills theory and also ensures the anomalous transformation laws under $U(1)_R$. Higher order perturbative corrections are absent. Instantons lead to new term. The anomaly and the instanton action suggest that

$$\mathcal{F} = i \frac{1}{2\pi} \mathcal{A}^2 \ln \frac{\mathcal{A}^2}{\Lambda^2} + \sum_{k=1}^{\infty} \mathcal{F}_k \left(\frac{\Lambda}{\mathcal{A}} \right)^{4k} \mathcal{A}^2, \quad (3.19)$$

where the k th term arises as a contribution of k instantons.

3.5 Duality

The first term of the effective Lagrangian (3.16) is

$$\frac{1}{4\pi} \int d^2\theta d^2\bar{\theta} K(A, \bar{A}), \quad (3.20)$$

where K is a Kähler potential:

$$K(A, \bar{A}) = \text{Im} \frac{\partial \mathcal{F}(A)}{\partial A} \bar{A}. \quad (3.21)$$

The Kähler metric is

$$ds^2 = \frac{\partial^2 K}{\partial a \partial \bar{a}} da d\bar{a} = \text{Im} \tau(a) da d\bar{a}, \quad (3.22)$$

where

$$\tau(a) = \frac{\partial^2 \mathcal{F}(a)}{\partial a^2}. \quad (3.23)$$

Since this theory is asymptotically free, for large $|a|$, $\mathcal{F}(a) \approx \mathcal{F}_{1\text{-loop}}$ (see Eq. (3.18)). In this limit,

$$\tau(a) \approx \frac{i}{\pi} \left(\ln \frac{a^2}{\Lambda^2} + 3 \right) \quad (3.24)$$

is a multi-valued function whose imaginary part is single-valued and positive. If $\text{Im} \tau(a)$ is globally defined, then it cannot be positive definite, since the

harmonic function $\text{Im}\tau(a)$ cannot have a minimum. This indicates that the above description of the metric must be valid only locally.

Now we define $a_D = \partial\mathcal{F}/\partial a$. The metric can then be written

$$ds^2 = \text{Im}da_D da = -\frac{i}{2}(da_D d\bar{a} - da d\bar{a}_D). \quad (3.25)$$

This formula is completely symmetric in a and a_D , so if we use a_D as the local parameter, then the metric will be in the same general form as Eq. (3.22), with a different harmonic function replacing $\text{Im}\tau$. As we will see, this transformation corresponds to electric-magnetic duality.

We treat a and a_D as functions of u , which is a local coordinate on a complex manifold $\mathcal{M}$ – the moduli space of the vacua of the theory. The metric is

$$ds^2 = -\frac{i}{2}\epsilon_{\alpha\beta}\frac{da^\alpha}{du}\frac{d\bar{a}^\beta}{d\bar{u}}dud\bar{u}, \quad (3.26)$$

where $a^\alpha = (a_D, a)$ $\alpha = 1, 2$, and let $\epsilon_{\alpha\beta}$ be the antisymmetric tensor with $\epsilon_{12} = 1$. This is manifestly invariant under the transformation:

$$v \rightarrow Mv + c, \quad M \in SL(2, \mathbb{R}), \quad (3.27)$$

where v is a column vector (a_D, a) , and c is a constant vector. Later, we will find M must be in $SL(2, \mathbb{Z})$ and c must vanish. The fact, that the metric (3.26) is invariant under the transformation (3.27), means that the functions $a_D(u)$ and $a(u)$ are multi-valued functions and M is a monodromy matrix.

$SL(2, \mathbb{R})$ is generated by the transformations

$$T_b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (3.28)$$

with real b . Therefore, it is sufficient to discuss the effect of only T_b and S . T_b acts as $a_D \rightarrow a_D + ba$ and $a \rightarrow a$. Since $\tau(a)$ is $\partial a_D / \partial a$, T_b shifts the θ angle by $2\pi b$. Generally, the allowed shifts in the θ angle are by integer multiples of 2π . So, b must be an integer, and $SL(2, \mathbb{R})$ is reduced to $SL(2, \mathbb{Z})$.

S acts as $a_D \rightarrow a$ and $a \rightarrow -a_D$. It means that A is transformed into

$$A_D \equiv \frac{d\mathcal{F}(A)}{dA} \quad (3.29)$$

and A_D is transformed into A . This transformation corresponds to electric-magnetic duality. To see this, define a holomorphic function $\mathcal{F}_D(A_D)$ by

$$\frac{d\mathcal{F}_D(A_D)}{dA_D} = -A. \quad (3.30)$$

This S duality transformations simply constitute a Legendre transformation $\mathcal{F}_D(A_D) = \mathcal{F}(A) - AA_D$. Using this relation, the effective Lagrangian (3.16) becomes [2]

$$\frac{1}{4\pi} \text{Im} \left[\int d^2\theta d^2\bar{\theta} \frac{\partial \mathcal{F}_D(A_D)}{\partial A_D} \bar{A}_D + \int d^2\theta \frac{1}{2} \frac{\partial^2 \mathcal{F}_D(A_D)}{\partial A_D^2} W_D W_D \right], \quad (3.31)$$

where W_D is a dual partner of W . The $-1/\tau(A)$ can be expressed in terms of A_D :

$$-\frac{1}{\tau(A)} = -\frac{1}{\mathcal{F}''(A)} = -\frac{dA}{dA_D} = \mathcal{F}_D''(A_D) \equiv \tau_D(A_D). \quad (3.32)$$

3.6 Quantum central charge

The Bogomol'nyi bound condition (3.9) must hold quantum mechanically. The remaining question is how the central charge Z_{SW} is modified by quantum effect. From classical result (3.8) and $\tau = \partial a_D / \partial a$, Seiberg and Witten [2] conclude

$$Z_{\text{SW}} = an_e + a_D n_m. \quad (3.33)$$

Since the central charge appears in the SUSY algebra, Z_{SW} must be invariant under the monodromies. The monodromy acts on a column vector $v = (a_D, a)^t$ as $v \rightarrow Mv + c$ (3.27). Since $n_e a + n_m a_D$ is not invariant under addition of a constant to a or a_D , and there is no way to compensate for this by any transformation of n_e or n_m , c must be zero. Moreover, under $v \rightarrow Mv$,

$$w \rightarrow wM^{-1}, \quad (3.34)$$

where w is the row vector $w = (n_m, n_e)$.

3.7 Structure of the moduli space

In this section we will study structure of the moduli space $\mathcal{M}$. Particularly we show that there are singular points in the moduli space.

3.7.1 The singularity at infinity

It is clear that there is a singularity at infinity. In the limit $u \rightarrow \infty$, due to asymptotic freedom, the perturbative expression is valid:

$$\mathcal{F}(a) \equiv \mathcal{F}_{1\text{-loop}} = i \frac{\mathcal{A}^2}{2\pi} \ln \frac{\mathcal{A}^2}{\Lambda^2} \quad (3.35)$$

$$a_D = \frac{\partial \mathcal{F}}{\partial a} \equiv \frac{2ia}{\pi} \ln \frac{a}{\Lambda} + i \frac{a}{\pi}. \quad (3.36)$$

It follows that a_D is not a single-valued function of a for large a . Since the physical parameter is really $u = a^2/2$ (at least for large u and a), the monodromy can be determined as follows. When u is taken around a counterclockwise contour of very large radius in the complex u -plane ($u \rightarrow e^{2\pi i}u$),

$$a_D \rightarrow -a_D + 2a \quad (3.37)$$

$$a \rightarrow -a. \quad (3.38)$$

Thus, there is a non-trivial monodromy at infinity in the u plane,

$$M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}. \quad (3.39)$$

3.7.2 Singularities at strong coupling

There is a singular point at infinity. The non-trivial monodromy at infinity implies that there must be other singular points. Indeed, there must be at least three singular points. If there were only two singular points ($u = \infty$ and $u = u_0$), then the monodromy around u_0 would be the same as M_∞ , and a^2 would not be affected by any monodromies. It would mean that a^2 were a good global complex coordinate ($u = a^2/2$), and the metric would be globally of the form (3.22) with a global harmonic function $\text{Im}\tau(a)$. But such a metric could not be positive.

Following Seiberg and Witten, we assume that there are only three singular points including $u = \infty$. Since there is a symmetry $u \leftrightarrow -u$ acting on the u plane (see Section 3.3), there are three singular points at $u = \infty$, $u = u_0$, and $u = -u_0$ ($u_0 \neq 0$). The most natural physical interpretation of singular points in the u plane is that some additional massless particles are appearing at a particular value of u . We assume a monopole becomes massless at $u = u_0$ and another particle becomes massless at $u = -u_0$. If a monopole becomes massless, then $a_D = 0$ and $a \neq 0$, because of the BPS mass formula $M = \sqrt{2}|Z_{\text{SW}}|$ and Eq. (3.33). Similarly, if a dyon becomes massless, $a + a_D = 0$ and $a, a_D \neq 0$.

3.7.3 Massless monopole

The monopole is described by hypermultiplet M of $\mathcal{N} = 2$ SUSY which couple locally to the A_D and W_D , and, near u_0 $a_D \sim \langle A_D \rangle$ is small. This theory is exactly $\mathcal{N} = 2$ SUSY QED with matter which have magnetic charge instead of electric charge. This theory is not asymptotically free, but has a β function given by

$$\mu \frac{dg_D}{d\mu} = \frac{g_D^3}{8\pi^2}, \quad (3.40)$$

where g_D is the coupling constant. But the scale μ is proportional to a_D and $4\pi i/g_D^2(a_D)$ is proportional to $\tau_D(\theta_D = 0)$, respectively. Therefore, for $u \approx u_0$ or $a_D \approx 0$,

$$a_D \frac{d}{da_D} \tau_D = -\frac{i}{\pi} \quad \Rightarrow \quad \tau_D = -\frac{i}{\pi} \ln a_D. \quad (3.41)$$

Since $\tau_D = -da/da_D$,

$$a \approx a_0 + \frac{i}{\pi} a_D \ln a_D \quad (u \approx u_0), \quad (3.42)$$

where $a_0 \neq 0$.

Now, since a_D is a good coordinate near that point,

$$a_D \approx c_0(u - u_0)^k \quad (k > 0) \quad (3.43)$$

with a constant c_0 . Later we will find that $k = 1$.

From Eqs. (3.42) and (3.43), we read the monodromy as u turns around u_0 counterclockwise, $u - u_0 \rightarrow e^{2\pi i}(u - u_0)$

$$a_D \rightarrow a'_D = e^{2\pi i k} a_D \quad (3.44)$$

$$\begin{aligned} a \rightarrow a' &= a_0 + \frac{i}{\pi} a'_D \ln a'_D \\ &= a - 2ke^{2\pi i k} a_D + \frac{i}{\pi} (e^{2\pi i k} - 1) a_D \ln a_D. \end{aligned} \quad (3.45)$$

Since this transformation should be an $SL(2, \mathbb{Z})$ transformation, $e^{2\pi i k} = 1$. Therefore the monodromy matrix is

$$M_{u_0}(k) = \begin{pmatrix} 1 & 0 \\ -2k & 1 \end{pmatrix}. \quad (3.46)$$

3.7.4 The third singularity

From two monodromies M_∞ and M_{u_0} , we can determine the third monodromy at $u = -u_0$. Since the contour around $u = \infty$ is equivalent to a counterclockwise contour of very large radius in the complex plane, This contour can be deformed into a contour encircling u_0 and a contour encircling $-u_0$, as shown in Fig. 3.2. Therefore

$$M_\infty = M_{u_0} M_{-u_0}, \quad (3.47)$$

and hence

$$M_{-u_0}(k) = \begin{pmatrix} -1 & 2 \\ -2k & 4k - 1 \end{pmatrix}. \quad (3.48)$$

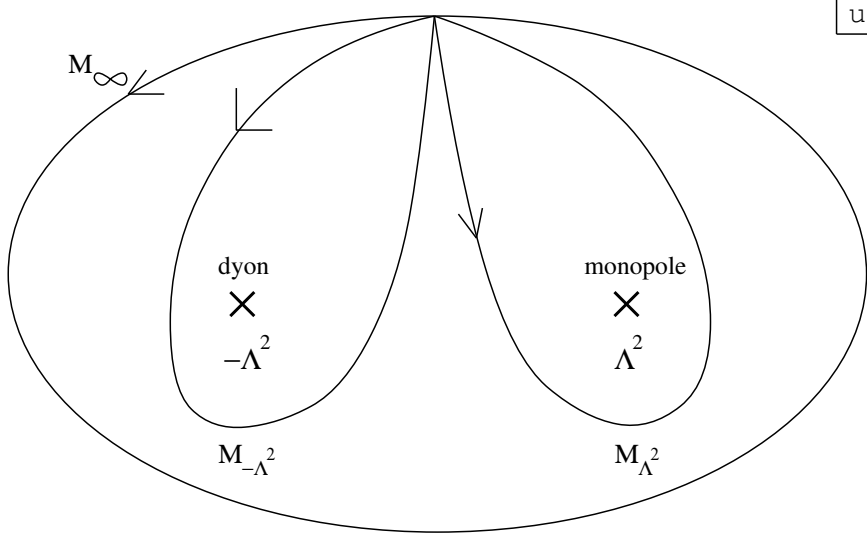


Figure 3.2: The quantum moduli space of vacua of $\mathcal{N} = 2$ $SU(2)$ Yang-Mills theory. We normalize u as $u_0 = \Lambda^2$. By the quantum effect, the singular point $u = 0$ of the classical moduli space (Fig. 3.1) split into the two singular points.

What kind of particle should become massless at $u = -u_0$? To answer this question, consider a monodromy transformation of charges $w = (n_m, n_e) : w \rightarrow wM^{-1}$, where $M \in SL(2, \mathbb{Z})$ (see Eq. (3.34)). The state of vanishing mass responsible for a singularity should be invariant under the monodromy [2]. For instance, it is clear that monopole $w = (1, 0)$ is a left eigenvector of M_{u_0} with unit eigenvalue:

$$(1, 0) = (1, 0)M_{u_0}. \quad (3.49)$$

It means that a monopole becomes massless at $u = u_0$. In the $u = -u_0$ case, if and only if $k = 1$, then there is a left eigenvector of M_{-u_0} with unit eigenvalue. The eigenvector is a dyon: $(n_m, n_e) = (1, -1)$. Consequently we conclude that k must be 1:

$$M_{u_0} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \quad M_{-u_0} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}, \quad (3.50)$$

and a dyon becomes massless at $u = -u_0$.

3.8 Exact solution

In this section, we will get the functions $a(u)$ and $a_D(u)$, exactly. Now we normalize u as $u_0 = 1$.

So far we have seen that $a_D(u)$ and $a(u)$ single-valued except for the monodromies around ∞ , 1 and -1 :

$$M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \quad M_{-1} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}, \quad (3.51)$$

and near $u = \infty$,

$$a \approx \sqrt{2u}, \quad a_D \approx i \frac{\sqrt{2u}}{\pi} \ln u. \quad (3.52)$$

Near $u = 1$

$$a \approx a_0 + \frac{i}{\pi} a_D \ln a_D, \quad a_D \approx c_0(u - 1), \quad (3.53)$$

where a_0 and c_0 are constant. The behavior near $u = -1$ is similar, with $a - a_D$ replacing a_D . From complex analysis, it is well-known that the $a(u)$ and $a_D(u)$ are really multi-valued functions with branch cut, the branch point being ∞ , 1 and -1 . Such functions are hypergeometric functions. A hypergeometric function can be represented as an integral along a contour on a Riemann surface. In order to get exact solutions, Seiberg and Witten considered an elliptic curve

$$y^2 = (x - 1)(x + 1)(x - u) \quad (y, x, u \in \mathbb{C}), \quad (3.54)$$

and the genus-one Riemann surface E_u corresponding to the elliptic curve (3.54)¹. The curve is so-called Seiberg-Witten curve. This elliptic curve equation describes a double cover of the x plane branched over $-1, 1, \infty$, and u . We can take cuts from -1 to 1 and from u to ∞ .

In the genus-one Riemann surface case, a homology group $H_1(E_u, \mathbb{C})$ is a finitely generated free Abelian group, whose rank is $2g$ ($=2$). Its basis (α, β) can be chosen as

$$\alpha \cdot \beta = 1, \quad (3.55)$$

¹ Seiberg and Witten explain: The reasons for considering the Riemann surface E_u are the following. Three monodromy matrices (3.51) do not generate all of $SL(2, \mathbb{Z})$ but only a certain subgroup $\Gamma(2)$ of matrices in $SL(2, \mathbb{Z})$ congruent to 1 modulo 2. In fact, M_∞ and M_1 do generate $\Gamma(2)$ [92]. Furthermore the u plane with punctures at $1, -1, \infty$ can be thought of as the quotient of the upper half plane H by $\Gamma(2)$. $H/\Gamma(2)$ parametrizes the elliptic curve (3.54).

where α encircles 1 and u , and β encircles 1 and -1 .

Seiberg and Witten show that exact solutions $a(u)$ and $a_D(u)$ can be represented by integrals of the differential λ along the two cycles α and β :

$$a_D(u) = \oint_{\alpha} \lambda, \quad a(u) = \oint_{\beta} \lambda. \quad (3.56)$$

If a different choice of cycles is made, the pair (a_D, a) will be transformed by an element of $SL(2, \mathbb{Z})$. If the residues of λ do not vanish, then $a_D(u)$ and $a(u)$ are shifted under the deformation of a closed path even across a pole of λ . $a_D(u)$ and $a(u)$ can not, however, be shifted under a monodromy transformation. Therefore the residues must vanish. One identifies $\lambda \sim \lambda + dw$ (with w a meromorphic integral, since the exact differential dw would not contribute to the contour integral (3.56)).

The Riemann surface E_u becomes singular when and only when two branch points coincide. For instance, when $u = 1$, the cycles α vanished and $a_D = 0$. It means that a monopole becomes massless at $u = 1$. When the cycle $\alpha + \beta$ vanishes, $a + a_D = 0$, a dyon becomes massless, as shown in Fig. 3.3.

On the Riemann surface, there is only one first-kind Abelian differential λ_1 :

$$\lambda_1 = \frac{dx}{y}. \quad (3.57)$$

Using λ_1 , one can define a 2×1 matrix, so-called period matrix,

$$\begin{pmatrix} B \\ A \end{pmatrix}, \quad (3.58)$$

where

$$A = \oint_{\alpha} \lambda_1, \quad B = \oint_{\beta} \lambda_1, \quad (3.59)$$

Now, when λ_1 is redefined as $\lambda'_1 = \lambda_1 B^{-1}$, the period matrix is

$$\begin{pmatrix} 1 \\ \tau_u \end{pmatrix}, \quad (3.60)$$

where

$$\tau_u = AB^{-1}. \quad (3.61)$$

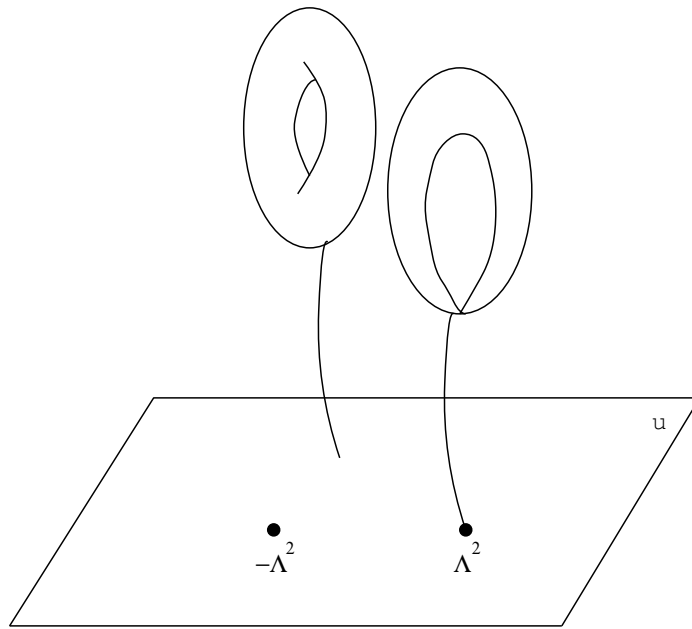


Figure 3.3: The moduli space and the Riemann surface (Seiberg-Witten curve). We normalize u as $u_0 = \Lambda^2$. At a singular point, the cycle on the Riemann surface vanishes, and the soliton, corresponding to the cycle, becomes massless.

This τ_u has the fundamental property

$$\text{Im}\tau_u > 0. \quad (3.62)$$

This inequality (3.62) suggests that the complex coupling constant τ is identified with τ_u . Eqs. (3.56) guarantee that $\tau = \tau_u$. From Eqs. (3.56), One has

$$\frac{da_D}{du} = \oint_{\alpha} \frac{d\lambda}{du}, \quad \frac{da}{du} = \oint_{\beta} \frac{d\lambda}{du}. \quad (3.63)$$

Suppose that

$$\frac{d\lambda}{du} = f(u)\lambda_1 = f(u)\frac{dx}{y}, \quad (3.64)$$

with $f(u)$ a function of u only. Then,

$$\frac{da_D}{du} = f(u)A, \quad \frac{da}{du} = f(u)B. \quad (3.65)$$

Consequently

$$\tau = \frac{da_D/du}{da/du} = \frac{A}{B} = \tau_u. \quad (3.66)$$

It remains to select $f(u)$. In order to get the desired asymptotic behavior of $a_D(u)$ and $a(u)$ near $u = 1, -1, \infty$, $f(u)$ must be $-\sqrt{2}/4\pi$. From Eq. (3.64),

$$\lambda = \frac{\sqrt{2}}{2\pi} \frac{dx\sqrt{x-u}}{\sqrt{x^2-1}}. \quad (3.67)$$

The exact solution $a_D(u)$ and $a(u)$ are

$$a_D(u) = \oint_{\alpha} \lambda = \frac{\sqrt{2}}{\pi} \int_1^u \sqrt{\frac{x-u}{x^2-1}} dx, \quad (3.68)$$

$$a(u) = \oint_{\beta} \lambda = \frac{\sqrt{2}}{\pi} \int_{-1}^1 \sqrt{\frac{x-u}{x^2-1}} dx. \quad (3.69)$$

From the differential equation $a_D = \partial\mathcal{F}(a)/\partial a$, we can obtain exact solution of $\mathcal{F}(a)$.

Chapter 4

$\mathcal{N} = 2$ $SU(N_c)$ SQCD

In the previous chapter, we have reviewed the Seiberg-Witten theory [2]. This work was generalized to theories including extra fundamental representation matter fields [3] and to other gauge groups $SU(N)$, $SO(N)$ and $Sp(2N)$ without or with matter fields in fundamental representation [7]–[14]. We shall call matter fields in the fundamental representation as quark. In this chapter, we will briefly review $\mathcal{N} = 2$ SUSY $SU(N_c)$ QCD (SQCD) with N_f flavors. In Section 4.1 and 4.2, we will discuss the classical and quantum moduli space of vacua of $\mathcal{N} = 2$ $SU(N_c)$ SQCD. In Section 4.3, we break the $\mathcal{N} = 2$ SUSY theory to $\mathcal{N} = 1$ SUSY by adding a perturbation $\Delta\mathcal{W}$ to the tree-level $\mathcal{N} = 2$ superpotential:

$$\Delta\mathcal{W} = \sum_{k=2}^{N_c} \frac{g_k}{k} \text{tr}(\Phi^k), \quad g_k \in \mathbb{C}, \quad (4.1)$$

where Φ is a chiral multiplet in terms of $\mathcal{N} = 1$ SUSY. g_2 is mass of Φ . Only singular points in moduli space of $\mathcal{N} = 2$ SUSY theory stay supersymmetric vacua by adding the perturbation. In Section 4.4 and 4.5, we introduce an examination of the Seiberg-Witten curve. The method of the examination is “integrating in” method. The “integrating in” method is used in various SUSY gauge theories. In Section 4.4, we will review general “integrating in” method. In Section 4.5, we will examine the Seiberg-Witten curve using the “integrating in” method.

4.1 Classical moduli space

$\mathcal{N} = 2$ $SU(N_c)$ SQCD is described in terms of $\mathcal{N} = 1$ superfields by a field strength chiral multiplet W_α and a scalar chiral multiplet Φ both in the

adjoint of the gauge group, and chiral multiplets Q^i in the $\mathbf{N}_c$, and $\tilde{Q}_i$ in the $\bar{\mathbf{N}}_c$ representations of the gauge group, where $i = 1, \dots, N_f$ are flavor indices. W and Φ together form a $\mathcal{N} = 2$ vector multiplet, while Q and $\tilde{Q}$ make up a hypermultiplet. We denote the complex scalar components of Φ , Q , and $\tilde{Q}$ and their vacuum expectation values by the same symbols. The Lagrangian in terms of $\mathcal{N} = 1$ superfields is

$$4\pi\mathcal{L} = \text{Im} \left[\tau \int d^2\theta d^2\bar{\theta} \text{tr} \left(\Phi^\dagger e^V \Phi + Q_i^\dagger e^V Q^i + \tilde{Q}^{\dagger i} e^V \tilde{Q}_i \right) + \tau \int d^2\theta \left(\frac{1}{2} \text{tr} W^2 + \mathcal{W} \right) \right], \quad (4.2)$$

where τ is the gauge coupling constant, $\mathcal{W}$ is the $\mathcal{N} = 2$ superpotential. In this chapter, τ is rescaled (following Ref. [13]):

$$\tau = \frac{\theta}{\pi} + \frac{8\pi i}{g^2}. \quad (4.3)$$

Here, V is

$$V = V^A T^A \quad (A = 1, \dots, N_c^2 - 1), \quad (4.4)$$

where T^A are the gauge group generators in the representation carried by Φ , Q^i or $\tilde{Q}_i$. The superpotential $\mathcal{W}_{\text{tree}}$ is

$$\mathcal{W}_{\text{tree}} = \sqrt{2} \tilde{Q}_i^a \Phi_a^b Q_b^i + \sqrt{2} m_j^i \tilde{Q}_i^a Q_a^j \quad (4.5)$$

$a, b = 1, \dots, N_c$ are color indices, and m_j^i is a quark mass matrix which must satisfy

$$[m, m^\dagger] = 0 \quad (4.6)$$

to preserve $\mathcal{N} = 2$ supersymmetry. The condition (4.6) implies that m can be diagonalized by a flavor rotation to a complex diagonal matrix

$$m = \text{diag}(m_1, \dots, m_{N_f}). \quad (4.7)$$

Classically and with no masses, the global symmetries are a $U(1)_B \times SU(N_f)$ flavor symmetry and a $U(1)_R \times SU(2)_R$ chiral R-symmetry which acts on the $\mathcal{N} = 2$ supersymmetry algebra. Mass terms and instanton corrections break $U(1)_R$. Under the unbroken $SU(2)_R$ the vector A_μ and the scalar Φ are singlets, while the vector multiplet fermions form a doublet. Similarly for the hypermultiplets, the fermions are singlets, while their scalar partners Q and $\tilde{Q}^\dagger$ form a doublet. Though the $SU(2)_R$ action cannot be made manifest

in terms of $\mathcal{N} = 1$ superfields, their transformations under an R-symmetry $U(1)_J \subset SU(2)_R$ are manifest in Lagrangian (4.2).

The β -function of this theory is

$$\beta(g) = -\frac{b}{16\pi^2}g^3, \quad (4.8)$$

where

$$b = 2N_c - N_f. \quad (4.9)$$

Therefore, when $N_f < 2N_c$, theory is asymptotically free and generates a strong coupling scale Λ . The instantons factor is proportional to

$$\exp\left(-\frac{8\pi^2}{g(E)^2}\right) = \left(\frac{\Lambda}{E}\right)^b, \quad (4.10)$$

and the $U(1)_R$ symmetry is anomalous, being broken by instantons down to a discrete $\mathbb{Z}_{2b}$ symmetry. For $N_f = 2N_c$, the theory is scale-invariant and the $U(1)_R$ is not anomalous. In the main part of this chapter, we concentrate on asymptotic free theories — $N_f < 2N_c$.

The $U(1)_R$ charges of Φ , m and Λ are 2, the charges of Q , $\tilde{Q}$ are 0. Since the $U(1)_R$ charges of all scalar fields and parameters are multiples of two, $\mathbb{Z}_{2b}$ can be regarded as $\mathbb{Z}_b$ ¹.

The symmetry charges for the scalar component fields and parameters are

	$SU(N_c)$	$\times$	$SU(N_f)$	$\times$	$U(1)_B$	$\times$	$U(1)_R$	$\times$	$U(1)_J$
Q	$\mathbf{N}_c$		$\mathbf{N}_f$		1		0		1
$\tilde{Q}$	$\bar{\mathbf{N}}_c$		$\bar{\mathbf{N}}_f$		-1		0		1
Φ	$\mathbf{adj}$		$\mathbf{1}$		0		2		0
m_A	$\mathbf{1}$		$\mathbf{adj}$		0		2		0
m_S	$\mathbf{1}$		$\mathbf{1}$		0		2		0
$\Lambda^{2N_c-N_f}$	$\mathbf{1}$		$\mathbf{1}$		0		$2(2N_c - N_f)$		0

(4.11)

where m_A is the traceless part of the mass matrix m , while m_S is the trace of the mass matrix.

¹ There is an exception. For the previous chapter ($N_c = 2, N_f = 0$) case, $\mathbb{Z}_{2b}$ can be regarded as $\mathbb{Z}_2$ not $\mathbb{Z}_4$ by following reasons. The gauge-invariant coordinates of the moduli space are not Φ but u_k , defined in Eq. (4.19). The $U(1)_R$ charges of u_k are $2k$. For $N_c = 2$, there is only u_2 , whose $U(1)_R$ charge is 4. For $N_f = 0$, there are no mass m whose charge is 2. Therefore, since the $U(1)_R$ charges of all gauge-invariant coordinates and parameters are multiples of four, $\mathbb{Z}_{2b}$ can be regarded as $\mathbb{Z}_2$ (see Section 3.3).

The classical vacua are the zeros of the scalar potential $V = (1/2)\text{tr}(D^2) + F^{i\dagger}F_i$ where $F_i = \partial\mathcal{W}/\partial\varphi^i$, $D^A = \varphi_i^\dagger(T^A)_j^i\varphi^j$, φ^i runs over all the scalar fields. Therefore, the classical vacua are found by setting the F -term and D -term to zero. The F -term equations are

$$\rho\delta_a^b = Q_a^i\tilde{Q}_i^b, \quad (4.12)$$

$$0 = Q_a^j m_j^i + \Phi_a^b Q_b^i, \quad (4.13)$$

$$0 = m_j^i \tilde{Q}_j^a + \tilde{Q}_i^b \Phi_b^a. \quad (4.14)$$

The D -term equation is

$$D^A = \Phi_c^{\dagger b}(t^A)_b^a \Phi_a^c - \Phi_c^b(t^A)_b^a \Phi_a^{\dagger c} + Q_i^{\dagger b}(t^A)_b^a Q_a^i - \tilde{Q}_i^b(t^A)_b^a \tilde{Q}_a^{\dagger i} = 0, \quad (4.15)$$

where $(t^A)_b^a$ are the gauge group generators in the fundamental representation. Note that the first and second terms of Eq. (4.15) are $SU(2)_R$ singlets, and the other terms are triplets. Therefore they must vanish separately. Hence,

$$0 = [\Phi, \Phi^\dagger], \quad (4.16)$$

$$\nu\delta_a^b = Q_a^i Q_i^{\dagger b} - \tilde{Q}_a^{\dagger i} \tilde{Q}_i^b, \quad \nu \in \mathbb{R}. \quad (4.17)$$

In the classical vacua, Φ , Q and $\tilde{Q}$ satisfies Eqs. (4.12)–(4.17). The solutions of these equations fall into various “branches” corresponding to different phases of the theory. The Coulomb phase is defined as the set of solutions with $Q = \tilde{Q} = 0$, the Higgs phase is defined as the set of solutions with $Q \neq 0$ and/or $\tilde{Q} \neq 0$. The Higgs phase is separated into two phases, the pure Higgs phase and the mixed phase. The pure Higgs phase is solution with $\Phi \neq 0$, and the mixed phase is those with non-vanishing Q , $\tilde{Q}$ and Φ .

In the Coulomb branch, Φ satisfies Eq. (4.16) with $Q = \tilde{Q} = 0$. Φ can be diagonalized by a color rotation to

$$\Phi = \text{diag}(\phi_1, \dots, \phi_{N_c}), \quad \sum_{a=1}^{N_c} \phi_a = 0. \quad (4.18)$$

Therefore, at a generic point in the Coulomb branch, the gauge symmetry $SU(N_c)$ is broken to $U(1)^{N_c-1}$.

The coordinates describing the Coulomb branch must be gauge-invariant. As gauge-invariant coordinates, we take

$$u_k = \frac{1}{k} \text{tr}(\Phi^k), \quad (k = 2, \dots, N_c). \quad (4.19)$$

(For $N_c = 2$ and $N_f = 0$, u in Eq. (3.11) is $2u_2$.) Alternative gauge-invariant coordinates which will be useful for $N_c > 3$ are defined at classical level as the symmetric polynomials s_k in ϕ_a

$$s_k = (-1)^k \sum_{a_1 < \dots < a_k} \phi_{a_1} \cdots \phi_{a_k}, \quad k = 2, \dots, N_c. \quad (4.20)$$

Using Eq. (4.20), we have

$$\prod_{a=1}^{N_c} (x - \phi_a) = x^{N_c} + \sum_{i=2}^{N_c} s_i x^{N_c-i}. \quad (4.21)$$

Using Eqs. (4.19) and (4.20) the sets s_k and u_k are related by Newton's formula [8]

$$k s_k + \sum_{i=1}^k i s_{k-i} u_i = 0, \quad k = 1, 2, \dots \quad (4.22)$$

where $s_0 = 1$, $s_1 = u_1 = 0$.

This classical Coulomb branch has two types of orbifold singularities. First type of singularities is along submanifolds where some of the ϕ_a 's are equal. It corresponds to the case where some of the non-Abelian gauge symmetry is restored. Second type of singularities is along submanifolds where $\phi_a + m_i = 0$. It corresponds to the case where the effective masses of the Q_a^i and $\tilde{Q}_i^a$ vanish which are given by the scalar potential (see Eq. (4.5)). The vanishing of one of these masses thus describes a complex codimension one submanifold in the Coulomb phase. The submanifold is called singular surface.

The Higgs branch and the mixed branch are discussed by Argyres, Plesser and Seiberg [13] in detail. The mixed branch is locally a product of a Coulomb branch for smaller gauge group and a smaller Higgs branch.

4.2 Quantum moduli space

In previous section, we have analyzed the moduli space of vacua at the classical level. In this section, we will analyze how this structure is modified quantum mechanically. The Higgs branch is not modified quantum mechanically [13]. But the Coulomb branch is modified. The mixed branch is still locally a product of a Coulomb branch and a Higgs branch [13].

We will analyze the quantum Coulomb branch. Seiberg and Witten determined the quantum moduli space of $\mathcal{N} = 2$ supersymmetric pure $SU(2)$

Yang-Mills theory as a certain one parameter family of elliptic curve [2]. The generalization of Ref. [2] to include matter hypermultiplets was carried out in Ref. [3]. For $SU(N_c)$ gauge theory without or with matter hypermultiplets, the moduli space was constructed as an $N_c - 1$ parameter family of hyperelliptic curves [8][9][10][11]. We will review their works.

As we have seen in the previous section, at a generic point in the Coulomb branch, the gauge symmetry is broken to $U(1)^{N_c-1}$ and the low energy Lagrangian can be written in term of $N_c - 1$ Abelian vector multiplet (A^i, W^i) ($i = 1, \dots, N_c - 1$) [2]:

$$\mathcal{L}_{\text{eff}} = \frac{1}{4\pi} \text{Im} \left[\int d^4\theta \frac{\partial \mathcal{F}}{\partial A^i} \bar{A}^i + \frac{1}{2} \int d^2\theta \frac{\partial^2 \mathcal{F}}{\partial A^i \partial A^j} W^i W^j \right]. \quad (4.23)$$

The metric is

$$\begin{aligned} ds^2 &= \text{Im} \frac{\partial^2 \mathcal{F}}{\partial a^i \partial a^j} da^i d\bar{a}^j \\ &= \text{Im} da_{D,j} d\bar{a}^i, \end{aligned} \quad (4.24)$$

where a^i are scalar components of A^i and

$$a_{D,j} = \frac{\partial \mathcal{F}}{\partial a_j}. \quad (4.25)$$

The mass of the BPS state is

$$M = \sqrt{2} |Z_{\text{SW}}|. \quad (4.26)$$

The bare masses break the flavor symmetry $SU(N_f) \rightarrow U(1)^{N_f-1}$, therefore states have associated quark number charge $S_j \in \mathbb{Z}$. The central charge Z_{SW} is

$$Z_{\text{SW}} = n_m^i a_{D,i} + n_{e,i} a^i + \sum_{j=1}^{N_f} S_j m_j, \quad (4.27)$$

where n_m^i , $n_{e,i}$ are the magnetic and electric charges and m_j are the eigenvalues of the diagonalized mass matrix (4.7). The monodromy transformation takes the form:

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \rightarrow M \begin{pmatrix} a_D \\ a \end{pmatrix} + c \quad M \in Sp(2N_c - 2, \mathbb{Z}), \quad (4.28)$$

where c is independent of the moduli parameters and depends on the masses of the quarks. The central charge must be invariant under the monodromy.

Therefore, c is an integral linear combination of the masses. a_D and a can be written as periods of a meromorphic one-form λ on a genus $N_c - 1$ hyperelliptic Riemann surface

$$a_{D,j} = \oint_{\alpha^j} \lambda, \quad a^i = \oint_{\beta_i} \lambda, \quad (4.29)$$

where α^i, β_i form a basis of homology basis, which can be chosen as

$$\alpha^i \cdot \alpha^j = 0 \quad \beta_i \cdot \beta_j = 0 \quad \alpha^i \cdot \beta_j = \delta_j^i. \quad (4.30)$$

The genus $N_c - 1$ hyperelliptic Riemann surface, so-called Seiberg-Witten curve, is

$$y^2 = \prod_{a=1}^{N_c} (x - \phi_a)^2 + 4\Lambda^{2N_c - N_f} \prod_{i=1}^{N_f} (x + m_i), \quad N_f \leq 2N_c - 2, \quad (4.31)$$

$$y^2 = \prod_{a=1}^{N_c} (x - \phi_a)^2 + 4\Lambda \prod_{i=1}^{N_f} \left(x + m_i + \frac{\Lambda}{N_c} \right), \quad N_f = 2N_c - 1. \quad (4.32)$$

The one-form λ is

$$\lambda = \frac{1}{2\pi i} \ln \left(\frac{F(x) - y}{F(x) + y} \right) dx, \quad (4.33)$$

$$F(x) = x^{N_c} + \sum_{i=2}^{N_c} s_i x^{N_c - i}, \quad N_f \leq 2N_c - 2, \quad (4.34)$$

$$F(x) = x^{N_c} + \sum_{i=2}^{N_c} s_i x^{N_c - i} - \frac{1}{4} \Lambda x^{N_c - 1}, \quad N_f = 2N_c - 1. \quad (4.35)$$

At generic point in the Coulomb branch, the Seiberg-Witten curve is a genus $N_c - 1$ hyperelliptic curve. There is codimension one “singular surface” in the Coulomb branch. On the singular surface, one or more cycles of the Seiberg-Witten curve shrink and genus becomes less than $N_c - 1$, as shown in Fig. 4.1. At the point, a soliton particle which corresponds to the vanishing cycle becomes massless.

4.3 Breaking to $\mathcal{N} = 1$ theory

In this section, we add the tree-level superpotential

$$\Delta\mathcal{W} = \sum_{k=2}^{N_c} g_k u_k \quad (4.36)$$

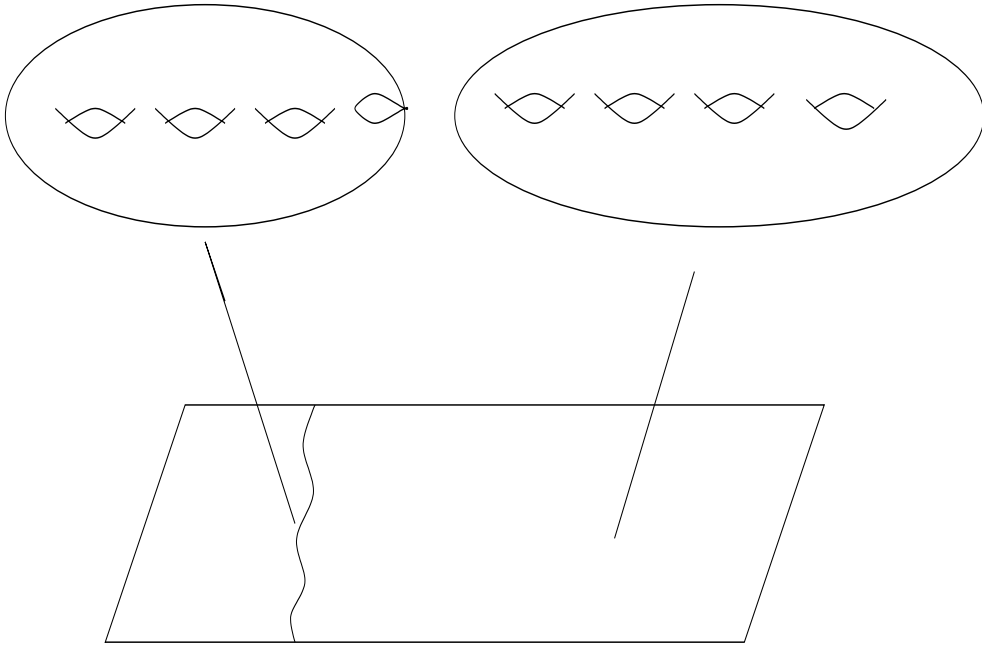


Figure 4.1: The singular surface of the Coulomb branch ($N_c = 5$). The singular surface is codimension one manifold in the Coulomb branch. At a point in the singular surface, a cycle of the Seiberg-Witten curve shrink and genus become less than $N_c - 1$,

to $\mathcal{N} = 2$ SQCD. The superpotential $\Delta\mathcal{W}$ explicitly breaks $\mathcal{N} = 2$ SUSY to $\mathcal{N} = 1$ SUSY.

The spectrum of the low energy $\mathcal{N} = 2$ SQCD at a generic point in the Coulomb branch consists of the $N_c - 1$ fields u_k and their associated photons. Perturbing by $\Delta\mathcal{W}$ lifts up the vacuum energy at the generic point in the Coulomb branch and does not give a SUSY ground state. The SUSY ground states obtained upon perturbing by $\Delta\mathcal{W}$ only occur where a soliton particle becomes massless and can condense. Consider the vicinity of a point where a single soliton becomes massless. The low energy theory has a superpotential which is approximately given by

$$\mathcal{W} = M(u_k)q\tilde{q} + \sum_{k=2}^{N_c} g_k u_k, \quad (4.37)$$

where q is the soliton chiral superfield, $\tilde{q}$ is the anti-soliton chiral superfield and $M(u_k)$ is a photon chiral superfield coupled with the soliton. There is a supersymmetric ground state if there is a solution $\langle u_k \rangle$ with

$$M(\langle u_k \rangle) = 0, \quad \text{and} \quad \frac{\partial M(\langle u_k \rangle)}{\partial u_k} \langle q\tilde{q} \rangle = -g_k. \quad (4.38)$$

Therefore one finds that the $\langle q\tilde{q} \rangle$ condenses provided $\partial M/\partial u_k \neq 0$. In such a vacuum, the condensate $\langle q\tilde{q} \rangle$ confines one of the photons and $N_c - 2$ are left massless. These vacua must correspond to the vacua in which the gauge group is classically broken to $SU(2) \times U(1)^{N_c-2}$ [93][94].

4.4 “Integrating in” method

We will introduce a consistency check of the Seiberg-Witten curve. The Coulomb branch of the moduli space of the $\mathcal{N} = 2$ SUSY vacua has codimension one singular surface where solitons such as monopoles or dyons become massless. When $\mathcal{N} = 2$ SQCD are broken to $\mathcal{N} = 1$ SUSY by perturbations of tree-level superpotentials $\Delta\mathcal{W}$, only these singular surfaces remain as $\mathcal{N} = 1$ SUSY vacua. Conversely, from a study of the low energy effective action of $\mathcal{N} = 1$ theory, we can derive some information on the singular surface of the moduli space of $\mathcal{N} = 2$ SQCD. In fact, Elitzur *et al.* [93] have developed a method to obtain the singular surface, using “integrating in” method [84]. This “integrating in” method has been extended to SUSY gauge theories including exceptional groups without or with quarks in the fundamental representation [94]–[101].

The “integrating in” method is used in various SUSY theories. In this section, we briefly review “integrating in” method generally. In the next

section, using the “integrating in” method, we will examine the Seiberg-Witten curve.

Consider two supersymmetric gauge theories. They are referred to as “up theory” and “down theory”. The down theory has matter superfields D_i . The nonperturbative superpotential of the down theory depends on the gauge singlets X_I , which are constructed out of the matter superfields D_i :

$$\mathcal{W}_d \equiv \mathcal{W}_{\text{down}}(X_I). \quad (4.39)$$

The up theory has a matter superfield U in addition to D_i . The nonperturbative superpotential of the up theory is

$$\mathcal{W}_u \equiv \mathcal{W}_{\text{up}}(X_I, M, Z_A), \quad (4.40)$$

where $M \sim U^2$ and Z_A are relevant gauge singlets constructed out of U and D_i .

We add to $\mathcal{W}_u$ the tree-level superpotential

$$\Delta\mathcal{W} = mM + \lambda^A Z_A, \quad (4.41)$$

where m is the mass of U . At low energies, the up theory is changed to down theory by integrating out U :

$$\begin{aligned} [\mathcal{W}_u + \Delta\mathcal{W}](X_I, M = \langle M \rangle(m, \lambda^A), Z_A = \langle Z_A \rangle(m, \lambda^A), m, \lambda^A) \\ = [\mathcal{W}_d + \mathcal{W}_i](X_I, m, \lambda), \end{aligned} \quad (4.42)$$

$$\equiv \mathcal{W}_L(X_I, m, \lambda), \quad (4.43)$$

where

$$\mathcal{W}_i \equiv \mathcal{W}_{\text{intermediate}}(X_I, m, \lambda). \quad (4.44)$$

We may now split the intermediate superpotential into two pieces:

$$\mathcal{W}_i = \Delta\mathcal{W}_d + \mathcal{W}_\Delta, \quad (4.45)$$

where $\Delta\mathcal{W}_d$ is the tree-level down superpotential:

$$\Delta\mathcal{W}_d \equiv \Delta\mathcal{W}(M = \langle M \rangle(m, \lambda^A), Z_A = \langle Z_A \rangle(m, \lambda^A), m, \lambda^A). \quad (4.46)$$

Up to now we have described the obvious procedure of “integrating out” U from the up theory to the down theory. But now, we will start with the down theory, and recover the dependence on the superfield U from the dependence on the parameters m and λ^A . We want to derive $\mathcal{W}_u$. If we know

$\mathcal{W}_\Delta$, then $\mathcal{W}_u$ can be derived by a Legendre transform of $\mathcal{W}_L$ with respect to m and λ^A :

$$\mathcal{W}_u(X_I, M, Z_A) = \mathcal{W}_L(X_I, m, \lambda^A) - \Delta\mathcal{W} \quad (4.47)$$

$$= [\mathcal{W}_d + \mathcal{W}_\Delta + \Delta\mathcal{W}_d - \Delta\mathcal{W}](X_I, M, Z_A), \quad (4.48)$$

$$\begin{aligned} M &= \frac{\partial}{\partial m} \mathcal{W}_L(X_I, m, \lambda), \\ Z_A &= \frac{\partial}{\partial \lambda^A} \mathcal{W}_L(X_I, m, \lambda). \end{aligned} \quad (4.49)$$

This procedure to recover the field U is called “integrating in”.

The additional term $\mathcal{W}_\Delta$ should obey [84]

$$\mathcal{W}_\Delta \rightarrow 0 \quad \text{as} \quad m \rightarrow \infty \quad \text{and} \quad \mathcal{W}_\Delta \rightarrow 0 \quad \text{as} \quad \Lambda_u \rightarrow 0, \quad (4.50)$$

where Λ_u is the scale of the up theory. When $\Lambda_u \rightarrow 0$, the theory becomes classical and the superpotential $\mathcal{W}_L$ collapses to $\Delta\mathcal{W}_d$. When $m \rightarrow \infty$, U become much heavier than Λ_u and decouple from the down theory. Using this limit condition (4.50), symmetry and holomorphy, we can sometimes set $\mathcal{W}_\Delta = 0$.

4.5 Examination

In this section, we will analyze the singular surface in the moduli space of the Coulomb branch by using the “integrating in” method, and examine the Seiberg-Witten curve.

The up theory is $\mathcal{N} = 2$ $SU(N_c)$ SQCD on the singular surface in the Coulomb branch. We now perturb the theory by adding a tree-level superpotential

$$\Delta\mathcal{W} = \sum_{k=2}^{N_c} g_k u_k. \quad (4.51)$$

In the Coulomb branch, $\mathcal{W}_{\text{tree}} = 0$. Thus, the superpotential is $\mathcal{W} = \Delta\mathcal{W}$. The classical vacuum expectation values of Φ are obtained from the classical equation of motion $\mathcal{W}'(\Phi) = 0$. After $SU(N_c)$ rotations the generic vacuum expectation value can be reduced to

$$\Phi_{\text{cl}} = \text{diag}(M, M, M_3, M_4, \dots, M_{N_c}), \quad (4.52)$$

where $M = g_{N_c-1}/g_{N_c}$ [96] and M_i are functions of g_k . In that case, the gauge group $SU(N_c)$ is broken to $SU(2) \times U(1)^{N_c-2}$. Therefore, the down theory is the $\mathcal{N} = 1$ $SU(2)$ Yang-Mills theory. Due to the gaugino condensation of

$\mathcal{N} = 1$ $SU(2)$ Yang-Mills theory, the nonperturbative superpotential of the $\mathcal{N} = 1$ $SU(2)$ Yang-Mills theory is [102][84]

$$\mathcal{W}_d = \pm 2\Lambda_L^3, \quad (4.53)$$

where Λ_L is a dynamical scale of $\mathcal{N} = 1$ $SU(2)$ Yang-Mills theory. It is related to the dynamical scale of the $\mathcal{N} = 2$ $SU(N_c)$ SQCD (up theory) [93][94]:

$$\Lambda_L^6 = \det(M\mathbf{1} + m)g_{N_c}^2\Lambda^{2N_c-N_f}. \quad (4.54)$$

Unfortunately, we cannot set $\mathcal{W}_\Delta = 0$ by holomorphy and symmetry. But We assume below that $\mathcal{W}_\Delta = 0$ [93]. Then

$$\mathcal{W}_L = \Delta\mathcal{W}[\Phi = \Phi_{\text{cl}}(g_k)] + 2g_{N_c}\sqrt{A(M)}, \quad (4.55)$$

where $A(x) = \Lambda^{2N_c-N_f} \det(x\mathbf{1} + m)$.

From Eq. (4.49), the vacuum expectation values of u_k are obtained by differentiating $\mathcal{W}_L$:

$$\langle u_k \rangle = \frac{\partial \mathcal{W}_L}{\partial g_k} \quad (4.56)$$

$$= u_k^{\text{cl}} \pm \delta_{k, N_c-1} \frac{A'(M)}{\sqrt{A(M)}} \pm \delta_{k, N_c} \frac{1}{\sqrt{A(M)}} (2A(M) - MA'(M)) \quad (4.57)$$

When $u_k = \langle u_k \rangle$, the Seiberg-Witten curve should become singular, since $\langle u_k \rangle$ are parameters of up theory, and the up theory is the theory on the singular surface. The Seiberg-Witten curve (4.31) is rewritten as

$$y^2 = f(x) \equiv P^2(x) - 4A(x). \quad (4.58)$$

When $u_k = \langle u_k \rangle$, $P(x) = \det(x - \Phi)$ becomes

$$P(x) = P_{\text{cl}}(x) \mp x \frac{A'(M)}{\sqrt{A(M)}} \mp \frac{1}{\sqrt{A(M)}} (2A(M) - MA'(M)), \quad (4.59)$$

where $P_{\text{cl}}(x) = \det(x - \Phi_{\text{cl}})$, and hence

$$P(M) = \mp 2\sqrt{A(M)}, \quad P'(M) = \mp \frac{A'(M)}{\sqrt{A(M)}}, \quad (4.60)$$

and

$$f(x = M) = f'(x = M) = 0. \quad (4.61)$$

Therefore, since

$$f(x) \propto (x - M)^2, \quad (4.62)$$

the Seiberg-Witten curve is singular when $u_k = \langle u_k \rangle$.

Chapter 5

Brane approach

In recent years, a deeper understanding of nonperturbative aspect of supersymmetric gauge theories in various dimensions has been gained in string theory framework. String theory contains p -dimensional objects, known as p -branes, in addition to fundamental strings (for review, see Ref. [15]). Branes are also solitonic BPS states. Supersymmetric gauge theories in $p+1$ dimensions are realized on the world-volumes of p -branes [16]–[37]. Witten [38] pointed out that intersecting brane configurations of type IIA string theory corresponding to $\mathcal{N} = 2$ SUSY gauge theories in four dimensions can be described by a single M theory five-brane wrapping around a Riemann surface. M theory is defined as the strong coupling physics of Type IIA string theory. Although the Seiberg-Witten curve is no more than an artificial object in field-theoretic framework, the Seiberg-Witten curve is interpreted as the Riemann surface in M theoretic framework. The brane approach has been applied to discuss various SUSY gauge theory [39]–[55].

In this chapter, we will briefly review part of the brane approach.

5.1 Type IIA theory

Type IIA string theory is $9+1$ dimensional theory, and has $(1,1)$ space-time SUSY; the spacetime supercharges generated by left and right moving worldsheet degrees of freedom Q_L and Q_R have opposite chirality:

$$\begin{aligned}\Gamma^0 \cdots \Gamma^9 Q_L &= +Q_L, \\ \Gamma^0 \cdots \Gamma^9 Q_R &= -Q_R,\end{aligned}\tag{5.1}$$

where Γ^μ are Dirac matrices.

The low energy effective theory of type IIA string theory is type IIA

supergravity. The bosonic massless states of type IIA supergravity are

$$\begin{aligned}
\phi & : \text{dilaton (NSNS)} \\
B_{\mu\nu} & : \text{two-form gauge potential (NSNS)} \\
g_{\mu\nu} & : \text{metric (NSNS)} \\
A_\mu & : \text{one-form gauge potential (RR)} \\
A_{\mu\nu\lambda} & : \text{three-form gauge potential (RR)}
\end{aligned}$$

($\mu, \nu, \lambda = 0, 1, \dots, 9$). In the standard Neveu-Schwarz-Ramond quantization [103] of superstring, the fields $g_{\mu\nu}$, $B_{\mu\nu}$ and ϕ originate in the same sector of the string Hilbert space, known as Neveu-Schwarz-Neveu-Schwarz (NSNS) sector, while A_μ and $A_{\mu\nu\lambda}$ are Ramond-Ramond (RR) sector fields. $(p+1)$ -form gauge potential couples electrically to p -branes and magnetically to $(d-4-p)$ -branes, where d is dimension ($d=10$). The NSNS sector field $B_{\mu\nu}$ couples electrically to fundamental string (1-brane) and magnetically to 5-brane, so-called NS5-brane. The RR sector fields A_μ and $A_{\mu\nu\lambda}$ couple to p -branes for even p . The p -branes are called Dp -branes. Fundamental strings can end on D-branes, as shown in Fig. 5.1.

D-brane and NS5-brane are BPS objects. A Dp -brane stretch along $(x^1, \dots, x^p)$ hyperplane preserves supercharges of the form

$$\epsilon_L = \Gamma^0 \Gamma^1 \dots \Gamma^p \epsilon_R. \quad (5.2)$$

A NS5-brane stretch along $(x^1, \dots, x^5)$ hyperplane preserves supercharges of the form

$$\epsilon_L = \Gamma^0 \Gamma^1 \dots \Gamma^5 \epsilon_L \quad \text{and} \quad \epsilon_R = \Gamma^0 \Gamma^1 \dots \Gamma^5 \epsilon_R. \quad (5.3)$$

The low energy world-volume theory on N parallel Dp -branes is $p+1$ dimensional $U(N)$ SUSY Yang-Mills. It describes the dynamics of the ground states of strings both of whose endpoints lie on the brane, as shown by Fig. 5.1(b). The N photons in the Cartan subalgebra of $U(N)$ are corresponding to strings both of whose endpoints lie on the same brane. The charged gauge bosons are corresponding to strings both of whose endpoints lie on the different branes. The masses of the charged gauge bosons are proportional to the distance between the different branes. For generic positions of the N branes, the charged gauge bosons are massive and the gauge symmetry $U(N)$ is broken to $U(1)^N$.

There is another p -dimensional object in the type IIA theory. It is called an orientifold p -plane (Op -plane). It can be thought as the fixed plane under a $\mathbb{Z}_2$ symmetry which acts on the spacetime coordinates and reverses the orientation of the string. The fixed plane of the $\mathbb{Z}_2$ transformation

$$x^\mu(z, \bar{z}) \leftrightarrow -x^\mu(\bar{z}, z) \quad \mu = p+1, \dots, 9 \quad (5.4)$$

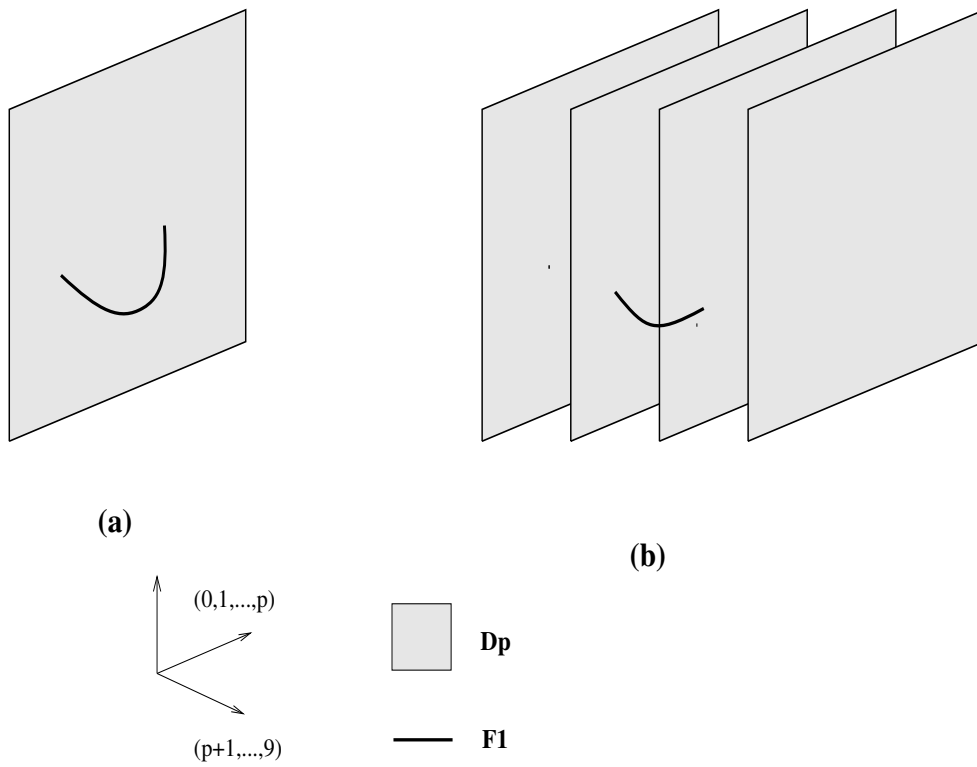


Figure 5.1: The D-Brane. “F1” represents fundamental string.

is an Op -plane extending in the $(x^0, x^1, \dots, x^p)$ directions. Here z parametrizes the string worldsheet. The Op -plane has the RR charge. The RR charge of the Op -plane is

$$Q_{O_p} = \pm 2^{p-4} Q_{D_p}, \quad (5.5)$$

where Q_{D_p} is the RR charges of Dp -brane. The positive orientifold charge $Q_{O_p} = +2^{p-4} Q_{D_p}$ gives rise to a symmetric projection on the ground states of strings stretched between a D-brane and a mirror image of a D-brane. The negative orientifold charge $Q_{O_p} = -2^{p-4} Q_{D_p}$ gives rise to an antisymmetric projection.

5.2 M theory

M theory is defined as strong coupling physics of type IIA string theory. M theory is an eleven dimensional quantum theory which includes gravity. At the low energy regime, M theory is described by eleven dimensional supergravity. The eleven dimensional supergravity includes the bosonic fields G_{MN} , which is the eleven dimensional metric, and A_{MNP} , which is three-form gauge potential ($M, N, P = 0, 1, \dots, 10$). The three-form gauge potential A_{MNP} couples electrically to membrane (M2-brane) and magnetically to five-brane (M5-brane). By dimensional reduction of eleven dimensional supergravity on a circle, we obtain type IIA supergravity, which is the low energy effective theory of the type IIA string theory. The metric G_{MN} gives rise to the metric $g_{\mu\nu}$, the one-form gauge potential $A_\mu = G_{\mu 10}$ and the dilaton $\phi = G_{10 10}$. The three-form gauge potential A_{MNP} gives rise to the three-form gauge potential $A_{\mu\nu\lambda}$ and two-form gauge potential $B_{\mu\nu} = A_{\mu\nu 10}$.

The electric and magnetic charges of A_{MNP} appear as central charges in the eleven dimensional superalgebra:

$$\{Q_\alpha, Q_\beta\} = (\Gamma^M C)_{\alpha\beta} P_M + \frac{1}{2} (\Gamma_{MN} C)_{\alpha\beta} Z^{MN} + \frac{1}{5!} (\Gamma_{MNPQR} C)_{\alpha\beta} Y^{MNPQR}, \quad (5.6)$$

where $\Gamma_{MN\dots}$ are antisymmetrized products of the 32×32 matrices in eleven dimensions, C is the charge conjugation matrix, Z^{MN} is the electric charge corresponding to A_{MNP} , and Y^{MNPQR} is the corresponding magnetic charge.

Type IIA branes have a natural interpretation in M theory.

- The fundamental string is an M2-brane wrapped around x^{10} . It couples electrically to $B_{\mu\nu} = A_{\mu\nu 10}$.
- The D0-brane is Kaluza-Klein (KK) mode of the graviton. It couples electrically to $A_\mu = G_{\mu 10}$.

- The D2-brane is an M2-brane unwrapped around x^{10} . It couples electrically to $A_{\mu\nu\lambda}$.
- The D4-brane is an M5-brane wrapped around x^{10} . It couples magnetically to $A_{\mu\nu\lambda}$.
- The NS5-brane is an M5-brane unwrapped around x^{10} . It couples magnetically to $B_{\mu\nu} = A_{\mu\nu 10}$.
- The D6-brane is a KK monopole. It couples magnetically to $A_\mu = G_{\mu 10}$.

By the dimensional reduction, the eleven dimensional superalgebra becomes

$$\begin{aligned} \{Q_\alpha, Q_\beta\} = & (\Gamma^\mu C)_{\alpha\beta} P_\mu + (\Gamma^{10} C)_{\alpha\beta} P_{10} + (\Gamma^\mu \Gamma^{10} C)_{\alpha\beta} Z_\mu + \frac{1}{2} (\Gamma^{\mu\nu} C)_{\alpha\beta} Z_{\mu\nu} \\ & + \frac{1}{4!} (\Gamma^{\mu\nu\rho\sigma} \Gamma^{10} C)_{\alpha\beta} Y_{\mu\nu\rho\sigma} + \frac{1}{5!} (\Gamma^{\mu\nu\rho\sigma\lambda} C)_{\alpha\beta} Y_{\mu\nu\rho\sigma\lambda}. \end{aligned} \quad (5.7)$$

The momentum in the eleventh direction P_{10} is reinterpreted as D0-brane charge. Z_μ , $Z_{\mu\nu}$, $Y_{\mu\nu\rho\sigma}$ and $Y_{\mu\nu\rho\sigma\lambda}$ are reinterpreted as fundamental string, D2-brane, D4-brane and NS5-brane charge, respectively.

5.3 QCD

In this section, we analyze the moduli space of vacua of $\mathcal{N} = 2$ SQCD by using the brane approach. Let us first consider type IIA string theory. The brane configuration depicted in Fig. 5.2 consists of two parallel NS5-branes with world-volume coordinates $(x^0, x^1, x^2, x^3, x^4, x^5)$, N_c D4-branes suspended between them with world-volume coordinates $(x^0, x^1, x^2, x^3, x^6)$, and N_f D6-branes with world-volume coordinates $(x^0, x^1, x^2, x^3, x^7, x^8, x^9)$. Since the D4-branes are finite in the x^6 direction, the low energy effective theory on the D4-branes is the four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory on its world-volume coordinates (x^0, x^1, x^2, x^3) . The theory has $SU(N_c)$ gauge group and N_f hypermultiplets in the fundamental representation of the gauge group. The string suspended between a D4-brane and a D6-brane gives a hypermultiplet in the fundamental representation of the gauge group. The mass of the hypermultiplet is proportional to the distance between the D4-brane and the D6-brane. The positions of D4-branes on (x^4, x^5) direction correspond to ϕ_i that are the vacuum expectation values of Φ . The gauge coupling constant is defined by

$$\frac{1}{g^2} \sim \frac{L}{g_s L_s}, \quad (5.8)$$

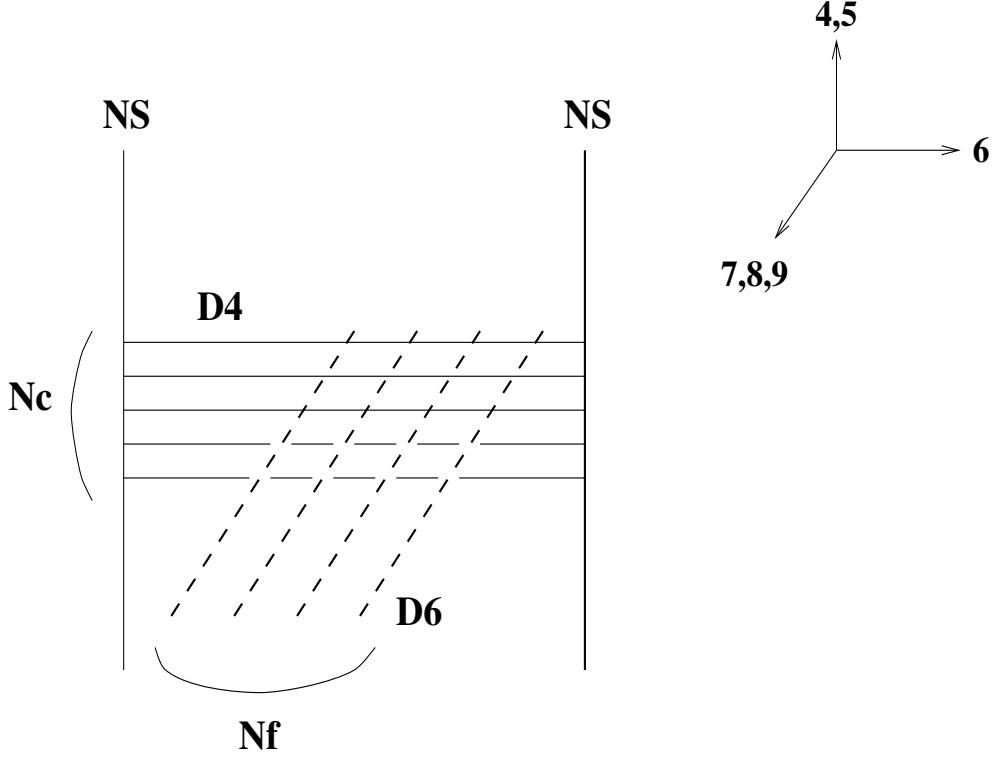


Figure 5.2: The brane configuration for $\mathcal{N} = 2$ $SU(5)$ SQCD with 4 flavors in type IIA picture.

where L is the distance between the two NS5-branes along the x^6 direction, g_s is the string coupling constant and L_s is the fundamental string length.

The tension of the D4-branes bends the two NS5-branes. The position of one of the NS5-branes in the x^6 direction depends on only x^4 and x^6 . For large $v = x^4 + ix^5$,

$$x^6 \sim \sum_{i=1}^{q_L} \log |v - a_i| - \sum_{j=1}^{q_R} \log |v - b_j|, \quad |v| \gg \max\{a_i, b_j\}, \quad (5.9)$$

where q_L and q_R are the numbers of D4-branes which end on the left and right side of the NS5-brane, and a_i and b_j are v values of D4-branes that end on the left and right side of the NS5-brane. For the left NS5-brane, $q_L = 0$, $q_R = N_c$ and $b_i = \phi_i$. For the right NS5-brane, $q_R = 0$, $q_L = N_c$ and $a_i = \phi_i$.

In M theory framework, We introduce eleventh coordinate x^{10} whose ra-

dius is $R = g_s L_s$. The generalization of (5.9) is

$$s = \sum_{i=1}^{q_L} \log(v - a_i) - \sum_{j=1}^{q_R} \log(v - b_j) + \text{constant}, \quad (5.10)$$

where $s = (x^6 + ix^{10})/R$. The fact that s varies holomorphically with v is required by supersymmetry. It is natural to identify the complexified gauge coupling of SUSY gauge theories as follows

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} = \frac{i}{2\pi} (s_L(v) - s_R(v)), \quad (5.11)$$

where $s_L(v)$ and $s_R(v)$ characterize the bending of the left and right NS5-branes.

The two NS5-branes and N_c D4-branes can be reinterpreted in M theory as a single M5-brane. The M5-brane world-volume is $\mathbb{R}^2 \times \Sigma$, where $\mathbb{R}^4$ spans the (x^0, x^1, x^2, x^3) directions, and Σ is a genus $N_c - 1$ Riemann surface, as shown in Fig. 5.3. The Riemann surface Σ is nothing but the Seiberg-Witten curve. The curve Σ is given by an algebraic equation:

$$t^2 - C_{N_c}(v, u_k)t + \Lambda^{2N_c - N_f} \prod_{i=1}^{N_f} (v + m_i) = 0, \quad (5.12)$$

where $t = \exp(-s)$, C_{N_c} is a degree N_c polynomial in v , $C_{N_c} = v^{N_c} + \dots$, with coefficients that depend on the moduli u_k , and m_i ($i = 1, \dots, N_f$) are the quark mass. The curve (5.12) is identified with the Seiberg-Witten curve (4.31) when $C_{N_c}(v, u_k) = \det(v - \Phi)$.

The D6-branes are Kaluza-Klein monopoles described by a Taub-NUT space [104]. One of the complex structure of this Taub-NUT space is the same as one of the complex structure of the ALE space of the A_{n-1} -type:

$$yx = \Lambda^{2N_c - N_f} \prod_{i=1}^{N_f} (v + m_i). \quad (5.13)$$

The D6-branes are located at $x = y = 0$, $v = -m_i$. In this framework, the Riemann surface Σ is defined as a curve in this surface given by

$$y + x = C_{N_c}(v, u_k). \quad (5.14)$$

We indentify y with t .

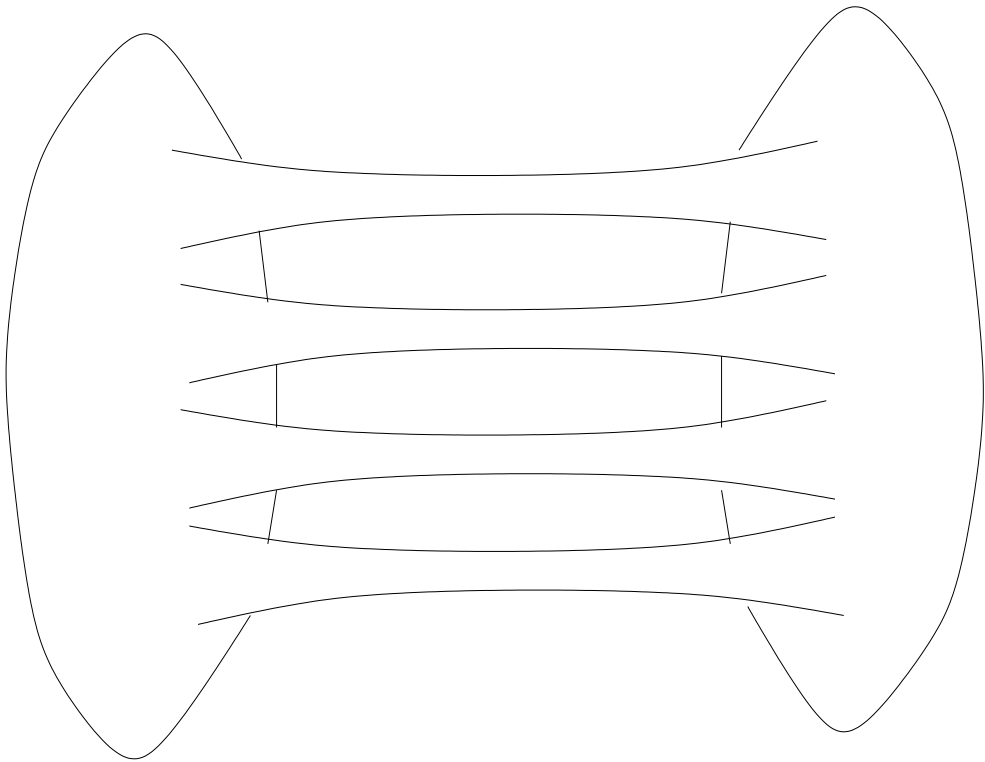


Figure 5.3: The brane configuration for $\mathcal{N} = 2$ $SU(4)$ SQCD in M theory picture.

Chapter 6

$\mathcal{N} = 2$ SYM with an antisymmetric tensor

In previous chapter, we have discussed the brane approach. Using the brane approach, one can understand various aspects of SUSY gauge theories that one cannot understand using field-theoretic approaches. For instance, one cannot understand the Seiberg-Witten curve of $\mathcal{N} = 2$ SUSY $SU(N_c)$ Yang-Mills (SYM) theory with an antisymmetric tensor matter. However, Landsteiner and Lopez [46] have proposed the Seiberg-Witten curve of the Coulomb branch of $\mathcal{N} = 2$ SUSY $SU(N_c)$ Yang-Mills theory with an antisymmetric tensor matter, using the brane approach. The Seiberg-Witten curve is a non-hyperelliptic curve unlike the case of SQCD with fundamental matters.

The purpose of this chapter is to check the non-hyperelliptic curve with the result of field-theoretic approach. We obtain the singularity surface of the $\mathcal{N} = 2$ SUSY $SU(N_c)$ Yang-Mills theory with an antisymmetric tensor matter by using the “integrating in” method. We find that the usual “integrating in” method assuming $\mathcal{W}_\Delta = 0$ gives a singularity surface which disagrees with the proposed M theoretic curve. By assuming that the brane configuration is correct, we find that there exists a nontrivial $\mathcal{W}_\Delta \neq 0$ which gives a singular surface consistent with the M theoretic curve at least up to certain high powers of the dynamical scale Λ of the gauge interactions. Our results can be regarded as an evidence for the necessity of nontrivial $\mathcal{W}_\Delta$.

Result of this chapter is based on my work [85] in collaboration with S. Tomizawa, N. Sakai and T. Sakai.

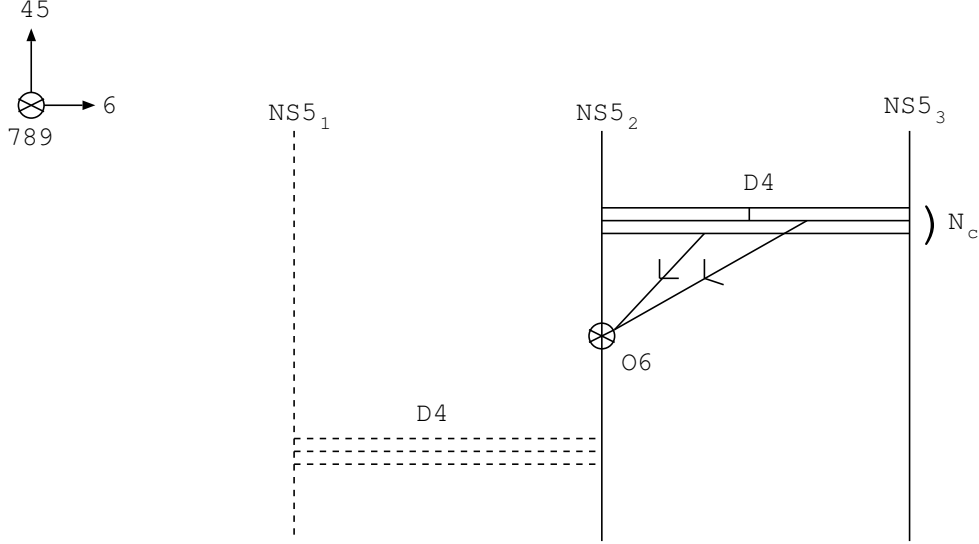


Figure 6.1: The brane configuration for $\mathcal{N} = 2$ $SU(N_c)$ SYM with an antisymmetric tensor in type IIA picture.

6.1 The brane configuration

In this section we briefly review the brane configuration in M theory describing the Coulomb branch of the $\mathcal{N} = 2$ SUSY $SU(N_c)$ Yang-Mills theory with an antisymmetric tensor matter [46]. Let us first examine the brane configuration in the type IIA string picture. Consider type IIA string theory in flat space-time. The brane configuration, as shown in Fig. 6.1, consists of an O6-plane of charge -4 with the world-volume coordinates $(x^0, x^1, x^2, x^3, x^7, x^8, x^9)$, NS5-branes with the world-volume coordinates $(x^0, x^1, x^2, x^3, x^4, x^5)$, and D4-branes with the world-volume coordinates $(x^0, x^1, x^2, x^3, x^6)$. The O6-plane sits at $x^4 = x^5 = x^6 = 0$. This means that the space-time should be identified under the transformation

$$(x^4, x^5, x^6) \rightarrow (-x^4, -x^5, -x^6). \quad (6.1)$$

One NS5-brane is placed on top of the O6-plane and the other NS5-brane is to the right of it. Further there are N_c D4-branes stretching in between the NS5-branes. In the left of the O6-plane we have of course the mirror image of these branes. The D4-branes have a finite extent in the x^6 direction. The four dimensional $\mathcal{N} = 2$ SUSY gauge theory we discuss is defined on the world-volume coordinates (x^0, x^1, x^2, x^3) of the D4-branes. When all N_c D4-branes coincide, the strings connecting N_c D4 branes in the left of the O6-plane give

the $SU(N_c)$ gauge vector multiplets. The strings connecting the left and right D4-branes give a hypermultiplet in the antisymmetric representation of the gauge group because of the presence of the O6-plane.

The brane configuration can be reinterpreted in M theory as a configuration of a single M5-brane embedded in the eleven-dimensional space-time $\mathbb{R}^7 \times S$ where S is the Atiyah-Hitchin space [105][106]. The complex structure of the Atiyah-Hitchin space is represented as

$$xy = \Lambda^{2N_c+4}v^{-4}, \quad (6.2)$$

where $v = x^4 + ix^5$. The $\mathbb{R}^7$ spans $(x^0, x^1, x^2, x^3, x^7, x^8, x^9)$ directions, while S spans (x^4, x^5, x^6) directions in the Type IIA limit and wraps around the circle in the eleventh direction x^{10} whose radius is denoted by R . The M5-brane world-volume becomes $\mathbb{R}^4 \times \Sigma$ where $\mathbb{R}^4$ spans (x^0, x^1, x^2, x^3) directions while Σ is a curve embedded in the Atiyah-Hitchin space S . For large y with x fixed, y tends to $t = \exp(-(x^6 + ix^{10})/R)$, while for large x with y fixed we have $x \sim t^{-1}$. In the M theoretic brane configuration, Λ represents the mass scale corresponding to the dynamical scale of gauge interaction in field theory. The curve Σ is not hyperelliptic, contrary to the case of the $\mathcal{N} = 2$ SUSY QCD where matter hypermultiplets are only in the fundamental representations.

Since there are three NS5-branes involved, the M theoretic curve describing the brane configuration becomes cubic in y . By using symmetry under $x \leftrightarrow y$, $v \leftrightarrow -v$ and other arguments, Landsteiner and Lopez has found the following curve Σ for the above brane configuration in M theory, and proposed it to describe the $\mathcal{N} = 2$ SUSY $SU(N_c)$ Yang-Mills gauge theory with an antisymmetric tensor matter field [46]

$$y^3 + y^2 (p(v) + 3\Lambda^{N_c+2}v^{-2}) + y\Lambda^{N_c+2}v^{-2} (q(v) + 3\Lambda^{N_c+2}v^{-2}) + \Lambda^{3N_c+6}v^{-6} = 0, \quad (6.3)$$

where

$$p(v) = \prod_{i=1}^{N_c} (v - a_i), \quad \text{and} \quad q(v) = p(-v). \quad (6.4)$$

The N_c parameters a_i represent the positions of the D4-branes in the IIA string picture.

We denote by Φ an $\mathcal{N} = 1$ chiral superfield in the adjoint representation in the $SU(N_c)$ gauge group. Together with the $\mathcal{N} = 1$ vector multiplet V in the adjoint representation, it forms an $\mathcal{N} = 2$ vector multiplet. In addition to them, we have an antisymmetric tensor matter A^{ij} and its conjugate $\tilde{A}_{ij}$, where $i, j = 1, 2, \dots, N_c$ are color indices. Both of them are the $\mathcal{N} = 1$ chiral

superfields and form together an $\mathcal{N} = 2$ hypermultiplet. The tree-level superpotential $\mathcal{W}_{\text{tree}}$ contains a tree-level mass parameter m of the antisymmetric tensor matter

$$\mathcal{W}_{\text{tree}} = \sqrt{2} \sum_{\substack{i < j \\ k < l}} \tilde{A}_{ij} (\Phi_k^i \delta_l^j + \delta_k^i \Phi_l^j + m \delta_k^i \delta_l^j) A^{kl}. \quad (6.5)$$

The distance between the average position of the D4-branes on the left and the average position of the D4-branes on the right is equal to the tree-level mass parameter m of the antisymmetric tensor matter

$$m = \frac{2}{N_c} \sum_{i=1}^{N_c} a_i. \quad (6.6)$$

The distance between the position of each D4-brane and the average position of the D4-branes on the left corresponds to the vacuum expectation value of the diagonal element ϕ_i of the adjoint matter Φ

$$a_i = \frac{m}{2} + \langle \phi_i \rangle, \quad (i = 1, \dots, N_c), \quad (6.7)$$

where the vacuum expectation value is denoted by $\langle \rangle$.

Rescaling and shifting $y \rightarrow (y - \Lambda^{N_c+2} v^{-1}) v^{-1}$, the curve (6.3) becomes $f(y, v) = 0$ where

$$\begin{aligned} f(y, v) \equiv & y^3 + y^2 v p(v) + y \Lambda^{N_c+2} (q(v) - 2p(v)) \\ & + \Lambda^{2N_c+4} v^{-1} (p(v) - q(v)). \end{aligned} \quad (6.8)$$

Notice that $q(v) = p(-v)$ and then $v^{-1} (p(v) - q(v))$ has no negative powers in v .

Although we do not know any field-theoretical method to obtain the curve for the case involving the antisymmetric tensor matter field, we can obtain rich information on the singular surface of the curve describing the Coulomb branch of the $\mathcal{N} = 2$ SUSY Yang-Mills gauge theories by the method of “integrating in” [93][84]. We will here compute the singular surface of the proposed curve, where the discriminant vanishes. Since the antisymmetric tensor representation in the $SU(3)$ gauge group is nothing but the antifundamental representation, we shall take the $SU(4)$ gauge group as the simplest nontrivial case. Using computer, in order to obtain the discriminant of the curve for the $SU(4)$ Yang-Mills theory with an antisymmetric matter, we calculate

$$\prod_{i=1} f(y_i, v_i), \quad (6.9)$$

where (y_i, v_i) are solutions of simultaneous equations

$$\frac{\partial f}{\partial y}(y, v) = 0, \quad (6.10)$$

$$\frac{\partial f}{\partial v}(y, v) = 0. \quad (6.11)$$

To perform an explicit calculation, we take the $m = 0$ case. The discriminant¹ is found to be

$$s_3^4 \Delta^2 \Delta_{\text{unphys}}, \quad (6.12)$$

where

$$\begin{aligned} \Delta = & 191102976s_3^2\Lambda^{18} + (-1327104s_4s_2^4 + 5308416s_2^2s_4^2 + 110592s_2^6 \\ & - 8957952s_3^4 - 39813120s_2s_4s_3^2 - 7077888s_4^3 + 9068544s_3^2s_2^3)\Lambda^{12} \\ & + (417792s_4^2s_2^5 - 1146880s_4^3s_2^3 + 139968s_3^6 + 245376s_3^4s_2^3 \\ & + 4096s_2^9 - 488448s_2^4s_4s_3^2 + 59904s_3^2s_2^6 + 2211840s_4^3s_3^2 \\ & + 442368s_4^2s_2^2s_3^2 - 67584s_4s_2^7 + 1179648s_4^4s_2 + 124416s_2s_4s_3^4)\Lambda^6 \\ & - 27648s_4^2s_3^4s_2^2 - 5632s_4^2s_3^2s_2^5 - 73728s_2s_4^4s_3^2 + 128s_3^2s_4s_2^7 \\ & - 256s_4^2s_2^8 - 16s_3^4s_2^6 + 65536s_2^2s_4^5 + 38912s_4^3s_2^3s_3^2 \\ & + 7776s_3^6s_4s_2 - 24576s_2^4s_4^4 - 729s_3^8 + 2016s_3^4s_4s_2^4 \\ & + 13824s_4^3s_3^4 - 65536s_4^6 - 216s_3^6s_2^3 + 4096s_4^3s_2^6, \end{aligned} \quad (6.13)$$

$$\Delta_{\text{unphys}} = (-s_4^3 + 27\Lambda^6s_3^2). \quad (6.14)$$

Here s_i are the moduli parameters (see Eqs. (4.20) and (4.21)). The factor Δ_{unphys} is believed to be unphysical [46]. On the other hand, the factor s_3^4 exhibits a singularity expected for the massless antisymmetric tensor matter field in the classical limit ($\Lambda \rightarrow 0$), as can be seen from the tree-level superpotential (6.5)

$$s_3^2 = \langle \prod_{i>j} (\phi_i + \phi_j) \rangle. \quad (6.15)$$

It is interesting to observe that this singularity is identical to the classical limit ($\Lambda \rightarrow 0$) even though we are not restricted to the weak coupling case.

¹ Note that the antisymmetric representation of $SU(4)$ is equivalent to the defining representation of $SO(6)$, for which the Seiberg-Witten curve has been derived. It turns out that the discriminant of the Seiberg-Witten curve for the $SO(6)$ with the defining representation contains the factor Δ_{massive} which reduces to $s_3^2\Delta$ in Eq. (6.13) for the massless case.

In order to see the singularity associated with the massless gauge fields, we shall take the classical limit ($\Lambda \rightarrow 0$). Then the factor Δ becomes

$$\Delta \rightarrow \langle \prod_{i>j} (\phi_i - \phi_j)^4 \rangle. \quad (6.16)$$

This is nothing but the classical singularity where the non-Abelian gauge symmetry is enhanced. We conclude that $s_3^2 \Delta$ correctly reproduces the singularities in the classical limit.

6.2 The “integrating in” method

In this section, we analyze the singular surface in the moduli space of the Coulomb branch by using the “integrating in” method in the field-theoretic framework. This method enables us to gain information on the singular surface in the Coulomb branch taking into account of nonperturbative quantum effects.

The $\mathcal{N} = 2$ SUSY is broken to $\mathcal{N} = 1$ by adding a perturbation $\Delta\mathcal{W}$ to the tree-level $\mathcal{N} = 2$ superpotential $\mathcal{W}_{\text{tree}}$ (6.5)

$$\Delta\mathcal{W} = \sum_{k=2}^{N_c} \frac{g_k}{k} \text{tr}(\Phi^k). \quad (6.17)$$

The classical vacuum expectation values of Φ are obtained from the classical equations of motion, $\partial(\mathcal{W}_{\text{tree}} + \Delta\mathcal{W})/\partial\Phi = 0$, and similarly for $A^{ij}, \tilde{A}_{ij}$. We are interested in the Coulomb branch, where $A^{ij} = \tilde{A}_{ij} = 0$. After $SU(N_c)$ rotations, the generic vacuum expectation value can be reduced to $\Phi_{\text{cl}} = \text{diag}(M, M, M_3, M_4, \dots, M_{N_c})$, where $M = g_{N_c-1}/g_{N_c}$. In that case, the gauge group $SU(N_c)$ is broken to $SU(2) \times U(1)^{N_c-2}$. The nonperturbative effects due to the gaugino condensation of the $SU(2)$ super Yang-Mills theory provides the additional superpotential

$$\mathcal{W}_d = \pm 2g_{N_c} (\Lambda_{\text{FT}}^{N_c+2} G)^{1/2}. \quad (6.18)$$

where

$$G = \prod_{p=3}^{N_c} (M_p + M + m), \quad (6.19)$$

and Λ_{FT} is the dynamical scale of the $SU(N_c)$ gauge theory with an anti-symmetric matter in field theory. The Λ_{FT} must be proportional to Λ of the M theory brane configuration in the previous section:

$$\Lambda_{\text{FT}} = c\Lambda \quad c \in \mathbb{C}, \quad (6.20)$$

where c is a renormalization-scheme-dependent constant. Following Elitzur *et al.* [93], we obtain the low-energy effective superpotential for the $\mathcal{N} = 1$ super Yang-Mills theory

$$\mathcal{W}_L = \Delta\mathcal{W}(\Phi = \Phi_{\text{cl}}(g_k)) + \mathcal{W}_d + \mathcal{W}_\Delta, \quad (6.21)$$

where $\mathcal{W}_\Delta$ is a possible additional superpotential constrained only by holomorphy and symmetry [84].

The vacuum expectation value of gauge invariants can be defined as

$$\langle u_k \rangle = \left\langle \frac{1}{k} \text{tr}(\Phi^k) \right\rangle, \quad (6.22)$$

which are obtained by differentiating $\mathcal{W}_L$,

$$\langle u_k \rangle = \frac{\partial \mathcal{W}_L}{\partial g_k}. \quad (6.23)$$

The Seiberg-Witten curve must be singular when $u_k = \langle u_k \rangle$. The vacuum expectation values are related to the moduli parameters s_i in Eq. (4.20) through the Newton formula (4.22).

As a simplest explicit example, we consider the $SU(4)$ case. Then, the vacuum expectation values and the gaugino condensation are given in terms of coupling parameters in the superpotential as

$$M = z_3 \quad (6.24)$$

$$M_3 = -z_3 + \sqrt{-z_3^2 - z_2} \quad (6.25)$$

$$M_4 = -z_3 - \sqrt{-z_3^2 - z_2} \quad (6.26)$$

$$G = m^2 + z_3^2 + z_2, \quad (6.27)$$

where z_3 and z_2 are complex parameters

$$z_3 \equiv \frac{g_3}{g_4}, \quad z_2 \equiv \frac{g_2}{g_4}. \quad (6.28)$$

In the remainder of this chapter, we will explore the singular surface by using the “integrating in” method, in order to compare it with curve obtained from the M5-brane. First, we assume $\mathcal{W}_\Delta = 0$ in this section. Then we find the vacuum expectation value including quantum effects using Eq. (6.23) as

$$\begin{aligned} \langle u_2 \rangle &= z_3^2 - z_2 \pm \Lambda_{\text{FT}}^3 G^{-1/2} \\ \langle u_3 \rangle &= 2z_3^3 + 2z_3 z_2 \pm 2z_3 \Lambda_{\text{FT}}^3 G^{-1/2} \\ \langle u_4 \rangle &= -\frac{3}{2} z_3^4 - 2z_3^2 z_2 + \frac{1}{2} z_2^2 \pm (2m^2 + z_2) \Lambda_{\text{FT}}^3 G^{-1/2}. \end{aligned} \quad (6.29)$$

These relations define a codimension one surface in the moduli space. It should correspond to the singular surface of the proposed curve (6.9) for $SU(4)$ with an antisymmetric tensor matter, namely the vanishing discriminant of the curve. We will find, however, the discriminant of the curve (6.9) does not vanish on $u_k = \langle u_k \rangle$ in Eq. (6.29) for the case $m = 0$.

We shall now test if the discriminant vanishes for any values of coupling parameters z_3 and z_2 by choosing an appropriate value for the renormalization-scheme-dependent factor c in Eq. (6.20). We find that the factor Δ in the discriminant (6.12), for instance, becomes on the codimension one surface (6.29) for $m = 0$

$$\begin{aligned}
& \Delta(u_k = \langle u_k \rangle) \\
&= -1024(c^6 - 4)(z_3^2 + z_2)^3(5z_3^2 + z_2)^6\Lambda^6 \\
&\quad + ((811c^6 - 3744)z_3^6 + (117c^6 - 768)z_3^4z_2 \\
&\quad + (-327c^6 + 1248)z_3^2z_2^2 + (47c^6 - 192)z_2^3) \\
&\quad \times 512(z_3^2 + z_2)^{3/2}(5z_3^2 + z_2)^3c^3\Lambda^9 \\
&\quad + \mathcal{O}(\Lambda^{12}) \\
&\neq 0.
\end{aligned} \tag{6.30}$$

To be more precise, there is no complex number c satisfying $\Delta(u_k = \langle u_k \rangle) \equiv 0$ for any values of z_3 and z_2 . Therefore the discriminant of the curve (6.9) does not vanish on the codimension one surface (6.29) obtained by assuming $\mathcal{W}_\Delta = 0$ in the “integrating in” method. Although we have no rigorous means to test the curve (6.3) obtained in M theory for general N_c , we are confident that the curve is correct at least for $N_c = 4$, since we have checked that the discriminant agrees with that of $SO(6)$ with the defining representation. Therefore we conclude that the assumption $\mathcal{W}_\Delta = 0$ in the case of the $SU(4)$ theory with an antisymmetric tensor matter leads us to inconsistent results and that the assumption $\mathcal{W}_\Delta = 0$ is not correct.

6.3 Non-zero $\mathcal{W}_\Delta$

In the previous section, we found that the M theory curve (6.9) is inconsistent with the codimension one surface obtained as a candidate for the singular surface in the moduli space assuming $\mathcal{W}_\Delta = 0$ in the “integrating in” method. In this section, we discuss the possibility of non-zero $\mathcal{W}_\Delta$ instead.

We first note that, in the classical limit ($\Lambda \rightarrow 0$), the discriminant of the M theory curve (6.9) vanishes on the codimension one surface (6.29) obtained by assuming $\mathcal{W}_\Delta = 0$ in the “integrating in” method. Eq. (6.30) shows that

the discriminant of the curve vanishes on the codimension one surface in the leading order of Λ , *i.e.*, up to order Λ^6 , provided $c^6 = 4$,

$$\Lambda_{\text{FT}}^6 = 4\Lambda^6, \quad \mathcal{W}_d = \pm 4g_4\Lambda^3 G^{1/2}. \quad (6.31)$$

Now we wish to explore to higher orders of Λ whether we can find a non-trivial $\mathcal{W}_\Delta$ which provides the singular surface consistent with the vanishing discriminant of the curve. Since the right-hand-side of Eq. (6.30) consists of terms with integer powers of Λ^3 , we need to introduce terms with Λ^{3n} $n \in \mathbb{N}$ only

$$\mathcal{W}_\Delta = \sum_{k=2}^{\infty} C_k (\Lambda^3)^k. \quad (6.32)$$

Since the Λ^3 term is given by $\mathcal{W}_d$, we assume $k \geq 2$.

From the additional superpotential $\mathcal{W}_\Delta$ condition (4.50), $\mathcal{W}_\Delta$ must satisfy the following conditions [84]

$$\begin{aligned} \mathcal{W}_\Delta \rightarrow 0 & \quad \text{as } g_2 \rightarrow \infty \quad \text{with } g_4\Lambda^3 G^{1/2} \text{ fixed,} \\ \mathcal{W}_\Delta \rightarrow 0 & \quad \text{as } \Lambda \rightarrow 0, \end{aligned} \quad (6.33)$$

and carry charge $(2, 2)$ under $U(1)_R \times U(1)_J$.

We list below the charge and mass dimension of the parameters.

	$U(1)_R$	$U(1)_J$	Dimension	
g_2	-2	2	1	
g_3	-4	2	0	
g_4	-6	2	-1	
z_3	2	0	1	(6.34)
z_2	4	0	2	
Λ	2	0	1	
G	4	0	2	
m	2	0	1	

From this table, we see that $\mathcal{W}_\Delta$ can be represented as

$$\mathcal{W}_\Delta = g_4 \sum_{k=2}^{\infty} f_k(z_3, z_2) \Lambda^{3k}, \quad (6.35)$$

where $f_k(z_3, z_2)$ is any function which carries the charge $(4 - 6k, 0)$. Note that we are discussing $SU(4)$ Yang-Mills theory with a massless ($m = 0$) antisymmetric tensor.

Now let us determine the lowest term of $\mathcal{W}_\Delta$ in order to make the singular surface consistent with the discriminant of the M theory curve. If $\mathcal{W}_\Delta = \mathcal{W}_2 \equiv g_4 f_2(z_3, z_2) \Lambda^6$, then

$$\begin{aligned}\langle u_2 \rangle &= z_3^2 - z_2 \pm 2\Lambda^3 G^{-1/2} + \frac{1}{g_4} \frac{\partial \mathcal{W}_2}{\partial z_2} \\ \langle u_3 \rangle &= 2z_3^3 + 2z_3 z_2 \pm 4z_3 \Lambda^3 G^{-1/2} + \frac{1}{g_4} \frac{\partial \mathcal{W}_2}{\partial z_3} \\ \langle u_4 \rangle &= -\frac{3}{2} z_3^4 - 2z_3^2 z_2 + \frac{1}{2} z_2^2 \pm 2z_2 \Lambda^3 G^{-1/2} \\ &\quad - \frac{z_2}{g_4} \frac{\partial \mathcal{W}_2}{\partial z_2} - \frac{z_3}{g_4} \frac{\partial \mathcal{W}_2}{\partial z_3} + \frac{\mathcal{W}_2}{g_4}.\end{aligned}\tag{6.36}$$

From this, we find

$$\begin{aligned}\Delta(u_k = \langle u_k \rangle) &= \mp 2048 (z_3^2 + z_2)^{3/2} (5z_3^2 + z_2)^6 (f_2(z_3, z_2) (z_3^2 + z_2) - 2) \Lambda^9 \\ &\quad + \mathcal{O}(\Lambda^{12}).\end{aligned}\tag{6.37}$$

Thus, in order for Δ to vanish up to Λ^9 , $\mathcal{W}_2$ must take the form

$$\begin{aligned}\mathcal{W}_2 &= 2g_4 (z_3^2 + z_2)^{-1} \Lambda^6 \\ &= 2g_4 G^{-1} \Lambda^6.\end{aligned}\tag{6.38}$$

This $\mathcal{W}_2$ (6.38) satisfies the conditions (6.33) and carries the charge (2, 2).

We can determine the next term of $\mathcal{W}_\Delta$ in a similar way and find

$$\mathcal{W}_d + \mathcal{W}_\Delta = \pm 4g_4 \Lambda^3 G^{1/2} + 2g_4 \Lambda^6 G^{-1} \mp 2g_4 \Lambda^9 G^{-5/2} + \mathcal{O}(\Lambda^{12}).\tag{6.39}$$

Inspired by the result (6.39), we restrict the form of $\mathcal{W}_\Delta$ in the following way

$$\mathcal{W}_d + \mathcal{W}_\Delta = g_4 \sum_{k=1}^{\infty} h_k G^2 (\Lambda^3 G^{-3/2})^k, \quad h_k \in \mathbb{C}.\tag{6.40}$$

By requiring for Δ to vanish, we work out h_k up to h_8

$$\begin{aligned}\mathcal{W}_d + \mathcal{W}_\Delta &= \pm 4g_4 \Lambda^3 G^{1/2} + 2g_4 \Lambda^6 G^{-1} \mp 2g_4 \Lambda^9 G^{-5/2} + 4g_4 \Lambda^{12} G^{-4} \\ &\quad \mp \frac{21}{2} g_4 \Lambda^{15} G^{-11/2} + 32g_4 \Lambda^{18} G^{-7} \mp \frac{429}{4} g_4 \Lambda^{21} G^{-17/2} \\ &\quad + 384g_4 \Lambda^{24} G^{-10} + \mathcal{O}(\Lambda^{27}),\end{aligned}\tag{6.41}$$

and we find that the discriminant vanishes up to the order Λ^{27} :

$$\Delta(u_k = \langle u_k \rangle) = \mathcal{O}(\Lambda^{30}). \quad (6.42)$$

Although we have only determined $\mathcal{W}_\Delta$ up to this order due to the increasing complexity of computation, we believe that the higher powers of Λ can be worked out with more efforts and that the form (6.40) will come out.

6.4 Discussion

In this chapter, we studied the singular surface of the moduli space of the $\mathcal{N} = 2$ SUSY $SU(N_c)$ gauge theory with an antisymmetric tensor matter from two points of view. One is to use a configuration of a single M5-brane and the other based on the “integrating in” method. It was discussed that the consistency between the two results requires $\mathcal{W}_\Delta \neq 0$, and the explicit form of it was worked out for $N_c = 4$. It is interesting that $\mathcal{W}_\Delta$ consists of terms with integer powers of $\pm\Lambda^3$ corresponding to two vacua of $SU(2)$ gaugino condensation. Using the dynamical scale $\Lambda_{SU(2)}$ of the unbroken $SU(2)$ SUSY Yang-Mills theory instead of Λ , the nonperturbative superpotential (6.40) is rewritten as

$$\mathcal{W}_d + \mathcal{W}_\Delta = g_4 \sum_{k=1} h_k G^2 \left(\pm \frac{\Lambda_{SU(2)}^3}{2g_4 G^2} \right)^k, \quad (6.43)$$

where the scale matching condition $\Lambda_{SU(2)}^3 = 2g_4 G^{1/2} \Lambda^3$ for $N_c = 4$ is used. This fact makes us suspect that the physical origin of $\mathcal{W}_\Delta$ might be understood as the gaugino condensation of $SU(2)$ SUSY Yang-Mills theory.

In order to break $\mathcal{N} = 2$ SUSY to $\mathcal{N} = 1$ SUSY, we added a perturbation (6.17) to the tree-level $\mathcal{N} = 2$ superpotential $\mathcal{W}_{\text{tree}}$ in the “integrating in” methods. Instead of this perturbation, we can consider another perturbation

$$\begin{aligned} \Delta\mathcal{W} = & \frac{g_2}{2} \text{tr}(\Phi^2) + \frac{g_3}{3} \text{tr}(\Phi^3) \\ & + g_4 \left(\frac{1}{4} \text{tr}(\Phi^4) - \alpha \left(\frac{1}{2} \text{tr}(\Phi^2) \right)^2 \right), \quad \alpha \in \mathbb{C}, \end{aligned} \quad (6.44)$$

to the tree-level $\mathcal{N} = 2$ superpotential $\mathcal{W}_{\text{tree}}$. It turns out that for $\alpha \neq 1/2$ there exist classical vacua where the gauge group is broken to $SU(2) \times U(1)^2$ [98]. One can then calculate $\mathcal{W}_\Delta$ for generic α by requiring that $\Delta = 0$ in the $u_k = \langle u_k \rangle$ surface. One of the most interesting results is that $\mathcal{W}_\Delta$ vanishes for $\alpha = 1/4$ [107] while $\mathcal{W}_\Delta \neq 0$ for any other value of α . This is presumably related to the fact that the antisymmetric representation of

$SU(4)$ is equivalent to the defining representation of $SO(6)$, for which the assumption $\mathcal{W}_\Delta = 0$ is found to be valid.

It is interesting to note that the “integrating in” methods have been applied under the assumption $\mathcal{W}_\Delta = 0$ [93]-[98], which provides consistent results in the Seiberg-Witten curves that are hyperelliptic. It has been known that the non-hyperelliptic Seiberg-Witten curves for the exceptional group cases are derived using the assumption $\mathcal{W}_\Delta = 0$ [95][98]. In contrast to these results, we have found that the assumption $\mathcal{W}_\Delta = 0$ is inconsistent in the case of $\mathcal{N} = 2$ SUSY $SU(N_c)$ gauge theory with an antisymmetric tensor matter, whose Seiberg-Witten curve is not of hyperelliptic type [46]. As another example of nontrivial $\mathcal{W}_\Delta$, we have studied also $\mathcal{N} = 2$ $SU(4)$ theory with a symmetric tensor whose Seiberg-Witten curve is proposed in Ref. [46]. In this case, we find that there are no complex number α in Eq. (6.44) to make $\mathcal{W}_\Delta$ vanish. On the other hand, we also find that for fundamental matters, $\mathcal{W}_\Delta$ vanishes for any value of α ($\alpha \neq 1/2$).

Higher values of N_c may be dealt with by a similar method with more computational efforts. We expect that $\mathcal{W}_\Delta \neq 0$ for $SU(N_c)$ with antisymmetric or symmetric representation for higher values of N_c also.

Chapter 7

Domain wall

So far, we have discussed the $(0,0)$ central charges and particle-like BPS states in $\mathcal{N} = 2$ SUSY field theories and the brane approach which is concerned with $\mathcal{N} = 2$ SUSY field theories. These central charges transform under the $(0,0)$ representation of Lorentz group. These central charges cannot exist in the $\mathcal{N} = 1$ SUSY theories. However, there are other types of central charges in $\mathcal{N} = 1$ SUSY theories. They are called $(1,0)$ central charges and $(1/2, 1/2)$ central charges, which transform under the $(1,0)$ and $(1/2, 1/2)$ representation of Lorentz group, respectively. The BPS states which are related to the $(1,0)$ and $(1/2, 1/2)$ central charges are not particle-like objects. The BPS objects are domain walls, BPS string and domain wall junctions.

- The typical BPS objects which are related to only the $(1,0)$ central charges are domain walls.
- The typical BPS objects which are related to only the $(1/2, 1/2)$ central charges are BPS string.
- The typical BPS objects which are related to both of the central charges are domain wall junctions.

In this chapter and the subsequent chapters, we will discuss these BPS objects. In this chapter, we discuss the BPS domain walls in various theories.

A domain wall is defined as an object which interpolates between two degenerate discrete minima of a potential and spread over two dimensions. For example, the potential of the scalar $T(x)$, illustrated in Fig. 7.1, have two minima at $T = T_-$ and T_+ . From $x = -\infty$ to $x = +\infty$, the field ϕ transits from T_- to T_+ , as shown in Fig. 7.2. Domain walls appear in many areas of physics. In supersymmetric unified models, domain walls can be formed during thermal evolution of our universe and often provide significant and

interesting constraints on model building. On the other hand, it has been found that domain walls in supersymmetric theories [57]–[64] can saturate the Bogomol’nyi bound in recent years.

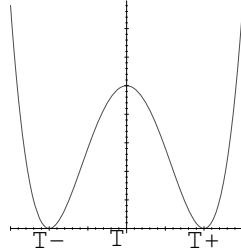


Figure 7.1: The potential of the scalar $T(x)$.

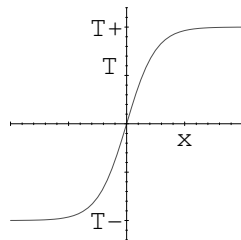


Figure 7.2: The domain wall solution.

7.1 Wess-Zumino models

In this section, we introduce the simplest model which has domain wall solutions. We start a non-gauge theory — the Wess-Zumino model. The La-

grangian is

$$\mathcal{L} = \frac{1}{4} \int d^4\theta \Phi \bar{\Phi} + \left\{ \frac{1}{2} \int d^2\theta \mathcal{W}(\Phi) + \text{h.c.} \right\}, \quad (7.1)$$

$$\mathcal{W}(\Phi) = \Lambda^2 \Phi - \Lambda^{2-n} \frac{\Phi^{n+1}}{n+1}, \quad (7.2)$$

where n is an integer ≥ 2 and Λ is a positive real scale. The vacuum expectation value of Φ is obtained from the equations of motions, $\partial \mathcal{W}(\Phi)/\partial \Phi = 0$. The model has n vacua.

$$\langle \Phi \rangle = \Phi_k \equiv \Lambda e^{2\pi i k/n}, \quad k = 0, 1, \dots, n-1. \quad (7.3)$$

For simplicity, we set $n = 2$. We will discuss the domain wall which interpolates between $\Phi = -\Lambda$ and Λ . We will assume that the BPS solution depends on only x^1 and the boundary condition is

$$\Phi = \Lambda, \quad \mathcal{W} = \frac{2}{3}\Lambda^3, \quad x^1 = \infty, \quad (7.4)$$

$$\Phi = -\Lambda, \quad \mathcal{W} = -\frac{2}{3}\Lambda^3, \quad x^1 = -\infty. \quad (7.5)$$

We compute the $(1, 0)$ central charges Z_k , using canonical commutators of fields at tree-level:

$$Z_k = 2 \int d^3x \partial_k \mathcal{W}^*(A^*), \quad (7.6)$$

$$\langle Z_k \rangle = \left(\frac{8}{3} \Lambda^3 \int dx^2 dx^3, 0, 0 \right). \quad (7.7)$$

The $(1/2, 1/2)$ central charges $\langle Y_k \rangle$ vanish. In this case, $H_{\text{I}} = H_{\text{II}} = \langle Z_1 \rangle$ and

$$\langle H \rangle \geq \langle Z_1 \rangle. \quad (7.8)$$

This is the Bogomol'nyi bound. The conserved supercharges are Eq. (2.24) and Eq. (2.27). Therefore, the BPS domain wall preserves 1/2 supersymmetry. The tension of the BPS domain wall is

$$T = \frac{\text{Total Energy}}{\text{Area}} = 2|\Delta \mathcal{W}|. \quad (7.9)$$

The BPS equations (2.30) become

$$\frac{d\Phi}{dx^1} = (\Lambda^2 - \Phi^2). \quad (7.10)$$

Thus, the BPS domain wall solution is

$$\Phi(x^1) = \Lambda \tanh(\Lambda x^1). \quad (7.11)$$

For $n > 2$ case, no exact analytic solution of the BPS domain wall solution is known.

7.2 Supersymmetric gluodynamics

Secondly, we will briefly review the domain wall of the pure $SU(N_c)$ Yang-Mills theory. The Lagrangian is

$$\mathcal{L} = \frac{1}{4g^2} \int d\theta^2 \text{tr} W^2 + \text{h.c.}, \quad (7.12)$$

$$= \frac{1}{g^2} \text{tr} \left[-\frac{1}{4} G^{\mu\nu} G_{\mu\nu} - i\lambda \sigma^\mu \partial_\mu \bar{\lambda} \right]. \quad (7.13)$$

The Lagrangian is invariant under $U(1)$ which is broken by the anomaly down to $\mathbb{Z}_{2N_c}$. The $\mathbb{Z}_{2N_c}$ chiral symmetry which is spontaneously broken down to the $\mathbb{Z}_2$ by the expectation value of the gaugino bilinear $\langle \text{tr}(\lambda\lambda) \rangle$. Thus, the $\mathcal{N} = 1$ SUSY $SU(N_c)$ Yang-Mills theory has N_c degenerate discrete vacua, each characterized by a different value of the chiral gaugino condensate [108] [109]

$$\langle \text{tr}(\lambda\lambda) \rangle = 16\pi^2 \Lambda^3 e^{2\pi i k / N_c}, \quad k = 0, \dots, N_c - 1. \quad (7.14)$$

The number of vacua labeled by the gluino condensate (7.14) is N_c . This is precisely the value of Witten's index [110] for the $\mathcal{N} = 1$ SUSY $SU(N_c)$ gluodynamics. Thus, the chiral asymmetric vacua saturate Witten's index. (On the other hand, Kovner and Shifman [111] argued that the theory admits also a chiral symmetric vacuum: $\langle \text{tr}(\lambda\lambda) \rangle = 0$ in which no mass gap develops. In this vacuum, the theory is presumably in the conformal regime; the energy density becomes positive in finite volume, so that the chiral symmetric vacuum does not contribute to Witten's index. It plays a role, however, in the domain wall dynamics.)

We will discuss the domain wall interpolating between distinct vacua at infinities. Since the theory is strongly coupled we have no idea how to describe the internal structure of the domain wall. But, the central charge is calculable. Unlike the Wess-Zumino models, the central charge Z_k must vanish at tree-level. However, at the loop level quantum anomaly modifies the central charges Z_k . In a generic SUSY gauge theory with a superpotential $\mathcal{W}$ and a set of the matter fields in the representations R_i of the gauge group, the complex conjugates of Z_k are [58]–[60]

$$Z_k^* = \frac{2}{3} \int d^3x \partial_k \left\{ 3\mathcal{W} - \sum_i Q_i \frac{\partial \mathcal{W}}{\partial Q_i} - \left[\frac{3T(G) - \sum_i T(R_i)}{16\pi^2} \text{tr} W^2 + \frac{1}{8} \sum_i \gamma_i \bar{D}^2 (\bar{Q}_i^\dagger e^V Q_i) \right] \right\}_{\theta=0}, \quad (7.15)$$

where $T(R)$ is the Dynkin index for representation R , $T(G)$ is the Dynkin index in the adjoint representation ($T(SU(N_c)) = N_c$), and γ_i are the anoma-

lous dimensions of the matter fields Q_i . We can drop total superderivative term in Eq. (7.15), since they do not contribute to the central charges. Eq. (7.15) become

$$Z_k^* = 2 \int d^3x \partial_k \left\{ \mathcal{W} - \frac{T(G) - \sum_i T(R_i)}{16\pi^2} \text{tr} W^2 \right\}_{\theta=0}. \quad (7.16)$$

For the pure $SU(N_c)$ Yang-Mills theory case, $\mathcal{W} = 0$, $T(G) = N_c$, $T(R_i) = 0$ and $W|_{\theta=0} = \lambda$. Therefore, the complex conjugates of Z_k are

$$Z_k^* = -\frac{N_c}{8\pi^2} \int d^3x \partial_k \text{tr}(\lambda\lambda). \quad (7.17)$$

Consequently, the domain wall tension T is

$$T = \frac{N_c}{8\pi^2} |\Delta \text{tr}(\lambda\lambda)|. \quad (7.18)$$

7.3 $U(1) \times U(1)'$ model

In this section, we will introduce the model which has the exact explicit analytic domain wall solution [64]. The model is a simplified toy model simulating the $\mathcal{N} = 2$ SUSY $SU(2)$ Yang-Mills theory (the Seiberg-Witten [2] theory) which is explicitly broken to $\mathcal{N} = 1$ by giving a mass term to the adjoint chiral superfield (see Section 4.3). The moduli space of the $\mathcal{N} = 2$ SUSY $SU(2)$ Yang-Mills theory has two singular points where monopole or dyon becomes massless respectively (see Section 3.7). In order to discuss the model in a simpler setting, Kaplunovsky *et al.* [64] have proposed the model which can be treated as a local field theory. They introduced two pairs of chiral superfields $\mathcal{M}$, $\tilde{\mathcal{M}}$ and $\mathcal{D}$, $\tilde{\mathcal{D}}$ simulating the monopole, anti-monopole and the dyon, anti-dyon of the Seiberg-Witten theory respectively. Instead of the modulus u of the Seiberg-Witten theory, they introduced a linearized analogue T as a neutral chiral superfield. The gauge group was chosen as $U(1) \times U(1)'$ simulating electric and magnetic gauge group and the quantum number of these chiral superfields are given by

	$\mathcal{M}$	$\tilde{\mathcal{M}}$	$\mathcal{D}$	$\tilde{\mathcal{D}}$	T
$U(1)$	0	0	1	-1	0
$U(1)'$	1	-1	1	-1	0

(7.19)

To mimic a massless monopole at $T = \Lambda$ and a massless dyon at $T = -\Lambda$, they consider a superpotential

$$\mathcal{W} = (T - \Lambda)\mathcal{M}\tilde{\mathcal{M}} + (T + \Lambda)\mathcal{D}\tilde{\mathcal{D}} - h^2T, \quad (7.20)$$

where the coupling parameter h^2 replaces the effect of the mass for the adjoint chiral superfield. Since the action is invariant under the three global $U(1)$ transformations

$$\mathcal{M} \rightarrow e^{i\delta_1} \mathcal{M}, \quad \tilde{\mathcal{M}} \rightarrow e^{-i\delta_1} \tilde{\mathcal{M}}, \quad \mathcal{D} \rightarrow e^{i\delta_2} \mathcal{D}, \quad \tilde{\mathcal{D}} \rightarrow e^{-i\delta_2} \tilde{\mathcal{D}}, \quad (7.21)$$

we can choose the vacuum configuration to be

$$\begin{aligned} \text{Vac.1 : } T &= \Lambda, \quad \mathcal{M} = \tilde{\mathcal{M}} = h, \quad \mathcal{D} = \tilde{\mathcal{D}} = 0, \\ \text{Vac.2 : } T &= -\Lambda, \quad \mathcal{D} = \tilde{\mathcal{D}} = h, \quad \mathcal{M} = \tilde{\mathcal{M}} = 0. \end{aligned} \quad (7.22)$$

We will consider a field configuration which is static and depends on only x_1 . In this section, we write x instead of x_1 .

The BPS state obeys the BPS equations. The BPS equations (2.31) and (2.30) become

$$K_{ij*} \frac{dA^i}{dx} = \Omega \frac{\partial \mathcal{W}^*}{\partial A^{*j}}, \quad \Omega = \frac{\langle Z_1^* \rangle}{|\langle iZ_1^* \rangle|}, \quad (7.23)$$

$$D = v_{\mu\nu} = 0. \quad (7.24)$$

For simplify, we assume a canonical Kähler metric $K_{ij*} = \delta_{ij}$, and the domain wall solution asymptotes to the first vacuum of (7.22) for $x \rightarrow -\infty$ and to the second vacuum for $x \rightarrow +\infty$. The BPS eqs. (7.23) become for the model

$$\begin{aligned} \frac{d\mathcal{M}}{dx} &= \Omega \tilde{\mathcal{M}}^* (T - \Lambda)^*, \\ \frac{d\mathcal{D}}{dx} &= \Omega \tilde{\mathcal{D}}^* (T + \Lambda)^*, \\ \frac{dT}{dx} &= \Omega \left(\mathcal{M} \tilde{\mathcal{M}} + \mathcal{D} \tilde{\mathcal{D}} - h^2 \right)^*, \\ |\mathcal{M}| &= |\tilde{\mathcal{M}}|, \quad |\mathcal{D}| = |\tilde{\mathcal{D}}|, \end{aligned} \quad (7.25)$$

where

$$\Omega = \frac{(h\Lambda^2)}{|h\Lambda^2|}, \quad (7.26)$$

since $\langle Z_1^* \rangle = 4h\Lambda^2$.

Eqs. (7.25) are invariant under the following global phase changes of parameters and fields

$$\begin{aligned} h &\rightarrow e^{i\beta} h, \quad \mathcal{M} \rightarrow e^{i\beta} \mathcal{M}, \quad \tilde{\mathcal{M}} \rightarrow e^{i\beta} \tilde{\mathcal{M}}, \quad \mathcal{D} \rightarrow e^{i\beta} \mathcal{D}, \quad \tilde{\mathcal{D}} \rightarrow e^{i\beta} \tilde{\mathcal{D}}, \\ \Lambda &\rightarrow e^{i\gamma} \Lambda, \quad T \rightarrow e^{i\gamma} T. \end{aligned} \quad (7.27)$$

The right hand sides of Eqs. (7.22) changes only the same overall phase $e^{-2i\beta-i\gamma}$. Hence, arbitrary complex parameters h and Λ can be obtained

from real-positive h and Λ by these phase changes. Therefore we shall take h and Λ to be real-positive in the following without loss of generality. Inspired by the condition (7.25), we wish to find a solution assuming

$$\mathcal{M}(x) = \tilde{\mathcal{M}}(x), \quad \mathcal{D}(x) = \tilde{\mathcal{D}}(x). \quad (7.28)$$

and that all of them are real-positive. The BPS eqs. become

$$\begin{aligned} \frac{d}{dx} \ln(\mathcal{M}\mathcal{D}) &= 2T, \\ \frac{d}{dx} \ln\left(\frac{\mathcal{M}}{\mathcal{D}}\right) &= 2\Lambda, \\ \frac{d}{dx} T &= (\mathcal{M}^2 + \mathcal{D}^2 - h^2). \end{aligned} \quad (7.29)$$

The first and second equations of Eqs. (7.29) can be rewritten as

$$\mathcal{M}(x) = h \exp\left(-\frac{1}{2}\xi(x) - \Lambda x\right), \quad (7.30)$$

$$\mathcal{D}(x) = h \exp\left(-\frac{1}{2}\xi(x) + \Lambda x\right), \quad (7.31)$$

$$T(x) = -\frac{1}{2} \frac{d\xi}{dx}. \quad (7.32)$$

Without loss of generality, we have placed the origin of the x coordinate in the center of the wall. In term of the $\xi(x)$, Eq. (7.29) for the T field becomes

$$\frac{d^2\xi}{dx^2} = -2 \frac{dT}{dx} = 2(h^2 - \mathcal{M}^2 - \mathcal{D}^2) \quad (7.33)$$

$$= 2h^2(1 - 2 \cosh(2\Lambda x) \exp(-\xi)). \quad (7.34)$$

It is clear that there is a unique solution $\xi(x_3)$ for any $h, \Lambda > 0$. However, for the special case of $h^2 = 2\Lambda^2$, Eq. (7.34) can be solved analytically. The solution is

$$\xi(x) = 2 \ln(2 \cosh 2\Lambda x), \quad (7.35)$$

and hence

$$\begin{aligned} \mathcal{M} = \tilde{\mathcal{M}} &= \frac{h}{1 + e^{2\Lambda x}}, \\ \mathcal{D} = \tilde{\mathcal{D}} &= \frac{h}{1 + e^{-2\Lambda x}}, \\ T &= -\Lambda \tanh \Lambda x. \end{aligned} \quad (7.36)$$

The solution is plotted in Fig. 7.3.

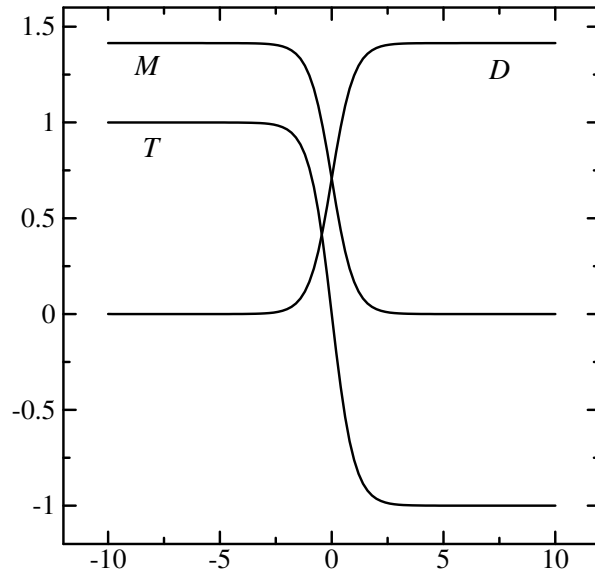


Figure 7.3: The domain wall solution of the $U(1) \times U(1)'$ model for $h = \sqrt{2}\Lambda$ case. We set $\Lambda = 1$ for simplicity.

Chapter 8

BPS string

In this chapter, we will discuss BPS strings. BPS strings are BPS objects which have non-zero $(1/2, 1/2)$ central charges and no non-zero $(1, 0)$ central charges. The BPS string preserves $1/2$ supersymmetry. The energy density of the BPS string is completely localized near one or some parallel axes.

Any four-dimensional $\mathcal{N} = 1$ theory can be dimensionally reduced to two dimensions, where it becomes $\mathcal{N} = 2$ theory. If the latter has topologically stable instantons, elevating the theory back to four dimensions gives us strings. Classical descriptions are totally equivalent. Distinctions occur at the level of quantum corrections, which are to be treated differently in two- and four-dimensional theories. The topological charge of the two-dimensional theory is related to the $(1/2, 1/2)$ central charges. This simple observation allows one to use a wealth of information regarding various two-dimensional models in analysis of saturated strings in four dimensions at the classical level.

In this chapter, we discuss the BPS string in various theories [65]-[69]. We will prove that the only gauge model admitting the BPS string is SQED with the Fayet-Iliopoulos term in weak coupling

8.1 SQED with the Fayet-Iliopoulos term

The simplest theory (and the only theory as is explained below) where saturated strings exist in the weak coupling regime is supersymmetric electrodynamics (SQED) [65]-[69], with two matter chiral superfield S and T , and the Fayet-Iliopoulos (FI) term. The charge of S and T are given by

$$\begin{array}{cc} S & T \\ U(1) & +1 \quad -1. \end{array} \tag{8.1}$$

In the superfield notation the Lagrangian of the model has the form

$$\mathcal{L} = \left\{ \frac{1}{4} \int d^2\theta W^2 + \text{h.c.} \right\} + \int d^2\theta d^2\bar{\theta} (S^\dagger e^{eV} S + T^\dagger e^{-eV} T) + 2\kappa \int d^2\theta d^2\bar{\theta} V(x, \theta, \bar{\theta}), \quad (8.2)$$

where e is the electric charge, and κ is the coefficient of the FI term. For simplify, we rewrite the FI term, $\kappa = -e\xi/2$. Since SQED with one chiral superfield is anomalous, we need to introduce two chiral superfields S and T with opposite charges.

In components the Lagrangian of SQED (8.2) has the form (in the Wess-Zumino gauge)

$$\mathcal{L} = -\frac{1}{4}v_{\mu\nu}v^{\mu\nu} - (\mathcal{D}_\mu\phi)^\dagger \mathcal{D}^\mu\phi - (\mathcal{D}_\mu\chi)^\dagger \mathcal{D}^\mu\chi - \frac{e^2}{8}(\phi^\dagger\phi - \chi^\dagger\chi - \xi)^2, + \text{fermions}, \quad (8.3)$$

where ϕ and χ are the lowest components of the superfields S and T , respectively, with the electric charges ± 1 , *e.g.*,

$$\mathcal{D}_\mu\phi = \partial_\mu\phi + i\frac{e}{2}v_\mu\phi, \quad \mathcal{D}_\mu\chi = \partial_\mu\chi - i\frac{e}{2}v_\mu\chi. \quad (8.4)$$

Without loss of generality we can assume that $\xi > 0$.

The classical moduli space of vacua is parameterized by one parameter u :

$$\langle\phi\rangle = e^{i\alpha}\sqrt{\xi}\cosh u, \quad (8.5)$$

$$\langle\chi\rangle = e^{i\beta}\sqrt{\xi}\sinh u. \quad (8.6)$$

The gauge symmetry is broken for any choice of vacuum. The kinetic term $\mathcal{L}_u$ of u becomes

$$\mathcal{L}_u = -\xi \cosh(2u) \{ (\partial_\mu u)(\partial^\mu u) + \frac{e^2}{4}v^\mu v_\mu \}. \quad (8.7)$$

The configuration of string-like object have the following structure:

$$\begin{cases} |\phi| \sim \sqrt{\xi} \cosh u_0, \\ |\chi| \sim \sqrt{\xi} \sinh u_0, \\ v_\mu \sim \frac{1}{r}, \end{cases} \quad r \rightarrow \infty \quad (8.8)$$

$$\begin{cases} |\phi| = 0, \\ |\chi| = 0, \\ v_\mu = 0, \end{cases} \quad r = 0. \quad (8.9)$$

However, u_0 must be zero. The reason is as follows. From the equation of motion of u ,

$$u \sim \ln r \quad r \rightarrow \infty. \quad (8.10)$$

So, u grows to infinity instead of approaching the non-zero constant. The boundary condition (8.8) cannot be satisfied. This argument does not work for a special choice $u_0 = 0$. Hence, we set $\chi(x) \equiv 0$.

For the static field configurations, assuming in addition that all fields depend only on x_1 and x_2 and $v_0 = v_3 = 0$, one gets the energy density functional $\mathcal{E}$ in the form

$$\begin{aligned} \mathcal{E} &= \int dx_1 dx_2 \left\{ \frac{1}{2} v_{12}^2 + \sum_{i=1,2} (\mathcal{D}_i \phi)^\dagger \mathcal{D}_i \phi + \frac{e^2}{8} (\phi^\dagger \phi - \xi)^2 \right\} \\ &= \int dx_1 dx_2 \left\{ \left| \frac{1}{\sqrt{2}} v_{12} + \frac{e}{2\sqrt{2}} (\phi^\dagger \phi - \xi) \right|^2 \right. \\ &\quad \left. + [(\mathcal{D}_1 + i\mathcal{D}_2) \phi]^\dagger (\mathcal{D}_1 + i\mathcal{D}_2) \phi \right\} + \mathcal{Q}, \\ &\geq \mathcal{Q}. \end{aligned} \quad (8.11)$$

where $\mathcal{Q}$ is the surface (topological) term,

$$\mathcal{Q} = \int dx_1 dx_2 \left\{ \frac{\xi e}{2} v_{12} - \frac{i}{2} \partial_i (\phi^\dagger \overleftrightarrow{\mathcal{D}}_j \phi) \varepsilon^{ij} \right\}, \quad i, j = 1, 2. \quad (8.12)$$

We will discuss the value of the surface term later.

The saturation equations are

$$\begin{aligned} v_{12} &= -\frac{e}{2} (\phi^\dagger \phi - \xi), \\ (\mathcal{D}_1 + i\mathcal{D}_2) \phi &= 0. \end{aligned} \quad (8.13)$$

The Ansatz which solves these equations is

$$\begin{aligned} \phi &= \sqrt{\xi} \eta e^{i\alpha}, \\ v_i &= -\frac{2a}{e} \frac{\partial \alpha}{\partial x^i}, \quad i = 1, 2, \end{aligned} \quad (8.14)$$

where

$$\alpha = \arg(x^1 + ix^2), \quad (8.15)$$

and η , a are some functions depending on r . This must be supplemented with the standard boundary conditions, namely

$$\eta(r), a(r) \longrightarrow \begin{cases} 0 & \text{at } r \rightarrow 0, \\ 1 & \text{at } r \rightarrow \infty. \end{cases} \quad (8.16)$$

For the given Ansatz the saturation equations (8.13) degenerate into a system of first-order equations

$$\begin{aligned} a' &= \frac{e^2}{4} \xi r (\eta^2 - 1), \\ \eta' &= \frac{\eta(1-a)}{r}, \end{aligned} \quad (8.17)$$

where the prime denotes differentiation over r . Its solution is well known.

It is instructive to compare the topological term in Eq. (8.12) with the central charge of the superalgebra. To derive the central charge one needs the expression for the supercurrent in SQED, which takes the form [69]. (We use the convention of Ref. [86], instead of the convention of Ref. [69].)

$$J_\alpha^\mu = -\frac{1}{2} \bar{\sigma}^{\mu\beta\dot{\beta}} J_{\alpha\beta\dot{\beta}} \quad (8.18)$$

$$\begin{aligned} J_{\alpha\beta\dot{\beta}} &= 2 (i v_{\beta\alpha} \bar{\lambda}_{\dot{\beta}} + \epsilon_{\beta\alpha} D \bar{\lambda}_{\dot{\beta}}) - 2\sqrt{2} \sum (\mathcal{D}_{\alpha\dot{\beta}} \phi^\dagger) \psi_\beta \\ &\quad + \frac{\sqrt{2}}{3} \sum \left[\partial_{\alpha\dot{\beta}} (\psi_\beta \phi^\dagger) + \partial_{\beta\dot{\beta}} (\psi_\alpha \phi^\dagger) - 3\epsilon_{\beta\alpha} \partial_{\dot{\beta}}^\gamma (\psi_\gamma \phi^\dagger) \right], \end{aligned} \quad (8.19)$$

where $D = -\frac{e}{2}(\phi^\dagger \phi - \chi^\dagger \chi - \xi)$. Above it is assumed that there is no superpotential. The sum runs over various matter supermultiplets, in particular, S and T in the case at hand. The second line may or may not be added, at will. The term is conserved by itself. Moreover, in the supercharge it is represented as an integral over the full derivative. The origin of the ambiguity is the ambiguity of the definition of Noether current j^μ :

$$j^\mu \rightarrow j'^\mu = j^\mu + \partial_\nu f^{\mu\nu}, \quad f^{\mu\nu} = -f^{\nu\mu}, \quad (8.20)$$

$$\partial_\mu j'^\mu = \partial_\mu j^\mu = 0. \quad (8.21)$$

From the supercurrent (8.19) the $(1/2, 1/2)$ central charges Y_k are given by:

$$Y_k = \int d^3x \epsilon_{kij} \left(\frac{\xi e}{2} \partial_i v_j - \sum \frac{i}{2} \partial_i (\bar{\phi} \overleftrightarrow{\mathcal{D}}_j \phi) \right) + \text{fermion}, \quad \epsilon_{123} = 1. \quad (8.22)$$

At weak coupling, the matter fermions do not contribute to Y_k . If $\langle Y_1 \rangle = \langle Y_2 \rangle = 0$ and $\langle Y_3 \rangle > 0$, then $H_{\text{II}} > H_{\text{I}}$, and the mass of BPS state is $\langle H \rangle = \langle Y_3 \rangle$. Therefore, the result (8.22) of SUSY algebra is consistent with Eq. (8.12). The BPS eqs. (2.35) are consistent with Eqs. (8.13).

Note that the coefficient of the $\bar{\phi} \overleftrightarrow{\mathcal{D}}_j \phi$ term of Eq. (8.22) is ambiguous – it depends on whether the second line in Eq. (8.19) is included in the

definition of the supercurrent. If we redefine supercurrent as “the minimal supercurrent” which do not have the second line in Eq. (8.19), then the coefficient of the $\vec{\phi}\vec{\mathcal{D}}_j\phi$ is zero. This ambiguity shows that the $\vec{\phi}\vec{\mathcal{D}}_j\phi$ term cannot contribute to the central charge under consideration. In fact, the Ansatz (8.14) shows that $\mathcal{D}_i\phi$ falls off sufficiently fast at $r \rightarrow \infty$. Then, $\vec{\phi}\vec{\mathcal{D}}_j\phi$ term of the surface term $\mathcal{Q}$ (8.12) cannot contribute, too.

Consequently, the energy density of BPS string is obviously proportional to ξ and to the magnetic flux of the string.

$$\mathcal{E} = \frac{\xi e}{2} \int dx_1 dx_2 v_{12}. \quad (8.23)$$

8.2 Generic non-Abelian Yang-Mills theory

8.2.1 Pure Yang-Mills theory

In this section, we consider the non-Abelian Yang-Mills theory. Firstly, we discuss the pure Yang-Mills theory. The Lagrangian is given by Eq. (7.13). Unlike the $(1, 0)$ central charges Z_k relevant for the domain walls, We can obtain the $(1/2, 1/2)$ central charges Y_k by the tree-level calculation [69]. (We use the convention of Ref. [86], instead of the convention of Ref. [69].)

$$Y_k = -\frac{1}{8}\epsilon^{knm} \int d^3x \partial_n \bar{\sigma}^{m\dot{\alpha}\alpha} R_{\dot{\alpha}\alpha}, \quad (8.24)$$

$$R_{\dot{\alpha}\alpha} = -\frac{4}{g^2} \text{tr}(\lambda_\alpha \bar{\lambda}_{\dot{\alpha}}). \quad (8.25)$$

The result is unambiguous, since the ambiguous term of the supercurrent do not contribute to this result.

Whether or not the chiral current $R_{\alpha\dot{\alpha}}$ actually vanishes is a dynamical issue which depends on the presence of the string-like solitons or domain wall junctions. SUSY gluodynamics is a strongly coupled theory. Therefore we cannot use quasiclassical considerations to analyze solitons.

8.2.2 Yang-Mills theory with matter

In the previous subsection, we obtained the unambiguous contribution from the gluino to the $(1/2, 1/2)$ central charges Y_k . However, the contribution from the matter is ambiguous [69], analogous to SQED.

The mass of the BPS string is related to only $(1/2, 1/2)$ central charges. The ambiguous term of the $(1/2, 1/2)$ central charges must not contribute to

the mass of the BPS string. Hence, only the gluino contributes to the mass of the BPS strings. (On the other hand, the mass of domain wall junction is related to $(1,0)$ and $(1/2,1/2)$ central charges. For the case of the domain wall junction solution, the ambiguity of the $(1/2,1/2)$ central charges can be canceled by the ambiguity of the $(1,0)$ central charges [69]. Hence, the ambiguity terms contribute to the energy of the BPS domain wall junction.)

8.2.3 BPS-saturated string in weak coupling

In previous subsections, We discussed generic non-Abelian gauge theory. Only the chiral current $R_{\alpha\dot{\alpha}}$ contribute to the mass of the BPS string.

In weak coupling (*i.e.*, for the string solitons in the quasiclassical treatment) the $(1/2,1/2)$ central charges must be saturated by only the term with the *bosonic* current. The current of matter *fermions* does not contribute at quasiclassical level, since the expectation values of fermions vanish at quasiclassical level. The chiral current $R_{\alpha\dot{\alpha}}$ (8.25) is fermion current. Thus, the BPS string of the non-Abelian gauge theories cannot be saturated at quasiclassical level.

At quasiclassical level, only the SQED with FI term can have the BPS string.

8.3 Strings in the Seiberg-Witten model

Here we will speculate on possible BPS strings at strong coupling. As we already know, such strings in non-Abelian gauge model do not appear in weak coupling. The $(1/2,1/2)$ central charge (appearing in the anticommutator $\{Q, \bar{Q}\}$) is not holomorphic – it need not depend holomorphically on the chiral parameters, in contradistinction with the $(1,0)$ charge. This means, that even if both the weak and strong coupling regimes are attainable in one and the same theory, generally speaking, nothing can be said regarding the BPS strings in the strong coupling regime from the behavior at weak coupling.

Turn now to discussion of the $\mathcal{N} = 2$ Yang-Mills theory. The exact solution for the low-energy effective action, as well as the exact spectrum of the BPS particles, is known [2] (see Chapter 3). Now we address the issue of possible stringy central charges. From the discussion above we saw that it can be attributed only to the gluino current (8.25), since there is no FI term in the model. Let us restrict ourselves to $SU(2)$ gauge group.

The absolute value of the variable $a(u)$ of the Seiberg-Witten theory has lower bound. The reason is as follows. The mass of the BPS saturated

electron, which has only electric charge, is proportional to $n_e a(u)$ (see Eq. (3.33)). If $a(u)$ could vanish, the electron would become massless. But, there is no singular point where the electron becomes massless in the moduli space of vacua (see Section 3.7). Thus, the variable $a(u)$ has lower bound:

$$|a(u)| > \text{constant } \Lambda. \quad (8.26)$$

Therefore $\pi_1(a(u)\text{-space})$ is nontrivial. Like instantons of two-dimensional model, the topologically stable objects with geometry are expected, provided a winds around the “forbidden” region.

The existence of such a stable object with the axial geometry would be a purely strong coupling effect since at the classical level the point $a = 0$ is attainable, and, correspondingly, π_1 is trivial. If the strings do exist, they may be BPS-saturated provided the term due to the gluino current R^μ (8.25) in the central charge is nonvanishing.

Chapter 9

Domain wall junction

In Chapter 7, we discussed BPS domain walls, which are BPS objects related to only $(1, 0)$ central charges. Domain walls occur in interpolating two discrete degenerate vacua in separate regions of space. In Chapter 8, we discussed BPS strings, which are BPS objects related to only $(1/2, 1/2)$ central charges. These objects preserve a half of the supersymmetry.

Recently another interesting possibility for BPS states has attracted much attention [70]–[73][69][74]–[83]. The BPS states are related to $(1, 0)$ and $(1/2, 1/2)$ central charges. The BPS states preserve a quarter of the supersymmetry. The BPS objects are junctions of BPS domain walls. If three or more different discrete vacua occur in separate regions of space, segments of domain walls separate each pair of the neighboring vacua. If the two spatial dimensions of all of these domain walls have one dimension in common, these domain walls meet at a one-dimensional junction. The configuration is called BPS domain wall junction. The configuration preserves a quarter of the supersymmetry.

Consider three domain walls meeting at a junction. We denote the tensions of the three domain walls as M_1 , M_2 and M_3 . If $M_3 > M_1 + M_2$, then the domain wall whose tension is M_3 becomes unstable against decay into the other two domain walls. Hence, when each tensions M_i are smaller than the sum of the the other two tensions [112], the domain wall junction is stable.

Result of part of Section 9.1 and the whole of Section 9.2 are based on my work [76] in collaboration with K. Ito, M. Naganuma and N. Sakai.

9.1 Wess-Zumino models

For simplicity, we start a non-gauge theory at weak coupling. I will concentrate on bosons, since the expectation values of fermions vanish at quasi-

classical level. Consider general Wess-Zumino models with arbitrary number of chiral superfields Φ^i and arbitrary superpotential $\mathcal{W}$. We compute the central charges from the chiral superfields at tree-level:

$$Z_k = 2 \int d^3x \partial_k \mathcal{W}^*(A^*), \quad (9.1)$$

$$Y_k = i\epsilon^{knm} \int d^3x K_{ij^*} \partial_n (A^{*j} \partial_m A^i) + \text{fermion}, \quad \epsilon^{123} = 1, \quad (9.2)$$

where the scalar component of the i -th chiral superfield Φ^i is denoted as A^i and the Kähler metric $K_{ij^*} = \partial^2 K(A^*, A) / \partial A^i \partial A^{*j}$ is obtained from the Kähler potential K .

These expressions of Z_k and Y_k are ambiguous — it depends on the definition of the supercurrents. We compute Eqs. (9.1) and (9.2), from the “the minimal supercurrent”, which does not have ambiguous term — the second line in Eq. (8.19). However, the energy of the domain wall junction solution is unambiguous, since the ambiguity of the $(1/2, 1/2)$ central charges can be canceled by the ambiguity of the $(1, 0)$ central charges [69]. Therefore, we will ignore the ambiguous term.

We assume that field configuration depends on x_1, x_2 only. The Bogomol’nyi bound condition becomes

$$\begin{aligned} \langle H \rangle &\geq \max\{H_I, H_{II}\}, \\ H_I &\equiv |\langle -iZ_1 - Z_2 \rangle| - \langle Y_3 \rangle, \\ H_{II} &\equiv |\langle iZ_1 - Z_2 \rangle| + \langle Y_3 \rangle. \end{aligned} \quad (9.3)$$

If $H_I > H_{II}$, the BPS eqs. become

$$2iK_{ij^*} \frac{\langle iZ_1 + Z_2 \rangle}{|\langle iZ_1 + Z_2 \rangle|} \partial_{\bar{z}} A^i = -\frac{\partial \mathcal{W}^*}{\partial A^{*j}}, \quad (9.4)$$

If $H_{II} > H_I$, the BPS eqs. become

$$2iK_{ij^*} \frac{\langle iZ_1 - Z_2 \rangle}{|\langle iZ_1 - Z_2 \rangle|} \partial_z A^i = -\frac{\partial \mathcal{W}^*}{\partial A^{*j}}. \quad (9.5)$$

Since $\langle Z_k \rangle$ and $\langle Y_k \rangle$ are given by total divergence as shown in Eqs. (9.1) and (9.2), they are fixed by boundary condition at spatial infinity. Therefore the boundary condition determines which of H_I and H_{II} is bigger.

We have derived the Bogomol’nyi bound condition (9.3) from the SUSY algebra. On the other hand, We can derive the same condition from the

Hamiltonian. The Hamiltonian for static configuration is

$$H = \int d^3x \left[K_{ij^*} \sum_{k=1}^3 \partial_k A^i \partial_k A^{*j} + K^{ij^*} \frac{\partial \mathcal{W}}{\partial A^i} \frac{\partial \mathcal{W}^*}{\partial A^{*j}} \right], \quad (9.6)$$

$$\begin{aligned} &= \int d^3x \left[K_{ij^*} \partial_3 A^i \partial_3 A^{*j} \right. \\ &\quad + K_{ij^*} \left((\partial_1 \pm i\partial_2) A^i - \Omega_{\pm} K^{ik^*} \frac{\partial \mathcal{W}^*}{\partial A^{*k}} \right) \left((\partial_1 \mp i\partial_2) A^{*j} - \Omega_{\pm}^* K^{lj^*} \frac{\partial \mathcal{W}}{\partial A^l} \right) \\ &\quad + (\partial_1 \pm i\partial_2)(\Omega_{\pm}^* \mathcal{W}) + (\partial_1 \mp i\partial_2)(\Omega_{\pm} \mathcal{W}^*) \\ &\quad \left. \pm K_{ij^*} i \partial_1 A^i \partial_2 A^{*j} \mp K_{ij^*} i \partial_2 A^i \partial_1 A^{*j} \right], \quad (9.7) \end{aligned}$$

where $K^{ik^*} K_{jk^*} = \delta_j^i$, $\Omega_{\pm} \in \mathbb{C}$ and $|\Omega_{\pm}| = 1$. The first line of Eq. (9.7) is zero, since we assumed the fields depend on only x_1, x_2 . The third line is

$$\Omega_{\pm} \frac{1}{2} (Z_1 \mp iZ_2) + \text{c.c.} \quad (9.8)$$

Now we fix $\Omega_{\pm}$ in order to maximize the third line of Eq. (9.7):

$$\Omega_{\pm} (Z_1 \mp iZ_2) = |Z_1 \mp iZ_2|. \quad (9.9)$$

The forth line is nothing but $\mp Y_3$. Hence, the Bogomol'nyi bound is

$$\langle H \rangle \geq |\langle Z_1 \rangle \mp i\langle Z_2 \rangle| \mp \langle Y_3 \rangle. \quad (9.10)$$

The BPS eqs. are

$$K_{ij^*} (\partial_1 \pm i\partial_2) A^i = \Omega_{\pm} \frac{\partial \mathcal{W}^*}{\partial A^{*j}} \quad (9.11)$$

Eqs. (9.10) and (9.11) are identified with Eqs. (9.3)–(9.5).

The general formula (9.2) for the case of many chiral superfields can be partially integrated as

$$Y_3 = \int dx_3 i \int d^2x [\partial_1 (K_i \partial_2 A^i) - \partial_2 (K_i \partial_1 A^i)] = \int dx_3 i \oint K_i dA^i, \quad (9.12)$$

where $K_i \equiv \partial K / \partial A^i$ and the last integral in the field space should be done as a map from a counter clockwise contour in the z plane. From this equation (9.12), We derive a following theorem [71][74][76]:

- When the counter clockwise contour in z is mapped to a *counter clockwise* path in the space of fields A^i , then $\frac{dY_3}{dx_3}$ is -2 times the area in field space spanned by the fields of solution, as measured by the Kähler metric.

- When the counter clockwise contour in z is mapped to a *clockwise* path in the space of fields A^i , then $\frac{dY_3}{dx_3}$ is +2 times the area in field space spanned by the fields of solution, as measured by the Kähler metric.

For simplify, we will discuss the simple nontrivial example. The superpotential is

$$\mathcal{W} = \Lambda^2 \Phi - \frac{\Phi^4}{4\Lambda}. \quad (9.13)$$

The theory has three supersymmetric vacua (Vac.I, Vac.II, Vac.III) in which the scalar component takes on vacuum expectation values:

$$\begin{aligned} \text{Vac.I} \quad \phi_{\text{I}} &= \Lambda, & \mathcal{W} &= \frac{3}{4}\Lambda^3, \\ \text{Vac.II} \quad \phi_{\text{II}} &= e^{2\pi i/3}\Lambda, & \mathcal{W} &= e^{2\pi i/3}\frac{3}{4}\Lambda^3, \\ \text{Vac.III} \quad \phi_{\text{III}} &= e^{-2\pi i/3}\Lambda, & \mathcal{W} &= e^{-2\pi i/3}\frac{3}{4}\Lambda^3. \end{aligned} \quad (9.14)$$

We set the boundary condition at spatial infinity as shown in Fig. 9.1. Since the central charges $\langle Z_k \rangle$ and $\langle Y_k \rangle$ are determined solely by the boundary condition at spatial infinity, we evaluate them on a large cylindrical region with a disk of large radius R ($R \gg |\Lambda|^{-1}$) centered at $z = 0$ and a height Δx_3 . Field configuration on the surface of the large cylinder approaches a step-function across domain walls. We find

$$\langle Z_1 \rangle \simeq \frac{9\sqrt{3}}{4}R\Lambda^3\Delta x_3, \quad (9.15)$$

$$\langle Z_2 \rangle \simeq -i\frac{9\sqrt{3}}{4}R\Lambda^3\Delta x_3, \quad (9.16)$$

$$\langle Y_3 \rangle \simeq -C|\Lambda|^2\Delta x_3, \quad C \in \mathbb{R}^+. \quad (9.17)$$

(We cannot find the numerical factor C of Y_3 . We need to find the exact profile of $\phi(x)$ on each domain wall in order to obtain the numerical factor C .) Therefore we obtain

$$H_{\text{I}} = |\langle -iZ_1 - Z_2 \rangle| - \langle Y_3 \rangle \simeq C|\Lambda|^2\Delta x_3, \quad (9.18)$$

$$H_{\text{II}} = |\langle iZ_1 - Z_2 \rangle| + \langle Y_3 \rangle \simeq \frac{9\sqrt{3}}{2}R|\Lambda|^3\Delta x_3 - C|\Lambda|^2\Delta x_3. \quad (9.19)$$

Hence, $H_{\text{II}} > H_{\text{I}}$. It is interesting to observe that H_{II} is larger than H_{I} primarily due to the different phases of $\langle Z_1 \rangle$ and $\langle Z_2 \rangle$ and not to the presence of $\langle Y_3 \rangle \neq 0$ term. This is in contrast to the case of a single domain wall where $H_{\text{I}} = H_{\text{II}}$ since $\langle Z_1 \rangle$ and $\langle Z_2 \rangle$ have the same phase factor and $\langle Y_3 \rangle = 0$. In

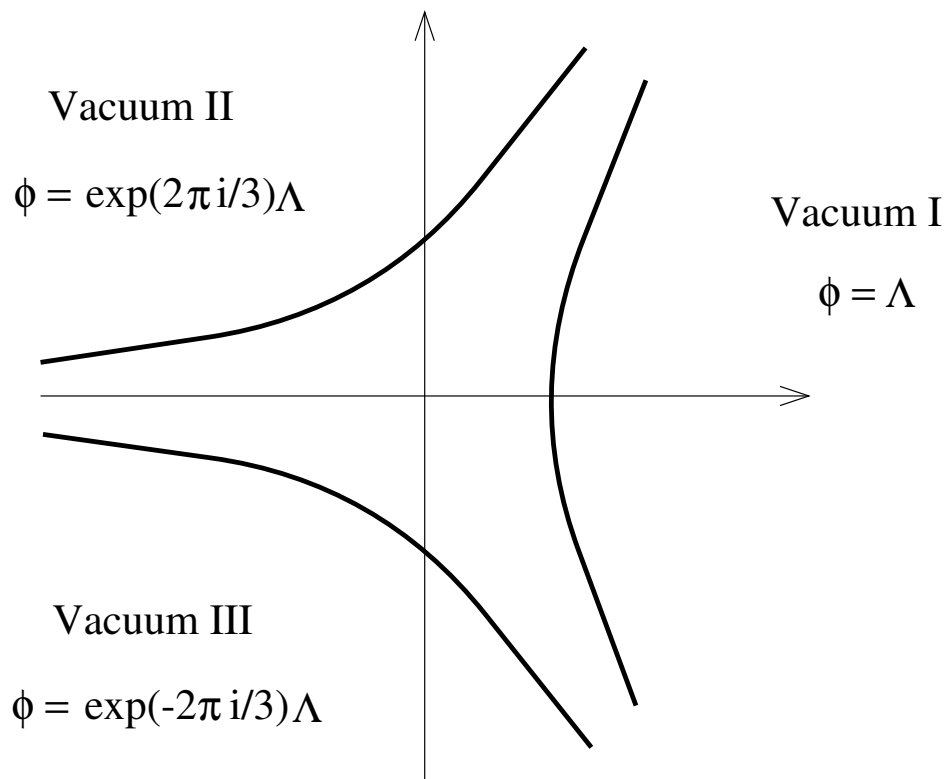


Figure 9.1: The field configuration of the domain wall junction.

fact we observe in Eq. (9.19) that the contribution of $\langle Y_3 \rangle$ to the mass of the domain wall junction is actually negative¹.

On each domain wall, we can analyze how supersymmetries are preserved.

$$(Q_1 + ie^{-i5\pi/3-i\alpha}\bar{Q}_1)|\text{Wall}_{\text{I,II}}\rangle = 0, \quad (9.20)$$

$$(Q_2 - ie^{-i\alpha}\bar{Q}_2)|\text{Wall}_{\text{I,II}}\rangle = 0, \quad (9.21)$$

$$(Q_1 - ie^{-i\alpha}\bar{Q}_1)|\text{Wall}_{\text{II,III}}\rangle = 0, \quad (9.22)$$

$$(Q_2 - ie^{-i\alpha}\bar{Q}_2)|\text{Wall}_{\text{II,III}}\rangle = 0, \quad (9.23)$$

$$(Q_1 + ie^{-i\pi/3-i\alpha}\bar{Q}_1)|\text{Wall}_{\text{III,I}}\rangle = 0, \quad (9.24)$$

$$(Q_2 - ie^{-i\alpha}\bar{Q}_2)|\text{Wall}_{\text{III,I}}\rangle = 0, \quad (9.25)$$

where $e^{i\alpha} = \Lambda^3/|\Lambda^3|$. Hence the entire configuration can preserve the supercharge

$$(Q_2 - ie^{-i\alpha}\bar{Q}_2)|\text{Junction}\rangle = 0. \quad (9.26)$$

It is equal to Eq. (2.27). Note that since an individual domain wall preserves 1/2 supersymmetry its low energy dynamics must be described by a $(2+1)$ -dimensional SUSY field theory with two supersymmetries. The two components of the spinor field of this effective theory are the coefficients of two Nambu-Goldstone fermions associated with the broken supersymmetries. The domain wall junction preserved only 1/4 supersymmetry, so there must be a total of three Nambu-Goldstone fermions. The fact that half of the domain wall's supersymmetry is preserved means that the junction's low energy dynamics is described by a $(1,0)$ -supersymmetric theory. This theory is chiral with one fermion that is either left-moving or right-moving [70].

Since $H_{\text{II}} > H_{\text{I}}$, the BPS eqs. (9.11)

$$2K_{\phi\phi^*}\partial_z\phi = \Omega_- \mathcal{W}'(\phi)^* \quad (9.27)$$

and

$$2K_{\phi\phi^*}\partial_{\bar{z}}\phi^* = \Omega_-^* \mathcal{W}'(\phi) \quad (9.28)$$

imply the relations

$$\partial_z \mathcal{W}(\phi) = \mathcal{W}'(\phi)\partial_z\phi = \frac{\Omega_-}{2} K^{\phi\phi^*} \mathcal{W}'(\phi) \mathcal{W}'(\phi)^* = \frac{\Omega_-}{2} V(\phi, \phi^*) \quad (9.29)$$

and

$$\partial_{\bar{z}} \mathcal{W}(\phi)^* = \frac{\Omega_-^*}{2} V(\phi, \phi^*) , \quad (9.30)$$

where V denotes the potential energy. We then integrate these equations over any large region S in the z - $\bar{z}$ plane; in particular, integrate over an

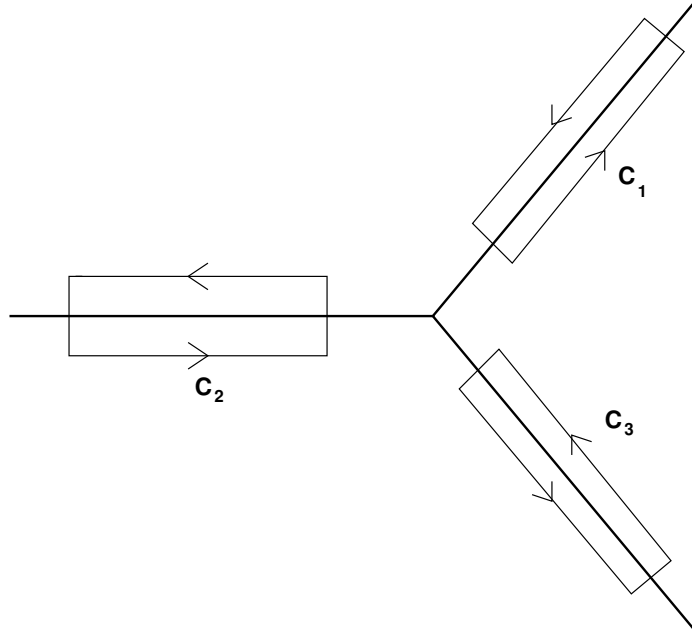


Figure 9.2: The three elongated rectangular contours in the z - $\bar{z}$ plane.

elongated rectangle containing a segment of one of the walls, whose long side has length $L \gg \Lambda^{-1}$, as shown in Fig. 9.2.

Using Stokes's theorem,

$$\int_S dz \wedge d\bar{z} (\partial_z v_{\bar{z}} - \partial_{\bar{z}} v_z) = \oint_{C \equiv \partial S} (dz v_z + d\bar{z} v_{\bar{z}}) , \quad (9.31)$$

and setting

$$\begin{aligned} v_{\bar{z}} &\equiv -i\Omega_-^* \mathcal{W} \\ v_z &= i\Omega_- \mathcal{W}^* , \end{aligned} \quad (9.32)$$

we obtain

$$-i \int_S dz \wedge d\bar{z} V = i \oint_C (\Omega_- \mathcal{W}^* dz - \Omega_-^* \mathcal{W} d\bar{z}) . \quad (9.33)$$

It is straightforward to check that for an isolated BPS wall, the kinetic and potential energies of the solution are equal, and so this must hold as well for the junction solution to order L^1 . Since $dz \wedge d\bar{z} = -2idx \wedge dy$, the equation above then says that the total mass enclosed by the contour C ,

¹ This fact seems to be against previous thoughts such as in Ref. [70].

for a 1/4-supersymmetric junction state, is given to order L^1 by the contour integral

$$M_{\text{enclosed}} = -i \oint_C (\Omega_- \mathcal{W}^* dz - \Omega_-^* \mathcal{W} d\bar{z}) = 2\text{Im} \left[\oint_C \Omega_- \mathcal{W}^* dz \right] \quad (9.34)$$

This reformulation of the BPS equations immediately yields a useful set of restrictions on the statics of wall junctions. Assuming the walls themselves to saturate the Bogomol'nyi bound $T = 2|\Delta\mathcal{W}|$ (see Eq. (7.9)), taking $L \rightarrow \infty$ and matching terms of order L^1 , we find that [71]

$$\Omega_- \frac{\Delta\mathcal{W}_{IJ}^*}{|\Delta\mathcal{W}_{IJ}|} \omega_{IJ} = 1, \quad (H_{\text{II}} > H_{\text{I}}), \quad (9.35)$$

where $\Delta\mathcal{W}_{IJ} = \mathcal{W}(\text{Vac}.J) - \mathcal{W}(\text{Vac}.I)$, and ω_{IJ} is a complex number of unit modulus which express the orientation of the domain wall from $\text{Vac}.I$ to $\text{Vac}.J$. On the other hand, if $H_{\text{I}} > H_{\text{II}}$, then

$$\Omega_+^* \frac{\Delta\mathcal{W}_{IJ}}{|\Delta\mathcal{W}_{IJ}|} \omega_{IJ} = -1, \quad (H_{\text{I}} > H_{\text{II}}). \quad (9.36)$$

Thus, since $\Delta\mathcal{W}_{I,II}$, $\Delta\mathcal{W}_{II,III}$, and $\Delta\mathcal{W}_{III,I}$ differ from each other in phase by 120° , this means that the orientations of the walls in the z - $\bar{z}$ plane must also differ by the same amount. Therefore, the integral forms of the BPS equations confirm reasonable physical expectations for the angles at which the walls must meet, based on the of forces on the junction. Although the $\mathbb{Z}_3$ symmetry of this particular case makes the statics particularly simple, it is straightforward to check that the integral form of the BPS equations yields the same consistent picture for an arbitrary superpotential with multiple vacua: the relative angles of walls at a junction must be arranged so as to cancel the total force on the junction point.

An interesting property of the domain walls is that the trajectory of fields of a single domain wall solution connecting two vacua always becomes a straight line segment when it is mapped to the complex plane of superpotential $\mathcal{W}$. The reason is as follows. Consider the domain wall from $\text{Vac}.I$ to $\text{Vac}.J$. On the domain wall, the field $\phi(z, \bar{z})$ depend on only $z = \omega_{IJ}t$, $t \in \mathbb{R}$. From the BPS eq. (9.27),

$$\frac{2K_{\phi\phi^*}}{\omega_{IJ}} \frac{d\phi}{dt} = \Omega_- \mathcal{W}'(\phi)^*. \quad (9.37)$$

Then multiplying both sides of (9.37) by $\omega_{IJ} \frac{d\phi^*}{dt}$

$$2K_{\phi\phi^*} \left| \frac{d\phi}{dt} \right|^2 = \Omega_- \omega_{IJ} \frac{d\mathcal{W}^*}{dt}. \quad (9.38)$$

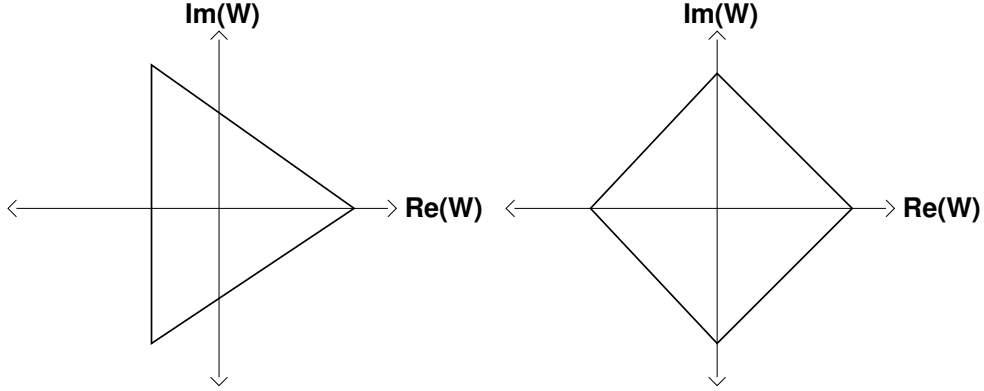


Figure 9.3: $\mathcal{W}$ -space for domain wall solutions of the models which have three and four supersymmetric vacua. Vertices are vacua, edges are domain walls, and the interior is the domain wall junction.

This shows that the imaginary component of $\Omega_{-}\omega_{IJ}\mathcal{W}^*$ is a constant, leading to domain walls tracing out straight lines in the $\mathcal{W}$ -plane. This is illustrated in Fig. 9.3.

9.2 $U(1) \times U(1)'$ model

9.2.1 The model

In previous section, we have discussed the Wess-Zumino model with one chiral matter and a superpotential (9.13). Although, this model is very simple, we cannot find exact analytical solution $\phi(z, \bar{z})$ from the BPS equation [75]. In this section, we will introduce another model which has exact analytical solution.

As we have discussed in Section 7.3, The $U(1) \times U(1)'$ model which has two matter $\mathcal{M}$ and $\mathcal{D}$ and superpotential (7.20) has the exact analytical solution of the domain wall.

We extend the $U(1) \times U(1)'$ model in Section 7.3 by adding an additional pair of chiral superfields $\mathcal{Q}, \tilde{\mathcal{Q}}$

$$\begin{array}{cc} & \mathcal{Q} & \tilde{\mathcal{Q}} \\ U(1) & 1 & -1 \\ U(1)' & 0 & 0 \end{array} \quad (9.39)$$

To make the quark massless at $T = m$ where m is the bare mass parameter

for the quark $\mathcal{Q}$, the superpotential is extended as

$$\mathcal{W} = (T - \Lambda)\mathcal{M}\tilde{\mathcal{M}} + (T + \Lambda)\mathcal{D}\tilde{\mathcal{D}} + (T - m)\mathcal{Q}\tilde{\mathcal{Q}} - h^2T. \quad (9.40)$$

This model simulates $\mathcal{N} = 2$ $SU(2)$ gauge theory with one flavor perturbed by an adjoint scalar mass. The chiral superfield $\mathcal{Q}$ and $\tilde{\mathcal{Q}}$ correspond to the quark and anti-quark. For large bare mass of the quark, the additional singularity corresponds to the situation where the effective mass of quark vanishes, whereas the $\mathbb{Z}_3$ symmetry among three singularities is realized in the limit of vanishing bare quark mass [3]. These three singularities become three discrete vacua of $\mathcal{N} = 1$ gauge theory when perturbed by the adjoint scalar mass [113].

This simple modification produces a model which possesses three distinct $\mathcal{N} = 1$ supersymmetric vacua and allows us to obtain an exact solution for junctions. Since the action is invariant under the three global $U(1)$ transformations

$$\begin{aligned} \mathcal{M} &\rightarrow e^{i\delta_1}\mathcal{M}, & \tilde{\mathcal{M}} &\rightarrow e^{-i\delta_1}\tilde{\mathcal{M}}, & \mathcal{D} &\rightarrow e^{i\delta_2}\mathcal{D}, & \tilde{\mathcal{D}} &\rightarrow e^{-i\delta_2}\tilde{\mathcal{D}}, \\ \mathcal{Q} &\rightarrow e^{i\delta_3}\mathcal{Q}, & \tilde{\mathcal{Q}} &\rightarrow e^{-i\delta_3}\tilde{\mathcal{Q}}, \end{aligned} \quad (9.41)$$

we can choose the vacuum configuration to be

$$\begin{aligned} \text{Vac.1 : } & T = m, & \mathcal{Q} = \tilde{\mathcal{Q}} = h, & \mathcal{M} = \tilde{\mathcal{M}} = \mathcal{D} = \tilde{\mathcal{D}} = 0, \\ \text{Vac.2 : } & T = \Lambda, & \mathcal{M} = \tilde{\mathcal{M}} = h, & \mathcal{Q} = \tilde{\mathcal{Q}} = \mathcal{D} = \tilde{\mathcal{D}} = 0, \\ \text{Vac.3 : } & T = -\Lambda, & \mathcal{D} = \tilde{\mathcal{D}} = h, & \mathcal{Q} = \tilde{\mathcal{Q}} = \mathcal{M} = \tilde{\mathcal{M}} = 0. \end{aligned} \quad (9.42)$$

We will consider a field configuration which is static and translationally invariant along x_3 direction. We assume that the three different vacua are realized in different directions at spatial infinity in x_1, x_2 plane.

9.2.2 The solution

The states which are saturated by the Bogomol'nyi bound obey Eqs. (2.25) and (2.26) or Eqs. (2.28) and (2.29). For simplicity, we now choose Eqs. (2.28) and (2.29). The BPS equation (2.29) for $U(1) \times U(1)'$ vector superfields can be satisfied trivially² by $v_\mu = 0$ and $D = 0$. The other BPS equation (2.28) becomes

$$2K_{ij^*}\frac{\partial A^i}{\partial z} = \Omega\frac{\partial \mathcal{W}^*}{\partial A^{*j}}, \quad \Omega = -i\frac{\langle iZ_1^* + Z_2^* \rangle}{|\langle iZ_1^* + Z_2^* \rangle|}. \quad (9.43)$$

² The $\mathbb{Z}_3$ symmetric case of the vanishing bare quark mass in the Seiberg-Witten theory yields a different charge assignment for the third singularity $(n_m, n_e) = (1, 2)$ instead of $(n_m, n_e) = (0, 1)$ [3]. Even if we use this charge assignment for the $\mathcal{Q}$ field, The vanishing D term condition gives the same result.

We will look for a solution of this partial differential equation. We observe that the phase of Ω can be absorbed by a rotation of field configuration, since the BPS equation (9.43) is invariant under a phase rotation : $\Omega \rightarrow e^{i\delta}\Omega$, $z \rightarrow e^{-i\delta}z$. Later, we will check that the solution satisfies $H_{\text{II}} > H_{\text{I}}$.

Since we assume a canonical Kähler metric $K_{ij*} = \delta_{ij}$, the BPS eqs. (9.43) become for our model

$$\begin{aligned} 2\frac{\partial\mathcal{M}}{\partial z} &= \Omega\tilde{\mathcal{M}}^*(T - \Lambda)^*, \\ 2\frac{\partial\mathcal{D}}{\partial z} &= \Omega\tilde{\mathcal{D}}^*(T + \Lambda)^*, \\ 2\frac{\partial\mathcal{Q}}{\partial z} &= \Omega\tilde{\mathcal{Q}}^*(T - m)^*, \\ 2\frac{\partial T}{\partial z} &= \Omega\left(\mathcal{M}\tilde{\mathcal{M}} + \mathcal{D}\tilde{\mathcal{D}} + \mathcal{Q}\tilde{\mathcal{Q}} - h^2\right)^*. \end{aligned} \quad (9.44)$$

Eqs. (9.42) and (9.44) are invariant under the following global phase changes of parameters, fields and complex coordinate z

$$\begin{aligned} h &\rightarrow e^{i\beta}h, \quad \mathcal{M} \rightarrow e^{i\beta}\mathcal{M}, \quad \tilde{\mathcal{M}} \rightarrow e^{i\beta}\tilde{\mathcal{M}}, \\ \mathcal{D} &\rightarrow e^{i\beta}\mathcal{D}, \quad \tilde{\mathcal{D}} \rightarrow e^{i\beta}\tilde{\mathcal{D}}, \quad \mathcal{Q} \rightarrow e^{i\beta}\mathcal{Q}, \quad \tilde{\mathcal{Q}} \rightarrow e^{i\beta}\tilde{\mathcal{Q}}, \end{aligned} \quad (9.45)$$

$$\Lambda \rightarrow e^{i\gamma}\Lambda, \quad T \rightarrow e^{i\gamma}T, \quad z \rightarrow e^{2i\beta+i\gamma}z. \quad (9.46)$$

Arbitrary complex parameters h and Λ can be obtained from real-positive h and Λ by these phase changes. Therefore we shall take h and Λ to be real-positive in the following without loss of generality. The BPS condition $D = 0$ yields $|\mathcal{M}(z, \bar{z})| = |\tilde{\mathcal{M}}(z, \bar{z})|$, $|\mathcal{D}(z, \bar{z})| = |\tilde{\mathcal{D}}(z, \bar{z})|$, and $|\mathcal{Q}(z, \bar{z})| = |\tilde{\mathcal{Q}}(z, \bar{z})|$. Inspired by this condition, we wish to find a solution assuming

$$\mathcal{M}(z, \bar{z}) = \tilde{\mathcal{M}}(z, \bar{z}), \quad \mathcal{D}(z, \bar{z}) = \tilde{\mathcal{D}}(z, \bar{z}), \quad \mathcal{Q}(z, \bar{z}) = \tilde{\mathcal{Q}}(z, \bar{z}). \quad (9.47)$$

and that all of them are real-positive in the entire complex plane. We shall see that this Ansatz gives a consistent solution.

We note that the model acquires a $\mathbb{Z}_3$ symmetry if we choose the bare mass m of $\mathcal{Q}$ as

$$m = i\sqrt{3}\Lambda. \quad (9.48)$$

In order to obtain the exact analytic solution of the domain wall junction, we specialize to this case, and shift the field T as $T' = T - i\frac{1}{\sqrt{3}}\Lambda$ to make $T' = 0$ as the origin of the $\mathbb{Z}_3$ rotation $T' \rightarrow e^{\pm i\frac{2\pi}{3}}T'$. The three vacua (9.42) and BPS equations (9.44) take manifestly $\mathbb{Z}_3$ symmetric forms

$$\text{Vac.1 : } T' = \frac{2}{\sqrt{3}}e^{i\frac{1}{2}\pi}\Lambda, \quad \mathcal{Q} = \tilde{\mathcal{Q}} = h, \quad \mathcal{M} = \tilde{\mathcal{M}} = \mathcal{D} = \tilde{\mathcal{D}} = 0,$$

$$\begin{aligned}\text{Vac.2 : } T' &= \frac{2}{\sqrt{3}} e^{-i\frac{1}{6}\pi} \Lambda, \quad \mathcal{M} = \tilde{\mathcal{M}} = h, \quad \mathcal{Q} = \tilde{\mathcal{Q}} = \mathcal{D} = \tilde{\mathcal{D}} = 0, \\ \text{Vac.3 : } T' &= \frac{2}{\sqrt{3}} e^{-i\frac{5}{6}\pi} \Lambda, \quad \mathcal{D} = \tilde{\mathcal{D}} = h, \quad \mathcal{Q} = \tilde{\mathcal{Q}} = \mathcal{M} = \tilde{\mathcal{M}} = 0,\end{aligned}\quad (9.49)$$

$$\begin{aligned}2\frac{\partial}{\partial z} \ln q_M &= \Omega \left(T'^* - \frac{2}{\sqrt{3}} e^{i\frac{1}{6}\pi} \Lambda \right), \\ 2\frac{\partial}{\partial z} \ln q_D &= \Omega \left(T'^* - \frac{2}{\sqrt{3}} e^{i\frac{5}{6}\pi} \Lambda \right), \\ 2\frac{\partial}{\partial z} \ln q &= \Omega \left(T'^* - \frac{2}{\sqrt{3}} e^{-i\frac{1}{2}\pi} \Lambda \right), \\ 2\frac{\partial}{\partial z} T' &= \Omega h^2 (q_M^2 + q_D^2 + q^2 - 1),\end{aligned}\quad (9.50)$$

where we have normalized the scalar fields by the nonzero expectation value h at vacua

$$\mathcal{M}(z, \bar{z}) = h q_M(z, \bar{z}), \quad \mathcal{D}(z, \bar{z}) = h q_D(z, \bar{z}), \quad \mathcal{Q}(z, \bar{z}) = h q(z, \bar{z}). \quad (9.51)$$

The first of Eq. (9.50) can be rewritten as

$$q_M = C(\bar{z}) \exp \left(\frac{1}{2} \eta - \frac{1}{\sqrt{3}} \Omega e^{i\frac{1}{6}\pi} \Lambda z \right), \quad (9.52)$$

$$\frac{\partial}{\partial z} \eta(z, \bar{z}) = \Omega T'^*(z, \bar{z}), \quad (9.53)$$

where the unknown function $C(\bar{z})$ is determined by the reality condition for q_M up to a constant which is absorbed into η

$$C(\bar{z}) = \exp \left(-\frac{1}{\sqrt{3}} \Omega^* e^{-i\frac{1}{6}\pi} \Lambda \bar{z} \right). \quad (9.54)$$

The remaining unknown function $\eta(z, \bar{z})$ should then be real. Consequently we obtain

$$q_M = \exp \left(\frac{1}{2} \eta + \frac{2}{\sqrt{3}} \Lambda \text{Re} \left(-\Omega e^{i\frac{1}{6}\pi} z \right) \right), \quad \eta(z, \bar{z}) = (\eta(z, \bar{z}))^*. \quad (9.55)$$

By an exactly similar procedure, we solve the second and third equations and obtain

$$q_D = \exp \left(\frac{1}{2} \eta + \frac{2}{\sqrt{3}} \Lambda \text{Re} \left(-\Omega e^{i\frac{5}{6}\pi} z \right) + C_D \right), \quad C_D \in \mathbb{R}, \quad (9.56)$$

$$q = \exp \left(\frac{1}{2} \eta + \frac{2}{\sqrt{3}} \Lambda \text{Re} \left(-\Omega e^{-i\frac{1}{2}\pi} z \right) + C \right), \quad C \in \mathbb{R}, \quad (9.57)$$

where C_D and C are integration constants. Let us assume that the origin $z = 0$ is the center of the domain wall junction and is $\mathbb{Z}_3$ symmetric. Therefore, $q_M = q_D = q$ at $z = 0$, which implies $C_D = C = 0$. Inserting Eq. (9.53) to the complex conjugate of the last of Eq. (9.50), we obtain

$$\begin{aligned} & 2 \frac{\partial^2}{\partial z \partial \bar{z}} \eta \\ &= -h^2 \left[1 - e^\eta \left\{ \exp \left(\frac{4\Lambda}{\sqrt{3}} \operatorname{Re} \left(-\Omega e^{i\frac{1}{6}\pi} z \right) \right) + \exp \left(\frac{4\Lambda}{\sqrt{3}} \operatorname{Re} \left(-\Omega e^{i\frac{5}{6}\pi} z \right) \right) \right. \right. \\ & \quad \left. \left. + \exp \left(\frac{4\Lambda}{\sqrt{3}} \operatorname{Re} \left(-\Omega e^{-i\frac{1}{2}\pi} z \right) \right) \right\} \right]. \end{aligned} \quad (9.58)$$

For the special case of $h^2 = 2\Lambda^2$, Eq. (9.58) can be solved analytically. Imposing the boundary conditions at infinity we obtain the solution

$$\begin{aligned} & \eta(z, \bar{z}) \\ &= -2 \ln \left[\exp \left(\frac{2\Lambda}{\sqrt{3}} \operatorname{Re} \left(-\Omega e^{i\frac{1}{6}\pi} z \right) \right) + \exp \left(\frac{2\Lambda}{\sqrt{3}} \operatorname{Re} \left(-\Omega e^{i\frac{5}{6}\pi} z \right) \right) \right. \\ & \quad \left. + \exp \left(\frac{2\Lambda}{\sqrt{3}} \operatorname{Re} \left(-\Omega e^{-i\frac{1}{2}\pi} z \right) \right) \right]. \end{aligned} \quad (9.59)$$

Therefore we find solutions for scalar fields as

$$\begin{aligned} \mathcal{M}(z, \bar{z}) &= \tilde{\mathcal{M}}(z, \bar{z}) = \frac{\sqrt{2}\Lambda s}{s+t+u}, \\ \mathcal{D}(z, \bar{z}) &= \tilde{\mathcal{D}}(z, \bar{z}) = \frac{\sqrt{2}\Lambda t}{s+t+u}, \\ \mathcal{Q}(z, \bar{z}) &= \tilde{\mathcal{Q}}(z, \bar{z}) = \frac{\sqrt{2}\Lambda u}{s+t+u}, \\ T'(z, \bar{z}) &= \frac{2\Lambda}{\sqrt{3}} \frac{e^{-i\frac{1}{6}\pi} s + e^{-i\frac{5}{6}\pi} t + e^{i\frac{1}{2}\pi} u}{s+t+u}, \end{aligned} \quad (9.60)$$

$$\begin{aligned} s &= \exp \left(\frac{2\Lambda}{\sqrt{3}} \operatorname{Re} \left(-\Omega e^{i\frac{1}{6}\pi} z \right) \right), \\ t &= \exp \left(\frac{2\Lambda}{\sqrt{3}} \operatorname{Re} \left(-\Omega e^{i\frac{5}{6}\pi} z \right) \right), \\ u &= \exp \left(\frac{2\Lambda}{\sqrt{3}} \operatorname{Re} \left(-\Omega e^{-i\frac{1}{2}\pi} z \right) \right). \end{aligned} \quad (9.61)$$

Now we will examine the solution more closely. The domain wall separating vacua I and J is characterized by a normal vector directing from I

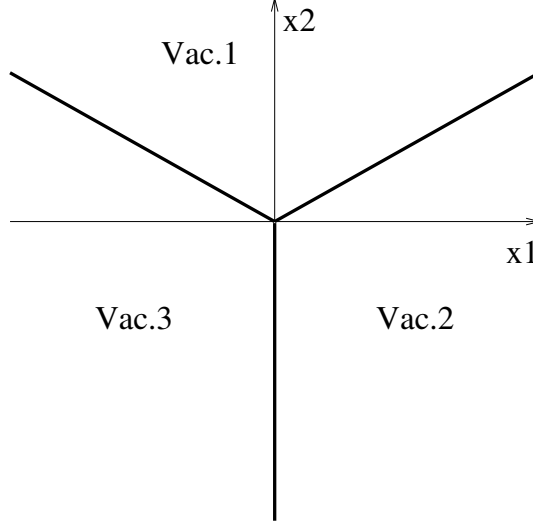


Figure 9.4: The field configuration of the domain wall junction for the case of $\Omega = -1$.

to J which is expressed as a complex number of unit modulus ω_{IJ} . If the difference of the superpotential $\mathcal{W}(\text{Vac}.I)$ at the vacuum I and $\mathcal{W}(\text{Vac}.J)$ at J is denoted as $\Delta\mathcal{W}_{IJ} = \mathcal{W}(\text{Vac}.J) - \mathcal{W}(\text{Vac}.I)$, the integral form of the BPS equation gives the condition on the direction of the domain walls.

$$\Omega \frac{\Delta\mathcal{W}_{IJ}^*}{|\Delta\mathcal{W}_{IJ}|} \omega_{IJ} = 1. \quad (9.62)$$

It is equal to Eq. (9.35). To orient the domain wall separating the vacuum 2 and 3 along the negative x_2 axis, as shown in Fig. 9.4, we choose $\Omega = -1$. The modulus of the field T' is plotted as a function of x_1 and x_2 in Fig. 9.5. From the BPS condition $v_\mu = D = 0$ and Eq. (9.7), the energy of the BPS domain wall solution is

$$H = \int d^3x \mathcal{H}, \quad (9.63)$$

$$\mathcal{H} = 2\text{Re}[(\partial_1 - i\partial_2)\Omega^*\mathcal{W}] - i(\partial_1 T')(\partial_2 T'^*) + i(\partial_2 T')(\partial_1 T'^*), \quad (9.64)$$

where $\Omega = -1$. The energy density $\mathcal{H}$ is plotted in Fig. 9.6.

Let us first examine the boundary conditions at spatial infinity $|z| \rightarrow \infty$.

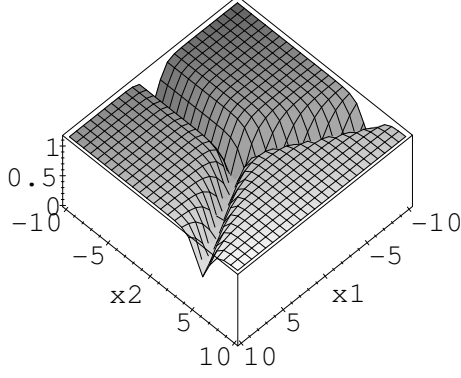


Figure 9.5: The modulus of the field T' as a function of x^1 and x^2 . We set $\Lambda = 1$ for simplicity.

From Eqs. (9.60) and (9.61), we find

$$\begin{aligned}
 &\text{when } -\frac{1}{2}\pi < \arg(z) < \frac{1}{6}\pi, \quad \text{then } s \gg t, u, \quad T' \rightarrow \frac{2}{\sqrt{3}}e^{-i\frac{1}{6}\pi}\Lambda, \quad (\text{vac.2}) \\
 &\text{when } \frac{1}{6}\pi < \arg(z) < \frac{5}{6}\pi, \quad \text{then } u \gg s, t, \quad T' \rightarrow \frac{2}{\sqrt{3}}e^{i\frac{1}{2}\pi}\Lambda, \quad (\text{vac.1}) \\
 &\text{when } \frac{5}{6}\pi < \arg(z) < \frac{3}{2}\pi, \quad \text{then } t \gg s, u, \quad T' \rightarrow \frac{2}{\sqrt{3}}e^{-i\frac{5}{6}\pi}\Lambda, \quad (\text{vac.3})
 \end{aligned} \tag{9.65}$$

Secondly, let us examine the asymptotic behavior along the region between two neighboring vacua. In the limit $x_2 \rightarrow -\infty$ with fixed x_1 , Eqs. (9.60) reduce to

$$\begin{aligned}
 \mathcal{M} &\rightarrow \frac{\sqrt{2}\Lambda e^{\Lambda x_1}}{e^{\Lambda x_1} + e^{-\Lambda x_1}}, \quad \mathcal{D} \rightarrow \frac{\sqrt{2}\Lambda e^{-\Lambda x_1}}{e^{\Lambda x_1} + e^{-\Lambda x_1}}, \quad \mathcal{Q} \rightarrow 0, \\
 T' &\rightarrow \Lambda \tanh \Lambda x_1 - \frac{\Lambda}{\sqrt{3}}i.
 \end{aligned} \tag{9.66}$$

Thus we recover the exact solution of domain wall (7.36) with x replaced by $-x_1$. By the $\mathbb{Z}_3$ symmetry, we also obtain respective exact domain wall solutions at the asymptotic region $x_1 = \pm\sqrt{3}x_2$ correctly.

Finally let us evaluate the central charges $\langle Z_k \rangle$ and $\langle Y_k \rangle$ and check $H_{\text{II}} > H_{\text{I}}$ to confirm that this solution is indeed realized as a 1/4 BPS state. Since

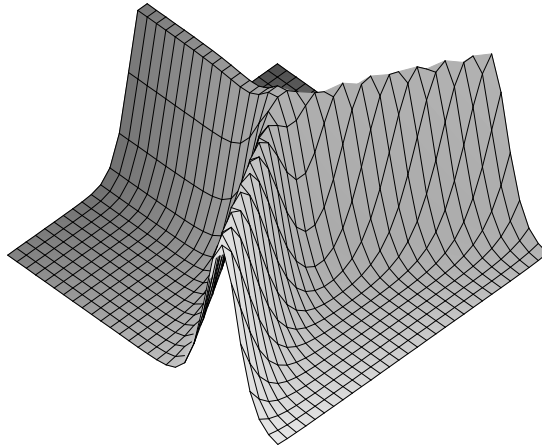


Figure 9.6: The energy density $\mathcal{H}$ of the domain wall junction. We set $\Lambda = 1$ for simplicity.

these charges are determined solely by the boundary condition at spatial infinity (9.65), we evaluate them on a large cylindrical region with a disk of large radius R ($R \gg \Lambda^{-1}$) centered at $z = 0$ and a height Δx_3 . Field configuration on the surface of the large cylinder approaches a step-function across domain walls. We find

$$\begin{aligned}\langle Z_1 \rangle &= -12\Lambda^3 R \Delta x_3, \\ \langle Z_2 \rangle &= i12\Lambda^3 R \Delta x_3, \\ \langle Y_3 \rangle &= -2\sqrt{3}\Lambda^2 \Delta x_3,\end{aligned}\tag{9.67}$$

with corrections suppressed exponentially as $R \rightarrow \infty$. Therefore we obtain

$$\begin{aligned}H_I &= |\langle -iZ_1 - Z_2 \rangle| - \langle Y_3 \rangle = 2\sqrt{3}\Lambda^2 \Delta x_3, \\ H_{II} &= |\langle iZ_1 - Z_2 \rangle| + \langle Y_3 \rangle = 24\Lambda^3 R \Delta x_3 - 2\sqrt{3}\Lambda^2 \Delta x_3.\end{aligned}\tag{9.68}$$

We see that $H_{II} > H_I$ confirming the correctness of the choice of the BPS equations (2.28) and (2.29).

In our case, we have contributions to $\langle Y_3 \rangle$ from the field T only, since Eq. (9.2) clearly shows that fields with real values do not contribute. Moreover the Kähler metric in our case is trivial and the counter clockwise contour in z is mapped to a counter clockwise contour in the field T . Therefore we obtain

$$\langle Y_3 \rangle = \int dx_3 (-2) \int d(\text{Re}T) d(\text{Im}T) = \int dx_3 (-2) \sqrt{3}\Lambda^2,\tag{9.69}$$

where the integration region in the field space T is the equilateral triangle whose vertices are the three vacuum field values. Let us also note that the domain walls correspond to straight lines in field space in our simple model. For general models, it has been shown that lines corresponding to domain walls are not straight lines in field space A^i , but become straight lines if mapped to the complex plane of superpotential $\mathcal{W}(A^i)$ as is derived in Section 9.1 (see Fig. 9.3).

Chapter 10

Conclusions

In this thesis, we have explored various BPS states in supersymmetric theories.

Originally, BPS states are defined in non-supersymmetric theories. Prasad and Sommerfield (1975) [4], and Bogomol'nyi (1976) [5] discussed the non-supersymmetric $SU(2)$ (or $O(3)$) Yang-Mills theory coupled to a Higgs field at the classical level. Prasad, Sommerfield and Bogomol'nyi argued that the mass of a dyon with electric and magnetic charges (n_e, n_m) is

$$M \geq \sqrt{2}|Z_{\text{SW}}|, \quad (10.1)$$

$$Z_{\text{SW}} = v(n_e + i4\pi/g^2 n_m), \quad (10.2)$$

where v is the vacuum expectation value of the Higgs and g is the gauge coupling constant. The inequality is called Bogomol'nyi bound. If the inequality is saturated, the state is called BPS state:

$$M_{\text{BPS}} = \sqrt{2}|Z_{\text{SW}}|. \quad (10.3)$$

Witten and Olive (1978) [6] reinterpreted the BPS states in supersymmetric theories. They pointed out that the same inequality must hold in $\mathcal{N} = 2$ supersymmetric $SU(2)$ pure Yang-Mills theory. They interpreted Z_{SW} as a central charge of $\mathcal{N} = 2$ supersymmetry algebra. We call it $(0, 0)$ central charge in this thesis. In the context of supersymmetric theories, the BPS states preserve some part of supersymmetries and, therefore, belong to special representations of the supersymmetry algebra known as “short” multiplets, which contain fewer states than standard (non-BPS) “long” multiplets of the superalgebra. The states which are BPS states at weak coupling cannot change into non-BPS states at strong coupling. Some of the properties of BPS states can be shown to be independent of the coupling constants, and

thus certain properties can be computed exactly by weak coupling methods. In fact, the inequality (10.1) and the equation (10.3) must hold at strong coupling.

The remaining question is how the central charge Z_{SW} is modified by quantum effect. By analogy with the classical result (1.2), Seiberg and Witten (1994) [2] assumed that at strong coupling the central charge Z_{SW} becomes

$$Z_{\text{SW}} = an_e + a_D n_m, \quad (10.4)$$

where a is the vacuum expectation value of the scalar component of $\mathcal{N} = 2$ supersymmetric vector multiplet in low energy effective theory, and a_D is a dual partner of a . Using this assumption and the equation (10.3) for the BPS states, Seiberg and Witten obtained exact result of the strong coupling physics of $\mathcal{N} = 2$ $SU(2)$ Yang-Mills theory.

On the other hand, the strong coupling physics of four-dimensional $\mathcal{N} = 1$ supersymmetric theories cannot be obtained. $\mathcal{N} = 1$ supersymmetric theories cannot have $(0, 0)$ central charges. However, $\mathcal{N} = 1$ supersymmetric theories have the other types of central charges — $(1, 0)$ and $(1/2, 1/2)$ central charges. There are BPS states related to the $(1, 0)$ and $(1/2, 1/2)$ central charges. Domain walls are BPS states which are related to the $(1, 0)$ central charges. BPS strings are BPS states which are related to the $(1/2, 1/2)$ central charges. Domain wall junctions are BPS states which are related to both of the central charges. The BPS domain walls and the BPS strings preserve a half of supersymmetry. The BPS domain wall junctions preserve a quarter of supersymmetry. These BPS states also cannot change into non-BPS states at stronger coupling. Therefore, we expect that the study of these BPS states will be useful for analyzing the strong coupling physics of $\mathcal{N} = 1$ supersymmetric theories.

In Chapter 2, we have summarized the $(0, 0)$, $(1, 0)$ and $(1/2, 1/2)$ central charges, and the BPS states. The BPS states must satisfy so-called BPS equations.

From Chapter 3 to Chapter 6, we have discussed $\mathcal{N} = 2$ supersymmetric theories and BPS states related to the $(0, 0)$ central charges. From Chapter 7 to Chapter 9, we have discussed $\mathcal{N} = 1$ supersymmetric theories and BPS states related to the $(1, 0)$ and $(1/2, 1/2)$ central charges.

In Chapter 3, we have reviewed the Seiberg-Witten theory. $\mathcal{N} = 2$ $SU(2)$ pure Yang-Mills theory is asymptotically free. Therefore, the coupling constant becomes strong at low energy scale. Low energy effective theory is defined as a theory whose modes are integrated down to an infrared cutoff in momentum regions. The low energy effective theory of $\mathcal{N} = 2$ $SU(2)$ pure Yang-Mills theory is $\mathcal{N} = 2$ $U(1)$ pure Yang-Mills theory which has

non-trivial prepotential. The low energy effective theory is singular when additional massless particle appears. There are at least two singular points at strong coupling region in moduli space of vacua. Seiberg and Witten assumed that there are two singular points at strong coupling region where a monopole and a dyon become massless. If a monopole becomes massless, then $a_D = 0$ because of Eqs. (10.3) and (10.4). Similarly, if a dyon becomes massless, $a + a_D = 0$. Seiberg and Witten showed that a and a_D can be represented by integrals of a differential λ along the two cycles α and β on a genus-one Riemann surface:

$$a_D = \oint_{\alpha} \lambda, \quad a = \oint_{\beta} \lambda. \quad (10.5)$$

The Riemann surface is called Seiberg-Witten curve. From a and a_D , we can obtain the exact prepotential of the low energy effective theory.

The Seiberg-Witten theory has been generalized to $\mathcal{N} = 2$ $SU(N_c)$ QCD in Chapter 4. The Seiberg-Witten curve for $\mathcal{N} = 2$ $SU(N_c)$ QCD is a genus $N_c - 1$ hyperelliptic curve. In Chapter 4 we have introduced a consistency check on the Seiberg-Witten theory. The method is called “integrating in” method.

Chapter 5 briefly reviewed brane approach. Using the brane approach, one can understand various aspects of supersymmetric gauge theories that one cannot understand using field-theoretic approaches. Although the Seiberg-Witten curve is no more than an artificial object in field-theoretic framework, the Seiberg-Witten curve is interpreted as the Riemann surface around which a single M theory five brane wraps in M theoretic framework.

One cannot derive the Seiberg-Witten curve of $\mathcal{N} = 2$ supersymmetric $SU(N_c)$ Yang-Mills theory with an antisymmetric tensor matter using field-theoretic approaches. However, Landsteiner and Lopez (1998) [46] have proposed the Seiberg-Witten curve of the Coulomb branch of $\mathcal{N} = 2$ supersymmetric $SU(N_c)$ Yang-Mills theory with an antisymmetric tensor matter, using the brane approach. The Seiberg-Witten curve is a non-hyperelliptic curve unlike the case of SQCD with fundamental matters. In Chapter 6, we have checked the non-hyperelliptic curve with the result of field-theoretic approach [85]. We have obtained the singularity surface of the $\mathcal{N} = 2$ supersymmetric $SU(N_c)$ Yang-Mills theory with an antisymmetric tensor matter by using the “integrating in” method. We have found that the usual “integrating in” method assuming $\mathcal{W}_{\Delta} = 0$ gives a singularity surface which disagrees with the proposed M theoretic curve. By assuming that the brane configuration is correct, we have found that there exists a nontrivial $\mathcal{W}_{\Delta} \neq 0$ which gives a singular surface consistent with the M theoretic curve at least up to certain high powers of the dynamical scale Λ of the gauge interactions.

Our results can be regarded as an evidence for the necessity of nontrivial $\mathcal{W}_\Delta$.

In Chapter 7, we have reviewed BPS domain walls. A domain wall is defined as an object which interpolates between two degenerate discrete minima of a potential and spread over two dimensions. Dvali and Shifman (1997) [57] began to discuss the BPS domain wall in supersymmetric theory. The domain wall solution depends on only one-dimensional spatial direction. The BPS domain wall is related to the $(1, 0)$ central charges.

In Chapter 8, we have reviewed BPS strings. A BPS string is defined as a field configuration whose energy density is localized near one or some parallel axes. The first researchers to give attention to BPS string was Edelstein *et al.* (1994) [65]. The BPS string is related to the $(1/2, 1/2)$ central charges. In weak coupling, the only gauge model admitting the BPS string is supersymmetric quantum electrodynamics with a Fayet-Iliopoulos term. However, other gauge theories may have BPS string solution at strong coupling.

Very recently, another interesting possibility for BPS states has attracted much attention [70]–[73][69][74]–[83]. The BPS states are related to $(1, 0)$ and $(1/2, 1/2)$ central charges. The BPS states preserve a quarter of the supersymmetry. The BPS objects are junctions of BPS domain walls. If three or more different discrete vacua occur in separate regions of space, segments of domain walls separate each pair of the neighboring vacua. If the two spatial dimensions of all of these domain walls have one dimension in common, these domain walls meet at a one-dimensional junction. The configuration is called BPS domain wall junction. The configuration preserves a quarter of the supersymmetry. In Chapter 9, we have discussed the BPS domain wall junctions. The BPS domain wall junction solution satisfies *half* of the BPS equations of the BPS domain wall solution. Therefore, the BPS domain wall junction solution can depend on two-dimensional spatial direction. It is difficult to solve the BPS equations, since there are only *half* of information of the BPS equations. We have introduced a model which has exact analytical domain wall junction solutions [76] in Section 9.2. The solution is the first exact analytical domain wall junction solution. The significance of our work [76] is that we have corrected misunderstanding about generic domain wall junction solutions. For instance, we have found that the $(1/2, 1/2)$ central charge Y_3 can give a negative contribution to the energy density of the domain wall junction.

It needs further investigation to understand how the study of these BPS domain walls, BPS strings and BPS domain wall junctions is concretely useful for analyzing the strong coupling physics of $\mathcal{N} = 1$ supersymmetric theories.

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Appendix A

Wess-Bagger notation and Supersymmetry

Throughout this thesis, we use Wess-Bagger notation [86], except that we use the Greek indices $(\mu, \nu, \dots)$ as spacetime indices: $\mu, \nu, \dots = 0, 1, 2, 3$, instead of the Latin indices $(m, n, \dots)$. We use the Latin indices as spatial indices: $m, n, \dots = 1, 2, 3$.

We summarize the notation and briefly review the supersymmetric theories.

Index The Greek indices $(\alpha, \beta, \dots, \dot{\alpha}, \dot{\beta}, \dots)$ denote two-component Weyl spinors: $\alpha, \beta, \dots, \dot{\alpha}, \dot{\beta}, \dots = 1, 2$. The capital Latin indices $(A, B, \dots)$ denote extended supersymmetry indices: $A, B = 1, \dots, \mathcal{N}$.

Metric

$$\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1). \quad (\text{A.1})$$

Antisymmetric $\epsilon^{\mu\nu\rho\sigma}$ tensor

$$\epsilon^{0123} = -\epsilon_{0123} = 1. \quad (\text{A.2})$$

Two-component spinor

$$\psi^\alpha = \epsilon^{\alpha\beta} \psi_\beta, \quad \psi_\alpha = \epsilon_{\alpha\beta} \psi^\beta, \quad (\text{A.3})$$

$$\epsilon_{21} = \epsilon^{12} = 1, \quad \epsilon_{12} = \epsilon^{21} = -1, \quad (\text{A.4})$$

$$\epsilon_{\alpha\beta} \epsilon^{\beta\gamma} = \delta_\alpha^\gamma, \quad (\text{A.5})$$

$$\begin{aligned}
\psi\chi &= \psi^\alpha\chi_\alpha = -\psi_\alpha\chi^\alpha = \chi^\alpha\psi_\alpha = \chi\psi, \\
\bar{\psi}\bar{\chi} &= \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} = -\bar{\psi}^{\dot{\alpha}}\bar{\chi}_{\dot{\alpha}} = \bar{\chi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}} = \bar{\chi}\bar{\psi}, \\
(\chi\psi)^\dagger &= (\chi^\alpha\psi_\alpha)^\dagger = \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} = \bar{\psi}\bar{\chi} = \bar{\chi}\bar{\psi}.
\end{aligned} \tag{A.6}$$

$\sigma_{\alpha\dot{\alpha}}$ matrices

$$\begin{aligned}
\sigma^0 &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\sigma^2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\end{aligned} \tag{A.7}$$

$$\bar{\sigma}^{\mu\dot{\alpha}\alpha} = \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}\sigma_{\beta\dot{\beta}}^\mu, \tag{A.8}$$

$$\begin{aligned}
\bar{\sigma}^0 &= \sigma^0, \\
\bar{\sigma}^{1,2,3} &= -\sigma^{1,2,3},
\end{aligned} \tag{A.9}$$

$$\begin{aligned}
P_{\alpha\dot{\alpha}} &= P_\mu\sigma_{\alpha\dot{\alpha}}^\mu = \begin{pmatrix} -P_0 + P_3 & P_1 - iP_2 \\ P_1 + iP_2 & -P_0 - P_3 \end{pmatrix}, \\
P^\mu &= -\frac{1}{2}\bar{\sigma}^{\mu\dot{\alpha}\alpha}P_{\alpha\dot{\alpha}},
\end{aligned} \tag{A.10}$$

$$\begin{aligned}
\sigma^{\nu\mu}{}_\alpha{}^\beta &= \frac{1}{4}(\sigma_{\alpha\dot{\alpha}}^\nu\bar{\sigma}^{\mu\dot{\alpha}\beta} - \sigma_{\alpha\dot{\alpha}}^\mu\bar{\sigma}^{\nu\dot{\alpha}\beta}), \\
\bar{\sigma}^{\nu\mu\dot{\alpha}}{}_{\dot{\beta}} &= \frac{1}{4}(\bar{\sigma}^{\nu\dot{\alpha}\alpha}\sigma_{\alpha\dot{\beta}}^\mu - \bar{\sigma}^{\mu\dot{\alpha}\alpha}\sigma_{\alpha\dot{\beta}}^\nu),
\end{aligned} \tag{A.11}$$

$$\begin{aligned}
\chi\sigma^\mu\bar{\psi} &= -\bar{\psi}\bar{\sigma}^\mu\chi, & (\chi\sigma^\mu\bar{\psi})^\dagger &= \psi\sigma^\mu\bar{\chi}, \\
\chi\sigma^\mu\bar{\sigma}^\nu\psi &= \psi\sigma^\nu\bar{\sigma}^\mu\chi, & (\chi\sigma^\mu\bar{\sigma}^\nu\psi)^\dagger &= \bar{\psi}\bar{\sigma}^\nu\sigma^\mu\bar{\chi}.
\end{aligned} \tag{A.12}$$

Four-component spinor

$$\Phi_D = \begin{pmatrix} \chi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}, \tag{A.13}$$

$$\Gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}. \tag{A.14}$$

Chiral superfield The chiral superfield Φ consists of a scalar field A , a spinor field ϕ , and an auxiliary field F :

$$\Phi(x, \theta) = A(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y), \quad (\text{A.15})$$

where $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$ and θ is Grassmann coordinate. The chiral superfield is characterized by the condition:

$$\bar{D}_{\dot{\alpha}}\Phi = 0, \quad (\text{A.16})$$

where $\bar{D}_{\dot{\alpha}}$ is differential operator:

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma_{\alpha\dot{\alpha}}^\mu\bar{\theta}^{\dot{\alpha}}\partial_\mu, \quad \bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha\sigma_{\alpha\dot{\alpha}}^\mu\partial_\mu. \quad (\text{A.17})$$

Their fields are transformed by supersymmetry transformation as:

$$\begin{aligned} \delta_\xi A &= \sqrt{2}\xi\phi, \\ \delta_\xi\phi &= i\sqrt{2}\sigma^\mu\bar{\xi}\partial_\mu A + \sqrt{2}\xi F, \\ \delta_\xi F &= i\sqrt{2}\bar{\xi}\bar{\sigma}^\mu\partial_\mu\phi. \end{aligned} \quad (\text{A.18})$$

The Lagrangian of the simplest model involving only chiral superfields is given by

$$\begin{aligned} \mathcal{L} &= \int d^2\theta d^2\bar{\theta} \Phi_i^\dagger \Phi_i \\ &+ \left[\int d^2\theta \left(\frac{1}{2}m_{ij}\Phi_i\Phi_j + \frac{1}{3}g_{ijk}\Phi_i\Phi_j\Phi_k + \lambda_i\Phi_i \right) + \text{h.c.} \right]. \end{aligned} \quad (\text{A.19})$$

In terms of component fields, $\mathcal{L}$ becomes

$$\begin{aligned} \mathcal{L} &= i\partial_\mu\bar{\psi}_i\bar{\sigma}^\mu\psi_i - \partial_\mu A_i^*\partial^\mu A_i + F_i^*F_i \\ &+ \left[m_{ij} \left(A_i F_j - \frac{1}{2}\psi_i\psi_j \right) \right. \\ &\quad \left. + g_{ijk}(A_i A_j F_k - \psi_i\psi_j A_k) + \lambda_i F_i + \text{h.c.} \right]. \end{aligned} \quad (\text{A.20})$$

The auxiliary fields F_i may be eliminated through their Euler equations. After the elimination, we can find that the potential $\mathcal{V}$ takes the form

$$\mathcal{V} = F_k^* F_k. \quad (\text{A.21})$$

This potential is always greater than or equal to zero. Thus, a supersymmetric vacuum expectation value of F_k must vanish.

General Wess-Zumino model The Lagrangian of the most general non-gauge model, so-called (general) Wess-Zumino model, takes form

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta} K(\Phi^i, \Phi^{\dagger j}) + \left[\int d^2\theta \mathcal{W}(\Phi^i) + \text{h.c.} \right], \quad (\text{A.22})$$

where $K(\Phi^i, \Phi^{\dagger j})$ is called Kähler potential, and $\mathcal{W}(\Phi^i)$ is called superpotential. It is important that the superpotential $\mathcal{W}$ is holomorphic.

In terms of component fields, $\mathcal{L}$ becomes

$$\mathcal{L} = -K_{ij^*} \partial_\mu A^i \partial^\mu A^{*j} - K^{ij^*} \frac{\partial \mathcal{W}}{\partial A^i} \frac{\partial \mathcal{W}^*}{\partial A^{*j}} + (\chi \text{ terms}), \quad (\text{A.23})$$

where K_{ij^*} , so-called Kähler metric, is

$$K_{ij^*} = \frac{\partial}{\partial A^i} \frac{\partial}{\partial A^{*j}} K, \quad \text{and} \quad K_{ij^*} K^{kj^*} = \delta_i^k. \quad (\text{A.24})$$

The potential $\mathcal{V}$ is

$$\mathcal{V} = K^{ij^*} F_i F_j^*, \quad F_i = \frac{\partial \mathcal{W}}{\partial A^i}. \quad (\text{A.25})$$

Vector superfield The vector superfield V consists of a gauge potential field v_μ , a spinor field λ , and an auxiliary field D in Wess-Zumino gauge:

$$V = -\theta\sigma^\mu\bar{\theta}v_\mu(x) + i\theta\theta\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x). \quad (\text{A.26})$$

The vector superfields satisfy the condition

$$V = V^\dagger. \quad (\text{A.27})$$

The supersymmetric field strength W_α is

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}D_\alpha V, \quad \bar{W}_{\dot{\alpha}} = -\frac{1}{4}DD\bar{D}_{\dot{\alpha}}V. \quad (\text{A.28})$$

The supersymmetric field strength W_α is a chiral superfield, and

$$\int d^2\theta W^\alpha W_\alpha = -2i\lambda\sigma^\mu\partial_\mu\bar{\lambda} - \frac{1}{2}v^{\mu\nu}v_{\mu\nu} + D^2 + \frac{i}{4}v^{\mu\nu}v^{\rho\sigma}\epsilon_{\mu\nu\rho\sigma}, \quad (\text{A.29})$$

where $v_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu$. The Lagrangian of supersymmetric pure $U(1)$ gauge model is

$$\mathcal{L} = \frac{1}{4} \int d\theta^2 W W + \text{h.c.} \quad (\text{A.30})$$

$$= \frac{1}{2}D^2 - \frac{1}{4}v^{\mu\nu}v_{\mu\nu} - i\lambda\sigma^\mu\partial_\mu\bar{\lambda}. \quad (\text{A.31})$$

$U(1)$ gauge theories The gauge transformation is

$$\begin{aligned}\Phi_i &\rightarrow \Phi'_i = e^{-ie_i\Lambda}\Phi_i, \\ V &\rightarrow V' = V + i(\Lambda - \Lambda^\dagger),\end{aligned}\tag{A.32}$$

where e_i are the $U(1)$ charges of the Φ_i .

The Lagrangian of the $U(1)$ gauge theories is

$$\begin{aligned}\mathcal{L} &= \int d^2\theta d^2\bar{\theta} \Phi_i^\dagger e^{e_i V} \Phi_i \\ &+ \left[\int d^2\theta \left(\mathcal{W}(\Phi) + \frac{1}{4} WW \right) + \text{h.c.} \right] \\ &+ 2\kappa \int d^2\theta d^2\bar{\theta} V,\end{aligned}\tag{A.33}$$

The κ term is called Fayet-Iliopoulos term. The potential $\mathcal{V}$ is

$$\mathcal{V} = \frac{1}{2} D^2 + F_k^* F_k,\tag{A.34}$$

where

$$D = -\kappa - \frac{e_i}{2} A_i^* A_i.\tag{A.35}$$

The gauge covariant derivatives are

$$\mathcal{D}_\mu A_i = \partial_\mu A_i + \frac{ie_i}{2} v_\mu A_i.\tag{A.36}$$

Non-Abelian gauge theories The gauge transformation is

$$\begin{aligned}\Phi &\rightarrow \Phi' = e^{-i\Lambda}\Phi, & \Lambda_{ij} &= T_{ij}^a \Lambda_a, \\ e^V &\rightarrow e^{V'} = e^{-i\Lambda^\dagger} e^V e^{i\Lambda}, & V_{ij} &= T_{ij}^a V_a,\end{aligned}\tag{A.37}$$

where T_{ij}^a are the hermitian generators of the gauge group in the representation defined by the chiral field Φ . In the adjoint representation, we normalized our generators as follows:

$$\text{tr}(T^a T^b) = k\delta^{ab}, \quad k > 0.\tag{A.38}$$

With this convention, the structure constants t^{abc}

$$[T^a, T^b] = it^{abc} T^c\tag{A.39}$$

are completely antisymmetric.

The supersymmetric field strength W_α is generalized to the non-Abelian case:

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}e^{-V}D_\alpha e^V, \quad (\text{A.40})$$

and transforms as follows:

$$W_\alpha \rightarrow W'_\alpha = e^{-i\Lambda}W_\alpha e^{i\Lambda}. \quad (\text{A.41})$$

The Lagrangian of the non-Abelian gauge theories is

$$\begin{aligned} \mathcal{L} = & \int d^2\theta d^2\bar{\theta} \Phi_i^\dagger e^{2gV} \Phi_i \\ & + \left[\int d^2\theta \left(\mathcal{W}(\Phi) + \frac{1}{4k} \text{tr}(WW) \right) + \text{h.c.} \right], \end{aligned} \quad (\text{A.42})$$

The potential $\mathcal{V}$ is

$$\mathcal{V} = \frac{1}{2}D^{(a)2} + F_k F_k^*. \quad (\text{A.43})$$

From Euler equations, we can find that

$$D^{(a)} = -g A^{i\dagger} T^{(a)} A^i. \quad (\text{A.44})$$

The supersymmetric transformation for A , ψ , F , $v_m^{(a)}$, $\lambda^{(a)}$ and $D^{(a)}$ in Wess-Zumino gauge is

$$\begin{aligned} \delta_\xi A &= \sqrt{2}\xi\psi, \\ \delta_\xi \psi &= i\sqrt{2}\sigma^\mu \bar{\xi} \mathcal{D}_\mu A + \sqrt{2}\xi F, \\ \delta_\xi F &= i\sqrt{2}\bar{\xi} \bar{\sigma}^\mu \mathcal{D}_\mu \psi + i2gT^{(a)} A \bar{\xi} \bar{\lambda}^{(a)}, \\ \delta_\xi v_m^{(a)} &= -i\bar{\lambda}^{(a)} \bar{\sigma}^\mu \xi + i\bar{\xi} \bar{\sigma}^\mu \lambda^{(a)}, \\ \delta_\xi \lambda^{(a)} &= \sigma^{\mu\nu} \xi v_{\mu\nu}^{(a)} + i\xi D^{(a)}, \\ \delta_\xi D^{(a)} &= -\xi \sigma^\mu \mathcal{D}_\mu \bar{\lambda}^{(a)} - \mathcal{D}_\mu \lambda^{(a)} \sigma^\mu \bar{\xi}, \end{aligned} \quad (\text{A.45})$$

where $\mathcal{D}_\mu$ are the gauge covariant derivatives:

$$\begin{aligned} \mathcal{D}_\mu A &= \partial_\mu A + igv_\mu^{(a)} T^{(a)} A, \\ \mathcal{D}_\mu \psi &= \partial_\mu \psi + igv_\mu^{(a)} T^{(a)} \psi, \\ \mathcal{D}_\mu \lambda^{(a)} &= \partial_\mu \lambda^{(a)} - gt^{abc} v_\mu^{(b)} \lambda^{(c)}, \\ v_{\mu\nu}^{(a)} &= \partial_\mu v_\nu^{(a)} - \partial_\nu v_\mu^{(a)} - gt^{abc} v_\mu^{(b)} v_\nu^{(c)}. \end{aligned} \quad (\text{A.46})$$

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