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DISSERTATION

Low energy effective theory on branes in six-dimensional warped flux compactifications

by

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Abstract

Cosmology based on general relativity explains phenomena of present universe successfully, such as present abundance of light elements and temperature (anisotropies) of cosmic microwave background. But when you try to understand the early universe such as in the period of creation of universe, 4-dimensional general relativity is not available anymore. More precisely, general relativity predicts the presence of the initial singularity, which leads violation of general relativity. This also means that our universe was extremely small in a very high-energy state at the beginning. The high-energy physics beyond the general relativity is thus essential. One of the candidates for such theories is superstring theory.

Recently there was a great discovery of D-brane, which is a membrane and defined by the collections of the endpoints of open strings. From this, we can have a new picture of the early universe. That is, gravity is higher dimensional rather than 4-dimensional and the matters are confined only on the membrane. This idea is called "braneworld". Therein our universe is membrane itself floating in the higher dimensional bulk space.

In braneworld model, Randall and Sundrum's one proposed in early 2000 is very attractive because the self-gravity of the membrane was carefully considered. Therefore, we can discuss the cosmology using it. Moreover, their model gives us a solution to the gauge hierarchy problem.

Though Randall-Sundrum model has been studied very intensively, its model is limited to 5-dimensional case. Noting that the string theory is formulated in 10 dimensions, the braneworld model with the total number of dimensions higher than 5 needs to be considered. Thus, as a next step we will proceed from the 5-dimensional "toy model" into the 6-dimensional model of braneworld.

There are two problems in constructing the higher dimensional model. One is the problem of the cause of the singularity and the other is the problem of the stabilization of extra dimensions. The former is generally known to arise by increasing the number of co-dimensions. We will solve these problems by introducing the capped regularization and flux. Then, we will derive the 4-dimensional effective theory using long-wavelength approximation and confirm the recovery of the conventional Einstein theory.

Contents

Chapter 1

Introduction

Where did our universe come from? Such naïve questions of the origin and nature of our universe have long been one of the greatest interests of human being since ancient era. With the developments of geometrical picture of gravity (General relativity) and particle physics, the scientific investigation of the very early universe becomes possible. According to the results of recent study, the very early universe might be governed by superstring theory which suggests that our universe should be dealt with higher dimensional space-time more than 4 dimensions. Original idea of extra dimension is first introduced by Kaluza and Klein [1] in rather different context, such as unification between gravitation (general relativity) and electromagnetism. They noticed that gravitational interaction and electromagnetic interactions are so alike, and treated as if both come from the same origin, that is, 5-dimensional Einstein equation. Subsequently, some extensions to non-Abelian gauge fields were considered. Though this type of unification of gravity and electromagnetism itself is not succeeded after all, the idea of compactification used in this Kaluza-Klein (KK) theory is still important for cosmology of the early universe, based on/inspired by superstring theory.

In particle physics theory, beyond the standard model of particles, superstring theory is proposed. The superstring theory is considered to be one of the candidates for the unification theories including gravity. In this theory, the fundamental objects are strings, not point particles. Superstring theories are formulated in 10 or 11 dimensions. In this context to obtain 4-dimensional theory like that of our world, 6 or 7 extra dimensions should be compactified somehow. If not, exotic particles appear easily and such particles are not permitted from experiments/observations. To avoid this disaster the size of extra dimensions should be very small. Thus, the compactification is very crucial in string theory.

Recent developments in string theory have large influences on the study of gravity and cosmology. Especially the D-brane which is the collection of endpoints of open strings (corresponding to matter), provides us with a remarkably new picture of our universe. This naturally implies that the matter should be localized on the D-brane and closed string (corresponding to graviton) need not localize on the brane. This picture will give "braneworld" idea, that is, our world (4-dimensional space-time) is regarded as a membrane existed in the higher dimensional space-time.

Let us see the first braneworld model proposed by Arkani-Hamed, Dimopoulos and Dvali (ADD) [5]. They bring this braneworld picture into solving the gauge hierarchy problem. The gauge hierarchy problem is the mystery of the huge discrepancy of two fundamental scales, weak mass scale ($\sim 10^3$[GeV]) and Planck mass scale ($\sim 10^{19}$[GeV]). In their model, this discrepancy can be explained by the volume factor of the large extra dimension which makes the original Planck scale smaller and by which the Planck scale is adjusted to TeV scale. Their assumption is that the gravity is propagating not only our world but also extra dimensions (≤ 0.1[mm]), while matters are confined to our world and never propagate away to the extra dimensional directions. This assumption is motivated by the nature of D-brane. In this model, below some scale specified by extra dimensions the law of gravitation should be modified. For example, in the 5-dimensional case, the extra-dimensional scale reaches almost $1[\mathrm{AU}] \sim 10^{13}[\mathrm{cm}]$, which contradicts our experimental facts. Thus, in Arkani-Hamed, Dimopoulos and Dvali (ADD) model the number of dimension is required to be 6 or higher. More detail will be discussed in the next chapter.

Now let us see another model proposed by Randall and Sundrum (RS) [6]. An essential difference is that they consider the self-gravity due to the existence of brane tension and the number of whole space-time is five. In their model, therefore, extra dimension is highly warped. Note that this model also makes the original Planck scale be the same order with the TeV scale but the relation of the volume factor is very different compare to the Arkani-Hamed, Dimopoulos and Dvali (ADD) model. The warped extra dimension gives exponential suppression of Planck scale, which allows to explain gauge hierarchy problem.

A lot of authors investigated Radall-Sundrum models because we can discuss the cosmology. However the string/M-theories are formulated in 10/11 dimensions. This means that the 5-dimensional picture of Randall-Sundrum (RS) model is a “toy model”. Therefore, we need to extend this model to higher dimensional models. We will investigate 6-dimensional braneworld as a next step.

There are two problems in constructing a higher dimensional model. One is the problem of the cause of the singularity and the other is the stabilization of extra dimensions. The former is due to the self-gravity of brane. It is generally known that higher co-dimension brane produces worse singularity and that in fact only pure tension brane can be accommodated on the co-dimension-2 brane. We will solve these problems by introducing the capped regularization and flux. The capped regularization is realized by introducing the co-dimension-1 wrapped brane, whose one extra dimension is compactified in the traditional Kaluza-Klein (KK) manner and where arbitrary matter can be accommodated. Introducing a certain action for the brane 5-dimensional general covariance will be broken spontaneously, so that we can essentially obtain 4-dimensional covariant theory. In flux compactification, repulsive flux and gravitational attraction are balanced, which gives the stabilization of the 2 extra dimensions. Then, we will derive the 4-dimensional effective theory using long-wavelength approximation and confirm that it is indeed the conventional Einstein theory.
The plan of this paper is as follows.

Chapter 2:
In the next chapter, we will briefly review Arkani-Hamed-Dimopoulos-Dvali (ADD) [5]. Then we focus on Randall-Sundrum (RS) models [6] in the details. First we will introduce the geometrical projection method [9] to understand the gravitational theory on the brane. In addition, we will explain the long-wavelength approximation which is useful for looking at the low energy regime.

The rest of this chapter is dedicated to the preparation for the discussion of 6-dimensional braneworld model. First, we will confirm that the naïve construction of co-dimension-2 brane is impossible through the example of [15]. Next we will consider the stabilization mechanism by flux based on [18] and [19]. Generally, this instability tends to be more noticeable in higher co-dimension space-time and this mechanism will be important for the construction of 4-dimensional effective theory from higher dimensional background. Thirdly, we will briefly review the procedure of regularization scheme. As seen from the concrete example in co-dimension-2 brane, singularity occurs when we put arbitrary matter on co-dimension-2 brane. We adopt the regularization scheme called “capped regularization” [28] which is described at the end of this chapter.

Chapter 3:

This chapter is the main part. We will construct the effective theory on the co-dimension-1 wrapped brane. Note that the extra dimension is stabilized by flux. Then we solve 6-dimensional Einstein-Maxwell system by long-wavelength approximation. We succeed in obtaining the conventional 4-dimensional effective equation on the brane.

Chapter 4:

In this chapter, we will summarize all the results and make some concluding remarks.

The contents of this dissertation is based on “Low energy effective theory on a regularized brane in six-dimensional flux compactifications”, S.Fujii, T.Kobayashi and T.Shiromizu, Phys.Rev.D76: 104052(2007) [35].

Chapter 2

Braneworld model and its extension

In this chapter we will give a review on braneworld models. Then we will discuss fundamental things in 6-dimensional space-time which will be useful for the discussion in the main part.

2.1 Review of braneworld

Here, we introduce two models, namely Arkani-Hamed, Dimopoulos and Dvali (ADD) model and Randall-Sundrum (RS) model. Although they are different from the original Kaluza-Klein (KK) theory, we briefly sketch the fundamental concept and terminology of Kaluza-Klein (KK) theory.

Kaluza-Klein compactification

We begin with the Kaluza-Klein (KK) theory in 5-dimensional space-time. Kaluza-Klein (KK) compactification is characterized as identification

$$y \sim y + L, \tag{2.1.1}$$

where y is the coordinate of the extra dimension and L is the compactification scale of one extra dimension. This compactification scale L should be very small. If not, unobserved extra modes of particles are excited easily. This seems to conflict with the results obtained through the experiments/observations. To see this, let us see the case of real massless scalar field in 5-dimensional flat space-times. Then, metric is

$$ds^2 = dy^2 + \eta_{\mu\nu} dx^\mu dx^\nu, \tag{2.1.2}$$

where $\eta_{\mu\nu}$ is the metric of 4-dimensional Minkowski space-time and the Greek indices running as $\mu = 0,1,2,3$. The y-coordinate is supposed to be periodic as the condition of Eq. (2.1.1). The action is given by

$$S = -\frac{1}{2}\int d^5x \sqrt{-{}^{(5)}g}\,\partial_A {}^{(5)}\phi(y,x)\partial^A {}^{(5)}\phi(y,x), \tag{2.1.3}$$

where capital Latin indices run 5 dimensions, $A = 0,1,2,3,4$ in this section.
From the periodicity, the real scalar field ${}^{(5)}\phi$ can be decomposed by the mode functions,

$$ {}^{(5)}\phi(y,x) = \sum_{n=-\infty}^{n=\infty} {}^{(4)}\phi_n(x) e^{\frac{2\pi i n}{L} y}. \tag{2.1.4} $$

Substituting this into the action and integrating over y, we have

$$ \begin{aligned} S = & -\frac{1}{2}\int d^4x \partial_\mu \hat{\phi}_0(x) \partial^\mu \hat{\phi}_0(x) \\ & - \int d^4x \sum_{n=1}^{n=\infty} \left\{ \partial_\mu \hat{\phi}_n(x) \partial^\mu \hat{\phi}_{-n}(x) + m^2_{KK(n)} \hat{\phi}_n(x) \hat{\phi}_{-n}(x) \right\}, \end{aligned} \tag{2.1.5} $$

where we define $\hat{\phi}_n := \sqrt{2\pi L}\, {}^{(4)}\phi_n(x)$ and

$$ m^2_{KK(n)} := \left(\frac{2\pi n}{L} \right)^2. \tag{2.1.6} $$

The first line and second line in Eq. (2.1.5) are corresponding to 4-dimensional massless and massive scalar fields respectively. The former massless mode is called zero-mode and the latter massive mode is called Kaluza-Klein (KK) mode. In very low energy only zero mode kinetic term is excited. But if we have large enough energy, say much larger than m_{KK}, KK modes will be excited. This may be regarded as exotic particles which we cannot usually observe in the low energy. To prevent the excitation of KK modes, the size of the extra dimension should be very small. As a summary we can say that to be consistent with no observational evidence of massive scalar particles, the size of the extra dimension should be very small.

2.1.1 Arkani-Hamed-Dimopoulos-Dvali (ADD) model

Arkani-Hamed, Dimopoulos and Dvali (ADD) [5] proposed the first version of braneworld model[1]. They explain gauge hierarchy model by assuming the large number of extra dimensions and the localization of standard model particles except "graviton". The localization of matter is motivated by superstring theory. The key object is D-brane. In the superstring theory, there exist open strings (matter) and closed strings (gravity). D-brane is corresponding to the collection of endpoints of open strings which turned out to be fundamental objects.

[1] The original braneworld model is made by Akama. But in the context of string theory Arkani-Hamed, Dimopoulos and Dvali (ADD) model can be regarded as the first.

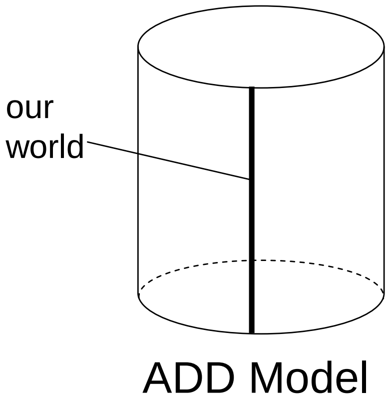


Figure 2.1.1: ADD model.

As in the Kaluza-Klein (KK) theory, we consider the space with topology $\mathscr{M} = X^4 \times R^q$. X^4 denote our world and R^q denotes q-dimensional extra dimension. At very small scale ($r \ll R$, where R is compactification scale), gravitational potential becomes

$$V(r) \sim \frac{G_{(4+q)} m_1 m_2}{r^{(q+1)}} \quad (r \ll R), \tag{2.1.7}$$

where $G_{(4+q)}$ is the gravitational constant in $4+q$-dimensional space-times. At much lager scale, (for $r \gg R$), the potential will be Newton's one

$$V(r) \sim \frac{G_{(4)} m_1 m_2}{r} \quad (r \gg R). \tag{2.1.8}$$

Note that this difference of two potentials can be explained as schematically shown in fig. 2.1.2.

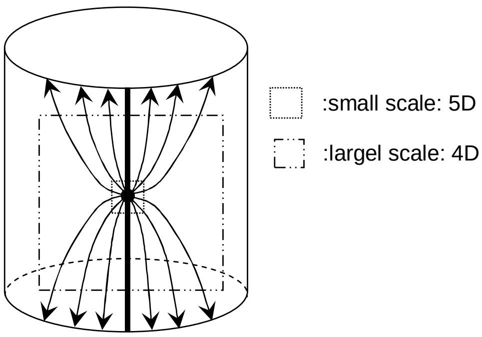


Figure 2.1.2: The behavior of gravitational force line.

At the intermediate scale ($r \sim R$) both potential will have the same value, that is

$$\frac{G_{(4+q)} m_1 m_2}{R^{q+1}} \sim \frac{G_{(4)} m_1 m_2}{R}. \tag{2.1.9}$$

From this we can have the relation between $G_{(4+q)}$ and $G_{(4)}$

$$G_{(4+q)} \sim G_{(4)} R^q. \tag{2.1.10}$$

Introducing the energy scales associated with the strength of the gravitational fields as $M_{(4+q)} \sim 1/G_{(4+q)}^{\frac{1}{2+q}}$ and $M_{(4)} \sim 1/G_{(4)}^{\frac{1}{2}}$, then we see that the above relation becomes

$$M_{(4)}^2 \sim M_{(4+q)}^{2+q} R^q. \tag{2.1.11}$$

Thus the gauge hierarchy problems can be solved by imposing

$$M_{(4+q)} \sim m_{EW}. \tag{2.1.12}$$

In this case, the size of extra dimensions is determined as

$$R \sim 10^{\frac{30}{q}-17} \times \left(\frac{1\text{TeV}}{m_{\text{EW}}}\right)^{1+\frac{2}{q}} [\text{cm}]. \tag{2.1.13}$$

Using this, we have the constraint on current model. For example, $q = 1$ case, we see

$$R \sim 10^{13}[\text{cm}] \sim 1[\text{AU}]. \tag{2.1.14}$$

This case is not permitted from the observation. For $q = 2$, the size of extra dimension becomes

$$R \sim 0.1[\text{mm}]. \tag{2.1.15}$$

This is marginally permitted for the tabletop experiments. For $q \geq 3$

$$R < 0.1[\text{mm}]. \tag{2.1.16}$$

This shows that modification of gravity may occur in the sub-millimeter scale.

2.1.2 Randall-Sundrum (RS) models

In Randall-Sundrum (RS) models the self-gravity of brane tension is taken account of and its whole space-time is 5 dimensions while in Arkani-Hamed-Dimopoulos-Dvali (ADD) model self-gravity of brane is not considered and the number of extra dimensions is arbitrary more than 2. Randall and Sundrum proposed two types of models. One is called RS1 model which is composed of two branes and solves the gauge hierarchy problem. And the other model is so called RS2 model which is composed only single brane in 5-dimensional space-time and it has infinite size of extra dimensions. Both cases propose quite a new picture of the universe and the extra dimension is strongly warped due to the self-gravity of branes.

Randall-Sundrum 1 (RS1) model

The total action for this system is given by

$$S = \frac{1}{2\kappa_{(5)}^2} \int d^5x \sqrt{-{}^{(5)}g} \left({}^{(5)}R - 2\Lambda_{(5)}\right) + S_+ + S_-, \tag{2.1.17}$$

where $\kappa^2_{(5)}$, ${}^{(5)}\!R$ and $\Lambda_{(5)}$ are 5-dimensional gravitational coupling, the 5-dimensional Ricci scalar, and the 5 dimensional (bulk) negative cosmological constant, respectively. Here, "bulk" means that the higher dimensional space-time where brane is imbedded in. S_+ and S_- are respectively the action of branes with positive and negative tension

$$S_+ = \int d^4x\sqrt{-\det{}^{(4)}\mathrm{g}}(-\lambda + \mathscr{L}^+_{\text{matter}}), \tag{2.1.18}$$

and

$$S_- = \int d^4x\sqrt{-\det{}^{(4)}\mathrm{g}}(\lambda + \mathscr{L}^-_{\text{matter}}), \tag{2.1.19}$$

where λ is positive constant which represents tension. $\mathscr{L}^{\pm}_{\text{matter}}$ is the Lagrangian for matters confined on the branes. As seen soon, the introduction of the bulk negative cosmological constant $\Lambda_{(5)}$ is essential to realize flat brane geometry. From now on, let us focus on the vacuum branes and we employ the following metric ansatz

$$ds^2 = dy^2 + a^2(y)\eta_{\mu\nu}(x)dx^\mu dx^\nu, \tag{2.1.20}$$

where y is the coordinate of the extra dimension, $\eta_{\mu\nu}$ is the metric of the 4-dimensional Minkowski space-time. In this coordinate the positive and negative tension branes are supposed to be located at $y=0$ and $y=L$, respectively. In addition, we assume Z_2-symmetry across each brane and the extra dimension is compactified to be S^1. The (μ,ν)-component of Einstein equation for the bulk space-time is

$$-\frac{1}{4}\partial_y\left(\frac{\partial_y a}{a}\right) - \left(\frac{\partial_y a}{a}\right)^2 = \frac{\Lambda_{(5)}}{6}. \tag{2.1.21}$$

The (y,y)-component of Einstein equation becomes

$$-\partial_y\left(\frac{\partial_y a}{a}\right) - \left(\frac{\partial_y a}{a}\right)^2 = \frac{\Lambda_{(5)}}{6}. \tag{2.1.22}$$

Using Eqs. (2.1.21) and (2.1.22) we have

$$\frac{\partial_y a}{a} = \sqrt{\frac{-\Lambda_{(5)}}{6}}, \tag{2.1.23}$$

and the solution is given by

$$a(y) = e^{-\frac{|y|}{\ell}}, \tag{2.1.24}$$

where $\ell := \sqrt{\frac{6}{-\Lambda_{(5)}}}$. As a result, the bulk space-time is 5-dimensional anti-de Sitter (AdS$_5$) space-time. The presence of branes implies the so-called junction condition

$$\left[\frac{\partial_y a}{a}\right]_{y=0} = -\frac{1}{3}\kappa^2_{(5)}\lambda, \tag{2.1.25}$$

$$\left[\frac{\partial_y a}{a}\right]_{y=L} = \frac{1}{3}\kappa_{(5)}^2\lambda, \tag{2.1.26}$$

which $[Q]_{y=0}$ is defined as the subtraction :$Q|_{y=0+\varepsilon} - Q|_{y=0-\varepsilon}$ for example. See appendix A for the derivation of the junction condition in general cases. Then, Z_2-symmetry and Eq. (2.1.24) implies the relation between the bulk curvature scale ℓ and the tension λ.

$$\frac{1}{\ell} = \frac{1}{6}\kappa_{(5)}^2\lambda, \tag{2.1.27}$$

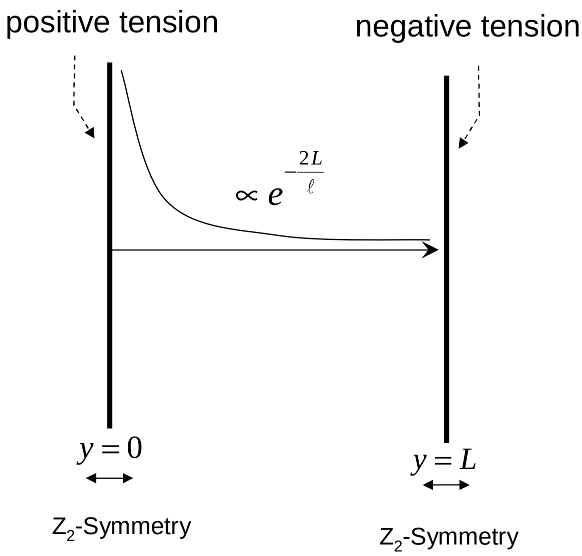


Figure 2.1.3: S^1/Z_2 compactified space and two branes.

Next we will look at the relation between 5-dimensional gravitational scale $M_{(5)} = \kappa_{(5)}^{-\frac{3}{2}}$ and the Planck scale. This consideration gives us how to solve the gauge hierarchy problem. To do so, we have to consider the fluctuation from the background space-times. For this purpose, we will employ the following metric ansatz without justification

$$ds^2 = dy^2 + e^{-\frac{2|y|}{\ell}} h_{\mu\nu}(x)dx^\mu dx^\nu, \tag{2.1.28}$$

where $h_{\mu\nu}(x)$ is the induced metric of the positive tension brane. Note that $q_{\mu\nu} = e^{-\frac{2|L|}{\ell}} h_{\mu\nu}(x)$ is the induced metric on the negative tension brane. Substituting this metric ansatz into the action and integrating over y, we obtain

$$S \sim \frac{M_{(5)}^3}{2}\ell(e^{\frac{2L}{\ell}} - 1)\int d^4x\sqrt{-\det{}^{(4)}g}\,{}^{(4)}R(q). \tag{2.1.29}$$

This equation can be regarded as the effective 4-dimensional gravitational action on the negative tension brane. From this we obtain the relation between $M_{\rm pl}$ and $M_{(5)}$ as

$$M_{\rm pl}^2 = M_{(5)}^3\ell(e^{\frac{2L}{\ell}} - 1). \tag{2.1.30}$$

By requiring following values

$$\ell \sim 0.1[\mathrm{mm}] \sim 10^{-3}[\mathrm{eV}] \quad \mathrm{M}_{(5)} \sim 1[\mathrm{TeV}], \tag{2.1.31}$$

we can obtain the following requirement for the ratio of the brane distance to the bulk curvature scale.

$$\frac{L}{\ell} \sim 19. \tag{2.1.32}$$

Then the gauge hierarchy problem is explained geometrically with the comparatively small components $\frac{L}{\ell} \sim O(10)$.

Randall-Sundrum 2 model and geometrical projection method

There is another type of Randall-Sundrum (RS) model. In this model only the positive tension brane exists and the extra dimension is not compactified. To see the feature of this model, we will employ the geometrical projection method developed by [9]. Then we will be able to have the 4-dimensional gravitational equation without any approximations.

Let us introduce the following coordinate locally

$$ds^2 = dY^2 + {}^{(4)}q_{\mu\nu}dx^\mu dx^\nu, \tag{2.1.33}$$

where Greek indices run 4 dimensions, $\mu = 0,1,2,3$, and capital Latin indices run 5 dimensions, $A = 0,1,2,3,4$. The brane is located at $Y = 0$ and its normal vector is described as $n^A = \left(\frac{\partial}{\partial Y}\right)^A$ and note that ${}^{(4)}q_{\mu\nu}(0,x)$ is the induced metric on the brane. The key relation is Gauss equation

$${}^{(4)}R^\alpha{}_{\beta\gamma\delta} = {}^{(5)}R^A{}_{BCD}q^\alpha_A q_\beta{}^B q_\gamma{}^C q_\delta{}^D + K^\alpha_\gamma K_{\beta\delta} - K^\alpha_\delta K_{\beta\gamma}, \tag{2.1.34}$$

where ${}^{(4)}R^\alpha{}_{\beta\gamma\delta}$ is the 4-dimensional Riemann tensor and ${}^{(5)}R^A{}_{BCD}$ is the 5-dimensional Riemann tensor. $K_{\mu\nu}$ is the extrinsic curvature defined by

$$\begin{aligned} K_{\mu\nu} &= \frac{1}{2}\mathscr{L}_n {}^{(4)}q_{\mu\nu} &(2.1.35)\\ &= {}^{(5)}\nabla_\mu {}^{(5)}n_\nu, &(2.1.36)\end{aligned}$$

where $\mathscr{L}_n Q$ denote the Lie derivative of Q with respect to the tangent vector n^A.

Taking the trace of Eq. (2.1.34), 5-dimensional Ricci tensor can be expressed as

$${}^{(4)}R_{\mu\nu} = {}^{(5)}R_{AB}q_\mu^A q_\nu{}^B - {}^{(5)}R^A_{BCD}n_A q_\mu{}^B n^C q_\nu{}^D + KK_{\mu\nu} - K_\mu{}^\alpha K_{\nu\alpha}, \tag{2.1.37}$$

Then, taking the combination of 4-dimensional Einstein tensors in the left hand side we have

$$\begin{aligned} {}^{(4)}G_{\mu\nu} &= \left({}^{(5)}R_{AB} - \frac{1}{2}{}^{(5)}g_{AB}{}^{(5)}R\right) q_\mu{}^A q_\nu{}^B + {}^{(5)}R_{AB}n^A n^B {}^{(4)}q_{\mu\nu} + KK_{\mu\nu} - K_\mu{}^\rho K_{\nu\rho} \\ &\quad -\frac{1}{2}q_{\mu\nu}\left(K^2 - K^{\alpha\beta}K_{\alpha\beta}\right) - {}^{(5)}R^A{}_{BCD}n_A n^C q_\mu{}^B q_\nu{}^D, \end{aligned} \tag{2.1.38}$$

where Einstein tensor is defined as ${}^{(4)}G_{\mu\nu} := {}^{(4)}R_{\mu\nu} - \frac{1}{2}{}^{(4)}g_{\mu\nu}{}^{(4)}R$. Using the 5-dimensional Einstein equation

$$ {}^{(5)}R_{AB} - \frac{1}{2}{}^{(5)}g_{AB}{}^{(5)}R = \kappa_{(5)}^2 T_{AB}, \tag{2.1.39} $$

we can replace the 5-dimensional Ricci tensor in Eq. (2.1.38) by the bulk energy-momentum tensor. And the last terms includes Weyl tensor ${}^{(5)}C_{ABCD}$, which characterizes gravitational wave degree of freedom. More precisely the Weyl tensor is included in the 5-dimensional Riemann tensor as

$$ {}^{(5)}R_{ABCD} = \frac{2}{3}\left({}^{(5)}g_{A[C}{}^{(5)}R_{D]B} - {}^{(5)}g_{B[C}{}^{(5)}R_{D]A}\right) - \frac{1}{6}{}^{(5)}g_{A[C}{}^{(5)}g_{D]B}{}^{(5)}R + {}^{(5)}C_{ABCD}, \tag{2.1.40} $$

where ${}^{(5)}g_{A[C}{}^{(5)}g_{D]B} = \frac{1}{2}({}^{(5)}g_{AC}{}^{(5)}g_{DB} - {}^{(5)}g_{AD}{}^{(5)}g_{CB})$ and ${}^{(5)}g_{A[C}{}^{(5)}R_{D]B} = \frac{1}{2}({}^{(5)}g_{AC}{}^{(5)}R_{DB} - {}^{(5)}g_{AD}{}^{(5)}R_{CB})$. By substituting the 5-dimensional Einstein equation and Eq. (2.1.40) into Eq. (2.1.38), we obtain

$$ \begin{aligned} {}^{(4)}G_{\mu\nu} &= \frac{2\kappa_{(5)}^2}{3}\left(T_{AB}q_\mu^{\ A}q_\nu^{\ B} + \left(T_{AB}n^A n^B - \frac{1}{4}\right)\right) + KK_{\mu\nu} - K_\mu^{\ \alpha}K_{\alpha\nu} \\ &\quad -\frac{1}{2}{}^{(4)}q_{\mu\nu}(K^2 - K^{\alpha\beta}K_{\alpha\beta}) - E_{\mu\nu}, \end{aligned} \tag{2.1.41} $$

where $E_{\mu\nu}$ is the projected Weyl tensor defined by

$$ E_{\mu\nu} := {}^{(5)}R^A_{\ BCD}n_A n^C q_\mu^{\ B} q_\nu^{\ D}. \tag{2.1.42} $$

Next we will consider the Codacci equation

$$ {}^{(4)}D_\alpha K^\alpha_{\ \mu} - {}^{(4)}D_\mu K = {}^{(5)}R_{AB}n^A q_\mu^{\ B}, \tag{2.1.43} $$

where ${}^{(4)}D_\mu$ is the covariant derivative with respect to ${}^{(4)}q_{\mu\nu}$. Using the 5-dimensional Einstein equation the Ricci tensor in the right-hand side is replaced by energy momentum tensor, T_{AB} as

$$ {}^{(4)}D_\alpha K^\alpha_{\ \mu} - {}^{(4)}D_\mu K = \kappa_{(5)}^2\,{}^{(5)}T_{AB}n^A q_\mu^{\ B}. \tag{2.1.44} $$

So far our argument is very general. From now on we will focus on the geometry on the brane. In our system we have (5-dimensional) bulk cosmological constant, $\Lambda_{(5)}$ and energy momentum tensor localized on the brane, $S_{\mu\nu}$. $S_{\mu\nu}$ is composed from brane tension λ and 4-dimensional energy momentum tensor $\tau_{\mu\nu}$.

$$ T_{AB} = -\frac{\Lambda_{(5)}}{\kappa_{(5)}^2}{}^{(5)}g_{AB} + S_{\mu\nu}q_A^{\ \mu}q_B^{\ \nu}\delta(Y), \tag{2.1.45} $$

$$ S_{\mu\nu} = -\lambda\,{}^{(4)}q_{\mu\nu} + \tau_{\mu\nu}. \tag{2.1.46} $$

Next let us see the junction condition for extrinsic curvature $K_{\mu\nu}$. (See appendix A for the derivation.)

$$ [K_{\mu\nu}]_{y=y_\pm} = -\kappa_{(5)}^2\left(S_{\mu\nu} - \frac{1}{3}{}^{(4)}q_{\mu\nu}S\right). \tag{2.1.47} $$

Due to the Z_2-symmetry, that is $K^+_{\mu\nu} = -K^-_{\mu\nu}$ and then

$$K^+_{\mu\nu} = -\frac{\kappa^2_{(5)}}{2}\left(S_{\mu\nu} - \frac{1}{3}{}^{(4)}q_{\mu\nu}S\right). \tag{2.1.48}$$

Substituting this and the bulk stress tensor into Eq. (2.1.41), we obtain [9]

$${}^{(4)}G_{\mu\nu} = -\Lambda_{\rm eff}{}^{(4)}q_{\mu\nu} + 8\pi G_{\rm eff}\tau_{\mu\nu} + \kappa^4_{(5)}\pi_{\mu\nu} - E_{\mu\nu}. \tag{2.1.49}$$

This may be regarded as the 4-dimensional effective Einstein equation. Each term is defined by

$$\Lambda_{\rm eff} := \frac{1}{2}\left(\Lambda_{(5)} + \frac{1}{6}\kappa^4_{(5)}\lambda^2\right), \tag{2.1.50}$$

$$G_{\rm eff} := \frac{\kappa^4_{(5)}\lambda}{48\pi}, \tag{2.1.51}$$

$$\pi_{\mu\nu} = -\frac{1}{4}\tau_{\mu\alpha}\tau^\alpha_\nu + \frac{1}{12}\tau\tau_{\mu\nu} + \frac{1}{8}{}^{(4)}q_{\mu\nu}\tau_{\alpha\beta}\tau^{\alpha\beta} - \frac{1}{24}{}^{(4)}q_{\mu\nu}\tau^2. \tag{2.1.52}$$

$\Lambda_{\rm eff}$ is the net 4-dimensional cosmological constant. If one wants to realize the flat brane, that is, $\Lambda_{\rm eff} = 0$, $\Lambda_{(5)} = -\frac{1}{6}\kappa^4_{(5)}\lambda^2$ is required. $G_{\rm eff}$ is the 4-dimensional effective gravitational constant.

To see the recovery of the 4-dimensional Einstein equation we examine the order of the magnitude. If one is interested in the energy scale of M we can evaluate some ratios as

$$\frac{|\kappa^4_{(5)}\pi_{\mu\nu}|}{|8\pi G_{\rm eff}\tau_{\mu\nu}|} \sim \frac{M^4}{M^4_\lambda}, \tag{2.1.53}$$

$$\frac{|E_{\mu\nu}|}{|8\pi G_{\rm eff}\tau_{\mu\nu}|} \sim \left(\frac{M}{M_\lambda}\right)^2\left(\frac{M_{(5)}}{M_\lambda}\right)^6, \tag{2.1.54}$$

where $M_\lambda \sim \lambda^{-\frac{1}{4}}$. $E_{\mu\nu}$ is evaluated by the corrected Newton potential in Randall-Sundrum 2 (RS2) model $V(R) \sim \frac{G_{(5)}m_1m_2}{r}\left(1 + \left(\frac{M^3_{(5)}}{M^4_\lambda}\frac{1}{r}\right)^2\right)$, [7], [8]. From these we can see that $\pi_{\mu\nu}$ and $E_{\mu\nu}$ are negligible at low energy scale ($\frac{M}{M_\lambda} \ll 1$)

$${}^{(4)}G_{\mu\nu} \simeq -\Lambda_{\rm eff}{}^{(4)}q_{\mu\nu} + 8\pi G_{\rm eff}\tau_{\mu\nu}. \tag{2.1.55}$$

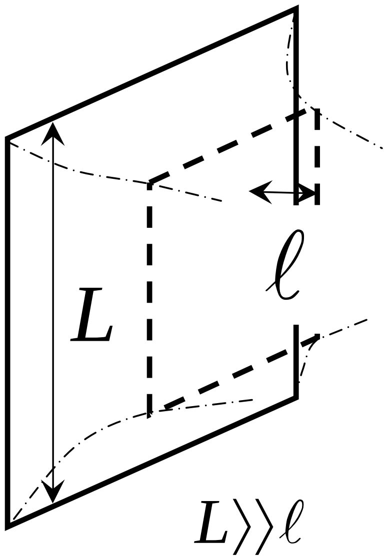
Long-wavelength approximation

Let us consider the Randall-Sundrum 1 (RS1) model again. For RS1 model it is known that $E_{\mu\nu}$ has main contribution to the effective theory at even low energy. Therefore, we have to evaluate $E_{\mu\nu}$ carefully. To do so, we should solve the full system somehow. We will describe the long-wavelength approximation which will be used in the next chapter. Here we will apply this method into RS1 model for simplicity.

Let us see the detail now. Here, we begin with definition of the expansion parameter

$$\varepsilon := \frac{{}^{(4)}R}{\partial_y K} \sim \left(\frac{\ell}{L}\right)^2 \ll 1. \tag{2.1.56}$$

where ℓ is the 5-dimensional (bulk) curvature scale and L is the typical scale in which one is interested.



For the present purpose we use the following metric ansatz

$$ds^2 = e^{2\varphi(y,x)}dy^2 + g_{\mu\nu}(y,x)dx^\mu dx^\nu, \tag{2.1.57}$$

where we suppose that positive and negative tension branes are located at $y = y_+$ and $y = y_-$, respectively. The metric and extrinsic curvature is expanded as

$$g_{\mu\nu}(y,x) = a^2(y)\left(h_{\mu\nu}(x) + \overset{(1)}{h}_{\mu\nu}(y,x) + \cdots\right), \tag{2.1.58}$$

$$:= \overset{(0)}{g}_{\mu\nu} + \overset{(1)}{g}_{\mu\nu} + \cdots \tag{2.1.59}$$

and

$$K^\mu_\nu = \overset{(0)}{K}{}^\mu_\nu + \overset{(1)}{K}{}^\mu_\nu + \cdots. \tag{2.1.60}$$

The equation which we have to solve is 5-dimensional Einstein equation with junction condition. The junction condition gives us the boundary condition on the brane.

To solve the bulk it is convenient decompose evolution equations into traceless/trace part. The traceless part of (μ,ν)-component of Einstein equation is

$$e^{-\varphi}\partial_y\tilde{K}^\mu_\nu + K\tilde{K}^\mu_\nu = {}^{(4)}\tilde{R}^\mu_\nu + \left({}^{(4)}D^\mu\,{}^{(4)}D_\nu\varphi + {}^{(4)}D^\mu\varphi\,{}^{(4)}D_\nu\varphi\right)_{\text{traceless}}, \tag{2.1.61}$$

where tilde denote the traceless part of quantities such as $\tilde{K}^\mu_\nu = K^\mu_\nu - \frac{1}{4}\delta^\mu_\nu K$ and ${}^{(4)}\tilde{R}^\mu_\nu = {}^{(4)}R^\mu_\nu - \frac{1}{4}\delta^\mu_\nu\,{}^{(4)}R$. ${}^{(4)}D_\mu$ is the covariant derivative with respect to induced metric ${}^{(4)}g_{\mu\nu}$. Since we assume Z_2-symmetry across the brane, the junction condition becomes

$$\left(K^\mu_\nu - \delta^\mu_\nu K\right)_{y=y_\pm} = -\frac{\kappa^2_{(5)}}{2}S^{\mu\pm}_\nu. \tag{2.1.62}$$

And the trace part of (μ,ν)-component Einstein equation is

$$e^{-\varphi}\partial_y K+\frac{1}{4}K^2+\tilde{K}^{\alpha}_{\beta}\tilde{K}_{\alpha}^{\ \beta}=-\frac{2}{3}\Lambda_{(5)}+{}^{(4)}\!R+\left({}^{(4)}\!D^2\varphi+({}^{(4)}\!D_\alpha\varphi)^2\right), \tag{2.1.63}$$

where $\Lambda_{(5)}=-\frac{6}{\ell^2}$, ${}^{(4)}\!R:=R^{\alpha}_{\ \alpha}$ and $K:={}^{(4)}\!K^{\alpha}_{\ \alpha}$. The (y,y)-component implies the Hamiltonian constraint

$$\frac{3}{4}K^2-\tilde{K}^{\alpha}_{\beta}\tilde{K}_{\alpha}^{\ \beta}={}^{(4)}\!R+\frac{12}{\ell^2}. \tag{2.1.64}$$

The (y,μ)-component does

$${}^{(4)}\!D_\alpha\tilde{K}^{\alpha}_{\ \mu}-\frac{3}{4}{}^{(4)}\!D_\mu K=0. \tag{2.1.65}$$

It is ready to solve the Einstein equation using the long-wavelength approximation.

At the 0th order, Eqs. (2.1.61), (2.1.63), (2.1.64) and (2.1.65) become

$$e^{-\varphi}\partial_y\overset{(0)}{\tilde{K}}{}^{\mu}_{\ \nu}+\overset{(0)}{K}\overset{(0)}{\tilde{K}}{}^{\mu}_{\ \nu}=0, \tag{2.1.66}$$

$$e^{-\varphi}\partial_y\overset{(0)}{K}+\frac{1}{4}+\overset{(0)}{\tilde{K}}{}^{\alpha}_{\ \beta}\overset{(0)}{\tilde{K}}{}_{\alpha}^{\ \beta}=-\frac{2}{3}\Lambda_{(5)}, \tag{2.1.67}$$

$$\frac{3}{4}\overset{(0)}{K}{}^2-\overset{(0)}{\tilde{K}}{}^{\alpha}_{\ \beta}\overset{(0)}{\tilde{K}}{}_{\alpha}^{\ \beta}=\frac{12}{\ell^2} \tag{2.1.68}$$

and

$${}^{(4)}\!D_\alpha\overset{(0)}{\tilde{K}}{}^{\alpha}_{\ \mu}-\frac{3}{4}{}^{(4)}\!D_\mu\overset{(0)}{K}=0, \tag{2.1.69}$$

where ${}^{(4)}\!D_\mu$ is the covariant derivative with respect to $g_{\mu\nu}$. From Eq. (2.1.61) Israel junction condition at the 0th order is

$$\left(\overset{(0)}{K}{}^{\mu}_{\ \nu}-\delta^{\mu}_{\ \nu}\overset{(0)}{K}\right)_{y=y_\pm}=\pm\frac{\kappa^2_{(5)}}{2}\lambda\delta^{\mu}_{\ \nu}. \tag{2.1.70}$$

Let us solve the above equations. From Eq. (2.1.66),

$$e^{-\varphi}\partial_y\overset{(0)}{\tilde{K}}{}^{\mu}_{\ \nu}=-\overset{(0)}{K}\overset{(0)}{\tilde{K}}{}^{\mu}_{\ \nu} \tag{2.1.71}$$

It is easy to check that the solution of traceless part. $\overset{(0)}{\tilde{K}}{}^{\mu}_{\ \nu}(y,x)$ is given by

$$\overset{(0)}{\tilde{K}}{}^{\mu}_{\ \nu}(y,x)=\frac{C^{\mu}_{\ \nu}(x)}{\sqrt{-{}^{(4)}g}}, \tag{2.1.72}$$

where $C^{\mu}_{\ \nu}(x)$ is the integral constant. But by the junction condition it becomes

$$\overset{(0)}{\tilde{K}}{}^{\mu}_{\ \nu}(y_{y_\pm},x)=0. \tag{2.1.73}$$

This implies

$$C^{\mu}_{\nu}(x) = 0. \tag{2.1.74}$$

Thus,

$$\overset{(0)}{\tilde{K}}{}^{\mu}_{\nu}(y,x) = 0. \tag{2.1.75}$$

From Eq. (2.1.64) the Hamiltonian constraint becomes

$$\frac{3}{4}\overset{(0)}{K}{}^{2} = \frac{12}{\ell^2}. \tag{2.1.76}$$

From Eq. (2.1.73), the trace part of the junction condition is

$$\overset{(0)}{K}(y_+,x) = -\frac{2}{3}\kappa^2_{(5)}\lambda. \tag{2.1.77}$$

Thus, we see

$$\overset{(0)}{K} = -\frac{4}{\ell} = -\frac{2}{3}\kappa^2_{(5)}\lambda. \tag{2.1.78}$$

As a short summary, we obtain

$$\overset{(0)}{K}{}^{\mu}_{\nu} = -\frac{1}{\ell}\delta^{\mu}_{\nu}. \tag{2.1.79}$$

We can also obtain metric at the 0th order $\overset{(0)}{g}_{\mu\nu}$ from this solution of extrinsic curvature. Remembering the definition

$$\overset{(0)}{K}_{\mu\nu} = \frac{1}{2}e^{-\varphi}\partial_y \overset{(0)}{g}_{\mu\nu}, \tag{2.1.80}$$

$\overset{(0)}{g}_{\mu\nu}$ is given by

$$\overset{(0)}{g}_{\mu\nu} = a^2(y)h_{\mu\nu}(x), \tag{2.1.81}$$

where $a(y) = e^{-\frac{d(y,x)}{\ell}}$ and $d(y,x) := \int_0^y e^{\varphi(y',x)}dy'$. Note that the 0th order solution (2.1.79) automatically satisfies momentum constraint at the 0th order.

Now let us move to calculation at the 1st order. At the 1st order Eqs. (2.1.61), (2.1.63), (2.1.64) and (2.1.65) become

$$e^{-\varphi}\partial_y \overset{(1)}{\tilde{K}}{}^{\mu}_{\nu} + \overset{(0)}{K}\overset{(1)}{\tilde{K}}{}^{\mu}_{\nu} = -{}^{(4)}\tilde{R}^{\mu}_{\nu}(\overset{(0)}{g}_{\mu\nu}) + e^{-\varphi^{(1)}}\left({}^{(4)}\overset{(0)}{D}{}^{\mu}\,{}^{(4)}\overset{(0)}{D}_{\nu}\,\varphi\right)_{\text{traceless}}, \tag{2.1.82}$$

$$e^{-\varphi}\partial_y \overset{(1)}{K} + \frac{1}{2}\overset{(0)}{K}\overset{(1)}{K} = {}^{(4)}R + e^{-\varphi^{(1)}}\left({}^{(4)}\overset{(0)}{D}{}^{2}\varphi\right), \tag{2.1.83}$$

$$\frac{3}{2}\overset{(0)}{K}\overset{(1)}{K} = {}^{(4)}R(\overset{(0)}{g}_{\mu\nu}) \tag{2.1.84}$$

and

$$ {}^{(4)}\overset{(0)}{D}_\alpha \overset{(1)}{\tilde{K}}{}^\alpha{}_\mu - \frac{3}{4} {}^{(4)}\overset{(0)}{D}_\mu \overset{(1)}{K} = 0. \tag{2.1.85} $$

The junction condition at the 1st order gives us the boundary condition

$$ \left(\overset{(1)}{K}{}^\mu{}_\nu - \delta^\mu_\nu \overset{(1)}{K} \right)_{y=y_\pm} = \mp \frac{\kappa^2_{(5)}}{2} \overset{(\pm)}{T}{}^\mu{}_\nu \tag{2.1.86} $$

Let us solve the above equations. Using the Hamiltonian constraint of Eq. (2.1.84), we can eliminate ${}^{(4)}R$ in Eq. (2.1.83), we obtain

$$ e^{-\varphi} \partial_y \overset{(1)}{K} - \frac{1}{2} \overset{(0)}{K} \overset{(1)}{K} = e^{-\varphi^{(1)}} \left({}^{(4)}\overset{(0)}{D}{}^2 \varphi \right). \tag{2.1.87} $$

where ${}^{(4)}D_\mu$ is the covariant derivative with respect to $g_{\mu\nu}$. Using

$$ {}^{(4)}R^\mu{}_\nu(g) = \frac{1}{a^2} \left\{ {}^{(4)}R^\mu{}_\nu(h) + \frac{2}{\ell} \overset{(h)}{D}{}^\mu \overset{(h)}{D}_\nu d + \frac{1}{\ell} \delta^\mu_\nu \overset{(h)}{D}{}^2 d + \frac{2}{\ell^2} \left(\overset{(h)}{D}{}^\mu \overset{(h)}{D}_\nu d - \delta^\mu_\nu (\overset{(h)}{D} d)^2 \right) \right\}, \tag{2.1.88} $$

and

$$ {}^{(4)}D^\mu {}^{(4)}D_\nu e^\varphi = \frac{1}{a^2} \partial_y \left\{ \overset{(h)}{D}{}^\mu \overset{(h)}{D}_\nu d + \frac{1}{\ell} \left(\overset{(h)}{D}{}^\mu d \overset{(h)}{D}_\nu d - \frac{1}{2} \delta^\mu_\nu (\overset{(h)}{D}_\alpha d)^2 \right) \right\}, \tag{2.1.89} $$

Eq. (2.1.82) becomes

$$ \partial_y (a^4 \overset{(1)}{\tilde{K}}{}^\mu{}_\nu) = -\frac{\ell}{2} \partial_y (a^2) {}^{(4)}\tilde{R}^\mu{}_\nu - \partial_y \left\{ a^2 \left(\overset{(h)}{D}{}^\mu \overset{(h)}{D}_\nu d + \frac{1}{\ell} \overset{(h)}{D}{}^\mu d \overset{(h)}{D}_\nu d \right)_{\text{traceless}} \right\}, \tag{2.1.90} $$

where $\overset{(h)}{D}_\mu$ is the covariant derivative with respect to $h_{\mu\nu}$ and 4-dimensional traceless part of Q^μ_ν is defined by $\{Q^\mu{}_\nu\}_{\text{traceless}} := Q^\mu{}_\nu - \frac{1}{4} \delta^\mu_\nu Q^\alpha_\alpha$. Its solution is obtained by simple integration

$$ \tilde{K}^\mu{}_\nu(y,x) = -\frac{\ell}{2} a^{-2} \tilde{R}^\mu{}_\nu(h) - a^{-2} \left(\overset{(h)}{D}{}^\mu \overset{(h)}{D}_\nu d + \frac{1}{\ell} \overset{(h)}{D}{}^\mu d \overset{(h)}{D}_\nu d \right)_{\text{traceless}} + \chi^\mu{}_\nu(x). \tag{2.1.91} $$

where $\chi^\mu{}_\nu(x)$ is the integral constant of the above traceless part of evolution equation. From the Hamiltonian constraint, we obtain the solution to the trace part of the extrinsic curvature as

$$ \overset{(1)}{K}(y,x) = -\frac{\ell}{6a^2} {}^{(4)}R(h) - \frac{1}{a^2} \left(\overset{(h)}{D}{}^2 d - \frac{1}{\ell} (\overset{(h)}{D} d)^2 \right). \tag{2.1.92} $$

These results are summarized as

$$ \overset{(1)}{K}{}^\mu{}_\nu - \delta^\mu_\nu \overset{(1)}{K} = \overset{(1)}{\tilde{K}}{}^\mu{}_\nu - \frac{3}{4} \delta^\mu_\nu \overset{(1)}{K} \tag{2.1.93} $$

$$
\begin{aligned}
&= -\frac{\ell}{2}a^{-2}\,{}^{(4)}G^{\mu}{}_{\nu}(h) - a^{-2}\left\{\overset{(h)}{D}{}^{\mu}\,\overset{(h)}{D}_{\nu}\,d - \delta^{\mu}_{\nu}\,\overset{(h)}{D}{}^{2}d + \frac{1}{\ell}\left(\overset{(h)}{D}{}^{\mu}d\,\overset{(h)}{D}_{\nu}\,d + \frac{1}{2}\delta^{\mu}_{\nu}(\overset{(h)}{D}_{\alpha}\,d)^{2}\right)\right\} \\
&\quad + \chi^{\mu}{}_{\nu}a^{-4}.
\end{aligned}
\tag{2.1.94}
$$

Let us consider the junction conditions. From the junction conditions on the $y = y_+$ brane, we obtain

$$
-\frac{\kappa^2_{(5)}}{2}\,\overset{(+)}{T}{}^{\mu}{}_{\nu} = -\frac{\ell}{2}\,{}^{(4)}G^{\mu}{}_{\nu}(h) + \chi^{\mu}{}_{\nu}. \tag{2.1.95}
$$

On the other hand, the junction condition at $y = y_-$ brane yields

$$
-\frac{\kappa^2_{(5)}}{2}\,\overset{(-)}{T}{}^{\mu}{}_{\nu}(g_{(-)}) = -\frac{\ell}{2}a_0^{-2}\,{}^{(4)}G^{\mu}{}_{\nu}(h) - a_0^{-2}\left(\overset{(h)}{D}{}^{\mu}d\,\overset{(h)}{D}_{\nu}\,d_0\right) + \chi^{\mu}{}_{\nu}. \tag{2.1.96}
$$

Eliminating $\chi^{\mu}{}_{\nu}$ from the above two equations, we obtain the gravitational equation at low energy scale

$$
\begin{aligned}
(1-a_0^2)\,{}^{(4)}G^{\mu}{}_{\nu}(h) &= \frac{\kappa^2_{(5)}}{\ell}\left(\overset{(+)}{T}{}^{\mu}{}_{\nu}(h) + a_0^{-2}\,\overset{(-)}{T}{}^{\mu}{}_{\nu}(h)\right) \\
&\quad + \frac{2}{\ell}a_0^2\left\{\overset{(h)}{D}{}^{\mu}d\,\overset{(h)}{D}_{\nu}\,d_0 - \delta^{\mu}_{\nu}\,\overset{(h)}{D}{}^{2}d_0 + \frac{1}{\ell}\left(\overset{(h)}{D}{}^{\mu}d_0\,\overset{(h)}{D}_{\nu}\,d_0 + \frac{1}{2}\delta^{\mu}_{\nu}(\overset{(h)}{D}{}^{\alpha}d)^{2}\right)\right\}.
\end{aligned}
\tag{2.1.97}
$$

We would like to remind reader that this procedure to obtain the low energy effective theory will be employed in the next chapter.

2.2 Toward the construction of higher-dimensional braneworld

Braneworld picture is motivated by superstring theory. The superstring theory is formulated in 10 dimensions. Thus, the well-studied 5-dimensional model is still “toy model”. As a next step, we will consider 6-dimensional model. And the rest of this chapter is dedicated to the study of this 6-dimensional gravity. In 6 dimension, in order to construct effective theory on the brane we have to overcome the singularity problem arise from the self-gravity of co-dimension-2 brane. We will discuss capped regularization scheme for a remedy of this problem. Additionally, there is another problem, that is, the size of extra dimension is unstable. We can stabilize the extra dimension by simply introducing flux in the system.

2.2.1 Occurrence of singularity and the idea of its avoidance

In co-dimension-2 braneworld, it is known that we cannot put arbitrary matter on the brane. Let us see the difficulty of co-dimension-2 braneworld using concrete model discussed in [15]. We begin with the following metric ansatz

$$ds^2 = -N^2(y)dt^2 + M^2(y)\delta_{ij}dx^i dx^j + B^2(y)dy^2 + L^2(y)d\phi^2. \tag{2.2.1}$$

where y and ϕ are the coordinate of extra dimensions. To see the feature of singularity, let us suppose that the most wrong behavior appears in $\partial_y^2 g_{AB}$ and we can ignore $\partial_y M$, $\partial_y L$, $\partial_y N$, $\partial_y B$. Then we obtain following equations (See appendix B for full equations)

$$-3\frac{\partial_y^2 M(y)}{M(y)} - \frac{\partial_y^2 L(y)}{L(y)} \simeq 8\pi G_{(6)}\frac{B^2(y)}{N^2(y)}\rho N^2(y)\delta^{(2)}(y), \tag{2.2.2}$$

$$2\frac{\partial_y^2 M(y)}{M(y)} + \frac{\partial_y^2 N(y)}{N(y)} + \frac{\partial_y^2 L(y)}{L(y)} \simeq 8\pi G_{(6)}\frac{B^2(y)}{M^2(y)}PM^2(y)\delta_{ij}\delta^{(2)}(y), \tag{2.2.3}$$

$$3\frac{\partial_y^2 M(y)}{M(y)} + \frac{\partial_y^2 N(y)}{N(y)} \simeq 0. \tag{2.2.4}$$

They are derived from $(0,0)$, (i,j) and (θ,θ)-components of 6-dimensional Einstein equation respectively. Note that we put the perfect fluid matter as localized one, not Nambu-Goto type. P is the pressure. From the above equations we have the following two

$$-4\frac{\partial_y^2 M(y)}{M(y)} = 8\pi G_{(6)}B^2(y)(\rho + P)\delta^{(2)}(y), \tag{2.2.5}$$

$$-2\left(\frac{\partial_y^2 M(y)}{M(y)} + \frac{\partial_y^2 L(y)}{L(y)}\right) = 8\pi G_{(6)}B^2(y)(\rho - P)\delta^{(2)}(y). \tag{2.2.6}$$

If the equation of state of tension is $P = -\rho := -\mu$, then

$$\partial_y^2 M(y) \simeq \partial_y N^2(y) \simeq 0, \tag{2.2.7}$$

$$\partial_y^2 L(y) \simeq 8\pi G_{(6)}\mu\delta^{(2)}(y). \tag{2.2.8}$$

The solution to Eq. (2.2.8) is given by Minkowski space-time with deficit angle

$$ds^2 = -C_1 dt^2 + C_2\delta_{ij}dx^i dx^j + C_3 dy^2 + (1 - 8\pi G_{(6)}\mu)y^2 d\phi^2. \tag{2.2.9}$$

This is the well-known solution for co-dimension-2 pure tension brane which can be analyzed. Therefore this case does not yield any problems.

What happens if other matters, such as $P \neq -\rho$ (non-Nambu-Goto type) are put on the 3-brane? In this case each second derivative will not vanish, that is

$$\frac{\partial_y^2 M}{M} \sim \frac{\partial_y^2 N}{N} \sim \frac{\partial_y^2 L}{L} \sim \delta^{(2)}(y) \sim \frac{1}{y}\delta(y), \tag{2.2.10}$$

which implies

$$M \propto y^2, \quad N \propto y^2, \quad L \propto y^2. \tag{2.2.11}$$

This solution is not under control of physics because the induced metric on the brane vanishes. From this example, we can say that arbitrary matters cannot be put on the co-dimension-2 brane.

Capped regularization

In order to treat co-dimension-2 braneworld model, we need some regularization scheme. Let us introduce the essence of capped regularization scheme used in the next chapter.

The procedure of this scheme is as follows. At first the singular region around the pole is removed (①,②). Then we introduce 5-dimensional special brane (③) and by way of the brane, new regular space is glued into original bulk space instead of the removed singular space(④). The detail will be described in the next chapter.

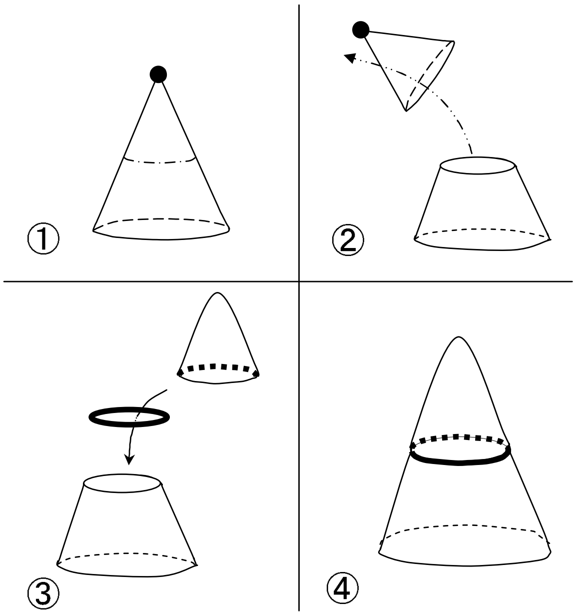


Figure 2.2.1: “Bulk surgery” : procedure for capped regularization.

2.2.2 Stabilization of extra dimensions by flux

When the number of extra dimension is increased, extra dimensions will be unstable in general. Here, we will introduce the Freund-Rubin stabilization mechanism by using flux [18]. This idea will be partially cured in the construction of 6-dimensional braneworld given in the next chapter. (See appendix C for some details)

The model we consider here is

$$S = \int d^{p+q}x \sqrt{-{}^{(p+q)}g} \left({}^{(p+q)}R - \frac{1}{2q!}F^2 \right), \tag{2.2.12}$$

where F is the q-form field. Let us start from following metric ansatz

$$\begin{aligned} ds^2 &= {}^{(p)}g_{\mu\nu}dx^\mu dx^\nu + {}^{(q)}g_{ij}dx^i dx^j &(2.2.13)\\ &= e^{-\frac{2q}{p-2}\phi(x)}h_{\mu\nu}dx^\mu dx^\nu + e^{2\phi(x)}\sigma_{ij}dx^i dx^j, &(2.2.14)\end{aligned}$$

where $g_{\mu\nu}$ is the p-dimensional space-time metric and σ_{ij} is the metric of q-dimensional unit sphere. The solution of Maxwell equation can be obtained as

$$F_{M_1\cdots M_q} = f\varepsilon_{M_1\cdots M_q}(\sigma), \tag{2.2.15}$$

where we assume the magnitude of flux f is constant, $\varepsilon_{M_1\cdots M_q}(\sigma)$ is the perfect anti-symmetrized tensor with respect to q-dimensional sphere.

Substituting Eq. (2.2.14) and (2.2.15) into Eq. (2.2.12) we obtain p-dimensional action

$$\begin{aligned} S_{\rm eff} &= \frac{1}{2}\int d^{p+q}x\sqrt{-{}^{(p+q)}g}\left({}^{(p+q)}R(g) - \frac{1}{2q!}F^2\right) &(2.2.16)\\ &= \frac{1}{2}\Omega_q \cdot \int d^p x\sqrt{-h}\Big({}^{(p)}R(h) \\ &\quad -\frac{q(p+q-2)}{p-2}(\overset{(h)}{D}_\alpha \phi)^2 + \frac{2q(p-1)}{p-2}\overset{(h)}{D}{}^2\phi - V(\phi)\Big), &(2.2.17)\end{aligned}$$

where Ω_q is the volume of q-dimensional sphere and

$$V(\phi) := \frac{f^2}{2}\left(-\frac{q(p-1)}{p+q-2}e^{-\frac{2(p+q-2)}{p-2}\phi(x)} + e^{-\frac{2(p-1)q}{p-2}\phi(x)}\right). \tag{2.2.18}$$

The typical shape of potential is depicted in Fig.2.2.2. Due to the tension of the magnetic flux , the size of extra dimensions are stabilized.

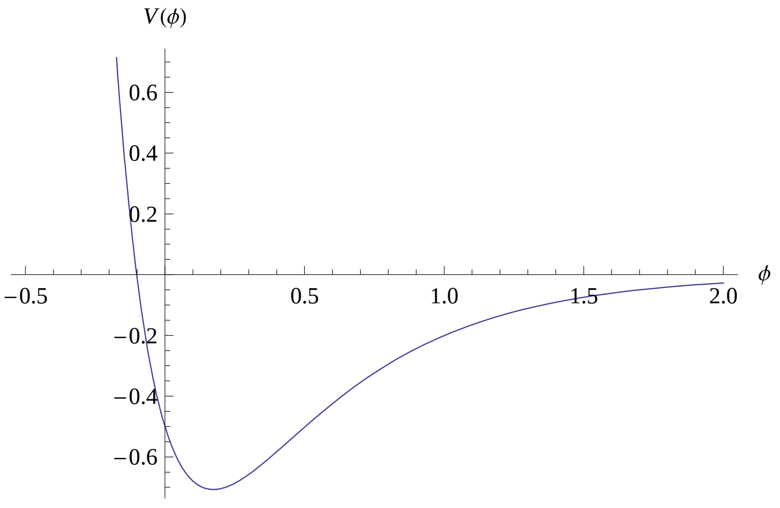


Figure 2.2.2: Effective potential $V(\phi)$ with flux. ($p = 4, q = 2$)

For comparison with the above case, let us see the potential without flux. The potential for ϕ becomes

$$V(\phi) = -\frac{f^2}{2}\frac{q(p-1)}{p+q-2}e^{-\frac{2(p+q-2)}{p-2}\phi(x)}. \tag{2.2.19}$$

As seen easily, the size of extra dimension is not stabilized without flux.

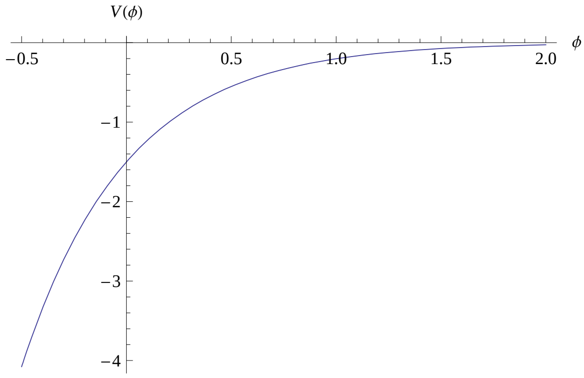


Figure 2.2.3: The effective potential $V(\phi)$ without flux ($p = 4, q = 2$).

Chapter 3

Low energy effective theory on a regularized brane in six-dimensional flux compactifications

We have seen the study of braneworld model in previous chapter through Arkani-Hamed-Dimopoulos-Dvali (ADD) model and Randall-Sundrum (RS) models. Randall-Sundrum model (RS) is attractive model due to including self-gravity in itself but it is limited to only 5-dimensional space-time. From the viewpoint of the string theory formulated in 10 dimensions we need higher dimensional braneworld model. Thus, we will study 6-dimensional model as a next step. We will employ capped regularization and flux stabilization mechanism to resolve some problems occurred in 6-dimensional models. Then we discuss the gravitational theory on the brane.

This chapter is composed of four sections. In next section 3.1, we describe the system and the basic equations. In section 3.2, using long-wavelength approximation, we approximately solve Einstein and Maxwell equations in the bulk. Then in section 3.3, we derive the junction conditions which give the part of boundary conditions for the above solutions. Finally, we will derive effective equations on the brane and we will confirm that the conventional 4-dimensional gravitational theory is recovered. Before closing this chapter we will discuss the counting of degree of freedom because we have a lot of equations, variables and boundary conditions.

3.1 Model and basic equations

In this section we will describe our 6-dimensional braneworld models. As the schematic picture drawn in Fig. 3.1.1, we employ the capped regularization and then introduce the two 4-branes for capped regularization. The full action is given by

$$S = \int d^6x\sqrt{-g}\left\{\frac{M_{(6)}^4}{2}\left({}^{(6)}R-\frac{1}{L_I^2}\right)-\frac{1}{4}F_{AB}F^{AB}\right\}+S^+_{\rm brane}+S^-_{\rm brane}, \tag{3.1.1}$$

where $F^2 := F_{AB}F^{AB}$ and $F_{AB} := \partial_A A_B - \partial_B A_A$ is the field strength of the $U(1)$ gauge field A_A. The capital Latin indices A numerate the 6-dimensional coordinates, $A = 0,1,2,3,4,5$ while hatted

Greek indices $\hat{\mu}$ are restricted to the 5-dimensional coordinates, $\hat{\mu} = 0,1,2,3,4$ and the Greek indices μ are restricted to the 4-dimensional coordinates, $\mu = 0,1,2,3$. This action describes the Einstein gravity and Maxwell field with the positive cosmological constant $\frac{1}{L_I^2}$. $M_{(6)}$ is the fundamental scale in 6 dimensions. $S^{\pm}_{\rm brane}$ are the action for 4-branes.

We consider the following action for each 4-brane [28]:

$$S^{\pm}_{\rm brane} \;:=\; -\int d^5x \left(\lambda_{\pm} + \frac{v_{\pm}^2}{2}(\partial_{\hat{\mu}}\Sigma_{\pm} - eA_{\hat{\mu}})(\partial_{\hat{\nu}}\Sigma_{\pm} - eA_{\hat{\nu}})^{(5)}g^{\hat{\mu}\hat{\nu}}\right) + S^{\pm}_{\rm matter}, \qquad (3.1.2)$$

where $\lambda_{\pm}$ is the brane tension, $v_{\pm}$ is the vacuum expectation value of the brane Higgs field and $\Sigma_{\pm}$ is its Goldstone mode. $S^{\pm}_m$ represents the matter action on the brane. The brane action necessarily couples to $A_{\hat{\mu}}$ in order to account for the jump of the Maxwell field at the brane. $S^{\pm}_{\rm mattrer}$ are the action for usual matters localized on the branes.

We assume the axial symmetric metric ansatz as follows

$$g_{AB}dx^A dx^B = \frac{L_I^2}{f(y)}e^{2\zeta(x)}dy^2 + \ell^2 e^{2\psi(x,y)}f(y)d\theta^2 + g_{\mu\nu}(x,y)dx^{\mu}dx^{\nu}, \qquad (3.1.3)$$

where ℓ determines the scale along the θ direction and we write down $g_{\mu\nu}(y,x)$ as

$$g_{\mu\nu}(y,x) = a^2(y)h_{\mu\nu}(x,y). \qquad (3.1.4)$$

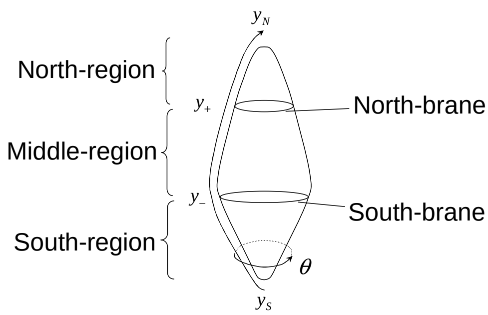


Figure 3.1.1: Map of two extra dimensions and two branes.

And y denotes the radial direction of two extra dimensions and θ denotes angular direction of two extra dimensions (See Fig.(3.1.1)). y_N and y_S denote the position of the poles where $f(y_N) = 0$ and $f(y_S) = 0$. y_+ and y_- are supposed to be the position of 4-branes.

The variation of (3.1.1) with respect to metric ${}^{(6)}g_{AB}$ leads 6-dimensional Einstein equations in the bulk

$${}^{(6)}G^A_{\ B} = -\frac{1}{2L_I^2}\delta^A_B + \frac{1}{M^4_{(6)}}\left(F^A_{\ C}F_B^{\ C} - \frac{1}{4}\delta^A_B F^{CD}F_{CD}\right) \qquad (3.1.5)$$

or we can write down as

$${}^{(6)}R^A_{\ B} = \frac{1}{4L_I^2}\delta^A_B + \frac{1}{M^4_{(6)}}\left(F^A_{\ C}F_B^{\ C} - \frac{1}{8}\delta^A_B F^{CD}F_{CD}\right). \qquad (3.1.6)$$

To prepare for solving the evolution along the y-direction, we decompose 6-dimensional Einstein equations into the y-direction plus the component orthogonal to y. Noting

$$ {}^{(6)}R^{\hat\mu}{}_{\hat\nu} = {}^{(5)}R^{\hat\mu}{}_{\hat\nu} - \frac{1}{n_y}\partial_y K^{\hat\mu}{}_{\hat\nu} - \hat K K^{\hat\mu}{}_{\hat\nu} - {}^{(5)}D^{\hat\mu}\,{}^{(5)}D_{\hat\nu}\zeta - {}^{(5)}D^{\hat\mu}\zeta\cdot{}^{(5)}D_{\hat\nu}\zeta, \tag{3.1.7} $$

the $(\hat\mu,\hat\nu)$-component Einstein equation becomes

$$ \begin{aligned} &{}^{(5)}R^{\hat\mu}{}_{\hat\nu} - \frac{1}{n_y}\partial_y K^{\hat\mu}{}_{\hat\nu} - \hat K K^{\hat\mu}{}_{\hat\nu} - {}^{(5)}D^{\hat\mu}\,{}^{(5)}D_{\hat\nu}\zeta - {}^{(5)}D^{\hat\mu}\zeta\cdot{}^{(5)}D_{\hat\nu}\zeta \\ &= \frac{1}{4L_I^2}\delta^{\hat\mu}{}_{\hat\nu} + \frac{1}{M_{(6)}^4}\left(F^{\hat\mu}{}_C F_{\hat\nu}{}^C - \frac{1}{8}\delta^{\hat\mu}{}_{\hat\nu}F^{CD}F_{CD}\right), \end{aligned} \tag{3.1.8} $$

where ${}^{(5)}D_{\hat\mu}$ is the covariant derivative with respect to ${}^{(5)}g_{\hat\mu\hat\nu}$, $n_y = \frac{L_I}{\sqrt{f}}e^{\zeta}$ is y-component of normal (dual) vector n_a, $K^{\hat\mu}{}_{\hat\nu}$ is the extrinsic curvature of $y=$constant hypersurfaces and $\hat K$ is its 5-dimensional trace. ${}^{(5)}R^{\hat\mu}{}_{\hat\nu}$ is the 5-dimensional Ricci tensor. The extrinsic curvature is defined by

$$ \begin{aligned} K_{\hat\mu\hat\nu} &= \frac{1}{2}\mathscr{L}_n\,{}^{(5)}g_{\hat\mu\hat\nu} && (3.1.9) \\ &= {}^{(5)}\nabla_{\hat\mu}\,{}^{(5)}n_{\hat\nu}, && (3.1.10) \end{aligned} $$

where n^a is the normal vector to the brane.

The (y,y)-component of Eq. (3.1.5), which leads Hamiltonian constraint

$$ {}^{(5)}R + K^{\hat\alpha}{}_{\hat\beta}K^{\hat\beta}_{\hat\alpha} - \hat K^2 = \frac{1}{L_I^2} - \frac{2}{M_{(6)}^4}\left(F^y{}_C F_y{}^C - \frac{1}{4}F^{CD}F_{CD}\right). \tag{3.1.11} $$

The 5-dimensional Ricci scalar ${}^{(5)}R$ can be written in terms of 4-dimensional quantities as

$$ {}^{(5)}R = {}^{(4)}R - 2\,{}^{(4)}D^{\alpha}\,{}^{(4)}D_{\alpha}\psi - 3\,{}^{(4)}D_{\alpha}\psi\cdot{}^{(4)}D^{\alpha}\psi, \tag{3.1.12} $$

where ${}^{(4)}D_\mu$ is the covariant derivative with respect to ${}^{(4)}g_{\mu\nu}$ and where we used

$$ \begin{cases} {}^{(5)}R^{\theta}{}_{\theta} = -{}^{(4)}D_{\alpha}\,{}^{(4)}D^{\alpha}\psi - 2\,{}^{(4)}D_{\alpha}\psi\cdot{}^{(4)}D^{\alpha}\psi, \\ {}^{(5)}R^{\mu}{}_{\nu} = {}^{(4)}R - {}^{(4)}D^{\mu}\,{}^{(4)}D_{\nu}\psi - {}^{(4)}D^{\mu}\psi\cdot{}^{(4)}D_{\nu}\psi. \end{cases} \tag{3.1.13} $$

The $(y,\hat\mu)$-component of Einstein equation gives us the momentum constraint

$$ {}^{(5)}D_{\hat\alpha}\left(K^{\hat\alpha}_{\hat\mu} - \delta^{\hat\alpha}_{\hat\mu}\hat K\right) = \frac{1}{M^4}F_{\hat\mu M}F^{yM}n_y. \tag{3.1.14} $$

Not only gravity but also Maxwell field should be solved at the same time. Maxwell equations are given by

$$ {}^{(6)}\nabla_N F^{NM} = 0, \tag{3.1.15} $$

where ${}^{(6)}\nabla_N$ is the covariant derivative with respect to the 6-dimensional metric.

Let us look at the Israel junction conditions

$$[K^{\hat{\mu}}_{\ \hat{\nu}} - \delta^{\hat{\mu}}_{\ \hat{\nu}}\hat{K}]_{y=y_\pm} = -\frac{1}{M^4_{(6)}}S^{\hat{\mu}}_{\ \hat{\nu}}. \tag{3.1.16}$$

$S^{\hat{\mu}}_{\ \hat{\nu}}$ is the total energy-momentum tensor computed from the brane action and then

$$S^{\rm brane}_{\hat{\mu}\hat{\nu}} := T^{\pm}_{\hat{\mu}\hat{\nu}} - \lambda_\pm {}^{(5)}g_{\hat{\mu}\hat{\nu}} + v^2_\pm\Big((\partial_{\hat{\mu}}\Sigma_\pm - eA_{\hat{\mu}})(\partial_{\hat{\nu}}\Sigma_\pm - eA_{\hat{\nu}}) - \frac{1}{2}{}^{(5)}g_{\hat{\mu}\hat{\nu}}(\partial_{\hat{\alpha}}\Sigma_\pm - eA_{\hat{\alpha}})^2\Big), \tag{3.1.17}$$

where $T^{\hat{\mu}}_{\pm\hat{\nu}}$ is the energy-momentum tensor of the usual matter fields on each brane. We assume that $T^{\mu}_{\pm\theta} = T^{\theta}_{\pm\nu} = 0$.

The junction condition for Maxwell field is given by

$$[n_y F^{y\hat{\mu}}]_{y=y_\pm} = -ev^2_\pm g^{\hat{\mu}\hat{\alpha}}(\partial_{\hat{\alpha}}\Sigma_\pm - eA_{\hat{\alpha}}). \tag{3.1.18}$$

3.2 Long-wavelength approximation

We employ the long-wavelength approximation to solve the 6-dimensional bulk geometry [33]. Let ℓ be a bulk curvature scale and L be a curvature scale on the brane, where ${}^{(5)}R \sim 1/L^2$. Assuming that ℓ is not so different from L_I, the small expansion parameter is defined as the ratio of the bulk curvature scale ℓ to the 4-dimensional intrinsic curvature scale L,

$$\varepsilon := \left(\frac{\ell}{L}\right)^2 = \ell^2|{}^{(4)}R|. \tag{3.2.1}$$

Then we expand all the variables as follows:

$$\begin{aligned}
&\psi = \psi^{(1)}(y,x) + \cdots, && \zeta(y,x) = \zeta^{(1)}(x) + \cdots, && g_{\hat{\mu}\hat{\nu}} = \overset{(0)}{g}_{\hat{\mu}\hat{\nu}} + \overset{(1)}{g}_{\hat{\mu}\hat{\nu}} \cdots, \\
&g_{yy} = \overset{(0)}{g}_{yy} + \overset{(1)}{g}_{yy} + \cdots, && K_{\mu\nu} = \overset{(0)}{K}_{\mu\nu} + \overset{(1)}{K}_{\mu\nu} + \cdots, && K_{\theta\theta} = \overset{(0)}{K}_{\theta\theta} + \overset{(1)}{K}_{\theta\theta} + \cdots, \\
&{}^{(5)}R_{\hat{\mu}\hat{\nu}} = 0 + \left\{{}^{(5)}R_{\hat{\mu}\hat{\nu}}\right\}^{(1)} + \cdots, && A_\mu = A_\mu^{(\frac{1}{2})} + \cdots, && A_\theta = A_\theta^{(0)} + A_\theta^{(1)} + \cdots, \\
&F_{y\theta} = \overset{(0)}{F}_{y\theta} + \overset{(1)}{F}_{y\theta}, && F_{\mu\theta} = \overset{(\frac{1}{2})}{F}_{\mu\theta} + \cdots, && F_{\mu y} = F_{\mu y}^{(\frac{1}{2})} + \cdots, \\
&F_{\mu\nu} = 0 + F^{(1)}_{\mu\nu} + \cdots.
\end{aligned}$$

From Eq. (3.1.13), 5-dimensional Ricci tensor can be expanded by the first order

$$ {}^{(5)}R^{\theta}_{\ \theta} = O(\varepsilon^{\frac{3}{2}}), \tag{3.2.2}$$

$$ {}^{(5)}R^{\alpha}_{\ \alpha} = \varepsilon\frac{1}{a^2}{}^{(4)}R(h^{(0)}) + O(\varepsilon^{\frac{3}{2}}). \tag{3.2.3}$$

Now we are ready to expand and solve the equations iteratively.

3.2.1 Bulk solutions at the 0th order

At zeroth order, the evolution equations and the Hamiltonian constraint are

$$\frac{f(y)}{L_I^2}\left\{\frac{\partial_y^2 a}{a}+\frac{\partial_y f\,\partial_y a}{f\,a}+3\left(\frac{\partial_y a}{a}\right)^2\right\}=-\frac{1}{4L_I^2}+\frac{1}{4M_{(6)}^4}\overset{(0)}{F}{}^{y\theta}\overset{(0)}{F}{}_{y\theta}, \tag{3.2.4}$$

$$\frac{1}{L_I^2}\left\{\frac{1}{2}\partial_y^2 f+\frac{2}{y}\partial_y f\right\}=-\frac{1}{4L_I^2}-\frac{3}{4}\frac{1}{M_{(6)}^4}\overset{(0)}{F}{}^{y\theta}\overset{(0)}{F}{}_{y\theta}, \tag{3.2.5}$$

$$\frac{f(y)}{L_I^2}\left\{-12\left(\frac{\partial_y a}{a}\right)^2-4\frac{\partial_y a}{a}\frac{\partial_y f}{f}\right\}=\frac{1}{L_I^2}-\frac{1}{M_{(6)}^4}\overset{(0)}{F}{}^{y\theta}\overset{(0)}{F}{}_{y\theta}\,. \tag{3.2.6}$$

Note that the momentum constraint is automatically satisfied at the 0th order.

The θ-component of Maxwell equation is

$$\partial_y F^{y\theta}+\overset{(0)}{K}{}^{\alpha}{}_{\alpha}\overset{(0)}{F}{}^{y\theta}=0. \tag{3.2.7}$$

Note that the only θ-component remains.

Let us solve equations. Eqs. (3.2.4) and (3.2.6) imply

$$\partial_y^2 a(y)=0, \tag{3.2.8}$$

and the solution is given by

$$a(y)=y. \tag{3.2.9}$$

Next we will solve Eq. (3.2.7). It is easy to see that the solution is

$$\overset{(0)}{F}{}_{y\theta}=M_{(6)}^2\ell\,\frac{Q}{y^4}, \tag{3.2.10}$$

where Q is the integral constant. Substitution of these solutions into Eq. (3.2.4) yields

$$\frac{f(y)}{L_I^2}\left\{\frac{\partial_y f(y)}{y\,f(y)}+3\frac{1}{y^2}\right\}=-\frac{1}{4L_I^2}+\frac{1}{4L_I^2}\frac{Q^2}{y^8}. \tag{3.2.11}$$

Then we have the solution to $f(y)$ as

$$f(y)=-\frac{y^2}{20}+\frac{\mu}{y^3}-\frac{Q^2}{12y^8}, \tag{3.2.12}$$

where μ and Q are integral constants.

We consider the parameter region of (Q,μ) so that $f(y)=0$ has two positive roots, y_N and $y_S(<y_N)$. As seen later, $y=y_N$ and y_S will be the position of north and south poles. Since

$$\overset{(0)}{F}{}_{y\theta}=\partial_y\overset{(0)}{A}{}_{\theta}=M_{(6)}^2\ell\frac{Q}{y^4}, \tag{3.2.13}$$

the vector potential $\overset{(0)}{A}_\theta$ becomes

$$\overset{(0)}{A}{}^{(I)}_\theta = \frac{\ell M_{(6)}^2}{3}\frac{Q}{y^3} + a_I^{(0)}. \tag{3.2.14}$$

where I specifies the region (see Fig.3.1.1) to which the solutions belong and $a_I^{(0)}$ is the integral constant. Imposing that A_θ vanishes at $y = y_N$, we arrive at

$$\overset{(0)}{A}{}^{(N)}_\theta = \frac{\ell M_{(6)}^2 Q}{3}\left(\frac{1}{y_N^3} - \frac{1}{y^3}\right). \tag{3.2.15}$$

3.2.2 Bulk solutions at the first order

Going to the first order in the long-wavelength approximation we will obtain the gravitational field equations that govern the behavior of the 4-dimensional metric $h^{(0)}_{\mu\nu}$. Although we start from the block-diagonal metric ansatz, only assuming axial symmetry the metric should still have non-diagonal components. In such case the equations at the first order may contain terms such as $\overset{(1/2)}{F}{}^{\mu y}\overset{(1/2)}{F}_{\nu y}$ and $\overset{(1/2)}{K}{}^\theta{}_\alpha \overset{(1/2)}{K}{}_\theta{}^\alpha$. However, in Appendix D we analyze the $O(\varepsilon^{\frac{1}{2}})$ equations and show that the long-wavelength approximation of $F_{\mu y}$ and $K^\mu{}_\theta$ in fact begins with $O(\varepsilon^{\frac{3}{2}})$. Therefore, in the following we will drop the contributions from such terms. At the first order, from Eq. (3.1.8) the (μ,ν)-component of the evolution equation becomes

$$\partial_y \overset{(1)}{K}{}^\mu{}_\nu + \left(\frac{4}{y} + \frac{\partial_y f}{2f}\right)\overset{(1)}{K}{}^\mu{}_\nu + \frac{1}{y}(\overset{(1)}{K}{}^\alpha{}_\alpha + \overset{(1)}{K}{}^\theta{}_\theta)\delta^\mu_\nu - \frac{L_I}{\sqrt{f}}{}^{(4)}\!R^\mu{}_\nu - \frac{1}{4}\frac{1}{M_{(6)}^4}\frac{L_I}{\sqrt{f}}\mathscr{F}^{(1)}\delta^\mu_\nu$$
$$+\frac{\sqrt{f}}{L_I}\zeta^{(1)}\left(\frac{4}{y^2} + \frac{1}{2y}\frac{\partial_y f}{f} + \frac{1}{4f} - \frac{1}{4}\frac{L_I^2}{M_{(6)}^4}\frac{1}{f}\overset{(0)}{F}{}^{y\theta}\overset{(1)}{F}_{y\theta}\right)\delta^\mu_\nu = 0, \tag{3.2.16}$$

where we defined the useful combination

$$\mathscr{F} := \frac{1}{M^4}\left(\overset{(0)}{F}_{y\theta}\overset{(1)}{F}{}^{y\theta} + \overset{(1)}{F}_{y\theta}\overset{(0)}{F}{}^{y\theta}\right). \tag{3.2.17}$$

Let us solve the traceless part of Eq. (3.2.16). Taking the traceless part of Eq. (3.2.16) yields

$$\frac{1}{y^4\sqrt{f}}\partial_y\left(y^4\sqrt{f}\,\overset{(1)}{\tilde{K}}{}^\mu{}_\nu\right) = \frac{L_I}{\sqrt{f}}{}^{(4)}\!\tilde{R}^\mu{}_\nu(a^2h). \tag{3.2.18}$$

Then, the solution is found to be

$$\overset{(1)}{\tilde{K}}{}^\mu{}_\nu = \frac{1}{3y\sqrt{f}}L_I\tilde{R}^\mu{}_\nu + \frac{1}{y^4\sqrt{f}}\tilde{C}^\mu{}_\nu(x), \tag{3.2.19}$$

where we defined the traceless part of the relevant tensors as $\overset{(1)}{\tilde{K}}{}^\mu{}_\nu := \overset{(1)}{K}{}^\mu{}_\nu - (1/4)\delta^\mu_\nu\overset{(1)}{K}{}^\alpha{}_\alpha$ and ${}^{(4)}\!\tilde{R}^\mu{}_\nu := {}^{(4)}\!R^\mu{}_\nu - (1/4)\delta^\mu_\nu{}^{(4)}\!R$. $\tilde{C}^\mu{}_\nu(x)$ is the integration "constant " to be fixed by the boundary conditions. Of course, $\tilde{C}^{\mu}_{\nu}(x)$ is traceless by the definition.

The 4-dimensional trace part of the evolution equations is

$$\partial_y \overset{(1)}{K}{}^{\alpha}_{\alpha} + \left(\frac{8}{y} + \frac{\partial_y f}{2f}\right) \overset{(1)}{K}{}^{\alpha}_{\alpha} + \frac{4}{y} \overset{(1)}{K}{}^{\theta}_{\theta}$$
$$-\frac{L_I}{\sqrt{f}} {}^{(4)}R(a^2 h) - \frac{1}{M^4_{(6)}} \frac{L_I}{\sqrt{f}} \mathscr{F}^{(1)} + \frac{\sqrt{f}}{L_I} \zeta^{(1)} \left(\frac{1}{f} + \frac{2}{y} \frac{\partial_y f}{f} + \frac{16}{y^2} - \frac{L_I^2}{M^4_{(6)}} \frac{1}{f} \overset{(0)}{F}{}^{y\theta} \overset{(1)}{F}_{y\theta} \right) = 0. \tag{3.2.20}$$

The (θ,θ)-component of the evolution equation is

$$\partial_y \overset{(1)}{K}{}^{\theta}_{\theta} + \frac{\partial_y f}{2f} \overset{(1)}{K}{}^{\alpha}_{\alpha} + \left(\frac{4}{y} + \frac{\partial_y f}{f}\right) \overset{(1)}{K}{}^{\theta}_{\theta} + \frac{3}{4} \frac{1}{M^4_{(6)}} \frac{L_I}{\sqrt{f}} \mathscr{F}^{(1)}$$
$$+\frac{\sqrt{f}}{L_I} \zeta^{(1)} \left(\frac{2}{y} \frac{\partial_y f}{f} + \frac{1}{4} \left(\frac{\partial_y f}{f}\right)^2 + \frac{1}{4f} + \frac{3}{4} \frac{L_I^2}{M^4_{(6)}} \frac{1}{f} \overset{(0)}{F}{}^{y\theta} \overset{(0)}{F}_{y\theta} \right) = 0. \tag{3.2.21}$$

The Hamiltonian constraint is

$${}^{(4)}R - \frac{\sqrt{f}}{L_I} \left\{ \frac{8}{y} \overset{(1)}{K}{}^{\theta}_{\theta} + (\frac{\partial_y f}{f} + \frac{6}{y}) \overset{(1)}{K}{}^{\alpha}_{\alpha} \right\} = -\frac{1}{M^4_{(6)}} \mathscr{F}^{(1)}. \tag{3.2.22}$$

Let us see how to determine the $\overset{(1)}{F}_{y\theta}$. Since we do not need concrete expression of this for deriving effective equation on the brane, we will briefly look at the procedure. At first, let us remember $\mathscr{F}^{(1)}$ which is

$$\mathscr{F}^{(1)} = \overset{(0)}{F}{}^{y\theta} \overset{(1)}{F}_{y\theta} + \overset{(1)}{F}{}^{y\theta} \overset{(0)}{F}_{y\theta} . \tag{3.2.23}$$

Noting

$$\overset{(1)}{F}{}^{y\theta} = \overset{(0)}{g}{}^{yy} \overset{(0)}{g}{}^{\theta\theta} \Big(-2 \overset{(1)}{\psi} \overset{(1)}{F}_{y\theta} - 2 \overset{(1)}{\zeta} \overset{(0)}{F}_{y\theta} + \overset{(1)}{F}_{y\theta} \Big). \tag{3.2.24}$$

$\mathscr{F}^{(1)}$ is expressed by $\overset{(1)}{F}_{y\theta}$ as

$$\mathscr{F}^{(1)} = 2 \overset{(0)}{F}{}^{y\theta} \Big(- \overset{(1)}{\zeta} \overset{(0)}{F}_{y\theta} - \overset{(1)}{\psi} \overset{(0)}{F}_{y\theta} + \overset{(1)}{F}_{y\theta} \Big). \tag{3.2.25}$$

This expression means that $\overset{(1)}{F}_{y\theta}$ is determined if we can determine $\mathscr{F}^{(1)}$ by solving the evolution equations (3.2.20), (3.2.21) and the Hamiltonian constraint (3.2.22). After finding the bulk profile of the extrinsic curvature and $\mathscr{F}(y)$, we will be ready to solve the Maxwell equations.

In order to derive an effective theory on the brane, we do not need to solve the complicated equations for $\overset{(1)}{K}{}^{\alpha}_{\alpha}$ and $\overset{(1)}{K}{}^{\theta}_{\theta}$. As will be shown in next section 3.3 it suffices to solve the bulk evolution with the combination of variables at the first order such as

$$\mathscr{K}^{(1)} := \frac{3}{4} \overset{(1)}{K}{}^{\alpha}_{\alpha} + \overset{(1)}{K}{}^{\theta}_{\theta} + \frac{1}{M^4} \frac{L_I}{\sqrt{f}} \overset{(0)}{F}{}^{y\theta} \overset{(1)}{A}_{\theta} + \frac{\sqrt{f}}{L_I} \left(\frac{\partial_y f}{2f} - \frac{1}{y} \right) \overset{(1)}{\psi} . \tag{3.2.26}$$

Note that this variable $\mathscr{K}^{(1)}$ is including gauge $\overset{(1)}{A_\theta}$ and coordinate degree of freedom $\overset{(1)}{\psi}$ in its definition. Then 4-dimensional trace part and (θ,θ)-evolution equation are combined to give one simple equation

$$\partial_y \mathscr{K}^{(1)} = \frac{1}{4}\frac{L_I}{\sqrt{f}}{}^{(4)}\!R(a^2h) - \frac{1}{y^4\sqrt{f}}\partial_y(y^4\sqrt{f})\mathscr{K}^{(1)}. \tag{3.2.27}$$

It is rearranged as

$$\frac{1}{y^4\sqrt{f}}\partial_y\left(y^4\sqrt{f}\mathscr{K}^{(1)}\right) = \frac{1}{4}\frac{L_I}{\sqrt{f}}{}^{(4)}\!R(a^2h). \tag{3.2.28}$$

The solution for this is given by

$$\mathscr{K}^{(1)} = \frac{1}{12y\sqrt{f}}L_I{}^{(4)}\!R + \frac{1}{y^4\sqrt{f}}\chi(x), \tag{3.2.29}$$

where $\chi(x)$ is an integral constant to be determined by the boundary conditions.

Then Eqs. (3.2.19) and (3.2.28) give us

$$\overset{(1)}{\tilde{K}}{}^\mu{}_\nu - \delta^\mu{}_\nu\mathscr{K}^{(1)} = \frac{L_I}{3y\sqrt{f}}\left({}^{(4)}\!R^\mu{}_\nu(h) - \frac{1}{2}\delta^\mu_\nu{}^{(4)}\!R(h)\right) + \frac{1}{y^4\sqrt{f}}\left(\tilde{C}^\mu{}_\nu - \delta^\mu{}_\nu\chi\right). \tag{3.2.30}$$

Before the closing this section, let us consider the momentum constraint. The μ-component of the momentum constraint at the first order is

$${}^{(4)}\!D_\alpha\overset{(1)}{\tilde{K}}{}^\alpha{}_\mu - {}^{(4)}\!D_\mu\left(\frac{3}{4}\overset{(1)}{K}{}^\alpha{}_\alpha + \overset{(1)}{K}{}^\theta{}_\theta\right) + \frac{\sqrt{f}}{L_I}\left(\frac{1}{y} - \frac{\partial_y f}{2f}\right){}^{(4)}\!D_\mu\overset{(1)}{\psi} = \frac{1}{M^4}\frac{L_I}{\sqrt{f}}\overset{(0)}{F}{}^{y\theta}\,{}^{(4)}\!D_\mu\overset{(1)}{A_\theta},$$

which can be simplified using $\mathscr{K}^{(1)}$ to

$${}^{(4)}\!D_\alpha\overset{(1)}{\tilde{K}}{}^\alpha{}_\mu - {}^{(4)}\!D_\mu\mathscr{K}^{(1)} = 0. \tag{3.2.31}$$

Eq. (3.2.31) and the Bianchi identity, ${}^{(4)}\!D_\alpha\left[{}^{(4)}\!R^\alpha_\mu - \frac{1}{2}\delta^\alpha_\mu{}^{(4)}\!R\right] = 0$, imply the constraint for the integration constants:

$${}^{(4)}\!D_\alpha\tilde{C}^\alpha_\mu - {}^{(4)}\!D_\mu\chi = 0. \tag{3.2.32}$$

So far we focused on solving the bulk Einstein/Maxwell field equations and we have not imposed the boundary conditions. In the next section we will give the boundary conditions carefully. Then we will obtain the low energy effective theory on the branes.

3.3 Boundary conditions: regularity and the junction condition

The metric function $f(y)$ vanishes at y_N and y_S. These points develop conical singularities in general and they are regarded as source of 3-branes. To avoid the singularities, we replace each of the conical branes with a wrapped 4-brane. The geometries of the north-, middle- and south-region are described by the 6-dimensional solutions obtained in the previous section with different cosmological constants (L_+ (L_-) for the north-region(south-region) and L_0 for the middle region). Near the pole $y = y_p$ ($p = N,S$), we have $f(y) \simeq \partial_y f|_{y=y_p}(y-y_p)$. The metric (3.1.3) can be approximated as

$$ds^2 \simeq \frac{L_I^2}{f'(y_p)} e^{2\zeta} \left(\frac{dy^2}{y-y_p} + \frac{\ell^2}{L_I^2} f'(y_p)^2 e^{2\psi-2\zeta}(y-y_p) d\theta^2 \right), \tag{3.3.1}$$

where $f'(y_p) := \partial_y f(y)|_{y=y_p}$. By transforming to the new coordinate $Y := \sqrt{y-y_p}$, we have

$$ds^2 \simeq \frac{4L_I^2}{f'(y_p)} e^{2\zeta} \left(dY^2 + \frac{\ell^2}{L_I^2} \frac{f'(y_p) e^{2\psi-2\zeta}}{4} Y^2 d\theta^2 \right), \tag{3.3.2}$$

To be regular at y_p, we impose

$$\frac{\ell}{L_I} \frac{|\partial_y f|}{2} e^{\psi-\zeta} \bigg|_{y=y_p} = 1, \tag{3.3.3}$$

where θ takes the value within $0 \leq \theta \leq 2\pi$. Clearly, it is required that $\psi(x,y_p) - \zeta(x) =$ constant. Without loss of generality we can set this constant contribution to be zero. Furthermore, we require $\psi(x,y_p)$=constant and $\zeta(x)$=constant. Then, we have $\psi^{(1)} \lesssim (y-y_p)$ and $\zeta^{(1)} \lesssim (y-y_p)$ near $y = y_p$.

Each of the regions is glued together so as to satisfy the Israel junction conditions [31] and the junction conditions for the Maxwell field. We start with assuming that the brane location is given by a x-dependent function $y = \varphi(x)$. The brane induced metric is given by $q_{\hat{\mu}\hat{\nu}} dx^{\hat{\mu}} dx^{\hat{\nu}} = \ell^2 f(\varphi(x)) d\theta^2 + \varphi^2(x) h^{(0)}_{\mu\nu} dx^\mu dx^\nu + O(\varepsilon)$.

Before discussing the junction conditions we have to solve the brane scalar field Σ. The equation for Σ is

$${}^{(5)}D_{\hat{\alpha}} \left(\partial^{\hat{\alpha}} \Sigma_\pm - eA^{\hat{\alpha}} \right) = 0. \tag{3.3.4}$$

The Eq. (3.3.4) at the 0th order becomes

$$\partial_\theta \left(\partial^\theta \overset{(0)}{\Sigma}_\pm \right) = 0, \tag{3.3.5}$$

and hence $\Sigma^{(0)}_\pm = n_\pm \theta + \sigma^{(0)}_\pm$. Here, $n_\pm$ must be integer because $\Sigma_\pm$ is the phase of the Higgs field and so $e^{i\Sigma(\theta+2\pi,x)} = e^{i\Sigma(\theta,x)}$. This property should hold at any order in the long-wavelength approximation. Therefore, the solution for Σ including corrections at the higher order should be of the form

$$\Sigma_\pm = n_\pm \theta + \sigma^{(0)}_\pm(x) + \sigma^{(1)}_\pm(x) + \cdots. \tag{3.3.6}$$

3.3.1 The junction conditions at the 0th order

We shall show now that the brane location is in fact independent of x. To do so, let us consider the θ-component of the Maxwell junction conditions at the zeroth order. From Eq. (3.1.18) the junction condition at the 0th order for the gauge field is

$$\left[n_y \overset{(0)}{F}{}^{y\theta}\right]_{y=y_\pm} = -ev_\pm^2 \overset{(0)}{g}{}^{\theta\theta}(\partial_\theta \overset{(0)}{\Sigma} - e \overset{(0)}{A}_\theta). \tag{3.3.7}$$

where $[A] := A|_{y=\varphi+\varepsilon} - A|_{y=\varphi-\varepsilon}$ and n^M is the unit normal to the brane. At this order, we have $n^M \simeq (n^y, 0)$. So the right hand side is independent of x and hence the brane location must be $y =$ constant $(=: y_\pm)$. The Israel junction conditions are given by Eqs. (3.1.16) and (3.1.17). The junction condition at the 0th order for gravity are

$$-\left[\overset{(0)}{K}{}^\mu{}_\nu - \delta^\mu_\nu \hat{K}^{(0)}\right]_{y=y_\pm} = -\frac{\lambda_\pm}{M_{(6)}^4} - \frac{v_\pm^2}{2M_{(6)}^4}(\partial_\theta \overset{(0)}{\Sigma} - e \overset{(0)}{A}_\theta)(\partial^\theta \overset{(0)}{\Sigma} - e \overset{(0)}{A}{}^\theta) \tag{3.3.8}$$

and

$$-\left[\overset{(0)}{K}{}^\theta{}_\theta - \hat{K}^{(0)}\right]_{y=y_\pm} = -\frac{\lambda_\pm}{M_{(6)}^4} + \frac{v_\pm^2}{2M_{(6)}^4}(\partial_\theta \overset{(0)}{\Sigma} - e \overset{(0)}{A}_\theta)(\partial^\theta \overset{(0)}{\Sigma} - e \overset{(0)}{A}{}^\theta), \tag{3.3.9}$$

where $\hat{K} := K^{\hat{\alpha}}_{\hat{\alpha}}$, $\overset{(0)}{K}{}^\mu{}_\nu - \delta^\mu_\nu \hat{K}^{(0)} = -\frac{\sqrt{f}}{L_I}(\frac{3}{y} + \frac{\partial_y f}{2f})$ and $\overset{(0)}{K}{}^\theta{}_\theta - \hat{K}^{(0)} = -\frac{\sqrt{f}}{L_I}\frac{4}{y}$.

For later use, let us derive the relation between tension $\lambda_\pm$ and the scalar/gauge field at the 0th order $\overset{(0)}{\Sigma}/\overset{(0)}{A}_\theta$. Using the explicit expression of extrinsic curvature, Eqs. (3.3.8) and (3.3.9) become

$$\left[\frac{1}{L_I}\right]_{y=y_\pm}\sqrt{f}(\frac{3}{y} + \frac{\partial_y f}{2f}) = -\frac{\lambda_\pm}{M_{(6)}^4} - \frac{v_\pm^2}{2M_{(6)}^4}(\partial_\theta \overset{(0)}{\Sigma} - e \overset{(0)}{A}_\theta)(\partial^\theta \overset{(0)}{\Sigma} - e \overset{(0)}{A}{}^\theta) \tag{3.3.10}$$

and

$$\left[\frac{1}{L_I}\right]_{y=y_\pm} = \frac{y}{4\sqrt{f}}\Big(-\frac{\lambda_\pm}{M_{(6)}^4} + \frac{v_\pm^2}{2M_{(6)}^4}(\partial_\theta \overset{(0)}{\Sigma} - e \overset{(0)}{A}_\theta)(\partial^\theta \overset{(0)}{\Sigma} - e \overset{(0)}{A}{}^\theta)\Big), \tag{3.3.11}$$

respectively. Eliminating $\left[\frac{1}{L_I}\right]_{y=y_\pm}$ from the aboves, we have

$$\begin{aligned}\frac{y}{4}\left(\frac{3}{y} + \frac{\partial_y f}{2f}\right)\left(-\frac{\lambda_\pm}{M_{(6)}^4} + \frac{v_\pm^2}{2M_{(6)}^4}(\partial_\theta \overset{(0)}{\Sigma} - e \overset{(0)}{A}_\theta)(\partial^\theta \overset{(0)}{\Sigma} - e \overset{(0)}{A}{}^\theta)\right) \\ = -\frac{\lambda_\pm}{M_{(6)}^4} - \frac{v_\pm^2}{2M_{(6)}^4}(\partial_\theta \overset{(0)}{\Sigma} - e \overset{(0)}{A}_\theta)(\partial^\theta \overset{(0)}{\Sigma} - e \overset{(0)}{A}{}^\theta).\end{aligned} \tag{3.3.12}$$

Thus, we can express $\lambda_\pm$ with respect to the quantities on the brane such as $\overset{(0)}{\Sigma}$ and $\overset{(0)}{A}{}^\theta$.

$$\lambda_\pm = -\frac{\frac{7}{4} + \frac{y}{8}\frac{\partial_y f}{f}}{\frac{1}{4} - \frac{y}{8}\frac{\partial_y f}{f}}\frac{v_\pm^2}{2}(\partial_\theta \overset{(0)}{\Sigma} - e \overset{(0)}{A}_\theta)(\partial^\theta \overset{(0)}{\Sigma} - e \overset{(0)}{A}{}^\theta)|_{y=y_\pm}. \tag{3.3.13}$$

This relation will be used to simplify the junction condition at the 1st order.

3.3.2 The junction conditions at the first order

We go on to specifying the boundary conditions at the first order at the poles and branes. As to the regularity conditions at the poles, it is required that

$$\overset{(1)}{\tilde{K}}{}^{\mu}{}_{\nu}, \quad \overset{(1)}{K}{}^{\alpha}{}_{\alpha}, \quad \overset{(1)}{K}_{\theta\theta} \lesssim |y-y_p|^{\frac{1}{2}}. \tag{3.3.14}$$

With this, the evolution equations for the extrinsic curvature (3.2.19), (3.2.20)[1], and (3.2.21) are regular at the poles. We also require that $|\overset{(1)}{F}_{y\theta}| < \infty$ at $y = y_p$. This implies that $\overset{(1)}{A}_{\theta} \lesssim (y-y_p)$ near the pole. Noting that $\overset{(1)}{\psi} \lesssim (y-y_p)$, we have $\mathscr{K}^{(1)} \lesssim |y-y_p|^{\frac{1}{2}}$ near $y = y_p$.

The (μ,ν)-component of Eq. (3.1.16) is

$$[K^{\mu}_{\nu} - \delta^{\mu}_{\nu}\hat{K}]^{(1)}_{y=y_\pm} = -\frac{1}{M^4_{(6)}}T^{\mu}_{\nu} - \frac{v^2_\pm}{M^4_{(6)}}\left\{e(\partial_\theta\Sigma^{(0)} - e\overset{(0)}{A}_{\theta})\overset{(0)}{g}{}^{\theta\theta}\overset{(1)}{A}_{\theta} + \overset{(1)}{\psi}(\partial_\theta\overset{(0)}{\Sigma} - e\overset{(0)}{A}_{\theta})^2\right\}. \tag{3.3.15}$$

The traceless part of the above becomes

$$\left[\overset{(1)}{\tilde{K}}{}^{\mu}{}_{\nu}\right]_{y_\pm} = -\frac{1}{M^4_{(6)}}\tilde{T}^{\mu}_{\nu}. \tag{3.3.16}$$

Using Eqs. (3.3.8), (3.3.13), (3.3.15) and (3.1.18), we can have the following compact form junction condition

$$\left[\overset{(1)}{\tilde{K}}{}^{\mu}{}_{\nu} - \delta^{\mu}{}_{\nu}\mathscr{K}^{(1)}\right]_{y=y_\pm} = -\frac{1}{M^4_{(6)}}T^{\mu}{}_{\nu}, \tag{3.3.17}$$

where $\mathscr{K}^{(1)}$ is defined as

$$\mathscr{K}^{(1)} := \frac{3}{4}\overset{(1)}{K}{}^{\alpha}{}_{\alpha} + \overset{(1)}{K}{}^{\theta}{}_{\theta} - \psi^{(1)}\frac{\sqrt{f}}{L_I}\left(\frac{1}{y} - \frac{1}{2}\frac{\partial_y f}{f}\right) + \frac{1}{M^4_{(6)}}\frac{L_I}{\sqrt{f}}\overset{(0)}{F}{}^{y\theta}\overset{(1)}{A}_{\theta}. \tag{3.3.18}$$

The trace part of Eq. (3.3.17) becomes

$$\left[\mathscr{K}^{(1)}\right]_{y_\pm} = \frac{1}{4M^4_{(6)}}T^{\alpha}{}_{\alpha}. \tag{3.3.19}$$

The junction conditions (3.3.16) and (3.3.19) together with the momentum constraint (3.2.31) imply the local conservation law for the energy-momentum tensor of brane localized matter:

$${}^{(4)}D_\alpha T^{\mu\alpha} = 0. \tag{3.3.20}$$

Also note that the Maxwell junction condition at the 1st order becomes

$$\left[\frac{\sqrt{f}}{L_I}(\overset{(1)}{F}{}^{y\theta} - \overset{(1)}{\zeta}\overset{(0)}{F}{}^{y\theta})\right]_{y=y_\pm} = -ev^2_\pm(\partial_\theta\overset{(1)}{\Sigma} - e\overset{(1)}{A}_{\theta}), \tag{3.3.21}$$

which should be satisfied for the solutions.

[1] As will be seen in next section, regularity condition for $\overset{(1)}{K}{}^{\alpha}{}_{\alpha}$ does not need to be imposed. There, we will confirm the behavior of $\overset{(1)}{K}{}^{\alpha}{}_{\alpha}$ is automatically converged.

3.3.3 The low energy effective theory on a brane

Now we are ready to fix the integration constants of original bulk solution (3.2.29) and then we derive the low energy effective theory on a brane.

From the regularity condition at the north pole one can determine the integration constant in the north-region and we have

$$\tilde{K}^{\mu}_{\ \nu} = \frac{y^3 - y_N^3}{3y^4\sqrt{f}} L_+ \tilde{R}^{\mu}_{\ \nu} \quad (y_+ < y \le y_N). \tag{3.3.22}$$

And the original bulk solution on middle-region is written as

$$\tilde{K}^{\mu}_{\ \nu} = \frac{1}{3y\sqrt{f}} L_0 {}^{(4)}\tilde{R}^{\mu}_{\ \nu} + \frac{1}{y^4\sqrt{f}} \tilde{C}^{\mu}_{\ \nu} \quad (y_- < y < y_+), \tag{3.3.23}$$

and imposing the Israel junction conditions at the north brane leads

$$\tilde{C}^{\mu}_{\ \nu} = -\frac{(y_N^2 - y_+^3)L_+ + y_+^3 L_0}{3} {}^{(4)}\tilde{R}^{\mu}_{\ \nu} + y_+^4 \sqrt{f_+} \frac{\tilde{T}^{\mu}_{+\nu}}{M^4}, \tag{3.3.24}$$

where $f_+ := f(y_+)$. Similarly, the regularity at the south pole determines the extrinsic curvature in the south-region as

$$\tilde{K}^{\mu}_{\ \nu} = \frac{y^3 - y_S^3}{3y^4\sqrt{f}} L_- {}^{(4)}\tilde{R}^{\mu}_{\ \nu} \quad (y_S \le y < y_-). \tag{3.3.25}$$

The Israel junction condition at $y = y_-$ implies

$$\frac{(y_-^3 - y_S^3)L_- - y_-^3 L_0}{3} {}^{(4)}\tilde{R}^{\mu}_{\ \nu} - \tilde{C}^{\mu}_{\ \nu} = y_-^4 \sqrt{f_-} \frac{\tilde{T}^{\mu}_{-\nu}}{M^4}. \tag{3.3.26}$$

Eliminating $\tilde{C}^{\mu}_{\ \nu}$ from Eqs. (3.3.24) and (3.3.26) we have

$$\frac{\ell_*^2}{\ell} {}^{(4)}\tilde{R}^{\mu}_{\ \nu} = \frac{1}{M^4} \left(y_+^4 \sqrt{f_+} \tilde{T}^{\mu}_{+\nu} + y_-^4 \sqrt{f_-} \tilde{T}^{\mu}_{-\nu} \right), \tag{3.3.27}$$

where

$$\ell_*^2 := \ell \int_{y_S}^{y_N} L_I y^2 dy \tag{3.3.28}$$

and $f_- := f(y_-)$.

Obviously, for the trace part we have the same relation as (3.3.27) with the substitution $\tilde{R}^{\mu}_{\ \nu} \to R/4$ and $\tilde{T}^{\mu}_{\pm\nu} \to -T^{\alpha}_{\pm\alpha}/4$ (and thus the constraint for the integration constants (3.2.32) is trivially satisfied). We have three bulk solutions as well,

$$\mathscr{K}^{(1)} = \begin{cases} \frac{y^3 - y_N^3}{12y^4\sqrt{f}} L_+ \frac{1}{4} {}^{(4)}R^{\alpha}_{\ \alpha} & (y_+ < y \le y_N) \\ \frac{1}{12y\sqrt{f}} L_0 {}^{(4)}R^{\alpha}_{\ \alpha} + \frac{1}{4y^4\sqrt{f}} \tilde{C}^{\alpha}_{(b)\alpha} & (y_- < y < y_+) \\ \frac{y^3 - y_S^3}{12y^4\sqrt{f}} L_- {}^{(4)}R^{\alpha}_{\ \alpha} & (y_S \le y < y_-) \end{cases}. \tag{3.3.29}$$

The junction condition $\left[\mathscr{K}^{(1)}\right]_{y=y_\pm} = \frac{T^\alpha_\alpha}{4M^4_{(6)}}$ gives two equations

$$C^\alpha_{\alpha(b)} = -\frac{(y_N^2 - y_+^3)L_+ + y_+^3 L_0}{12} {}^{(4)}\!R^\alpha_\alpha(h) - y_+^4\sqrt{f_+}\frac{\overset{(+)}{T}{}^\alpha_\alpha}{4M^4} \tag{3.3.30}$$

(3.3.31)

and

$$\frac{(y_-^3 - y_S^3)L_- - y_-^3 L_0}{12} {}^{(4)}\!R^\alpha_\alpha(h) - C^\alpha_{\alpha(b)} = -y_-^4\sqrt{f_-}\frac{\overset{(-)}{T}{}^\alpha_\alpha}{4M^4}. \tag{3.3.32}$$

Then eliminating $C^\alpha_{\alpha(b)}$, we have

$$\frac{\ell_*^2}{4\ell} {}^{(4)}\!R^\alpha_\alpha = -\frac{1}{4M^4}\left(y_+^4\sqrt{f_+}\,\overset{(+)}{T}{}^\alpha_\alpha + y_-^4\sqrt{f_-}\,\overset{(-)}{T}{}^\alpha_\alpha\right). \tag{3.3.33}$$

Since the induced metric on the north brane is $y_+^2 h_{\mu\nu}$, the Ricci tensor on the north brane is given by $\mathscr{R}^\mu_\nu := {}^{(4)}\!R^\mu_\nu / y_+^2$. Using Eqs. (3.3.27) and (3.3.33), we finally arrive at the low energy effective gravitational equation on the north brane

$$\mathscr{R}^\mu_\nu(g_+) - \frac{1}{2}\delta^\mu_\nu\mathscr{R}(g_+) = \kappa_+^2\,\overset{(+)}{\overline{T}}{}^\mu_\nu + \frac{y_-^2}{y_+^2}\kappa_-^2\,\overset{(-)}{\overline{T}}{}^\mu_\nu, \tag{3.3.34}$$

where we defined the 4-dimensional gravitational coupling at each brane as

$$\kappa_\pm^2 := \frac{y_\pm^2}{2\pi\ell_*^2 M^4}, \tag{3.3.35}$$

and the energy-momentum tensor integrated along the θ-direction as $\overset{(\pm)}{\overline{T}}{}^\mu_\nu := 2\pi\ell\sqrt{f_\pm}\,\overset{(\pm)}{T}{}^\mu_\nu$.

Eq. (3.3.34) shows that gravity at low energies is described by 4-dimensional general relativity (when matter on the south brane can be neglected.) This generalizes the perturbative analysis of [28, 34] to the nonlinear regime. The effect of the extra scalar mode that appears in [28, 34] is at higher order in the long-wavelength approximation and hence is not observed in the present leading order analysis.

It is interesting to note that we can still have the conventional 4-dimensional effective theory even if one takes the limit of the brane location to the poles keeping the averaged stress tensor $\overline{T}^\mu_\nu$ finite.

3.4 Consistency check

For the derivation of the effective theory on the branes, it was enough to compute $\mathscr{K}^{(1)}$ and $\overset{(1)}{\tilde{K}}{}^\mu_\nu$. We did not need to solve the Maxwell equation at the first order explicitly. Therefore, we have to check no extra conditions from the gauge fields. We first check the behavior of other geometrical quantities, $\overset{(1)}{K}{}^\mu_\nu$ and $\overset{(1)}{K}{}^\theta_\theta$ because they are contained in $\mathscr{K}^{(1)}$ together with $\overset{(1)}{A}_\mu$.

3.4.1 Behavior of $\overset{(1)}{K}{}^{\alpha}_{\alpha}$ and $\overset{(1)}{K}{}^{\theta}_{\theta}$

While we know that $\mathscr{K}^{(1)}$ can be regulated at the poles, it is not clear whether $\overset{(1)}{K}{}^{\alpha}_{\alpha}$ and $\overset{(1)}{K}{}^{\theta}_{\theta}$ well behaves or not at the poles. Using Eqs. (3.2.20) and (3.2.21), we have the following equation

$$\partial_y^2\Omega - \frac{\partial_y^2 I}{\partial_y I}\partial_y\Omega = \frac{3L_I R(h)}{y^3 f\partial_y\left(\frac{\sqrt{f}}{y}\right)} - \frac{1}{L_I}\zeta^{(1)}\frac{4}{\partial_y\left(\frac{\sqrt{f}}{y}\right)}\left\{\frac{1}{16}\frac{\partial_y^2 f\cdot\partial_y f}{f^2} + \frac{5}{8y}\left(\frac{\partial_y f}{f}\right)^2\right.$$
$$\left. + \frac{109}{8y^2}\frac{\partial_y f}{f} + \frac{63}{2y^3} + \frac{\partial_y f}{8f^2} + \frac{3}{yf} + \frac{7}{4y}\frac{\partial_y^2 f}{f} - \frac{1}{8}\frac{\partial_y^3 f}{f}\right\}, \tag{3.4.1}$$

where

$$\Omega := \frac{\overset{(1)}{K}{}^{\alpha}_{\alpha}}{\sqrt{f}\partial_y\left(\frac{\sqrt{f}}{y}\right)}, \tag{3.4.2}$$

and

$$\partial_y I := \frac{1}{y^9\sqrt{f}(y\partial_y f - 2f)^2}. \tag{3.4.3}$$

Then it turned out that the solution is given by

$$\Omega = C_1(x) + C_2(x)I(y) + \int^y dy_1\partial_{y_1}I(y_1)\int^{y_1}\frac{dy_2}{\partial_{y_2}I(y_2)}\frac{3L_I{}^{(4)}R(h)}{y_2^3 f(y_2)\partial_{y_2}\left(\frac{\sqrt{f(y_2)}}{y_2}\right)}$$
$$-\frac{\zeta^{(1)}(x)}{L_I}\int^y dy_1\partial_{y_1}I(y_1)\int^{y_1}\frac{dy_2}{\partial_{y_2}I(y_2)}\frac{4}{\partial_{y_2}\left(\frac{\sqrt{f(y_2)}}{y_2}\right)}\left\{\frac{1}{16}\frac{\partial_{y_2}^2 f\cdot\partial_{y_2}f}{f^2} + \frac{5}{8y_2}\left(\frac{\partial_{y_2}f}{f}\right)^2\right.$$
$$\left. + \frac{109}{8y_2^2}\frac{\partial_{y_2}f}{f} + \frac{63}{2y_2^3} + \frac{\partial_{y_2}f}{8f^2} + \frac{3}{y_2 f} + \frac{7}{4y_2}\frac{\partial_{y_2}^2 f}{f} - \frac{1}{8}\frac{\partial_{y_2}^3 f}{f}\right\}. \tag{3.4.4}$$

Once we obtain the solution to Ω, the solution for $\overset{(1)}{K}{}^{\theta}_{\theta}$ can be also obtained through Eq. (3.2.20) with Eq. (3.2.22).

At first, let us check the behavior of $\overset{(1)}{K}{}^{\alpha}_{\alpha}$ near the poles. Since $f(y)\sim f'(y_p)(y-y_p)$ near the poles, $\overset{(1)}{K}{}^{\alpha}_{\alpha}$ is approximated as

$$\overset{(1)}{K}{}^{\alpha}_{\alpha} \simeq C_1\cdot k_1 + C_2\cdot k_2(y-y_p)^{\frac{1}{2}} + 3L_I R(h)k_3(y-y_p)^{\frac{3}{2}}$$
$$+\frac{\zeta^{(1)}}{L_I}\left(k_4(y-y_p)^{\frac{1}{2}} + k_5(y-y_p)^{\frac{3}{2}} + k_6(y-y_p)^{\frac{5}{2}} + k_7(y-y_p)^{\frac{7}{2}}\right), \tag{3.4.5}$$

where

$$\begin{aligned}
&k_1(y_p) = \tfrac{f'(y_p)}{2y_p}, && k_2 = \frac{4}{y_p^{12} f'(y_p)^{\frac{3}{2}}}, && k_3 = \frac{2}{3y_p^3 f'(y_p)^{\frac{1}{2}}}, \\
&k_4 = \frac{y_p f''(y_p)+38f'(y_p)-2y_p}{f'(y_p)^{\frac{1}{2}}}, && k_5 = \frac{109f'(y_p)+y_p f'''(y_p)}{3y_p f'(y_p)^{\frac{1}{2}}}, && k_6 = \frac{126f'(y_p)^{\frac{1}{2}}}{5y_p^3}, \\
&k_7 = \tfrac{8}{7}.
\end{aligned} \tag{3.4.6}$$

From this we see that there are no divergence in $\overset{(1)}{K}{}^{\alpha}{}_{\alpha}$. So it is automatically regular at the poles.

Next, let us see the behavior of $\overset{(1)}{K}{}^{\theta}{}_{\theta}$. From Eq. (3.3.33) we have

$$\begin{aligned}
\overset{(1)}{K}{}^{\theta}{}_{\theta} \simeq \frac{y_p}{4}\Bigg\{ & -k_1 C_1 (y-y_p)^{-1} - \frac{1}{2}k_2 (y-y_p)^{-\frac{1}{2}} + \frac{k_3}{2}(y-y_p)^{\frac{1}{2}} \cdot 3L_I R(h) \\
& + \frac{\zeta^{(1)}}{L_I}\left(k_8 (y-y_p)^{-1} + \frac{13}{y_p^2} - \frac{k_4}{2}(y-y_p)^{-\frac{1}{2}} \right) \Bigg\},
\end{aligned} \tag{3.4.7}$$

where $k_8 = \frac{y_p f''(y_p)+2y_p+6f'(y_p)}{2y_p f'(y_p)}$. From this, we can see that $\overset{(1)}{K}{}^{\theta}{}_{\theta}$ diverges at the poles. Thus, we have to impose the regularity condition on $\overset{(1)}{K}{}^{\theta}{}_{\theta}$.

3.4.2 Trace part consistency check

In the previous subsection we saw that the regularity condition for $\overset{(1)}{K}{}^{\theta}{}_{\theta}$ at the poles, but not for $\overset{(1)}{K}{}^{\alpha}{}_{\alpha}$. Then we must check the consistency between the number of the regularity/continuity/junction conditions and the total number of independent variables. Since traceless evolution equation is rather trivial, we will focus on trace part hereafter.

Integral constants:

5 (for $\overset{(1)}{K}{}^{\theta}_{\theta}, \overset{(1)}{K}{}^{\alpha}_{\alpha}, \overset{(1)}{A}_{\theta}, \overset{(1)}{\psi}, \overset{(1)}{\zeta}$) × 3 regions = 15

Boundary conditions:

3 regularity	(for $\overset{(1)}{K}{}^{\theta}_{\theta}, \overset{(1)}{A}_{\theta}, \overset{(1)}{\psi}$)	× 2 poles	= 6
+2 continuity	(for $\overset{(1)}{A}_{\theta}, \overset{(1)}{\psi}$)	× 2 branes	= 4
+3 junction	(for $\overset{(1)}{K}{}^{\theta}_{\theta}, \overset{(1)}{K}{}^{\alpha}_{\alpha}, \overset{(1)}{F}{}^{y\theta}$)	× 2 branes	= 6
-1 junction	(for $\overset{(1)}{K}{}^{\alpha}_{\alpha}$)	× 1	= -1
			15

Figure 3.4.1: Summary of counting

So far, we have 5 variables such as $\overset{(1)}{K}{}^{\alpha}{}_{\alpha}(y,x)$, $\overset{(1)}{K}{}^{\theta}{}_{\theta}(y,x)$, $\overset{(1)}{A_\theta}(y,x)$, $\overset{(1)}{\psi_I}(y,x)$ and $\overset{(1)}{\zeta_I}(x)$. Each variable has one integral constant for each region (north/south/middle region). Thus the net number of variables or integral constants amounts to $5 \times 3 = 15$.

On the other hand, let us count the number of boundary conditions. We have three kinds of conditions, that is, continuity, regularity and junction conditions. As in the discussion of previous subsection, the regularity conditions are required except for $\overset{(1)}{K}{}^{\alpha}{}_{\alpha}(y,x)$. Then we have $3 \times 2 = 6$ conditions for regularity. About the continuity conditions, $\overset{(1)}{A_\theta}(y,x)$ and $\overset{(1)}{\psi_I}(y,x)$ are imposed on. Thus we have $2 \times 2 = 4$ continuity conditions. About the junction conditions, three conditions for $\overset{(1)}{K}{}^{\theta}{}_{\theta}(y,x)$, $\overset{(1)}{K}{}^{\alpha}{}_{\alpha}(y,x)$ and $\overset{(1)}{F_{y\theta}}$ per north/south brane. Therefore the total number of junction conditions is $3 \times 2 = 6$. However one of them is used for the derivation of the effective theory. Thus, $6 - 1 = 5$ junction conditions can determine the integral constants/variables. As a result the total number of the boundary conditions is $6 + 4 + 5 = 15$. This is exactly the same with the number of the integral constants. Summary of this subsection is given by Fig. 3.4.1.

Chapter 4

Summary and discussion

In order to study the physics of the early universe, the high-energy physics such as the superstring theory cannot be ignored. Motivated by the recent progress in superstring theory, a new picture of our universe was proposed. Therein, our universe is a brane located in the higher dimensional space-times and then a lot of models of this braneworld have been discussed. Among them Randall-Sundrum (RS) model is intensively studied. This is because the effect of self-gravity is seriously treated and then we can discuss the cosmology in this model. However RS model is 5-dimensional model and on the other hand, superstring theory is formulated in 10/11 dimensions. Our study is to extend RS braneworld model into higher dimensional model. As the first step we studied the 6-dimensional model.

The raising of the co-dimensions will arise some problems, that is, the singularity problem due to the self-gravity of higher co-dimension branes and the instability problem in the size of extra dimensions. These problem are resolved by introducing flux and capped regularization scheme. The regularization scheme is developed by making use of the well-behaved property of the co-dimension-1 brane. The brane action we adopt includes the property needed for the surgery of the bulk with flux and gravity, and the property of spontaneously breaking the 5-dimensional general covariance which makes the theory 4-dimensional one. This model has been actively investigated [28], [34]. But these studies were focused on linear perturbation analysis. Since we are interested in the non-linear regime, we employed another approximation scheme, that is, long-wavelength approximation which is useful for the low energy scales. The small parameter is taken to be the ratio of the curvature length of the extra dimension to the 4-dimensional scale which we are interested in. As a result, up to the first order of the long-wavelength approximation, we confirmed that the conventional 4-dimensional theory is recovered even at the non-linear regime. Interestingly, we can have the conventional effective theory even if the brane is taken to the poles keeping the averaged stress tensor on the brane finite.

As remaining problem, we have to check the higher order correction terms of long-wavelength approximation. Can we still construct the regular model up to higher orders? This is the rather non-trivial question. Especially, in the case of the limit where the co-dimension-1 brane is taken to the poles.

This model is partly motivated by the warped flux compactification model in superstring theory [38]. However the self-gravity has not been treated carefully in such models and the validity of the discussion for cosmology is still debatable. So our study may be regarded as a toy-model for this model. The extension of our toy model to this direction may give some information to the string inspired cosmology. Although our model is still far from the original model of string cosmology, the further extension of effective theory on the brane is urgently expected.

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Appendix A

Derivation of Israel junction condition

In this appendix, the junction condition is derived. To discuss concretely, we show the derivation of Israel junction condition in the general case where the number of the dimensions of whole space-time is d with co-dimension-1 ($d-1$-dimensional) brane.

Using the decomposition of d-dimensional Ricci tensor, the Israel junction conditions can be obtained. Let us take the block-diagonal metric ansatz (A.0.1) as in [9]

$$ds^2 = n_y^2 dY^2 + {}^{(d-1)}q_{\mu\nu}dx^\mu dx^\nu. \tag{A.0.1}$$

where normal vector to y =constant hypersurfaces is $n^a = n^y \left(\frac{\partial}{\partial Y}\right)^a$ and ${}^{(4)}q_{\mu\nu}$ is the induced metric on Y =constant hypersurfaces. This induced metric satisfies

$$q_{AB} := -n_A n_B + g_{AB}, \tag{A.0.2}$$

where $n^A n_A = 1$ and $n^A q_{AB} = 0$. Then generally, we have

$${}^{(d)}R_{CD}q_\mu{}^C q_\nu{}^D = {}^{(d-1)}R_{\mu\nu} - \mathscr{L}_n K_{\mu\nu} - KK_{\mu\nu} + 2K_\mu{}^\alpha K_{\alpha\nu}, \tag{A.0.3}$$

where capital Latin indices runs d-dimensional space-time, $A = 0, \cdots, d-1$ and Greek indices run $(d-1)$-dimensional space-time, $\mu = 0, \cdots, d-2$. $K_{\mu\nu}$ is the extrinsic curvature defined by

$$K_{\mu\nu} = \frac{1}{2}\mathscr{L}_n \overset{(d-1)}{q_{\mu\nu}} \tag{A.0.4}$$

$$= \frac{1}{2}\frac{1}{n_y}\partial_y {}^{(d-1)}q_{\mu\nu}. \tag{A.0.5}$$

Then, ${}^{(d)}R^\mu_\nu$ is decomposed as

$${}^{(d)}R^\mu{}_\nu = {}^{(d-1)}R^\mu{}_\nu - \frac{1}{n_y}q^{\mu\alpha}\partial_y K_{\alpha\nu} - KK^\mu{}_\nu + 2K^{\mu\alpha}K_{\alpha\nu} \tag{A.0.6}$$

$$= {}^{(d-1)}R^\mu{}_\nu - \frac{1}{n_y}\partial_y K^\mu{}_\nu - KK^\mu{}_\nu. \tag{A.0.7}$$

In Randall-Sundrum (RS1) case, the total action is

$$S = \frac{1}{2\kappa_{(d)}^2}\int dy d^{d-1}x\sqrt{-{}^{(d)}g}\left({}^{(d)}R - 2\Lambda_{(d)}\right) - \lambda\int_{y=y_+} d^{d-1}x\sqrt{-{}^{(d-1)}g_+}$$

$$+\lambda \int_{y=y_-} d^{d-1}x\sqrt{-{}^{(d-1)}g_-} + \overset{(+)}{S}_{\text{matter}} + \overset{(-)}{S}_{\text{matter}}, \tag{A.0.8}$$

where positive (negative) tension brane is located at $y = y_+$ ($y = y_-$). Then Einstein equation becomes

$$\begin{aligned}&{}^{(d)}R_{AB} - \frac{1}{2}{}^{(d)}g_{AB}{}^{(d)}R = -\Lambda_{(d)}{}^{(d)}g_{AB} - \kappa^2_{(d)}\lambda q^{(+)}_{AB}\frac{1}{n_y}\delta(y-y_+) + \kappa^2_{(d)}\lambda q^{(-)}_{AB}\frac{1}{n_y}\delta(y-y_-)\\ &+\kappa^2_{(d)}T^{(+)}_{\mu\nu}q^{\mu}{}_{A}q^{\nu}{}_{B}\frac{1}{n_y}\delta(y-y_+) + \kappa^2_{(d)}T^{(-)}_{\mu\nu}q^{\mu}{}_{A}q^{\nu}{}_{B}\frac{1}{n_y}\delta(y-y_+),\end{aligned} \tag{A.0.9}$$

where $\frac{\sqrt{-{}^{(d-1)}g}}{\sqrt{-{}^{(d)}g}} = \frac{1}{n_y}$ is used. Note that $q_{yy} = 0$ and $q_{\mu y} = 0$ by the definition Eq. (A.0.2). The (μ,ν)-component of the above can be re-written as

$$\begin{aligned}&{}^{(d-1)}R^{\mu}{}_{\nu} - \frac{1}{n_y}\partial_y K^{\mu}{}_{\nu} - KK^{\mu}{}_{\nu} = \frac{2}{d-2}\Lambda_{(d)}\delta^{\mu}_{\nu} + \frac{\kappa^2_{(d)}\lambda}{d-2}\delta^{\mu}_{\nu}\frac{1}{n_y}\delta(y-y_+) - \frac{\kappa^2_{(d)}\lambda}{d-2}\delta^{\mu}_{\nu}\frac{1}{n_y}\delta(y-y_-)\\ &+\kappa^2_{(d)}\left(\overset{(+)}{T}{}^{\mu}{}_{\nu} - \frac{1}{d-2}\delta^{\mu}_{\nu}\overset{(+)}{T}{}^{\alpha}{}_{\alpha}\right)\frac{1}{n_y}\delta(y-y_+) + \kappa^2_{(d)}\left(\overset{(-)}{T}{}^{\mu}{}_{\nu} - \frac{1}{d-2}\delta^{\mu}_{\nu}\overset{(-)}{T}{}^{\alpha}{}_{\alpha}\right)\frac{1}{n_y}\delta(y-y_-).\end{aligned} \tag{A.0.10}$$

By integration of Eq. (A.0.10) within infinitesimally small interval around brane we obtain

$$\left[K^{\mu}{}_{\nu}\right]_{y=y_\pm} = \mp\frac{\kappa^2_{(d)}\lambda}{d-2}\delta^{\mu}_{\nu} - \kappa^2_{(d)}\left(\overset{(\pm)}{T}{}^{\mu}{}_{\nu} - \frac{1}{d-2}\delta^{\mu}_{\nu}\overset{(\pm)}{T}{}^{\alpha}{}_{\alpha}\right). \tag{A.0.11}$$

Now let us consider symmetric case. When the metric has reflection symmetry around the brane, the metric is said to have the Z_2-symmetry at the location of brane. Now we impose Z_2-symmetry on this system. Then

$$\left[K^{\mu}{}_{\nu}\right]_{y=y_+} = 2K^{\mu}{}_{\nu}|_{y=y_+}, \tag{A.0.12}$$

$$\left[K^{\mu}{}_{\nu}\right]_{y=y_-} = -2K^{\mu}{}_{\nu}|_{y=y_-}. \tag{A.0.13}$$

Thus the junction condition becomes

$$K^{\mu}{}_{\nu}|_{y=y_\pm} = \mp\frac{\kappa^2_{(d)}\lambda}{2(d-2)}\delta^{\mu}_{\nu} - \frac{\kappa^2_{(d)}}{2}\left(\overset{(\pm)}{T}{}^{\mu}{}_{\nu} - \frac{1}{d-2}\delta^{\mu}_{\nu}\overset{(\pm)}{T}{}^{\alpha}{}_{\alpha}\right) \tag{A.0.14}$$

$$= -\frac{\kappa^2_{(d)}}{2}\left(\overset{(\pm)}{S}{}^{\mu}{}_{\nu} - \frac{1}{d-2}\delta^{\mu}_{\nu}\overset{(\pm)}{S}{}^{\alpha}{}_{\alpha}\right), \tag{A.0.15}$$

where $\overset{(\pm)}{S}_{\mu\nu} := \pm\lambda\,{}^{(d-1)}q_{\mu\nu} + T_{\mu\nu}$.

Appendix B

Appendix of Sec. 2.2.1

In this appendix, the derivations of Eqs. (2.2.2), (2.2.3) and (2.2.4) in section 2.2.1 are given. Following [15] we will consider co-dimension-2 braneworld model

$$S = \int d^6x \sqrt{-{}^{(6)}g}\,{}^{(6)}R + \int d^4x \sqrt{-{}^{(4)}g}\,\mathscr{L}_{\rm matter}. \tag{B.0.1}$$

The second term represents the matter on the co-dimension-2 brane. Note that if the matter is limited to pure tension case, $p = -\rho$ the brane action becomes Nambu-Goto type, $\int d^4x \sqrt{-{}^{(4)}g}\,\mathscr{L}_{\rm matter} = -\mu \int d^4x \sqrt{-{}^{(4)}g}$. Here we include more general case with the equation of state $p \neq -\rho$.

We assume axial symmetry and diagonal type metric ansatz

$$ds^2 = -N^2(y)dt^2 + M^2(y)\delta_{ij}dx^i dx^j + B^2(y)dy^2 + L^2(y)d\theta^2, \tag{B.0.2}$$

where (y,θ) denote two extra dimensions. From this model, we have the 6-dimensional Einstein equation

$${}^{(6)}G_{AB} = \delta_A{}^{\mu}\delta_B{}^{\nu}T_{\mu\nu}\delta^{(2)}(y). \tag{B.0.3}$$

Each component becomes

$$\begin{aligned}{}^{(6)}G_{00} \sim \frac{N^2(y)}{B^2(y)}\Big\{&-3\frac{\partial_y^2 M(y)}{M(y)} - \frac{\partial_y M(y)\,\partial_y L(y)}{M(y)L(y)} - \frac{\partial_y^2 L(y)}{L(y)} - 3\frac{\partial_y M(y)\,\partial_y M(y)}{M^2(y)} \\ &+3\frac{\partial_y B\,\partial_y M}{B(y)\,M(y)} + \frac{\partial_y B\,\partial_y L(y)}{B(y)\,L(y)}\Big\} \\ =\ & \rho N^2(y)\delta^{(2)}(y),\end{aligned} \tag{B.0.4}$$

$$\begin{aligned}{}^{(6)}G_{ij} \sim \frac{M^2(y)}{B^2(y)}\Big\{&2\frac{\partial_y^2 M(y)}{M(y)} + \frac{\partial_y M(y)\,\partial_y L(y)}{M(y)\,L(y)} + \frac{\partial_y^2 N(y)}{N(y)} + \frac{\partial_y N(y)\,\partial_y L(y)}{N(y)\,L(y)} + \frac{\partial_y^2 L(y)}{L(y)} - \frac{\partial_y B\,\partial_y L(y)}{B(y)\,L(y)} \\ &-2\frac{\partial_y B(y)\,\partial_y M(y)}{B(y)\,M(y)} + \left(\frac{\partial_y M(y)}{M(y)}\right)^2 + \frac{\partial_y B(y)\,\partial_y N(y)}{B(y)\,N(y)} + 2\frac{\partial_y M(y)\,\partial_y N(y)}{M(y)N(y)}\Big\} \\ =\ & PM^2(y)\delta_{ij}\delta^{(2)}(y),\end{aligned} \tag{B.0.5}$$

$$
{}^{(6)}G_{\theta\theta} \sim \frac{L^2(y)}{B^2(y)}\Big\{3\frac{\partial_y^2 M(y)}{M(y)} + \frac{\partial_y^2 N(y)}{N(y)} - 3\frac{\partial_y B(y)\ \partial_y M(y)}{B(y)\ M(y)} + 3\left(\frac{\partial_y M(y)}{M(y)}\right)^2 - \frac{\partial_y B(y)\ \partial_y N(y)}{B(y)\ N(y)}
+3\frac{\partial_y M(y)\ \partial_y N(y)}{M(y)\ N(y)}\Big\} = 0 \tag{B.0.6}
$$

and

$$
G_{yy} = 3\Big\{\left(\frac{\partial_y M(y)}{M(y)}\right)^2 + \frac{\partial_y M\ \partial_y L(y)}{M(y)\ L(y)}\Big\} + \frac{\partial_y N(y)\ \partial_y L(y)}{N(y)\ L(y)} + 3\frac{\partial_y M(y)\ \partial_y N(y)}{M(y)\ N(y)} = 0 \tag{B.0.7}
$$

In order to see the behavior of this solution, we pick up only the main contributing term, that is, $\partial_y^2 g_{AB}$. Ignoring the first order derivative terms, we can obtain Eqs. (2.2.2), (2.2.3) and (2.2.4)

$$
-3\frac{\partial_y^2 M(y)}{M(y)} - \frac{\partial_y^2 L(y)}{L(y)} \simeq 8\pi G_{(6)}\frac{B^2(y)}{N^2(y)}\rho N^2(y)\delta^{(2)}(y), \tag{B.0.8}
$$

$$
2\frac{\partial_y^2 M(y)}{M(y)} + \frac{\partial_y^2 N(y)}{N(y)} + \frac{\partial_y^2 L(y)}{L(y)} \simeq 8\pi G_{(6)}\frac{B^2(y)}{M^2(y)}PM^2(y)\delta_{ij}\delta^{(2)}(y) \tag{B.0.9}
$$

and

$$
3\frac{\partial_y^2 M(y)}{M(y)} + \frac{\partial_y^2 N(y)}{N(y)} \simeq 0. \tag{B.0.10}
$$

Note that G_{yy}-component yields trivial relation in this evaluation.

Appendix C

Some calculations in the Freund-Rubin compactification

In this appendix we will derive Eqs. (2.2.15), (2.2.17) and (2.2.18) in Sec. 2.2.2. Let us remember the total action

$$S = \frac{1}{2}\int d^{p+q}x\sqrt{-{}^{(p+q)}\hat{g}}\left({}^{(p+q)}R(g) - \frac{1}{2q!}F^2\right). \tag{C.0.1}$$

The metric ansatz is

$$\begin{aligned} ds^2 &= e^{-\frac{2q}{p-2}\phi(x)}h_{\mu\nu}dx^\mu dx^\nu + e^{2\phi(x)}\sigma_{ij}dx^i dx^j && \text{(C.0.2)}\\ &:= g_{\mu\nu}dx^\mu dx^\nu + g_{ij}dx^i dx^j, && \text{(C.0.3)}\end{aligned}$$

where whole space-time is $p+q$-dimension. The number of extra dimensions is q and its topology is q-sphere. Note that scalar field $\phi(x)$[1] determines the size of q-sphere and induced metric on unit q-sphere, σ_{ij} depends on only angular components. Latin indices such as $i, j, \cdots$ denote q-dimensional extra space and Greek indices $\mu, \nu, \cdots$ shows p-dimensional space-time.

First, we write down some useful results for Ricci tensor

$${}^{(p+q)}R_{ij}(g) = {}^{(q)}R_{ij}(g) + e^{\frac{2(p+q-2)}{p-2}\phi(x)}\overset{(h)}{D}{}^2\phi\sigma_{ij}, \tag{C.0.4}$$

$${}^{(q)}R_{ijk\ell}(\sigma) = \frac{1}{R^2}(\sigma_{ik}\sigma_{j\ell} - \sigma_{jk}\sigma_{i\ell}), \tag{C.0.5}$$

$${}^{(p+q)}R_{\mu\nu}(g) = {}^{(p)}R_{\mu\nu}(g) - q\overset{(h)}{D}_\nu\overset{(h)}{D}_\mu\phi + \frac{q^2}{p-2}(-2\overset{(h)}{D}_\mu\phi\cdot\overset{(h)}{D}_\nu\phi - h_{\mu\nu}(\partial_\alpha\phi)^2) - q\overset{(h)}{D}_\mu\phi\cdot\overset{(h)}{D}_\nu\phi, \tag{C.0.6}$$

$${}^{(p)}R_{\mu\nu}(g) = {}^{(p)}R_{\mu\nu}(h) + q\overset{(h)}{D}_\mu\overset{(h)}{D}_\nu\phi + \frac{q}{p-2}h_{\mu\nu}\overset{(h)}{D}{}^2\phi + \frac{q^2}{p-2}\overset{(h)}{D}_\mu\phi\cdot\overset{(h)}{D}_\nu\phi - \frac{q^2}{p-2}(\overset{(h)}{D}_\alpha\phi)^2, \tag{C.0.7}$$

where R is the radius of q-sphere. Then from Eqs. (C.0.6) and (C.0.7) we have

$${}^{(p+q)}R_{\mu\nu}(g) = {}^{(p)}R_{\mu\nu}(h) - \frac{q(p+q-2)}{p-2}\overset{(h)}{D}_\mu\phi\overset{(h)}{D}_\nu\phi + \frac{q}{p-2}\overset{(h)}{D}{}^2\phi\cdot h_{\mu\nu}. \tag{C.0.8}$$

[1] As we are only interested in zero mode of this metric, y-dependence on ϕ is ignored to be $\phi(x)$, [19].

Then from Eqs. (C.0.4), (C.0.5) and (C.0.8) we have

$$\begin{aligned} {}^{(p+q)}R(\hat{g}) &:= g^{\mu\nu\,(p+q)}R_{\mu\nu}(g) + g^{ij\,(p+q)}R_{ij}(g), \\ &= e^{\frac{2q}{p-2}\phi(x)\,(p)}R(h) + e^{-2\phi(x)\,(q)}R(\sigma) \\ &\quad + e^{\frac{2q}{p-2}\phi(x)}\Big(-\frac{q(p+q-2)}{p-2}(\overset{(h)}{D}_\alpha\,\phi)^2 + \frac{2q(p-1)}{p-2}\overset{(h)}{D}{}^2\phi\Big). \end{aligned} \tag{C.0.9}$$

We start from $p+q$-dimensional Einstein equations

$$ {}^{(p+q)}R_{AB} - \frac{1}{2}g_{AB}\,{}^{(p+q)}R = \frac{1}{2(q-1)!}\left(F_{AM_2\cdots M_q}F_B{}^{M_2\cdots M_q} - \frac{1}{2q}g_{AB}F^2\right). \tag{C.0.10}$$

The (μ,ν)-component can be written as

$$ {}^{(p+q)}R_{\mu\nu} = -\frac{q-1}{q!(p+q-2)}\,{}^{(p)}g_{\mu\nu}F^2. \tag{C.0.11}$$

The (i,j)-component becomes

$$ {}^{(p+q)}R_{ij} = \frac{1}{2(q-1)!}\left(F_{im_2\cdots m_q}F_j{}^{m_2\cdots m_q} - \frac{q-1}{q!(p+q-2)}g_{ij}F^2\right). \tag{C.0.12}$$

$$ \tag{C.0.13}$$

Secondly, let us solve Maxwell equation

$$ \partial_M\left(\sqrt{-{}^{(p+q)}g}F^{M_1M_2\cdots M_q}\right) = 0. \tag{C.0.14}$$

Then this solution can be obtained easily

$$ F_{M_1\cdots M_q} = f\varepsilon_{M_1\cdots M_q}(\sigma), \tag{C.0.15}$$

where f is constant and $\varepsilon_{M_1\cdots M_q}(\sigma)$ is the Levi-Civita tensor of unit q-dimensional sphere. We have one more relation between the size of q-sphere, R and the strength of flux, f

$$ f^2 = \frac{2(q-1)(p+q-2)}{R^2(p-1)}, \tag{C.0.16}$$

which can be adjusted by using stabilized background solution. Substituting Eqs. (C.0.9), (C.0.15) into the original action we have

$$\begin{aligned} S_{\text{eff}} &= \frac{1}{2}\int d^{p+q}x\sqrt{-{}^{(p+q)}g}\left({}^{(p+q)}R(g) - \frac{1}{2q!}F^2\right) \qquad (\text{C.0.17}) \\ &= \frac{1}{2}\int d^qx\sqrt{-\sigma}\cdot\int d^px\sqrt{-h}\left({}^{(p)}R(h) - \frac{f^2}{2}\left(-\frac{q(p-1)}{p+q-2}e^{\frac{2(p+q-2)}{p-2}\phi(x)} + e^{-\frac{2(p-1)q}{p-2}\phi(x)}\right)\right) \end{aligned} \tag{C.0.18}$$

Appendix D

The leading order of K^{μ}_{θ} and $F^{\theta\mu}$

Let us see (μ,θ)-evolution equation noted in Sec. 3.2.2 more concretely. Here we consider the following non-diagonal metric ansatz

$$ds^2 = \frac{L_I^2}{f(y)} e^{2\zeta(x)} dy^2 + \ell^2 e^{2\psi(y,x)} f(y) d\theta^2 + 2\ell b_\mu(y,x) d\theta dx^\mu + g_{\mu\nu}(x,y) dx^\mu dx^\nu \qquad \text{(D.0.1)}$$

with the assumptions

$$b_\mu = \varepsilon^{1/2} b_\mu^{(1/2)} + \cdots, \qquad K^{\mu}_{\theta} = \varepsilon^{1/2} \overset{(1/2)}{K^{\mu}_{\theta}} + \cdots, \quad F_{y\theta} = \overset{(0)}{F}_{y\theta} + \varepsilon \overset{(1)}{F}_{y\theta} + \cdots .$$

We shall show in this appendix that the leading order terms in the gradient expansion of K^{μ}_{θ} and $F^{\mu y}$ are in fact $O(\varepsilon^{3/2})$.

The μ-component of the $O(\varepsilon^{1/2})$ Maxwell equations reads

$$\partial_y\left(y^4 \overset{(1/2)}{F^{y\mu}}\right) = 0, \qquad \text{(D.0.2)}$$

and thus we have

$$\overset{(1/2)}{F^{\mu y}} = M_{(6)}^2 \frac{C_1^\mu(x)}{y^4}. \qquad \text{(D.0.3)}$$

The $O(\varepsilon^{1/2})$ evolution equation reduces to

$$\begin{aligned} \frac{1}{L_I}\frac{1}{y^4}\partial_y\left(y^4\sqrt{f}\,\overset{(1/2)}{K^{\mu}_{\theta}}\right) &= -\frac{1}{M^4}\overset{(0)}{F}_{\theta y}\overset{(1/2)}{F^{\mu y}} \\ &= \ell Q \frac{C_1^\mu(x)}{y^8}, \end{aligned} \qquad \text{(D.0.4)}$$

which can be integrated to give

$$\overset{(1/2)}{K^{\mu}_{\theta}} = \frac{1}{y^4\sqrt{f}}\left[-\frac{L_I \ell Q}{3}\frac{C_1^\mu(x)}{y^3} + C_2^\nu(x)\right]. \qquad \text{(D.0.5)}$$

The $O(\varepsilon)$ evolution equations contain terms like $\overset{(1/2)}{F}_{\nu y}\overset{(1/2)}{F}{}^{\mu y} \propto h_{\nu\lambda} C_1^{\lambda} C_1^{\mu}/[y^6 f(y)]$. Thus, in the cap regions it is required that $C^{\mu}_{1(\text{north})} = 0$ and $C^{\mu}_{1(\text{south})} = 0$ because otherwise this term would show a singular behavior at the poles. Similarly, to ensure the regular behavior of $\overset{(1/2)}{K}{}^{\mu}_{\theta}$ at the poles we impose $C^{\mu}_{2(\text{north})} = 0$ and $C^{\mu}_{2(\text{south})} = 0$ in the cap regions. To fix the integration constants in the middle-region, we invoke the junction conditions at the branes. The junction conditions of the Maxwell field and Israel junction conditions imply, respectively,

$$\left[n_y \overset{(1/2)}{F}{}^{y\mu}\right]_{y=y_\pm} = -\mathrm{e}v^2 \left(\partial^{\mu}\Sigma - \mathrm{e}A^{\mu}\right)^{(1/2)}, \tag{D.0.6}$$

$$\left[\overset{(1/2)}{K}{}^{\mu}_{\theta}\right]_{y=y_\pm} = -\frac{v^2}{M^4}\left(\partial_\theta \Sigma^{(0)} - \mathrm{e}A^{(0)}_{\theta}\right)\left(\partial^{\mu}\Sigma - \mathrm{e}A^{\mu}\right)^{(1/2)}. \tag{D.0.7}$$

Combining these two equations and noting that $\overset{(1/2)}{K}{}^{\mu}_{\theta} = 0$, $\overset{(1/2)}{F}{}^{y\mu} = 0$ in each cap, we obtain two linear algebraic equations for $C^{\mu}_{1(\text{middle})}$ and $C^{\mu}_{2(\text{middle})}$:

$$\left[\overset{(1/2)}{K}{}^{\mu}_{\theta} - \frac{1}{\mathrm{e}M^4}\left(\partial_\theta \Sigma^{(0)} - \mathrm{e}A^{(0)}_{\theta}\right) n_y \overset{(1/2)}{F}{}^{y\mu}\right]_{y=y_\pm} = 0. \tag{D.0.8}$$

Thus, it is now clear that both $C^{\mu}_{1(\text{middle})}$ and $C^{\mu}_{2(\text{middle})}$ vanish in the bulk.

To sum up, we have shown in this appendix that

$$K^{\mu}_{\theta} = \varepsilon^{3/2} \overset{(3/2)}{K}{}^{\mu}_{\theta} + \cdots, \tag{D.0.9}$$

$$F^{y\mu} = \varepsilon^{3/2} \overset{(3/2)}{F}{}^{y\mu} + \cdots, \tag{D.0.10}$$

and that

$$\left(\partial^{\mu}\Sigma - \mathrm{e}A^{\mu}\right)^{(1/2)} = 0 \quad \text{on the branes.} \tag{D.0.11}$$

Appendix E

Brief summary of derivation of basic equations and junction conditions for the gauge fields

In this appendix, we will give a brief derivation of each basic equation and junction condition in Sec. 3.1. The model we start is the extended Randall-Sundrum type model in 6-dimensional space-time

$$S = \int d^6x \sqrt{-{}^{(6)}g} \left\{ \frac{M_{(6)}^4}{2} \left({}^{(6)}R - \frac{1}{L_I^2} \right) - \frac{1}{4} F_{AB} F^{AB} \right\} + S_{\text{brane}}^{+} + S_{\text{brane}}^{-}, \qquad \text{(E.0.1)}$$

where

$$S_{\text{brane}}^{\pm} = - \int d^5x \sqrt{-{}^{(5)}g} \left(\lambda_{\pm} + \frac{v_{\pm}^2}{2} g^{\hat{\mu}\hat{\nu}} (\partial_{\hat{\mu}} \Sigma_{\pm} - eA_{\hat{\mu}})(\partial_{\hat{\nu}} \Sigma_{\pm} - eA_{\hat{\nu}}) \right). \qquad \text{(E.0.2)}$$

The metric ansatz is

$$ds^2 = L_I^2 e^{2\zeta(y,x)} \frac{dy^2}{f(y)} + \ell^2 e^{2\psi(y,x)} f(y) d\theta^2 + a^2(y) h_{\mu\nu} dx^\mu dx^\nu, \qquad \text{(E.0.3)}$$

$$:= {}^{(6)}g_{AB} dx^A dx^B, \qquad \text{(E.0.4)}$$

where ${}^{(6)}g := \det(\mathrm{g_{AB}})$. By variation of the total action (E.0.1) with respect to metric ${}^{(6)}g_{AB}$ and $\delta_g S = 0$ leads 6-dimensional Einstein equations,

$${}^{(6)}G^A_B = -\frac{1}{2L_I^2} \delta^A_B + \frac{1}{M_{(6)}^4} \left(F^A_C F_B{}^C - \frac{1}{4} \delta^A_B F^{CD} F_{CD} \right) \qquad \text{(E.0.5)}$$

then

$${}^{(6)}R^A_B = \frac{1}{4L_I^2} \delta^A_B + \frac{1}{M_{(6)}^4} \left(F^A_C F_B{}^C - \frac{1}{8} \delta^A_B F^{CD} F_{CD} \right). \qquad \text{(E.0.6)}$$

From the $(\hat{\mu}, \hat{\nu})$-component of Eq. (E.0.6), we can have the evolution equation

$$\frac{1}{n_y} \partial_y K^{\hat{\mu}}_{\hat{\nu}} = {}^{(5)}R^{\hat{\mu}}_{\hat{\nu}} - \hat{K} K^{\hat{\mu}}_{\hat{\nu}} - {}^{(5)}D^{\hat{\mu}} {}^{(5)}D_{\hat{\nu}} \zeta - {}^{(5)}D^{\hat{\mu}} \zeta \cdot {}^{(5)}D_{\hat{\nu}} \zeta - \frac{1}{4L_I^2} \delta^{\hat{\mu}}_{\hat{\nu}}$$

$$-\frac{1}{M_{(6)}^4}\left(F^{\hat{\mu}}_{C}F_{\hat{\nu}}{}^{C}-\frac{1}{8}\delta^{\hat{\mu}}_{\hat{\nu}}F^{CD}F_{CD}\right) \tag{E.0.7}$$

We also note other useful expressions

$$^{(6)}R_{yAy}{}^{A}=-\frac{1}{n_y}\partial_y\hat{K}-K^{\hat{\alpha}}_{\hat{\beta}}K^{\hat{\beta}}_{\hat{\alpha}}-{}^{(5)}D^{\hat{\alpha}}\zeta\cdot{}^{(5)}D_{\hat{\alpha}}\zeta-{}^{(5)}D_{\hat{\alpha}}{}^{(5)}D^{\hat{\alpha}}\zeta. \tag{E.0.8}$$

$$^{(6)}R={}^{(5)}R-\frac{2}{n_y}\partial_y K^{\hat{\alpha}}_{\hat{\alpha}}-\hat{K}K^{\hat{\alpha}}_{\hat{\alpha}}-K^{\hat{\alpha}}_{\hat{\beta}}K^{\hat{\beta}}_{\hat{\alpha}}\ -2\left({}^{(5)}D^{\hat{\alpha}(5)}D_{\hat{\alpha}}\zeta+{}^{(5)}D^{\hat{\alpha}}\zeta\cdot{}^{(5)}D_{\hat{\alpha}}\zeta\right) \tag{E.0.9}$$

$$^{(6)}R_{\hat{\mu}Ay}{}^{A}=\frac{1}{n^y}\left({}^{(5)}D_{\hat{\alpha}}K^{\hat{\alpha}}_{\hat{\mu}}-{}^{(5)}D_{\hat{\mu}}\hat{K}\right) \tag{E.0.10}$$

By Eqs. (E.0.8) and (E.0.9),

$$^{(6)}G_{yy}\ =\ \frac{1}{2}g_{yy}\left({}^{(6)}R^{y}_{y}-{}^{(6)}R^{\hat{\alpha}}_{\hat{\alpha}}\right)=-\frac{1}{2}g_{yy}\left({}^{(5)}R-\hat{K}^2+K^{\hat{\alpha}}_{\hat{\beta}}K^{\hat{\beta}}_{\hat{\alpha}}\right). \tag{E.0.11}$$

Using the above equations, 6-dimensional Einstein equation yields following three equations.

Results of 5+1 decomposition

Evolution equations is

$$\frac{1}{n_y}\partial_y K^{\hat{\mu}}_{\hat{\nu}}\ =\ {}^{(5)}R^{\hat{\mu}}_{\hat{\nu}}-\hat{K}K^{\hat{\mu}}_{\hat{\nu}}-{}^{(5)}D^{\hat{\mu}(5)}D_{\hat{\nu}}\zeta-{}^{(5)}D^{\hat{\mu}}\zeta\cdot{}^{(5)}D_{\hat{\nu}}\zeta-\frac{1}{4L_I^2}\delta^{\hat{\mu}}_{\hat{\nu}}$$
$$-\frac{1}{M_{(6)}^4}\left(F^{\hat{\mu}}_{C}F_{\hat{\nu}}{}^{C}-\frac{1}{8}\delta^{\hat{\mu}}_{\hat{\nu}}F^{CD}F_{CD}\right) \tag{E.0.12}$$

Hamiltonian constraint is

$$^{(5)}R+K^{\hat{\alpha}}_{\hat{\beta}}K^{\hat{\beta}}_{\hat{\alpha}}-\hat{K}^2=\frac{1}{L_I^2}-\frac{2}{M_{(6)}^4}\left(F^{y}_{C}F_{y}{}^{C}-\frac{1}{4}F^{CD}F_{CD}\right). \tag{E.0.13}$$

Momentum constraint is

$$^{(5)}D_{\hat{\alpha}}K^{\hat{\alpha}}_{\hat{\mu}}-{}^{(5)}D_{\hat{\mu}}\hat{K}=\frac{1}{M_{(6)}^4}F_{\hat{\mu}C}F^{yC}n_y. \tag{E.0.14}$$

Next we will briefly look at the junction condition for the gauge fields. The variation of total action (E.0.1) with respect to the gauge field A_A and its extremum yield another junction condition

$$\delta_A S\ =\ -\frac{1}{4}\int d^6x\left(\sqrt{-{}^{(6)}g}\,4F^{AB}\right)\partial_A\delta A_B \tag{E.0.15}$$
$$-\int d^6x\sqrt{-{}^{(6)}g}\frac{v_{\pm}^2}{2}\cdot 2g^{B\hat{\nu}}(\partial_{\hat{\nu}}\Sigma_{\pm}-eA_{\hat{\nu}})(-eA\delta A_B)\frac{\sqrt{-{}^{(5)}g}}{\sqrt{-{}^{(6)}g}}\delta(y). \tag{E.0.16}$$

Note that

$$-\int d^6x\sqrt{-{}^{(6)}g}F^{AB}\partial_A\delta A_B\ =\ \int d^6x\sqrt{-{}^{(6)}g}\left(\partial_A F^{AB}+\frac{1}{2}g^{CD}\partial_A g_{CD}F^{AB}\right)\delta A_B, \tag{E.0.17}$$

$$
\begin{aligned}
= \quad & \int d^6x \sqrt{-{}^{(6)}g} \Big((n^y \partial_y F^{yB} + n^y \partial_y n_y F^{yB} + n_y \hat{K} F^{yB}) + \partial_\alpha F^{\alpha B} \\
& + \frac{1}{2} g^{CD} \partial_\alpha g_{CD} F^{\alpha B} \Big) \delta A_B. \qquad \text{(E.0.18)}
\end{aligned}
$$

Then $\delta_A S = 0$ yields

$$
n^y \partial_y \left(n_y F^{yB} \right) + (\cdots) + e v_\pm^2 g^{B\alpha} (\partial_{\hat{\alpha}} \Sigma_\pm - e A_{\hat{\alpha}}) \frac{1}{n_y} \delta(y) = 0, \qquad \text{(E.0.19)}
$$

then

$$
\partial_y \left(n_y F^{yB} \right) + (\cdots) + e v_\pm^2 g^{B\alpha} (\partial_{\hat{\alpha}} \Sigma_\pm - e A_{\hat{\alpha}}) \delta(y) = 0, \qquad \text{(E.0.20)}
$$

where terms $(\cdots)$ in Eq. (E.0.20) vanish by the integration. Integration around brane leads junction condition for 6-dimensional Maxwell fields

Junction condition for Maxwell field

$$
[n_y F^{yB}]_{y=y_\pm} = -e v_\pm^2 g^{B\alpha} (\partial_{\hat{\alpha}} \Sigma_\pm - e A_{\hat{\alpha}}). \qquad \text{(E.0.21)}
$$

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