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POISSON STOCHASTIC ANALYSIS

Infinite Markov Particle Systems
and Their Applications

By

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CHAPTER I

Introduction

1 Purposes and Motivations

Let us consider a particle system of independent identically distributed Markov processes which have an invariant measure m . It is well-known that this particle system is also stationary by letting the initial positions be the m -Poisson random measure, i.e., the Poisson random measure with intensity m . This property was first discovered by Doob [5] for Brownian motion and by Derman [3] for countable state space Markov chains. This particle system is called an *equilibrium process*. Port [11] and Spitzer [14] gave the limit theorems related with the number of particles hitting a subset by some time as the time goes to infinity. Furthermore the ergodic properties were studied by Shiga and Takahashi in [15]. It is also known by Dobrushin [4] and Stone [16] that even if the initial positions are not m -Poisson, then under some proper conditions the particle distribution becomes m -Poisson as time goes to infinity. In the non-Poisson case for a transient aperiodic random walk, Weiss [17] gave the limit theorems like Port [11].

A typical example of infinite Markov particle systems is an independent Brownian particle system in $\mathbf{R}^d$. Its formal generator $\mathcal{L}$ is given by the following: Let $\mathcal{M}(\mathbf{R}^d)$ be the totality of counting measures $\mu = \sum_n \delta_{x_n}$ on $\mathbf{R}^d$. For a suitable functional $\Phi(\mu) = F(\langle \mu, f_1 \rangle, \dots, \langle \mu, f_n \rangle)$ of $\mu \in \mathcal{M}(\mathbf{R}^d)$,

$$\begin{aligned} \mathcal{L}\Phi(\mu) &= \frac{1}{2} \sum_{k=1}^d \langle \mu, \mathcal{D}_k^2 \Phi(\mu; \cdot) \rangle, \\ &= \sum_{i=1}^n \partial_i F(\langle \mu, f_1 \rangle, \dots, \langle \mu, f_n \rangle) \langle \mu, A f_i \rangle \\ &\quad + \frac{1}{2} \sum_{i,j=1}^n \partial_{ij}^2 F(\langle \mu, f_1 \rangle, \dots, \langle \mu, f_n \rangle) \langle \mu, \Gamma(f_i, f_j) \rangle, \end{aligned}$$

where $\langle \mu, f_i \rangle$ denotes integration of a function f_i on $\mathbf{R}^d$ by μ and $\mathcal{D}_k$ is a kind of differential operator defined as

$$\mathcal{D}_k \Psi(\mu; x) = \lim_{h \rightarrow 0} \frac{1}{h} \left[\Psi(\mu + \delta_{x_k(h)} - \delta_x; x_k(h)) - \Psi(\mu; x) \right]$$

with $x_k(h) = (x_1, \dots, x_k + h, x_{k+1}, \dots, x_d)$, and $\mathcal{D}_k^2 = \mathcal{D}_k \circ \mathcal{D}_k$ for $k = 1, 2, \dots, d$. Note that if $\Psi(\mu; x) = \Phi(\mu)$, then

$$\mathcal{D}_k \Phi(\mu; x) = \lim_{h \rightarrow 0} \frac{1}{h} \left[\Phi(\mu + \delta_{x_k(h)} - \delta_x) - \Phi(\mu) \right].$$

Moreover $\partial_i = \partial / \partial x_i$, A is the generator of Brownian motion in $\mathbf{R}^d$ and $\Gamma(f, g) = A(fg) - fAg - gAf$, i.e., $Af = \frac{1}{2} \Delta_x f$ and $\Gamma(f, g) = \nabla_x f \cdot \nabla_x g$.

This process is constructed as follows: Let $(w(t), P_x)$ be a Brownian motion starting from x in $\mathbf{R}^d$. P_x can be regarded as a measure on the continuous path space W_c from $[0, \infty)$ to $\mathbf{R}^d$. Let the following be given:

Ω is the space of counting measures $\omega = \sum_n \delta_{w_n}, w_n \in W_c$,
 $X_t(\omega) = \omega(t) = \sum_n \delta_{w_n(t)}$ if $\omega = \sum_n \delta_{w_n} \in \Omega$,
 $\mathcal{F} = \sigma(X_s : s < \infty), \mathcal{F}_t = \sigma(X_s : s \leq t)$.

Define

$$\mathbf{P}_\mu = \bigotimes_{n \geq 1} P_{x_n} \quad \text{if } \mu = \sum_{n \geq 1} \delta_{x_n} \in \mathcal{M}(\mathbf{R}^d).$$

Then $(X_t, \mathbf{P}_\mu)$ is an $\mathcal{M}(\mathbf{R}^d)$ -valued Markov process starting from μ satisfying that

$$\mathbf{E}_\mu \left[e^{-\langle X_{t+s}, f \rangle} \middle| \mathcal{F}_s \right] = \exp[-\langle X_s, V_t f \rangle] \quad \text{for measurable functions } f \geq 0,$$

where

$$V_t f(x) = -\log E_x[\exp(-f(w(t)))].$$

Furthermore let $m(dx) = dx$ be the Lebesgue measure on $\mathbf{R}^d$, Π_m be the m -Poisson measure on $\mathcal{M}(\mathbf{R}^d)$, i.e., the distribution of the Poisson random measure with intensity $m(dx)$ and set

$$\mathbf{P} = \int \Pi_m(d\mu) \mathbf{P}_\mu.$$

Then, under $\mathbf{P}$, $X = \{X_t\}_{t \geq 0}$ is an $\mathcal{M}(\mathbf{R}^d)$ -valued stationary Markov process with an initial measure Π_m , because $w(t)$ is stationary under the σ -finite measure $P_m = \int dx P_x$;

$$\begin{aligned} \mathbf{E}[\exp\{-\langle X_t, f \rangle\}] &= \int e^{-\langle \mu, V_t f \rangle} \Pi_m(d\mu) \\ &= \exp\left[-\langle m, 1 - e^{-V_t f} \rangle\right] \\ &= \exp\left[-\langle P_m, 1 - e^{-f(w(t))} \rangle\right] \\ &= \exp\left[-\langle m, 1 - e^{-f} \rangle\right]. \end{aligned}$$

Note that $\mathbf{P}$ is also induced by the P_m -Poisson measure Π_{P_m} on Ω . $(X_t, \mathbf{P})$ is called the *equilibrium process associated with* $(w(t), P_m)$ (see [5], [11]).

For this process we can get limit theorems related with number of particles hitting a compact set by applying the following result: For a three-dimensional Brownian motion F. Spitzer [13] showed that if B is a compact set with a positive capacity $C(B)$, then

$$P_m(0 < T_B < t) = t C(B) + \sqrt{t} 4(2\pi)^{-3/2} C(B)^2 + o(\sqrt{t})$$

as $t \rightarrow \infty$, where T_B is the first hitting time of B . Moreover in case of $d \geq 4$ (see also [6]) it holds that

$$P_m(0 < T_B < t) - t C(B) = \begin{cases} \log t C(B)^2 / (4\pi)^2 + o(\log t) & \text{if } d = 4, \\ \int_{\mathbf{R}^d} P_x(0 < T_B < \infty)^2 dx + o(1) & \text{if } d \geq 5 \end{cases}$$

as $t \rightarrow \infty$, where $\int_{\mathbf{R}^d} P_x(0 < T_B < \infty)^2 dx$ is finite whenever $d \geq 5$.

Set $N_t^B = \{w : 0 < T_B(w) < t\}$. Then $X(N_t^B)$ denotes the number of different particles hitting B during the time interval $(0, t)$ and $\mathbf{E}[X(N_t^B)] = P_m(N_t^B)$. By the subadditive ergodic theorem [8] it holds that $X(N_t^B)/t \rightarrow C(B)$ $\mathbf{P}$ -a.s. and in $L^1(\mathbf{P})$ as $t \rightarrow \infty$. Furthermore

$$\frac{X(N_t^B) - t C(B)}{\sqrt{t}} \rightarrow \begin{cases} N(M^B, C(B)) & \text{if } d = 3, \\ N(0, C(B)) & \text{if } d \geq 4 \end{cases}$$

in law as $t \rightarrow \infty$, where $M^B = 4C(B)^2/(2\pi)^{3/2}$ and $N(u, v)$ is the Gaussian distribution with mean u and variance v .

On the other hand for a Markov process which does not have an invariant measure, we can construct a stationary Markov particle system. In fact by extending the probability space and considering new paths which are born at random time, it is possible to realize a stationary Markov process with the same transition probability. The distribution (which may not be a probability measure in general) is called a *Kuznetsov measure* ([10]). By using this measure we can construct the stationary Markov particle system. This particle system is realized by letting new particles immigrate according to a Poisson distribution. In order to investigate such a particle system we consider a simple model, that is, the *infinite Markov particle system with singular immigration associated with absorbing Brownian motion* $(w^0(t), P_x^0)$ in a half space $H = \mathbf{R}^{d-1} \times (0, \infty)$. Note that in general “immigration” means particles immigrate from interior of the space. However in the present paper “singular immigration” means infinitely many particles immigrate from the boundary of H . In particular, we shall deal with the case that particles immigrate uniformly from the boundary at random time. We want to characterize this Markov particle system as a diffusion, i.e., a counting measure valued continuous Markov process. So we introduce a suitable subspace of counting measures with an appropriate topology. We shall further discuss the sample path properties and the influence of the immigration part for Hölder continuity, finiteness and infiniteness of the number of particles near the boundary. We shall also study on a hitting time for a compact subset under the Kuznetsov measure in order to obtain the same type limit theorems as above.

The formal generator of an infinite Markov particle system can be applied to Malliavin calculus for jump-type Markov processes. Malliavin calculus is one of powerful methods to study on the existence and smoothness of transition densities for Markov processes determined by stochastic differential equations. We give an outline of Malliavin calculus according to [2]. Let $\Phi = (\Phi^1, \dots, \Phi^d)$ be an $\mathbf{R}^d$ -valued random variable on a probability space $(\Omega, \mathcal{F}, P)$. The fundamental lemma is the following: If there exists a constant $C > 0$ such that for each $k = 1, \dots, d$ and all C^∞ -functions with compact support $F \in C_c^\infty(\mathbf{R}^d)$, $|E[\partial_k F(\Phi)]| \leq C\|F\|_\infty$, then Φ admits a density $p(x)$, i.e., $P(\Phi \in dx) = p(x)dx$ (the smoothness of $p(x)$ can be obtained by a similar estimate). To get the above inequality it is enough to show that for each $k = 1, \dots, d$, there is a random variable $G_k \in L^1(P)$ such that

$$(1.1) \quad E[\partial_k F(\Phi)] = E[F(\Phi)G_k] \quad \text{for all } F \in C_c^\infty(\mathbf{R}^d).$$

Malliavin operator $\mathcal{L}$ is useful to obtain this formula (the definition will be given in §2.1 of Chapter III). $\mathcal{L} : H_\infty \subset \bigcap_{p < \infty} L^p(P) \rightarrow \bigcap_{p < \infty} L^p(P)$; is a symmetric linear operator

such that $E[\Phi \mathcal{L}\Psi] = E[\Psi \mathcal{L}\Phi] = -E[\Gamma(\Phi, \Psi)]$ for $\Phi, \Psi \in H_\infty$, where $\Gamma(\Phi, \Psi) \equiv \{\mathcal{L}(\Phi\Psi) - \Phi\mathcal{L}\Psi - \Psi\mathcal{L}\Phi\}/2$ is a Malliavin covariance of (Φ, Ψ) . If $\Phi = (\Phi^1, \dots, \Phi^d) \in (H_\infty)^d$, $\Psi \in H_\infty$ and $F \in C_p^2(\mathbf{R}^d)$ (the totality of C^2 -functions with polynomial growth derivatives up to order 2), then

$$\Gamma(F \circ \Phi, \Psi) = \sum_i \partial_i F(\Phi) \Gamma(\Phi^i, \Psi).$$

Hence we can easily get the following *integration-by-parts formula*: For each $j = 1, \dots, d$,

$$E \left[\Psi \sum_i \partial_i F(\Phi) \Gamma(\Phi^i, \Phi^j) \right] = -E \left[F(\Phi) \{ \Psi \mathcal{L}\Phi^j + \Gamma(\Psi, \Phi^j) \} \right].$$

Let $\sigma_\Phi = (\Gamma(\Phi^i, \Phi^j))_{i,j \leq d}$ be a $d \times d$ -matrix. If σ_Φ is invertible a.s., then Φ admits a density. In fact by taking (j, k) -component $(\sigma_\Phi^{-1})_{jk}$ of the inverse matrix σ_Φ^{-1} as Ψ and by summing up with respect to j we have the equation (1.1) with $G_k = -\sum_j \{(\sigma_\Phi^{-1})_{jk} \mathcal{L}\Phi^j + \Gamma((\sigma_\Phi^{-1})_{jk}, \Phi^j)\}$. Of course to show the existence of the density we need the condition $(\sigma_\Phi^{-1})_{ji} \in H_\infty$, but this can be taken away (see the proof of Th. 4-7 in [2]).

As $\Phi = (\Phi^1, \dots, \Phi^d)$ we want to consider a Markov process (X_t^x, P) which is determined by the following stochastic differential equation (SDE): Let E be a domain of $\mathbf{R}^n$. For functions $a(x) : \mathbf{R}^d \rightarrow \mathbf{R}^d$, $f(x, z) : \mathbf{R}^d \times E \rightarrow \mathbf{R}^d$,

$$(1.2) \quad X_t^x = x + \int_0^t a(X_s^x) ds + \int_0^t \int_E f(X_{s-}^x, z) \tilde{\mu}(ds dz),$$

where $\tilde{\mu} = \mu - \nu$ is a compensated Poisson random measure and μ is a Poisson random measure with intensity $\nu(ds dz) = ds dz$, which is realized on a *Poisson space* $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, i.e., for a fixed $T > 0$, Ω is the collection of counting measures $\mu = \sum_n \delta_{(t_n, z_n)}$ on $[0, T] \times E$ such that $\mu([0, T] \times K) < \infty$ for any compact set K in E , $\mu(\{t\} \times E) \leq 1$ for all $t \in [0, T]$ and $\mu(\{0\} \times E) = 0$. Set $\mathcal{F}_t^0 = \sigma(\mu(A) : A \in \mathcal{B}([0, t]) \otimes \mathcal{E})$ for $t \in [0, T]$, $\mathcal{F}_t = \bigcap_{u>t} \mathcal{F}_u^0$ for $t \in [0, T]$, and $\mathcal{F} = \mathcal{F}_T^0$.

Therefore we need a Malliavin operator $\mathcal{L}$ for *Poisson functionals* (random variables on the Poisson space). It can be constructed as follows: Consider a diffusion $z(t)$ in E defined by

$$dz(t) = \sqrt{2\rho(z(t))} I dB_t + \nabla_z \rho(z(t)) dt, \quad z(0) = z \in E,$$

where I is the $n \times n$ -unit matrix, $\{B_t\}$ is an n -dimensional Brownian motion and $\rho \in C_b^\infty(E \rightarrow (0, \infty))$ is a fixed function satisfying the condition such that the above stochastic equation has the solution $(z(t))_{t \geq 0}$. The generator of this process is given by

$$Af(z) = \rho(z) \Delta_z f(z) + \nabla_z \rho(z) \cdot \nabla_z f(z) \quad (= \Delta_z(\rho f) - f \Delta_z \rho(z)).$$

Then we can consider the same type infinite Markov particle system as before. Its formal generator $\mathcal{L}$ is used as a Malliavin operator.

In 1987, Bichteler, Gravereaux and Jacod [2] considered the following Lévy generator:

$$L\varphi(x) = \nabla_x \varphi(x) \cdot a(x) + \int_{\{|y|<1\}} [\varphi(x+y) - \varphi(x) - \nabla_x \varphi(x) \cdot y] K(x, dy)$$

with a Lévy kernel

$$K(x, A) = \int_E 1_A(f(x, z)) dz.$$

They gave sufficient conditions on (a, f) for the existence and smoothness of the transition density $p_t(x, y)$ of the corresponding Markov process (X_t^x, P) to the Lévy generator L . This process is given by the SDE (1.2). They also considered the case that the support of the Lévy kernel is decomposed several curves;

$$K(x, A) = \sum_{i=1}^n \int_0^1 1_A(y_i(x, u)) g_i(x, u) du,$$

where $y_i(x, u) : \mathbf{R}^d \times [0, 1] \rightarrow \mathbf{R}^d$ are C^1 -curves starting from 0; $y_i(x, 0) = 0$ and $g_i(x, u) : \mathbf{R}^d \times (0, 1] \rightarrow (0, \infty)$ are C^1 -weights. However this class does not include Markov processes that have the following generator:

$$L = -(-\Delta_x)^{\alpha(x)/2},$$

where $\alpha(x) : \mathbf{R}^d \rightarrow \mathbf{R}$ is Dini continuous and satisfies the inequalities

$$(1.3) \quad 0 < \zeta_1 \leq \alpha(x) \leq \zeta_2 < 2 \quad \text{for all } x \in \mathbf{R}^d \text{ with some constants } \zeta_1 \text{ and } \zeta_2.$$

In this case the Lévy kernel of this operator is expressed as

$$(1.4) \quad K(x, A) = \theta_{\alpha(x)} \int_{\mathbf{S}^{d-1}} d\sigma \int_0^\infty 1_{A \setminus \{0\}}(u\sigma) u^{-1-\alpha(x)} du,$$

where

$$\theta_\alpha = \sin \frac{\pi\alpha}{2} \Gamma(1+\alpha) \Gamma\left(\frac{d+\alpha}{2}\right) \pi^{-(d+1)/2} \Gamma\left(\frac{1+\alpha}{2}\right)^{-1}.$$

Bass [1] proved the well-posedness of the martingale problem for this generator. The Markov process corresponding to this generator is called a *stable-like process* following Bass [1]. So we extend the operator L and the Lévy kernel $K(x, A)$ to more general type which includes a class of stable like processes and discuss the existence and smoothness of the transition densities for the corresponding Markov processes.

2 Problems and results

In Chapter II, we consider the infinite Markov particle system $(X_t, \mathbf{P}_\mu)$ with singular immigration starting from a counting measure $\mu = \sum_n \delta_{x_n}$ on a half space $H = \mathbf{R}^{d-1} \times (0, \infty)$, associated with absorbing Brownian motion $(w^0(t), P_x^0)$ (and also Brownian excursion law Q^0) in H . Q^0 is given by

$$Q^0 = \lim_{r \downarrow 0} \int_H \nu_r(dx) P_x^0,$$

where $\nu_r(dx) = \nu_r(x_d) dx$, $r > 0$, is the entrance law corresponding to the transition semi-group (P_t^0) of $(w^0(t), P_x^0)$, with $\nu_r(y) = y / \sqrt{2\pi r^3} \exp[-y^2/(2r)]$, $r, y > 0$. We use the same

notation P_t^0 and P_x^0 for the semigroup and the probability, for it is convenient and it has no confusion. Let $\mathcal{M} = \mathcal{M}(H)$ denote the space of all counting measures $\mu = \sum_n \delta_{x_n}$ on H . The process $(X_t, \mathbf{P}_\mu)$ is constructed as follows: $X_t = X_t^0 + X_t^I$ with

$$\begin{aligned} X_t^0(\omega) : & \quad \text{the no immigration part,} \\ X_t^I(\omega) : & \quad \text{the immigration part,} \end{aligned}$$

and

$$\mathbf{P}_\mu = \mathbf{P}_\mu^0 \otimes \mathbf{P}^I,$$

where

$$\mathbf{P}_\mu^0 = \bigotimes_{n \geq 1} P_{x_n}^0 \quad \text{if} \quad \mu = \sum_{n \geq 1} \delta_{x_n} \in \mathcal{M}$$

and

$$\mathbf{P}^I \quad \text{is the} \quad \int_0^\infty \theta_{-s}(Q^0) ds \text{-Poisson measure,}$$

i.e., the distribution of the Poisson random measure with intensity $\int_0^\infty \theta_{-s}(Q^0) ds$. From this construction the generator $\mathcal{L}$ of this process should be expressed as the sum of no immigration part $\mathcal{L}^0$ and immigration part $\mathcal{L}^I$. That is, for some suitable functional $\Phi(\mu)$ of counting measures μ ,

$$\begin{aligned} \mathcal{L}^0 \Phi(\mu) &= \frac{1}{2} \sum_{k=1}^d \langle \mu, \mathcal{D}_k^2 \Phi(\mu; \cdot) \rangle, \\ \mathcal{L}^I \Phi(\mu) &= \frac{1}{2} \langle \widetilde{m}, \mathcal{D}_d \Phi(\mu; \cdot)|_{x_d=0+} \rangle, \end{aligned}$$

where $\widetilde{m} = d\tilde{x}$ is the Lebesgue measure on $\mathbf{R}^{d-1}$ and $\mathcal{D}_k, \mathcal{D}_k^2$ are defined by the same way as in §1. The transition probability of this Markov process is given by the following: For measurable functions $f \geq 0$,

$$\mathbf{E}_\mu \left[e^{-\langle X_t, f \rangle} \right] = \exp \left[-\langle \mu, V_t f \rangle - \int_0^t \langle \nu_r, 1 - e^{-f} \rangle dr \right],$$

where $V_t f(x) = -\log E_x^0[\exp(-f(w^0(t)))]$.

First we want to deal with this process as a counting measure valued diffusion and give a characterization by the operator $\mathcal{L}$. However, the space $\mathcal{M}$ is too big. So we have to find a suitable subspace of $\mathcal{M}$ with an appropriate topology.

Fix $p > d$. We introduce the following subspace $\overline{\mathcal{M}}_p$: Let $f \in D_p$ if $f \in C^2$, $f(\cdot, 0+) = \partial_d^2 f(\cdot, 0+) = 0$, and all partial derivatives of order from 0 to 2 have asymptotic $O(|x|^{-p})$ at infinity. Then D_p is stable under the semi-group (P_t^0) . We define $\mathcal{M}_p \subset \overline{\mathcal{M}}_p (\subset \mathcal{M})$ by $\mu \in \mathcal{M}_p$ if $\langle \mu, g_p \rangle < \infty$ with $g_p(x) = (1 + |x|^2)^{-p/2}$, and $\mu \in \overline{\mathcal{M}}_p$ if $\langle \mu, g_{p,0} \rangle < \infty$ with a fixed positive function $g_{p,0}(x) \in D_p$ such that $g_{p,0}(x) = g_p(x)$ for large x and $\partial_d g_{p,0}(\tilde{x}, 0+) = g_p(\tilde{x}, 0+)$ ($\tilde{x} \in \mathbf{R}^{d-1}$). The topology of $\overline{\mathcal{M}}_p$ is defined by

$$\mu_n \rightarrow \mu \text{ in } \overline{\mathcal{M}}_p \iff \langle \mu_n, f \rangle \rightarrow \langle \mu, f \rangle \text{ for all } f \in C_c \text{ and } f = g_{p,0} \text{ as } n \rightarrow \infty.$$

Now set

$$\mathbf{D}_p = \{ \Phi(\mu) = F(\langle \mu, f_1 \rangle, \dots, \langle \mu, f_n \rangle) : F \in C^2(\mathbf{R}^n), |F^{(i)}(x)| \leq C(1 + |x|)^k, \\ i = 0, 1, 2 \text{ for some } C > 0, k \geq 0, \text{ and } f_j \in D_p, j = 1, 2, \dots, n \},$$

and for $\Phi(\mu) = F(\langle \mu, f_1 \rangle, \dots, \langle \mu, f_n \rangle) \in \mathbf{D}_p, \mu \in \mathcal{M}_p$, we define an operator $\mathcal{L}$ by

$$\begin{aligned} \mathcal{L}\Phi(\mu) &= \sum_{i=1}^n \partial_i F(\langle \mu, f_1 \rangle, \dots, \langle \mu, f_n \rangle) \left\{ \langle \mu, Af_i \rangle + \frac{1}{2} \langle \widetilde{m}, \partial_d f_i(\cdot, 0+) \rangle \right\} \\ &\quad + \frac{1}{2} \sum_{i,j=1}^n \partial_{ij}^2 F(\langle \mu, f_1 \rangle, \dots, \langle \mu, f_n \rangle) \langle \mu, \Gamma(f_i, f_j) \rangle, \end{aligned}$$

where A is the generator of (P_t^0) and $\Gamma(f, g) = A(fg) - fAg - gAf$, i.e., $Af = \frac{1}{2} \Delta_x f$ and $\Gamma(f, g) = \nabla_x f \cdot \nabla_x g$ for $f, g \in D_p$.

Moreover set $\Omega_p = C([0, \infty) \rightarrow \overline{\mathcal{M}}_p)$, which is a Polish space.

We first give a martingale characterization for $(X_t, \mathbf{P}_\mu)$ by $(\mathcal{L}, \mathbf{D}_p)$ (cf. [7], [9]).

Theorem 2.1. (Theorem 2.1 in Chapter II)

Fix $p > d$ and $\mu \in \overline{\mathcal{M}}_p$. Then under $\mathbf{P}_\mu$, $\{X_t\}$ is an $\overline{\mathcal{M}}_p$ -valued Markov process having a continuous sample paths relative to the topology in $\overline{\mathcal{M}}_p$ and satisfies that

- (i) $\mathbf{P}_\mu(X_0 = \mu) = 1$,
- (ii) for each $\Phi \in \mathbf{D}_p$, $t > 0$ and $k \geq 1$, $\Phi(X_t), \int_0^t |\mathcal{L}\Phi(X_s)| ds \in L^k(\mathbf{P}_\mu)$,
- (iii) for each $\Phi \in \mathbf{D}_p$, $M_t^\Phi \equiv \Phi(X_t) - \Phi(X_0) - \int_0^t \mathcal{L}\Phi(X_s) ds$ is a $\mathbf{P}_\mu$ -martingale.

Moreover if a probability measure $\tilde{\mathbf{P}}_\mu$ on Ω_p satisfies (i) and if M_t^Φ is a local $\tilde{\mathbf{P}}_\mu$ -martingale for any $\Phi(\mu) = \langle \mu, f \rangle$ and $\langle \mu, f \rangle^2$ with $f \in D_p$, then $\tilde{\mathbf{P}}_\mu = \mathbf{P}_\mu$ on Ω_p .

REMARK 2.2. (i) When $\mu \in \overline{\mathcal{M}}_p - \mathcal{M}_p$, $\mathcal{L}\Phi(\mu)$ may not be well-defined because of $\langle \mu, \Gamma(f_i, f_i) \rangle$ being possibly infinite. But for all $t > 0$, $\int_0^t \mathcal{L}\Phi(X_s) ds$ is well-defined $\mathbf{P}_\mu$ -a.s..

(ii) It is easily seen that $\langle X_t, g_p \rangle, \int_0^t \langle X_s, g_p \rangle ds \in L^1(\mathbf{P}_\mu)$ for all $t > 0$. Hence

$$\mathbf{P}_\mu(\langle X_t, g_p \rangle < \infty \text{ for all } t \in \mathbf{Q} \cap (0, \infty) \text{ and } dt\text{-a.a. } t) = 1.$$

However we can show that for any time interval $(a, b); 0 \leq a < b$,

$$\mathbf{P}_\mu(\langle X_t, g_p \rangle = \infty \text{ for some } t \in (a, b)) = 1 \text{ even if } \mu = 0,$$

that is, the time points t of $\langle X_t, g_p \rangle = \infty$ exist densely. We can also show that for any $T > 0$,

$$\mathbf{P}_0 \left(\sup_{t \in \mathbf{Q} \cap (0, T)} \langle X_t, g_p \rangle = \infty \right) = 1.$$

These results come from the immigration part X^I (see §2.4 in Chapter II).

We further discuss Hölder continuity of $\langle X_t, f \rangle, f \in D_p$.

Theorem 2.3. (Theorem 2.3 in Chapter II)

Fix $p > d$. Set $\Gamma f = |\nabla_x f|^2$.

(i) If $\mu \in \overline{\mathcal{M}}_p$, then X_t is locally $(1/4 - \epsilon)$ -Hölder continuous in $t \in [0, \infty)$ for any sufficiently small $\epsilon > 0$ under $\mathbf{P}_\mu$, i.e., for each fixed $f \in D_p$, $\langle X_t, f \rangle, t \geq 0$ is locally $(1/4 - \epsilon)$ -Hölder continuous $\mathbf{P}_\mu$ -a.s.

(ii) If $\mu \in \mathcal{M}_p$, then X_t is locally $(1/2 - \epsilon)$ -Hölder continuous in $t \in [0, \infty)$ for any sufficiently small $\epsilon > 0$ under $\mathbf{P}_\mu$.

(iii) Even if $\mu \in \overline{\mathcal{M}}_p - \mathcal{M}_p$, X_t is locally $(1/2 - \epsilon)$ -Hölder continuous in $t \in (0, \infty)$ for any sufficiently small $\epsilon > 0$ under $\mathbf{P}_\mu$.

(iv) If $\mu \in \overline{\mathcal{M}}_p$, then for $f \in D_p$ such that $\langle \mu, \Gamma f \rangle > 0$, $\langle X_t, f \rangle$ is not $1/2$ -Hölder continuous at $t = 0$. Furthermore if $f \in D_p$ such that $\{\Gamma f = 0\}$ is a compact subset in H , and if $\mu \in \overline{\mathcal{M}}_p$ satisfies one of the following conditions:

$$\begin{cases} d = 1 \Rightarrow \mu(\{x \geq a\}) = \infty & \text{for some } a > 0, \\ d \geq 2 \Rightarrow \langle \mu, \Gamma f \rangle > 0, \end{cases}$$

then $\langle X_t, f \rangle, t \geq 0$, is nowhere $1/2$ -Hölder continuous under $\mathbf{P}_\mu$, i.e.,

$$\left\{ \omega : |\langle X_{t+h}(\omega) - X_t(\omega), f \rangle| \leq C(\omega)\sqrt{h} \text{ for some } t \in [0, \infty) \text{ and for all } h \in [0, u(\omega)] \right\}$$

is a $\mathbf{P}_\mu$ -null set, where $C(\omega)$ and $u(\omega)$ are a.s. positive random variables.

When $\mu \in \overline{\mathcal{M}}_p - \mathcal{M}_p$, it seems to be difficult to completely determine the exponent of Hölder continuity at $t = 0$. However we can give some sufficient conditions for them. Set $S_a = \{|x_i| \leq a, i = 1, \dots, d\} \cap H$.

Theorem 2.4. (Theorem 2.4 in Chapter II)

Let $\mu \in \overline{\mathcal{M}}_p - \mathcal{M}_p$.

(i) Suppose that for some $T > 0, 1 \leq \theta < 2$,

$$(2.5) \quad \langle \mu|_{S_1}, P_t^0 1 \rangle \asymp \int_{S_1} \frac{dx}{x_d^\theta} P_t^0 1(x) \text{ for all } 0 < t \leq T,$$

where $g(t) \asymp h(t), 0 < t \leq T$ means $Ch(t) \leq g(t) \leq C'h(t), 0 < t \leq T$ for some positive constants C, C' depending only on T , then X_t is $((3 - \theta)/4 - \epsilon)$ -Hölder continuous at $t = 0$ for any sufficiently small $\epsilon > 0$, moreover for each $f \in D_p$ such that $\Gamma f(\tilde{0}, 0+) > 0$, $\langle X_t, f \rangle$ is not $(3 - \theta)/4$ -Hölder continuous at $t = 0$ under $\mathbf{P}_\mu$.

(ii) Suppose that for some $T > 0$ and $1 < \theta < 2, \eta > 0$ or $\theta = 2, \eta > 1$,

$$(2.6) \quad \langle \mu|_{S_{1/\epsilon}}, P_t^0 1 \rangle \asymp \int_{S_{1/\epsilon}} \frac{dx}{x_d^\theta (\log 1/x_d)^\eta} P_t^0 1(x) \text{ for all } 0 < t \leq T,$$

then X_t is $(3 - \theta)/4$ -Hölder continuous at $t = 0$, moreover for each $f \in D_p$ such that $\Gamma f(\tilde{0}, 0+) > 0$, $\langle X_t, f \rangle$ is not $((3 - \theta)/4 + \epsilon)$ -Hölder continuous at $t = 0$ for any sufficiently $\epsilon > 0$ under $\mathbf{P}_\mu$.

REMARK 2.5. In the later half of (ii) we can show the following:

$$\limsup_{t \downarrow 0} \frac{|\langle X_t, f \rangle|}{t^{(3-\theta)/4} (\log 1/t)^{-\eta/2}} = \infty \quad \mathbf{P}_\mu\text{-a.s. if } 1 < \theta < 2, \eta > 0,$$

$$\limsup_{t \downarrow 0} \frac{|\langle X_t, f \rangle|}{t^{1/4} (\log 1/t)^{-(\eta-1)/2}} = \infty \quad \mathbf{P}_\mu\text{-a.s. if } \theta = 2, \eta > 1.$$

These Hölder continuities mainly depend on the no immigration part X_t^0 .

Here we give some examples of $\mu = \sum_{n \geq 2} \delta_{x_n} \in \overline{\mathcal{M}}_p - \mathcal{M}_p$ which satisfy the conditions (2.5), (2.6): For simplicity let $d = 1$.

- (a) $x_n = r^n$ ($0 < r < 1$) $\Rightarrow \theta = 1$ in (2.5),
- (b) $x_n = n^{-p}$ ($p > 1$) $\Rightarrow \theta = 1 + 1/p$ in (2.5),
- (c) $x_n = n^{-p}(\log n)^{-q}$ ($p > 1, q > 0$ or $p = 1 < q$) $\Rightarrow \theta = 1 + 1/p, \eta = q/p$ in (2.6).

By using this process we can construct an *equilibrium process with immigration* $(X_t, \mathbf{P})$, which is a stationary Markov particle system with immigration associated with $(w^0(t), P_x^0)$ in H and the Lebesgue measure $m(dx) = dx$ on H . That is, let Π_m be the m -Poisson measure on $\mathcal{M}$, i.e., the distribution of the Poisson random measure with intensity m . Define

$$\mathbf{P} = \int \Pi_m(d\mu) \mathbf{P}_\mu.$$

Then $(X_t, \mathbf{P})$ is a stationary Markov process because

$$\begin{aligned} \mathbf{E}[\exp\{-\langle X_t, f \rangle\}] &= \exp\left[-\langle m, 1 - e^{-V_t f} \rangle - \int_0^t \langle \nu_r, 1 - e^{-f} \rangle\right] \\ &= \exp\left[-\langle m, 1 - e^{-f} \rangle\right]. \end{aligned}$$

This process is also constructed by using a *Kuznetsov measure* Q_m , which is a stationary σ -finite Markov measure with the same transition law as absorbing Brownian motion in H ([10]). This measure is defined as the integration of the time-shift of excursion law Q^0 with respect to the time in $\mathbf{R}^1$. We define W as the set of all maps $w : \mathbf{R} \rightarrow H_\Delta$ such that there is an open interval $(\alpha(w), \beta(w))$ on which w is H -valued and continuous, with $w(t) = \Delta$ if $t \leq \alpha(w)$ or $t \geq \beta(w)$, and a constant map $[\Delta]$, i.e., $[\Delta](t) = \Delta$ for all t . Set

$$Q_m = \int_{\mathbf{R}} \theta_{-s}(Q^0) ds \quad \text{on } W.$$

Then $(w(t), Q_m)$ is a stationary Markov process with the same transition probability (P_t^0) as absorbing Brownian motion. Q_m is called the *Kuznetsov measure associated with* $(w^0(t), P_x^0, m)$. If we restrict the time interval to $[0, \infty)$, then $\mathbf{P}$ can be also defined as the Q_m -Poisson measure on Ω . $(X_t, \mathbf{P})$ is called the *equilibrium process with immigration associated with* $(w^0(t), P_x^0, m)$ or $(w(t), Q_m)$.

Secondly we consider a hitting rate for Brownian excursions and give the asymptotic behavior under Q_m . That is, for a fixed compact set B in H , we define the following hitting time:

$$\sigma_B(w) = \inf\{t > 0 : w(t) \in B\},$$

where the infimum of the empty set is plus infinity, and $\pi_B(dx)$ be the capacitary measure of B , that is, π_B is supported on ∂B and satisfies that

$$P_x^0(T_B < \infty) = \int g^0(x, y) \pi_B(dy),$$

with the potential kernel

$$g^0(x, y) = \int_0^\infty p_t^0(x, y) dt,$$

where $p_t^0(x, y)$ is the transition density of $(w^0(t), P_x^0)$. The *capacity* is defined by

$$C^0(B) = \pi_B(1) = \int_{\partial B} \pi_B(dx).$$

Moreover if we set $\tau_B(w) = \inf\{t \in \mathbf{R} : w(t) \in B\} (= \infty \text{ if } \{\cdot\} = \emptyset)$, then

$$C^0(B) = Q_m(0 < \tau_B < 1) = Q^0(\sigma_B < \infty)$$

We have the following result:

Theorem 2.6. (Theorem 3.1 in Chapter II)

Let B be a compact subset of H with a positive capacity. Then it holds that

$$Q_m(0 \leq \sigma_B < t) = t C^0(B) + f(t)$$

with

$$f(t) = \begin{cases} \sqrt{\frac{2t}{\pi}} - \frac{2}{3}b + c + o(1) & \text{if } d = 1, \\ \log t \Phi(B) + o(\log t) & \text{if } d = 2, \\ O(1) & \text{if } d \geq 3 \end{cases}$$

as $t \rightarrow \infty$, where

$$b = \inf\{x : x \in B\} \quad \text{and} \quad c = \sup\{x : x \in B\} \quad \text{if } d = 1$$

and

$$\Phi(B) = \frac{1}{\pi} \int_{\partial B} x_2 \pi_B(dx) Q^0[w^{(2)}(\sigma_B) : \sigma_B < \infty] = \frac{1}{\pi} \left(\int_{\partial B} x_2 \pi_B(dx) \right)^2$$

with $w = (w^{(1)}, w^{(2)})$ if $d = 2$.

REMARK 2.7. In one-dimensional case we have more explicit forms; $g^0(x, y) = 2(x \wedge y)$, $\pi_B(dx) = \delta_b(dx)/(2b)$ and $C^0(B) = 1/(2b)$.

Let $(\{X_t\}_{t \in \mathbf{R}}, \mathbf{P})$ be the equilibrium process associated with $(w(t), Q_m)$. Set $N_t^B = \{w \in W : 0 \leq \sigma_B(w) < t\}$. Then $X(N_t^B)$ is the number of particles hitting B during time interval $[0, t]$. Now we have the following result:

Theorem 2.8. (Theorem 3.3 in Chapter II)

Let B be a compact subset of H with a positive capacity. Then

$$\frac{X(N_t^B)}{t} \rightarrow C^0(B) \quad \mathbf{P}\text{-a.s. and in } L^1(\mathbf{P})$$

as $t \rightarrow \infty$. Moreover

$$\frac{X(N_t^B) - tC^0(B)}{\sqrt{t}} \rightarrow \begin{cases} N(\sqrt{2/\pi}, C^0(B)) & \text{if } d = 1, \\ N(0, C^0(B)) & \text{if } d \geq 2 \end{cases}$$

in law as $t \rightarrow \infty$.

In Chapter III, we extend the Malliavin operator $\mathcal{L}$ for functionals of μ in order to apply to more general jump-type Markov processes as follows.

In 1988, R. F. Bass [1] showed the well-posedness of the martingale problem for a fairly wide class of the Lévy generator L :

$$(2.7) \quad L\varphi(x) = \nabla_x \varphi(x) \cdot a(x) + \int [\varphi(x+y) - \varphi(x) - \nabla_x \varphi(x) \cdot y 1_{(|y|<1)}] K(x, dy).$$

In fact, his class includes the stable-like processes.

Our concern is to investigate the absolute continuity of the transition probability and the smoothness of the transition density for the Markov process on $\mathbf{R}^d$ associated with the Lévy generator L of (2.7). But it seems to be very difficult, so we mainly consider the following Lévy generator obtained by cutting off big jumps in (2.7):

$$(2.8) \quad L\varphi(x) = \nabla_x \varphi(x) \cdot a(x) + \int [\varphi(x+y) - \varphi(x) - \nabla_x \varphi(x) \cdot y] K(x, dy)$$

with

$$(2.9) \quad K(x, A) = \int_S M(d\sigma) \int_0^{u_0} 1_{A \setminus \{0\}}(y(x, u, \sigma)) g(x, u, \sigma) du,$$

where $u_0 > 0$ is a fixed constant and $(S, \mathcal{S}, M(d\sigma))$ is a finite measure space. We assume that

(i) $y(x, u, \sigma)$ is a bounded measurable function on $\mathbf{R}^d \times [0, u_0] \times S$, of class C^1 in u on $[0, u_0]$ and $y(x, 0, \sigma) = 0$.

(ii) $g(x, u, \sigma) = u^{-1-\alpha(x)} m(x, \sigma)$

with $0 \leq \alpha(x) < 2$ and a bounded positive function $m(x, \sigma)$. In particular, if $S = \mathbf{S}^{d-1}$, $y(x, u, \sigma) = u\sigma$ and $m(x, \sigma) = \theta_{\alpha(x)}$, then the corresponding Markov process should be called a *stable-like process with bounded jumps*, since big jumps are suppressed in (1.4).

From the result for α -stable Lévy processes (cf. [12]), in the case where the condition (1.3) holds, it would be expected that the transition probability $P_t(x, dy)$ has a density $p_t(x, y)$ which is smooth in y or in (x, y) for every $t > 0$ under a regularity condition and a non-degeneracy condition on $K(x, dy)$. Furthermore we shall also discuss the Lévy kernel (2.9) with $\alpha(x) \equiv 0$. In this case the result on L -processes [18] would suggest that for each $r \geq 0$, the transition density $p_t(x, y)$ would be of class C^r in y or in (x, y) only for sufficiently large t 's.

We give an assumption on (a, y, m, α) . Let $r \geq 1$.

Assumption $(L - r)$. (i) $a(x)$ is a C_b^r -function on $\mathbf{R}^d$;

(ii) $y(x, u, \sigma)/u$ is r -times differentiable in (x, u) on $\mathbf{R}^d \times [0, u_0]$, $\nabla_x^i \nabla_u^j (y(x, u, \sigma)/u)$ is bounded in (x, u, σ) for all i, j with $0 \leq i + j \leq r$ and $\nabla_u y(x, 0, \sigma)$ is continuous at every $\sigma \in \text{supp} M(d\sigma)$;

(iii) $m(x, \sigma)$ is r -times differentiable in x such that $0 < \xi_1 \leq m(x, \sigma) \leq \xi_2 < \infty$ and $\nabla_x^k m(x, \sigma)$ is bounded in (x, σ) for all k with $0 \leq k \leq r$;

(iv) $\alpha(x) \equiv 0$, or $\alpha(x)$ is a C_b^r -function on $\mathbf{R}^d$ and satisfies the inequality $0 < \zeta_1 \leq \alpha(x) \leq \zeta_2 < 2$ with some constants ζ_1, ζ_2 .

We always assume $(L - 1)$, then there exists a Markov process (X_t^x, P) in $\mathbf{R}^d$ with initial condition $X_0^x = x$ corresponding to the Lévy generator L . In fact, if we define the following functions:

$$G(x, u, \sigma) = \int_u^{u_0} g(x, v, \sigma) dv \quad \text{for } u \in (0, u_0),$$

$$\hat{G}(x, \cdot, \sigma) \quad \text{is the inverse function of } G(x, \cdot, \sigma)$$

and

$$(2.10) \quad f(x, z, \sigma) = y(x, \hat{G}(x, z, \sigma), \sigma) \quad \text{for } z \in E = (0, \infty).$$

Then K is expressed as

$$K(x, A) = \int_S M(d\sigma) \int_0^\infty 1_{A \setminus \{0\}}(f(x, z, \sigma)) dz$$

and X_t^x is realized as a unique solution to the following stochastic differential equation driven by the Poisson random measure $\mu(ds dz d\sigma)$ with intensity $\nu(ds dz d\sigma) = ds dz M(d\sigma)$:

$$(2.11) \quad X_t^x = x + \int_0^t a(X_s^x) ds + \int_0^t \int_{E \times S} f(X_{s-}^x, z, \sigma) \tilde{\mu}(ds dz d\sigma) \quad (x \in \mathbf{R}^d),$$

where $\tilde{\mu} = \mu - \nu$ is a compensated Poisson random measure.

Since $X^x = (X_t^x)$ defines a standard Markov process in $\mathbf{R}^d$, we denote its transition probability by $P_t(x, dy)$. The next assumption is a non-degeneracy condition for $K(x, dy)$:

Assumption (N) .

$$\text{span} \{ \nabla_u y(x, 0, \sigma) : \sigma \in \text{supp} M(d\sigma) \} = \mathbf{R}^d$$

for every $x \in \mathbf{R}^d$.

Note that under $(L-1)$ and (N) , $K(x, dy)$ satisfies $K(x, \mathbf{R}^d) = \infty$ and $\int |y|^2 K(x, dy) < \infty$.

The first result of this chapter is the following:

Theorem 2.9. (Existence of transition densities, Theorem 3.1 in Chapter III)

Assume that $(L-4)$ and (N) hold. Then $P_t(x, dy)$ has a density $p_t(x, \cdot)$ for all $t > 0, x \in \mathbf{R}^d$.

Assumption (SN) . There exist constants $\epsilon > 0$, a finite number of points $\sigma_i \in \text{supp} M(d\sigma), i = 1, \dots, N$ and disjoint neighborhoods S_i of $\sigma_i, i = 1, \dots, N$ such that for all $x, y \in \mathbf{R}^d$,

$$\sum_{i=1}^N \inf_{\sigma \in S_i} |y \cdot \nabla_u y(x, 0, \sigma)|^2 \geq \epsilon |y|^2.$$

This is stronger than (N) . The following assumption are rather artificial, but they are required by technical reasons (see [2]). Let $f(x, z, \sigma)$ be defined by (2.10).

Assumption (C) . $\inf_{x,z,\sigma} |\det(I + \nabla_x f(x, z, \sigma))| > 0$.

Assumption (SC) . $\inf_{v \in [0,1], x,z,\sigma} |\det(I + v \nabla_x f(x, z, \sigma))| > 0$.

These assumptions hold, for instance, if either $|y/u|, |\nabla_x y/u|, |\nabla_u(y/u)|$, or $|\nabla_x y/u|, |\nabla_x m|, |\nabla_x \alpha|$ are sufficiently small, or if u_0 is sufficiently small (see Remark 3.6 in Chapter III).

Now we state the main result of this chapter.

Theorem 2.10. (Smoothness of transition densities, Theorem 3.3 in Chapter III)

Assume that (SN) and (C) hold. Let $r \geq 1$ and let $M_0 = \min_{1 \leq i \leq N} M(S_i)$, where S_i is found in (SN) .

a) Let $0 < \zeta_1 \leq \alpha(x) \leq \zeta_2 < 2$.

(i) Under $(L - (r + 5))$, $P_t(x, dy)$ has a density $p_t(x, y)$ of class C^r in y for all $x \in \mathbf{R}^d, t > 0$.

(ii) Under $(L - (r + 2d + 5))$, $p_t(x, y)$ is of class C^r in (x, y) for all $t > 0$.

(iii) Assume further that (SC) holds. Then, under $(L - (2r + 4d + 8))$, $p_t(x, y)$ is of class C^r in (t, x, y) of $(0, \infty) \times \mathbf{R}^d \times \mathbf{R}^d$.

(iv) In the cases (i), (ii) and (iii), if $(L - r)$ is fulfilled for all $r \geq 1$, then the term C^r can be replaced by C^∞ .

b) Let $\alpha(x) \equiv 0$, and $\eta = 1/\zeta_1$.

(i) Under $(L - (r + d + 5))$ (resp. $(L - (r + 5))$), $P_t(x, dy)$ has a density $p_t(x, y)$ of class C^r in y , if $t > \{4d(r + d + 1)\eta + 2\eta^2\}/M_0$ (resp. $t > \{4d^2(r + 1)\eta + 2\eta^2\}/M_0$).

(ii) Under $(L - (r + 2d + 5))$, $p_t(x, y)$ is of class C^r in (x, y) , if $t > \{4d(r + 2d + 1)\eta + 2\eta^2\}/M_0$.

(iii) Assume further that (SC) holds. Then, under $(L - (2r + 4d + 8))$, $p_s(x, y)$ is of class C^r in (s, x, y) of $(t, \infty) \times \mathbf{R}^d \times \mathbf{R}^d$, if $t > \{8d(r + 2d + 2)\eta + 2\eta^2\}/M_0$.

Finally we shall apply above results to a stable-like process with bounded jumps and a stable-like process itself. Recall that the generator of a stable-like process is decomposed into $L = A^0 + A^1$, where

$$\begin{aligned} A^0 \varphi(x) &= \int_{|y| < u_0} [\varphi(x+y) - \varphi(x) - \nabla_x \varphi(x) \cdot y] K(x, dy), \\ A^1 \varphi(x) &= \int_{|y| \geq u_0} [\varphi(x+y) - \varphi(x)] K(x, dy) \end{aligned}$$

and $K(x, dy)$ is given by (1.4).

Assume that $\alpha(x)$ is a C_b^1 -function satisfying the inequalities (1.3). Since A^0 is a special type of (2.8), there corresponds a Markov process on $\mathbf{R}^d$ which is defined as the unique solution of the SDE (2.11) with $f(x, z, \sigma)$ of (2.10) on a Poisson space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ over $[0, T] \times (0, \infty) \times \mathbf{S}^{d-1}$. We denote by $P_t^0(x, dy)$ its transition probability.

On the other hand Bass [1] establishes the well-posedness of the martingale problem for L , so that L determines a Markov process on $\mathbf{R}^d$ uniquely. We denote by $P_t(x, dy)$ its transition probability. Then we obtain the following result:

Theorem 2.11. (Theorem 3.5 in Chapter III)

Suppose that $\alpha(x)$ is a C_b^j -function on $\mathbf{R}^d$ for some j , and satisfies (1.3). Let $r \geq 0$.

a) (Stable-like process with bounded jumps)

(i) *If $j \geq 4$, then $P_t^0(x, dy)$ has a density $p_t^0(x, y)$ for all $t > 0$ and $x \in \mathbf{R}^d$.*

(ii) *Assume (C). If $j \geq r + 5$, then $p_t^0(x, y)$ is of class C^r in y for all $t > 0, x \in \mathbf{R}^d$.*

If $j \geq r + 2d + 5$, then $p_t^0(x, y)$ is of class C^r in (x, y) for $t > 0$.

(iii) *Assume (SC). If $j \geq 2r + 4d + 8$, then $p_t^0(x, y)$ is of class C^r in (t, x, y) of $(0, \infty) \times \mathbf{R}^d \times \mathbf{R}^d$.*

b) (Stable-like process)

If $j \geq 4$, then $P_t(x, dy)$ has a density $p_t(x, y)$ given by

$$(2.12) \quad p_t(x, y) = p_t^0(x, y) + \int_0^t ds \int_{\mathbf{R}^d} P_{t-s}(x, dy_1) A^1 p_s^0(y_1, y)$$

for all $t > 0$ and $x \in \mathbf{R}^d$.

REMARK 2.12. To obtain the smoothness results of $P_t(x, dy)$ for stable-like process, the formula (2.12) might be useful. But we need more information on integrability of derivatives of $p_t^0(x, y)$, which seems rather difficult.

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CHAPTER II

Infinite Markov Particle Systems with Singular Immigration; Martingale Problems and Limit Theorems

1 Introduction

In this chapter we are mainly concerned with infinite Markov particle systems $(X_t, \mathbf{P}_\mu)$ with singular immigration associated with absorbing Brownian motion $(w^0(t), P_x^0)$ in a half space $H = \mathbf{R}^{d-1} \times (0, \infty)$, starting from $\mu \in \mathcal{M}$. Here $\mathcal{M} = \mathcal{M}(H)$ is the space of all counting measures $\mu = \sum_n \delta_{x_n}$ on H . It is constructed out of infinitely many independent absorbing Brownian particles starting from points of the support of μ and another independent particles which immigrate uniformly from boundary at random time and move according to the excursion law Q^0 . The immigration part is obtained as the limit by putting the starting points of independent absorbing Brownian particles which immigrate in H at random times, close to the boundary with infinite mass. From this construction the generator $\mathcal{L}$ of this process should be expressed as the sum of no immigration part $\mathcal{L}^0$ and immigration part $\mathcal{L}^I$. That is, for some suitable functional $\Phi(\mu)$ of counting measures μ ,

$$\begin{aligned}\mathcal{L}^0\Phi(\mu) &= \frac{1}{2} \sum_{k=1}^d \langle \mu, \mathcal{D}_k^2 \Phi(\mu; \cdot) \rangle, \\ \mathcal{L}^I\Phi(\mu) &= \frac{1}{2} \langle \widetilde{m}, \mathcal{D}_d \Phi(\mu; \cdot)|_{x_d=0+} \rangle,\end{aligned}$$

where $\widetilde{m} = d\widetilde{x}$ is the Lebesgue measure on $\mathbf{R}^{d-1}$, $\mathcal{D}_k$ is a kind of differential operator defined as

$$\mathcal{D}_k \Psi(\mu; x) = \lim_{h \rightarrow 0} \frac{1}{h} \left[\Psi(\mu + \delta_{x_k(h)} - \delta_x; x_k(h)) - \Psi(\mu; x) \right]$$

with $x_k(h) = (x_1, \dots, x_k + h, x_{k+1}, \dots, x_d)$, and $\mathcal{D}_k^2 = \mathcal{D}_k \circ \mathcal{D}_k$ for $k = 1, 2, \dots, d$. Note that if $\Psi(\mu; x) = \Phi(\mu)$, then

$$\mathcal{D}_k \Phi(\mu; x) = \lim_{h \rightarrow 0} \frac{1}{h} \left[\Phi(\mu + \delta_{x_k(h)} - \delta_x) - \Phi(\mu) \right].$$

We first formulate this operator and give the martingale characterization for $(X_t, \mathbf{P}_\mu)$. To treat this process as a diffusion, we introduce a subspace $\overline{\mathcal{M}}_p$ ($p > d$) of $\mathcal{M}$ with a topology so that X_t can be continuous in it. Moreover we investigate Hölder continuity of the sample paths. It is very interesting that the exponent of Hölder continuity changes from $1/4$ to $1/2$ according to the starting point μ .

By using this process we can construct an *equilibrium process with immigration* $(X_t, \mathbf{P})$, which is a stationary Markov particle system with immigration associated with $(w^0(t), P_x^0)$ in H and the Lebesgue measure $m(dx) = dx$ on H . This process is also constructed by using a *Kuznetsov measure* Q_m , which is a stationary σ -finite Markov measure

with the same transition law as the absorbing Brownian motion in H . This measure is defined as the integration of the time-shift of excursion law Q^0 with respect to the time in $\mathbf{R}^1$.

On the other hand for Brownian motion $(w(t), P_x)$ in $\mathbf{R}^d$, F. Spitzer [17] showed that, in case of $d = 3$, if B is a compact set with a positive capacity $C(B)$, then

$$P_m(0 < T_B < t) = t C(B) + \sqrt{t} 4(2\pi)^{-3/2} C(B)^2 + o(\sqrt{t})$$

as $t \rightarrow \infty$, where $P_m = \int dx P_x$ and T_B is the first hitting time for B (note that, in case of $d = 4$ the second term is $\log t C(B)^2 / (4\pi)^2$, which is given by R. K. Gettoor [3], and in case of $d \geq 5$ it is $\int P_x(0 < T_B < \infty)^2 dx < \infty$, which is also given in [17]). This result can be applied to an *equilibrium process* $(X_t, \mathbf{P})$ (see [13], [18]), which is a stationary Markov particle system associated with the Lebesgue measure $m(dx) = dx$ on $\mathbf{R}^d$. Let $X_t(N_t^B)$ denote the number of particles of the equilibrium process hitting B during the time interval $(0, t)$, where $N_t^B = \{0 < T_B < t\}$. By the subadditive ergodic theorem [10], it holds that $X(N_t^B)/t \rightarrow C(B)$ $\mathbf{P}$ -a.s. and in $L^1(\mathbf{P})$ as $t \rightarrow \infty$. Furthermore

$$\frac{X(N_t^B) - t C(B)}{\sqrt{t}} \rightarrow \begin{cases} N(M^B, C(B)) & \text{if } d = 3, \\ N(0, C(B)) & \text{if } d \geq 4 \end{cases}$$

in law as $t \rightarrow \infty$, where $M^B = 4C(B)^2 / (2\pi)^{3/2}$ and $N(u, v)$ is the Gaussian distribution with mean u and variance v .

So secondly we consider a hitting rate for Brownian excursions and give the asymptotic behavior under Q_m . For a fixed compact set B in H , let

$$\sigma_B(w) = \inf\{t > 0 : w(t) \in B\}.$$

If the capacity $C^0(B)$ associated with $(w^0(t), P_x^0, m)$ is positive, then

$$Q_m(0 \leq \sigma_B < t) = t C^0(B) + f(t)$$

with $f(t) = \sqrt{2t/\pi} + a(B) + o(1)$ ($d = 1$), $\Phi(B) \log t + o(\log t)$ ($d = 2$), $O(1)$ ($d \geq 3$) as $t \rightarrow \infty$, where $a(B), \Phi(B)$ are certain constants. By applying the result to the equilibrium process with immigration, we derive limit theorems for it (see §3.1).

Now we define infinite Markov particle systems $(X_t, \mathbf{P}_\mu)$ with immigration starting from μ associated with absorbing Brownian motion in a half space H .

Fix an extra point $\Delta \notin H$ and set $H_\Delta = H \cup \{\Delta\}$. Let W^0 be the set of all mappings w^0 from $[0, \infty)$ to H_Δ such that $w^0 : [0, \zeta(w^0)) \rightarrow H$ is continuous and $w^0(t) = \Delta$ if $t \geq \zeta(w^0)$ for a certain constant $\zeta(w^0) > 0$. Let $(w^0(t), P_x^0)$ be absorbing Brownian motion starting from x in H with the transition semi-group (P_t^0) . We use the same notation P_x^0 and P_t^0 for the probability and the semi-group, for it is convenient and there is no possibility of confusion. Let W^I be the set of all mappings w^I from $[0, \infty)$ to H_Δ such that $w^I : (\alpha(w^I), \beta(w^I)) \rightarrow H$ is continuous and $w^I(t) = \Delta$ if $t \notin (\alpha(w^I), \beta(w^I))$ for a certain non empty open interval $(\alpha(w^I), \beta(w^I))$. Set $W = W^0 \cup W^I$. For $r > 0, u > 0$, let

$$\nu_r(u) = \frac{u}{\sqrt{2\pi r^3}} \exp \left[-\frac{u^2}{2r} \right] \left(= \frac{1}{2} \frac{\partial}{\partial v} p_r^0(u, v) \Big|_{v=0} \right),$$

where $p_r^0(u, v)$ is the transition density of absorbing Brownian motion on $(0, \infty)$, and

$$\nu_r(dx) = \nu_r(x_d)dx \quad \text{on } H.$$

Then $\nu_r P_s^0 = \nu_{r+s}$ holds for $r > 0, s \geq 0$, thus $\nu = (\nu_r)_{r>0}$ is an *entrance law* for (P_t^0) . Moreover we define the following σ -finite measure on W :

$$Q^0 = \lim_{r \downarrow 0} \int_H \nu_r(dx) P_x^0$$

Then Q^0 is supported on $\{\alpha = 0\}$ (see [7]) and governs excursions starting uniformly from boundary of H . Q^0 is called the *Brownian excursion law*. Let the following be given:

$$\begin{aligned} \Omega &\text{ is the space of counting measures } \omega = \sum_n \delta_{w_n}, w_n \in W, \\ X_t(\omega) &= \omega(t)|_H \text{ for } \omega \in \Omega, \\ \mathcal{F} &= \sigma(X_s : s < \infty), \mathcal{F}_t = \sigma(X_s : s < t). \end{aligned}$$

We define a probability measure $\mathbf{P}_\mu$ for $\mu \in \mathcal{M}$ as follows: For $\omega = \sum_n \delta_{w_n} \in \Omega$, let ω^0 (resp. ω^I) denote $\sum_{n:w_n \in W^0} \delta_{w_n}$ (resp. $\sum_{n:w_n \in W^I} \delta_{w_n}$), and set $\Omega^0 = \{\omega^0 \in \Omega : \omega \in \Omega\}$ and $\Omega^I = \{\omega^I \in \Omega : \omega \in \Omega\}$,

$$\begin{aligned} X_t^0(\omega) &= \omega^0(t)|_H : && \text{the no immigration part,} \\ X_t^I(\omega) &= \omega^I(t)|_H : && \text{the immigration part.} \end{aligned}$$

Hence $X_t = X_t^0 + X_t^I$ and we can identify

$$\Omega \equiv \Omega^0 \oplus \Omega^I \cong \Omega^0 \otimes \Omega^I.$$

So we set

$$\mathbf{P}_\mu = \mathbf{P}_\mu^0 \otimes \mathbf{P}^I,$$

where

$$\mathbf{P}_\mu^0 = \bigotimes_{n \geq 1} P_{x_n}^0 \quad \text{if} \quad \mu = \sum_{n \geq 1} \delta_{x_n}, x_n \in H$$

and

$$\mathbf{P}^I \quad \text{is the } \int_0^\infty \theta_{-s}(Q^0) ds \text{-Poisson measure,}$$

i.e., the distribution of the Poisson random measure with intensity $\int_0^\infty \theta_{-s}(Q^0) ds$. Then it satisfies that

$$\mathbf{E}_\mu \left[e^{-\langle X_{t+s}, f \rangle} \middle| \mathcal{F}_s \right] = \exp \left[-\langle X_s, V_t f \rangle - \int_0^t \langle \nu_r, 1 - e^{-f} \rangle dr \right]$$

for positive measurable functions f on H , where

$$V_t f(x) = -\log E_x^0[\exp(-f(w^0(t)))] = -\log\{1 - P_t^0(1 - e^{-f})(x)\}.$$

Under $\mathbf{P}_\mu$, $\{X_t\}$ is an $\mathcal{M}$ -valued Markov process starting from μ with immigration from boundary. Each particles dies when it reaches boundary of H , and infinitely many particles are born from the boundary. Thus $(X_t, \mathbf{P}_\mu)$ is called the *infinite Markov particle system with singular immigration associated with* $(w^0(t), P_x^0, Q^0)$. Note that X_t^I can be also defined as the following:

$$X_t^I = \int_0^t \int_W \delta_{w(t-s)} N^0(ds dw) \Big|_H,$$

where

$$N^0(\omega; ds dw) = \# \{n : \alpha(w_n) \in ds, w_n(\cdot - \alpha(w_n)) \in dw\} \quad \text{if} \quad \omega = \sum_{n \geq 1} \delta_{w_n}.$$

Then $N^0(ds dw)$ is a Poisson random measure with intensity $ds Q^0(dw)$. Let $m(dx) = dx$ be the Lebesgue measure on H and Π_m be the m -Poisson measure on $\mathcal{M}$, i.e., the distribution of the Poisson random measure with intensity m . Define

$$\mathbf{P} = \int \Pi_m(d\mu) \mathbf{P}_\mu.$$

Then $(X_t, \mathbf{P})$ is a stationary Markov process because

$$\begin{aligned} \mathbf{E}[\exp\{-\langle X_t, f \rangle\}] &= \exp \left[-\langle m, 1 - e^{-V_t f} \rangle - \int_0^t \langle \nu_r, 1 - e^{-f} \rangle \right] \\ &= \exp \left[-\langle m, 1 - e^{-f} \rangle \right]. \end{aligned}$$

This process is also defined by using a stationary σ -finite measure Q_m . We extend W to the set of all maps $w : \mathbf{R} \rightarrow H_\Delta$ such that there is a nonempty open interval $(\alpha(w), \beta(w))$ on which w is H -valued and continuous, with $w(t) = \Delta$ if $t \leq \alpha(w)$ or $t \geq \beta(w)$, and a constant map $[\Delta]$, i.e., $[\Delta](t) = \Delta$ for all t . We use the same notation W for this set.

$$Q_m = \int_{\mathbf{R}} \theta_{-s}(Q^0) ds.$$

Then $(w(t), Q_m)$ is a stationary Markov process with the same transition probability (P_t^0) as absorbing Brownian motion. Q_m is called the *Kuznetsov measure associated with* $(w^0(t), P_x^0, m)$. If we restrict the time interval to $[0, \infty)$, then $\mathbf{P}$ can be also defined as the Q_m -Poisson measure on Ω . $(X_t, \mathbf{P})$ is called the *equilibrium process with immigration associated with* $(w^0(t), P_x^0, m)$ or $(w(t), Q_m)$.

In §2 we give a martingale characterization for $(X_t, \mathbf{P}_\mu)$ (cf. [8], [11]) and consider Hölder continuity of sample paths.

In §3 we give asymptotic behavior of hitting rates for Brownian excursions and their applications to the equilibrium process with immigration. We first investigate asymptotic behavior of $Q_m(0 \leq \sigma_B < t)$ as $t \rightarrow \infty$ and limit theorems for the equilibrium process with immigration associated with $(w(t), Q_m)$. We also consider some asymptotic behaviors corresponding to P_m^0 and Q^0 (see §3.3). Moreover since the excursion law Q^0 governs excursions starting uniformly from the boundary of H , we also discuss the non-uniform case and give some results (see §3.4).

A notion of Kuznetsov measures is introduced by S. E. Kuznetsov [12] and recently studied by many authors in relation with capacity theory, e.g., Fitzsimmons and Maisonneuve [2], Gettoor [5], and Gettoor and Steffens [7]. They treat the Kuznetsov measures in more general situation, thus we give definitions of infinite Markov particle systems with immigration and equilibrium processes with immigration in the general situation (see §2.2). We can also consider the same problem as Spitzer for the general Kuznetsov measure and give the first term of the asymptotic for it (see §3.2).

2 Martingale Problems and Hölder Continuity

In this section we consider the infinite Markov particle system $(X_t, \mathbf{P}_\mu)$ with immigration starting from a counting measure μ , associated with absorbing Brownian motion $(w^0(t), P_x^0)$ in the half space H and the Lebesgue measure $m(dx) = dx$ on H . We give martingale characterization of $(X_t, \mathbf{P}_\mu)$ and investigate the Hölder continuity of sample paths.

2.1 Results

First we define some spaces of functions and measures. Fix $p > d$. Set

$$g_p(x) = (1 + |x|^2)^{-p/2} \quad \text{and} \quad g_{p,0}(x) = g_p(x)h_0(x_d),$$

where $h_0(v)$ is a fixed C^∞ -function in $v > 0$ satisfying the following properties:

- (a) h_0 is non-decreasing and $0 < h_0 \leq 1$,
- (b) $h_0(v) = v$ for $v \in (0, 1/2]$ and $h_0 = 1$ on $[2, \infty)$.

Then $g_{p,0}(x)$ satisfies that for $x = (\tilde{x}, x_d) \in \mathbf{R}^{d-1} \times (0, \infty)$, $g_{p,0}(\tilde{x}, 0+) = \partial_d^2 g_{p,0}(\tilde{x}, 0+) = 0$ and $\partial_d g_{p,0}(\tilde{x}, 0+) = g_p(\tilde{x}, 0+)$, where $\partial_d = \partial/\partial x_d$.

Let $C = C(H)$ be the space of all continuous functions in H .

$$f \in C_p = C_p(H) \iff f \in C, \|f/g_p\|_\infty < \infty.$$

$$f \in C_{p,0} = C_{p,0}(H) \iff f \in C, \|f/g_{p,0}\|_\infty < \infty.$$

The following function space D_p is stable under P_t^0 (see §2.3):

$$f \in D_p = D_p(H) \iff f \in C^2, f, \partial_d^2 f \in C_{p,0} \text{ and other partial derivatives are in } C_p.$$

We denote $C_c = C_c(H)$ the space of all continuous functions with compact support and $C_c^\infty = C_c^\infty(H)$ the space of all C^∞ -functions with compact support.

Let $\mathcal{M} = \mathcal{M}(H)$ be the space of all counting measures on H .

$$\begin{aligned}\mu \in \mathcal{M}_p &= \mathcal{M}_p(H) \Rightarrow \mu \in \mathcal{M}, \langle \mu, g_p \rangle < \infty. \\ \mu \in \overline{\mathcal{M}}_p &= \overline{\mathcal{M}}_p(H) \Rightarrow \mu \in \mathcal{M}, \langle \mu, g_{p,0} \rangle < \infty.\end{aligned}$$

Then $\mathcal{M}_p \subset \overline{\mathcal{M}}_p \subset \mathcal{M}$. The topology of $\overline{\mathcal{M}}_p$ is defined by the weak topology with respect to $C_{p,0}$, i.e.,

$$\mu_n \rightarrow \mu \text{ in } \overline{\mathcal{M}}_p \Leftrightarrow \langle \mu_n, f \rangle \rightarrow \langle \mu, f \rangle \text{ for all } f \in C_{p,0} \text{ as } n \rightarrow \infty.$$

This is equivalent to that $\langle \mu_n, f \rangle \rightarrow \langle \mu, f \rangle$ for all $f \in C_c$ and $f = g_{p,0}$ as $n \rightarrow \infty$. Then $\overline{\mathcal{M}}_p$ is Polish, i.e., metrizable, complete and separable.

Now fix $p > d$. Set

$$\begin{aligned}\mathbf{D}_p &= \{ \Phi(\mu) = F(\langle \mu, f_1 \rangle, \dots, \langle \mu, f_n \rangle) : F \in C^2(\mathbf{R}^n), |F^{(i)}(x)| \leq C(1 + |x|)^k, \\ &\quad i = 0, 1, 2 \text{ for some } C > 0, k \geq 0, \text{ and } f_j \in D_p, j = 1, 2, \dots, n \},\end{aligned}$$

and for $\Phi(\mu) = F(\langle \mu, f_1 \rangle, \dots, \langle \mu, f_n \rangle) \in \mathbf{D}_p, \mu \in \mathcal{M}_p$, we define an operator $\mathcal{L}$ by

$$\begin{aligned}\mathcal{L}\Phi(\mu) &= \sum_{i=1}^n \partial_i F(\langle \mu, f_1 \rangle, \dots, \langle \mu, f_n \rangle) \left\{ \langle \mu, A f_i \rangle + \frac{1}{2} \langle \widetilde{m}, \partial_d f_i(\cdot, 0+) \rangle \right\} \\ &\quad + \frac{1}{2} \sum_{i,j=1}^n \partial_{ij}^2 F(\langle \mu, f_1 \rangle, \dots, \langle \mu, f_n \rangle) \langle \mu, \Gamma(f_i, f_j) \rangle,\end{aligned}$$

where $\partial_i = \partial / \partial x_i$, $\widetilde{m} = d\widetilde{x}$ is the $(d-1)$ -dimensional Lebesgue measure, A is the generator of (P_t^0) and $\Gamma(f, g) = A(fg) - fAg - gAf$, i.e., $Af = \frac{1}{2} \Delta_x f$ and $\Gamma(f, g) = \nabla_x f \cdot \nabla_x g$ for $f, g \in D_p$.

Moreover set $\Omega_p = C([0, \infty) \rightarrow \overline{\mathcal{M}}_p)$, which is a Polish space.

Theorem 2.1. *Fix $p > d$ and $\mu \in \overline{\mathcal{M}}_p$. Then under $\mathbf{P}_\mu$, $\{X_t\}$ is an $\overline{\mathcal{M}}_p$ -valued Markov process having a continuous sample paths relative to the topology in $\overline{\mathcal{M}}_p$ and satisfies that*

- (i) $\mathbf{P}_\mu(X_0 = \mu) = 1$,
- (ii) for each $\Phi \in \mathbf{D}_p$, $t > 0$ and $k \geq 1$, $\Phi(X_t), \int_0^t |\mathcal{L}\Phi(X_s)| ds \in L^k(\mathbf{P}_\mu)$,
- (iii) for each $\Phi \in \mathbf{D}_p$, $M_t^\Phi \equiv \Phi(X_t) - \Phi(X_0) - \int_0^t \mathcal{L}\Phi(X_s) ds$ is a $\mathbf{P}_\mu$ -martingale.

Moreover if a probability measure $\tilde{\mathbf{P}}_\mu$ on Ω_p satisfies (i) and if M_t^Φ is a local $\tilde{\mathbf{P}}_\mu$ -martingale for any $\Phi(\mu) = \langle \mu, f \rangle$ and $\langle \mu, f \rangle^2$ with $f \in D_p$, then $\tilde{\mathbf{P}}_\mu = \mathbf{P}_\mu$ on Ω_p .

REMARK 2.2. (i) When $\mu \in \overline{\mathcal{M}}_p - \mathcal{M}_p$, $\mathcal{L}\Phi(\mu)$ may not be well-defined because of $\langle \mu, \Gamma(f_i, f_i) \rangle$ being possibly infinite. But for all $t > 0$, $\int_0^t \mathcal{L}\Phi(X_s) ds$ is well-defined $\mathbf{P}_\mu$ -a.s..

(ii) It is easily seen that $\langle X_t, g_p \rangle, \int_0^t \langle X_s, g_p \rangle ds \in L^1(\mathbf{P}_\mu)$ for all $t > 0$ (see §2.3). Hence

$$\mathbf{P}_\mu(\langle X_t, g_p \rangle < \infty \text{ for all } t \in \mathbf{Q} \cap (0, \infty) \text{ and } dt\text{-a.a. } t) = 1.$$

However we can show that for any time interval $(a, b); 0 \leq a < b$,

$$\mathbf{P}_\mu(\langle X_t, g_p \rangle = \infty \text{ for some } t \in (a, b)) = 1 \text{ even if } \mu = 0,$$

that is, the time points t of $\langle X_t, g_p \rangle = \infty$ exist densely. We also have

$$\mathbf{P}_0 \left(\sup_{t \in \mathbf{Q} \cap (a, b)} \langle X_t, g_p \rangle = \infty \right) = 1.$$

We shall prove these results in §2.4

Theorem 2.3. Fix $p > d$. Set $\Gamma f = \Gamma(f, f) = |\nabla_x f|^2$ for $f \in D_p$.

(i) If $\mu \in \overline{\mathcal{M}}_p$, then X_t is locally $(1/4 - \epsilon)$ -Hölder continuous in $t \in [0, \infty)$ for all $0 < \epsilon < 1/4$ under $\mathbf{P}_\mu$, i.e., for each $\gamma = 1/4 - \epsilon$, $f \in D_p$ and $T > 0$, there is a constant $C = C(\mu, f, T, \gamma) > 0$ such that

$$\mathbf{P}_\mu \left(\omega : \sup_{0 < |t-s| \leq \eta(\omega), s, t \in [0, T]} \frac{|\langle X_t(\omega) - X_s(\omega), f \rangle|}{|t - s|^\gamma} \leq C \right) = 1,$$

where $\eta(\omega)$ is an a.s. positive random variable.

(ii) If $\mu \in \mathcal{M}_p$, then X_t is locally $(1/2 - \epsilon)$ -Hölder continuous in $t \in [0, \infty)$ for all $0 < \epsilon < 1/2$ under $\mathbf{P}_\mu$.

(iii) Even if $\mu \in \overline{\mathcal{M}}_p - \mathcal{M}_p$, X_t is locally $(1/2 - \epsilon)$ -Hölder continuous in $t \in (0, \infty)$ for all $0 < \epsilon < 1/2$ under $\mathbf{P}_\mu$.

(iv) If $\mu \in \overline{\mathcal{M}}_p$, then for $f \in D_p$ such that $\langle \mu, \Gamma f \rangle > 0$, $\langle X_t, f \rangle$ is not $1/2$ -Hölder continuous at $t = 0$. Furthermore if $f \in D_p$ such that $K_f \equiv \{\Gamma f = 0\}$ is a compact subset in H , and if $\mu \in \overline{\mathcal{M}}_p$ satisfies one of the following conditions:

$$\begin{cases} d = 1 \Rightarrow \mu(\{x \geq a\}) = \infty \text{ for some } a > 0, \\ d \geq 2 \Rightarrow \langle \mu, \Gamma f \rangle > 0, \text{ i.e., } \mu(K_f^c) \geq 1, \end{cases}$$

then $\langle X_t, f \rangle, t \geq 0$, is nowhere $1/2$ -Hölder continuous under $\mathbf{P}_\mu$, i.e.,

$$\left\{ \omega : |\langle X_{t+h}(\omega) - X_t(\omega), f \rangle| \leq C(\omega)\sqrt{h} \text{ for some } t \in [0, \infty) \text{ and for all } h \in [0, u(\omega)] \right\}$$

is a $\mathbf{P}_\mu$ -null set, where $C(\omega)$ and $u(\omega)$ are a.s. positive random variables.

When $\mu \in \overline{\mathcal{M}}_p - \mathcal{M}_p$, it seems to be difficult to completely determine the exponent of Hölder continuity at $t = 0$. However we can give some sufficient conditions for them. Set $S_a = \{|x_i| \leq a, i = 1, \dots, d\} \cap H$.

Theorem 2.4. Let $\mu \in \overline{\mathcal{M}}_p - \mathcal{M}_p$.

(i) Suppose that for some $T > 0, 1 \leq \theta < 2$,

$$(2.1) \quad \langle \mu|_{S_1}, P_t^0 1 \rangle \asymp \int_{S_1} \frac{dx}{x_d^\theta} P_t^0 1(x) \text{ for all } 0 < t \leq T,$$

where $g(t) \asymp h(t), 0 < t \leq T$ means $Ch(t) \leq g(t) \leq C'h(t), 0 < t \leq T$ for some positive constants C, C' depending only on T , then X_t is $((3 - \theta)/4 - \epsilon)$ -Hölder continuous at $t = 0$ for all $0 < \epsilon < (3 - \theta)/4$, moreover for each $f \in D_p$ such that $\Gamma f(\tilde{0}, 0+) > 0$, $\langle X_t, f \rangle$ is not $(3 - \theta)/4$ -Hölder continuous at $t = 0$ under $\mathbf{P}_\mu$.

(ii) Suppose that for some $T > 0$ and $1 < \theta < 2, \eta > 0$ or $\theta = 2, \eta > 1$,

$$(2.2) \quad \langle \mu|_{S_{1/\epsilon}}, P_t^0 1 \rangle \asymp \int_{S_{1/\epsilon}} \frac{dx}{x_d^\theta (\log 1/x_d)^\eta} P_t^0 1(x) \text{ for all } 0 < t \leq T,$$

then X_t is $(3 - \theta)/4$ -Hölder continuous at $t = 0$, moreover for each $f \in D_p$ such that $\Gamma f(\tilde{0}, 0+) > 0$, $\langle X_t, f \rangle$ is not $((3 - \theta)/4 + \epsilon)$ -Hölder continuous at $t = 0$ for all $\epsilon > 0$ under $\mathbf{P}_\mu$.

REMARK 2.5. In the later half of (ii) we can show the following:

$$\limsup_{t \downarrow 0} \frac{|\langle X_t, f \rangle|}{t^{(3-\theta)/4}(\log 1/t)^{-\eta/2}} = \infty \quad \mathbf{P}_\mu\text{-a.s. if } 1 < \theta < 2, \eta > 0,$$

$$\limsup_{t \downarrow 0} \frac{|\langle X_t, f \rangle|}{t^{1/4}(\log 1/t)^{-(\eta-1)/2}} = \infty \quad \mathbf{P}_\mu\text{-a.s. if } \theta = 2, \eta > 1.$$

Here we give some examples of $\mu = \sum_{n \geq 2} \delta_{x_n} \in \overline{\mathcal{M}}_p - \mathcal{M}_p$ which satisfy the conditions (2.1), (2.2): For simplicity let $d = 1$.

- (a) $x_n = r^n$ ($0 < r < 1$) $\Rightarrow \theta = 1$ in (2.1),
- (b) $x_n = n^{-p}$ ($p > 1$) $\Rightarrow \theta = 1 + 1/p$ in (2.1),
- (c) $x_n = n^{-p}(\log n)^{-q}$ ($p > 1, q > 0$ or $p = 1 < q$) $\Rightarrow \theta = 1 + 1/p, \eta = q/p$ in (2.2).

Now it is possible to determine the exponent of Hölder continuity of sample paths of the equilibrium process with immigration associated with absorbing Brownian motion in H . Because under $\mathbf{P}$, X_t has an initial distribution $\Pi_m(d\mu)$ and $\langle m, g_p \rangle < \infty$ for all $p > d$. Set $D_{d+} = \bigcup_{p > d} D_p$, $\overline{\mathcal{M}}_{d+} = \bigcap_{p > d} \overline{\mathcal{M}}_p$ and $\Omega_{d+} = \bigcap_{p > d} \Omega_p = C([0, \infty) \rightarrow \overline{\mathcal{M}}_{d+})$. Then we have the following:

Corollary 2.6. *$(X_t, \mathbf{P})$ is an $\overline{\mathcal{M}}_{d+}$ -valued stationary Markov process having continuous sample paths relative to the topology in $\overline{\mathcal{M}}_{d+}$ with initial distribution $\Pi_m(\mu)$. Furthermore $X_t, t \geq 0$ is locally $(1/2 - \epsilon)$ -Hölder continuous for all $0 < \epsilon < 1/2$, and for each $f \in D_{d+}$ such that $K_f = \{\Gamma f = 0\}$ is compact in H , $\langle X_t, f \rangle, t \geq 0$ is nowhere $1/2$ -Hölder continuous.*

2.2 Kuznetsov Measures and Infinite Markov Particle Systems with Immigration

In this subsection we give a general theory of Kuznetsov measures according to [2], [5] and [7], and define the infinite Markov particle systems with immigration induced by Kuznetsov measures. We also discuss the Markov property of equilibrium processes with immigration.

Let S be a Lusin space, i.e., a Borel measurable subset of a separable metric space, and fix a point $\Delta \notin S$. Set $S_\Delta = S \cup \{\Delta\}$. Let $(W^0, \mathcal{F}^0, w^0(t), P_x^0), t \geq 0, x \in S$, be a Borel right process with transition semi-group $(P_t^0)_{t \geq 0}$. That is, $(P_t^0)_{t \geq 0}$ is a Borel right semi-group, W^0 is the set of all right-continuous paths $w^0 : [0, \infty) \rightarrow S_\Delta, \theta_t w^0(s) = w^0(t+s), \mathcal{F}^0 = \sigma(w^0(t) : t \geq 0), \mathcal{F}_t^0 = \sigma(w^0(s) : 0 \leq s \leq t)$ and $\zeta(w^0) = \inf\{t : w^0(t) = \Delta\}$ is the lifetime of w^0 . Set $P_\Delta^0 = \delta_{[\Delta]}$.

Let W be the set of all maps $w : \mathbf{R} \rightarrow S_\Delta$ such that there is a nonempty open interval $(\alpha(w), \beta(w))$ on which w is S -valued and right-continuous, with $w(t) = \Delta$ if $t \leq \alpha(w)$ or $t \geq \beta(w)$, and a constant map $[\Delta]$, i.e., $[\Delta](t) = \Delta$ for all t . Set $\mathcal{G}^0 = \sigma(w(t) : t \in \mathbf{R})$ and $\mathcal{G}_t^0 = \sigma(w(s) : -\infty < s \leq t)$. Moreover let $\gamma_t, t \in \mathbf{R}$, be mappings from W to W^0 defined by

$$\gamma_t w(s) = \begin{cases} w(t+s) & \text{for } s \geq 0 \text{ if } t > \alpha(w), \\ \Delta & \text{for } s \geq 0 \text{ if } t \leq \alpha(w). \end{cases}$$

Then the following hold: $\gamma_t = \gamma_0 \circ \theta_t$ for $t \in \mathbf{R}$. If $\alpha < t < \beta$, then $\zeta \circ \gamma_t = \beta \circ \theta_t$. Moreover γ_t is $\mathcal{G}_{s+t}^0 | \mathcal{F}_s^0$ -measurable for $s \geq 0, t \in \mathbf{R}$.

A family of σ -finite measures $\xi = (\xi_t)_{t \in \mathbf{R}}$ on S is called an *entrance rule* if $\xi_s P_{t-s}^0 \leq \xi_t$ for $s < t$ and $\xi_s P_{t-s}^0 \uparrow \xi_t$ as $s \uparrow t$. Moreover $\xi = (\xi_t)_{t \in \mathbf{R}}$ is called an *entrance law* if $\xi = (\xi_t)_{t \in \mathbf{R}}$ is an entrance rule and satisfies that $\xi_s P_{t-s}^0 = \xi_t$ for $s < t$.

The following result by Kuznestov [12] is well-known (see [2], [5] and [7]): For an entrance rule $\xi = (\xi_t)$ there exists a unique σ -finite measure $Q_\xi(dw)$ on W not charging $[\Delta]$ such that for $-\infty < t_1 < t_2 < \dots < t_n < +\infty$

$$\begin{aligned} Q_\xi(\alpha < t_1, w(t_1) \in dy_1, w(t_2) \in dy_2, \dots, w(t_n) \in dy_n, t_n < \beta) \\ = \xi_{t_1}(dy_1) P_{t_2-t_1}^0(y_1, dy_2) \cdots P_{t_n-t_{n-1}}^0(y_{n-1}, dy_n) \end{aligned}$$

In particular $(w(t), \mathcal{G}_t^0, Q_\xi)$ is Markov in the sense that if $t \in \mathbf{R}$ and $V \in \mathcal{F}^0$, then

$$Q_\xi(V \circ \gamma_t | \mathcal{G}_t^0) = P_{w(t)}^0(V) \quad \text{on } \{\alpha < t\}.$$

DEFINITION 2.7. Q_ξ is called the *Kuznetsov measure associated with* $(w^0(t), P_x^0, \xi)$.

If $\xi_t = m$ a σ -finite measure on S for all $t \in \mathbf{R}$, then we write Q_m for Q_ξ , and then Q_m is stationary, i.e., $Q_m[f(w(t))] = m(f)$. We denote the class of excessive (resp. invariant, purely excessive) measures by **Exc** (resp. **Inv**, **Pur**). That is, for a σ -finite measure m

$$\begin{aligned} m \in \mathbf{Exc} & \quad \text{if} \quad m P_t^0 \leq m \quad \text{for all } t \geq 0, \\ m \in \mathbf{Inv} & \quad \text{if} \quad m P_t^0 = m \quad \text{for all } t \geq 0, \\ m \in \mathbf{Pur} & \quad \text{if} \quad m \in \mathbf{Exc} \quad \text{and} \quad \lim_{t \rightarrow \infty} m P_t^0 = 0. \end{aligned}$$

$m \in \mathbf{Exc}$ may be decomposed uniquely as $m = m_i + m_p$, where $m_i \in \mathbf{Inv}$ and $m_p \in \mathbf{Pur}$. If $m \in \mathbf{Pur}$, then there exists a unique entrance law $\nu = (\nu_t)_{t>0}$ such that

$$m = \int_0^\infty \nu_t dt.$$

From now on we fix $m \in \mathbf{Exc}$ (may be infinite) and also $m_i, m_p, \nu = (\nu_t)_{t>0}$, which are uniquely determined by m as above. Then there are Kuznetsov measures

$$Q_m, Q_{m_i}, Q_{m_p}, \quad \text{and} \quad Q^0 \equiv Q_\nu \quad \text{on } W,$$

which satisfy that

$$Q_m = Q_{m_i} + Q_{m_p}$$

and

$$Q_{m_p} = \int_{\mathbf{R}} \theta_{-s}(Q^0) ds = \int_{\mathbf{R}} \theta_s(Q^0) ds.$$

It is shown that $Q_{m_i}(\alpha > -\infty) = 0$ in [2] and $Q^0(\alpha \neq 0) = 0$ in [7].

Now we consider the infinite Markov particle systems induced by Kuznetsov measures. We restrict the time interval to $[0, \infty)$ and the following are defined by the same way as in case of absorbing Brownian motion (see §1): $\mu = \sum_{n \geq 1} \delta_{x_n} \in \mathcal{M} = \mathcal{M}(S)$, $\omega = \sum_n \delta_{w_n} \in \Omega$ ($w_n \in W$), $X_t(\omega) = \omega(t)|_S$, $\mathcal{F}$, $\mathcal{F}_t$, $\mathbf{P}_\mu \equiv \mathbf{P}_\mu^0 \otimes \mathbf{P}^I$, $\Pi_m(d\mu)$ and $\mathbf{P} \equiv \int \Pi_m(d\mu) \mathbf{P}_\mu$. Then $(X_t, \mathbf{P}_\mu)$ is an $\mathcal{M}$ -valued Markov process starting from μ and $(X_t, \mathbf{P})$ is an $\mathcal{M}$ -valued stationary Markov process with a stationary measure Π_m .

DEFINITION 2.8. (i) $(X_t, \mathbf{P}_\mu)$ is called the *infinite Markov particle system associated with $(w^0(t), P_x^0)$* . In particular if $Q^0 \neq 0$, then $(X_t, \mathbf{P}_\mu)$ is called the *infinite Markov particle system with immigration associated with $(w^0(t), P_x^0, Q^0)$* .

(ii) $(X_t, \mathbf{P})$ is called the *equilibrium process associated with $(w^0(t), P_x^0, m)$* or $(w(t), Q_m)$. In particular if $m \in \mathbf{Exc} \setminus \mathbf{Inv}$, then $(X_t, \mathbf{P})$ is called the *equilibrium process with immigration associated with $(w^0(t), P_x^0, m)$* or $(w(t), Q_m)$.

REMARK 2.9. (i) $\mathbf{P}$ is also defined as the Q_m -Poisson measure Π_{Q_m} , i.e, $\mathbf{P} = \Pi_{Q_m}$ in the sense of finite-dimensional distributions.

(ii) When $m \in \mathbf{Inv}$, we can identify $Q_m = P_m^0 (= \int m(dx) P_x^0)$, so the definition of equilibrium processes is the same as in [18] by T. Shiga and Y. Takahashi.

The Markov property of $\mathbf{P}_\mu$ and the identification of $\mathbf{P} = \Pi_{Q_m}$ can be proved similarly to [18].

2.3 Proofs

Before proceeding the proofs we give two fundamental properties of the transition semi-group (P_t^0) of absorbing Brownian motion in H . Note that P_t^0 is given by $P_t^0 f(x) = \int P_t^0(x, dy) f(y)$ with $P_t^0(x, dy) = p_t^0(x, y) dy$ and

$$p_t^0(x, y) = p_t(x_1, y_1) \cdots p_t(x_{d-1}, y_{d-1}) p_t(x_d, y_d),$$

where for $u, v \in \mathbf{R}$,

$$p_t(u, v) = \frac{1}{\sqrt{2\pi t}} \exp \left[-\frac{(v-u)^2}{2t} \right]$$

and for $u, v > 0$,

$$\begin{aligned} p_t^0(u, v) &= p_t(u, v) - p_t(u, -v) \\ &= \frac{1}{\sqrt{2\pi t}} \left\{ \exp \left[-\frac{(v-u)^2}{2t} \right] - \exp \left[-\frac{(v+u)^2}{2t} \right] \right\} \\ &= \int_{-u}^u dz \frac{v+z}{\sqrt{2\pi t^3}} \exp \left[-\frac{(v+z)^2}{2t} \right] \end{aligned}$$

The following first claim is easily obtained from the proof of Proposition 2.3 in [8]: Let $p > d$. If $f \in C_p$, $f \geq 0$ and $\lim_{|x| \rightarrow \infty} |x|^p f(x) = l$ exists, then

$$(2.3) \quad \lim_{|x| \rightarrow \infty} \sup_{t \geq 0} |x|^p P_t^0 f(x) = l.$$

The second is that if $f \in C^2$ and $f(\tilde{x}, 0+) = 0$, then

$$P_t^0 f(x) = x_d \partial_d (P_t^0 f)(\tilde{x}, 0+) + \frac{1}{2} x_d^2 P_t^0 (\partial_d^2 f)(\tilde{x}, \theta x_d)$$

for some $\theta \in (0, 1)$. This follows from that $P_t^0 f(\cdot, 0+) = 0$ if $t > 0$ and $f \in L^1(H, dx)$, and that $\partial_d^2 (P_t^0 f) = P_t^0 (\partial_d^2 f)$ if $f \in C^2$; $f(\cdot, 0+) = 0$. Thus we also see that D_p is stable under P_t^0 , i.e., $P_t^0(D_p) \subset D_p$.

From these facts we get the following:

$$(2.4) \quad \sup_{t \geq 0} \|g_p^{-1} P_t^0 g_p\|_\infty < \infty,$$

$$(2.5) \quad \sup_{t \geq 0} \|g_{p,0}^{-1} P_t^0 g_{p,0}\|_\infty < \infty$$

and for all $s > 0$, $P_s^0 g_p \in C_{p,0}$, hence

$$(2.6) \quad \sup_{s \leq t < \infty} \|g_{p,0}^{-1} P_t^0 g_p\|_\infty < \infty.$$

Proof of Theorem 2.1.

(i) is trivial. The integrabilities in (ii) are reduced to the following: For each $j, k \geq 0$ and $t > 0$,

$$(2.7) \quad \mathbf{E}_\mu \left[\langle X_t, g_p \rangle^k \right], \mathbf{E}_\mu \left[\left(\int_0^t \langle X_u, g_{p,0} \rangle^j \langle X_u, g_p \rangle du \right)^k \right] < \infty.$$

Lemma 2.10. *Let $p > d$. Fix $t > 0, \epsilon > 0$. Then, for each $k \geq 0$,*

$$\sup_{0 \leq a \leq \epsilon} \|g_{p,0}^{-1} \frac{\partial^k}{\partial a^k} V_t(ag_p)\|_\infty < \infty.$$

Proof. This is deduced to (2.6) by computing $(\partial^k / \partial a^k) V_t(ag_p)$. ■

From this lemma it is easy to see that if $\mu \in \overline{\mathcal{M}}_p$, then the Laplace transform

$$\begin{aligned} \varphi_t(a; f) &\equiv \mathbf{E}_\mu \left[e^{-a \langle X_t, f \rangle} \right] \\ &= \exp \left[-\langle \mu, V_t(af) \rangle - \int_0^t \langle \nu_r, 1 - e^{-af} \rangle dr \right] \end{aligned}$$

is C^∞ at $a = 0+$ if $t > 0$ and $f \in D_p$. Hence the first claim follows. For instance, we have for $f, g \in D_p$,

$$\mathbf{E}_\mu \langle X_t, f \rangle = \langle \mu, P_t^0 f \rangle + \int_0^t \langle \nu_r, f \rangle dr,$$

$$\mathbf{E}_\mu[\langle X_t, f \rangle \langle X_t, g \rangle] = \mathbf{E}_\mu \langle X_t, f \rangle \mathbf{E}_\mu \langle X_t, g \rangle + \langle \mu, P_t^0(fg) - P_t^0 f P_t^0 g \rangle + \int_0^t \langle \nu_r, fg \rangle dr.$$

More generally we can give the explicit formula of $\mathbf{E}_\mu[\langle X_t, f_1 \rangle \cdots \langle X_t, f_n \rangle]$. But it's too tedious to describe it. By using the formula one can easily see the following:

Proposition 2.11. *For all $0 \leq t_1 \leq \cdots \leq t_n$ and $f_i \in D_p; f_i \geq 0, i = 1, 2, \dots, n$,*

$$\begin{aligned} \mathbf{E}_\mu [\langle X_{t_1}, f_1 \rangle \cdots \langle X_{t_n}, f_n \rangle] &\leq \prod_{i=1}^n \langle \mu, P_{t_i}^0 f_i \rangle + C_1^{(n)} \sum_{i=1}^n \prod_{j \neq i} \langle \mu, P_{t_j}^0 f_j \rangle \\ &+ C_2^{(n)} \sum_{i_1 \neq i_2} \prod_{j \neq i_1, i_2} \langle \mu, P_{t_j}^0 f_j \rangle + \cdots + C_{n-1}^{(n)} \sum_{j=1}^n \langle \mu, P_{t_j}^0 f_j \rangle + C_n^{(n)}, \end{aligned}$$

where $C_k^{(n)}, k = 1, \dots, n$ are positive constants, independent of $\{t_i\}$.

Proof. In case of $t_1 = \cdots = t_n = t$ it is easily obtained. Then by using the Markov property we can get the general case, of course by induction on n . ■

To show the second claim in (2.7), we need the following lemma:

Lemma 2.12. *Let $\mu \in \overline{\mathcal{M}}_p$. For each $T > 0$, there exists a constant $C_T > 0$ such that*

$$\langle \mu, P_t^0 g_p \rangle \leq C_T / \sqrt{t} \quad (0 < t \leq T).$$

Proof. For simplicity we only consider the case of $d = 1$. Suppose that $\langle \mu, g_{p,0} \rangle < \infty$. It is clear that for all $\epsilon > 0$, $\sup_t \langle \mu, 1_{(\epsilon, \infty)} P_t^0 g_p \rangle < \infty$ by (2.3) and it is easy to see that

$$\sup_{t \geq 0, 0 < x \leq \epsilon} \frac{\sqrt{t}}{x} P_t^0 g_p(x) \leq \sqrt{\frac{2}{\pi}}.$$

The general cases $d \geq 2$ can be proved by a similar way. ■

Hence by (2.5) and Proposition 2.11 we have (2.7). Thus (ii) in Theorem 2.1 is proved. Moreover it is easily seen that

Lemma 2.13. *Let $\mu \in \overline{\mathcal{M}}_p$. Fix $T > 0$. For each $k \geq 1$,*

$$\mathbf{E}_\mu \left[\left(\int_0^t \langle X_u, g_p \rangle du \right)^k \right] \leq C_T t^{k/2} \quad (0 \leq t \leq T).$$

This lemma is used to show the Hölder continuity of X_t later.

REMARK 2.14. When $\mu \in \overline{\mathcal{M}}_p - \mathcal{M}_p$, it holds that

$$\mathbf{E}_\mu \left[\int_0^t \langle X_u, g_p \rangle^k du \right] \begin{cases} < \infty & \text{if } k = 1, \\ = \infty & \text{if } k \geq 2. \end{cases}$$

On the other hand if $\mu \in \mathcal{M}_p$, then for each $T > 0$ and $k \geq 1$, $\sup_{0 \leq t \leq T} \langle \mu, P_t^0 g_p \rangle$ is finite because of (2.4). Thus $C_T = \sup_{0 \leq t \leq T} \mathbf{E}_\mu[\langle X_t, g_p \rangle^k]$ is finite and it holds that

$$\begin{aligned} \mathbf{E}_\mu \left[\left(\int_0^t \langle X_u, g_p \rangle du \right)^k \right] &\leq t^{k-1} \int_0^t \mathbf{E}_\mu[\langle X_u, g_p \rangle^k] du \\ &\leq C_T t^k \quad (0 \leq t \leq T). \end{aligned}$$

Next we show the martingale property (iii) in Theorem 2.1.

Lemma 2.15.

- (i) $\sup_{t \geq 0} \|g_p^{-1} \frac{\partial}{\partial t} V_t f\|_\infty < \infty$ for each $f \in D_p$,
- (ii) $\sup_{0 < r \leq T} \langle \nu_r, g_{p,0} \rangle < \infty$ for each $T > 0$.

Proof. Since for $h = 1 - e^{-f}$ and $\Gamma f = \Gamma(f, f) = |\nabla_x f|^2$,

$$\frac{\partial}{\partial t} V_t f = \frac{P_t^0 A h}{1 - P_t^0 h} \left(= \frac{A P_t^0 h}{1 - P_t^0 h} = A V_t f - \frac{1}{2} \Gamma V_t f \right)$$

and $\|1 - P_t^0 h\|^{-1} \leq \exp[\|f\|_\infty] < \infty$, (i) is deduced to (2.4). (ii) follows from $\sup_{r > 0} \int_0^1 y \nu_r(y) dy = 1/2$ and $\int_1^\infty y^{-p} \nu_r(y) dy \leq \int_1^\infty \nu_r(y) dy \leq 1/\sqrt{2\pi r}$. $\blacksquare$

Let $f \in D_p$. Since

$$\lim_{r \downarrow 0} \langle \nu_r, 1 - e^{-f} \rangle = \frac{1}{2} \langle \widetilde{m}, \partial_d f(\cdot, 0+) \rangle,$$

if $\mu \in \mathcal{M}_p$, then

$$\begin{aligned} \mathcal{L} e^{-\langle \cdot, f \rangle}(\mu) &= - \left\{ \langle \mu, A f - \frac{1}{2} \Gamma f \rangle + \frac{1}{2} \langle \widetilde{m}, \partial_d f(\cdot, 0+) \rangle \right\} e^{-\langle \mu, f \rangle} \\ &= \lim_{t \downarrow 0} \frac{1}{t} \left\{ \mathbf{E}_\mu \left[e^{-\langle X_t, f \rangle} \right] - e^{-\langle \mu, f \rangle} \right\}. \end{aligned}$$

When $\mu \in \overline{\mathcal{M}}_p$, $\mathcal{L} e^{-\langle \cdot, f \rangle}(\mu)$ may not be well-defined. However it holds that $\mathbf{P}_\mu(X_u \in \mathcal{M}_p \text{ for } du\text{-a.a. } u > 0) = 1$ as mentioned in Remark 2.2. Hence by Lemma 2.15, noting that

$$\lim_{r \downarrow 0} \langle \nu_r, f^n \rangle = \begin{cases} \frac{1}{2} \langle \widetilde{m}, \partial_d f(\cdot, 0+) \rangle & (n = 1), \\ 0 & (n \geq 2), \end{cases}$$

if $\mu \in \overline{\mathcal{M}}_p$ and $\Phi \in \mathbf{D}_p$, then we can get

$$\frac{\partial}{\partial u} \mathbf{E}_\mu[\Phi(X_u)|X_s] = \mathbf{E}_\mu[\mathcal{L}\Phi(X_u)|X_s] \quad \text{for } du\text{-a.a. } u > s, \quad \mathbf{P}_\mu\text{-a.s.}$$

Therefore from this equation and by using the Markov property we can see that for each $\Phi \in \mathbf{D}_p$ and $0 \leq s < t$,

$$\mathbf{E}_\mu \left[\Phi(X_t) - \Phi(X_s) - \int_s^t \mathcal{L}\Phi(X_u) du \middle| \mathcal{F}_s \right] = 0.$$

That is, (iii) in Theorem 2.1 holds. In particular, we have

Lemma 2.16. *For each $f \in D_p$,*

$$\begin{aligned} M_t(f) &= \langle X_t, f \rangle - \langle X_0, f \rangle - \int_0^t \mathcal{L}\langle \cdot, f \rangle(X_u) du \\ &= \langle X_t, f \rangle - \langle X_0, f \rangle - \int_0^t \langle X_u, Af \rangle du - \frac{t}{2} \langle \widetilde{m}, \partial_d f(\cdot, 0+) \rangle \end{aligned}$$

is a square integrable $\mathbf{P}_\mu$ -martingale with quadratic variation

$$\langle\langle M(f) \rangle\rangle_t = \int_0^t \langle X_u, \Gamma f \rangle du.$$

Now we prove that X_t is continuous relative to the topology in $\overline{\mathcal{M}}_p$. We use an approximation by finite particle systems. Set $H_n = \{x \in H : |x| \leq n, x_d \geq 1/n\}$ for $n \geq 1$. Then $\mu(H_n) < \infty$ whenever $\mu \in \overline{\mathcal{M}}_p$. Fix $T > 0$ and for each $n \geq 1$, we set

$$W_n = \{w : 0 \leq \alpha(w) < T, \beta(w) - \alpha(w) \geq 1/n, |w(\alpha+)| \leq n\},$$

$$Q_m^{(n)} = Q_m|_{W_n} \quad \text{and} \quad Q_n^0 = Q^0(\cdot; \beta > 1/n)$$

Then $Q_m^{(n)} = \int_0^T \theta_{-s}(Q_n^0) ds$ and $Q_m^{(n)}(W) = T\nu_{1/n}(\{|x| \leq n\}) < \infty$ (note that $Q^0(\alpha \neq 0) = 0$). Now for each $n \geq 1$, we define

$$X^{(n)}(\omega) = \sum_{k:w_k(0) \in H_n} \delta_{w_k} + \sum_{k:w_k \in W_n} \delta_{w_k} \quad \text{if } \omega = \sum_k \delta_{w_k}.$$

Then under $\mathbf{P}_\mu$, $X_t^{(n)}$ is a finite Markov particle system, that is, $\mathbf{P}_\mu(X^{(n)}(W) < \infty) = 1$ and

$$\mathbf{E}_\mu \left[e^{-\langle X_{t+s}^{(n)}, f \rangle} \middle| \mathcal{F}_s \right] = \exp \left[-\langle X_s^{(n)}, V_t f \rangle - \int_0^t \langle Q_n^0, 1 - e^{-f_r} \rangle dr \right].$$

Clearly $X_t^{(n)}$ is continuous relative to the topology in $\overline{\mathcal{M}}_p$ and $X_t^{(n)}(\omega)$ converges to $X_t(\omega)$ as $n \rightarrow \infty$ for every ω and $t \geq 0$. Fix $f \in D_p$ and let $M_t^{(n)} = M_t^{(n)}(f)$, $M_t = M_t(f)$ be the

martingale part of $X_t^{(n)}, X_t$ respectively defined as in lemma 2.16. By Doob's maximal inequality and Lebesgue's dominated convergence theorem we see that

$$\begin{aligned} \mathbf{E}_\mu[\sup_{t \leq T} |M_t^{(n)} - M_t^{(m)}|^2] &\leq 4\mathbf{E}_\mu[|M_T^{(n)} - M_T^{(m)}|^2] \\ &\rightarrow 0 \quad (n, m \rightarrow \infty). \end{aligned}$$

Hence there is a continuous process $\widetilde{M}_t = \widetilde{M}_t(f)$ such that $\lim_{k \rightarrow \infty} \sup_{t \leq T} |M_t^{(n_k)} - \widetilde{M}_t| = 0, \mathbf{P}_\mu$ -a.s. for some suitable subsequence $\{M_t^{(n_k)}\}_{k=1}^\infty$. This implies $\mathbf{P}_\mu(M_t = \widetilde{M}_t \text{ for all } t \leq T) = 1$, that is, M_t is continuous. This implies the continuity of X_t in $\overline{\mathcal{M}}_p$.

Finally we prove the uniqueness of $\mathbf{P}_\mu$ on Ω_p for the martingale problem.

Lemma 2.17. *If $f \in C_c^\infty$, then $V_t f \in C^\infty \cap D_p$ for every $t \geq 0, p > d$, and the following hold: For each $t \geq 0$,*

- (i) $\frac{\partial}{\partial t} V_t f$ is continuous in t with respect to the norm $\|\cdot\|_{g_{p,0}}$,
- (ii) $\Gamma V_t f$ is continuous in t with respect to the norm $\|\cdot\|_{g_p}$.

Proof. Set $h = 1 - e^{-f} \in C_c^\infty$. Since $Ah \in C_c^\infty$,

$$\frac{\partial}{\partial t} V_t f = \frac{P_t^0 Ah}{1 - P_t^0 h} \quad \text{and} \quad \|1 - P_t^0 h\|_\infty \leq e^{\|f\|_\infty} < \infty,$$

(i) is reduced to the continuity of $P_t^0 f$ for $f \in C_c^\infty$, i.e., $\|(P_t^0 f - f)/g_{p,0}\|_\infty \rightarrow 0 \quad (t \rightarrow 0)$. To prove this it suffices to show that, for each $f \in C_c$,

$$\sup_{t \geq 0} \|g_{p,0}^{-1} P_t^0 f\|_\infty < \infty,$$

however it is not difficult. (ii) is reduced to the following: For each $f \in C_c^\infty$,

$$\|(P_t^0 f - P_{t_0}^0 f)/g_{p,0}\|_\infty, \|(\partial_i P_t^0 f - \partial_i P_{t_0}^0 f)/g_{p,0}\|_\infty \rightarrow 0 \quad i = 1, 2, \dots, d \quad (t \rightarrow t_0).$$

The second follows from the continuity of the transition semi-group $(P_t)_{t \geq 0}$ of Brownian motion in $\mathbf{R}^d$. Because if we extend f to on $\mathbf{R}^d$ by setting $f = 0$ on H^c , then

$$\partial_i P_t^0 f(x) = \begin{cases} P_t^0(\partial_i f)(x) & \text{for } i \neq d, \\ P_t(\partial_d f)(x) + P_t(\partial_d \bar{f})(x) & \text{for } i = d, \end{cases}$$

where $\bar{f}(x) = f(\tilde{x}, -x_d)$. One can more easily see that $\|(P_t f - f)/g_p\|_\infty \rightarrow 0 \quad (t \rightarrow 0)$ for each $f \in C_c(\mathbf{R}^d)$. ■

Let $\tilde{\mathbf{P}}_\mu$ be a probability measure on Ω_p , under which $X_0 = \mu$ a.s. and $\Phi(X_t) - \Phi(X_0) - \int_0^t \mathcal{L}\Phi(X_s)ds$ is a local martingale for $\Phi(\mu) = \langle X_t, f \rangle, \langle X_t, f \rangle^2 \quad (f \in D_p)$. We first assume that $\langle X_t, g_{p,0} \rangle, \int_0^t \langle X_s, g_p \rangle ds \in L^2(\tilde{\mathbf{P}}_\mu)$ for each $t > 0$ and that the above local martingale is a martingale. Then by using Lemma 2.17 carefully and noting $\langle \nu_{T-s}, 1 - e^{-f} \rangle = \langle \nu_r, V_{T-s} f \rangle_{r=0}$, we see that for each $f \in C_c^\infty$ and $T > 0$,

$$\langle X_t, V_{T-t}f \rangle - \langle X_0, V_Tf \rangle - \frac{1}{2} \int_0^t \langle X_s, \Gamma V_{T-s}f \rangle ds - \int_0^t \langle \nu_{T-s}, 1 - e^{-f} \rangle ds$$

is a square integrable continuous $\tilde{\mathbf{P}}_\mu$ -martingale with quadratic variation

$$\int_0^t \langle X_s, \Gamma V_{T-s}f \rangle ds$$

in $0 \leq t \leq T$. Hence by using Ito's formula we see that

$$\exp \left[-\langle X_t, V_{T-t}f \rangle + \int_0^t \langle \nu_{T-r}, 1 - e^{-f} \rangle dr \right]$$

is a $\tilde{\mathbf{P}}_\mu$ -martingale. Thus by taking $T = t$ we have

$$\tilde{\mathbf{E}}_\mu \left[e^{-\langle X_t, f \rangle} | \mathcal{F}_s \right] = \exp \left[-\langle X_s, V_{t-s}f \rangle - \int_0^{t-s} \langle \nu_u, 1 - e^{-f} \rangle du \right].$$

Therefore $\tilde{\mathbf{P}}_\mu = \mathbf{P}_\mu$ in the sense of finite-dimensional distributions, hence it holds on Ω_p , because Ω_p is Polish. In case of a local martingale by using a localization method we can also show $\tilde{\mathbf{P}}_\mu = \mathbf{P}_\mu$ on Ω_p .

Proof of Theorem 2.3.

We first show that $X_t, t \geq 0$ is locally γ -Hölder continuous under $\mathbf{P}_\mu$ in each cases of (i) $\mu \in \overline{\mathcal{M}}_p, \gamma \in (0, 1/4)$ and (ii) $\mu \in \mathcal{M}_p, \gamma \in (0, 1/2)$. We apply the following lemma which can be easily proved (cf. Theorem 2.2.8 in [9]):

Lemma 2.18. *Fix $T > 0$. Suppose that for each $f \in D_p$, there are some constants $a, b, C > 0$ such that*

$$\mathbf{E}_\mu [|\langle X_t - X_s, f \rangle|^a] \leq C(t-s)^{1+b} \quad \text{for all } 0 \leq s < t \leq T.$$

Then $X_t, 0 \leq t \leq T$ is locally $(b/a - \epsilon)$ -Hölder continuous for every $\epsilon \in (0, b/a)$ in the sense of (i) in Theorem 2.3.

Now fix $\mu \in \overline{\mathcal{M}}_p$. Then for each $k \geq 1, T > 0$ and $f \in D_p$ there is a constant $C_T = C_T(f) > 0$ such that

$$(2.8) \quad \mathbf{E}_\mu \left[\langle X_t - X_s, f \rangle^{2k} \right] \leq C_T(t-s)^{k/2}$$

for all $0 \leq s < t \leq T$. In fact, for any fixed $f \in D_p$ and $s \geq 0, N_t^s(f) = M_{t \vee s}(f) - M_s(f)$ is a continuous $(\mathcal{F}_t)$ -martingale with quadratic variation $\langle\langle N^s(f) \rangle\rangle_t = \langle\langle M(f) \rangle\rangle_{t \vee s} - \langle\langle M(f) \rangle\rangle_s$. Hence by the martingale moment inequality and Lemma 2.13 we have

$$\begin{aligned} \mathbf{E}_\mu \left[\{M_t(f) - M_s(f)\}^{2k} \right] &\leq C'_T \mathbf{E}_\mu \left[\{\langle\langle M(f) \rangle\rangle_t - \langle\langle M(f) \rangle\rangle_s\}^k \right] \\ &= C'_T \mathbf{E}_\mu \left[\left(\int_s^t \langle X_u, \Gamma f \rangle du \right)^k \right] \\ &\leq C_T(t-s)^{k/2} \quad \text{for } 0 \leq s < t \leq T \end{aligned}$$

Furthermore by $|Af| \leq Cg_{p,0}$ and Remark 2.14 we have

$$\mathbf{E}_\mu \left[\left(\int_s^t \langle X_u, |Af| \rangle du \right)^{2k} \right] \leq C_T (t-s)^{2k} \quad \text{for } 0 \leq s < t \leq T.$$

Therefore inequality (2.8) holds. Moreover if $\mu \in \mathcal{M}_p$, then it can be seen that

$$\mathbf{E}_\mu [\langle X_t - X_s, f \rangle^{2k}] \leq C_T (t-s)^k$$

for all $0 \leq s < t \leq T$.

Next we show (iii). Let $\mu \in \overline{\mathcal{M}}_p - \mathcal{M}_p$. Since for each $r > 0$, $\mathbf{P}_\mu(X_r \in \mathcal{M}_p) = 1$, it holds that, by (i) in Theorem 2.3, $X_t, t \geq 0$ is locally $(1/2 - \epsilon)$ -Hölder continuous for every $\epsilon \in (0, 1/2)$ under $\mathbf{P}_{X_r(\omega)}$ for $\mathbf{P}_\mu$ -a.a. ω . Thus letting $r \downarrow 0$, the claim follows.

Finally we prove (iv). It suffices to consider the martingale part $M_t(f) = \langle X_t, f \rangle - \langle X_0, f \rangle - \int_0^t \mathcal{L}(\cdot, f)(X_s) ds$. First let $\mu \in \overline{\mathcal{M}}_p$ and fix $f \in D_p$ such that $\langle \mu, \Gamma f \rangle > 0$. We also fix $f_0 \in C_{p,0}$ such that $0 \leq f_0 \leq \Gamma f$ and $\langle \mu, f_0 \rangle > 0$. By the continuity of $\langle X_t, f_0 \rangle$ there exist a.s. positive random variables C, u such that $\langle M(f) \rangle_t \geq \int_0^t \langle X_s, f_0 \rangle ds \geq Ct$ for all $0 \leq t < u$, $\mathbf{P}_\mu$ -a.s. By the time change for martingale, $M_t(f) = B_{\langle M(f) \rangle_t}$ holds a.s., where B_t is a standard one-dimensional Brownian motion. Since $B_t, t \geq 0$ is nowhere $1/2$ -Hölder continuous, $\langle X_t, f \rangle$ is not $1/2$ -Hölder continuous at $t = 0$. Next fix $f \in D_p$ such that $K = K_f \equiv \{\Gamma f = 0\}$ is compact in H . We also fix $f_0 \in C_{p,0}$ such that $0 \leq f_0 \leq \Gamma f$ and $f_0 > 0$ on K^c . Let $\mu \in \overline{\mathcal{M}}_p$ such that $\mu(\{x \geq a\}) = \infty$ for some $a > 0$ if $d = 1$, and that $\mu(K^c) \geq 1$ if $d \geq 2$. Then $\langle \mu, f_0 \rangle > 0$ for every $d \geq 1$, more strongly we have

Lemma 2.19. *Under the above conditions, it holds that*

$$I_T(f_0) \equiv \inf_{0 \leq t \leq T} \langle X_t, f_0 \rangle > 0 \quad \mathbf{P}_\mu\text{-a.s. for each } T > 0.$$

Proof. Recall that $X_t = X_t^0 + X_t^I$ and $\mathbf{P}_\mu = \mathbf{P}_\mu^0 \otimes \mathbf{P}^I$ (see §1). For $a > 0$, set $H_a = \{x_d \geq a\}$. Since

$$\mathbf{E}^I [e^{-X_t^I(H_a)}] = \exp \left[- \int_0^t \langle \nu_r, 1 - e^{-1_{H_a}} \rangle dr \right]$$

and $\nu_r(H_a) = \infty$ if and only if $d \geq 2$ for each $r > 0$, it holds that $\mathbf{P}^I(X_t^I(H_a) = \infty) = 1$ for every $t > 0, a > 0$ if $d \geq 2$. Moreover if $\mu \in \overline{\mathcal{M}}_p$ such that $\mu(H_a) = \infty$ for some $a > 0$, then

$$\mathbf{P}_\mu(X_t(H) = \infty \quad \text{for all } t \geq 0) = 1.$$

Because for each fixed $T > 0$, $\mathbf{P}_\mu^0(X_T^0(H) = 0) = (\otimes_{k \geq 1} P_{x_k}^0)(\zeta \leq T) = \prod_{k \geq 1} P(|B_T| \geq x_{k,d}) \leq \prod_{k: x_{k,d} \geq a} P(|B_T| \geq a) = 0$, where (B_t, P) is a standard one-dimensional Brownian motion and $x_{k,d}$ is d -th coordinate of x_k . By a similar way it can be easily seen that for each integer $n \geq 0$, $\mathbf{P}_\mu(X_T^0(H) = n) = 0$. Thus $\mathbf{P}_\mu(X_t(H) = \infty \quad \text{for all } t \leq T) \geq$

$\mathbf{P}_\mu(X_T^0(H) = \infty) = 1$. Since $T > 0$ is arbitrary, the above equation holds. Therefore under the given assumption on μ , we have

$$\begin{aligned} \mathbf{P}_\mu(X_t(H) = \infty \text{ for all } t > 0) &= \lim_{n \rightarrow \infty} \mathbf{P}_\mu(X_t(H) = \infty \text{ for all } t \geq 1/n) \\ &= \lim_{n \rightarrow \infty} \mathbf{E}_\mu[\mathbf{P}_{X_{1/n}}(X_t(H) = \infty \text{ for all } t \geq 0)] \\ &= 1. \end{aligned}$$

On the other hand $\mathbf{P}_\mu(X_t(K) < \infty \text{ for all } t \geq 0) = 1$ by the $\overline{\mathcal{M}}_p$ -continuity of X_t . Now if $I_T(f_0) = 0$, then by the continuity of $\langle X_t, f_0 \rangle$ and $\langle \mu, f_0 \rangle > 0$ we can find a number $t \in (0, T]$ such that $\langle X_t, f_0 \rangle = 0$, i.e., $X_t(K) = X_t(H)$. These facts imply that $\mathbf{P}_\mu(I_T(f_0) = 0) = 0$. $\blacksquare$

REMARK 2.20. When $d = 1$, the condition on μ in Lemma 2.19 can not be weakened to the same as in $d \geq 2$. For example if μ is finite or $\mu = \sum_{n \geq 1} \delta_{x_n}$ with $x_n = n^p$ ($p > 1$), then $\mathbf{P}_\mu(X_t(H) = 0) > 0$ for each $t > 0$, which implies $\mathbf{P}_\mu(I_T(f_0) = 0) > 0$ for every $T > 0$.

In virtue of this lemma, we see that $\langle X_t, f \rangle$ is nowhere $1/2$ -Hölder continuous in $t \in [0, \infty)$.

Proof of Theorem 2.4.

We shall prove (i). Suppose that $\mu \in \overline{\mathcal{M}}_p - \mathcal{M}_p$ satisfies the condition (2.1) with $T > 0, 1 \leq \theta < 2$. Then for each $a > 0$, there exists a small $0 < T' < 1$ such that

$$\begin{aligned} (2.9) \quad \langle \mu, P_t^0 g_p \rangle &\asymp \langle \mu, P_t^0 1_{S_a} \rangle \\ &\asymp \langle \mu|_{S_1}, P_t^0 1 \rangle \\ &\asymp \begin{cases} t^{-(\theta-1)/2} & (1 < \theta < 2), \\ \log 1/t & (\theta = 1), \end{cases} \end{aligned}$$

for all $0 < t \leq T'$. In fact by (2.3) we see that $\sup_t \langle \mu|_{S_1^c}, P_t^0 g_p \rangle$ and $\sup_t \langle \mu, P_t^0 (g_p 1_{S_a^c}) \rangle$ are finite for each $a > 0$. By (2.1),

$$\begin{aligned} \langle \mu|_{S_1}, P_t^0 g_p \rangle &\asymp \langle \mu|_{S_1}, P_t^0 1 \rangle \\ &\asymp \int_{S_1} \frac{dx}{x_d^\theta} P_t^0 1(x) \\ &= \int_0^1 \frac{dx_d}{x_d^\theta} \int_0^{x_d} \sqrt{\frac{2}{\pi t}} \exp\left[-\frac{y^2}{2t}\right] dy. \end{aligned}$$

When $1 < \theta < 2$, the right hand side is equal to

$$\begin{aligned} \frac{1}{\theta-1} \sqrt{\frac{2}{\pi t}} \int_0^1 dy (y^{1-\theta} - 1) \exp\left[-\frac{y^2}{2t}\right] &\asymp \int_0^{1/\sqrt{t}} (\sqrt{t}v)^{1-\theta} \exp\left[-\frac{v^2}{2}\right] dv \\ &\asymp t^{-(\theta-1)/2}. \end{aligned}$$

When $\theta = 1$, the asymptotic $\log(1/t)$ is similarly obtained. Hence by the same way as in the proof of Theorem 2.3, we can show the $((3-\theta)/4 - \epsilon)$ -Hölder continuity at $t = 0$ for every $\epsilon \in (0, (3-\theta)/4)$.

On the other hand the non-Hölder continuity is proved in the following manner: When $\theta = 1$, we've already shown the non $1/2$ -Hölder continuity. So it is enough to consider the case $1 < \theta < 2$. Fix $f \in D_p$ such that $\Gamma f(\tilde{0}, 0+) > 0$ and let $M_t = M_t(f)$ be the martingale part of $\langle X_t, f \rangle$. By the same way as in the proof of Proposition 2.11 if $0 \leq s < u$, then for $f_0 = \Gamma f$, we have

$$\mathbf{E}_\mu[\langle X_s, f_0 \rangle \langle X_u, f_0 \rangle] = \mathbf{E}_\mu \langle X_s, f_0 \rangle \mathbf{E}_\mu \langle X_u, f_0 \rangle + V(s, u)$$

with

$$\begin{aligned} V(s, u) &= \langle \mu, P_s^0(f_0 P_{u-s}^0 f_0) - (P_s^0 f_0)(P_u^0 f_0) \rangle \\ &\quad + \int_0^s \langle \nu_r, f_0 P_{u-s}^0 f_0 \rangle dr + \mathbf{E}_\mu \langle X_s, f_0 \rangle \int_0^{u-s} \langle \nu_r, f_0 \rangle dr. \end{aligned}$$

Hence for all $0 \leq t \leq T$,

$$\begin{aligned} \mathbf{E}_\mu[\langle M \rangle_t^2] - (\mathbf{E}_\mu \langle M \rangle_t)^2 &= 2 \int \int_{0 \leq s < u \leq t} V(s, u) ds du \\ &\leq C \int \int_{0 \leq s < u \leq t} \langle \mu, P_s^0(f_0 P_{u-s}^0 f_0) \rangle ds du \\ &\leq C' t \int_0^t \langle \mu, P_s^0 \Gamma f \rangle ds \\ &= C' t \mathbf{E}_\mu \langle M \rangle_t, \end{aligned}$$

where C, C' are constants, independent of $0 \leq t \leq T$. Moreover by (2.9) and $\Gamma f \geq c$ on S_a for some constants $a, c > 0$, we have

$$\begin{aligned} \mathbf{E}_\mu \langle M \rangle_t &\asymp \int_0^t \langle \mu, P_s^0 1_{S_a} \rangle ds \\ &\asymp t^{(3-\theta)/2}. \end{aligned}$$

Therefore for some $C_T > 0$,

$$\mathbf{E}_\mu \left[\left| \frac{\langle M \rangle_t}{\mathbf{E}_\mu \langle M \rangle_t} - 1 \right|^2 \right] \leq C_T \frac{t}{\mathbf{E}_\mu \langle M \rangle_t} \asymp t^{(\theta-1)/2} \quad (\rightarrow 0 \quad \text{as } t \downarrow 0).$$

Thus for a sequence $t_n = 1/2^n$, we have

$$\lim_{n \rightarrow \infty} \frac{\langle M \rangle_{t_n}}{\mathbf{E}_\mu \langle M \rangle_{t_n}} = 1 \quad \mathbf{P}_\mu\text{-a.s..}$$

Hence it is easily seen that

$$\liminf_{t \downarrow 0} \frac{\langle M \rangle_t}{\mathbf{E}_\mu \langle M \rangle_t} > 0 \quad \mathbf{P}_\mu\text{-a.s..}$$

Now by

$$\frac{|M_t|}{t^{(3-\theta)/4}} = \frac{|B_{\langle M \rangle_t}|}{\sqrt{\langle M \rangle_t}} \left(\frac{\langle M \rangle_t}{t^{(3-\theta)/2}} \right)^{1/2}$$

and the well-known result

$$\limsup_{t \downarrow 0} \frac{B_t}{\sqrt{t}} = \infty \quad \mathbf{P}_\mu\text{-a.s.},$$

it holds that

$$(2.10) \quad \limsup_{t \downarrow 0} \frac{|M_t|}{t^{(3-\theta)/4}} = \infty \quad \mathbf{P}_\mu\text{-a.s..}$$

Hence $M_t(f)$ is not $(3-\theta)/4$ -Hölder continuous at $t=0$, so is not $\langle X_t, f \rangle$.

Next we show (ii). Suppose that $\mu \in \overline{\mathcal{M}}_p - \mathcal{M}_p$ satisfies the condition (2.2) with $T > 0, 1 < \theta \leq 2$ and $\eta > 0$ ($1 < \theta < 2$) or $\eta > 1$ ($\theta = 2$). Then for $0 < a < 1$, there exists $0 < T' < 1$ such that

$$(2.11) \quad \begin{aligned} \langle \mu, P_t^0 g_p \rangle &\asymp \langle \mu, P_t^0 1_{S_a} \rangle \\ &\asymp \langle \mu|_{S_a}, P_t^0 1 \rangle \\ &\asymp \begin{cases} t^{-(\theta-1)/2} (\log 1/t)^{-\eta} & (1 < \theta < 2, \eta > 0), \\ t^{-1/2} (\log 1/t)^{-(\eta-1)} & (\theta = 2, \eta > 1), \end{cases} \end{aligned}$$

for all $0 < t \leq T'$. In fact in this case for a small $0 < a < 1$,

$$\langle \mu|_{S_a}, P_t^0 1 \rangle \asymp \int_0^a \frac{dx_d}{x_d^\theta (\log 1/x_d)^\eta} \int_0^{x_d} \sqrt{\frac{2}{\pi t}} \exp \left[-\frac{y^2}{2t} \right] dy.$$

Let $p = 1/(\theta - 1), q = \eta/(\theta - 1)$ and set $x = z^{-p}(\log z)^{-q}$. We have $p > 1, q > 0$ or $p = 1 < q$ and

$$\begin{aligned} dx &= -\frac{p \log z + q}{z^{p+1} (\log z)^{q+1}} dz \\ &\asymp -x^{1+1/p} (\log 1/x)^{q/p} dz, \end{aligned}$$

that is, $x^{-\theta} (\log 1/x)^{-\eta} dx \asymp -dz$. Hence for some $b > 1; a = b^{-p} (\log b)^{-q}$,

$$\langle \mu|_{S_a}, P_t^0 1 \rangle \asymp \int_b^\infty dz \int_0^{z^{-p} (\log z)^{-q}} \sqrt{\frac{2}{\pi t}} \exp \left[-\frac{y^2}{2t} \right] dy.$$

If $\sqrt{t} = z^{-p} (\log z)^{-q}$, then $z = h(t) \asymp t^{-1/(2p)} (\log 1/t)^{-q/p} = t^{-(\theta-1)/2} (\log 1/t)^{-\eta}$. We divide the integral area of z at $h(t)$. Note that $z \in (b, h(t))$ if and only if $z^{-p} (\log z)^{-q} > \sqrt{t}$. Thus it easy to see that

$$\frac{1}{\sqrt{t}} \int_b^{h(t)} dz \int_0^{z^{-p} (\log z)^{-q}} \exp \left[-\frac{y^2}{2t} \right] dy \asymp h(t)$$

and

$$\begin{aligned} \frac{1}{\sqrt{t}} \int_{h(t)}^\infty dz \int_0^{z^{-p} (\log z)^{-q}} \exp \left[-\frac{y^2}{2t} \right] dy &\asymp \frac{1}{\sqrt{t}} \int_{h(t)}^\infty \frac{dz}{z^p (\log z)^q} \\ &\asymp \frac{1}{\sqrt{t}} \int_{\log h(t)}^\infty \frac{1}{v^q} e^{-(p-1)v} dv. \end{aligned}$$

If $p > 1, q > 0$, i.e., $1 < \theta < 2, \eta > 0$, then the right hand side is equal to or less than

$$\begin{aligned} \frac{1}{\sqrt{t}(\log h(t))^q} \int_{\log h(t)}^{\infty} e^{-(p-1)v} dv &\asymp t^{-1/2}(\log h(t))^{-q} h(t)^{-(p-1)} \\ &\asymp h(t). \end{aligned}$$

If $p = 1 < q$, i.e., $\theta = 2, \eta = q > 1$, then the right hand side is equal to

$$\begin{aligned} \frac{1}{\sqrt{t}} \int_{\log h(t)}^{\infty} \frac{dv}{v^q} &\asymp t^{-1/2}(\log h(t))^{-(q-1)} \\ &\asymp t^{-1/2}(\log 1/t)^{-(\eta-1)}. \end{aligned}$$

Therefore (2.11) holds. Now noting that

$$\left(t^{(3-\theta)/2}(\log 1/t)^{-\eta}\right)' \asymp t^{-(\theta-1)/2}(\log 1/t)^{-\eta}$$

and

$$\left(\sqrt{t}(\log 1/t)^{-(\eta-1)}\right)' \asymp t^{-1/2}(\log 1/t)^{-(\eta-1)},$$

we have the following: For each $0 < a < 1$, there exists a constant $0 < T' < 1$ such that

$$\begin{aligned} \int_0^t \langle \mu, P_s^0 g_p \rangle ds &\asymp \int_0^t \langle \mu, P_s^0 1_{S_a} \rangle ds \\ &\asymp \int_0^t \langle \mu|_{S_a}, P_s^0 1 \rangle ds \\ &\asymp \begin{cases} t^{(3-\theta)/2}(\log 1/t)^{-\eta} & (1 < \theta < 2, \eta > 0), \\ t^{1/2}(\log 1/t)^{-(\eta-1)} & (\theta = 2, \eta > 1). \end{cases} \end{aligned}$$

We shall show that for each $f \in D_p$,

$$(2.12) \quad \limsup_{t \downarrow 0} \frac{|M_t(f)|}{t^{(3-\theta)/4}} = 0,$$

which implies the $(3-\theta)/4$ -Hölder continuity of X_t at $t = 0$. We first consider the case $1 < \theta < 2, \eta > 0$. For simplicity of notations we set $M_t = M_t(f), \gamma = (3-\theta)/4$. Fix an integer $N \geq 1$ such that $N\eta > 1$. By a martingale moment inequality we have

$$\begin{aligned} \mathbf{P}_{\mu}(\sup_{s \leq t} |M_s| \geq \epsilon) &\leq \frac{C}{\epsilon^{2N}} \mathbf{E}_{\mu}[\langle M \rangle_t^N] \\ &\leq \frac{C'}{\epsilon^{2N}} \frac{t^{2N\gamma}}{(\log 1/t)^{N\eta}} \end{aligned}$$

for all $\epsilon > 0$ and small $t > 0$, where C, C' are independent of (ϵ, t) . For each fixed $k, n \geq 1$, set $t = 1/2^n$ and $\epsilon = t^{\gamma}/k$. Then the right hand side is equal to $C'k^{2N}n^{-N\eta}(\log 2)^{-N\eta}$. This is the general term of a convergent series. Hence by Borel-Cantelli's lemma, for $\mathbf{P}_{\mu}$ -a.a. ω , it holds that for each $k \geq 1$, there is an integer-valued random variable $N(\omega)$ such that if $n \geq N(\omega)$, then $\sup_{s \leq 2^{-n}} |M_s(\omega)| \leq 2^{-n\gamma}k^{-1}$, that is,

$$\sup_{s \in (2^{-n-1}, 2^{-n}]} \frac{|M_s(\omega)|}{s^\gamma} \leq \frac{2^\gamma}{k}.$$

Therefore the equation (2.12) follows. The case of $\theta = 2, \eta > 1$ can be proved by the same way.

We can show the equations in Remark 2.5 by the same way as the proof of (2.10). These imply the non-Hölder continuity in (ii) of Theorem 2.4. .

Corollary 2.6 immediately follows from Theorem 2.1 and Theorem 2.3.

The proofs are complete.

2.4 Further Remarks

In this subsection we give some more properties of X_t^0 and X_t^I . Also we shall prove the infiniteness and unboundedness of $\langle X_t, g_p \rangle$.

(a) The results of Theorem 2.1, Theorem 2.3 (except the later half in (iv)) and Theorem 2.4 are also valid for the infinite Markov particle system X_t^0 with no immigration. Of course all terms related to the immigration part are deleted. Moreover if we assume $\mu(\{x_d \geq a\}) = \infty$ for some $a > 0$ and each $d \geq 1$, then the same result as the later half in (iv) of Theorem 2.3 is also valid, because this guarantee $\mathbf{P}_\mu^0(X_t^0(H) = \infty \text{ for all } t \geq 0) = 1$, and hence Lemma 2.19 holds.

(b) For any (short) time interval, infinitely many particles are born and die, i.e., for all $0 \leq s < t$, $\mathbf{P}^I(X^I(s \leq \alpha < \beta \leq t) = \infty) = 1$. In fact, in case of $d = 1$. Since

$$Q_m(s \leq \alpha < t) = \int_0^\infty Q^0(s \leq \alpha + u < t) du = (t - s)Q^0(W) = \infty,$$

we see that the number of particles which are born in the time interval $[s, t)$ is infinite, i.e., $\mathbf{P}^I(X^I(s \leq \alpha < t) = \infty) = 1$. Moreover

$$Q_m(s \leq \alpha < t < \beta) = \int_s^t Q^0(w(u) \in H) du = \sqrt{\frac{2}{\pi}}(\sqrt{t} - \sqrt{s}) < \infty$$

implies that the number of particles which are born after time s and survive until time t is finite. Thus our claim follows. In general case ($d \geq 2$), restrict on $\{|w(\alpha+)| \leq n\}$.

To prove the infiniteness of $\langle X_t, g_p \rangle$ mentioned in (ii) of Remark 2.2, it suffices to show the following result: Let $d = 1$.

Theorem 2.21. *For any $T > 0$, $\mathbf{P}^I(X_t^I((0, 1)) = \infty \text{ for some } t \in (0, T)) = 1$.*

REMARK 2.22. We can also show that for any $T > 0$, $\mathbf{P}^I(\sup_{t \in \mathbf{Q} \cap (0, T)} X_t^I((0, 1)) = \infty) = 1$.

Let $W_0 = W \cap \{\alpha = 0\}$ be the totality of excursions in $H = (0, \infty)$. $X_t^I((0, 1))$ can be expressed as

$$N_t = N^0(D_t) \quad \text{with} \quad D_t = \{(s, w) \in [0, \infty) \times W_0 : w(t-s) \in (0, 1), 0 \leq s < t\}.$$

For each fixed $k \geq 1$, we define a smaller process $S_{k,t}$ as follows: Let $a_k = 1/2^k$, $t_j^k = j/4^k$ ($j = 1, 2, \dots$) and

$$\xi^k = N^0(V^k)$$

with

$$V^k = \left\{ (s, w) : s \in [0, a_k^2), w(a_k^2 - s), w(2a_k^2 - s) \in [a_k, 2a_k), \right. \\ \left. w(3a_k^2 - s) = \Delta, \tau_{2a_k}(w) > \beta(w) \right\},$$

where $\tau_a(w)$ is the passage time to $a > 0$ of w . For each $j \geq 1$, if $t_j^k \leq t < t_{j+1}^k$, then set

$$\xi_t^k = N^0(V_j^k) \quad \text{with} \quad V_j^k = \theta_{-t_{j-1}^k}(V^k)$$

(note that ξ_t^k is undefined for $0 \leq t < t_1^k$). It holds that $\xi_t^k \stackrel{(d)}{=} \xi^k$. In particular, if we set $\xi_{t_j}^k = \xi_{t_j^k}^k$, then $\{\xi_{t_j}^k : j = 1, 2, \dots, k = 1, 2, \dots\}$ are independent.

REMARK 2.23. ξ^k denotes the number of particles which are born during the time interval $[0, a_k^2)$, stay in $[a_k, 2a_k)$ at time points $a_k^2, 2a_k^2$ and die during the time interval $(2a_k^2, 3a_k^2]$, and also which never hit $2a_k$.

Now for each $k \geq 1$, set $S_{k,t} = \sum_{n=k}^{\infty} \xi_t^n$. Clearly if $t \geq t_1^k (= a_k^2 = 1/4^k)$, then $S_{k,t} \leq N_t$. Hence to prove Theorem 2.21 it is enough to show the following proposition:

Proposition 2.24. *For each $k, i \geq 1$, $\mathbf{P}^I(S_{k,t} = \infty \text{ for some } t_1^k \leq t < t_{i+1}^k) = 1$.*

Proof. We define a random variable $U_k^{*,i}$ for each $k, i \geq 1$ as follows: Set

$$U_k^{1,i} = \max_{1 \leq j \leq i} \xi_{t_j}^k$$

and

$$I_1 = \{j = 1, 2, \dots, i : \xi_{t_j}^k = U_k^{1,i}\}, \\ J_1 = 4^k + [(4I_1 - 3) \cup (4I_1 - 2) \cup (4I_1 - 1) \cup 4I_1].$$

Also define

$$U_k^{2,i} = U_k^{1,i} + \max_{j \in J_1} \xi_{t_j}^{k+1}.$$

If we have $I_n, J_n, U_k^{n+1,i}$, then set

$$\begin{aligned} I_{n+1} &= \{j \in J_n : \xi_{t_j}^{k+n} = U_k^{n,i}\}, \\ J_{n+1} &= 4^{k+n} + [(4I_{n+1} - 3) \cup (4I_{n+1} - 2) \cup (4I_{n+1} - 1) \cup 4I_{n+1}] \end{aligned}$$

and

$$U_k^{n+2,i} = U_k^{n+1,i} + \max_{j \in J_{n+1}} \xi_{t_j}^{k+n+1}.$$

So we define

$$U_k^{*,i} = \lim_{n \rightarrow \infty} U_k^{n,i}.$$

We can show the following two claims:

Claim 1. $\lambda_k \equiv \mathbf{E}^I \xi^k = C/2^k$ for all $k \geq 1$ ($C = \mathbf{E}^I \xi^0 > 0$).

Claim 2. For each $k, i \geq 1$, $\mathbf{P}^I(U_k^{*,i} = \infty) = 1$.

Obviously Claim 2 implies Proposition 2.24.

Proof of Claim 1. Let V_s^k denote the s -section of V^k .

$$\begin{aligned} \lambda_k &= \mathbf{E}^I N^0(V^k) = \int_0^{a_k^2} ds Q^0(V_s^k) \\ &= \int_0^{a_k^2} ds \int_{a_k}^{2a_k} Q^0(w(a_k^2 - s) \in dy_1; a_k^2 - s < \tau_{2a_k}) \\ &\quad \int_{a_k}^{2a_k} P_{y_1}^0(w^0(a_k^2) \in dy_2; a_k^2 < T_{2a_k}) P_{y_2}^0(\zeta \leq a_k^2, \zeta < T_{2a_k}) \\ &= \int_0^{a_k^2} du \int_{a_k}^{2a_k} dy_1 \lim_{r \downarrow 0} \int_0^{a_k} dx \nu_r(x) p_u^{0,2a_k}(x, y_1) \\ &\quad \int_{a_k}^{2a_k} dy_2 p_{a_k^2}^{0,2a_k}(y_1, y_2) P_{y_2}(T_0 \leq a_k^2, T_0 < T_{2a_k}) \\ &= \int_{a_k}^{2a_k} dy_1 \int_{a_k}^{2a_k} dy_2 p_{a_k^2}^{0,2a_k}(y_1, y_2) \\ &\quad \int_0^{a_k^2} du \lim_{r \downarrow 0} \int_0^{a_k} dx \nu_r(x) p_u^{0,2a_k}(x, y_1) P_{y_2}(T_0 \leq a_k^2, T_0 < T_{2a_k}) \\ &= a_k^2 \int_1^2 dz_1 \int_1^2 dz_2 p_{a_k^2}^{0,2a_k}(a_k z_1, a_k z_2) \\ &\quad \int_0^{a_k^2} du \lim_{r \downarrow 0} \int_0^{a_k} dx \nu_r(x) p_u^{0,2a_k}(x, a_k z_1) P_{a_k z_2}(T_0 \leq a_k^2, T_0 < T_{2a_k}), \end{aligned}$$

where $p_t^{0,b}(x, y)$ denotes the transition density for Brownian motion absorbed at 0 and $b > 0$ ($x, y > 0$). Note that

$$\begin{aligned} p_t^{0,b}(x, y) &\left(\begin{aligned} &= P_x^0(w^0(t) = y; t < T_b) = P_x(B_t = y; t < T_0 \wedge T_b) \end{aligned} \right) \\ &= \sum_{n=-\infty}^{\infty} p_t^0(x, y + 2nb) \end{aligned}$$

and

$$P_x(T_0 \in dt; T_0 < T_b) = \frac{dt}{\sqrt{2\pi t^3}} \sum_{n=-\infty}^{\infty} (x + 2nb) \exp \left[-\frac{(x + 2nb)^2}{2t} \right].$$

By using the scaling properties; for any $a > 0$,

$$\nu_{a^2 s}(ay) = \frac{1}{a^2} \nu_s(y) \quad \text{and} \quad p_{a^2 v}^{0,2a}(ay, az) = \frac{1}{a} p_v^{0,2}(y, z),$$

we can get

$$\begin{aligned} \mathbf{E}^I \xi^k &= a_k \int_1^2 dz_1 \int_1^2 dz_2 p_1^{0,2}(z_1, z_2) \int_0^1 dv \lim_{s \downarrow 0} \int_0^1 dy \nu_s(y) p_v^{0,2}(y, z_1) P_{z_2}(T_0 \leq 1, T_0 < T_2) \\ &= a_k \mathbf{E}^I \xi^0. \end{aligned}$$

Proof of Claim 2. We shall show that for each $k, i \geq 1$ and $m \geq 0$, $\mathbf{P}^I(U_k^{*,i} \leq m) = 0$ by mathematical induction.

(1) For each $k, i \geq 1$, $\mathbf{P}^I(U_k^{*,i} = 0) = 0$.

In fact if $U_k^{*,i} = 0$, then $\xi_{t_j}^{k+n} = 0$ for all $n \geq 0, 4^n \leq j < 4^n(i+1)$. But the sum of these expectations is given as $i(\lambda_k + 4\lambda_{k+1} + 4^2\lambda_{k+2} + \dots)$ and this is infinity by Claim 1. Hence the probability of this event is 0.

(2) If we assume that $\mathbf{P}^I(U_k^{*,i} \leq m-1) = 0$ for all $k, i \geq 1$, then

$$\begin{aligned} \mathbf{P}^I(U_k^{*,i} \leq m) &= \sum_{m_k=0}^m \mathbf{P}^I(\xi^k = m_k)^i \mathbf{P}^I(U_{k+1}^{*,4i} \leq m - m_k) \\ &\quad + \sum_{j=1}^{i-1} \binom{i}{j} \sum_{m_k=1}^m \mathbf{P}^I(\xi^k = m_k)^j \mathbf{P}^I(\xi^k \leq m_k - 1)^{i-j} \mathbf{P}^I(U_{k+1}^{*,4j} \leq m - m_k) \\ &= \mathbf{P}^I(\xi^k = 0)^i \mathbf{P}^I(U_{k+1}^{*,4i} \leq m) \\ &= \mathbf{P}^I(\xi^k = 0)^i \mathbf{P}^I(\xi^{k+1} = 0)^{4i} \mathbf{P}^I(U_{k+2}^{*,4^2 i} \leq m) \\ &\leq \mathbf{P}^I(\xi^k = 0)^i \mathbf{P}^I(\xi^{k+1} = 0)^{4i} \dots \mathbf{P}^I(\xi^{k+n} = 0)^{4^n i} \rightarrow 0 \quad (n \rightarrow \infty) \end{aligned}$$

by Claim 1. This implies $\mathbf{P}^I(U_k^{*,i} \leq m) = 0$ for all $k, i \geq 1$.

(3) From the above results (1) and (2) we have Claim 2. ■

Proof of Remark 2.22. For each $k \geq 1$, set

$$S_k^* \equiv \lim_{n \rightarrow \infty} \max_{t_1^k \leq t < t_2^k} \sum_{j=k}^{k+n} \xi_t^j.$$

It holds that for all $m \geq 0$,

$$\mathbf{P}^I(S_k^* \leq m) = \sum_{m_k=0}^m \mathbf{P}^I(\xi^k = m_k) \mathbf{P}^I(S_{k+1}^* \leq m - m_k)^4.$$

Hence by the same way as the case of $U_k^{*,i}$ we can show $\mathbf{P}^I(S_k^* \leq m) = 0$ for all $m \geq 0$. This implies $\mathbf{P}^I(\sup_{t \in \mathbf{Q} \cap [t_1^k, t_2^k)} X_t^I((0, 1)) = \infty) = 1$ for all $k \geq 1$. ■

3 Asymptotic Behavior of Hitting Rates for Brownian Excursions; Applications to Equilibrium Processes

In this section we consider the Kuznetsov measure $Q_m(dw)$ and the canonical process $\{w(t)\}_{t \in \mathbf{R}}$ associated with absorbing Brownian motion $(w^0(t), P_x^0)$ in the half space H and the Lebesgue measure $m(dx) = dx$ on H . We give the asymptotic behavior of $Q_m(0 \leq \sigma_B < t)$ as $t \rightarrow \infty$ for the hitting time $\sigma_B(w) = \inf\{t > 0 : w(t) \in B\} (= \infty \text{ if } \{\cdot\} = \emptyset)$ of a compact subset B in H . Moreover by applying the result to the equilibrium process with immigration $(X_t, \mathbf{P})$ associated with $(w(t), Q_m)$, we also give limit theorems for it.

3.1 Main Results

Let B be a compact subset of H and $\pi_B(dx)$ be the capacitary measure of B , that is, π_B is supported on ∂B and satisfies that

$$P_x^0(T_B < \infty) = \int g^0(x, y) \pi_B(dy),$$

with the potential kernel

$$g^0(x, y) = \int_0^\infty p_t^0(x, y) dt,$$

where $p_t^0(x, y)$ is the transition density of $(w^0(t), P_x^0)$. The *capacity* is defined by

$$C^0(B) = \pi_B(1) = \int_{\partial B} \pi_B(dx).$$

Moreover if we set $\tau_B(w) = \inf\{t \in \mathbf{R} : w(t) \in B\} (= \infty \text{ if } \{\cdot\} = \emptyset)$, then

$$C^0(B) = Q_m(0 < \tau_B < 1) = Q^0(\sigma_B < \infty)$$

(see §3.3).

We shall use a symbol $Q^0[\cdot]$ as the integral by the measure Q^0 .

Our main result is the following:

Theorem 3.1. *Let B be a compact subset of H with a positive capacity. Then it holds that*

$$Q_m(0 \leq \sigma_B < t) = t C^0(B) + f(t)$$

with

$$f(t) = \begin{cases} \sqrt{\frac{2t}{\pi}} - \frac{2}{3}b + c + o(1) & \text{if } d = 1, \\ \log t \Phi(B) + o(\log t) & \text{if } d = 2, \\ O(1) & \text{if } d \geq 3 \end{cases}$$

as $t \rightarrow \infty$, where

$$b = \inf\{x : x \in B\} \quad \text{and} \quad c = \sup\{x : x \in B\} \quad \text{if } d = 1$$

and

$$\Phi(B) = \frac{1}{\pi} \int_{\partial B} x_2 \pi_B(dx) Q^0[w^{(2)}(\sigma_B) : \sigma_B < \infty] = \frac{1}{\pi} \left(\int_{\partial B} x_2 \pi_B(dx) \right)^2$$

with $w = (w^{(1)}, w^{(2)})$ if $d = 2$.

REMARK 3.2. In one-dimensional case we have more explicit forms; $g^0(x, y) = 2(x \wedge y)$, $\pi_B(dx) = \delta_b(dx)/(2b)$ and $C^0(B) = 1/(2b)$.

Let $(\{X_t\}_{t \in \mathbf{R}}, \mathbf{P})$ be the equilibrium process associated with $(w(t), Q_m)$. Set $N_t^B = \{w \in W : 0 \leq \sigma_B(w) < t\}$. Then $X(N_t^B)$ is the number of particles hitting B during time interval $[0, t)$. Now we have the following result:

Theorem 3.3. *Let B be a compact subset of H with a positive capacity. Then*

$$\frac{X(N_t^B)}{t} \rightarrow C^0(B) \quad \mathbf{P}\text{-a.s. and in } L^1(\mathbf{P})$$

as $t \rightarrow \infty$. Moreover

$$\frac{X(N_t^B) - tC^0(B)}{\sqrt{t}} \rightarrow \begin{cases} N(\sqrt{2/\pi}, C^0(B)) & \text{if } d = 1, \\ N(0, C^0(B)) & \text{if } d \geq 2 \end{cases}$$

in law as $t \rightarrow \infty$.

3.2 Kuznetsov Measures and Capacity Theory

In this subsection we consider the asymptotic of $Q_m(0 \leq \tau_B < t)$ in more general situation and give the first term of the asymptotic.

The situation is the same as in §2.2. That is, S is a Lusin space, $S_\Delta \equiv S \cup \{\Delta\} (\Delta \notin S)$, $(W^0, \mathcal{F}^0, \mathcal{F}_t^0, w^0(t), P_x^0)$, $t \geq 0$, $x \in S$, is a Borel right process with transition semi-group $(P_t^0)_{t \geq 0}$ and let W be the set of all maps $w : \mathbf{R} \rightarrow S_\Delta$ such that w is S -valued right continuous on some $(\alpha(w), \beta(w))$ and $w(t) = \Delta$ for $t \notin (\alpha(w), \beta(w))$.

We fix $m = m_i + m_p \in \mathbf{Exc}$. It is decomposed uniquely as $m = m_i + m_p \in \mathbf{Inv} \oplus \mathbf{Pur}$. Moreover there exists a unique entrance law $\nu = (\nu_t)_{t > 0}$ such that

$$m_p = \int_0^\infty \nu_t dt.$$

Then there are Kuznetsov measures

$$Q_m, Q_{m_i}, Q_{m_p} \quad \text{and} \quad Q^0 = Q_\nu \quad \text{on } W,$$

which satisfy that

$$Q_m = Q_{m_i} + Q_{m_p}, \quad \text{and} \quad Q_{m_p} = \int_{\mathbf{R}} \theta_{-s}(Q^0) ds = \int_{\mathbf{R}} \theta_s(Q^0) ds.$$

Note that Q_{m_i} is supported on $\{\alpha = -\infty\}$ and Q^0 is supported on $\{\alpha = 0\}$ ([2], [7]).

Let B be a Borel subset in S . Recall that $\tau_B(w) = \inf\{t \in \mathbf{R} : w(t) \in B\}$ and note that $\tau_B \circ \theta_s = \tau_B - s$ for all $s \in \mathbf{R}$. According to [7] B is *co-transient* if $Q_m(\tau_B = -\infty) = 0$, and for such B we define the following *co-capacities*:

$$\begin{aligned} \hat{C}(B) &= Q_m(0 < \tau_B < 1), \\ \hat{C}^i(B) &= Q_{m_i}(0 < \tau_B < 1), \\ \hat{C}^p(B) &= Q_{m_p}(0 < \tau_B < 1). \end{aligned}$$

Then

$$\hat{C}(B) = \hat{C}^i(B) + \hat{C}^p(B) \quad \text{and} \quad \hat{C}^p(B) = Q^0(\tau_B < \infty)$$

The last equation is due to

$$\begin{aligned} Q_{m_p}(0 < \tau_B < t) &= \int_{\mathbf{R}} ds \int_s^{s+t} Q^0(\tau_B \in du) \\ &= \int_{\mathbf{R}} Q^0(\tau_B \in du) \int_{u-t}^u ds \\ &= t Q^0(\tau_B < \infty). \end{aligned}$$

Now recall $\sigma_B(w) = \inf\{t > 0 : w(t) \in B\}$ and note that under Q^0 , $\tau_B = \sigma_B$ and $\tau_B \circ \theta_{-s} = \sigma_B \circ \theta_{-s} = \sigma_B + s$ for $s \geq 0$, but in general $\tau_B \circ \theta_s \neq \sigma_B \circ \theta_s$ for $s > 0$. We have the following result:

Theorem 3.4. *If B is a co-transient set with a positive and finite co-capacity, then*

$$Q_m(0 \leq \sigma_B < t) = t \hat{C}(B) + o(t) \quad \text{as } t \rightarrow \infty.$$

Proof. One can easily see that

$$Q_m(0 \leq \sigma_B < t) = Q_{m_i}(0 \leq \sigma_B < t) + Q_{m_p}(0 \leq \sigma_B < t),$$

$$Q_{m_i}(0 \leq \sigma_B < t) = P_{m_i}^0(T_B < t)$$

and

$$\begin{aligned} Q_{m_p}(0 \leq \sigma_B < t) &= \int_0^\infty Q^0(0 \leq \sigma_B \circ \theta_s < t) ds + \int_0^t Q^0(-s \leq \sigma_B < t-s) ds \\ &= \int_0^\infty dt \int_S \nu_s(dx) P_x^0(\sigma_B < t) + \int_0^t Q^0(\sigma_B < t-s) ds \\ &= P_{m_p}^0(T_B < t) + \int_0^t Q^0(\sigma_B < s) ds. \end{aligned}$$

That is,

$$(3.1) \quad Q_m(0 \leq \sigma_B < t) = P_m^0(T_B < t) + \int_0^t Q^0(\sigma_B < s) ds.$$

By Spitzer's formula ([7])

$$\lim_{t \rightarrow \infty} \frac{1}{t} P_m^0(T_B < t) = \widehat{C}^i(B),$$

and clearly

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t Q^0(\sigma_B < s) ds = Q^0(\sigma_B < \infty) = Q^0(\tau_B < \infty) = \widehat{C}^p(B).$$

Therefore our claim follows. ■

REMARK 3.5. According to [7] the capacity of B is defined by $C(B) = Q_m(0 < \lambda_B < 1)$, where $\lambda_B = \sup\{t \in \mathbf{R} : w(t) \in B\} (= -\infty \text{ if } \{\cdot\} = \emptyset)$ is the last exit time of B for $w(t)$. For a Borel set B in S , B is *transient* if $Q_m(\lambda_B = \infty) = 0$. If B is both transient and co-transient, then $C(B) = \widehat{C}(B)$. Let $L_B(w^0) = \sup\{t > 0 : w^0(t) \in B\} (= 0 \text{ if } \{\cdot\} = \emptyset)$ be the last exit time for w^0 . B is *strongly transient* if $P_m^0(L_B \geq \zeta) = 0$, where $\zeta(w^0)$ is the life time of w^0 . If (w^0, P_x^0) and $(\widehat{w}^0, \widehat{P}_x^0)$ are transient standard processes in weak duality relative to an excessive measure m and if B is strongly transient with finite capacity, then there is a measure

$$\pi_B(dx) = \lim_{t \downarrow 0} \frac{1}{t} P_x^0(0 < L_B \leq t) m(dx) = \lim_{t \downarrow 0} \frac{1}{t} P_x^0(T_B \leq t, T_B \circ \theta_t = \infty) m(dx) \quad \text{on } S$$

such that $P_x^0(T_B < \infty) m(dx) = \int \widehat{G}^0(y, dx) \pi_B(dy)$, where $\widehat{G}^0(y, dx) = \int_0^\infty dt \widehat{P}_t^0(y, dx)$ [4]. Moreover $Q_m(\lambda_B \in dt, w(\lambda_B -) \in dy) = dt \pi_B(dy)$ and hence $Q_m(0 < \lambda_B < 1) = \pi_B(1)$. If B is strongly co-transient then $Q_m(\tau_B \in dt, w(\tau_B) \in dy) = dt \widehat{\pi}_B(dy)$ and $Q_m(0 < \tau_B < 1) = \widehat{\pi}_B(1)$ hold [6].

3.3 Proof of Main Results

Recall that $H = \{x_d > 0\}$, $m(dx) = dx$ on H , $(w^0(t), P_x^0)$ is absorbing Brownian motion in H , $(P_t^0)_{t \geq 0}$ is its transition semi-group. Q^0 is a Brownian excursion law, Q_m is a Kuznetsov measure associated with $(w^0(t), P_x^0, m)$, and $\nu_r(dx) = \nu_r(x_d) dx$ with

$$\nu_r(u) = \frac{u}{\sqrt{2\pi r^3}} \exp\left[-\frac{u^2}{2r}\right]$$

for $r > 0, u > 0$. Here we give some useful results for the entrance law:

$$\lim_{r \downarrow 0} \int_\epsilon^\infty \nu_r(dx_d) = 0, \quad 2 \lim_{r \downarrow 0} \int_0^\epsilon x_d^n \nu_r(dx_d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n \geq 2 \end{cases}$$

for all $\epsilon > 0$, and

$$\int_H \nu_r(dx) p_t^0(x, y) = \int_0^\infty \nu_r(dx_d) p_t^0(x_d, y_d) = \nu_{r+t}(y_d),$$

where $\nu_r(dx_d) = \nu_r(x_d)dx_d$ on $(0, \infty)$.

Note that the facts in Remark 3.5 and that w^0 is symmetric relative to m , i.e., $w^0 = \hat{w}^0$. For each compact subset B of H , it is strongly transient, it holds that

$$C^0(B) = \pi_B(1) = Q_m(0 < \tau_B < 1) = Q^0(\sigma_B < \infty)$$

and this is finite (cf. Theorem 6.5.3 in [15]). We can show $\pi_B(1) = Q^0(\sigma_B < \infty)$ directly. In fact, since $\int_H p_t^0(x, y)\nu_r(dx) = \nu_{r+t}(y_d)$ and $\int_0^\infty \nu_r(y_d)dr = 1$, we have $Q^0(\sigma_B < \infty) = \lim_{r \downarrow 0} \int_H \nu_r(dx)P_x^0(T_B < \infty) = \lim_{r \downarrow 0} \int_{\partial B} \pi_B(dy) \int_H g^0(x, y)\nu_r(dx) = \pi_B(1)$. Moreover we see that

$$\begin{aligned} \pi_B(dy) &= \hat{\pi}_B(dy) = Q_m[w(\tau_B) \in dy : 0 < \tau_B < 1] \\ &= \int_0^\infty Q^0[w(\tau_B) \in dy : s < \tau_B < s+1]ds + \int_0^1 Q^0[w(\tau_B) \in dy : \tau_B < 1-s]ds \\ &= Q^0[w(\tau_B) \in dy : \tau_B < \infty]. \end{aligned}$$

By a simple computation one can get

$$\int_0^t Q^0(\sigma_B < s)ds = t Q^0(\sigma_B < \infty) - Q^0[\sigma_B \wedge t : \sigma_B < \infty].$$

Thus by the equation (3.1), noting that $m \in \mathbf{Pur}$, we have

$$(3.2) \quad Q_m(0 \leq \sigma_B < t) = t C^0(B) + P_m^0(T_B < t) - Q^0[\sigma_B \wedge t : \sigma_B < \infty].$$

Now Theorem 3.1 can be proved as follows: First we consider the one-dimensional case. In this case our claim is immediately obtained by the following proposition:

Proposition 3.6. *Let $d = 1$ and B be a non-empty compact subset of $H = (0, \infty)$. Set $b = \inf B$ and $c = \sup B$. Then*

$$P_m^0(T_B < t) = \sqrt{\frac{2t}{\pi}} - \frac{b}{2} + c + o(1)$$

as $t \rightarrow \infty$, and

$$Q^0[\sigma_B : \sigma_B < \infty] = \frac{b}{6}.$$

Proof. It is easy to see that

$$\int_0^\infty P_x^0(T_B < t)dx = \int_0^b P_x^0(T_b < t)dx + \int_b^c P_x^0(T_B < t)dx + \int_c^\infty P_x^0(T_c < t)dx.$$

When $t \rightarrow \infty$, the first term of the right hand side goes to

$$\int_0^b P_x^0(T_b < \infty)dx = \int_0^b \frac{x}{b}dx = \frac{b}{2}$$

and the second term goes to

$$\int_b^c P_x^0(T_B < \infty) dx = c - b.$$

The last term is equal to

$$\int_0^\infty P_x(T_0 < t) dx = \int_0^\infty 2P_0(B_t > x) dx = \sqrt{\frac{2t}{\pi}},$$

where (B_t, P_x) is Brownian motion in $\mathbf{R}$. Hence the first claim follows. To prove the second claim we need some well-known results (cf. [9]). If $0 \leq x \leq b$, then

$$E_x^0[e^{-\lambda T_b} : T_b < \infty] = E_x[e^{-\lambda T_b} : T_b < T_0] = \frac{\sinh(x\sqrt{2\lambda})}{\sinh(b\sqrt{2\lambda})}$$

and

$$P_x^0(T_b < \infty) = P_x(T_b < T_0) = x/b.$$

Thus for all $0 < \epsilon \leq b$ it holds that

$$\begin{aligned} \lim_{u \downarrow 0} \int_0^\epsilon \nu_u(dx) E_x^0[e^{-\lambda T_b} : T_b < \infty] &= \frac{\sqrt{\lambda}}{\sqrt{2} \sinh(b\sqrt{2\lambda})} \\ &= \left(b \int_{-1}^1 e^{b\sqrt{2\lambda}v} dv \right)^{-1} \end{aligned}$$

and

$$\lim_{u \downarrow 0} \int_0^\epsilon \nu_u(dx) P_x^0(T_b < \infty) = 1/(2b).$$

Now since $\lambda^{-1}(1 - e^{-\lambda\sigma_B}) = \lambda^{-2} \int_0^{\sigma_B} e^{-\lambda v} dv \nearrow \sigma_B$ as $\lambda \downarrow 0$ we have

$$\begin{aligned} Q^0[\sigma_B : \sigma_B < \infty] &= \lim_{\lambda \downarrow 0} \frac{1}{\lambda} Q^0[1 - e^{-\lambda\sigma_B} : \sigma_B < \infty] \\ &= \lim_{\lambda \downarrow 0} \frac{1}{\lambda} \lim_{u \downarrow 0} \int_0^b \nu_u(dx) E_x^0[1 - e^{-\lambda T_b} : T_b < \infty] \\ &= \lim_{\lambda \downarrow 0} \frac{1}{\lambda} \left\{ \frac{1}{2b} - \left(b \int_{-1}^1 e^{b\sqrt{2\lambda}v} dv \right)^{-1} \right\} \\ &= \frac{b}{6}. \end{aligned}$$

Therefore our claim follows. ■

We proceed with the proof of Theorem 3.1 in the higher dimensional case ($d \geq 2$). In this case we use a technique which is a kind of approximation by the capacitary measure. We further decompose the equation (3.2) to

$$\begin{aligned} Q_m(0 \leq \sigma_B < t) - t C^0(B) \\ = P_m^0(T_B < t) - t Q^0(t \leq \sigma_B < \infty) - Q^0[\sigma_B : \sigma_B < t]. \end{aligned}$$

Moreover

$$P_m^0(T_B < t) = \int_H dx \int_{\partial B} \pi_B(dy) \int_0^t p_s^0(x, y) ds + P_m^0(T_B < t, T_B \circ \theta_t < \infty),$$

$$Q^0(t \leq \sigma_B < \infty) = Q^0(\sigma_B \circ \theta_t < \infty) - Q^0(\sigma_B < t, \sigma_B \circ \theta_t < \infty)$$

and

$$Q^0[\sigma_B : \sigma_B < t] = \lim_{u \downarrow 0} \int_H \nu_u(dx) \int_{\partial B} \pi_B(dy) \int_0^t s p_s^0(x, y) ds + g(t)$$

with

$$g(t) = Q^0[\sigma_B : \sigma_B < t] - \lim_{u \downarrow 0} \int_H \nu_u(dx) \int_{\partial B} \pi_B(dy) \int_0^t s p_s^0(x, y) ds.$$

The second equality is shown as follows:

$$\begin{aligned} P_m^0(T_B < t) - \int_H dx \int_{\partial B} \pi_B(dy) \int_0^t p_s^0(x, y) ds \\ &= \int_H dx \left\{ P_x^0(T_B < t) - \int_{\partial B} \pi_B(dy) \int_0^t p_s^0(x, y) ds \right\} \\ &= \int_H dx \left\{ \int_{\partial B} \pi_B(dy) \int_t^\infty p_s^0(x, y) ds - P_x^0(t \leq T_B < \infty) \right\} \\ &= \int_H dx \left\{ P_x^0(T_B \circ \theta_t < \infty) - P_x^0(t \leq T_B < \infty) \right\} \\ &= P_m^0(T_B < t, T_B \circ \theta_t < \infty). \end{aligned}$$

Furthermore we see that

$$\begin{aligned} &\int_H dx \int_{\partial B} \pi_B(dy) \int_0^t p_s^0(x, y) ds \\ &= \int_{\partial B} \pi_B(dy) \int_0^t ds \int_0^\infty dr \int_H \nu_r(dx) p_s^0(x, y) \\ &= \int_{\partial B} \pi_B(dy) \int_0^t ds \int_s^\infty \nu_u(y_d) du \\ &= \int_{\partial B} \pi_B(dy) \left\{ t \int_t^\infty \nu_u(y_d) du + \int_0^t u \nu_u(y_d) du \right\}, \end{aligned}$$

$$\begin{aligned} Q^0(\sigma_B \circ \theta_t < \infty) &= \int_H \nu_t(dx) P_x^0(T_B < \infty) \\ &= \int_H \nu_t(dx) \int_{\partial B} \pi_B(dy) \int_0^\infty p_s^0(x, y) ds \\ &= \int_{\partial B} \pi_B(dy) \int_0^\infty ds \int_0^\infty \nu_t(dx_d) p_s^0(x_d, y_d) \\ &= \int_{\partial B} \pi_B(dy) \int_t^\infty \nu_u(y_d) du \\ &\left(= \sqrt{\frac{2}{\pi t}} \int_{\partial B} y_d \pi_B(dy) + O\left(\frac{1}{\sqrt{t^3}}\right) \right) \end{aligned}$$

and

$$\begin{aligned}
& \lim_{u \downarrow 0} \int_H \nu_u(dx) \int_{\partial B} \pi_B(dy) \int_0^t s p_s^0(x, y) ds \\
&= \int_{\partial B} \pi_B(dy) \int_0^t s \nu_s(y_d) ds \\
&\left(= \sqrt{\frac{2t}{\pi}} \int_{\partial B} y_d \pi_B(dy) - \int_{\partial B} y_d^2 \pi_B(dy) + o(1) \right)
\end{aligned}$$

Therefore we have

$$\begin{aligned}
& Q_m(0 \leq \sigma_B < t) - t C^0(B) \\
&= P_m^0(T_B < t, T_B \circ \theta_t < \infty) + t Q^0(\sigma_B < t, \sigma_B \circ \theta_t < \infty) - g(t).
\end{aligned}$$

Now we can prove that

$$(3.3) \quad P_m^0(T_B < t, T_B \circ \theta_t < \infty) = o(1),$$

$$(3.4) \quad t Q^0(\sigma_B < t, \sigma_B \circ \theta_t < \infty) = \begin{cases} \Phi(B) + o(1) & \text{if } d = 2, \\ O(1/\sqrt{t}) & \text{if } d \geq 3 \end{cases}$$

and

$$(3.5) \quad g(t) = \begin{cases} -\log t \Phi(B) + o(\log t) & \text{if } d = 2, \\ O(1) & \text{if } d \geq 3 \end{cases}$$

as $t \rightarrow \infty$. Hence our claim follows.

To prove the above equations (3.3), (3.4) and (3.5) we need several lemmas.

Lemma 3.7. *For each $h > 0$, $C_h \equiv \int_H y_d P_y^0(T_B < h) dy$ is finite and it holds that*

$$P_m^0(t \leq T_B < t + h) \leq C_h \sqrt{\frac{2}{\pi t}}, \quad Q^0(t \leq \sigma_B < t + h) \leq C_h \frac{1}{\sqrt{2\pi t^3}}$$

for all $t > 0$.

Proof. Since $w^0(t)$ is bounded on a finite time interval, for each $t > 0$, there is a bounded set K_t in H such that for all $x \in B$, $P_x^0(w^0(s) \in K_t)$ for all $s \in [0, t] > 1/2$. Hence for each $x \in H$, $P_x^0(w^0(t) \in K_t) \geq P_x^0(w^0(t) \in K_t, T_B < t) \geq E_x^0[P_{w^0(T_B)}^0(w^0(t-s) \in K_t) |_{s=T_B} : T_B < t] \geq P_x^0(T_B < t)/2$, that is, $P_x^0(T_B < t) \leq 2P_x^0(w^0(t) \in K_t)$. Therefore

$$\begin{aligned}
C_h &\leq 2 \int_H x_d P_x^0(w^0(h) \in K_h) dx \\
&= 2 \int_H dx x_d \int_{K_h} p_h^0(x, y) dy \\
&= 2 \int_{K_h} dy \int_0^\infty x_d p_h^0(x_d, y_d) dx_d \\
&= 2 \int_{K_h} y_d dy < \infty
\end{aligned}$$

and

$$\begin{aligned}
P_m^0(t \leq T_B < t+h) &\leq \int_H P_x^0(T_B \circ \theta_t < h) dx \\
&= \int_H dx \int_H dy p_t^0(x, y) P_y^0(T_B < h) \\
&= \int_H dy P_y^0(T_B < h) \int_0^\infty dx_d p_t^0(x_d, y_d) \\
&= \int_H dy P_y^0(T_B < h) \int_{-y_d}^{y_d} \frac{1}{\sqrt{2\pi t}} \exp\left[-\frac{z^2}{2t}\right] dz \\
&\leq \sqrt{\frac{2}{\pi t}} \int_H y_d P_y^0(T_B < h) dy.
\end{aligned}$$

On the other hand

$$\begin{aligned}
Q^0(t \leq \sigma_B < t+h) &\leq Q^0(\sigma_B \circ \theta_t < h) \\
&= \int_H \nu_t(dx) P_x^0(T_B < h) \\
&\leq \frac{1}{\sqrt{2\pi t^3}} \int_H x_d P_x^0(T_B < h) dx.
\end{aligned}$$

■

REMARK 3.8. From this lemma we see that $P_m^0(T_B < t) = O(\sqrt{t})$ as $t \rightarrow \infty$.

Lemma 3.9.

$$\int_t^\infty p_s^0(x, y) ds \begin{cases} \leq \frac{4x_d y_d}{(2\pi)^{d/2} dt^{d/2}} & \text{for all } t > 0, \\ = \frac{4x_d y_d}{(2\pi)^{d/2} dt^{d/2}} + O\left(\frac{1}{t^{d/2+1}}\right) & \text{as } t \rightarrow \infty, \end{cases}$$

where the $O(t^{-d/2-1})$ -constant is bounded whenever (x, y) is bounded.

Proof. Since

$$\int_t^\infty p_s^0(x, y) ds = \int_t^\infty ds (2\pi s)^{-d/2} \exp\left[-\frac{|\tilde{x} - \tilde{y}|^2}{2s}\right] \int_{(x_d - y_d)^2/(2s)}^{(x_d + y_d)^2/(2s)} e^{-z} dz,$$

where $\tilde{x} = (x_1, \dots, x_{d-1}) \in \mathbf{R}^{d-1}$, our first claim immediately follows. Moreover by using Taylor's formula

$$\exp\left[-\frac{|\tilde{x} - \tilde{y}|^2}{2s} - z\right] = 1 - \left(\frac{|\tilde{x} - \tilde{y}|^2}{2s} + z\right) e^{-\theta},$$

where $\theta > 0$ depends on (s, z, x, y) , we see that

$$\begin{aligned}
0 &\leq \frac{4x_dy_d}{(2\pi)^{d/2}dt^{d/2}} - \int_t^\infty p_s^0(x, y)ds \\
&\leq \int_t^\infty ds(2\pi s)^{-d/2} \int_{(x_d-y_d)^2/(2s)}^{(x_d+y_d)^2/(2s)} \left(\frac{|\tilde{x} - \tilde{y}|^2}{2s} + z \right) dz \\
&= t^{-d/2-1} \frac{2x_dy_d}{(2\pi)^{d/2}(d+2)} (|x-y|^2 + 2x_dy_d).
\end{aligned}$$

■

Using these lemmas we can prove the following:

Lemma 3.10. *Let $d \geq 2$.*

$$P_m^0(T_B < t, T_B \circ \theta_t < \infty) = \begin{cases} O\left(\frac{\log t}{\sqrt{t}}\right) & \text{if } d = 2, \\ O(1/\sqrt{t}) & \text{if } d \geq 3 \end{cases}$$

and

$$Q^0(\sigma_B < t, \sigma_B \circ \theta_t < \infty) = \begin{cases} \Phi(B)/t + o(1/t) & \text{if } d = 2, \\ O(1/\sqrt{t^3}) & \text{if } d \geq 3 \end{cases}$$

as $t \rightarrow \infty$.

Proof. We first prove the two-dimensional case.

$$\begin{aligned}
&P_m^0(T_B < t, T_B \circ \theta_t < \infty) \\
&\leq \int_H dx P_x^0(T_B < [t] - 1, T_B \circ \theta_t < \infty) + \int_H dx P_x^0([t] - 1 \leq T_B < t) \\
&= \int_H dx \int_H P_x^0(w^0(t) \in dy : T_B < [t] - 1) P_y^0(T_B < \infty) + O(1/\sqrt{t}) \quad \text{by Lemma 3.7} \\
&= \int_{\partial B} \pi_B(dz) \int_H dx \int_{v \in \partial B} \int_0^{[t]-1} d_s P_x^0(w^0(T_B) \in dv : T_B \leq s) \int_H P_{t-s}^0(v, dy) g^0(y, z) \\
&\quad + O(1/\sqrt{t}) \\
&= \int_{\partial B} \pi_B(dz) \int_H dx \int_{v \in \partial B} \int_0^{[t]-1} d_s P_x^0(w^0(T_B) \in dv : T_B \leq s) \int_{t-s}^\infty p_u^0(v, z) du \\
&\quad + O(1/\sqrt{t}).
\end{aligned}$$

By Lemma 3.9 the first term is equal or less than

$$\begin{aligned}
&\frac{1}{\pi} \int_{\partial B} \pi_B(dz) z_2 \int_H dx \int_{v \in \partial B} v_2 \int_0^{[t]-1} d_s P_x^0(w^0(T_B) \in dv : T_B \leq s) \frac{1}{t-s} \\
&\leq \frac{c^2}{\pi} C^0(B) \int_H dx \int_0^{[t]-1} \frac{P_x^0(T_B \in ds)}{t-s} \quad (c = \sup\{x_2 > 0 : x \in \partial B\}) \\
&= \frac{c^2}{\pi} C^0(B) \int_H dx \left\{ \frac{P_x^0(T_B \leq t)}{t} + \int_0^{[t]-1} \left(\frac{1}{t-s} - \frac{1}{t} \right) P_x^0(T_B \in ds) \right\} \\
&= \frac{c^2}{\pi} C^0(B) \left\{ \frac{P_m^0(T_B \leq t)}{t} + \int_0^{[t]-1} \frac{s}{(t-s)t} P_m^0(T_B \in ds) \right\}.
\end{aligned}$$

Now by using Lemma 3.7 the first term and the second term are equal to $O(1/\sqrt{t})$ and $O(\log t/\sqrt{t})$ as $t \rightarrow \infty$, respectively. In fact,

$$\begin{aligned}
\int_0^{[t]-1} \frac{s}{t-s} P_m^0(T_B \in ds) &\leq \sum_{k=1}^{[t]-1} \frac{k}{t-k} P_m^0(k-1 \leq T_B < k) \\
&= O\left(\sum_{k=1}^{[t]-1} \frac{k}{t-k} \cdot \frac{1}{\sqrt{k}}\right) \\
&= O\left(\int_1^{t/2} \frac{ds}{\sqrt{s}} + \sqrt{t} \int_1^{t/2} \frac{ds}{s}\right) \\
&= O(\sqrt{t} \log t).
\end{aligned}$$

Therefore

$$P_m^0(T_B < t, T_B \circ \theta_t < \infty) = O(\log t/\sqrt{t}) \quad \text{if } d = 2.$$

Moreover

$$\begin{aligned}
&Q^0(\sigma_B < t, \sigma_B \circ \theta_t < \infty) \\
&= Q^0(\sigma_B < [t] - 1, \sigma_B \circ \theta_t < \infty) + Q^0([t] - 1 \leq \sigma_B < t, \sigma_B \circ \theta_t < \infty) \\
&= \lim_{r \downarrow 0} \int_H \nu_r(dx) \int_{\partial B} \pi_B(dz) \int_{v \in \partial B} \int_0^{[t]-1} d_s P_x^0(w^0(T_B) \in dv : T_B \leq s) \int_{t-s}^\infty p_u^0(v, z) du \\
&\quad + O(1/\sqrt{t^3}).
\end{aligned}$$

Thus

$$\begin{aligned}
&Q^0(\sigma_B < t, \sigma_B \circ \theta_t < \infty) - \frac{1}{\pi t} \int_{\partial B} z_2 \pi_B(dz) \int_{\partial B} v_2 Q^0(w(\sigma_B) \in dv : \sigma_B \leq [t] - 1) \\
&= \lim_{r \downarrow 0} \int_H \nu_r(dx) \int_{\partial B} \pi_B(dz) \int_{v \in \partial B} \int_0^{[t]-1} d_s P_x^0(w^0(T_B) \in dv : T_B \leq s) \\
&\quad \times \left(\int_{t-s}^\infty p_u^0(v, z) du - \frac{z_2 v_2}{\pi t} \right) + O\left(\frac{1}{\sqrt{t^3}}\right).
\end{aligned}$$

Since by Lemma 3.9

$$\int_{t-s}^\infty p_u^0(v, z) du - \frac{z_2 v_2}{\pi t} = \frac{1}{\pi} z_2 v_2 \left(\frac{1}{t-s} - \frac{1}{t} \right) + O\left(\frac{1}{(t-s)^2}\right)$$

as $t \rightarrow \infty$. Hence we have

$$\begin{aligned}
&\left| Q^0(\sigma_B < t, \sigma_B \circ \theta_t < \infty) - \Phi(B)/t \right| \\
&\leq \frac{c^2}{\pi} C^0(B) \lim_{u \downarrow 0} \int_H \nu_u(dx) \int_0^{[t]-1} \left(\frac{1}{t-s} - \frac{1}{t} \right) P_x^0(T_B \in ds) \\
&\quad + C \lim_{u \downarrow 0} \int_H \nu_u(dx) \int_0^{[t]-1} (t-s)^{-2} P_x^0(T_B \in ds) + o\left(\frac{1}{t}\right),
\end{aligned}$$

where C is a finite constant. By using Lemma 3.7 we can also see that the first term is equal to $O(\log t/\sqrt{t^3})$ and the second term is to $O(1/\sqrt{t^3})$ as $t \rightarrow \infty$. Thus

$$Q^0(\sigma_B < t, \sigma_B \circ \theta_t < \infty) = \Phi(B)/t + o(1/t) \quad \text{if } d = 2.$$

In the higher dimensional cases ($d \geq 3$) by a similar way our claim immediately follows. In fact, by using Lemma 3.9 it is easy to see that

$$\int_0^{[t]-1} \left((t-s)^{-d/2} - t^{-d/2} \right) P_m^0(T_B \in ds) = O\left(\frac{1}{\sqrt{t}}\right)$$

and

$$\lim_{u \downarrow 0} \int_H \nu_u(dx) \int_0^{[t]-1} \left((t-s)^{-d/2} - t^{-d/2} \right) P_x^0(T_B \in ds) = O\left(\frac{1}{\sqrt{t^3}}\right)$$

as $t \rightarrow \infty$. From this we can get

$$P_m^0(T_B < t, T_B \circ \theta_t < \infty) = O\left(1/\sqrt{t}\right)$$

and

$$Q^0(\sigma_B < t, \sigma_B \circ \theta_t < \infty) = O\left(1/\sqrt{t^3}\right)$$

as $t \rightarrow \infty$. ■

From this lemma the equations (3.3) and (3.4) are obtained.

Note that in the above computations we also have the following result:

Theorem 3.11. *Let B be a compact subset of H with a positive capacity. Then*

$$P_m^0(T_B < t) = \sqrt{\frac{2t}{\pi}} \int_H 2x_d \pi_B(dx) + \begin{cases} -\frac{b}{2} + c + o(1) & \text{if } d = 1, \\ -\int_H x_d^2 \pi_B(dx) + o(1) & \text{if } d \geq 2 \end{cases}$$

and

$$Q^0(t \leq \sigma_B < \infty) + \begin{cases} o(1/t) & \text{if } d = 1, \\ \sqrt{\frac{2}{\pi t}} \int x_d \pi_B(dx) - \begin{cases} \Phi(B)/t + o(1/t) & \text{if } d = 2, \\ O(1/\sqrt{t^3}) & \text{if } d \geq 3 \end{cases} \end{cases}$$

as $t \rightarrow \infty$.

Note that if $d = 1$, then $\int_H 2x_d \pi_B(dx) = 1$ by Remark 3.2, and that $Q^0(\sigma_B < t) \rightarrow C^0(B)$ as $t \rightarrow \infty$. And if $d \geq 2$, then

$$\begin{aligned} P_m^0(T_B < t) &= \int \pi_B(dx) \int_0^t ds \int_0^\infty \nu_u(x_d) du + P_m^0(T_B < t, T_B \circ \theta_t < \infty), \\ Q^0(t \leq \sigma_B < \infty) &= \int \pi_B(dx) \int_t^\infty \nu_u(x_d) du - Q^0(\sigma_B < t, \sigma_B \circ \theta_t < \infty). \end{aligned}$$

We prove the equation (3.5). It is enough to show that for all $h > 0$

$$g(t+h) - g(t) = \begin{cases} -\frac{h}{t}\Phi(B) + o\left(\frac{h}{t}\right) & \text{if } d = 2, \\ O\left(\frac{h}{\sqrt{t^3}}\right) & \text{if } d \geq 3 \end{cases}$$

as $t \rightarrow \infty$. This can be proved by a similar way to the proof of Lemma 3.10. Since

$$P_x^0(T_B < t) - \int_{\partial B} \pi_B(dy) \int_0^t p_s^0(x, y) ds = P_x^0(T_B < t, T_B \circ \theta_t < \infty),$$

we have

$$\begin{aligned} g(t+h) - g(t) &= \lim_{u \downarrow 0} \int_H \nu_u(dx) \left\{ E_x^0[T_B : t \leq T_B < t+h] - \int_{\partial B} \pi_B(dy) \int_t^{t+h} p_s^0(x, y) ds \right\} \\ &\leq \lim_{u \downarrow 0} \int_H \nu_u(dx) \left\{ (t+h)P_x^0(t \leq T_B < t+h) - t \int_{\partial B} \pi_B(dy) \int_t^{t+h} p_s^0(x, y) ds \right\} \\ &= t \lim_{u \downarrow 0} \int_H \nu_u(dx) \left\{ P_x^0(t \leq T_B < t+h) - \int_{\partial B} \pi_B(dy) \int_t^{t+h} p_s^0(x, y) ds \right\} \\ &\quad + h \lim_{u \downarrow 0} \int_H \nu_u(dx) P_x^0(t \leq T_B < t+h) \\ &= t \lim_{u \downarrow 0} \int_H \nu_u(dx) \left\{ P_x^0(T_B < t+h, T_B \circ \theta_{t+h} < \infty) - P_x^0(T_B < t, T_B \circ \theta_t < \infty) \right\} \\ &\quad + h Q^0(t \leq \sigma_B < t+h) \\ &= t \left\{ Q^0(\sigma_B < t+h, \sigma_B \circ \theta_{t+h} < \infty) - Q^0(\sigma_B < t, \sigma_B \circ \theta_t < \infty) \right\} \\ &\quad + h Q^0(t \leq \sigma_B < t+h) \\ &= \begin{cases} -\frac{h}{t}\Phi(B) + o\left(\frac{h}{t}\right) & \text{if } d = 2, \\ O\left(\frac{h}{\sqrt{t^3}}\right) & \text{if } d \geq 3 \end{cases} \end{aligned}$$

as $t \rightarrow \infty$ by Lemma 3.7 and Lemma 3.10. A lower estimate is given by the same way:

$$\begin{aligned} g(t+h) - g(t) &\geq t \left\{ Q^0(\sigma_B < t+h, \sigma_B \circ \theta_{t+h} < \infty) - Q^0(\sigma_B < t, \sigma_B \circ \theta_t < \infty) \right\} \\ &\quad - h \left\{ Q^0(\sigma_B \circ \theta_t < \infty) - Q^0(\sigma_B \circ \theta_{t+h} < \infty) \right\} \\ &= \begin{cases} -\frac{h}{t}\Phi(B) + o\left(\frac{h}{t}\right) & \text{if } d = 2, \\ O\left(\frac{h}{\sqrt{t^3}}\right) & \text{if } d \geq 3 \end{cases} \end{aligned}$$

as $t \rightarrow \infty$. Therefore the equation (3.5) follows.

Finally we prove Theorem 3.3. Recall $N_t^B = \{0 \leq \sigma_B < t\}$. The first claim follows by the subadditive ergodic theorem [10] and $\mathbf{E}[X(N_t^B)] = Q_m(N_t^B)$. For the second claim it is enough to show that if

$$Q_m(N_t^B) = t C_1 + \sqrt{t} C_2 + o(\sqrt{t})$$

as $t \rightarrow \infty$, where $C_1 > 0$ and $C_2 \geq 0$, then

$$\mathbf{E} \exp \left[iz \frac{X(N_t^B) - tC_1}{\sqrt{t}} \right] \rightarrow \exp \left[iC_2 z - \frac{C_1}{2} z^2 \right]$$

as $t \rightarrow \infty$. However this can be easily obtained by using the following formula:

$$\mathbf{E} \exp[iz \langle X, F \rangle] = \exp \left\{ - \int_W (1 - e^{izF(w)}) Q_m(dw) \right\}$$

for all L^1 -functions $F(w)$ on (W, Q_m) . In fact, as $t \rightarrow \infty$,

$$- \int_W \left(1 - \exp \left[\frac{iz}{\sqrt{t}} 1_{N_t^B} \right] \right) Q_m(dw) - iz\sqrt{t}C_1 \rightarrow izC_2 - \frac{z^2}{2}C_1.$$

3.4 Further Results

The Brownian excursion law Q^0 can be also defined by the following: For $x = (\tilde{x}, x_d) \in \mathbf{R}^{d-1} \times (0, \infty) = H$,

$$Q_x^0 = \lim_{r \downarrow 0} \int_0^\infty \nu_r(dx_d) P_x^0 \quad \text{and} \quad Q^0 = \int_{\mathbf{R}^{d-1}} d\tilde{x} Q_{\tilde{x}}^0.$$

So in this final subsection we change the Lebesgue measure $d\tilde{x}$ on $\mathbf{R}^{d-1} \cong \partial H$, the boundary of H , to an arbitrary σ -finite measure $\lambda(d\tilde{x})$ and consider the same problem as above. Let $p_r^{(d-1)}(\tilde{y}, \tilde{x})$ denote the transition density of the $(d-1)$ -dimensional Brownian motion. Define

$$Q_\lambda^0 = \int_{\mathbf{R}^{d-1}} \lambda(d\tilde{x}) Q_{\tilde{x}}^0 = \lim_{r \downarrow 0} \int_H \nu_r^\lambda(dx) P_x^0,$$

where $\nu_r^\lambda(dx) = \nu_r^\lambda(x)dx$ with

$$\nu_r^\lambda(x) = \int_{\mathbf{R}^{d-1}} p_r^{(d-1)}(\tilde{y}, \tilde{x}) \lambda(d\tilde{y}) \nu_r(x_d).$$

Then $\nu^\lambda = (\nu_r^\lambda)_{r>0}$ is an entrance law for $(P_t^0)_{t \geq 0}$, i.e., $\nu_r^\lambda P_t^0 = \nu_{r+t}^\lambda$. Moreover set

$$m^\lambda(dx) = \int_0^\infty dr \nu_r^\lambda(dx) \quad \text{and} \quad Q_{m^\lambda} = \int_{-\infty}^\infty \theta_{-s} (Q_\lambda^0) ds,$$

Note that $m^\lambda(dx) = m^\lambda(x)dx$ with

$$m^\lambda(x) = \int_0^\infty \nu_r^\lambda(x) dr = \frac{\Gamma(d/2)}{(2\pi)^{d/2}} x_d \int_{\mathbf{R}^{d-1}} \left(\frac{2}{|\tilde{x} - \tilde{y}|^2 + x_d^2} \right)^{d/2} \lambda(d\tilde{y}),$$

where Γ is gamma function. Then $m^\lambda \in \mathbf{Pur}$ and Q_{m^λ} is the Kuznetsov measure associated with $(w^0(t), P_x^0, m^\lambda)$. For this measure m^λ , we have the following: The transition density $p_t^{m^\lambda}(x, y) = p_t^0(x, y)/m^\lambda(x)$, the capacitary measure $\pi_B^{m^\lambda}(dx) = m^\lambda(x)\pi_B(dx)$, the capacity $C^{m^\lambda}(B) = \pi^{m^\lambda}(1)$, the co-capacitary measure $\hat{\pi}_B^{m^\lambda}(dx) = Q_{m^\lambda}(w(\tau_B) \in dx : 0 < \tau_B < 1) = Q_\lambda^0(w(\sigma_B) \in dx : \sigma_B < \infty)$ and the co-capacity $\hat{C}^{m^\lambda}(B) = Q_{m^\lambda}(0 < \tau_B < 1) = Q_\lambda^0(\sigma_B < \infty)$. In this case w^0 is not symmetric relative to m^λ , thus in general $C^{m^\lambda}(B) \neq \hat{C}^{m^\lambda}(B)$.

Now fix a compact subset B of H with positive co-capacity. By the same computation as in §3.3, we can get

$$\begin{aligned} & Q_{m^\lambda}(0 \leq \sigma_B < t) - t \hat{C}^{m^\lambda}(B) \\ &= P_{m^\lambda}^0(T_B < t) - t Q_\lambda^0(t \leq \sigma_B < \infty) - Q_\lambda^0[\sigma_B : \sigma_B < t] \\ &= P_{m^\lambda}^0(T_B < t, T_B \circ \theta_t < \infty) + t Q_\lambda^0(\sigma_B < t, \sigma_B \circ \theta_t < \infty) - g_\lambda(t), \end{aligned}$$

where

$$g_\lambda(t) = Q_\lambda^0[\sigma_B : \sigma_B < t] - \lim_{r \downarrow 0} \int_H \nu_r^\lambda(dx) \int_{\partial B} \pi_B(dy) \int_0^t p_s^0(x, y) ds.$$

We only consider the case that $\lambda(d\tilde{x}) = \delta_{x^{(k-1)}(0)}(dx^{(k-1)}) dx^{(d-k)}$ for a certain $k = 1, 2, \dots, d$ and a certain fixed point $x^{(k-1)}(0) \in \mathbf{R}^{k-1}$. Then we have the following:

$$\nu_r^\lambda(x) = \frac{x_d}{(2\pi)^{k/2} r^{1+k/2}} \exp \left[-\frac{|x^{(k-1)} - x^{(k-1)}(0)|^2 + x_d^2}{2r} \right]$$

and

$$m^\lambda(x) = \frac{\Gamma(k/2)}{\pi^{k/2}} \frac{x_d}{\{|x^{(k-1)} - x^{(k-1)}(0)|^2 + x_d^2\}^{k/2}}.$$

Note that if $k = 1$, then $\nu_r^\lambda = \nu_r$ and $m^\lambda(x) = 1$. So it suffices to consider the case of $2 \leq k \leq d$. In this case it can be seen that

$$P_{m^\lambda}^0(T_B < t, T_B \circ \theta_t < \infty) = \begin{cases} O\left(\frac{\log t}{t}\right) & (k = d = 2), \\ O(1/t) & (k = 2, d \geq 3), \\ O(1/t^{k/2}) & (3 \leq k \leq d) \end{cases}$$

and

$$Q_\lambda^0(\sigma_B < t, \sigma_B \circ \theta_t < \infty) = \begin{cases} \Phi_\lambda(B)/t + o(1/t) & (d = 2), \\ O(1/\sqrt{t^3}) & (d = 3), \\ O(1/t^2) & (d \geq 4), \end{cases}$$

where

$$\begin{aligned} \Phi_\lambda(B) &= \frac{1}{\pi} \int_{\partial B} x_2 \pi_B(dx) Q_\lambda^0[w^{(2)}(\sigma_B) : \sigma_B < \infty] \\ &= \frac{1}{\pi} \int_{\partial B} x_2 m^\lambda(x)^{-1} \pi_B^{m^\lambda}(dx) \int_{\partial B} y_2 \hat{\pi}_B^{m^\lambda}(dy). \end{aligned}$$

More exactly when $d \geq 4$, we have

$$Q_\lambda^0(\sigma_B < t, \sigma_B \circ \theta_t < \infty) = \begin{cases} O\left(1/t^{1+k/2}\right) & (k \leq d-3), \\ O\left(1/t^{d/2}\right) & (k = d-2, d-1, d). \end{cases}$$

Furthermore

$$g_\lambda(t) = \begin{cases} -\log t \Phi_\lambda(B) + o(\log t) & (d=2), \\ O(1) & (d \geq 3) \end{cases}$$

as $t \rightarrow \infty$. Therefore we have the following result: For each $2 \leq k \leq d$,

$$Q_{m^\lambda}(0 \leq \sigma_B < t) = t \hat{C}^{m^\lambda}(B) + \begin{cases} \log t \Phi_\lambda(B) + o(\log t) & (d=2), \\ O(1) & (d \geq 3) \end{cases}$$

and $P_{m^\lambda}^0(T_B < t)$ is equal to

$$\begin{cases} \log t \int \frac{x_d}{(2\pi)^{k/2}} \pi_B(dx) + O(1) & (k=2), \\ \frac{\Gamma(k/2-1)}{2\pi^{k/2}} \int x_d \left\{ |x^{(k-1)} - x^{(k-1)}(0)|^2 + x_d^2 \right\}^{1-k/2} \pi_B(dx) + o(1) & (k \geq 3) \end{cases}$$

as $t \rightarrow \infty$. Moreover since

$$\begin{aligned} Q_\lambda^0(\sigma_B \circ \theta_t < \infty) &= \frac{2t^{-k/2}}{k(2\pi)^{k/2}} \int x_d \pi_B(dx) \\ &\quad - \frac{2t^{-k/2-1}}{(k+2)(2\pi)^{k/2}} \int x_d \left\{ |x^{(k-1)} - x^{(k-1)}(0)|^2 + x_d^2 \right\} \pi_B(dx) + O\left(t^{-k/2-2}\right), \end{aligned}$$

we can get the asymptotic behavior of

$$Q_\lambda^0(t \leq \sigma_B < \infty) = Q_\lambda^0(\sigma_B \circ \theta_t < \infty) - Q_\lambda^0(\sigma_B < t, \sigma_B \circ \theta_t < \infty).$$

However it is too tedious to describe it.

Of course we can also apply the above result to the equilibrium process with immigration and obtain a similar result to Theorem 3.3.

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CHAPTER III

Existence and Smoothness of Transition Densities for Jump-type Markov Processes; Applications of Malliavin Calculus

1 Introduction

Let X_t be a d -dimensional stable Lévy process with exponent $0 < \alpha < 2$, defined on a probability space $(\Omega, \mathcal{F}, P)$. Then its Lévy measure is given by

$$(1.1) \quad K(A) = \int_{\mathbf{S}^{d-1}} M(d\sigma) \int_0^\infty 1_{A \setminus \{0\}}(u\sigma) u^{-1-\alpha} du \quad (A \in \mathcal{B}(\mathbf{R}^d)),$$

where $M(d\sigma)$ is a bounded Borel measure on a $(d-1)$ -dimensional unit sphere $\mathbf{S}^{d-1}$. For each $t > 0$, denote by $P_t(dx)$, the distribution of X_t . It is well-known that under the following non-degeneracy condition on $M(d\sigma)$:

$$(1.2) \quad M(\{\sigma : x \cdot \sigma \neq 0\}) > 0 \quad \text{for every } x \in \mathbf{R}^d \setminus \{0\},$$

$P_t(dx)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbf{R}^d$, and moreover its density $p_t(x)$ is a C^∞ -function of $x \in \mathbf{R}^d$ for all $t > 0$ (cf. [7]).

Next let Y_t be a d -dimensional Lévy process of class L (simply called an L -process). Then its Lévy measure is expressed by

$$K(A) = \int_{\mathbf{S}^{d-1}} M(d\sigma) \int_0^\infty 1_{A \setminus \{0\}}(u\sigma) u^{-1} k(u, \sigma) du,$$

where $M(d\sigma)$ is a bounded Borel measure on $\mathbf{S}^{d-1}$ and $k(u, \sigma)$ is a non-negative measurable function on $(0, \infty) \times \mathbf{S}^{d-1}$, non-increasing, right continuous in u satisfying that $0 < k(u, \sigma) \leq \infty$ for each $\sigma \in \mathbf{S}^{d-1}$ and $\int_{\mathbf{S}^{d-1}} M(d\sigma) \int_0^\infty (u \wedge u^{-1}) k(u, \sigma) du < \infty$. Then the distribution $P_t(dx)$ of Y_t has a density $p_t(x)$ with respect to the Lebesgue measure on $\mathbf{R}^d$ (henceforth we simply say “a density $p_t(x)$ ”) for $t > 0$ under the condition (1.2) ([8]). Moreover for any $r \geq 0$, $p_t(x)$ can be chosen as a C^r -function of $x \in \mathbf{R}^d$, provided t is larger than some constant depending on $(M(d\sigma), k(0+, \sigma), d, r)$ (cf. [7], [10]).

On the other hand Bass [1] discussed the martingale problem for the Lévy generator L :

$$(1.3) \quad L\varphi(x) = \nabla_x \varphi(x) \cdot a(x) + \int [\varphi(x+y) - \varphi(x) - \nabla_x \varphi(x) \cdot y 1_{(|y|<1)}] K(x, dy)$$

and proved the well-posedness of the martingale problem for a fairly wide class of the Lévy generator L . In fact, his class includes the following case:

$$(1.4) \quad L = -(-\Delta_x)^{\alpha(x)/2},$$

where $\alpha(x) : \mathbf{R}^d \rightarrow \mathbf{R}$ is Dini continuous and satisfies the inequalities

$$(1.5) \quad 0 < \zeta_1 \leq \alpha(x) \leq \zeta_2 < 2 \quad \text{for all } x \in \mathbf{R}^d \text{ with some constants } \zeta_1 \text{ and } \zeta_2.$$

In this case the Lévy kernel of this operator is expressed as

$$(1.6) \quad K(x, A) = \theta_{\alpha(x)} \int_{\mathbf{S}^{d-1}} d\sigma \int_0^\infty 1_{A \setminus \{0\}}(u\sigma) u^{-1-\alpha(x)} du,$$

where

$$\theta_\alpha = \sin \frac{\pi\alpha}{2} \Gamma(1+\alpha) \Gamma\left(\frac{d+\alpha}{2}\right) \pi^{-(d+1)/2} \Gamma\left(\frac{1+\alpha}{2}\right)^{-1}.$$

To the Lévy generator (1.4) there corresponds a Markov process on $\mathbf{R}^d$, which is called a *stable-like process* following Bass [1].

Generalizing the Lévy kernels (1.1) and (1.6), we introduce the following class of Lévy kernels:

$$(1.7) \quad K(x, A) = \int_S M(d\sigma) \int_0^\infty 1_{A \setminus \{0\}}(y(x, u, \sigma)) u^{-1-\alpha(x)} m(x, u, \sigma) du,$$

where

- (i) M is a bounded Borel measure on a topological space S ,
- (ii) $y(x, u, \sigma) : \mathbf{R}^d \times [0, \infty) \times S \rightarrow \mathbf{R}^d$ is measurable, of class C^1 in u on $[0, \infty)$ and $y(x, 0, \sigma) = 0$.
- (iii) $m(x, u, \sigma) : \mathbf{R}^d \times [0, \infty) \times S \rightarrow \mathbf{R}^d$ is a bounded non-negative measurable function.
- (iv) $\alpha(x) : \mathbf{R}^d \rightarrow [0, 2)$ is a measurable function.

Our main concern in this chapter is to investigate the absolute continuity of the transition probability and the smoothness of the transition density for the Markov process on $\mathbf{R}^d$ associated with the Lévy generator L of (1.3) with (1.7). In particular, in the case where the condition (1.5) holds, it would be expected that the transition probability $P_t(x, dy)$ has a density $p_t(x, y)$ which is smooth in y or in (x, y) for every $t > 0$ under a regularity condition and a non-degeneracy condition on $K(x, dy)$. Furthermore we shall also discuss the Lévy kernel (1.7) with $\alpha(x) \equiv 0$. In this case the result on one-dimensional L -processes ([10]) would suggest that for each $r \geq 0$, the transition density $p_t(x, y)$ would be of class C^r in y or in (x, y) only for sufficiently large t 's.

To avoid some technical difficulties, we shall mainly consider the following Lévy generator obtained by cutting off big jumps in (1.7):

$$(1.8) \quad L\varphi(x) = \nabla_x \varphi(x) \cdot a(x) + \int [\varphi(x+y) - \varphi(x) - \nabla_x \varphi(x) \cdot y] K(x, dy)$$

with

$$(1.9) \quad K(x, A) = \int_S M(d\sigma) \int_0^{u_0} 1_{A \setminus \{0\}}(y(x, u, \sigma)) g(x, u, \sigma) du,$$

where $u_0 > 0$ is a fixed constant and we assume that

- (i) $y(x, u, \sigma)$ is a bounded measurable function on $\mathbf{R}^d \times [0, u_0] \times S$, of class C^1 in u on $[0, u_0]$ and $y(x, 0, \sigma) = 0$.
- (ii) $g(x, u, \sigma) = u^{-1-\alpha(x)} m(x, \sigma)$

with $0 \leq \alpha(x) < 2$ and a bounded positive function $m(x, \sigma)$. In particular, if $S = \mathbf{S}^{d-1}$, $y(x, u, \sigma) = u\sigma$ and $m(x, \sigma) = \theta_{\alpha(x)}$, then the corresponding Markov process should be called a *stable-like process with bounded jumps*, since big jumps are suppressed in (1.6).

Our main results in this chapter are summarized as follows: Suppose that a smoothness condition on $a(x)$, $y(x, u, \sigma)$, $m(x, \sigma)$ and $\alpha(x)$, and a non-degeneracy condition on $M(d\sigma)$ and $y(x, u, \sigma)$ are fulfilled. Then

(a) if $\alpha(x)$ satisfies (1.5), the transition probability $P_t(x, dy)$ has a density $p_t(x, y)$ which is smooth in y or in (x, y) for every $t > 0$.

(b) if $\alpha(x) \equiv 0$, then also $P_t(x, dy)$ has a density $p_t(x, y)$, and for a given integer $r \geq 0$, it is of class C^r in y or in (x, y) , provided t is larger than a constant depending on $(y(x, u, \sigma), M(d\sigma), m(x, \sigma), d, r)$.

The smoothness and non-degeneracy conditions are rather complicated in the general case (which are given in §3.1); however we can simply describe the result for a stable-like process with bounded jumps: the transition probability has a C^∞ -density if $\alpha(x)$ is a C^∞ -function whose derivatives of all order are bounded, it satisfies (1.5) and $|\nabla_x \alpha(x)|$ is sufficiently small. We remark that these results should, at least partially, hold for a stable-like process itself because its Lévy generator is a perturbation from that of a stable-like process with bounded jumps by adding a bounded operator.

The proof of our results is essentially based on Malliavin calculus on a Poisson space which was developed by Bichteler, Gravereaux and Jacod [2], but we have to modify their framework to apply it to our class including stable-like processes with bounded jumps. Furthermore we have to emphasize our smoothness result for the transition density holds for every $t > 0$, when $\alpha(x)$ satisfied (1.5), although [2] discusses the smoothness of the density only for sufficiently large t 's under a different situation from ours.

This chapter is organized as follows: In §2 we introduce a Poisson space and a notion of derivatives of Poisson functionals with some modifications of [2], and by using it we define a Malliavin operator (see §2.1). Moreover we apply Malliavin calculus to stochastic differential equations and summarize the results used later (see §2.2). In §3 we state the main results for the generator L defined by (1.8) with the kernel $K(x, dy)$ defined by (1.9) and for a stable-like process (see §3.1). The proofs of our main results are given by using the results in §2.2 (see §3.2).

Throughout this chapter we shall use the following notations:

$A^\top$: the transpose of a matrix A ,

$|E|$: the n -dimensional Lebesgue measure of $E \subset \mathbf{R}^n$,

C^r : the totality of r -times continuously differentiable functions,

C_0^r : the totality of C^r -functions with compact support,

C_b^r : the totality of C^r -functions with bounded derivatives of all order between 0 and

r ,

$C = C^0, C_0 = C_0^0, C_b = C_b^0$,

$C^\infty \cap_{r \geq 0} C^r, C_0^\infty \cap_{r \geq 0} C_0^r, C_b^\infty \cap_{r \geq 0} C_b^r$

$\partial_i = \partial / \partial x_i$.

2 Malliavin Calculus for Jump-type Stochastic Differential Equations

In this section we define a Malliavin operator $(\mathcal{L}, H_\infty)$ for Poisson functionals (random variables on a Poisson space), and give existence and smoothness results of transition

densities for jump-type stochastic differential equations by applying Malliavin calculus (which are obtained by according to [2]).

2.1 Differential calculus on a Poisson space

Let E be an open subset in the n -dimensional Euclidean space $\mathbf{R}^n$ with $|E| = \infty$ and let $\mathcal{E}$ be the Borel σ -field on E . Fix $T > 0$ and let $(S, \mathcal{S}, M(d\sigma))$ be a topological finite measure space with $\mathcal{S} = \mathcal{B}(S)$ (the Borel σ -field of S). Next we take a Poisson space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ associated with a Poisson random measure distributing on $[0, T] \times E \times S$ with the intensity measure $\nu(ds dz d\sigma) = ds dz M(d\sigma)$ in the following way. Let Ω denotes the collection of point measures $\mu = \sum_n \delta_{(t_n, z_n, \sigma_n)}$ on $[0, T] \times E \times S$ such that $\mu([0, T] \times K \times S) < \infty$ for any compact set K in E , $\mu(\{t\} \times E \times S) \leq 1$ for all $t \in [0, T]$ and $\mu(\{0\} \times E \times S) = 0$. Set $\mathcal{F}_t^0 = \sigma(\mu(A) : A \in \mathcal{B}([0, t]) \otimes \mathcal{E} \otimes \mathcal{S})$ for $t \in [0, T]$, $\mathcal{F}_t = \bigcap_{u>t} \mathcal{F}_u^0$ for $t \in [0, T]$, and $\mathcal{F} = \mathcal{F}_T^0$. Then there exists a unique probability measure P on $(\Omega, \mathcal{F})$ such that, under P , μ is a Poisson random measure distributing on $[0, T] \times E \times S$ with the intensity measure $\nu(ds dz d\sigma)$. As usual we extend $\mathcal{F}_t$ by adding all negligible events, and we use the same notation $\mathcal{F}_t$ as for the original. Then $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ is our basic Poisson space. We define a σ -finite measure Q on $\Omega \times [0, T] \times E \times S$ by $Q(d\mu ds dz d\sigma) = P(d\mu) \mu(ds dz d\sigma)$.

Let $\Phi(\mu)$ be a Poisson functional, that is, a random variable on the Poisson space $(\Omega, \mathcal{F}, P)$. For $\mu \in \Omega$ and $(s, z, \sigma) \in \text{supp} \mu$, we set $\mu^\theta(s, z, \sigma) = \mu - \delta_{(s, z, \sigma)} + \delta_{(s, z+\theta, \sigma)}$, which belongs to Ω if $z + \theta \in E$.

DEFINITION 2.1. We define an m -th derivative of Φ by

$$\mathcal{D}^m \Phi(\mu; s, z, \sigma) = \nabla_\theta^m \Phi(\mu^\theta(s, z, \sigma))|_{\theta=0} \quad (n^m\text{-vector})$$

if it is well defined for Q -a.e. (μ, s, z, σ) , where ∇_θ^m is the m -th differential operator in θ (in the usual sense). Moreover define

$$\mathcal{D}^k(\mathcal{D}^m \Phi)(\mu; s, z, \sigma) = \nabla_\theta^k (\mathcal{D}^m \Phi(\mu^\theta(s, z, \sigma); s, z + \theta, \sigma))|_{\theta=0};$$

then $\mathcal{D}^k \mathcal{D}^m = \mathcal{D}^{k+m}$.

Next we introduce basic functional spaces $\mathcal{R}^{(m)}$ ($m = 0, 1, 2, \dots, \infty$). Let $C_{0,E}^m$ denote the totality of real-valued measurable functions $f(s, z, \sigma)$ on $[0, T] \times E \times S$ those which vanish outside $[0, T] \times K \times S$ for some compact set K in E , of class C^m in z and their derivatives in z up to order m are bounded in (s, z, σ) . We set

$$\begin{aligned} \mathcal{R}^{(m)} &= \left\{ \Phi(\mu) = F(\langle \mu, f_1 \rangle, \dots, \langle \mu, f_k \rangle) : \right. \\ &\quad \left. F \in C_p^m(\mathbf{R}^k), f_i \in C_{0,E}^m, i = 1, \dots, k, k = 1, 2, \dots \right\}, \end{aligned}$$

where $\langle \mu, f \rangle = \int_{[0,T] \times E \times S} f(s, z, \sigma) \mu(ds dz d\sigma)$ and $C_p^m(\mathbf{R}^k)$ is the totality of real-valued C^m -functions on $\mathbf{R}^k$ with polynomial growth derivatives up to order m . Then $\mathcal{R}^{(m)}$ is dense in $L^p = L^p(\Omega, P)$ for all $1 \leq p < \infty$, stable under C_p^m -composition (i.e., if $\Phi = (\Phi^1, \dots, \Phi^k)$, $\Phi^i \in \mathcal{R}^{(m)}$ and $F \in C_p^m(\mathbf{R}^k)$, then $F(\Phi) \in \mathcal{R}^{(m)}$) and $\mathcal{F}_T^0 = \sigma(\mathcal{R}^{(m)})$. Clearly every element of $\mathcal{R}^{(m)}$ is m -times differentiable in the sense of Definition 2.1.

Now we fix an auxiliary function $\rho(z) \in C_b^\infty(E \rightarrow (0, \infty))$ and introduce some higher order symmetric differential operators.

DEFINITION 2.2. (i) For $m \geq 1$ and $\Phi \in \mathcal{R}^{(2m)}$, set

$$\mathcal{L}^{(m)}\Phi = \langle \mu, \sum_{k=0}^m \binom{m}{k} \nabla_z^k \rho(\mathcal{D}^k (\text{trace} \mathcal{D}^2)^{m-k} \Phi)^\top \rangle,$$

where $\text{trace} \mathcal{D}^2 \Phi(\mu; s, z, \sigma) = \Delta_\theta \Phi(\mu^\theta(s, z, \sigma))|_{\theta=0}$ (Δ_θ is the Laplacian with respect to the variable θ). $\mathcal{L}^{(m)}\Phi$ is called the $\mathcal{L}^{(m)}$ -derivative of Φ .

For $m \geq 1$ and $\Phi, \Psi \in \mathcal{R}^{(2m)}$, set

$$\langle \mathcal{D}^m \Phi, \mathcal{D}^m \Psi \rangle_\rho = \langle \mu, \rho \mathcal{D}^m \Phi \mathcal{D}^m \Psi^\top \rangle.$$

Then we have the following proposition by a similar argument to Proposition 9-3 of [2].

Proposition 2.3. For $m \geq 1$, $\mathcal{L}^{(m)} : \mathcal{R}^{(2m)} \rightarrow \bigcap_{p < \infty} L^p$ is a linear operator such that for $\Phi, \Psi \in \mathcal{R}^{(2m)}$,

$$(2.1) \quad E[\Phi \mathcal{L}^{(m)} \Psi] = E[\Psi \mathcal{L}^{(m)} \Phi] = (-1)^m E[\langle \mathcal{D}^m \Phi, \mathcal{D}^m \Psi \rangle_\rho].$$

Set $\mathcal{L} = \mathcal{L}^{(1)}$ and $\mathcal{R} = \mathcal{R}^{(2)}$; then it holds that for $\Phi = F(\langle \mu, f_1 \rangle, \dots, \langle \mu, f_k \rangle)$ and $\Psi = H(\langle \mu, h_1 \rangle, \dots, \langle \mu, h_q \rangle) \in \mathcal{R}$,

$$\begin{aligned} \mathcal{D}\Phi &= \sum_{i=1}^k \partial_i F(\langle \mu, f_1 \rangle, \dots) \nabla_z f_i, \\ \mathcal{L}\Phi &= \langle \mu, \rho \text{trace} \mathcal{D}^2 \Phi + \nabla_z \rho \mathcal{D}\Phi^\top \rangle \\ &= \sum_{i=1}^k \partial_i F(\langle \mu, f_1 \rangle, \dots) \langle \mu, \rho \Delta_z f_i + \nabla_z f_i^\top \rangle \\ &\quad + \sum_{i,j=1}^k \partial_{ij}^2 F(\langle \mu, f_1 \rangle, \dots) \langle \mu, \rho \nabla_z f_i \nabla_z f_j^\top \rangle, \\ \langle \mathcal{D}\Phi, \mathcal{D}\Psi \rangle_\rho &= \langle \mu, \rho \mathcal{D}\Phi \mathcal{D}\Psi^\top \rangle \\ &= \sum_{i=1}^k \sum_{j=1}^q \partial_i F(\langle \mu, f_1 \rangle, \dots) \partial_j H(\langle \mu, h_1 \rangle, \dots) \langle \mu, \rho \nabla_z f_i \nabla_z h_j^\top \rangle. \end{aligned}$$

Moreover if $\Phi = (\Phi^1, \dots, \Phi^d)$, $\Phi^i \in \mathcal{R}$ and $F \in C_p^2(\mathbf{R}^d)$, then

$$\mathcal{D}(F \circ \Phi) = \sum_{i=1}^d \partial_i F(\Phi) \mathcal{D}\Phi^i,$$

$$\mathcal{L}(F \circ \Phi) = \sum_{i=1}^d \partial_i F(\Phi) \mathcal{L}\Phi^i + \sum_{i,j=1}^d \partial_{ij}^2 F(\Phi) \langle \mathcal{D}\Phi^i, \mathcal{D}\Phi^j \rangle_\rho.$$

In particular, for $\Phi, \Psi \in \mathcal{R}$,

$$\mathcal{L}(\Phi \Psi) = \Phi \mathcal{L}\Psi + 2\langle \mathcal{D}\Phi, \mathcal{D}\Psi \rangle_\rho + \Psi \mathcal{L}\Phi.$$

Proof. We shall show only (2.1) since the proofs of the other parts are quite routine. Let $\Phi = F(\langle \mu, f_1 \rangle, \dots, \langle \mu, f_k \rangle)$ and $\Psi = H(\langle \mu, h_1 \rangle, \dots, \langle \mu, h_q \rangle) \in \mathcal{R}^{(2m)}$, and K be a compact set in E such that $[0, T] \times K \times S$ contains all the supports of f_i 's and h_i 's. Now set $(S_i, Z_i, A_i)_{i \leq N} = \text{supp} \cap ([0, T] \times K \times S)$ with $S_1 < \dots < S_N$, where $N = \mu([0, T] \times K \times S) \geq 0$. Then N is a Poisson random variable with a parameter $T|K|$, and the random variables $(Z_i)_{i \leq N}$ are conditionally independent on the σ -field $\mathcal{G} \equiv \sigma(N, S_1, \dots, S_N)$ with the uniform distribution over K , $U_K(dz) = dz/|K|$; that is, for each $r \geq 1$ and bounded Borel functions G on K^r , it holds

$$E[g(Z_1, \dots, Z_r) 1_{(N=r)} | \mathcal{G}] = \int_{K^r} g(z_1, \dots, z_r) U_K(dz_1) \cdots U_K(dz_r) 1_{(N=r)}.$$

So it is enough to show that

$$(2.2) \quad E[\langle \mathcal{D}^m \Phi, \mathcal{D}^m \Psi \rangle_\rho | \mathcal{G}] = (-1)^m E[\Phi \mathcal{L}^{(m)} \Psi | \mathcal{G}].$$

For simplicity of notations we set $f_i^j(z) = f_i(S_j, z_j, A_j)$ and $h_i^j(z) = h_i(S_j, z_j, A_j)$. For a fixed $\mu \in \Omega$, set $N = N(\mu)$ and

$$g(z_1, \dots, z_N) = \sum_{r=1}^N \rho(z_r) \nabla_{z_r}^m \left(F \left(\sum_{s=1}^N f_1^s(z_s), \dots, \sum_{s=1}^N f_k^s(z_s) \right) \right) \nabla_{z_r}^m \left(H \left(\sum_{s=1}^N h_1^s(z_s), \dots, \sum_{s=1}^N h_k^s(z_s) \right) \right)^\top;$$

then we have $\langle \mathcal{D}^m \Phi, \mathcal{D}^m \Psi \rangle_\rho = \langle \mu, \rho \mathcal{D}^m \Phi \mathcal{D}^m \Psi^\top \rangle = g(Z_1, \dots, Z_N)$. Moreover

$$\begin{aligned} E[\langle \mathcal{D}^m \Phi, \mathcal{D}^m \Psi \rangle_\rho | \mathcal{G}] &= \int_{K^N} U_K(dz_1) \cdots U_K(dz_N) g(z_1, \dots, z_N) \\ &= \sum_{r=1}^N \sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_{K^N} U_K(dz_1) \cdots U_K(dz_N) \\ &\quad \rho(z_r) \nabla_{z_r}^\alpha \left(F \left(\sum_{s=1}^N f_1^s(z_s), \dots \right) \right) \nabla_{z_r}^\alpha \left(H \left(\sum_{s=1}^N h_1^s(z_s), \dots \right) \right), \end{aligned}$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ is multi-index and $\alpha! = \alpha_1! \cdots \alpha_n!$. Since $\text{supp} f_i^j$ and $\text{supp} h_i^j$ are contained in K , $\nabla_{z_r}^\alpha (F(\dots))$ and $\nabla_{z_r}^\alpha (H(\dots))$ vanish in K^c if $|\alpha| \geq 1$. Hence by using the classical integration-by-parts formula we get

$$\begin{aligned} E[\langle \mathcal{D}^m \Phi, \mathcal{D}^m \Psi \rangle_\rho | \mathcal{G}] &= (-1)^m \sum_{r=1}^N \sum_{|\alpha|=m} \frac{m!}{\alpha!} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \int_{K^N} U_K(dz_1) \cdots U_K(dz_N) \\ &\quad F \left(\sum_{s=1}^N f_1^s(z_s), \dots \right) \nabla_{z_r}^\beta \rho(z_r) \nabla_{z_r}^{2\alpha-\beta} \left(H \left(\sum_{s=1}^N h_1^s(z_s), \dots \right) \right). \end{aligned}$$

On the other hand it is easy to see that

$$\begin{aligned} \mathcal{L}^{(m)} \Psi &= \langle \mu, \sum_{k=0}^m \binom{m}{k} \nabla_z^k \rho(\mathcal{D}^k (\text{trace} \mathcal{D}^2)^{m-k} \Psi)^\top \rangle \\ &= \sum_{r=1}^N \sum_{|\alpha|=m} \frac{m!}{\alpha!} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \nabla_{z_r}^\beta \rho(z_r) \nabla_{z_r}^{2\alpha-\beta} \left(H \left(\sum_{s=1}^N h_1^s(z_s), \dots \right) \right). \end{aligned}$$

Therefore (2.2) holds, and this completes the proof of (2.1). $\blacksquare$

REMARK 2.4. $(\mathcal{L}, \mathcal{R})$ is essentially the same one as in [2]. Note that $\mathcal{L}^{(i)} : \mathcal{R}^{(2m)} \rightarrow \mathcal{R}^{(2m-2i)}$, $i = 1, \dots, m$, but $\mathcal{L}^{(2)} \neq \mathcal{L}^2 \equiv \mathcal{L} \circ \mathcal{L}$.

Next we want to extend the domains of $\mathcal{D}^m$ and $\mathcal{L}$, which are of interest for themselves, although they will be later used only in the case $m = 1$.

Let $\mathcal{H}$ be the completion of $\mathcal{F} \otimes \mathcal{B}([0, T]) \otimes \mathcal{E} \otimes \mathcal{S}$ by Q . For each $k \geq 1$ and $p \in (0, \infty)$, denote by $L_{(k)}^p$ the set of all $(\mathbf{R}^n)^k$ -valued $\mathcal{H}$ -measurable functions F on $\Omega \times [0, T] \times E \times S$ with the norm $|F|_p = \|\langle \mu, \rho | F|^2 \rangle^{1/2}\|_{L^p} < \infty$, where $\|\cdot\|_{L^p}$ denotes the L^p -norm in $L^p = L^p(\Omega, P)$. For each $k \geq 1$ and $p \in [1, \infty)$, $(L_{(k)}^p, |\cdot|_p)$ is a Banach space by identifying two functions which are equal Q -a.e. In particular, $L_{(k)}^2$ is a Hilbert space with the inner product $(F, G)_L = E[\langle F, G \rangle_\rho]$, where $\langle F, G \rangle_\rho = \langle \mu, \rho F G^\top \rangle$.

Let us introduce several spaces of Poisson functionals in a similar way to the case of the Wiener space ([9]).

DEFINITION 2.5. For $p_0, p_1, \dots, p_m \geq 2$, it is said that Φ belongs to $H(p_0, p_1, \dots, p_m)$ (resp. $H(p_0|p_1, \dots, p_m)$) if there is a sequence $\{\Phi_k\}$ in $\mathcal{R}^{(m)}$ (resp. $\mathcal{R}^{(2m)}$) such that $\Phi_k \rightarrow \Phi$ in L^{p_0} and, for each $i = 1, \dots, m$, $\{\mathcal{D}^i \Phi_k\}_{k=1}^\infty$ (resp. $\{\mathcal{L}^{(i)} \Phi_k\}_{k=1}^\infty$) is a Cauchy sequence in $L_{(i)}^{p_i}$ (resp. L^{p_i}). Denote the limit by $\mathcal{D}^i \Phi$ (resp. $\mathcal{L}^{(i)} \Phi$), which is called the i -th weak derivative (resp. the $\mathcal{L}^{(i)}$ -weak derivative) of Φ . Moreover set, for $p_0, p_1, \dots, p_m, q_1, \dots, q_r \geq 2$,

$$H(p_0, p_1, \dots, p_m | q_1, \dots, q_r) = H(p_0, p_1, \dots, p_m) \cap H(p_0 | q_1, \dots, q_r).$$

By using (2.1) we shall show that $\mathcal{D}^m \Phi$ and $\mathcal{L}^{(m)} \Phi$ are well defined for every $\Phi \in H(p_0, p_1, \dots, p_m)$ and $\Phi \in H(p_0 | p_1, \dots, p_m)$ respectively. It is easy to see that the approximating sequences can be chosen in $\mathcal{R}^{(\infty)}$. Suppose that $\Phi_k \in \mathcal{R}^{(\infty)} \rightarrow 0$ in L^2 and $\mathcal{D}^i \Phi_k \rightarrow F_i$ in $L_{(i)}^2$; then for any $\Psi \in \mathcal{R}^{(\infty)}$,

$$\begin{aligned} (F_i, \mathcal{D}^i \Psi)_L &= \lim_{k \rightarrow \infty} (\mathcal{D}^i \Phi_k, \mathcal{D}^i \Psi)_L \\ &= (-1)^i \lim_{k \rightarrow \infty} E[\Phi_k \mathcal{L}^{(i)} \Psi] \quad (\text{by (2.1)}) \\ &= 0. \end{aligned}$$

Thus $|F_i|_2^2 = (F_i, F_i)_L = \lim_{k \rightarrow \infty} (F_i, \mathcal{D}^i \Phi_k)_L = 0$, i.e., $F_i = 0$ Q -a.e.. Also suppose that $\mathcal{L}^{(i)} \Phi_k \rightarrow \theta_i$ in L^2 ; then for all $\Psi \in \mathcal{R}^{(\infty)}$,

$$E[\theta_i \Psi] \lim_{k \rightarrow \infty} E[\Psi \mathcal{L}^{(i)} \Phi] = \lim_{k \rightarrow \infty} E[\Phi_k \mathcal{L}^{(i)} \Psi] = 0.$$

Since $\mathcal{R}^{(\infty)}$ is dense in L^2 , we have $\theta_i = 0$ a.s.

Clearly the above spaces in Definition 2.5 are Banach spaces with the following norms respectively:

$$\|\Phi\|_{H(p_0, \dots, p_m)} = \|\Phi\|_{L^{p_0}} + \sum_{i=1}^m |\mathcal{D}^i \Phi|_{p_i},$$

$$\|\Phi\|_{H(p_0|p_1, \dots, p_m)} = \|\Phi\|_{L^{p_0}} + \sum_{i=1}^m \|\mathcal{L}^{(i)}\Phi\|_{L^{p_i}}$$

and

$$\|\Phi\|_{H(p_0, \dots, p_m|q_1, \dots, q_r)} = \|\Phi\|_{L^{p_0}} + \sum_{i=1}^m |\mathcal{D}^i\Phi|_{p_i} + \sum_{i=1}^r \|\mathcal{L}^{(i)}\Phi\|_{L^{q_i}}.$$

In particular, from (2.3) of the following proposition it follows that

$$H_2^{(m)} \equiv H(2, \underbrace{2, \dots, 2}_m | \underbrace{2, \dots, 2}_m) = H(2 | \underbrace{2, \dots, 2}_m).$$

Proposition 2.6. a) $\mathcal{L}^{(m)} : H_2^{(m)} \rightarrow L^2$ is a symmetric linear operator;

b) Set $\Gamma^m(\Phi, \Psi) = \langle \mu, \rho \mathcal{D}^m \Phi \mathcal{D}^m \Psi^\top \rangle$, then $\Gamma^m : H_2^{(m)} \times H_2^{(m)} \rightarrow L^1$ is a continuous bilinear symmetric non-negative operator satisfying

$$(2.3) \quad E[\Phi \mathcal{L}^{(m)} \Psi] = E[\Psi \mathcal{L}^{(m)} \Phi] = (-1)^m E[\Gamma^m(\Phi, \Psi)].$$

The proof is obvious.

By using them we can define a Malliavin operator $(\mathcal{L}, H_\infty)$ as in [2].

DEFINITION 2.7. For $q \in [2, \infty)$, denote by H_q the Banach space $H(q, q|q)$ with the norm $\|\Phi\|_{H_q} = \|\Phi\|_{L^q} + \|\mathcal{L}\Phi\|_{L^q} + |\mathcal{D}\Phi|_q$, and set $H_\infty \equiv \bigcap_{q \geq 2} H_q$, which is a Fréchet space. Then $(\mathcal{L}, H_\infty)$ (resp. $\Gamma(\Phi, \Psi) \equiv \Gamma^1(\Phi, \Psi)$) is called a *Malliavin operator* (resp. *Malliavin covariance* of (Φ, Ψ)).

We here list several properties of the Malliavin operator and the Malliavin covariance which can be checked by the same arguments as in Theorem 8-18 of [2].

Theorem 2.8. Let $(\mathcal{L}, H_\infty)$ be the Malliavin operator.

- a) $\mathcal{L} : H_\infty \subset \bigcap_{p < \infty} L^p$ is a linear operator and satisfies the following properties:
- (i) H_∞ is stable under C_p^2 -composition and $\mathcal{F}_T^0 = \sigma(H_\infty)$;
 - (ii) $\mathcal{L}$ is symmetric in L^2 , i.e., $E[\Phi \mathcal{L} \Psi] = E[\Psi \mathcal{L} \Phi]$;
 - (iii) $\Gamma : H_\infty \times H_\infty \rightarrow \bigcap_{p < \infty} L^p$ is a continuous bilinear symmetric non-negative operator;
 - (iv) if $\Phi = (\Phi^1, \dots, \Phi^d)$, $\Phi^i \in H_\infty$ and $F \in C_p^2(\mathbf{R}^d)$, then

$$\mathcal{D}(F \circ \Phi) = \sum_{i=1}^d \partial_i F(\Phi) \mathcal{D}\Phi^i,$$

$$\mathcal{L}(F \circ \Phi) = \sum_{i=1}^d \partial_i F(\Phi) \mathcal{L}\Phi^i + \sum_{i,j=1}^d \partial_{ij}^2 F(\Phi) \Gamma(\Phi^i, \Phi^j).$$

In particular, for $\Phi, \Psi \in H_\infty$,

$$\mathcal{L}(\Phi\Psi) = \Phi\mathcal{L}\Psi + 2\Gamma(\Phi, \Psi) + \Psi\mathcal{L}\Phi.$$

b) (i) If $\Phi = (\Phi^1, \dots, \Phi^d)$, $\Phi^i \in H_\infty$, $F \in C_p^2(\mathbf{R}^d)$ and $\Psi \in H_\infty$, then

$$\Gamma(F \circ \Phi, \Psi) = \sum_{i=1}^d \partial_i F(\Phi) \Gamma(\Phi^i, \Psi);$$

(ii) $\mathcal{L}1 = 0$, $E[\mathcal{L}\Phi] = 0$ and $E[\Phi\mathcal{L}\Psi] = E[\Psi\mathcal{L}\Phi] = -E[\Gamma(\Phi, \Psi)]$.

c) For $\Phi = (\Phi^1, \dots, \Phi^d) \in (H_\infty)^d$, $F \in C_p^2(\mathbf{R}^d)$ and $\Psi \in H_\infty$, the integration-by-parts formula holds:

$$E \left[\Psi \sum_{i=1}^d \partial_i F(\Phi) \Gamma(\Phi^i, \Phi^j) \right] = -E \left[F(\Phi) \left\{ \Psi \mathcal{L}\Phi^j + \Gamma(\Phi^j, \Psi) \right\} \right].$$

2.2 Applications of Malliavin calculus to stochastic differential equations

Now exactly in the same way as in [2], we can apply the Malliavin operator to a solution of the following stochastic differential equation driven by the Poisson random measure $\mu(ds dz d\sigma)$:

$$(2.4) \quad X_t^x = x + \int_0^t a(X_s^x) ds + \int_0^t \int_{E \times S} f(X_{s-}^x, z, \sigma) \tilde{\mu}(ds dz d\sigma) \quad (x \in \mathbf{R}^d),$$

where $\tilde{\mu} = \mu - \nu$ is a compensated Poisson random measure and

$$\begin{aligned} a(x) &= (a^i(x)) && : \mathbf{R}^d \rightarrow \mathbf{R}^d, \\ f(x, z, \sigma) &= (f^i(x, z, \sigma)) && : \mathbf{R}^d \times E \times S \rightarrow \mathbf{R}^d \end{aligned}$$

are measurable functions. We also write (2.4) in the differential form

$$dX^x = a(X^x)dt + f(X_-^x)d\tilde{\mu}, \quad X_0^x = x.$$

We give also an assumption on $a(x)$ and $f(x, z, \sigma)$. Let $r \geq 1$.

Assumption $(A - r)$. (i) $a(x)$ is a C_b^r -function on $\mathbf{R}^d$;

(ii) $f(x, z, \sigma)$ is r -times differentiable in (x, z) on $\mathbf{R}^d \times E$ and moreover

$$\begin{aligned} f(0, \cdot, \cdot) &\in \cap_{2 \leq p \leq \infty} L^p(E \times S, dz M(d\sigma)) \\ \sup_x |\nabla_x^i f(x, \cdot, \cdot)| &\in \cap_{2 \leq p \leq \infty} L^p(E \times S, dz M(d\sigma)) \quad \text{for } 1 \leq i \leq r, \\ \sup_{x, z, \sigma} |\nabla_x^i \nabla_z^j f(x, z, \sigma)| &< \infty \quad \text{for } 1 \leq i + j \leq r \text{ and } j \leq 1. \end{aligned}$$

We shall always assume at least $(A - r)$. Then (2.4) has a unique solution X_t^x , which satisfies $\sup_{t \leq T} |X_t^x| \in \cap_{p < \infty} L^p$ and defines a standard strong Markov Process X^x with the generator L :

$$L\varphi(x) = \nabla_x \varphi(x) \cdot a(x) + \int [\varphi(x+y) - \varphi(x) - \nabla_x \varphi(x) \cdot y] K(x, dy),$$

where K is given by

$$K(x, A) = \int_{E \times S} 1_{A \setminus \{0\}}(f(x, z, \sigma)) dz M(d\sigma).$$

From now on, we fix a Malliavin operator with an auxiliary function ρ satisfying

Condition (ρ) . $\rho : E \rightarrow (0, \infty)$ is a C_b^∞ -function such that

- (i) $\rho(z)/d(z)^2, |\nabla_z \rho(z)|/d(z) \in \bigcap_{1 \leq p < \infty} L^p(E, dz)$, where $d(z) \equiv \inf_{z \in E^c} |z - y| \wedge 1$ (if $E^c \equiv \mathbf{R}^n \setminus E = \emptyset$), $\equiv 1$ (if $E = \mathbf{R}^n$);
- (ii) for $r \geq 0, |\nabla_x^r \rho| \in L^1(E, dz)$.

For simplicity we consider the equation (2.4) with a fixed initial value $x \in \mathbf{R}^d$ and we denote the solution by $X_t = X_t^x$. then we have

Theorem 2.9. (cf. [2], Theorem 10-3). Assume (A-3). Then for each $t \leq T$, the components X_t^i of the solution X_t belong to H_∞ . Moreover set $U_t^{ij} = \Gamma(X_t^i, X_t^j)$ and $V_t^i = \mathcal{L}X_t^i$; then the processes $U = (U^{ij})_{i,j \leq d}$ and $V = (V^i)_{i \leq d}$ are solutions to the following equations, respectively:

$$\begin{cases} dU = \left[U \nabla_x a(X)^\top + \nabla_x a(X) U \right] dt \\ \quad + \left[U_- \nabla_x f(X_-)^\top + \nabla_x f(X_-) U_- \right] d\tilde{\mu} \\ \quad + \left[\rho \nabla_z f(X_-) \nabla_z f(X_-)^\top + \nabla_x f(X_-) U_- \nabla_x f(X_-)^\top \right] d\mu, \\ U_0 = 0, \end{cases}$$

and

$$\begin{cases} dV = \nabla_x a(X) V dt + \nabla_x f(X_-) V_- d\tilde{\mu} \\ \quad + \sum_{i,j=1}^d \left[\partial_{ij}^2 a(X) U^{ij} dt + \partial_{ij}^2 f(X_-) U_-^{ij} d\tilde{\mu} \right] \\ \quad + \left[\rho \Delta_z f(X_-) + \nabla_z \rho \nabla_z f(X_-)^\top \right] d\mu, \\ V_0 = 0, \end{cases}$$

where $\partial_{ij}^2 = \partial^2 / (\partial x_i \partial x_j)$.

Finally we modify the results of [2] according to the present situation, which will be used to prove our main results in §3.1.

Denote by ∇X_t^x the Fréchet derivative of a map $x \in \mathbf{R}^d \rightarrow X_t^x \in L^p$. Then under (A-2), ∇X_t^x exists for each $x \in \mathbf{R}^d$ and $t \in [0, T]$, and it is a unique solution to the following linear stochastic differential equation:

$$d\nabla X^x = \nabla_x a(X^x) \nabla X^x dt + \nabla_x f(X_-^x) \nabla X_-^x d\tilde{\mu}, \quad \nabla X_0^x = I$$

([2] Theorem 6-29). Let us introduce an $\mathbf{R}^d \times \mathbf{R}^d$ -valued process K^x defined by

$$dK^x = \nabla_x a(X^x) dt + \nabla_x f(X_-^x) d\tilde{\mu}, \quad K_0^x = 0.$$

Then

$$d\nabla X^x = dK^x \nabla X_-^x, \quad \nabla X_0^x = I.$$

Define a sequence of stopping times:

$$T_0^x = 0, T_n^x = \inf \left\{ t > T_{n-1}^x : \det(I + \Delta K_t^x) = 0 \right\} \quad (= T \text{ if } \{\cdot\} = \emptyset),$$

where $\Delta K_t^x = K_t^x - K_{t-}^x$. We further define processes $\nabla X^x(n)$ by

$$d\nabla X^x(n) = dK^x(n) \nabla X^x(n)_-, \quad \nabla X^x(n)_0 = I,$$

where $\Delta K_t^x = K_t^x - K_{t \wedge T_{n-1}^x}^x$. Clearly $K^x(1) = K^x$ and $\nabla X^x(1) = \nabla X^x$. Since K^x is a right continuous process with left-hand limits and $\{0 < t < T : \det(I + \Delta X_t^x) = 0\}$ is a finite set, $T_n^x \uparrow T$. It is also well-known that ([5])

$$\begin{aligned} \nabla X^x(n)_t &= I \quad \text{if } t \leq T_{n-1}^x, \\ \nabla X^x(n)_t &\text{ is invertible if } t < T_n^x, \\ \nabla X^x(n)_t &\text{ is not invertible if } t \geq T_n^x, \\ \nabla X^x(n)_{t-} &\text{ is invertible if } t \leq T_n^x. \end{aligned}$$

Moreover we define the process H^x by

$$dH^x = \rho \nabla_z f \nabla_z f^\top (X_-^x) d\mu \quad \text{and} \quad H_0^x = 0.$$

Then it is shown in [2] (Proposition 10-22) that under (A-3)

$$\begin{aligned} U_t^x &= \nabla X^x(n)_t \left(U_{T_{n-1}^x}^x + \int_{T_{n-1}^x}^t \int_{E \times S} \left\{ \nabla X^x(n)_{s-}^{-1} (I + \nabla_x f)^{-1} \nabla_z f \nabla_z f^\top \right. \right. \\ &\quad \left. \left. (I + \nabla_x f)^{-1, \top} (X_{s-}^x, z, \sigma) \nabla X^x(n)_{s-}^{-1, \top} \rho(z) \right\} \mu(ds dz d\sigma) \right) \nabla X^x(n)_t^\top \\ &\quad \text{if } T_{n-1}^x \leq t < T_n^x, \\ U_{T_n^x}^x &= \Delta H_{T_n^x}^x + (I + \Delta K_{T_n^x}^x) U_{(T_n^x)-}^x (I + \Delta K_{T_n^x}^x)^\top \quad \text{and } U_0^x = 0. \end{aligned}$$

Therefore, by Theorem 10-30 (a) of [2], we see that, under (A-4), if U_t^x is invertible a.s., then the distribution of X_t^x has a density. Furthermore, using the above formulae and the arguments in §11 of [2], we can obtain the following lemma, which will be used to prove Theorem 3.1.

Lemma 2.10. *Let $t > 0$. Under (A-4), if for each $n \geq 1$,*

$$\begin{aligned} &\int_{T_{n-1}^x}^t \int_{E \times S} \left[\nabla X^x(n)_{s-}^{-1} (I + \nabla_x f)^{-1} \nabla_z f \right. \\ &\quad \left. \nabla_z f^\top (I + \nabla_x f)^{-1, \top} (X_{s-}^x, z, \sigma) \nabla X^x(n)_{s-}^{-1, \top} \rho(z) \right] \mu(ds dz d\mu) \end{aligned}$$

is invertible a.s. on $\{\mu : T_{n-1}^x(\mu) < t < T_n^x(\mu)\}$, then the distribution of X_t^x has a density.

To obtain the smoothness results, we introduce a function $h(z)$ on E following [2], which is related to the integrability of $(U_t^x)^{-1}$.

DEFINITION 2.11. Let $\zeta, \theta > 0$ be given. A measurable function $h : E \rightarrow [0, \infty)$ is called (ζ, θ) -broad if

$$\int_0^\infty ds s^{\zeta-1} \exp \left[-\theta \int_E (1 - e^{-sh(z)}) dz \right] < \infty.$$

Example 2.12. Let $E = (\eta, \infty)$ for some $\eta \in \mathbf{R}$. Let $\zeta, \theta > 0$ be given and set $h(z) = |z|^\gamma e^{-\delta|z|}$ ($z \in E$) for some constants $\gamma, \delta \in \mathbf{R}$. If $\delta < \theta/\zeta$, then h is (ζ, θ) -broad for all $\gamma \in \mathbf{R}$. If $\delta = \theta/\zeta$, then h is also (ζ, θ) -broad for $\gamma > \delta/\theta = 1/\zeta$. Otherwise h is not (ζ, θ) -broad. In particular, if $\delta = 0$, that is, $h(z) = |z|^\gamma$, then h is (ζ, θ) -broad for all $\zeta, \theta > 0$ and $\gamma \in \mathbf{R}$.

The assumption $(SB - (\zeta, \theta))$ of [2] is modified as follows:

Assumption $(B - (\zeta, \theta))$. There exist constants $\epsilon > 0, \delta \geq 0$, a (ζ, θ) -broad function h , a function ρ satisfying the condition (ρ) and a family of disjoint sets $\{S_i\}_{i=1}^N \subset \mathcal{S}$ with $M(S_i) = M_0 > 0$ such that for each $x, y \in \mathbf{R}^d$,

$$\sum_{i=1}^N \inf_{z: h(z) > 0, \sigma \in S_i} \frac{\rho(z)}{h(z)} y^\top F(x, z, \sigma) y \geq \epsilon \frac{|y|^2}{1 + |x|^\delta},$$

where F is the function with values in the set of symmetric non-negative definite $d \times d$ -matrices defined by

$$(2.5) \quad F(x, z, \sigma) = \begin{cases} (I + \nabla_x f)^{-1} \nabla_z f & \nabla_z f^\top (I + \nabla_x f)^{-1, \top} (x, z, \sigma) \\ & \text{if } \det(I + \nabla_x f(x, z, \sigma)) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Now we can modify the results of [2] (Theorem 2-27, 2-28 and 2-29) in the following way.

Theorem 2.13. Let $r \geq 0$ and $t > 0$. Assume $(A - j)$ and $(S - (\zeta, \theta))$ for some j and $\zeta, \theta > 0$. Assume also $\inf_{x, z, \sigma} |\det(I + \nabla_x f(x, z, \sigma))| > 0$. Then

a) the distribution of X_t^x has a density $p_t(x, \cdot)$ of class C^r for each $x \in \mathbf{R}^d$, provided either

(i) $j \geq r + d + 5, \theta \leq M_0 t$ and $\zeta > 2d(r + d + 1)/[M_0 t/\theta]$ (where $[a]$ denotes the integer part of a), or

(ii) $j \geq r + 5, \theta \leq M_0 t$ and $\zeta > 2d^2(r + 1)/[M_0 t/\theta]$.

b) Moreover the function: $(x, y) \rightarrow p_t(x, y)$ is of class C^r , provided that $j \geq r + 2d + 5, \theta \leq M_0 t$ and $\zeta > 2d(r + 2d + 1)/[M_0 t/\theta]$ hold.

c) Furthermore assume $\inf_{v \in [0, 1], x, z, \sigma} |\det(I + v \nabla_x f(x, z, \sigma))| > 0$. If $j \geq 2r + 4d + 8, \theta \leq M_0 t$ and $\zeta > 4d(r + 2d + 2)/[M_0 t/\theta]$, then the function: $(s, x, y) \rightarrow p_s(x, y)$ is of class C^r on $(t, T) \times \mathbf{R}^d \times \mathbf{R}^d$.

REMARK 2.14. In the above theorem, we only give the result corresponding to the first assertion (i) of Theorem 2-28 of [2]: the proof of the second assertion (ii) of Theorem 2-28 includes trivial mistakes (see the equations (4-24) and (4-37) in [2]) and it is hard to correct them. Here we need two more degrees of differentiability on the coefficients (a, f) than that in [2], because we use Malliavin's approach (see Remark 10-33 of [2]). However if we take Bismut's approach the regularity assumption on (a, f) would be improved.

3 Existence and Smoothness of Transition Densities for Lévy Generators

In this section we give our main results in this chapter and the proofs.

3.1 Main results

By using Theorem 2.13 and Lemma 2.10 in §2.2 we give results for the existence and smoothness of a density for the transition probability $P_t(x, dy)$ corresponding to the Lévy generator L defined by (1.8). Its Lévy kernel $K(x, dy)$ is given by (1.9) with the following functions:

$$\begin{aligned} y(x, u, \sigma) : \mathbf{R}^d \times [0, u_0] \times S &\rightarrow \mathbf{R}^d, y(x, 0, \sigma) = 0 && \text{(curve);} \\ g(x, u, \sigma) : \mathbf{R}^d \times (0, u_0] \times S &\rightarrow (0, \infty) && \text{(weight).} \end{aligned}$$

We restrict the function $g(x, u, \sigma)$ as follows:

$$g(x, u, \sigma) = u^{-1-\alpha(x)} m(x, \sigma),$$

where $m(x, \sigma) > 0$ and $0 \leq \alpha(x) < 2$.

In order to relate L and the equation (2.4) we introduce the following functions:

$$(3.1) \quad G(x, u, \sigma) = \int_u^{u_0} g(x, v, \sigma) dv \quad \text{for } u \in (0, u_0),$$

$\hat{G}(x, \cdot, \sigma)$ is the inverse function of $G(x, \cdot, \sigma)$

and

$$(3.2) \quad f(x, z, \sigma) = y(x, \hat{G}(x, z, \sigma), \sigma).$$

Then K is expressed as

$$K(x, A) = \int_S M(d\sigma) \int_0^\infty 1_{A \setminus \{0\}}(f(x, z, \sigma)) dz.$$

We here choose $E = (0, \infty)$ and a Poisson space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ over $[0, T] \times E \times S$ with the intensity measure $dt dz M(d\mu)$. Then we can consider the stochastic differential equation (2.4) associated with $a(x)$ and $f(x, z, \sigma)$ of (3.2) on the Poisson space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$; its solution solves the martingale problem for the Lévy generator L .

Now we give an assumption on (a, y, m, α) . Let $r \geq 1$.

Assumption $(L - r)$. (i) $a(x)$ is a C_b^r -function on $\mathbf{R}^d$;
(ii) $y(x, u, \sigma)/u$ is r -times differentiable in (x, u) on $\mathbf{R}^d \times [0, u_0]$, $\nabla_x^i \nabla_u^j (y(x, u, \sigma)/u)$ is bounded in (x, u, σ) for all i, j with $0 \leq i + j \leq r$ and $\nabla_u y(x, 0, \sigma)$ is continuous at every $\sigma \in \text{supp} M(d\sigma)$;
(iii) $m(x, \sigma)$ is r -times differentiable in x such that $0 < \xi_1 \leq m(x, \sigma) \leq \xi_2 < \infty$ and $\nabla_x^k m(x, \sigma)$ is bounded in (x, σ) for all k with $0 \leq k \leq r$;
(iv) $\alpha(x) \equiv 0$, or $\alpha(x)$ is a C_b^r -function on $\mathbf{R}^d$ and satisfies the inequality $0 < \zeta_1 \leq \alpha(x) \leq \zeta_2 < 2$ with some constants ζ_1, ζ_2 .

Under the assumption $(L - 1)$ the equation (2.4) has a unique solution X_t^x for each initial condition $X_0^x = x$. Since $X^x = (X_t^x)$ defines a standard Markov process on $\mathbf{R}^d$, we denote its transition probability by $P_t(x, dy)$. The next assumption is a non-degeneracy condition for $K(x, dy)$:

Assumption (N) .

$$\text{span} \{ \nabla_u y(x, 0, \sigma) : \sigma \in \text{supp} M(d\sigma) \} = \mathbf{R}^d$$

for every $x \in \mathbf{R}^d$.

Note that under $(L - 1)$ and (N) , $K(x, dy)$ satisfies $K(x, \mathbf{R}^d) = \infty$ and $\int |y|^2 K(x, dy) < \infty$.

The first result of this chapter is the following.

Theorem 3.1. (Existence of transition densities). *Assume that $(L - 4)$ and (N) hold. Then $P_t(x, dy)$ has a density $p_t(x, \cdot)$ for all $t > 0, x \in \mathbf{R}^d$.*

REMARK 3.2. When S is a finite set, this assertion was shown in Theorem 3-8 of [2]. Léandre [6] also discussed this problem under a weaker non-degeneracy condition than (N) , in the case where S is finite and $g(x, u, \sigma)$ is independent of x . We here generalize S and $g(x, u, \sigma)$, which enables us to treat a stable-like process. We shall prove the above theorem by combining two methods in [2] and [6].

Assumption (SN) . There exist constants $\epsilon > 0$, a finite number of points $\sigma_i \in \text{supp} M(d\sigma)$, $i = 1, \dots, N$ and disjoint neighborhoods S_i of σ_i , $i = 1, \dots, N$ such that for all $x, y \in \mathbf{R}^d$,

$$\sum_{i=1}^N \inf_{\sigma \in S_i} |y^\top \nabla_u y(x, 0, \sigma)|^2 \geq \epsilon |y|^2.$$

This is stronger than (N) . The following assumption are rather artificial, but they are required by technical reasons (see [2]). Let $f(x, z, \sigma)$ be defined by (3.2).

Assumption (C) . $\inf_{x, z, \sigma} |\det(I + \nabla_x f(x, z, \sigma))| > 0$.

Assumption (SC) . $\inf_{v \in [0, 1], x, z, \sigma} |\det(I + v \nabla_x f(x, z, \sigma))| > 0$.

These assumptions hold, for instance, if either $|y/u|, |\nabla_x y/u|, |\nabla_u(y/u)|$, or $|\nabla_x y/u|, |\nabla_x m|, |\nabla_x \alpha|$ are sufficiently small, or if u_0 is sufficiently small. These conditions will be discussed in Remark 3.6.

Now we can state the main result of this chapter.

Theorem 3.3. (Smoothness of transition densities). Assume that (SN) and (C) hold. Let $r \geq 1$ and let $M_0 = \min_{1 \leq i \leq N} M(S_i)$, where S_i is found in (SN).

a) Let $0 < \zeta_1 \leq \alpha(x) \leq \zeta_2 < 2$.

(i) Under $(L - (r + 5))$, $P_t(x, dy)$ has a density $p_t(x, y)$ of class C^r in y for all $x \in \mathbf{R}^d, t > 0$.

(ii) Under $(L - (r + 2d + 5))$, $p_t(x, y)$ is of class C^r in (x, y) for all $t > 0$.

(iii) Assume further that (SC) holds. Then, under $(L - (2r + 4d + 8))$, $p_t(x, y)$ is of class C^r in (t, x, y) of $(0, \infty) \times \mathbf{R}^d \times \mathbf{R}^d$.

(iv) In the cases (i), (ii) and (iii), if $(L - r)$ is fulfilled for all $r \geq 1$, then the term C^r can be replaced by C^∞ .

b) Let $\alpha(x) \equiv 0$, and $\eta = 1/\xi_1$.

(i) Under $(L - (r + d + 5))$ (resp. $(L - (r + 5))$), $P_t(x, dy)$ has a density $p_t(x, y)$ of class C^r in y , if $t > \{4d(r + d + 1)\eta + 2\eta^2\}/M_0$ (resp. $t > \{4d^2(r + 1)\eta + 2\eta^2\}/M_0$).

(ii) Under $(L - (r + 2d + 5))$, $p_t(x, y)$ is of class C^r in (x, y) , if $t > \{4d(r + 2d + 1)\eta + 2\eta^2\}/M_0$.

(iii) Assume further that (SC) holds. Then, under $(L - (2r + 4d + 8))$, $p_s(x, y)$ is of class C^r in (s, x, y) of $(t, \infty) \times \mathbf{R}^d \times \mathbf{R}^d$, if $t > \{8d(r + 2d + 2)\eta + 2\eta^2\}/M_0$.

REMARK 3.4. (i) The result corresponding to (b) in Theorem 3.3 is found in Theorem 3-12 of [2], but their assumption involves $f(x, z, \sigma)$. However it would be desirable to give the assumption in terms of the Lévy generator (1.8) with (1.9), without using $f(x, z, \sigma)$.

(ii) As mentioned in Remark 2.14 it is possible to drop one or two degrees of differentiability on the coefficients (a, y, m, α) in our results. However it is not of interest for us.

Finally we shall apply above results to a stable-like process with bounded jumps and a stable-like process itself. Note that the generator of a stable-like process is decomposed into $L = A^0 + A^1$, where

$$A^0\varphi(x) = \int_{|y| < u_0} [\varphi(x + y) - \varphi(x) - \nabla_x \varphi(x) \cdot y] K(x, dy),$$

$$A^1\varphi(x) = \int_{|y| \geq u_0} [\varphi(x + y) - \varphi(x)] K(x, dy)$$

and $K(x, dy)$ is given by (1.6).

Assume that $\alpha(x)$ is a C_b^1 -function satisfying the inequalities (1.5). Since A^0 is a special type of (1.8), there corresponds a Markov process on $\mathbf{R}^d$ which is defined as the unique solution of the SDE (2.4) with $f(x, z, \sigma)$ of (3.2) on a Poisson space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ over $[0, T] \times (0, \infty) \times \mathbf{S}^{d-1}$. We denote by $P_t^0(x, dy)$ its transition probability.

On the other hand Bass [1] establishes the well-posedness of the martingale problem for L , so that L determines a Markov process on $\mathbf{R}^d$ uniquely. We denote by $P_t(x, dy)$ its transition probability. Then we obtain the following result:

Theorem 3.5. Suppose that $\alpha(x)$ is a C_b^j -function on $\mathbf{R}^d$ for some j , and satisfies (1.5). Let $r \geq 0$.

a) (Stable-like processes with bounded jumps)

- (i) If $j \geq 4$, then $P_t^0(x, dy)$ has a density $p_t^0(x, y)$ for all $t > 0$ and $x \in \mathbf{R}^d$.
(ii) Assume (C). If $j \geq r + 5$, then $p_t^0(x, y)$ is of class C^r in y for all $t > 0, x \in \mathbf{R}^d$.
If $j \geq r + 2d + 5$, then $p_t^0(x, y)$ is of class C^r in (x, y) for $t > 0$.
(iii) Assume (SC). If $j \geq 2r + 4d + 8$, then $p_t^0(x, y)$ is of class C^r in (t, x, y) of $(0, \infty) \times \mathbf{R}^d \times \mathbf{R}^d$.
b) (Stable-like processes)
If $j \geq 4$, then $P_t(x, dy)$ has a density $p_t(x, y)$ given by

$$(3.3) \quad p_t(x, y) = p_t^0(x, y) + \int_0^t ds \int_{\mathbf{R}^d} P_{t-s}(x, dy_1) A^1 p_s^0(y_1, y)$$

for all $t > 0$ and $x \in \mathbf{R}^d$.

Proof. a) It is easy to check that $m(x, \sigma) = \theta_{\alpha(x)}$ satisfies $(L - j)$ -(iii). Hence the claims follow from Theorem 3.1 and Theorem 3.3 (a).

b) Let $C_\infty = C_\infty(\mathbf{R}^d)$ be the Banach space of continuous functions on $\mathbf{R}^d$ vanishing at infinity with supremum norm. It is easy to see that $P_t^0(x, dy)$ induces a s-continuous conservative Markov semi-group $\{T_t^0\}$ acting on C_∞ , and that A^1 is a bounded generator of a Markov semi-group acting on C_∞ . Let $(G^0, \mathcal{D}(G^0))$ be the generator of $\{T_t^0\}$, and set $G = G^0 + A^1$ and $\mathcal{D} \equiv \mathcal{D}(G) = \mathcal{D}(G^0)$. Then $(G, \mathcal{D}(G))$ generates an s-continuous conservative Markov semi-group $\{T_t\}$ acting on C_∞ . Furthermore the following perturbation formula holds:

$$(3.4) \quad T_t = T_t^0 + \int_0^t T_{t-s} A^1 T_s^0 ds \quad \text{on } C_\infty$$

(see Theorem 3.1 of [3], also Chap. 1 and 4 of [4]). Then there is a Markov process (X_t, P_x) associated with the Markov semi-group $\{T_t\}$. Since $\mathcal{D} \supset C_0^2(\mathbf{R}^d)$ and $G = L$ on $C_0^2(\mathbf{R}^d)$, the probability law of (X_t) under P_x is a solution of the martingale problem for $(L, C_0^2(\mathbf{R}^d))$. Hence by the well-posedness of the martingale problem, $\{T_t\}$ coincides with the semi-group associated with $P_t(x, dy)$. Therefore (3.3) follows from (3.4). ■

REMARK 3.6. a) To obtain the smoothness results of $P_t(x, dy)$ for stable-like process, the formula (3.3) might be useful. But we need more information on integrability of derivatives of $p_t^0(x, y)$, which seems rather difficult.

b) For the assumptions (C) and (SC), we can give a sufficient condition as follows: Set $n(x, u, \sigma) = y(x, u, \sigma)/u$. Then under $(L - r)$, $n(x, u, \sigma)$ is r -times differentiable in (x, u) on $\mathbf{R}^d \times [0, u_0]$, its derivatives are bounded in (x, u, σ) up to order r , and

$$f(x, z, \sigma) = \hat{G}(x, z, \sigma) n(x, \hat{G}(x, z, \sigma), \sigma).$$

- (i) In case of $\alpha(x) \equiv 0$ we have $\hat{G}(x, z, \sigma) = u_0 \exp[-z/m(x, \sigma)]$. Then we get

$$\begin{aligned} |\nabla_x f(x, z, \sigma)| &\leq u_0 \left((|n| + |\nabla_u n|)(x, \hat{G}(x, z, \sigma), \sigma) |\nabla_x m(x, \sigma)| / \xi_1 \right. \\ &\quad \left. + |\nabla_x n(x, \hat{G}(x, z, \sigma), \sigma)| \right). \end{aligned}$$

Form this we deduce that if either $|n|, |\nabla_u n|, |\nabla_x n|$ or $|\nabla_x m|, |\nabla_x n|$ are sufficiently small, then (SC) holds. For instance, suppose that all the functions α, n, m are independent of $(x_2, \dots, x_d)$. Then $\det(I + v \nabla_x f) = 1 + v \partial_1 f$ for $v \in [0, 1]$. In this case (C) and (SC)

hold if $u_0 < \xi_1 / ((\|n\| + \|\nabla_u n\|)\|\nabla_x m\| + \xi_1\|\nabla_x n\|)$, where $\|\cdot\|$ denotes the supremum norm with respect to the variables x, u and σ .

(ii) In case of $0 < \zeta_1 \leq \alpha(x) \leq \zeta_2 < 2$ we have $\hat{G}(x, z, \sigma) = (\alpha(x)/m(x, \sigma)z + u_0^{-\alpha(x)})^{-1/\alpha(x)}$. It is easy to see that

$$\begin{aligned} |\nabla_x \hat{G}(x, z, \sigma)| &\leq |\nabla_x \alpha(x)| / \zeta_1^2 (1 \vee \zeta_2 u_0 \log u_0) \\ &\quad + u_0 / \zeta_1 (\{|\nabla_x \alpha(x)| / \zeta_1 + |\nabla_x m(x, \sigma)| / \xi_1\} \vee |\nabla_x \alpha(x)| \log u_0) \\ &\quad \text{if } u_0 \geq 1, \end{aligned}$$

$$\begin{aligned} |\nabla_x \hat{G}(x, z, \sigma)| &\leq |\nabla_x \alpha(x)| / \zeta_1^2 u_0^{\zeta_1(1/\zeta_2 - 1/e)} \\ &\quad + u_0 / \zeta_1 (\{|\nabla_x \alpha(x)| / \zeta_1 + |\nabla_x m(x, \sigma)| / \xi_1\} \vee |\nabla_x \alpha(x)| \log u_0) \\ &\quad \text{if } u_0 < 1. \end{aligned}$$

Moreover

$$\begin{aligned} |\nabla_x f(x, z, \sigma)| &\leq (|n| + |\nabla_u n|)(x, \hat{G}(x, z, \sigma), \sigma) |\nabla_x \hat{G}(x, z, \sigma)| \\ &\quad + |\nabla_x n(x, \hat{G}(x, z, \sigma), \sigma)| |\hat{G}(x, z, \sigma)|. \end{aligned}$$

From these inequalities we deduce that if either $|n|, |\nabla_u n|, |\nabla_x n|$ or $|\nabla_x \alpha|, |\nabla_x m|, |\nabla_x n|$ are sufficiently small, or if u_0 is sufficiently small, then (SC) holds. In particular, for a stable-like process with bounded jumps, we have $n(x, u, \sigma) = \sigma$ and $m(x, \sigma) = \theta_{\alpha(x)}$. Let

$$\begin{aligned} \xi_1 &= \left(\sin \frac{\pi}{2} \zeta_1 \right) \wedge \left(\sin \frac{\pi}{2} \zeta_2 \right) \\ &\quad \Gamma(1 + \zeta_1) \Gamma((d + \zeta_1)/2) \pi^{-(d+1)/2} \Gamma((1 + \zeta_2)/2)^{-1}, \end{aligned}$$

$$\xi_2 = \Gamma(1 + \zeta_2) \Gamma((d + \zeta_2)/2) \pi^{-(d+1)/2} \Gamma((1 + \zeta_1)/2)^{-1}$$

and

$$\begin{aligned} \eta &= \left(\pi \cos \frac{\pi}{2} \zeta_1 \Gamma(1 + \zeta_2) \Gamma((d + \zeta_2)/2) + \gamma_1 \Gamma((d + \zeta_2)/2) \right. \\ &\quad \left. + \gamma_2 \Gamma(1 + \zeta_2) + \gamma_3 \Gamma(1 + \zeta_2) \Gamma((d + \zeta_2)/2) \right) \left(2\pi^{(d+1)/2} \Gamma((1 + \zeta_1)/2) \right)^{-1} \end{aligned}$$

with $\gamma_1 = 2 \sup_x \Gamma'(1 + \alpha(x))$, $\gamma_2 = \sup_x \Gamma'((d + \alpha(x))/2)$ and $\gamma_3 = \sup_x \Gamma'((1 + \alpha(x))/2)$. Then $0 < \xi_1 \leq \theta_{\alpha(x)} \leq \xi_2 < \infty$ and $|\nabla_x \theta_{\alpha(x)}| \leq \eta |\nabla_x \alpha(x)|$. Let $u_0 = 1$. In this case we have

$$|\nabla_x f(x, z, \sigma)| \leq \left(1/\zeta_1^2 + (1/\zeta_1 + \eta/\xi_1)\zeta_1 \right) |\nabla_x \alpha(x)|.$$

Therefore if $\alpha(x) = \alpha(x_1)$ and $\sup_x |\nabla_x \alpha(x)| < \zeta_1^2 \xi_1 / (2\xi_1 + \zeta_1 \eta)$, then (SC) holds.

3.2 Proofs

This subsection will be devoted to the proofs of Theorem 3.1 and Theorem 3.3. Recall that $\hat{G}(x, z, \sigma)$ and $f(x, z, \sigma)$ are defined by (3.1) and (3.2) respectively, and $E = (0, \infty)$. The following lemma is obtained by elementary calculation.

Lemma 3.7. *For $r \geq 1$, $(L - r)$ implies $(A - r)$ and that for every $0 \leq i + j \leq r$,*

$$\sup_{x, \sigma} |\nabla_x^i \nabla_z^j f(x, z, \sigma)| \in \bigcap_{2 \leq p \leq \infty} L^p(E, dz) \quad \text{is bounded.}$$

Furthermore, if $r \geq 2$, then $\lim_{z \rightarrow \infty} \sup_{x, \sigma} |\nabla_x f(x, z, \sigma)| = 0$ and for each $\epsilon > 0$, there is a constant $\delta = \delta(\epsilon) > 0$ such that

$$\sup_{z > \delta, x, \sigma} |\nabla_u y(x, \hat{G}(x, z, \sigma), \sigma) - \nabla_u y(x, 0, \sigma)| \leq \epsilon.$$

Proof of Theorem 3.1. It suffices to show that the distribution of the solution X_t^x of (2.4) with $E = (0, \infty)$ has a density for every $0 < t \leq T$ and $x \in \mathbf{R}^d$. For simplicity we drop the super script “ x ” for X_t^x, T_n^x , etc. Let $F(x, z, \sigma)$ be the same as in (2.5) and we define the process R_t with values in the set of all symmetric non-negative definite $d \times d$ -matrices as

$$R_t = \int_0^t \int_{E \times S} \rho(z) F(X_{s-}, z, \sigma) \mu(ds dz d\sigma).$$

This is well-defined by Lemma 7-4 in [2] and satisfies the condition

$$y^\top R_t y \geq y^\top R_s y \geq 0 \quad \text{for all } y \in \mathbf{S}^{d-1} \text{ if } 0 \leq s \leq t.$$

By lemma 2.10 it is enough to show that for any fixed $n \geq 1$ and $0 \leq u < t$,

$$(3.5) \quad \int_u^t \nabla X(n)_s^{-1} dR_s \nabla X(n)_{s-}^{-1, \top} \quad \text{is invertible a.s. on } \{T_n > t\}.$$

Here we restrict the Poisson space and the time interval to $\Omega' = \Omega \cap \{T_n > t\}$ and $[u, t]$ respectively. That is, we take the restricted Poisson space $(\Omega', \mathcal{F}', (\mathcal{F}'_s)_{s \in [u, t]}, P')$ defined by $P'(\cdot) = P(\cdot | \Omega')$, $\mathcal{F}'_s = (\cap_{r > s} \sigma(\{\mu(A) : A \in \mathcal{B}([u, r]) \otimes \mathcal{E} \otimes \mathcal{S}\} \cup \mathcal{N}))|_{\Omega'}$ and $\mathcal{F}' = \mathcal{F}'_t$, where $\mathcal{N}$ is the totality of the P -null sets of $\mathcal{F}$. Then $(\mathcal{F}'_s)_{s \in [u, t]}$ is right-continuous and $\mathcal{F}'_u$ is the trivial field. The proof of (3.5) will be divided into three steps.

(Step 1). We fix $x \in \mathbf{R}^d$ and $\tilde{y} \in \mathbf{S}^{d-1}$ arbitrarily. By the assumption (N) there is a point $\sigma_0 \in \text{supp} M(d\sigma)$ such that $\nabla_u y^\top(x, 0, \sigma_0) \tilde{y} \neq 0$. Note that $\nabla_z f(x, z, \sigma) = -(\nabla_u y/g)(x, \hat{G}(x, z, \sigma), \sigma)$. Then by Lemma 3.7 and the continuity of $\nabla_u y(x, 0, \sigma)$ at σ_0 , there exist a constant $L > 0$ and a neighborhood S_0 of σ_0 such that $\nabla_z f(I + \nabla_x f)^{-1, \top}(x, z, \sigma) \tilde{y} \neq 0$ for every $z \geq L$ and $\sigma \in S_0$.

(Step 2). Next we prove that for each $y \in \mathbf{S}^{d-1}$ and $s \in (u, t]$,

$$(3.6) \quad Q^y(s) \equiv \int_u^s y^\top \nabla X(n)_-^{-1} dR \nabla X(n)_-^{-1, \top} y > 0 \quad P'\text{-a.s..}$$

Suppose that (3.6) does not hold. That is, there exist a vector $y_0 \in \mathbf{S}^{d-1}$ and a time point $s \in (u, t]$ such that $P'(Q^{y_0}(s) = 0) > 0$. For $\beta > 0$, we introduce the local martingale:

$$Y_\beta(s) = \exp \left[-\beta Q^{y_0}(s) + \int_u^s \int_{E \times S} \left\{ 1 - \exp[-\beta \rho(z) \tilde{F}(r, z, \sigma)] \right\} dr dz M(d\sigma) \right],$$

where $\tilde{F}(s, z, \sigma) = y_0^\top \nabla X(n)_{s-}^{-1} F(X_{s-}, z, \sigma) \nabla X(n)_{s-}^{-1, \top} y_0$. First we suppose that $Y_\beta(s)$ is a martingale. If $Q^{y_0}(s) = 0$, then

$$\lim_{\beta \rightarrow \infty} Y_\beta(s) \exp \left[\int_u^s \int_{\{\rho(z) \tilde{F}(r, z, \sigma) \neq 0\}} dz M(d\sigma) \right].$$

By Fatou's lemma we have

$$E' \left[\exp \left[\int_u^s \int_{\{\rho(z) \tilde{F}(r, z, \sigma) \neq 0\}} dz M(d\sigma) \right] \right] \leq 1.$$

Hence it holds almost surely on $\{Q^{y_0}(s) = 0\}$ that

$$(3.7) \quad \int_{\{\rho(z) \tilde{F}(r, z, \sigma) \neq 0\}} dz M(d\sigma) \text{ is finite for almost every } r \in (u, s].$$

In the case where $Y_\beta(s)$ is not a martingale, we also have the same result by using a stopping time argument. Since $\rho > 0$ on E and $\tilde{F}(s, z, \sigma) = |\nabla_z f^\top (I + \nabla_x f)^{-1, \top} (X_{s-}, z, \sigma) \nabla X(n)_{s-}^{-1, \top} y_0|^2$, (3.7) contradicts the result in Step 1. Therefore (3.6) holds.

(Step 3). Finally we prove (3.5). Let

$$K_s = \ker \int_u^s \nabla X(n)^{-1} dR \nabla X(n)^{-1, \top} \quad \text{for } s \in (u, t]$$

and $K_{u+} \cup_{s > u} K_s$ (note that $K_r \supset K_s$ if $r \leq s$). Then (3.6) implies that for each $y \in \mathbf{S}^{d-1}$ and any $s > u$, $P'(y \in K_s) = 0$, i.e., $P'(y \in K_{u+}) = 0$. From this we can deduce that $P'(K_{u+} = \{0\}) = 1$ as follows. Since $\text{rank} \int_u^s \nabla X(n)^{-1} dR \nabla X(n)^{-1, \top}$ is $\mathcal{F}'_s$ -measurable, it follows that $\mathcal{F}'_u$ is the trivial field, we have some $0 \leq k \leq d$ such that $P'(\dim K_{u+} = k) = 1$. If $k \geq 1$, then for each $n \geq 1$, we can choose an $\mathcal{F}'_{u+1/n}$ -measurable unit random vector e_n which belongs to $K_{u+1/n}$, P' -a.s.. Denote by $CL(\{e_n\})$ the totality of cluster points of $\{e_m : m \geq n\}$, that is, $CL(\{e_n\}) = \bigcap_{n \geq 1} \overline{\{e_m : m \geq n\}}$. Then from the triviality of $\mathcal{F}'_u$ we see that there is a deterministic non-empty compact subset C of $\mathbf{S}^{d-1}$ such that $P'(CL(\{e_n\}) = C) = 1$ (note that we can regard $C_n = \overline{\{e_m : m \geq n\}}$ as an $\mathcal{F}'_{u+1/n}$ -measurable random variable with values in the set of all non-empty compact subsets of $\mathbf{S}^{d-1}$, which is a compact metric space with the Hausdorff metric). Hence for any $y \in C$, $P'(y \in K_{u+}) = 1$, which contradicts (3.6). Thus we have $K_t \subset K_{u+} = \{0\}$, P' -a.s., from this (3.5) follows. Therefore the proof of Theorem 3.1 is complete. $\blacksquare$

Next we proceed to the smoothness results. The following inequalities can be easily obtained.

Lemma 3.8. (i) *In case of $0 < \zeta_1 \leq \alpha(x) \leq \zeta_2 < 2$, there are constants $\eta, \eta' > 0$ and $\eta'' \geq 0$ such that*

$$\sup_{x,\sigma} g(x, \widehat{G}(x, z, \sigma), \sigma) \leq \eta'(1+z)^\eta \quad \text{for all } z > \eta''.$$

(ii) In case of $\alpha(x) \equiv 0$, there are constants $\eta, \eta' > 0$ and $\eta'' \geq 0$ such that

$$\sup_{x,\sigma} g(x, \widehat{G}(x, z, \sigma), \sigma) \leq \eta' \exp[\eta z] \quad \text{for all } z > \eta''$$

(in this case we can take $\eta = 1/\xi_1$).

Proof of Theorem 3.3. First note that for S_i in (SN) , we may assume $M(S_i) = M_0 > 0$ for $i = 1, \dots, N$ by modifying $M(d\sigma)$ and $m(x, \sigma)$ as follows:

$$\overline{M}(d\sigma) = M_0 \sum_{i=1}^N M(d\sigma \cap S_i) / M(S_i) + M(d\sigma \setminus S'),$$

where $S' = \sum_{i=1}^N S_i$, and

$$\overline{m}(x, \sigma) = \begin{cases} m(x, \sigma) M(S_i) / M_0 & \text{if } \sigma \in S_i, \\ m(x, \sigma) & \text{otherwise,} \end{cases}$$

(then $\overline{m}(x, \sigma) \geq m(x, \sigma) \geq \xi_1 > 0$).

Next we verify the assertion a) by using Theorem 2.13 together with Lemma 3.7. For this end, it suffices to prove that for any $\zeta, \theta > 0$, the assumption $(B - (\zeta, \theta))$ holds (see §2.2). By the same argument as in the proof of Theorem 3-12 of [2] we see that there exist positive constants ϵ', ϕ and γ such that if $E' = (\phi, \infty)$, then for all $z, y \in \mathbf{R}^d$,

$$(3.8) \quad \sum_{i=1}^N \inf_{z \in E', \sigma \in S_i} y^\top F(x, z, \sigma) y (1+z)^\gamma \geq \epsilon' |y|^2 / 2.$$

In fact, set $\psi = \sup\{|\nabla_u y(x, 0, \sigma)| : x \in \mathbf{R}^d, \sigma \in \sum_{i=1}^N S_i\}$; then $0 < \psi < \infty$ by $(L-1)$ and (N) . Since $\lim_{z \rightarrow \infty} \sup_{x,\sigma} |\nabla_x f(x, z, \sigma)| = 0$, for $V \equiv (I + \nabla_x f)^{-1} - I$, there is a constant $\phi_1 > 0$ such that

$$(3.9) \quad \|V(x, z, \sigma)\| \leq \left(\frac{1}{4\psi} \sqrt{\frac{\epsilon}{N}} \right) \wedge \frac{1}{4} \quad \text{for all } z > \phi_1,$$

where $\|\cdot\|$ denotes the operator norm. Let

$$U(x, z, \sigma) = (I + \nabla_x f(x, z, \sigma))^{-1} \nabla_z f(x, z, \sigma),$$

which is well-defined by (C). Then

$$\begin{aligned} U(x, z, \sigma) &= (I + V(x, z, \sigma)) \nabla_z f(x, z, \sigma) \\ &= -g(x, \widehat{G}(x, z, \sigma), \sigma)^{-1} (\nabla_u y(x, 0, \sigma) + V(x, z, \sigma) \nabla_u y(x, 0, \sigma) \\ &\quad + (I + V(x, z, \sigma)) [\nabla_u y(x, \widehat{G}(x, z, \sigma), \sigma) - \nabla_u y(x, 0, \sigma)]). \end{aligned}$$

Thus Lemma 3.8 (i), (3.8) and (3.9) yield that if $\phi = \phi_1 \vee \delta\left(\frac{1}{5}\sqrt{\frac{\epsilon}{N}}\right) \vee \eta''$, then for all $z > \phi, \sigma \in \sum_{i=1}^N S_i$ and $x, y \in \mathbf{R}^d$,

$$|y^\top U(x, z, \sigma)| \geq \frac{1}{\eta'}(1+z)^{-\eta} \left(|y^\top \nabla_u y(x, 0, \sigma)| - |y| \frac{1}{2} \sqrt{\frac{\epsilon}{N}} \right).$$

Now we fix $x, y \in \mathbf{R}^d$ with $y \neq 0$. From (SN) there is an index i such that $\inf_{\sigma \in S_i} |y^\top \nabla_u y(x, 0, \sigma)|^2 \geq \epsilon |y|^2 / N$. Hence if $z > \phi$ and $\sigma \in S_i$, then

$$|y^\top U(x, z, \sigma)| \geq \frac{1}{\eta'}(1+z)^{-\eta} \frac{1}{2} \sqrt{\frac{\epsilon}{N}} |y|.$$

Therefore, for all $x, y \in \mathbf{R}^d$, (3.8) holds (note that $y^\top F(x, z, \sigma)y = |y^\top U(x, z, \sigma)|^2$).

Now we can choose a function $\rho(z) : E = (0, \infty) \rightarrow (0, \infty)$ such that it satisfies the condition (ρ) and $\rho(z) = (1+z)^{-2}$ for $z > \phi$. If we set $h(z) = (1+z)^{-\gamma} \rho(z) 1_{E'}(z) = (1+z)^{-\gamma-2} 1_{E'}(z)$, then for any $\zeta, \theta > 0$, $h(z)$ is a (ζ, θ) -broad function on E' , which turns to a (ζ, θ) -broad function on E (see Example 2.12). Since

$$\inf_{z: h(z) > 0} \frac{\rho(z)}{h(z)} y^\top F(x, z, \sigma)y = \inf_{z \in E'} y^\top F(x, z, \sigma)y (1+z)^\gamma,$$

our claim follows.

Finally we prove the assertion b). Let $\eta = 1/\xi_1$ and denote by H the constant $2d(r+d+1)/\eta$ (resp. $2d^2(r+1)/\eta$ in (i), $2d(r+2d+1)/\eta$ in (ii), and $4d(r+2d+2)/\eta$ in (iii), respectively. By Theorem 2.13, for the proof of b) it suffices to prove that $(B - (\zeta, \theta))$ holds for $\zeta = \eta, \theta = M_0 t / [H + 1]$. This is proved by a similar argument to the proof of a). In fact, (3.8) holds with $\exp[2\eta z]$ instead of $(1+z)^\gamma$ by Lemma 3.8 (ii). Let ρ be a function satisfying the condition (ρ) and $\rho(z) = z^{-2}$ for $z > \phi$, and set $h(z) = e^{-2\eta z} \rho(z) 1_{E'}(z) = z^{-2} e^{-2\eta z} \rho(z) 1_{E'}(z)$. Since $2\eta < M_0 t / (\eta[H + 1]) = \theta/\eta = \theta/\zeta$ provided $t > 2\eta^2(H + 1)/M_0 (\geq 2\eta^2[H + 1]/M_0)$, h is (ζ, θ) -broad on E . Therefore we see that the assumption $(B - (\zeta, \theta))$ holds. $\blacksquare$

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