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T H E S I S

Computations of trace fields of hyperbolic 3-manifolds

by

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Contents

1. Introduction

For an orientable complete hyperbolic 3-dimensional orbifold M of finite volume, there is a unique Kleinian group Γ of finite covolume satisfying $M \cong \mathbb{H}^3/\Gamma$ up to conjugacy. The minimal field containing $\mathbb{Q}$ and the set of the traces of all elements of Γ is called the trace field of Γ and denoted by $\mathbb{Q}(\mathrm{tr}\Gamma)$. Since $\mathbb{Q}(\mathrm{tr}\Gamma)$ is invariant under conjugations, it does not depend on the choice of Γ.

By the Mostow-Prasad Rigidity Theorem, if complete hyperbolic 3-manifold M of finite volume is homeomorphic to $\mathbb{H}^3/\Gamma_1$ and $\mathbb{H}^3/\Gamma_2$, then Γ_1 is conjugate to Γ_2. Hence $\mathbb{Q}(\mathrm{tr}\Gamma)$ is a topological invariant. It is known that $\mathbb{Q}(\mathrm{tr}\Gamma)$ has finite degree over $\mathbb{Q}$. Since $\mathbb{Q}(\mathrm{tr}\Gamma)$ is not contained in the real field, $\mathbb{Q}(\mathrm{tr}\Gamma)$ is a non-totally real algebraic number field.

Naturally, the following question arises.

Question 1.1. *Does every non-totally real algebraic number field occur as the trace field of some Kleinian group of finite covolume?*

For example, this question is raised in [10]. In this paper, we compute trace fields for some families of hyperbolic 3-manifolds and orbifolds and give some examples of trace fields. Moreover, we consider moduli of cusp cross-sections of Kleinian groups of finite covolume. This paper proceeds as follows:

In section 3, we consider the degrees of trace fields. Hoste and Shanahan proved that for each $n \in \mathbb{N} - \{1\}$, there exists an algebraic number field of degree n which occurs as a trace field in [5]. This result follows from their computations of trace fields for a family of the complements of links.

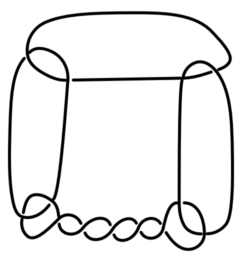

Figure 1. $C(p, s)$ for $p = 4, s = 5$

Let $C(p, s)$ be the complement of a p chain link with s left half twists in $\mathbb{S}^3$ shown in Figure 1. In [14], Neumann and Reid proved that $C(p, s)$ has a complete hyperbolic structure of finite volume if and only if $\{|p + s|, |s|\} \not\subset \{0, 1, 2\}$. Moreover, they computed the trace fields of $C(p, s)$ for $|p + s|, |s| \leq 13$. Based on their work, Hoste and Shanahan computed the trace fields of $C(1, s)$ in [5].

One of the purpose of section 3 is to compute the trace fields of $C(2,s)$. To state it, define $\psi_s(x)$ to be

$$\begin{cases} (x^{\frac{s}{2}+1} - x^{-(\frac{s}{2}+1)}) + (x^{\frac{s}{2}} + x^{-\frac{s}{2}}) & (\frac{s}{2}\text{: even}) \\ (x^{\frac{s}{2}+1} + x^{-(\frac{s}{2}+1)}) + (x^{\frac{s}{2}} - x^{-\frac{s}{2}}) & (\frac{s}{2}\text{: odd}) \\ (x^{s+1} + x^{-(s+1)}) + 2\sum_{j=0}^{\frac{s-1}{2}} (-1)^j (x^{s-2j} - x^{-(s-2j)}) & (s\text{: odd}). \end{cases}$$

Since $x^{2j-1} - x^{-(2j-1)}$ and $x^{2j} + x^{-2j}$ can be written as a polynomial in $z = x - x^{-1}$, $\psi_s(x)$ can be written in $z = x - x^{-1}$, which we denote by $\Psi_s(z)$. The following theorem is the first main result in this paper.

Theorem 1.2. (1) *$\Psi_s(z)$ is irreducible.*
(2) *For $s \in \mathbb{Z}\setminus\{-2,-1,0\}$, the trace field of $C(2,s)$ is $\mathbb{Q}(w)$, where w is a root of $\Psi_s(z)$ for $s > 0$, or $\Psi_{-(2+s)}(z)$ for $s < -2$.*

Remark 1.3. The explicit form of $\Psi_s(z)$ is in section 3.

According to the computation done by Hoste and Shanahan, the degree of the trace field of $C(1,s)$ runs over all elements of $\mathbb{N}\setminus\{1\}$ as s runs over all elements of $\mathbb{Z}$. By Theorem 1.2, we have

Corollary 1.4. *The degree of the trace field of $C(2,s)$ runs over all elements of $\mathbb{N}\setminus\{1\}$ as s runs over all elements of $\mathbb{Z}$.*

Chain links can be obtained by performing Dehn fillings on a cusp of the Whitehead link complement. Jeon proved that for each cusped hyperbolic 3-manifold M and $N \in \mathbb{N}$, there exist only finitely many Dehn fillings of M whose trace fields have degree $< N$ in [7]. Thus, the set of p's such that the degree of the trace field of $C(p,s)$ runs over all elements of $\mathbb{N}\setminus\{1\}$ as s runs over all elements of $\mathbb{Z}$ is finite. By the work of T. Chinburg, such p's are contained in the set $\{1,2,3,4,5,6,8,12\}$ (the proof can be found in [14]). We prove that such p's are contained in the set $\{1,2,4,8\}$.

In section 4, we consider the question whether cyclotomic fields occur as trace fields or not. It is known that some cyclotomic fields occur as trace fields. For example, $\mathbb{Q}(e^{2\pi i/4})$ occurs as the trace field of the complement of the Whitehead link, and $\mathbb{Q}(e^{2\pi i/5})$ occurs as the trace field of some Fibonacci manifold (see [10], Theorem 4.8.2).

Hilden, Lozano and Montesinos have considered a three parameter family $\hat{B}(m,n,p)$, $3 \le m,n,p \le \infty$, of hyperbolic orbifolds, which are called Borromean orbifolds, with singular loci along the Borromean rings (the cone angles at the three components are $\frac{2\pi}{m}, \frac{2\pi}{n}, \frac{2\pi}{p}$). They have computed the trace fields of Borromean orbifolds in [4].

We consider Borromean orbifolds to prove that infinitely many cyclotomic fields occur as trace fields. Based on works in [4], we decide whether $\mathbb{Q}(\mathrm{tr}B(m,\infty,\infty))$ is identical to some cyclotomic field or not. The following theorem is the second main result in this paper.

Theorem 1.5. (1) *If m is even, then $\mathbb{Q}(\mathrm{tr}B(m,\infty,\infty)) = \mathbb{Q}(e^{\pi i/m})$.*

(2) *If m is odd and larger than 3, then $\mathbb{Q}(\mathrm{tr}B(m,\infty,\infty))$ is not a cyclotomic field.*
(3) $\mathbb{Q}(\mathrm{tr}B(3,\infty,\infty)) = \mathbb{Q}(i)$.

Next we consider the case when both m and n are in $\mathbb{N}$. By this theorem, if k is odd, then $\mathbb{Q}(e^{\pi i/k})$ does not occur as $\mathbb{Q}(\mathrm{tr}B(m,\infty,\infty))$ for any m. However, $\mathbb{Q}(e^{\pi i/k})$ may occur as $\mathbb{Q}(\mathrm{tr}B(m,n,\infty))$ for some $m,n \in \mathbb{N}$. We want to find cyclotomic fields which do not occur as trace fields of non-compact Borromean orbifolds. For this purpose, we establish the following.

Theorem 1.6. *There exists an algorithm to decide whether $\mathbb{Q}(e^{\pi i/q})$ occurs as the trace field of some non-compact Borromean orbifold or not for a prime number q.*

Applying the algorithm in Theorem 1.6, we have checked that $\mathbb{Q}(e^{\pi i/q})$ does not occur as the trace field of some non-compact Borromean orbifold for prime numbers up to 23.

In section 5, we consider moduli of cusp cross-sections of hyperbolic 3-manifolds. If a hyperbolic 3-manifold has finite volume, then each cusp cross-section admits a structure of a complex torus. Hence we can assign an element of $\mathbb{H}/\mathrm{PSL}(2,\mathbb{Z})$ to each cusp cross-section. It is known that if Γ is non-cocompact, then Γ is conjugate to a subgroup of $\mathrm{PSL}(2,\mathbb{Q}(\mathrm{tr}\Gamma))$. Thus, the representatives of the moduli of cusp cross-sections of Γ are contained in $\mathbb{Q}(\mathrm{tr}\Gamma)$.

Nimershiem proved the set of moduli of cusp cross-sections of hyperbolic 3-manifolds of finite volume is dense in $\mathbb{H}/\mathrm{PSL}(2,\mathbb{Z})$ in [13]. Furthermore, McReynolds proved that of arithmetic hyperbolic 3-manifolds is dense in $\mathbb{H}/\mathrm{PSL}(2,\mathbb{Z})$ in [12].

We consider the following question.

Question 1.7. *Does every modulus of complex torus whose representatives are in some non-totally real algebraic number field appears as the modulus of a cusp cross-section of some hyperbolic 3-manifold of finite volume?*

At first, we answer a special case of Question 1.7.

Theorem 1.8. *Every element of an imaginary quadratic number field arises as a representative of the modulus of a cusp cross-section of an arithmetic hyperbolic 3-manifold.*

It is known that the modulus of a cusp cross-section of an arithmetic hyperbolic 3-manifold has its representatives in an imaginary quadratic field. Hence a complex torus arises as a cusp cross-section of an arithmetic hyperbolic 3-manifold if and only if the torus has its representatives in an imaginary quadratic field.

Furthermore, we prove the following.

Theorem 1.9. *Let K be a non-totally real algebraic number field of degree 3. If there is a non-cocompact finite covolume Kleinian group Γ with integral traces satisfying $K = \mathbb{Q}(\mathrm{tr}\Gamma)$, then every element of K arises as a representative of the modulus of a cusp cross-section of a hyperbolic 3-manifold of finite volume.*

It is known that there is a non-totally real algebraic number field of degree 3 satisfying the conditions in Theorem 1.9. Hence, the set of moduli of cusp cross-sections of non-arithmetic hyperbolic 3-manifolds of finite volume is dense in the moduli space $\mathbb{H}/\mathrm{PSL}(2,\mathbb{Z})$.

2. Preliminaries

2.1. Trace fields and invariant trace fields.

2.1.1. *Kleinian groups.* The group $\mathrm{PSL}(2,\mathbb{C})$ is the quotient of the group $\mathrm{SL}(2,\mathbb{C})$ of all 2×2 matrices with complex entries and determinant 1 by its center $\{\pm I\}$. $\mathrm{PSL}(2,\mathbb{C})$ acts on $\widehat{\mathbb{C}} = \mathbb{C}\cup\{\infty\}$ by Möbius transformations. By the Poincaré extension, every Möbius transformation extends to an orientation-preserving isometry on the upper half 3-space $\mathbb{H}^3$ equipped with the hyperbolic metric (cf.[18]). Moreover, every orientation-preserving isometry is obtained by the Poincaré extension of some Möbius transformation. Hence $\mathrm{PSL}(2,\mathbb{C})$ can be identified with the group of orientation-preserving isometries of $\mathbb{H}^3$.

Definition 2.1. A discrete subgroup Γ of $\mathrm{PSL}(2,\mathbb{C})$ is called a Kleinian group.

If Γ is torsion-free, then $\mathbb{H}^3/\Gamma$ is an orientable hyperbolic 3-manifold. If Γ is a conjugate of Γ' in $\mathrm{PSL}(2,\mathbb{C})$, then $\mathbb{H}^3/\Gamma$ and $\mathbb{H}^3/\Gamma'$ are isometric. Conversely every orientable hyperbolic 3-manifold M has the form $\mathbb{H}^3/\Gamma$, where Γ is a torsion-free Kleinian group, uniquely determined by the isometry class of M up to conjugacy.

Definition 2.2. A Kleinian group is said to be elementary if Γ has a finite orbit in its action on $\hat{\mathbb{C}}$. If not, Γ is said to be non-elementary.

Definition 2.3. Kleinian groups Γ_1 and Γ_2 are said to be commensurable if Γ_1 and Γ_2 have a common subgroup of finite index. Γ_1 and Γ_2 are said to be commensurable in the wide sense if Γ_1 is conjugate to a Kleinian group Γ_3 such that Γ_2 and Γ_3 are commensurable.

2.1.2. *Trace fields and invariant trace fields.*

Definition 2.4. Let Γ be a Kleinian group. $\mathbb{Q}(\{\mathrm{tr}\gamma \mid \gamma\in\Gamma\})$ is called the trace field of Γ and denoted by $\mathbb{Q}(\mathrm{tr}\Gamma)$.

By the definition, $\mathbb{Q}(\mathrm{tr}\Gamma)$ is invariant under conjugations.

Theorem 2.5. *Let Γ be a Kleinian group of finite covolume. Then $\mathbb{Q}(\mathrm{tr}\Gamma)$ is a finite extension of $\mathbb{Q}$.*

Theorem 2.6 ([10], Theorem 3.3.4)**.** *Let $\Gamma^{(2)}$ be the subgroup of Γ generated by the set $\{\gamma^2 \mid \gamma \in \Gamma\}$. Then $\mathbb{Q}(\mathrm{tr}\Gamma^{(2)})$ is an invariant of the commensurability class of Γ.*

Definition 2.7. $\mathbb{Q}(\mathrm{tr}\Gamma^{(2)})$ is called the invariant trace field of Γ.

Definition 2.8. For a hyperbolic 3-manifold $M = \mathbb{H}^3/\Gamma$, $\mathbb{Q}(\mathrm{tr}\Gamma)$ and $\mathbb{Q}(\mathrm{tr}\Gamma^{(2)})$ are called the trace field of M and the invariant trace field of M, respectively. We denote the invariant trace field of M by kM.

For the complement of a link in $\mathbb{S}^3$, the trace field coincides with the invariant trace field by the following theorem.

Theorem 2.9 ([10], Corollary 4.2.2)**.** *If $M = \mathbb{H}^3/\Gamma$ is the complement of a link in a $\mathbb{Z}_2$-homology sphere, then the trace field coincides with the invariant trace field.*

Remark 2.10. The trace field and the invariant trace field of M is well-defined since trace is invariant under conjugations.

Definition 2.11. If for each $\gamma \in \Gamma$, $\mathrm{tr}\gamma$ is an algebraic integer, then Γ is said to have integral traces.

Lemma 2.12. *Let Γ be a non-cocompact Kleinian group of finite covolume, and K be the trace field of Γ with the ring of integers $\mathcal{O}_K$. If Γ has integral traces, then Γ is conjugate in $\mathrm{PSL}(2,\mathbb{C})$ to a subgroup of*

$$\mathrm{PSL}(2,\mathcal{O}_K;J) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,d \in \mathcal{O}_K, b \in J^{-1}, c \in J \right\}$$

for some fractional ideal J of K.

Proof. See the proof of Theorem 3.3.8 and Lemma 5.2.4 in [10]. □

2.1.3. *Volumes and degrees of trace fields.*

Theorem 2.13 (Jeon)**.** *For each $D > 0$ and $N \in \mathbb{N}$, there exist only finitely many orientable complete hyperbolic 3-manifolds of degree less than N and volume less than D.*

We now recall the following theorem to describe the theorem from another viewpoint.

Theorem 2.14. *For each $D > 0$, there exist only finitely many complete hyperbolic 3-manifolds $M_1, \cdots, M_m$ such that every complete hyperbolic 3-manifold of volume less then D can be obtained by performing Dehn surgery on some cusps of M_i.*

Theorem 2.15 (Jeon)**.** *Let M be a complete hyperbolic 3-manifold of finite volume with toral ends $T_1, \cdots, T_n$. Then for each N the set of pairs (p_i, q_i) such that the orbifold obtained by performing (p_i, q_i)-Dehn surgery on T_i has the trace field of degree $< N$ is finite.*

Hodgson proved it in the case M has a single cusp. The proof can be found in [9]. Jeon had conjectured it in [6] and proved it in [7].

Theorem 2.16 ([10], Theorem 8,2,3)**.** *Let Γ be a non-cocompact arithmetic Kleinian group. Then Γ is commensurable in the wide sense with a Bianchi group.*

3. Invariant trace fields of chain links

3.1. **Chain links.** Exposition of this subsection mainly follows that of [14].

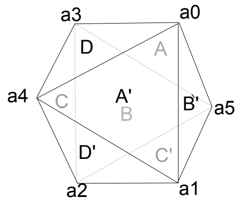

Figure 2. octahedron

3.1.1. *Whitehead link.* We denote the complement of the Whitehead link in $\mathbb{S}^3$ by W. W can be obtained by identifying the faces of an ideal octahedron, gluing A to A', B to B' and so on, as shown in Figure 2. Now we consider the orbifold $W(p,q)$ obtained by performing (p,q)

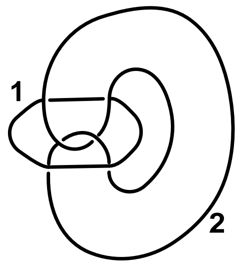

Figure 3. Whitehead link

Dehn surgery of W at the toral end 1 in Figure 3 for $p, q \in \mathbb{Z}$. We do not assume that p and q are coprime, so that $W(p,q)$ can be an orbifold.

Let (u_1, v_1) and (u_2, v_2) be analytic Dehn surgery parameters for the toral ends 1 and 2. Since we only perform Dehn surgery on the toral end 1, we have $u_2 = 0$ and $v_2 = 0$, by the requirement that the hyperbolic structure on the toral end 2 is complete.

To represent (u_1, v_1) in terms of shapes of octahedra, we consider a special ideal octahedron in $\mathbb{H}^3$ with vertices at 0, 1, ∞, -1, x, and x^{-1} on $\partial\mathbb{H}^3 = \widehat{\mathbb{C}}$.

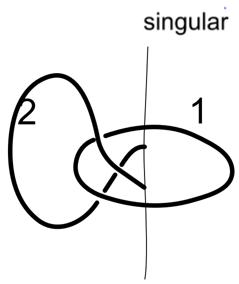

FIGURE 4. Orbifold W'

In [14], Neumann and Reid obtained the following equations.

$$\begin{aligned} u_1 &= \log x + \log(x+1) - \log(x-1) && (1)\\ v_1 &= 4\log x - 2\pi i. && (2) \end{aligned}$$

Here log denotes the standard branch of natural log on the complex plane splitting along $(-\infty, 0]$.

$$e = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{PGL}(2, \mathbb{C})$$

rotates W about the axis through a_1 and a_3. We denote the quotient space $W/\{e\}$ by W'. Then W' inherits a structure as a hyperbolic orbifold for each $x \in \mathbb{H}$.

For the orbifold W', we can obtain

$$\begin{aligned} u_1' &= \log x + \log(x+1) - \log(x-1) && (3)\\ v_1' &= 2\log x - \pi i, && (4) \end{aligned}$$

where (u_1', v_1') is an analytic Dehn surgery parameter for the toral end 1 of W' in the case that the hyperbolic structure on the toral end 2 is complete.

For each $x \in \mathbb{H}$, the real Dehn surgery parameter $(p_1(x), q_1(x))$ takes their value in $\mathbb{R}^2 - \mathcal{N}$ where $\mathcal{N}$ denotes the closed parallelogram in $\mathbb{R}^2$ with vertices $\pm(-4, 1)$, $\pm(0, 1)$. Hence $W(p, q)$ has a hyperbolic structure for integer pair $(p, q) \in \mathbb{R}^2 - \mathcal{N}$. In this paper, we only consider the case $W(p, q)$ for $(p, q) \in \mathbb{Z}^2$. The following theorem allows us to compute the invariant trace fields of $W(p, q)$ and $W'(p, 2q)$.

Theorem 3.1 (Neumann-Reid[14], Theoem 6.2)**.**

$$kW(p, q) = \mathbb{Q}(\alpha - \alpha^{-1}) \ (q \in \mathbb{Z})$$

$$kW'(p, 2q) = \mathbb{Q}(\alpha - \alpha^{-1}) \ (q \in \frac{1}{2}\mathbb{Z}).$$

3.1.2. *Chain links.* Let $C(p,s)$ denote the complement of a p chain link in $\mathbb{S}^3$ with s left half twists. $C(p,2q)$ is homeomorphic to the manifold obtained by performing (p,q) Dehn filling at the toral end 1 of W, and then taking the p-fold cover of the resulting manifold or orbifold. Hence, if $W(p,q)$ admits a complete hyperbolic structure, then $C(p,2q)$ admits a complete hyperbolic structure. Therefore, there exist Kleinian groups Γ and Γ' such that $\mathbb{H}^3/\Gamma$ and $\mathbb{H}^3/\Gamma'$ are isometric to $C(p,2q)$ and $W(p,q)$ respectively. Since $C(p,2q)$ is a cover of $W(p,q)$, Γ' is conjugate to a finite index subgroup of Γ. Hence we have $kW(p,q) = kC(p,2q)$.

To obtain $C(p,s)$ for s odd, we use W'. We obtain the orbifold $W'(p,s)$ by performing (p,s) Dehn filling of the toral end of W'. This orbifold is a quotient of $C(p,s)$. Neumann and Reid proved the following theorem in [14].

Theorem 3.2 (Neumann-Reid[10], Theorem 5.1)**.**
(1) *$C(p,s)$ has a hyperbolic structure if and only if $\{|p+s|,|s|\} \not\subset \{0,1,2\}$.*
(2) *$C(p,s)$ and $C(p',s')$ have the same invariant trace field if $(p'+s',s') = \pm(p+s,s)$ or $(p'+s',s') = \pm(-s,p+s)$.*

By this theorem, if we obtain the invariant trace fields of $C(2,s)$ in the case $s > 0$, then we can obtain the invariant trace fields of $C(2,s)$ in the case $s < -2$. Hence we compute the invariant trace field (which coincides with the trace field by Theorem 2.9) of $C(2,s)$ only in the case $s > 0$ in the next subsection.

3.2. **Proof of Theorem 1.2.** In this subsection, we compute the trace fields of $C(2,s)$. To prove $\Psi_s(z)$ is indeed irreducible, we need the lemma proved by Hoste and Shanahan. Hence we first recall the lemma with their proof.

Definition 3.3. Let $p(x) \in \mathbb{Z}[x]$ be a polynomial with no repeated roots. $p(x)$ is called complete if $p(\alpha) = 0$ implies $p(-\alpha^{-1}) = 0$.

The following lemma is a part of the lemma 3 in [5]. To obtain Theorem 1.2, we recall it.

Lemma 3.4 (Hoste-Shanahan)**.** *Let $p(x)$ be a complete polynomial which has no roots on the imaginary axis. If the leading and the next coefficients of $p(x)$ are 1 and the norm of every root in the first quadrant is less than 1, then $p(x)$ does not factor into two complete factors in $\mathbb{Z}[x]$.*

Before we prove Theorem 1.2, we prove the following lemma.

Lemma 3.5. *The polynomial $f(x) = (x+1)^p x^{p+4q} - (x-1)^p \in \mathbb{Z}[x]$ has no repeated roots*

Proof. Not open to the public. □

First, we prove Theorem 1.2 in the case s is even. In this case, we can apply Lemma 3.4 directly.

Theorem 3.6. *If $s \in \mathbb{Z}$ is even, then the trace field of $C(2,s)$ is $\mathbb{Q}(w)$, where w is a root of the irreducible polynomial $\Psi_s(z)$ in Theorem 1.2.*

Proof. Not open to the public. □

Next we prove Theorem 1.2 in the case that s is odd. In this case, we cannot apply Lemma 3.4 directly, so we extend Lemma 3.4 in the proof of Theorem 3.7.

Theorem 3.7. *If $s \in \mathbb{Z}$ is odd, then the trace field of $C(2,s)$ is $\mathbb{Q}(w)$, where w is a root of the irreducible polynomial $\Psi_s(z)$.*

Proof. Not open to the public. □

By Theorem 3.6 and 3.7, we obtain Theorem 1.2.

In the rest of this section, we prove the following theorem.

Theorem 3.8. *If for fixed $p \in \mathbb{Z}$ the degree of the trace field of $C(p,s)$ runs over all elements of $\mathbb{N} \setminus \{1\}$ as s runs over all elements of $\mathbb{Z}$, then p is contained in the set $\{1,2,4,8\}$.*

To prove Theorem 3.8, we first prove the following theorem.

Theorem 3.9. *Fix $p \in \mathbb{N}$. For any $n \in \mathbb{N}$ there exists s_0 such that if s is larger than s_0, then the degree of $kC(p,s)$ is larger than n.*

Proof. Not open to the public. □

We now prove Theorem 3.8.

Proof. Not open to the public. □

Remark 3.10. If p is 1 or 2, then the degree of the trace field of $C(p,s)$ runs over all elements of $\mathbb{N} \setminus \{1\}$ as s runs over all elements of $\mathbb{Z}$. However I don't know, in the case p is 4 or 8, whether this property holds or not.

At the end of this section, we describe the explicit form of $\Psi_s(z)$ in Theorem 1.2.

$$\begin{aligned} z^{\frac{s}{2}+1} &+ z^{\frac{s}{2}} + \sum_{i=1}^{\frac{s}{4}} \left\{ \sum_{n=0}^{i} \sum_{i_1+\cdots+i_n=i} C_{i_1\cdots i_n} \right\} z^{\frac{s}{2}+1-2i} \\ &+ \sum_{i=1}^{\frac{s}{4}} \left\{ \sum_{n=0}^{i} (-1)^n \sum_{i_1+\cdots+i_n=i} D_{i_1\cdots i_n} \right\} z^{\frac{s}{2}-2i} \ (\frac{s}{2}\text{: even}) \end{aligned}$$

$$\begin{aligned} z^{\frac{s}{2}+1} &+ z^{\frac{s}{2}} + \sum_{i=1}^{\frac{s+2}{4}} \left\{ \sum_{n=0}^{i} (-1)^n \sum_{i_1+\cdots+i_n=i} C_{i_1\cdots i_n} \right\} z^{\frac{s}{2}+1-2i} \\ &+ \sum_{i=1}^{\frac{s-2}{4}} \left\{ \sum_{n=0} \sum_{i_1+\cdots+i_n=i} D_{i_1\cdots i_n} \right\} z^{\frac{s}{2}-2i} \ (\frac{s}{2}\text{: odd}) \end{aligned}$$

$$
\begin{aligned}
z^{s+1} \quad &+ \quad \sum_{i=1}^{\frac{s+1}{2}} \left\{ \sum_{n=0}^{i} \sum_{i_1+\cdots+i_n=i} E_{i_1\cdots i_n} \right\} z^{s+1-2i} \\
&+ \quad \sum_{i=1}^{\frac{s-1}{2}} \left\{ 2 \sum_{j}^{\frac{s-1}{2}} \sum_{n=0}^{i} \sum_{i_1+\cdots+i_n=i} F_{i_1\cdots i_n} \right\} z^{s-2i} \ (s\text{: odd}).
\end{aligned}
$$

where

$$C_{i_1\cdots i_n} = (-1)^{1+i_1} \cdots (-1)^{1+i_n} \begin{pmatrix} \frac{s}{2}+1 \\ i_1 \end{pmatrix} \cdots \begin{pmatrix} \frac{s}{2}+1-2i_1-\cdots-2i_{n-1} \\ i_n \end{pmatrix},$$

$$D_{i_1\cdots i_n} = \begin{pmatrix} \frac{s}{2} \\ i_1 \end{pmatrix} \cdots \begin{pmatrix} \frac{s}{2}-2i_1-\cdots-2i_{n-1} \\ i_n \end{pmatrix},$$

$$E_{i_1\cdots i_n} = (-1)^n \begin{pmatrix} s+1 \\ i_1 \end{pmatrix} \cdots \begin{pmatrix} s+1-2i_1-\cdots-2i_{n-1} \\ i_n \end{pmatrix},$$

and

$$F_{i_1\cdots i_n} = (-1)^j(-1)^{1+i_1} \cdots (-1)^{1+i_n} \begin{pmatrix} s-2j \\ i_1 \end{pmatrix} \cdots \begin{pmatrix} s-2j-2i_1-\cdots-2i_{n-1} \\ i_n \end{pmatrix}.$$

These polynomials can be simplified by letting $z = x - x^{-1}$ as follows:

$$
\begin{cases}
(x^{\frac{s}{2}+1} - x^{-(\frac{s}{2}+1)}) + (x^{\frac{s}{2}} + x^{-\frac{s}{2}}) & (\frac{s}{2}\text{: even}) \\
(x^{\frac{s}{2}+1} + x^{-(\frac{s}{2}+1)}) + (x^{\frac{s}{2}} - x^{-\frac{s}{2}}) & (\frac{s}{2}\text{: odd}) \\
(x^{s+1} + x^{-(s+1)}) + 2\sum_{j=0}^{\frac{s-1}{2}} (-1)^j (x^{s-2j} - x^{-(s-2j)}) & (s\text{: odd})
\end{cases}
$$

4. Trace fields of non-compact Borromean orbifolds

The aim of this section is to prove Theorem 1.5 and Theorem 1.6. For the proofs, we recall the work of Hilden, Lozano and Montesinos in [4].

4.1. **Borromean orbifolds.** For further details for this subsection, the reader can refer to [4].

4.1.1. *Borromean orbifolds.* A Borromean orbifold $\hat{B}(m,n,p)$ is an orbifold with underlying space S^3 and singular locus along the Borromean rings such that the isotropy groups of the components of the singular locus are cyclic of order m,n,p. If m,n,p are larger than 2, then $\hat{B}(m,n,p)$ admits a hyperbolic structure. In this case, for each $\hat{B}(m,n,p)$, there exists a Kleinian group $B(m,n,p)$ satisfying $\hat{B}(m,n,p) \cong \mathbb{H}^3/B(m,n,p)$.

Hilden, Lozano, and Montesinos have proved the following theorem.

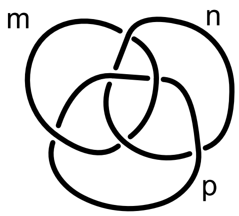

FIGURE 5. Borromean orbifold

Theorem 4.1 (Hilden-Lozano-Montesinos, [4], Proposition 5.4.). *If m, n, p are integers larger than 2, then*

$$\mathbb{Q}(\mathrm{tr}B(m,n,p)) = \mathbb{Q}\left(\cos\frac{\pi}{m}, \cos\frac{\pi}{n}, \cos\frac{\pi}{p}, iABC\right),$$

where A, B, C satisfy

$$C^2(A^2-1) = \tan^2\frac{\pi}{m},$$
$$A^2(B^2-1) = \tan^2\frac{\pi}{n} \quad \text{and}$$
$$B^2(C^2-1) = \tan^2\frac{\pi}{p}.$$

This theorem allows us to compute trace fields of Borromean orbifolds.

Hereafter, we only consider non-compact Borromean orbifolds and assume p to be ∞. In this case, $C = 1$ and so we have

$$\tan^2\frac{\pi}{m} = A^2 - 1, \tag{5}$$
$$\tan^2\frac{\pi}{n} = A^2(B^2-1). \tag{6}$$

Hence by Theorem 4.1, we obtain

$$\mathbb{Q}(\mathrm{tr}B(m,n,\infty)) = \mathbb{Q}\left(\cos\frac{\pi}{m}, \cos\frac{\pi}{n}, iAB\right).$$

Following a similar procedure, we have

$$\mathbb{Q}(\mathrm{tr}B(m,\infty,\infty)) = \mathbb{Q}\left(\cos\frac{\pi}{m}, iA\right).$$

4.2. **Proof of Theorem 1.5.** In this subsection, we first compute the degree of $\mathbb{Q}(\mathrm{tr}B(m,n,\infty))$ for $3 \le m, n \le \infty$. Moreover, we decide if $\mathbb{Q}(\mathrm{tr}B(m,\infty,\infty))$ is identical to some cyclotomic field for each m.

Remark 4.2. Since $\hat{B}(\infty,\infty,\infty)$ is the complement of the Borromean rings, $\mathbb{Q}(\mathrm{tr}B(\infty,\infty,\infty)) = \mathbb{Q}(i)$.

Lemma 4.3. *Let $\hat{B}(m,n,p)$ be a Borromean orbifold and $B(m,n,p)$ a Kleinian group satisfying $\hat{B}(m,n,p) \cong \mathbb{H}^3/B(m,n,p)$.*

(1) $\mathbb{Q}(\mathrm{tr}B(m,n,\infty)) = \mathbb{Q}\left(\cos\frac{\pi}{m}, \cos\frac{\pi}{n}, \sqrt{-\cos^2\frac{\pi}{m}\tan^2\frac{\pi}{n} - 1}\right)$.

(2) $\mathbb{Q}(\mathrm{tr}B(m,\infty,\infty)) = \mathbb{Q}\left(\cos\frac{\pi}{m}, i\right)$.

Proof. Not open to the public. □

The following lemma allows us to compute the degree of $\mathbb{Q}(\mathrm{tr}B(m,n,\infty))$ over $\mathbb{Q}$.

Lemma 4.4. *Let l be the least common multiple of m and n. Denote the greatest common divisor by (m,n).*

(1) *If $(m,n) \neq 1$, then $\mathbb{Q}\left(\cos\frac{\pi}{l}\right) = \mathbb{Q}\left(\cos\frac{\pi}{m}, \cos\frac{\pi}{n}\right)$.*

(2) *If $(m,n) = 1$, then $\mathbb{Q}\left(\cos\frac{\pi}{l}\right)$ has degree 2 over $\mathbb{Q}\left(\cos\frac{\pi}{m}, \cos\frac{\pi}{n}\right)$.*

Proof. Not open to the public. □

As a corollary, we obtain the following:

Corollary 4.5. *Let l be the least common multiple of m and n. The degree of $\mathbb{Q}(\mathrm{tr}B(m,n,\infty))$ is as follows:*

$$\begin{cases} \frac{1}{2}\varphi(2l) & \text{if } (m,n) = 1, \\ \varphi(2l) & \text{if } (m,n) \neq 1, \end{cases}$$

where φ is the Euler function.

Next, we prove Theorem 1.5.

Proof of Theorem 1.5 (1). Not open to the public. □

In order to complete the proof of Theorem 1.5, we need the following lemma.

Lemma 4.6. *Assume that m is odd. Let $G = \{id, \tau\}$ be a subgroup of $Gal(\mathbb{Q}(e^{\pi i/2m}), \mathbb{Q})$, where τ is the involution in $Gal(\mathbb{Q}(e^{\pi i/2m}), \mathbb{Q})$ which sends $e^{\pi i/2m}$ to $e^{(2m-1)\pi i/2m}$. Then, $\mathbb{Q}\left(\cos\frac{\pi}{m}, i\right)$ is the G-invariant subfield of $\mathbb{Q}(e^{\pi i/2m})$, namely*

$$\mathbb{Q}\left(\cos\frac{\pi}{m}, i\right) = \{x \in \mathbb{Q}(e^{\pi i/2m}) \mid \forall g \in G, g(x) = x\}.$$

Proof. Not open to the public. □

Proofs of Theorem 1.5 (2) and (3). Not open to the public. □

4.3. **Proof of Theorem 1.6.** In this subsection, we prove Theorem 1.6. We first show the following:

Proposition 4.7. *Let q be an odd prime number. Then $\mathbb{Q}(e^{\pi i/q})$ occurs as the trace field of a non-compact Borromean orbifold if and only if either*

$$D = \sqrt{(2 - e^{2\pi i/q} - e^{-2\pi i/q})(6 - e^{2\pi i/q} - e^{-2\pi i/q})},$$

or

$$E = \sqrt{(2 - e^{2\pi i/q} - e^{-2\pi i/q})(10 + 3e^{2\pi i/q} + 3e^{-2\pi i/q})}$$

is contained in $\mathbb{Q}\left(\cos\frac{\pi}{q}\right)$.

The proof of this proposition requires the following three lemmas.

Lemma 4.8. *Let q be an odd prime number. Then $\mathbb{Q}(e^{\pi i/q})$ occurs as the trace field of a non-compact Borromean orbifold if and only if $\sqrt{-1-\sin^2\frac{\pi}{q}}$ or $\sqrt{-\frac{1}{4}\tan^2\frac{\pi}{q}-1}$ is contained in $\mathbb{Q}(e^{\pi i/q})$.*

Proof. Not open to the public. □

Lemma 4.9. *$\sqrt{-1-\sin^2\frac{\pi}{q}}$ is in $\mathbb{Q}(e^{\pi i/q})$ if and only if $\sqrt{\frac{1+\sin^2\frac{\pi}{q}}{\sin^2\frac{\pi}{q}}}$ is in $\mathbb{Q}\left(\cos\frac{\pi}{q}\right)$.*

Proof. Not open to the public. □

Similarly, we can prove the following lemma.

Lemma 4.10. *$\sqrt{-1-\frac{1}{4}\tan^2\frac{\pi}{q}}$ is in $\mathbb{Q}(e^{\pi i/q})$ if and only if $\sqrt{\frac{4+\tan^2\frac{\pi}{q}}{4\sin^2\frac{\pi}{q}}}$ is in $\mathbb{Q}\left(\cos\frac{\pi}{q}\right)$.*

Now we prove Propsition 4.7.

Proof of Proposition 4.7. Not open to the public. □

Remark 4.11. $\mathbb{Q}(e^{\pi i/q}) = \mathbb{Q}(e^{2\pi i/q})$ for an odd prime number q.

Next we prove the following:

Lemma 4.12. *Let ζ denote $e^{2\pi i/q}$, and let $Y = a_1\zeta + a_2\zeta^2 + \cdots + a_{q-2}\zeta^{q-2} + a_{q-1}\zeta^{q-1}$ be an element of $\mathbb{Q}\left(\cos\frac{2\pi}{q}\right)$. Then Y^2 has the form*

$$c_0 + c_1\zeta + \cdots + c_{q-2}\zeta^{q-2},$$

where c_0 satisfies $c_0 = a_1^2 + (a_1-a_2)^2 + (a_2-a_3)^2 + \cdots + (a_{\frac{q-1}{2}-1} - a_{\frac{q-1}{2}})^2$.

Remark 4.13. $\{\zeta, \cdots, \zeta^{q-1}\}$ is a basis of $\mathbb{Q}(e^{2\pi i/q})$ over $\mathbb{Q}$.

Proof. Not open to the public. □

We are finally ready to prove Theorem 1.6.

Proof of Theorem 1.6. Not open to the public. □

By utilizing Mathematica, we have checked that $\mathbb{Q}(e^{2\pi i/q})$ does not occur for q up to 23. However, we don't know whether there exists any other prime number q such that $\mathbb{Q}(e^{2\pi i/q})$ occurs as the trace field of some non-compact Borromean orbifold or not.

5. On the moduli of cusps of Kleinian groups

The aim of this section is to prove Theorem 1.8 and 1.9.

5.1. **Moduli of cusp cross-sections.** In this subsection, we explain the method for assigning an element of $\mathcal{M}_1$ to a cusp cross-section.

5.1.1. *The moduli space of complex tori.*

Definition 5.1. Let $\{\omega_1, \omega_2\}$ be a basis of $\mathbb{C}$ as a real vector space of rank 2. A $\mathbb{Z}$-module generated by ω_1, ω_2 is called a lattice.

A quotient space of $\mathbb{C}$ by a lattice is called a complex torus. If a complex torus has the form $\mathbb{C}/\langle\omega_1, \omega_2\rangle$, then $\{\omega_1, \omega_2\}$ is called a basis of the complex torus.

Definition 5.2. The moduli space $\mathcal{M}_1$ of complex tori is the space of the equivalence classes of tori; here two tori are equivalent if there is a biholomorphic map between them.

Let M be a complex torus with a basis $\{\omega_1, \omega_2\}(\mathrm{Im}(\omega_2/\omega_1) > 0)$. We define the map from $\mathcal{M}_1$ to $\mathbb{H}/\mathrm{PSL}(2, \mathbb{Z})$ by

$$M \mapsto \frac{\omega_2}{\omega_1}.$$

This map is well-defined and bijective (cf.[8]). Hence the moduli space of complex tori can be identified with $\mathbb{H}/\mathrm{PSL}(2, \mathbb{Z})$.

5.1.2. *Moduli of cusp cross-sections.* Let Γ be a torsion-free Kleinian group of finite covolume.

Definition 5.3. A point $x \in \widehat{\mathbb{C}} = \partial\mathbb{H}^3$, the sphere at infinity, is called a cusp of a Kleinian group Γ if the stabilizer Γ_x is a free abelian group of rank 2.

We denote the set of cusps of Γ by $\mathrm{Cusp}(\Gamma)$. Let x be a cusp of Γ and σ an element of $\mathrm{PSL}(2, \mathbb{C})$ which maps x to ∞. Then σ induces a biholomorphic map from $\{\widehat{\mathbb{C}}\backslash\{x\}\}/\Gamma_x$ to $\{\widehat{\mathbb{C}}\backslash\{\infty\}\}/\sigma\Gamma_x\sigma^{-1}$. Since $\sigma\Gamma_x\sigma^{-1}$ has the form $\left\langle \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \right\rangle$, $\{\widehat{\mathbb{C}}\backslash\{\infty\}\}/\sigma\Gamma_x\sigma^{-1}$ is biholomorphic to a complex torus. Hence if x is a cusp of Γ, then $\{\widehat{\mathbb{C}}\backslash\{x\}\}/\Gamma_x$ is biholomorphic to a complex torus. Hence we can assign an element of $\mathbb{H}/\mathrm{PSL}(2, \mathbb{Z})$ to each cusp. We call the assigned element of $\mathbb{H}/\mathrm{PSL}(2, \mathbb{Z})$ the modulus of a cusp cross-section and denote it by $\mathrm{Mod}(x)$.

If two points x_1, x_2 of $\mathrm{Cusp}(\Gamma)$ are in a same orbit by the action of Γ, $\{\widehat{\mathbb{C}}\backslash\{x_1\}\}/\Gamma_{x_1}$ is biholomorphic to $\{\widehat{\mathbb{C}}\backslash\{x_2\}\}/\Gamma_{x_2}$. Hence Mod induces the map of $\mathrm{Cusp}(\Gamma)/\Gamma$ to $\mathbb{H}/\mathrm{PSL}(2, \mathbb{Z})$. Moreover, if Γ' is conjugate to Γ, then the diagram

$$\begin{tikzcd}[column sep=tiny] \mathrm{Cusp}(\Gamma)/\Gamma \arrow[rr, leftrightarrow] \arrow[dr] & & \mathrm{Cusp}(\Gamma')/\Gamma' \arrow[dl] \\ & \mathbb{H}/\mathrm{PSL}(2, \mathbb{Z}) & \end{tikzcd}$$

is commutative.

Lemma 5.4. *Let K be a non-totally real algebraic number field. If a Kleinian group Γ is conjugate to a subgroup Γ' of $\mathrm{PSL}(2,K)$ in $\mathrm{PSL}(2,\mathbb{C})$, then for each cusp of Γ' there is $\sigma \in \mathrm{PSL}(2,\mathbb{C})$ which move the cusp to ∞. Moreover the modulus of a cusp cross-section of Γ has its representatives in K.*

Proof. Not open to the public. □

It is known that every Kleinian group of finite covolume is conjugate to a subgroup of $\mathrm{PSL}(2,\mathbb{A})$ in $\mathrm{PSL}(2,\mathbb{C})$, where $\mathbb{A}$ is the set of the algebraic numbers (cf.[10]). Hence the modulus of a cusp cross-section has its representatives in some non-totally real algebraic number field by the fact. Moreover, every non-cocompact arithmetic Kleinian group is conjugate to a group commensurable with some Bianchi group $\mathrm{PSL}(2,\mathcal{O}_d)$ in $\mathrm{PSL}(2,\mathbb{C})$ (cf.[10]). Hence the modulus of a cusp cross-section of an arithmetic hyperbolic 3-manifold has its representatives in some imaginary quadratic number field.

5.2. **Proofs of Theorem 1.8 and 1.9.** At first, we prove the following two lemmas.

Lemma 5.5. *Let Λ be a $\mathbb{Z}$-module of rank n. Let α, β be elements of Λ which are linearly independent. If $a, b \in \mathbb{Z}\langle\alpha,\beta\rangle$ are linearly independent, then there is $N \in \mathbb{N}$ such that $\mathbb{Z}\langle\alpha,\beta\rangle \cap N\Lambda$ is contained in $\mathbb{Z}\langle a,b\rangle$.*

Proof. Not open to the public. □

Lemma 5.6. *Let K be a non-totally real algebraic number field and $\mathcal{O}_K$ be the ring of integers of K. If $\mathrm{PSL}(2,\mathcal{O}_K;J)$ has a torsion free subgroup Γ such that*

$$\Gamma_\infty = \left\langle \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \right\rangle$$

for some fractional ideal J of K, then for any $a, b \in \mathbb{Z}\langle\alpha,\beta\rangle$ which are linearly independent, Γ has a subgroup Γ' of finite index such that

$$\Gamma'_\infty = \left\langle \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right\rangle.$$

Proof. Not open to the public. □

Proof of Theorem 1.8. Not open to the public. □

Lemma 5.7. *Let L be a non-totally real algebraic number field of degree 3 and $\{\alpha,\beta,\gamma\}$ be a basis of L over $\mathbb{Q}$. For all $x \in L$, there exist $a_1, a_2, a_3, a_4 \in \mathbb{Z}$ such that*

$$x = \frac{a_1\alpha + a_2\beta}{a_3\alpha + a_4\beta}.$$

Proof. Not open to the public. □

By Lemma 5.6 and Lemma 5.7, the following lemma is deduced.

Lemma 5.8. *Let K be a non-totally real algebraic number field of degree 3 over $\mathbb{Q}$ and $\mathcal{O}_K$ be the ring of integers of K. If* $\mathrm{PSL}(2, \mathcal{O}_K; J)$ *has a torsion-free subgroup Γ of finite covolume such that*

$$\Gamma_\infty = \left\langle \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \right\rangle$$

for some fractional ideal J of K, then every element of K arises as a representative of the modulus of a cusp cross-section of some hyperbolic 3-manifold of finite volume.

By this lemma and Lemma 2.12 and Lemma 5.4, we obtain Theorem 1.9. As a corollary, we have the following:

Corollary 5.9. *Let K be a non-totally real algebraic number field of degree 3. If there is a non-cocompact Kleinian group Γ of finite covolume contained in* $\mathrm{PSL}(2, \mathcal{O}_K)$*, then every element of K arises as a representative of the modulus of a cusp cross-section of a hyperbolic 3-manifold of finite volume.*

The complement of the two-bridge knot $(7/3)$ in S^3 admits a complete hyperbolic structure of finite volume. Thus there is a Kleinian group Γ such that $\mathbb{H}^3/\Gamma$ is isomorphic to the complement in S^3 of the two-bridge knot $(7/3)$. It is known that

$$\Gamma = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \right\rangle$$

is one of such groups, where z satisfies the equation $x^3+x^2+2x+1=0$ (see [17]). Hence $\mathbb{Q}(z)$ satisfies the conditions in Theorem 1.9. We finally list some examples of hyperbolic 3-manifolds whose corresponding Kleinian groups satisfying the conditions in Theorem 1.9. All contents can be found in the Appendix in [10].

1: Manifold $m006$.
Minimum polynomial: $x^3 + 2x - 1$.
Discriminant: -59.

2: Manifold 015.
Minimum polynomial: $x^3 - x^2 + 1$.
Discriminant: -23.

3: Manifold 034.
Minimum polynomial: $x^3 + x - 1$.
Discriminant: -31.

4: Manifold 035.
Minimum polynomial: $x^3 - x^2 + x + 1$.
Discriminant: -44.

5: Manifold 045.
Minimum polynomial: $x^3 - x^2 + 3x - 2$.
Discriminant: -107.

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