Tokyo Tech Reseanch Repository

## 論文／著書情報 <br> Article／Book Information

| Title | Buchberger stratification on the space of polynomials |
| :---: | :---: |
| Authors | Fumiharu Kato，Kei Ohnoshi |
| Citation | Kumamoto Journal of Mathematics，Vol．28，，pp．1－10 |
| Pub．date | 2015， 4 |
| URL | http：／／www．sci．kumamoto－u．ac．jp \kkjm／ |

# Buchberger stratification on the space of polynomials 

Fumiharu Kato and Kei Ohnoshi

(Received October 8, 2014)
(Accepted February 3, 2015)


#### Abstract

In this paper, we consider the stratification on the space of polynomials, with a fixed set of monomials, induced from the notion of G-types, which encodes the process of Buchberger algorithm for obtaining Gröbner bases. We show that this indeed gives a finite stratification by locally closed subspaces, from which we deduce several finiteness results on Gröbner bases.


## 1 Introduction

The notion of Gröbner bases and the algorithm to obtain them are extremely useful, both in theories and for computational purposes, and have been a subject of extensive studies in various fields of mathematics. One of the interesting aspects of Gröbner bases is the several finiteness and boundedness properties, e.g., upperbound of the number of polynomials in a Gröbner base. The existence of such an upper-bound is already known; actually an upper-bound itself for the maximal degree has been obtained in recent works such as [2][4]. But our point of the interests lies rather in, whatever the upper-bound is, the geometric structures behind these finiteness and boundedness properties. This naturally leads us to study the space of ideals (with some data fixed), which we regard as the 'moduli space' of Buchberger algorithms.

In this paper, we are going to consider the space of polynomials $X=X\left(n, r,\left\{P_{i}\right\}_{i=1}^{r}\right)$, where $n, r$ are positive integers and $\left\{P_{i}\right\}_{i=1}^{r}$ a collection of finite sets of monomials, which parametrizes all sequences $\boldsymbol{F}=\left(F_{1}, \ldots, F_{r}\right)$ of $n$-variable polynomials such that each $F_{i}$ has the monomials in $P_{i}$. Our object of study is the partition $\mathscr{S}=\left\{S_{\lambda}\right\}_{\lambda \in \Lambda}$ of this space induced from the notion of $G$-types of $\boldsymbol{F}=\left(F_{1}, \ldots, F_{r}\right)(\S 2.6)$. Roughly, the G-type collects all types of the polynomials that appear in the whole process of the Buchberger algorithm (with any term order fixed once for all), where by the type of a polynomial we mean the set of all monomials with non-zero coefficients. In other words, G-types drop off all the coefficients, but keeping the knowledge about whether being zero or non-zero, of

[^0]the polynomials that appear in the process, and thus provide a feasible way to encode the 'process type' of the algorithm. This gives a partition of the space $X$ into at most countably many subsets, called $G$-isotypic loci. An easy, but perhaps requiring a little care, fact is that each G-isotypic locus is actually a locally closed subspace of $X$ (Proposition 3.3). This is of course not a surprising result, but combined with a simple observation from algebraic geometry (Lemma 3.4), one can show that the number of the subsets in $\mathscr{S}=\left\{S_{\lambda}\right\}_{\lambda \in \Lambda}$ is actually finite (Theorem 3.5 ); in other words, once the data $n, r$ and $\left\{P_{i}\right\}$ as above are fixed, then there are only finitely many possible types of the flow charts, i.e., G-types, of the Buchberger processes starting from the sequences of polynomials in $X$, which seems highly non-trivial. We thus obtain a finite stratification $\mathscr{S}=\left\{S_{\lambda}\right\}_{\lambda \in \Lambda}$ on the space $X$ by G -isotypic loci. We call this stratification the Buchberger stratification.

As a corollary, one immediately sees that the number (resp. the maximal degree) of the polynomials that appear in the Gröbner basis of the ideal generated by $F_{1}, \ldots, F_{r}$ from $X$ is upper-bounded by a universal constant that only depends on the data $n, r,\left\{P_{i}\right\}$, and the term ordering (Corollary 3.8). Unfortunately, our method cannot give the explicit bound, but could give the existence in a quite elementary way.

In the final section, we will consider a simple and explicit example, and enumerate all the G-types therein.

After submiting the first draft of this paper, the authors are informed from the referee that what we call Buchberger stratification in this paper, or at least a notion similar to it, has already been developed by Weispfenning [5]; see [3] for more information. In spite of this similarity, we think that our scope of this paper differs from these works, mainly in that we regard the existence of the stratifications as one of the most important geometric backgrounds for the existence of the upperbounds; notice that the key for the bounded-ness lies not in the computationtheoretic aspects, but rather in the geometric backgrounds, as the simple algebraicgeometry lemma (Lemma 3.4) shows.

Acknowledgements. The authors thanks Professor Takeshi Abe in Kumamoto University for the valuable discussions and comments. We also thank the referee for his careful reading and for informing the authors about the research status on Comprehensive Gröbner bases.

## 2 G-types

### 2.1 Notation and conventions

Let $n \geq 1$ be a positive integer, and consider the ring $\mathbb{C}\left[X_{1}, \ldots, X_{n}\right]$ of polynomials of $n$ variables. We will employ the multi-index notation as follows: For any $\nu=\left(\nu_{1}, \ldots, \nu_{n}\right) \in \mathbb{N}^{n}$, where $\mathbb{N}=\mathbb{Z}_{\geq 0}$ is the set of non-negative integers, we write $X^{\nu}=X_{1}^{\nu_{1}} \cdots X_{n}^{\nu_{n}}$ and $|\nu|=\nu_{1}+\cdots+\nu_{n}$; each polynomial $F=F(X) \in \mathbb{C}\left[X_{1}, \ldots, X_{n}\right]$ can be written as $F=\sum_{\nu \in \mathbb{N}^{n}} a_{\nu} X^{\nu}$, where $a_{\nu}=0$ for all but finitely many $\nu$ 's.

We fix once for all a term order $\leq$ on $\mathbb{N}^{n}$. For any non-zero $F \in \mathbb{C}\left[X_{1}, \ldots, X_{n}\right]$,
we denote by $\nu(F)=\left(\nu(F)_{1}, \ldots, \nu(F)_{n}\right), \operatorname{LT}(F)$, and $\operatorname{LM}(F)=X^{\nu(F)}$, the leading degree, the leading term, and the leading monomial, respectively, of $F$.

For a non-zero $F \in \mathbb{C}\left[X_{1}, \ldots, X_{n}\right]$, we denote by $C_{F}=\nu(F)+\mathbb{N}^{n}$ the cone in $\mathbb{N}^{n}$ generated by the leading degree of $F$. For an ordered sequence $\boldsymbol{F}=\left(F_{1}, \ldots, F_{r}\right)$ of non-zero polynomials, we set $C_{\boldsymbol{F}}=\bigcup_{i=1}^{r} C_{F_{i}}$.

### 2.2 Type of polynomials

For a polynomial $F=\sum_{\nu \in \mathbb{N}^{n}} a_{\nu} X^{\nu} \in \mathbb{C}\left[X_{1}, \ldots, X_{n}\right]$, the type of $F$ is defined to be the finite set

$$
\tau(F)=\left\{\nu \in \mathbb{N}^{n} \mid a_{\nu} \neq 0\right\}
$$

Clearly, we have:

- $F=0$ if and only if $\tau(F)=\emptyset$;
- if $F \neq 0$, its degree $\operatorname{deg}(F)$ is equal to $\max \{|\nu| \mid \nu \in \tau(F)\}$.

Two polynomials $F, G \in \mathbb{C}\left[X_{1}, \ldots, X_{n}\right]$ are said to be isotypic, written $F \stackrel{\tau}{\sim} G$, if $\tau(F)=\tau(G)$. It is clear that the relation $\stackrel{\tau}{\sim}$ is an equivalence relation on the set $\mathbb{C}\left[X_{1}, \ldots, X_{n}\right]$.

The type of an ordered set $\boldsymbol{F}=\left(F_{1}, \ldots, F_{r}\right)$ of polynomials is defined to be

$$
\tau(\boldsymbol{F})=\left(\tau\left(F_{1}\right), \ldots, \tau\left(F_{r}\right)\right)
$$

and two ordered sets $\boldsymbol{F}, \boldsymbol{G}$ of polynomials are said to be isotypic, written $\boldsymbol{F} \stackrel{\tau}{\sim} \boldsymbol{G}$, if $\tau(\boldsymbol{F})=\tau(\boldsymbol{G})$.

### 2.3 Division algorithm

Let $\boldsymbol{F}=\left(F_{1}, \ldots, F_{r}\right)$ be an ordered set of non-zero polynomials in $\mathbb{C}\left[X_{1}, \ldots, X_{n}\right]$, and $H \in \mathbb{C}\left[X_{1}, \ldots, X_{n}\right]$ a polynomial. We say $H$ is $\boldsymbol{F}$-reduced if $\tau(H) \cap C_{\boldsymbol{F}}=\emptyset$. For any $H \in \mathbb{C}\left[X_{1}, \ldots, X_{n}\right]$ we can always find $Q_{1}, \ldots, Q_{r} \in \mathbb{C}\left[X_{1}, \ldots, X_{n}\right]$ such that $R=H-\left(Q_{1} F_{1}+\cdots+Q_{r} F_{r}\right)$ is $\boldsymbol{F}$-reduced. Such an $R$ is, however, not in general uniquely determined by $H$. Hence, for the sake of uniformity, we need to fix an explicit process to obtain it.

To do this, let us illustrate a single division step as follows: If $H$ is not $\boldsymbol{F}$ reduced, let $\nu_{0}=\max \tau(H) \cap C_{\boldsymbol{F}}$ be the maximal (with respect to the fixed term order $\leq)$ degree in $\tau(H) \cap C_{\boldsymbol{F}}$, and take the minimal $i=i_{0}$ in the set $\left\{i \mid \nu_{0} \in C_{F_{i}}\right\}$. Then we set

$$
\begin{equation*}
H^{\prime}=H-\frac{\mathrm{T}_{\nu_{0}}(H)}{\operatorname{LT}\left(F_{i_{0}}\right)} F_{i_{0}} \tag{*}
\end{equation*}
$$

where $\mathrm{T}_{\nu_{0}}(H)$ is the degree- $\nu_{0}$ term of $H$. This process will be denoted by $H \underset{\underset{\sim}{\boldsymbol{F}}}{H^{\prime}}$, and the pair $\left(\nu_{0}, i_{0}\right)$ as above will be called the initial index. One can iterate this process until we get the $\boldsymbol{F}$-reduced result in the end (cf. [1, §1.5]):

$$
\begin{equation*}
H=H_{0} \underset{\rightsquigarrow}{\boldsymbol{F}} H_{1} \underset{\rightsquigarrow}{\underset{w}{\boldsymbol{F}}} \ldots \stackrel{\boldsymbol{F}}{\rightsquigarrow} H_{s}, \tag{**}
\end{equation*}
$$

often written $H \underset{{\underset{\sim}{*}}^{\boldsymbol{F}}}{+} R$ for short, where $R=H_{s}$ is $\boldsymbol{F}$-reduced. We define the division type of $H$ with respect to $\boldsymbol{F}$ to be

$$
\tau_{\mathrm{D}}(H, \boldsymbol{F})=\left(\tau(\boldsymbol{F}) ; \tau\left(H_{0}\right), \tau\left(H_{1}\right), \ldots, \tau\left(H_{s}\right)\right),
$$

referring to the diagram $(* *)$.

## $2.4 \quad S$-polynomial

For two non-zero polynomials $F, G \in \mathbb{C}\left[X_{1}, \ldots, X_{n}\right]$, the so-called $S$-polynomial of $F$ and $G$ is defined as follows: Define $\mu=\left(\mu_{1}, \ldots, \mu_{n}\right) \in \mathbb{N}^{n}$ by $\mu_{i}=\max \left\{\nu(F)_{i}, \nu(G)_{i}\right\}$ for $i=1, \ldots, n$. Then set

$$
S(F, G)=\frac{X^{\mu} \cdot F}{\operatorname{LT}(F)}-\frac{X^{\mu} \cdot G}{\operatorname{LT}(G)} .
$$

If an ordered set $\boldsymbol{F}=\left(F_{1}, \ldots, F_{r}\right)$ of non-zero polynomials in $\mathbb{C}\left[X_{1}, \ldots, X_{n}\right]$ is given, one has the $\boldsymbol{F}$-reduction of $S(F, G)$, which we denote by $\bar{S}_{\boldsymbol{F}}(F, G)$, i.e.,

$$
S(F, G) \stackrel{F}{\rightsquigarrow}+\bar{S}_{\boldsymbol{F}}(F, G)
$$

in the notation as in $\S 2.3$.

### 2.5 Gröbner basis

A sequence $\boldsymbol{F}=\left(F_{1}, \ldots, F_{r}\right)$ of non-zero polynomials is called a Gröbner basis (of the ideal generated by $F_{1}, \ldots, F_{r}$ ) if, for any $i, j=1, \ldots, n$ with $i<j$, we have $\bar{S}_{\boldsymbol{F}}\left(F_{i}, F_{j}\right)=0$. If $\boldsymbol{F}=\left(F_{1}, \ldots, F_{r}\right)$ is not a Gröbner basis, one has $(i, j)$ such that $\bar{S}_{\boldsymbol{F}}\left(F_{i}, F_{j}\right) \neq 0$. Take such $\left(i_{0}, j_{0}\right)$ that is minimal in the set $\{(i, j) \mid i, j=1, \ldots, r, i<j\}$ with respect to the lexicographical ordering, and define a new sequence of polynomials $\boldsymbol{F}^{\prime}=\left(F_{1}, \ldots, F_{r+1}\right)$ by

$$
F_{r+1}=\bar{S}_{F}\left(F_{i_{0}}, F_{j_{0}}\right) .
$$

Let us denote this process by $\boldsymbol{F} \stackrel{\mathrm{B}}{\rightsquigarrow} \boldsymbol{F}^{\prime}$, and call the index $\left(i_{0}, j_{0}\right)$ as above the initial index; if $\boldsymbol{F}$ is a Gröbner basis, we set $\left(i_{0}, j_{0}\right)=\infty$. A classical but significant fact is that, starting from arbitrary $\boldsymbol{F}=\boldsymbol{F}_{0}=\left(F_{1}, \ldots, F_{r}\right)$, one eventually obtains a Gröbner basis $\boldsymbol{F}_{s}$ by iterating the above process (cf. $[1, \S 1.7]$ ):

$$
\begin{equation*}
\boldsymbol{F}=\boldsymbol{F}_{0} \stackrel{\stackrel{\mathrm{~B}}{\rightsquigarrow} \boldsymbol{F}_{1} \stackrel{\mathrm{~B}}{\rightsquigarrow} \ldots \stackrel{\mathrm{~B}}{\rightsquigarrow} \boldsymbol{F}_{s}, ~, ~, ~}{\text {, }} \tag{*}
\end{equation*}
$$

often written $\boldsymbol{F}_{0} \stackrel{\text { B }}{\rightsquigarrow} \boldsymbol{F}_{s}$ for short.

### 2.6 G-type

Consider the single step $\boldsymbol{F} \stackrel{\text { B }}{\rightsquigarrow} \boldsymbol{F}^{\prime}$ as above. We define the $B$-type of $\boldsymbol{F}$ to be the following data:

$$
\tau_{\mathrm{B}}(\boldsymbol{F})=\left(\tau_{\mathrm{D}}\left(S\left(F_{i}, F_{j}\right), \boldsymbol{F}\right)\right)_{(i, j) \leq\left(i_{0}, j_{0}\right)},
$$

where $(i, j)$ runs over the set $\left\{(i, j) \mid i, j=1, \ldots, r, i<j,(i, j) \leq\left(i_{0}, j_{0}\right)\right\}$ (ordered by the lexicographical order) and $\left(i_{0}, j_{0}\right)$ is the initial index of $\boldsymbol{F}$. Finally, referring to the diagram $(*)$ in $\S 2.5$, we define the $G$-type of $\boldsymbol{F}$ to be

$$
\tau_{\mathrm{G}}(\boldsymbol{F})=\left(\tau_{\mathrm{B}}\left(\boldsymbol{F}_{0}\right), \ldots, \tau_{\mathrm{B}}\left(\boldsymbol{F}_{s}\right)\right)
$$

Two ordered sequences $\boldsymbol{F}=\left(F_{1}, \ldots, F_{r}\right)$ and $\boldsymbol{G}=\left(G_{1}, \ldots, G_{s}\right)$ are said to be $G$-isotypic, written $\boldsymbol{F} \stackrel{\mathcal{G}}{\sim} \boldsymbol{G}$, if we have $\tau_{\mathrm{G}}(\boldsymbol{F})=\tau_{\mathrm{G}}(\boldsymbol{G})$. It is clear that the relation $\stackrel{G}{\sim}$ is an equivalence relation on the set of all ordered sequences of non-zero polynomials. Notice that $\boldsymbol{F} \stackrel{\mathcal{G}}{\sim} \boldsymbol{G}$ in particular requires that $\tau(\boldsymbol{F})=\tau(\boldsymbol{G})$, hence that $r=s$ and $\tau\left(F_{i}\right)=\tau\left(G_{i}\right)$ for any $i=1, \ldots, r$.

## 3 Buchberger stratification

### 3.1 The space of polynomials

Let $r \geq 1$ be a positive integer, and $P_{1}, \ldots, P_{r}$ non-empty finite subsets of $\mathbb{N}^{n}$. The typical choices of $P_{i}$ 's are as follows:
(1) (Inhomogenous case) let $d_{1}, \ldots, d_{r}$ be a sequence of non-negative integers and $P_{i}=\left\{\nu \in \mathbb{N}^{n}| | \nu \mid \leq d_{i}\right\}$ for $i=1, \ldots, r ;$
(2) (Homogenous case) let $d_{1}, \ldots, d_{r}$ be a sequence of non-negative integers and $P_{i}=\left\{\nu \in \mathbb{N}^{n}| | \nu \mid=d_{i}\right\}$ for $i=1, \ldots, r$.

Let $V_{i}$ for $i=1, \ldots, r$ be the $\mathbb{C}$-vector subspace of $\mathbb{C}\left[X_{1}, \ldots, X_{n}\right]$ spanned by $\left\{X^{\nu} \mid \nu \in P_{i}\right\}$. Consider the associated projective space $\mathbb{P}\left(V_{i}\right)$ for each $i=$ $1, \ldots, r$; each point of $\mathbb{P}\left(V_{i}\right)$ corresponds to a non-zero polynomial $F_{i}$, considered up to multiplication by non-zero constants, such that $\tau\left(F_{i}\right) \subseteq P_{i}$. Set

$$
X\left(n, r,\left\{P_{i}\right\}_{i=1}^{r}\right)=\mathbb{P}\left(V_{1}\right) \times \cdots \times \mathbb{P}\left(V_{r}\right),
$$

whose points are in one to one correspondence with ordered $r$-tuples $\boldsymbol{F}=\left(F_{1}, \ldots, F_{r}\right)$ of non-zero polynomials, each considered up to multiplication by non-zero constants, such that $\tau\left(F_{i}\right) \subset P_{i}$ for $i=1, \ldots, r$.

### 3.2 Subsets by G-types

The equivalence relation $\stackrel{G}{\sim}$ defined in $\S 2.6$ gives a set-theoretic partition $\mathscr{S}=$ $\left\{S_{\lambda}\right\}_{\lambda \in \Lambda}$ of $X=X\left(n, r,\left\{P_{i}\right\}_{i=1}^{r}\right)$ by G-isotypic classes, where the set $\Lambda$ indexes all possible G-types of ordered $r$-tuples $\boldsymbol{F}=\left(F_{1}, \ldots, F_{r}\right)$ corresponding to points of $X$.

### 3.3 Proposition

The subset $S_{\lambda}$ for each $\lambda \in \Lambda$ is a locally closed set in the projective variety $X=X\left(n, r,\left\{P_{i}\right\}_{i=1}^{r}\right)$ with respect to the Zariski topology, i.e., the intersection of a Zariski closed subset and a Zariski open subset.

Proof. We may suppose $S_{\lambda} \neq \emptyset$. Take $\boldsymbol{F}=\left(F_{1}, \ldots, F_{r}\right) \in S_{\lambda}$, and write $F_{i}=\sum_{\nu \in \tau\left(F_{i}\right)} a_{i, \nu} X^{\nu}$ for $i=1, \ldots, r$; notice that coefficients $a_{i, \nu}$ are all non-zero. Any point of $X$ corresponds to $\widetilde{\boldsymbol{F}}=\left(\widetilde{F}_{1}, \ldots, \widetilde{F}_{r}\right)$ with $\widetilde{F}_{i}=F_{i}+\sum_{\nu \in P_{i}} \xi_{i, \nu} X^{\nu}$ for $i=1, \ldots, r$. We need to show that the condition for $\widetilde{\boldsymbol{F}}$ to have the same G-type as $\boldsymbol{F}$ is given by finitely many zero's and non-zero's of polynomials of $\left\{\xi_{i, \nu}\right\}$, i.e.,
$(*)$ there exist a finite set of polynomials $\left\{\Phi_{\alpha}\right\}_{\alpha}$ and a set of finitely many non-zero polynomials $\left\{\Psi_{\beta}\right\}_{\beta}$ of $\left\{\xi_{i, \nu}\right\}$ such that the condition is given by $\Phi_{\alpha}\left(\xi_{i, \nu}\right)=0$ for all $\alpha$ and $\Psi_{\beta}\left(\xi_{i, \nu}\right) \neq 0$ for all $\beta$.

Notice that the condition $\tau(\boldsymbol{F})=\tau(\widetilde{\boldsymbol{F}})$ already requires $\xi_{i, \nu}=0$ for $\nu \notin \tau\left(F_{i}\right)$. In order for the following inductive argument to make sense, we need to put $\widetilde{F}_{i}$ to have more general form: $\widetilde{F}_{i}=\sum_{\nu \in \tau\left(F_{i}\right)} \phi_{i, \nu}(\xi) X^{\nu}$ for $i=1, \ldots, r$, where $\phi_{i, \nu}(\xi)$ is a non-zero rational function of $\xi_{i, \nu}$ 's such that $\phi_{i, \nu}(0)=a_{i, \nu} \neq 0$.

Set $\tau_{\mathrm{G}}(\boldsymbol{F})=\left(\tau_{\mathrm{B}}\left(\boldsymbol{F}_{0}\right), \ldots, \tau_{\mathrm{B}}\left(\boldsymbol{F}_{s}\right)\right)$ and $\tau_{\mathrm{G}}(\widetilde{\boldsymbol{F}})=\left(\tau_{\mathrm{B}}\left(\widetilde{\boldsymbol{F}}_{0}\right), \ldots, \tau_{\mathrm{B}}\left(\widetilde{\boldsymbol{F}}_{\widetilde{s}}\right)\right)$, and we will always indicate by tilde the corresponding objects constructed from $\widetilde{\boldsymbol{F}}$. Our condition $\tau_{\mathrm{G}}(\boldsymbol{F})=\tau_{\mathrm{G}}(\widetilde{\boldsymbol{F}})$ especially implies $s=\widetilde{s}$. If we show that the first condition $\tau_{\mathrm{B}}\left(\boldsymbol{F}_{0}\right)=\tau_{\mathrm{B}}\left(\widetilde{\boldsymbol{F}}_{0}\right)$ is equivalent to a condition of the form $(*)$ above, then the proposition follows by induction with respect to $s$ (since $\tau_{\mathrm{B}}\left(\boldsymbol{F}_{0}\right)=\tau_{\mathrm{B}}\left(\widetilde{\boldsymbol{F}}_{0}\right)$ implies that $\left.\tau\left(\boldsymbol{F}_{1}\right)=\tau\left(\widetilde{\boldsymbol{F}}_{1}\right)\right)$. Consider, then, the condition $\tau_{\mathrm{B}}(\boldsymbol{F})=\tau_{\mathrm{B}}(\widetilde{\boldsymbol{F}})$ (under the hypothesis $\tau(\boldsymbol{F})=\tau(\widetilde{\boldsymbol{F}}))$. If we have $\tau_{\mathrm{D}}\left(S\left(F_{i}, F_{j}\right), \boldsymbol{F}\right)=\tau_{\mathrm{D}}\left(S\left(\widetilde{F}_{i}, \widetilde{F}_{j}\right), \widetilde{\boldsymbol{F}}\right)$ for all $(i, j)<\left(i_{0}, j_{0}\right)$, then it follows, in particular, that the initial index $\left(i_{0}, j_{0}\right)$ of $\boldsymbol{F}$ is equal to the initial index of $\widetilde{\boldsymbol{F}}$. Hence it suffices to show that the condition $\tau_{\mathrm{D}}\left(S\left(F_{i}, F_{j}\right), \boldsymbol{F}\right)=\tau_{\mathrm{D}}\left(S\left(\widetilde{F}_{i}, \widetilde{F}_{j}\right), \widetilde{\boldsymbol{F}}\right)$ is equivalent to a condition of the form $(*)$ above for any $(i, j) \leq\left(i_{0}, j_{0}\right)$.

Therefore, the proof boils down to showing the following claims: Let $\boldsymbol{F}$ and $\widetilde{\boldsymbol{F}}$ be as above, and $H=\sum_{\nu} b_{\nu} X^{\nu}$ and $\widetilde{H}=\sum_{\nu \in \tau(H)} \psi_{\nu}(\xi) X^{\nu}$, where $\psi_{\nu}(\xi)$ is a rational function of $\xi_{i, \nu}$ 's such that $\psi_{\nu}(0)=b_{\nu} \neq 0$; then
(a) the coefficients of $\tau\left(S\left(\widetilde{F}_{i}, \widetilde{F}_{j}\right)\right)$ of each degree $\nu$ is a rational function $\eta_{i j}$ of $\xi_{i, \nu}$ 's; if $\nu \in \tau\left(S\left(F_{i}, F_{j}\right)\right)$, then $\eta_{i j}(0)$ is equal to the degree- $\nu$ coefficient of $S\left(F_{i}, F_{j}\right)$, hence non-zero;
(b) if $H \stackrel{F}{\rightsquigarrow} H^{\prime}$ and $\widetilde{H} \underset{\rightsquigarrow}{\widetilde{F}} \widetilde{H}^{\prime}$, then the coefficients of $\widetilde{H}^{\prime}$ of each degree $\nu$ is a rational function $\zeta_{i j}$ of $\xi_{i, \nu}$ 's; if $\nu \in \tau\left(H^{\prime}\right)$, then $\zeta_{i j}(0)$ is equal to the degree- $\nu$ coefficient of $H^{\prime}$, hence non-zero.

Notice that, in (b), the initial indices of $H \underset{\rightsquigarrow}{\boldsymbol{F}} H^{\prime}$ and $\widetilde{H} \underset{\rightsquigarrow}{\widetilde{F}} \widetilde{H}^{\prime}$ must be the same, since $\tau(H)=\tau(\widetilde{H})$ and $\tau(\boldsymbol{F})=\tau(\widetilde{\boldsymbol{F}})$. Both assertions (a) and (b) can be verified by an easy calculation, and hence the proof is done.

Thus the G-types give a partition $\mathscr{S}=\left\{S_{\lambda}\right\}_{\lambda \in \Lambda}$ of $X=X\left(n, r,\left\{P_{i}\right\}_{i=1}^{r}\right)$ by locally closed subspaces. On the other hand, there are only countably many possible G-types that arise from $X$. The following easy lemma shows that, in fact, the partition is finite:

### 3.4 Lemma

Let $X$ be an algebraic variety over $\mathbb{C}$, and $X=\coprod_{\lambda \in \Lambda} S_{\lambda}$ a countable partition of $X$ by locally closed subsets. Then the partition is finite; i.e., there exists a finite set $\left\{\lambda_{1}, \ldots, \lambda_{r}\right\} \subseteq \Lambda$ such that $X=\coprod_{i=1}^{r} S_{\lambda_{i}}$.

Proof. We show the lemma by induction with respect to the maximal dimension $n$ of the irreducible components of $X$. If $n=0$, then $X$ consists of finitely many points, and the claim is trivial. In general, we may assume that $X$ is irreducible. Moreover, considering the irreducible decomposition of each $S_{\lambda}$, we may assume that every $S_{\lambda}$ is irreducible. Since the complement of any non-empty open subset consists of irreducible components of dimension $<n$, we may assume that $X$ is affine. Suppose there exist no $S_{\lambda}$ that is open in $X$. Then every $S_{\lambda}$ is an irreducible subvariety of $X$ of dimension $<n$. By Noether's normalization theorem, we have a finite surjective morphism $\pi: X \rightarrow \mathbb{A}^{n}$ to the affine $n$-space. It follows that the countably many subvarieties $\pi\left(S_{\lambda}\right)$ of dimension $<n$ cover the affine space $\mathbb{A}^{n}$, which is absurd. Hence there must exist an open $S_{\lambda}$. By induction, the complement $X \backslash S_{\lambda}$ is covered by finitely many $S_{\mu}$ 's, which shows the claim.

### 3.5 Theorem

The partition $\mathscr{S}=\left\{S_{\lambda}\right\}_{\lambda \in \Lambda}$ gives a stratification on the variety $X=X\left(n, r,\left\{P_{i}\right\}_{i=1}^{r}\right)$ by finitely many locally closed subspaces.

We call this stratification the Buchberger stratification.

### 3.6 Remark

The theorem shows that there exists a unique Zariski open dense stratum among $S_{\lambda}$ 's, which implies that there is a unique process type of the algorithm that applies to general sequences $\boldsymbol{F}=\left(F_{1}, \ldots, F_{r}\right)$ of polynomials. The process thus characterized should be appropriately called the generic process with respect to $n$, $r$, and $\left\{P_{i}\right\}$.

### 3.7 Corollary

For fixed $n$ ( $=$ the number of variables), $r$ ( $=$ the number of polynomials), and non-empty finite subsets $P_{1}, \ldots, P_{r}$, there are only finitely many possible $G$-types of $\boldsymbol{F}=\left(F_{1}, \ldots, F_{r}\right) \in X$.

Since the number and the degrees of polynomials that appear in the Buchberger algorithm are preserved by passage to their types, one has in particular the following:

### 3.8 Corollary

There exists a number $N=N\left(n, r,\left\{P_{i}\right\}, \leq\right)\left(\right.$ resp. $\left.D=D\left(n, r,\left\{P_{i}\right\}, \leq\right)\right)$, depending only on $n, r,\left\{P_{i}\right\}$, and the term ordering $\leq$ such that, for any $\boldsymbol{F}=$
$\left(F_{1}, \ldots, F_{r}\right) \in X=X\left(n, r,\left\{P_{i}\right\}_{i=1}^{r}\right)$, the number (resp. the maximal degree) of the polynomials that appear in the Gröbner basis of the ideal generated by $F_{1}, \ldots, F_{r}$ is upper-bounded by $N$ (resp. D).

## 4 An example

## 4.1

Let us consider the case $n=r=2$, and $P_{1}=\{(2,0),(0,2)\}, P_{2}=\{(2,0),(1,1)\}$, with the lexicographical ordering $\leq$. The sequences $\boldsymbol{F}=\boldsymbol{F}_{0}$ in question are of the form $\boldsymbol{F}=\left(F_{1}=a_{1} x^{2}+c_{1} y^{2}, F_{2}=a_{2} x^{2}+b_{2} x y\right)$, where we write $x=X_{1}$ and $y=X_{2}$. Notice that the corresponding space $X=X\left(2,2,\left\{P_{i}\right\}\right)$ is isomorphic to $\mathbb{P}^{1} \times \mathbb{P}^{1}$ with bi-homogenous coordinate $\left(a_{1}: c_{1}\right) \times\left(a_{2}: b_{2}\right)$.

## 4.2

First, we assume that the coefficients $a_{1}, c_{1}, a_{2}, b_{2}$ are all non-zero, that is, $\tau\left(\boldsymbol{F}_{0}\right)=$ $(\{(2,0),(0,2)\},\{(2,0),(1,1)\})$. We have $\tau_{\mathrm{D}}\left(S\left(F_{1}, F_{2}\right), \boldsymbol{F}_{0}\right)=\left(\tau\left(\boldsymbol{F}_{0} ;\{(1,1),(0,2)\}\right)\right.$, $\tau_{\mathrm{B}}\left(\boldsymbol{F}_{0}\right)=\left(\tau_{\mathrm{D}}\left(S\left(F_{1}, F_{2}\right), \boldsymbol{F}_{0}\right)\right)$, and $F_{3}=\bar{S}_{\boldsymbol{F}_{0}}\left(F_{1}, F_{2}\right)=S\left(F_{1}, F_{2}\right)=-b_{3} x y+c_{3} y^{2}$, where $b_{3}=b_{2} / a_{2} \neq 0$ and $c_{3}=c_{1} / a_{1} \neq 0$. Then we set $\boldsymbol{F}_{1}=\left(F_{1}, F_{2}, F_{3}\right)$. Since $S\left(F_{1}, F_{2}\right)=F_{3}$ reduces to 0 by $\boldsymbol{F}_{1}$, we have $\tau_{\mathrm{D}}\left(S\left(F_{1}, F_{2}\right), \boldsymbol{F}_{1}\right)=\left(\tau\left(\boldsymbol{F}_{1}\right) ;\{(1,1),(0,2)\}, \emptyset\right)$. Since

$$
S\left(F_{1}, F_{3}\right)=\frac{c_{3}}{b_{3}} x y^{2}+\frac{c_{1}}{a_{1}} y^{3} \xrightarrow[\rightsquigarrow]{F_{1}}\left(\frac{c_{1}}{a_{1}}+\frac{c_{3}^{2}}{b_{3}^{2}}\right) y^{3},
$$

we have two possibilities

$$
\tau_{\mathrm{D}}\left(S\left(F_{1}, F_{3}\right), \boldsymbol{F}_{1}\right)= \begin{cases}\left(\tau\left(\boldsymbol{F}_{1}\right) ;\{(1,2),(0,3)\},\{(0,3)\}\right) & \text { if } a_{1} b_{2}^{2}+c_{1} a_{2}^{2} \neq 0 \\ \left(\tau\left(\boldsymbol{F}_{1}\right) ;\{(1,2),(0,3)\}, \emptyset\right) & \text { if } a_{1} b_{2}^{2}+c_{1} a_{2}^{2}=0\end{cases}
$$

In both cases, we have $\tau_{\mathrm{B}}\left(\boldsymbol{F}_{1}\right)=\left(\tau_{\mathrm{D}}\left(S\left(F_{1}, F_{2}\right), \boldsymbol{F}_{1}\right), \tau_{\mathrm{D}}\left(S\left(F_{1}, F_{3}\right), \boldsymbol{F}_{1}\right)\right)$.
(1) If $a_{1} b_{2}^{2}+c_{1} a_{2}^{2} \neq 0$, then we set $F_{4}=c_{4} y^{3}$, where $c_{4}=c_{1} / a_{1}+c_{3}^{2} / b_{3}^{2} \neq 0$, and $\boldsymbol{F}_{2}=\left(F_{1}, F_{2}, F_{3}, F_{4}\right)$. We have $\tau_{\mathrm{D}}\left(S\left(F_{1}, F_{2}\right), \boldsymbol{F}_{2}\right)=\left(\tau\left(\boldsymbol{F}_{2}\right) ;\{(1,1),(0,2)\}, \emptyset\right)$, $\tau_{\mathrm{D}}\left(S\left(F_{1}, F_{3}\right), \boldsymbol{F}_{2}\right)=\left(\tau\left(\boldsymbol{F}_{2}\right) ;\{(1,2),(0,3)\},\{(0,3)\}, \emptyset\right)$, and $\tau_{\mathrm{D}}\left(S\left(F_{1}, F_{4}\right), \boldsymbol{F}_{2}\right)=$ $\left(\tau\left(\boldsymbol{F}_{2}\right) ;\{(0,5)\}, \emptyset\right)$. Since

$$
S\left(F_{2}, F_{3}\right)=\left(\frac{b_{2}}{a_{2}}+\frac{c_{3}}{b_{3}}\right) x y^{2} \stackrel{\boldsymbol{F}_{2}}{\sim}\left(\frac{b_{2}}{a_{2}}+\frac{c_{3}}{b_{3}}\right) \frac{c_{3}}{b_{3}} y^{3} \stackrel{\boldsymbol{F}_{2}}{\leadsto} 0,
$$

and $b_{2} / a_{2}+c_{3} / b_{3}=\left(a_{1} b_{2}^{2}+c_{1} a_{2}^{2}\right) / a_{1} a_{2} b_{2} \neq 0$, we have

$$
\left.\tau_{\mathrm{D}}\left(S\left(F_{2}, F_{3}\right), \boldsymbol{F}_{2}\right)=\left(\tau\left(\boldsymbol{F}_{2}\right) ;\{(1,2)\},\{(0,3)\}, \emptyset\right\}\right)
$$

Then one goes on calculating to obtain

$$
\begin{aligned}
& \left.\tau_{\mathrm{D}}\left(S\left(F_{2}, F_{4}\right), \boldsymbol{F}_{2}\right)=\left(\tau\left(\boldsymbol{F}_{2}\right) ;\{(1,4)\},\{(0,5)\}, \emptyset\right\}\right), \\
& \left.\tau_{\mathrm{D}}\left(S\left(F_{3}, F_{4}\right), \boldsymbol{F}_{2}\right)=\left(\tau\left(\boldsymbol{F}_{2}\right) ;\{(0,4)\}, \emptyset\right\}\right),
\end{aligned}
$$

whereby concluding that $\boldsymbol{F}_{2}$ is the Gröbner basis. Thus we have exactly one G-type in this case.
(2) If $a_{1} b_{2}^{2}+c_{1} a_{2}^{2}=0$, then we proceed to $S\left(F_{2}, F_{3}\right)$; since

$$
S\left(F_{2}, F_{3}\right)=\left(\frac{b_{2}}{a_{2}}+\frac{c_{3}}{b_{3}}\right) x y^{2}=0
$$

we have

$$
\tau_{\mathrm{D}}\left(S\left(F_{2}, F_{3}\right), \boldsymbol{F}_{1}\right)=\left(\tau\left(\boldsymbol{F}_{1}\right) ; \emptyset\right) .
$$

Then we go on calculating the other S-polynomials to see that $\boldsymbol{F}_{1}$ is actually a Gröbner basis. Hence, we have exactly one G-type in this situation.

In total, we have the two G-types, which correspond to the Zariski open dense stratum $a_{1} b_{2}^{2}+c_{1} a_{2}^{2} \neq 0$, which corresponds to the generic process (cf. Remark $3.6)$, and the stratum given by $a_{1} b_{2}^{2}+c_{1} a_{2}^{2}=0$.

## 4.3

If one (and only one) of $a_{1}, c_{1}, a_{2}, b_{2}$ in the bi-homogenous coordinate $\left(a_{1}: c_{1}\right) \times$ $\left(a_{2}: b_{2}\right)$ is zero, then, by an easy calculation, one has the single possible G-type in each case, and hence we have in total four strata on $X$, each given by a line in $X \cong \mathbb{P}^{1} \times \mathbb{P}^{1}$. If two of $a_{1}, c_{1}, a_{2}, b_{2}$ are zero, then $F_{1}$ and $F_{2}$ are monomials and hence $\boldsymbol{F}_{0}$ itself is a Gröbner basis. This comprises four G-types, which correspond to points of $X$.

Summing up all this, we conclude that the Buchberger stratification on $X$ consist of 10 strata, one 2-dimensional (Zariski open strata), five 1-dimensional, and four 0 -dimensional.

## References

[1] Adams, William W.; Loustaunau, P.: An introduction to Gröbner bases. Graduate Studies in Mathematics, 3. American Mathematical Society, Providence, RI, 1994.
[2] Dubé, T.W.: The structure of polynomial ideals and Gröbner bases. SIAM J. Comput. 19 (1990), no. 4, 750-775.
[3] Manubens, M.; Montes, A.: Minimal canonical comprehensive Gröbner systems. J. Symb. Comp. 44, 463-478.
[4] Mayr, E.W.; Ritscher, S.: Dimension-dependent bounds for Gröbner bases of polynomial ideals. J. Symbolic Comput. 49 (2013), 78-94.
[5] Weispfenning, V.: Comprehensive Gröbner basis. J. Symb. Comp. 36 (3-4), 669-683.

Fumiharu Kato
Department of Mathematics, Kumamoto University
Kurokami Kumamoto 860-8555, Japan
e-mail: kato@sci.kumamoto-u.ac.jp
Kei Ohnoshi
Department of Mathematics, Kumamoto University Kurokami Kumamoto 860-8555, Japan
e-mail: onoshi.kei@gmail.com


[^0]:    Mathematical Subject Classification (2010): Primary 13P10; Secondary 13P05
    Key words: Gröbner basis, stratification

