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OUTLINE OF THE THESIS "GENERATORS AND PRESENTATIONS FOR THE MAPPING CLASS GROUP OF A NON-ORIENTABLE SURFACE AND ITS SUBGROUPS"

GENKI OMORI

Let $\Sigma_{g,n}$ be a compact connected oriented surface of genus $g \geq 0$ with $n \geq 0$ boundary components and we put $\Sigma_g := \Sigma_{g,0}$. The *mapping class group* $\mathcal{M}(\Sigma_{g,n})$ of $\Sigma_{g,n}$ is the group of isotopy classes of orientation preserving self-diffeomorphisms on $\Sigma_{g,n}$ fixing the boundary pointwise. By Dehn [3], it is proved that $\mathcal{M}(\Sigma_g)$ is generated by Dehn twists. After that, simplified generating sets for $\mathcal{M}(\Sigma_g)$ by Dehn twists were given by Mumford [22], Lickorish [18], Humphries [12]. For $n \geq 1$, $\mathcal{M}(\Sigma_{g,n})$ is also generated by Dehn twists (for instance, see [5]). Presentations for $\mathcal{M}(\Sigma_{g,n})$ were given by Hatcher-Thurston [9], Harer [8], Wajnryb [34], Gervais [6, 7], Luo [20] and Labruère-Paris [16].

Let $N_{g,n}$ be a compact connected non-orientable surface of genus $g \geq 1$ with $n \geq 0$ boundary components. The surface $N_g := N_{g,0}$ is a connected sum of g real projective planes. The mapping class group $\mathcal{M}(N_{g,n})$ of $N_{g,n}$ is the group of isotopy classes of self-diffeomorphisms on $N_{g,n}$ fixing the boundary pointwise. For $g \geq 2$, Lickorish proved that $\mathcal{M}(N_g)$ is not generated by Dehn twists in [17], and $\mathcal{M}(N_g)$ is generated by Dehn twists and a "Y-homeomorphism" in [17, 19]. After that, simplified generating sets for $\mathcal{M}(N_g)$ were given by Chillingworth [2] and Szepietowski [32]. By Stukow [29], for $g \geq 1$, $\mathcal{M}(N_{g,n})$ is also Dehn twists and a Y-homeomorphism. Presentations for $\mathcal{M}(N_{g,n})$ were given by Paris-Szepietowski [25] and Stukow [30] when $n \in \{0,1\}$.

In this thesis, we give a presentation for $\mathcal{M}(N_{g,n})$ and generators for some subgroups of $\mathcal{M}(N_{g,n})$. We explain their precise results and contents of this thesis in the following.

In Section 1, we state an introduction to this thesis, and in Section 2, we prepare some propositions and lemmas for later sections.

In Section 3, we give a simple infinite presentation for $\mathcal{M}(N_{g,n})$ (Theorem 3.1). This result is obtained in our papers [15] and [23]. In the case of orientable surfaces, Gervais [6] gave a simple infinite presentation for $\mathcal{M}(\Sigma_{g,n})$ by using finite presentations of Hatcher-Thurston [9], Harer [8] and Wajnryb [34]. After that, Luo [20] improved Gervais' presentation (Theorem 3.2). In the case of non-orientable surfaces, for $n \in \{0,1\}$, a finite presentation for $\mathcal{M}(N_{g,n})$ was given by Lickorish [17], Birman-Chillingworth [1] and Stukow [26] for low genus cases, and by Paris-Szepietowski [25] for general genus cases. Note that $\mathcal{M}(N_1)$ and $\mathcal{M}(N_{1,1})$ are trivial (see [4, Theorem 3.4]) and $\mathcal{M}(N_2)$ is finite (see [17, Lemma 5]). After that, Stukow [30] rewrote Paris-Szepietowski's presentation into a finite presentation with Dehn twists and a Y-homeomorphism as generators (Theorem 3.3). In Section 3, we apply Gervais' argument to Stukow's finite presentation to give a simple infinite presentation for $\mathcal{M}(N_{g,n})$ when $n \in \{0,1\}$. A finite presentation for $\mathcal{M}(N_{g,n})$ is not known for $n \geq 2$. In Section 3, we also construct a finite presentation for

$\mathcal{M}(N_{g,n})$ when $n \geq 2$ in Proposition 3.4 and apply Gervais' argument to our finite presentation for $\mathcal{M}(N_{g,n})$ to give a simple infinite presentation for $\mathcal{M}(N_{g,n})$ when $n \geq 2$.

In Section 4, we give a small generating set for the twist subgroup of the mapping class group of a non-orientable surface by Dehn twists (Theorem 4.1). This result is obtained in our paper [24]. Let S be either $N_{g,n}$ or $\Sigma_{g,n}$. The *twist subgroup $\mathcal{T}(S)$ of $\mathcal{M}(S)$* is the subgroup of $\mathcal{M}(S)$ which is generated by all Dehn twists. Dehn [3] proved that $\mathcal{M}(\Sigma_g) = \mathcal{T}(\Sigma_g)$. In fact, $\mathcal{M}(\Sigma_{g,n}) = \mathcal{T}(\Sigma_{g,n})$ for any $n \geq 0$ (for instance, see [5]). However, $\mathcal{T}(N_{g,n})$ is an index 2 subgroup of $\mathcal{M}(N_{g,n})$ by Lickorish [18] and Stukow [27, Corollary 6.4]. In particular, $\mathcal{T}(N_{g,n})$ is a proper subgroup of $\mathcal{M}(N_{g,n})$.

In the case of orientable surfaces, Dehn [3] proved that $\mathcal{M}(\Sigma_g) = \mathcal{T}(\Sigma_g)$ is generated by $2g(g-1)$ Dehn twists. The generating set includes Dehn twists along separating simple closed curves. Mumford [22] showed that $\mathcal{M}(\Sigma_g)$ is generated by Dehn twists along non-separating simple closed curves, and Lickorish [18] gave a finite generating set for $\mathcal{M}(\Sigma_g)$ by $3g-1$ Dehn twists along non-separating simple closed curves. By an argument in the proof of Theorem 4.13 in [5], $\mathcal{M}(\Sigma_{g,1})$ is also generated by $3g-1$ Dehn twists along non-separating simple closed curves. After that, Humphries [12] proved that $\mathcal{M}(\Sigma_{g,n})$ is generated by a subset of Lickorish's generating set whose cardinality is $2g+1$ for $g \geq 2$ and $n \in \{0,1\}$, and he also proved that the generating set is minimal in generating sets for $\mathcal{M}(\Sigma_{g,n})$ by Dehn twists.

In the case of non-orientable surfaces, Lickorish [17] showed that $\mathcal{M}(N_2)$ is generated by a Dehn twist and a Y-homeomorphism. In generally, Chillingworth [2] gave a finite generating set for $\mathcal{M}(N_g)$ which consists of $\frac{3g-5}{2}$ (resp. $\frac{3g-6}{2}$) Dehn twists and a Y-homeomorphism for odd (resp. even) g. After that, Szepietowski [33] proved that $\mathcal{M}(N_g)$ is generated by a subset of Chillingworth's generating set which consists of g Dehn twists and a Y-homeomorphism, and Hirose [11] showed that the generating set is minimal in generating sets for $\mathcal{M}(N_g)$ by Dehn twists and Y-homeomorphisms. By Stukow's finite presentation for $\mathcal{M}(N_{g,n})$ in [30] when $n \in \{0,1\}$ and an argument in [11], $\mathcal{M}(N_{g,1})$ also has a minimal generating set by Dehn twists and Y-homeomorphisms which consists of g Dehn twists and a Y-homeomorphism.

In the case of the twist subgroup, Chillingworth [2] showed that $\mathcal{T}(N_g)$ is generated by a Dehn twist for $g = 2$, two Dehn twists for $g = 3$, $\frac{3g-1}{2}$ Dehn twists for the other odd g and $\frac{3g}{2}$ Dehn twists for the other even g. By an argument as in [12], we can reduce the number of Chillingworth's generators to $g+2$ for odd $g > 3$ and $g+3$ for even $g > 3$. For $n \in \{0,1\}$, Stukow [31] gave a finite presentation for $\mathcal{T}(N_{g,n})$ whose generators are $g+2$ Dehn twists essentially by relations of the presentation. In Section 4, we prove that $\mathcal{T}(N_{g,n})$ is generated by $g+1$ Dehn twists for $g \geq 4$ (Theorem 4.1). The generating set is a proper subset of the generating set of Stukow's finite presentation in [31]. By applying Hirose's argument in [11], the difference between the number of the generators in Theorem 4.1 and a lower bound of numbers of generators for $\mathcal{T}(N_{g,n})$ by Dehn twists is one (see Remark 4.3). The author does not know whether the generating set for $\mathcal{T}(N_{g,n})$ in Theorem 4.1 is minimal in generating sets for $\mathcal{T}(N_{g,n})$ by Dehn twists or not.

In Section 5, we give a finite generating set for the level 2 twist subgroup of the mapping class group of a closed non-orientable surface (Theorem 5.1). This result is obtained in our paper [14]. Let S be either $N_{g,n}$ or $\Sigma_{g,n}$, and put $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$. For $n = 0$ or 1, we denote by $\Gamma_2(S)$ the subgroup of $\mathcal{M}(S)$ which consists of elements of $\mathcal{M}(S)$ acting trivially on $H_1(S;\mathbb{Z}_2)$. $\Gamma_2(S)$ is called the *level 2 mapping class group* of S. For a group G, a normal subgroup H of G and a subset X of H, H is *normally generated in G by X* if H is the normal closure of X in G. In particular, for $X = \{x_1, \dots, x_n\}$, if H is the normal closure of X in G, we also say that H is *normally generated in G by $x_1, \dots, x_n$*.

In the case of orientable surfaces, Humphries [13] proved that $\Gamma_2(\Sigma_{g,n})$ is normally generated in $\mathcal{M}(\Sigma_{g,n})$ by the square of the Dehn twist along a non-separating simple closed curve for $g \geq 1$ and $n = 0$ or 1. In the case of non-orientable surfaces, Szepietowski [32] proved that $\Gamma_2(N_g)$ is normally generated in $\mathcal{M}(N_g)$ by a Y-homeomorphism for $g \geq 2$. For a group G, the first homology group $H_1(G)$ of G is isomorphic to the abelianization of G. In [33], he also gave an explicit finite generating set for $\Gamma_2(N_g)$ and proved that $H_1(\Gamma_2(N_4)) \cong \mathbb{Z}_2^{10}$. This generating set is minimal for $g = 3$, 4. Hirose-Sato [10] gave a minimal generating set for $\Gamma_2(N_g)$ and showed that $H_1(\Gamma_2(N_g)) \cong \mathbb{Z}_2^{\binom{g}{3}+\binom{g}{2}}$ for $g \geq 5$.

We denote by $\mathcal{T}_2(N_g)$ the subgroup of $\mathcal{T}(N_g)$ which consists of elements acting trivially on $H_1(N_g;\mathbb{Z}_2)$ and we call $\mathcal{T}_2(N_g)$ the *level 2 twist subgroup of $\mathcal{M}(N_g)$*. Chillingworth [2] proved that $\mathcal{T}(N_2) \cong \mathbb{Z}_2$ and $\mathcal{T}(N_2)$ is generated by the Dehn twist along the non-separating two-sided simple closed curve. Hence $\mathcal{T}_2(N_2)$ is a trivial group because Dehn twists along non-separating two-sided simple closed curves induce nontrivial actions on $H_1(N_g;\mathbb{Z}_2)$. Let $\mathrm{Aut}(H_1(N_g;\mathbb{Z}_2),\cdot)$ be the group of automorphisms on $H_1(N_g;\mathbb{Z}_2)$ preserving the intersection form $\cdot$ on $H_1(N_g;\mathbb{Z}_2)$. Since the action of $\mathcal{M}(N_g)$ on $H_1(N_g;\mathbb{Z}_2)$ preserves the intersection form $\cdot$, there is the natural homomorphism from $\mathcal{M}(N_g)$ to $\mathrm{Aut}(H_1(N_g;\mathbb{Z}_2),\cdot)$. McCarthy-Pinkall [21] showed that the restriction of the homomorphism to $\mathcal{T}(N_g)$ is surjective. Thus $\mathcal{T}_2(N_g)$ is finitely generated.

We give an explicit finite generating set for $\mathcal{T}_2(N_g)$ (Theorem 5.1) in Section 5. The generating set consists of "crosscap pushing maps" along non-separating two-sided simple loops and squares of Dehn twists along non-separating two-sided simple closed curves. In the last part of Section 5.1, we also give the smaller finite generating set for $\mathcal{T}_2(N_g)$ (Theorem 5.2). However, the generating set consists of crosscap pushing maps along non-separating two-sided simple loops, squares of Dehn twists along non-separating two-sided simple closed curves and squares of Y-homeomorphisms. As applications of Theorem 5.1, we give a small normal generating set for $\mathcal{T}_2(N_g)$ for $g \geq 3$ (Theorem 5.3) and determine the abelianization of $\mathcal{T}_2(N_g)$ except $g = 4$ (Theorem 5.4).

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