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**The Asymptotic Behavior of Eisenstein Series and
a Comparison of the Weil-Petersson and
the Zograf-Takhtajan Metrics**

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Chapter 1

Non-completeness of Zograf-Takhtajan's Kähler
metric for Teichmüller space of punctured
Riemann surfaces

Non-Completeness of Zograf-Takhtajan's Kähler metric for Teichmüller space of punctured Riemann surfaces

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ABSTRACT. A new Kähler metric on the Teichmüller space $T_{g,n}$ ($n > 0$), Zograf and Takhtajan introduced in calculating the first Chern form of the Quillen metric for families of $\bar{\partial}$ -operators. The metric is described in terms of the Eisenstein-Maass series. We prove that it is incomplete. And we also give an alternative proof of non-completeness of the Weil-Petersson metric. For that, we use the pinching family, constructed by Wolpert, whose tangent vectors are always represented by using the relative Poincaré series associated with the pinched geodesic.

§0. INTRODUCTION

We consider the Teichmüller space $T_{g,n}$ and the Teichmüller curve $\mathcal{T}_{g,n}$ of Riemann surfaces of type (g, n) (i.e. of genus g with $n \geq 0$ punctures). We shall assume that $2g - 2 + n > 0$. The projection $\pi : \mathcal{T}_{g,n} \rightarrow T_{g,n}$ forms a holomorphic family. Let $d\pi : T\mathcal{T}_{g,n} \rightarrow TT_{g,n}$ be its differential. We denote the kernel of $d\pi$ by $T_v\mathcal{T}_{g,n}$. The bundle $\rho : T_v\mathcal{T}_{g,n} \rightarrow \mathcal{T}_{g,n}$ is defined as the vertical tangent bundle of the fibration $\pi : \mathcal{T}_{g,n} \rightarrow T_{g,n}$. Its restriction to the fibre of π (i.e. a Riemann surface of type (g, n)) corresponds to the holomorphic tangent vector space of the surface. Thus the Poincaré metric on the Riemann surface defines a metric in the bundle ρ , by which ρ may be considered as a holomorphic hermitian line bundle.

Now, we are interested in the family of $\bar{\partial}_k$ -operators acting on k -differentials on Riemann surfaces (that is cross-sections of $T_v^{-k}\mathcal{T}_{g,n}|_{\pi^{-1}(p)} \rightarrow \pi^{-1}(p)$). Then, for the determinant line bundle $\lambda_k = \det(\text{ind } \bar{\partial}_k) = \wedge^{\max} \text{Ker } \bar{\partial}_k \otimes (\wedge^{\max} \text{Coker } \bar{\partial}_k)^{-1}$ naturally defined on $T_{g,n}$, Quillen ([6]) introduced a metric, called the Quillen metric (in the case $n > 0$, as the determinant of the Laplacian we shall use $Z(k)$, values of the Selberg-zeta function on positive integers). When $n > 0$, Zograf and Takhtajan ([7], [8]) discovered the following surprising formula

$$c_1(\lambda_k) = \frac{6k^2 - 6k + 1}{12\pi^2} \omega_{WP} - \frac{1}{9} \omega_{ZT}$$

Here, ω_{WP} denotes the 2-form of the Weil-Petersson metric, and ω_{ZT} denotes the 2-form of a new Kähler metric, represented by using the Eisenstein-Maass series, that they defined first in their papers. Also they showed that the new metric (after this we denote it by Z-T metric for simplicity) was Kählerian and invariant under

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the natural action of the Teichmüller modular group $Mod_{g,n}$ as the Weil-Petersson metric (we denote it by W-P metric) was. The W-P metric is known to have a negative curvature and be incomplete ([9], [10]). The following problem that I have considered seems to remain unsolved yet.

Conjecture: The curvature of the Z-T metric is negative.

In the present paper, we shall show that the Z-T metric is incomplete too.

We consider that it may be difficult to apply the method of Wolpert [9] directly for the Z-T metric. For the W-P metric, he evaluated the norm of the tangent vector of a curve approaching $\partial T_{g,n}$ with using the duality of the spaces of Beltrami differentials and quadratic differentials on Riemann surfaces (originally using the duality, the inner product of the W-P metric was defined). But it is hard to get the estimate for the Z-T metric because the inner product of the metric contains a term of the Eisenstein-Maass series ((1.4), (1.5)). It also makes the problem harder that in defining the inner product for the Z-T metric we must use the tangent vectors represented by harmonic Beltrami differentials. Furthermore another difficulty lies in examining how the Eisenstein-Maass series may depend on a parameter of a pinching deformation.

Here we shall pay attention to a new fascinating pinching family constructed in [11]. The length of a simple geodesic on Riemann surfaces in the family moves so that $l(t) = le^t$ ($t; 0 \rightarrow -\infty$) and the tangent vector field formed by the family is always represented by the harmonic Beltrami differential which is described as the dual of the quadratic differential expressed as a relative Poincaré series concerning the pinched geodesics. The series has a striking property that values of the norm of it as quadratic differential reflect the behavior of the length of the pinched geodesic. Then using the pinching deformation, we shall give another proof of non-completeness of the W-P metric (§2.) and get the main theorem that the Z-T metric is incomplete too.

The Z-T metric can be defined only in the case $n > 0$. Thus using the existence of cusps, we consider a uniformized model describing the pinching family in the upper half plane H , where we shall analyze how the relative Poincaré series depends on the pinching parameter t and the coordinate of H (§3. Lemma 3., 5.). In §4, we estimate the size of the tangent vector for the Z-T metric and show that the length of the curve formed by the pinching deformation is finite.

In fact as in (1.5), the inner product for the Z-T metric can be rewritten in the form of an integral containing no terms of the Eisenstein-Maass series. This makes it easy to estimate the integral over a cusp corresponding to each Eisenstein-Maass series (Lemma 5.). Also we must show how the Eisenstein-Maass series on the "half-width" collar around the pinched geodesics depends on the pinching parameter (Lemma 4.).

From now on, we should next consider whether the Z-T metric can be extended to the compactified Moduli space in any form. For this, the author thinks that it is important to discover a formula of the representation of the Z-T 2-form by the Fenchel-Nielsen coordinate like the Wolpert's formula for the W-P metric [3] and believes that such a formula exists.

The content of this paper is the following.

In §1., we recall necessary facts from Teichmüller theory and introduce the W-P and Z-T metric. And we refer to a lemma in hyperbolic geometry necessary for §2. (the collar lemma for non-compact Riemann surfaces) and some properties of the relative Poincaré series associated with simple geodesics.

In §2., we mention the construction of the new pinching deformation by Wolpert and its properties, and give another proof of non-completeness of the W-P metric.

In §3., we restrict ourselves to the case $n > 0$. Here we will construct an uniformized model and give some estimates in it for the Eisenstein-Maass series and the relative Poincaré series.

In §4., we shall give the main theorem. Our estimates give different orders of magnitude for the W-P and Z-T lengths of the same curves. The information could help in a future study of the Z-T kähler form.

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§1. NOTATIONS AND SOME FACTS

Let X be a Riemann surface of type (g, n) with $2g - 2 + n > 0$. We can represent it as a quotient H/Γ of the upper half plane $H = \{z \in \mathbb{C} \mid \text{Im} z > 0\}$ by the action of the first kind Fuchsian group $\Gamma \subset \text{PSL}(2, \mathbb{R})$; Γ is described as follows.

$$\Gamma = \left\langle A_k, B_k, S_i \in \text{PSL}(2, \mathbb{R}) \mid \prod_{k=1}^g [A_k, B_k] \prod_{i=1}^n S_i = Id \right\rangle$$

Here let A_k, B_k be $2g$ hyperbolic transformations, let S_i be n parabolic transformations ($k = 1, 2, \dots, g, i = 1, 2, \dots, n$).

$\Gamma_i = \langle S_i \rangle$ ($i = 1, 2, \dots, n$) are cyclic subgroups of Γ . The fixed point of S_i is denoted by $z_i \in \mathbb{R} \cup \{\infty\}$. We can select $\sigma_i \in \text{PSL}(2, \mathbb{R})$ so that $\sigma_i(\infty) = z_i$, $\langle \sigma_i^{-1} S_i \sigma_i \rangle = \langle z \mapsto z + 1 \rangle$.

A Riemann surface $X_0 \simeq H/\Gamma_0$ introduces the Teichmüller space with the base point X_0 , denoted by $T_{g,n}(\Gamma_0)$. The Teichmüller modular group $\text{Mod}_{g,n}(\Gamma_0)$ acts on $T_{g,n}(\Gamma_0)$ properly discontinuously. The Moduli space can be represented as a quotient space

$$\mathcal{M}_{g,n} \simeq T_{g,n}(\Gamma_0) / \text{Mod}_{g,n}(\Gamma_0)$$

. Here we define the following several spaces of automorphic forms.

A complex valued function f on H is called an automorphic form with the weight $(2l, 2m)$ for Γ if for any $z \in H$, and $\gamma \in \Gamma$,

$$f(\gamma z) \gamma'(z)^l \overline{\gamma'(z)^m} = f(z).$$

Such a form corresponds to a tensor of type (l, m) on the Riemann surface $X \simeq H/\Gamma$.

$\rho(z)|dz|^2$ denotes the Poincaré metric $y^{-2}(dx^2 + dy^2)$ on H . Let's denote by $\mathcal{H}^{l,m}$ the Hilbert space of automorphic forms of the weight $(2l, 2m)$ with the scalar product

$$\langle f_1, f_2 \rangle_\Gamma = \int \int_{H/\Gamma} f_1 \overline{f_2} \rho^{-l-m+1} = \int \int_{H/\Gamma} f_1 \overline{f_2} y^{2l+2m-2} dx dy.$$

We denote by $\Omega_2(\Gamma)$ the space of all holomorphic cusp forms of weight 4 (correspond to meromorphic 2-differentials on the Riemann surface $\overline{H/\Gamma}$ with poles of order not exceeding 1 only at punctures).

$\Omega_2(\Gamma)$ is a Hilbert subspace of $\mathcal{H}^{2,0}$ and $(3g-3+n)$ dimensional as we easily get from the Riemann-Roch theorem.

Denote by $B(\Gamma)$ the Banach space of all measurable Γ -invariant forms of weight $(-2,2)$ with supremum norm

$$\|\mu\|_\infty = \text{ess. sup}_{z \in H} |\mu(z)|$$

The elements of $B(\Gamma)$ are called Beltrami differentials.

Let $HB(\Gamma)$ be the space of all elements of $B(\Gamma)$ that can be represented as $y^2 \bar{\varphi}$ with $\varphi \in \Omega_2(\Gamma)$. $HB(\Gamma)$ is a Banach subspace of $B(\Gamma)$ and its elements are called harmonic Beltrami differentials.

Here we consider the natural Kodaira-Serre pairing

$$B(\Gamma) \times \Omega_2(\Gamma) \longrightarrow \mathbb{C} ; (\mu, \varphi) \longmapsto \iint_{H/\Gamma} \mu \varphi.$$

$N(\Gamma) = \Omega_2(\Gamma)^\perp$ is the space of all elements of $B(\Gamma)$ so that for any $\varphi \in \Omega_2(\Gamma)$,

$$\int_{H/\Gamma} \mu \varphi = 0$$

holds. Then, It follows that ([3])

$$(1.1) \quad B(\Gamma) = HB(\Gamma) \oplus N(\Gamma), \text{ in particular } HB(\Gamma) \simeq B(\Gamma)/N(\Gamma).$$

Next we consider the Bers projection.

By $B_1(\Gamma)$ we denote the space of all elements of $B(\Gamma)$ with the norm < 1 .

The Bers projection $\Phi : B_1(\Gamma_0) \longrightarrow T_{g,n}(\Gamma_0)$ is a holomorphic submersion. Its differential $d\Phi|_0 : B(\Gamma_0) \longrightarrow T_0(T_{g,n}(\Gamma_0))$ is a surjective homomorphism. Here the following fact is known ([3]).

$$(1.2) \quad \begin{aligned} \text{Ker } d\Phi|_0 &= N(\Gamma_0), \\ \text{in particular,} \\ d\Phi|_0 : B(\Gamma_0)/N(\Gamma_0) &\simeq T_0(T_{g,n}(\Gamma_0)). \end{aligned}$$

From (1.1) and (1.2) we get the following diagram

$$(1.3) \quad \begin{array}{ccc} T_0(T_{g,n}(\Gamma_0)) \times T_0^*(T_{g,n}(\Gamma_0)) & \longrightarrow & \mathbb{C} \\ \wr \downarrow & & \wr \downarrow \\ B(\Gamma_0)/N(\Gamma_0) \times \Omega_2(\Gamma_0) & \longrightarrow & \mathbb{C}. \end{array}$$

Now choose a basis $\{\mu_1, \mu_2, \dots, \mu_{3g-3+n}\}$ in $HB(\Gamma_0)$ and let $\kappa(t) = t_1\mu_1 + t_2\mu_2 + \dots + t_{3g-3+n}\mu_{3g-3+n}$ ($t_j \in \mathbb{C}$, $j = 1, 2, \dots, 3g-3+n$).

Let U be an enough small neighborhood of the origin in $\mathbb{C}^{3g-3+n}$. For $\kappa(t)$ with $t = (t_1, t_2, \dots, t_{3g-3+n}) \in U$, there exists unique diffeomorphisms $f^{\kappa(t)} : H \rightarrow H$ satisfying the Beltrami equations

$$\frac{\partial f^{\kappa(t)}}{\partial \bar{z}} = \mu \frac{\partial f^{\kappa(t)}}{\partial z}$$

and fixing the points $0, 1, \infty$.

Set $\Gamma^{\kappa(t)} = f^{\kappa(t)}\Gamma_0 (f^{\kappa(t)})^{-1}$ and $X^{\kappa(t)} = H/\Gamma^{\kappa(t)}$. By $[X^{\kappa(t)}]$ we denote the point in $T_{g,n}(\Gamma_0)$ whose representative is a pair $(X^{\kappa(t)}, f^{\kappa(t)})$.

The map $U \ni (t_1, t_2, \dots, t_{3g-3+n}) \mapsto [X^{\kappa(t)}] \in T_{g,n}(\Gamma_0)$ defines a biholomorphic coordinate in a neighbourhood of the origin in $T_{g,n}(\Gamma_0)$ (Ahlfors-Weill's theorem).

The Weil-Petersson metric dg^2 , the Z-T metric (is defined only in $n > 0$) $dh^{(i)2}, dh^{ZT2}$ are defined from the following inner products ([7], [8]).

(1.4)

$$g_{j,\bar{k}}(0) = g \left(\frac{\partial}{\partial t_j}, \frac{\partial}{\partial t_k} \right) = \iint_{H/\Gamma_0} \mu_j(z) \overline{\mu_k(z)} y^{-2} dx dy = \langle \mu_j, \mu_k \rangle_{\Gamma_0}^{\text{WP}}$$

(1.5)

$$\begin{aligned} h_{j,\bar{k}}^{(i)}(0) &= h^{(i)} \left(\frac{\partial}{\partial t_j}, \frac{\partial}{\partial t_k} \right) = \iint_{H/\Gamma_0} E^{(i)}(z, 2) \mu_j(z) \overline{\mu_k(z)} y^{-2} dx dy \\ &= \int_0^\infty \int_0^1 \mu_j(\sigma_i z) \overline{\mu_k(\sigma_i z)} dx dy = \langle \mu_j, \mu_k \rangle_{\Gamma_0}^{(i)} \\ h_{j,\bar{k}}^{ZT}(0) &= h^{ZT} \left(\frac{\partial}{\partial t_j}, \frac{\partial}{\partial t_k} \right) = \sum_{i=1}^n h_{j,\bar{k}}^{(i)}(0) = \sum_{i=1}^n \langle \mu_j, \mu_k \rangle_{\Gamma_0}^{(i)} = \langle \mu_j, \mu_k \rangle_{\Gamma_0}^{ZT} \end{aligned}$$

Here, $E^{(i)}(z, 2) = \sum_{\gamma \in (S_i) \setminus \Gamma} \text{Im}(\sigma_i^{-1} \gamma z)^2$ is the Eisenstein-Maass series with respect to each cusp z_i ([5], [8]). This series is a sum of positive terms, hence the pairings $h^{(i)}, h^{ZT}$ are positive definite. The series has the asymptotic behavior near cusps z_i

$$E^{(i)}(\sigma_j z, 2) = \delta_{ij} y^2 + \phi_{ij}(2) y^{-1} + O(e^{-2\pi y}) \quad \text{for } z \text{ near } \infty.$$

These metrics have the following properties ([8], [9], [10]).

- a) $dg^2, dh^{(i)2}, dh^{ZT2}$ are all Kählerian.
- b) dg^2 is incomplete.
- c) dg^2 has negative curvature.
- d) dg^2, dh^{ZT2} are invariant under the natural action of $Mod_{g,n}(\Gamma_0)$, while we can not define the action of $Mod_{g,n}(\Gamma_0)$ for $dh^{(i)2}$. (Because the action can provide a permutation of punctures.)

In the present paper, we prove that the claim b) with dg^2 replaced by $dh^{(i)2}, dh^{ZT2}$ is true (§ 4.). Still the author conjectures that the claim c) with dg^2 replaced by $dh^{(i)2}, dh^{ZT2}$ would be true, but this has not been solved yet.

Now consider only the case $n > 0$. Any element of $\Omega_2(\Gamma)$ is called a cusp form for Γ , has the following Fourier expansion around each cusp z_i ([4]). Let $\mu = y^2 \overline{\varphi}, \nu = y^2 \overline{\psi}$ for $\varphi, \psi \in \Omega_2(\Gamma)$. Then,

$$\varphi(\sigma_i z) \sigma_i'(z)^2 = \sum_{l=1}^{\infty} a_l^{(i)} e^{2\pi \sqrt{-1} l z}, \quad \psi(\sigma_i z) \sigma_i'(z)^2 = \sum_{l=1}^{\infty} b_l^{(i)} e^{2\pi \sqrt{-1} l z}, \quad (1, 2, \dots, n),$$

where, $a_l^{(i)} = \int_0^1 \varphi(\sigma_i z) \sigma_i'(z)^2 e^{-2\pi\sqrt{-1}lz} dz$, $b_l^{(i)} = \int_0^1 \psi(\sigma_i z) \sigma_i'(z)^2 e^{-2\pi\sqrt{-1}lz} dz$.
In fact by using the Rankin L-series

$$L^{(i)}(\mu, \nu; s) = \sum_{l=1}^{\infty} \frac{a_l^{(i)} \overline{b_l^{(i)}}}{l^s},$$

(1.4), (1.5) can be described as follows ([8]).

$$(1.6) \quad \langle \mu, \nu \rangle_{\Gamma}^{WP} = \frac{3!}{(4\pi)^3} \left(g - 1 + \frac{n}{2} \right) \text{Res}_{s=4} L^{(i)}(\mu, \nu; s)$$

$$\langle \mu, \nu \rangle_{\Gamma}^{(i)} = \frac{4!}{(4\pi)^5} L^{(i)}(\mu, \nu; 5)$$

Next, from [1] we quote the collar lemma for non-compact Riemann surfaces necessary for § 2..

The collar lemma for non-compact Riemann surfaces. *Let X be any Riemann surface of type (g, n) with $2g - 2 + n > 0$. The length of a simple closed geodesics γ on X with respect to the Poincaré metric denotes $l(\gamma)$.*

$C(\gamma)$, the collar neighbourhood of γ consists of all points p satisfying

$$d(p, \gamma) \leq \sinh^{-1} \left(1 / \sinh \frac{1}{2} l(\gamma) \right)$$

, where $d(\cdot, \cdot)$ denotes the hyperbolic distance. It is embedded in X .
Then the next assertions follow.

- (i) X has n cusp regions $C^1, C^2, \dots, C^n$ which are embedded in X . These are disjoint.
- (ii) let γ any simple closed geodesic on X . Then $C(\gamma), C^1, C^2, \dots, C^n$ are disjoint each other.
- (iii) Denote by $\beta_1, \beta_2, \dots, \beta_k$ all simple closed geodesics on X having length $< 2\sinh^{-1}1$. Then $C(\beta_1), C(\beta_2), \dots, C(\beta_k)$ are all disjoint.

Moreover, for any $p \in X - (C(\beta_1) \cup \dots \cup C(\beta_k) \cup C^1 \cup \dots \cup C^n)$ the injectivity radius of p satisfies $r_p(X) > \sinh^{-1}1$.

In particular, from the above we have the next claim.

Any non-trivial simple closed curve α with length $< 2\sinh^{-1}1$, is included in $C(\beta_1) \cup C(\beta_2) \cup \dots \cup C(\beta_k) \cup C^1 \cup C^2 \cup \dots \cup C^n$.

We call $2\sinh^{-1}1$ the *short geodesics constant*. And geodesics with length $< 2\sinh^{-1}1$ are called *short geodesics*. We shall end this section with introducing the relative Poincaré series associated to a closed geodesic to construct a pinching family of Riemann surfaces in §2. ([3]). The series was considered first by H. Petersson.

We choose an element $A \in \Gamma$ that covers a closed geodesic α on $X \simeq H/\Gamma$. Set

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $\det A = 1$, $\operatorname{tr} A^2 \neq 4$. Define

$$\omega_A = \frac{\operatorname{tr}^2 A - 4}{(cz^2 + (d-a)z - b)^2}.$$

Denote by p, q the fixed points of A . Then ω_A above can be also represented in the following form

$$\omega_A = \frac{(p-q)^2}{(z-p)^2(z-q)^2}.$$

Moreover for any $B \in \operatorname{PSL}(2, \mathbb{R})$, the identity

$$(1.7) \quad \omega_{B^{-1}AB} = (\omega_A \circ B) B'^2$$

holds. In particular, if $p = 0, q = \infty$, ω_A can be reduced to

$$(1.8) \quad \omega_A = \frac{1}{z^2}.$$

Consider the relative Poincaré series

$$(1.9) \quad \Theta_\alpha = \sum_{B \in \langle A \rangle \setminus \Gamma} \omega_{B^{-1}AB} = \sum_{B \in \langle A \rangle \setminus \Gamma} (\omega_A \circ B) B'^2.$$

Denote by l_α the hyperbolic length of α . When we consider l_α as a function on $T_{g,n}(\Gamma)$, the next follows ([3])

$$(1.10) \quad dl_\alpha(\mu) = (dl_\alpha, \mu) = \frac{2}{\pi} \operatorname{Re} \iint_{H/\Gamma_0} \mu \Theta_\alpha, \text{ for any } \mu \in B(\Gamma).$$

The series (1.9) converges absolutely and uniformly on any compact set and is an element of $\Omega_2(\Gamma)$. Here we quote Wolpert's result (Lemma 2.2, 2.3 in [11]).

Lemma 1. *Let $X \simeq H/\Gamma$ be any Riemann surface of type (g, n) with $2g - 2 + n > 0$.*

Select any short geodesic σ on X with hyperbolic length l_σ . Denote by A_σ the one of elements of Γ that cover σ and by $\tilde{\sigma}$ the axis of A_σ (i.e. the geodesic arc joining between the fixed points of A_σ in the upper half plane H).

We take a standard collar of σ as follows

$$C(\sigma) = \left\{ z \in H \mid d(z, \tilde{\sigma}) \leq \sinh^{-1} \left(1 / \sinh \frac{1}{2} l_\sigma \right) \right\} / \langle A_\sigma \rangle.$$

For any $z \in C(\sigma)$, let $\rho(z)$ be the injectivity radius at z in $C(\sigma)$ and let $\delta(z)$ be the distance from z to the boundary of $C(\sigma)$.

Now there is an absolute constant C_0 such that for any short geodesic σ (i.e. $l_\sigma < 2 \sinh^{-1} 1$), then

$$(1.11) \quad \rho(z) e^{\delta(z)} \geq C_0.$$

Furthermore there exist absolute constants C_1, C_2 such that

$$\sum_{\gamma \in G} |\omega_{\gamma^{-1}A_\sigma\gamma}(Imz)^2| \leq C_1 l_\sigma \rho(z)^{-1} e^{-d(z, \sigma)}, \text{ for } G = \langle A_\sigma \rangle \setminus \Gamma, z \in H,$$

Moreover,

$$(1.12) \quad \text{for } z \notin C(\sigma), \quad \rho(z)^{-1} e^{-d(z, \sigma)} \leq C_2 l_\sigma, \quad \text{holds and}$$

$$(1.13) \quad \text{for } z \in C(\sigma), \text{ the sum over } G = \langle A_\sigma \rangle \setminus \Gamma - \langle A_\sigma \rangle \text{ is bounded by } C_1 l_\sigma^2.$$

Remark. 1. From now on, assume that "absolute constants" should imply the universal constants independent of g and n , and Moduli.

2. In (1.13), we can not replace $C(\sigma)$ by the collar for any other element $\in \Gamma - \langle A_\sigma \rangle$.

Lemma 2. Let X, σ be as in Lemma 1. Then

$$(1.14) \quad \langle \Theta_\sigma, \Theta_\sigma \rangle_\Gamma = \frac{\pi}{2} l_\sigma + O(l_\sigma^3), \quad \text{with an absolute for the } O\text{-term}$$

proof. We may assume that $A_\sigma(z) = e^{l_\sigma} z$. According to (1.8),

$$\iint_{H/\Gamma} |\Theta_\sigma|^2 (\text{Im} z)^2 dx dy = \iint_{1 < |z| < e^{l_\sigma}} \Theta_\sigma \frac{1}{\bar{z}^2} (\text{Im} z)^2 dx dy$$

Set $z = r e^{i\theta}$ and expand the integral,

$$\begin{aligned} &= \iint_{1 < |z| < e^{l_\sigma}} \sin^2 \theta \frac{dr}{r} d\theta + \iint_{1 < |z| < e^{l_\sigma}} \sum_{\langle A_\sigma \rangle \setminus \Gamma - \langle A_\sigma \rangle} \omega_{\gamma^{-1} A_\sigma \gamma} \left(\frac{\text{Im} z}{\bar{z}} \right)^2 dx dy \\ &\leq \frac{\pi}{2} l_\sigma + C_1 l_\sigma^2 \iint_{1 < |z| < e^{l_\sigma}} \frac{dx dy}{|z|^2} = \frac{\pi}{2} l_\sigma + \pi C_1 l_\sigma^3. \end{aligned}$$

Here we used Lemma 1(1.13). $\square$

We concluded our preparation.

§2. A CONSTRUCTION OF THE PINCHING FAMILY BY WOLPERT

Set $X_0 = H/\Gamma_0$. We consider the Teichmüller space $T_{g,n}(\Gamma_0)$ with the base surface X_0 . Without losing generality, we may assume that X_0 has short geodesics. Let σ be one of short geodesics on X_0 . Following [11], we now construct a smooth pinching family

$$\{[X_t] \in T_{g,n}(\Gamma_0) \mid \infty < t \leq 0, [X_0] = [H/\Gamma_0]\}$$

such that the length of the geodesic σ_t on X_t (corresponds to σ on X_0) changes so that $l(t) = l_\sigma(X_t) = l e^t$. Denote by Γ_t a Fuchsian group uniformizing X_t .

Definition. The tangent vector $\tau(t) \in HB(\Gamma_t)$ can be defined as follows

$$(2.1) \quad \tau(t) = p(t) (dA_t)^{-1} \overline{\Theta_\sigma^t} \quad (p(t) \in \mathbb{R}), \quad \text{Re} \iint_{X_t} \tau(t) \Theta_\sigma^t = \frac{\pi}{2} l(t),$$

where dA_t is a hyperbolic area element, Θ_σ^t is the series of (1.9) on X_t associated with σ_t, Γ_t .

We now explain that the definition above is well-defined. We easily get $p(t) = 1 + O(l(t)^2)$. For due to Lemma 2, we have

$$\begin{aligned} \frac{\pi}{2} l(t) &= p(t) \iint_{X_t} |\Theta_\sigma^t|^2 (dA_t)^{-1} = p(t) \langle \Theta_\sigma^t, \Theta_\sigma^t \rangle_{\Gamma_t} \\ &= p(t) \left\{ \frac{\pi}{2} l(t) + O(l(t)^3) \right\}. \end{aligned}$$

it follows that

$$(2.2) \quad p(t) = (1 + O(l(t)^2))^{-1} = 1 + O(l(t)^2).$$

Further from (1.10), (2.1), we get

$$\frac{dl(t)}{dt} = (dl(t), \tau) = \frac{2}{\pi} \operatorname{Re} \iint_{H/\Gamma_t} \tau \Theta_\sigma^t = l(t).$$

Integrating from $-\infty$ to t , we have $l(t) = le^t$.

Therefore, we see that $\tau(t)$ is defined in $-\infty < t \leq 0$ and it induces the smooth pinching family suitable for our purpose. It gives us a simple proof of non-completeness of the W-P metric.

Corollary. *The Weil-Petersson metric is incomplete.*

Proof. The above constructed family forms the curve in $T_{g,n}(\Gamma_0)$ as denoted by $c(t)$. Let $|\frac{dc}{dt}|_{WP}$ be the line element associated with the W-P metric. Due to (1.4), (2.1), we get

$$\left| \frac{dc}{dt} \right|_{WP}^2 = \iint_{H/\Gamma_t} |p(t) \Theta_\sigma^t(dA_t^{-1})|^2 dA_t = p(t) \frac{\pi}{2} le^t = O(e^t).$$

It follows that

$$\int_{-\infty}^0 \left| \frac{dc}{dt} \right|_{WP} dt \leq \text{Const.} \int_{-\infty}^0 e^{\frac{t}{2}} dt < \infty.$$

Thus the length of the curve $c(t)$ for the W-P metric is finite. $\square$

§3. A UNIFORMIZED MODEL OF THE PINCHING FAMILY AND SOME ESTIMATES

In this section, we will prepare for the proof of non-completeness of the Z-T metric. From now on, always assume that $n > 0$. First, we choose X_0 to contain a short geodesics. If necessary, by taking a conjugation, we can assume that the Fuchsian group Γ_0 uniformizing X_0 contains a linear transformation $p(z) = z + 1$. Next choose an element of Γ_0 covering a short geodesic σ and denote it by the same letter σ , too. Let a_0 (resp. b_0) be the repelling (resp. attractive) fixed point of σ . If necessary, by taking the inverse, we can assume that $b_0 < a_0$. Further without losing generality, assume that $b_0 < 1 < a_0$.

Now choose an unique lift of a quasiconformal mapping associated with X_t such that $f^t : H \rightarrow H$ fixes $0, 1, \infty$. Set $\Gamma_t = f^t \Gamma_0 (f^t)^{-1}$. Then X_t can be given by H/Γ_t . And set $p_t = f^t \circ p \circ (f^t)^{-1}$, $\sigma_t = f^t \circ \sigma \circ (f^t)^{-1}$, $a_t = f^t(a_0)$, $b_t = f^t(b_0)$. Noticing that p_t is a parabolic transformation of $\operatorname{PSL}(2, \mathbb{R})$ and satisfies $p_t(\infty) = \infty$, $p_t(0) = 1$, at last we know that $p_t(z) = p(z) = z + 1$. σ_t can be described as

$$(3.1) \quad \frac{\sigma_t(z) - a_t}{\sigma_t(z) - b_t} = e^{l(t)} \frac{z - a_t}{z - b_t} \quad (-\infty < t \leq 0).$$

Moreover from considering the action of Γ_t on the axis of σ_t , we easily get

$$(3.2) \quad \begin{aligned} 0 < b_t < 1 < a_t < 2, \quad a_t - b_t < 1 \\ f^t(m) = m \quad \text{for any } m \in \mathbb{Z}. \end{aligned}$$

As above we got an uniformized model (H/Γ_t) $(-\infty < t \leq 0)$ (Fig. 1).

The Eisenstein-Maass series associated with cusp ∞ , $E_t^\infty(z, 2)$, the relative Poincaré series of (1.9) can be described as

$$E_t^\infty(z, 2) = \sum_{\delta \in \langle p_t \rangle \setminus \Gamma_t} (\text{Im} \delta z)^2,$$

$$\Theta_t(z) = \sum_{\gamma \in \langle \sigma_t \rangle \setminus \Gamma_t} \omega_{\gamma^{-1} \sigma_t \gamma}(z).$$

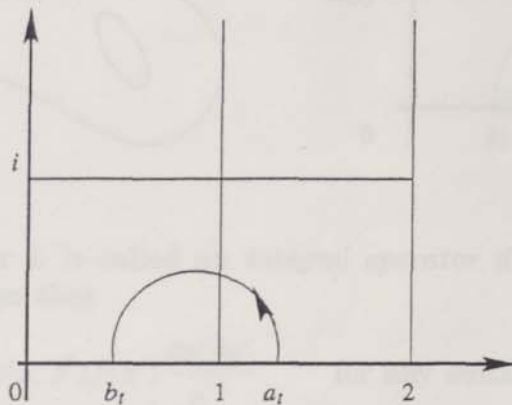


Figure 1.

Remark.

Let f^t, g^t be two normalized lifts of different quasiconformal maps defining $[X_t]$. Then $f^t|_{\mathbb{R}} = g^t|_{\mathbb{R}}$, for any $\gamma \in \Gamma$, $f^t \circ \gamma \circ (f^t)^{-1} = g^t \circ \gamma \circ (g^t)^{-1}$ are known to hold ([3]).

From now on, all objects and discussions depending on t are determined only by $f^t|_{\mathbb{R}}$ and Γ_t . Therefore, they do not depend on which quasiconformal maps associated with the family we will choose. Here, one will present three lemmata on prior estimates.

Lemma 3. *There exists an absolute constant C_3 such that $a_t - b_t \leq C_3 e^t$ ($t \rightarrow -\infty$) In particular, it follows that $\lim_{t \rightarrow -\infty} a_t = \lim_{t \rightarrow -\infty} b_t = 1$.*

Proof. As in Fig. 2, the distance between the geodesic σ_t and the boundary of the cusp region associated with ∞ is greater than the length of the width of the collar $C(\sigma_t) = \sinh^{-1}(1/\sinh \frac{1}{2} l(\gamma))$. (from the collar lemma for non-compact type). In the uniformized model (H, Γ_t) above, the neighbourhood

$$\{z \in \mathbb{C} | d(z, \{\text{Im} w = 1\}) \leq -\log l(t)\}$$

of the boundary of the cusp region corresponding to ∞ must not touch the orbits of the axis of σ_t by $\langle p(z) \rangle$ (Fig. 3). Therefore we get $\frac{a_t - b_t}{2} \leq l(t)$. From this, the claim of the lemma easily follows. $\square$

Due to Lemma 3, we can control several values in the model (H, Γ_t) . Next we shall understand at some degree how the Eisenstein-Maass series behaves as $t \rightarrow -\infty$. Here from Kubota's book, we quote the remarkable theorem about the eigenfunction of the Laplacian ([5], Theorem 1.3.2).

A linear operator T on the space of complex valued functions on H is called an *invariant operator* if for any function f on H , any $\gamma \in \text{PSL}(2, \mathbb{R})$,

$$T(f(\gamma z)) = (Tf)(\gamma z), \quad z \in H$$

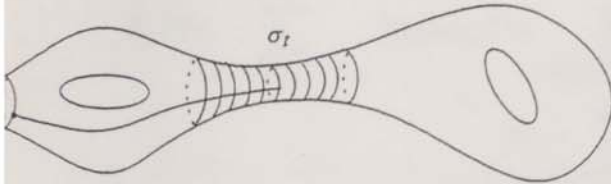


Figure 2.

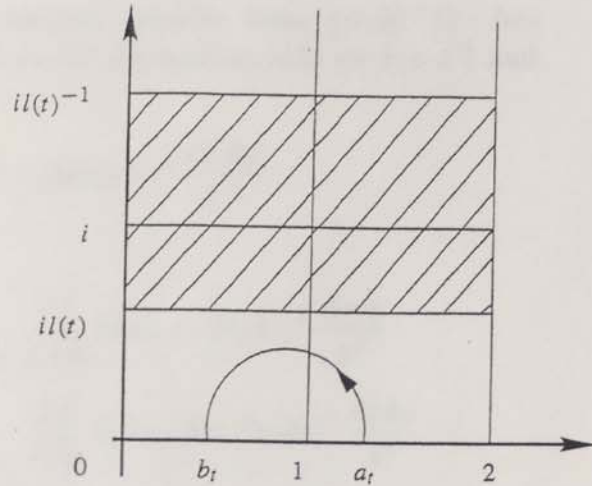


Figure 3.

holds. A linear operator L is called an *integral operator* if it is represented by a kernel function $k(z, z')$ so that

$$(Lf)(z) = \iint_H k(z, z') f(z') \frac{dx' dy'}{y'^2} \quad \text{for any suitable function } f.$$

Thus we easily know that a kernel function $k(z, z')$ defines an invariant integral operator if and only if k is invariant under the diagonal action of $\text{PSL}(2, \mathbb{R})$, i.e. $k(\gamma z, \gamma z') = k(z, z')$ holds for any $\gamma \in \text{PSL}(2, \mathbb{R})$.

Remark.

Now it is enough for our purpose to assume that k is smooth, $\text{PSL}(2, \mathbb{R})$ -invariant and $k(\cdot, z')$ has a compact support for z' fixed.

Fact. Let $f(z)$ be a twice differentiable function on H satisfying $Df = \lambda f$, where $D = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$ is the hyperbolic Laplacian and λ is a complex number and let L be an invariant operator. Then there exists a constant $\Lambda_L(\lambda)$ depending on L and λ not on the individual function f such that $Lf = \Lambda_L(\lambda)f$.

From using this, we can show the next lemma about the behavior of the Eisenstein series on the "half-width collar" of the pinched geodesic.

Lemma 4. Let (H, Γ_t) be the uniformized model of the pinching family, constructed in §2.. Let $C'(\sigma_t)$ be a "half-width collar" around the geodesic σ_t

$$C'(\sigma_t) = \left\{ z \in H \mid d(z, \bar{\sigma}_t) \leq \frac{1}{2} \sinh^{-1} \left(1 / \sinh \frac{1}{2} l_\gamma \right) \right\} / \langle \sigma_t \rangle \quad (\text{refer to §1}).$$

Then the Eisenstein-Maass series associated with ∞ is estimated from above so that

$$E_t^\infty(z, 2) = \sum_{\delta \in \langle P_t \rangle \setminus \Gamma_t} (\text{Im} \delta z)^2 \leq C_4 e^{-\frac{1}{2}} \quad \text{on } z \in C'(\sigma_t).$$

Proof. First we may assume that $l(t) < 2 \sinh^{-1} 1$. Notice that for any $\sigma \in \text{PSL}(2, \mathbb{R})$ $D\{(\text{Im} \sigma z)^2\} = 2(\text{Im} \sigma z)^2$ holds. Here we define a $\text{PSL}(2, \mathbb{R})$ -invariant kernel function

$$k(z, z') = \begin{cases} 1, & \text{if } d(z, z') < \epsilon \\ 0, & \text{otherwise} \end{cases}$$

where $\epsilon > 0$ is an enough small fixed constant (smaller than $\frac{1}{2}\sinh^{-1}1$). According to Fact. above, there is a constant $\Lambda_\epsilon(2)$ depending only on $k(z, z')$ and eigenvalue 2 such that

$$\Lambda_\epsilon(2)\text{Im}(\sigma z)^2 = \iint_H k(z, z')\text{Im}(\sigma z')^2 \frac{dx'dy'}{y'^2}.$$

For any z_0 , we get

$$\begin{aligned} \Lambda_\epsilon(2) \sum_{\delta \in \langle P_t \rangle \setminus \Gamma_t} (\text{Im} \delta z_0)^2 &= \sum_{\delta \in \langle P_t \rangle \setminus \Gamma_t} \iint_H k(z_0, z) (\text{Im} \delta z)^2 \frac{dx dy}{y^2} \\ &= \sum_{\delta \in \langle P_t \rangle \setminus \Gamma_t} \iint_H k(\delta z_0, \delta z) (\text{Im} \delta z)^2 \frac{dx dy}{y^2} \\ &= \sum_{\delta \in \langle P_t \rangle \setminus \Gamma_t} \iint_H k(\delta z_0, z) (\text{Im} z)^2 \frac{dx dy}{y^2} \\ &= \sum_{\delta \in \langle P_t \rangle \setminus \Gamma_t} \iint_{\{z \in H | d(\delta z_0, z) < \epsilon\}} dx dy = (*). \end{aligned}$$

Here we set $B(\delta z_0, \epsilon) = \{z \in H | d(\delta z_0, z) < \epsilon\}$, $\mathcal{B} = \bigcup_{\delta \in \langle P_t \rangle \setminus \Gamma_t} B(\delta z_0, \epsilon)$.

Noticing that if $B(z_0, \epsilon) \cap B(\delta z_0, \epsilon) \neq \emptyset$ then $d(z_0, \delta z_0) < 2\epsilon$, we know that the multiplicity of the projection $B(z_0, \epsilon) \rightarrow B(z_0, \epsilon)/\Gamma$ is at most $c\rho^{-1}(z_0)$ for a fixed $z_0 \in C'(\sigma_t) (\subset \{-1 < \text{Re} z \leq 2\})$ (according to the collar lemma). From simple consideration, the multiplicity of the points of the union $\mathcal{B}$ are at most $c\rho^{-1}(z_0)$. Then,

$$(*) \leq c\rho^{-1}(z_0) \iint_{\mathcal{B} \cap \{-1 < \text{Re} z < 2\}} dx dy,$$

Noticing that $\mathcal{B}$ is included in a set (refer to Lemma 3)

$$\mathcal{R} = \left\{ w \in H \mid d(w, \{\text{Im} z = 1\}) \geq -\frac{1}{2} \log l(t) - \epsilon \right\},$$

we have the further bounds

$$\begin{aligned} &\leq c\rho^{-1}(z_0) \iint_{\mathcal{R} \cap \{-1 < \text{Re} z < 2\}} dx dy = c\rho^{-1}(z_0) \iint_{\substack{-1 \leq x < 2 \\ 0 < y < e^{\epsilon} \sqrt{l(t)}}} dx dy \\ &= C' \rho^{-1}(z_0) \sqrt{l(t)}, \text{ following Lemma 1. (1.11),} \\ &\leq C' e^{\delta(z_0)} \sqrt{l(t)} \\ &\leq C' l(t)^{-1} \sqrt{l(t)} = C' l(t)^{-\frac{1}{2}} = O(e^{-\frac{t}{2}}). \end{aligned}$$

Here we used $\delta(z_0) \leq -\log l(t)$ (t : small enough). $\square$

The last lemma concerns about the behavior of the relative Poincaré series (1.9) near cusp ∞ . Since the series $\Theta_t(z)$ is a cusp form and invariant under the action of $p_t(z) = z + 1$, it has the Fourier expansion around ∞ as stated in §1..

$$\Theta_t(z) = \sum_{n=1}^{\infty} a_n^t e^{2\pi\sqrt{-1}nz}, \quad a_n^t = \int_0^1 \Theta_t(z) e^{-2\pi\sqrt{-1}nz} dz, \quad z \in H$$

The next estimates hold.

Lemma 5. (1) $a_n \leq C_1 l(t)^2 e^{2\pi n}$ ($n = 1, 2, \dots, n$), where C_1 is the absolute constant in Lemma 1..

(2) $|\Theta_t(z)(\text{Im}z)^2| \leq C_5 l(t)^2 y^2 e^{-2\pi y}$ holds for $\text{Im}z = y \geq 2$, where C_5 is an absolute constant.

Proof. (1) now consider only the case $\text{Im}z = y \geq 1$. Since $\{\text{Im}z \geq 1\}$ is a cusp region in particular is outside of the collar $C(\sigma_t)$ of the geodesic σ_t , we get, according to Lemma 1 (1.12),

$$|a_n^t| \leq \int_0^1 |\Theta_t(z)| e^{2\pi n y} dx \leq C_1 \frac{l(t)^2}{y^2} e^{2\pi n}.$$

We note that a function $G(y) = \frac{e^{2\pi n y}}{y^2}$ ($y \geq 1$) has a minimum $G(1) = e^{2\pi n}$. Finally we get $|a_n^t| \leq C_1 l(t)^2 e^{2\pi n}$.

(2) now restrict ourselves to $\{\text{Im}z \geq 2\}$. From the Fourier expansion,

$$\begin{aligned} |\Theta_t(z)(\text{Im}z)^2| &\leq y^2 \sum_{n=1}^{\infty} |a_n^t| e^{-2\pi n y}, \text{ following (1) ,} \\ &\leq y^2 \sum_{n=1}^{\infty} C_1 l(t)^2 e^{2\pi n} e^{-2\pi n y} \\ &= C_1 l(t)^2 y^2 \sum_{n=1}^{\infty} \left\{ e^{2\pi(1-y)} \right\}^n \\ &= C_1 l(t)^2 y^2 e^{2\pi(1-y)} \left\{ 1 - e^{2\pi(1-y)} \right\}^{-1} \end{aligned}$$

and since the last term is bounded on $\{y \geq 2\}$,

$$\leq C_5 l(t)^2 y^2 e^{-2\pi y}.$$

□

§4. NON-COMPLETENESS OF ZOGRAF-TAKHTAJAN'S METRIC

We begin a proof of the main theorem.

Theorem. *Zograf-Takhtajan's metrics $dh^{(i)2}, dh^{ZT2}$ ($i = 1, 2, \dots, n$) are incomplete.*

Proof. It is enough to consider only the case where a cusp $z_i = \infty$. Let $c(t)$ be the curve in $T_{g,n}(\Gamma_0)$, formed by the family $\{X_t\}$, and let $\left| \frac{dc(t)}{dt} \right|_{\infty}$ be the line element of the tangent vector with respect to $dh^{\infty 2}$. Then we divide the next integral into three parts.

$$\begin{aligned} \left| \frac{dc(t)}{dt} \right|_{\infty}^2 &= \iint_{H/\Gamma_t} \left| p(t) \overline{\Theta_t(z)} (\text{Im}z)^2 \right|^2 E_t^{\infty}(z, 2) \frac{dx dy}{y^2} \\ &= \iint_{C'(\sigma_t)} + \iint_{C(\sigma_t) - C'(\sigma_t)} + \iint_{H/\Gamma_t - C(\sigma_t)} = \text{I} + \text{II} + \text{III}. \end{aligned}$$

From Lemma 2. and 4., it follows that

$$\begin{aligned} I &\leq C_4 e^{-\frac{t}{2}} p(t)^2 \iint_{C'(\sigma_t)} |\Theta_t(z)|^2 \operatorname{Im} z^2 dx dy \\ &\leq \operatorname{Const.} e^{-\frac{t}{2}} \iint_{H/\Gamma_t} |\Theta_t(z)|^2 \operatorname{Im} z^2 dx dy \\ &\leq \operatorname{Const.} e^{-\frac{t}{2}} \left(\frac{\pi}{2} l e^t + O(e^{3t}) \right) = \operatorname{Const.} e^{\frac{t}{2}}. \end{aligned}$$

Next consider

$$\Pi = \iint_{C(\sigma_t) - C'(\sigma_t)} |\Theta_t(z) \operatorname{Im} z^2|^2 dx dy = \iint_{\sum_{\delta \in \langle p_t \rangle \setminus \Gamma_t} \{\delta(C(\sigma_t) - C'(\sigma_t))\}} |\Theta_t(z) \operatorname{Im} z^2|^2 dx dy,$$

here we use Lemma 1.(1.12). By using (1.11), we get for $z \in C(\sigma_t) - C'(\sigma_t)$,

$$\rho(z)^{-1} e^{-d(z, \sigma_t)} \leq \operatorname{Const.} e^{\delta(z) - d(z, \sigma_t)} \leq \operatorname{Const.} e^0.$$

Using this, we get

$$\sum_{\gamma \in \langle \sigma_t \rangle \setminus \Gamma_t} |\omega_{\gamma^{-1} \sigma_t \gamma} \operatorname{Im}^2(z)| \leq C_1 l(t) \rho(z)^{-1} e^{-d(z, \sigma_t)} \leq \operatorname{Const.} l(t).$$

It follows that

$$\Pi \leq \iint_{\substack{0 \leq \operatorname{Re} z \leq 1 \\ 0 \leq \operatorname{Im} z \leq 1}} \operatorname{Const.} l(t)^2 dx dy = \operatorname{Const.} l(t)^2 = \operatorname{Const.} e^{2t}.$$

Consider the third term

$$\begin{aligned} \text{III} &= \iint_{H/\Gamma_t - C(\sigma_t)} |\Theta_t(z) (\operatorname{Im} z)^2|^2 dx dy \\ &= \iint_{\sum_{\delta \in \langle p_t \rangle \setminus \Gamma_t} \{\delta(H/\Gamma_t - C(\sigma_t))\} \cap \{\operatorname{Im} z < 2\}} |\Theta_t(z) (\operatorname{Im} z)^2|^2 dx dy \\ &\quad + \iint_{\{\operatorname{Im} z \geq 2\}} |\Theta_t(z) (\operatorname{Im} z)^2|^2 dx dy. \end{aligned}$$

From (1.12), Lemma 5.(2), we get

$$= \operatorname{Const.} l(t)^4 + \operatorname{Const.} l(t)^4 \leq \operatorname{Const.} e^{4t}.$$

From the three estimates, we finally get

$$\left| \frac{dc(t)}{dt} \right|_{\infty}^2 \leq \text{I} + \text{II} + \text{III} \leq \operatorname{Const.} (e^{\frac{t}{2}} + e^{2t} + e^{4t}) = O(e^{\frac{t}{2}}).$$

Therefore,

$$\int_{-\infty}^0 \left| \frac{dc(t)}{dt} \right|_{\infty} dt \leq \operatorname{Const.} \int_{-\infty}^0 e^{\frac{t}{4}} dt < \infty.$$

□

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Chapter 2

The asymptotic behavior of Eisenstein series and
a comparison of the Weil-Petersson and the
Zograf-Takhtajan metrics

The Asymptotic Behavior of Eisenstein Series and a Comparison of the Weil-Petersson and the Zograf-Takhtajan Metrics

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ABSTRACT. The asymptotic behavior of Eisenstein series for degenerating hyperbolic surfaces with cusps is of interest to us. In order to investigate it we use integral representations of eigenfunctions for the Laplacian, the collar lemma, the interior Schauder estimates, the maximum principles for subharmonic functions and the Harnack Inequalities. As an application, we will compare the Weil-Petersson and the Zograf-Takhtajan metrics near the boundary of moduli spaces.

§0. INTRODUCTION

The Quillen metric defined for the determinant line bundle of the Laplacian over the Teichmüller space T_g of compact hyperbolic surfaces with genus g has played an important role in the moduli theory ([14]). The metric is described as the product of a special value of the Selberg zeta function and the usual L^2 -fibre metric with respect to the Poincaré metric. The first Chern form of the metric is represented by the Weil-Petersson two-form for T_g , whose formula has been shown by various methods.

Zograf and Takhtajan proved the formula by quasiconformal deformation theory. Moreover they defined the regularized metric for the determinant bundle of the Laplacian for the Teichmüller space $T_{g,n}$ of hyperbolic surfaces with cusps of type (g, n) and calculated its first Chern form, which is described in terms of the Weil-Petersson metric and a new Kähler metric as called Zograf-Takhtajan metric. They showed in [16] that the Zograf-Takhtajan metric is Kählerian and invariant under the action of the mapping class group as the Weil-Petersson metric is. It has been recently shown that the Zograf-Takhtajan metric is incomplete for $T_{g,n}$ as the Weil-Petersson metric is ([13]). The proof has been accomplished by showing that the length of a curve approaching the boundary of $T_{g,n}$ with respect to the Zograf-Takhtajan metric is finite. The construction of the curve is due to Wolpert ([25] II).

On the other hand, recently Fujiki and Weng shed new light on the geometry of the moduli spaces of punctured Riemann surfaces and the Zograf-Takhtajan metric ([4],[19]). From Arakelov geometric points of view, Weng found an arithmetic Riemann-Roch theorem for singular metrics, and established a generalization

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of Mumford type isometries. Fujiki and Weng have observed that the Zograf-Takhtajan metric is algebraic, and Weng proposed a general arithmetic factorization in terms of the Weil-Petersson metric and the Zograf-Takhtajan metric and Selberg zeta functions.

Therefore the asymptotic behavior of the Zograf-Takhtajan metric near the boundary of moduli space is of importance and of interest for studying compactifications of moduli spaces of punctured Riemann surfaces. In the previous paper ([13]), we observed that the metric is incomplete. In that proof we obtained an estimate of Eisenstein series of index 2 just around pinching geodesics, which is regrettably very rough and far from the precise asymptotic behavior, and a rough estimate of the Zograf-Takhtajan metric ([13]).

In this paper we find the asymptotic behavior of Eisenstein series of index s with $\text{Re } s > 1$ in Theorem 1 and 2. As a simple application, we improve an estimate of the Zograf-Takhtajan metric near the boundary of the moduli space (Theorem 3). As a result it turns out that the magnitude of the Zograf-Takhtajan norm is less than or equal to that of the Weil-Petersson norm.

We outline the content of this paper. In §1, we review elementary properties of Eisenstein series and definitions of the Weil-Petersson metric and the Zograf-Takhtajan metric and Wolf's degenerating family of Riemann surfaces with cusps by constructing infinite-energy harmonic maps which are defined to be maps from a hyperbolic surface with nodes into smooth hyperbolic surfaces (Though he has constructed the harmonic map for the degenerating family of compact Riemann surfaces in [21], we can use it for punctured Riemann surfaces. cf. [26]). One of the important features of Wolf's construction is that the term of the Eisenstein series of index 2 associated to the nodes appears in the asymptotic behavior of the pull-back of hyperbolic metrics of target surfaces by the infinite-energy harmonic maps (1.7). The infinite-energy harmonic map $w^l : S_0 \rightarrow S_l$ converges uniformly to id in the C^k -norm on compact subsets of S_0 ([21]). And $(w^l)^* \Delta_l$, the pull-back of the negative hyperbolic Laplacian Δ_l on S_l by w^l converges uniformly to Δ_0 , the Laplacian on S_0 also in the same sense as above ([24], [25] II, [26], cf.[9]). We will apply the nice parameterization of the degenerating family to a search of the asymptotic behavior of Eisenstein series (Theorem 1 and 2).

In §2, we shall prove two fundamental lemmas. In Lemma 1, we give an estimate of the Eisenstein series for punctured hyperbolic surfaces on any horocycle around the associated cusp. Amazingly it is independent of the complex structure and the topological type of the surfaces. To prove this, we use integral representations of eigenfunctions of the Laplacian that turn out to be very powerful ([10]). By a similar way we give an estimate of the Eisenstein series on a collar neighborhood of a separating pinching geodesic, which shows the order of convergence of the Eisenstein series to 0 on the component of S_0 not containing the associated cusp (cf. Theorem 1 (2)(ii), Theorem 2).

In §3, we shall prove one of the main theorems. Now for simplicity, we assume here that there exists just one pinching geodesic l , separating S^l into two parts. Let $E_p^l(z, s)$ be the Eisenstein series associated to a cusp p for S^l . We consider $(w^l)^* E_p^l(z, s)$, the pull-back of $E_p^l(z, s)$ by $w^l : S_0 \rightarrow S_l \setminus \{l\}$. S_0 is now divided into two parts; $S_{0,1}$ containing p , and the other component $S_{0,2}$. Then Theorem 1 claims that $(w^l)^* E_p^l(z, s)$ converges on $S_{0,1}$ to E_p^0 , the Eisenstein series associated to p for S_0 , and $(w^l)^* E_p^l(z, s)$ converges to 0 on $S_{0,2}$.

The outline of the proof is as follows. Denote by $C(a)$ the cusp region around p with length $1/a$ horocycle. First of all we can obtain the uniform boundedness of the sup-norm of $(w^l)^* E_p^l(z, s)$ over a region $S_0 - C(a)$ by using the maximum principles for subharmonic functions. Then using the interior Schauder estimates, taking an exhausting family of compact sets for S_0 and by the standard diagonal argument, we have obtained a convergent subsequence to a limit function $F_0(z, s)$. Noticing the zeroth Fourier coefficient of Eisenstein series and $F_0(z, s)$ around all cusps of S_0 , we can see that $F_0(z, s)$ that is independent of how to choose subsequences. In Theorem 2, we shall show that a subsequence of $(w^l)^* E_p^l(z, s)$ multiplied with some constants K_l , which could go to ∞ , converges on $S_{0,2}$ to the Eisenstein series for $S_{0,2}$ associated to the new cusp arising from the pinching geodesic l .

In §4, we shall apply the asymptotic behavior of the Eisenstein series and compare the magnitudes of two norms near the boundary of the moduli space along the degenerating family constructed by S. Wolpert. It follows that the magnitude of the Zograf-Takhtajan metric is less than or equal to that of the Weil-Petersson metric. The author believes that the two metrics are comparable near the boundary of the moduli space, which has not been proved. The most difficult point is to analyze the asymptotic behavior of $(w^l)^* E_p^l(z, s)$ near the pinching geodesic l because $E_p^0(z, s)$ assumes 0 at the node of S_0 . This theme is what we should investigate in the future.

We close this chapter with surveying and proposing some approaches to studying the asymptotic behavior of the Eisenstein series. Wolpert has investigated Eisenstein series $E(z, s)$ with $\{s \in \mathbb{C} | \operatorname{Re} s = \frac{1}{2}, s \neq \frac{1}{2}\}$, while we shall investigate that for $\operatorname{Re} s > 1$ ([25] I). He has shown that a subsequence of $\hat{E}(z, s)$, the normalized Eisenstein series, the L^2 -norm of which on a thick part of S_l could be constant, converges to a non-trivial sum of the Eisenstein series for S_0 . The beautiful proof has been accomplished by investigating Legendre functions and showing his original Schauder-type inequalities. It seems hard to apply our method directly to the case where $\operatorname{Re} s \leq 1$. The reason for difficulty is that we can not use the maximum principles because $E(z, s)$ with $\operatorname{Re} s \leq 1$ is not subharmonic, and we can not extend Lemma 1 to the case of $\operatorname{Re} s \leq 1$, and $E(z, s)$ has poles on $\{s \in [0, 1]\}$ (For example, we have observed that the constants $M_1(\operatorname{Re} s, a)$ assumes the infinity at $\operatorname{Re} s = 1$). What we have to study seems the behavior of the scattering matrix $\Phi(s)$ in (1.4).

The reader is also referred to the recent manuscript of T. Falliero ([3]). She has investigated the asymptotic behavior of hyperbolic Eisenstein series, which is introduced by S. Kudla and J. Millson, for pinching the associated simple closed geodesic.

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§1. PRELIMINARIES

1.1 Eisenstein Series.

Let S be a punctured hyperbolic surface of type (g, n) ($n > 0$). It can be represented as a quotient H/Γ of the upper half plane $H = \{z \in \mathbb{C} | \operatorname{Im} z > 0\}$ by the action of a torsion free finitely generated Fuchsian group $\Gamma \in \operatorname{PSL}_2(\mathbb{R})$.

The group is generated by $2g$ hyperbolic transformations $A_1, B_1, \dots, A_g, B_g$ and parabolic transformations $P_1, \dots, P_n$ satisfying the relation

$$A_1 B_1 A_1^{-1} B_1^{-1} \dots A_g B_g A_g^{-1} B_g^{-1} P_1 \dots P_n = 1.$$

The fixed points of the parabolic elements $P_1, \dots, P_n$ will be denoted by $z_1, z_2, \dots, z_n \in \mathbb{R} \cup \{\infty\}$ respectively and called inequivalent cusps. The projection of the cusps $z_1, z_2, \dots, z_n$ are the punctures $p_1, p_2, \dots, p_n$ of S . For each $i = 1, \dots, n$, denote by Γ_i the stabilizer of z_i in Γ that is the cyclic subgroup of Γ generated by P_i . Pick $\sigma_i \in \text{PSL}_2(\mathbb{R})$ such that $\sigma_i \infty = z_i$ and $\langle \sigma_i^{-1} P_i \sigma_i \rangle = \langle z \mapsto z + 1 \rangle$. Then, for $a > 1$, the a -cusp region $C_i(a)$ associated to p_i is represented as a quotient $\langle \sigma_i^{-1} P_i \sigma_i \rangle \backslash \{z \in H | \text{Im} z > a\} \simeq \Gamma \backslash \{z \in H | \text{Im} z > a\}$, equipped with the metric $ds^2 = (dy^2 + dx^2)/y^2$;

$$C_i(a) \simeq [a, \infty) \times S^1.$$

Let $\Delta : C^\infty(S) \rightarrow C^\infty(S)$ be the negative hyperbolic Laplacian of S . Regarded as an operator in $L^2(S)$ with domain $C_0^\infty(S)$, Δ is essentially self-adjoint. Denote by $\overline{\Delta}$ the unique self-adjoint extension (that is, the Friedrichs extension). Then the continuous spectrum of $\overline{\Delta}$ can be described in terms of the Eisenstein series ([6]Chap.VII, [10]Chap.V, [17]§3.2).

The Eisenstein series attached to z_i is defined by

$$E_i(z, s) = \sum_{\gamma \in \langle P_i \rangle \backslash \Gamma} \text{Im}(\sigma_i^{-1} \gamma z)^s, \quad \text{Re } s > 1.$$

We remark that the sum is independent of how we choose each representative in the coset decomposition above. The series is absolutely convergent in the upper half-plane and in the half-plane $\text{Re } s > 1$ and it satisfies

$$(1.1) \quad \Delta E_i(z, s) = s(s-1)E_i(z, s).$$

A. Selberg originally showed that the series admits meromorphic continuation to the whole complex s -plane, is continuous on $\{\text{Re } s = \frac{1}{2}\}$, and satisfies a system of functional equations ([15]§7). Several mathematicians also verified this by the various methods ([2], [6] Th.11.6, [10]pp.23–46, [12]). $E_i(z, s)$ has Fourier expansions at the punctures p_j , ([6]Prop.8.6, [10]§2.2, [11]§8, [17]§3.1)

$$(1.2) \quad E_i(\sigma_j z, s) = \delta_{ij} y^s + \phi_{ij}(s) y^{1-s} + \sum_{m \neq 0} c_m(s) y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi |m| y) e^{2\pi \sqrt{-1} m x},$$

$K_{s-\frac{1}{2}}(z)$ the MacDonald-Bessel function ([18], p.78) which has the following asymptotics ([18], p.202)

$$(1.3) \quad y^{\frac{1}{2}} K_{s-\frac{1}{2}}(y) \sim \sqrt{\frac{\pi}{2}} e^{-y}, \text{ as } y \in \mathbb{R} \rightarrow \infty, \quad \text{for any complex } s.$$

The scattering matrix $\Phi(s) = (\phi_{ij}(s))$ enters in the functional equations ([6]Th.11.8, [10]Th.4.4.2, [15](7.36), [17]Th.3.5.1),

$$(1.4) \quad \mathbb{E}(z, s) = \Phi(s) \mathbb{E}(z, 1-s), \quad \Phi(s) \Phi(1-s) = 1,$$

where $\mathbb{E}(z, s)$ is the vector of the Eisenstein series.

Thanks to Y. C. de Verdière's keen observation, the Eisenstein series turns out to have the following characterization ([2], [12]).

Claim. For $\operatorname{Re} s > \frac{1}{2}, s \notin (\frac{1}{2}, 1]$ $E_i(z, s)$ is a unique solution of the equation $\Delta E_i(z, s) = s(s-1)E_i(z, s)$ such that $E_i(z, s) - \operatorname{Im}(\sigma_i^{-1}z)^s \chi|_{C_i(1)}$ is square integrable on the whole surface, where $\chi|_{C_i(1)}$ is the characteristic function of $C_i(1)$.

Remark. We will use the above Claim just for the case where $\operatorname{Re} s > 1$ (Theorem 1 and 2).

1.2 The Weil-Petersson and the Zograf-Takhtajan Metrics.

Denote by $T_{g,n}$ Teichmüller space of hyperbolic surfaces of type (g, n) . Now we consider the tangent and cotangent spaces at S of $T_{g,n}$. The cotangent space is $Q(S)$, the integrable holomorphic quadratic differentials on S . Let $B(S)$ be the L^∞ -closure of Γ -invariant, bounded, $(-1, 1)$ -forms, i.e. the Beltrami differentials for S . For $\mu \in B(S), \varphi \in Q(S)$ the integral $\int_S \mu \varphi$ defines a paring, let $Q(S)^\perp$ be the annihilator of $Q(S)$. The tangent space at S to $T_{g,n}$ is $B(S)/Q(S)^\perp \simeq HB(S)$, the Serre dual space of $Q(S)$, i.e. the harmonic Beltrami differentials on S . Then for $\mu, \nu \in HB(S)$, the Weil-Petersson and the Zograf-Takhtajan metrics are defined as follows ([16]),

$$(1.5) \quad \langle \mu, \nu \rangle_{\text{WP}} = \iint_S \mu(z) \overline{\nu(z)} \rho(z) \frac{\sqrt{-1}}{2} dz \wedge d\bar{z}$$

$$(1.6) \quad \langle \mu, \nu \rangle_{(i)} = \iint_S E_i(z, 2) \mu(z) \overline{\nu(z)} \rho(z) \frac{\sqrt{-1}}{2} dz \wedge d\bar{z}$$

$$\langle \mu, \nu \rangle_{\text{ZT}} = \sum_{i=1}^n \langle \mu, \nu \rangle_{(i)},$$

where $\rho(z)|dz|^2$ denotes the Poincaré metric on S .

Both the Weil-Petersson and the Zograf-Takhtajan metric are Kählerian and incomplete ([13], [16]).

1.3 Degenerating Parameters and Infinite-Energy Harmonic Maps.

In this part, we consider degeneration of hyperbolic surfaces. Denote by $(S_l (l > 0), \rho_l(w)|dw|^2)$ a degenerating family of hyperbolic surfaces of type (g, n) . We assume that several disjoint simple closed geodesics $l_1, l_2, \dots, l_m$ on S_l will be pinched (We denote their hyperbolic lengths by the same notations). Let Δ_l be the negative Laplacian of S_l . To compare functions on the limit surface $(S_0, \rho(z)|dz|^2)$ and $(S_l, \rho_l(w)|dw|^2)$, we use the infinite-energy harmonic maps $w^l : S_0 \rightarrow S_l \setminus \{l_1, l_2, \dots, l_m\}$ constructed by M. Wolf ([9], [21], [26]). A node on S_0 is a pair of cusps and distinct nodes involve distinct cusps.

We can select simple closed geodesics $\{l_{m+1}, \dots, l_{3g-3+n}\}$ so that the set $\{l_1, \dots, l_m\} \cup \{l_{m+1}, \dots, l_{3g-3+n}\}$ is a pair of pants decomposition for S_l . Let $\{\theta_1, \dots, \theta_m, \theta_{m+1}, \dots, \theta_{3g-3+n}\}$ be twist angles around the corresponding geodesics. Then $\{l_1, \dots, l_{3g-3+n}, \theta_1, \dots, \theta_{3g-3+n}\}$ represent the Fenchel-Nielsen coordinates that are global real analytic for $T_{g,n}$. Here we set $\vec{l} = (l_1, l_2, \dots, l_{3g-3+n})$, and $\vec{\theta} = (\theta_1, \dots, \theta_{3g-3+n})$.

Then precise real-analytic parameterization is obtained by Wolf ([21], [26])

$$(1.7) \quad (w^l)^* \rho_l |dw|^2 = \Psi(\vec{l}, \vec{\theta}) dz^2 + (H(\vec{l}, \vec{\theta}) + L(\vec{l}, \vec{\theta})) \rho |dz|^2 + \overline{\Psi(\vec{l}, \vec{\theta})} d\bar{z}^2,$$

where $\Psi(\vec{l}, \vec{\theta}) = \rho_l w_z^l \overline{w_z^l}$, $H(\vec{l}, \vec{\theta}) = [\rho_l(w^l(z)/\rho(z))] |w_z^l|^2$, $L(\vec{l}, \vec{\theta}) = [\rho_l(w^l(z)/\rho(z))] |w_{\bar{z}}^l|^2$.

Choose a real basis $\{\Psi_1, \dots, \Psi_m, \Psi_{m+1}, \dots, \Psi_{6g-6+n-m}\}$ of quadratic differentials on S_0 with the property that, for $1 \leq i \leq m$, Ψ_i has a second order pole at the i -th node with leading coefficient equal to one and is otherwise regular or has at worst first order poles at the other nodes and punctures, and is the only basis that has a second pole at the i -th node; and furthermore, for $i > m$, $\{\Psi_{m+1}, \dots, \Psi_{6g-6+n-m}\}$ forms a real basis of integrable holomorphic quadratic differentials on S_0 .

Wolf finds also that $\Psi(\vec{l}, \vec{\theta})$ in (1.7) can be described in terms of the above basis as

$$(1.8) \quad \Psi(\vec{l}, \vec{\theta}) = \sum_{i=1}^m \frac{l_i^2}{4} \Psi_i + \sum_{j=m+1}^{6g-6+n-m} t_j(\vec{l}, \vec{\theta}) \Psi_j,$$

where $t_j(\vec{l}, \vec{\theta})$ is real analytic in $(\vec{l}, \vec{\theta})$ ([21], p.535).

The Beltrami differential of w^l is $\mu_l = w_z^l / w_{\bar{z}}^l = \bar{\Psi} / \rho H$. Wolf shows that when S_l degenerates to S_0 , the Beltrami differential μ_l converges uniformly to zero on compact subsets of S_0 , and the harmonic map w^l converges uniformly to id on compact subsets of S_0 .

Of interest to us now is the behavior of Δ_l . We review the discussion of Wolpert ([24], [25], [26]). There exists a basic map $\sigma_l : L^2(S_l) \rightarrow L^2(S_0)$, for $f \in L^2(S_l)$, $\sigma_l(f) = f(w^l)$. Define $(w^l)^* \Delta_l = \sigma_l \Delta_l \sigma_l^{-1}$. Then $(w^l)^* \Delta_l$ is real-analytic family ([24], p.450, [25], p.98, [26], pp.254-258). From [24] Lemma 5.3, we easily see that for any $k \in \mathbb{N}$ in the C^k -norm on compact subsets of components of S_0 , $(w^l)^* \Delta_l$ converges uniformly to Δ_0 that is defined to be the formal sum of the hyperbolic Laplacians for components of S_0 .

Many mathematicians investigated, by various parameterizations, degeneration of hyperbolic surfaces and the asymptotic behavior of several functions; for example, Green's functions ([7], [8], [9], [20]), eigenfunctions of the Laplacian and its eigenvalues ([7], [8], [9], [25], [26]), the Riemann matrix and Faltings invariant ([20]).

Remark. Basically those various parameterizations turn out to be almost the same powerful tools. But what we should pay attention to is that the parameters by the infinite-energy harmonic maps are independent of twist-angles around the pinching geodesics. Nevertheless, the family S_l with pairs of opened collars glued by adequate twist-angles agrees with R_l constructed by Wolpert [25], pp.103-104 ([23], Appendix, [26], pp.251-252).

§ 2. SOME ELEMENTARY ESTIMATES OF EISENSTEIN SERIES

We give key lemmas which play important roles in the proof of the main theorem (cf. [13], Lemma 4).

Lemma 1. *We use the same notations as in § 1. Let the index of Eisenstein series $Re\ s > 1$. For any $i = 1, 2, \dots, n$ and any $a > 1$,*

$$|E_i(z, s)| < M_1(Re\ s, a), \quad \text{on } \partial C_i(a).$$

Here $M_1(Re\ s, a)$ is a constant depending only on $Re\ s, a$, independent of complex structure and topological type of the surface.

Proof. We recall a fundamental fact. For $\epsilon > 0$, set a $\text{PSL}_2(\mathbb{R})$ -invariant kernel function

$$k_\epsilon(z, z') = \begin{cases} 1, & \text{if } d(z, z') < \epsilon \\ 0, & \text{otherwise.} \end{cases}$$

Then there exists a constant $\Lambda_\epsilon(s)$ depending only on ϵ and the index s such that for any $\sigma \in \text{PSL}_2(\mathbb{R})$,

$$(2.1) \quad \Lambda_\epsilon(s) \text{Im}(\sigma z)^s = \iint_H k_\epsilon(z, z') \text{Im}(\sigma z')^s \frac{dx' dy'}{y'^2}, \quad (z' = x' + y').$$

([10], Theorem 1.3.2).

Now without losing generality, we may assume that $\langle P_i \rangle = \langle z \mapsto z + 1 \rangle$.

For any $z_0 \in \partial C_i(a)$, we get

$$\begin{aligned} \Lambda_\epsilon(s) E_i(z_0, s) &= \Lambda_\epsilon(s) \sum_{\delta \in \langle P_i \rangle \setminus \Gamma} (\text{Im} \delta z_0)^s \\ &= \sum_{\delta \in \langle P_i \rangle \setminus \Gamma} \iint_H k_\epsilon(z_0, z) (\text{Im} \delta z)^s \frac{dx dy}{y^2} \\ &= \sum_{\delta \in \langle P_i \rangle \setminus \Gamma} \iint_H k_\epsilon(\delta z_0, \delta z) (\text{Im} \delta z)^s \frac{dx dy}{y^2} \\ &= \sum_{\delta \in \langle P_i \rangle \setminus \Gamma} \iint_H k_\epsilon(\delta z_0, z) (\text{Im} z)^s \frac{dx dy}{y^2} \\ &= \sum_{\delta \in \langle P_i \rangle \setminus \Gamma} \iint_{B(\delta z_0, \epsilon)} y^{s-2} dx dy. \end{aligned}$$

Here we set $B(\delta z_0, \epsilon) = \{z \in H \mid d(\delta z_0, z) < \epsilon\}$.

Due to the Shimizu-Leutbecher lemma, we select $\epsilon_0 = \epsilon_0(a) > 0$ depending only on $a > 1$ so that the injectivity radius at any point on $\partial C_i(a)$ could be larger than ϵ_0 .

Because $B(\delta z_0, \epsilon_0)$ ($\delta \in \langle P_i \rangle \setminus \Gamma$) are mutually disjoint and $\sum_{\delta \in \langle P_i \rangle \setminus \Gamma} B(\delta z_0, \epsilon_0) \subset \{z \in H \mid -1 < \text{Re } z < 2, 0 < \text{Im } z < 2a\}$ (replacing ϵ_0 by a smaller positive number if necessary, and taking appropriate representatives δ if necessary), we see

$$\begin{aligned} \Lambda_{\epsilon_0}(\text{Re } s) |E_i(z_0, s)| &\leq \Lambda_{\epsilon_0}(\text{Re } s) E_i(z_0, \text{Re } s) = \iint_{\sum_{\delta \in \langle P_i \rangle \setminus \Gamma} B(\delta z_0, \epsilon_0)} y^{\text{Re } s - 2} dx dy \\ &\leq \iint_{\substack{-1 < x < 2 \\ 0 < y < 2a}} y^{\text{Re } s - 2} dx dy \\ &= \frac{3 \cdot 2^{\text{Re } s - 1}}{\text{Re } s - 1} a^{\text{Re } s - 1}. \end{aligned}$$

Finally we conclude the proof of this lemma with setting

$$M_1(\text{Re } s, a) = \frac{3 \cdot 2^{\text{Re } s - 1} a^{\text{Re } s - 1}}{(\text{Re } s - 1) \Lambda_{\epsilon_0}(\text{Re } s)}.$$

□

Let $l_1, \dots, l_m$ be pinching geodesics on S_l . For $0 < k \leq 1$ and $j = 1, \dots, m$, set

$$N_{l_j}(k) = \{p \in S_l \mid d(p, l_j) < k \sinh^{-1}(1/\sinh \frac{l_j}{2})\},$$

the collar neighborhood around l_j in S_l ([1], 4.1). Here we quote an important claim due to S. Wolpert ([25] II, Lemma 2.1).

Claim. *Let $\rho(z)$ be the injectivity radius of S_l at z . There is an absolute positive constant C_0 such that for $l < 2\sinh^{-1}1$, and any $z \in N_l(1)$, then $\rho(z)e^{d(z, \partial N_l(1))} \geq C_0$.*

We improve the estimate of the Eisenstein series in [13], Lemma 4 just for the case where there exist separating pinching geodesics on S_l . Let $E_i^l(z, s)$ be the Eisenstein series attached to p_i for S_l .

Lemma 2. *Let the index $\text{Re } s > 1$. Assume that there is only one pinching geodesic $l = l_1$ on S_l , separating S_l into two parts; $S_{l,1}$ containing the puncture p_i and the other component $S_{l,2}$. Then for $l < 2\sinh^{-1}1$,*

$$|E_i^l(z, s)| < M_2(\text{Re } s) l^{\text{Res}(1+k)-2}, \quad \text{on } \partial N_l(k) \cap S_{l,2}.$$

Here $M_2(\text{Re } s)$ is an absolute constant depending only on $\text{Re } s$.

Proof. Without losing generality, we may assume that $\langle P_i \rangle = \langle z \mapsto z + 1 \rangle$. First we fix a constant $\epsilon = \epsilon_1 < \sinh^{-1}1$. As in the proof of Lemma 1, for any $z_1 \in \partial N_l(k) \cap S_{l,2}$, we apply (2.1) for ϵ_1 . We have

$$\Lambda_{\epsilon_1}(s) E_i^l(z_1, s) = \sum_{\delta \in \langle P_i \rangle \setminus \Gamma} \iint_{B(\delta z_1, \epsilon_1)} y^{s-2} dx dy.$$

Next we repeat the same discussion in the proof of [13], Lemma 4. Notice that if $B(z_1, \epsilon_1) \cap B(\delta z_1, \epsilon_1) \neq \emptyset$ then $d(z_1, \delta z_1) < 2\epsilon_1$. And we easily see that the multiplicity of $B(z_1, 2\epsilon_1) \rightarrow B(z_1, 2\epsilon_1)/\Gamma$ is at most $c\rho^{-1}(z_1)$ (c is an absolute constant). Here we set $\mathcal{B} = \bigcup_{\delta \in \langle P_i \rangle \setminus \Gamma} B(\delta z_1, \epsilon_1)$.

Thus from simple consideration, the multiplicity of the points of $\mathcal{B}$ are at most $c\rho^{-1}(z_1)$. Another important fact is that $\mathcal{B}$ is included in the set

$$\mathcal{R} = \{ z \in H \mid d(z, \{\text{Im } z = 1\}) \geq -(1+k) \log l - \epsilon_1 \} \quad (\text{See [25]II, p.102}).$$

Combining the above arguments, we have obtained

$$\begin{aligned}
\Lambda_{\epsilon_1}(\operatorname{Re} s) |E_i^l(z_1, s)| &\leq \Lambda_{\epsilon_1}(\operatorname{Re} s) E_i^l(z_1, \operatorname{Re} s) = \sum_{\delta \in \langle P_i \rangle \setminus \Gamma} \iint_{B(\delta z_1, \epsilon_1)} y^{\operatorname{Res}-2} dx dy \\
&\leq c \rho(z_1)^{-1} \iint_{\mathcal{B}} y^{\operatorname{Res}-2} dx dy \\
&\leq c \rho(z_1)^{-1} \iint_{\mathcal{R} \cap \{-1 < \operatorname{Re} z < 2\}} y^{\operatorname{Res}-2} dx dy \\
&\leq c \rho(z_1)^{-1} \iint_{\substack{-1 < x < 2 \\ 0 < y < e^{\epsilon_1} l^{1+k}}} y^{\operatorname{Res}-2} dx dy \\
&\leq c' e^{d(z_1, \partial N_l(1))} \frac{3 \cdot e^{(\operatorname{Res}-1)\epsilon_1}}{\operatorname{Re} s - 1} l^{(\operatorname{Res}-1)(1+k)} \quad (\text{cf. Claim in §2}) \\
&\leq c' l^{-(1-k)} \frac{3 \cdot e^{(\operatorname{Res}-1)\epsilon_1}}{\operatorname{Re} s - 1} l^{(\operatorname{Res}-1)(1+k)} \quad (\text{By } d(z_1, \partial N_l(1)) \approx -(1-k) \log l) \\
&= c' \frac{3 \cdot e^{(\operatorname{Res}-1)\epsilon_1}}{\operatorname{Re} s - 1} l^{\operatorname{Res}(1+k)-2}.
\end{aligned}$$

We may set

$$M_2(\operatorname{Re} s) = c' \frac{3 \cdot e^{(\operatorname{Res}-1)\epsilon_1}}{(\operatorname{Re} s - 1) \Lambda_{\epsilon_1}(\operatorname{Re} s)}. \quad \square$$

Remark. For any s with $\operatorname{Re} s > 1$, there is $0 < k \leq 1$ such that $\operatorname{Re} s (1+k) - 2 > 0$.

§ 3. THE ASYMPTOTIC BEHAVIOR OF EISENSTEIN SERIES

Our aim is to prove one of the main theorems. From now on, the cusps of S_0 that arise from the cusps of S_l are called the *old cusps* and the cusps of S_0 that arise from the pinching geodesics of S_l are called the *new cusps*.

Theorem 1. *We set the same notations as in §1. Let the index $\operatorname{Re} s > 1$.*

(1) *If $\{l_1, \dots, l_m\}$ do not separate S_l , then for any $i = 1, \dots, n$, as $l_1, \dots, l_m \rightarrow 0$,*

$$(3.1) \quad (w^l)^* E_i^l(z, s) \longrightarrow E_i^0(z, s)$$

uniformly on any compact subset of S_0 . Here $E_i^0(z, s)$ is the Eisenstein series attached to the old puncture p_i for S_0 .

(2) *Assume that $\{l_1, \dots, l_m\}$ separate S_l . Denote by $S_{0,1}^i$ and $S_{0,2}^i$ respectively the component of S_0 containing p_i and the union of the components of S_0 not containing p_i . Let q_j ($j = 1, \dots, m$) be the new cusp arising from l_j . Denote by $C_j(b)$ ($b > 1$) be the cusp region around q_j in S_0 , each composed of usual two b -cusp regions. Then*

(i) *For any $i = 1, \dots, n$, as $l_1, \dots, l_m \rightarrow 0$,*

$$(3.2) \quad (w^l)^* E_i^l(z, s) \longrightarrow E_i^0(z, s)$$

uniformly on any compact subset of $S_{0,1}^i$. Here $E_i^0(z, s)$ is the Eisenstein series attached to p_i for $S_{0,1}^i$.

(ii) For any $i = 1, \dots, n$ and any $b > 1$, as $l_1, \dots, l_m \rightarrow 0$,

$$(3.3) \quad (w^l)^* E_i^l(z, s) \longrightarrow 0$$

uniformly on $S_{0,2}^i - \bigcup_{j=1}^m C_j(b)$.

Furthermore, for $b > 1$ fixed,

$$|(w^l)^* E_i^l(z, s)| = O \left(\max_{j=1, \dots, m} l_j^{(2-\delta)\text{Re } s - 2} \right), \quad \text{for any small } \delta > 0$$

on $S_{0,2}^i - \bigcup_{j=1}^m C_j(b)$.

Proof. (1) First of all, we estimate $|E_i^l(z, s)|$ on $S_l - C_i(a)$. It is easily seen that $|E_i^l(z, s)| \leq E_i^l(z, \text{Re } s)$. Since the index $\text{Re } s > 1$, $E_i^l(z, \text{Re } s)$ clearly turns out to be a subharmonic function of z (Cf. (1.1)). Thanks to this property, we can apply the maximum principle and Lemma 1. Then observing that $E_i^l(z, s)$ vanishes at all the cusps except for the i -th cusp since $\text{Re } s > 1$, we obtain

$$(3.4) \quad \begin{aligned} \|E_i^l(z, s)\|_{C^0(S_l - C_i(a))} &\leq \|E_i^l(z, \text{Re } s)\|_{C^0(S_l - C_i(a))} \\ &\leq \sup_{z \in \partial C_i(a)} E_i^l(z, \text{Re } s) \leq M_1(\text{Re } s, a). \end{aligned}$$

For any region $\Omega \subset\subset S_0$, we select $a = a(\Omega) > 1$ such that $w^l(\Omega) \subset\subset S_l - C_i(a)$. From (3.4), for sufficiently small $l_1, \dots, l_m$,

$$(3.5) \quad \|(w^l)^* E_i^l(z, s)\|_{C^0(\Omega)} < M_1(\text{Re } s, a).$$

For $z \in \Omega$, from the definition of $(w^l)^* \Delta_l$ in §1.1.3,

$$((w^l)^* \Delta_l - s(s-1)) (w^l)^* E_i^l(z, s) = 0.$$

As stated in §1.1.3, $(w^l)^* \Delta_l$ converges uniformly to Δ_0 in the C^2 -norm on compact subsets of Ω as $l_1, \dots, l_m \rightarrow 0$. Then by (3.5) and the interior Schauder estimates ([5], Theorem 6.2), for any $0 < \alpha < 1$ and $\Omega' \subset\subset \Omega$, there exists a positive constant $L' = L'(\Omega', \Omega, s, \alpha)$ such that

$$\|(w^l)^* E_i^l(z, s)\|_{C^{2,\alpha}(\Omega')} \leq L'.$$

By the standard compactness argument ([5]), and taking an exhausting family of compact subsets of S_0 , the diagonal method and (3.4), there is a subsequence $\{l_1^j, \dots, l_m^j\}$ such that

$$\lim_{j \rightarrow \infty} (w^{l^j})^* E_i^{l^j}(z, s) = F_0(z, s)$$

in the C^2 -norm on compact subsets of S_0 for a function $F_0(z, s) \in C^2(S_0)$, which satisfies on S_0 ,

$$(3.6) \quad \begin{aligned} (\Delta_0 - s(s-1)) F_0(z, s) &= 0. \\ \|F_0(z, s)\|_{C^0(S_0 - C_i(a))} &\leq M_1(\text{Re } s, a). \end{aligned}$$

By elliptic regularity, it turns out that $F_0(z, s) \in C^\infty(S_0)$. We have to show just that the limit function $F_0(z, s)$ equals to $E_i^0(z, s)$ and is independent of how we choose subsequences.

From [10], Theorem 3.2.1, we see that the Fourier expansion of $F_0(z, s)$ at any cusp of Γ_0 (we set $S_0 \simeq H/\Gamma_0$) is of the form

$$c_0 y^s + c'_0 y^{1-s} + O(e^{-2\pi\sqrt{-1}y}), \quad \text{as } y \rightarrow \infty.$$

It should be noted that w^l converges uniformly to id on compact subsets of S_0 (§1, 1.3), and the 0-th Fourier coefficients of $E_i^l(\sigma_j z, s)$ at the old cusps (1.2) are of the form

$$\delta_{ij} y^s + \phi_{ij}(s) y^{1-s} = \int_0^1 E_i^l(\sigma_j z, s) dx.$$

Thus by the above remarks, Lemma 1, (1.2) and (3.6), we easily see that the 0-th coefficients of $F_0(z, s)$ at the old cusp p_i and at the other old and the new cusps are respectively of the form $y^s + cy^{1-s}$, $c'y^{1-s}$ with constants c, c' . The conclusion follows from the above observation, (3.6), Claim in §1.

(2) (i) We can show (3.2) in much the same way as in the proof of (3.1).

(2) (ii) First we prove the convergence on compact subsets of S_0 as in (1). We can obtain the limit function $f_0(z, s)$ of a subsequence of $\{(w^l)^* E_i^l(z, s)\}$, satisfying

$$(3.7) \quad \begin{aligned} &(\Delta_0 - s(s-1)) f_0(z, s) = 0. \\ &\|f_0(z, s)\|_{C^0(S_0 - C_i(a))} \leq M_1(\operatorname{Re} s, a). \end{aligned}$$

From the inequality in (3.7), we can show that the 0-th Fourier coefficients of $f_0(z, s)$ at all cusps of $S_{0,2}^i$ are of the form $b'' y^{1-s}$, which means that $f_0(z, s)$ is square integrable. Thus $f_0(z, s)$ is clearly included in the domain of the self-adjoint extension of the Laplacian $\overline{\Delta_0}$. Since $-s(s-1)$ is not an eigenvalue of $\overline{\Delta_0}$, $(\overline{\Delta_0} - s(s-1))$ has a densely defined bounded inverse. By the above arguments and the equation in (3.7), we obtain $f_0(z, s) = 0$.

Next we prove the uniform convergence on $S_{0,2}^i - \bigcup_{j=1}^m C_j(b)$ for any $b > 1$. We already know

$$\begin{aligned} \sup_{z \in S_{0,2}^i - \bigcup_{j=1}^m C_j(b)} |(w^l)^* E_i^l(z, s)| &\leq \sup_{z \in S_{0,2}^i - \bigcup_{j=1}^m C_j(b)} (w^l)^* E_i^l(z, \operatorname{Re} s) \\ &\leq \max_{j=1, \dots, m} \left\{ \sup_{z \in \partial C_j(b)} (w^l)^* E_i^l(z, \operatorname{Re} s) \right\}. \end{aligned}$$

Because we have observed that the last term in the above inequalities converges to 0, the conclusion easily follows.

Finally we give another proof of (2)(ii) that gives us the order of convergence. As is stated in the remark following lemma 2, for any given $s = s_1$ with $1 < \operatorname{Re} s_1$, we can find $0 < k = k_1 < 1$ such that $\operatorname{Re} s_1(1 + k_1) - 2 > 0$. Then for any $b = b_1 > 1$, we easily see that for sufficiently small $l_1, \dots, l_m$,

$$w^l(S_{0,2}^i - \bigcup_{j=1}^m C_j(b_1)) \subset w^l(S_{0,2}^i) - \bigcup_{j=1}^m N_{l_j}(k_1).$$

Then

$$\begin{aligned}
 \sup_{z \in S_{0,2}^i - \bigcup_{j=1}^m C_j(b_1)} |(w^l)^* E_i^l(z, s_1)| &\leq \sup_{w \in w^l(S_{0,2}^i) - \bigcup_{j=1}^m N_{l_j}(k_1)} E_i^l(w, \operatorname{Re} s_1) \\
 &\leq \max_{j=1, \dots, m} \sup_{\partial N_{l_j}(k_1) \cap w^l(S_{0,2}^i)} E_i^l(w, \operatorname{Re} s_1) \\
 (3.8) \quad &\leq M_2(\operatorname{Re} s_1) \max_{j=1, \dots, m} l_j^{\operatorname{Res}_1(1+k_1)-2}.
 \end{aligned}$$

From this we clearly obtain that $(w^l)^* E_i^l(z, s_1)$ converges uniformly to 0 on $S_{0,2}^i - \bigcup_{j=1}^m C_j(b_1)$ for any $b = b_1 > 1$ with the order of its convergence as in (3.8). $\square$

Theorem 2. *We set the same assumption and notations as in Theorem 1 (2). Pick R one of the components of $S_{0,2}^i$, which has the new cusps $q_1, q_2, \dots, q_t$, arising from the pinching geodesics $l_1, \dots, l_m$ on S_l , and has the old cusps which may be denoted by $p_1, \dots, p_u$, differing from p_i , replacing the enumeration if necessary. Denote by $E_{q_j}(z, s)$ the Eisenstein series attached to q_j for R ($j = 1, \dots, t$).*

Then there exist some constants $K_l \rightarrow \infty$ and a subsequence $l_1^{(h)} = \dots = l_m^{(h)} = l^{(h)} \rightarrow 0$,

$$K_{l^{(h)}}(w^{l^{(h)}})^* E_i^{l^{(h)}}(z, s) \longrightarrow G_0(z, s)$$

on any compact subset of R , where $G_0(z, s)$ is a non-trivial smooth function on R satisfying

$$(\Delta_0 - s(s-1)) G_0(z, s) = 0 \quad \text{on } R.$$

And $\lim_{l \rightarrow 0} K_l l^{2(\operatorname{Res}-1)-\delta} = \infty$, for any $\delta > 0$.

Moreover $G_0(z, s)$ is of the form,

$$(3.9) \quad G_0(z, s) = \sum_{j=1}^t B_j E_{q_j}(z, s),$$

where B_j ($j = 1, \dots, t$) are some constants.

Proof. Let $C_v^o(a), C_j^n(a)$ ($a \geq 1$) be respectively the a -cusp regions around p_v, q_j in R ($v = 1, \dots, u, j = 1, \dots, t$). For any $a \geq 1$, set $\Omega(a) = R - \bigcup_{v=1}^u C_v^o(a) - \bigcup_{j=1}^t C_j^n(a)$.

Now we set constants $K_l > 0$ such that

$$K_l \sup_{z \in \Omega(1)} (w^l)^* E_i^l(z, \operatorname{Re} s) = K_l \times \sup_{z \in \bigcup_{v=1}^u \partial C_v^o(1) \cup \bigcup_{j=1}^t \partial C_j^n(1)} (w^l)^* E_i^l(z, \operatorname{Re} s) = 1.$$

Then we can show the following.

Lemma 3. *For any $a > 1$ fixed,*

$$L_l(a) = K_l \sup_{z \in \Omega(a)} (w^l)^* E_i^l(z, \operatorname{Re} s) = K_l \times \sup_{z \in \bigcup_{v=1}^u \partial C_v^o(a) \cup \bigcup_{j=1}^t \partial C_j^n(a)} (w^l)^* E_i^l(z, \operatorname{Re} s)$$

is uniformly bounded as $l_1 = \dots = l_m = l \rightarrow 0$.

Proof of Lemma 3. let $a > 1$ be fixed. Then taking a region $\Omega' \supset \supset \Omega(a)$ and applying Cor. 8.21. in [5], we have a constant $C = C(\Omega(a), \Omega')$ such that for l small enough, the Harnack inequalities

$$\sup_{\Omega(a)} K_l(w^l)^* E_i^l(z, \operatorname{Re} s) \leq C \inf_{\Omega(a)} K_l(w^l)^* E_i^l(z, \operatorname{Re} s)$$

hold. Thus we get

$$\begin{aligned} L_l(a) &= \sup_{\Omega(a)} K_l(w^l)^* E_i^l(z, \operatorname{Re} s) \leq C \inf_{\Omega(a)} K_l(w^l)^* E_i^l(z, \operatorname{Re} s) \\ &\leq C \inf_{\Omega(1)} K_l(w^l)^* E_i^l(z, \operatorname{Re} s) \\ &\leq C \sup_{\Omega(1)} K_l(w^l)^* E_i^l(z, \operatorname{Re} s) \\ &= C < \infty. \end{aligned}$$

We have concluded the proof of Lemma 3.

We continue to the proof of Theorem 3. In the same way in the proof of Theorem 1, we take an exhausting family of compact subsets of R and apply the maximum principles, the interior Schauder estimates and the standard diagonal method to $\{K_l(w^l)^* E_i^l(z, \operatorname{Re} s)\}$. Thus we observe that there is a subsequence $l_1^{(h)} = \dots = l_m^{(h)} = l^{(h)} \rightarrow 0$ and $K_{l^{(h)}}$,

$$\lim_{h \rightarrow \infty} K_{l^{(h)}}(w^{l^{(h)}})^* E_i^{l^{(h)}}(z, s) = G_0(z, s)$$

uniformly on any compact subset of R , for a non-trivial function $G_0(z, s)$ satisfying

$$(3.10) \quad (\Delta_0 - s(s-1)) G_0(z, s) = 0.$$

Take a region D in R such that for sufficiently small $l_1, \dots, l_m$, $w^l(D) \subseteq w^l(R) - \bigcup_{j=1}^m N_{l_j}(1)$. By Lemma 2, we see

$$\begin{aligned} \sup_{z \in D} |(w^l)^* E_i^l(z, s)| &\leq \max_{j=1, \dots, m} \sup_{\partial N_{l_j}(1)} E_i^l(w, \operatorname{Re} s) \\ &\leq M_2(1, \operatorname{Re} s) \max_{j=1, \dots, m} l_j^{2(\operatorname{Re} s - 1)}. \end{aligned}$$

By the above inequality, it is necessary that $K_l \geq \hat{c} l^{-2(\operatorname{Re} s - 1)}$ for a constant $\hat{c}$.

All we have to show is just that $G_0(z, s)$ is of the form (3.9). From [10], Theorem 3.2.1, we see that the Fourier expansion of $G_0(z, s)$ at any cusp of Γ_0 (we set $R \simeq H/\Gamma_0$) is of the form

$$c_0 y^s + c'_0 y^{1-s} + \sum_{m \neq 0} c_m(s) y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|m|y) e^{2\pi\sqrt{-1}mx},$$

on $y > 1$. Here $K_{s-\frac{1}{2}}$ is the MacDonald-Bessel function.

Because for any $v = 1, \dots, u$,

$$\sup_{z \in C^o(a)} K_l |(w^l)^* E_i^l(z, s)| \leq \sup_{z \in \partial C_v^o(a)} K_l (w^l)^* E_i^l(z, \operatorname{Re} s) \leq L_l(a)$$

holds, we see that $G_0(z, s)$ is bounded in $C_v^o(a)$.

Therefore by (3.10), Claim in §1, the discussion of the proof of Theorem 1 (2) (ii) and the above observations, we have seen that $G_0(z, s)$ is a linear sum of the Eisenstein series attached to the new cusps $q_1, \dots, q_t$. It follows that there are constants $B_1, \dots, B_t$ such that

$$G_o(z, s) = \sum_{j=1}^t B_j E_{q_j}(z, s). \quad \square$$

§ 4. A COMPARISON OF THE W-P AND THE Z-T METRICS

We review the construction of the degenerating family of punctured surfaces $\{c(t) = [S_{l(t)}] \in T_{g,n} \mid -\infty < t < 0\}$ in [13] (cf. [25] II). For simplicity assume that there is only one pinching geodesic on $S_{l(t)}$ as denoted by $l(t)$. The feature of the construction is to initiate with defining the vector field of the degenerating family that is represented as harmonic Beltrami differential as follows.

Set $\Gamma_t \subset \operatorname{PSL}_2(\mathbb{R})$ such that $S_{l(t)} \simeq H/\Gamma_t$. Let A_t be the one of primitive hyperbolic elements of Γ_t that cover the pinching geodesic $l(t)$. Define a relative Poincaré series associated to $l(t)$,

$$\Theta_t = \sum_{B \in \langle A_t \rangle \setminus \Gamma_t} (\omega_{A_t} \circ B) B'^2,$$

where $\omega_{A_t} = 1/z^2$, if A_t is normalized to fix 0, ∞ . We remark that Θ_t is independent of how we choose each representative in the coset decomposition above. Then we define a tangent vector $\tau(t) \in HB(\Gamma_t)$,

$$\frac{dc(t)}{dt} = \tau(t) = p(t) (d\rho_t)^{-1} \overline{\Theta_t}, \quad \iint_{S_{l(t)}} \tau(t) \Theta_t = \frac{\pi}{2} l(t),$$

where $d\rho_t$ is the hyperbolic area element for $S_{l(t)}$. It should be noted here that $p(t)$ is defined by the integration above. Then as is discussed in [13], §2, we have obtained the resulting behavior of $p(t)$ and $l(t)$ as follows,

$$(4.1) \quad \begin{aligned} p(t) &= 1 + O(l(t)^2) \\ \frac{dl(t)}{dt} &= (dl(t), \tau) = \frac{2}{\pi} \iint_{S_{l(t)}} \tau \Theta_t = l(t), \quad l(t) = le^t. \end{aligned}$$

Theorem 3. *The Weil-Petersson and the Zograf-Takhtajan metrics have the following behavior near the boundary of $T_{g,n}$, along the degenerating family of hyperbolic surfaces constructed by Wolpert. That is, let τ be the vector field formed by the degenerating family with only one pinching geodesic for simplicity. Then the norms of τ with respect to the Weil-Petersson and Zograf-Takhtajan metrics satisfy*

$$\|\tau\|_{ZT} \leq n\tilde{c}\|\tau\|_{WP} \quad \text{as } l \rightarrow 0.$$

where $\tilde{c}$ is an absolute constant.

Proof. Evaluate the norms of the tangent vectors $\tau(t)$ in terms of the Weil-Petersson and the Zograf-Takhtajan metrics respectively,

$$(4.2) \quad \begin{aligned} \left\| \frac{dc}{dt} \right\|_{\text{WP}}^2 &= \iint_{S_{l(t)}} |\tau(t)|^2 d\rho_t \\ \left\| \frac{dc}{dt} \right\|_i^2 &= \iint_{S_{l(t)}} E_i^l(z, 2) |\tau(t)|^2 d\rho_t. \end{aligned}$$

From ([25] II Lemma 2.3 and [13] Corollary), we have seen that

$$(4.3) \quad \left\| \frac{dc}{dt} \right\|_{\text{WP}}^2 \approx l(t), \quad \text{as } l(t) \rightarrow 0.$$

Therefore we have to just show that the second norm in (4.2) has the same order as (4.3). Divide the Zograf-Takhtajan's norm in (4.2) into three parts as in [13] Theorem,

$$\left\| \frac{dc}{dt} \right\|_i^2 = \iint_{N_l(\frac{1}{2})} + \iint_{N_l(1) - N_l(\frac{1}{2})} + \iint_{S_{l(t)} - N_l(1)} = \text{I} + \text{II} + \text{III}.$$

Since we have already shown in [13] that

$$\text{II} = O(l(t)^2), \quad \text{III} = O(l(t)^4), \quad \text{as } l(t) \rightarrow 0$$

(that is, small enough), we may just concentrate on estimating the main term I.

From Lemma 1 and (4.1), it follows that

$$\begin{aligned} \text{I} &= \iint_{N_l(\frac{1}{2})} E_i^l(z, 2) |\tau(t)|^2 d\rho_t \\ &= \iint_{N_l(\frac{1}{2})} E_i^l(z, 2) |p(t)(d\rho_t)^{-1} \overline{\Theta_t}|^2 d\rho_t \\ &\leq M_1(2, 1) \iint_{S_{l(t)}} 2|(d\rho_t)^{-1} \overline{\Theta_t}|^2 \approx l(t), \quad \text{as } l(t) \rightarrow 0. \quad \square \end{aligned}$$

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