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**Goldman-Turaev Lie Bialgebras for Some Types of
Surfaces
and
Graphical Calculus for Colored Jones Polynomials**

Wataru Yuasa

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CHAPTER 1

Introduction

This thesis treats two main objects. The one is Goldman-Turaev Lie bialgebra for surfaces and another colored Jones polynomials for knots and links. In Part 1, we define Goldman-Turaev Lie bialgebras for 2-orbifolds and bordered surface. We also give a deformation quantization of the Goldman Lie algebra of a bordered surface by using the Kauffman bracket skein algebra for the surface.

The Goldman Lie algebra of a closed oriented surface S was defined by Goldman [Gol86]. He defined a Lie bracket, called the *Goldman bracket*, on the free $\mathbb{Z}$ -module $\mathbb{Z}\hat{\pi}(S)$ spanned by the set of homotopy classes of oriented loops on S where $\mathbb{Z}$ is the ring of integers. The Goldman bracket is defined topologically by using smoothings of the union of two loops and the local intersection number at intersection points. He also defined the Goldman Lie algebra $\mathbb{Z}\bar{\pi}(S)$ of unoriented loops and constructed an embedding of $\mathbb{Z}\bar{\pi}(S)$ into $\mathbb{Z}\hat{\pi}(S)$ as Lie algebras. We remark that the Lie algebra $\mathbb{Z}\bar{\pi}(S)$ was implicitly considered by Wolpert [Wol83].

The space $\text{Hom}(\pi_1(S), G)/G$ of the conjugacy classes of representations of the fundamental group of S to a Lie group G satisfying some general conditions has a symplectic structure introduced by Goldman [Gol84]. The symplectic structure is a generalization of the Weil-Petersson form on the Teichmüller space of S . Then $C^\infty(\text{Hom}(\pi_1(S), G)/G)$ has a Poisson bracket. Goldman constructed Lie algebra homomorphisms $\mathbb{Z}\hat{\pi}(S) \rightarrow C^\infty(\text{Hom}(\pi_1(S), G)/G)$ and $\mathbb{Z}\bar{\pi}(S) \rightarrow C^\infty(\text{Hom}(\pi_1(S), G')/G')$ for some Lie groups G and G' .

The Goldman Lie algebras $\mathbb{Z}\hat{\pi}(S)$ and $\mathbb{Z}\bar{\pi}(S)$ have also a knot-theoretical aspect. Turaev [Tur89, Tur91] gave quantizations of the Poisson algebras associated with $\mathbb{Z}\hat{\pi}(S)$ and $\mathbb{Z}\bar{\pi}(S)$. A quantization of a Poisson $\mathbb{C}$ -algebra B is a $\mathbb{C}[[h]]$ -linear map from $\mathbb{C}[[h]]$ -algebra A to B where $\mathbb{C}[[h]]$ is the ring of complex formal power series in one variable h . This map p sends the non-commutativity $ab - ba$ of two elements a and b of A modulo h to the Poisson bracket of $p(a)$ and $p(b)$. Turaev used skein algebras of S to quantize $\mathbb{Z}\hat{\pi}(S)$ and $\mathbb{Z}\bar{\pi}(S)$. The skein algebra of S is a quotient of the module spanned by the regular isotopy classes of links in $S \times [0, 1]$ by the skein relation. The multiplication of the skein algebra is induced by superposition of links. The Jones-Conway skein relation was used for quantization of $\mathbb{Z}\hat{\pi}(S)$, the Kauffman bracket skein relation was used for the quantization of $\mathbb{Z}\bar{\pi}(S)$. Hoste-Przytycki [HP90] also considered the quantization of the Goldman Lie algebras. Turaev defined one more operation on the module $\mathbb{Z}\hat{\pi}'(S)$ where $\mathbb{Z}\hat{\pi}'(S)$ is the quotient of $\mathbb{Z}\hat{\pi}(S)$ by the submodule spanned by the trivial loop. The operation, called the *Turaev cobracket*, was defined by using smoothing of one loop and the local intersection number at self-intersection points. He also gave a co-quantization of the co-Poisson coalgebra associated with the Turaev co-Lie coalgebra $\mathbb{Z}\hat{\pi}'(S)$ by using the skein algebra of S . A co-quantization of the co-Poisson coalgebra B' is a $\mathbb{C}[[h]]$ -linear map from a $\mathbb{C}[[h]]$ -coalgebra A' to B' . The map measures the difference $\Delta(a) - \sigma \circ \Delta(a)$ of the comultiplication and opposite comultiplication of an element a modulo h by the

Poisson cobracket. We remark that the comultiplication of the skein algebra is the Jaeger's composition product [Jac89]. Turaev also proved that the Goldman bracket and the Turaev cobracket satisfy compatibility conditions of a Lie bialgebra. Moreover the quantization and the co-quantization satisfy some conditions. Thus he gave a bi-quantization of the bi-Poisson bialgebra associated with the Goldman-Turaev Lie bialgebra.

The author studied Goldman-Turaev Lie bialgebras of two kinds of surfaces: a orientable two orbifold and a bordered surface. One of the main reasons why we consider these Goldman Lie algebras is due to the Teichmüller Theory. As we have already mentioned, The Poisson structure of the Goldman Lie algebra of a closed surface is compatible with one of the function space of the Teichmüller space of the surface. Penner [Pen87] introduced the notion of λ -length and constructed the decorated Teichmüller spaces of punctured surfaces and bordered surfaces. (See Penner's book [Pen12] for details). Moreover, Teichmüller spaces of Riemann surfaces with orbifold points were also considered by Chekhov and Shapiro [CS14], Chekhov and Mazzocco [MC09] and Chekhov [Che09b]. These Teichmüller spaces have also Poisson structures. Thus we can obtain Poisson algebras of the functions on these Teichmüller spaces. It is natural to consider the Goldman Lie algebra of these types of surfaces. Roger and Yang [RY14] defined the Goldman Lie algebra of unoriented curves on punctured surfaces. They also gave a quantization of the Goldman Lie algebra and constructed a Poisson homomorphism of it to the space of λ -length functions on the decorated Teichmüller space. Chas and Gadgil [CG12] defined the Goldman Lie algebra of an oriented two orbifold by using hyperbolic group-theoretical method.

In Part 2, we will give explicit formulas for A_1 and A_2 web spaces. These formulas are obtained by diagrammatic calculation by using skein relations corresponding to quantum representations of $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$. We will give explicit formulas of the $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ colored Jones polynomials for 2-bridge links. Moreover, we give two explicit formulas for the "limit" of the $\mathfrak{sl}_3$ colored Jones polynomials for the $(2, 2m)$ -torus link from our formulas about A_2 web spaces.

V. F. R. Jones discovered a celebrated polynomial invariant of a knot, the Jones polynomial, in his paper [Jon85]. After Kauffman gave a reformulation of the Jones polynomial of a link using the Kauffman bracket in [Kau87], the skein theory has developed in various directions and many quantum invariants have been discovered. Main objects in Part 2 are the clasped A_1 and A_2 web spaces introduced by Kuperberg [Kup96]. The A_1 web space W_n is the vector space spanned by regular homotopy classes of crossingless unoriented arcs, called a *basis web*, on the disk D_n with n marked points. The A_1 bracket $\langle \cdot \rangle_2$, it is also called the Kauffman bracket, is the linear map from the vector space spanned by tangles in D_n to W_n defined by the Kauffman bracket skein relations. The A_1 clasp is denoted as a white box $\langle -\square^n \rangle_2 \in W_{n+n}$ where the arc decorated by n means n -parallelized arcs. It is also called the Jones-Wenzl idempotent and the magic element. The A_2 web space, the A_2 bracket and the A_2 clasp are defined in a similar way. An A_2 basis web is a non-elliptic embedded bipartite trivalent graph with oriented edges. The vertices are divided into sinks  and sources . A non-elliptic A_2 web means that internal polygons have no 0-, 2-, 4-gons.

The author calculates A_1 and A_2 webs with clasps and obtained some formulas: the skein relations with clasps, the m -full twists formula for two strands with clasps, and the bubble skein formula. These formulas are useful to compute the $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ colored Jones polynomials. Especially, it is difficult to compute the $\mathfrak{sl}_3$ colored Jones polynomial

and known a few examples. The formula for trefoil knot was given by Lawrence [Law03], more generally for the torus knot $T(2, k)$ was given by Garoufalidis, Morton and Vuong [GMV13] through a method of the representation theory. By using our formulas, We explicitly compute the $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ colored Jones polynomials for 2-bridge knots and links.

The *tail* is a q -series which is a kind of limit of the colored Jones polynomial. For the $\mathfrak{sl}_2$ colored Jones polynomial, the existence of $\mathfrak{sl}_2$ tails for some class of links was known and gave some explicit formulas of $\mathfrak{sl}_2$ tails. However, an existence theorem and explicit formulas for $\mathfrak{sl}_3$ tails were unknown. We give two explicit formulas of the $\mathfrak{sl}_3$ tail for the $(2, 2m)$ -torus link. One of these formulas is obtained by using the m -full twists formula. The other is obtained by calculation using $\mathfrak{sl}_3$ colored trivalent graphs. The method of graphical calculus using colored trivalent graphs are well-known for A_1 webs but not for A_2 webs. To be precise, the value of the Θ -graph and the quantum 6- j symbols for $\mathfrak{sl}_3$ are unknown. These values are necessary for calculations of colored Jones polynomials using colored trivalent graph. We evaluated the Θ -graph and the quantum 6- j symbols for A_2 webs in a special case and computed the $\mathfrak{sl}_3$ colored Jones polynomial for the $(2, 2m)$ -torus link. We derived two formulas for the $\mathfrak{sl}_3$ tail for the $(2, 2m)$ -torus link from the explicit presentations of the $\mathfrak{sl}_3$ colored Jones polynomial. Of course, these two $\mathfrak{sl}_3$ tails are identical. Thus, I obtain a q -series identity. This q -series identity is a knot theoretical generalization of the Andrews-Gordon identities for the Ramanujan false theta function. Recently, it was confirmed that the left-hand side of the identity is written by the false theta function for $\mathfrak{sl}_3$ in [BKM17].

This thesis is organized as follows. Part 1 deals with Goldman-Turaev Lie bialgebras for surfaces. In Chapter 2, we review definitions of the Goldman bracket and the Turaev cobracket of a surface. In Chapter 3, we define a Goldman-Turaev Lie bialgebra of oriented loops on an oriented two orbifold. In Chapter 4, we define a Poisson bracket and a Poisson cobracket of curves on a bordered surface. A bordered surface is the surface with marked points in its boundary and a curve on the bordered surface is a loop or an arc with endpoints in the marked points. We also construct a quantization of the Poisson algebra of unoriented curves on bordered surface by using the skein algebra defined by Muller [Mul12]. In Chapter 5, we describe a relationship between a finite Galois covering space of surfaces and the Goldman Lie algebras. Part 2 deals with web spaces and colored Jones polynomials. In Chapter 6, we derive some formulas for A_1 and A_2 web spaces from graphical calculus using skein relations. As an application, the $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ colored Jones polynomials for 2-bridge knots and links are given. In Chapter 7 we give two explicit formulas of the tail of the $\mathfrak{sl}_3$ colored Jones polynomials for the $(2, 2m)$ -torus link. These formulas derive a generalization of the Andrews-Gordon identities for the Ramanujan false theta function.

Part 1

On Goldman-Turaev Lie bialgebras of 2-orbifolds and bordered surfaces

CHAPTER 2

Goldman-Turaev Lie bialgebras of surfaces

In this chapter, we review the definition of the original Goldman-Turaev Lie bialgebra of a surface according to Goldman [Gol86] and Turaev [Tur89, Tur91]. See Section 2 of Chapter 5 about the definition of the Lie bialgebra.

Let S be a connected compact oriented smooth surface and $\mathbb{Z}$ the ring of integers.

1. Goldman bracket of surface

Let us introduce the Goldman Lie algebra of a surface $\mathbb{Z}\hat{\pi}(S)$ and $\mathbb{Z}\bar{\pi}(S)$. We use the following notations.

- $\hat{\pi}(S)$ (resp. $\bar{\pi}(S)$) denotes the set of all free homotopy classes of oriented (resp. unoriented) loops on S .
- $\mathbb{Z}\hat{\pi}(S)$ (resp. $\mathbb{Z}\bar{\pi}(S)$) is the free $\mathbb{Z}$ -module spanned by $\hat{\pi}(S)$ (resp. $\bar{\pi}(S)$).
- $|a|$ denotes the free homotopy classes of a where a is an oriented or unoriented loop.
- $\varepsilon(p; a, b)$ denotes the local intersection number at p of a and b for any transverse double intersection point p of oriented curves a and b , see Figure 2.1.

Let a and b be immersed oriented loops on S such that $a \cup b$ is *generic*, that is, all intersection points of $a \cup b$ are finite and transverse double points. Then we define the Goldman bracket of these oriented loops by

$$[a, b] = \sum_{p \in a \cap b} \varepsilon(p; a, b) |a_p b_p| \in \mathbb{Z}\hat{\pi}(S),$$

where a_p and b_p are closed paths based at p obtained from a and b respectively and $a_p b_p$ is the product of these paths.

THEOREM 1.1 (Goldman [Gol86]). The Goldman bracket $[\cdot, \cdot]$ is well-defined on $\hat{\pi}(S)$ and the linear extension of this bracket defines the Lie algebra structure on $\mathbb{Z}\hat{\pi}(S)$.

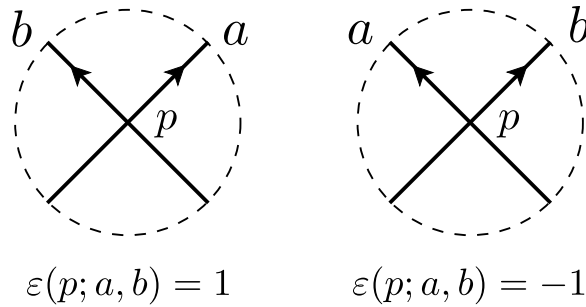
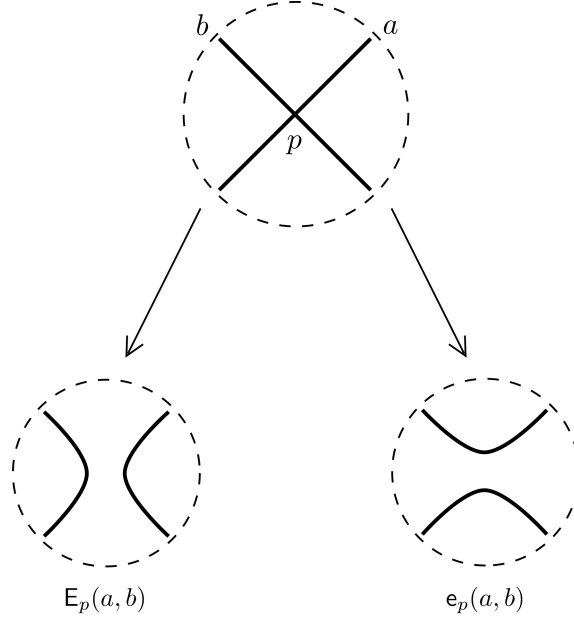


FIGURE 2.1. the local intersection number

FIGURE 2.2. unoriented curves $E_p(a, b)$ and $e_p(a, b)$

Next, we define the Goldman bracket of unoriented loops on S . Let a and b be unoriented loops on S such that $a \cup b$ is generic. Then we define two unoriented loops $E_p(a, b)$ and $e_p(a, b)$ for any intersection point p of a and b by Figure 3.5. Then the Goldman bracket of unoriented curves on S is defined by

$$[|a|, |b|] = \sum_{p \in a \cap b} |E_p(a, b)| - |e_p(a, b)|.$$

THEOREM 1.2 (Goldman [Gol86]). The above bracket is well-defined on $\bar{\pi}(S)$ and the $\mathbb{Z}$ -linear extension of this bracket defines the Lie algebra structure on $(\mathbf{KG})\bar{\pi}(S)$.

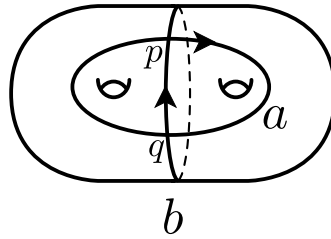
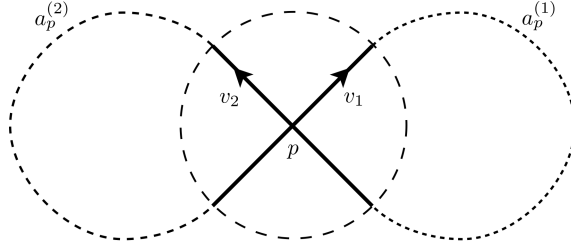


FIGURE 2.3. two loops on the closed surface of genus 2

EXAMPLE 1.3. We show examples of the Goldman brackets on $\mathbb{Z}\hat{\pi}(S)$ and $\mathbb{Z}\bar{\pi}(S)$ of two loops illustrated in Figure 2.3.

The Goldman bracket of oriented loops a and b is

FIGURE 2.4. subloops a_p^1 and a_p^2

$$[|a|, |b|] = \begin{array}{c} \text{Diagram 1} \\ |a_p b_p| \end{array} - \begin{array}{c} \text{Diagram 2} \\ |a_q b_q| \end{array}.$$

The Goldman bracket of unoriented loops obtained by forgetting orientations of a and b is

$$[|a|, |b|] = \begin{array}{c} \text{Diagram 1} \\ |E_p(a, b)| \end{array} - \begin{array}{c} \text{Diagram 2} \\ |e_p(a, b)| \end{array} + \begin{array}{c} \text{Diagram 3} \\ |E_q(a, b)| \end{array} - \begin{array}{c} \text{Diagram 4} \\ |e_q(a, b)| \end{array}.$$

2. Turaev cobracket of surface

The Goldman Lie algebra of a surface S has a direct sum decomposition of the Lie algebra such that

$$\mathbb{Z}\hat{\pi}(S) = \mathbb{Z}1 \oplus \mathbb{Z}\hat{\pi}'(S)$$

where 1 is the element in $\hat{\pi}(S)$ represented by the trivial loop and $\hat{\pi}'(S)$ is the other elements (see Goldman [Gol86], Kawazumi-Kuno [KK14]). Turaev [Tur89, Tur91] defined a cobracket on $\mathbb{Z}\hat{\pi}'(S) = \mathbb{Z}\hat{\pi}(S)/\mathbb{Z}1$. Let a be a homotopically non-trivial generic immersed loop on S . Denote the set of self intersection points of a by $\mathfrak{h} a$. For any p in $\mathfrak{h} a$, we assume that two distinct points q_1 and q_2 in the domain of $a: S^1 \rightarrow S$ satisfies $a(q_1) = a(q_2) = p$. In addition, the pair of tangent vectors $(da(v_1), da(v_2))$ coincides the orientation of S where v_i is the tangent vector at q_i . Then we denote $a|_{[q_1, q_2]}$ and $a|_{[q_2, q_1]}$ by a_p^1 and a_p^2 respectively, see Figure 2.4.

The Turaev cobracket $\delta: \mathbb{Z}\hat{\pi}'(S) \rightarrow \mathbb{Z}\hat{\pi}'(S) \otimes \mathbb{Z}\hat{\pi}'(S)$ is defined by the $\mathbb{Z}$ -linear extension of the following map:

$$(2.1) \quad \delta(\|a\|) = \sum_{p \in \mathfrak{h} a} (\|a_p^1\| \otimes \|a_p^2\| - \|a_p^2\| \otimes \|a_p^1\|)$$

where the map $\| \cdot \|$ is the composition of $| \cdot |$ and the projection $\mathbb{Z}\hat{\pi}(S) \rightarrow \mathbb{Z}\hat{\pi}'(S)$.

THEOREM 2.1 (Turaev [Tur89, Tur91]). The Turaev cobracket δ is well-defined and defines the co-Lie coalgebra structure on $\mathbb{Z}\hat{\pi}'(S)$.

See Turaev [Tur89, Tur91] for details about coalgebra structure on $\mathbb{Z}\hat{\pi}'(S)$.

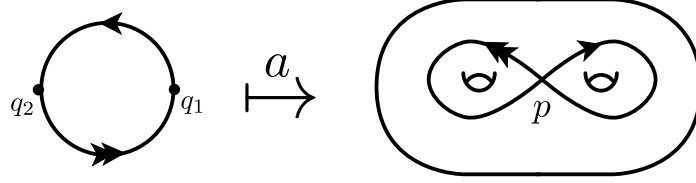


FIGURE 2.5. a loop on the closed surface of genus 2

EXAMPLE 2.2. The Turaev cobracket of the loop a illustrated in Figure 2.5 is

$$\begin{aligned} \delta(\|a\|) = & \begin{array}{c} \text{Diagram 1} \otimes \text{Diagram 2} \\ \parallel a_p^1 \parallel \quad \parallel a_p^2 \parallel \end{array} \\ & - \begin{array}{c} \text{Diagram 3} \otimes \text{Diagram 4} \\ \parallel a_p^2 \parallel \quad \parallel a_p^1 \parallel \end{array} . \end{aligned}$$

The Goldman Lie bracket naturally induces a bracket on $\mathbb{Z}\hat{\pi}'(S)$ by the projection $\mathbb{Z}\hat{\pi}(S) \rightarrow \mathbb{Z}\hat{\pi}'(S)$.

THEOREM 2.3 (Turaev [Tur89, Tur91]). The Goldman bracket and the Turaev cobracket on $\mathbb{Z}\hat{\pi}'(S)$ satisfy the conditions of a Lie bialgebra.

We call this Lie bialgebra the *Goldman-Turaev Lie bialgebra* of S . We remark that the Lie bialgebra is involutive (see Chas [Cha04] for details).

Goldman-Turaev Lie bialgebras of two orbifolds

1. Introduction

The Goldman Lie algebra has many aspects and relationships between other mathematical objects. For example, The unoriented Goldman Lie algebra of a surface is related to the Poisson geometry of $SL_2(\mathbb{C})$ -characters of the fundamental group of the surface, the Weil-Peterson geometry on the Teichmüller space of the surface. (See Goldman [Gol86], Turaev [Tur89, Tur91] and Bullock, Frohman and Kania-Bartoszyńska [BFKB99]). Moreover, The Kauffman bracket skein module of the thickened surface [Kau88] is a quantization of the Poisson algebra constructed from the Lie algebra. (See Turaev [Tur89, Tur91], Hoste and Przytycki [HP90]). The definition of the Goldman Lie algebra is extended to various types of surfaces. For example, Roger and Yang [RY14] defined the unoriented Goldman Lie algebra of punctured surfaces and the author [Yua15] the oriented and unoriented Goldman Lie algebra of bordered surfaces. The above theory related to the Goldman Lie algebra also develops for each types of surfaces. For example, These Goldman Lie algebras also have skein quantizations, see Roger and Yang [RY14], Muller [Mul12] and the author [Yua15] and the decorated Teichmüller theory of punctured and bordered surface developed by Penner [Pen87, Pen04] etc.

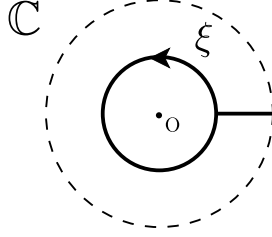
This chapter addresses a definition of the Goldman bracket and the Turaev cobracket of 2-orbifolds. First, we extend the definition of the Goldman Lie algebra to an oriented 2-dimensional orbifold. Let S be an oriented 2-orbifold with singular set ΣS . Then main results of this chapter is the following.

THEOREM. A quotient module $\mathbb{Z}\hat{\pi}(S, \Sigma S)$ of $\mathbb{Z}\hat{\pi}(\Sigma)$ has a Lie algebra structure induced from the oriented Goldman bracket of surface $S - \Sigma S$.

Next, We also define the Turaev co-Lie coalgebra of the 2-orbifold $(S, \Sigma S)$. The Turaev cobracket of a surface was defined on the quotient vector space of the Goldman Lie algebra of the surface by the trivial loop. In the case of a 2-orbifold, we use a quotient $\mathbb{Z}\hat{\pi}'(S, \Sigma S)$ of the Goldman Lie algebra of the 2-orbifold by the elements of $\mathbb{Z}\hat{\pi}(S, \Sigma S)$ represented by elliptic elements.

THEOREM. The quotient Lie algebra $\mathbb{Z}\hat{\pi}'(S, \Sigma S)$ has a co-Lie coalgebra structure induced from the Turaev cobracket of surface $S - \Sigma S$.

This chapter is organized as follows. In Section 2, we define the Goldman Lie algebra of a 2-orbifold by the quotient of the Goldman Lie algebra of the surface obtained from removing all singular points. In Section 3 We introduce the Turaev cobracket of a 2-orbifold. In section 4 we define the unoriented Goldman Lie algebra and observe the skein relations of it.

FIGURE 3.1. ξ

2. the Goldman bracket for an orbifold

A orbifold was defined by Satake [Sat56] (under the name of “V-manifold”) and Thurston [Thu79]. For details about orbifolds, see Bonahon-Siebenmann [BS85] and Matsumoto-Montesinos [MM91] etc.

In this chapter, we only consider a compact oriented 2-dimensional orbifold S with 0-dimensional isolated singularity. We denote the singular set of S by ΣS and a connected oriented smooth surface with punctures $S - \Sigma S$ by S_* . A singular point p in ΣS has a folding chart $(\mathbb{D}, C_p, \varphi_p, U_p)$. The folding map $\varphi_p : \mathbb{D} \rightarrow U_p$ is a continuous map with $\varphi_p(0) = p$ where $\mathbb{D}$ is the open unit disk in $\mathbb{C}$ and U_p an open neighborhood of p . The cyclic group C_p of order m_p acts on $\mathbb{D}$ by $2\pi/m_p$ -rotation. We denote the orbifold by $(S, \Sigma S)$ for the sake of simplicity. If all isotropy groups are trivial, that is $\Sigma S = \emptyset$, then S is an compact connected oriented smooth surface. Let p_0 be a base point in S_* . Fix a reference path u_p from p_0 to $\varphi_p(1)$ in $S - \bigcup_{p \in \Sigma S} U_p$ for each p in P . Let ξ be a closed path in $\mathbb{D}$ defined by

$$\xi(t) = \begin{cases} 1 - 2t & (0 \leq t \leq 1/4), \\ (1/2) \exp(4\pi i(t - 1/4)) & (1/4 \leq t \leq 3/4), \\ 2t - 1 & (3/4 \leq t \leq 1), \end{cases}$$

see figure 3.1. We define a closed path $\xi_p = u_p(\varphi_p \circ \xi)\bar{u}_p$ by connecting the above paths where $\bar{u}_p$ is the inverse path of u_p . This closed path represents m_p -th power of a “meridian” of a singular point p . $\langle \xi \rangle_S$ denotes a subgroup of $\pi_1(S_*, p_0)$ which normally generated by $\{\xi_p \mid p \in \Sigma S\}$. We call the subgroup $\langle \xi \rangle_S$ the characteristic subgroup of $\pi_1(S_*, p_0)$. The orbifold fundamental group of $(S, \Sigma S)$ can be defined by $\pi_1(S_*, p_0)/\langle \xi \rangle_S$ and we denote it by $\pi_1^{\text{orb}}(S, p_0)$.

We define the Goldman Lie algebra of the above orbifold $(S, \Sigma S)$ as a quotient Lie algebra of the Goldman Lie algebra $\mathbb{Z}\hat{\pi}(S_*)$. $I(S, \Sigma S)$ denotes the submodule of $\mathbb{Z}\hat{\pi}(S_*)$ generated by the set

$$\{|\ell| - |\ell\xi_p| \mid \ell \in \pi_1(S_*, p_0), p \in \Sigma S\}$$

as a module. We remark that the definition of $I(S, \Sigma S)$ is independent of choice of reference paths. Because, if we take a new reference path v_p , then

$$\begin{aligned} |\ell| - |\ell v_p(\varphi_p \circ \xi)\bar{v}_p| &= |\ell| - |\ell(v_p\bar{u}_p)u_p(\varphi_p \circ \xi)\bar{u}_p(u_p\bar{v}_p)| \\ &= |(u_p\bar{v}_p)\ell(v_p\bar{u}_p)| - |(u_p\bar{v}_p)\ell(v_p\bar{u}_p)u_p(\varphi_p \circ \xi)\bar{u}_p| \\ &= |\ell'| - |\ell'u_p(\varphi_p \circ \xi)\bar{u}_p| \end{aligned}$$

where ℓ' is $(u_p\bar{v}_p)\ell(v_p\bar{u}_p)$ in $\pi_1(S_*, p_0)$.

LEMMA 2.1. $I(S, \Sigma S)$ is an ideal of $\mathbf{K}\hat{\pi}'(S_*)$.

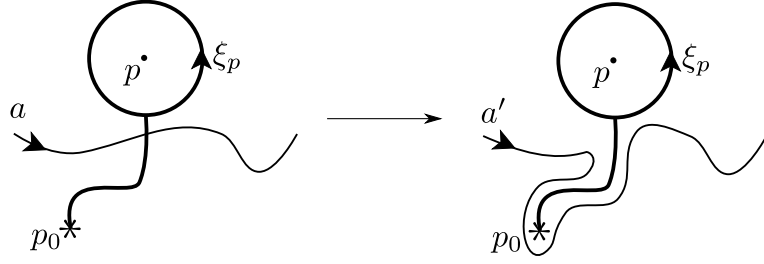


FIGURE 3.2. replacement of a loop

PROOF. We show that $[|a|, |\ell| - |\ell\xi_p|]$ is an element in I_P for any a in $\pi_1(S_*)$. We take a representative of $|a|$ which intersects $\ell\xi_p$ transversely. If the representative intersects at points on ξ_p , then we replace it by a homotopy to keep away from a neighborhood of $U_p \cup u_p$, see Figure 3.2. We denote the representative by a' . We remark that the representative has no intersection points with ξ_p . Therefore we obtain the following equality.

$$\begin{aligned}
[|a|, |\ell| - |\ell\xi_p|] &= [|a'|, |\ell| - |\ell\xi_p|] \\
&= \sum_{q \in a' \cap \ell} \varepsilon(q; a', \ell) |a'_q \ell_q| - \sum_{q \in a' \cap \ell\xi_p} \varepsilon(q; a', \ell\xi_p) |a'_q (\ell\xi_p)_q| \\
&= \sum_{q \in a' \cap \ell} \varepsilon(q; a', \ell) (|a'_q \ell_q| - |a'_q (\ell\xi_p)_q|) \\
&= \sum_{q \in a' \cap \ell} \varepsilon(q; a', \ell) (|\ell_{p_0q} a'_q \ell_q \bar{\ell}_{p_0q}| - |(\ell_{p_0q} a'_q \ell_q \bar{\ell}_{p_0q}) \xi_p|) \in I_P
\end{aligned}$$

where ℓ_{p_0q} is the path from p_0 to q obtained by restricting ℓ to $[0, \ell^{-1}(q)]$ and $\bar{\ell}_{p_0q}$ is its inverse path. $\square$

DEFINITION 2.2. The Goldman Lie algebra of $(S, \Sigma S)$ is defined by $\mathbb{Z}\hat{\pi}(S_*)/I(S, \Sigma S)$. We denote it by $\mathbb{Z}\hat{\pi}(S, \Sigma S)$.

REMARK 2.3. Let S be a 2-dimensional manifold with a set of specified points $P \subset S$. We define ζ_p as a meridian of p in P and the characteristic subgroup $\langle \zeta \rangle_S$ of $\pi_1(S - P)$ by considering $p \in P$ as a “order 1 singular point”. Then $\mathbb{Z}\hat{\pi}(S, P) = \mathbb{Z}\hat{\pi}(S - P)/\langle \zeta \rangle_S$ is isomorphic to the Goldman Lie algebra of S .

Let G be a group and $\hat{G}$ the set of conjugacy classes of G . $\hat{g}$ denote the conjugacy class represented by g in G . $\mathbb{Z}\hat{G}$ is the free $\mathbb{Z}$ -module spanned by $\hat{G}$. The Goldman Lie algebra of a surface S can be considered as $\widehat{\mathbb{Z}\hat{\pi}_1(S, p_0)}$ by a canonical bijection between $\widehat{\pi_1(S, p_0)}$ and $\hat{\pi}(S)$. The canonical bijection is given by regarding $|a|$ as $\hat{a}$ for any a in $\pi_1(S)$.

PROPOSITION 2.4. There is an isomorphism between $\mathbb{Z}\hat{\pi}(S, \Sigma S)$ and $\widehat{\mathbb{Z}\pi_1^{\text{orb}}(S, p_0)}$.

PROOF. We denote $\pi_1^{\text{orb}}(S, p_0)$ by Γ . The quotient map $\Phi_{\#}: \pi_1(S_*) \rightarrow \Gamma$ induces a $\mathbb{Z}$ -linear map $\Phi_{\#}: \mathbb{Z}\hat{\pi}(S_*) \rightarrow \mathbb{Z}\hat{\Gamma}$. The kernel of $\Phi_{\#}$ is generated by a subset $\{|a| - |b| \mid |\Phi_{\#}(a)| = |\Phi_{\#}(b)| \text{ and } a, b \in \pi_1(S_*)\}$ of $\mathbb{K}\hat{\pi}(S_*)$. The condition $|\Phi_{\#}(a)| = |\Phi_{\#}(b)|$ means that $\Phi_{\#}(a)$ is conjugate to $\Phi_{\#}(b)$ in Γ . Therefore $a^{-1}gbg^{-1}$ is contained in $\langle \xi \rangle_S$

for some g in $\pi_1(S_*)$. We can denote $a^{-1}gbg^{-1}$ by $\prod_{i=1}^n h_i \xi_{p_i} h_i^{-1}$ where h_i is in $\pi_1(S_*)$ and p_i in ΣS for each i . Then the generator

$$\begin{aligned} |a| - |b| &= |a| - \left| a \prod_{i=1}^n h_i \xi_{p_i} h_i^{-1} \right| \\ &= |a| - |a h_1 \xi_{p_1} h_1^{-1}| + \sum_{i=1}^{n-1} \left(\left| a \prod_{j=1}^i h_j \xi_{p_j} h_j^{-1} \right| - \left| a \prod_{j=1}^{i+1} h_j \xi_{p_j} h_j^{-1} \right| \right) \\ &= \sum_{i=1}^n |a_i| - |a_i \xi_{p_i}| \end{aligned}$$

where $a_1 = h_1^{-1} a h_1$ and a_i is defined inductively by $h_i^{-1} a_{i-1} h_i$. Consequently, The ideal $I(S, \Sigma S)$ includes the kernel of Φ_* . The reverse inclusion is clear. Therefore Φ_* induce an isomorphism $\Phi: \mathbb{Z}\hat{\pi}(S, \Sigma S) \rightarrow \mathbb{Z}\hat{\Gamma}$. $\square$

Chas and Gadgil [CG12] also gave a definition of the Goldman bracket for an hyperbolic orbifolds by use of hyperbolic geometry. They defined a Lie bracket on $\mathbb{Z}\hat{\Gamma}$ where Γ was a discrete subgroup of the orientation preserving isometric group of the upper half plane $\mathbb{H}$ with the hyperbolic metric. The Jacobi identity of the bracket was proved by careful calculations using a relationship between two intersecting axes of hyperbolic elements described in the book of Beardon [Bea83].

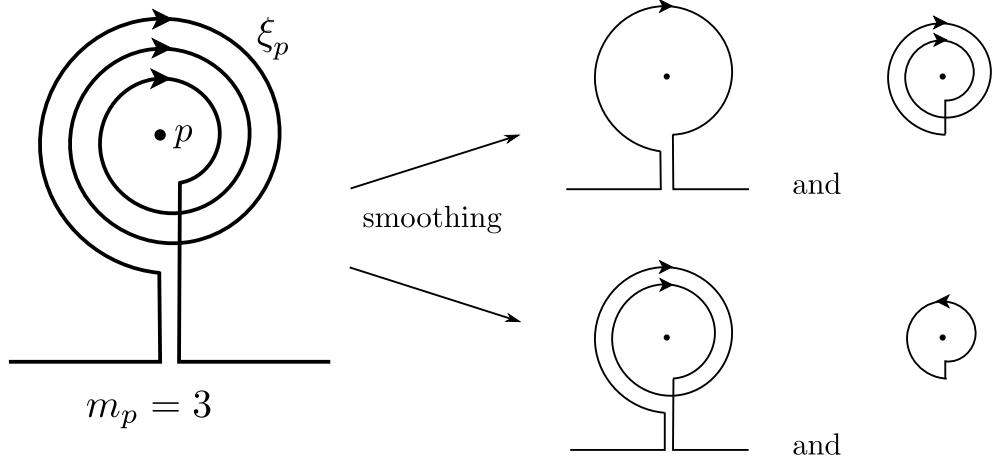
REMARK 2.5. We denote the projection by $\varpi: \mathbb{H} \rightarrow S_\Gamma$ and $\mathbb{H} - \varpi^{-1}(\Sigma S_\Gamma)$ by $\mathbb{H}_0$ where S_Γ is an orbifold $\mathbb{H}/\Gamma$. We take a base point $\tilde{p}_0$ in $\mathbb{H}_0$ and $p_0 = \varpi(\tilde{p}_0)$. For any isometry x in Γ , we take a path $\gamma_x^{p_0}$ from x to $x(p_0)$ and the projection $\varpi(\gamma_x^{p_0})$ is a loop on S_Γ . We can obtain an isomorphism between Γ and $\pi_1^{\text{orb}}(S, p_0)$ by sending x to the element represented by $\varpi(\gamma_x^{p_0})$. The reason for the independence of choice of the path $\gamma_x^{p_0}$ is that the characteristic subgroup of $\pi_1((S_\Gamma)_*, p_0)$ is equal to $\varpi_\#(\pi_1(\mathbb{H}_0, \tilde{p}_0))$. (Refer to [Kat87], [MM91], [Thu79] etc.) By the above fact and Proposition 2.4 give a linear isomorphism map $\Phi: \mathbb{Z}\hat{\Gamma} \rightarrow \mathbf{K}\hat{\pi}(S_\Gamma, \Sigma S_\Gamma)$. But we couldn't prove that Φ induces an isomorphism between Lie algebras $\mathbb{C}\hat{\Gamma}$ and $\mathbf{K}\hat{\pi}(S_\Gamma, \Sigma S_\Gamma)$. The reason is that Chas and Gadgil [CG12] consider not only loops on $S_\Gamma - \Sigma S_\Gamma$ but also loops intersecting singular points. We couldn't make sure how describe the bracket of two loops intersecting singular points in our Lie algebra.

3. the Turaev cobracket for orbifolds

We will define a cobracket on the Goldman Lie algebra of an orbifold. We use the same notation as in section 2. Let $(S, \Sigma S)$ be an orbifold. We define a vector subspace $J(S, \Sigma S)$ of $\mathbb{Z}\hat{\pi}(S, \Sigma S)$ generated by elliptic elements in $\pi_1^{\text{orb}}(S)$. More precisely, $J(S, \Sigma S)$ is defined as the following. The meridian ζ_p of a singular point p is a loop running along u_p from p_0 , once going round p and returning along $\bar{u}_p$. We consider ζ_p as an element in $\pi_1^{\text{orb}}(S, p_0)$ and $\langle \zeta \rangle_S$ is a subgroup of $\pi_1^{\text{orb}}(S, p_0)$ which normally generated by $\{ \zeta_p \mid p \in \Sigma S \}$. We define a submodule

$$J(S, \Sigma S) = \widehat{\mathbb{Z}\langle \zeta \rangle_S}$$

of $\mathbb{Z}\hat{\pi}(S, \Sigma S)$. The Turaev cobracket of an orbifold $(S, \Sigma S)$ is defined on the quotient $\mathbb{Z}\hat{\pi}(S, \Sigma S)'$ of $\mathbb{Z}\hat{\pi}(S, \Sigma S)$ by $J(S, \Sigma S)$. We define $\delta: \mathbb{Z}\hat{\pi}(S_*) \rightarrow \mathbb{Z}\hat{\pi}(S, \Sigma S)' \otimes \mathbb{Z}\hat{\pi}(S, \Sigma S)'$ by the same way as the formula (2.1). The well-definedness of δ is guaranteed because $J(S, \Sigma S)$ contains $\mathbb{Z}1$. Next, we observe the difference between $\delta(|\ell|)$ and $|\ell \xi_p|$ by the

FIGURE 3.3. Decomposed curves on the right are in $J(S, \Sigma S)$

same way as the proof of Lemma 2.1. δ produce the sum of alternative differences of loops obtained by smoothing at self-intersection points. We decompose $|\ell|$ into two loops $|a_q^1| \cup |a_q^2|$ by smoothing at a self-intersection point q of ℓ . Then the decomposition of $|\ell\xi_p|$ at q is denoted by $|a_q^1\xi_p| \cup |a_q^2|$ or $|a_q^1| \cup |a_q^2\xi_p|$. Moreover, one of the two loops obtained by smoothing at the self-intersection points of ξ is an element in $\widehat{\langle \zeta \rangle}_S$. (See an example in Figure 3.3) We can confirm that the decomposed loops of an element in $\langle \zeta \rangle_S$ are also in $\widehat{\langle \zeta \rangle}_S$ by the same way. Therefore, we can define a cobracket $\delta: \mathbb{Z}\hat{\pi}(S, \Sigma S)' \rightarrow \mathbb{Z}\hat{\pi}(S, \Sigma S)' \otimes \mathbb{Z}\hat{\pi}(S, \Sigma S)'$.

THEOREM 3.1. The pair $(\mathbb{Z}\hat{\pi}(S, \Sigma S)', \delta)$ is a co-Lie algebra.

COROLLARY 3.2. The triple $(\mathbb{Z}\hat{\pi}(S, \Sigma S)', [\cdot, \cdot]', \delta)$ is a involutive Lie bialgebra.

PROOF. We can easily see that $J(S, \Sigma S)$ is contained in the center of the Lie algebra $\mathbb{Z}\hat{\pi}(S, \Sigma S)$. The Goldman bracket on $\mathbb{Z}\hat{\pi}(S, \Sigma)$ induce a Lie algebra structure on $\mathbb{Z}\hat{\pi}(S, \Sigma S)'$. The involutive Lie bialgebra $(\mathbb{Z}\hat{\pi}(S_*)', [\cdot, \cdot]', \delta)$ induce the conditions of the Lie bialgebra of $(\mathbb{Z}\hat{\pi}(S, \Sigma S)', [\cdot, \cdot]', \delta)$. $\square$

4. unoriented Goldman Lie algebra of orbifolds

Goldman [Gol86] also introduced the Goldman Lie algebra of unoriented curves on a surface (implicitly considered by Wolpert [Wol83]). We observe a definition of it for orbifolds. We mainly consider quotients of the symmetric algebra of the Goldman Lie algebra of unoriented curves on surfaces. The symmetric algebra of a Lie algebra naturally has a Poisson structure, that is, the Lie bracket $\{\cdot, \cdot\}$ and the multiplication of symmetric algebra satisfy the Leibniz rule;

$$\{a, bc\} = \{a, b\}c + b\{a, c\}.$$

We directly give the definition of the Poisson algebra composed of the Goldman Lie algebra of unoriented curves. Let S be a connected oriented smooth surface.

DEFINITION 4.1.

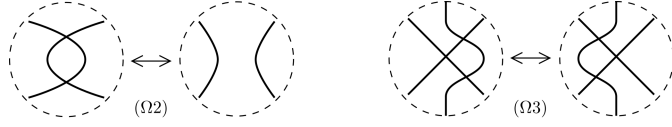
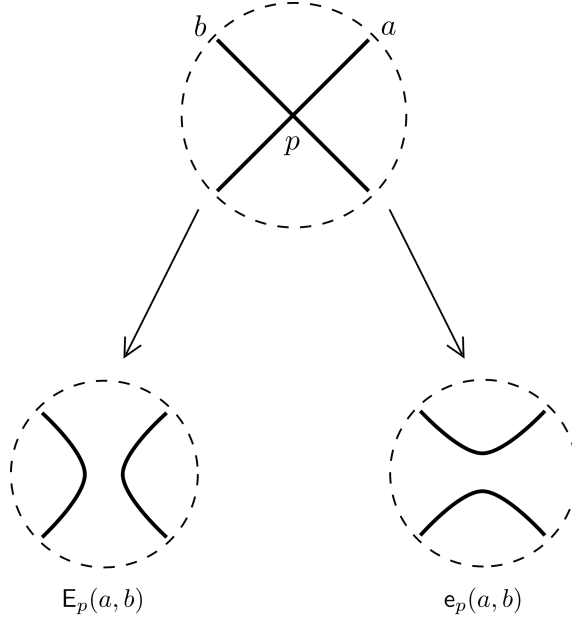


FIGURE 3.4. regular homotopy

FIGURE 3.5. unoriented curves $E_p(a, b)$ and $e_p(a, b)$

- A curve on S is an generic immersion of a finite disjoint union of circles. An *unoriented curves* on S is obtained by identifying arbitrary orientations of a curves on S .
- Unoriented curves D and D' on S are *regularly homotopic* if D is obtained from D' by a sequence of ambient isotopies of S and local moves $(\Omega 2)$ and $(\Omega 3)$ illustrated in Figure 3.4.

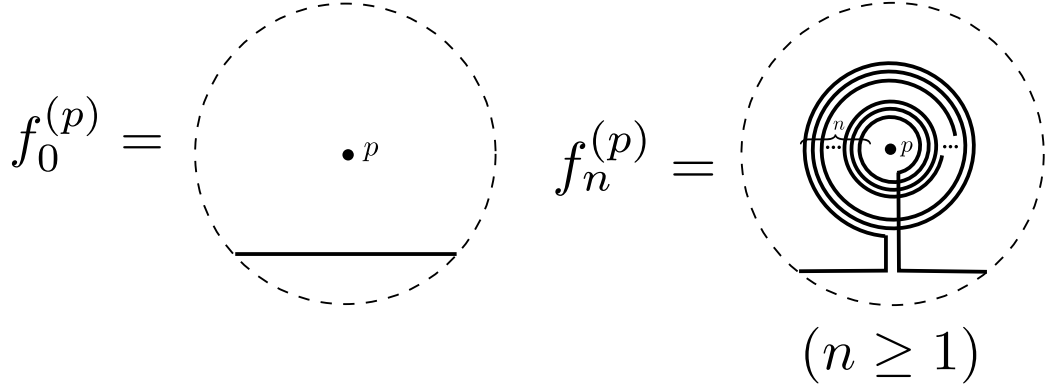
We denote the $\mathbb{C}$ -vector space spanned by the set of the regular homotopy classes of unoriented curves on S by $\mathcal{C}(S)$. We define the Goldman bracket $\{\cdot, \cdot\}: \mathcal{C}(S) \times \mathcal{C}(S) \rightarrow \mathcal{C}(S)$ by the following. Let a and b be unoriented curves of S such that $a \cup b$ is generic. Then we define two unoriented curves $E_p(a, b)$ and $e_p(a, b)$ for any intersection points p of a and b by Figure 3.5. Then the Goldman bracket of unoriented curves on S is

$$\{|a|, |b|\} = \sum_{p \in a \cap b} |E_p(a, b)| - |e_p(a, b)|$$

where $|D|$ is the regular homotopy class of an unoriented curve D on S .

THEOREM 4.2 (Goldman [Gol86]). The above bracket is well-defined Lie bracket. Thus $(\mathcal{C}(S), \{\cdot, \cdot\})$ is a Lie algebra.

REMARK 4.3.

FIGURE 3.6. unoriented curve $f_n^{(p)}$

- Precisely, Goldman defined the bracket for the vector space spanned by the homotopy classes of unoriented curves.
- $\mathcal{C}(S)$ has a well-defined multiplication defined by a union of two unoriented curves and it makes $(\mathcal{C}(S), \{\cdot, \cdot\})$ a Poisson algebra. We call it the unoriented Goldman Poisson algebra of S .

Let $(S, \Sigma S)$ be an orbifold defined in section 2 and $f_n^{(p)}$ an unoriented curve which have the same neighborhood illustrated in Figure 3.6 where p a singular point.

DEFINITION 4.4. We define the unoriented Goldman Poisson algebra $\mathcal{C}(S, \Sigma S)$ of the orbifold $(S, \Sigma S)$ as a quotient of the vector space $\mathcal{C}(S_*)$ by the subspace $\bar{I}(S, \Sigma S)$ generated by

$$\{ f_{m_p}^{(p)} - f_0^{(p)} \mid p \in \Sigma S \}$$

The diagrams picture a neighborhood of a singular point and these diagrams are identical in the exterior of the neighborhood.

LEMMA 4.5. The vector subspace $\bar{I}(S, \Sigma S)$ is a Poisson ideal.

PROOF. We can choose a representative of a regular homotopy class of an unoriented curve without intersecting the neighborhoods of singular points. Therefore $\bar{I}(S, \Sigma S)$ is an ideal of algebra. By the same way as Lemma 2.1, we can show that $\bar{I}(S, \Sigma S)$ is a Lie ideal. $\square$

REMARK 4.6. We can construct a Poisson homomorphism from $\mathcal{C}(S, \Sigma S)$ to the symmetric algebra of $\mathbb{C}\hat{\pi}(S, \Sigma S)$ as in [Gol86]. The homomorphism send the unoriented curve D to the summation of all orientated curves obtained from D divided by $2^{\#D-1}$ where $\#D$ is the number of components of D .

At the end of this section, we consider a quotient Poisson algebra $\text{Sk}(S)$ of $\mathcal{C}(S)$.

DEFINITION 4.7. $\text{Sk}(S)$ is the quotient by the following relations:

- (1)  =  +  (the skein relation);
- (2)  = -2 (the framing relation).

These relations are naturally derived from the Kauffman bracket skein module of $S \times [0, 1]$. In fact, The quantization of the Poisson algebra $\text{Sk}(S)$ is the Kauffman bracket skein algebra of $S \times [0, 1]$. Moreover, $\text{Sk}(S)$ and the Poisson algebra of the SL_2 -characters of the fundamental group of S is deeply related as mentioned in Section 1. Especially, Roger and Yang [RY14] define a Poisson algebra $\text{Sk}(S, P)$ of a surface S with a set of marked points (punctures) P in $\text{int } S$. The Poisson algebra $\text{Sk}(S, P)$ has regular homotopy classes of arcs joining punctures in P as elements of it. The Poisson algebra $\text{Sk}(\Sigma)$ has additional relations *the puncture skein relation* and *the puncture-framing relation*. Roger and Yang construct a homomorphism from $\text{Sk}(S, P)$ to the Poisson algebra of geodesic length functions on the decorated Teichmüller space.

On the other hands, Chekhov [Che09b, Che09a] studies the Teichmüller theory of orbifolds with singular points of order 2 by using the fat graph technique and defines a Poisson bracket of the algebra of geodesic length functions. Chekhov and Mazzocco [MC09] extend the theory to orbifolds with singular points of order 2 and 3. Finally, Chekhov and Shapiro [CS14] define the Teichmüller space of orbifolds with singular points of arbitrary order.

Let $(S, \Sigma S)$ be an orbifold defined in section 2. It seems to be natural to define the Poisson algebra $\text{Sk}(S, \Sigma S)$ as a quotient of $\mathcal{C}(S_*)$ by the skein relation, the framing relation and “*the singular-framing relation*”. For any singular point p and some δ_p in $\mathbb{C}$, the singular-framing relation is

$$\begin{array}{c} \text{---} \\ \bigcirc \cdot p \\ \text{---} \end{array} = \delta_p.$$

By using the relations, $f_n^{(p)}$ decompose into two loops as the following:

$$f_n^{(p)} = a_n f_0^{(p)} + b_n f_1^{(p)}.$$

Then we get recurrence relations (the Chebyshev polynomials):

$$\begin{aligned} a_n &= \delta_p a_{n-1} - a_{n-2}, & a_1 &= 0, & a_2 &= 1; \\ b_n &= \delta_p b_{n-1} - b_{n-2}, & b_1 &= 1, & b_2 &= \delta_p. \end{aligned}$$

Therefore,

$$f_n^{(p)} = \frac{x_p^{n-1} - x_p^{-n+1}}{x_p - x_p^{-1}} f_0^{(p)} + \frac{x_p^n - x_p^{-n}}{x_p - x_p^{-1}} f_1^{(p)}.$$

where $\delta_p = x_p + x_p^{-1}$. (For example, see example Remark 7 of Kauffman and Lins [KL94a])

By the definition of $\mathcal{C}(S, \Sigma S)$, the equation $f_{m_p}^{(p)} = f_0^{(p)}$ must be satisfied, that is,

$$\begin{aligned} \frac{x_p^{m_p-1} - x_p^{-m_p+1}}{x_p - x_p^{-1}} &= 1 \\ \frac{x_p^{m_p} - x_p^{-m_p}}{x_p - x_p^{-1}} &= 0. \end{aligned}$$

Therefore $x_p = \exp(\pi i / m_p)$ where m_p is the order of the singular point p and $\delta_p = \cos(\pi / m_p)$.

QUESTION 4.8. Construct a Poisson homomorphism from $\text{Sk}(S, \Sigma)$ to a Poisson algebra of geodesic length functions on $(S, \Sigma S)$.

Poisson algebras of curves on bordered surfaces and skein quantization

1. Introduction

In this Chapter, we will mainly discuss Goldman Lie algebras of an compact connected oriented smooth surface S with marked points M in its boundary, we call it a bordered surface $\Sigma = (S, M)$.

As we described in Chapter 1, the Goldman Lie algebra of a surface is related to two topics;

- a relation to the Poisson geometry on the representation space of the surface group and
- quantizations by using skein algebras of the thickened surface.

In the case of closed surfaces, Goldman [Gol86], Turaev [Tur89, Tur91], Hoste and Przytycki [HP90], Bullock, Frohman and Kania-Bartoszyńska [BFKB99] and many people studied the above topics.

For a punctured surface (S, P) , a connected oriented smooth surface S with punctures P in $\text{int } S$, Roger and Yang [RY14] studied a Poisson algebra $\mathcal{C}(S, P)$ of unoriented loops on (S, P) . They defined a Poisson algebra $\mathcal{C}(S, P)$ by generalizing the Goldman bracket to unoriented arcs with endpoints in P . They constructed a Poisson homomorphism $\mathcal{C}(S, P) \rightarrow C^\infty(\mathcal{T}^d(S))$ by using an explicit description of the Weil-Petersson form on the decorated Teichmüller space $\mathcal{T}^d(S)$ described by Mondello [Mon09]. A quantization of $\mathcal{C}(S, P)$ was also given in [RY14] by introducing the Kauffman bracket skein algebra of (S, P) .

Main results of this chapter is as follows. Let oriented curves on a bordered surface $\Sigma = (S, M)$ be oriented loops and arcs with endpoints in M .

THEOREM. We define a Poisson algebra $\text{PS}(\Sigma)^+$ and a co-Poisson coalgebra $\text{CS}(\Sigma)^+$ of the regular homotopy classes of oriented curves on a bordered surface $\Sigma = (S, M)$.

The Poisson bracket and the co-Poisson cobracket are generalizations of the Goldman bracket and the Turaev cobracket. In fact, the Poisson algebra and the co-Poisson coalgebra of the homotopy classes of oriented curves on Σ are induced by $\text{PS}(\Sigma)^+$ and $\text{CS}(\Sigma)^+$. The homotopy version of the Poisson bracket and the Poisson cobracket are coincide with the Goldman bracket and the Turaev cobracket if $M = \emptyset$. Furthermore, this Poisson bracket is also a generalization of the swapping algebra defined by Labourie [Lab12]. In order to state the second result, we give a definition of a quantization of a Poisson algebra. Let $\mathbb{C}$ be the field of complex numbers and $\mathbb{C}[h]$ the complex polynomial ring in one variable h . For a complex vector space V , we denote by $V[h]$ the set of all polynomials in one variable h with coefficient in V .

DEFINITION. A *quantization* of a Poisson $\mathbb{C}$ -algebra B with the Poisson bracket $\{\cdot, \cdot\}_B$ is an associative $\mathbb{C}[h]$ -algebra A such that

- (1) A is isomorphic to $B[h]$ as $\mathbb{C}[h]$ -modules (A is a free $\mathbb{C}[h]$ -module),
- (2) A/hA is isomorphic to B as a $\mathbb{C}$ -algebras and the isomorphism induces the Poisson algebra isomorphism.

If we denote the quotient map $p: A \rightarrow A/hA$, the above Poisson structure on A/hA is defined by

$$\{p(a), p(b)\}_{A/hA} = p\left(\frac{ab - ba}{h}\right)$$

for any a and b in A . A quantization of B over the ring of complex formal power series $\mathbb{C}[[h]]$ is similarly defined by making A a topological algebra and replacing the first condition with topological freeness of A as $\mathbb{C}[[h]]$ -module. (See Drinfeld [Dri87] for details about the quantization.)

We define a Poisson algebra $\overline{\text{PS}}(\Sigma)^+$ of unoriented curves on Σ and construct a quantization of a quotient Poisson algebra $\text{Sk}(\Sigma)$ of $\overline{\text{PS}}(\Sigma)^+$. This quantization is given by using the skein algebra $\text{Sk}_q(\Sigma)$ defined by Muller [Mul12]. We denote by $\mathcal{K}_h(\Sigma)$ the $\mathbb{C}[[h]]$ -algebra obtained by substituting $q = \exp(h/2)$ into $\mathbb{C}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ -algebra $\text{Sk}_q(\Sigma)$

THEOREM (5.9). The map $\tilde{p}: \mathcal{K}_h(\Sigma) \rightarrow \text{Sk}(\Sigma)$ gives a quantization of $\text{Sk}(\Sigma)$.

We remark that a relation between $\text{Sk}_q(\Sigma)$ and the quantum cluster algebra $\mathcal{A}_q(\Sigma)$ obtained from a triangulation of Σ was shown by Muller [Mul12]. (See Fomin and Zelevinsky [FZ02] and Fomin, Shapiro and Thurston [FST08] for details about cluster algebras and Berenstein and Zelevinsky [BZ05] about quantum cluster algebras.)

THEOREM (Theorem 9.8 of Muller [Mul12]). If a bordered surface Σ is triangulable and has at least two marked points in each boundary component, then

$$\mathcal{A}_q(\Sigma) = \text{Sk}_q^o(\Sigma).$$

The above $\text{Sk}_q^o(\Sigma)$ is the localized skein algebra defined by Muller [Mul12]. The specialization $q = 1$ of $\mathcal{A}_q(\Sigma)$ becomes a commutative cluster algebra and $\mathcal{A}_1(\Sigma) = \text{Sk}_1^o(\Sigma)$, see Corollary 11.5(1) of Muller [Mul12].

This chapter is organized as follows. Section 2 is preliminary definitions. Section 3 and 4 give the definitions of the Poisson algebra and the co-Poisson coalgebra of the regular homotopy classes of oriented curves on a bordered surface. In Section 5, we define the Poisson algebra of unoriented curves on a bordered surface and give its quantization.

2. Preliminaries

This section gives definitions of main targets and notations which we treat in this chapter.

We consider an oriented, compact, connected and smooth surface S with boundary. The boundary ∂S contains a finite set M of marked points. For each marked point in M , we fix an inward normal vector to ∂S . We call the pair $\Sigma = (S, M)$ a *bordered surface*. We allow a bordered surface which has boundary components with no marked points. We define curves on a bordered surface $\Sigma = (S, M)$.

DEFINITION 2.1.

- A *loop* on Σ is an oriented immersed loop on $\text{int } S = S \setminus \partial S$ whose self-intersection points consists only of transverse double points. The set of loops on Σ is denoted by $\text{Loops}(\Sigma)$.

- An *arc* on Σ is an oriented immersed arc on S with endpoints in M whose interior is disjoint from ∂S . Moreover, we require that self-intersection points of the arc in $\text{int } S$ are transverse double points and the arc is tangent to inward normal vector on M at the endpoints. The set of arcs on Σ is denoted by $\text{Arcs}(\Sigma)$.
- A *curve* on Σ is a loop or an arc on Σ . We denote the set of curves by $\text{Curves}(\Sigma) = \text{Loops}(\Sigma) \cup \text{Arcs}(\Sigma)$.
- A subset X of $\text{Curves}(\Sigma)$ is *generic* or *generic curves* if X has only finite transverse double points in $\text{int } S$.

REMARK 2.2. Dylan Thurston call generic subsets of $\text{Curves}(\Sigma)$ curve diagrams on Σ and elements of $\text{Curves}(\Sigma)$ connected curve diagrams in his paper [Thu14].

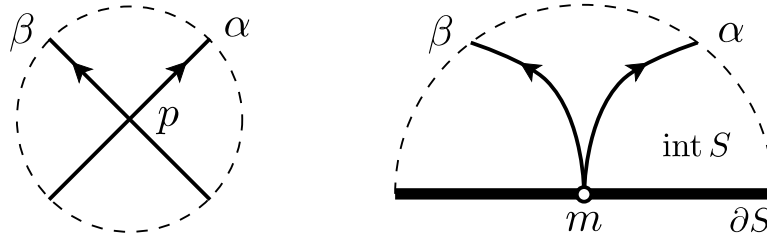


FIGURE 4.1. intersections of curves

We consider the following equivalence relation on $\text{Curves}(\Sigma)$. For generic subset $\{\alpha, \beta\}$ of $\text{Curves}(\Sigma)$, the curve α is *regularly homotopic* to β if there is a finite sequence of generic curves $\{f_1, f_2, \dots, f_n\} \subset \text{Curves}(\Sigma)$ such that f_{i+1} is obtained from f_i by ambient isotopy of S fixing a collar of ∂S or one of the moves $(\Omega 2-1)$, $(\Omega 2-2)$, $(\Omega 3)$ and $(\Omega b2-1)$ – $(\Omega b2-4)$ illustrated in Figure 4.2. We denote the set of regular homotopy classes of curves on Σ by $\hat{\pi}(\Sigma)^+$.

We define the local intersection number of two curves on Σ . Let $\{\alpha, \beta\} \subset \text{Curves}(\Sigma)$ be generic. We define the local intersection number of α and β at an intersection point p .

DEFINITION 2.3 (the local intersection number). If p is in $S \setminus \partial S$,

$$\varepsilon(p; \alpha, \beta) = \begin{cases} 1 & \text{if } \begin{array}{c} \beta \\ \nearrow \\ \alpha \end{array}, \\ -1 & \text{otherwise.} \end{cases}$$

If p is a marked point in M ,

$$\varepsilon(p; \alpha, \beta) = \begin{cases} \frac{1}{2} & \text{if } \begin{array}{c} \beta \\ \nearrow \\ \alpha \end{array}, \begin{array}{c} \alpha \\ \nearrow \\ \beta \end{array}, \begin{array}{c} \alpha \\ \searrow \\ \beta \end{array}, \begin{array}{c} \beta \\ \searrow \\ \alpha \end{array}, \\ -\frac{1}{2} & \text{otherwise.} \end{cases}$$

We next introduce notations which represent subarcs or based subloops of curves on Σ . We take a curve α in $\text{Curves}(\Sigma)$ and distinct points p and q on α .

- $\alpha_p^q: I = [0, 1] \rightarrow S$ is obtained along α such that $\alpha_p^q(0) = p$, $\alpha_p^q(1) = q$ and $p, q \notin \alpha_p^q(I \setminus \partial I)$.

If $\alpha \in \text{Loops}(\Sigma)$ and p is not a self-intersection point of α , then

- α_p is the based loop at p obtained from α .

If p is a self-intersection point of α , then we take two tangent vectors v_1 and v_2 of α at p such that (v_1, v_2) coincides with orientation of S . For $i = 1, 2$, We can obtain the based

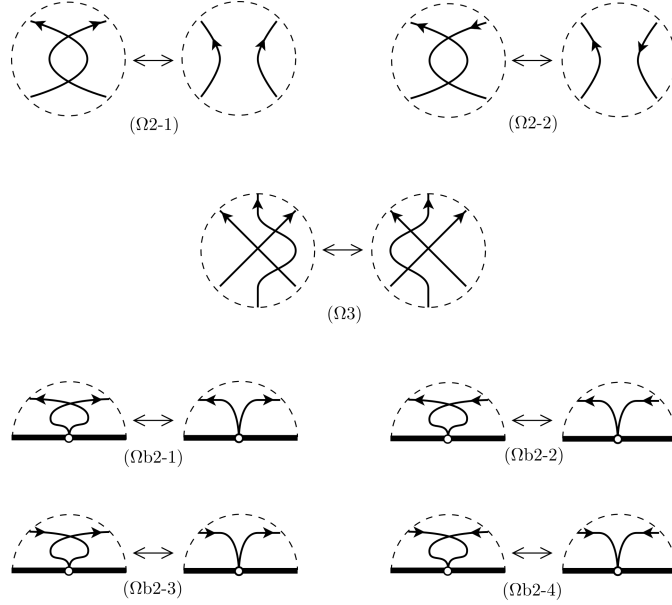


FIGURE 4.2. local moves of curves

loop $\alpha_p^{(i)}$ which goes from p in the direction of v_i along α and stops at the first meeting p . Let α be in $\text{Arcs}(\Sigma)$ and p a self-intersection point of α in $\text{int } S$, then we can define either $\alpha_p^{(1)}$ or $\alpha_p^{(2)}$. We call the self-intersection point p is type(I) (resp. type(II)) if we can define $\alpha_p^{(1)}$ (resp. $\alpha_p^{(2)}$). (See Figure 2.4) For x, y in M , we can decompose an arc $\alpha = \alpha_x^y$ on Σ to three parts α_x^p , $\alpha_p^{(1)}$ (resp. $\alpha_p^{(2)}$) and α_p^y if p is type(I) (resp. type(II)).

3. Poisson algebras of curves on bordered surfaces

We will define a Poisson bracket for two regular homotopy classes of curves on Σ in this section. Let us denote by $\mathbf{K}$ a commutative associative ring with unit containing the field of rational numbers, $\langle t \rangle$ the infinite cyclic group generated by t and $\mathbf{K}[t, t^{-1}] = \mathbf{K}\langle t \rangle$ the Laurent polynomial ring.

Let $\mathbf{K}\hat{\pi}(\Sigma)^+$ be the free $\mathbf{K}$ -module on $\hat{\pi}(\Sigma)^+$. The Laurent polynomial ring $\mathbf{K}[t, t^{-1}]$ acts on $\mathbf{K}\hat{\pi}(\Sigma)^+$ such that the action of t (resp. t^{-1}) on a curve on Σ is an insertion of a positively (resp. negatively) oriented monogon into interior of the curve. Thus $\mathbf{K}\hat{\pi}(\Sigma)^+$ is a $\mathbf{K}[t, t^{-1}]$ -module. Then we define the $\mathbf{K}[t, t^{-1}]$ -module $Z(\Sigma)^+$ as the quotient of $\mathbf{K}\hat{\pi}(\Sigma)^+$ by the $\mathbf{K}[t, t^{-1}]$ -submodule C_M^+ generated by monogons with vertices in M . The monogons are contractible arcs on Σ which have no self-intersection points. We denote by $|\alpha|$ the element of $Z(\Sigma)$ represented by α in $\text{Curves}(\Sigma)$. Let us denote by $S(\Sigma)^+$ the symmetric $\mathbf{K}[t, t^{-1}]$ -algebra $\text{Sym}(Z(\Sigma)^+)$.

REMARK 3.1.

- (1) Let $\hat{\pi}(\Sigma)$ be the set of homotopy classes of curves on Σ and C_M the $\mathbf{K}$ -submodule of $\mathbf{K}\hat{\pi}(\Sigma)$ generated by contractible arcs on Σ . Then $Z(\Sigma)^+/(t-1)Z(\Sigma)^+ = Z(\Sigma) = \mathbf{K}\hat{\pi}(\Sigma)/C_M$. We denote the symmetric $\mathbf{K}$ -algebra of $Z(\Sigma)$ by $S(\Sigma)$.
- (2) A curve diagram is an generic immersion of a 1-manifold $(X, \partial X) \rightarrow (S, M)$ which defined by Thurston [Thu14]. In our own words, the set of curve diagrams

is the set of all generic curves on Σ . He define the commutative associative monoid $\mathbf{C}(\Sigma)$ of regular homotopy classes of curve diagrams on Σ whose unit is the empty diagram. The multiplication of two curve diagrams is given by the union of two diagrams. A subset X of $\text{Curves}(\Sigma)$ defines an element of $\text{Sym}(\mathbf{K}\hat{\pi}(\Sigma)^+)$ by multiplication of regular homotopy classes of elements in X . The monoid ring $\mathbf{KC}(\Sigma)$ is isomorphic to $\text{Sym}(\mathbf{K}\hat{\pi}(\Sigma)^+)$ through the above map.

We next define a bracket on $S(\Sigma)^+$. We take curves a and b in $\text{Loops}(\Sigma)$ and $\gamma = \gamma_x^y$ and $\eta = \eta_z^w$ in $\text{Arcs}(\Sigma)$ such that $\{a, b, \gamma, \eta\}$ is generic and x, y, z and w are in M . We define a bracket $\{\cdot, \cdot\}': \text{Curves}(\Sigma) \times \text{Curves}(\Sigma) \rightarrow S(\Sigma)^+$ as the following.

$$(3.1) \quad \{a, b\}' = \sum_{p \in a \cap b} \varepsilon(p; a, b) |a_p b_p|,$$

$$(3.2) \quad \{a, \eta\}' = \sum_{p \in a \cap \eta} \varepsilon(p; a, \eta) |\eta_z^p a_p \eta_p^w|,$$

$$(3.3) \quad \{\gamma, b\}' = \sum_{p \in \gamma \cap b} \varepsilon(p; \gamma, b) |\gamma_x^p b_p \gamma_p^y|,$$

$$(3.4) \quad \begin{aligned} \{\gamma, \eta\}' &= \sum_{p \in \gamma \cap \eta} \varepsilon(p; \gamma, \eta) |\gamma_x^p \eta_p^w| |\eta_z^p \gamma_p^y| \\ &= \sum_{p \in \gamma \cap \eta \cap \text{int } S} \varepsilon(p; \gamma, \eta) |\gamma_x^p \eta_p^w| |\eta_z^p \gamma_p^y| \\ &\quad + \delta_{x,z} \varepsilon(x; \gamma, \eta) |\gamma| |\eta| + \delta_{y,w} \varepsilon(y; \gamma, \eta) |\eta| |\gamma|, \end{aligned}$$

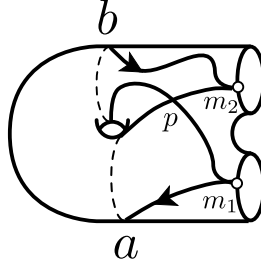
where δ is the Kronecker delta function. We know that (3.1)–(3.4) are invariant under local moves $(\Omega 2-1)$ and $(\Omega 3)$. (See Goldman [Gol86].) We can also confirm that (3.1)–(3.4) are invariant under $(\Omega 2-2)$ and (3.4) is invariant under $(\Omega b2-1)$ – $(\Omega b2-4)$ by comparing terms of the bracket coming from intersection points illustrated in diagrams of local moves. For example,

$$\begin{aligned} \text{the terms coming from } \{\alpha, \beta\}' \text{ of } \begin{array}{c} \beta \\ \text{---} \circlearrowleft \alpha \end{array} &= \begin{array}{c} \beta \\ \text{---} \circlearrowleft \alpha \end{array} - \begin{array}{c} \beta \\ \text{---} \circlearrowright \alpha \end{array} \\ &= t \begin{array}{c} \beta \\ \text{---} \circlearrowleft \alpha \end{array} - t \begin{array}{c} \beta \\ \text{---} \circlearrowright \alpha \end{array} \\ &= 0, \end{aligned}$$

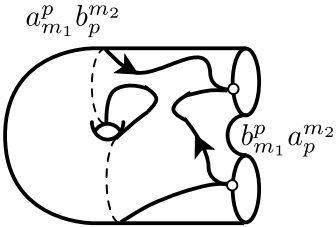
$$\begin{aligned} \text{the terms coming from } \{\alpha, \beta\}' \text{ of } \begin{array}{c} \beta \\ \text{---} \circlearrowleft \alpha \end{array} &= \begin{array}{c} \beta \\ \text{---} \circlearrowleft \alpha \end{array} - \frac{1}{2} \begin{array}{c} \beta \\ \text{---} \circlearrowleft \alpha \end{array} \\ &= \frac{1}{2} \begin{array}{c} \beta \\ \text{---} \circlearrowleft \alpha \end{array} \\ &= \text{the terms coming from } \{\alpha, \beta\}' \text{ of } \begin{array}{c} \beta \\ \text{---} \circlearrowleft \alpha \end{array}. \end{aligned}$$

When γ or η is a monogon with vertex in M , then the bracket vanishes. Therefore we can define the $\mathbf{K}[t, t^{-1}]$ -bilinear map $\nabla' = \{\cdot, \cdot\}': Z(\Sigma)^+ \otimes_t Z(\Sigma)^+ \rightarrow S(\Sigma)^+$. The symbol $\otimes_t$ denotes the tensor product over $\mathbf{K}[t, t^{-1}]$. Using the Leibniz rule, the bracket is extended to $\nabla = \{\cdot, \cdot\}: S(\Sigma)^+ \otimes_t S(\Sigma)^+ \rightarrow S(\Sigma)^+$.

EXAMPLE 3.2. We show an example of the Goldman bracket where S is the compact connected oriented surface of genus 1 with two boundary components. We take two

FIGURE 4.3. two curves a and b on a bordered surface

marked points $\{m_1, m_2\}$ and two oriented arcs a and b as illustrated in Figure 4.3. Then the Goldman bracket of a and b is

$$\begin{aligned}
 \{|a|, |b|\} &= |a_{m_1}^p b_p^{m_2}| |b_{m_1}^p a_p^{m_2}| \\
 &\quad - \frac{1}{2} |a| |b| - \frac{1}{2} |b| |a| \\
 &= |a_{m_1}^p b_p^{m_2}| |b_{m_1}^p a_p^{m_2}| \\
 &\quad - |a| |b|
 \end{aligned}$$


We can easily prove the following lemma.

LEMMA 3.3.

- (1) If ∇' is skew symmetric, then ∇ is skew symmetric.
- (2) If ∇' is skew symmetric and satisfies the Jacobi identity, then ∇ satisfies the Jacobi identity.

PROOF. Use the Leibniz rule and an induction on the degree of symmetric power. $\square$

PROPOSITION 3.4. ∇' is skew symmetric and satisfies the Jacobi identity.

PROOF. We can see ∇' is skew symmetric by the definition of the bracket and the local intersection number. We prove the Jacobi identity. We take curves a, b and c in $\text{Loops}(\Sigma)$ and $\gamma = \gamma_x^y, \eta = \eta_z^w$ and $\zeta = \zeta_u^v$ in $\text{Arcs}(\Sigma)$ where x, y, z, w, u and v are in M . We assume that $\{a, b, c, \gamma, \eta, \zeta\}$ is generic. For a, b and c , the Jacobi identity is satisfied because the definition of ∇' for loops coincides with the Goldman bracket. We have to prove three equalities:

$$(3.5) \quad \{\{a, b\}', \gamma\}' + \{\{b, \gamma\}', a\}' + \{\{\gamma, a\}', b\}' = 0,$$

$$(3.6) \quad \{\{a, \gamma\}', \eta\}' + \{\{\gamma, \eta\}', a\}' + \{\{\eta, a\}', \gamma\}' = 0,$$

$$(3.7) \quad \{\{\gamma, \eta\}', \zeta\}' + \{\{\eta, \zeta\}', \gamma\}' + \{\{\zeta, \gamma\}', \eta\}' = 0.$$

We can prove these equalities by direct calculation. We leave the readers the calculation of (3.5) because it is simpler than (3.6) and (3.7). We will calculate (3.6) and (3.7) according

to the definition of the bracket using the Kronecker delta function. First, we calculate (3.6).

$$\begin{aligned}
 & \{\{a, \gamma\}', \eta\}' \\
 (3.8) \quad &= \sum_{p \in a \cap \gamma} \sum_{q \in \gamma_x^p \cap \eta \cap \text{int } S} \varepsilon(p; a, \gamma) \varepsilon(q; \gamma, \eta) |\gamma_x^q \eta_q^w| |\eta_z^q \gamma_q^p a_p \gamma_p^y| \\
 (3.9) \quad &+ \sum_{p \in a \cap \gamma} \sum_{q \in a \cap \eta} \varepsilon(p; a, \gamma) \varepsilon(q; a, \eta) |\gamma_x^q a_q^p \eta_q^w| |\eta_z^q a_q^p \gamma_p^y| \\
 (3.10) \quad &+ \sum_{p \in a \cap \gamma} \sum_{q \in \gamma_x^p \cap \eta \cap \text{int } S} \varepsilon(p; a, \gamma) \varepsilon(q; \gamma, \eta) |\gamma_x^p a_p \gamma_p^q \eta_q^w| |\eta_z^q \gamma_q^y| \\
 (\delta 1) \quad &+ (\text{ the terms of degree 1 of the Kronecker delta function}),
 \end{aligned}$$

$$\begin{aligned}
 & \{\{\gamma, \eta\}', a\}' \\
 (3.11) \quad &= \sum_{p \in \gamma \cap \eta \cap \text{int } S} \sum_{q \in \gamma_x^p \cap a} \varepsilon(p; \gamma, \eta) \varepsilon(q; \gamma, a) |\gamma_x^q a_q \gamma_q^p \eta_p^w| |\eta_z^p \gamma_p^y| \\
 (3.12) \quad &+ \sum_{p \in \gamma \cap \eta \cap \text{int } S} \sum_{q \in \eta_x^p \cap a} \varepsilon(p; \gamma, \eta) \varepsilon(q; \eta, a) |\gamma_x^p \eta_p^q a_q \eta_q^w| |\eta_z^p \gamma_p^y| \\
 (3.13) \quad &+ \sum_{p \in \gamma \cap \eta \cap \text{int } S} \sum_{q \in \eta_z^p \cap a} \varepsilon(p; \gamma, \eta) \varepsilon(q; \eta, a) |\gamma_x^p \eta_p^w| |\eta_z^q a_q \gamma_q^p \eta_p^y| \\
 (3.14) \quad &+ \sum_{p \in \gamma \cap \eta} \sum_{q \in \gamma_x^p \cap a} \varepsilon(p; \gamma, \eta) \varepsilon(q; \gamma, a) |\gamma_x^p \eta_p^w| |\eta_z^p \gamma_p^q a_q \gamma_q^y| \\
 (\delta 2) \quad &+ (\text{ the terms of degree 1 of the Kronecker delta function}),
 \end{aligned}$$

$$\begin{aligned}
 & \{\{\eta, a\}', \gamma\}' \\
 (3.15) \quad &= \sum_{p \in \eta \cap a} \sum_{q \in \eta_z^p \cap \gamma \cap \text{int } S} \varepsilon(p; \eta, a) \varepsilon(q; \eta, \gamma) |\eta_z^q \gamma_q^y| |\gamma_x^q \eta_q^p a_p \eta_p^w| \\
 (3.16) \quad &+ \sum_{p \in \eta \cap a} \sum_{q \in a \cap \gamma} \varepsilon(p; \eta, a) \varepsilon(q; a, \gamma) |\eta_z^p a_q^p \gamma_q^y| |\gamma_x^q a_q^p \eta_p^w| \\
 (3.17) \quad &+ \sum_{p \in \eta \cap a} \sum_{q \in a \cap \gamma} \varepsilon(p; \eta, a) \varepsilon(q; a, \gamma) |\eta_z^p a_q^p \gamma_q^y| |\gamma_x^q a_q^p \eta_p^w| \\
 (\delta 3) \quad &+ (\text{ the terms of degree 1 of the Kronecker delta function}).
 \end{aligned}$$

The sum of (3.8) and (3.10) is the sum of subset of curves obtained from $a \cup \gamma \cup \eta$ by smoothing at any pair of intersection points $p \in a \cap \gamma$ and $q \in \gamma \cap \eta$ with the sign $\varepsilon(p; a, \gamma) \varepsilon(q; \gamma, \eta)$. We remark that a subset $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ of curves on Σ gives the element $|\alpha_1| |\alpha_2| \dots |\alpha_n|$ of $S(\Sigma)^+$. The sum of (3.11) and (3.14) is also the sum of the curves with the opposite sign $\varepsilon(q; \gamma, \eta) \varepsilon(p; \gamma, a)$. Therefore, these sums cancel each other out. Similarly, the sum of (3.15) and (3.17) cancels out the sum of (3.12) and (3.13). Clearly, (3.9) is opposite in sign to (3.16). It is only necessary to confirm that the terms of degree 1 of the Kronecker delta function vanishes.

$$\begin{aligned}
 (\delta 1) = & \sum_{p \in a \cap \gamma} \delta_{y,w} \varepsilon(p; a, \gamma) \varepsilon(y; \gamma, \eta) |\gamma_x^p a_p \gamma_p^y| |\eta| \\
 & + \sum_{p \in a \cap \gamma} \delta_{x,z} \varepsilon(p; a, \gamma) \varepsilon(x; \gamma, \eta) |\eta| |\gamma_x^p a_p \gamma_p^y|,
 \end{aligned}$$

$$\begin{aligned}
(\delta 2) = & \sum_{p \in \gamma \cap a} \delta_{y,w} \varepsilon(p; \gamma, a) \varepsilon(y; \gamma, \eta) \left| \gamma_x^p a_p \gamma_p^y \right| |\eta| \\
& + \sum_{p \in \eta \cap a} \delta_{y,w} \varepsilon(p; \eta, a) \varepsilon(y; \gamma, \eta) |\gamma| \left| \eta_z^p a_p \eta_p^y \right| \\
& + \sum_{p \in \eta \cap a} \delta_{x,z} \varepsilon(p; \eta, a) \varepsilon(x; \gamma, \eta) \left| \eta_z^p a_p \eta_p^w \right| |\gamma| \\
& + \sum_{p \in \gamma \cap a} \delta_{x,w} \varepsilon(p; \gamma, a) \varepsilon(x; \gamma, \eta) |\eta| \left| \gamma_x^p a_p \gamma_p^y \right|,
\end{aligned}$$

$$\begin{aligned}
(\delta 3) = & \sum_{p \in \eta \cap a} \delta_{w,y} \varepsilon(p; \eta, a) \varepsilon(w; \eta, \gamma) \left| \eta_z^p a_p \eta_p^w \right| |\gamma| \\
& + \sum_{p \in \eta \cap a} \delta_{z,x} \varepsilon(p; \eta, a) \varepsilon(z; \eta, \gamma) \left| \eta_z^p a_p \eta_p^w \right| |\gamma|.
\end{aligned}$$

From the above, we see that the sum of $(\delta 1)$ and $(\delta 3)$ cancels out $(\delta 2)$. Consequently, the Jacobi identity (3.6) holds. Finally, we prove (3.7).

(3.18)

$$\{\{\gamma, \eta\}' \zeta\}' = \sum_{p \in \gamma \cap \eta \cap \text{int } S} \sum_{q \in \gamma_x^p \cap \zeta \cap \text{int } S} \varepsilon(p; \gamma, \eta) \varepsilon(q; \gamma, \zeta) \left| \gamma_x^q \zeta_q^v \right| \left| \zeta_u^q \gamma_q^w \right| \left| \eta_z^p \gamma_p^y \right|$$

(3.19)

$$+ \sum_{p \in \gamma \cap \eta \cap \text{int } S} \sum_{q \in \eta_p^w \cap \zeta \cap \text{int } S} \varepsilon(p; \gamma, \eta) \varepsilon(q; \eta, \zeta) \left| \gamma_x^p \eta_p^q \zeta_q^v \right| \left| \zeta_u^q \eta_q^w \right| \left| \eta_z^p \gamma_p^y \right|$$

(3.20)

$$+ \sum_{p \in \gamma \cap \eta \cap \text{int } S} \sum_{q \in \eta_z^p \cap \zeta \cap \text{int } S} \varepsilon(p; \gamma, \eta) \varepsilon(q; \eta, \zeta) \left| \gamma_x^p \eta_p^w \right| \left| \eta_z^q \zeta_q^v \right| \left| \zeta_u^q \eta_q^p \gamma_p^y \right|$$

(3.21)

$$+ \sum_{p \in \gamma \cap \eta \cap \text{int } S} \sum_{q \in \gamma_p^y \cap \zeta \cap \text{int } S} \varepsilon(p; \gamma, \eta) \varepsilon(q; \gamma, \zeta) \left| \gamma_x^p \eta_p^w \right| \left| \eta_z^p \gamma_p^q \zeta_q^v \right| \left| \zeta_u^q \gamma_q^y \right|$$

($\delta 4$)

+ (the terms of degree 1 of the Kronecker delta function)

($\delta \delta$)

+ (the terms of degree 2 of the Kronecker delta function).

The sum of (3.18) and (3.21) appears in a cyclic permutation of the symbols (γ, η, ζ) at the sum of (3.19) and (3.20) with the opposite sign. We can confirm that the terms of (3.7) which has no Kronecker delta function vanishes in the same way. We can calculate

remaining parts $(\delta 4)$ and $(\delta \delta)$ as the following.

$$(3.22) \quad (\delta 4) = \sum_{p \in \gamma \cap \zeta \cap \text{int } S} \delta_{x,z} \varepsilon(x; \gamma, \eta) \varepsilon(p; \gamma, \zeta) |\gamma_x^p \zeta_p^v| |\zeta_u^p \gamma_p^y| |\eta|$$

$$(3.23) \quad + \sum_{p \in \eta \cap \zeta \cap \text{int } S} \delta_{x,z} \varepsilon(x; \gamma, \eta) \varepsilon(p; \eta, \zeta) |\gamma| |\eta_z^p \zeta_p^v| |\zeta_u^p \eta_p^w|$$

$$(3.24) \quad + \sum_{p \in \gamma \cap \zeta \cap \text{int } S} \delta_{y,w} \varepsilon(y; \gamma, \eta) \varepsilon(p; \gamma, \zeta) |\eta| |\gamma_x^p \zeta_p^v| |\zeta_u^p \gamma_p^y|$$

$$(3.25) \quad + \sum_{p \in \eta \cap \zeta \cap \text{int } S} \delta_{y,w} \varepsilon(y; \gamma, \eta) \varepsilon(p; \eta, \zeta) |\eta_z^p \zeta_p^v| |\zeta_u^p \eta_p^w| |\gamma|$$

$$(3.26) \quad + \sum_{p \in \gamma \cap \eta \cap \text{int } S} \delta_{x,u} \varepsilon(x; \gamma, \zeta) \varepsilon(p; \gamma, \eta) |\gamma_x^p \eta_p^w| |\zeta| |\eta_z^p \gamma_p^y|$$

$$(3.27) \quad + \sum_{p \in \gamma \cap \eta \cap \text{int } S} \delta_{w,v} \varepsilon(w; \eta, \zeta) \varepsilon(p; \gamma, \eta) |\zeta| |\gamma_x^p \eta_p^w| |\eta_z^p \gamma_p^y|$$

$$(3.28) \quad + \sum_{p \in \gamma \cap \eta \cap \text{int } S} \delta_{z,u} \varepsilon(z; \eta, \zeta) \varepsilon(p; \gamma, \eta) |\gamma_x^p \eta_p^w| |\eta_z^p \gamma_p^y| |\zeta|$$

$$(3.29) \quad + \sum_{p \in \gamma \cap \eta \cap \text{int } S} \delta_{x,z} \varepsilon(y; \gamma, \zeta) \varepsilon(p; \gamma, \eta) |\gamma_x^p \eta_p^w| |\zeta| |\eta_z^p \gamma_p^y|.$$

The opposite sign of (3.22) coincides with a cyclic permutation of (γ, η, ζ) at (3.28). Similarly, (3.23), (3.24) and (3.25) correspond to (3.26), (3.27) and (3.29), respectively.

$$(3.30) \quad (\delta \delta) = \delta_{x,z} \delta_{x,u} \varepsilon(x; \gamma, \eta) \varepsilon(x; \gamma, \zeta) |\gamma| |\zeta| |\eta|$$

$$(3.31) \quad + \delta_{x,z} \delta_{y,v} \varepsilon(x; \gamma, \eta) \varepsilon(y; \gamma, \zeta) |\zeta| |\gamma| |\eta|$$

$$(3.32) \quad + \delta_{x,z} \delta_{z,u} \varepsilon(x; \gamma, \eta) \varepsilon(z; \eta, \zeta) |\gamma| |\eta| |\zeta|$$

$$(3.33) \quad + \delta_{x,z} \delta_{w,v} \varepsilon(x; \gamma, \eta) \varepsilon(w; \eta, \zeta) |\gamma| |\zeta| |\eta|$$

$$(3.34) \quad + \delta_{y,w} \delta_{x,u} \varepsilon(y; \gamma, \eta) \varepsilon(x; \gamma, \zeta) |\eta| |\gamma| |\zeta|$$

$$(3.35) \quad + \delta_{y,w} \delta_{y,v} \varepsilon(y; \gamma, \eta) \varepsilon(y; \gamma, \zeta) |\eta| |\zeta| |\gamma|$$

$$(3.36) \quad + \delta_{y,w} \delta_{z,u} \varepsilon(y; \gamma, \eta) \varepsilon(z; \eta, \zeta) |\eta| |\zeta| |\gamma|$$

$$(3.37) \quad + \delta_{y,w} \delta_{w,v} \varepsilon(y; \gamma, \eta) \varepsilon(w; \eta, \zeta) |\zeta| |\eta| |\gamma|.$$

In a similar way, (3.30), (3.31), (3.33) and (3.35) correspond to (3.32), (3.36), (3.34) and (3.37), respectively. We finish proving the Jacobi identity (3.7). $\square$

Lemma 3.3 and Proposition 3.4 tell us that the pair $(S(\Sigma)^+, \nabla)$ is a Poisson algebra. We denote it by $\text{PS}(\Sigma)^+$. When we put $t = 1$, we can obtain the Poisson structure on $S(\Sigma)$. We denote it by $\text{PS}(\Sigma)$.

REMARK 3.5.

- (1) If M is the empty set, then the Poisson algebra $\text{PS}((S, M))$ is the Goldman Lie algebra of S .
- (2) Kawazumi-Kuno[KK12] defined an action of the Goldman Lie algebra of S on the module generated by the homotopy set of $\text{Arcs}(\Sigma)$. We can consider the action is the adjoint action of $\text{Loops}(\Sigma)$ on $\text{Arcs}(\Sigma)$ in $\text{PS}(\Sigma)$.
- (3) Labourie[Lab12] defined the swapping algebra $\mathcal{L}_k(M)$ where M is a set of points of the boundary of a disk $\mathbb{D} = \{z \in \mathbb{C} \mid |z| \leq 1\}$ and k any real number. $\text{PS}((\mathbb{D}, M))$ coincides with the swapping algebra $\mathcal{L}_0(M)$.

4. co-Poisson coalgebras of curves on bordered surfaces

Let $C_{\mathcal{O}}^+$ be the $\mathbf{K}[t, t^{-1}]$ -submodule of $\mathbf{K}\hat{\pi}(\Sigma)^+$ generated by a trivial loop with no self-intersection. The submodule $C_{\mathcal{O}}^+$ is uniquely determined although we have two choices of such a trivial loop, the positive or negative curl. We denote the quotient $\mathbf{K}[t, t^{-1}]$ -module $\mathbf{K}\hat{\pi}(\Sigma)^+ / (C_M^+ + C_{\mathcal{O}}^+)$ by $Z(\Sigma)_0^+$, the quotient map $\text{Curves}(\Sigma) \rightarrow Z(\Sigma)_0^+$ by $\|\cdot\|$. The symmetric $\mathbf{K}[t, t^{-1}]$ -algebra $S(\Sigma)_0^+ = \text{Sym}(Z(\Sigma)_0^+)$ has a commutative and associative multiplication m and a cocommutative and coassociative comultiplication Δ which define a bialgebra structure on it.

In this section, we define a cobracket of $S(\Sigma)_0^+$ which is a generalization of Turaev cobracket [Tur89, Tur91]. We show that the cobracket and the bialgebra structure on $S(\Sigma)_0^+$ are compatible.

First, we define a cobracket on $S(\Sigma)_0^+$. For any curve α on Σ , we denote the set of all self-intersection points of α by $\mathfrak{h} \alpha$. If α is in $\text{Arcs}(\Sigma)$, then $\mathfrak{h} \alpha$ decomposes to two subsets type(I) and type(II). We denote the subset of type(I) (resp. type(II)) self-intersection points by $\mathfrak{h}_{(I)} \alpha$ (resp. $\mathfrak{h}_{(II)} \alpha$). Let a in $\text{Loops}(\Sigma)$ and γ in $\text{Arcs}(\Sigma)$ be generic. We define $\delta': \text{Curves}(\Sigma) \rightarrow S(\Sigma)_0^+ \otimes_t S(\Sigma)_0^+$ as the following.

$$(4.1) \quad \delta'(a) = \sum_{p \in \mathfrak{h} a} \left\| a_p^{(1)} \right\| \otimes_t \left\| a_p^{(2)} \right\| - \left\| a_p^{(2)} \right\| \otimes_t \left\| a_p^{(1)} \right\|,$$

$$(4.2) \quad \begin{aligned} \delta'(\gamma) &= \sum_{p \in \mathfrak{h}_{(I)} \gamma} \left\| \gamma_p^{(1)} \right\| \otimes_t \left\| \gamma_x^p \gamma_p^y \right\| - \left\| \gamma_x^p \gamma_p^y \right\| \otimes_t \left\| \gamma_p^{(1)} \right\| \\ &\quad + \sum_{p \in \mathfrak{h}_{(II)} \gamma} \left\| \gamma_x^p \gamma_p^y \right\| \otimes_t \left\| \gamma_p^{(2)} \right\| - \left\| \gamma_p^{(2)} \right\| \otimes_t \left\| \gamma_x^p \gamma_p^y \right\| \\ &= \sum_{p \in \mathfrak{h}_{(I)} \gamma \cap \text{int } S} \left\| \gamma_p^{(1)} \right\| \otimes_t \left\| \gamma_x^p \gamma_p^y \right\| - \left\| \gamma_x^p \gamma_p^y \right\| \otimes_t \left\| \gamma_p^{(1)} \right\| \\ &\quad + \sum_{p \in \mathfrak{h}_{(II)} \gamma \cap \text{int } S} \left\| \gamma_x^p \gamma_p^y \right\| \otimes_t \left\| \gamma_p^{(2)} \right\| - \left\| \gamma_p^{(2)} \right\| \otimes_t \left\| \gamma_x^p \gamma_p^y \right\|. \end{aligned}$$

We can confirm that δ' defines the same element in $S(\Sigma)_0^+ \otimes_t S(\Sigma)_0^+$ for local moves $(\Omega 2-1)$, $(\Omega 2-2)$, $(\Omega 3)$, $(\Omega b2-2)$ and $(\Omega b2-3)$. For example,

the terms coming from $\delta'(\alpha)$ of  =  =  $\otimes_t$  -  $\otimes_t$  +  $\otimes_t$  -  $\otimes_t$  =  $\otimes_t$  -  $\otimes_t$  +  $\otimes_t$  -  $\otimes_t$  = 0.

We do not need to calculate local moves $(\Omega b2-1)$ and $(\Omega b2-4)$ because connected arcs have no such self-intersections. If α is a curl or a monogon with vertex in M , then the value $\delta'(\alpha)$ vanishes. Therefore the map δ' extends to $\delta: Z(\Sigma)^+ \rightarrow S(\Sigma)_0^+ \otimes_t S(\Sigma)_0^+$ and the

extended map δ is a $\mathbf{K}[t, t^{-1}]$ -linear map. Furthermore, we extend it to the $\mathbf{K}[t, t^{-1}]$ -linear map on $S(\Sigma)^+$ by using the (m, δ) -compatibility. (See Appendix A.) Consequently, we obtain the $\mathbf{K}[t, t^{-1}]$ -linear map $\delta: S(\Sigma)_0^+ \rightarrow S(\Sigma)_0^+ \otimes_t S(\Sigma)_0^+$.

LEMMA 4.1.

- (1) If δ' is co-skew symmetric, then δ is co-skew symmetric.
- (2) If δ' satisfies the co-Leibniz rule, then δ satisfies the co-Leibniz rule.
- (3) If δ' is co-skew symmetric and satisfies the co-Leibniz rule and the co-Jacobi identity, then δ satisfies the co-Jacobi identity.

PROOF. Use the (m, δ) -compatibility and an induction on the degree of symmetric power. $\square$

PROPOSITION 4.2. δ' is co-skew symmetric and satisfies the co-Jacobi identity.

PROOF. It is clear that the map δ' is co-skew symmetric by definition. We directly calculate $(\tau^2 + \tau + \text{id}^{\otimes 3}) \circ (\text{id} \otimes_t \delta') \circ \delta'(\alpha)$ to prove the co-Jacobi identity. If α is a loop on Σ , then the map δ' coincides with the Turaev cobracket. We have to confirm the co-Jacobi identity when α is an arc on Σ .

Let $\gamma = \gamma_x^y$ is an arc on Σ with endpoints x and y in M . Then,

$$\begin{aligned} & (\text{id} \otimes_t \delta') \circ \delta'(\gamma_x^y) \\ &= \sum_{p \in \mathfrak{H}(\mathfrak{U})} \sum_{q \in \mathfrak{H}(\mathfrak{U})} \left(\left\| \gamma_p^{(1)} \right\| \otimes_t \left\| (\gamma_x^p \gamma_p^y)^{(1)} \right\| \otimes_t \left\| (\gamma_x^p \gamma_p^y)^q (\gamma_x^p \gamma_p^y)^y \right\| \right. \\ & \quad \left. - \left\| \gamma_p^{(1)} \right\| \otimes_t \left\| (\gamma_x^p \gamma_p^y)^q (\gamma_x^p \gamma_p^y)^y \right\| \otimes_t \left\| (\gamma_x^p \gamma_p^y)^{(1)} \right\| \right) \end{aligned} \quad (4.3)$$

$$\begin{aligned} & + \sum_{p \in \mathfrak{H}(\mathfrak{U})} \sum_{q \in \mathfrak{H}(\mathfrak{U})} \left(\left\| \gamma_p^{(1)} \right\| \otimes_t \left\| (\gamma_x^p \gamma_p^y)^q (\gamma_x^p \gamma_p^y)^y \right\| \otimes_t \left\| \gamma_p^{(2)} \right\| \right. \\ & \quad \left. - \left\| \gamma_p^{(1)} \right\| \otimes_t \left\| \gamma_p^{(2)} \right\| \otimes_t \left\| (\gamma_x^p \gamma_p^y)^q (\gamma_x^p \gamma_p^y)^y \right\| \right) \end{aligned} \quad (4.4)$$

$$\begin{aligned} & - \sum_{p \in \mathfrak{H}(\mathfrak{U})} \sum_{q \in \mathfrak{H}(\mathfrak{U})} \left(\left\| \gamma_x^p \gamma_p^y \right\| \otimes_t \left\| (\gamma_p^{(1)})_q^{(1)} \right\| \otimes_t \left\| (\gamma_p^{(1)})_q^2 \right\| \right. \\ & \quad \left. - \left\| \gamma_x^p \gamma_p^y \right\| \otimes_t \left\| (\gamma_p^{(1)})_q^2 \right\| \otimes_t \left\| (\gamma_p^{(1)})_q^{(1)} \right\| \right) \end{aligned} \quad (4.5)$$

$$\begin{aligned} & + \sum_{p \in \mathfrak{H}(\mathfrak{U})} \sum_{q \in \mathfrak{H}(\mathfrak{U})} \left(\left\| \gamma_x^p \gamma_p^y \right\| \otimes_t \left\| (\gamma_p^{(2)})_q^{(1)} \right\| \otimes_t \left\| (\gamma_p^{(2)})_q^2 \right\| \right. \\ & \quad \left. - \left\| \gamma_x^p \gamma_p^y \right\| \otimes_t \left\| (\gamma_p^{(2)})_q^2 \right\| \otimes_t \left\| (\gamma_p^{(2)})_q^{(1)} \right\| \right) \end{aligned} \quad (4.6)$$

$$\begin{aligned} & - \sum_{p \in \mathfrak{H}(\mathfrak{U})} \sum_{q \in \mathfrak{H}(\mathfrak{U})} \left(\left\| \gamma_p^{(2)} \right\| \otimes_t \left\| (\gamma_x^p \gamma_p^y)^{(1)} \right\| \otimes_t \left\| (\gamma_x^p \gamma_p^y)^q (\gamma_x^p \gamma_p^y)^y \right\| \right. \\ & \quad \left. - \left\| \gamma_p^{(1)} \right\| \otimes_t \left\| (\gamma_x^p \gamma_p^y)^q (\gamma_x^p \gamma_p^y)^y \right\| \otimes_t \left\| (\gamma_x^p \gamma_p^y)^{(1)} \right\| \right) \end{aligned} \quad (4.7)$$

$$\begin{aligned} & - \sum_{p \in \mathfrak{H}(\mathfrak{U})} \sum_{q \in \mathfrak{H}(\mathfrak{U})} \left(\left\| \gamma_p^{(2)} \right\| \otimes_t \left\| (\gamma_x^p \gamma_p^y)^q (\gamma_x^p \gamma_p^y)^y \right\| \otimes_t \left\| \gamma_p^{(2)} \right\| \right. \\ & \quad \left. - \left\| \gamma_p^{(1)} \right\| \otimes_t \left\| \gamma_p^{(2)} \right\| \otimes_t \left\| (\gamma_x^p \gamma_p^y)^q (\gamma_x^p \gamma_p^y)^y \right\| \right) \end{aligned} \quad (4.8)$$

where $\cap \gamma_p^{(i)}$ in (4.5) and (4.6) is the set of self-intersection points of the free loop obtained by $\gamma_p^{(i)}$ for $i = 1, 2$. Setting $\gamma_x^p \cap_{(1)} \gamma_p^y = \cap_{(1)} \gamma_x^y \cap \gamma_p^p \setminus \{p\}$, we can obtain the following decomposition $\cap_{(1)} \gamma_x^p \gamma_p^y = (\cap_{(1)} \gamma_x^p) \cup (\cap_{(1)} \gamma_p^y) \cup (\gamma_x^p \cap_{(1)} \gamma_p^y)$. By using the decomposition,

$$(4.3) = \sum_{p \in \cap_{(1)} \gamma} \sum_{q \in \cap_{(1)} \gamma_p^p} \left(\|\gamma_p^{(1)}\| \otimes_t \|\gamma_q^{(1)}\| \otimes_t \|\gamma_x^q \gamma_p^p \gamma_p^y\| - \|\gamma_p^{(1)}\| \otimes_t \|\gamma_x^q \gamma_p^p \gamma_p^y\| \otimes_t \|\gamma_q^{(1)}\| \right) \\ + \sum_{p \in \cap_{(1)} \gamma} \sum_{q \in \cap_{(1)} \gamma_p^y} \left(\|\gamma_p^{(1)}\| \otimes_t \|\gamma_q^{(1)}\| \otimes_t \|\gamma_x^p \gamma_p^q \gamma_p^y\| - \|\gamma_p^{(1)}\| \otimes_t \|\gamma_x^p \gamma_p^q \gamma_p^y\| \otimes_t \|\gamma_q^{(1)}\| \right) \\ + \sum_{p \in \cap_{(1)} \gamma} \sum_{q \in \gamma_x^p \cap_{(1)} \gamma_p^y} \left(\|\gamma_p^{(1)}\| \otimes_t \|\gamma_q^{(1)}\| \otimes_t \|\gamma_p^p \gamma_q^q\| - \|\gamma_x^q \gamma_p^y\| \otimes_t \|\gamma_x^q \gamma_p^y\| \otimes_t \|\gamma_p^p \gamma_q^q\| \right).$$

We calculate (4.4)–(4.8) in the same way and we can confirm the co-Jacobi identity. $\square$

By Lemma 4.1 and Proposition 4.2, the pair $(S(\Sigma)_0^+, \delta)$ is a co-Poisson coalgebra. We denote it by $CS(\Sigma)^+$.

REMARK 4.3. We can define the co-Poisson coalgebra $CS(\Sigma)$ by substituting $t = 1$. To explain more precisely, we denote by C_O the $\mathbf{K}$ -submodule of $\mathbf{K}\hat{\pi}(\Sigma)$ generated by the trivial loop. The quotient $Z(\Sigma)_0 = \mathbf{K}\hat{\pi}(\Sigma)/(C_M + C_O)$ is $Z(\Sigma)_0^+/(t-1)Z(\Sigma)_0^+$. Then we can define the cobracket on the symmetric $\mathbf{K}$ -algebra $S(\Sigma)_0 = \text{Sum}(Z(\Sigma)_0)$ which induced from the above δ . We denote the co-Poisson coalgebra by $CS(\Sigma)$.

REMARK 4.4. The bracket on $PS(\Sigma)^+$ induces a Poisson structure on $S(\Sigma)_0^+$. Although $S(\Sigma)_0^+$ has the bracket and the cobracket, they do not satisfy the compatibility conditions of a bi-Poisson bialgebra. To be precise, the (∇, Δ) -compatibility is not satisfied for two arcs on Σ . We show a counter-example in Figure 4.4.

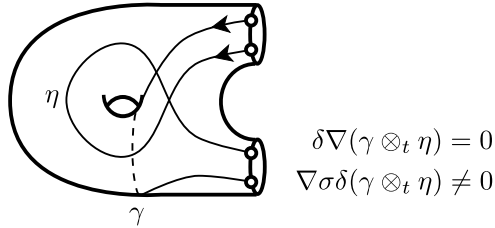


FIGURE 4.4. a counter-example of the (∇, Δ) -compatibility

5. Poisson algebras of unoriented curves and its quantization

Let $\Sigma = (S, M)$ be a bordered surface. We define unoriented curves on Σ by identifying oriented curves on Σ and its inverse. We denote the set of unoriented curves, loops and arcs by $\overline{\text{Curves}}(\Sigma)$, $\overline{\text{Loops}}(\Sigma)$ and $\overline{\text{Arcs}}(\Sigma)$ respectively. We denote the set of regular homotopy classes of unoriented curves on Σ by $\pi(\Sigma)^+$. Let $G = \langle T \mid T^2 = 1 \rangle$ be the cyclic group of order 2, $\mathbf{K}G$ the group ring and $(\mathbf{K}G)\pi(\Sigma)^+$ the free $\mathbf{K}G$ -module with basis $\pi(\Sigma)^+$. The submodule $\bar{C}_T$ of $(\mathbf{K}G)\pi(\Sigma)^+$ is generated by the elements $T|\alpha| - |\alpha'|$ for any α in $\overline{\text{Curves}}(\Sigma)$ where α' is a curve obtained by inserting a monogon into interior of α . We remark that we have two choices of α' because α is two-sided and these elements in $\pi(\Sigma)^+$ may be different. The submodule $\bar{C}_M^+$ of $(\mathbf{K}G)\pi(\Sigma)^+$ is generated by the unoriented monogons with vertices in M . Then the $\mathbf{K}G$ -module $\bar{Z}(\Sigma)^+$

is the quotient of $(\mathbf{K}G)\bar{\pi}(\Sigma)^+$ by $\bar{C}_T + \bar{C}_M^+$ and $\bar{\mathcal{P}\mathcal{S}}(\Sigma)^+ = \text{Sym}(\bar{Z}(\Sigma)^+)$. We use the same symbol $|\cdot|$ for representing the quotient map $\overline{\text{Curves}}(\Sigma) \rightarrow \bar{Z}(\Sigma)^+$. For any subset $X = \{a_1, a_2, \dots, a_n\}$ of $\overline{\text{Curves}}(\Sigma)$, $|X|$ denotes the element $|a_1| |a_2| \cdots |a_n|$ of $\bar{\mathcal{P}\mathcal{S}}(\Sigma)^+$.

Let $\{a_1, a_2, \dots, a_n\}$ be a generic subset of $\overline{\text{Curves}}(\Sigma)$. We define local moves of $a_1 \cup a_2 \cup \cdots \cup a_n$ for n -tuple of generic unoriented curves $(a_1, a_2, \dots, a_n)$. Let p be an intersection point of a_i and a_j where $1 \leq i < j \leq n$. A neighborhood of p has two subarcs a'_i of a_i and a'_j of a_j . We give an orientation of the subarcs such that the pair of tangent vectors at p obtained from (a'_i, a'_j) coincides with the orientation of the surface S . $E_p(a_1, a_2, \dots, a_n)$ denotes unoriented curves obtained by smoothing the crossing at p under the given orientation. We define $e_p(a_1, a_2, \dots, a_n)$ as the unoriented curves obtained by exchanging a_i and a_j of $E_p(a_1, a_2, \dots, a_n)$. We illustrate the two eliminations E_p and e_p at the intersection point p in Figure 3.5.

Let $p_1, p_2, \dots, p_m$ be intersection points of distinct two curves in $\bigcup_{1 \leq i \leq n} a_i$. Let us denote by $X_{p_1} X_{p_2} \cdots X_{p_m}(a_1, a_2, \dots, a_n)$ unoriented curves obtained by performing the eliminations X_{p_i} under the above rule where the symbol X is E or e .

For any a and b in $\overline{\text{Curves}}(\Sigma)$, We define a bracket on $\bar{S}(\Sigma)^+$ by

$$\begin{aligned} \{a, b\} = & \sum_{p \in a \cap b \cap \text{int } S} |E_p(a, b)| - |e_p(a, b)| \\ (5.1) \quad & + \frac{1}{2}(n_+(a, b) - n_-(a, b)) |a| |b|. \end{aligned}$$

In the above, $n_+(a, b)$ (resp. $n_-(a, b)$) is the number of pairs of endpoints of a and b such that the pairs of endpoints lie in the same marked point m in M and a is locally located on the right (resp. left) side of b in the neighborhood of m . We can confirm that the value of the bracket is invariant under the unoriented local moves $(\bar{\Omega}2)$, $(\bar{\Omega}3)$ and $(\bar{\Omega}b2)$ illustrated in Figure 4.5. Thus the bracket is defined on $\hat{\pi}(\Sigma)^+$ and we extend it $\mathbf{K}G$ -

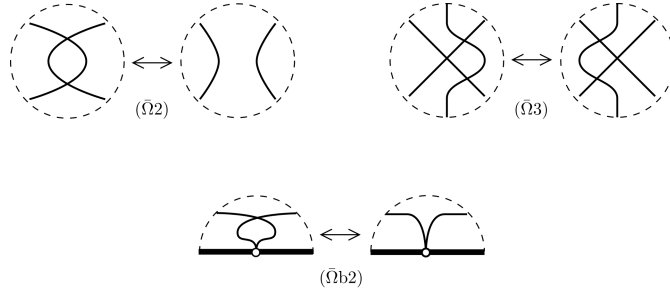


FIGURE 4.5. the local moves of unoriented curves

bilinearly. We obtain $\{\cdot, \cdot\}: (\mathbf{K}G)\bar{\pi}(\Sigma) \otimes_G (\mathbf{K}G)\bar{\pi}(\Sigma) \rightarrow \bar{S}(\Sigma)^+$ where the symbol $\otimes_G$ denotes the tensor product over $\mathbf{K}G$. If a or b is a monogon with vertex in M , then the bracket vanishes. We can also confirm that the bracket vanishes on $(\mathbf{K}G)\bar{\pi}(\Sigma) \otimes_G \bar{C}_T + \bar{C}_T \otimes_G (\mathbf{K}G)\bar{\pi}(\Sigma)$ because the bracket preserves monogons in the curves. Therefore, we can define the bracket on $\bar{Z}(\Sigma)^+$ and extend to $\bar{S}(\Sigma)^+$ by the Leibniz rule.

EXAMPLE 5.1. Let us forget orientations of the arcs a and b in Figure 4.3. We can see that $n_+(a, b) = 0$ and $n_-(a, b) = 2$. Therefore, the Goldman bracket of unoriented curves a and b is

$$\begin{aligned}
\{|a|, |b|\} = & \quad \underbrace{\quad}_{|E_p(a,b)|} \quad - \quad \underbrace{\quad}_{|e_p(a,b)|} \\
& - \quad \underbrace{\quad}_{|a| |b|} .
\end{aligned}$$

We can easily show the following lemma by definition.

LEMMA 5.2. Let $\{a_1, a_2, \dots, a_n\}$ and $\{a, b\}$ be generic subsets of $\overline{\text{Curves}}(\Sigma)$.

- (1) $E_p(a_1, a_2, \dots, a_i, \dots, a_j, \dots, a_n) = e_p(a_1, a_2, \dots, a_j, \dots, a_i, \dots, a_n)$ for an intersection point p of a_i and a_j .
- (2) $n_+(a, b) = n_-(a, b)$.

PROPOSITION 5.3. The pair $(\bar{S}(\Sigma)^+, \{\cdot, \cdot\})$ is a Poisson algebra.

PROOF. Lemma 5.2 implies skew-symmetry of the bracket. We have to show the Jacobi identity. Let us denote $n_+(a, b) - n_-(a, b)$ by $N(a, b)$. Let a, b and c be in $\overline{\text{Curves}}(\Sigma)$.

$$\begin{aligned}
\{|a|, |b|\}, |c|\} = & \sum_{p \in a \cap b \cap \text{int } S} \{|E_p(a, b)|, |c|\} - \{|e_p(a, b)|, |c|\} \\
& + \frac{1}{2} N(a, b) \{|a| |b|, |c|\}.
\end{aligned}$$

For any p in $a \cap b \cap \text{int } S$,

$$\begin{aligned}
\{|e_p(a, b)|, |c|\} = & \sum_{q \in a \cap c \cap \text{int } S} |E_q e_p(a, b, c)| - |e_q e_p(a, b, c)| \\
& + \sum_{q \in b \cap c \cap \text{int } S} |E_q e_p(a, b, c)| - |e_q e_p(a, b, c)| \\
& + \frac{1}{2} (N(a, c) + N(b, c)) |E_p(a, b)| |c|,
\end{aligned}$$

$$\begin{aligned}
\{|E_p(a, b)|, |c|\} = & \sum_{q \in a \cap c \cap \text{int } S} |E_q E_p(a, b, c)| - |e_q E_p(a, b, c)| \\
& + \sum_{q \in b \cap c \cap \text{int } S} |E_q E_p(a, b, c)| - |e_q E_p(a, b, c)| \\
& + \frac{1}{2} (N(a, c) + N(b, c)) |e_p(a, b)| |c|.
\end{aligned}$$

$$\begin{aligned}
\{|a| |b|, |c|\} &= \sum_{q \in a \cap c \cap \text{int } S} |E_q(a, c)| |b| - |e_q(a, c)| |b| \\
&+ \sum_{q \in b \cap c \cap \text{int } S} |E_q(b, c)| |a| - |e_q(b, c)| |a| \\
&+ \frac{1}{2}(N(a, c) + N(b, c)) |a| |b| |c|.
\end{aligned}$$

Taking the above calculations together,

$$\{\{|a|, |b|\}, |c|\}$$

(5.2)

$$= \sum_{p \in a \cap b \cap \text{int } S} \sum_{q \in a \cap c \cap \text{int } S} |E_q E_p(a, b, c)| - |e_q E_p(a, b, c)| - |E_q e_p(a, b, c)| + |e_q e_p(a, b, c)|$$

(5.3)

$$+ \sum_{p \in a \cap b \cap \text{int } S} \sum_{q \in b \cap c \cap \text{int } S} |E_q E_p(a, b, c)| - |e_q E_p(a, b, c)| - |E_q e_p(a, b, c)| + |e_q e_p(a, b, c)|$$

(5.4)

$$+ \sum_{p \in a \cap b \cap \text{int } S} \frac{1}{2}(N(a, c) + N(b, c))(|E_p(a, b)| |c| - |e_p(a, b)| |c|)$$

(5.5)

$$+ \sum_{p \in a \cap c \cap \text{int } S} \frac{1}{2}N(a, b)(|E_p(a, c)| |b| - |e_p(a, c)| |b|)$$

(5.6)

$$+ \sum_{p \in b \cap c \cap \text{int } S} \frac{1}{2}N(a, b)(|E_p(b, c)| |a| - |e_p(b, c)| |a|)$$

(5.7)

$$+ \frac{1}{4}N(a, b)(N(a, c) - N(b, c)) |a| |b| |c|.$$

Replacing the symbols (a, b, c) of (5.2) with (b, c, a) ,

$$\begin{aligned}
&\sum_{p \in b \cap c \cap \text{int } S} \sum_{q \in b \cap a \cap \text{int } S} |E_q E_p(b, c, a)| - |e_q E_p(b, c, a)| - |E_q e_p(b, c, a)| + |e_q e_p(b, c, a)| \\
&= \sum_{q \in a \cap b \cap \text{int } S} \sum_{p \in b \cap c \cap \text{int } S} |e_q E_p(a, b, c)| - |E_q E_p(a, b, c)| - |e_q e_p(a, b, c)| + |E_q e_p(b, c, a)| \\
&= -(5.3)
\end{aligned}$$

by using Lemma 5.2. Consequently, (5.2) and (5.3) vanish in $\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\}$. We can show that (5.4) of $\{b, \{c, a\}\}$ cancels the sum of (5.6) and (5.5) of $\{c, \{a, b\}\}$ by similar calculation. Finally, We can confirm that (5.7) of $\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\}$ vanishes by using $N(a, b) = -N(b, a)$. $\square$

We denote the Poisson algebra $(\bar{S}(\Sigma)^+, \{\cdot, \cdot\})$ by $\overline{PS}(\Sigma)^+$.

We can consider variations of $\overline{PS}(\Sigma)^+$ by substituting $T = 1$ and $T = -1$ by the same way as substituting $t = 1$ in $PS(\Sigma)^+$ and $CS(\Sigma)^+$. Let us denote these Poisson $\mathbf{K}$ -algebras by $\overline{PS}(\Sigma)^+|_{T=1}$ and $\overline{PS}(\Sigma)^+|_{T=-1}$. If we substitute $T = 1$, then $\mathbf{K}G$ is replaced with K , $\hat{Z}^+(\Sigma)$ with $\hat{Z}(\Sigma) = \mathbf{K}\hat{\pi}(\Sigma)/\bar{C}_M$ where $\hat{\pi}(\Sigma)$ is the set of homotopy classes of curves on Σ and $\bar{C}_M$ is the submodule of $\mathbf{K}\hat{\pi}(\Sigma)$ generated by the contractible arcs. Therefore

$\overline{\text{PS}}(\Sigma)^+|_{T=-1}$ is the symmetric $\mathbf{K}$ -algebra of $\bar{Z}(\Sigma)$ with the Poisson bracket induced from $\overline{\text{PS}}(\Sigma)^+$. We denote the Poisson algebra by $\overline{\text{PS}}(\Sigma)$.

We will discuss about the Poisson algebra $\overline{\text{PS}}(\Sigma)^+|_{T=-1}$. Concretely speaking, we will give a quantization of a quotient Poisson algebra of $\overline{\text{PS}}(\Sigma)^+|_{T=-1}$. As a first step, we define the skein algebra of Σ defined by Muller [Mul12]. The skein algebra $\text{Sk}_q(\Sigma)$ is given by the quotient of the algebra of framed links in $\Sigma \times [0, 1]$ by the skein relators. Next, we give the Poisson algebra $\text{Sk}(\Sigma)$ which is a quotient of $\overline{\text{PS}}(\Sigma)^+|_{T=-1}$ by the relations corresponding to the skein relations. Finally, we will give a quantization homomorphism from $\text{Sk}_q(\Sigma)$ to $\text{Sk}(\Sigma)$ by $q \rightarrow 1$.

We give the definition of framed links on a bordered surface Σ combinatorially by using link diagrams according to Muller [Mul12].

DEFINITION 5.4. Let $D = \{a_1, a_2, \dots, a_n\} \subset \overline{\text{Curves}}(\Sigma)$ be generic unoriented curves on Σ . We consider D as the immersion $a_1 \cup a_2 \cup \dots \cup a_n$ from the disjoint union of domains to S . We give a strict order “ $<$ ” on $D^{-1}(p)$ for any intersection point p of D in $\text{int } S$. We give a total order “ $\leq$ ” on $D^{-1}(m)$ for any intersection point m of D in M . We define the *link diagram* D on Σ as generic curves D equipped with these ordered set, the *crossing data* of D .

If the strictly ordered set $D^{-1}(p) = \{p_1, p_2\}$ has the order $p_1 < p_2$, then we can also describe diagrammatically it in a neighborhood of $p \in \text{int } S$ as Figure 4.6. We illustrate the neighborhood of $p \in \text{int } S$ by gapping the strand containing p_2 at p . We call the strand containing p_1 (resp. p_2) is the *overcrossing* (resp. the *undercrossing*) of p_1 (resp. p_2) at p .

If we take two elements m_1 and m_2 from the total ordered set $D^{-1}(m)$, we can also describe the strand of m_1 and m_2 diagrammatically in a neighborhood of m in the same way. We illustrate as the real crossing for $m_1 = m_2$. We call the strand of m_1 is the *simultaneous-crossing* with m_2 at m .

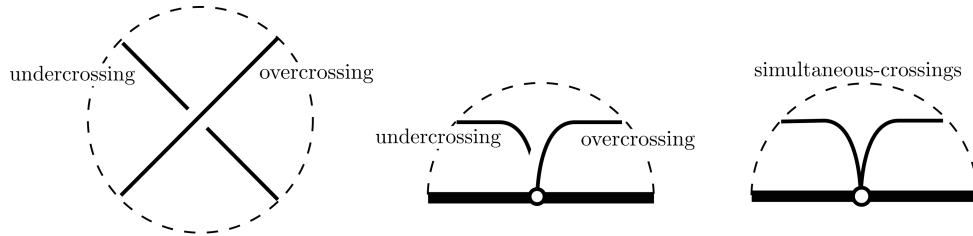


FIGURE 4.6. the crossing data

Let us denote by $\mathcal{D}(\Sigma)$ the set of link diagrams on Σ . We define an equivalence relation, *regular isotopy*, on $\mathcal{D}(\Sigma)$ according to the approach of Kauffman [Kau88].

DEFINITION 5.5. Link diagrams D and D' are *regularly isotopic* if D is obtained from D' by a finite sequence of ambient isotopy of S fixing collar of ∂S and preserving crossing data and Reidemeister moves (R1'), (R2), (R3), (Rb2-1) and (Rb2-2). See Figure 7.1. The set of regular isotopy classes of link diagrams on Σ is denoted by $\mathcal{L}(\Sigma)^+$. A *framed link* is an element in $\mathcal{L}(\Sigma)^+$ and we denote by $[D]$ the framed link represented by the link diagram D .

We remark that $\mathcal{D}(\Sigma)$ has the empty diagram $\emptyset$ and $\mathcal{L}(\Sigma)^+$ has the empty link $[\emptyset]$. We can define the multiplication of two link diagrams D_1 and D_2 by superposition of link

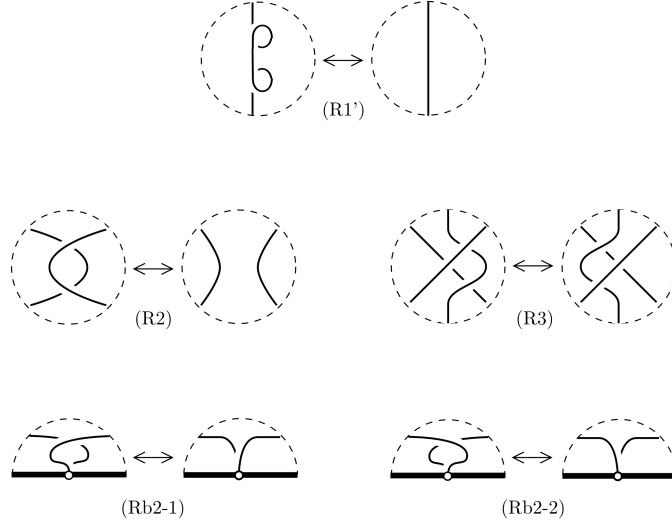


FIGURE 4.7. the Reidemeister moves

diagrams if $D_1 \cup D_2$ is generic. Then we give crossing data on $D_1 \cap D_2$ such that all strands of D_1 contained in neighborhoods of $D_1 \cap D_2$ are overcrossings and strands of D_2 are undercrossings. We denote the multiplication by $D_1 \cdot D_2$. Let L_1 and L_2 be framed links on Σ . We can take representatives D_1 of L_1 and D_2 of L_2 such that $D_1 \cup D_2$ is generic curves on Σ . Then $\mathcal{L}(\Sigma)^+$ has the multiplicative structure defined by $L_1 L_2 = [D_1 \cdot D_2]$.

Next, we introduce the skein algebra of Σ . Let us denote by $\mathbf{K}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ the ring of Laurent polynomials in one variable $q^{\frac{1}{2}}$ and $\mathbf{K}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]\mathcal{L}(\Sigma)^+$ the free $\mathbf{K}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ -module on $\mathcal{L}(\Sigma)^+$. The free module $\mathbf{K}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]\mathcal{L}(\Sigma)^+$ has the multiplication induced from $\mathcal{L}(\Sigma)^+$. We prepare two sets $r(b)$ and $\ell(b)$ to define a relation of the skein algebra for any strand b contained in a neighborhood of M of a link diagram D . The elements of $r(b)$ (resp. $\ell(b)$) is the simultaneous-crossings with b such that these simultaneous-crossings are located on the right (resp. left) side of the simultaneous-crossing of b .

Let D be a link diagram. For an intersection point $p \in \text{int } S$ of D , we define link diagrams D_0^p and D_∞^p by eliminating the crossing at p as illustrated in Figure 4.8. The crossing data on the intersection points of D_0 and D_∞ are naturally induced from the crossing data of D . We assume that the crossing data $D^{-1}(m)$ on $m \in M$ has the order

$$\cdots < a_1 = a_2 = \cdots = a_i < b = b_1 = b_2 = \cdots = b_j < c_1 = c_2 = \cdots = c_k < \cdots.$$

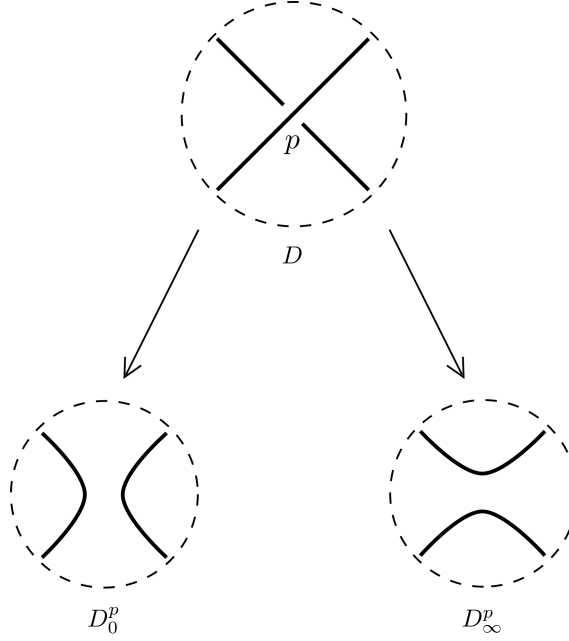
Then we define link diagrams D_+^b and D_-^b by changing the order of $D^{-1}(m)$. The link diagrams D_+^b and D_-^b have the same underlying immersion with one of D . The crossing data of $(D_+^b)^{-1}(m)$ is

$$\cdots < a_1 = a_2 = \cdots = a_i < b_1 = b_2 = \cdots = b_j < b < c_1 = c_2 = \cdots = c_k < \cdots$$

and the crossing data of $(D_-^b)^{-1}(m)$

$$\cdots < a_1 = a_2 = \cdots = a_i < b < b_1 = b_2 = \cdots = b_j < c_1 = c_2 = \cdots = c_k < \cdots.$$

DEFINITION 5.6. *The skein algebra $\text{Sk}_q(\Sigma)$ on Σ is the quotient of $\mathbf{K}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]\mathcal{L}(\Sigma)$ by the ideal generated by the following relations:*

FIGURE 4.8. link diagrams D_0^p and D_∞^p

- (1) $[D] = q[D_0^p] + q^{-1}[D_\infty^p]$ for any link diagram D and its intersection point p in $\text{int } S$ (the Kauffman bracket skein relation),
- (2) $q^{\frac{1}{2}(\#\ell(b) - \#r(b))} [D_-^b] = [D] = q^{\frac{1}{2}(\#r(b) - \#\ell(b))} [D_+^b]$ for any link diagram D and its intersection point m in M (the boundary skein relation),
- (3) $[\mathcal{O}] = -(q^2 + q^{-2}) [\emptyset]$ where $\mathcal{O}$ is the curl on $\text{int } S$,
- (4) $[\mathcal{O}_m] = 0$ for any m in M where $\mathcal{O}_m$ is the monogon with vertex m .

We remark that the ideal generated by the relation $A = B$ means that the ideal generated by $A - B$.

We prepare a Poisson algebra whose quantization is $\text{Sk}_q(\Sigma)$. The Poisson algebra denoted by $\text{Sk}(\Sigma)$ is obtained from $\overline{\text{PS}}(\Sigma)^+|_{T=-1}$.

DEFINITION 5.7. The Poisson $\mathbf{K}$ -algebra $\text{Sk}(\Sigma)$ is the quotient of the algebra $\overline{\text{S}}(\Sigma)^+|_{T=-1}$ by the ideal generated by the following relations:

- (1) $|D| = |D_0^p| + |D_\infty^p|$ for any underlying immersion of a link diagram D and its intersection point $p \in \text{int } S$,
- (2) $|\mathcal{O}| = -2$.

We remark that the relation (1) is independent of the crossing data of D . We can easily confirm that the ideal is the Poisson ideal of $\overline{\text{PS}}(\Sigma)^+|_{T=-1}$. Therefore, we can obtain a Poisson structure on $\text{Sk}(\Sigma)$.

We calculate the Poisson bracket on $\text{Sk}(\Sigma)$ according to the definition of Poisson bracket (5.1). Let us denote by $|D|$ the element of $\text{Sk}(\Sigma)$ represented by generic curves D . Let $\{a, b\} \subset \overline{\text{Curves}}(\Sigma)$ be generic curves. We assume that the intersection points of a and

b in $\text{int } S$ is $\{p_1, p_2, \dots, p_n\}$. Then,

$$\begin{aligned} |E_{p_i}(a, b)| &= \sum_{X \in \{E, e\}} |X_1 X_2 \cdots X_{i-1} E_i X_{i+1} X_{i+2} \cdots X_n(a, b)|, \\ |e_{p_i}(a, b)| &= \sum_{X \in \{E, e\}} |X_1 X_2 \cdots X_{i-1} e_i X_{i+1} X_{i+2} \cdots X_n(a, b)|, \\ |a| |b| &= \sum_{X \in \{E, e\}} |X_1 X_2 \cdots X_n(a, b)|. \end{aligned}$$

Therefore,

$$\begin{aligned} \{|a|, |b|\} &= \sum_{i=1}^n \sum_{X \in \{E, e\}} |X_1 \cdots X_{i-1} E_i X_{i+1} \cdots X_n(a, b)| - |X_1 \cdots X_{i-1} e_i X_{i+1} \cdots X_n(a, b)| \\ &\quad + \sum_{X \in \{E, e\}} \frac{1}{2} (n_+(a, b) - n_-(a, b)) |X_1 X_2 \cdots X_n(a, b)|. \end{aligned}$$

We observe a correspondence between the set of symbols $X = \{X_1 X_2 \cdots X_n \mid X = E, e\}$ and $E = \bigsqcup_{i=1}^n \{X_1 X_2 \cdots X_{i-1} E_i X_{i+1} \cdots X_n \mid X = E, e\}$. Fix an element $Y_1 Y_2 \cdots Y_n \in X$, then the number of elements in E which coincide with $Y_1 Y_2 \cdots Y_n$ is the number of symbols E_i appearing in $Y_1 Y_2 \cdots Y_n$. Consequently, we can describe the bracket of $\text{Sk}(\Sigma)$ as the following,

$$\begin{aligned} \{|a|, |b|\} &= \sum_{X \in \{E, e\}} (\#\{X_i = E_i\} - \#\{X_i = e_i\}) |X_1 X_2 \cdots X_n(a, b)| \\ (5.8) \quad &\quad + \sum_{X \in \{E, e\}} \frac{1}{2} (n_+(a, b) - n_-(a, b)) |X_1 X_2 \cdots X_n(a, b)| \end{aligned}$$

where $\#\{X_i = E_i\} = \{i \in \{1, 2, \dots, n\} \mid X_i = E_i\}$ and $\#\{X_i = e_i\} = \{i \in \{1, 2, \dots, n\} \mid X_i = e_i\}$.

Let $\mathbb{C}[[h]]$ be the ring of complex formal power series in one variable h . We consider the skein algebra of Σ with coefficient in $\mathbb{C}[[h]]$, that is, the quotient of $\mathbb{C}[[h]] \cdot \mathcal{L}(\Sigma)^+$ such that its skein relations are obtained by substituting $q = \exp(h/2)$. We denote by $\mathcal{K}_h(\Sigma)$ the skein algebra with coefficient in $\mathbb{C}[[h]]$. We also use the same symbol as a framed link $[D]$ to represent the element in $\mathcal{K}_h(\Sigma)$.

A *simple multicurve* D is an immersion of generic curves on Σ with no intersection points in $\text{int } S$, no curls and no monogons with vertices in M . We define the framed link of the simple multicurve D as a framed link $[D]$ with only simultaneous-crossings. We denote by $\text{SMulti}(\Sigma)$ the set of links of simple multicurves on Σ .

LEMMA 5.8 (Muller [Mul12] Lemma 4.1.). The skein algebra $\mathcal{K}_h(\Sigma)$ is a topologically free module.

PROOF. We construct a $\mathbb{C}[[h]]$ -linear map from $\mathcal{K}_h(\Sigma)$ to the set of formal power series with coefficient in $\mathbb{C}\text{SMulti}(\Sigma)$ where $\mathbb{C}\text{SMulti}(\Sigma)$ is the complex vector space spanned by $\text{SMulti}(\Sigma)$. For any link diagram D , we expand $[D]$ in a series of links of simple multicurves in $\mathcal{K}_h(\Sigma)$. First, we can eliminate intersection points in $\text{int } S$ of D by using the Kauffman bracket skein relation. Next, the crossings of D on M can be deformed into simultaneous-crossings by using the boundary skein relation. Finally, we vanish curls and monogons by relation (3) and (4) in Definition 5.6. Consequently, $[D]$ can be expanded

in a formal power series with coefficient in $\mathbb{C}\text{SMulti}(\Sigma)$. The local moves of link diagrams is realized by the skein relation and the boundary skein relation;

$$\begin{aligned}
\text{Diagram 1} &= -q^3 \text{Diagram 2}, \\
\text{Diagram 3} &= -q^{-3} \text{Diagram 4}, \\
\text{Diagram 5} &= q \text{Diagram 6} + q^{-1} \text{Diagram 7} \\
&= q^2 \text{Diagram 8} + \text{Diagram 9} + \text{Diagram 10} + q^{-2} \text{Diagram 11} \\
&= q^2 \text{Diagram 12} - (q^2 + q^{-2}) \text{Diagram 13} + \text{Diagram 14} + q^{-2} \text{Diagram 15} \\
&= \text{Diagram 16}, \\
\text{Diagram 17} &= q \text{Diagram 18} + q^{-1} \text{Diagram 19} \\
&= q \text{Diagram 20} + q^{-1} \text{Diagram 21} \\
&= q \text{Diagram 22} + q^{-1} \text{Diagram 23} \\
&= \text{Diagram 24}, \\
\text{Diagram 25} &= q \text{Diagram 26} + q^{-1} \text{Diagram 27} \\
&= q \text{Diagram 28} \\
&= \text{Diagram 29}.
\end{aligned}$$

Therefore, we get the same expansion of $[D]$ for another link diagram D' which is a representative of the link $[D]$.

We can describe any element $f \in \mathcal{K}_h(\Sigma)$ as

$$f = \sum_{n \geq 0} \left(\sum_{D \in \mathcal{D}(\Sigma)} a_D^{(n)} [D] \right) h^n$$

where $a_D^{(n)} \in \mathbb{C}$. By expanding $[D]$ as the above, the element in $\mathbb{C}\text{SMulti}(\Sigma)[[h]]$ is uniquely determined. We denote by $\Phi: \mathcal{K}_h(\Sigma) \rightarrow \mathbb{C}\text{SMulti}(\Sigma)[[h]]$ the $\mathbb{C}[[h]]$ -linear extension of the above map where $\mathbb{C}\text{SMulti}(\Sigma)[[h]]$ is the formal power series with coefficient in $\mathbb{C}\text{SMulti}(\Sigma)$. The inverse map $\Psi: \mathbb{C}\text{SMulti}(\Sigma)[[h]] \rightarrow \mathcal{K}_h(\Sigma)$ is induced by the composed map of the inclusion $\text{SMulti}(\Sigma)$ into $\mathcal{L}(\Sigma)^+$ and the projection of $\mathcal{L}(\Sigma)^+$ to $\mathcal{K}_h(\Sigma)$. It is clear that $\Phi \circ \Psi = \text{id}_{\mathcal{K}_h(\Sigma)}$ and $\Psi \circ \Phi = \text{id}_{\mathbb{C}\text{SMulti}(\Sigma)[[h]]}$. $\square$

THEOREM 5.9. The skein algebra $\mathcal{K}_h(\Sigma)$ gives a quantization of the Poisson algebra $\text{Sk}(\Sigma)$.

PROOF. We can construct the $\mathbb{C}$ -linear isomorphism $\text{Sk}(\Sigma) \rightarrow \text{CSMulti}(\Sigma)$ in the same way as the construction of the map Φ in the proof of Lemma 5.8. Therefore, $\mathcal{K}_h(\Sigma)$ is isomorphic to $\text{Sk}(\Sigma)[[h]]$ through $\text{CSMulti}(\Sigma)[[h]]$.

Comparing the relations of $\mathcal{K}_h(\Sigma)$ and $\text{Sk}(\Sigma)$, the algebra homomorphism $\tilde{p}: \mathcal{K}_h(\Sigma) \rightarrow \text{Sk}(\Sigma)$ is obtained by $h \mapsto 0$ and $[D] \mapsto |D|$. The algebra isomorphism $p: \mathcal{K}_h(\Sigma)/h\mathcal{K}_h(\Sigma) \rightarrow \text{Sk}(\Sigma)$ is induced by $\tilde{p}$ because $\ker \tilde{p} \cong \ker \tilde{p} \circ \Psi = h\text{CSMulti}(\Sigma)[[h]] \cong h\mathcal{K}_h(\Sigma)$. We need to confirm the equation $\{\tilde{p}(x), \tilde{p}(y)\} = \tilde{p}((xy - yx)/h)$ for any $x, y \in \mathcal{K}_h(\Sigma)$. For any link diagram $D \in \mathcal{D}(\Sigma)$, we can describe $[D] \in \mathcal{K}_h(\Sigma)$ as a sum of multiplications of loops and arcs by using the skein relation and the boundary skein relation. Therefore, we only have to calculate the commutator $\tilde{p}([a][b] - [b][a])/h$ such that a and b are generic curves in Σ . We define a link diagram $D(a, b)$ as a link diagram replacing over and under crossings of $a \cdot b$ with simultaneous-crossings at marked points. We denote the strands contained in the neighborhood of M of $D(a, b)$ by a_1, a_2, b_1 and b_2 where a_i (resp. b_i) is contained in a (resp. b) for $i = 1, 2$. Then we obtain the following equations by using boundary skein relation:

$$\begin{aligned} [a][b] &= [a \cdot b] = q^{\frac{1}{2}(\#\ell(a_1) - \#r(a_1))} q^{\frac{1}{2}(\#\ell(a_2) - \#r(a_2))} [D(a, b)], \\ [b][a] &= [b \cdot a] = q^{\frac{1}{2}(\#r(a_1) - \#\ell(a_1))} q^{\frac{1}{2}(\#r(a_2) - \#\ell(a_2))} [D(b, a)]. \end{aligned}$$

We denote the intersection points of a and b in $\text{int } S$ by $\{p_1, p_2, \dots, p_n\}$. Then we eliminate the crossings at p_i of $D(a, b)$ and $D(b, a)$ by using the skein relation;

$$\begin{aligned} [D(a, b)] &= \sum_{\varepsilon_i \in \{0, \infty\}} q^{\#\{\varepsilon_i=0\} - \#\{\varepsilon_i=\infty\}} [(\dots ((D(a, b)_{\varepsilon_1}^{p_1})_{\varepsilon_2}^{p_2}) \dots)_{\varepsilon_n}^{p_n}], \\ [D(b, a)] &= \sum_{\varepsilon'_i \in \{0, \infty\}} q^{\#\{\varepsilon'_i=0\} - \#\{\varepsilon'_i=\infty\}} [(\dots ((D(b, a)_{\varepsilon'_1}^{p_1})_{\varepsilon'_2}^{p_2}) \dots)_{\varepsilon'_n}^{p_n}]. \end{aligned}$$

The elimination $\varepsilon_i = 0$ (resp. $\varepsilon_i = \infty$) of p_i of $D(a, b)$ coincides with the elimination $\varepsilon'_i = \infty$ (resp. $\varepsilon'_i = 0$) of p_i of $D(b, a)$ at the neighborhood of p_i . Therefore,

$$(5.9) \quad [a][b] - [b][a] = \sum_{\varepsilon_i \in \{0, \infty\}} (q^{\varepsilon + \frac{1}{2}N} - q^{-\varepsilon - \frac{1}{2}N}) [(\dots ((D(a, b)_{\varepsilon_1}^{p_1})_{\varepsilon_2}^{p_2}) \dots)_{\varepsilon_n}^{p_n}]$$

where $\varepsilon = \#\{\varepsilon_i = 0\} - \#\{\varepsilon_i = \infty\}$ and $N = \sum_{i=1}^2 \#\{\ell(a_i)\} - \#\{r(a_i)\}$. We calculate (5.9) by substituting $q = \exp(h/2)$ modulo $h^2 \mathcal{K}_h(\Sigma)$. Then,

$$[a][b] - [b][a] = \sum_{\varepsilon_i \in \{0, \infty\}} \left(\varepsilon + \frac{N}{2} \right) h [(\dots ((D(a, b)_{\varepsilon_1}^{p_1})_{\varepsilon_2}^{p_2}) \dots)_{\varepsilon_n}^{p_n}] \pmod{h^2 \mathcal{K}_h(\Sigma)}.$$

Considering underlying immersion of $[(\dots ((D(a, b)_{\varepsilon_1}^{p_1})_{\varepsilon_2}^{p_2}) \dots)_{\varepsilon_n}^{p_n}]$, we get

$$\tilde{p}([(\dots ((D(a, b)_{\varepsilon_1}^{p_1})_{\varepsilon_2}^{p_2}) \dots)_{\varepsilon_n}^{p_n}]) = |X_{p_n} \cdots X_{p_2} X_{p_1}(a, b)|$$

where $X_{p_i} = E_{p_i}$ if $\varepsilon_i = 0$ and $X_{p_i} = e_{p_i}$ if $\varepsilon_i = \infty$. We can also show that

$$\varepsilon = \#\{X_{p_i} = E_{p_i}\} - \#\{X_{p_i} = e_{p_i}\}$$

and

$$N = n_+(a, b) - n_-(a, b).$$

Therefore,

$$\begin{aligned}
 \tilde{p}\left(\frac{[a][b] - [b][a]}{h}\right) &= \sum_{\mathbf{X} \in \{\mathbf{E}, \mathbf{e}\}} (\#\{\mathbf{X}_i = \mathbf{E}_i\} - \#\{\mathbf{X}_i = \mathbf{e}_i\}) |\mathbf{X}_1 \mathbf{X}_2 \cdots \mathbf{X}_n(a, b)| \\
 &\quad + \sum_{\mathbf{X} \in \{\mathbf{E}, \mathbf{e}\}} \frac{1}{2} (n_+(a, b) - n_-(a, b)) |\mathbf{X}_1 \mathbf{X}_2 \cdots \mathbf{X}_n(a, b)| \\
 &= \{ |a|, |b| \} \quad \text{by (5.8).}
 \end{aligned}$$

Thus, the map $\tilde{p}$ induces an isomorphism of Poisson algebras $\mathcal{K}_h(\Sigma)/h\mathcal{K}_h(\Sigma)$ and $\text{Sk}(\Sigma)$. $\square$

Appendix

1. Goldman Lie algebras and finite Galois coverings

In this section, we describe a relationship between Goldman Lie algebras of finite Galois coverings. More precisely, the action of the covering transformation group Γ of a finite unbranched Galois covering $\tilde{S} \rightarrow S$ on the total space $\tilde{S}$ induces an action of Γ on $\mathbf{K}\hat{\pi}(\tilde{S})$. Then the Γ -invariant part $\mathbf{K}\hat{\pi}(\tilde{S})^\Gamma$ is embedded in the Goldman Lie algebra $\mathbf{K}\hat{\pi}(S)$ of the base space.

Let $\tilde{S}$ and S be connected oriented smooth surfaces. An orientation preserving self-diffeomorphism γ of $\tilde{S}$ induces a bijective map from $a \cap b$ to $\gamma a \cap \gamma b$ where a and b are generic immersed loops on $\tilde{S}$. The local intersection number of a and b at p agrees with that of γa and γb at $\gamma(p)$ because γ preserves orientation of $\tilde{S}$. Therefore we can obtain an automorphism γ_* of the Goldman Lie algebra $\mathbf{K}\hat{\pi}(\tilde{S})$ given by $\gamma_*(|x|) = |\gamma x|$ for any x in $\pi_1(\tilde{S})$.

Let $f: \tilde{S} \rightarrow S$ be a finite Galois covering with no branched points and Γ the covering transformation group of it. We know that Γ acts on $\mathbf{K}\hat{\pi}(\tilde{S})$ from the previous discussion. Let $\mathbf{K}\hat{\pi}(\tilde{S})^\Gamma$ denote the Γ -invariant part of $\mathbf{K}\hat{\pi}(\tilde{S})$.

LEMMA 1.1. $\mathbf{K}\hat{\pi}(\tilde{S})^\Gamma$ is a Lie subalgebra of $\mathbf{K}\hat{\pi}(\tilde{S})$.

PROOF. $\gamma_*[v, w] = [\gamma_*v, \gamma_*w] = [v, w]$ for any v and w in $\mathbf{K}\hat{\pi}(\tilde{S})^\Gamma$ and γ in Γ . Therefore $\mathbf{K}\hat{\pi}(\tilde{S})^\Gamma$ is a Lie subalgebra of $\mathbf{K}\hat{\pi}(\tilde{S})$. $\square$

We remark that the Lie subalgebra $\mathbf{K}\hat{\pi}(\tilde{S})^\Gamma$ is generated by the set $\Gamma\hat{\pi}(\tilde{S}) = \{\Gamma(a) \mid a \in \pi_1(\tilde{S})\}$ where $\Gamma(a)$ denotes $\sum_{\gamma \in \Gamma} |\gamma a|$. We denote the vector subspace by $\mathbb{C}\Gamma\hat{\pi}(\tilde{S})$. If $u = \sum_{a \in \pi_1(\tilde{S})} r_a |a|$ is an element in $\mathbf{K}\hat{\pi}(\tilde{S})^\Gamma$, then $u = \sum_{a \in \pi_1(\tilde{S})} (r_a / \#\Gamma) \Gamma(a)$ where $\#\Gamma$ is the number of elements in Γ . Therefore the vector subspace generated by $\Gamma\hat{\pi}(\tilde{S})$ includes $\mathbf{K}\hat{\pi}(\tilde{S})^\Gamma$. The reverse inclusion is obvious. Consequently, $\mathbf{K}\hat{\pi}(\tilde{S})^\Gamma$ is isomorphic to $\mathbb{C}\Gamma\hat{\pi}(\tilde{S})$. If we take the coefficient of the Goldman Lie algebra in $\mathbb{Z}$, then we cannot show the isomorphism.

We define a linear map $\hat{f}: \mathbf{K}\hat{\pi}(\tilde{S})^\Gamma \rightarrow \mathbf{K}\hat{\pi}(S)$ by $\hat{f}(\Gamma(a)) = |f_\#(a)|$ where $f_\#: \pi_1(\tilde{S}) \rightarrow \pi_1(S)$ is the injective homomorphism induced by f .

LEMMA 1.2. $\mathbf{K}\hat{\pi}(\tilde{S})^\Gamma$ is the free vector space on $\Gamma\hat{\pi}(\tilde{S})$.

PROOF. If $\hat{f}(\Gamma(a)) = \hat{f}(\Gamma(b))$, that is $|f_\#(a)| = |f_\#(b)|$, hold for any a and b in $\pi_1(\tilde{S})$, then there exist γ in Γ which satisfies $|a| = |\gamma b|$. We obtain $\Gamma(a) = \Gamma(b)$. This implies that $|f_\#(a)|$ is not freely homotopic to $|f_\#(b)|$ if $\Gamma(a) \neq \Gamma(b)$. Let $s_1\Gamma(a_1) + s_2\Gamma(a_2) + \dots + s_k\Gamma(a_k) = 0$ for all $s_1, \dots, s_k \in \mathbb{C}$ where $\Gamma(a_1), \dots, \Gamma(a_k)$ are distinct elements in $\Gamma\hat{\pi}(\tilde{S})$. Then $s_1|f_\#(a_1)| + \dots + s_k|f_\#(a_k)| = 0$ holds in $\mathbf{K}\hat{\pi}(S)$. Therefore $s_1 = \dots = s_k = 0$ $\square$

THEOREM 1.3. $\hat{f}: \mathbf{K}\hat{\pi}(\tilde{S})^\Gamma \rightarrow \mathbf{K}\hat{\pi}(S)$ is an injective Lie algebra homomorphism.

PROOF. We can see that $\hat{f}: \Gamma\hat{\pi}(\tilde{S}) \rightarrow \hat{\pi}(S)$ is injective because of the proof of Lemma 1.2. We show that $\hat{f}$ is a Lie algebra homomorphism. For any a and b in $\pi_1(\tilde{S})$,

$$[\Gamma(a), \Gamma(b)] = \sum_{\gamma, \gamma' \in \Gamma} [|\gamma'a|, |\gamma b|] = \sum_{\gamma, \gamma' \in \Gamma} \gamma'_* [|a|, |\gamma'^{-1}\gamma b|] = \sum_{\gamma, \gamma' \in \Gamma} \gamma'_* [|a|, |\gamma b|].$$

Therefore we obtain that

$$\hat{f}([\Gamma(a), \Gamma(b)]) = \sum_{\gamma \in \Gamma} f_*([|a|, |\gamma b|])$$

where $f_*: \mathbf{K}\hat{\pi}(\tilde{S}) \rightarrow \mathbf{K}\hat{\pi}(S)$ is the induced linear map obtained from f . We take generic immersed loops $|a|$ and $|b|$ as representatives of a and b respectively. Let $H: (S^1 \sqcup S^1) \times [0, 1] \rightarrow S$ be a homotopy such that

$$H(\cdot, 0) = f \circ |a| \sqcup f \circ |b|, H(\cdot, 1) = a' \sqcup b'$$

where $a' \sqcup b'$ is generic immersion. We remark that $|a| \sqcup \gamma \circ |b|$ is a lift of $f \circ |a| \sqcup f \circ |b|$ for each $\gamma \in \Gamma$. We can obtain a lift of H such that

$$\tilde{H}_\gamma(\cdot, 0) = |a| \sqcup \gamma \circ |b|, \tilde{H}_\gamma(\cdot, 1) = \tilde{a}' \sqcup \tilde{b}'_\gamma$$

because of the homotopy lifting property. $a' \sqcup b'$ intersects tangentially if $\tilde{a}' \sqcup \tilde{b}'_\gamma$ intersects tangentially. $a' \sqcup b'$ has triple points if $\tilde{a}' \sqcup \tilde{b}'_\gamma$ has triple points. Therefore $\tilde{a}' \sqcup \tilde{b}'_\gamma$ is generic immersion. Next, we observe a correspondence between intersection points of $a' \sqcup b'$ and $\tilde{a}' \sqcup \tilde{b}'_\gamma$. We show that there uniquely exists a lift of intersection point p of a' and b' on $\tilde{a}' \cap \tilde{b}'_\gamma$. We take a lift $\tilde{p}$ on $\tilde{a}'$ for each intersection point p of a' and b' . There is γ in Γ such that $\tilde{p}$ is on $\tilde{b}'_\gamma$ because the action of Γ is transitive. If there is another lift on $\tilde{a}'$, then p is a triple (or more multiple) point. Therefore the lift $\tilde{p}$ is uniquely determined. There is no another lift of b' such that $\tilde{p}$ is on it for the same reason. Accordingly, we obtain a bijective map between $\cup_{\gamma \in \Gamma} \tilde{a}' \cap \tilde{b}'_\gamma$ and $a' \cap b'$ by restricting f . The following equality holds because f is an orientation preserving local diffeomorphism.

$$f \circ (\tilde{a}'_\gamma \tilde{b}'_{\gamma\tilde{p}}) = a'_p b'_p, \varepsilon(\tilde{p}; \tilde{a}', \tilde{b}'_\gamma) = \varepsilon(p; a', b').$$

Therefore

$$\begin{aligned} \hat{f}[\Gamma(a), \Gamma(b)] &= \sum_{\gamma \in \Gamma} f_*([|a|, |\gamma b|]) \\ &= \sum_{\gamma \in \Gamma} f_*([|\tilde{a}'|, |\tilde{b}'_\gamma|]) \\ &= \sum_{\gamma \in \Gamma} f_* \left(\sum_{\tilde{p} \in \tilde{a}' \cap \tilde{b}'_\gamma} \varepsilon(\tilde{p}; \tilde{a}', \tilde{b}'_\gamma) |\tilde{a}'_\gamma \tilde{b}'_{\gamma\tilde{p}}| \right) \\ &= \sum_{\tilde{p} \in \cup_{\gamma \in \Gamma} \tilde{a}' \cap \tilde{b}'_\gamma} \varepsilon(\tilde{p}; \tilde{a}', \tilde{b}'_\gamma) |f \circ (\tilde{a}'_\gamma \tilde{b}'_{\gamma\tilde{p}})| \\ &= \sum_{p \in a' \cap b'} \varepsilon(p; a', b') |a'_p b'_p| \\ &= [a', b'] = [f_\#(a), f_\#(b)] = [\hat{f}(\Gamma(a)), \hat{f}(\Gamma(b))]. \end{aligned}$$

From the above, $\hat{f}: \mathbf{K}\hat{\pi}(\tilde{S})^\Gamma \rightarrow \mathbf{K}\hat{\pi}(S)$ is a homomorphism. $\square$

QUESTION 1.4. Is Theorem 1.3 true for an orbifold covering space $f: (\tilde{S}, \Sigma\tilde{S}) \rightarrow (S, \Sigma S)$ such that $f|_{\tilde{S}_*}: \tilde{S}_* \rightarrow S_*$ is a finite Galois covering space.

2. bi-Poisson bialgebra

In this section, we review the definition of a (bi-, co-)Poisson (bi-, co-)algebra.

Let R be a commutative ring with unit and S a module over R . We assume that S is a commutative associative algebra equipped with the multiplication $m: S \otimes S \rightarrow S$ and the unit $u: R \rightarrow S$. Moreover, S equips a linear map $\nabla = \{\cdot, \cdot\}: S \otimes S \rightarrow S$. The symbol $\otimes$ denotes the tensor product over R in this section.

DEFINITION 2.1. The pair (S, ∇) is a Poisson algebra if it satisfies the following conditions.

- (1) $\nabla \circ \sigma_{1,2} = -\nabla$ (the skew symmetry),
- (2) $\nabla \circ (\nabla \otimes \text{id}) \circ (\tau^2 + \tau + \text{id}_{S \otimes S}) = 0$ (the Jacobi identity),
- (3) $\nabla \circ (\text{id} \otimes m) = m \circ (\nabla \otimes \text{id} + (\text{id} \otimes \nabla) \circ \sigma_{1,2})$ (the Leibniz rule)

where $\sigma_{i,j}$ is a linear operator which exchanges i -th and j -th tensor components and $\tau = \sigma_{2,3} \circ \sigma_{1,2}$. We call the linear map ∇ the Poisson bracket. We can describe above conditions explicitly as the following. For any a, b, c in S

- (1) $\{a, b\} = -\{b, a\}$,
- (2) $\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0$,
- (3) $\{a, bc\} = \{a, b\}c + b\{a, c\}$.

EXAMPLE 2.2. The symmetric algebra $\text{Sym}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ over R is a Poisson algebra. By using the Leibniz rule, the Lie bracket of $\mathfrak{g}$ is extended to the Poisson bracket on $\text{Sym}(\mathfrak{g})$.

We assume that S is a cocommutative coassociative coalgebra equipped with the comultiplication $\Delta: S \rightarrow S \otimes S$ and the counit $\epsilon: S \rightarrow R$. Moreover, S equips a linear map $\delta: S \rightarrow S \otimes S$.

DEFINITION 2.3. The pair (S, δ) is a co-Poisson coalgebra if it satisfies the following conditions.

- (4) $\sigma_{1,2} \circ \delta = -\delta$ (the co-skew symmetry),
- (5) $(\tau^2 + \tau + \text{id}_{S \otimes S}) \circ (\text{id} \otimes \delta) \circ \delta = 0$ (the co-Jacobi identity),
- (6) $(\text{id} \otimes \Delta) \circ \delta = (\delta \otimes \text{id} + \sigma_{1,2} \circ (\text{id} \otimes \delta)) \circ \Delta$ (the co-Leibniz rule).

We call the linear map δ the Poisson cobracket. The definition of a co-Poisson coalgebra is the dual of a Poisson algebra.

DEFINITION 2.4. A bialgebra $(S, m, u, \Delta, \epsilon)$ is a bi-Poisson bialgebra if S equips the above linear maps ∇ and δ which satisfy condition (1)-(6) and the following compatibility conditions.

- (7) $\Delta \circ \nabla = \nabla_{S \otimes S} \circ \sigma_{2,3} \circ \Delta_{S \otimes S}$ (the (Δ, ∇) -compatibility),
- (8) $\delta \circ m = m_{S \otimes S} \circ \sigma_{2,3} \circ \delta_{S \otimes S}$ (the (m, δ) -compatibility),
- (9) $\delta \circ \nabla = \nabla_{S \otimes S} \circ \sigma_{2,3} \circ \delta_{S \otimes S}$ (the (∇, δ) -compatibility)

where $m_{S \otimes S} = (m \otimes m) \circ \sigma_{2,3}$, $\Delta_{S \otimes S} = \sigma_{2,3} \circ (\Delta \otimes \Delta)$, $\nabla_{S \otimes S} = (\nabla \otimes m + m \otimes \nabla) \circ \sigma_{2,3}$ and $\delta_{S \otimes S} = \sigma_{2,3} \circ (\delta \otimes \Delta + \Delta \otimes \delta)$.

EXAMPLE 2.5. The symmetric algebra $\text{Sym}(\mathfrak{g})$ of a Lie coalgebra $(\mathfrak{g}, \delta)$ over R is a co-Poisson bialgebra. The comultiplication of $\text{Sym}(\mathfrak{g})$ is defined by extending $\Delta(x) = x \otimes 1 + 1 \otimes x$ for any x in $\mathfrak{g}$ as an algebra homomorphism. By using the (m, δ) -compatibility, the Lie cobracket of $\mathfrak{g}$ is extended to the Poisson cobracket on $\text{Sym}(\mathfrak{g})$.

Part 2

Graphical calculus for the $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ colored Jones polynomial

The $\mathfrak{sl}_3$ colored Jones polynomials for 2-bridge links

1. Introduction

After Kauffman [Kau87] gave a reformulation of the Jones polynomial [Jon85] of a link using the Kauffman bracket, the linear skein theory has developed in various directions. This theory is related to quantum representations and treats modules whose elements are diagrams in a disk such as graphs, tangles, webs and so on. Lickorish [Lic91, Lic93, Lic92] introduced the linear skein theory based on quantum $\mathfrak{sl}_2$ representations. He constructed the quantum $SU(2)$ invariants of closed 3-manifolds suggested by Witten [Wit89], rigorously defined by Reshetikhin and Turaev [RT91]. This linear skein theory is developed based on the Kauffman bracket [Kau87] and the Kauffman bracket skein module. The Kauffman bracket also gives polynomial invariants of knots and links called the colored Jones polynomials. (We call it the $\mathfrak{sl}_2$ colored Jones polynomials in this paper.) Through the linear skein theory for the Kauffman bracket, we can define and calculate quantum $SU(2)$ invariants of closed 3-manifolds and links graphically (see details in [KL94b], Chapter 13, 14 of [Lic97]).

Many quantum invariants and corresponding skein theory have been constructed for other quantum groups. Kuperberg [Kup94] defined brackets for Lie algebras A_2 , G_2 and C_2 and corresponding quantum invariants of regular isotopy classes of link diagrams. He also introduced web spaces for simple Lie algebra of rank 2 in [Kup96]. Web spaces are generalizations of the Kauffman bracket skein module. The Kauffman bracket skein module corresponds to the A_1 web space. The linear skein theory associated with A_n was also introduced by Murakami, Ohtsuki and Yamada [MOY98] and Sikora [Sik05]. It gives a reformulation of the HOMFLY polynomial [FYH⁺85, PT88]. The quantum invariant of 3-manifolds was also given by using these linear skein theories in a similar approach to Lickorish [Lic91, Lic93] (see Ohtsuki and Yamada [OY97] for $SU(3)$ and Yokota [Yok97] for $SU(n)$).

We will treat the skein theory for A_1 and A_2 . The A_1 web spaces have particular elements called the Jones-Wenzl idempotents defined in [Jon83, Wen87]. The Jones-Wenzl idempotents play an important role to construct the quantum $SU(2)$ invariants of 3-manifolds and the $\mathfrak{sl}_2$ colored Jones polynomials of knots and links. These idempotents are generalized to the A_2 case (see [Kup96] and [OY97]). We call them the A_2 clasps. The A_2 clasps also play an important role to construct quantum $SU(3)$ invariants of 3-manifolds and the $\mathfrak{sl}_3$ colored Jones polynomials of knots and links. Many people have calculated clasped A_1 web spaces and gave explicit formulas of $\mathfrak{sl}_2$ colored Jones polynomials for some knots and links by graphical calculus. These explicit formulas are useful for case studies of conjectures related to quantum invariants of knots and links: the volume conjecture [Kas97, MM01], the AJ conjecture [Gar04], the slope conjecture [Gar11] and so on.

On the other hand, there are few examples, see Kim [Kim06, Kim07], of graphical calculus for clasped A_2 web spaces. Only two explicit formulas of the $\mathfrak{sl}_3$ colored Jones polynomial were obtained using the representation theory of quantum groups. The formula for trefoil knot was given by Lawrence [Law03], more generally for the torus knot $T(2, k)$ was given by Garoufalidis, Morton and Vuong [GMV13]. As far as the author knows, there is no example of graphical calculus of the $\mathfrak{sl}_3$ colored Jones polynomial for a non-trivial link.

In this paper, we will give some formulas for clasped A_1 and A_2 web spaces. These formulas explicitly give the $\mathfrak{sl}_2$ colored Jones polynomials for a 2-bridge link and the $\mathfrak{sl}_3$ colored Jones polynomials of type $(n, 0)$ for it. We remark that the $\mathfrak{sl}_3$ colored Jones polynomial treated in [Law03] and [GMV13] is type (n, m) .

The paper is organized as follows. We firstly introduce the A_1 and A_2 web spaces and clasps based on Kuperberg [Kup96] in Section 2. The colored skein relations and twist formulas for A_1 and A_2 web spaces are given in Section 3. The A_1 web space is the Kauffman bracket skein module. In this case, Hajij gave the colored Kauffman bracket skein relation in [Haj14b]. We will show it in another method using a lemma from the theory of integer partitions. This method is used to show the colored A_2 bracket skein relations and twist formulas. In Section 4, we will give the A_2 bracket bubble skein expansion formula. This formula is an A_2 version of the Kauffman bracket bubble skein expansion formula in [Haj14a]. In Section 5, we give an explicit formulas for the $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ colored Jones polynomials for 2-bridge knots and links. In this paper, we only treat the $\mathfrak{sl}_3$ colored Jones polynomials of type $(n, 0)$.

2. Preliminaries

In this section, we review definitions of two vector spaces, A_1 web spaces and A_2 web spaces, introduced by Kuperberg [Kup96]. For each web space, we inductively define a particular element called a clasp according to [Kup96, OY97].

2.1. Quantum integers and q -Pochhammer symbol. First, we organize notations of quantum integers and the q -Pochhammer symbol which we use in this paper.

Let k be an integer and q an indeterminate. We denote $q^{\frac{k}{2}} - q^{-\frac{k}{2}}$ by $\{k\}$. A quantum integer is defined by

$$[k] = \frac{\{k\}}{\{1\}}.$$

Let us denote $\{k\}! = \prod_{l=1}^k \{l\}$ and $[k]! = \prod_{l=1}^k [l]$. Let n be a non-negative integer. Then, a version of q -binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]! [n-k]!} = \frac{\{n\}!}{\{k\}! \{n-k\}!}$$

for $k \leq n$. If $k > n$, we define it by 0.

A q -Pochhammer symbol is defined by

$$(q; q)_k = \prod_{l=1}^k (1 - q^l).$$

We sometimes abbreviate it as $(q)_k$. Another version of q -binomial coefficient is defined by

$$\binom{n}{k}_q = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}$$

for $k \leq n$. If $k > n$, we define it by 0. We also define a q -multinomial coefficient as

$$\binom{n}{n_1, n_2, \dots, n_m}_q = \frac{(q)_n}{(q)_{n_1} (q)_{n_2} \cdots (q)_{n_m}},$$

where $n_1, n_2, \dots, n_m$ are non-negative integers such that $n_1 + n_2 + \cdots + n_m = n$.

It is easy to show the following transformation formulas:

- $\{k\}! = (-1)^k q^{-k(k+1)/4} (q; q)_k$,
- $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] = q^{-(n-k)k/2} \binom{n}{k}_q$,
- $(q; q)_k = (-1)^k q'^{-\frac{1}{2}k(k+1)} (q'; q')_k$,
- $\binom{n}{k}_q = q'^{k^2 - nk} \binom{n}{k}_{q'}$,

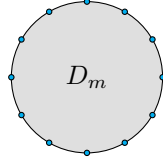
where $q' = q^{-1}$. We use the following formulas for quantum integers in this paper.

LEMMA 2.1. For any integers a, b and c ,

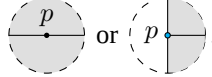
- (1) $[a][b] = \sum_{i=1}^a [a+b-(2i-1)] = \sum_{i=1}^b [a+b-(2i-1)]$,
- (2) $[a][b] - [a-c][b-c] = [a+b-c][c]$,
- (3) $[a][b-c] + [c][a-b] = [b][a-c]$

PROOF. By easy calculation using the definition of quantum integers. $\square$

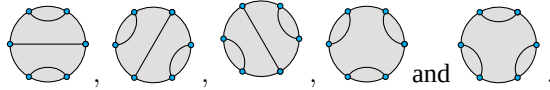
2.2. The A_1 web spaces. Let D_m denote the disk $\{z \in \mathbb{C} \mid |z| \leq 1\}$ with the set $E_m = \{\exp(2\pi\sqrt{-1}/m)^{j-1} \mid j = 1, 2, \dots, m\}$ of marked points on the boundary:



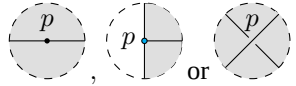
An A_1 basis web on D_m is the boundary-fixing isotopy class of a proper embedding of arcs into D_m with no intersection points such that each endpoint lies in E_m . For any point p on the arcs, its neighborhood is either



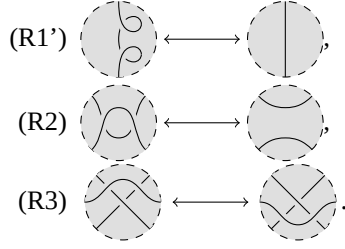
Let B_m be the set of A_1 basis webs on D_m . We consider B_0 has a single element $\emptyset$ called the empty disk. For example, B_6 consists of the following A_1 basis webs:



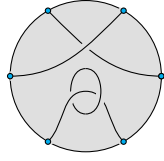
The A_1 web space W_m is the $\mathbb{Q}(q^{\frac{1}{4}})$ -vector space spanned by B_m , where $\mathbb{Q}(q^{\frac{1}{4}})$ is the field of rational functions in one variable $q^{\frac{1}{4}}$. We next define tangles in D_m and the A_1 bracket, also known as the Kauffman bracket. A *tangle diagram* in D_m is a proper immersion of 1-manifolds into D_m such that any intersection point is transverse double points with crossing data. For any point p on a tangle diagram, it has one of the following neighborhoods:



In particular, a tangle diagram in D_0 is a link diagram in the disk. A tangle diagram G is *regularly isotopic* to G' if G is obtained from G' by a finite sequence of boundary-fixing isotopies of D_m and the following moves:



In the above pictures, the outside of the left tangle diagram is coincide with the outside of the right tangle diagram. A *tangle* is the regular isotopy class of a tangle diagram and T_m denotes the set of tangles in D_m . The diagram below is an example of a tangle diagram in D_6 :



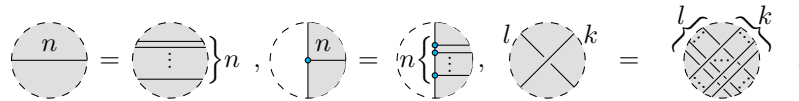
DEFINITION 2.2 (The Kauffman bracket). We define a $\mathbb{Q}(q^{\frac{1}{4}})$ -linear map $\langle \cdot \rangle_2 : \mathbb{Q}(q^{\frac{1}{4}})T_m \rightarrow W_m$ by the following.

- $\langle \text{crossing} \rangle_2 = q^{\frac{1}{4}} \langle \text{no crossing} \rangle_2 + q^{-\frac{1}{4}} \langle \text{no crossing} \rangle_2$,
- $\langle G \sqcup \text{loop} \rangle_2 = -[2] \langle G \rangle_2$,

where G is any tangle in T_m . We call this linear map the A_1 bracket or the Kauffman bracket.

For any tangle diagram, we can obtain a sum of tangles with no crossings by using the Kauffman bracket relation and eliminate trivial loops from these tangles. Therefore, we can define the map from the set of tangle diagrams to W_m . We can easily confirm that this map does not change under the moves of regular isotopy.

We next define A_1 clasps which also called *Jones-Wenzl projectors*, *magic elements* etc. We consider an A_1 web space $W_{n+n} = W_{2n}$. We use a tangle diagram whose components are decorated with non-negative integers. A components decorated by n means n parallelization of it. A tangle diagram with decoration is defined by the following local pictures:



We omit the boundary of the disk D_m from tangle diagrams in D_m later in the paper.

For example, a tangle  is denoted by $\times$.

We define an A_1 clasp $\langle \text{clasp}^n \rangle_2 \in W_{n+n}$ inductively by the following.

DEFINITION 2.3 (The A_1 clasps, The Jones-Wenzl idempotents etc.).

$$(2.1) \quad \begin{aligned} \left\langle -\left[\begin{array}{c} 1 \\ \hline \end{array} \right] \right\rangle_2 &= \frac{1}{1} \in W_{1+1} \\ \left\langle -\left[\begin{array}{c} n \\ \hline \end{array} \right] \right\rangle_2 &= \left\langle -\left[\begin{array}{c} n-1 \\ \hline 1 \end{array} \right] \right\rangle_2 + \frac{[n-1]}{[n]} \left\langle \frac{n-1}{1} \left[\begin{array}{c} n-2 \\ \hline \end{array} \right] \left[\begin{array}{c} n-1 \\ \hline 1 \end{array} \right] \right\rangle_2 \in W_{n+n} \end{aligned}$$

The A_1 clasp has the following properties.

LEMMA 2.4 (Kauffman-Lins [KL94b] etc.). For any positive integer n ,

- $\left\langle -\left[\begin{array}{c} n \\ \hline \end{array} \right] \left[\begin{array}{c} n \\ \hline \end{array} \right] \right\rangle_2 = \left\langle -\left[\begin{array}{c} n \\ \hline \end{array} \right] \right\rangle_2$,
- $\left\langle -\left[\begin{array}{c} n-k-2 \\ \hline 1 \\ \hline k \end{array} \right] \right\rangle_2 = 0 \quad (k = 0, 1, \dots, n-2).$

It is easy to calculate the following.

LEMMA 2.5. For $k = 0, 1, \dots, n$,

- $\left\langle -\left[\begin{array}{c} n \\ \hline \end{array} \right] \left[\begin{array}{c} k \\ \hline n-k \end{array} \right] \right\rangle_2 = q^{\frac{k(n-k)}{4}} \left\langle -\left[\begin{array}{c} n \\ \hline \end{array} \right] \right\rangle_2$, $\left\langle -\left[\begin{array}{c} n \\ \hline \end{array} \right] \left[\begin{array}{c} k \\ \hline n-k \end{array} \right] \right\rangle_2 = q^{-\frac{k(n-k)}{4}} \left\langle -\left[\begin{array}{c} n \\ \hline \end{array} \right] \right\rangle_2$,
- $\left\langle -\left[\begin{array}{c} n \\ \hline \end{array} \right] \left[\begin{array}{c} n-k \\ \hline k \end{array} \right] \right\rangle_2 = (-1)^k \frac{[n+1]}{[n-k+1]} \left\langle -\left[\begin{array}{c} n-k \\ \hline \end{array} \right] \right\rangle_2$,
- $\left\langle -\left[\begin{array}{c} n \\ \hline \end{array} \right] \left[\begin{array}{c} n \\ \hline \end{array} \right] \right\rangle_2 = (-1)^n q^{\frac{n^2+2n}{4}} \left\langle -\left[\begin{array}{c} n \\ \hline \end{array} \right] \right\rangle_2$, $\left\langle -\left[\begin{array}{c} n \\ \hline \end{array} \right] \left[\begin{array}{c} n \\ \hline \end{array} \right] \right\rangle_2 = (-1)^n q^{-\frac{n^2+2n}{4}} \left\langle -\left[\begin{array}{c} n \\ \hline \end{array} \right] \right\rangle_2$.

Let $N = (n_1, n_2, \dots, n_k)$ be a k -tuple of positive integers.

DEFINITION 2.6 (Clasped A_1 web spaces). We define a subspace W_N of $W_{n_1+n_2+\dots+n_k}$ called a clasped A_1 web space as the following:

$$W_N = \left\{ \left\langle \left[\begin{array}{c} n_3 \\ \hline \end{array} \right] \left[\begin{array}{c} n_2 \\ \hline \end{array} \right] \left[\begin{array}{c} n_1 \\ \hline \end{array} \right] \left[\begin{array}{c} n_k \\ \hline \end{array} \right] w \right\rangle_2 \mid w \in W_{n_1+n_2+\dots+n_k} \right\}.$$

For example, if $k = 3, n_1 = 1, n_2 = 2, n_3 = 3$ and $w = \left\langle \left[\begin{array}{c} 3 \\ \hline \end{array} \right] \left[\begin{array}{c} 2 \\ \hline \end{array} \right] \left[\begin{array}{c} 1 \\ \hline \end{array} \right] \right\rangle_2 = q^{\frac{1}{4}} \left[\begin{array}{c} 3 \\ \hline \end{array} \right] \left[\begin{array}{c} 2 \\ \hline \end{array} \right] \left[\begin{array}{c} 1 \\ \hline \end{array} \right] +$

$q^{-\frac{1}{4}} \left[\begin{array}{c} 3 \\ \hline \end{array} \right] \left[\begin{array}{c} 2 \\ \hline \end{array} \right] \left[\begin{array}{c} 1 \\ \hline \end{array} \right]$, then the above diagram means

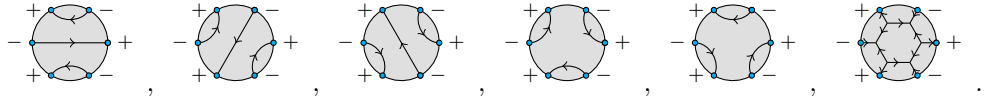
$$\left\langle \left[\begin{array}{c} 3 \\ \hline \end{array} \right] \left[\begin{array}{c} 2 \\ \hline \end{array} \right] \left[\begin{array}{c} 1 \\ \hline \end{array} \right] \right\rangle_2 = q^{\frac{1}{4}} \left\langle \left[\begin{array}{c} 3 \\ \hline \end{array} \right] \left[\begin{array}{c} 2 \\ \hline \end{array} \right] \left[\begin{array}{c} 1 \\ \hline \end{array} \right] \right\rangle_2 + q^{-\frac{1}{4}} \left\langle \left[\begin{array}{c} 3 \\ \hline \end{array} \right] \left[\begin{array}{c} 2 \\ \hline \end{array} \right] \left[\begin{array}{c} 1 \\ \hline \end{array} \right] \right\rangle_2 = q^{\frac{1}{4}} \left\langle \left[\begin{array}{c} 3 \\ \hline \end{array} \right] \left[\begin{array}{c} 2 \\ \hline \end{array} \right] \left[\begin{array}{c} 1 \\ \hline \end{array} \right] \right\rangle_2.$$

2.3. The A_2 web spaces. We define the A_2 web spaces. Let $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m)$ be a m -tuple of signs $+$ or $-$. Let D_ε denote D_m whose marked point $\exp(2\pi\sqrt{-1}/m)^{j-1}$ is decorated by ε_j for $j = 1, 2, \dots, m$. A bipartite uni-trivalent graph G is a directed graph such that every vertex is either trivalent or univalent and these vertices are divided into sinks or sources. A sink (resp. source) is a vertex such that all edges adjoining to the

vertex point into (resp. away from) it. A *bipartite trivalent graph* G in D_ε is an embedding of a uni-trivalent graph into D_ε such that for any vertex v has the following neighborhoods:

-  or  if v is a sink,
-  or  if v is a source.

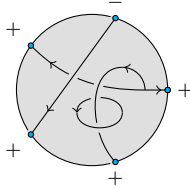
An A_2 *basis web* is the boundary-fixing isotopy class of a bipartite trivalent graph G in D_ε , where any internal face of $D \setminus G$ has at least six sides. Let us denote B_ε is the set of A_2 basis webs in D_ε . For example, $B_{(+,-,+,+,-)}$ has the following A_2 basis webs:



The A_2 *web space* W_ε is the $\mathbb{Q}(q^{\frac{1}{6}})$ -vector space spanned by B_ε . A *tangled trivalent graph diagram* in D_ε is an immersed bipartite uni-trivalent graph in D_ε whose intersection points are only transverse double points of edges with crossing data  or . Tangled trivalent graph diagrams G and G' are regularly isotopic if G is obtained from G' by a finite sequence of boundary-fixing isotopies and (R1'), (R2), (R3) and (R4) moves with some direction of edges.

$$(R4) \quad \begin{array}{c} \text{Diagram 1} \end{array} \longleftrightarrow \begin{array}{c} \text{Diagram 2} \end{array}, \quad \begin{array}{c} \text{Diagram 3} \end{array} \longleftrightarrow \begin{array}{c} \text{Diagram 4} \end{array}.$$

Tangled trivalent graphs in D_ε are regular isotopy classes of tangled trivalent graph diagrams in D_ε . We denote T_ε the set of tangled trivalent graphs in D_ε . The diagram below is an example of a tangled trivalent graph diagram in $D_{(+,-,+,+,-)}$.



DEFINITION 2.7 (The A_2 bracket [Kup96]). We define a $\mathbb{Q}(q^{\frac{1}{6}})$ -linear map $\langle \cdot \rangle_3 : \mathbb{Q}(q^{\frac{1}{6}})T_\varepsilon \rightarrow W_\varepsilon$ by the following.

- $\langle \text{Diagram 1} \rangle_3 = q^{\frac{1}{3}} \langle \text{Diagram 2} \rangle_3 - q^{-\frac{1}{6}} \langle \text{Diagram 3} \rangle_3,$
- $\langle \text{Diagram 4} \rangle_3 = q^{-\frac{1}{3}} \langle \text{Diagram 5} \rangle_3 - q^{\frac{1}{6}} \langle \text{Diagram 6} \rangle_3,$
- $\langle \text{Diagram 7} \rangle_3 = \langle \text{Diagram 8} \rangle_3 + \langle \text{Diagram 9} \rangle_3,$
- $\langle \text{Diagram 10} \rangle_3 = [2] \langle \text{Diagram 11} \rangle_3,$

$$\bullet \left\langle G \sqcup \begin{array}{c} \text{circle with a dot} \end{array} \right\rangle_3 = [3] \langle G \rangle_3.$$

We can confirm that this map is well-defined as with the Kauffman bracket.

We next consider A_2 web space $W_{n^++n^-} = W_{(+,+, \dots, +, -, -, \dots, -)}$ whose first n marked points are decorated with $+$ and next n marked points are decorated with $-$. We define A_2 clasps $\left\langle \begin{array}{c} \text{clasp diagram} \end{array} \right\rangle_3 \in W_{n^++n^-}$ inductively by the following.

DEFINITION 2.8. (The A_2 clasps)

$$(2.2) \quad \begin{aligned} \left\langle \begin{array}{c} \text{clasp diagram} \end{array} \right\rangle_3 &= \begin{array}{c} \text{clasp diagram} \end{array} \in W_{1^++1^-} \\ \left\langle \begin{array}{c} \text{clasp diagram} \end{array} \right\rangle_3 &= \left\langle \begin{array}{c} \text{clasp diagram} \end{array} \right\rangle_3 - \frac{[n-1]}{[n]} \left\langle \begin{array}{c} \text{clasp diagram} \end{array} \right\rangle_3 \in W_{n^++n^-} \end{aligned}$$

A_2 clasps have the following properties.

LEMMA 2.9 (Properties of A_2 clasps). For any positive integer n ,

$$\begin{aligned} \bullet \left\langle \begin{array}{c} \text{clasp diagram} \end{array} \right\rangle_3 &= \left\langle \begin{array}{c} \text{clasp diagram} \end{array} \right\rangle_3, \\ \bullet \left\langle \begin{array}{c} \text{clasp diagram} \end{array} \right\rangle_3 &= 0 \quad (k = 0, 1, \dots, n-2). \end{aligned}$$

We can easily calculate the following.

LEMMA 2.10. For $k = 0, 1, \dots, n$,

$$\begin{aligned} \bullet \left\langle \begin{array}{c} \text{clasp diagram} \end{array} \right\rangle_3 &= q^{\frac{k(n-k)}{3}} \left\langle \begin{array}{c} \text{clasp diagram} \end{array} \right\rangle_3, \left\langle \begin{array}{c} \text{clasp diagram} \end{array} \right\rangle_3 = q^{-\frac{k(n-k)}{3}} \left\langle \begin{array}{c} \text{clasp diagram} \end{array} \right\rangle_3, \\ \bullet \left\langle \begin{array}{c} \text{clasp diagram} \end{array} \right\rangle_3 &= \frac{[n+1][n+2]}{[n-k+1][n-k+2]} \left\langle \begin{array}{c} \text{clasp diagram} \end{array} \right\rangle_3, \\ \bullet \left\langle \begin{array}{c} \text{clasp diagram} \end{array} \right\rangle_3 &= q^{\frac{n^2+3n}{3}} \left\langle \begin{array}{c} \text{clasp diagram} \end{array} \right\rangle_3, \left\langle \begin{array}{c} \text{clasp diagram} \end{array} \right\rangle_3 = q^{-\frac{n^2+3n}{3}} \left\langle \begin{array}{c} \text{clasp diagram} \end{array} \right\rangle_3. \end{aligned}$$

Let n^+ be an n -tuple of $+$ and n^- an n -tuple of $-$. For a k -tuple of signs $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k)$, We define $N^\varepsilon = (n_1^{\varepsilon_1}, n_2^{\varepsilon_2}, \dots, n_k^{\varepsilon_k})$. We use a notation $n_1^{\varepsilon_1} + n_2^{\varepsilon_2} + \dots + n_k^{\varepsilon_k}$ to represent $(n_1 + n_2 + \dots + n_k)$ -tuple of signs. It is defined in the following manner: the first n_1 signs are ε_1 , the next n_2 signs are ε_2 , $\dots$, and the last n_k signs are ε_k .

DEFINITION 2.11 (A clasped A_2 web space). We define a subspace W_{N^ε} of $W_{n_1^{\varepsilon_1} + n_2^{\varepsilon_2} + \dots + n_k^{\varepsilon_k}}$ called the *clasped A_2 web space* as follows:

$$W_{N^\varepsilon} = \left\{ \left\langle \begin{array}{c} \text{clasp diagram} \end{array} \right\rangle_3 \mid w \in W_{n_1^{\varepsilon_1} + n_2^{\varepsilon_2} + \dots + n_k^{\varepsilon_k}} \right\}.$$

3. Colored skein relations

In this section, we introduce colored skein relations for clasped A_1 and A_2 web spaces. Although the formula for a clasped A_1 web space is already known by Yamada [Yam89, Yam92] and Hajij [Haj14b], we prove this formula by another method using the theory of

integer partitions. We use the same method to prove colored skein relations for clasped A_2 web spaces.

3.1. Colored Kauffman bracket skein relations. We review the colored Kauffman bracket skein relations and give another proof by using theory of integer partitions. Let us consider clasped A_1 web spaces.

PROPOSITION 3.1. Let n be non-negative integers.

$$(1) \left\langle \begin{array}{c} \text{Diagram 1} \end{array} \right\rangle_2 = \sum_{k=0}^n q^{\frac{-n^2+2k^2}{4}} \binom{n}{k}_q \left\langle \begin{array}{c} \text{Diagram 2} \end{array} \right\rangle_2 \quad (\text{the colored Kauffman bracket skein relation by Hajij [Haj14b]}),$$

$$(2) \left\langle \bigcirc^n \right\rangle_2 = (-1)^n [n+1] \emptyset.$$

The second equality is a special case of the second equality of Lemma 2.5. We prove the colored Kauffman bracket skein relation by using a well-known identity from the theory of integer partitions.

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_s)$ be a partition of an integer into s parts, that is, λ is the s -tuples of positive integers such that $\lambda_i \geq \lambda_{i+1}$ for $i = 1, 2, \dots, s-1$, and $|\lambda|$ denotes $\lambda_1 + \lambda_2 + \dots + \lambda_s$. For given non-negative integers k and l , $\mathcal{P}(k, l)$ denotes the set of partitions λ such that $0 \leq \lambda_1 \leq k$ and $0 \leq s \leq l$.

LEMMA 3.2 (Andrews and Eriksson [AE04] etc.).

$$\binom{k+l}{k}_q = \sum_{\lambda \in \mathcal{P}(k, l)} q^{|\lambda|}.$$

PROOF OF PROPOSITION 3.1 (1).

$$(3.1) \quad \left\langle \begin{array}{c} \text{Diagram 1} \end{array} \right\rangle_2 = \left\langle \begin{array}{c} \text{Diagram 2} \end{array} \right\rangle_2$$

$$= q^{\frac{1}{4}} \left\langle \begin{array}{c} \text{Diagram 3} \end{array} \right\rangle_2 + q^{-\frac{1}{4}} \left\langle \begin{array}{c} \text{Diagram 4} \end{array} \right\rangle_2$$

$$(3.2) \quad = q^{\frac{2n-1}{4}} \left\langle \begin{array}{c} \text{Diagram 5} \end{array} \right\rangle_2 + q^{-\frac{2n-1}{4}} \left\langle \begin{array}{c} \text{Diagram 6} \end{array} \right\rangle_2.$$

We used the Kauffman bracket skein relation in (3.1) and Lemma 2.5 in (3.2). We define a clasped A_1 web $\langle \sigma(k, l; n) \rangle_2$ as follows:

$$\langle \sigma(k, l; n) \rangle_2 = \left\langle \begin{array}{c} \text{Diagram 7} \end{array} \right\rangle_2.$$

By the above calculation,

$$(3.3) \quad \langle \sigma(k, l; n) \rangle_2 = q^{\frac{2(n-k-l)-1}{4}} \langle \sigma(k+1, l; n) \rangle_2 + q^{-\frac{2(n-k-l)-1}{4}} \langle \sigma(k, l+1; n) \rangle_2.$$

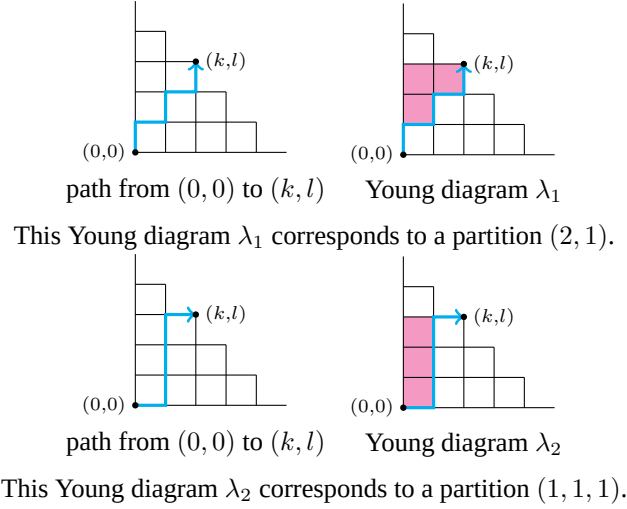
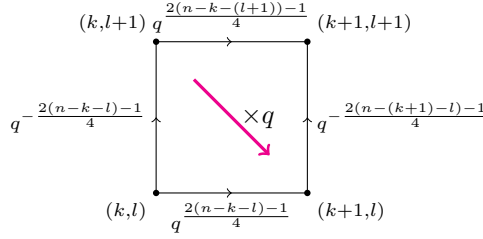
FIGURE 6.1. Two examples for $n = 5$, $(k, l) = (2, 3)$.

FIGURE 6.2. Shift of a path.

We make $\langle \sigma(k, l; n) \rangle_2$ correspond to a lattice point (k, l) in $\mathbb{Z} \times \mathbb{Z}$ for each non-negative integers k and l such that $0 \leq k + l \leq n$. We decorate vectors $(1, 0)$ and $(0, 1)$ from (k, l) with coefficients of $\langle \sigma(k+1, l; n) \rangle_2$ and $\langle \sigma(k, l+1; n) \rangle_2$ of (3.3), respectively. The left-hand side of the colored Kauffman bracket skein relation is $\langle \sigma(0, 0; n) \rangle_2$ and clasped A_1 webs appear in the right-hand side are $\langle \sigma(k, l; n) \rangle_2$ such that $k + l = n$. Fix a pair of k and l such that $k + l = n$, then the coefficient of $\langle \sigma(k, l; n) \rangle_2$ of the right-hand side of the colored Kauffman bracket skein relation is determined as follows:

- (1) Take a lattice path from $(0, 0)$ to (k, l) constructed from vectors $(1, 0)$ and $(0, 1)$.
- (2) Product all decorations of vectors appearing in the path.
- (3) Sum up the product of decorations for all paths from $(0, 0)$ to (k, l) .

A Young diagram is obtained by cutting out the upper side of a path from $(0, 0)$ to (k, l) from a rectangle $\{(x, y) \mid 0 \leq x \leq k, 0 \leq y \leq l\}$. The Young diagram corresponds to a partition of an integer (see two examples in Fig. 6.1). If a path from $(0, 0)$ to (k, l) with $k + l = n$ shifts downward by a box, then the product of decorations is multiplied by q . In fact, a product of decorations of $(k, l) \rightarrow (k, l+1) \rightarrow (k+1, l+1)$ is $q^{-\frac{1}{2}}$ and $(k, l) \rightarrow (k+1, l) \rightarrow (k+1, l+1)$ is $q^{\frac{1}{2}}$ (see Fig. 6.2). The product of decorations of the top path is $\prod_{i=0}^{l-1} q^{-\frac{2(n-i)-1}{4}} \prod_{j=0}^{k-1} q^{\frac{2(n-l-j)-1}{4}}$. Therefore, the coefficient of $\langle \sigma(k, l; n) \rangle_2$

with $k + l = n$ is

$$\prod_{i=0}^{l-1} q^{-\frac{2(n-i)-1}{4}} \prod_{j=0}^{k-1} q^{\frac{2(n-l-j)-1}{4}} \sum_{\lambda \in \mathcal{P}(k,l)} q^{|\lambda|} = q^{\frac{-n^2+2k^2}{4}} \binom{n}{k}_q$$

by Lemma 3.2. $\square$

The colored Kauffman bracket skein relation is the expansion of a half twist of two strands colored by n . We also give the expansion of a full twist of two strands colored by n .

PROPOSITION 3.3 (The full twist formula [Mas03, Lemma 4.1]).

$$\left\langle \begin{array}{c} n \\ \text{[Diagram: Full twist of two strands]} \\ n \end{array} \right\rangle_2 = \sum_{k=0}^n (-1)^{n-k} q^{\frac{2k^2-n^2+k-n}{2}} \frac{(q)_n}{(q)_k} \binom{n}{k}_q \left\langle \begin{array}{c} n \quad k \\ \text{[Diagram: Two parallel strands with crossings]} \\ n-k \quad n \end{array} \right\rangle_2$$

PROOF. We prove this formula by the same argument as the proof of Proposition 3.1. First, the following equation is obtained by using the Kauffman bracket skein relation and Lemma 2.5.

$$\begin{aligned} \left\langle \begin{array}{c} n \\ \text{[Diagram: Full twist of two strands]} \\ n \end{array} \right\rangle_2 &= \left\langle \begin{array}{c} n \\ \text{[Diagram: Full twist of two strands with a crossing]} \\ n-1 \end{array} \right\rangle_2 \\ &= q^{\frac{1}{4}} \left\langle \begin{array}{c} n \\ \text{[Diagram: Full twist of two strands with a crossing]} \\ n-1 \end{array} \right\rangle_2 + q^{-\frac{1}{4}} \left\langle \begin{array}{c} n \\ \text{[Diagram: Full twist of two strands]} \\ n-1 \end{array} \right\rangle_2 \\ &= q^{\frac{1}{2}} \left\langle \begin{array}{c} n \\ \text{[Diagram: Full twist of two strands with a crossing]} \\ n-1 \end{array} \right\rangle_2 + (1 - q^{-1}) q^{-\frac{1}{2}(n-1)} \left\langle \begin{array}{c} n-1 \\ \text{[Diagram: Full twist of two strands]} \\ 1 \end{array} \right\rangle_2. \end{aligned}$$

This equation implies

(3.4)

$$q^{-\frac{1}{2}(n-i-1)} \left\langle \begin{array}{c} n \\ \text{[Diagram: Full twist of two strands]} \\ n-i-1 \end{array} \right\rangle_2 - q^{-\frac{1}{2}(n-i)} \left\langle \begin{array}{c} n \\ \text{[Diagram: Full twist of two strands]} \\ n-i \end{array} \right\rangle_2 = (1-q) q^{-n-\frac{1}{2}} q^i \left\langle \begin{array}{c} n-1 \\ \text{[Diagram: Full twist of two strands]} \\ 1 \end{array} \right\rangle_2$$

for $i = 0, 1, \dots, n-1$. Thus, we obtain

$$\left\langle \begin{array}{c} n \\ \text{[Diagram: Full twist of two strands]} \\ n \end{array} \right\rangle_2 = q^{\frac{n}{2}} \left\langle \begin{array}{c} n \\ \text{[Diagram: Two parallel strands]} \\ n \end{array} \right\rangle_2 - (1 - q^n) q^{-\frac{n+1}{2}} \left\langle \begin{array}{c} n-1 \\ \text{[Diagram: Full twist of two strands]} \\ 1 \end{array} \right\rangle_2$$

by taking the sum of both sides of (3.4) for $i = 0, 1, \dots, n-1$. Also,

$$\left\langle \begin{array}{c} j \\ \text{[Diagram: Full twist of two strands]} \\ i \end{array} \right\rangle_2 = q^{\frac{j}{2}} \left\langle \begin{array}{c} j \\ \text{[Diagram: Full twist of two strands]} \\ i \end{array} \right\rangle_2 - (1 - q^j) q^{-\frac{i+j}{2}} \left\langle \begin{array}{c} j-1 \\ \text{[Diagram: Full twist of two strands]} \\ i \end{array} \right\rangle_2$$

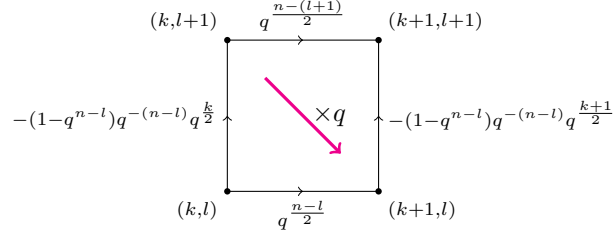


FIGURE 6.3. Shift of a path.

for any non-negative integers i and j . We define a clasped A_1 web $\langle \sigma^2(k, l; n) \rangle_2$ as follows:

$$\langle \sigma^2(k, l; n) \rangle_2 = \left\langle \begin{array}{c} \text{Diagram of a clasped } A_1 \text{ web with } n \text{ strands, } k \text{ clasps on the left, and } l \text{ clasps on the right.} \end{array} \right\rangle_2.$$

Then, we obtain

$$(3.5) \quad \langle \sigma^2(k, l; n) \rangle_2 = q^{\frac{n-l}{2}} \langle \sigma^2(k+1, l; n) \rangle_2 - (1-q^{n-l})q^{-(n-l)}q^{\frac{k}{2}} \langle \sigma^2(k, l+1; n) \rangle_2$$

for non-negative integers k and l such that $k+l \leq n$. In the same way as the proof of Proposition 3.1 (1), we make $\langle \sigma^2(k, l; n) \rangle_2$ correspond to a lattice point (k, l) and decorate edges with coefficients by using the resolution (3.5). If a path from $(0, 0)$ to (k, l) with $k+l = n$ shifts downward by a box, then the product of decorations is multiplied by q (see Fig. 6.3). Therefore, the coefficient of $\langle \sigma^2(k, l; n) \rangle_2$ with $k+l = n$ for the expansion of a full twist is

$$\prod_{i=0}^{l-1} -(1-q^{n-i})q^{-(n-i)} \prod_{j=0}^{k-1} q^{\frac{n-l}{2}} \binom{n}{k}_q = (-1)^l q^{\frac{2k^2-n^2+k-n}{2}} \frac{(q; q)_n}{(q; q)_{n-l}} \binom{n}{k}_q.$$

□

We next give expansions of m half twists and m full twists of two strands colored by n .

LEMMA 3.4 (Masbaum [Mas03, Lemma 4.3]). For any non-negative integers $k \neq n$,

$$\left\langle \begin{array}{c} \text{Diagram of a clasped } A_1 \text{ web with } n \text{ strands, } k \text{ clasps on the left, and } n-k \text{ clasps on the right.} \end{array} \right\rangle_2 = (-1)^{n-k} q^{-\frac{n^2-k^2+2n-2k}{4}} \left\langle \begin{array}{c} \text{Diagram of a clasped } A_1 \text{ web with } n-k \text{ strands, } n-k \text{ clasps on the left, and } k \text{ clasps on the right.} \end{array} \right\rangle_2.$$

PROOF. Slide two right-side A_1 clasps to the left and calculate resulting clasped A_1 web using Lemma 2.5. □

Let n, m be non-negative integers.

PROPOSITION 3.5 (The m half twists formula).

$$\left\langle \begin{array}{c} \text{[Diagram: } m \text{ half twists]} \\ n \end{array} \right\rangle_2 = (-1)^{mn} q^{-\frac{m}{4}(n^2+2n)} \times \sum_{0 \leq k_m \leq \dots \leq k_1 \leq n} (-1)^{n-k_m} q^{\frac{n-k_m}{2}} (-1)^{\sum_{i=1}^m k_i} q^{\frac{1}{2} \sum_{i=1}^m (k_i^2 + k_i)} \binom{n}{k'_1, k'_2, \dots, k'_m, k_m}_q \left\langle \begin{array}{c} \text{[Diagram: } m \text{ full twists]} \\ n \end{array} \right\rangle_2,$$

where k_i, k'_i are integers such that $k_0 = n, k'_{i+1} = k_i - k_{i+1}$ for $i = 0, 1, \dots, m-1$.

PROOF. Let k_i, k_{i+1} as above. We define a clasped A_1 web $\langle \sigma^{m-i}(k_i; n) \rangle_2$ for any $k_i = 0, 1, \dots, n$ as

$$\langle \sigma^{m-i}(k_i; n) \rangle_2 = \left\langle \begin{array}{c} \text{[Diagram: } m-i \text{ half twists]} \\ n-k_i \end{array} \right\rangle_2.$$

We apply Proposition 3.1 (1) to a rightmost half twist and use Lemma 3.4 $m-i-1$ times. Thus, we obtain

$$\begin{aligned} \langle \sigma^{m-i}(k_i; n) \rangle_2 &= \sum_{k_{i+1}=0}^{k_i} (-1)^{(m-i-1)(k_i-k_{i+1})} q^{\frac{1}{4}(-k_i^2+2k_{i+1}^2)} q^{\frac{m-i-1}{4}(k_{i+1}^2-k_i^2+2k_{i+1}-2k_i)} \binom{k_i}{k_{i+1}}_q \langle \sigma^{m-i-1}(k_{i-1}; n) \rangle_2. \end{aligned}$$

The left-hand side of the m half twists formula is $\langle \sigma^m(k_0; n) \rangle_2$. Therefore, we can obtain the m half twist formula by using the above equation for $i = 0, 1, \dots, m-1$ in turn and calculation of the exponent sum of q . $\square$

PROPOSITION 3.6 (m full twists formula [Mas03]).

$$\left\langle \begin{array}{c} \text{[Diagram: } m \text{ full twists]} \\ n \end{array} \right\rangle_2 = q^{-\frac{m}{2}(n^2+2n)} \sum_{0 \leq k_m \leq \dots \leq k_1 \leq n} (-1)^{n-k_m} q^{\frac{n-k_m}{2}} q^{\sum_{i=1}^m (k_i^2 + k_i)} \times \frac{(q)_n}{(q)_{k_m}} \binom{n}{k'_1, k'_2, \dots, k'_m, k_m}_q \left\langle \begin{array}{c} \text{[Diagram: } m \text{ full twists]} \\ n \end{array} \right\rangle_2$$

PROOF. We prove this formula by the same way as the proof of Proposition 3.5. We only have to use Proposition 3.3 instead of Proposition 3.1 (1). $\square$

REMARK 3.7.

- Twist formulas in this section treat only right-handed twists. Left-handed versions of twist formulas can be obtained by substituting q^{-1} for q .
- We can easily calculate twist formulas for an n -colored strand and m -colored strand by using twist formulas for two n -colored strands.

3.2. Colored A_2 bracket skein relations. Let us consider clasped A_2 web spaces. We use the following graphical notations to represent certain A_2 webs.

DEFINITION 3.8. For positive integers n and m , a *colored 4-valent vertex*

$$\begin{array}{c} \uparrow m \\ \begin{array}{|c|} \hline \rightarrow n \\ \hline \rightarrow n \\ \hline \downarrow m \end{array} \end{array} \in W_{n^++m^++n^-+m^-}$$

is defined as follows: $\begin{array}{c} \uparrow m \\ \begin{array}{|c|} \hline \rightarrow n \\ \hline \rightarrow n \\ \hline \downarrow m \end{array} \end{array} = \begin{array}{c} \rightarrow \\ \vdots \\ \rightarrow \end{array} \begin{array}{c} \vdots \\ \rightarrow \\ \vdots \end{array} \begin{array}{c} \rightarrow \\ \vdots \\ \rightarrow \end{array} \begin{array}{c} \rightarrow \\ \vdots \\ \rightarrow \end{array} \in W_{n^++1^++n^-+1^-}$ for $m = 1$, $\begin{array}{c} \uparrow m \\ \begin{array}{|c|} \hline \rightarrow n \\ \hline \rightarrow n \\ \hline \downarrow m \end{array} \end{array} =$

$\begin{array}{c} \uparrow m-1 \\ \begin{array}{|c|} \hline \rightarrow n \\ \hline \rightarrow n \\ \hline \downarrow m-1 \end{array} \end{array}$ for $m > 1$. We also define $\begin{array}{c} \downarrow m \\ \begin{array}{|c|} \hline \rightarrow n \\ \hline \rightarrow n \\ \hline \downarrow m \end{array} \end{array} \in W_{n^++m^-+n^-+m^+}$ in the same way.

DEFINITION 3.9. For positive integer n , a *colored trivalent vertex*

$$\begin{array}{c} \nearrow n \\ \triangle \\ \searrow n \end{array} \in W_{n^++n^++n^+}$$

is defined as follows: $\begin{array}{c} \nearrow n \\ \triangle \\ \searrow n \end{array}$ for $n = 1$, $\begin{array}{c} \nearrow n \\ \triangle \\ \searrow n \end{array} = \begin{array}{c} 1 \\ \leftarrow \begin{array}{|c|} \hline \rightarrow n-1 \\ \hline \rightarrow n-1 \\ \hline \downarrow n-1 \end{array} \triangle \rightarrow n-1 \end{array}$ for $n > 1$. We

also define $\begin{array}{c} \nwarrow n \\ \triangle \\ \swarrow n \end{array} \in W_{n^-+n^-+n^-}$ in the same way.

We sometimes omit orientations of A_2 webs when the orientations obviously turn out from the previous A_2 webs.

LEMMA 3.10.

- (1) $\begin{array}{c} m_1 \quad m_2 \\ \begin{array}{|c|} \hline \rightarrow n \\ \hline \rightarrow n \\ \hline \downarrow m_1 \quad m_2 \end{array} \end{array} = \begin{array}{c} m_1+m_2 \\ \begin{array}{|c|} \hline \rightarrow n \\ \hline \rightarrow n \\ \hline \downarrow m_1+m_2 \end{array} \end{array},$
- (2) $\begin{array}{c} \uparrow n \\ \begin{array}{|c|} \hline \rightarrow n \\ \hline \rightarrow n \\ \hline \downarrow n \end{array} \end{array} = \begin{array}{c} \rightarrow \\ \triangle \\ \rightarrow \end{array} \begin{array}{c} \rightarrow \\ \triangle \\ \rightarrow \end{array} \begin{array}{c} \rightarrow \\ \triangle \\ \rightarrow \end{array} \begin{array}{c} \rightarrow \\ \triangle \\ \rightarrow \end{array},$
- (3) $\begin{array}{c} \rightarrow \\ \begin{array}{|c|} \hline \rightarrow n \\ \hline \rightarrow n \\ \hline \downarrow m \end{array} \end{array} = \begin{array}{c} \rightarrow \\ \begin{array}{|c|} \hline \rightarrow n \\ \hline \rightarrow n \\ \hline \downarrow m \end{array} \end{array},$
- (4) $\left\langle \begin{array}{c} 1 \\ \begin{array}{|c|} \hline \rightarrow n \\ \hline \rightarrow n \\ \hline \downarrow n \end{array} \end{array} \right\rangle_3 = \begin{array}{c} n \\ \rightarrow \end{array} + \sum_{i=0}^{n-1} \begin{array}{c} n-i-1 \\ \rightarrow \end{array} \begin{array}{c} 1 \\ \begin{array}{|c|} \hline \rightarrow i \\ \hline \rightarrow i \\ \hline \downarrow i \end{array} \end{array},$
- (5) $\left\langle \begin{array}{c} n+m \\ \begin{array}{|c|} \hline \rightarrow n \\ \hline \rightarrow n \\ \hline \downarrow m \end{array} \end{array} \right\rangle_3 = \left\langle \begin{array}{c} n+m \\ \begin{array}{|c|} \hline \rightarrow n \\ \hline \rightarrow n \\ \hline \downarrow m \end{array} \end{array} \right\rangle_3,$
- (6) $\left\langle \begin{array}{c} n \\ \begin{array}{|c|} \hline \rightarrow n \\ \hline \rightarrow n \\ \hline \downarrow n \end{array} \end{array} \right\rangle_3 = \left\langle \begin{array}{c} n \\ \begin{array}{|c|} \hline \rightarrow n \\ \hline \rightarrow n \\ \hline \downarrow n \end{array} \end{array} \right\rangle_3.$

The above equations also hold for the opposite orientations.

PROOF. (1) and (2) are obtained by definitions of colored tri- and 4-valent vertices. Let us show (3) by the induction on m . We can easily confirm in the case of $m = 1$. If $m > 1$, then

$$(3.6) \quad \begin{array}{c} \begin{array}{c} \text{Diagram 1: A vertex with three strands labeled } m, n, m. \end{array} \\ = \begin{array}{c} \text{Diagram 2: A vertex with three strands labeled } m-1, n, m-1. \end{array} \\ = \begin{array}{c} \text{Diagram 3: A vertex with three strands labeled } m-1, n, m-1. \end{array} \\ = \begin{array}{c} \text{Diagram 4: A vertex with three strands labeled } m-1, n, m-1. \end{array} \\ = \begin{array}{c} \text{Diagram 5: A vertex with three strands labeled } m, n, m. \end{array} \end{array}$$

We used the induction hypothesis in (3.6). We can show (4) by an easy calculation for $n = 1$. If $n > 1$, then

$$\begin{aligned} \left\langle \begin{array}{c} \text{Diagram 1: A vertex with three strands labeled } 1, n, 1. \end{array} \right\rangle_3 &= \left\langle \begin{array}{c} \text{Diagram 2: A vertex with three strands labeled } 1, n-1, 1. \end{array} \right\rangle_3 = \left\langle \begin{array}{c} \text{Diagram 3: A vertex with three strands labeled } 1, n-1, 1. \end{array} \right\rangle_3 + \left\langle \begin{array}{c} \text{Diagram 4: A vertex with three strands labeled } 1, n-1, 1. \end{array} \right\rangle_3 \\ &= \begin{array}{c} \text{Diagram 5: A vertex with three strands labeled } 1, n-1, 1. \end{array} + \sum_{i=0}^{n-2} \begin{array}{c} \text{Diagram 6: A vertex with three strands labeled } 1, n-i-2, 1. \end{array} + \begin{array}{c} \text{Diagram 7: A vertex with three strands labeled } 1, n-1, 1. \end{array} \end{aligned}$$

by the induction on n . (5) is easily showed by the induction on m . In order to prove (6), we only have to show the following:

$$\left\langle \begin{array}{c} \text{Diagram 1: A vertex with three strands labeled } n, k, n-k-2. \end{array} \right\rangle_3 = 0$$

for $k = 1, 2, \dots, n-2$ through the recursive definition of A_2 clasps. Furthermore,

$$\left\langle \begin{array}{c} \text{Diagram 1: A vertex with three strands labeled } n, k, n-k-2. \end{array} \right\rangle_3 = \left\langle \begin{array}{c} \text{Diagram 2: A vertex with three strands labeled } n, k, n-k-2. \end{array} \right\rangle_3 = \left\langle \begin{array}{c} \text{Diagram 3: A vertex with three strands labeled } n, k, n-k-2. \end{array} \right\rangle_3$$

by Lemma 3.10 (5).

$$\begin{aligned} \left\langle \begin{array}{c} \text{Diagram 1: A vertex with three strands labeled } n, k, n-k-1. \end{array} \right\rangle_3 &= \left\langle \begin{array}{c} \text{Diagram 2: A vertex with three strands labeled } n, k, n-k-1. \end{array} \right\rangle_3 = \left\langle \begin{array}{c} \text{Diagram 3: A vertex with three strands labeled } n, k, n-k-1. \end{array} \right\rangle_3 \\ &= \left\langle \begin{array}{c} \text{Diagram 4: A vertex with three strands labeled } n, k, n-k-1. \end{array} \right\rangle_3 = 0 \end{aligned}$$

by Lemma 3.10 (4). We can prove (1)–(6) with the opposite oriented A_2 webs in the same way. $\square$

THEOREM 3.11. Let n be a positive integer.

$$(1) \quad \left\langle \begin{array}{c} \text{Diagram 1: A vertex with three strands labeled } n, n, n. \end{array} \right\rangle_3 = \sum_{k=0}^n (-1)^k q^{\frac{2n^2-6nk+3k^2}{6}} \binom{n}{k}_q \left\langle \begin{array}{c} \text{Diagram 2: A vertex with three strands labeled } n, k, n-k. \end{array} \right\rangle_3,$$

$$\begin{aligned}
(2) \quad & \left\langle \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right\rangle_3 = \sum_{k=0}^n (-1)^k q^{\frac{-2n^2+3k^2}{6}} \binom{n}{k}_q \left\langle \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} \right\rangle_3, \\
(3) \quad & \left\langle \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} \right\rangle_3 = \sum_{k=0}^n \left\langle \begin{array}{c} \text{Diagram 7} \\ \text{Diagram 8} \end{array} \right\rangle_3, \\
(4) \quad & \left\langle \begin{array}{c} \text{Diagram 9} \\ \text{Diagram 10} \end{array} \right\rangle_3 = [n+1] \left\langle \begin{array}{c} \text{Diagram 11} \\ \text{Diagram 12} \end{array} \right\rangle_3, \\
(5) \quad & \left\langle \begin{array}{c} \text{Diagram 13} \\ \text{Diagram 14} \end{array} \right\rangle_3 = \frac{[n+1][n+2]}{[2]} \emptyset.
\end{aligned}$$

We can obtain the following by easy calculation.

LEMMA 3.12. For $k = 0, 1, \dots, n$,

$$\left\langle \begin{array}{c} \text{Diagram 15} \\ \text{Diagram 16} \end{array} \right\rangle_3 = \frac{[n+1][n+2]}{[n-k+1][n-k+2]} \left\langle \begin{array}{c} \text{Diagram 17} \\ \text{Diagram 18} \end{array} \right\rangle_3.$$

This lemma gives Theorem 3.11 (5) when $k = n$. Theorem 3.11 (1) is obtained from a calculation of Theorem 3.11 (2) substituting q^{-1} with q . Thus, we only need to prove Theorem 3.11 (2)–(4). To prove Theorem 3.11 (2)–(4), we prepare lemmas.

LEMMA 3.13. For $n \geq 2$ and $0 \leq k \leq n-1$,

$$(3.7) \quad \left\langle \begin{array}{c} \text{Diagram 19} \\ \text{Diagram 20} \end{array} \right\rangle_3 = \left\langle \begin{array}{c} \text{Diagram 21} \\ \text{Diagram 22} \end{array} \right\rangle_3.$$

PROOF. From the definition of the colored trivalent graph and Lemma 3.10 (5), we obtain

$$\left\langle \begin{array}{c} \text{Diagram 23} \\ \text{Diagram 24} \end{array} \right\rangle_3 = \left\langle \begin{array}{c} \text{Diagram 25} \\ \text{Diagram 26} \end{array} \right\rangle_3 = \left\langle \begin{array}{c} \text{Diagram 27} \\ \text{Diagram 28} \end{array} \right\rangle_3.$$

We apply the following identity (and the mirror image of the A_2 webs) to the left-hand side of (3.7):

$$\left\langle \begin{array}{c} \text{Diagram 29} \\ \text{Diagram 30} \end{array} \right\rangle_3 = \left\langle \begin{array}{c} \text{Diagram 31} \\ \text{Diagram 32} \end{array} \right\rangle_3$$

We compute the following:

$$\begin{aligned}
\left\langle \begin{array}{c} \text{Diagram 33} \\ \text{Diagram 34} \end{array} \right\rangle_3 &= \left\langle \begin{array}{c} \text{Diagram 35} \\ \text{Diagram 36} \end{array} \right\rangle_3 = \left\langle \begin{array}{c} \text{Diagram 37} \\ \text{Diagram 38} \end{array} \right\rangle_3 \\
&= \left\langle \begin{array}{c} \text{Diagram 39} \\ \text{Diagram 40} \end{array} \right\rangle_3 + \left\langle \begin{array}{c} \text{Diagram 41} \\ \text{Diagram 42} \end{array} \right\rangle_3
\end{aligned}$$

and the second term vanishes by a property of the A_2 clasp. We slide  to the left side repeatedly as the above computation. Consequently, we obtain the equation (3.7). $\square$

LEMMA 3.14. For $n \geq 2$,

$$\left\langle \begin{array}{c} n \rightarrow \text{[diagram]} \leftarrow n \\ \uparrow n \end{array} \right\rangle_3 = \left\langle \begin{array}{c} n \rightarrow \text{[diagram]} \leftarrow n-1 \\ \uparrow 1 \quad \uparrow n-1 \end{array} \right\rangle_3 - \frac{[n-1]}{[n]} \left\langle \begin{array}{c} n \rightarrow \text{[diagram]} \leftarrow n-1 \\ \uparrow 1 \quad \uparrow n-1 \end{array} \right\rangle_3.$$

We define the right most A_2 web as $\left\langle \begin{array}{c} \text{[diagram]} \\ \uparrow \end{array} \right\rangle_3$ for $n = 2$. The above equation also holds for the opposite orientations.

PROOF. By using the definition of an A_2 clasp and Lemma 3.10 (6),

$$\begin{aligned} \left\langle \begin{array}{c} n \rightarrow \text{[diagram]} \leftarrow n \\ \uparrow n \end{array} \right\rangle_3 &= \left\langle \begin{array}{c} n \rightarrow \text{[diagram]} \leftarrow n-1 \\ \uparrow 1 \quad \uparrow n-1 \end{array} \right\rangle_3 \\ &= \left\langle \begin{array}{c} n \rightarrow \text{[diagram]} \leftarrow n-1 \\ \uparrow 1 \quad \uparrow n-1 \end{array} \right\rangle_3 - \frac{[n-1]}{[n]} \left\langle \begin{array}{c} n \rightarrow \text{[diagram]} \leftarrow n-1 \\ \uparrow 1 \quad \uparrow n-1 \end{array} \right\rangle_3. \end{aligned}$$

The A_2 web of the second term is computed as follows:

$$\begin{aligned} \left\langle \begin{array}{c} n \rightarrow \text{[diagram]} \leftarrow n-1 \\ \uparrow 1 \quad \uparrow n-1 \end{array} \right\rangle_3 &= \left\langle \begin{array}{c} n \rightarrow \text{[diagram]} \leftarrow n-1 \\ \uparrow 1 \quad \uparrow n-1 \end{array} \right\rangle_3 = \left\langle \begin{array}{c} n \rightarrow \text{[diagram]} \leftarrow n-1 \\ \uparrow 1 \quad \uparrow n-1 \end{array} \right\rangle_3 \\ &= \left\langle \begin{array}{c} n \rightarrow \text{[diagram]} \leftarrow n-1 \\ \uparrow 1 \quad \uparrow n-1 \end{array} \right\rangle_3 = \left\langle \begin{array}{c} n \rightarrow \text{[diagram]} \leftarrow n-1 \\ \uparrow 1 \quad \uparrow n-1 \end{array} \right\rangle_3 \\ &= \left\langle \begin{array}{c} n \rightarrow \text{[diagram]} \leftarrow n-1 \\ \uparrow 1 \quad \uparrow n-1 \end{array} \right\rangle_3 = \left\langle \begin{array}{c} n \rightarrow \text{[diagram]} \leftarrow n-1 \\ \uparrow 1 \quad \uparrow n-1 \end{array} \right\rangle_3. \end{aligned}$$

We used Lemma 3.10 (5) and (6) for the first three equalities, Lemma 3.10 (4) for the last equality. $\square$

PROOF OF THEOREM 3.11 (2). We prove Theorem 3.11 (2) in a similar way to the proof of the colored Kauffman bracket skein relation. By using skein relations the A_2

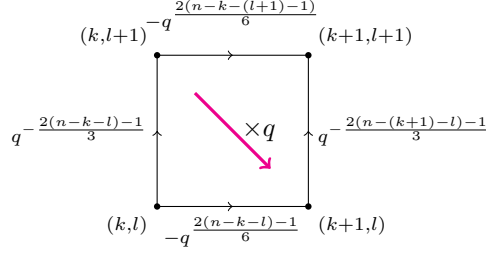


FIGURE 6.4. Shift of a path.

bracket and properties of A_2 clasps, we can easily calculate as follows:

$$\begin{aligned}
 \left\langle \begin{array}{c} \text{Diagram 1} \end{array} \right\rangle_3 &= \left\langle \begin{array}{c} \text{Diagram 2} \end{array} \right\rangle_3 \\
 &= q^{-\frac{1}{3}} \left\langle \begin{array}{c} \text{Diagram 3} \end{array} \right\rangle_3 - q^{\frac{1}{6}} \left\langle \begin{array}{c} \text{Diagram 4} \end{array} \right\rangle_3 \\
 &= q^{-\frac{2n-1}{3}} \left\langle \begin{array}{c} \text{Diagram 5} \end{array} \right\rangle_3 - q^{\frac{2n-1}{6}} \left\langle \begin{array}{c} \text{Diagram 6} \end{array} \right\rangle_3.
 \end{aligned}$$

We define clasped A_2 webs $\langle \sigma(k, l; n) \rangle_3$ as follows:

$$\langle \sigma(k, l; n) \rangle_3 = \left\langle \begin{array}{c} \text{Diagram 7} \end{array} \right\rangle_3.$$

The above calculation and Lemma 3.13 imply

$$\langle \sigma(k, l; n) \rangle_3 = q^{-\frac{1}{3}(2(n-k-l)-1)} \langle \sigma(k, l+1; n) \rangle_3 - q^{\frac{1}{6}(2(n-k-l)-1)} \langle \sigma(k+1, l; n) \rangle_3.$$

We can calculate the coefficient of $\langle \sigma(k, l; n) \rangle_3$ with $k + l = n$:

$$\prod_{i=0}^{l-1} q^{-\frac{1}{3}(2(n-i)-1)} \prod_{j=0}^{k-1} (-q^{\frac{1}{6}(2(n-j-l)-1)}) \left(\sum_{\lambda \in \mathcal{P}(k, l)} q^{|\lambda|} \right) = (-1)^k q^{\frac{-2n^2+3k^2}{6}} \binom{n}{k}_q,$$

in a similar way to the proof of Proposition 3.1 (1) (see Fig. 6.4). $\square$

PROOF OF THEOREM 3.11 (3) AND (4). We firstly give a proof of (3). By attaching “Y” to the lower clasp of the left hand side of Lemma 3.14, we obtain the following

equation:

$$(3.8) \quad \left\langle \begin{array}{c} n \leftarrow \begin{array}{c} \text{---} \xrightarrow{1} \text{---} \\ \text{---} \xrightarrow{n-1} \text{---} \\ \text{---} \xrightarrow{n-2} \text{---} \end{array} \rightarrow n-1 \\ \text{---} \xrightarrow{1} \text{---} \\ \text{---} \xrightarrow{n-2} \text{---} \end{array} \right\rangle_3 = \frac{[n-1]}{[n]} \left\langle \begin{array}{c} n \leftarrow \begin{array}{c} \text{---} \xrightarrow{1} \text{---} \\ \text{---} \xrightarrow{n-1} \text{---} \\ \text{---} \xrightarrow{n-2} \text{---} \end{array} \rightarrow n-1 \\ \text{---} \xrightarrow{1} \text{---} \\ \text{---} \xrightarrow{n-2} \text{---} \end{array} \right\rangle_3 \\ = \left\langle \begin{array}{c} n \leftarrow \begin{array}{c} \text{---} \xrightarrow{1} \text{---} \\ \text{---} \xrightarrow{n-1} \text{---} \\ \text{---} \xrightarrow{n-2} \text{---} \end{array} \rightarrow n-1 \\ \text{---} \xrightarrow{1} \text{---} \\ \text{---} \xrightarrow{n-2} \text{---} \end{array} \right\rangle_3.$$

The above equation also holds for the opposite orientations and the mirror image of the A_2 webs. We prove (3) by an induction on n . In the case of $n = 1$, it is the definition of the A_2 bracket. Apply Lemma 3.14 to the upper left and lower right colored trivalent vertex in turn:

$$\begin{aligned} \text{LHS of (3)} &= \left\langle \begin{array}{c} \text{---} \xrightarrow{n} \text{---} \\ \text{---} \xrightarrow{n} \text{---} \\ \text{---} \xrightarrow{n} \text{---} \end{array} \right\rangle_3 = \left\langle \begin{array}{c} \text{---} \xrightarrow{n} \text{---} \\ \text{---} \xrightarrow{n} \text{---} \\ \text{---} \xrightarrow{n} \text{---} \end{array} \right\rangle_3 \\ &= \left\langle \begin{array}{c} \text{---} \xrightarrow{n} \text{---} \\ \text{---} \xrightarrow{n} \text{---} \\ \text{---} \xrightarrow{n} \text{---} \end{array} \right\rangle_3 - \frac{[n-1]}{[n]} \left\langle \begin{array}{c} \text{---} \xrightarrow{n} \text{---} \\ \text{---} \xrightarrow{n} \text{---} \\ \text{---} \xrightarrow{n} \text{---} \end{array} \right\rangle_3 \\ &= \left\langle \begin{array}{c} \text{---} \xrightarrow{n} \text{---} \\ \text{---} \xrightarrow{n} \text{---} \\ \text{---} \xrightarrow{n} \text{---} \end{array} \right\rangle_3 - \left\langle \begin{array}{c} \text{---} \xrightarrow{n} \text{---} \\ \text{---} \xrightarrow{n} \text{---} \\ \text{---} \xrightarrow{n} \text{---} \end{array} \right\rangle_3. \end{aligned}$$

The last equation is obtained by applying (3.8) to the upper right, the lower left and the upper left colored trivalent vertex in turn. Then, we use the induction hypothesis and Lemma 3.13:

$$\begin{aligned} \text{LHS of (3)} &= \sum_{k=0}^{n-1} \left\langle \begin{array}{c} \text{---} \xrightarrow{1} \text{---} \\ \text{---} \xrightarrow{k} \text{---} \\ \text{---} \xrightarrow{k} \text{---} \\ \text{---} \xrightarrow{1} \text{---} \end{array} \right\rangle_3 - \sum_{k=1}^{n-1} \left\langle \begin{array}{c} \text{---} \xrightarrow{n-k} \text{---} \\ \text{---} \xrightarrow{k} \text{---} \\ \text{---} \xrightarrow{k} \text{---} \\ \text{---} \xrightarrow{n-k} \text{---} \end{array} \right\rangle_3 \\ &= \sum_{k=0}^{n-1} \left\langle \begin{array}{c} \text{---} \xrightarrow{n-k-1} \text{---} \\ \text{---} \xrightarrow{k+1} \text{---} \\ \text{---} \xrightarrow{k+1} \text{---} \\ \text{---} \xrightarrow{n-k-1} \text{---} \end{array} \right\rangle_3 + \sum_{k=0}^{n-1} \left\langle \begin{array}{c} \text{---} \xrightarrow{n-k} \text{---} \\ \text{---} \xrightarrow{k} \text{---} \\ \text{---} \xrightarrow{k} \text{---} \\ \text{---} \xrightarrow{n-k} \text{---} \end{array} \right\rangle_3 - \sum_{k=1}^{n-1} \left\langle \begin{array}{c} \text{---} \xrightarrow{n-k} \text{---} \\ \text{---} \xrightarrow{k} \text{---} \\ \text{---} \xrightarrow{k} \text{---} \\ \text{---} \xrightarrow{n-k} \text{---} \end{array} \right\rangle_3 \\ &= \sum_{k=0}^n \left\langle \begin{array}{c} \text{---} \xrightarrow{n-k} \text{---} \\ \text{---} \xrightarrow{k} \text{---} \\ \text{---} \xrightarrow{k} \text{---} \\ \text{---} \xrightarrow{n-k} \text{---} \end{array} \right\rangle_3. \end{aligned}$$

The second equation used

$$\left\langle \begin{array}{c} \text{---} \xrightarrow{n-k} \text{---} \\ \text{---} \xrightarrow{n-k} \text{---} \\ \text{---} \xrightarrow{n-k} \text{---} \\ \text{---} \xrightarrow{n-k} \text{---} \end{array} \right\rangle_3 = \left\langle \begin{array}{c} \text{---} \xrightarrow{n-k-1} \text{---} \\ \text{---} \xrightarrow{1} \text{---} \\ \text{---} \xrightarrow{1} \text{---} \\ \text{---} \xrightarrow{n-k-1} \text{---} \end{array} \right\rangle_3 + \left\langle \begin{array}{c} \text{---} \xrightarrow{n-k} \text{---} \\ \text{---} \xrightarrow{n-k} \text{---} \\ \text{---} \xrightarrow{n-k} \text{---} \\ \text{---} \xrightarrow{n-k} \text{---} \end{array} \right\rangle_3.$$

The above is obtained by straightforward calculation using the A_2 bracket skein relation and a property of the A_2 clasp.

We prove (4) by an induction on n . When $n = 1$, it is trivial because of the definition of the A_2 bracket. By using Lemma 3.14 and (3.8),

$$\begin{aligned}
 \text{LHS of (4)} &= \left\langle \begin{array}{c} \text{Diagram 1} \end{array} \right\rangle_3 - \frac{[n-1]}{[n]} \left\langle \begin{array}{c} \text{Diagram 2} \end{array} \right\rangle_3 \\
 &= \left\langle \begin{array}{c} \text{Diagram 3} \end{array} \right\rangle_3 - \frac{[n-1]}{[n]} \left\langle \begin{array}{c} \text{Diagram 4} \end{array} \right\rangle_3 \\
 &= \left\langle \begin{array}{c} \text{Diagram 5} \end{array} \right\rangle_3 - \left(\frac{[n-1][2]}{[n]} - \frac{[n-2]}{[n]} \right) \left\langle \begin{array}{c} \text{Diagram 6} \end{array} \right\rangle_3 \\
 &= \left\langle \begin{array}{c} \text{Diagram 7} \end{array} \right\rangle_3 - \left\langle \begin{array}{c} \text{Diagram 8} \end{array} \right\rangle_3.
 \end{aligned}$$

Finally, we use the induction hypothesis and $\left\langle \begin{array}{c} \text{Diagram 9} \end{array} \right\rangle_3 = [2] \left\langle \begin{array}{c} \text{Diagram 10} \end{array} \right\rangle_3$. $\square$

We also give a full twist and m full twists formula for the A_2 bracket.

PROPOSITION 3.15 (a full twist formula).

$$\left\langle \begin{array}{c} \text{Diagram 11} \end{array} \right\rangle_3 = q^{\frac{n^2}{3}} \sum_{k=0}^n q^{k^2 - n^2 + k - n} \frac{(q)_n}{(q)_k} \binom{n}{k}_q \left\langle \begin{array}{c} \text{Diagram 12} \end{array} \right\rangle_3.$$

PROOF. We prove the above formula in a similar way to the proof of Proposition 3.3.

$$\begin{aligned}
 \left\langle \begin{array}{c} \text{Diagram 13} \end{array} \right\rangle_3 &= \left\langle \begin{array}{c} \text{Diagram 14} \end{array} \right\rangle_3 \\
 &= -q^{\frac{1}{6}} \left\langle \begin{array}{c} \text{Diagram 15} \end{array} \right\rangle_3 + q^{-\frac{1}{3}} \left\langle \begin{array}{c} \text{Diagram 16} \end{array} \right\rangle_3 \\
 &= q^{\frac{1}{3}} \left\langle \begin{array}{c} \text{Diagram 17} \end{array} \right\rangle_3 - q^{-\frac{1}{6}} \left\langle \begin{array}{c} \text{Diagram 18} \end{array} \right\rangle_3 + q^{-\frac{5}{3} - \frac{2}{3}(n-1)} \left\langle \begin{array}{c} \text{Diagram 19} \end{array} \right\rangle_3 \\
 &= q^{\frac{1}{3}} \left\langle \begin{array}{c} \text{Diagram 20} \end{array} \right\rangle_3 + q^{-\frac{2}{3}n} (q^{-1} - 1) \left\langle \begin{array}{c} \text{Diagram 21} \end{array} \right\rangle_3.
 \end{aligned}$$

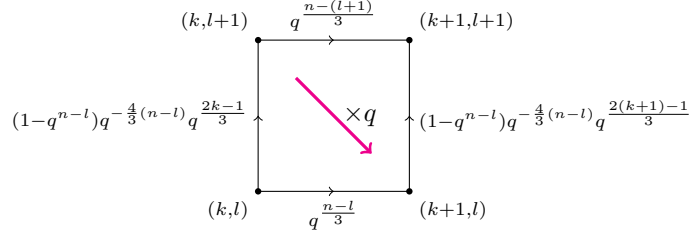


FIGURE 6.5. Shift of a path.

Then, we obtain

(3.9)

$$q^{-\frac{1}{3}(n-i)} \left\langle \begin{array}{c} n \\ \text{web} \\ 1 \end{array} \right\rangle_3 - q^{-\frac{1}{3}(n-i-1)} \left\langle \begin{array}{c} n \\ \text{web} \\ 1 \end{array} \right\rangle_3 = (1-q)q^{-n-1}q^i \left\langle \begin{array}{c} n \\ \text{web} \\ 1 \end{array} \right\rangle_3$$

for $i = 0, 1, \dots, n-1$. By taking the sum of both sides of (3.9) for $i = 0, 1, \dots, n-1$,

$$\left\langle \begin{array}{c} n \\ \text{web} \\ 1 \end{array} \right\rangle_3 = q^{\frac{n}{3}} \left\langle \begin{array}{c} n \\ \text{web} \\ 1 \end{array} \right\rangle_3 + (1-q^n)q^{-\frac{2n}{3}-1} \left\langle \begin{array}{c} n \\ \text{web} \\ 1 \end{array} \right\rangle_3.$$

From the above equation,

$$\left\langle \begin{array}{c} j \\ \text{web} \\ i \end{array} \right\rangle_3 = q^{\frac{j}{3}} \left\langle \begin{array}{c} j \\ \text{web} \\ i \end{array} \right\rangle_3 + (1-q^j)q^{-\frac{2}{3}(i+j)-\frac{1}{3}} \left\langle \begin{array}{c} j \\ \text{web} \\ i \end{array} \right\rangle_2$$

for any non-negative integers i and j . We set a clasped A_2 web $\langle \sigma^2(k, l; n) \rangle_3$ as

$$\langle \sigma^2(k, l; n) \rangle_3 = \left\langle \begin{array}{c} n-k-l \\ \text{web} \\ n \end{array} \right\rangle_3,$$

and obtain

(3.10)

$$\langle \sigma^2(k, l; n) \rangle_3 = q^{\frac{n-l}{3}} \langle \sigma^2(k+1, l; n) \rangle_3 + (1-q^{n-l})q^{-\frac{4}{3}(n-l)}q^{\frac{2k-1}{3}} \langle \sigma^2(k, l+1; n) \rangle_3$$

for non-negative integers k and l such that $k+l \leq n$. We make $\langle \sigma^2(k, l; n) \rangle_3$ correspond to lattice point (k, l) in a similar way to the proof of Proposition 3.3. The coefficient of $\langle \sigma^2(k, l; n) \rangle_3$ with $k+l = n$ can be calculated by using the resolution (3.10) and Fig. 6.5:

$$\left(\prod_{i=0}^{l-1} (1-q^{n-i})q^{-\frac{4}{3}(n-i)-\frac{1}{3}} \right) \left(\prod_{j=0}^{k-1} q^{\frac{n-l}{3}} \right) \left(\sum_{\lambda \in \mathcal{P}(k, l)} q^{|\lambda|} \right) = q^{\frac{n^2}{3}} q^{k^2-n^2+k-n} \frac{(q; q)_n}{(q; q)_k} \binom{n}{k}_q.$$

□

LEMMA 3.16.

$$\left\langle \begin{array}{c} n \\ \text{[diagram]} \\ n \end{array} \right\rangle_3 = q^{-\frac{n^2-k^2+3n-3k}{3}} \left\langle \begin{array}{c} n \\ \text{[diagram]} \\ n-k \end{array} \right\rangle_3$$

PROOF. We can prove it in a similar way to the proof of Lemma 3.4. $\square$

THEOREM 3.17 (m full twists formula for the A_2 bracket).

$$\begin{aligned} \left\langle \begin{array}{c} n \\ \text{[diagram]} \\ n \end{array} \right\rangle_3 &= q^{-\frac{2m}{3}(n^2+3n)} \sum_{0 \leq k_m \leq \dots \leq k_1 \leq n} q^{n-k_m} q^{\sum_{i=1}^m (k_i^2+2k_i)} \\ &\quad \times \frac{(q)_n}{(q)_{k_m}} \binom{n}{k'_1, k'_2, \dots, k'_m, k_m}_q \left\langle \begin{array}{c} n \\ \text{[diagram]} \\ n-k_m \end{array} \right\rangle_3, \end{aligned}$$

where k_i, k'_i are integers such that $k_0 = n, k'_{i+1} = k_i - k_{i+1}$ for $i = 0, 1, \dots, m-1$.

PROOF. We can prove it in the same way as proof of Proposition 3.5 by use of Proposition 3.15 and Lemma 3.16. $\square$

4. Bubble skein expansion formulas

In this section, we consider the bubble skein expansion formula. In the case of clasped A_1 web spaces, Hajij [Haj14a] proved the formula. First, we rewrite coefficients of the formula by using quantum binomial coefficients. Next, we give the bubble skein expansion formula for clasped A_2 web spaces.

4.1. The Kauffman bracket bubble skein expansion formula. Let Δ_n denote the coefficients of $\left\langle \bigcirc^n \right\rangle_2$, that is, $\Delta_n = (-1)^n [n+1]$.

THEOREM 4.1 (The Kauffman bracket bubble skein expansion formula by Hajij [Haj14a]). Let $m, n \geq k, l$ be positive integers.

$$\left\langle \begin{array}{c} n-k \\ \text{[diagram]} \\ m-k \end{array} \right\rangle_2 = \sum_{t=\max\{k,l\}}^{\min\{k+l,n,m\}} (-1)^{t-k-l} \frac{[n][m][t][t][n+m-t+1]}{[k][k][l][l][n+m-k-l+1]} \left\langle \begin{array}{c} n-k \\ \text{[diagram]} \\ m-k \end{array} \right\rangle_2$$

PROOF. Hajij gave this formula in [Haj14a] as follows. Let M, N, M', N' be non-negative integers and $k \geq l \geq 1$ positive integers. Then,

$$\begin{aligned} &\left\langle \begin{array}{c} N \\ \text{[diagram]} \\ M \end{array} \right\rangle_2 \\ &= \sum_{i=0}^{\min\{M,N,l\}} (-1)^{i(i-l)} \frac{(\prod_{j=0}^{l-i-1} \Delta_{k-j-1} \Delta_{M+N+k-i-j}) (\prod_{j=0}^{i-1} \Delta_{N-j-1} \Delta_{M-j-1})}{\prod_{j=0}^{l-1} \Delta_{N+k-j-1} \Delta_{M+k-j-1}} \left[\begin{array}{c} l \\ i \end{array} \right] \left\langle \begin{array}{c} N \\ \text{[diagram]} \\ M \end{array} \right\rangle_2. \end{aligned}$$

We can easily rewrite coefficients of the above formula and obtain our formula. $\square$

4.2. The A_2 bracket bubble skein expansion formula.

THEOREM 4.2 (The A_2 bracket bubble skein expansion formula).

$$\left\langle \begin{array}{c} n-k \quad n \quad n-l \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ m-k \quad m \quad m-l \end{array} \right\rangle_3 = \sum_{t=\max\{k,l\}}^{\min\{k+l,n,m\}} \frac{[n][m][t][t][n+m-t+2]}{[k][k][l][l][n][m]} \left\langle \begin{array}{c} n-k \quad n-t \quad n-l \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ m-k \quad m-t \quad m-l \end{array} \right\rangle_3$$

Firstly, we calculate the A_2 web appearing in the left-hand side of Theorem 2.7, we call it an A_2 bubble skein element, when $k = 1$ or $l = 1$.

LEMMA 4.3.

$$\begin{aligned} (1) \quad & \left\langle \begin{array}{c} n-1 \quad n \quad n-l \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ m-1 \quad m \quad m-l \end{array} \right\rangle_3 \\ &= \frac{[n+m-l+2][l]}{[n][m]} \left\langle \begin{array}{c} n-1 \quad n-l \\ \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \\ m-1 \quad m-l \end{array} \right\rangle_3 + \frac{[n-l][m-l]}{[n][m]} \left\langle \begin{array}{c} n-1 \quad n-l-1 \quad n-l \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ m-1 \quad m-l-1 \quad m-l \end{array} \right\rangle_3 \\ (2) \quad & \left\langle \begin{array}{c} n-k \quad n \quad n-1 \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ m-k \quad m \quad m-1 \end{array} \right\rangle_3 \\ &= \frac{[n+m-k+2][k]}{[n][m]} \left\langle \begin{array}{c} n-k \quad n-1 \\ \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \\ m-k \quad m-1 \end{array} \right\rangle_3 + \frac{[n-k][m-k]}{[n][m]} \left\langle \begin{array}{c} n-k \quad n-k-1 \quad n-1 \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ m-k \quad m-k-1 \quad m-1 \end{array} \right\rangle_3 \end{aligned}$$

PROOF. We only prove (1). For any integers k and l such that $1 \leq k, l \leq \min\{m, n\}$,

$$\begin{aligned}
\left\langle \begin{array}{c} n-k \quad n \quad n-l \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ m-k \quad m \quad m-l \end{array} \right\rangle_3 &= \frac{[m+2]}{[m]} \left\langle \begin{array}{c} n \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ m \end{array} \right\rangle_3 - \frac{[n-1]}{[n]} \left\langle \begin{array}{c} n-k \quad n-1 \quad n-l \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ m-k \quad m \quad m-l \end{array} \right\rangle_3 \\
&= \left(\frac{[m+2]}{[m]} - [2] \frac{[n-1]}{[n]} \right) \left\langle \begin{array}{c} n \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ m \end{array} \right\rangle_3 \\
&\quad + \frac{[n-1][m-1]}{[n][m]} \left\langle \begin{array}{c} n-k \quad n-1 \quad n-l \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ m-k \quad m \quad m-l \end{array} \right\rangle_3 \\
&= \left(\frac{[n][m+2] - [2][n-1][m] + [n-1][m-1]}{[n][m]} \right) \left\langle \begin{array}{c} n-k \quad n-1 \quad n-l \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ m-k \quad m-1 \quad m-l \end{array} \right\rangle_3 \\
&\quad + \frac{[n-1][m-1]}{[n][m]} \left\langle \begin{array}{c} n-k \quad n-2 \quad n-l \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ m-k \quad m-2 \quad m-l \end{array} \right\rangle_3.
\end{aligned}$$

We use the definition of A_2 clasp and Lemma 2.10 in the first equality. By Lemma 2.1 (1) and (2), we obtain $[2][m] = [m+1] + [m-1]$ and $[n][m+2] - [n-1][m+1] = [n+m+1]$. Thus,

$$(4.1) \quad \left\langle \begin{array}{c} n-k \quad n \quad n-l \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ m-k \quad m \quad m-l \end{array} \right\rangle_3 = \frac{[n+m+1]}{[n][m]} \left\langle \begin{array}{c} n-k \quad n-1 \quad n-l \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ m-k \quad m-1 \quad m-l \end{array} \right\rangle_3 + \frac{[n-1][m-1]}{[n][m]} \left\langle \begin{array}{c} n-k \quad n-2 \quad n-l \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ m-k \quad m-2 \quad m-l \end{array} \right\rangle_3.$$

We substitute $k = 1$ in (4.1). The A_2 web appearing in the second term of the right-hand side has only an A_2 bubble skein element decorated with 1 and $l-1$. By using (4.1) of $k = 1$ repeatedly, we can obtain

$$\begin{aligned}
\left\langle \begin{array}{c} n-1 \quad n \quad n-l \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ m-1 \quad m \quad m-l \end{array} \right\rangle_3 &= \frac{[n+m+1] + [n+m-1] + \cdots + [n+m-2l+3]}{[n][m]} \left\langle \begin{array}{c} n-1 \quad n-l \\ \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \\ m-1 \quad m-l \end{array} \right\rangle_3 \\
&\quad + \frac{[n-l][m-l]}{[n][m]} \left\langle \begin{array}{c} n-1 \quad n-l-1 \quad n-l \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ m-1 \quad m-l-1 \quad m-l \end{array} \right\rangle_3,
\end{aligned}$$

and confirm that $[n+m+1] + [n+m-1] + \cdots + [n+m-2l+3] = [n+m-l+2][l]$ (see Lemma 2.1 (1)). $\square$

PROOF OF THEOREM 2.7. We assume that $0 \leq k \leq l \leq \min\{n, m\}$. By substituting Lemma 4.3 (1) for (4.1),

$$(4.2) \quad \left\langle \begin{array}{ccc} n-k & n & n-l \\ \boxed{\leftarrow} & \boxed{\leftarrow} & \boxed{\leftarrow} \\ & \boxed{\leftarrow} & \\ k & \boxed{\leftarrow} & l \\ & \boxed{\leftarrow} & \\ m-k & m & m-l \\ \boxed{\rightarrow} & \boxed{\rightarrow} & \boxed{\rightarrow} \end{array} \right\rangle_3 = \frac{[n+m-l+2][l]}{[n][m]} \left\langle \begin{array}{ccc} n-k & n-1 & n-l \\ \boxed{\leftarrow} & \boxed{\leftarrow} & \boxed{\leftarrow} \\ & \boxed{\leftarrow} & \\ k-1 & \boxed{\leftarrow} & l-1 \\ & \boxed{\leftarrow} & \\ m-k & m-1 & m-l \\ \boxed{\rightarrow} & \boxed{\rightarrow} & \boxed{\rightarrow} \end{array} \right\rangle_3 \\ + \frac{[n-l][m-l]}{[n][m]} \left\langle \begin{array}{ccc} n-k & n-l-1 & n-l \\ \boxed{\leftarrow} & \boxed{\leftarrow} & \boxed{\leftarrow} \\ & \boxed{\leftarrow} & \\ k-1 & \boxed{\leftarrow} & l \\ & \boxed{\leftarrow} & \\ m-k & m-l-1 & m-l \\ \boxed{\rightarrow} & \boxed{\rightarrow} & \boxed{\rightarrow} \end{array} \right\rangle_3.$$

We used

$$\begin{aligned} [n+m+1] + [n+m-l+1][l-1] &= [n+m+1][1] - [n+m+1-l][1-l] \\ &= [n+m-l+2][l] \end{aligned}$$

in the above equation (see Lemma 2.1 (2)). We prove this theorem by the induction on $\max\{m, n\}$. From (4.2) and the induction hypothesis,

$$\begin{aligned} &\left\langle \begin{array}{ccc} n-k & n & n-l \\ \boxed{\leftarrow} & \boxed{\leftarrow} & \boxed{\leftarrow} \\ & \boxed{\leftarrow} & \\ k & \boxed{\leftarrow} & l \\ & \boxed{\leftarrow} & \\ m-k & m & m-l \\ \boxed{\rightarrow} & \boxed{\rightarrow} & \boxed{\rightarrow} \end{array} \right\rangle_3 \\ &= \frac{[n+m-l+2][l]}{[n][m]} \sum_{t=l-1}^{\min\{k+l-2, n-1, m-1\}} \frac{[n-1][m-1][t][t][n+m-t]}{[k-1][k-1][l-1][l-1]} \left\langle \begin{array}{ccc} n-k & n-t-1 & n-l \\ \boxed{\leftarrow} & \boxed{\leftarrow} & \boxed{\leftarrow} \\ & \boxed{\leftarrow} & \\ t-k+1 & \boxed{\leftarrow} & t-l+1 \\ & \boxed{\leftarrow} & \\ m-k & m-t-1 & m-l \\ \boxed{\rightarrow} & \boxed{\rightarrow} & \boxed{\rightarrow} \end{array} \right\rangle_3 \\ &+ \frac{[n-l][m-l]}{[n][m]} \sum_{t=l}^{\min\{k+l-1, n-1, m-1\}} \frac{[n-1][m-1][t][t][n+m-t]}{[k-1][k-1][l][l]} \left\langle \begin{array}{ccc} n-k & n-t-1 & n-l \\ \boxed{\leftarrow} & \boxed{\leftarrow} & \boxed{\leftarrow} \\ & \boxed{\leftarrow} & \\ t-k+1 & \boxed{\leftarrow} & t-l+1 \\ & \boxed{\leftarrow} & \\ m-k & m-t-1 & m-l \\ \boxed{\rightarrow} & \boxed{\rightarrow} & \boxed{\rightarrow} \end{array} \right\rangle_3 \end{aligned}$$

For $t > l$ and $\min\{k+l, n, m\} \neq k+l$,

$$\begin{aligned}
& \text{the coefficient of } \left\langle \begin{array}{c} n-k \quad n-t \quad n-l \\ \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \\ t-k \quad t-l \\ m-k \quad m-t \quad m-l \end{array} \right\rangle_3 \\
&= \left(\frac{[n+m-l+2][l]}{[n][m]} \right) \left(\frac{[n-1][m-1][t-1][t-1][n+m-t+1]}{[k-1][k-1][l-1][l-1][n+m-k-l+2]} \right) \\
&\quad + \left(\frac{[n-l][m-l]}{[n][m]} \right) \left(\frac{[n-1][m-1][t-1][t-1][n+m-t+1]}{[k-1][k-1][l][l][n+m-k-l+1]} \right) \\
&= \left(\frac{[n+m-l+2][k+l-t]+[t-l][n+m-k-l+2]}{[n+m-k-l+2]} \right) \\
&\quad \times \left(\frac{[l]}{[n][m]} \right) \left(\frac{[n-1][m-1][t-1][t-1][n+m-t+1]}{[k-1][k-1][l-1][l-1]} \right) \\
&= \left(\frac{[n+m+2-t][k]}{[n+m-k-l+2]} \right) \left(\frac{[l]}{[n][m]} \right) \left(\frac{[n][m]}{[k][l]} \right) \left(\frac{[n][m][t][t][n+m-t+1]}{[k][k][l][l][n+m-k-l+1]} \right) \\
&= \frac{[n][m][t][t][k][l][n+m-t+2]}{[k][k][l][l]}.
\end{aligned}$$

We used

$$[n+m-l+2][k-(t-l)] + [t-l][n+m-l+2-k] = [k][n+m+2-t]$$

in the third equality (see Lemma 2.1 (3)). If $t = l$, then the coefficient of $\left\langle \begin{array}{c} n-k \quad n-l \\ \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \\ l-k \quad l \\ m-k \quad m-l \end{array} \right\rangle_3$

is

$$\left(\frac{[n+m-l+2][l]}{[n][m]} \right) \left(\frac{[n-1][m-1][l-1][l-1][n+m-l+1]}{[k-1][k-1][l-1][l-1]} \right) = \frac{[n][m][l][k][n+m-l+2]}{[k][k][l][l]}.$$

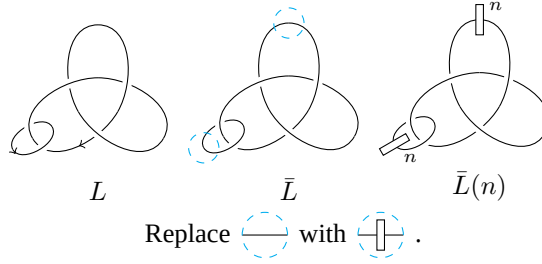
When $\min\{k+l, n, m\} = k+l$, we need to confirm the coefficient of $\left\langle \begin{array}{c} n-k \quad n-k-l \quad n-l \\ \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \\ l \quad k \\ m-k \quad m-k-l \quad m-l \end{array} \right\rangle_3$:

$$\left(\frac{[n-l][m-l]}{[n][m]} \right) \left(\frac{[n-1][m-1][k+l-1][k+l-1]}{[k-1][k-1][l][l]} \right) = \frac{[n][m][k+l][k+l]}{[k][k][l][l]}.$$

We remark that we can prove when $0 \leq l \leq k \leq \min\{n, m\}$ through the use of Lemma 4.3 (2) and (4.1). $\square$

REMARK 4.4. We can prove directly Theorem 4.1 in a similar way to the proof of Theorem 2.7.

$$\boxed{m} = \begin{cases} \text{right-handed } m \text{ half twists} & \text{if } m > 0, \\ \text{left-handed } m \text{ half twists} & \text{if } m < 0, \\ \text{ } & \text{if } m = 0. \end{cases}$$

FIGURE 6.6. m half twists.FIGURE 6.7. A Link diagram colored by n .

5. Colored Jones polynomials of 2-bridge links

In this section, we compute $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ colored Jones polynomials for 2-bridge links by using twist formulas. An explicit formula for the $n + 1$ dimensional $\mathfrak{sl}_2$ Jones polynomial of twist knots was given by Masbaum [Mas03] and, more generally, for 2-bridge knots was given by Takata [Tak08]. Both of these formulas were derived from the linear skein theory for the Kauffman bracket. We will give a more explicit formula of the $\mathfrak{sl}_2$ Jones polynomial for 2-bridge links in a similar way to [Mas03].

5.1. 2-bridge knot and link diagrams. We briefly recall link diagrams of the 2-bridge knots and links. You can find details on definitions of the 2-bridge knots and links and the classification by Schubert [Sch56] in [BZ03] and [Mur96]. Let m be an integer. A boxed m implies m half twists (see Fig. 6.6).

LEMMA 5.1. The 2-bridge knots and links have the following standard presentation.

$$\begin{aligned} l \text{ is odd : } & \overbrace{\boxed{2a_1} \boxed{2a_2} \boxed{2a_3} \boxed{2a_4} \cdots \boxed{2a_l}}^{\text{link diagram}} \\ l \text{ is even : } & \overbrace{\boxed{2a_1} \boxed{2a_2} \boxed{2a_3} \boxed{2a_4} \cdots \boxed{2a_l}}^{\text{link diagram}} \end{aligned}$$

The above $a_1, a_2, \dots, a_l$ are non-zero integers. We denote this link diagram by $[2a_1, 2a_2, \dots, 2a_l]$.

PROOF. See, for example, [Kaw96, Chapter 2]. $\square$

5.2. The $\mathfrak{sl}_2$ colored Jones polynomial. We introduce the $\mathfrak{sl}_2$ colored Jones polynomial. Let L be an oriented link diagram in D_0 with ordered link components $(L_1, L_2, \dots, L_r)$. We denote by $\bar{L} = (\bar{L}_1, \bar{L}_2, \dots, \bar{L}_r)$ an unoriented link diagram obtained by forgetting the orientation of L . For each link component $\bar{L}_i$ ($i = 1, 2, \dots, r$) of $\bar{L}$, we cut away a short strand from $\bar{L}_i$ and substitute $\boxed{n}$ for it. Thus, we obtain the unoriented link diagram colored by n . We denote it by $\bar{L}(n)$. (See Fig. 6.7.)

DEFINITION 5.2. The $\mathfrak{sl}_2$ colored Jones polynomial $J_{n+1}^{\mathfrak{sl}_2}(L; q)$ of a link represented by L is defined by

$$J_{n+1}^{\mathfrak{sl}_2}(L; q) = ((-1)^n q^{\frac{n^2+2n}{4}})^{-w(L)} \langle \bar{L}(n) \rangle_2 / \langle \bigoplus^n \rangle_2,$$

where $w(L)$ is the writhe of L .

LEMMA 5.3.

$$\left\langle \begin{array}{c} \text{Diagram of a link with two components, each a full twist of } n-k \text{ strands.} \\ \text{The top component has } n-k \text{ strands, and the bottom component has } n-k \text{ strands.} \\ \text{The strands are labeled } n-k \text{ and } n. \end{array} \right\rangle_2 = (-1)^{n-k} q^{\frac{k-n}{2}} \frac{(1-q^{n+1})}{(1-q^{k+1})} \left\langle \bigoplus^n \right\rangle_2.$$

THEOREM 5.4. Let $a_1, a_2, \dots, a_l$ be non-zero integers.

$$\begin{aligned} & J_{n+1}^{\mathfrak{sl}_2}([2a_1, 2a_2, \dots, 2a_l]; q) \\ &= \sum_{k_{|a_1|}^{(1)}=0}^{K_0} \sum_{k_{|a_2|}^{(2)}=0}^{K_1} \dots \sum_{k_{|a_l|}^{(l)}=0}^{K_{l-1}} \\ & \left(\prod_{j=0}^{l-1} \left(\sum_{\substack{k_{|a_{j+1}|}^{(j+1)} \leq k_{|a_{j+1}|}^{(j+1)} \leq \dots \leq k_1^{(j+1)} \leq K_j}} (-1)^{K_j - k_{|a_{j+1}|}^{(j+1)}} q^{a_{j+1}(n^2+2n)} q^{\frac{\varepsilon_{j+1}}{2}(K_j - k_{|a_{j+1}|}^{(j+1)})} \right. \right. \\ & \quad \times q^{\varepsilon_{j+1} \sum_{i=1}^{|a_{j+1}|} (k_i^{(j+1)^2} + k_i^{(j+1)})} \frac{(q^{\varepsilon_{j+1}})_{K_j}}{(q^{\varepsilon_{j+1}})_{k_{|a_{j+1}|}^{(j+1)}}} \binom{n}{k_1^{(j+1)'}, k_2^{(j+1)'}, \dots, k_{|a_{j+1}|}^{(j+1)'}, k_{|a_{j+1}|}^{(j+1)}}_{q^{\varepsilon_{j+1}}} \Bigg) \\ & \quad \times (-1)^{n-K_l} q^{\frac{K_l-n}{2}} \frac{1-q^{n+1}}{1-q^{K_l+1}} \Bigg), \end{aligned}$$

where $\varepsilon_{j+1} = \frac{a_{j+1}}{|a_{j+1}|}$, $K_0 = n$, $K_j = n - k_{|a_j|}^{(j)}$ and $k_0^{(j)} = K_j$, $k_{i+1}^{(j+1)'} = k_i^{(j)} - k_{i+1}^{(j)}$.

PROOF. We calculate the following clasped A_1 web using Lemma 3.4 and Proposition 3.6.

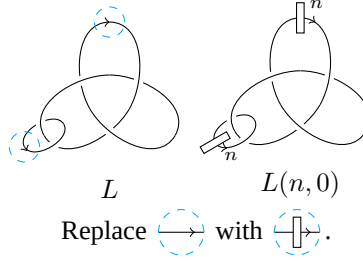
$$\begin{aligned}
& \left\langle \begin{array}{c} \text{Diagram: A box with } 2a_{j+1} \text{ strands, } K_j \text{ strands on the left, and } n-K_j \text{ strands on the right.} \end{array} \right\rangle_2 \\
&= \delta_n(K_j; q^{\varepsilon_{j+1}})_2^{2|a_{j+1}|} \left\langle \begin{array}{c} \text{Diagram: A box with } 2a_{j+1} \text{ strands, } K_j \text{ strands on the left, and } n-K_j \text{ strands on the right.} \end{array} \right\rangle_2 \\
&= \sum_{k_{|a_{j+1}|}^{(j+1)}=0}^{K_j} \left(\sum_{k_{|a_{j+1}|}^{(j+1)} \leq k_{|a_{j+1}|-1}^{(j+1)} \dots \leq k_1^{(j+1)} \leq K_j} \delta_n(K_j; q^{\varepsilon_{j+1}})_2^{2|a_{j+1}|} \gamma_{K_j}(k_1^{(j+1)}, k_2^{(j+1)}, \dots, k_{|a_{j+1}|}^{(j+1)}; q^{\varepsilon_{j+1}})_2 \right. \\
&\quad \times \left. \left\langle \begin{array}{c} \text{Diagram: A box with } 2a_{j+1} \text{ strands, } k_{|a_{j+1}|}^{(j+1)} \text{ strands on the left, and } n-k_{|a_{j+1}|}^{(j+1)} \text{ strands on the right.} \end{array} \right\rangle_2 \right) \\
&= \sum_{k_{|a_{j+1}|}^{(j+1)}=0}^{K_j} \left(\sum_{k_{|a_{j+1}|}^{(j+1)} \leq k_{|a_{j+1}|-1}^{(j+1)} \dots \leq k_1^{(j+1)} \leq K_j} \delta_n(K_j; q^{\varepsilon_{j+1}})_2^{2|a_{j+1}|} \gamma_{K_j}(k_1^{(j+1)}, k_2^{(j+1)}, \dots, k_{|a_{j+1}|}^{(j+1)}; q^{\varepsilon_{j+1}})_2 \right. \\
&\quad \times \left. \left\langle \begin{array}{c} \text{Diagram: A box with } 2a_{j+1} \text{ strands, } K_{j+1} \text{ strands on the left, and } n-K_{j+1} \text{ strands on the right.} \end{array} \right\rangle_2 \right),
\end{aligned}$$

where

$$\begin{aligned}
\delta_n(K_j; q)_2 &= (-1)^{n-K_j} q^{-\frac{n^2+2n-K_j^2-2K_j}{4}} \quad \text{and} \\
\gamma_{K_j}(k_1^{(j+1)}, k_2^{(j+1)}, \dots, k_{|a_{j+1}|}^{(j+1)}; q)_2 &= (-1)^{K_j-k_{|a_{j+1}|}^{(j+1)}} q^{-\frac{|a_{j+1}|}{2}(k_{|a_{j+1}|}^{(j+1)} + 2k_{|a_{j+1}|}^{(j+1)})} \\
&\quad \times q^{\frac{1}{2}(K_j-k_{|a_{j+1}|}^{(j+1)})} q^{\sum_{i=1}^{|a_{j+1}|} (k_i^{(j+1)} + k_i^{(j+1)})} \\
&\quad \times \frac{(q)_{K_j}}{(q)_{k_{|a_{j+1}|}^{(j+1)}}} \binom{n}{k_1^{(j+1)'}, k_2^{(j+1)'}, \dots, k_{|a_{j+1}|}^{(j+1)'}, k_{|a_{j+1}|}^{(j+1)'}}_q.
\end{aligned}$$

We also obtain the following in the same way.

$$\begin{aligned}
& \left\langle \begin{array}{c} \text{Diagram: A box with } 2a_{j+1} \text{ strands, } K_j \text{ strands on the left, and } n-K_j \text{ strands on the right.} \end{array} \right\rangle_2 \\
&= \sum_{k_{|a_{j+1}|}^{(j+1)}=0}^{K_j} \left(\sum_{k_{|a_{j+1}|}^{(j+1)} \leq k_{|a_{j+1}|-1}^{(j+1)} \dots \leq k_1^{(j+1)} \leq K_j} \delta_n(K_j; q^{\varepsilon_{j+1}})_2^{2|a_{j+1}|} \gamma_{K_j}(k_1^{(j+1)}, k_2^{(j+1)}, \dots, k_{|a_{j+1}|}^{(j+1)}; q^{\varepsilon_{j+1}})_2 \right. \\
&\quad \times \left. \left\langle \begin{array}{c} \text{Diagram: A box with } 2a_{j+1} \text{ strands, } K_{j+1} \text{ strands on the left, and } n-K_{j+1} \text{ strands on the right.} \end{array} \right\rangle_2 \right).
\end{aligned}$$

FIGURE 6.8. A Link diagram colored by $(n, 0)$.

Therefore,

$$\begin{aligned}
 & \langle \bar{L}(n) \rangle_2 \\
 &= \sum_{k_{|a_1|}^{(1)}=0}^{K_0} \sum_{k_{|a_2|}^{(2)}=0}^{K_1} \cdots \sum_{k_{|a_l|}^{(l)}=0}^{K_{l-1}} \\
 & \left(\prod_{j=0}^{l-1} \left(\sum_{k_{|a_{j+1}|}^{(j+1)} \leq k_{|a_{j+1}|-1}^{(j+1)} \leq \cdots \leq k_1^{(j+1)} \leq K_j} \delta_n(K_j; q^{\varepsilon_{j+1}})_2^{2|a_{j+1}|} \gamma_{K_j}(k_1^{(j+1)}, k_2^{(j+1)}, \dots, k_{|a_{j+1}|}^{(j+1)}; q^{\varepsilon_{j+1}})_2 \right) \right) \\
 & \times \left\langle \begin{array}{c} \text{Diagram with } n-K_l \text{ and } n \text{ boxes} \\ \text{Diagram with } n-K_l \text{ and } n \text{ boxes} \end{array} \right\rangle_2
 \end{aligned}$$

for $L = [2a_1, 2a_2, \dots, 2a_l]$. The writhe of $[2a_1, 2a_2, \dots, 2a_l]$ is $-2(a_1 + a_2 + \dots + a_l)$. Lemma 5.3 and explicit calculation of the coefficient imply the formula in this theorem. $\square$

5.3. The $\mathfrak{sl}_3$ colored Jones polynomials. We introduce the $\mathfrak{sl}_3$ colored Jones polynomial of type $(n, 0)$. Let L be an oriented link diagram in D_0 with ordered link components $(L_1, L_2, \dots, L_r)$. We replace a part of L_i with $\begin{array}{c} \boxed{n} \\ \rightarrow \end{array}$ for $i = 1, 2, \dots, r$ (see Fig. 6.8). We denote this oriented link diagram decorated with white boxes by $L(n, 0)$.

DEFINITION 5.5. The colored $\mathfrak{sl}_3$ Jones polynomial $J_{(n,0)}^{\mathfrak{sl}_3}(L; q)$ of a link represented by a link diagram L is defined by

$$J_{(n,0)}^{\mathfrak{sl}_3}(L; q) = (q^{\frac{n^2+3n}{3}})^{-w(L)} \langle L(n, 0) \rangle_3 / \langle \bigoplus^n \rangle_3,$$

where $w(L)$ is the writhe of L .

LEMMA 5.6.

$$\left\langle \begin{array}{c} \text{Diagram with } n-k \text{ and } n \text{ boxes} \\ \text{Diagram with } n-k \text{ and } n \text{ boxes} \end{array} \right\rangle_3 = q^{-(n-k)} \frac{(1-q^{n+1})(1-q^{n+2})}{(1-q^{k+1})(1-q^{k+2})} \left\langle \bigoplus^n \right\rangle_3.$$

PROOF. Use Lemma 2.10. $\square$

THEOREM 5.7.

$$\begin{aligned}
& J_{(n,0)}^{\mathfrak{sl}_3}([2a_1, 2a_2, \dots, 2a_l]; q) \\
&= \sum_{k_{|a_1|}^{(1)}=0}^{K_0} \sum_{k_{|a_2|}^{(2)}=0}^{K_1} \cdots \sum_{k_{|a_l|}^{(l)}=0}^{K_{l-1}} \\
& \left(\prod_{j=0}^{l-1} \left(\sum_{\substack{k_{|a_{j+1}|}^{(j+1)} \leq k_{|a_{j+1}|}^{(j+1)} - 1 \leq \dots \leq k_1^{(j+1)} \leq K_j}} q^{\varepsilon_{j+1}(K_j - k_{|a_{j+1}|}^{(j+1)})} q^{\varepsilon_{j+1} \sum_{i=1}^{|a_{j+1}|} (k_i^{(j+1)})^2 + 2k_i^{(j+1)}} \right) \right. \\
& \quad \times \frac{(q^{\varepsilon_{j+1}})_{K_j}}{(q^{\varepsilon_{j+1}})_{k_{|a_{j+1}|}^{(j+1)}}} \left(k_1^{(j+1)'}, k_2^{(j+1)'}, \dots, k_{|a_{j+1}|}^{(j+1)'}, k_{|a_{j+1}|}^{(j+1)} \right)_{q^{\varepsilon_{j+1}}} \Bigg) \\
& \quad \times q^{-(n-K_l)} \frac{(1-q^{n+1})(1-q^{n+2})}{(1-q^{K_l+1})(1-q^{K_l+2})},
\end{aligned}$$

where $\varepsilon_{j+1} = \frac{a_{j+1}}{|a_{j+1}|}$, $K_0 = n$, $K_j = n - k_{|a_j|}^{(j)}$ and $k_0^{(j)} = K_j$, $k_{|a_{j+1}|}^{(j+1)'} = k_i^{(j)} - k_{i+1}^{(j)}$.

PROOF. It is proved in the same way as the proof of Theorem 5.4 using Proposition 2.6 and Lemma 5.6. Instead of $\delta_n(K_j; q)_2$ and $\gamma_{K_j}(k_1^{(j+1)}, k_2^{(j+1)}, \dots, k_{|a_{j+1}|}^{(j+1)}; q)_2$, we use

$$\begin{aligned}
\delta_n(K_j; q)_3 &= q^{-\frac{n^2+3n-K_j^2-3K_j}{3}} \quad \text{and} \\
\gamma_{K_j}(k_1^{(j+1)}, k_2^{(j+1)}, \dots, k_{|a_{j+1}|}^{(j+1)}; q)_3 &= q^{-\frac{2|a_{j+1}|}{3}(k_{|a_{j+1}|}^{(j+1)})^2 + 3k_{|a_{j+1}|}^{(j+1)}} \\
& \quad \times q^{K_j - k_{|a_{j+1}|}^{(j+1)}} q^{\sum_{i=1}^{|a_{j+1}|} (k_i^{(j+1)})^2 + 2k_i^{(j+1)}} \\
& \quad \times \frac{(q)_{K_j}}{(q)_{k_{|a_{j+1}|}^{(j+1)}}} \left(k_1^{(j+1)'}, k_2^{(j+1)'}, \dots, k_{|a_{j+1}|}^{(j+1)'}, k_{|a_{j+1}|}^{(j+1)} \right)_q,
\end{aligned}$$

respectively. $\square$

REMARK 5.8. Wedrich gave a formula for the HOMFLY polynomials of 2-bridge links colored with one-column Young diagrams in [Wed14]. We consider that our formula should coincide with specialized his formula. He also suggest us the relationship between our calculus and the calculus of MOY graphs [MOY98] thorough a work of Tubbenhauer, Vaz and Wedrich [TVW15]. However, we have not confirmed the specific relationship.

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A q -series identity via the $\mathfrak{sl}_3$ colored Jones polynomials for the $(2, 2m)$ -torus link

1. Introduction

The Rogers-Ramanujan identities were first discovered and proved by Rogers [Rog94]. After that, these identities appeared in Ramanujan's first letter to Hardy without proof. To this day, the Rogers-Ramanujan identities have been generalized by many people and have been proved in many ways. One of these generalizations is the Andrews-Gordon identities.

THEOREM 1.1 (The Andrews-Gordon identities for the Ramanujan theta function [And74]).

$$f(-q^{2m}, -q) = (q; q)_\infty \sum_{k_m \leq \dots \leq k_2 \leq k_1} \frac{q^{\sum_{j=1}^{m-1} k_j(k_j+1)}}{\prod_{j=1}^{m-1} (q; q)_{k_j - k_{j+1}}},$$

where $m > 0$ and $f(a, b) = \sum_{i=0}^{\infty} a^{\frac{i(i+1)}{2}} b^{\frac{i(i-1)}{2}} + \sum_{i=1}^{\infty} a^{\frac{i(i-1)}{2}} b^{\frac{i(i+1)}{2}}$ is Ramanujan's general theta function (see, for example, [AB05]).

In knot theory, the Andrews-Gordon identities appear through the $\mathfrak{sl}_2$ colored Jones polynomial. The colored Jones polynomial is a q -polynomial invariant of knots and links. Dasbach and Lin showed the stability of some coefficients of the colored Jones polynomials for alternating knots in [DL06, DL07]. They conjectured that the k -th coefficient of the $n+1$ dimensional colored Jones polynomial $J_n(K; q)$ for an alternating knot K coincides up to sign for $n \geq k$. This suggests the existence of q -series $\sum_{k=0}^{\infty} a_k q^k$ for an alternating link K called the *tail*. The coefficient a_k is given by the k -th coefficient of $J_n(K; q)$ with $n \geq k$. Armond [Arm13] proved the existence of tails for adequate links, in particular for alternating knots. Independently, for alternating knots, Garoufalidis and Lê [GL15] showed a more general stability of coefficients of $J_n(K; q)$. In [AD11], Armond and Dasbach showed two explicit formulas for the tails of the colored Jones polynomial for the $(2, 2m+1)$ -torus knot. These two formulas conclude the left- and right-hand side of the Andrews-Gordon identities for the Ramanujan theta function. They used the formula for the colored Jones polynomial of the $(2, 2m+1)$ -torus knot in [Mor95] and obtained by the method of Armond [Arm14]. Hajij [Haj16] also showed the Andrews-Gordon identities for the Ramanujan theta function and the Ramanujan false theta function using two expressions of the tail of the $(2, 2m+1)$ -torus knot and the $(2, 2m)$ -torus link, respectively.

THEOREM 1.2 (The Andrews-Gordon identities for the Ramanujan false theta function [Haj16]).

$$\Psi(q^{2m-1}, q) = (q; q)_\infty \sum_{k_{m-1} \leq \dots \leq k_2 \leq k_1} \frac{q^{\sum_{j=1}^{m-1} k_j(k_j+1)}}{(q; q)_{k_{m-1}}^2 \prod_{j=1}^{m-2} (q; q)_{k_j - k_{j+1}}},$$

where $m > 1$ and $\Psi(a, b) = \sum_{i=0}^{\infty} a^{\frac{i(i+1)}{2}} b^{\frac{i(i-1)}{2}} - \sum_{i=0}^{\infty} a^{\frac{i(i-1)}{2}} b^{\frac{i(i+1)}{2}}$ is Ramanujan's general false theta function (see, for example, [MSZ09]).

He used formulas for the colored Jones polynomial obtained by using a colored trivalent graph representation of the Kauffman bracket skein elements (see, for example, [KL94b]) and the Kauffman bracket bubble skein expansion formula in [Haj14a].

The main subject of this paper is the tail of the colored $\mathfrak{sl}_3$ Jones polynomials for the $(2, 2m)$ -torus link. The existence of tails of the $\mathfrak{sl}_3$ colored Jones polynomials for any class of knots and links is unknown. Moreover, there is no example of an explicit formula for a tail of the $\mathfrak{sl}_3$ colored Jones polynomial. We will give two explicit formulas of “the $\mathfrak{sl}_3$ tail” for the $(2, 2m)$ -torus link. As the result, the following q -series identity holds.

THEOREM 4.3.

$$\sum_{i=0}^{\infty} q^{-2i} q^{m(i^2+2i)} \frac{(1 - q^{i+1})^3 (1 + q^{i+1})}{1 - q} = (q)_{\infty} \sum_{0 \leq k_m \leq \dots \leq k_2 \leq k_1} \frac{q^{-2k_m} q^{\sum_{j=1}^m (k_i^2 + 2k_i)}}{(q)_{k_m}^2 (q)_{k_1 - k_2} \dots (q)_{k_{m-1} - k_m}}.$$

The right-hand side of the above identity is obtained from the full twist formula in [Yua16]. The left-hand side is obtained by using of colored trivalent graph presentation of A_2 webs.

The paper is organized as follows. We first review definitions and formulas related to the A_2 web space in section 2. The A_2 web space is a generalization of the Kauffman bracket skein module. In section 3, we introduce a method to represent some types of A_2 webs using colored trivalent graphs. Furthermore, we give values of some θ -graphs and quantum $6j$ symbols. As an application, we explicitly give the $\mathfrak{sl}_3$ colored Jones polynomial for a 2-bridge link. In section 4, we derive two q -series, that is “the $\mathfrak{sl}_3$ tails”, by using explicit formulas for the $\mathfrak{sl}_3$ colored Jones polynomial for the $(2, 2m)$ -torus link in section 3 and in [Yua16].

2. The A_2 web space and some formulas

The skein theory has been developed with a quantum representation of Lie algebra $\mathfrak{g}$. If $\mathfrak{g} = \mathfrak{sl}_2$, the skein theory consists of the Kauffman bracket skein module, which is called the Temperley-Lieb algebra and the A_1 web space, and the Kauffman bracket. Kuperberg constructed the skein theory for Lie algebras of rank 2 in [Kup94, Kup96]. In this section, we review definitions of the A_2 web space, the A_2 bracket and the A_2 clasp.

Let $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m)$ be an m -tuple of signs $+$ or $-$. Let D_{ε} denote the unit disk with signed marked points $\{\exp(2\pi\sqrt{-1}/m)^{j-1} \mid j = 1, 2, \dots, m\}$ on its boundary. The sign of $\exp(2\pi\sqrt{-1}/m)^{j-1}$ is given by ε_j for $j = 1, 2, \dots, m$. A *bipartite uni-trivalent graph* G is a directed graph such that each vertex is either trivalent or univalent and the vertices are divided into the sinks and the sources. A sink (resp. source) is a vertex such that all edges adjoining to the vertex point into (resp. away from) it. A *bipartite trivalent graph* G in D_{ε} is an embedding of a uni-trivalent graph into D_{ε} such that any vertex v has the following neighborhoods:



An A_2 *basis web* is the boundary-fixing isotopy class of a bipartite trivalent graph G in D_{ε} , where any internal face of $D_{\varepsilon} \setminus G$ has at least six sides. Let us denote B_{ε} the set of A_2

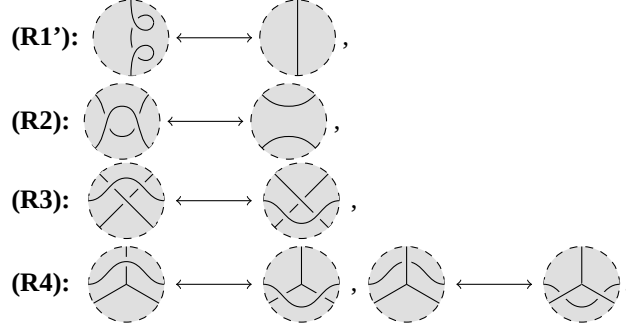
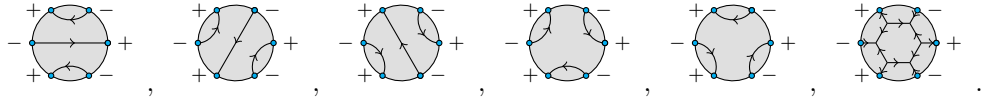


FIGURE 7.1. The Reidemeister moves for tangled trivalent graph diagrams

basis webs in D_ε . For example, $B_{(+,-,+,-,+,-)}$ has the following A_2 basis webs:



The A_2 web space W_ε is the $\mathbb{Q}(q^{\frac{1}{6}})$ -vector space spanned by B_ε . A *tangled trivalent graph diagram* in D_ε is an immersed bipartite uni-trivalent graph in D_ε whose intersection points are only transverse double points of edges with crossing data or . Tangled trivalent graph diagrams G and G' are regularly isotopic if G is obtained from G' by a finite sequence of boundary-fixing isotopies and Reidemeister moves, see Figure 7.1, with some direction of edges.

Tangled trivalent graphs in D_ε are regular isotopy classes of tangled trivalent graph diagrams in D_ε . We denote T_ε the set of tangled trivalent graphs in D_ε .

DEFINITION 2.1 (The A_2 bracket [Kup96]). We define a $\mathbb{Q}(q^{\frac{1}{6}})$ -linear map $\langle \cdot \rangle_3: \mathbb{Q}(q^{\frac{1}{6}})T_\varepsilon \rightarrow W_\varepsilon$ by the following.

- $\langle \text{crossing} \rangle_3 = q^{\frac{1}{3}} \langle \text{loop} \rangle_3 - q^{-\frac{1}{6}} \langle \text{trivalent vertex} \rangle_3$,
- $\langle \text{crossing} \rangle_3 = q^{-\frac{1}{3}} \langle \text{loop} \rangle_3 - q^{\frac{1}{6}} \langle \text{trivalent vertex} \rangle_3$,
- $\langle \text{trivalent vertex} \rangle_3 = \langle \text{loop} \rangle_3 + \langle \text{trivalent vertex} \rangle_3$,
- $\langle \text{loop} \rangle_3 = [2] \langle \text{trivalent vertex} \rangle_3$,
- $\langle G \sqcup \text{loop} \rangle_3 = [3] \langle G \rangle_3$,

where $[n] = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$ is a quantum integer.

We remark that this map is invariant under the Reidemeister moves for tangled trivalent graphs.

We next consider the A_2 web space $W_{n^++n^-} = W_{(+,+, \dots, +, -, \dots, -)}$ whose first n marked points are decorated with $+$ and next n marked points are decorated with $-$. We define A_2 clasps $\text{---} \boxed{n} \text{---} \in W_{n^++n^-}$ inductively by the following.

DEFINITION 2.2. (The A_2 clasps)

$$(2.1) \quad \begin{aligned} & \text{---} \boxed{\text{---}} \xrightarrow{1} = \text{---} \xrightarrow{1} \in W_{1^+ + 1^-} \\ & \text{---} \boxed{\text{---}} \xrightarrow{n} = \left\langle \text{---} \boxed{\text{---}} \xrightarrow{n-1} \right\rangle_3 - \frac{[n-1]}{[n]} \left\langle \text{---} \boxed{\text{---}} \xrightarrow{n-2} \right\rangle_3 \in W_{n^+ + n^-} \end{aligned}$$

A_2 clasps have the following properties.

LEMMA 2.3 (Properties of A_2 clasps). For any positive integer n ,

- $\left\langle \text{---} \boxed{\text{---}} \xrightarrow{n} \right\rangle_3 = \text{---} \boxed{\text{---}} \xrightarrow{n}$,
- $\left\langle \text{---} \boxed{\text{---}} \xrightarrow{n-k-2} \right\rangle_3 = 0 \quad (k = 0, 1, \dots, n-2)$.

We also define the A_2 clasp of type (n, m) according to Ohtsuki and Yamada [OY97].

DEFINITION 2.4 (the A_2 clasp of type (n, m)).

$$\left\langle \text{---} \boxed{\text{---}} \xrightarrow{n} \right\rangle_3 = \sum_{k=0}^{\min\{m, n\}} (-1)^k \frac{[n][m]}{[n+m+1]} \left\langle \text{---} \boxed{\text{---}} \xrightarrow{n-k} \right\rangle_3$$

LEMMA 2.5 (Property of A_2 clasps of type (m, n)).

$$\left\langle \text{---} \boxed{\text{---}} \xrightarrow{n-1} \right\rangle_3 = 0.$$

We review some formulas for clasped A_2 web spaces in [Yua16]. We define a q -Pochhammer symbol as

$$(q; q)_k = \prod_{l=1}^k (1 - q^l).$$

We abbreviate it as $(q)_k$. A q -binomial coefficient is defined by

$$\binom{n}{k}_q = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}$$

for $k \leq n$. If $k > n$, we define it by 0. We also define a q -multinomial coefficient as

$$\binom{n}{n_1, n_2, \dots, n_m}_q = \frac{(q)_n}{(q)_{n_1} (q)_{n_2} \cdots (q)_{n_m}},$$

where $n_1, n_2, \dots, n_m$ are non-negative integers such that $n_1 + n_2 + \cdots + n_m = n$.

THEOREM 2.6 (The m full twists formula [Yua16]).

$$\begin{aligned} \left\langle \text{---} \boxed{\text{---}} \xrightarrow{n} \right\rangle_3 &= q^{-\frac{2m}{3}(n^2+3n)} \sum_{0 \leq k_m \leq \cdots \leq k_1 \leq n} q^{n-k_m} q^{\sum_{i=1}^m (k_i^2+2k_i)} \\ &\times \frac{(q)_n}{(q)_{k_m}} \binom{n}{k'_1, k'_2, \dots, k'_m, k_m}_q \left\langle \text{---} \boxed{\text{---}} \xrightarrow{n-k_m} \right\rangle_3, \end{aligned}$$

where k_i, k'_i are integers such that $k_0 = n, k'_{i+1} = k_i - k_{i+1}$ for $i = 0, 1, \dots, m-1$.

THEOREM 2.7 (The A_2 bracket bubble skein expansion formula [Yua16]).

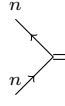
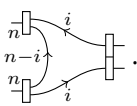
$$\left\langle \begin{array}{c} n-k \quad n \quad n-l \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ m-k \quad m \quad m-l \end{array} \right\rangle_3 = \sum_{t=\max\{k,l\}}^{\min\{k+l,n,m\}} \frac{[n][m][t][t][n+m-t+2]}{[k][k][l][l][n+m-k-l+2]} \left\langle \begin{array}{c} n-k \quad n-t \quad n-l \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ m-k \quad m-t \quad m-l \end{array} \right\rangle_3,$$

where $\begin{bmatrix} a \\ b \end{bmatrix} = \frac{[a]!}{[b]![a-b]!}$ for $a \geq b$ and $[a]! = \prod_{i=1}^a [i]$.

3. Trivalent graphs for clasped A_2 web spaces

For the Kauffman bracket skein module, a method for representing its elements by colored trivalent graphs is well known. Formulas related to colored trivalent graphs play a very important role in defining the quantum $SU(2)$ invariants for 3-manifolds and the quantum representations for mapping class groups of surfaces, etc. (See, for example, Kauffman and Lins [KL94b], Turaev and Viro [TV92] and Roberts [Rob94].) In this section, we represent a certain clasped A_2 web space by using colored trivalent graphs and give some formulas.

DEFINITION 3.1. Let n be a non-negative integer. For $0 \leq i \leq n$, we define a colored

trivalent graph  by .

We use the following notations:

- $\Delta(m, n) = \left\langle \begin{array}{c} m \quad n \\ \text{---} \text{---} \\ \text{---} \end{array} \right\rangle_3$,
- $\theta(n, n, (i, i)) = \left\langle \begin{array}{c} n \\ \text{---} \text{---} \\ n \end{array} \right\rangle_3$,
- $\text{Tet} \begin{bmatrix} n & n & (j, j) \\ n & n & (i, i) \end{bmatrix} = \left\langle \begin{array}{c} n \quad n \\ \text{---} \text{---} \\ n \end{array} \right\rangle_3$,
- $\left\{ \begin{array}{c} n & n & (j, j) \\ n & n & (i, i) \end{array} \right\} = \frac{\text{Tet} \begin{bmatrix} n & n & (j, j) \\ n & n & (i, i) \end{bmatrix} \Delta(j, j)}{\theta(n, n, (j, j))^2}$,

where m, n are any non-negative integers and $0 \leq i, j \leq n$.

LEMMA 3.2.

- (1) $\Delta(i, j) = \frac{[i+1][j+1][i+j+2]}{[2]}$,
- (2) $\theta(n, n, (i, i)) = \sum_{k=0}^i (-1)^k \frac{[i]^2}{[2i+1]} \frac{\Delta(n, 0)^2}{\Delta(n-i+k, 0)} = \frac{[n+i+2]}{[n]^2} \Delta(i, i)$,
- (3) $\text{Tet} \begin{bmatrix} n & n & (j, j) \\ n & n & (i, i) \end{bmatrix} = \sum_{k=0}^i (-1)^k \frac{[i]^2}{[2i+1]} \frac{[n-j]}{[i-k]} \frac{[n+j+2]}{[i-k]} \theta(n, n, (j, j))$.

PROOF. We only show (2) and (3). The first equation of (2) is obtained by expanding the double-lined edge using Definition 2.4. $\theta(n, n, (i, i))$ is also represented by a colored

graph . This clasped A_2 web has a bubble skein element colored by $n - i$

and we use Theorem 2.7. Thus, we obtain the second equation of (2). The proof of (3) is obtained by using Definition 2.4 and Theorem 2.7. We expand a double-lined edge colored by s . Then, the clasped A_2 web appearing in each term has two bubble skein elements. Finally, $\theta(n, n, (j, j))$ is obtained by expanding one of these bubble skein elements. $\square$

LEMMA 3.3.

$$\left\langle \begin{array}{c} i \\ \text{---} \circlearrowleft \text{---} \\ j \\ n \end{array} \right\rangle_3 = \delta_{ij} \frac{\theta(n, n, (i, i))}{\Delta(i, i)} \left\langle \begin{array}{c} i \\ \text{---} \\ \text{---} \end{array} \right\rangle_3,$$

where $\text{---} \text{---}$ denotes $\begin{array}{c} i \\ \text{---} \text{---} \\ i \end{array}$ and δ_{ij} is the Kronecker delta function.

PROOF. Theorem 2.7 and Lemma 2.5 show the value is zero if $i \neq j$. If $i = j$, then the coefficient is obtained by closing double-lined edges of both sides. $\square$

PROPOSITION 3.4 (Recoupling Theorem).

$$\left\langle \begin{array}{c} n \\ \nearrow \text{---} \nwarrow \\ i \\ \text{---} \text{---} \\ n \end{array} \right\rangle_3 = \sum_{j=0}^n \left\{ \begin{array}{ccc} n & n & (j, j) \\ n & n & (i, i) \end{array} \right\} \left\langle \begin{array}{c} n \\ \nearrow \text{---} \nwarrow \\ j \\ \text{---} \text{---} \\ n \end{array} \right\rangle_3$$

PROOF. We can prove this proposition in the same way as the recoupling theory of Temperley-Lieb algebra, see [KL94b, Chapter 7], using Definition 2.4 and Lemma 3.3. $\square$

We give a formula of the $\mathfrak{sl}_3$ colored Jones polynomial $J_{(n,0)}^{\mathfrak{sl}_3}([2a_1, 2a_2, \dots, 2a_l]; q)$ for a 2-bridge link $[2a_1, 2a_2, \dots, 2a_l]$ using colored trivalent graphs. In this paper we define the $\mathfrak{sl}_3$ colored Jones polynomial for framed links obtained by a link diagram with the blackboard framing.

DEFINITION 3.5.

$$J_{(n,0)}^{\mathfrak{sl}_3}([2a_1, 2a_2, \dots, 2a_l]; q) = \begin{cases} \left\langle \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ n \text{---} 2a_1 \text{---} 2a_2 \text{---} 2a_3 \text{---} 2a_4 \text{---} \dots \text{---} 2a_l \\ n \end{array} \right\rangle_3 / \Delta(n, 0) & \text{if } l \text{ is odd,} \\ \left\langle \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ n \text{---} 2a_1 \text{---} 2a_2 \text{---} 2a_3 \text{---} 2a_4 \text{---} \dots \text{---} 2a_l \\ n \end{array} \right\rangle_3 / \Delta(n, 0) & \text{if } l \text{ is even,} \end{cases}$$

where, $a_1, a_2, \dots, a_l$ are non-zero integers and

$$\boxed{m} = \begin{cases} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} & \text{if } m > 0, \\ \text{right-handed } m \text{ half twists} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} & \text{if } m < 0. \\ \text{left-handed } m \text{ half twists} \end{cases}$$

THEOREM 3.6.

$$\begin{aligned} & J_{(n,0)}^{\mathfrak{sl}_3}([2a_1, 2a_2, \dots, 2a_l]; q) \\ &= \sum_{0 \leq i_1, i_2, \dots, i_l \leq n} \frac{\Delta(i_1, i_1)}{\Delta(n, 0)} \frac{\theta(n, n, (i_l, i_l))}{\theta(n, n, (i_1, i_1))} q^{-\frac{2}{3}(n^2+3n)(\sum_{k=1}^l a_k)} q^{\sum_{k=1}^l a_k(i_k^2+2i_k)} \\ & \quad \times \prod_{k=1}^{l-1} \left\{ \begin{array}{ccc} n & n & (i_{k+1}, i_{k+1}) \\ n & n & (i_k, i_k) \end{array} \right\}. \end{aligned}$$

LEMMA 3.7.

$$\left\langle \begin{array}{c} i \\ \text{---} \\ n \end{array} \right\rangle_3 = q^{-\frac{2}{3}(n^2+3n)+i^2+2i} \left\langle \begin{array}{c} n \\ \text{---} \\ n \end{array} \right\rangle_3$$

PROOF. It is shown by using the following formula [Yua16, Lemma 3.16]:

$$\left\langle \begin{array}{c} i \\ \text{---} \\ n-i \\ \text{---} \\ i \end{array} \right\rangle_3 = q^{-\frac{n^2+3n-i^2-3i}{3}} \left\langle \begin{array}{c} i \\ \text{---} \\ i \\ \text{---} \\ n-i \end{array} \right\rangle_3,$$

and Theorem 2.6 with $m = 1$. □

PROOF OF THEOREM 3.6.

$$\begin{aligned} (3.1) \quad \left\langle \begin{array}{c} n \\ \text{---} \\ 2a_k \\ \text{---} \\ n \end{array} \right\rangle_3 &= \left\langle \begin{array}{c} n \\ \text{---} \\ 0 \\ \text{---} \\ n \end{array} \right\rangle_3 = \sum_{j=0}^n \left\{ \begin{array}{cc} n & n \\ n & n \end{array} \begin{array}{c} (i_k, i_k) \\ (0, 0) \end{array} \right\} \left\langle \begin{array}{c} n \\ \text{---} \\ i_k \\ \text{---} \\ n \end{array} \right\rangle_3 \\ &= q^{-\frac{2a_k}{3}(n^2+3n)+a_k(i_k^2+2i_k)} \frac{\Delta(i_k, i_k)}{\theta(n, n, (i_k, i_k))} \left\langle \begin{array}{c} n \\ \text{---} \\ i_k \\ \text{---} \\ n \end{array} \right\rangle_3. \end{aligned}$$

This equation is derived from Proposition 3.4, Lemma 3.7 and $\left\{ \begin{array}{cc} n & n \\ n & n \end{array} \begin{array}{c} (i_k, i_k) \\ (0, 0) \end{array} \right\} = \frac{\Delta(i_k, i_k)}{\theta(n, n, (i_k, i_k))}$. By Proposition 3.4 and Lemma 3.3,

$$\begin{aligned} (3.2) \quad \left\langle \begin{array}{c} n \\ \text{---} \\ i_k \\ \text{---} \\ n \end{array} \right\rangle_3 &= \sum_{s=0}^n \left\{ \begin{array}{cc} n & n \\ n & n \end{array} \begin{array}{c} (s, s) \\ (i_k, i_k) \end{array} \right\} \left\langle \begin{array}{c} n \\ \text{---} \\ s \\ \text{---} \\ i_{k+1} \\ \text{---} \\ n \end{array} \right\rangle_3 \\ &= \left\{ \begin{array}{cc} n & n \\ n & n \end{array} \begin{array}{c} (i_{k+1}, i_{k+1}) \\ (i_k, i_k) \end{array} \right\} \left\langle \begin{array}{c} n \\ \text{---} \\ i_{k+1} \\ \text{---} \\ n \end{array} \right\rangle_3. \end{aligned}$$

First, we apply (3.1) to boxed $2a_k$ for all $0 \leq k \leq l$ of the two bridge link diagram. Next, we repeatedly apply (3.2) to the leftmost doubled edge. Finally, we obtain $\theta(n, n, (i_l, i_l))$. □

4. The tail of the $\mathfrak{sl}_3$ colored Jones polynomial for $T(2, 2m)$

We discuss a stability of coefficients of the $\mathfrak{sl}_3$ colored Jones polynomial for the torus link $T(2, 2m)$ in this section for positive integer m . Two explicit formulas of $J_{(n,0)}^{\mathfrak{sl}_3}(T(2, 2m))$ are obtained from Theorem 3.6 and [Yua16, Theorem 5.7] because $T(2, 2m)$ has a presentation $[2m]$ as a 2-bridge link. We write these two formulas below. Let m be a positive integer.

From Theorem 3.6,

$$\begin{aligned} \psi_n^{(m)}(q) &= J_{(n,0)}^{\mathfrak{sl}_3}(T(2, 2m)) = q^{-\frac{2m}{3}(n^2+3n)} \sum_{i=0}^n \frac{\Delta(i, i)}{\Delta(n, 0)} q^{m(i^2+2i)} \\ &= q^{-\frac{2m}{3}(n^2+3n)+n} \sum_{i=0}^n q^{-2i} q^{m(i^2+2i)} \frac{(1-q^{i+1})^3(1+q^{i+1})}{(1-q)(1-q^{n+1})(1-q^{n+2})}. \end{aligned}$$

From [Yua16, Theorem 5.7],

$$\begin{aligned} g_n^{(m)}(q) &= J_{(n,0)}^{\mathfrak{sl}_3}(T(2, 2m)) \\ &= q^{-\frac{2m}{3}(n^2+3n)+n} \sum_{0 \leq k_m \leq \dots \leq k_2 \leq k_1 \leq n} q^{-2k_m} q^{\sum_{j=1}^m (k_i^2+2k_i)} \\ &\quad \times \frac{(q)_n^2}{(q)_{k_m}^2 (q)_{n-k_1} (q)_{k_1-k_2} \dots (q)_{k_{m-1}-k_m}} \frac{(1-q^{n+1})(1-q^{n+2})}{(1-q^{n-k_m+1})(1-q^{n-k_m+2})}. \end{aligned}$$

DEFINITION 4.1. Suppose $f(q), f_n(q) \in \mathbb{Z}[[q]]$ for $n \geq 1$. The limit of $\{f_n(q)\}_n$ is $f(q)$, denoted $\lim_{n \rightarrow \infty} f_n(q) = f(q)$, means that $f_n(q) = f(q)$ in $\mathbb{Z}[[q]]/q^{n+1}\mathbb{Z}[[q]]$ for all n .

We can see that the minimum degree of $J_{(n,0)}^{\mathfrak{sl}_3}(T(2, 2m))$ is $q^{-\frac{2m}{3}(n^2+3n)+n}$ and consider the limit of $\Psi_n^{(m)}(q) = q^{\frac{2m}{3}(n^2+3n)-n} \psi_n(q)$ and $G_n^{(m)}(q) = q^{\frac{2m}{3}(n^2+3n)-n} g_n(q)$. The following lemma ensures the existence of the limit.

LEMMA 4.2. $\Psi_n^{(m)}(q) = \Psi_{n+1}^{(m)}(q)$ in $\mathbb{Z}[[q]]/q^{n+1}\mathbb{Z}[[q]]$.

PROOF. It is clear from the expression of $\Psi_{n+1}^{(m)}(q)$. □

We explicitly give the limit of $\{\Psi_n^{(m)}(q)\}_n$ and $\{G_n^{(m)}\}_n$. First, we can easily obtain the following:

$$(4.1) \quad \lim_{n \rightarrow \infty} \Psi_n^{(m)}(q) = \sum_{i=0}^{\infty} q^{-2i} q^{m(i^2+2i)} \frac{(1-q^{i+1})^3(1+q^{i+1})}{1-q}.$$

Next, we consider the limit of $\{G_n^{(m)}\}_n$. In $\mathbb{Z}[[q]]/q^{n+1}\mathbb{Z}[[q]]$,

$$\begin{aligned} \frac{q^{k_m^2}}{1-q^{n-k_m+1}} &= q^{k_m} (1 + q^{n-k_m+1} + q^{2(n-k_m+1)} + \dots) \\ &= q^{k_m^2} + q^{n+1+k_m^2-k_m} + (\text{higher-order terms}) \\ &= q^{k_m^2} \end{aligned}$$

because $k_m^2 - k_m \geq 0$. In a similar way, we obtain $\frac{q^{k_m^2}}{1-q^{n-k_m+2}} = q^{k_m^2}, q^{k_1^2+2k_1} \frac{(q)_n}{(q)_{n-k_1}} = q^{k_1^2+2k_1}$. From the above, we conclude that the limit of $\{G_n^{(m)}\}_n$ is

$$(4.2) \quad \lim_{n \rightarrow \infty} G_n^{(m)}(q) = (q)_\infty \sum_{0 \leq k_m \leq \dots \leq k_2 \leq k_1} \frac{q^{-2k_m} q^{\sum_{j=1}^m (k_i^2+2k_i)}}{(q)_{k_m}^2 (q)_{k_1-k_2} \dots (q)_{k_{m-1}-k_m}}.$$

Consequently, we have the following identity of q -series.

THEOREM 4.3.

$$\sum_{i=0}^{\infty} q^{-2i} q^{m(i^2+2i)} \frac{(1-q^{i+1})^3(1+q^{i+1})}{1-q} = (q)_\infty \sum_{0 \leq k_m \leq \dots \leq k_2 \leq k_1} \frac{q^{-2k_m} q^{\sum_{j=1}^m (k_i^2+2k_i)}}{(q)_{k_m}^2 (q)_{k_1-k_2} \dots (q)_{k_{m-1}-k_m}}.$$

The above identity is a knot-theoretical generalization of the Andrews-Gordon identities for the Ramanujan false theta function.

REMARK 4.4. Recently, Bringmann, Kaszian and Milas gave an explicit formula for the $\mathfrak{sl}_3$ false theta function (see eq. (1.7) of [BKM17]). The left-hand side of the identity of Theorem 4.3 $\mathfrak{sl}_3$ tail can be identified with the summand of the $\mathfrak{sl}_3$ false theta function up to the factor $1 - q$ (see Sec. 9 (5) in [BKM17]).

We compute the expansion of q -series $\Psi^{(m)}(q) = \lim_{n \rightarrow \infty} \Psi_n^{(m)}(q)$ up to order 150 by Mathematica 11.0 [**Inc16**] in Table 1. It is conjectured that the collection of coefficients of the expansion of $\Psi^{(m)}(q)$ except for zeros is $\{1, -1, -1, 1, 1, 1, -1, -1, -1, -1, 1, 1, 1, 1, \dots\}$ for $m > 0$. Sequences of non-zero coefficients and zero coefficients appear in the expansion of $\Psi^{(m)}(q)$ alternatively. The n -th sequence of non-zero coefficients has length $4n$ and the n -th sequence of zero coefficients length $(2n + 1)(m - 2)$ for $m > 0$.

m	$\Psi^{(m)}(q)$
1	$1 + O(q^{151})$
2	$ \begin{aligned} &1 - q - q^2 + q^3 + q^4 + q^5 - q^6 - q^7 - q^8 - q^9 + q^{10} + q^{11} + q^{12} + q^{13} + q^{14} - q^{15} \\ &- q^{16} - q^{17} - q^{18} - q^{19} - q^{20} + q^{21} + q^{22} + q^{23} + q^{24} + q^{25} + q^{26} + q^{27} - q^{28} - q^{29} \\ &- q^{30} - q^{31} - q^{32} - q^{33} - q^{34} - q^{35} + q^{36} + q^{37} + q^{38} + q^{39} + q^{40} + q^{41} + q^{42} + q^{43} \\ &+ q^{44} - q^{45} - q^{46} - q^{47} - q^{48} - q^{49} - q^{50} - q^{51} - q^{52} - q^{53} - q^{54} + q^{55} + q^{56} + q^{57} \\ &+ q^{58} + q^{59} + q^{60} + q^{61} + q^{62} + q^{63} + q^{64} + q^{65} - q^{66} - q^{67} - q^{68} - q^{69} - q^{70} - q^{71} \\ &- q^{72} - q^{73} - q^{74} - q^{75} - q^{76} - q^{77} + q^{78} + q^{79} + q^{80} + q^{81} + q^{82} + q^{83} + q^{84} + q^{85} \\ &+ q^{86} + q^{87} + q^{88} + q^{89} + q^{90} - q^{91} - q^{92} - q^{93} - q^{94} - q^{95} - q^{96} - q^{97} - q^{98} - q^{99} \\ &- q^{100} - q^{101} - q^{102} - q^{103} - q^{104} + q^{105} + q^{106} + q^{107} + q^{108} + q^{109} + q^{110} + q^{111} \\ &+ q^{112} + q^{113} + q^{114} + q^{115} + q^{116} + q^{117} + q^{118} + q^{119} - q^{120} - q^{121} - q^{122} - q^{123} \\ &- q^{124} - q^{125} - q^{126} - q^{127} - q^{128} - q^{129} - q^{130} - q^{131} - q^{132} - q^{133} - q^{134} - q^{135} \\ &+ q^{136} + q^{137} + q^{138} + q^{139} + q^{140} + q^{141} + q^{142} + q^{143} + q^{144} + q^{145} + q^{146} + q^{147} \\ &+ q^{148} + q^{149} + q^{150} + O(q^{151}) \end{aligned} $
3	$ \begin{aligned} &1 - q - q^2 + q^3 + q^7 + q^8 - q^9 - q^{10} - q^{11} - q^{12} + q^{13} + q^{14} + q^{20} + q^{21} + q^{22} - q^{23} \\ &- q^{24} - q^{25} - q^{26} - q^{27} - q^{28} + q^{29} + q^{30} + q^{31} + q^{39} + q^{40} + q^{41} + q^{42} - q^{43} - q^{44} \\ &- q^{45} - q^{46} - q^{47} - q^{48} - q^{49} - q^{50} + q^{51} + q^{52} + q^{53} + q^{54} + q^{64} + q^{65} + q^{66} + q^{67} \\ &+ q^{68} - q^{69} - q^{70} - q^{71} - q^{72} - q^{73} - q^{74} - q^{75} - q^{76} - q^{77} - q^{78} + q^{79} + q^{80} + q^{81} \\ &+ q^{82} + q^{83} + q^{95} + q^{96} + q^{97} + q^{98} + q^{99} + q^{100} - q^{101} - q^{102} - q^{103} - q^{104} - q^{105} \\ &- q^{106} - q^{107} - q^{108} - q^{109} - q^{110} - q^{111} - q^{112} + q^{113} + q^{114} + q^{115} + q^{116} + q^{117} \\ &+ q^{118} + q^{132} + q^{133} + q^{134} + q^{135} + q^{136} + q^{137} + q^{138} - q^{139} - q^{140} - q^{141} - q^{142} \\ &- q^{143} - q^{144} - q^{145} - q^{146} - q^{147} - q^{148} - q^{149} - q^{150} + O(q^{151}) \end{aligned} $
4	$ \begin{aligned} &1 - q - q^2 + q^3 + q^{10} + q^{11} - q^{12} - q^{13} - q^{14} - q^{15} + q^{16} + q^{17} + q^{28} + q^{29} + q^{30} \\ &- q^{31} - q^{32} - q^{33} - q^{34} - q^{35} - q^{36} + q^{37} + q^{38} + q^{39} + q^{54} + q^{55} + q^{56} + q^{57} - q^{58} \\ &- q^{59} - q^{60} - q^{61} - q^{62} - q^{63} - q^{64} - q^{65} + q^{66} + q^{67} + q^{68} + q^{69} + q^{88} + q^{89} + q^{90} \\ &+ q^{91} + q^{92} - q^{93} - q^{94} - q^{95} - q^{96} - q^{97} - q^{98} - q^{99} - q^{100} - q^{101} - q^{102} + q^{103} \\ &+ q^{104} + q^{105} + q^{106} + q^{107} + q^{130} + q^{131} + q^{132} + q^{133} + q^{134} + q^{135} - q^{136} - q^{137} \\ &- q^{138} - q^{139} - q^{140} - q^{141} - q^{142} - q^{143} - q^{144} - q^{145} - q^{146} - q^{147} + q^{148} + q^{149} \\ &+ q^{150} + O(q^{151}) \end{aligned} $
5	$ \begin{aligned} &1 - q - q^2 + q^3 + q^{13} + q^{14} - q^{15} - q^{16} - q^{17} - q^{18} + q^{19} + q^{20} + q^{36} + q^{37} + q^{38} \\ &- q^{39} - q^{40} - q^{41} - q^{42} - q^{43} - q^{44} + q^{45} + q^{46} + q^{47} + q^{69} + q^{70} + q^{71} + q^{72} - q^{73} \\ &- q^{74} - q^{75} - q^{76} - q^{77} - q^{78} - q^{79} - q^{80} + q^{81} + q^{82} + q^{83} + q^{84} + q^{112} + q^{113} \\ &+ q^{114} + q^{115} + q^{116} - q^{117} - q^{118} - q^{119} - q^{120} - q^{121} - q^{122} - q^{123} - q^{124} - q^{125} \\ &- q^{126} + q^{127} + q^{128} + q^{129} + q^{130} + q^{131} + O(q^{151}) \end{aligned} $
6	$ \begin{aligned} &1 - q - q^2 + q^3 + q^{16} + q^{17} - q^{18} - q^{19} - q^{20} - q^{21} + q^{22} + q^{23} + q^{44} + q^{45} + q^{46} \\ &- q^{47} - q^{48} - q^{49} - q^{50} - q^{51} - q^{52} + q^{53} + q^{54} + q^{55} + q^{84} + q^{85} + q^{86} + q^{87} - q^{88} \\ &- q^{89} - q^{90} - q^{91} - q^{92} - q^{93} - q^{94} - q^{95} + q^{96} + q^{97} + q^{98} + q^{99} + q^{136} + q^{137} \\ &+ q^{138} + q^{139} + q^{140} - q^{141} - q^{142} - q^{143} - q^{144} - q^{145} - q^{146} - q^{147} - q^{148} - q^{149} \\ &- q^{150} + O(q^{151}) \end{aligned} $

TABLE 1. Expansion of $\Psi^{(m)}(q)$ up to order 150 for $m = 1, 2, \dots, 6$.

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