論文 / 著書情報
Article / Book Information

Title	Kernel-based collocation methods for Zakai equations
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URL	https://arxiv.org/abs/1710.09090
Pub. date	2017, 10

Kernel-based collocation methods for Zakai equations

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October 26, 2017

Abstract

We examine an application of the kernel-based interpolation to numerical solutions for Zakai equations in nonlinear filtering, and aim to prove its rigorous convergence. To this end, we find the class of kernels and the structure of collocation points explicitly under which the process of iterative interpolation is stable. This result together with standard argument in error estimation shows that the approximation error is bounded by the order of the square root of the time step and the error that comes from a single step interpolation. Our theorem is well consistent with the results of numerical experiments.

Key words: Zakai equations, kernel-based interpolation, stochastic partial differential equations, radial basis functions.

AMS MSC 2010: 60H15, 65M70, 93E11.

1 Introduction

We are concerned with numerical methods for Zakai equations, linear stochastic partial differential equations of the form

$$du(t,x) = L_0 u(t,x)dt + \sum_{k=1}^{m} L_k u(s,x) dW_k(t), \quad 0 \leqslant t \leqslant T, \tag{1.1}$$

with initial condition $u(0,x) = u_0(x)$, where the process $\{W(t) = (W_1(t), \dots, W_m(t))\}_{0 \leqslant t \leqslant T}$ is an m-dimensional standard Wiener processes on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Here, for each $k = 0, 1, \dots, m$, the partial differential operator L_k is given by

$$L_0 f(x) = \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j} (a_{ij}(x) f(x)) + \sum_{i=1}^{d} \frac{\partial}{\partial x_i} (b_i(x) f(x)),$$

$$L_k f(x) = \beta_k(x) f(x) + \sum_{i=1}^{d} \frac{\partial}{\partial x_i} (\gamma_{ik}(x) f(x)), \quad k = 1, \dots, m$$

where $a = (a_{ij})$ is $\mathbb{R}^{d\times d}$-valued, $b = (b_i)$ is $\mathbb{R}^d$-valued, $\beta = (\beta_k)$ is $\mathbb{R}^m$-valued, $\gamma = (\gamma_{ik})$ is $\mathbb{R}^{d\times m}$-valued, and u_0 is $\mathbb{R}$-valued, all of which are defined on $\mathbb{R}^d$. The conditions for these functions are described in Section 2 below.

It is well known that solving Zakai equations is amount to computing the optimal filter for diffusion processes. We refer to Rozovskii [17], Kunita [13], Liptser and Shiryaev [14], Bensoussan [4], Bain and Crisan [2], and the references therein for Zakai equations and its relation with nonlinear filtering. It is also well known that for linear diffusion processes the optimal filters allow for finite dimensional realizations, i.e., they can be represented by some stochastic and deterministic differential equations in finite dimensions. For nonlinear diffusion processes, it is difficult to obtain such realizations except some special cases (see Beneš [3] and [4]). Thus one may be led to numerical approach to Zakai equations for computing the optimal filter. Several efforts have been made to obtain approximation methods for the equations during the past several decades. For example, the finite difference method (see Yoo [20], Gyöngy [9] and the references therein), the particle method (see Crisan et al. [6]), a series expansion approach (Lototsky et al. [15]), Galerkin type approximation (Ahmed and Radaideh [1] and Frey et al. [7]) and the splitting up method (Bensoussan et al. [5]).

In this paper, we examine the approximation of $u(t,x)$ by a collocation method with kernel-based interpolation. Given a points set $\Gamma = \{x_1,\dots,x_N\} \subset \mathbb{R}^d$ and a positive definite function $\Phi : \mathbb{R}^d \to \mathbb{R}$, the function

$$I(f)(x) := \sum_{j=1}^{N} (A^{-1} f|_\Gamma)_j \Phi(x - x_j), \quad x \in \mathbb{R}^d,$$

interpolates f on Γ. Here, $A = \{\Phi(x_j - x_\ell)\}_{j,\ell=1,\dots,N}$, $f|_\Gamma$ is the column vector composed of $f(x_j)$, $j = 1,\dots,N$, and $(A^{-1}z)_j$ denotes the j-th component of $A^{-1}z$ for $z \in \mathbb{R}^N$. Thus, with time grid $\{t_0,\dots,t_n\}$, the function u^h recursively defined by

$$\begin{aligned} u^h(t_i,x) &= u^h(t_{i-1},x) + L_0 I(u^h(t_{i-1},\cdot)(x)(t_i - t_{i-1}) \\ &\quad + \sum_{k=1}^{m} L_k I(u^h(t_{i-1},\cdot)(x)(W_k(t_i) - W_k(t_{i-1})), \quad i = 0,\dots,n, \;\; x \in \mathbb{R}^d, \end{aligned}$$

is a good candidate for an approximate solution of (1.1). The approximation above can be seen as a kernel-based (or meshfree) collocation method for stochastic partial differential equations. The meshfree collocation method is proposed by Kansa [11], where deterministic partial differential equations are concerned. Since then many studies on numerical experiments and practical applications for this method are generated. As for rigorous convergence, Schaback [18] and Nakano [16] study the case of deterministic linear operator equations and fully nonlinear parabolic equations, respectively. However, at least for parabolic equations, there is little known about explicit examples of the grid structure and kernel functions that ensure rigorous convergence. An exception is Hon et.al [10], where an error bound is obtained for a special heat equation in one dimension. A main difficulty lies in handling the process of the iterative kernel-based interpolation. A straightforward estimates for $|I(f)(x)|$ involves the condition number of the matrix A, which in general rapidly diverges to infinity (see Wendland [19]). Thus we need to take a different route. Our main idea is to introduce a condition on the decay of $|A_{ij}^{-1}|$ when $|i-j|$ becomes large and to choose an appropriate approximation domain whose radius goes to infinity such that the interpolation is still effective. From this together with standard argument in error estimation we find that the approximation error is bounded by the order of the square root of the time step and the error that comes from a single step interpolation. See Lemma 3.7 and Theorem 3.4 below.

The structure of this paper is as follows: Section 2 introduces some notation, and describes the basic results for Zakai equations and the kernel-based interpolation, which are used in this paper. We derive an approximation method for the filter and prove its convergence in Section 3. Numerical experiments are performed in Section 4.

2 Preliminaries

2.1 Notation

Throughout this paper, we denote by a^{T} the transpose of a vector or matrix a. For $a = (a_i) \in \mathbb{R}^\ell$ we set $|a| = (\sum_{i=1}^\ell (a_i)^2)^{1/2}$. For a multiindex $\alpha = (\alpha_1, \dots, \alpha_d)$ of nonnegative integers, the differential operator D^α is defined as usual by

$$D^\alpha = \frac{\partial^{|\alpha|_1}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}$$

with $|\alpha|_1 = \alpha_1 + \cdots + \alpha_d$. For an open set $\mathcal{O} \subset \mathbb{R}^d$, we denote by $C^\kappa(\mathcal{O})$ the space of continuous real-valued functions on $\mathcal{O}$ with continuous derivatives up to the order $\kappa \in \mathbb{N}$, with the norm

$$\|f\|_{C^\kappa(\mathcal{O})} = \max_{|\alpha|_1 \leqslant \kappa} \sup_{x \in \mathcal{O}} |D^\alpha f(x)|.$$

Further, we denote by $C_0^\infty(\mathbb{R}^d)$ the space of infinitely differentiable functions on $\mathbb{R}^d$ with compact supports. For any $p \in [1, \infty)$ and any open set $\mathcal{O} \subset \mathbb{R}^d$, we denote by $L^p(\mathcal{O})$ the space of all measurable functions $f : \mathcal{O} \to \mathbb{R}$ such that

$$\|f\|_{L^p(\mathcal{O})} := \left\{ \int_{\mathcal{O}} |f(x)|^p dx \right\}^{1/p} < \infty.$$

For $\kappa \in \mathbb{N}$, we write $H^\kappa(\mathcal{O})$ for the space of all measurable functions f on $\mathcal{O}$ such that the generalized derivatives $D^\alpha f$ exist for all $|\alpha|_1 \leqslant \kappa$ and that

$$\|f\|_{H^\kappa(\mathcal{O})}^2 := \sum_{|\alpha|_1 \leqslant \kappa} \|D^\alpha f\|_{L^2(\mathcal{O})}^2 < \infty.$$

In addition, for $0 < r < 1$, we write $H^{\kappa+r}(\mathcal{O})$ for the space of all measurable functions f on $\mathcal{O}$ such that the generalized derivatives $D^\alpha u$ exist for all $|\alpha|_1 \leqslant \kappa$ and that

$$\|f\|_{H^{\kappa+r}(\mathcal{O})}^2 := \|f\|_{H^\kappa(\mathcal{O})}^2 + \sum_{|\alpha|_1 = \kappa} \int_{\mathcal{O}} \int_{\mathcal{O}} \frac{|D^\alpha f(x) - D^\alpha f(y)|^2}{|x-y|^{d+2r}} dx dy < \infty.$$

For $x \in \mathbb{R}$ we use the notation $\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leqslant x\}$. By C we denote positive constants that may vary from line to line and that are independent of h introduced below.

2.2 Zakai equations

We impose the following conditions for the equation (1.1):

Assumption 2.1. (i) All components of the functions a, b, β, γ, and u_0 are infinitely differentiable with bounded continuous derivatives of any order.

(ii) For any $x \in \mathbb{R}^d$,

$$\xi^{\mathsf{T}}(a(x) - \gamma(x)\gamma(x)^{\mathsf{T}})\xi \geqslant 0, \quad \xi \in \mathbb{R}^d.$$

It follows from Assumption 2.1(i) and Gerencsér et.al [8, Theorem 2.1] that there exists a unique predictable process $\{u(t)\}_{0\leqslant t\leqslant T}$ such that the following are satisfied:

(i) $u(t,\cdot,\omega) \in H^{\nu}(\mathbb{R}^d)$ for any $(t,\omega) \in [0,T] \times \Omega_0$, where $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 1$ and for any $\nu \in \mathbb{N}$;

(ii) for $\varphi \in C_0^{\infty}(\mathbb{R}^d)$,

(2.1)
$$(u(t),\varphi) = (u_0,\varphi) + \int_0^t (u(s,\cdot), L_0^*\varphi)ds + \sum_{k=1}^m \int_0^t (u(s,\cdot), L_k^*\varphi)dW_k(s), \quad 0 \leqslant t \leqslant T, \text{ a.s.}$$

Here, $(\cdot,\cdot)$ denots the inner product in $L^2(\mathbb{R}^d)$, and for each $k = 0,1,\ldots,m$, the partial differential operator L_k^* is the formal adjoint of L_k. Moreover, $u(t,x)$ satisfies

$$\mathbb{E}\left[\sup_{0\leqslant t\leqslant T} \|u(t,\cdot)\|^2_{H^{\nu}(\mathbb{R}^d)}\right] \leqslant C\|\pi_0\|^2_{H^{\nu}(\mathbb{R}^d)}, \quad \nu \in \mathbb{N}.$$

Further, as in [17, Proposition 3, Section 1.3, Chapter 4], there exists a version $\tilde{u}$, with respect to x, of u such that $\tilde{u}(t,x,\omega) \in C^{\infty}(\mathbb{R}^d)$ for $(t,\omega) \in [0,T] \times \Omega$ and that for any $\kappa \in \mathbb{N}$ and $|\alpha|_1 \leqslant \kappa$,

(2.2)
$$D^{\alpha}\tilde{u}(t,x) = D^{\alpha}\pi_0(x) + \int_0^t D^{\alpha}L_0\tilde{u}(s,x)dx + \sum_{k=1}^m \int_0^t D^{\alpha}L_k\tilde{u}(s,x)dW_k(s), \quad \text{a.s.}, \quad (t,x) \in [0,T]\times\mathbb{R}^d.$$

In particular, $\tilde{u}$ is a solution to the Zakai equation in the strong sense, i.e., $\tilde{u}$ satisfies

$$\tilde{u}(t,x) = \tilde{u}_0(x) + \int_0^t L_0\tilde{u}(s,x)dx + \sum_{k=1}^m \int_0^t L_k\tilde{u}(s,x)dW_k(s), \quad \text{a.s.}, \quad (t,x) \in [0,T] \times \mathbb{R}^d.$$

We remark that in (2.2) the stochastic integral is taken to be a continuous version with respect to (t,x). With this version, (2.2) holds with probability one uniformly on $[0,T] \times \mathbb{R}^d$.

2.3 Kernel-based interpolation

In this subsection, we recall the basis of the interpolation theory with positive definite functions. We refer to [19] for a complete account. Let $\Phi : \mathbb{R}^d \to \mathbb{R}$ be a radial and positive definite function, i.e., $\Phi(x) = \Phi(-x)$ for all $x \in \mathbb{R}^d$ and for every $\ell \in \mathbb{N}$, for all pairwise distinct $y_1, \ldots, y_\ell \in \mathbb{R}^d$ and for all $\alpha = (\alpha_i) \in \mathbb{R}^\ell \backslash \{0\}$, we have

$$\sum_{i,j=1}^{\ell} \alpha_i \alpha_j \Phi(y_i - y_j) > 0.$$

Let $\Gamma = \{x_1, \cdots, x_N\}$ be a finite subset of $\mathbb{R}^d$ and put $A = \{\Phi(x_i - x_j)\}_{1 \leqslant i,j \leqslant N}$. Then A is invertible and thus for any $g : \mathbb{R}^d \to \mathbb{R}$ the function

$$I(g)(x) = \sum_{j=1}^{N} (A^{-1}(g|_\Gamma))_j \Phi(x - x_j), \quad x \in \mathbb{R}^d,$$

interpolates g on Γ. If $\Phi \in C(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$, then

$$\mathcal{N}_\Phi(\mathbb{R}^d) := \left\{ f \in C(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) : \widehat{f}/\sqrt{\widehat{\Phi}} \in L^2(\mathbb{R}^d) \right\},$$

called the native space, is a real Hilbert space with inner product

$$(f, g)_{\mathcal{N}_\Phi(\mathbb{R}^d)} = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \frac{\widehat{f}(\xi)\overline{\widehat{g}(\xi)}}{\widehat{\Phi}(\xi)} d\xi$$

and the norm $\|f\|^2_{\mathcal{N}_\Phi(\mathbb{R}^d)} := (f, f)_{\mathcal{N}_\Phi(\mathbb{R}^d)}$. Here, for $f \in L^1(\mathbb{R}^d)$, the function $\widehat{f}$ is the Fourier transform of f, defined as usual by

$$\widehat{f}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-\sqrt{-1}x^\mathsf{T}\xi} dx, \quad \xi \in \mathbb{R}^d.$$

Moreover, $\mathbb{R}^d \times \mathbb{R}^d \ni (x, y) \mapsto \Phi(x - y)$ is a reproducing kernel for $\mathcal{N}_\Phi(\mathbb{R}^d)$. If $\widehat{\Phi}$ satisfies

$$c_1(1 + |\xi|^2)^{-\kappa} \leqslant \widehat{\Phi}(\xi) \leqslant c_2(1 + |\xi|^2)^{-\kappa}, \quad \xi \in \mathbb{R}^d, \tag{2.3}$$

for some constants $c_1, c_2 > 0$ and $\kappa > d/2$, then we have from Corollary 10.13 in [19] that $H^\kappa(\mathbb{R}^d) = \mathcal{N}_\Phi(\mathbb{R}^d)$ and

$$c_1\|f\|_{H^\kappa(\mathbb{R}^d)} \leqslant \|f\|_{\mathcal{N}_\Phi(\mathbb{R}^d)} \leqslant c_2\|f\|_{H^\kappa(\mathbb{R}^d)}, \quad f \in H^\kappa(\mathbb{R}^d). \tag{2.4}$$

Namely, the native space $\mathcal{N}_\Phi(\mathbb{R}^d)$ coincides with the Sobolev space $H^\kappa(\mathbb{R}^d)$ with equivalent norm. Further, we mention that (2.4) and Corollary 10.25 in [19] implies

$$\|I(g)\|_{H^\kappa(\mathbb{R}^d)} \leqslant C\|g\|_{H^\kappa(\mathbb{R}^d)}, \quad \|g - I(g)\|_{H^\kappa(\mathbb{R}^d)} \leqslant C\|g\|_{H^\kappa(\mathbb{R}^d)}, \quad g \in H^\kappa(\mathbb{R}^d). \tag{2.5}$$

The so called Wendland kernel is a typical example of Φ satisfying (2.3)–(2.5), which is defined as follows: for a given $\tau \in \mathbb{N}$, set the function $\Phi_{d,\tau}$ satisfying $\Phi_{d,\tau}(x) = \phi_{d,\tau}(|x|)$, $x \in \mathbb{R}^d$, where

$$\phi_{d,\tau}(r) = \int_r^\infty r_\tau \int_{r_\tau}^\infty r_{\tau-1} \int_{r_{\tau-1}}^\infty \cdots r_2 \int_{r_2}^\infty r_1 \max\{1-r_1, 0\}^\nu dr_1 dr_2 \cdots dr_\tau, \quad r \geqslant 0$$

with $\nu = \lfloor d/2 \rfloor + \tau + 1$. For example,

$$\begin{aligned}
&\phi_{1,2}(r) \doteq \max\{1-r,0\}^5(8r^2+5r+1),\\
&\phi_{1,3}(r) \doteq \max\{1-r,0\}^7(315r^3+285r^2+105r+15),\\
&\phi_{1,4}(r) \doteq \max\{1-r,0\}^9(5760r^4+6795r^3+3555r^2+945r+105),\\
&\phi_{2,4}(r) \doteq \max\{1-r,0\}^{10}(9009r^4+9450r^3+4410r^2+1050r+105),\\
&\phi_{2,5}(r) \doteq \max\{1-r,0\}^{12}(215040r^5+283185r^4+172620r^3+59430r^2+11340r+945),
\end{aligned}$$

where $\doteq$ denotes equality up to a positive constant factor.

Then, $\Phi_{d,\tau} \in C^{2\tau}(\mathbb{R}^d)$ and $\mathcal{N}_{\Phi_{d,\tau}}(\mathbb{R}^d) = H^{\tau+(d+1)/2}(\mathbb{R}^d)$. Furthermore, $\Phi_{d,\tau}$ satisfies (2.3)–(2.5) with $\kappa = \tau + (d+1)/2$.

3 Collocation method for Zakai equations

Let us describe the collocation methods for (2.1). In what follows, we always consider the version of u, and thus by abuse of notation, we write u for $\tilde{u}$. Moreover, we restrict ourselves to the class of Wendland kernels described in Section 2.3. Suppose that the open rectangle $(-R,R)^d$ for some $R > 0$ is the set of points at which the approximate solution is to be computed. We take a set of grid points That is, we choose a set $\Gamma = \{x_1, \ldots, x_N\}$ consisting of pairwise distinct points such that

$$\Gamma = \{x_1, \ldots, x_N\} \subset (-R,R)^d.$$

To construct an approximate solution of Zakai equation, we first take a set of time discretized points such that $0 = t_0 < t_1 < \cdots < t_n = T$. The solution u of the Zakai equation approximately satisfies

$$u(t_i, x) \simeq u(t_{i-1}, x) + \sum_{k=0}^m L_k u(t_{i-1}, x)\Delta W_k(t_i),$$

where $W_0(t) = t$ and $\Delta W_k(t_i) = W_k(t_i) - W_k(t_{i-1})$. Since $L_k u(t_{i-1}, x) \simeq L_k I(u(t_{i-1}, \cdot))(x)$, we see

$$u(t_i, x) \simeq u(t_{i-1}, x) + \sum_{k=0}^m L_k I(u(t_{i-1}, \cdot))(x)\Delta W_k(t_i).$$

Thus, we define the function u^h, a candidate of an approximate solution parametrized with a parameter $h > 0$, by

$$\begin{aligned}
&u^h(t_0, x) = u_0(x), \quad x \in \mathbb{R}^d,\\
&u^h(t_i, x) = u^h(t_{i-1}, x) + \sum_{k=0}^m L_k I(u^h(t_{i-1}, \cdot))(x)\Delta W_k(t_i), \quad x \in \mathbb{R}^d, \ \ i = 1, \ldots, n.
\end{aligned} \tag{3.1}$$

With this definition, the N-dimensional vector $u_i^h = (u_{i,1}^h, \ldots, u_{i,N}^h)^\mathsf{T}$ of the collocation points satisfies

$$u_0^h = (u_0(x_1), \ldots, u_0(x_N))^\mathsf{T},$$
$$u_i^h = u_{i-1}^h + \sum_{k=0}^{m} (A_k A^{-1} u_{i-1}^h) \Delta W_k(t_i), \quad i = 1, \ldots, n.$$

Here, we have set $A_k = (A_{k,j\ell})_{1 \leqslant j,\ell \leqslant N}$ with $A_{k,j\ell} = L_k \Phi(x - x_\ell)|_{x=x_j}$. This follows from

$$L_k u^h(t_i, x_j) = \sum_{\ell=1}^{N} (A^{-1} u_i^h)_\ell L_k \Phi(x - x_\ell)|_{x=x_j} = (A_k A^{-1} u_i^h)_j.$$

To discuss the error of the approximation above, set $\Delta t = \max_{1 \leqslant i \leqslant n}(t_i - t_{i-1})$ and consider the Hausdorff distance $\Delta_1 x$ between Γ and $(-R, R)^d$, and the separation distance $\Delta_2 x$ defined respectively by

$$\Delta_1 x = \sup_{x \in (-R,R)^d} \min_{j=1,\ldots,N} |x - x_j|, \quad \Delta_2 x = \frac{1}{2} \min_{i \neq j} |x_i - x_j|.$$

Then suppose that Δt, R, N, $\Delta_1 x$ and $\Delta_2 x$ are functions of h.

Assumption 3.1. (i) The parameters Δt, R, N, and $\Delta_1 x$ satisfy $\Delta t \to 0$, $R \to \infty$, $N \to \infty$, and $\Delta_1 x \to 0$ as $h \searrow 0$.

(ii) There exist c_1, c_2, c_3, c_4 and λ, positive constants independent of h, such that

$$|(A^{-1})_{ij}| \leqslant c_1 \frac{(\Delta_2 x)^d}{N}$$

for $i, j \in \{1, \ldots, N\}$ with $|i - j| > c_2 (\Delta_2 x)^{-\lambda d}$, and that

$$c_3 (\Delta_2 x)^{-(1+\lambda)d} \leqslant R^{1/2} \leqslant c_4 (\Delta_1 x)^{-(\tau - 3/2)/d}.$$

Remark 3.2. Notice that $\Delta_2 x \leqslant C \Delta_1 x$. Thus the condition $\Delta_1 x \to 0$ implies $\Delta_2 x \to 0$ as $h \searrow 0$.

Remark 3.3. Suppose that Γ is quasi-uniform in the sense that

$$c_5 R N^{-1/d} \leqslant \Delta_2 x \leqslant c_6 R N^{-1/d}, \quad c_5' R N^{-1/d} \leqslant \Delta_1 x \leqslant c_6' R N^{-1/d}$$

hold for some positive constants c_5, c_6, c_5', c_5'. In this case, a sufficient condition for which the latter part of Assumption 3.1 (ii) holds is

$$c_7 N^{(1 - 1/(1 + 2d(1+\lambda)))\frac{1}{d}} \leqslant R \leqslant c_8 N^{(1 - d/(d + 2\tau - 3))\frac{1}{d}}$$

with $\tau \geqslant 3/2 + (1 + \lambda) d^2$, for some positive constants c_7 and c_8.

The approximation error for the Zakai equation is estimated as follows:

Theorem 3.4. *Suppose that Assumptions* 2.1 *and* 3.1 *hold. Suppose moreover that* $\tau \geqslant 3$. *Then, there exists* $h_0 > 0$ *such that*

$$\max_{i=1,\dots,n} \sup_{x\in(-R,R)^d} \mathbb{E}\left[|u(t_i,x)-u^h(t_i,x)|^2\right] \leqslant C\left(\Delta t + (\Delta_1 x)^{2\tau-3}\right), \quad h \leqslant h_0.$$

The rest of this section is devoted to the proof of Theorem 3.4. To this end, for every $x \in (-R,R)^d$, put

$$\mathcal{I}(x) = \{i \in \{1,\dots,N\} : |x - x_i| \leqslant 1\}.$$

In what follows, $\#\mathcal{K}$ denotes the cardinality of a finite set $\mathcal{K}$.

Lemma 3.5. *Suppose that* $\Delta_2 x \leqslant 1$. *Then there exists* $\nu \in \mathbb{N}$ *such that*

$$\#\mathcal{I}(x) \leqslant \nu\lfloor q^{-d}\rfloor, \quad x \in (-R,R)^d.$$

Proof. Fix $x \in (-R,R)^d$. Put $q = \Delta_2 x$ for notational simplicity. It follows from the definition of $\mathcal{I}(x)$ that

$$\bigcup_{i\in\mathcal{I}(x)} B_{q/2}(x_i) \subset B_{3/2}(x).$$

Further, $\{B_{q/2}(x_i)\}_{i\in\mathcal{I}(x)}$ is disjoint. Indeed, otherwise, there exists $y \in \mathbb{R}^d$, $i,j \in \mathcal{I}(x)$ such that $|y-x_i| \leqslant q/2$ and $|y-x_j| \leqslant q/2$. This implies $|x_i-x_j| \leqslant q = (1/2)\min_{i'\neq j'}|x_{i'}-x_{j'}|$, and so $x_i = x_j$. Since we have assumed that x_i's are pairwise distinct, we have $i = j$. Denote by $\mathrm{Leb}(A')$ for the Lebesgue measure for A'. Then we have $C\#\mathcal{I}(x)q^d = \mathrm{Leb}(\cup_{i\in\mathcal{I}(x)}B_{q/2}(x_i)) \leqslant \mathrm{Leb}(B_{3/2}(x))$. Thus, $\#\mathcal{I}(x) \leqslant \tilde{C}q^{-d}$ for some $\tilde{C} > 0$ that is independent of x. □

Lemma 3.6. *Suppose that Assumption* 3.1 (i) *and* $\tau \geqslant 3$ *hold. Then, there exists* $h_0 > 0$ *such that for any multi-index* α *with* $|\alpha|_1 \leqslant 2$ *and* $f \in H^{\tau+(d+1)/2}(\mathbb{R}^d)$, *we have*

$$\|D^\alpha f - D^\alpha I(f)\|_{L^\infty(-R,R)^d} \leqslant C(\Delta_1 x)^{\tau+1/2-|\alpha|_1}\|f\|_{H^{\tau+(d+1)/2}(\mathbb{R}^d)}, \quad h \leqslant h_0.$$

Proof. This result is reported in [19, Corollary 11.33] for more general domains. However, a simple application of that result leads to an ambiguity of the dependence of the constant C on R. Here we will confirm that we can take C to be independent of R.

Let $f \in H^{\tau+(d+1)/2}(\mathbb{R}^d)$ with $f|_\Gamma = 0$. Set $\tilde{\Gamma} = \{x_1/R,\dots,x_N/R\}$ and $\tilde{f}(y) = f(Ry)$, $y \in (-1,1)^d$. Then, $\tilde{f}|_{\tilde{\Gamma}} = f|_\Gamma = 0$ and

$$\sup_{y\in(-1,1)^d}\min_{\xi\in\tilde{\Gamma}}|\xi - y| = \sup_{y\in(-R,R)^d}\min_{j=1,\dots,N}\left|\frac{x_j}{R}-\frac{y}{R}\right| = \frac{\Delta_1 x}{R}.$$

Since $\Delta_1 x/R \to 0$ as $h \searrow 0$ and $\tau \geqslant 3$, we can apply [19, Theorem 11.32] to $\tilde{f}$ to obtain

$$|D^\alpha\tilde{f}(y)| \leqslant C(\Delta_1 x/R)^{\tau+1/2-|\alpha|_1}|\tilde{f}|_{H^{\tau+(d+1)/2}((-1,1)^d)}, \quad h \leqslant h_0 \tag{3.2}$$

for some $h_0 > 0$. Here, for an open set $\mathcal{O}$ and $g \in H^{\tau+(d+1)/2}(\mathcal{O})$,

$$|g|^2_{H^{(d+1)/2+\tau}(\mathcal{O})} = \sum_{|\alpha|=k}\int_{\mathcal{O}}\int_{\mathcal{O}}\frac{|D^\alpha g(x)-D^\alpha g(y)|^2}{|x-y|^{d+2s}}dxdy$$

with k being the smallest integer that exceeds $\tau + d/2 - 1/2$ and $s = k - (\tau + d/2 - 1/2)$. It is straightforward to see that

$$D^\alpha \tilde{f}(y) = R^{|\alpha|_1}(D^\alpha f)(Ry), \quad |\tilde{f}|_{H^{k+s}((-1,1)^d)} = R^{k+s-d/2}|f|_{H^{k+s}((-R,R)^d)}.$$

Substituting these relations into (3.2), we have

$$|D^\alpha f(y)| \leqslant C(\Delta_1 x)^{\tau+1/2-|\alpha|_1}|f|_{H^{\tau+(d+1)/2}((-R,R)^d)}. \tag{3.3}$$

This and (2.5) yield

$$\begin{aligned}\|D^\alpha f - D^\alpha I(f)\|_{L^\infty((-R,R)^d)} &\leqslant C(\Delta_1 x)^{\tau+1/2-|\alpha|_1}|f - I(f)|_{H^{\tau+(d+1)/2}((-R,R)^d)}\\ &\leqslant C(\Delta_1 x)^{\tau+1/2-|\alpha|_1}\|f\|_{H^{\tau+(d+1)/2}(\mathbb{R}^d)}.\end{aligned}$$

Thus the lemma follows. □

Observe that for any $f : \mathbb{R}^d \to \mathbb{R}$,

$$I(f)(x) = \sum_{j=1}^N (A^{-1}f|_\Gamma)_j \Phi(x - x_j) = \sum_{j=1}^N f(x_j)Q_j(x), \quad x \in \mathbb{R}^d,$$

where

$$Q_j(x) = \sum_{i=1}^N (A^{-1})_{ij}\Phi(x - x_i), \quad j = 1, \ldots, N.$$

The following result tells us that the process of iterative kernel-based interpolation is stable, which is a key to our convergence analysis.

Lemma 3.7. *Suppose that Assumption* 3.1 *and* $\tau \geqslant 3$ *hold. Then,*

$$\sup_{h>0} \sup_{x\in(-R,R)^d} \sum_{j=1}^N |D^\alpha Q_j(x)| < \infty, \quad |\alpha|_1 \leqslant 2.$$

Proof. Fix $\tilde{x} \in (-R, R)^d$ and $|\alpha|_1 \leqslant 2$. Set $q = \Delta_2 x$ for simplicity. First consider the set

$$\mathcal{J}(\tilde{x}) := \{j \in \{1, \ldots, N\} : |i - j| > c_2 q^{-\lambda d}, \ i \in \mathcal{I}(\tilde{x})\}.$$

Then, by Assumption 3.1 (ii),

$$|A^{-1}_{ij}| \leqslant c_1 \frac{q^d}{N}, \quad j \in \mathcal{J}(\tilde{x}), \ i \in \mathcal{I}(\tilde{x}).$$

This together with Lemma 3.5 leads to

$$\begin{aligned}\sum_{j\in\mathcal{J}(\tilde{x})} |D^\alpha Q_j(\tilde{x})| &= \sum_{j\in\mathcal{J}(\tilde{x})} |\sum_{i\in\mathcal{I}(\tilde{x})} (A^{-1})_{ij} D^\alpha \Phi(\tilde{x} - x_i)| \leqslant C \sum_{j\in\mathcal{J}(\tilde{x})} \sum_{i\in\mathcal{I}(\tilde{x})} |(A^{-1})_{ij}|\\ &\leqslant C \sum_{j\in\mathcal{J}(\tilde{x})} \sum_{i\in\mathcal{I}(\tilde{x})} \frac{q^d}{N} \leqslant CNq^{-d}\frac{q^d}{N} \leqslant C.\end{aligned} \tag{3.4}$$

Now again by Assumption 3.1 (ii) and Lemma 3.5, there exists $\tilde{\nu} \in \mathbb{N}$ such that

$$\#\{j : j \notin \mathcal{J}(\tilde{x})\} \leqslant \tilde{\nu}\lfloor R^{1/2} \rfloor.$$

Then, by Kergin interpolation (see Kergin [12]) there exists a polynomial p on $\mathbb{R}^d$ with degree at most $\tilde{\nu}\lfloor R^{1/2} \rfloor$ that interpolates $\operatorname{sgn}(D^\alpha Q_j(\tilde{x}))$ at x_j for all $j \notin \mathcal{J}(\tilde{x})$. This leads to

$$\sum_{j \notin \mathcal{J}(\tilde{x})} |D^\alpha Q_j(\tilde{x})| = \sum_{j \notin \mathcal{J}(\tilde{x})} \operatorname{sgn}(D^\alpha Q_j(\tilde{x})) D^\alpha Q_j(\tilde{x}) = \sum_{j \notin \mathcal{J}(\tilde{x})} p(x_j) D^\alpha Q_j(\tilde{x}). \tag{3.5}$$

By Corollary 3.9 in [19], this polynomial p satisfies $|p(x)| \leqslant 2$ for $x \in (-R, R)^d$. This together with Proposition 11.6 in [19] and Assumption 3.1 (ii) implies that

$$\max_{|\alpha|_1 \leqslant 1} \sup_{x \in (-R,R)^d} |D^\alpha p(x)| \leqslant C_0$$

for some $C_0 > 0$ that is independent of R and $\tilde{x}$. In particular, p is Lipschitz continuous on $(-R, R)^d$ with Lipschitz coefficient C_0. Hence the function

$$\tilde{p}(x) := \inf_{y \in (-R,R)^d} \{p(y) + C_0|x - y|\}, \quad x \in \mathbb{R}^d,$$

is Lipschitz continuous on $\mathbb{R}^d$ with the same Lipschitz coefficient and satisfies $\tilde{p} = p$ on $(-R, R)^d$. Further, for $\varepsilon > 0$ to be specified later, define the function $\bar{p}$ on $\mathbb{R}^d$ by

$$\bar{p}(x) = \varepsilon^{-d} \int_{\mathbb{R}^d} \tilde{p}(y) \zeta\left(\frac{x - y}{\varepsilon}\right) dy,$$

where ζ is a C^∞-function such that $0 \leqslant \zeta(x) \leqslant 1$ for $x \in \mathbb{R}^d$, $\zeta(x) = 1$ for $|x| \leqslant 1$, and $\zeta(x) = 0$ for $|x| > 1 + \tilde{c}$ for some $\tilde{c} > 0$. It is straightforward to verify that this function satisfies

$$|\bar{p}(x) - \tilde{p}(x)| \leqslant C_1\varepsilon, \quad |D^\alpha \bar{p}(x)| \leqslant C_1 \varepsilon^{-|\alpha|_1}, \quad |\alpha|_1 \leqslant \nu_1, \ \ x \in \mathbb{R}^d \tag{3.6}$$

for some $C_1 > 0$ that is independent of R and ε. Here $\nu_1 = \tau + \min\{\kappa \in \mathbb{Z} : \kappa \geqslant (d + 1)/2\}$. Then consider the function $\hat{p} \in H^{\nu_1}(\mathbb{R}^d)$ defined by $\hat{p}(x) = \tilde{p}(x)\zeta(x/R)$, $x \in \mathbb{R}^d$. With these modifications and in view of (3.4)–(3.6), we obtain

$$\begin{aligned}
\sum_{j=1}^N |D^\alpha Q_j(\tilde{x})| &\leqslant \sum_{j \notin \mathcal{J}(\tilde{x})} p(x_j) D^\alpha Q_j(\tilde{x}) + C = \sum_{j=1}^N p(x_j) D^\alpha Q_j(\tilde{x}) - \sum_{j \in \mathcal{J}(\tilde{x})} p(x_j) D^\alpha Q_j(\tilde{x}) + C \\
&\leqslant \sum_{j=1}^N p(x_j) D^\alpha Q_j(\tilde{x}) + 2 \sum_{j \in \mathcal{J}(\tilde{x})} |D^\alpha Q_j(\tilde{x})| + C \\
&\leqslant \sum_{j=1}^N \tilde{p}(x_j) D^\alpha Q_j(\tilde{x}) + C \leqslant \sum_{j=1}^N \hat{p}(x_j) D^\alpha Q_j(\tilde{x}) + C_1\varepsilon \sum_{j=1}^N |D^\alpha Q_j(\tilde{x})| + C.
\end{aligned}$$

Setting ε to satisfy $C_1\varepsilon < 1$, we obtain

$$\sum_{j=1}^{N} |D^\alpha Q_j(\tilde{x})| \leqslant \frac{1}{1-C_1\varepsilon}|D^\alpha I(\hat{p})(\tilde{x})| + \frac{C}{1-C_1\varepsilon}.$$

Moreover, by Lemma 3.6 and (3.6),

$$\begin{aligned}
|D^\alpha I(\hat{p})(\tilde{x})| &\leqslant |D^\alpha I(\hat{p})(\tilde{x}) - D^\alpha \hat{p}(\tilde{x})| + |D^\alpha \bar{p}(\tilde{x})| \\
&\leqslant C(\Delta_1 x)^{\tau-3/2}\|\hat{p}\|_{H^{\tau+(d+1)/2}(\mathbb{R}^d)} + C \\
&\leqslant C(\Delta_1 x)^{\tau-3/2}\|\bar{p}\|_{H^{\nu_1}((-(1+\tilde{c})R,(1+\tilde{c})R)^d)} + C \\
&\leqslant C\|\bar{p}\|_{C^{\nu_1}(\mathbb{R}^d)} R^{d/2}(\Delta_1 x)^{\tau-3/2} + C.
\end{aligned}$$

Assumption 3.1 (ii) and the boundedness of $\|\bar{p}\|_{C^{\nu_1}(\mathbb{R}^d)}$ now lead to the conclusion of the lemma. □

Proof of Theorem 3.4. First, for $i = 0, \ldots, n-1$ and $x \in \mathbb{R}^d$, we have

$$\begin{aligned}
(u(t_{i+1}, x) - u^h(t_{i+1}, x))^2 = (u(t_i, x) &- u^h(t_i, x))^2 + (S_{i+1}(x))^2 + (\Theta_{i+1}(x))^2 \\
&+ 2(u(t_i, x) - u^h(t_i, x))S_{i+1}(x) + 2\Theta_{i+1}(x)S_{i+1}(x) \\
&+ 2(u(t_i, x) - u^h(t_i, x))\Theta_{i+1}(x).
\end{aligned}$$

Here, for $i = 0, \ldots, n-1$ and $x \in \mathbb{R}^d$,

$$\begin{aligned}
S_{i+1}(x) &= \sum_{k=0}^{m} L_k I(u(t_i) - u^h(t_i))(x)\Delta W^k_{t_{i+1}}, \\
\Theta_{i+1}(x) &= \sum_{k=0}^{m} \int_{t_i}^{t_{i+1}} (L_k u(s, x) - L_k I(u(t_i))(x))dW^k_s.
\end{aligned}$$

It is straightforward to see that

$$\begin{aligned}
&\mathbb{E}(S_{i+1}(x))^2 \\
&= \mathbb{E}|L_0 I(u(t_i) - u^h(t_i))(x)|^2(t_{i+1} - t_i)^2 + \sum_{k=1}^{m} \mathbb{E}|L_k I(u(t_i) - u^h(t_i))(x)|^2(t_{i+1} - t_i) \\
&\leqslant C \sum_{|\alpha|_1 \leqslant 2} \mathbb{E}|D^\alpha I(u(t_i) - u^h(t_i))|^2 \Delta t.
\end{aligned}$$

By Cauchy-Schwartz inequality and Lemma 3.7,

$$
\begin{aligned}
&\mathbb{E}|D^\alpha I(u(t_i)-u^h(t_i))(x)|^2 = \mathbb{E}\left|\sum_{j=1}^N f(x_j)D^\alpha Q_j(x)\right|^2 \\
&= \sum_{j,\ell=1}^N \mathbb{E}[(u(t_i,x_j)-u^h(t_i,x_j))(u(t_i,x_\ell)-u^h(t_i,x_\ell))]D^\alpha Q_j(x)D^\alpha Q_\ell(x) \\
&\leqslant \sum_{j,\ell=1}^N (\mathbb{E}|u(t_i,x_j)-u^h(t_i,x_j)|^2)^{1/2}(\mathbb{E}|u(t_i,x_\ell)-u^h(t_i,x_\ell)|^2)^{1/2}|D^\alpha Q_j(x)||D^\alpha Q_\ell(x)| \\
&\leqslant \sup_{y\in(-R,R)^d}\mathbb{E}|u(t_i,y)-u^h(t_i,y)|^2\left(\sum_{j=1}^N |D^\alpha Q_j(x)|\right)^2 \\
&\leqslant C\sup_{y\in(-R,R)^d}\mathbb{E}|u(t_i,y)-u^h(t_i,y)|^2.
\end{aligned}
$$

Hence,

$$
\mathbb{E}(S_{i+1}(x))^2 \leqslant C\sup_{y\in(-R,R)^d}\mathbb{E}|u(t_i,y)-u^h(t_i,y)|^2\Delta t. \tag{3.7}
$$

Next, it follows from Itô isometry that

$$
\begin{aligned}
&\mathbb{E}|\Theta_{i+1}(x)|^2 \\
&\leqslant 2\mathbb{E}\left|\int_{t_i}^{t_{i+1}}(L_0u(s,x)-L_0I(u(t_i))(x))ds\right|^2 + 2\mathbb{E}\left|\sum_{k=1}^m\int_{t_i}^{t_{i+1}}(L_ku(s,x)-L_kI(u(t_i))(x))dW_s^k\right|^2 \\
&\leqslant 2\Delta t\mathbb{E}\int_{t_i}^{t_{i+1}}|L_0u(s,x)-L_0I(u(t_i))(x)|^2ds + 2\sum_{k=1}^m\mathbb{E}\int_{t_i}^{t_{i+1}}|L_ku(s,x)-L_kI(u(t_i))(x)|^2ds \\
&\leqslant C\sum_{|\alpha|_1\leqslant 2}\mathbb{E}\int_{t_i}^{t_{i+1}}|D^\alpha u(s,x)-D^\alpha I(u(t_i))(x)|^2ds.
\end{aligned}
$$

Again by Itô isometry,

$$
\begin{aligned}
\mathbb{E}|D^\alpha u(s,x)-D^\alpha u(t_i,x)|^2 &= \mathbb{E}\left|\sum_{k=0}^m\int_{t_i}^s D^\alpha u(r,x)dW_r^k\right|^2 \leqslant C\sum_{k=0}^m\mathbb{E}\int_{t_i}^s|D^\alpha L_ku(r,x)|^2dr \\
&\leqslant C\mathbb{E}\sup_{0\leqslant t\leqslant T}\|u(t)\|^2_{C^4(\mathbb{R}^d)}(s-t_i).
\end{aligned}
$$

This and Lemma 3.6 means

$$
\mathbb{E}|D^\alpha u(t_i,x)-D^\alpha I(u(t_i))(x)|^2 \leqslant C(\Delta_1 x)^{2\tau-3}\mathbb{E}\|u\|^2_{H^{\tau+(d+1)/2}(\mathbb{R}^d)}.
$$

Thus,

$$
\mathbb{E}|\Theta_{i+1}(x)|^2 \leqslant C(\Delta t)^2 + C(\Delta_1 x)^{2\tau-3}\Delta t. \tag{3.8}
$$

The arguments used in the estimations above yield

$$
\begin{aligned}
&\mathbb{E}(u(t_i,x)-u^h(t_i,x))S_{i+1}(x)\\
&= \mathbb{E}(u(t_i,x)-u^h(t_i,x))L_0 I(u(t_i)-u^h(t_i))(x)(t_{i+1}-t_i)\\
&\leqslant (\mathbb{E}|u(t_i,x)-u^h(t_i,x)|^2)^{1/2}(\mathbb{E}|L_0 I(u(t_i)-u^h(t_i))(x)|^2)^{1/2}\Delta t\\
&\leqslant C \sup_{y\in(-R,R)^d} \mathbb{E}|u(t_i,y)-u^h(t_i,y)|^2\Delta t.
\end{aligned}
\tag{3.9}
$$

Furthermore, we obtain

$$
\begin{aligned}
&\mathbb{E}(u(t_i,x)-u^h(t_i,x))\Theta_{i+1}(x)\\
&= \mathbb{E}(u(t_i,x)-u^h(t_i,x))\int_{t_i}^{t_{i+1}}(L_0u(s,x)-L_0I(u(t_i))(x))ds\\
&\leqslant (\mathbb{E}|u(t_i,x)-u^h(t_i,x)|^2)^{1/2}\left(\mathbb{E}\left|\int_{t_i}^{t_{i+1}}(L_0u(s,x)-L_0I(u(t_i))(x))ds\right|^2\right)^{1/2}\\
&\leqslant (\mathbb{E}|u(t_i,x)-u^h(t_i,x)|^2)^{1/2}\left(\Delta t\mathbb{E}\int_{t_i}^{t_{i+1}}|L_0u(s,x)-L_0I(u(t_i))(x)|^2\,ds\right)^{1/2}\\
&= (\mathbb{E}|u(t_i,x)-u^h(t_i,x)|^2\Delta t)^{1/2}\left(\mathbb{E}\int_{t_i}^{t_{i+1}}|L_0u(s,x)-L_0I(u(t_i))(x)|^2\,ds\right)^{1/2}\\
&\leqslant 2\mathbb{E}|u(t_i,x)-u^h(t_i,x)|^2\Delta t+2\mathbb{E}\int_{t_i}^{t_{i+1}}|L_0u(s,x)-L_0I(u(t_i))(x)|^2\,ds.
\end{aligned}
$$

and so

$$
\begin{aligned}
&\mathbb{E}(u(t_i,x)-u^h(t_i,x))\Theta_{i+1}(x)\\
&\leqslant 2\sup_{y\in(-R,R)^d}\mathbb{E}|u(t_i,y)-u^h(t_i,y)|^2)\Delta t + C(\Delta t)^2 + C(\Delta_1 x)^{2\tau-3}\Delta t.
\end{aligned}
\tag{3.10}
$$

Then, from (3.7)–(3.10) we have, for $i=0,\ldots,n-1$,

$$
\begin{aligned}
&\sup_{x\in(-R,R)^d}\mathbb{E}|u(t_{i+1},x)-u^h(t_{i+1},x)|^2\\
&\leqslant (1+C\Delta t)\sup_{x\in(-R,R)^d}\mathbb{E}|u(t_i,x)-u^h(t_i,x)|^2 + C(\Delta t)^2 + C(\Delta_1 x)^{2\tau-3}\Delta t.
\end{aligned}
$$

A simple application of the discrete Gronwall lemma now leads to what we aim to prove. □

4 Numerical experiments

In this section, we apply our collocation method to the one-dimentional Zakai equation

$$
\begin{cases}
du(t,x) = \left(\dfrac{1}{2}\dfrac{\partial^2}{\partial x^2}u(t,x) - \dfrac{\partial}{\partial x}(\tanh(x)u(t,x))\right)dt + u(t,x)dW(t), \quad 0\leqslant t\leqslant 1,\\
u(0,x) = \dfrac{1}{\sqrt{2\pi}}\cosh(x)e^{-|x|^2/2}.
\end{cases}
\tag{4.1}
$$

The unique solution $u(t,x)$ to (4.1) is given by

$$u(t,x) = \frac{1}{\sqrt{2\pi}} \cosh(x) \exp\left(W(t) - \frac{3t}{2} - \frac{|x|^2}{2(1+t)} \right).$$

We use the Wendland kernel $\phi_{1,4}$ scaled by some positive constant for the performance test. As suggested in Remark 3.3, we take the uniform grid points on $[-R+2R/(N+1), R-2R/(N+1)]$ where $R = (1/5)N^{(1-1/(2\tau-2))}$. To check the validity of Assumption 3.1 (ii), we plot

$$\iota(N) = \min\{|i-j| : |A^{-1}_{ij}| < \Delta_2 x/N\}$$

in Figure 4.1. We can see that $\iota(N) > 1/(2\Delta_2 x)$ for $30 \leqslant N \leqslant 100$. Thus, Assumption 3.1

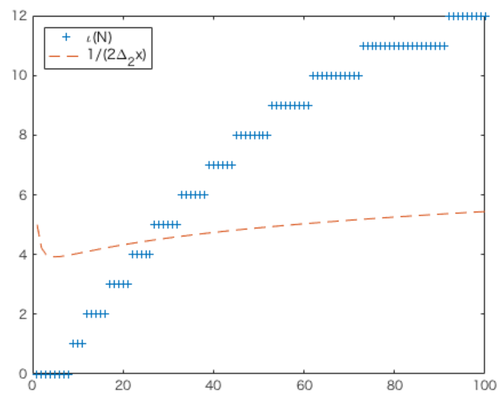

Figure 4.1: Plotting $\iota(N)$ and $1/(2\Delta_2 x)$ for $N = 1, 2, \ldots, 100$.

(ii) is satisfied with $c_1 = 1$, $c_2 = 1/2$, and $\lambda = 1$ for the sequence of the tuning parameters defined by N from 30 at least to 100. This is consistent with the condition $\tau \geqslant 3/2+(1+\lambda)d^2$ in Remark 3.3.

Figure 4.2 plots sample paths of $u(\cdot,x)$ and $u^h(\cdot,x)$ for several spacial points. Figure 4.3 plots snapshots of the time evolutions of $u(t,\cdot)$ and $u^h(t,\cdot)$. The both show that our collocation method yields a good approximation as well as the accumulation of the error near the time maturity cannot be negligible. To compare an averaged performance, we compute the root squared mean errors averaged over 10000 samples, defined by

$$\mathrm{RSME} := \sqrt{\frac{1}{10000 \times 41(n+1)} \sum_{i=0}^{n} \sum_{j=1}^{41} \sum_{\ell=1}^{10000} |u_\ell(t_i,\xi_j) - u^h_\ell(t_i,\xi_j)|^2},$$

for several values of N and n. Here, u_ℓ and u^h_ℓ are the exact solution and approximate solution at ℓ-th trial, respectively, and $\{\xi_1,\ldots,\xi_{41}\}$ is the set of evaluation points consisting of the equi-spaced grid points on $[-2,2]$. Table 4.1 shows that the resulting root squared mean errors are sufficiently small for all pairs (N,n) although its decrease is nonmonotonic. We can conclude that Theorem 3.4 is well consistent with the results of our experiments especially for the case of $N \geqslant 2^5$.

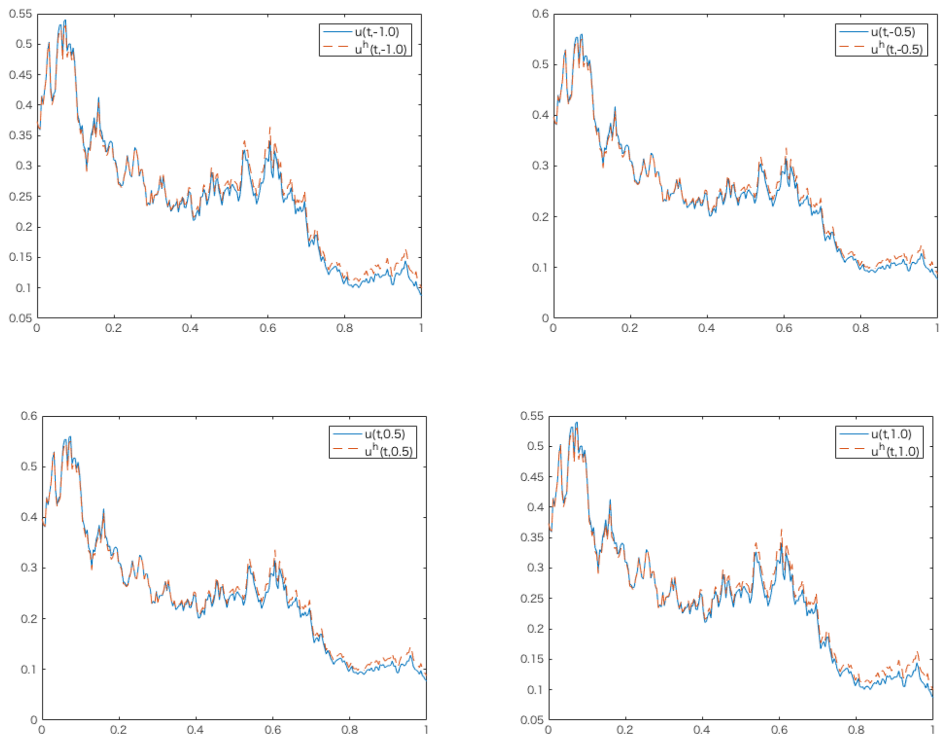

Figure 4.2: Comparing the exact solution (solid line) and the approximated one (dashed line) at $x = -1, -1/2, 1/2, 1$, in the case of $N = 2^5$ and $n = 2^8$.

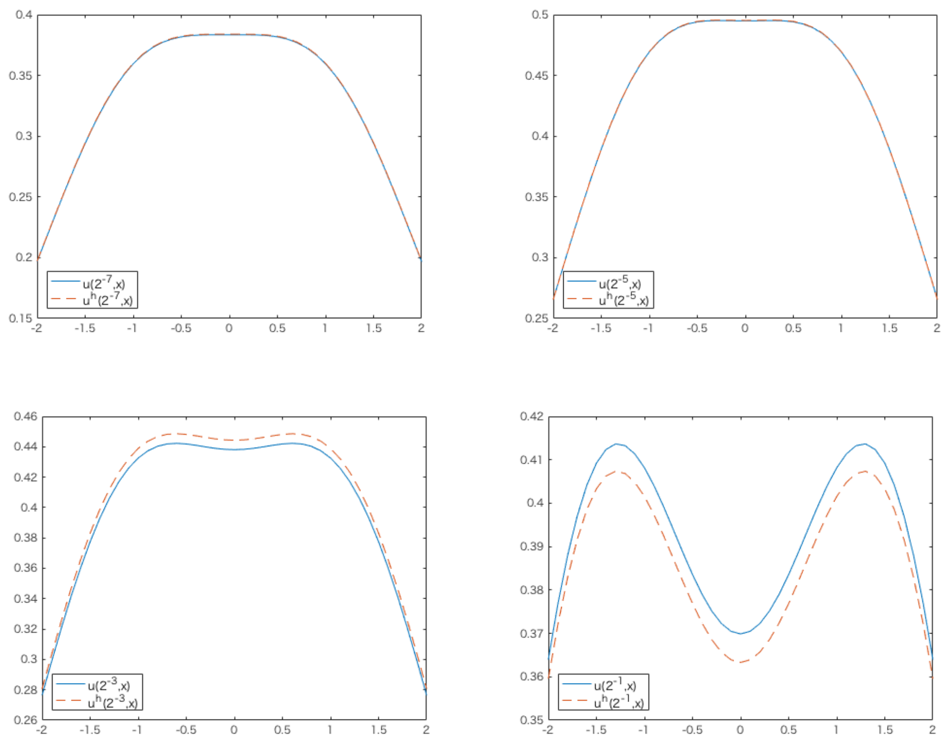

Figure 4.3: Comparing the exact solution (solid line) and the approximated one (dashed line) at time $= 2^1\Delta t, 2^3\Delta t, 2^5\Delta t, 2^7\Delta t$, in the case of $N = 2^5$ and $n = 2^8$.

N	R	$(\Delta_1 x)^{\tau-3/2}$	n	$\sqrt{\Delta t}$	RSME
2^4	2.0159	0.0274	2^6	0.1250	0.0714
			2^8	0.0625	0.0726
			2^{10}	0.0312	0.0745
2^5	3.5919	0.0221	2^6	0.1250	0.0323
			2^8	0.0625	0.0252
			2^{10}	0.0312	0.0234
2^6	6.4000	0.0172	2^6	0.1250	0.0333
			2^8	0.0625	0.0261
			2^{10}	0.0312	0.0241

Table 4.1: The resulting RSME's for several pairs (N, n).

Acknowledgements

This study is partially supported by JSPS KAKENHI Grant Number JP17K05359.

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