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Applications of game theory in many-to-one matching problems

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I would like to dedicate this thesis to my loving parents.

Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. This dissertation is my own work and contains nothing which is the outcome of work done in collaboration with others, except as specified in the text.

Chengyue Li
February 2018

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Abstract

We study two-sided matching problems where one side consists of institutions and the other side individuals. These problems have prolonged social and economic influence since they associate with lots of real-world matching practice such as job matching, school admission and so on. Theoretically, such a problem can be formulated as a many-to-one matching model. Previous and current research on this model has been focusing on mainly three aspects: stability concept, equilibrium analysis and algorithmic solution. Our research aims at exploring to which extent Gale and Shapley's classical model can be enhanced to solve complex many-to-one matching problems, and then contributes to fixing market failures by providing solutions in these three aspects. Specifically, we look into a labor market with externalities among firms, a preference revelation game which is defined on a school choice market and a house allocation problem with existing tenants.

We first study a labor market with externalities among firms. In this market, each firm cares about not only which workers it hires but also which workers the other firms hire. Under this assumption, the standard definition of stable matching is not applicable any more since firms have preferences over matchings instead of workers. Moreover, we assume that whenever a firm hires new workers or fires its current workers, the matching of the rest firms and workers is not kept fixed. Then different outcomes will in turn affect this firm's strategic choice. In order to formulate each firm's expectation on such uncertainty, we apply a tool called estimation function which represents all possible matchings it believes will happen. Particularly, we enhance the current notion of estimation function by defining a function which assigns a numeric rank to each matching. Among all these matchings, we assume that there exists a matching which determines a firm will deviate or not according to whether this matching is more preferred to the current one. In this sense, we name such a matching as threshold matching. Based on it, we propose a new stability concept called T-stability. Finally, we provide two restrictions on firms' preferences which are called T-substitutability and induced preferences. When these restrictions are satisfied, we show that T-stable matchings always exist.

Then we analyze a preference revelation game which is defined on a school choice market. In this market, each school's preferences are open to the public due to laws, regulations

and so on, but each student is able to report preferences strategically. This fact enables us to formulate this market as a preference revelation game of students and then analyze the equilibrium outcome. Previous research has demonstrated the existence of two Nash equilibria which are called strong Nash equilibrium and strictly strong Nash equilibrium. Moreover, two types of strictly strong Nash equilibria have been characterized according to the way by which they are derived. In our research, we refine the current result by proposing an even stronger equilibrium concept called passively-strictly strong Nash equilibrium. It rules out a deviating coalition which includes students who were threatened to join. Simply speaking, such a student deviates not for achieving a better outcome but for avoiding an even worse one which can be caused by the rest coalition members. Then we show two existence results about this Nash equilibrium. One is that if two types of strictly strong Nash equilibria are not equivalent, then none of them is a passively-strictly strong Nash equilibrium. The other one is that if the strictly strong Nash equilibrium derived by an iterative deferred acceptance algorithm satisfies a condition called irrelevance of low-tier agents, then it is also a passively-strictly strong Nash equilibrium.

Finally, we investigate a house allocation problem with existing tenants. This problem is not a many-to-one matching problem, but is closely related since it adopts a popular mechanism which has been demonstrated to be a special case of the student-optimal stable mechanism which has important application in many-to-one matching problems. Named as the MIT-NH4 mechanism, this mechanism satisfies several nice properties including individual rationality, fairness and strategy-proofness. But it does not satisfy Pareto efficiency and easily causes large efficiency loss. Witnessing this fact, our research aims at recovering efficiency loss for the MIT-NH4 mechanism. We first partition houses into underdemanded house, quasi-underdemanded house and overdemanded house according to the outcome determined by the standard MIT-NH4 mechanism. Then we show that students who are matched with underdemanded houses and quasi-underdemanded houses are Pareto unimprovable. But nevertheless, if they give consent to certain priority violation, then Pareto efficiency of the other students' assignment can be improved. Based on these results, we propose an efficiency-adjusted MIT-NH4 mechanism which recovers efficiency loss for the standard MIT-NH4 mechanism. Furthermore, we show two properties of the new mechanism. One is that the consent granted by Pareto unimprovable students will not worsen their own assignments. The other one is that the efficiency-adjusted MIT-NH4 mechanism proceeds along a monotonic improving path.

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Chapter 1

Introduction

1.1 Market and market design

1.1.1 Two-sided matching market

Li [25] provided two definitions of what counts as a market. He first argued that a market often refers to a set of agents who seek to transact with each other. A stock market where some agents buy stocks and some agents sell stocks is a typical example. In this case, what has been transacted is explicit, and any particular agent may be a buyer at one price and a seller at another. However, As Kominers et al.[24] pointed out, the definitions of what counts as a transaction are quite expansive. For example, allocating students to schools is also a transaction, even though it seems to involve no transaction at all. On the other hand, Li [25] argued that a market also refers to a rule-governed institution that facilitates transactions. When taking this definition, some researchers use the word “marketplace” to identify a market. Then a school choice market consists of a clearing house which allocates students to schools according to certain rules.

A key feature of the school choice market is two-sidedness. That is, schools and students belong to two disjoint sets. Some schools may leave the market due to reasons such as they do not plan to launch any education program, but none will become students. Similar examples include marriage market where agents are men and women, labor market where agents are firms and workers, medical intern market where agents are hospitals and medical students, etc.

Theoretical investigation on two-sided matching markets and related problems was pioneered by Gale and Shapley [15]. They studied a marriage market where men and women were seeking spouses, and proposed a solution concept for this market called stable matching. Simply speaking, it is a matching under which none of the men and women would prefer another person to his or her current partner. Gale and Shapley [15] further demonstrated that stable matchings always exist, and designed a computational approach called the deferred acceptance algorithm (DA) to find out at least one stable matching for a given market. Based on Gale and Shapley's work, extensive study on more complicated matching markets and related problems has been conducted.

1.1.2 Market design

When speaking of a two-sided matching market, the word “market" usually means a marketplace that we have introduced previously. A marketplace basically consists of two ingredients — the rules that guide market transactions and the infrastructure that enables those transactions to take place. For example, Gale and Shapley's marriage market can be regarded as a platform which gathers men and women (infrastructure), and then determines the matching for them according to the DA (rule). Therefore, once a market does not work well, we should check out whether failures have occurred with the infrastructure or with the rules.

The fundamental purpose of doing market design is to fix market failures and pursue better performance of market. Roth [34] summarized that well-functioning marketplaces should be able to provide thickness, overcome the congestion that thickness brings, and make it safe and sufficiently simple to participants. Specifically, thickness implies that a marketplace is able to attract a large enough proportion of the potential participants to come and transact with one another. Particularly, providing thickness will blur the distinction between centralized markets and decentralized markets. On the other hand, thickness usually results in congestion. Hence it requires a marketplace to overcome this obstacle by providing enough time or by making transactions fast enough so that participants can consider enough alternative possible transactions to arrive at satisfactory ones. Finally, a marketplace needs to make it safe and sufficiently simply to participants as opposed to transacting outside of the marketplace or engaging in costly and risky strategic behavior.

1.2 Research Motivation

1.2.1 Research motivation

Since Gale and Shapley's seminal paper, two-sided matching markets and related problems under more complicated assumptions have been extensively studied in economic literature. Complexity in turn brings challenge of fixing more difficult market failures. Therefore, our research generally aims at exploring to which extent Gale and Shapley's classical model can be enhanced to handle complex matching problems, and then contributes to fixing market failures caused by them.

Although there are many kinds of matching markets which associate with a spectrum of real-world matching problems, our research will mainly focus on a single type called many-to-one matching. This type has wide applications in analyzing lots of economic phenomena since it is the theoretical model of two-sided matching markets and related problems where one side consists of institutions (e.g. firms, schools, hospitals) and the other side individuals (e.g. workers, students, medical interns). The name “many-to-one" roots in the nature that firms hire many workers, schools admit many students and hospitals enroll many interns, but workers typically work for only one firm, and so forth.

Generally speaking, most of the previous and current research on many-to-one matching markets and related problems has provided solutions to fix market failures in these three aspects: stability concept, equilibrium analysis, and algorithmic solution. Therefore, our research is organized according to these three topics.

1.2.2 Motivation for stability concept

Let us have a look at Gale and Shapley’s stability concept. It is not hard to notice that this concept is not sufficient for some more complicated assumptions. Recall the marriage market. Now suppose that each man or woman cares about not only with whom he or she is matched, but also with whom the other men and women are. Such sort of assumption is especially important when competitions occur. For example, in a labor market, suppose that there is a firm planning to lure an experienced worker from one of its rival firms. Once put into action, the rival firm may hire another worker who is possibly even better or worse, or just leave this position empty. Different expectations on the rival firm’s reactions will in turn affect this firm’s strategic choice. Theoretically, such influence caused by the others’ behavior is called externality. Once externalities are present, firms will have preferences over matchings instead of workers. Apparently, Gale and Shapley’s stability concept is not compatible with this case. Therefore, new stability concept needs to be defined and its existence problem needs to be examined.

1.2.3 Motivation for equilibrium analysis

Let us recall the matching process. We can see that the outcome greatly depends on which preferences the agents from each side report. In many real-world matching problems, if one side consists of institutions (e.g. firms, schools, and hospitals), then their preferences are usually open to public because of laws, regulations and so on. However, individuals on the other side (e.g. workers, students, and medical interns) are able to report their preferences freely. This fact enables individuals to manipulate the outcome by misreprenting their preferences independently or cooperatively. The preferences to report can be regarded as strategies to play, and such a problem can be formulated as a strategic game. Instead of stable matchings, equilibrium analysis becomes the main focus of this kind of research. Until now, the existence of several equilibria with different sense of strength has been successfully demonstrated. We are wondering whether the current equilibrium concepts could be further refined. Such sort of work is especially meaningful when multiple equilibria exist.

1.2.4 Motivation for algorithmic solution

As we have introduced, the DA algorithm proposed by Gale and Shapley can find out at least one stable matching for a given two-sided matching market. Because of this feature, DA and other similar algorithms play an important role in real-world matching programs. However, some research has pointed out that although the stable matching determined by DA is always fair, it may not be Pareto efficient. When the size of a matching program becomes large, this flaw will be magnified, and consequently result in tremendous efficiency loss. Other similar algorithms which are based on DA also suffer from this problem. Therefore, we are wondering whether such efficiency loss can be recovered. Besides the efficiency problem, we encounter the challenge of designing new algorithms whenever new stability concept or equilibrium concept is proposed. Both tasks require comprehensive study on DA itself and its extensions.

1.3 Research Objectives

In summary, the main objective of our research is to enhance the adaptability of Gale and Shapley's two-sided matching model in complex many-to-one problems and then contributes to fixing market failures. We will particularly focus on the three aspects that we have introduced, and set up the following goals for each.

(i) Extending Gale and Shapley's stability concept to incorporate externalities, and then examining under which conditions such stable matchings always exist.

(ii) Refining the current equilibrium concept for a preference revelation game which is defined on a many-to-one matching market. This requires defining an even stronger equilibrium concept and then examining its existence problem.

(iii) Analyzing the efficiency recovery problem of DA and other algorithms which are based on DA.

1.4 Outline of this thesis

This chapter introduces our research topic, and then summarizes the research motivation and objectives. Chapter 2 gives a brief review of the literature on two-sided matching markets and related problems, especially those on many-to-one matching which is the main focus of our work. Particularly, the history and developments are organized in three aspects which are stability concept, equilibrium analysis and algorithmic solution.

In Chapter 3, we study a labor market with externalities among firms. In this market, each firm cares about not only which workers it hires but also which workers the other firms hire. Therefore, each firm has to make expectations on the other firms' reaction once it plans to hire or fire workers. To describe these expectations, we apply a useful tool called estimation function. Then we propose a new stability concept called T-stability which depends on firms' estimation functions. Finally, we provide two restrictions on firms' preferences. When they are satisfied, we show that T-stable matchings always exist.

In Chapter 4, we study a preference revelation game which is defined on a school choice market. In real-world practice, schools' preferences are usually open to the public due to laws, regulations and so on, but students are able to report their preferences strategically. Therefore, we can formulate a school choice market as a preference revelation game of students and then analyze the equilibrium outcome. Previous research has demonstrated the existence of a strong Nash equilibrium and a strictly strong Nash equilibrium. We propose an even stronger equilibrium concept called passively-strictly strong Nash equilibrium. Then we show that when two restrictions are satisfied, this equilibrium always exists.

In Chapter 5, we study a house allocation problem with existing tenants which is closely related to many-to-one matching problems. Among the real-world mechanisms which are adopted by this house allocation problem, the MIT-NH4 mechanism has attracted research interest since it satisfies several nice properties. But this mechanism suffers from a vital flaw that it easily results in Pareto inefficient outcomes. Witnessing this fact, we made attempts to recover efficiency loss for this mechanism. We first characterize two types of houses which are called underdemanded house and quasi-underdemanded house. Then we show that although students who are matched with those houses are Pareto unimprovable, they play an important role in Pareto recovery solutions. Finally, we propose an efficiency-adjusted MIT-NH4 mechanism which recovers efficiency loss for the standard one.

In Chapter 6, we summarize the main contributions of our work and discuss some possible directions for the future research.

In Appendix, we give a brief review of some recent applications of market design theory in solving real-world problems.

Chapter 2

Research background

2.1 Literature Review

2.1.1 General review of two-sided matching markets

Theoretical investigation on two-sided matching markets was pioneered by Gale and Shapley [15]. Instead of the many-to-one matching market that we are going to focus on, they studied a similar but simpler type which is called one-to-one matching market. This type is typically represented by a marriage market where a set of men and a set of women each constitutes one side. The function of this market is to match them with a fundamental assumption that each man can marry at most one woman and each woman can marry at most one man. A related work, Shapley and Shubik [41], studied a class of games called assignment games.

As the research on one-to-one matching market develops, people naturally started wondering whether the achieved findings can be carried on to a more complicated type which is called many-to-one matching market. In such a market, one side usually consists of institutions (e.g. firms, schools, hospitals) and the other side individuals (e.g. workers, students, medical interns). A central issue here is how to model the preferences of the institutions, since these involve comparisons of different groups of individuals. No comparable question arose in the marriage market where preferences over individuals were sufficient to determine preferences over matchings. Kelso and Crawford [22] studied a labor market which consists of firms and workers. They proposed a restriction on firms' preferences called substitutability. Then they demonstrated that when firms' preferences satisfy substitutability, the results achieved by Gale and Shapley [15] still hold for many-to-one matching market.

Besides the labor market, there are some real-world many-to-one matching problems which were originally decentralized but later benefited from centralized matching programs. Two among them have drawn researcher's great interest since they have prolonged social and economic influence. The first one is called school choice problem. In many countries, children were automatically sent to a school in their neighborhoods. Lots of complaints arose since the placements were always far from students' and their parents' expectation. In order to improve this situation, attempts to design new matching mechanisms were made in many countries. In the U.S., more and more cities are employing school choice programs: school authorities take students' and their parents' preferences into account. Such transform aims at achieving more desirable outcomes, and making matching mechanisms more understandable and accessible for the participants. Abdulkadiroğlu and Sönmez [4] reviewed the history of school matching mechanisms in the U.S., and showed that mechanisms in many cities such as Boston have two significant flaws. One is that the mechanism is manipulable. That is, students may benefit from lying about their preferences. The other one is that the result may be neither fair nor efficient. Then they advocated a variation of the DA for use by school choice programs. Abdulkadiroğlu et al. [2] studied a similar centralized matching mechanism which was adopted by the New York City schools. Abdulkadiroğlu et al. [3] studied another mechanism that was being evaluated by the Boston schools.

The other problem is known as the hospital-intern matching. In many countries, medical students work as interns at hospitals. For example, in the U.S., more than 20,000 medical students and 4,000 hospitals are matched. Beginning around 1900, the American hospital-intern market was decentralized, and suffered from unraveling of appointment dates. Since medical interns' quality and interests were unknown, inefficiency arose, and doctors and hospitals tried to change their system. In 1952, a centralized matching mechanism called *NIMP* (National Internship Matching Program, was renamed the National Resident Matching Program, *NRMP*) was established. An early discussion in the medical literature of the algorithm adopted in that market can be found in Stalnaker [45]. A game-theoretical analysis on this market was presented by Roth [30]. This work explained the orderly operation and longevity of *NRMP*, in contrast to the turmoil that characterized various earlier short-lived attempts to organize the market. Some further issues concerning rural hospitals were discussed in Roth [32]. Roth and Peranson [35] explained how a two-sided matching procedure has been adapted to match 20,000 doctors per year to medical residency programs.

In addition, a description and analysis of the many small regional markets for similar entry-level medical positions in England, Scotland, and Wales is contained in Roth [33]. These markets were also investigated by Roth and Sotomayor [38]. In fact, the British medical intern market is a well-known example for another type of two-sided matching market which is known as many-to-many matching market. Unlike those who are in the U.S., medical interns in the U.K. are required to work for more than one hospital throughout their internship programs. This means each medical intern needs to sign multiple contracts with several hospitals. Another similar example is the assignment of teachers to high schools in some countries. For example, As Echenique and Oviedo [13] pointed out, 35 percent of teachers in Argentina work in more than one school. The assignment of teachers to high schools is a clear candidate for a centralized solution guided by theory. Finally, one can view many-to-many matching as an abstract model of contracting between down-stream firms and up-stream providers.

Since we are going to focus on many-to-one matching markets and related problems, we will give a more detailed review of the works in this field. As we have mentioned in the previous chapter, they generally focused on exploring three aspects which are stability concept, equilibrium analysis and algorithmic solution. In this research, we contribute to each aspect by studying a labor market with externalities, a preference revelation game which is defined on a school choice market and a house allocation problem with existing tenants, respectively.

2.1.2 Many-to-one matching markets with externalities

Our research contributes to the work of stability concept by proposing and investigating a new stability concept for a labor market with externalities among firms. As we have mentioned in the previous chapter, a labor market is a typical example of many-to-one matching market. In the standard model, there is a fundamental assumption that each firm's preferences depend on the workers it hires and each worker's preferences depend on the firm he or she works for.

However, in some real-world situations, a firm may care about not only the workers whom it hires but also the workers its rival firms hire. For example, suppose there is a firm planning to lure a worker from one of its rival firms. Once put into action, this firm has to take the rival firm's reaction into account — it may leave the position empty or hire a new worker, possibly even better or even worse. In economic theory, we call such influence caused by others' behavior externalities.

Previous research on two-sided matching markets with externalities mainly involved the following three problems. The first problem is to clarify the domain of externalities. Specifically, we need to make it clear whether externalities exist among all agents or among only part of the agents before starting the formal study. Sasaki and Toda [39], Hafalir [17] and Mumcu and Saglam [28] assumed that externalities exist among all agents in their research on a marriage market. Bando [7], who studied a labor market, assumed that externalities exist among only firms. The second problem is to define a stability concept which is able to incorporate these externalities. Finally, the last problem comes to be finding out whether such stable matchings always exist.

Since externalities are caused by the others' behavior, how to incorporate externalities depends on how to formulate other agents' behavior. It can be assumed fixed at a certain state or completely unpredictable. Under the fixed assumption, Mumcu and Saglam [28] studied a marriage market with externalities. Particularly, their model allowed for the prospect of being single. Then they proposed a stability concept, and showed under which conditions it always exist. Moreover, the core in this market was discussed. Bando [7] extended Mumcu and Saglam [28]'s model to a labor market with externalities among firms. He first extended the standard notion of choice set to incorporate externalities, and then proposed a stability concept based on it. Particularly, his stability concept is a farsighted one which is resistant to firms' myopic behavior. Then he provided several restrictions under which such a stable matching always exist. A related research, Bando [8], proposed an algorithm which is based on the fixed-point theorem. It can find out all the stable matchings that were defined in Bando [7].

On the other hand, some research assumed that the other agents' behavior is not kept fixed. This added difficulty to formulating their behavior. Sasaki and Toda [39] proposed a useful tool called estimation function to describe an agent's expectation on the other agents' behavior. With the help of estimation function, they proposed a stability concept which incorporates externalities. Under this stability concept, a man and a woman deviate from the current matching as soon as a more preferred matching exists in their estimation functions. Finally, Sasaki and Toda [39] demonstrated that such stable matchings always exist under the universal estimation function which includes all possible matchings. Particularly, Bando et al. [10] named this stability concept as O-stability since these estimation functions are quite optimistic. Based on the results of Sasaki and Toda [39], Hafalir [17] proposed a stability concept under which a man and a woman deviate from the current matching if all matchings in their estimation functions are more preferred. Then he showed that under the sophisticated estimation function which depends on agents' preferences, such stable matchings always exist. Since each agent behaves in a very conservative way in this case, Bando et al. [10] named this stability concept as C-stability.

In addition, a recent work Fisher and Hafalir [14] studied moderately sized one-to-one matching markets with aggregate externalities. Such externalities do not change very much in response to changes in two individuals' actions. Based on this feature, they proposed a primary tool called "Auxiliary (Matching) Game" where players regard the level of externality as fixed. Then they use this tool to define a correspondence whose set of fixed points is equal to the set of stable matchings in the full game.

2.1.3 Preference revelation game based on many-to-one matching market

Our research contributes to the work of equilibrium analysis by formulating a school choice market as a preference revelation game and proposing a new equilibrium concept. This sort of research basically aims at analyzing agents' strategic behavior and has great importance in designing real-world matching programs. In the hospital-intern matching and school-student matching programs, hospitals' and schools' preferences are usually open to the public due to laws, regulations and so on. But medical interns and students are able to report any preferences including fake ones. This flexibility enables them to strategically determine which preferences to report while thinking about whether a better outcome could consequently be achieved. This fact motivated researchers to formulate such a matching problem as a preference revelation game of medical interns or students and then study their strategic behavior.

Preference revelation games defined on school choice market were first studied by Dubins and Freedman [12]. They demonstrated that none of the student groups can make all its members strictly better off by changing their joint strategies. This result directly implies that the truth-telling strategy profile is always a strong Nash equilibrium (*SN*). Two later research, Martínez et al. [26] and Hatfield and Kojima[18] extended this result to more general models. Sotomayor [44] focused on the stability of the equilibrium outcomes. She proposed an equilibrium concept called Nash equilibrium (*NE*) in the strong sense. Then she demonstrated that the random stable matching rule and any stable matching rule implement the set of stable matching via *NE* in the strong sense. Bando [9] proposed an equilibrium concept called strictly strong Nash equilibrium (*SSN*). This concept is stronger than Dubins and Freedmen's *SN* in the sense that it requires none of the student groups is able to make all its members weakly better off and at least one member strictly better off by changing their joint strategies. Then he demonstrated the existence of two *SSNs*. The first one is obtained by iteratively applying DA so that we simply call it *DA-SSN*. This *SSN* has a very nice property that it always exists. Moreover, two alternative algorithms for computing *DA-SSN* can be found in Kesten [23] and Tang and Yu [46]. The other *SSN* is obtained by constructing a corresponding house allocation market which is defined in Shapley and Scarf [40] and then identifying the strict core. Hence we simply call it *Core-SSN*. Unlike *DA-SSN*, its existence depends on whether the strict core exists. Besides these results, Bando [9] showed that an *SSN* coincides with the student-optimal von Neumann-Morgenstern stable matching in a one-to-one matching market.

2.1.4 House allocation problem with existing tenants

Our research contributes to the work of matching algorithms by recovering efficiency loss for a real-world algorithm which is applied to a house allocation problem with existing tenants. This problem is not a many-to-one matching problem but is closely related to it.

In the previous chapter, we have addressed that Gale and Shapley [15] not only demonstrated the existence of stable matchings, but also designed the DA which is able to find out at least one stable matching for a given matching problem. Since the DA can be easily coded, it has been adapted to support many real-world matching programs such as *NRMP* and school matching in some U.S. cities. The performance of these programs in turn motivated theoretical research which aims at studying the properties of the DA and other similar algorithms. Abdulkadiroğlu and Sönmez [4] reviewed school matching programs in the U.S. cities, and gave a summary of their pros and cons. They proposed a mechanism called the student-optimal stable matching mechanism (*SOSM*) which satisfies many nice properties. Also, this mechanism adopts an algorithm which is actually a modification of the DA which let students choose schools first. Particularly, it always leads to a stable matching that is weakly preferred by all students to any other stable matchings. Therefore, this algorithm is called student-optimal deferred acceptance algorithm (*SODA*). But on the other hand, they pointed out that *SODA* has a flaw which can hardly be overlooked. That is, it easily causes Pareto inefficient outcomes. Observing these facts, Kesten [23] and Tang and Yu [46] made attempts to recover efficiency loss for *SOSM*. Kesten [23] constructed a school choice model with consents. Here “consents" means consents which are granted by some or all students to priority violations. By utilizing these consents, Kesten [23] proposed an efficiency-adjusted deferred acceptance mechanism (*EADAM*) which recovers efficiency loss for *SOSM*. Tang and Yu [46] improved Kesten [23]'s model by introducing a simplified algorithm which leads to exactly the same result as the *EADAM* does. Under the *EADAM* and the simplified *EADAM*, a Pareto efficient or constraint Pareto efficient matching can be achieved with violating other nice properties at a minimal level.

Witnessing these results, we started wondering whether other DA-like algorithms are also suffering from the efficiency loss problem. Then we noticed a real-world house allocation problem. It is the theoretical model of a situation that lots of colleges encounter. At the beginning of each school year, freshmen come and apply for dormitories. On the other hand, seniors who have already been assigned dormitories in the previous year can either continue staying in the current dormitory or apply for a different one. Abdulkadiroğlu and Sönmez [5] first studied this model, and formally named it as house allocation with existing tenants. An “existing tenant" refers to a senior student. In contrast, a freshman is called a “new applicant". They compared the pros and cons of several real-world mechanisms which have been applied to this house allocation problem, and then proposed a mechanism called "You request my house — I get your turn" (YRMH-IGYT) mechanism. Two related research, Sönmez and Ünver [42] and Sönmez and Ünver [43], extensively studied the properties and alternatives of the YRMH-IGYT mechanism.

Among the real-world mechanisms, there is a mechanism which has been under operation at Massachusetts Institute of Technology for about three decades. It is applied to the allocation of a dormitory called the New House 4, so it is known as the MIT-NH4 mechanism (the NH4 mechanism for short). Abdulkadiroğlu and Sönmez [5] were the first to theoretically study this mechanism. They pointed out that it sometimes produces Pareto inefficient allocations and thus may result in large efficiency loss. Guillen and Kesten [16] further demonstrated that the NH4 mechanism has several nice properties including individually rationality, fairness and strategy-proofness. Moreover, they revealed some connections between the house problem with existing tenants and the school choice problem. An important finding is that the house allocation problem is actually a special case of the school choice problem where efficiency recovery has been extensively studied. This result motivated us to learn from these attempts and contribute to solving the same problem for the house allocation problem with existing tenants.

2.2 Symbols and definitions

Our research is going to study a labor market, a school choice market and a house allocation problem with existing tenants. In this subsection, we introduce the basic symbols and definitions which will be frequently used in each model.

2.2.1 A labor market

We denote a labor market be a tuple $(F, W, \succeq_F, \succeq_W, q_F)$. The agents in this market are a set of firms F and a set of workers W. Since each firm $f \in F$ is able to hire multiple workers up to its capacity, we denote its quota be q_f and let $q_F = (q_f)_{f \in F}$.

We assume that each firm has strict preferences over workers and each worker has strict preferences over firms.

Definition 1 (Preferences in a labor market)**.** *In a labor market, each firm's and each worker's preferences are defined as below:*

- *Given $f \in F$ and $w, w' \in W$, $w \succeq_f w'$ means that f weakly prefers w to w'; When $w \neq w'$ holds, we use $w \succ_f w'$ to represent that f strictly prefers w to w'.*

- *Given $w \in W$ and $f, f' \in F$, $f \succeq_w f'$ means that w weakly prefers f to f'; When $f \neq f'$ holds, we use $f \succ_w f'$ to represent that w strictly prefers f to f'.*

Although we are going to study different many-to-one matching problems, we denote a matching be μ throughout this thesis for the sake of convenience and consistency. But nevertheless, we endow μ with slightly different definitions under different settings. Here is the definition of matching in a labor market.

Definition 2 (Matching in a labor market)**.** *In a labor market, a matching μ is defined as a function from $F \cup W$ into $2^{F \cup W}$ such that for all $f \in F$ and for all $w \in W$:*

- $\mu(w) \in F$ *and* $|\mu(w)| \leq 1$,
- $\mu(f) \in 2^W$,
- $\mu(w) = \{f\}$ *if and only if* $w \in \mu(f)$.

Since firms are able to hire multiple workers, each firm chooses its favorite subset of workers when faced with a set of workers who are applying for it.

Definition 3 (Choice set in a labor market)**.** *In a labor market, given $G \subseteq W$ and $f \in F$, each firm f's choice set $Ch_f(G| \succeq_f)$ satisfies:*

- $Ch_f(G| \succeq_f) \subseteq G$,
- $Ch_f(G| \succeq_f) \succeq_f G'$ *for all* $G' \subseteq G$.

The classical solution concept for two-sided matching market is called stable matching, and it was first proposed by Gale and Shapley [15] for a one-to-one matching market. Kelso and Crawford [22] extended this definition to many-to-one matching market. Simply speaking, it asks for the following two conditions.

Definition 4 (Stable matching in a labor market)**.** *In a labor market, a stable matching is evaluated from the following two aspects.*

(i) (Individual rationality) A matching μ is individually rational

- *for workers if* $\mu(w) \succeq_w w$ *for all* $w \in W$,
- *for firms if* $Ch_f(\mu^{-1}(f)| \succeq_f) = \mu^{-1}(f)$ *for all* $f \in F$.

(ii) (blocking) A matching μ is blocked by $(f, G) \in F \times 2^W$ if

- $f \succ_w \mu(w)$,
- $w \in Ch_f(\mu^{-1}(f) \cup \{w\}| \succeq_f)$ *for all* $w \in G$.

We say that a matching is stable if it is individually rational for both firms and workers, and is not blocked.

2.2.2 A school choice market

We denote a school choice market be a tuple $(I, C, \succeq_I, \succeq_C, q_C)$. The agents in this market is a set of students I and a set of schools C. Since each school $c \in C$ is able to admit multiple students up to its capacity, we denote its quota be q_c and let $q_C = (q_c)_{c \in C}$.

We assume that each school has strict preferences over students and each student has strict preferences over schools.

Definition 5 (Preferences in a school choice market)**.** *In a school choice market, each school's and each student's preferences are defined as below:*

- *Given $c \in C$ and $i, i' \in I$, $i \succeq_c i'$ means that c weakly prefers i to i'; When $i \neq i'$ holds, we use $i \succ_c i'$ to represent that c strictly prefers i to i'.*
- *Given $i \in I$ and $c, c' \in C$, $c \succeq_i c'$ means that i weakly prefers c to c'; When $c \neq c'$ holds, we use $c \succ_i c'$ to represent that i strictly prefers c to c'.*

In a school choice market, we define a matching in a slightly different way from that we did in a labor market.

Definition 6 (Matching in a school choice market)**.** *In a school choice market, a matching μ is defined as a function from I into $C \cup I$ such that*

- *for each $c \in C$, $|\mu^{-1}(c)| \leq q_c$,*
- *for each $i \in I$, $\mu(i) \in I$ implies $\mu(i) = i$.*

Let us recall the definition of matching in a labor market and then compare these two. The difference between them roots in the nature of these two markets. We assumed an abstract labor market which is usually decentralized. This means no authority (such as a government) interferes with the market behavior of firms and workers. Therefore, we define a matching be a function from the union of the set of firms and the set of workers to all its subsets with the purpose to mimic their interactive matching process. In contrast, a school choice program is launched by an authority and takes the form of a centralized market. The authority takes charge of the matching programs, and assigns students to different schools. Therefore, we define a matching be a function from the set of students to its union with the set of schools in order to mimic this process which is lead by a third party.

On the other hand, the choice set and the stable matching are defined in the same way as those in the labor market.

Definition 7 (Choice set in a school choice market)**.** *In a school choice market, given $S \subseteq I$ and $c \in C$, each school c's choice set $Ch_c(S| \succeq_c)$ satisfies:*

- $Ch_c(S| \succeq_c) \subseteq S$,
- $Ch_c(S| \succeq_c) \succeq_c S'$ *for all* $S' \subseteq S$.

Definition 8 (Stable matching in a school choice market)**.** *In a school choice market, a stable matching is evaluated from the following two aspects.*

(i) (Individual rationality) A matching μ *is individually rational*

- *for students if* $\mu(i) \succeq_i i$ *for all* $i \in I$,
- *for schools if* $Ch_c(\mu^{-1}(c)| \succeq_c) = \mu^{-1}(c)$ *for all* $c \in C$.

(ii) (blocking) A matching μ *is blocked by* $(c,S) \in C \times 2^I$ *if*

- $c \succ_i \mu(i)$,
- $i \in Ch_c(\mu^{-1}(c) \cup \{i\}| \succeq_c)$ *for all* $i \in S$.

We say that a matching is stable if it is individually rational for both schools and students, and is not blocked.

2.2.3 A house allocation problem with existing tenants

In a house allocation problem with existing tenants, the basic elements are a set of students and a set of houses. Particularly, students and houses can be further divided into the following sets:

- a finite set of existing tenants I_E,
- a finite set of newcomers I_N,
- a finite set of occupied houses $H_O = \{h_i\}_{i \in I_E}$,
- a finite set of vacant houses H_V.

Let $I = I_E \cup I_N$ be the set of all students and $H = H_O \cup H_V \cup \{h_0\}$ be the set of all houses plus the null house. The null house $\{h_0\}$ means the no house option for each student.

We assume that each student $i \in I$ has strict preferences over houses.

Definition 9 (Students' preferences in a house allocation problem with existing tenants)**.** *In a house allocation problem with existing tenants, given* $i \in I$ *and* $h, h' \in H$, hR_ih' *means that* i *weakly prefers* h *to* h'*; When* $h \neq h'$ *holds, we use* hP_ih' *to represent that* i *strictly prefers* h *to* h'.

In this market, an allocation α is defined as below.

Definition 10 (Allocation in a house allocation problem with existing tenants)**.** *In a house allocation problem with existing tenants, an allocation (or sometimes it is called a matching)* α *is a function from* I *into* H *such that* $|\alpha^{-1}(h)| \in \{0,1\}$ *for each* $h \in H_O \cup H_V$*.*

In words, it is a list of assignments which satisfies (1) every student is assigned at most one house, and (2) no house other than the null house can be assigned to more than one student.

2.2.4 The deferred acceptance algorithm (DA)

In this section, we give a formal description of the deferred acceptance algorithm which was proposed by Gale and Shapley [15], as well as its variation—the student-optimal deferred acceptance algorithm (*SODA*). Although different researchers may use slightly different descriptions for these two algorithms in their works, ours is generally based on Roth [34].

The deferred acceptance algorithm (DA)

Suppose we have a marriage market in which a set of men and a set of women are seeking for spouse. Suppose men's and women's preferences are strict.

- Step 1. Each man proposes to his 1st choice among the set of women (if he has any acceptable choices). Each woman rejects any unacceptable proposals and, if more than one acceptable proposal is received, tentatively accepts the most preferred one and rejects all others.

- Step k. Any man rejected at Step $k-1$ makes a new proposal to its most preferred acceptable mate who has not yet rejected him. If no acceptable choices remain, he makes no proposal. Each woman tentatively accepts her most preferred proposal, and rejects the rest.

- Stop when no further proposals are made, and match each woman to the man (if any) whose proposal she is holding.

In fact, this algorithm determines a stable matching which is weakly preferred by all men to any other stable matchings ("M-optimal stable matching"). If we let women make proposals first in the same algorithm, then we will obtain a stable matching which is weakly preferred by all women to any other stable matchings ("W-optimal stable matching").

The student-optimal stable algorithm (SODA)

Next we introduce an variation of the DA which is applied in school choice problems.

Suppose we have a school choice market where each school can admit multiple students up to its quota q and each student can attend at most one school. Suppose schools' and students' preferences are strict.

- Step 1. Each student applies to his or her 1st choice among the set of schools (If he or she has any acceptable choice). Each school tentatively accepts the q most preferred applications, and rejects all others.
- Step k. Any student rejected in Step $k-1$ makes a new application to its most acceptable school which has not yet reject him or her. If no acceptable choices remain, he or she makes no application. Each school tentatively accepts the q most preferred applications, and rejects the rest.
- Stop when no further applications are made, and match each student to the school (if any) which has accepted his or her application.

This algorithm determines a stable matching which is weakly preferred by all students to any other stable matchings.

Chapter 3

A new stability concept for a labor market with externalities among firms

3.1 Introduction

In a standard model of a labor market, there is a fundamental assumption that each firm's preferences depend on the workers it hires and each worker's preferences depend on the firm he or she works for.

This chapter studies an interesting extension which takes externalities into account. Generally speaking, there are two kinds of externalities which have been extensively studied in economic literature. One is externalities among workers. That is, each worker cares about not only which firm he or she works for but also whom he or she works with. The other one is externalities among firms. That is, each firm cares about not only whom it hires but also whom its rival firms hire.

We examine a labor market with externalities among firms. The key difference between this model and a standard model is firms' preferences. The presence of externalities brings two problems. The first one is how to define the stability concept since the standard definition of stable matching proposed by Gale and Shapley [15] is not compatible with externalities. The other one is to examine under which conditions the re-defined stable matchings always exist.

To solve the first problem, our research follows the estimation function approach which was proposed by Sasaki and Toda [39]. Since workers do not have externalities, we define estimation functions on firms unilaterally. Based on the estimation functions, we propose a new stability concept which is called T-stability. Specifically, a firm blocks the current matching if a particular matching called the threshold matching is more preferred to the current one. By introducing threshold matching, we allow each firm to block flexibly according to their evaluation towards different sets of workers. Because of this feature, two classical solutions – O-stability proposed by Sasaki and Toda [39] and C-stability proposed by Hafalir [17] can be regarded as special cases of T-stability. Then we continue to examine under which conditions T-stable matchings always exist. We propose a restriction called T-substitutability. It is an extension of the standard substitutability which is proposed by Kelso and Crawford [22]. Besides T-substitutability, we propose one more restriction called induced preferences. When firms' preferences satisfy these two restrictions, we show the existence T-stable matchings. Also, we provide an example of a specific estimation function that is compatible with T-stable matchings.

The rest of this chapter is organized as follows. Section 3.2 explains the settings of our model. Section 3.3 introduces a new stability concept called T-stable matching. Section 3.4 introduces the restrictions on firms' preferences. Section 3.5 shows the existence of T-stable matchings and provides a specific example. Section 3.6 concludes.

3.2 The model

3.2.1 Firms' preferences with externalities

Suppose there is a labor market as defined in Chapter 2. Let $M(F,W)$ be the set of all matchings in this market.

Given $f \in F$ and $G \subseteq W$, define $M(f,G) = \{\mu \in M(F,W) | \mu(f) = G \text{ and } \mu(w) = \{f\}$ for all $w \in G\}$. This is the set of all matchings in which firm f hires a set of workers G and every worker in G works for firm f.

Since externalities exist among firms, each firm has a strict preference relation $\succeq_f$ over $M(F,W)$. On the other hand, each worker's preferences $\succeq_w$ do not have externalities and they follow the standard assumption which is given in Chapter 2. We denote a labor market with externalities among firms be a tuple $(F,W,(\succeq_f)_{f\in F},(\succeq_w)_{w\in W},q_F)$.

3.2.2 Estimation function

Since we assume that externalities exist among only firms, the estimation function is defined unilaterally on firms. Formally, let $f \in F$, and define the estimation function of f be a correspondence φ_f which associates a nonempty subset of $M(f,G)$ with a nonempty subset $G \subseteq W$. Then $\varphi_f(G)$ is the set of matchings that f believes will happen when it hires G. We note that $\varphi_f(G)$ does not necessarily include all matchings in $M(f,G)$. Denote $\Phi_f = \cup_{G\subseteq W}\varphi_f(G)$ be the estimation profile of f .

Since f has strict preferences over $M(f,G)$, the matchings in $\varphi_f(G)$ can be ordered. We define a one-to-one function $\gamma : \Phi_f \to 2^W \times \{1,2,...,K\}$ where $K = Max_{G \subseteq W}|\varphi_f(G)|$ and $\gamma(\mu) = (\mu(f),k)$ such that $\mu \in \Phi_f$ and $1 \leq k \leq K$. γ assigns each matching in $\varphi_f(G)$ a numeric rank which represents to which extent f prefers this matching when it hires G. For example, suppose $\mu \in \varphi_f(G)$ and $\gamma(\mu) = (G,1)$. Then μ is f's most preferred matching in $\varphi_f(G)$. Similarly, suppose $\mu' \in \varphi_f(G')$ and $\gamma(\mu') = (G',k)$ such that $1 \leq k \leq |\varphi_f(G')|$. Then μ' is f's kth preferred matching in $\varphi_f(G')$.

Each rank k can be interpreted as f's evaluation on a matching according to a certain criterion such as salary. For example, consider $\mu \in \varphi_f(G)$. Then $\gamma(\mu) = (G,k)$ with a smaller k suggests that f hires G under a lower salary. Hence this matching is ranked higher.

Particularly, we note that any $\mu \notin \Phi_f$ does not have a rank since it is not included in f's estimation functions. Because each firm may rule out some situations which they believe will hardly occur when hiring a particular set of workers. On the other hand, this suggests a firm may have different estimation when hiring different sets of workers. Below we use a simple example to illustrate how a firm's estimation is like when hiring different sets of workers.

Example 1. *Consider a simple labor market with externalities which consists of two firms $\{f_1, f_2\}$ and three workers $\{w_1, w_2, w_3\}$. Suppose each firm hires at least one worker (in order to guarantee production at a minimal level, regardless of the quality of workers). Then there are 12 possible matchings in this market.*

$$\mu_1 = (\{w_1\},\{w_2\})$$
$$\mu_2 = (\{w_1\},\{w_3\})$$
$$\mu_3 = (\{w_1\},\{w_2,w_3\})$$
$$\mu_4 = (\{w_2\},\{w_1\})$$
$$\mu_5 = (\{w_2\},\{w_3\})$$
$$\mu_6 = (\{w_2\},\{w_1,w_3\})$$
$$\mu_7 = (\{w_3\},\{w_1\})$$
$$\mu_8 = (\{w_3\},\{w_2\})$$
$$\mu_9 = (\{w_3\},\{w_1,w_2\})$$
$$\mu_{10} = (\{w_1,w_2\},\{w_3\})$$
$$\mu_{11} = (\{w_1,w_3\},\{w_2\})$$
$$\mu_{12} = (\{w_2,w_3\},\{w_1\})$$

Now pick up $\{w_1\}$, $\{w_2\}$ and $\{w_3\}$. Obviously they are different sets of workers. Then suppose f_1 has the following estimation functions on each set.

$\varphi_{f_1}(w_1) = \{\mu_1, \mu_3, \mu_2\}$
$\varphi_{f_1}(w_2) = \{\mu_4, \mu_5, \mu_6\}$
$\varphi_{f_1}(w_3) = \{\mu_7, \mu_9\}$

There are three matchings in f_1's estimation when it hires $\{w_1\}$ or $\{w_2\}$. These matchings cover all possibilities when f_1 hires these two sets of workers. However, there are only two matchings in f_1's estimation when it hires $\{w_3\}$. The possible outcome μ_8 in which f_2 hires w_2 is not included. This means f_1 believes that f_2 will never hire a single w_2 as long as w_1 is available.

In real-world practice, the number of firms and workers may be tremendous and thus taking all possible outcomes into account will become a quite tough task for firms. Therefore, this setting of estimation function enables each firm to refine its estimation by eliminating the matchings that it believes will hardly occur. In the next subsection, we will introduce a concept called threshold which allows each firm to further refine its estimation.

On the other hand, when each f is indifferent with all $\mu \in \varphi_f(G)$, the market becomes a standard labor market.

3.3 Stability concept

3.3.1 Two classical stability concepts

In this section, we propose a stability concept which depends on firms' estimation functions.

When externalities exist among firms, Definition 4 is not applicable any more since the preferences of firms are defined over matchings. Sasaki and Toda [39] and Hafalir [17] proposed two extensions which depend on estimation functions and incorporate externalities. However, their stability concepts are defined for a marriage market with externalities where the matching is one-to-one. We extend their stability concepts to our many-to-one matching market.

First, we need to define a special category of matchings. Given a matching $\mu \in M(F,W)$, for each $(f,G) \in F \times 2^W$ such that $\mu(f) = G$ and $\mu(w) = \{f\}$ for all $w \in G$, we define a matching $\bar{\mu}$ which satisfies: (i) $\bar{\mu}(f) \subseteq G$; (ii) $\bar{\mu} \in \varphi_f(\bar{\mu}(f))$; and (iii) For each $w \in G \backslash \bar{\mu}(f)$, $\bar{\mu} = \varnothing$. It is a matching in which f unilaterally fires some (or all) of its current workers G. In the meantime, the matchings of the other firms and workers possibly change. Denote the set of all such $\bar{\mu}$ be $M(f|\mu)$.

Definition 11 (Sasaki and Toda [39]). *A matching $\mu \in M(F,W)$ is individually rational if there does not exist $f \in F$ and a matching $\mu' \in M(f|\mu)$ such that $\mu' \succ_f \mu$, and $\mu(w) \succeq_w \varnothing$ for each $w \in W$.*

Definition 12 (Sasaki and Toda [39]). *A matching $\mu \in M(F,W)$ is blocked by a pair $(f,G) \in F \times 2^W$ if there exists a matching $\mu' \in \varphi_f(G)$, $\mu' \succ_f \mu$ and $f \succeq_w \mu(w)$ for each $w \in G$.*

Under this stability concept, a firm blocks the current matching as long as a desirable result exists in its estimation. Bando et al.[10] named these two notions as optimistic individual rationality (O-IR) and optimistic blocking (O-blocking) since such estimation is quite optimistic. A matching is O-stable if it satisfies O-IR and is not O-blocked.

Definition 13 (Hafalir [17]). *A matching $\mu \in M(F,W)$ is individually rational if there does not exist $f \in F$ such that for all matching μ' with $\mu' \in M(f|\mu)$, $\mu' \succ_f \mu$ and $\mu(w) \succeq_w \varnothing$ for each $w \in W$.*

Definition 14 (Hafalir [17]). *A matching $\mu \in M(F,W)$ is blocked by a pair $(f,G) \in F \times 2^W$ if for all matching μ' with $\mu' \in \varphi_f(G)$, $\mu' \succ_f \mu$ and $f \succeq_w \mu(w)$ for each $w \in G$.*

Under this stability concept, each firm is very cautious. It blocks the current matching only when the worst outcome in its estimation still turns out to be more preferred. Bando et al. [10] named these two notions as conservative individual rationality (C-IR) and conservative blocking (C-blocking) since firms behave in a conservative way in this case. A matching is C-stable if it satisfies C-IR and is not C-blocked.

3.3.2 Threshold

In our model, we assume that each firm has a specific expectation based on which it determines whether to block the current matching or not. We use the following example to show how this assumption works.

Example 2. *Consider Example 1 again. Suppose the current matching is $\mu = \mu_4$, and f_1's estimation function on $\{w_1\}$ and $\{w_3\}$ are as listed in Example 1.*

$$\varphi_{f_1}(w_1) = \{\mu_1, \mu_3, \mu_2\}$$
$$\varphi_{f_1}(w_3) = \{\mu_7, \mu_9\}$$

Now suppose f_1 is thinking about fires its current worker w_2, and hires w_1 or w_3 to fill out this position. We assume that (i) f_1 will fire w_2 and hire w_1 if it believes μ_1 will occur, and (ii) f_1 will fire w_2 and hire w_3 if it believes μ_9 will occur.

We provide a real-world situation which can be supported by this example. Suppose both f_1 and f_2 evaluate education most in their recruitment programs. On the other hand, w_1, w_3 and w_2 each has the highest, the second highest and the worst education background. Particularly, f_1 thinks w_1 is obviously better than both w_2 and w_3, but w_3 is just slightly better than w_2. Then f_1 believes that once it hires w_1, it is almost for sure that f_2 will hire w_3. But even if f_2 turns out to hire w_2 (or w_2 and w_3 together), it does not make much difference for f_1 as long as it has secured its favorite worker w_1. On the other hand, since w_3 is not as desirable as w_1, f_1 has to carefully evaluate all the possibilities when it hires w_3, and will not make the decision unless the worst outcome still turns out to be better than the current matching.

In Example 2, the two matchings $\mu_1 \in \varphi_{f_1}(w_1)$ such that $\gamma(\mu_1) = (w_1, 1)$ and $\mu_9 \in \varphi_{f_1}(w_3)$ such that $\gamma(\mu_9) = (w_3, 3)$ can be regarded as "boundary" conditions for f to block. We name such a matching as the threshold matching, and name its rank as the threshold. Then we say f_1's threshold when hiring w_1 is 1, and its threshold when hiring w_3 is 3.

Since each firm may have different thresholds for different sets of workers, we denote each threshold be $t^{f,G}$ such that $f \in F$, $G \subseteq W$ and $1 \leq t^{f,G} \leq |\varphi_f(G)|$. It depends on f and the set G that it wants to hire.

Then we propose a stability concept based on estimation function and threshold.

Definition 15 (T-individual rationality)**.** *A matching $\mu \in M(F,W)$ is individually rational if there does not exist $f \in F$ and a matching $\mu' \in M(f|\mu)$ with $\gamma(\mu') = (\mu'(f), t^{f,\mu'(f)})$, $\mu' \succ_f \mu$ and $\mu(w) \succeq_w \varnothing$ for each $w \in W$.*

Definition 16 (T-blocking)**.** *A matching $\mu \in M(F,W)$ is T-blocked by a pair $(f,G) \in F \times 2^W$ if there exists a matching $\mu' \in \varphi_f(G)$ with $\gamma(\mu') = (G, t^{f,G})$, $\mu' \succ_f \mu$ and $f \succeq_w \mu(w)$ for each $w \in G$.*

We name these two notions as T-individual rationality (T-IR) and T-blocking. A matching is T-stable if it satisfies T-IR and is not T-blocked.

A larger $t^{f,G}$ suggests that f is more cautious when blocking μ with G . In contrast, a smaller $t^{f,G}$ indicates that f is more optimistic when doing so. Particularly, when $t^{f,G} = |\varphi_f(G)|$ holds for all $f \in F$ and $G \subseteq W$, this stability concept becomes the C-stability that is proposed by Hafalir [17]. On the other hand, when there is no specific restriction on $t^{f,G}$, this stability concept becomes the O-stability that is proposed by Sasaki and Toda [39]. Denote the set of T-stable matchings be S_T . In the next sections, we examine under which conditions S_T is always nonempty.

3.4 Restrictions on preferences

3.4.1 Substitutability

In the study of a standard labor market, Kelso and Crawford [22] proposed a restriction on the firms' preferences which is called substitutability. With this restriction, they demonstrated that stable matchings under Definition 4 always exist in this market.

Definition 17 (Substitutability, Kelso and Crowford [22])**.** *A firm f's preferences over sets of workers has the property of substitutability if, for any set of workers G that contains workers w and w' such that $w \neq w'$, if $w \in Ch_f(G)$ then $w \in Ch_f(G)\backslash\{w'\}$.*

In words, if f chooses w from a set of workers which contains w', then it will still choose w when w' becomes unavailable.

Theorem 1 (Kelso and Crowford [22])**.** *When each firm's preferences satisfy substitutability, the set of stable matchings is nonempty.*

Once externalities are present, Definition 17 will not be applicable since each firm's choice set will depend on how the other firms and workers are matched. Next we define a choice set which incorporates externalities and then propose a new substitutability concept.

Define $R(f,G) = \{\mu \in M(F,W) | \mu(f) = \{\varnothing\}$ and $\mu(w) = \varnothing$ for all $w \in G\}$. This is the set of matchings in which firm f does not hire any worker and all workers in G are unemployed.

In a labor market with externalities, for each $\mu \in R(f,G)$, define $Ch_f(G|\mu)$ as each firm f's choice set such that (i) $Ch_f(G|\mu) \subseteq G$ and (ii) $\mu' \succeq_f \mu''$ where $\gamma(\mu') = (Ch_f(G|\mu), t^{f,Ch_f(G|\mu)})$ and $\gamma(\mu'') = (G', t^{f,G'})$ such that $G' \subseteq G$.

That is, given a set of workers, a firm's choice is a subset such that this firm prefers the threshold matching when hiring this subset of workers to the threshold matching when hiring any other subset of workers.

Based on this choice set, we define the following notion of substitutability.

Definition 18 (T-substitutability)**.** *In a labor market with externalities among firms, a firm f's preferences over matchings has the property of T-substitutability if for any $G \subseteq W$ and $\mu \in R(f,G)$, if $w, w' \in Ch_f(G|\mu)$ and $w \neq w'$, then $w \in Ch_f(G|\mu)\backslash\{w'\}$.*

Recall Definition 17 which is the standard definition of substitutability. The above definition conveys the same meaning, but differs in the way by which a firm's choice set is defined. When externalities do not exist, a firm's choice set upon a given set of workers is just the most preferred subset. But once externalities are present, how to compare these subsets and then pick out the favorite one becomes the question. We assume that each firm decides the choice set according to the threshold matching when hiring different subsets of workers. Therefore, our definition of substitutability depends on threshold and is named as T-substitutability.

On the other hand, Bando [7] proposed the same notion of substitutability as ours. But the choice set is defined in a different way in his work.

3.4.2 Induced preferences

In Sasaki and Toda [39], each firm ordered different sets of workers according to the best outcome in the estimation function when hiring this set of workers. Hafalir [17] assumed that each firm generates this ordering according to the worst outcome in the estimation function. Since the best/worst matching in $\varphi_f(G)$ must exist, an induced preference ordering over 2^W is well defined. Then Gale and Shapley's result on the existence of stable matching is directly applicable.

We follow this approach with an extension: we consider more than the extreme cases. In the previous section, we have introduced the threshold which represents a "boundary" condition for firms to block. We assume that each firm generates a preference ordering on the sets of workers according to the threshold matching when hiring different sets of workers. This technique enables us to convert a matching market with externalities into one without externalities.

Definition 19 (Induced preferences)**.** *Each firm $f \in F$ has an induced preference ordering $\succeq_f^{\varphi}$ which is determined in the following way: for any $G, G' \subseteq W$ such that $G \neq G'$, $G \succeq_f^{\varphi} G'$ if and only if $\mu \succeq_f \mu'$ where $\gamma(\mu) = (G, t^{f,G})$ and $\gamma(\mu') = (G', t^{f,G'})$.*

In words, f prefers G to G' if and only if it prefers the threshold matching in $\varphi_f(G)$ to the threshold matching in $\varphi_f(G')$. Since firms have strict preferences over $M(F,W)$ and $t^{f,G}$ must exist, the induced preference ordering $\succeq_f^{\varphi}$ is well defined and has no externalities.

Denote the new market where each firm has the induced preference ordering be
$(F, W, (\succeq_f^{\varphi})_{f \in F}, (\succeq_w)_{w \in W}, q_F)$.

Next we show that substitutability is preserved in this market.

Lemma 1. *If firms' preferences satisfy T-substitutability in $(F, W, (\succeq_f)_{f \in F}, (\succeq_w)_{w \in W}, q_F)$, then they satisfy substitutability in $(F, W, (\succeq_f^{\varphi})_{f \in F}, (\succeq_w)_{w \in W}, q_F)$.*

Proof. Since firms' preferences satisfy T-substitutability, given any $\mu \in R(f, G)$, if $w, w' \in Ch_f(G|\mu)$ and $w \neq w'$, then $w \in Ch_f(G \backslash \{w'\}|\mu)$. By the definition of choice set with externalities, we know that $\gamma^{-1}[(Ch_f(G|\mu), t^{f,Ch_f(G|\mu)})] \succeq_f \gamma^{-1}[(G', t^{f,G'})]$ where $G' \subseteq G$ and

$G' \neq Ch_f(G|\mu)$. This suggests $Ch_f(G|\mu) \succeq_f^{\varphi} G'$. Similarly, we obtain $Ch_f(G\backslash\{w'\}|\mu) \succeq_f^{\varphi} G''$ where $G'' \subseteq G\backslash\{w'\}$ and $G'' \neq Ch_f(G\backslash\{w'\}|\mu)$. Therefore, we have $Ch_f(G) = Ch_f(G|\mu)$ where $w, w' \in Ch_f(G|\mu)$, and $Ch_f(G\backslash\{w'\}) = Ch_f(G\backslash\{w'\}|\mu)$ where $w \in Ch_f(G\backslash\{w'\})|\mu)$. This indicates firms' preferences satisfy substitutability in $(F, W, (\succeq_f^{\varphi})_{f\in F}, (\succeq_w)_{w\in W}, q_F)$. □

3.5 Existence

3.5.1 Existence of T-stable matching

In the previous section, Lemma 1 actually connects a labor market with externalities to a standard labor market. Immediately, we have the following lemma.

Lemma 2. *If firms' preferences satisfy T-substitutability in $(F, W, (\succeq_f)_{f\in F}, (\succeq_w)_{w\in W}, q_F)$, then the set of stable matchings is nonempty in $(F, W, (\succeq_f^{\varphi})_{f\in F}, (\succeq_w)_{w\in W}, q_F)$.*

Proof. According to Lemma 1, we know that firms' preferences satisfy substitutability in $(F, W, (\succeq_f^{\varphi})_{f\in F}, (\succeq_w)_{w\in W}, q_F)$. This is a standard a labor market where externalities do not present. Then applying Theorem 1, we know that the set of stable matchings is nonempty. □

Denote the set of these stable matchings be I^{φ} . Specifically,

$I^{\varphi} = \{\mu \in M(F, W) | \nexists (f, G) s.t. \mu(f) \neq G, G \succ_f^{\varphi} \mu(f)$ and $f \succeq_w \mu(w)$ for each $w \in G\}$.

We note that $G \succ_f^{\varphi} \mu(f)$ if and only if $\gamma^{-1}[(G, t^{f,G})] \succ_f \gamma^{-1}[(\mu(f), t^{f,\mu(f)})]$ according to Definition 19. Then we have an equivalent form:

$I^{\varphi} = \{\mu \in M(F, W) | \nexists (f, G) s.t. \mu(f) \neq G, \gamma^{-1}[(G, t^{f,G})] \succ_f \gamma^{-1}[(\mu(f), t^{f,\mu(f)})]$ and $f \succeq_w \mu(w)$ for each $w \in G\}$.

Moreover, define:

$I^{\varphi}(f, G) = \{\mu \in I^{\varphi} | \mu(f) = G$ and $\mu(w) = f$ for all $w \in G\}$.

This is the set of stable matchings where f and G are matched.

Then we have the following proposition.

Proposition 1. *For all $(f, G) \in F \times 2^W$, if $\mu \in \varphi_f(G) \cap I^{\varphi}(f, G)$, then $\mu \in S_T$.*

Proof. Suppose there exists $\mu \in \varphi_f(G) \cap I^{\varphi}(f, G)$ and μ is T-blocked. Then there exists G' such that $\gamma^{-1}[(G', t^{f,G'})] \succ_f \gamma^{-1}[(G, t^{f,G})]$ and $f \succeq_w \mu(w)$ for all $w \in G$. This indicates $G' \succ_f^{\varphi} G$ and $f \succeq_w \mu(w)$ for all $w \in G$. Therefore, we should have $\mu \notin I^{\varphi}(f, G)$. Immediately, a contradiction occurs. □

3.5.2 An estimation function which is compatible with T-stable matchings

Although Proposition 1 has shown the existence of T-stable matchings, a natural extension is to find out which specific estimation function is compatible with these matchings. In fact, any estimation function which includes matchings satisfying the sufficient condition of Proposition 1 will work. This means there is no unique way to construct such an estimation function. In this subsection, we provide an example. The algorithm is modified from Hafalir[17].

When $|F| = 1$ and $|W| = 1$, there is no externality and the set of stable matchings is nonempty. Then assume that in a market where $|F'| = p < m$ and $|W'| = q < n$, the set of T-stable matchings $S_T(F',W')$ is nonempty. We want to show that in a market with $|F| = p+1$ and $|W| = q+1$, the set of T-stable matchings $S_T(F,W)$ is nonempty.

Let $\phi_f^1(G) = S_T(F',W')$. Suppose that when a firm and a set of workers (f,G) blocks, f will believe that the worst matching in $\phi_f^1(G)$ will occur. In other words, f tentatively sets $t^{f,G} = |\varphi_f(G)|$ for any $G \subseteq W$. Then each f has an induced preference ordering $\succeq_f^{\phi,1}$ over 2^W. Thus a matching game without externalities is defined. When the T-substitutability is satisfied, the set of stable matchings I^{ϕ^1} in this game is nonempty according to Lemma 2. This set and the set $\phi_f^1(G)$ together form a new estimation function which is denoted by $\phi_f^2(G)$.

Then starting from $\phi_f^2(G)$, each f blocks with G according to its threshold $t^{f,G}$. Therefore, each firm adjusts its induced preference ordering $\succeq_f^{\phi,1}$ to a new induced preference ordering $\succeq_f^{\phi,2}$. Again, a matching game without externalities is defined and we can obtain the set of stable matchings I^{ϕ^2} . This set and the set $\phi_f^2(G)$ together form a new estimation function. Denoted it be $\phi_f^3(G)$.

Starting from $\phi_f^3(G)$, each f blocks with G according to its threshold $t^{f,G}$. We proceed in this way until no firm adjusts its induced preferences any more. At this point, the set of stable matchings in the induced game gives nothing new.

Formally, let $\phi_f^1(G) = S_\phi(F',W')$ and for $k = 1,2,3....$ Inductively define:

$\phi_f^{k+1}(G) = \phi_f^k(G) \cup I^{\phi^k}(f,G)$

Since $S_\phi(F',W')$ is assumed to be nonempty, $\phi_f^k(G)$ and $I^{\phi^k}(f,G)$ are well defined and nonempty.

Then, for all $(f,G) \in F \times 2^W$ and for $k = 1,2,3,...$, we note that:

$\phi_f^k(G) \subseteq \phi_f^{k+1}(G)$ and $\phi_f^k(G) \in M(F,W)$.

The sequence of sets is monotone and the set of matchings is finite. Therefore, ϕ_f^k has a limit. Let:

$\phi_f(G) = \lim_{k\to\infty} \phi_f^k(G)$.

Denote I^ϕ be the set of stable matchings in $(F, W, (\succeq^\phi)_{f\in F}, (\succeq_w)_{w\in W}, q_F)$. We note that $I^\phi(f,G) \subset \phi_f(G)$ for all $(f,G) \in F \times 2^W$.

Then each $\mu \in I^\phi(f,G)$ is a T-stable matching according to Proposition 1.

3.5.3 Example

In this subsection, we provide a simple example. It shows how to construct the estimation function that is introduced in the previous subsection, and shows the existence of a T-stable matching.

Example 3. *Consider a one-to-one matching market with externalities with three firms $F = \{f_1, f_2, f_3\}$ and three workers $W = \{w_1, w_2, w_3\}$. T-substitutability is trivially satisfied since this matching market is one-to-one. For the sake of simplicity, we assume that only full matchings (no one is matched with $\varnothing$) are considered. Then there are six matchings as listed below.*

$\mu_1 = (\{w_1\}, \{w_2\}, \{w_3\})$
$\mu_2 = (\{w_2\}, \{w_3\}, \{w_1\})$
$\mu_3 = (\{w_3\}, \{w_1\}, \{w_2\})$
$\mu_4 = (\{w_1\}, \{w_3\}, \{w_2\})$
$\mu_5 = (\{w_3\}, \{w_2\}, \{w_1\})$
$\mu_6 = (\{w_2\}, \{w_1\}, \{w_3\})$

Suppose each firm's estimation when it hires different sets of workers include all possibilities. Moreover, suppose each firm has the following thresholds:

$t^{f_1,w_1} = 1, t^{f_1,w_2} = |\phi_{f_1}(w_2)|, t^{f_1,w_3} = |\phi_{f_1}(w_3)|$
$t^{f_2,w_1} = |\phi_{f_2}(w_1)|, t^{f_2,w_2} = |\phi_{f_2}(w_2)|, t^{f_2,w_3} = |\phi_{f_2}(w_3)|$
$t^{f_3,w_1} = |\phi_{f_3}(w_1)|, t^{f_3,w_2} = |\phi_{f_3}(w_2)|, t^{f_3,w_3} = |\phi_{f_3}(w_3)|$

The preferences of firms and workers are listed below:

$f_1 : \mu_5 \succeq \mu_4 \succeq \mu_2 \succeq \mu_3 \succeq \mu_6 \succeq \mu_1$
$f_2 : \mu_3 \succeq \mu_1 \succeq \mu_5 \succeq \mu_6 \succeq \mu_4 \succeq \mu_2$
$f_3 : \mu_2 \succeq \mu_4 \succeq \mu_6 \succeq \mu_1 \succeq \mu_5 \succeq \mu_3$
$w_1 : f_1 \succeq f_2 \succeq f_3$
$w_2 : f_1 \succeq f_2 \succeq f_3$
$w_3 : f_1 \succeq f_2 \succeq f_3$

We first determine $\phi_f^1(G) = S_T(F', W')$.

$\phi^1_{f_1}(w_1) = \mu_1, \phi^1_{f_1}(w_2) = \mu_6, \phi^1_{f_1}(w_3) = \mu_3$
$\phi^1_{f_2}(w_1) = \mu_3, \phi^1_{f_2}(w_2) = \mu_5, \phi^1_{f_2}(w_3) = \mu_4$
$\phi^1_{f_3}(w_1) = \mu_5, \phi^1_{f_3}(w_2) = \mu_4, \phi^1_{f_3}(w_3) = \mu_6$

We tentatively set $t^{f_i,w_j} = |\phi_{f_i}(w_j)|$ for all $i \in \{1,2,3\}$ and $j \in \{1,2,3\}$. Then we obtain the induced preference ordering $\succeq^{\phi,1}_f$ for each firm as listed below. We omit the preferences of workers since they are fixed throughout.

$f_1 : w_3 \succeq^{\phi,1} w_2 \succeq^{\phi,1} w_1$
$f_2 : w_1 \succeq^{\phi,1} w_2 \succeq^{\phi,1} w_3$
$f_3 : w_2 \succeq^{\phi,1} w_3 \succeq^{\phi,1} w_1$

Then we obtain the set of stable matchings $I^{\phi^1} = \{\mu_3\}$. We need to add this matching to $\phi^1_{f_3}(w_2)$. Then each $\phi^2_{f_i}(w_j)$ is listed below.

$\phi^2_{f_1}(w_1) = \mu_1, \phi^2_{f_1}(w_2) = \mu_6, \phi^2_{f_1}(w_3) = \mu_3$
$\phi^2_{f_2}(w_1) = \mu_3, \phi^2_{f_2}(w_2) = \mu_5, \phi^2_{f_2}(w_3) = \mu_4$
$\phi^2_{f_3}(w_1) = \mu_5, \phi^2_{f_3}(w_2) = \{\mu_4, \mu_3\}, \phi^2_{f_3}(w_3) = \mu_6$

Now each f adjusts its induced preference ordering according to its threshold. Then we obtain $\succeq^{\phi,2}_f$ as listed below.

$f_1 : w_3 \succeq^{\phi,2} w_2 \succeq^{\phi,2} w_1$
$f_2 : w_1 \succeq^{\phi,2} w_2 \succeq^{\phi,2} w_3$
$f_3 : w_3 \succeq^{\phi,2} w_1 \succeq^{\phi,2} w_2$

The set of stable matchings is $I^{\phi^2} = \{\mu_3\}$.

At this point, no firm adjusts its induced preferences ordering any more. Therefore, we have $S_T = \mu_3$.

Let us have a look at matching μ_3. For f_3, since $\mu_6 \succeq_{f_3} \mu_5 \succeq_{f_3} \mu_3$ where $\gamma(\mu_5) = (w_1, t^{f_3,w_1})$ and $\gamma(\mu_6) = (w_3, t^{f_3,w_3})$, f_3 wants to block μ_3 with either w_1 or w_3 . However, both of w_1 and w_3 will reject f_3 ; For f_2 , μ_3 is the most preferred matching. This means f_2 will not block; For f_1 , $\mu_4 \succeq_{f_1} \mu_2 \succeq_{f_1} \mu_3$. However, $\gamma(\mu_4) \neq (w_1, t^{f_1,w_1})$ and $\gamma(\mu_2) \neq (w_2, t^{f_1,w_2})$. This means f_1 will not block. Therefore, μ_3 is T-stable.

3.6 Summary

This chapter studies a labor market with externalities among firms. The presence of externalities makes the existence of a stable matching under the definition of Gale and Shapley[15] extremely difficult. In other words, the matchings among firms and workers are highly “unstable". At an unstable matching, there exist firm-worker(s) pairs who would like to cancel their current matching and then match up together. As a result, these firms who lost their workers and these workers who got fired have to seek new matchings. Lots of workers may simultaneously apply to the same firm, and some firms may regret having hired their current workers once they receive the application from more preferred workers. Such chaos is a symptom of congestion, and the congestion failure will become severe with the growth of market thickness.

In order to fix the congestion failure, we propose a new stability concept called T-stability. This stability concept depends on the estimation functions and the threshold matchings of each firm. Then we provide two restrictions on firms' preferences. The first one is an extension of the standard notion of substitutability and depends on threshold matchings. Hence it is named as T-substitutability. The other one is called induced preferences. When these restrictions are satisfied, we show the existence of T-stable matchings under the beliefs which are generated from threshold matchings.

Chapter 4

A new equilibrium concept for a preference revelation game under SODA

4.1 Introduction

In this chapter, we formulate a school choice market as a preference revelation game of students and analyze the equilibrium outcomes. Such sort of study is especially important for designing real-world matching programs such as and the school-student matching and the hospital-intern matching which were introduced in Chapter 1. In those programs, institutions' preferences are usually open to the public due to certain regulations such as law, but individuals are able to report preferences strategically. As a result, analyzing individuals' strategic behavior plays an important role in a successful design.

As we have addressed in Chapter 2, the existence of two Nash equilibria—*SN* and *SSN* has already been demonstrated in the previous works. Moreover, *SSN* can be further categorized as *DA-SSN* and *Core-SSN* according to the way that it is derived. Our reseach aims at refining those results by proposing a new equilibrium concept called passively-strictly strong Nash equilibrium (*P-SSN*). It is stronger than both *DA-SSN* and *Core-SSN*. It rules out a deviation called passively weak deviation. Such a deviating coalition includes members who become strictly worse off. That is, such a student was actually unwilling to deviate. But if not doing so, she will receive an even worse outcome caused by the unilateral deviation of the rest members. In this sense, there exist students who "passively" joined the deviating coalition.

Then we show two preliminary existence results about *P-SSN*. (i) If the *DA-SSN* and the *Core-SSN* are not equivalent, then neither of them is a *P-SSN*. (ii) If the matching determined by the *DA-SSN* under *SODA* satisfies a property called irrelevance of low-tier agents, then it is also a *P-SSN*. In general, *P-SSN* is a quite restrictive equilibrium concept. But nevertheless, we will discuss some extensions with relaxed assumptions that give inspiration to the refinement problem when multiple *SSNs* exist.

The rest of this chapter is organized as follows. Section 4.2 formulates a related preference revelation game which is based on a school choice market. Section 4.3 defines our equilibrium concept with an example. Section 4.4 shows two preliminary results about its existence problem. Section 4.5 concludes.

4.2 The model

4.2.1 Restrictions on schools' preferences

Suppose we have a school choice market $(I, C, \succeq_I, \succeq_C, q_C)$ as defined in Chapter 2. Let $\succeq_I = (\succeq_i)_{i \in I}$ be the students' preference profile and let $\succeq_C = (\succeq_c)_{c \in C}$ be the schools' preference profile. Particularly, let $\succeq_c^*$ be the restriction of $\succeq_c$ to singleton sets and the empty set. For each $i \in I$ and $c \in C$, we say that i is acceptable for c if $i \succ_c^* \varnothing$ ($i \succ_c \varnothing$).

We assume that each school c's preferences satisfy responsiveness with quota q_c which is defined in Roth [31].

Definition 20 (Responsiveness, Roth [31])**.** *A school c's preferences $\succeq_c$ satisfy responsiveness with quota c if (i) for any $i, i' \succ_c \varnothing$ and any $S \succ_c \varnothing$ such that $i \in S, i' \notin S$ and $|S| < q_c$, $S \backslash \{i\} \cup \{i'\} \succ_c S$ if and only if $i' \succ_c i$, (ii) for any $i \in I$ and any $S \succ_i \varnothing$ with $i \notin S$ and $|S| < q_c$, $S \cup \{i\} \succ_c S$ if and only if $i \succ_c \varnothing$, and (iii) for any $S \subseteq I$ with $|S| > q_c$, $\varnothing \succ_c S$.*

That is, c is strictly better off by replacing any student with a more preferred acceptable student in $\succeq_c^*$, and if c has a vacant seat, then it is strictly better off by adding an acceptale student and worse off by adding an unacceptable student. Then we identify $\succeq_c$ with $\succeq_c^*$ because only $\succeq_c^*$ is relevant to our analysis.

4.2.2 Preference revelation game

Let $(I, (D_i(C))_{i \in I}, \succeq_I, DA(\cdot, \succeq_c, q_C))$ be a strategic game defined on a school choice market $(I, C, \succeq_I, \succeq_C, q_C)$. I is the set of students who are the players. Given a strategy profile $\succeq_I'$, the outcome of this game is determined by *SODA* and is denoted be $DA(\succeq_I', \succeq_C, q_C)$. Each student $i \in I$ evaluates the outcome according to her true preferences $\succeq_i$.

For each $i \in I$, define $D_i(C)$ as the set of her strict preferences over $C \cup \{i\}$. A coalition $S \subseteq I$ is a nonempty subset of students . Define $D_S(C) = \times_{i \in S} D_i(C)$ as coalition S's joint strategies.

SN and *SSN* are two solution concepts that have been extensively studied in the previous research. Given a strategy profile $\succeq_I^*$, we say that a coalition S has a weak deviation at $\succeq_I^*$ if there exists $\succeq'_S \in D_S(C)$ such that (i) $\nu(i) \succeq_i \mu(i)$ for all $i \in S$ and (ii) $\nu(i) \succ_i \mu(i)$ for some $i \in S$, where $\nu = DA((\succeq'_S, \succeq^*_{I\setminus S}), \succeq_C, q_C)$ and $\mu = DA(\succeq_I^*, \succeq_C, q_C)$. A strategy profile $\succeq_I^*$ is an *SSN* if each coalition does not have any weak deviation. When $\nu(i) \succ_i \mu(i)$ holds for all $i \in S$, we say that S has a strong deviation at $\succeq_I^*$, and $\succeq_I^*$ is an *SN* if each coalition does not have any strong deviation.

Dubins and Freedman [12] showed that the truth-telling strategy profile $\succeq_I$ is always an *SN*, but it may not be an *SSN*. Hence the existence problem of *SSN* becomes one of the research interest in the later works. Bando [9] demonstrated the existence of the following two *SSNs*.

DA-SSN

Given a market $M = (I, C, \succeq_I, \succeq_C, q_C)$, for each $I' \subseteq I$ and $c \in C$, let $\succeq_c |^{I'}$ be the restriction of $\succeq_c$ to $2^{I'}$. That is, $\succeq_c |^{I'}$ is the strict preferences over $2^{I'}$ such that for any $S, S' \subseteq I'$, $S \succeq_c |^{I'} S'$ if and only if $S \succeq_c S'$. Let $\succeq_C |^{I'} = (\succeq_c |^{I'})_{c\in C}$ be the profile of the restricted preferences. Then the iterative *DA* is defined as follows.

- Step 0: Let $M_0 = (I, C, \succeq_I, \succeq_C, q_C)$ and $DA(M_0) = \mu_0$. Let L_0 be the set of last proposers under $DA(M_0)$. For each $i \in L_0$, define: $\succeq_i^*$: $\mu_0(i), i$. If $I \setminus L_0 = \varnothing$, then the alogrithm terminates. If $I \setminus I_0 \neq \varnothing$, then set $I_1 = I \setminus L_0$ and proceed to the next step.
- Step k ($k \geq 1$): Let $M_k = (I_k, C, \succeq_{I_k}, \succeq_C |^{I_k}, q_C)$ and $DA(M_k) = \mu_k$. Let L_k be the set of last proposers under $DA(M_k)$. For each $i \in L_k$, define: $\succeq_i^*$: $\mu_k(i), \mu_{k-1}(i), ..., \mu_0(i), i$. If $I \setminus (L_0 \cup ... \cup L_k) = \varnothing$, then the algorithm terminates. If $I \setminus (L_0 \cup ... \cup L_k) \neq \varnothing$, then set $I_{k+1} = I \setminus (L_0 \cup ... \cup L_k)$ and proceed to the next step.

This algorithm terminates in finite steps and yields a strategy profile $\succeq_I^{DA} = (\succeq_i^*)_{i\in I}$, which is the *DA-SSN*.

Core-SSN

Given a market $M = (I, C, \succeq_I, \succeq_C, q_C)$, let μ be a stable matching in this market. Then let $(I^*, \geqslant_{I^*}, \mu)$ be a corresponding house allocation market which is defined in Shapley and Scarf [40]. I^* is the set of students such that $I^* = \{i \in I | \mu(i) \in C\}$. Each i's initial endowment is $\mu(i)$. For each $i \in I^*$, $\geqslant_i$ is a preference relationship for I^* satisfying: (i) for any $j \in I^*$, if i is unacceptable for $\mu(j)$, then $i >_i j$, (ii) for any $j, j' \in I^*$ where i is acceptable for $\mu(j)$ and $\mu(j')$, $j \geq_i j'$ if and only if $\mu(j) \succeq_i \mu(j')$.

An allocation is defined as a bijection $x : I^* \to I^*$. Define a matching μ_x such that (i) if $i \in I^*$, then $\mu_x(i) = \mu(x(i))$; and (ii) if $i \notin I^*$, then $\mu_x(i) = i$.

If x is a strict core allocation, then the strategy profile $\succeq_I^{Core} = (\succeq_i^{Core})_{i \in I}$ such that $\succeq_i^{Core}$: $\mu_x(i), \mu(i), i$ is the *Core-SSN*. Particularly, we note that this result depends on whether the strict core exists.

Under *SODA*, both of the *DA-SSN* and the *Core-SSN* yield a matching which is Pareto efficient, and weakly Pareto dominates the matching determined by students' true preferences.

4.3 Equilibrium concept

In this section, we propose a new equilibrium concept which is stronger than *SSN* (and thus it is stronger than both *DA-SSN* and *Core-SSN*). First of all, we use a simple example to show how it is defined.

Example 4. *Let* $I = \{i_1, i_2, i_3, i_4, i_5\}$, $C = \{c_1, c_2, c_3, c_4\}$ *and* $q_{c_1} = q_{c_2} = q_{c_3} = q_{c_4} = 1$. *The students' preferences and the schools' preferences are given in the following table.*

$\succeq_{i_1}: c_2, c_4, c_1, i_1$ $\quad$ $\succeq_{c_1}: i_1, i_5, i_4, i_3, \varnothing$

$\succeq_{i_2}: c_3, c_2, c_4, i_2$ $\quad$ $\succeq_{c_2}: i_4, i_2, i_1, \varnothing$

$\succeq_{i_3}: c_1, c_3, i_3$ $\quad$ $\succeq_{c_3}: i_3, i_2, i_4, \varnothing$

$\succeq_{i_4}: c_3, c_1, c_4, c_2, i_4$ $\quad$ $\succeq_{c_4}: i_2, i_1, i_4, \varnothing$

$\succeq_{i_5}: c_1, i_5$

The student-optimal stable matching μ is :

$$\begin{matrix} c_1 & c_2 & c_3 & c_4 & \varnothing \\ i_1 & i_4 & i_3 & i_2 & i_5 \end{matrix}$$

Next we look into the *DA-SSN* and the *Core-SSN* for this example.

DA-SSN. Applying the iterative *DA* which is introduced in Section 4.2.2, we obtain the following strategy profile $\succeq_I^{DA}$:

$\succeq_{i_1}^{DA}: c_4, c_1, i_1$ $\quad$ $\succeq_{i_2}^{DA}: c_2, c_4, i_2$ $\quad$ $\succeq_{i_3}^{DA}: c_1, c_3, i_3$ $\quad$ $\succeq_{i_4}^{DA}: c_3, c_1, c_2, i_4$

$\succeq_{i_5}^{DA}: i_5$.

The corresponding matching $\mu^* = DA(\succeq_I^{DA}, \succeq_C, q_C)$ is:

$$\begin{matrix} c_1 & c_2 & c_3 & c_4 & \varnothing \\ i_3 & i_2 & i_4 & i_1 & i_5 \end{matrix}$$

Consider the following deviating strategies $\succeq'_{i_1}$ for i_1 and $\succeq'_{i_2}$ for i_2:

$$\succeq'_{i_1}: c_2, c_1, i_1 \quad \succeq'_{i_2}: c_3, c_2, c_4, i_2$$

We can check that $DA(\succeq'_{\{i_1,i_2\}}, \succeq^{DA}_{\{i_3,i_4,i_5\}}, \succeq_C, q_C)$ yields a matching ν which is equivalent to μ.

However, if i_4 agrees to deviate and play the following strategy:

$$\succeq'_{i_4}: c_4, c_2, i_4$$

then $DA(\succeq'_{\{i_1,i_2,i_4\}}, \succeq^{DA}_{\{i_3,i_5\}}, \succeq_C, q)$ yields the following matching ν':

$$\begin{matrix} c_1 & c_2 & c_3 & c_4 & \varnothing \\ i_3 & i_1 & i_2 & i_4 & i_5 \end{matrix}$$

We note that i_4 is strictly worse off in ν' than in μ^* since $\nu'(i_4) = c_4$, $\mu^*(i_4) = c_3$ and $c_3 \succ_{i_4} c_4$. But nevertheless, she is strictly better off in ν' than in ν since $\nu'(i_4) = c_4$, $\nu(i_4) = c_2$ and $c_4 \succ_{i_4} c_2$. On the other hand, we note that $\nu'(i_1) \succ_{i_1} \mu^*(i_1)$ and $\nu'(i_2) \succ_{i_2} \mu^*(i_2)$. In this sense, we can regard $\succeq'_{\{i_1,i_2\}}$ as i_1's and i_2's "threatening" strategy profile which forces i_4 to join the deviating coalition in order to avoid an even worse outcome caused by the unilateral deviation of them.

Core-SSN. By constructing a house allocation market as introduced in Section 4.2.2, we obtain the following strategy profile $\succeq_I^{Core}$:

$$\succeq^{Core}_{i_1}: c_2, c_1, i_1 \quad \succeq^{Core}_{i_2}: c_4, i_2 \quad \succeq^{Core}_{i_3}: c_1, c_3, i_3 \quad \succeq^{Core}_{i_4}: c_3, c_2, i_4$$
$$\succeq^{Core}_{i_5}: i_5.$$

The corresponding matching $\mu' = DA(\succeq_I^{Core}, \succeq_C, q_C)$ is:

$$\begin{matrix} c_1 & c_2 & c_3 & c_4 & \varnothing \\ i_3 & i_1 & i_4 & i_2 & i_5 \end{matrix}$$

Consider the following deviating strategy $\succeq''_{i_2}$ for i_2:

$$\succeq''_{i_2}: c_2, c_4, i_2$$

We can check that $DA(\succeq''_{i_2}, \succeq^{Core}_{\{i_1,i_3,i_4,i_5\}}, \succeq_C, q_C)$ yields a matching ν'' which is equivalent to μ.
However, if i_1 agrees to join i_2 and play the following deviating strategy:

$$\succeq''_{i_1}: c_4, c_1, i_1$$

then $DA(\succeq''_{\{i_1,i_2\}}, \succeq^{Core}_{\{i_3,i_4,i_5\}}, \succeq_C, q_C)$ yields the following matching ν''':

$$\begin{matrix} c_1 & c_2 & c_3 & c_4 & \varnothing \\ i_3 & i_2 & i_4 & i_1 & i_5 \end{matrix}$$

Similarly, we observe that $\nu'''(i_2) \succ_{i_2} \mu'(i_2)$ and $\mu'(i_1) \succ_{i_1} \nu'''(i_1) \succ_{i_1} \nu''(i_1)$. We can regard $\succeq''_{i_2}$ as i_2's threatening strategy which forces i_1 to deviate in order to avoid an even worse outcome caused by her unilateral deviation.

Example 4 presents a kind of deviating coalition in which some members were actually unwilling to deviate. They deviate not for achieving a better outcome, but for avoiding an even worse outcome which is caused by the unilateral deviation of the rest members. We name this deviation as passively weak deviation and give the formal definition as follows.

Definition 21. *(Passively weak deviation) Given a strategy profile $\succeq^*_I \in D_I(C)$ and a coalition $S \subseteq I$, we say that S has a passively weak deviation at $\succeq^*_I$ if there exists $\succeq'_S \in D_S(C)$ such that*

(i) $\nu(i) \succ_i \mu(i)$, $\exists i \in S$

(ii) $\nu(i) \succeq_i \nu'(i)$, $\forall i \in S$

where $\mu = DA(\succeq^*_I, \succeq_C, q_C)$, $\nu = DA(\succeq'_S, \succeq^*_{I\setminus S}, \succeq_C, q_C)$ and $\nu' = DA(\succeq'_T, \succeq^*_{I\setminus T}, \succeq_C, q_C)$ such that $T = \{i \in S | \nu(i) \succeq_i \mu(i)\}$.

In a passively weak deviating coalition S, only a sub-group of members T are at least weakly better off. For each member in $S\setminus T$, although joining S will bring her an outcome which is worse than that in an *SSN*, she has to do so because the outcome caused by T's unilateral deviation is even worse for her. In this sense, each student in $S\setminus T$ “passively" chooses to deviate.

We say that a strategy profile $\succeq^*_I$ is a passively-strictly strong Nash equilibrium (*P-SSN*) if each coalition does not have any passively weak deviation.

We note that $T = S$ holds when $\nu(i) \succeq_i \mu(i)$ holds for all $i \in S$, and thus $\nu'(i) = \nu(i)$ holds for all $i \in S$ in this case. This means a weak deviation must also be a passively weak deviation, and thus a *P-SSN* must be an *SSN*. However, Example 4 shows that the converse may not be true.

4.4 Existence

In this section, we study the existence problem of *P-SSN*. Particularly, we examine under which conditions a *DA-SSN* or a *Core-SSN* would become a *P-SSN*.

We first introduce two strategies that will support our next results. One is the c-bottom strategy proposed by Bando [9]. Given a market $(I, C, \succeq_I, \succeq_C, q_C)$, for each $\succeq'_i \in D_i(C)$ and $c \in C \cup \{i\}$, define $U(c, \succeq'_i) = \{c' \in C \cup \{i\} | c' \succeq'_i c\}$ as the set of colleges that i weakly prefers to c under $\succeq'_i$. Then each i's c-bottom strategy $\succeq^c_i$ is defined as follows.

Definition 22. *(c-bottom strategy, Bando [9])*

(i) $c \succeq'_i i$ *and* $c \succeq^c_i i$,

(ii) $U(c, \succeq^c_i) \subseteq U(c, \succeq'_i)$,

(iii) there exists no $c' \in C$ *such that* $c \succ^c_i c' \succ^c_i i$.

Let $\mu = DA(\succeq_I, \succeq_C, q_C)$. We are interested in $\mu(i)$-bottom strategy related to the true preference $\succeq_i$ ($\mu(i)$-bottom strategy for short). Bando [9] further demonstrated the following results about $\mu(i)$-bottom strategy.

Lemma 3. *(Bando [9]) Let* $M = (I, C, \succeq_I, \succeq_C, q_C)$ *be a school choice market. Let* μ *be a stable matching in* M*. For each* $i \in I$*, let* $\succeq^*_i$ *be the* $\mu(i)$*-bottom strategy. Consider a coalition* S *and* $\succeq'_S \in D_S(C)$ *such that (i)* $\nu(i) \succeq_i \mu^*(i)$ *for all* $i \in S$ *and (ii)* $\succeq'_i$: $\nu(i), i$ *for all* $i \in S$*, where* $\nu = DA(\succeq'_S, \succeq^*_{I \setminus S}, \succeq_C, q_C)$ *and* $\mu^* = DA(\succeq^*_I, \succeq_C, q_C)$*. Then we have that*

(i) $\mu^*(i) \succeq_i \mu(i)$ *for all* $i \in I$

(ii) $\nu(i) \succeq_i \mu(i)$ *for all* $i \in I$

(iii) $|\nu(i)| = |\mu(i)|$ *for all* $i \in I$ *and* $|\nu^{-1}(c)| = |\mu^{-1}(c)|$ *for all* $c \in C$*, where* $|\mu(i)| = 1$ *if* $\mu(i) \in C$ *and* $|\mu(i)| = 0$ *if* $\mu(i) = i$.

Here we note that the deviating coalition's joint strategies have a quite simple structure. Roth [29] and Bando [9] demonstrated that if a coalition S has a successful deviation (either in a strong sense or in a weak sense), then it equals to using a simplified strategy that each student in S reports the deviating outcome as her first choice. Therefore, we also consider this kind of simple deviation in this research.

Based on the previous results, we are ready to introduce our first finding. We show that if the *DA-SSN* and the *Core-SSN* yield different matchings under *SODA*, then neither of them is a *P-SSN*. In other words, we can always construct a passively weak deviation at either the *DA-SSN* or the *Core-SSN*.

Proposition 2. *Let $(I,C,\succeq_I,\succeq_C,q_C)$ be a school choice market. Let $\mu = DA(\succeq_I,\succeq_C,q_C)$. Let $\mu^* = DA(\succeq_I^{DA},\succeq_C,q_C)$. Suppose that the Core-SSN exists, and let $\mu' = DA(\succeq_I^{Core},\succeq_C,q_C)$. If $\mu^* \neq \mu'$, then neither $\succeq_I^{Core}$ nor $\succeq_I^{DA}$ is a P-SSN.*

Proof. (i) We first show that we can construct a passively weak deviation at $\succeq_I^{Core}$. By assumption, there exists $i \in I$ such that $\mu^*(i) \succ_i \mu'(i)$. Otherwise $\mu'(i) \succeq_i \mu^*(i)$ holds for all $i \in I$. Since $\mu^* \neq \mu'$, a contradiction occurs with the Pareto efficiency of μ^*. Let $S^+ = \{i \in I | \mu^*(i) \succ_i \mu'(i)\}$, and let $S^- = \{i \in I | \mu'(i) \succeq_i \mu^*(i)\}$. For each $c \in C$, define $A_c(\mu,\succeq_I) = \{i \in I | c \succeq_i \mu(i)\}$ as the set of students who weakly prefer c to their assignments.

Consider the strategy $\succeq_i'$: $\mu^*(i),\mu(i),i$ for all $i \in S^+$. Let $\nu = DA((\succeq_{S^+}',\succeq_{S^-}^{Core}),\succeq_C,q_C)$. We will show that $\nu(i) = \mu(i)$ holds for all $i \in I$. Suppose that there exists $i' \in I$ such that $\nu(i') \neq \mu(i')$. Note that $(\succeq_{S^+}',\succeq_{S^-}^{Core})$ consists of $\mu(i)$-bottom strategies. Hence by Lemma 3, $\nu(i) \succeq_i \mu(i)$ holds for all $i \in I$. This implies that $\nu(i') \succ_{i'} \mu(i')$. Let $\nu(i') = c$. By the stability of ν, we have that $i' \in Ch_c(A_c(\nu,(\succeq_{S^+}',\succeq_{S^-}^{Core}))| \succeq_c)$. On the other hand, recall the construction of $\succeq_I^{Core}$ and $\succeq_I^{DA}$. We note that $\nu(i) \in A_{\mu(i)}(\mu,\succeq_I)$ for all $i \in S^+$ and $\nu(j) \in A_{\mu(j)}(\mu,\succeq_I)$ for all $j \in S^-$. This implies that ν is also stable in the market $(I,C,\succeq_I,\succeq_C,q_C)$. However, μ is the student-optimal matching in this market, which implies that $\mu(i) \succeq_i \nu(i)$ holds for all $i \in I$. Hence the only possibility is that $\nu(i) = \mu(i)$ for all $i \in I$.

Let $S' = \{i \in I | \mu^*(i) \succ_i \mu'(i)\}$. Then consider $\succeq_{i''}'$: $\mu^*(i''),\mu(i''),i''$ for all $i'' \in S'$. Let $\nu' = DA((\succeq_{S^+}',\succeq_{S'}',\succeq_{S^-\setminus S'}^{Core}),\succeq_C,q_C)$. Then $\nu'(i) = \mu^*(i) \succeq_i \mu(i)$ for all $i \in I$. Therefore, $S^+ \cup S'$ constitutes a passively weak deviation where each $i''' \in S^+ \cup S'$ plays stratgy $\succeq_{i'''}'$: $\mu^*(i'''),\mu(i'''),i'''$.

(ii) Similarly, let $S^+ = \{i \in I | \mu'(i) \succ_i \mu^*(i)\}$, and let $S^- = \{i \in I | \mu^*(i) \succeq_i \mu'(i)\}$. Then consider the strategy $\succeq_i''$: $\mu'(i),\mu(i),i$ for all $i \in S^+$. The proof goes almost the same as that in (i), and thus we omit it here. □

However, the *Core-SSN* usually fails to exist since it depends on the existence of the strict core in the corresponding house allocation market. Therefore, our second result focuses on the *DA-SSN*. We show that under *SODA*, if the *DA-SSN* yields a matching which satisfies a condition called irrelevance of low-tier agents, then it is also a *P-SSN*.

Definition 23. *(Irrelevance of low-tier agents) Let $M = (I, C, \succeq_I, \succeq_C, q_C)$ be a school choice market. Let $\mu^* = DA(\succeq_I^{DA}, \succeq_C, q_C)$. Suppose that the iterative DA ends in K steps such that $K \geq 1$. Denote the set of last proposers in each step be L_k such that $0 \leq k \leq K$. We say that μ^* satisfies irrelevance of low-tier agents if and only if pick any $i \in I$ such that $i \in L_k$, we have that $\mu^*(i) \succ_i \mu^*(i')$ for all i' such that $i' \in L_{k'}$ and $k' < k$.*

One may refer to Bando [9] for a detailed description of the iterative DA which finds out a *DA-SSN*. Simply speaking, this algorithm iteratively applies the DA, and then determines the matching for one student who turns out the last proposer in each round. In other words, if a student is identified as the last proposer in an earlier round, then her matching will be determined earlier. This condition requires each student not to desire the assignment of any student who is identified as the last proposer earlier than her.

The next lemma is a direct application of Lemma 3.

Lemma 4. *Let $M = (I, C, \succeq_I, \succeq_C, q_C)$ be a market. Let $\mu = DA(\succeq_I, \succeq_C, q_C)$. Denote $\succeq_I^{DA}$ be $\succeq_I^*$ and let $\mu^* = DA(\succeq_I^*, \succeq_C, q_C)$. Consider a coalition S and $\succeq_S' \in D_S(C)$ such that (i) $\nu(i) \succeq_i \mu^*(i)$ for some $i \in S$, (ii) $\nu(i) \succeq_i \nu'(i)$ for all $i \in S$, and (iii) $\succeq_i'$: $\nu(i), \mu(i), i$ for all $i \in S$, where $\nu = DA(\succeq_S', \succeq_{I \setminus S}^*, \succeq_C, q_C)$ and $\nu' = DA(\succeq_T', \succeq_{I \setminus T}^*, \succeq_C, q_C)$ such that $T = \{i \in S | \nu(i) \succeq_i \mu^*(i)\}$. Then we have that*

(i) $\nu'(i) \succeq_i \mu(i)$ for all $i \in I$,

(ii) $|\nu(i)| = |\mu(i)|$ for all $i \in I$ and $|\nu^{-1}(c)| = |\mu^{-1}(c)|$ for all $c \in C$, where $|\mu(i)| = 1$ if $\mu(i) \in C$ and $|\mu(i)| = 0$ if $\mu(i) = i$.

Since $(\succeq_T', \succeq_{I \setminus T}^*)$ consists of $\mu(i)$-bottom strategies, Lemma 4 immediately follows from Lemma 3

Then we have the following result.

Proposition 3. *Let $(I, C, \succeq_I, \succeq_C, q_C)$ be a school choice market. Let $\mu = DA(\succeq_I, \succeq_C, q_C)$. Let $\mu^* = DA(\succeq_I^{DA}, \succeq_C, q_C)$. If μ^* satisfies irrelevance of lower-tier agents, then $\succeq_I^{DA}$ is a P-SSN.*

Proof. Suppose that $\succeq_I^{DA}$ is not a *P-SSN*. Then there exists a coalition S and $\succeq_S' \in D_S(C)$ such that (i) $\nu(i) \succ_i \mu^*(i)$ for some $i \in S$, (ii) $\nu(i) \succeq_i \nu'(i)$ for all $i \in S$, and (iii) $\succeq_i'$: $\nu(i), \mu(i), i$ for all $i \in S$, where $\nu = DA(\succeq_S', \succeq_{I \setminus S}^{DA}, \succeq_C, q_C)$ and $\nu' = DA(\succeq_T', \succeq_{I \setminus T}^{DA}, \succeq_C, q_C)$ such that $T = \{i \in S | \nu(i) \succeq_i \mu^*(i)\}$. Moreover, $S \setminus T \neq \varnothing$ holds. Otherwise, S has a weak deviation from $\succeq_I^{DA}$ and it cannot be an *SSN*,.

Let $T^+ = \{i \in S | \nu(i) \succ_i \mu^*(i)\}$. Suppose that the iterative *DA* ends in K rounds such that $K \geq 0$. Then it suffice to show that $L_k \cap T^+ = \varnothing$ for each $k \in \{0, 1, ..., K\}$.

Suppose that there exists $i \in L_k$ such that $\nu(i) \succ_i \mu^*(i)$. Since $|\nu(i)| = |\mu^*(i)|$ holds for all $i \in I$ according to Lemma 4, we know that $\nu(i) \in C$ and $\mu(i) \in C$. Let $\nu(i) = c$. Then $|\mu^{*-1}(c)| = q_c$ holds. Otherwise, we have that $\mu^{*-1}(c) \cup \{i\} \succ_c \mu^{*-1}(c)$ by responsiveness. This implies $i \in Ch_c(\mu^{*-1}(c) \cup \{i\}| \succeq_c)$ and hence a contradiction occurs with the stability of μ^*. On the other hand, since μ^* satisfies irrelevance of low-tier agents, we know that $c \in \mu^*(L_{k+1} \cup L_{k+2} \cup ... \cup L_K)$. Then there exists j such that $\mu^*(j) \in \mu^*(L_{k+1} \cup L_{k+2} \cup ... \cup L_K)$ and $\nu(j) \in \mu^*(L_0 \cup L_1 \cup ... \cup L_k)$. However, we have that $\mu(j) \succ_j \nu(j)$ again by the irrelevance of low-tier agents. Since $\nu(j) \succeq_j \mu(j)$ by Lemma 4, a contradiction occurs. □

4.5 Summary

This chapter analyzes a preference revelation game which is defined on a school choice market. Although it seems to be a rare case in real-world situations, students may form coalitions and cooperatively decide strategies in order to achieve more preferred outcomes. If a mechanism is not immune to such sort of cooperation, then students have to be engaged in costly and risky strategic behavior. Therefore, analyzing the equilibrium outcomes of this game becomes important.

We refine the current equilibrium concepts by proposing a new one called passively-strictly strong Nash equilibrium (*P-SSN*). It rules out a deviating coalition called passively weak deviating coalition which includes members who become strictly worse off. In other words, they were actually unwilling to deviate. But if not doing so, they will receive an even worse outcome which is caused by the unilateral deviation of the rest members. In this sense, some students were threatened to join the deviating coalition. Then we show two preliminary existence results about *P-SSN*: (i) If the *DA-SSN* and the *Core-SSN* are not equivalent, then neither of them is a *P-SSN*. (ii) If the matching determined by the *DA-SSN* satisfies a property called irrelevance of low-tier agents, then it is also a *P-SSN*.

Chapter 5

An efficiency-adjusted NH4 mechanism for a house allocation problem with existing tenants

5.1 Introduction

Instead of many-to-one matching markets, this chapter studies a closely related house allocation problem. Generally speaking, there are two kinds of house allocation models which are categorized according to the initial ownership. The first one was proposed by Shapley and Scarf [40]. They considered a house allocation market where each agent is initially endowed with a house. Recall the *Core-SSN* that we have extensively discussed in Chapter 4. It is just derived through constructing a house allocation market of this type for a school choice market. The other model was proposed by Hylland and Zeckhauser [19]. In contrast, they assumed that none of the agents initially owns a house.

An interesting extension is to think about a mixed model. That is, some agents initially own houses and some do not. This model can be used to explain lots of real-world problems, among which the best-known one is an on-campus housing problem. At the beginning of each school year, freshmen come and apply for dormitories. On the other hand, seniors who have been assigned dormitories in the previous year can either choose to continue staying in the current dormitory or apply for a different one. Abdulkadiroğlu and Sönmez [5] first studied this problem, and formally name it as house allocation with existing tenants. An “existing tenant" refers to a senior student. In contrast, a freshman is called a “new applicant".

In this problem, the allocation is usually determined through the following procedure. First of all, each existing tenant decides whether to join the allocation program together with the new applicants or not. If so, her current house will be cancelled and added back to the pool. If not, she simply keeps staying in her current house and this house will not be available for any other student. In the next step, each participant reports a strict preference ordering over the available houses to a central clearing house. The central clearing house first decides a priority ordering among all students according to an exogenous rule (e.g. lottery, GPA, etc.). Then based on this priority ordering and students' preferences, the allocation is determined through an allocation mechanism.

However, the above procedure involves two key problems. One is the participation of existing tenants. Abdulkadiroğlu and Sönmez [5] pointed out that once lots of existing tenant choose to keep her current house rather than join the allocation program, large efficiency loss may occur due to the waste of housing resource. Therefore, assuring each existing tenant a house which is at least as good as her current one is essential to boost the participation rate. Theoretically, this condition is named as individual rationality for existing tenants. We must take this condition into account when designing the allocation mechanism. The other problem is exactly how to design the allocation mechanism.

In real-world practice, there is a mechanism which has been under operation at Massachusetts Institute of Technology for about three decades to allocate a dormitory called New House 4. Hence it is called the MIT-NH4mechanism (the NH4 mechanism for short) . This mechanism solved the problem of individual rationality, and it realizes several nice properties including fairness and strategy-proofness but it easily causes large efficiency loss. This fact motivated us to explore solutions which aim at recovering efficiency loss for it.

On the other hand, Guillen and Kesten [16] demonstrated that the NH4 mechanism is actually a special case of the student-optimal stable mechanism (*SOSM*) which has important applications in the school choice problem. Similar as the NH4 mechanism, the *SOSM* satisfies individual rationality, fairness and strategy-proofness, but not Pareto efficiency (Abdulkadiroğlu and Sönmez [4]). Unlike the NH4 mechanism, several attempts to recover efficiency loss for *SOSM* have been done. We extend those results to house allocation problem with existing tenants, and then propose an efficiency-adjusted NH4 mechanism which recovers efficiency loss for the standard NH4 mechanism. Simply speaking, it is an iterative process of recovering efficiency loss which is caused by resolving squatting conflicts. We adapt the notions of underdemanded school and essentially underdemanded school which were proposed by Tang and Yu [46] to new definitions of underdemanded house and quasi-underdemanded house. Then we show that students who are assigned these houses are Pareto unimprovable. But nevertheless, once some of these students give consents to priority violations, Pareto improvements upon the standard NH4 outcome would become possible. Moreover, we show that such consents do not change the assignments of the consenting students, which is important for students' motivation to grant such consents.

The rest of this chapter is organized as follows. Section 5.2 introduces a house allocation problem with existing tenants and some key research focuses. Section 5.3 explains the standard NH4 algorithm and the efficiency-adjusted NH4 algorithm that we propose. Section 5.4 provides detailed examples to show how our algorithm works. Section 5.5 concludes.

5.2 The model

5.2.1 Preliminaries

Suppose we have a house allocation problem with existing tenants as defined in Chapter 2. We simply call it a house allocation problem in the following sections.

Let $\mathscr{A}$ be the set of all allocations. Since this problem has an initial state that some students have houses and some do not, we define the initial allocation as α_0 which satisfies (1) $\alpha_0 \in \mathscr{A}$; (2) For each $i \in I_E$, there exists $h_i \in H_O$ such that $\alpha_0(i) = h_i$; and (3) For each $j \in I_N$, $\alpha_0(j) = h_0$. In words, every existing tenant is initially endowed with a house, but each new applicant is not.

In the previous section, we have introduced that the allocation is determined through a mechanism according to not only the students' preferences but also a priority ordering among all students. The priority ordering f is usually exogenously determined by a certain rule such as lottery. Below is the formal definition.

Definition 24 (Priority ordering)**.** *An ordering among students is a one-to-one and onto function f from $\{1,2,...,|I_E \cup I_N|\}$ to $I_E \cup I_N$.*

For example, student $f(1)$ has the highest priority among all students. Let F denote the set of all priority orderings.

Let us denote each student's preferences profile be P_i, and denote their products be $P = \times_{i \in I} P_i$. Then we can simply characterize a house allocation problem with existing tenants as (f, P).

A mechanism Φ is a systematic procedure that chooses an allocation $\Phi(f, P)$ for each problem (f, P). In practice, this procedure is usually implemented by an algorithm.

5.2.2 Properties of a desirable allocation

We expect the obtained allocation to satisfy some nice properties. A basic property that is usually considered is individual rationality.

Definition 25. *An allocation* $\alpha \in \mathscr{A}$ *is individually rational if* $\alpha(i)R_i\alpha_0(i)$ *for each* $i \in I$.

This means each student obtains a house that is at least as good as her initial assignment. Specifically, for every $i \in I_N$ who is a new applicant, we require $\alpha(i)R_ih_0$. For every $j \in I_E$ who is an existing tenant, we require $\alpha(j)R_jh_j$. A mechanism is individually rational if it always chooses an individually rational allocation for any given problem.

Pareto efficiency and fairness are among the most popular properties when doing resource allocation problems.

Definition 26. *An allocation* $\alpha \in \mathscr{A}$ *is Pareto efficient if there does not exist* $\mu \in \mathscr{A}$ *such that* $\mu(i)R_i\alpha(i)$ *for all* $i \in I$ *and* $\mu(j)P_j\alpha(j)$ *for some* $j \in I$.

A mechanism is Pareto efficient if it always chooses a Pareto efficient allocation for any given problem.

Definition 27. *An allocation* $\alpha \in \mathscr{A}$ *is fair if for every* $i, j \in I$, *if* $\alpha(j)P_i\alpha(i)$, *then either (i)*$f^{-1}(j) < f^{-1}(i)$ *or (ii)* $j \in I_E$ *with* $\alpha(j) = h_j$.

We note that this definition of fairness is different from the standard one which only requires condition (i) to hold. In fact, condition (ii) incorporates individual rationality for existing tenants. This extension was first proposed by Guillen and Kesten [16]. A mechanism is fair if it always chooses a fair allocation for any given problem.

Unfortunately, individual rationality, Pareto efficiency and fairness are not compatible.

Proposition 4 (Guillen and Kesten [16])**.** *No mechanism is individually rational, Pareto efficient, and fair.*

A possible solution is to achieve all properties with certain constraints. Kesten [23] and Tang and Yu [46] have made such effort under the school choice model. In their work, efficiency loss was recovered by utilizing students' consents to priority violations. In this research, we extend this idea to house allocation problem. Specifically, we propose an algorithm which recovers efficiency loss for the NH4 mechanism.

5.3 The Algorithm

5.3.1 The standard NH4 mechansim

First of all, we need to know about the standard NH4 mechanism which was first introduced in Abdulkadiroğlu and Sönmez [5]. It works as the following process.

1. A priority ordering f is chosen.

2. The first student is tentatively assigned her top choice among all houses, the next student is tentatively assigned her top choice among the remaining houses, and so on, until a squatting conflict occurs.

3. A squatting conflict occurs if it is the turn of an existing tenant but every remaining house is worse than her current house. That means someone else, the conflicting agent, is tentatively assigned the existing tenant's current house. When this happen: (1) the existing tenant is assigned her current house and removed from the process, and (2) all tentative assignments starting with the conflicting agent and up to the existing tenant are erased. At this point the squatting conflict is resolved and the process starts over again with the conflicting agent. Every squatting conflict that occurs afterwards is resolved in a similar way.

4. The process is over when there are no houses or agents left. At this point all tentative assignments are finalized.

Let the allocation determined by the NH4 algorithm be φ. Particularly, we note that this algorithm proceeds by resolving squatting conflicts. We formally define a squatter below.

Definition 28. *A student i is called a squatter at* φ *if (i)* $i \in I_E$*; (ii)* $\varphi(i) = h_i$*; and (iii) there exists* $j \in I$ *such that* $f^{-1}(j) < f^{-1}(i)$ *and* $h_i P_j \varphi(j)$*.*

We first note that a squatter must be an existing tenant. And she is matched with her initial assignment at the NH4 allocation φ. However, we need to be careful that even if an existing tenant is matched with her initial assignment at φ, she is not necessarily a squatter. We name the houses which are assigned to squatters at φ as squatting houses and denote the set be SQ.

We are going to carefully study the properties of students who are assigned houses in SQ and students who are assigned houses in $H \backslash SQ$. They are going to play an important role in our efficiency-recovery attempts.

5.3.2 The efficiency-adjusted NH4 mechanism

First of all, we note that φ is non-wasteful. That is, whenever a student prefers another house to her assignment, this house must have been assigned to another student. The formal proof can be referred to Guillen and Kesten [16]. This property is important for our next results.

Theorem 2 (Guillen and Kesten [16])**.** *The NH4 mechanism is non-wasteful.*

In the previous section, we have learned that the NH4 algorithm depends on identifying and resolving squatting conflicts. The following lemma shows that once squatting conflict does not occur, the obtained allocation must be Pareto efficient.

Lemma 5. *Given a house allocation problem (f,P) and run the NH4 algorithm for it. If no squatting conflict occurs, then φ is Pareto efficient.*

Proof. Suppose there exists $\alpha \in \mathscr{A}$ such that $\alpha(i)R_i\varphi(i)$ for all $i \in I$ and $\alpha(j)P_j\varphi(j)$ for at least one $j \in I$. Since φ is non-wasteful and student j desires $\alpha(j)$ at φ, $\alpha(j)$ must have been assigned to another student. Since no squatting conflict has occurred, it must be the case that $\alpha(j) \in \cup_{k=1}^{f^{-1}(j)-1}\{\varphi[f(k)]\}$. Let this set be $H^{f^{-1}(j)-1}$ for the sake of simplicity. Since student j obtains a house in $H^{f^{-1}(j)-1}$ at α, there must be at least one student j' such that $\varphi(j') \in H^{f^{-1}(j)-1}$ and $\alpha(j') \notin H^{f^{-1}(j)-1}$. Moreover, $\alpha(j')P_{j'}\varphi(j')$ according to the assumption. Again, $\alpha(j') = \varphi(j'')$ such that $j'' \neq j'$ due to non-wastefulness. Since $f^{-1}(j') < f^{-1}(j'')$, there should have been a squatting conflict between j'' and j', or a squatting conflict between j'' and a student who has a higher priority than j'. Either case is a contradiction. □

On the other hand, when φ is Pareto inefficient, at least one squatting conflict must have occurred throughout the NH4 algorithm.

The algorithm we propose in this research aims at recovering efficiency loss which is caused by resolving squatting conflicts. Particularly, we note that re-allocation of a house happens only when a squatting conflict occurs and there are two cases. One is being occupied by a squatter. The other is being re-allocated due to the cancellation of the previous assignments. On the other hand, some houses have been assigned to only one student throughout. We show that the assignments of these houses are Pareto unimprovable.

We name such a house as an underdemanded house. This concept directly follows the idea of underdemanded school that is proposed by Tang and Yu [46] under the school choice model.

Definition 29 (Underdemanded house)**.** *A house $h \in H$ is underdemanded at φ if and only if $\varphi(i)P_ih$ for any $i \in I$ such that $\varphi(i) \neq h$.*

That is, every student who is not matched with an underdemanded house strictly prefers her assignment at φ.

Definition 30 (Quasi-underdemanded house)**.** *A house $h \in H$ is quasi-underdemanded at φ if and only if for any $i \in I$ such that $hP_i\varphi(i)$, $\varphi(i)$ is underdemanded.*

That is, a quasi-underdemanded house is desired only by students who are matched with underdemanded houses.

Proposition 5. *All students who are matched with underdemanded houses and quasi-underdemanded houses at φ are Pareto unimprovable.*

Proof. Suppose $i \in I$ is assigned an underdemanded house at φ. Let $\alpha \in \mathscr{A}$ be an allocation which Pareto dominates φ. We show that $\alpha(i) = \varphi(i)$ at any such α. Without loss of generality, assume that $\alpha(i)P_i\varphi(i)$. Since φ is non-wasteful, $\alpha(i)$ must be assigned to another student at φ. Denote this student be j_1 such that $j_1 \in I\backslash\{i\}$. Then at α, j_1 receives $\alpha(j_1)$ such that $\alpha(j_1) \neq \varphi(j_1)$. By assumption, $\alpha(j_1)P_{j_1}\varphi(j_1)$. According to non-wastefulness, $\alpha(j_1)$ must be assigned to student j_2 such that $j_2 \in I\backslash\{i, j_1\}$ at φ. Then at α, j_2 receives $\alpha(j_2)$ such that $\alpha(j_2) \neq \varphi(j_2)$. By assumption, $\alpha(j_2)P_{j_2}\varphi(j_2)$. According to non-wastefulness, $\alpha(j_2)$ must be assigned to student j_3 such that $j_3 \in I\backslash\{i, j_1, j_2\}$ at φ......Continue this process and finally, there must be a student $j_k \in I\backslash\{i, j_1, j_2, ..., j_{k-1}\}$ such that $\alpha(j_k) = \varphi(i)$ and $\alpha(j_k)P_{j_k}\varphi(j_k)$. In other words, $\{i, j_1, j_2, ..., j_{k-1}, j_k\}$ constitutes an improvement cycle upon φ in which each student receives the next student's assignment (j_k receives i's). Therefore, we obtain $\varphi(i)P_{j_k}\varphi(j_k)$. This contradicts the assumption that $\varphi(i)$ is underdemanded.

Now suppose $j \in I$ is assigned a quasi-underdemanded house at φ. Similarly, let $\alpha' \in \mathscr{A}$ be an allocation which Pareto dominates φ and assume $\alpha'(j)P_j\varphi(j)$. Apply the argument in the previous proof, we should again obtain an improvement cycle $\{j, j'_1, j'_2, ..., j'_{k-1}, j'_k\}$ where each student receives the next student's assignment (j'_k receives j's). Since $\varphi(j)$ is quasi-underdemanded and $\varphi(j)P_j\varphi(j'_k)$, we know that $\varphi(j'_k)$ is underdemanded. Since $\varphi(j'_k)P_{j'_{k-1}}\varphi(j'_{k-1})$, a contradiction occurs. □

In fact, this proof is applicable to any non-wasteful allocation mechanism, because any Pareto improvement upon a non-wasteful allocation must take the form of improvement cycles. Neither does an underdemanded house nor a quasi-underdemanded house can be included in any of these cycles. We take the set of underdemanded houses and the set of quasi-underdemanded houses as a whole and denote the set be *UD*. In contrast, a house which is neither underdemanded nor quasi-underdemanded is called a overdemanded house. We donte the set of these houses be *OD*.

Based on the previous results, we are ready to introduce an efficiency-adjusted NH4 algorithm which recovers efficiency loss by iteratively preventing squatting conflicts from happening. It works as follows.

Round 0 Run the NH4 algorithm for a given problem (f, P).

Round $k, k \geq 1$ This round consists of three steps:

- 1. Identify the houses which have been assigned to only one student throughout Round 0 (In other words, such a house has never been re-allocated);

- 2. For each student who is matched with such a house: If this student consents, modify this house to the top of her preference list and then proceed to the next step; If this student does not consent, then before modify this house to the top of her preference list, remove all of her more preferred houses from the preference list of any student who has a lower priority and also prefers these houses. Then proceed to the next step.

- 3. Rerun the NH4 algorithm for the problem with the modified preference lists.

Stop when no squatting conflict occurs.

Because the numbers of students and houses are finite, the algorithm terminates in a finite number of rounds. The allocation obtained in the final round is the outcome of the algorithm.

Observation 1. *Given a house allocation problem (f,P) and apply the efficiency-adjusted NH4 mechanism to it. Suppose the process terminates in Round R such that $R \geq 1$. Denote each round be r such that $0 \leq r \leq R$ and let the set UD obtained in each round be UD^r. If a student $i \in I$ is assigned $h \in UD^r$ in Round r, then she is still assigned this house in Round $r+1$.*

Proof. Let student i's preference list in each round be P_i^r and let the allocation obtained in each round be ϕ^r. Here $0 \leq r \leq R$. If $\phi^r(i) = h$ such that h is underdemanded in Round r, then h will be upgraded to the top of P_i^{r+1} in Round $r+1$. In the meantime, for any $j \in I$ such that $j \neq i$, $\phi^r(j) P_j^r h$, and the relative ordering of $\phi^r(j)$ and h are unchanged on P_j^{r+1}. This indicates $\phi^r(j) P_j^{r+1} h$. Therefore, h is still underdemanded in Round $r+1$ and $\phi^{r+1}(i) = \phi^r(i) = h$. If $\phi^r(i) = h'$ such that h' is quasi-underdemanded in Round r, then there exists $j' \in I$ such that $h' P_{j'}^r \phi^r(j')$ and for such a j', $\phi^r(j')$ is underdemanded in Round r. In Round $r+1$, $\phi^r(j')$ is upgraded to the top of $P_{j'}^{r+1}$. This suggests $\phi^r(j') P_{j'}^{r+1} h'$. Therefore, h' becomes underdemanded in Round $r+1$ and $\phi^{r+1}(i) = \phi^r(i) = h'$. □

In words, once a student is assigned an underdemanded house or a quasi-underdemanded house in a round, she is going to be assigned the same house in the following round. We will show such a student's consent can be used to recover efficiency loss.

Here “consent" means consent to priority violations. Next we show that the efficiency-adjusted NH4 algorithm actually proceeds along a monotonic improving path. That is, the allocation determined in each round weakly Pareto dominates the allocation determined in the previous round.

Lemma 6. *Given a house allocation problem (f,P) and apply the efficiency-adjusted NH4 mechanism to it. The allocation obtained at the end of Round r such that $r \geq 1$ assigns each student a house that is at least as good for her as the house she was assigned at the end of Round $r-1$.*

Proof. First of all, we note that all squatters in Round $r-1$ cannot be worse off in Round r according to the individual rationality of the NH4 mechanism. Then We only need to examine the students who are assigned $h \in H \backslash SQ^{r-1}$.

Suppose that there are a problem (f,P), a round r such that $r \geq 1$, of the efficiency-adjusted NH4 algorithm, and a student $i_1 \in H \backslash SQ^{r-1}$ such that the house student i_1 obtains in Round r is worse for her than the house h_1^{r-1} she obtained in Round $r-1$. According to non-wastefulness, h_1^{r-1} must have been assigned to another student in Round r. This means when we run the NH4 algorithm in Round r, there is a student $i_2 \in I \backslash \{i_1\}$ who is assigned house h_1^{r-1} in Round r and who was assigned house h_2^{r-1} in Round $r-1$ that is better for her than h_1^{r-1} (Otherwise i_2 would have applied to and obtained h_1^{r-1} in Round $r-1$). Similarly, this means there is a student $i_3 \in I \backslash \{i_1, i_2\}$ who is assigned house h_2^{r-1} in Round r and who was assigned house h_3^{r-1} in Round $r-1$ that is better for her than h_2^{r-1}, and so on. Thus, there must be a student $i_k \in I \backslash \{i_1, i_2, ..., i_{k-1}\}$ such that $k \geq 2$ who is the first student to apply to and obtain a house h_{k-1}^{r-1} that is worse for her than the house h_k^{r-1} she was assigned in Round $r-1$.

We first note that it cannot be the case that $h_k^{r-1} = \mu_0(i_k)$. Otherwise i_k can obtain this house whenever she wants to.

Let UD^{r-1} denote the set of underdemanded houses and quasi-underdemanded houses in Round $r-1$, and let the allocation obtained in this round be ϕ^{r-1}.

When $\phi^{r-1}(i_k) = h_k^{r-1} \in UD^{r-1}$, it should be $\phi^{r-1}(i_k) = \phi^r(i_k) = h_k^{r-1}$ according to Observation 1. This contradicts the assumption that $\phi^r(i_k) = h_{k-1}^{r-1}$.

When $\phi^{r-1}(i_k) = h_k^{r-1} \notin UD^{r-1}$, since $h_k^{r-1} P_{i_k}^r h_{k-1}^{r-1}$, we know that h_k^{r-1} remains on $P_{i_k}^r$ in Round r. Moreover, due to non-wastefulness, h_k^{r-1} must have been assigned to $j' \in I$ such that (1) $f^{-1}(j') < f^{-1}(i_k)$ or (2) $f^{-1}(i_k) < f^{-1}(j')$ and $\mu_0(j') = h_k^{r-1}$. If it is case (1), then j' must prefer $\phi^{r-1}(j')$ to h_k^{r-1} in Round $r-1$ since $\phi^{r-1}(j') \neq h_k^{r-1}$. Therefore, j' applies to and obtains a worse house in Round r. If it is case (2), then there should have been a squatting conflict between j' and i_k or between j' and a student who has a higher priority than i_k does. Since j' did not choose to squat in Round $r-1$, j' applies to and obtains a worse house in Round r. Either case contradicts the assumption that i_k is the first student who applies to and obtains a house in Round r that is worse than her house in Round $r-1$. □

When all students consent, we show that the efficiency-adjusted NH4 mechanism always selects a Pareto efficient allocation.

Theorem 3. *The efficiency-adjusted NH4 mechanism is Pareto efficient when all students consent.*

Proof. Let Round K be the terminal round of the efficiency-adjusted NH4 algorithm and denote the eventual allocation be $\phi = \phi_K$. Suppose there exists $\mu \in \mathscr{A}$ such that μ Pareto dominates ϕ. Then μ Pareto dominates $\phi_1, \phi_2, ..., \phi_{K-1}$ according to Lemma 6. In Round K, there is again an improvement cycle $(i_1, i_2, ..., i_k)$ upon ϕ such that $k \geq 2$ and each student prefers and obtains the next student's house (for student i_k it is i_1's). Also, none of these students' houses belongs to UD^k such that $0 \leq k \leq K$ according to Proposition 5.

In this cycle, let us pick up three students $(i_{k'-1}, i_{k'}, i_{k'+1})$ in the order such that $2 \leq k' \leq k-1$. When it comes to student i_k's turn, $\phi(i_{k'+1})$ must have already been assigned to student $i_{k'+1}$ such that $f^{-1}(i_{k'+1}) < f^{-1}(i_{k'})$. On the other hand, since $i_{k'-1}$ prefers $\phi(i'_k)$, it must be $f^{-1}(i_{k'}) < f^{-1}(i_{k'-1})$. Otherwise $\phi(i_{k'})$ should have been re-allocated from $i_{k'-1}$ to $i_{k'}$ through a squatting conflict and this cannot be the terminal round. Therefore, we have $f^{-1}(i_{k'+1}) < f^{-1}(i_{k'}) < f^{-1}(i_{k'-1})$. This indicates that the priority ordering for students in the improvement cycle $(i_1, i_2, ..., i_k)$ should be $f^{-1}(i_k) < ... < f^{-1}(i_2) < f^{-1}(i_1)$. Since i_k prefers $\phi(i_1)$ to $\phi(i_k)$, a squatting conflict occurs between them. This contradicts our assumption that Round K is the terminal round. □

Finally, we show that whether a student consents or not does not affect her own assignment.

Theorem 4. *Under the efficiency-adjusted NH4 mechanism, the assignment of any student does not change whether she consents or not.*

Proof. This result can be obtained directly from the following two observations. (1) A student's consent will be used only if her assignment is Pareto unimprovable. (2) The consent of each student affects only other students' assignments. □

We note that this result is important for a student's motivation of consenting. Only when consenting is not going to make her worse off, there is a possibility for her to do so. Moreover, we note that whether a student obtains an underdemanded house (or a quasi-underdemanded house) can be observed only after the algorithm starts.

In practice, "consent" can be implemented by asking each consenting student to sign an agreement before entering the allocation program. Once necessary, her consent will be utilized to improve the efficiency of the eventual allocation.

5.3.3 An alternative proof

Next we provide an althernative approach to demonstrate the Pareto efficiency of the efficiency-adjusted NH4 mechanism.

We first have the following observation.

Observation 2. *Given a house allocation problem* (f, P) *and run the NH4 algorithm for it. If no squatting conflict occurs, then the NH4 mechanism is equivalent to the simple serial dictatorship mechanism induced by the priority ordering* f.[1]

Suppose that ϕ is not Pareto efficient, and we want to do Pareto improvement upon it. Then according to Proposition 5, such an improvement can not involve any student i with $\phi(i) \in UD^r \subseteq UD$. Recall Observation 1. It suggests that she will end up obtaining the assignment $\phi(i)$ such that $\phi(i) = \phi^R(i) = \phi^{R-1} = ... = \phi^r(i)$. In other words, Pareto improvements upon ϕ are possible through only the allocation of OD.

Therefore, based on the original house allocation problem (f, P), we are able to construct a sub-problem $(\tilde{H}, \tilde{I}, \tilde{f}, \tilde{P})$ which satisfies:

- $\tilde{H} = OD$,
- $\tilde{I} = \{i \in I | \phi(i) \in OD\}$,
- $\tilde{f}$ is a bijection $\tilde{f} : \{1, 2, ..., |\tilde{I}|\} \to \tilde{I}$ such that for any $i, j \in \tilde{I}$ with $i \neq j$, $\tilde{f}^{-1}(i) < \tilde{f}^{-1}(j)$ if and only if $f^{-1}(i) < f^{-1}(j)$,
- $\tilde{P}_i = P_i$ for all $i \in \tilde{I}$.

In words, this problem consists of the overdemanded houses and the students who are assigned these houses under ϕ in the original problem. In the meantime, these students' preferences and the priority ordering among them do not change.

Apply the efficiency-adjusted NH4 mechanism to $(\tilde{H}, \tilde{I}, \tilde{f}, \tilde{P})$. We note that it is actually a replica of the terminal round of the original problem (f, P) with the assignments of UD excluded. Moreover, it is equivalent to a simple serial dictatorship mechansim induced by the priority ordering $\tilde{f}$. Denote its outcome be $\psi^{\tilde{f}}$.

Immediately, some of the results achieved in the previous research are applicable. Let $\mathscr{F}'$ be the class of all bijections like $\tilde{f}$. Let $\mathscr{A}'$ be the set of all allocations, and then define $\Psi^{\mathscr{F}'} = \{\alpha \in \mathscr{A}' | \ \psi^{\tilde{f}} = \alpha \text{ for some } \tilde{f} \in \mathscr{F}'\}$. It is the set of allocations determined by the simple serial dictatorship mechanism induced by the priority ordering $\tilde{f}$.

Lemma 7. *(Abdulkadiroğlu and Sönmez [1])* $\Psi^{\mathscr{F}'} \subseteq \mathscr{E}$.

Here $\mathscr{E}$ is the set of Pareto efficient allocations in this problem. Lemma 7 suggests that the allocation of OD is Pareto efficient under the efficiency-ajusted NH4 mechanism since it is equivalent to the induced simple serial dictatorship mechanism.

[1] One may refer to Abdulkadiroğlu and Sönmez [1] for a comprehensive introduction of the random serial dictatorship mechanism and its induced simple serial dictatorship mechanism.

According to the previous results, we are able to show that the efficiency-adjusted NH4 mechanism always selects a Pareto efficient allocation when all students consent.

Theorem 5. *The efficiency-adjusted NH4 mechanism is Pareto efficient when all students consent.*

Proof. Given a house allocation problem (f, P) and apply the efficiency-adjusted NH4 mechanism to it. Then define the sub-problem $(\tilde{H}, \tilde{I}, \tilde{f}, \tilde{P})$. Since $\tilde{f} \in \mathscr{F}'$, we obtain $\psi^{\tilde{f}} \in \Psi^{\tilde{\mathscr{F}}} \subseteq \mathscr{E}$. Then according to Proposition 5 and Observation 1, we know that none of the houses in $UD = H \backslash H'$ can be used to do Pareto improvements upon ϕ. Therefore, ϕ is Pareto efficient. □

5.4 Example

In this section, we provide two simple examples to show how the efficiency-adjusted NH4 mechanism works.

Example 5 (All students consent). *Let $I_N = \{i_1, i_2\}$, $I_E = \{i_3, i_4, i_5\}$, $H_V = \{h_1, h_2\}$ and $H_O = \{h_3, h_4, h_5\}$. Each existing tenant i_k occupies the house h_k for $k = \{3,4,5\}$. Let the priority ordering f be $i_1 - i_2 - i_3 - i_4 - i_5$. The following table lists the preferences of students. Suppose all students consent.*

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	P_{i_5}
h_5	h_4	h_5	h_3	h_4
h_4	h_3	h_4	h_4	h_3
h_2	h_5	h_3	:	h_5
h_1	h_2	:	:	:
h_3	h_1	:	:	:

Round (0). *Run the standard NH4 algorithm.*

First student i_1 is tentatively assigned h_5, next student i_2 is tentatively assigned h_4, next student i_3 is tentatively assigned h_3, and next is student i_4's turn and a squatting conflict occurs. The conflicting agent is i_2 who currently owns h_4, and the squatter is i_4 who was initially assigned h_4. i_3's tentative assignment is erased. i_4 is assigned her current house h_4 and removed from the process. This resolves the squatting conflict. Resolve later squatting conflicts in this way and we can finally obtain allocation $\phi^0 = (h_2, h_1, h_3, h_4, h_5)$.

We

that has been assigned to only one student throughout Round 0 and $\phi^0(i_2) = h_1$. Since i_2 consents, modify h_1 to the top of her preference list and go to the next round.

Round (1)**.** *The modified preferences are listed below. Rerun the standard NH4 algorithm.*

$P^1_{i_1}$	$P^1_{i_2}$	$P^1_{i_3}$	$P^1_{i_4}$	$P^1_{i_5}$
h_5	h_1	h_5	h_3	h_4
h_4	:	h_4	h_4	h_3
h_2	:	h_3	:	h_5
h_1	:	:	:	:
h_3	:	:	:	:

We still obtain allocation $\phi^1 = (h_2, h_1, h_3, h_4, h_5)$ through the standard NH4 algorithm. Moreover, we need to proceed to the next round since squatting conflicts occurred. Besides h_1, h_2 is the house that has been assigned to only one student throughout Round 1 and $\phi^1(i_1) = h_2$. Since i_1 consents, modify h_2 to the top of her preference list and go to the next round.

Round (2)**.** *The modified preferences are listed below. Rerun the standard NH4 algorithm.*

$P^2_{i_1}$	$P^2_{i_2}$	$P^2_{i_3}$	$P^2_{i_4}$	$P^2_{i_5}$
h_2	h_1	h_5	h_3	h_4
:	:	h_4	h_4	h_3
:	:	h_3	:	h_5
:	:	:	:	:
:	:	:	:	:

We obtain allocation $\phi^2 = (h_2, h_1, h_5, h_3, h_4)$ through the standard NH4 algorithm. Since no squatting conflict occurred, we terminate the process and obtain allocation $\phi = \phi^2 = (h_2, h_1, h_5, h_3, h_4)$

Obviously, $\phi = (h_2, h_1, h_5, h_3, h_4)$ Pareto dominates the standard NH4 allocation $\varphi = (h_2, h_1, h_3, h_4, h_5)$. Moreover, we note that i_1's assignment and i_2's assignment are not affected by their consenting behavior.

Example 6 (Not all students consent)**.** *Let us consider Example 5 again. Suppose student i_1 does not consent and student i_2, i_3, i_4, i_5 consent.*

Round (0)**.** *The same as in Example 5.*

Round (1)**.** *The same as in Example 5.*

Round 0 and Round 1 are exactly the same as in Example 5. But since i_1 does not consent, before modifying h_2 to the top of i_1's preference list, we need to remove any h such that $hP^1_{i_1}h_2$ from i_2's, i_3's, i_4's and i_5's preference lists if they also prefer this house. For example, since $h_5P^1_{i_3}h_3$ and $h_4P^1_{i_3}h_3$, we need to remove h_5 and h_4 from $P^1_{i_3}$ to obtain $P^2_{i_3}$. We treat $P^1_{i_4}$ and $P^1_{i_5}$ in the same way.

Round (2)**.** *The modified preferences are listed below. Rerun the standard NH4 algorithm.*

$P^2_{i_1}$	$P^2_{i_2}$	$P^2_{i_3}$	$P^2_{i_4}$	$P^2_{i_5}$
h_2	h_1	h_3	h_3	h_3
:	:	:	h_4	h_5
:	:	:	:	:
:	:	:	:	:
:	:	:	:	:

We obtain allocation $\phi^2 = (h_2, h_1, h_3, h_4, h_5)$ without encountering any squatting conflict. Therefore, we terminate the algorithm and obtain $\phi = \phi^2 = (h_2, h_1, h_3, h_4, h_5)$. This result is equivalent to φ. We cannot make Pareto improvement since a crucial student does not accept any priority violation.

5.5 Summary

This chapter investigates a house allocation problem with both existing tenants and new applicants. When designing the allocation mechanism, we first need to solve the thickness problem which means the market should be able to attract a sufficiently large number of participants. Particularly, we shall guarantee the participation rate of existing tenants. Otherwise the pool of available houses will be quite small, which will make it hard to improve the overall welfare due to the shortage of housing resource. Besides the thickness problem, we need to think about what kind of allocation we would like to achieve. If an allocation is not Pareto efficient, then there exist students who are able to receive a better house without making other students worse. Then some students may exchange houses among themselves, and a student whose house is preferred by multiple students may receive multiple requests for exchange. Such circumstance is a symptom of the congestion failure.

We provide a solution by proposing an efficiency-adjusted NH4 mechanism. The NH4 mechanism is a real-world allocation mechanism which proceeds by iteratively resolving squatting conflicts between two students. It is an appealing mechanism since it satisfies individual rationality, fairness and strategy-proofness. But it fails Pareto efficiency. Based on these observations, we propose an efficiency-adjusted NH4 algorithm that allows each student to consent a certain priority violation that has no effect on her own assignment. Such consent can help recover efficiency loss which is caused by resolving squatting conflicts. Moreover, we show that only the consents granted by students who are assigned underdemanded houses or quasi-underdemanded houses will be helpful.

Chapter 6

Conclusions and future work

6.1 Summary of contributions

In this research, we have extensively studied many-to-one matching markets and related problems under complex assumptions. We have contributed to fixing market failures by proposing solutions in three aspects which are stability concept, equilibrium analysis and algorithmic solution.

- In our first research, we study a labor market with externalities. Once externalities are present, stable matching under the standard definition easily fails to exist. In an unstable matching, the firms whose workers resigned and the workers who got fired have to seek new matchings and congestion will consequently occur. Our work contributes to solving this failure by proposing a new stability concept and showing its existence under two restrictions on firms' preferences. Particularly, our stability concept depends on a notion called threshold. It enhances the adaptability of estimation function which is proposed in the previous research and is used to formulate externalities. Moreover, our stability concept remains compatible with two classical stability concepts as they turn out to be special cases of ours.

- In our second research, we analyze a preference revelation game which is defined on a school choice market. In this market, students are able to form coalitions and then cooperatively determine which preferences to report with the purpose of achieving better outcomes. Therefore, a mechanism which is not immune to such coalitional behavior will force students to engage in costly and risky strategic behavior. Our work refines the current equilibrium concept by taking the credibility of each agent's behavior into account. Even under the strongest equilibrium concept that we currently have, there may exist multiple equilibria. Therefore, our result delivers new ideas for selecting a certain outcome among them. Moreover, our result enriches the theoretical framework of the equilibrium analysis of this game.

- In our last research, we investigate a house allocation problem with existing tenants. The NH4 mechanism is a popular real-world mechanism which solves this house allocation problem, but it suffers from a flaw that it easily results in Pareto inefficient allocations. If an allocation is not Pareto efficient, then some students will be able to receive a better house without making other students worse. Congestion will occur since some students may exchange houses among themselves, and a student whose house is preferred by multiple students may receive multiple requests for exchange. In order to solve this problem, we propose an efficiency-adjusted NH4 mechanism. By applying this mechanism, Pareto efficiency is able to be achieved with violating the other desirable properties at a minimal level. This work is also useful to deal with the efficiency problem of other DA-like algorithms.

In conclusion, our work has enhanced the adaptability of Gale and Shapley's classical two-sided matching model in complex problems and has contributed to fixing market failures caused by them.

6.2 Future work

In the future research, we plan to continue enhancing the adaptability of Gale and Shapley's classical model in many-to-one matching markets in the three aspects that we have focused on. Specifically,

- In the study of stability concept, we hope to apply the estimation function approach to interpret another type of externality. That is, each worker cares about with whom he or she works. We hope to propose a new stability concept which incorporates these externalities and then examine its existence problem.

- In the study of equilibrium analysis, we hope to explore other equilibrium concepts since *P-SSN* is a quite restrictive one which easily fails to exist. We are considering the following two extensions: (i) Students who have threatened others to join the deviating coalition shall not become worse off even if they deviate unilaterally. That is, their threatening strategy will not hurt themselves back even if the target students do not cooperate; (ii) each student who deviates passively should receive a strictly better outcome than that caused by the rest students' unilateral deviation.

- In the study of algorithmic solution, we hope to examine an important property called strategy-proofness for DA and other DA-like algorithms. If this property is satisfied, then reporting truthful preferences will become a dominate strategy for agents who are able to freely report their preferences. Although Guillen and Kesten [16] have demonstrated that no mechanism can realize individual rationality, Pareto efficiency, fairness and strategy-proofness at the same time, we have the option to explore mechanisms which give strategy-proofness the highest priority and achieve other properties with acceptable restrictions. Moreover, collaboration with researchers in other disciplines to design algorithms which can be easily coded and understood is also expected.

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Appendix

Recent developments of market design

As we have mentioned in the first two chapters, market design has attracted lots of research interest since 1980s, which means it has been developing for almost four decades. In fact, the labor market, the school choice market and the house allocation problem that we have extensively studied in this research are among the classical models of this theory. Nowadays, more and more markets and related problems are benefiting from this theory under the joint effort of market designers including economists, operations researchers, computer scientists etc. Below we summarize some new developments which are especially supported by market design theory.

Kidney exchange

Kidney transplantation is the treatment of choice for end-stage renal disease. Moreover, medical evidence shows that a transplanted kidney from a live donor survives significantly longer than a deceased kidney. Hence live donation is always the first choice for a patient. A live donor is usually a relative or a friend of a patient, so the donor agrees to donate only if this particular patient is the recipient. However, most patients are unable to receive the donor's kidney since they are biologically incompatible in blood-type or the patient's immune system has already produced antibodies to one of the donor's proteins. On the other hand, even cadaver kidneys are in a grave shortage. Therefore, market designers are being challenged to increase the number of transplants. Particularly, there is an important constraint that monetary incentives are forbidden in this problem.

Generally speaking, three exchange mechanisms were proposed by the previous work. The first one is an exchange between two patient-donor pairs. It applies when the patient in each of two incompatible patient-donor pairs was compatible with the donor in the other pair. The second one is a cycle of incompatible patient-donor pairs of any size such that the donor in the first pair donates a kidney to the patient in the second, the second pair donates to the third and so on until the cycle closed, with the last pair donating to the first. Also, pairs donate a kidney for higher priority on the cadaver waiting list could be integrated with the exchange pool by having them donate to another incompatible pair in a chain that would end with donation to the waiting list. One may refer to Roth et al.[36] for a comprehensive introduction of these two mechanisms. The last one is a cycle which is integrated with altruistic donors who are willing to donate a kidney without requiring another donor kidney for exchange. Specifically, it has the altruistic donor donate to some pair that is willing to exchange a kidney and has that pair donate to someone on the cadaver waiting list. One may refer to Roth et al. [37] for a comprehensive introduction of this mechanism.

Radio spectrum realloction

The development of smartphones, tablets and such sort of new devices is resulting in a drastic increase of mobile broadband traffic. Without additional airwaves to handle the booming demand, consumers will face more dropped calls, connection delays and slower downloads of data. Aims at creating new high-speed Internet networks, the U.S. Federal Communications Commission (FCC) proved a proposal to reclaim public airwaves now used for broadcast television and auction them off for use in wireless broadband networks, with a portion of the proceeds paid to broadcasters. This complex auction consists of four steps: (1) “reverse auction" which buys TV broadcast rights by offering descending prices to sellers; (2) “forward auction" which sells mobile broadband rights by offering ascending prices to buyers; (3) “clearing system" which jointly determines how many channels to clear and licenses to sell in the two auctions; (4) Broadcasters that do not sell can be moved to new channels. A set of bids is acceptable only if the rejected bidders and nonparticipating stations can be “repacked” into the reduced set of TV band. However, each feasibility checking step is NP-hard and the auction requires about 7,5000 such steps.

Milgrom [27] designed a mechanism called “deferred-acceptance (DA) clock auctions" which is closely related with the DA proposed by Gale and Shapley [15]. This mechanism has four advantages. First, if bidders are substitutes, then the DA clock auction can optimize an objective like maximum welfare of minimum cost. Second, not only it is a dominant strategy for a bidder to bid truthfully, but truthful bidding is optimal even if the bidder does not understand the auctioneer’s price reduction rule or does not trust the auctioneer to adhere to the rule. Third, any DA auction can be modified to respect the auctioneer’s budget constraint by adding rounds in which prices continue to fall. Fourth, winning bidders in a DA clock auction reveal only the minimal information about their values needed to prove that they should be winning.

Refugee resettlement

Forced by the escalating conflict in the Middle East during the recent year, those people displaced from countries such as Syria are deemed to be refugees under the mandate of the United Nations High Commissions for Refugees (UNHCR). One of the main destinations for refugees who were seeking asylum is Europe. The drastically increasing scale of refugee arrival consequently caused the European refugee crisis which began in 2015. However, except for the Dubins Regulation which requires that the member state in which an asylum seeker enters first is obligated to render asylum, there has been no systematic ways to relocate refugees among member states. This added great pressure to member states which are located at the external border of EU. In an attempt to reduce pressure on these states, the European commission proposed relocation scheme for refugees. However, it did not specify which refugees should be relocated to which member states. Hence designing more systematic ways to relocate refugees to member states becomes a big task. On the other hand, hosting countries have begun to reconsider the systems they use to register, process, and allocate refugees to local areas.

In general, the refugee resettlement problem involves relocation of refugees at two levels. The first is relocation to EU member states and other areas (such as U.K.). Jones and Teytelboym [21] proposed a two-sided matching system which assigns refugees to member states. The second is relocation to local areas of hosting countries. Jones and Teytelboym [20] described in general terms how a two-sided matching system can be constructed to meet the British government's commitment to resettle 20,000 Syrian refugees by 2020. Delacretaz et al. [11] developed three different refugee resettlement systems that can be used by hosting countries under different circumstances. Unlike Jones and Teytelboym [20] and Delacretaz et al. [11] which mainly aimed at improving matching efficiency and reducing internal movement of refugees across localities, Andersson and Ehlers [6] extensively studied the Swedish case and proposed a mechanism with focuses on stable maximum matchings.