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Doctoral Thesis

Analysis of $\mathcal{N} = 3$ supersymmetric theories by using string junctions

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Abstract

I investigate the supersymmetry enhancement from $\mathcal{N} = 3$ to $\mathcal{N} = 4$ conjectured by Aharony and Tachikawa, which is expected to occur in special class of N=3 theories. In general, $\mathcal{N} = 3$ theories are strongly coupled and Lagrangian description is not known. All known examples of $\mathcal{N} = 3$ theories are constructed by using string theory. I consider the world volume theories on D3-branes in S-fold backgrounds. I study the spectrum of BPS states of $\mathcal{N} = 3$ theories by using string junctions, and I find the agreement of the spectrum with that of $\mathcal{N} = 4$ theories. This fact strongly suggests the supersymmetry enhancement.

There is another construction of $\mathcal{N} = 3$ theories using codimension-2 branes in 6d maximal supergravity. For the purpose of generalizing this construction, I consider codimension-2 brane solutions in various dimensions. I construct codimension-2 brane solutions in 9d, 8d, and 7d maximal supergravities, and I classify preserved supersymmetries on the branes.

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Contents

Chapter 1

Introduction

Strongly coupled quantum field theories often appear and play an important role in investigations of field theories. The strongly coupled field theories are difficult to analyze because perturbation method cannot be used for the theories. For instance, the quantum chromodynamics (QCD) is an asymptotically free theory and its coupling constant becomes large in low energy region. One of the important problems for particle physics is the color confinement problem of QCD. Only hadrons are observable and color-charged particles like as quarks and gluons cannot be found. Lattice QCD also suggests that the color confinement should occur. However, the color confinement has not been confirmed analytically so far. The understanding of strongly coupled quantum field theories is essential for the color confinement.

Although analysis of strongly coupled theories is difficult in realistic model, there has been much progress in supersymmetric field theories. The supersymmetry relates bosonic states and fermionic states, and generators of supersymmetry are spinors. Supersymmetric field theories are classified by the number of irreducible spinor generators conventionally denoted by $\mathcal{N}$. In 4d field theory without gravity, the number is limited to $\mathcal{N} \leq 4$. The supersymmetry imposes constraints on theories and the analysis becomes easier as $\mathcal{N}$ increases.

4d supersymmetric field theories have been studied for a long time. Historically, $\mathcal{N} = 1$ supersymmetric field theory has been studied as a candidate beyond the standard model. Because $\mathcal{N} = 1$ supersymmetry imposes a relatively weak constraint on theories, there are many kinds of 4d $\mathcal{N} = 1$ supersymmetric models. On the other hand, in the case of $\mathcal{N} = 4$, the supersymmetry severely restricts the theory, and we cannot use it for phenomenological models. It is actually believed that $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) is the only theory with $\mathcal{N} = 4$ supersymmetry. However, $\mathcal{N} = 4$ SYM is an interesting toy-model to analyze the strong coupling region because large supersymmetry suppresses quantum corrections. Moreover, the β-function for the Yang-Mills coupling constant vanishes, and the $\mathcal{N}=4$ SYM has superconformal symmetry.

4d $\mathcal{N}=2$ supersymmetry does not impose a strong restriction on a theory enough to fix uniquely the theory, while it enables us to determine low-energy effective action. The construction of the low-energy effective action of $\mathcal{N}=2$ theory was done in 1994 by Seiberg and Witten[1, 2]. After the discovery of the Seiberg-Witten solutions, the knowledge of $\mathcal{N}=2$ supersymmetric field theories has been improved. Especially, it has been recognized the existence of non-Lagrangian theories. Significant examples of the non-Lagrangian theories are Argyres-Douglas (AD) theories[3, 4], which are 4d $\mathcal{N}=2$ superconformal theories (SCFT). A feature of AD theories is that the theory contains electric and magnetic particles simultaneously as light degrees of freedom. In such systems, we cannot write the Lagrangian with local operators. Note that there is no proof that non-Lagrangian theories will never have any Lagrangian descriptions. In fact, there are Lagrangian theories which flow to non-Lagrangian theories at an IR fixed point[5]. In this thesis, we define non-Lagrangian theories as theories which have no tunable parameters that we can use to take weak coupling limit because for such a theory no systematic prescription to write down the Lagrangian is known. We will define the concept of the non-Lagrangian theory in more detail in chapter 2.

Because the non-Lagrangian theories cannot be studied by perturbative methods, we have to use other methods to investigate such theories. In the last few decades, it has been recognized that string theory and dualities are powerful tools to investigate field theories. Especially, it is useful to study strongly coupled field theories, which are difficult to analyze by perturbative methods.

A duality is an equivalence between two theories which are described in different ways. For analysis of strongly coupled theories, dualities, especially S-duality, play an important role. S-duality is a duality which connects a strongly coupled theory with weakly coupled theory. The S-duality holds in 4d $\mathcal{N}=4$ SYM. From the field theory viewpoint, it is difficult to confirm the duality. However, once we accept the S-duality, we can study a strongly coupled field theory by using the dual description that is weakly coupled.

One of the advantages using string theory is it provides us a dual description of a field theory. AdS/CFT correspondence[6] is an example of such a duality. The AdS/CFT correspondence claims that a d-dimensional conformal field theory is equivalent to a theory in $d+1$-dimensional Anti-de Sitter (AdS) space. Although this proposal is highly non-trivial, in the context of string theory, it is convincing because these two descriptions are obtained from the same set up including so-called D-branes. When we consider open strings attached to the D-branes as elementary degrees of freedom, the world-volume theory of the D-branes is a gauge theory[7]. On the other hand, when we regard D-branes as a background geometry of the theory of closed strings, the geometry close to the D-branes is AdS[8]. Thanks to the AdS/CFT correspondence, we can analyze some strongly coupled field theories by using the corresponding weakly coupled gravity theory.

In this thesis we focus on a certain class of non-Lagrangian theories: 4d $\mathcal{N} = 3$ supersymmetric theories. Strictly speaking, $\mathcal{N} = 4$ supersymmetric theories are special cases of $\mathcal{N} = 3$ theories. We would like to consider genuine $\mathcal{N} = 3$ supersymmetric theories which do not have $\mathcal{N} = 4$ supersymmetry. Unlike 4d $\mathcal{N} = 1, 2, 4$ supersymmetric theories, it is known that all genuine $\mathcal{N} = 3$ theories are non-Lagrangian in the sense that we mentioned above. Until recently it had not been clear whether such theories exist. In 2015, García-Etxebarria and Regalado demonstrated that 4d genuine $\mathcal{N} = 3$ theories can be realized by using S-fold background in the framework of string theory[9]. They explicitly showed that one of the $\mathcal{N} = 4$ supercharges is projected out in the S-fold and then genuine $\mathcal{N} = 3$ supersymmetry is realized. We call a theory realized in the S-fold background an S-fold theory. This construction motivated analyses of genuine $\mathcal{N} = 3$ theories. One of the development is that Aharony and Tachikawa suggested that in certain situations the supersymmetry of S-fold theories is enhanced to $\mathcal{N} = 4$[10]. Although the construction of S-fold theories removes one of the $\mathcal{N} = 4$ supercharges, consistency from superconformal algebra requires that in such a situation $\mathcal{N} = 4$ supersymmetry must somehow reappears. Namely, although the theory seemingly has genuine $\mathcal{N} = 3$ by construction, it must be, in fact, an $\mathcal{N} = 4$ theory. This supersymmetry enhancement is important to understand the physics of genuine $\mathcal{N} = 3$ and $\mathcal{N} = 4$ theories. A purpose of this thesis is to investigate the supersymmetry enhancement by using string junctions.

We also study codimension-2 brane solutions in maximal supergravities. It is known that some 4d $\mathcal{N} = 3$ theories are regarded as the world-volume theories of codimension-2 branes[11]. Thus, we expect that construction of new codimension-2 brane solutions gives us some hints for realizations of new non-Lagrangian theories.

The goal of this doctoral thesis is to study non-Lagrangian field theories, especially 4d $\mathcal{N} = 3$ theories by using string and M-theory. We review some basic elements of superstring theories and relations between string theory and supersymmetric field theories in the remaining part of this chapter.

In Chapter 2, we review 4d superconformal field theories with focusing on the superconformal algebra and their representations. By using superconformal symmetry, we can investigate SCFTs even if we do not know the Lagrangians of the theories. We also derive some basic properties of $\mathcal{N} = 3$ SCFTs by using the algebra.

In Chapter 3, we investigate the supersymmetry enhancement proposed by Aharony and Tachikawa[10]. First, we review the construction of 4d $\mathcal{N} = 3$ theories given by García-Etxebarria and Regalado[9]. The $\mathcal{N} = 3$ supersymmetry is realized on D3-branes in the special background geometry which is called S-fold. After that, we use string junctions in S-fold in order to check the consistency of the spectrum to the supersymmetry enhancement. This chapter is based on [12].

In Chapter 4, after giving a short review of codimension-2 brane construction of 4d $\mathcal{N} = 3$ theories[11], we construct codimension-2 brane solutions in 9d, 8d and 7d maximal supergravity theories. This chapter is based on [13].

Chapter 5 is devoted to the conclusions and discussions.

We note some notations used in this thesis in Appendix.

1.1 String theory and M-theory

It is known that there are five different anomaly free superstring theories: type I, type IIA, type IIB, heterotic $SO(32)$, and heterotic $E_8 \times E_8$. These theories are related to each other by string dualities.

In this thesis we use only type IIA and type IIB theories. We first review these theories.

1.1.1 Type IIA and Type IIB theories

The superstring theory is the theory of strings propagating in ten-dimensional Minkowski spacetime. In the spacetime, a string sweeps two-dimensional surface called the world-sheet. Let $\sigma^i (i = 0, 1)$ be the coordinates on the world-sheet. (The σ^0 and σ^1 are the temporal and the spacial coordinates, respectively.) The motion of a string is specified by giving 10d coordinates $X^\mu(\sigma)$ ($\mu = 0, 1, \ldots, 9$) as functions of σ^i. Mathematically, $X^\mu(\sigma)$ give a map from the 2d Minkowski space into the 10d target space.

There are two types of strings: open strings and closed strings. In both cases the length of a string is finite, and we normalize σ^1 so that $\sigma^1 \in [0, \pi]$. In addition to $X^\mu(\sigma)$, there are fermionic fields $\psi^\mu(\sigma)$ which are the superpartners of X^μ. The bosonic part of the string theory action is given by

$$S_{\rm str} = -\frac{1}{4\pi l_s^2} \int d^2\sigma \sqrt{-h} h^{ij} \partial_i X^\mu \partial_j X_\mu. \tag{1.1.1}$$

The parameter l_s is called string length which is the unique parameter in string theories. The metric $h_{ij}(\sigma)$ describes the geometry of the world-sheet.

As we mentioned above, there are two kinds of strings: open strings and closed strings. For a closed string, we impose the periodic boundary condition $X^\mu(\sigma+\pi) = X^\mu(\sigma)$. For an open string, we have to specify boundary conditions for X^μ at the two endpoints. There are two kinds of boundary conditions:

$$X^\mu|_{\sigma^1=0,\pi} = \text{const} : \text{Dirichlet boundary condition} \tag{1.1.2}$$

$$\frac{\partial}{\partial\sigma^1} X^\mu|_{\sigma^1=0,\pi} = 0 : \text{Neumann boundary condition} \tag{1.1.3}$$

One of these boundary conditions should be chosen for each μ separately. When the Dirichlet boundary condition $X^I(\sigma^1 = 0,\pi) = L^I(\text{const})$ is imposed on the $9-p$ bosonic fields X^I $(I = p+1,\ldots,9)$ and the Neumann boundary condition $\frac{\partial}{\partial\sigma^1}X^\alpha(\sigma^1 = 0,\pi) = 0$ is imposed on the other bosonic fields X^α $(\alpha = 0,\ldots,p)$, the string endpoints can only move on the p-dimensional hypersurface

$$x^I = L^I \quad (I = 9-p,\ldots,9). \tag{1.1.4}$$

We call such a hypersurface Dp-brane.

For the fermionic fields we should impose appropriate boundary conditions. We do not explain them in detail. We only note that the choice of boundary conditions makes a difference in string spectra, and these spectra are classified into two different sectors: Ramond (R) sector and Neveu-Schwarz (NS) sector.

By quantizing fields on the world-sheet, we obtain a finite number of massless particles as well as the infinite tower of massive ones. The masses are proportional to the string tension $T_s = \frac{1}{2\pi l_s^2}$. When we consider the low energy effective theory of the string theory, only massless spectrum survives, and in the following, we omit the massive spectrum. The massless spectrum of a closed string contains spin-2 graviton. Thus, the low energy effective theories of type II string theories are supergravity theories.

The massless bosonic fields appearing in type IIA and type IIB theories are listed in Table.1.1. These fields are classified into NS-NS sector and R-R sector [1]. NS-

Table 1.1: The field contents in type IIA and type IIB

	NS-NS	R-R
Type IIA	$g_{\mu\nu}, \phi, B_{\mu\nu}$	$C_\mu, C_{\mu\nu\rho}$
Type IIB	$g_{\mu\nu}, \phi, B_{\mu\nu}$	$C, C_{\mu\nu}, C_{\mu\nu\rho\sigma}$

[1]A closed string motion can be separated by left-moving mode and right-moving mode. The boundary condition can be imposed on left and right-moving mode independently to each other.Thus a closed superstring excitation mode has four sectors, NS-NS, NS-R, R-NS, and R-R. The NS-R and R-NS sectors contain fermionic fields in ten dimensions.

NS fields are common in type IIA and type IIB and contains the graviton $g_{\mu\nu}$, the dilaton ϕ, and the anti-symmetric tensor field $B_{\mu\nu}$ that we sometimes call B-field. The contents of the R-R sector is different between type IIA and type IIB theories. Type IIA theory has the 1-form and the 3-form fields and type IIB theory has the scalar, the 2-form, and the 4-form fields. The B-field and the R-R fields are gauge potentials coupling to different charged objects.

In general, an electric particle coupling to a vector gauge field by $e\int A_1$. This is generalized to the following coupling between an n-form field A_n and an $(n-1)$-brane.

$$S_{\text{ele}} \propto \int_{\text{world}-\text{vol}} A_n. \tag{1.1.5}$$

We call branes which are coupled by A_n through the action (1.1.5) electrically charged objects. We also consider magnetically charged objects. They couple to a dual gauge field $\widetilde{A}_{D-n-2}$, which is defined by $d\widetilde{A}_{D-n-2} = *F_{n+1} = *dA_n$. The magnetic charge for A_n can also be defined as the integral of the flux $F_{n+1} = dA_n$:

$$\mathcal{Q}_{\text{mag}} = \oint F_{n+1}, \tag{1.1.6}$$

where the integral is taken over S^{n+1} surrounding the object. To summarize the electric and magnetic objects for A_n are

$$\begin{aligned} \text{electric}: &\ (n-1)\text{-brane}, \\ \text{magnetic}: &\ (D-n-3)\text{-brane}. \end{aligned}$$

The B-field electrically couples to string world-sheet by

$$S_{B_2} = \int B_2. \tag{1.1.7}$$

The integral is taken over the string world-sheet. Magnetically charged objects for B-field are spatially five-dimensional objects called NS5-branes.

Polchinski [14] showed that D-branes are charged objects for the R-R fields. A Dp-brane couples to the R-R $(p+1)$-form field by

$$S_{C_{p+1}} = \mu_p \int_{p+1} C_{p+1}, \tag{1.1.8}$$

where the integral is taken over the world-volume of the Dp-brane. Thus type IIA theory contains Dp-branes with even p and type IIB contains Dp-branes with odd p. Dp-branes and the R-R fields coupling to the branes are listed in Table 1.2. The

Table 1.2: R-R fields and electric and magnetic charged D-branes for the fields

	R-R fields	(ele, mag)
Type IIA	C_1	(D0, D6)
	C_3	(D2, D4)
Type IIB	C_0	(D(−1), D7)
	C_2	(D1, D5)
	C_4	(D3, D3)

Dp-brane charge μ_p is given by

$$\mu_p = \frac{2\pi}{(2\pi l_s)^{p+1}}. \tag{1.1.9}$$

The massless spectrum of an open string contains spin-1 gauge fields and scalar fields. The scalar fields can be regarded as fluctuation modes of the D-branes. Due to these scalar fields, when there are D-branes in the string theories, a gauge theory is realized on the D-branes.

Dp-branes are dynamical objects although Dp-branes are introduced as hypersurfaces defined by boundary conditions for open strings. The motion of a Dp-brane is specified by dynamical scalar fields $X^\mu(\xi^\alpha)$ which is a map from $(p+1)$-dimensional world-volume with coordinate ξ^α to ten-dimensional target space. The action of a Dp-brane can be divided into two parts as

$$S_{\mathrm{D}p} = S_{\mathrm{DBI}} + S_{\mathrm{CS}}. \tag{1.1.10}$$

S_{DBI} is the Dirac-Born-Infeld(DBI) action

$$S_{\mathrm{DBI}} = -\mu_p \int d^{p+1}\xi e^{-\phi}\sqrt{-\det(G_{\alpha\beta} + B_{\alpha\beta} + 2\pi l_s^2 F_{\alpha\beta})}, \tag{1.1.11}$$

that gives the coupling of a Dp-brane to the NS-NS fields in the bulk.

$G_{\alpha\beta}, B_{\alpha\beta}$ and ϕ in (1.1.11) are pull back of the ten dimensional metric, the B-field and the dilaton. $F_{\alpha\beta}$ is the field strength of the gauge field A_α which appears as open string massless modes. The Dp-brane charge μ_p is related to the tention T_p by

$$T_p = \frac{\mu_p}{g_s}, \quad g_s = e^{<\phi>} \tag{1.1.12}$$

where g_s is the coupling constant of type II theories which is determined by the vacuum expectation value of the dilaton ϕ.

$S_{\rm CS}$ contains the Chern-Simons(CS) type terms. The Chern-Simons type terms can be written by

$$S_{\rm CS} = \mu_p \int \left[e^{B_2 + 2\pi l_s^2 F_2} \wedge \sum_n C_n \right]_{p+1}, \tag{1.1.13}$$

which describe couplings of Dp-branes with the R-R fields and the B-field. The square bracket $[\quad]_{p+1}$ means we pick up only $(p+1)$-form.

1.1.2 Dualities

T-duality

T-duality is the equivalence between two compactified string theories. Let us consider type IIA theory with x^9 direction being compactified on a circle of radius R and type IIB theory compactified on a circle of radius $\widetilde{R} = l_s^2/R$. We denote these circles of compactified directions as $S^1(R)$ and $S^1(\widetilde{R})$. These two theories have the same energy spectra of strings. First, we consider the closed string spectra. On the type IIA side, the momentum along x^9 direction p is quantized by

$$p = \frac{m}{R} \quad (m \in \mathbb{Z}), \tag{1.1.14}$$

due to the compactification. In the compactified background there are winding closed strings around S^1. For a string with winding number n, the mass is given by

$$M^2 = \left(\frac{m}{R}\right)^2 + \left(\frac{nR}{l_s^2}\right)^2 + (\text{oscillation}) \quad (m, n \in \mathbb{Z}), \tag{1.1.15}$$

where the 1st, 2nd, and 3rd terms on the right hand side are the contribution of the momentum (1.1.14), the winding, and the internal oscillation of the string. Similarly, on the type IIB side,

$$M^2 = \left(\frac{m'}{\widetilde{R}}\right)^2 + \left(\frac{n'\widetilde{R}}{l_s^2}\right)^2 + (\text{oscillation}) \quad (m', n' \in \mathbb{Z}), \tag{1.1.16}$$

These two mass spectra are the same under the following identification:

$$\widetilde{R} = \frac{l_s^2}{R}, \quad (m, n) = (n', m'). \tag{1.1.17}$$

This agreement of the spectra strongly suggests the equivalence of the two systems, and actually it is known that the equivalence holds not only for the spectra but also for the interactions.

The T-duality holds even if D-branes exist. T-duality relates a Dp-brane wrapped on S^1 on one system and a D$(p-1)$-brane that is not wrapped on S^1 on the other system. As a simple check let us compare the spectra of open strings attached to the D-branes.

For concreteness, we consider a type IIA Dp-brane wrapped on $S^1(R)$ and a type IIB D$(p-1)$-brane that is not wrapped on $S^1(\widetilde{R})$. On the type IIA side, an open string can propagate along x^9 with quantized momentum $p = m/R$, and its mass is given by

$$M^2 = \left(\frac{m}{R}\right)^2 + (\text{oscillation}) \quad (m \in \mathbb{Z}). \tag{1.1.18}$$

On the type IIB side, a string ending on the unwrapped D-brane can wind the $S^1(\widetilde{R})$ and its mass is given by

$$M^2 = \left(\frac{n'\widetilde{R}}{l_s^2}\right)^2 + (\text{oscillation}) \quad (n' \in \mathbb{Z}). \tag{1.1.19}$$

These two spectra are the same under the identification

$$\widetilde{R} = \frac{l_s^2}{R}, \quad m = n'. \tag{1.1.20}$$

S-duality

Another important duality in string theory is the S-duality. The S-duality is a strong-weak duality between type IIB with the coupling g_s and that with $\widetilde{g}_s = 1/g_s$. Because the string coupling g_s is not a parameter of the theory but a vacuum expectation value of a dilaton ϕ, the S-duality is a $\mathbb{Z}_2$ symmetry in type IIB theory. This is extended to $SL(2,\mathbb{Z})$ symmetry which is also called S-duality.

The action of type IIB supergravity[2] is

$$\begin{aligned} S =& \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g}\left(R - \frac{\partial_\mu\bar{\tau}\partial^\mu\tau}{2(\mathrm{Im}\,\tau)^2} - \frac{1}{2}\mathcal{M}_{IJ}|F_3^I \cdot F_3^J| - \frac{1}{4}|\widetilde{F}_5|^2\right) \\ &- \frac{1}{8\kappa_{10}^2}\varepsilon_{IJ}\int C_4 \wedge F_3^I \wedge F_3^J. \end{aligned} \tag{1.1.21}$$

[2]We consider only massless fields and decouple a gauge field. Thus this theory is a type IIB supergravity.

We combine two 2-form fields B_2 and C_2 into C_2^I as

$$C_2^I = \begin{pmatrix} B_2 \\ C_2 \end{pmatrix}, \quad F_3^I = dC_2^I. \tag{1.1.22}$$

$\mathcal{M}_{IJ}$ is the 2×2 matrix defined from the scalar fields C and ϕ by

$$\mathcal{M}_{IJ} = \frac{1}{\operatorname{Im}\tau} \begin{pmatrix} |\tau|^2 & -\operatorname{Re}\tau \\ -\operatorname{Re}\tau & 1 \end{pmatrix}, \quad \tau = C + ie^{-\phi}. \tag{1.1.23}$$

The field strength $\widetilde{F}_5$ of the 4-form field C_4 is defined by

$$\widetilde{F}_5 = dC_4 + \frac{1}{2}\varepsilon_{IJ} C_2^I \wedge F_3^J, \tag{1.1.24}$$

and satisfies the self-dual condition

$$\widetilde{F}_5 = *\widetilde{F}_5. \tag{1.1.25}$$

The action (1.1.21) is invariant under the following $SL(2,\mathbb{R})$ transformations[3]

$$C_2^I \to C_2'^I = \Lambda^I{}_J C_2^J, \quad \Lambda^I{}_J = \begin{pmatrix} d & c \\ b & a \end{pmatrix} \in SL(2,\mathbb{R}), \tag{1.1.26}$$

$$C_4 \to C_4' = C_4, \tag{1.1.27}$$

$$g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu}, \tag{1.1.28}$$

$$\tau \to \tau' = \frac{a\tau + b}{c\tau + d}, \tag{1.1.29}$$

$$\mathcal{M} \to \mathcal{M}' = (\Lambda^{-1})^T \mathcal{M} \Lambda^{-1}. \tag{1.1.30}$$

The transformation of gauge fields C_2^I requires the transformation of the corresponding electric charges $q_I = (q_{F1}, q_{D1})$, and the magnetic charges $m^I = (m_{NS5}, m_{D5})$.

Fundamental strings (F1-branes) and D1-branes are electrically charged objects for B_2 and C_2. We also have bound states of two kinds of strings. A one-dimensional object having a charge q_I is regarded as a bound state of q_{F1} F1s and q_{D1} D1-branes. We call this object (q_{F1}, q_{D1})-string. The coupling of (q_{F1}, q_{D1})-string and the C_2^I is given by.

$$q_I \int C_2^I. \tag{1.1.31}$$

[3]In this section, we use the conventions of Polchinski[15].

The charges $m^I = (m_{NS5}, m_{D5})$ are magnetic charges for C_2^I and defined by integrating the fluxes:

$$m^I = \int_{S^3} F_3^I. \tag{1.1.32}$$

For the invariance of (1.1.31), these charges should be transformed under the $SL(2,\mathbb{R})$ as

$$q_I \to q'_I = q_J(\Lambda^{-1})^J{}_I, \quad m^I \to m'^I = \Lambda^I{}_J m^J. \tag{1.1.33}$$

Because q_I and m^I are integers the transformation matrix Λ must be an elemnt of $SL(2,\mathbb{Z}) \subset SL(2,\mathbb{R})$. Thus, the $SL(2,\mathbb{R})$ symmetry breaks into $SL(2,\mathbb{Z})$ in type IIB superstring theory. We note that this $SL(2,\mathbb{Z})$ transformation mixes fundamental strings and D1-branes.

The transformation (1.1.29) contains strong-weak duality. Let us assume $C = 0$ for simplicity. Then the $SL(2,\mathbb{Z})$ transformation with

$$\Lambda = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{1.1.34}$$

for the dilaton field is

$$\phi \to \phi' = -\phi. \tag{1.1.35}$$

Because the string coupling g_s is related to ϕ by $g_s = e^{<\phi>}$, (1.1.35) gives the transformation

$$g_s \to g'_s = \frac{1}{g_s}. \tag{1.1.36}$$

1.1.3 M-theory

M-theory is the eleven-dimensional theory defined as the strong coupling limit $g_s \to \infty$ of type IIA string theory[16]. Although its definition as a quantum theory has not been known yet, its low energy effective theory is believed to be the 11d supergravity.

The 11d supergravity was constructed in [17]. This theory contains graviton $g_{MN}(L, M, N = 0, 1, \ldots, 9, \sharp)$, gravitino Ψ_M, and anti-symmetric tensor field (3-form field) A_3. The 3-form field A_3 is a gauge field and there are electrically charged objects and magnetically charged objects. The electrically charged objects are called M2-branes. The M2-branes couple with A_3 by the action

$$S_{A_3} = \mu_{\mathrm{M2}} \int A_3. \tag{1.1.37}$$

The integral is taken over $(1+2)$-dimensional world-volume of the M2-branes. The magnetically charged objects are called M5-branes. The magnetic charge is defined by

$$\mu_{M5} = \oint_{S^4} F_4, \tag{1.1.38}$$

where $F_4 = dA_3$ and the integral taken over the world-volume of the M5-brane.

Type IIA string theory is obtained from M-theory by S^1 compactification. We can easily show that the correspondence of massless fields. If we compactify the $\sharp$ direction, the fields are divided as

$$g_{MN} \to g_{\mu\nu}, g_{\mu\sharp}, g_{\sharp\sharp} \tag{1.1.39}$$

$$A_{LMN} \to A_{\mu\nu\rho}, A_{\mu\nu\sharp}, \tag{1.1.40}$$

$$\Psi_M \to \Psi_\mu, \Psi_\sharp. \tag{1.1.41}$$

The fields on the right hand side of (1.1.39)$\sim$(1.1.41) are the same as the massless fields in type IIA. For the bosonic fields (1.1.39) and (1.1.40) match the fields in Table 1.1 by the correspondence

$$g_{\mu\nu}, \quad g_{\mu\sharp} \leftrightarrow C_\mu, \quad g_{\sharp\sharp} \leftrightarrow e^\phi, \tag{1.1.42}$$

$$A_{\mu\nu\rho} \leftrightarrow C_{\mu\nu\rho}, \quad A_{\mu\nu\sharp} \leftrightarrow B_{\mu\nu}. \tag{1.1.43}$$

We can also check the relations of branes. In the S^1 compactified background, there are the following four types of branes:

- wrapped M2-branes
- unwrapped M2-branes
- wrapped M5-branes
- unwrapped M5-branes.

When the size of S^1 is small, these branes are regarded as 1-branes, 2-branes, 4-branes, and 5-branes. These are identified with fundamental strings, D2-branes, D4-branes, and NS5-branes, respectively in type IIA string theory.

D0 and D6-branes in type IIA also have the corresponding objects in M-theory. These branes are coupled to the 1-form field C_μ, and because of $C_\mu \sim g_{\mu\sharp}$ these branes should be related to geometric objects from the viewpoint of 11d supergravity. The R-R 1-form C_1 in type IIA appears in the 11d metric as

$$ds^2_{11d} = g_{\mu\nu}dx^\mu dx^\nu + (dx^\sharp + C_\mu dx^\mu)^2. \tag{1.1.44}$$

This is invariant under the coordinate transformation

$$x^\sharp \to x'^\sharp = x^\sharp + \chi(x^\mu) \tag{1.1.45}$$

with C_1 transformed by

$$C_\mu \to C'_\mu = C_\mu - \partial_\mu \chi. \tag{1.1.46}$$

This is nothing but the $U(1)$ gauge transformation of C_1. The transformation (1.1.45) is translation along $x^\sharp$ direction and the corresponding conserved charge is the momentum $p_\sharp$, which couples to $U(1)$ gauge field C_μ. Namely, the Kaluza-Klein(KK) modes electrically couple to the gauge field C_μ and we identify D0-branes and KK modes. We can also show that D6-branes correspond to Kaluza-Klein monopoles, which are solitonic objects in general relativity. We summarize the correspondence of the branes between M-theory and type IIA theory in Table 1.3.

Table 1.3: The correspondence of objects between M-theory and type IIA theory

M-theory	type IIA theory
M2	D2, F1
M5	D4, NS5
KK mode	D0
KK monopole	D6

From the correspondence of D0-branes and KK modes, we can obtain a relation between parameters. From (1.1.9), the mass of D0-brane is

$$T_0 = \frac{1}{g_s l_s}. \tag{1.1.47}$$

On the other hand, in M-theory compactified on S^1 with radius R_M the minimum unit of the KK momentum is

$$p_\sharp = \frac{1}{R_M}. \tag{1.1.48}$$

By equating these two, we obtain the relation

$$R_M = g_s l_s. \tag{1.1.49}$$

The relation (1.1.49) shows that the strong coupled limit of type IIA string theory corresponds to the decompactification limit of M-theory.

We have shown that type II theories and M-theory are related by dualities. It is also known that other theories (type I and two heterotic strings) are also related by dualities, which we do not explain here.

1.2 4d $\mathcal{N}=4$ super Yang-Mills theory

A supersymmetry in four dimensions (4d) is specified by the number of supercharges, $\mathcal{N}\leq 4$. $\mathcal{N}=4$ supersymmetry is the maximal supersymmetry for theories without gravity. The only known 4d $\mathcal{N}=4$ supersymmetric theory is supersymmetric Yang-Mills theory (SYM). The β-function of 4d $\mathcal{N}=4$ SYM is $\beta=0$. Namely, this is a superconformal field theory (SCFT). We will explain SCFT in Chapter 2 in detail.

The 4d $\mathcal{N}=4$ SYM has a vector multiplet, which is the only free field multiplet of the $\mathcal{N}=4$ superconformal group. The field contents are listed in Table 1.4. Because $\mathcal{N}=4$ supersymmetry imposes strong constraint on the theory, the

Table 1.4: Field components of 4d $\mathcal{N}=4$ SYM. The index $I=1,\dots,4$ is fundmental representation of $SU(4)_R$ R-symmetry.

Fields	$SU(4)_R$
ϕ^{IJ}	$\mathbf{6}$
λ_I	$\overline{\mathbf{4}}$
$\overline{\lambda}^I$	$\mathbf{4}$
$F_{\mu\nu}$	$\mathbf{1}$

theory is determined uniquely by choosing the gauge group. The bosonic part of the Lagrangian is

$$\mathcal{L}=-\frac{1}{4g_{\mathrm{YM}}^2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}+2(D_\mu\phi^{IJ})^2-[\phi^{IJ},\phi^{KL}]^2)+\frac{\theta}{32\pi^2}\mathrm{Tr}F_{\mu\nu}\widetilde{F}^{\mu\nu},\quad \widetilde{F}^{\mu\nu}=\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}. \tag{1.2.1}$$

This theory has $SU(4)_R$ R-symmetry acting on the four supercharges Q^I.

It is convenient to define the complex coupling constant τ as

$$\tau\equiv\frac{\theta}{2\pi}+\frac{4\pi i}{g_{\mathrm{YM}}^2}. \tag{1.2.2}$$

It is believed that this theory has S-duality (Montonen-Olive duality) [18, 19, 20]. That claims that the theory with gauge group G and coupling τ is equivalent to the theory with dual gauge group $\widehat{G}$ and coupling $-1/\tau$. This duality is closely related to the S-duality in the type IIB string theory. We explain the Montonen-Olive duality in detail and the relation between type IIB string theory and 4d $\mathcal{N}=4$ SYM.

1.2.1 Montonen-Olive duality

The Montonen-Olive duality is an electromagnetic duality. Their original conjecture was for an $SU(2)$ gauge theory, and it has been generalized to the case of an arbitrary compact Lie group[21]. According to Montonen and Olive, we show that duality in an $SU(2)$ Yang-Mills theory.

Let us consider the $SU(2)$ gauge theory with adjoint scalar field $\Phi = \frac{1}{2}\sigma^a\Phi^a$, which causes spontaneous symmetry breaking from $SU(2)$ to $U(1)$. The Lagrangian density is given by

$$\mathcal{L} = -\frac{1}{2e^2}[\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} - 2\mathrm{Tr}D_\mu\Phi D^\mu\Phi - 2V(\Phi)], \tag{1.2.3}$$

$$V(\Phi) = -\mu^2\mathrm{Tr}\Phi^2 + \frac{\lambda}{e^2}(\mathrm{Tr}\Phi^2)^2. \tag{1.2.4}$$

For the existence of a lower bound for the energy, λ must be positive, and we assume $\mu^2 > 0$. This theory is called Georgi-Glashow model. The covariant derivative (for the adjoint representation) and the field strength are defined by

$$D_\mu\Phi = \partial_\mu\Phi + i[A_\mu, \Phi], \tag{1.2.5}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]. \tag{1.2.6}$$

We define the electric field and magnetic field by

$$E_i = F_{0i}, \quad B_i = \frac{1}{2}\varepsilon_{ijk}F_{jk} \quad (i, j, k = 1, 2, 3). \tag{1.2.7}$$

We separate the Lagrangian $L = \int d^3x\mathcal{L}$ into the kinetic energy T and the potential energy U_{pot}

$$T = \frac{1}{e^2}\int d^3x(\mathrm{Tr}E_iE_i + \mathrm{Tr}(D_0\Phi)^2), \tag{1.2.8}$$

$$U_{pot} = \frac{1}{e^2}\int d^3x(\mathrm{Tr}B_iB_i + \mathrm{Tr}(D_i\Phi)^2 + V(\Phi)). \tag{1.2.9}$$

A vacuum solution satisfies $U_{pot} = 0$, which holds if and only if

$$B_i = 0, \quad D_i\Phi = 0, \quad V(\Phi) = 0. \tag{1.2.10}$$

The potential (1.2.4) has classical minima with

$$(\Phi^a)^2 = v^2, \quad v \equiv e\sqrt{\frac{\mu^2}{\lambda}}. \tag{1.2.11}$$

This vacuum expectation value preserves only $U(1) \subset SU(2)$. We choose the vacuum solution

$$\Phi(x) = \Phi_0 \equiv \frac{v}{2}\sigma^3, \tag{1.2.12}$$

$$A_\mu(x) = 0, \tag{1.2.13}$$

and let us consider a fluctuation from the vacuum solutions (1.2.12) and (1.2.13).

$$\Phi(x) = \Phi_0 + \widetilde{\Phi}(x), \tag{1.2.14}$$

$$A_\mu = \widetilde{A}_\mu(x). \tag{1.2.15}$$

Because of the vacuum expectation value, the off-diagonal parts of $\widetilde{\Phi}$ and $\widetilde{A}_\mu$ are all massive and the gauge field for the unbroken $U(1)$ is the third component of the $SU(2)$ gauge field A^3_μ. By substituting (1.2.14) and (1.2.15) to the Lagrangian (1.2.3), we can calculate the masses of the fields. The physical fields are

$$\mathcal{A}_\mu = A^3_\mu, \tag{1.2.16}$$

$$W_\mu = \frac{1}{\sqrt{2}}(A^1_\mu + iA^2_\mu), \tag{1.2.17}$$

$$\varphi = \Phi^3. \tag{1.2.18}$$

$\mathcal{A}_\mu, W_\mu$, and φ are massless photon, W-bosons with mass $m_W = ev$ and electric charge $\pm e$, and neutral scalar with mass $m_H = \sqrt{2}\mu$, respectively.

In addition to the W-bosons, we have another type of massive particles realized as solitons. Solitons are classical solutions which have finite energy. A finite energy solution should approach a vacuum solution as $r \to 0$. Let us take the following ansatz for the asymptotic form of Φ.

$$\lim_{r\to\infty} \Phi(r,\theta,\phi) = \Phi_\infty(\theta,\phi) \equiv U(\theta,\phi)\Phi_0 U^{-1}(\theta,\phi). \tag{1.2.19}$$

Because Φ_0 belongs to adjoint representation, $\Phi_\infty(\theta,\phi)$ can be regarded as a map $S^2 \to SU(2) \approx S^2$. This map can be classified by the homotopy group $\pi_2(S^2) = \mathbb{Z}$. Correspondingly, configurations of the scalar field are classified by the winding number

$$w = \frac{1}{8\pi}\varepsilon_{ijk}\varepsilon_{abc}\int d^2S_i\hat{\phi}^a\partial_j\hat{\phi}^b\partial_k\hat{\phi}^c, \quad \hat{\phi}^a = \frac{\Phi^a}{\sqrt{(\Phi^a)^2}}. \tag{1.2.20}$$

The vacuum solution (1.2.12) has $w = 0$. The asymptotic form of the scalar field must be

$$\Phi^a = v\hat{\phi}^a, \tag{1.2.21}$$

and the covariant derivative $D_i\Phi^a$ must fall faster than $r^{-3/2}$. In the asymptotic region

$$D_i\Phi^a = \partial_i\hat{\phi}^a - \varepsilon_{abc}A_i^b\hat{\phi}^c = 0, \tag{1.2.22}$$

and this equation can be solved by

$$A_i^a = \varepsilon_{abc}\hat{\phi}^b\partial_i\hat{\phi}^c + f_i\hat{\phi}^a, \tag{1.2.23}$$

where f_i is an arbitrary function. The magnetic field obtained from the gauge potential (1.2.23) is

$$B_i^d = \frac{1}{2}\varepsilon_{ijk}\left[\varepsilon_{abc}\hat{\phi}^a\partial_j\hat{\phi}^b\partial_k\hat{\phi}^c + (\partial_j f_k - \partial_k f_j)\right]\hat{\phi}^d. \tag{1.2.24}$$

The magnetic field for the unbroken $U(1)$ is defined by the projection by $\hat{\phi}^a$,

$$\hat{\phi}^a B_i^a = \frac{1}{2}\varepsilon_{ijk}\left[\varepsilon_{abc}\hat{\phi}^a\partial_j\hat{\phi}^b\partial_k\hat{\phi}^c + (\partial_j f_k - \partial_k f_j)\right]. \tag{1.2.25}$$

The magnetic charge is defined by

$$Q_m \equiv \frac{1}{e}\int d^2S_i\hat{\phi}^a B_i^a. \tag{1.2.26}$$

If we substitute (1.2.25) into (1.2.26), the first term in (1.2.25) is proportional to winding number, while the second term vanishes due to Gauss's theorem. We obtain

$$Q_m = \frac{4\pi}{e}w. \tag{1.2.27}$$

The monopole mass is the energy $E = T + U_{pot}$ of this solution. Because the solution is static, we set $T = 0$. By doing a little deformation of U_{pot}, we obtain

$$U_{pot} = \frac{1}{e^2}\int d^3x[(B_i^a - D_i\Phi^a)^2 + V(\Phi)] + \frac{1}{e^2}\int d^3x B_i^a D_i\Phi^a. \tag{1.2.28}$$

For a solution with the asymptotic form (1.2.21), the second integral become

$$\frac{v}{e^2}\int d^3x B_i^a D_i\hat{\phi}^a = \frac{v}{e^2}\int dS_i B_i^a\hat{\phi}^a = \frac{v}{e}Q_m. \tag{1.2.29}$$

Thus the energy has the lower bound $E \geq \frac{v}{e}Q_m$. In the limit $\mu^2 \to 0, \lambda \to 0$ with $v^2 = \mu^2 e^2/\lambda$ held fixed, the monopole solution saturates this bound. This limit is called Bogomol'nyiPrasadSommerfield (BPS) limit. In the BPS limit, $V(\Phi) = 0$ and when the solution satisfies $B_i = D_i\Phi^a$ it saturates the energy lower bound. Hence after taking the BPS limit a monopole has the mass $vQ_m = 4\pi v/e^2$.

The spectrum of elementary fields and (anti-)monopoles are listed in Table 1.5. In the BPS limit, this theory seems to have the symmetry which simultaneously

Table 1.5: The particle masses and charges

	Mass	Q_e	Q_m
photon	0	0	0
Φ	0	0	0
$W^{\pm}$	v	$\pm e$	0
Monopole	$\frac{4\pi v}{e^2}$	0	$\pm\frac{4\pi}{e}$

exchange the electric charge Q_e and the magnetic charge Q_m, and the coupling constant e and $4\pi/e$. This result suggests that the theory described by the Lagrangian (1.2.3) has a dual description which contains monopoles as elementary particles.

Unfortunately, careful analysis shows that this naive expectation is not correct. The W-boson has spin 1 while we can show that the monopole has spin 0. In this case, the exchange of W-bosons and monopoles does not keep the theory invariant.

This deficit can be fixed by introducing extended supersymmetry. When we consider $\mathcal{N} = 4$ supersymmetry, the W-boson becomes a member of the supermultiplet which also contains five scalars and eight fermionic states. The monopole also has degenerate states, which are caused by fermionic zero modes. It is known that $\mathcal{N} = 4$ SYM has 16 degenerate monopole states and these states form the same supermultiplet as W-bosons[20]. In the $\mathcal{N} = 4$ SYM, strong-weak duality transformation of the coupling constant g_{YM} is given by

$$g_{\text{YM}}^2 \to g_{\text{YM}}'^2 = \frac{4\pi^2}{g_{\text{YM}}^2}. \tag{1.2.30}$$

The Montonen-Olive duality has been generalized to the case of an arbitrary compact Lie group by Goddard, Nuyts and Olive (GNO)[21]. They suggest that a gauge theory with gauge group G in a strong coupling region has a dual description with gauge group $G^\vee$ in a weak coupling region. A dual gauge group $G^\vee$ is defined by a Lie algebra $g^\vee$ whose roots are the dual roots $\boldsymbol{\alpha}^\vee = 2\boldsymbol{\alpha}/\boldsymbol{\alpha}^2$ of g.

Table 1.6: Some examples of Lie group G and its dual group $G^\vee$

G	$G^\vee$
$U(N)$	$U(N)$
$SU(MN)/\mathbb{Z}_N$	$SU(MN)/\mathbb{Z}_M$
$SO(2N+1)$	$Sp(N)$

1.2.2 Brane realization of W-bosons and monopoles

The low energy behavior of Dp-branes are described by $(p+1)$-dimensional SYM[7]. This fact gives us many insights of SYM in strong couple region. We start considering the DBI action (1.1.11) and the Chern-Simons like term (1.1.13) of D3-branes and we show that the low energy effective action of D-branes becomes Yang-Mills theory.

Let us considering a flat D3-brane in the flat 10d spacetime with the metric $g_{\mu\nu} = \eta_{\mu\nu}$. We assume that the anti-symmetric tensor fields B_2, C_2 and C_4 vanish and the scalar fields $e^{<\phi>} = g_s$ and C_0 are constant. For simplicity, we consider only bosonic fields. We can gauge fix the reparametrization symmetry on the brane by

$$X^\alpha(\xi) = \xi^\alpha \quad (\alpha = 0, 1, 2, 3) \tag{1.2.31}$$

The remaining six scalar fields $X^a(\xi^\alpha)$ $(a = 4, \ldots, 9)$ have physical degrees of freedom. The induced metric $G_{\alpha\beta}$ is

$$G_{\alpha\beta} = \eta_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu \approx \eta_{\alpha\beta} + \partial_\alpha X^a \partial_\beta X^a + O((\partial X)^4). \tag{1.2.32}$$

The DBI action (1.1.11) becomes

$$S^0_{D3} = -\frac{\mu_3}{g_s}V_3 - \frac{1}{4(2\pi g_s)}\int d^4\xi \left[F_{\alpha\beta}F^{\alpha\beta} + \frac{2}{(2\pi l_s^2)^2}\partial_\alpha X^a \partial^\alpha X^a\right] + O(F^4), \tag{1.2.33}$$

where V_3 is a D3-brane world-volume and the first term is merely constant which does not affect the equations of motion. By comparing (1.2.33) and the $\mathcal{N} = 4$ SYM Lagrangian (1.2.1), we can read off the Yang-Mills coupling constant $g_{\rm YM}$

$$g^2_{\rm YM} = 2\pi g_s, \tag{1.2.34}$$

and the scalar fields ϕ^{IJ} and X^a are related by

$$\phi^{IJ} \sim \frac{1}{2\pi l_s^2} X^a. \tag{1.2.35}$$

The CS term (1.1.13) gives

$$S_\theta = \frac{\theta}{8\pi^2}\int F_2 \wedge F_2, \tag{1.2.36}$$

where the θ-angle is related to C_0 by

$$C_0 = \frac{\theta}{2\pi}. \tag{1.2.37}$$

On a single D3-brane, $\mathcal{N}=4$ $U(1)$ SYM is realized.

It is known that N coincident D3-branes low-energy effective theory is the $\mathcal{N}=4$ SYM with gauge group $U(N)$[7]. We label the D-branes by an index i ($i=1,\dots,N$). All fields are expressed by $N\times N$ matrices, and (i,j) component ($i,j=1,\dots,N$) of a field corresponds to a string connecting i-th and j-th D3-branes. The low energy effective action (1.2.33) and (1.2.36) are generalized to

$$\begin{aligned} S_{\rm D3} = &-\frac{1}{4g_{\rm YM}^2}\int d^4\xi {\rm Tr}\left[F_{\alpha\beta}F^{\alpha\beta} + \frac{2}{(2\pi l_s^2)^2}(D_\alpha X^a)^2 - \frac{1}{(2\pi l_s^2)^4}[X^a, X^b]^2\right] \\ &+ \frac{\theta}{8\pi^2}\int F_2\wedge F_2. \end{aligned} \tag{1.2.38}$$

When there is only one D3-brane, the scalar fields X^a are interpreted as the position of the D3-brane in six-dimensional transverse space. Now there are N D3-branes and the scalar fields X^a are $N\times N$ matrices. In this situation, we interpret the N diagonal components of X^a as the positions of N D3-branes.

We can read off more interesting information for the SYM from the D-brane configuration. For simplicity, we consider two parallel D3-branes. Because of the string tension T_s, the open string stretched between the two D3-branes has the energy

$$T_s L = \frac{1}{2\pi l_s^2}L, \tag{1.2.39}$$

where L is the distance between two D3-branes. If two D3-branes are coincident, the open string has massless states and such a state corresponds to the gauge field for unbroken $U(2)$ gauge symmetry. We regard the open string with mass $\frac{1}{2\pi l_s^2}L$ as a W-boson state. Furthermore, we can interpret the separation of two D3-branes corresponds to the spontaneous symmetry breaking from $U(2)$ to $U(1)^2$.

The following fact is important for the analysis in Chapter 3.

The open string connecting separated two D-branes corresponds to the massive gauge field which is called W-boson. We can also find objects in the D-brane setup corresponding to monopoles. We can identify monopoles with D1-branes stretched between D3-branes. Let us consider the system of a D1-brane ending on the D3-branes. For a D3-brane the CS like action (1.1.13) contains the following term:

$$S_m = \frac{1}{(2\pi l_s)^2} \int F_2 \wedge C_2. \tag{1.2.40}$$

This term shows us the field strength F_2 contributes to the D1-charge current. On the other hand, the integral of F_2 gives a magnetic charge

$$Q_{\text{mag}} = \oint_{S^2} F_2, \tag{1.2.41}$$

where the integral is taken over S^2 on the world-volume of one of the D3-branes. Therefore, in order to conserve the D1 charge current the magnetic monopole and the endpoint of the D1-brane must be at the same point. Namely, we can regard the endpoint of D1-brane as a monopole in the world-volume theory. The tension of D1-brane is given by $T_1 = \frac{\mu_1}{g_s} = \frac{1}{2\pi l_s^2 g_s}$. The monopole mass is reproduced as the energy of the D1-brane

$$T_1 L = \frac{1}{2\pi l_s^2 g_s} L = \frac{4\pi}{e^2} v. \tag{1.2.42}$$

As a result, a W-boson corresponds to a string stretching between separating two D3-branes, and a monopole corresponds to a D1-brane stretching between separating two D3-branes[4].

We can also consider a (p, q)-string. The (p, q)-string stretched between D3-branes corresponds to a dyon in 4d SYM with electric charge p and magnetic charge q. The tension of a (p, q)-string is

$$T_{(p,q)} = \frac{1}{2\pi l_s^2} |p + \tau q|, \quad \tau = \frac{\theta}{2\pi} + i\frac{2\pi}{g_s}. \tag{1.2.43}$$

By multiplying by the distance between the D3-branes L, we can reproduce the dyon mass.

Through the D3-brane realization of $\mathcal{N} = 4$ SYM, the S-duality in type IIB corresponds to the Montonen-Olive duality. For example, the transformation of the string coupling constant (1.1.36) is equivalent to (1.2.30) under the identification (1.2.34).

[4] In the case of two parallel D3-branes, we identify the traceless part of the 2×2 matrices with the fundamental representation of $SU(2)$. The normalization of the electric coupling constant by $e^2 = 2g_{\text{YM}}^2 = 4\pi g_s$[22].

1.2.3 BPS states

In subsection 1.2.1, we took the BPS limit and found the monopole solution which saturates the energy lower bound $E \geq vQ_m$. More generally, the mass of a dyon has a similar lower bound depending on its electric and magnetic charges [23];

$$E \geq \frac{v}{g_{\rm YM}}\sqrt{Q_e^2+Q_m^2}. \tag{1.2.44}$$

The states saturating this bound are called BPS states. Note that by using coupling constant g and integral charges p and q, the electric charge and magnetic charge appearing in this bound are given by

$$Q_e = g_{\rm YM}\left(p+\frac{\theta}{2\pi}q\right), \quad Q_m = \frac{4\pi}{g_{\rm YM}}q. \tag{1.2.45}$$

The same BPS bound appears in $\mathcal{N}=2$ supersymmetric field theories. Supercharges Q_α^I and $\overline{Q}_{\dot\alpha I}(I=1,2)$ satisfy the anti-commutation relations

$$\{Q_\alpha^I,\overline{Q}_{\dot\alpha J}\} = 2(\sigma^\mu)_{\alpha\dot\alpha}P_\mu\delta^I{}_J, \tag{1.2.46}$$

$$\{Q_\alpha^I,Q_\beta^J\} = 2\varepsilon_{\alpha\beta}\varepsilon^{IJ}Z, \tag{1.2.47}$$

$$\{\overline{Q}_{\dot\alpha I},\overline{Q}_{\dot\beta J}\} = 2\varepsilon_{\alpha\beta}\varepsilon_{IJ}Z^*, \tag{1.2.48}$$

where Z is the complex charge called the central charge. By using $U(1)_r$ R-symmetry, which acts on the supercharges Q^I with charge -1, we can choose the Z as a real number. Let us define new supercharges

$$A_\alpha = \frac{1}{2}(Q_\alpha^1+\varepsilon_{\alpha\beta}(Q_\beta^2)^\dagger), \quad B_\alpha = \frac{1}{2}(Q_\alpha^1-\varepsilon_{\alpha\beta}(Q_\beta^2)^\dagger). \tag{1.2.49}$$

They satisfy

$$\{A_\alpha,A_\beta^\dagger\} = (M+Z)\delta_{\alpha\beta}, \quad \{B_\alpha,B_\beta^\dagger\} = (M-Z)\delta_{\alpha\beta}. \tag{1.2.50}$$

Since all physical states must have positive norm, (1.2.50) leads to the energy bound

$$M \geq Z. \tag{1.2.51}$$

States related by the action of supercharges form a supermultiplet. Because the supercharges Q_α^I and $\overline{Q}_{\dot\alpha I}$ commute with the Hamiltonian $H=P^0$, all states in a supermultiplet have the same energy. If a state has the mass $M=Z$, anti-commutator of B_α and $B_\beta^\dagger$ becomes $\{B_\alpha,B_\beta^\dagger\}=0$. Then, the degeneracy is reduced.

Actually, the bound (1.2.51) is the BPS bound and the states satisfying the bound is called half-BPS ($\frac{1}{2}$-BPS) states, which are eliminated by half of the supersymmetry. If the Lagrangian of an $\mathcal{N} = 2$ theory is given, we can define the supercharges Q and $\overline{Q}$ in terms of fundamental fields, and also we can calculate the anti-commutator of them. Then we can find that the central charge can be written by the electric and the magnetic charges as follows;

$$Z = \frac{v}{g_{\mathrm{YM}}}(Q_e + iQ_m) = v(p + \tau q), \quad \tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g_{\mathrm{YM}}^2} \quad (p, q \in \mathbb{Z}). \tag{1.2.52}$$

The energy bound (1.2.51) of this central charge reproduces the bound (1.2.44).

1.2.4 String junction

We have explained that a string connecting D-branes corresponds to a BPS particles (W-boson, monopole, and dyon). From the viewpoint of supersymmetric field theory, these BPS states are $\frac{1}{2}$-BPS states. Since (p, q)-string corresponds to a dyon, the central charge of (p, q)-string can be found in the configuration of D-branes. When a (p, q)-string connects two D3-branes (we denote their positions as x_1 and x_2), the central charge is given by

$$Z_{(p,q)} = \frac{x_1 - x_2}{2\pi l_s^2}(p + \tau q). \tag{1.2.53}$$

Actually, the mass of (p, q)-string, which is given by tension (1.2.43) times $|x_1 - x_2|$, coincides with $|Z|$.

It is known that in a theory with gauge group of rank ≥ 2, there are quarter BPS ($\frac{1}{4}$-BPS) states, which are eliminated by a quarter of the supersymmetry. In the string theory, the $\frac{1}{4}$-BPS states can be realized by string junctions [24, 25, 26, 27, 28]. A string junction has three-pronged configuration and endpoints are attached on different D-branes (See Figure 1.2.4). Let (p_i, q_i) $(i = 1, 2, 3)$ be the charges of the three strings. The charge conservation requires

$$\sum_{i=1}^{3} p_i = \sum_{i=1}^{3} q_i = 0. \tag{1.2.54}$$

For instance, a $(1, 0)$-string (fundamental string) and a $(0, 1)$-string (D1-brane) can join to become a $(1, 1)$-string.

The shape of a string junction is determined by the balance of tensions at the intersection point, and it is confirmed that its mass is the same as the mass of $\frac{1}{4}$-BPS dyon calculated in the field theory[28].

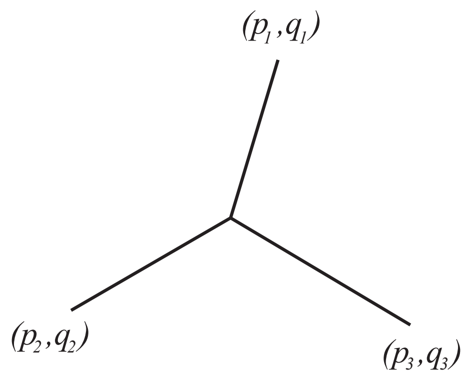

Figure 1.1: A string junction in the general case

1.2.5 Moduli space

The moduli space of a theory is defined as the set of vacua. Let us consider a field theory with potential term $V(\phi)$ depending on scalar fields ϕ. We assume that the minimum is given by $V(\phi) = 0$. Then, vacua is specified by the vacuum expectation value of ϕ satisfying $V(\phi) = 0$. For gauge theories points ϕ and ϕ' related by a gauge transformation $\phi \to \phi' = g\phi$ must be identified. Thus, the moduli space $\mathcal{M}$ is defined by

$$\mathcal{M} \equiv \{\phi | V(\phi) = 0\}/G, \tag{1.2.55}$$

where G is the gauge group of the theory. Let H be the subgroup of G which keeps a vacuum invariant. In general, H depends on the point in $\mathcal{M}$ we choose.

We have already seen the example of moduli space in 4d $\mathcal{N} = 4$ $U(N)$ SYM. The theory has the potential

$$V(X) = \frac{1}{(2\pi l_s^2)^4}[X^a, X^b]^2, \tag{1.2.56}$$

and potential minima are given by diagonal matrices $X^a = \mathrm{diag}(X_i^a)$ $(i = 1, \ldots, N)$. Thus moduli space of the theory is $(\mathbb{R}^6)^N/S_N$ which corresponds to the space where N D3-branes can move. S_N is the permutation group of order N, which is the Weyl group of $U(N)$. In this situation, gauge symmetry breaks into $H = U(1)^N$ at a generic point in the moduli space. In general, we call the subspace of moduli space in which rank r gauge group G brakes into $U(1)^r$ the Coulomb branch. For relatively simple Lagrangian theories, the Coulomb branch is parametrized by adjoint scalar fields.

Chapter 2

Superconformal field theory

In this chapter, we review the four-dimensional (4d) superconformal field theory (SCFT) in order to derive some basic properties of 4d $\mathcal{N} = 3$ superconformal field theories. Theories with $\mathcal{N} = 1, 2$ and 4 superconformal symmetry have been studied extensively in the last few decades, and played essential roles in the development of field theories. However, $\mathcal{N} = 3$ theories had not attracted much attention until quite recently. A reason for this is that the $\mathcal{N} = 3$ vector multiplet, which is the only $\mathcal{N} = 3$ multiplet of free fields, has the same field contents as the $\mathcal{N} = 4$ vector multiplet, and if we try making an $\mathcal{N} = 3$ theory in a perturbative way we end up with an $\mathcal{N} = 4$ theory. It had not been clear if there are genuine $\mathcal{N} = 3$ theories which does not have $\mathcal{N} = 4$ supersymmetry until concrete examples were constructed by using string theory.

In the following sections, we will first explain superconformal algebra, CFT operators and how to construct representations of superconformal symmetry. Next we explain the reason why genuine $\mathcal{N} = 3$ SCFTs are difficult to study. After that, we classify representations of $\mathcal{N} = 2$ and 3 superconformal symmetries and comment on some properties of $\mathcal{N} = 3$ SCFT.

2.1 Superconformal symmetry

2.1.1 Conformal symmetry

First, we consider the 4d conformal symmetry and its representations. The 4d conformal algebra is generated by translations P_μ, Lorentz generators $M_{\mu\nu}$, special conformal transformations K_μ, and the dilatation D. The non-vanishing commutation relations are

$$[M_{\mu\nu}, P_{\rho},] = i(\eta_{\mu\rho}P_{\nu} - \eta_{\nu\rho}P_{\mu}), \quad (2.1.1)$$
$$[M_{\mu\nu}, K_{\rho}] = i(\eta_{\mu\rho}K_{\nu} - \eta_{\nu\rho}K_{\mu}), \quad (2.1.2)$$
$$[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}), \quad (2.1.3)$$
$$[D, P_{\mu}] = iP_{\mu}, \quad (2.1.4)$$
$$[D, K_{\mu}] = -iK_{\mu}, \quad (2.1.5)$$
$$[P_{\mu}, K_{\nu}] = 2iM_{\mu\nu} + 2i\eta_{\mu\nu}D. \quad (2.1.6)$$

We define the generators M_{AB} $(A, B = 0, 1, \ldots, 5)$ as

$$M_{AB} = \begin{pmatrix} M_{\mu\nu} & -\frac{1}{2}(P_{\mu} - K_{\mu}) & -\frac{1}{2}(P_{\mu} + K_{\mu}) \\ \frac{1}{2}(P_{\mu} - K_{\mu}) & 0 & D \\ \frac{1}{2}(P_{\mu} + K_{\mu}) & -D & 0 \end{pmatrix}. \quad (2.1.7)$$

Then the commutation relations (2.1.1)∼(2.1.6) are unified into the following single commutation relation with $\eta_{AB} = \mathrm{diag}(-1, 1, 1, 1, 1, -1)$:

$$[M_{AB}, M_{CD}] = i(\eta_{AC}M_{BD} - \eta_{AD}M_{BC} - \eta_{BC}M_{AD} + \eta_{BD}M_{AC}). \quad (2.1.8)$$

Thus the 4d conformal algebra is isomorphic to $so(2, 4)$.

When there is a state $|\mathcal{O}\rangle$ in a CFT, we can consider the local operator $\mathcal{O}(x)$ corresponding to the state $|\mathcal{O}\rangle$. The relation between the state and the operator is

$$|\mathcal{O}\rangle = \mathcal{O}(0)\,|0\rangle\,, \quad (2.1.9)$$

where $|0\rangle$ is the conformal invariant vacuum state of the CFT. For an operator $\mathcal{O}^a(0)$ which belongs to an irreducible representation of the rotation group $Spin(4)$, the conformal generators D and $M_{\mu\nu}$ act on it as follows,

$$[D, \mathcal{O}^a(0)] = i\Delta\mathcal{O}^a(0), \quad (2.1.10)$$
$$[M_{\mu\nu}, \mathcal{O}^a(0)] = (S_{\mu\nu})_b{}^a\mathcal{O}^b(x), \quad (2.1.11)$$

where $S_{\mu\nu}$ are spin matrices, and Δ is the conformal dimension, which is the scale dimension of the operator. Note that the conformal generators act on local operators by the commutator. Sometimes we use shorthand notation where commutators of charges with local operators are implicit,

$$[\mathcal{Q}, \mathcal{O}] \to \mathcal{Q}\mathcal{O} \quad (\mathcal{Q} : \text{any charge}). \quad (2.1.12)$$

Conformal representations

Let $\mathcal{O}$ be an operator with dimension Δ. From the conformal algebra, we can show that K_μ play the role of lowering operators for the dimension,

$$DK_\mu\mathcal{O}(0) = i(\Delta - 1)K_\mu\mathcal{O}(0). \tag{2.1.13}$$

Thus, we can obtain arbitrarily low-dimensional operators by repeatedly acting K_μ on a given state as far as the state does not vanish. Because the dimension is bounded from below in a unitary theory, the action of K_μ must eventually terminate. Namely, in any unitary representation of conformal algebra there are operators such that $[K_\mu, \mathcal{O}(0)] = 0$. Such operators are called primary operators. From a given primary operator, we can construct operators of higher dimension by acting P_μ,

$$\mathcal{O}(0)\ (\dim \Delta) \quad \rightarrow \quad P_{\mu_1}\dots P_{\mu_n}\mathcal{O}(0)\ (\dim \Delta + n). \tag{2.1.14}$$

We call such operators descendant operators. The primary operator and the tower of descendant operators form a multiplet of the conformal group.

To summarize, a priamry operator satisfies

$$[D, \mathcal{O}(0)] = i\Delta\mathcal{O}(0), \tag{2.1.15}$$
$$[M_{\mu\nu}, \mathcal{O}(0)] = S_{\mu\nu}\mathcal{O}(0), \tag{2.1.16}$$
$$K_\mu, \mathcal{O}(0) = 0, \tag{2.1.17}$$

and a conformal representation is uniquely specified by giving the conformal dimension Δ and a pair of half integers $(j, \bar{j})$ representing spins of $SU(2)_L \times SU(2)_R \subset SO(4)$. We use the notation

$$[j, \bar{j}]_\Delta \tag{2.1.18}$$

to label the conformal representations.

An operator inserted at an arbitrary point x is given by

$$\mathcal{O}(x) = e^{ix\cdot P}\mathcal{O}(0)e^{-ix\cdot P}. \tag{2.1.19}$$

By using (2.1.1)∼(2.1.6) and (2.1.15)∼(2.1.17), we obtain the following transformation of primary operator $\mathcal{O}^a(x)$,

$$[P_\mu, \mathcal{O}^a(x)] = -i\partial_\mu\mathcal{O}^a(x), \tag{2.1.20}$$
$$[M_{\mu\nu}, \mathcal{O}^a(x)] = i(x_\mu\partial_\nu - x_\nu\partial_\mu)\mathcal{O}^a(x) + (S_{\mu\nu})_b{}^a\mathcal{O}^b(x), \tag{2.1.21}$$
$$[K_\mu, \mathcal{O}^a(x)] = i(-2x_\mu(x\cdot\partial) + x^2\partial_\mu - 2\Delta x_\mu)\mathcal{O}^a(x) - 2x^\nu(S_{\mu\nu})_b{}^a\mathcal{O}^b(x), \tag{2.1.22}$$
$$[D, \mathcal{O}^a(x)] = i(x\cdot\partial + \Delta)\mathcal{O}^a(x). \tag{2.1.23}$$

Long and short representations

As we mentioned above, we can construct the representation of conformal group by acting P_μ on a primary operator. However, descendant states are not always physical states.

For instance, let us consider a free scalar field ϕ as the primary operator. This has quantum numbers $\Delta = 1$ and $(j, \overline{j}) = (0, 0)$, and thus the representation constructed from this operator is $[0, 0]_1$. The descendant operators are

$$\phi \to \partial_\mu \phi \to \partial_\mu \partial_\nu \phi \to \cdots . \tag{2.1.24}$$

Because ϕ satisfies the equation of motion $\partial^2 \phi = 0$, $\partial^2 \phi$ and its derivatives are not physical operators. Such operators correspond to zero-norm states, and in order to obtain a unitary representation, we have to remove the tower of descendants starting from $\partial^2 \phi$.

If zero-norm states appear in descendants such a representation is called short representation. Otherwise, the representation is called long representation.

2.1.2 Superconfomal symmetry

It is known that the 4d conformal group can be extended to the superconformal algebra by including the supercharges. In addition to the generators of conformal algebra, the superconformal algebra includes the supercharges $Q^I_\alpha, \overline{Q}_{\dot{\alpha} I}$, and $S^\alpha_I, \overline{S}^{\dot{\alpha} I}$ $(I = 1, 2, \ldots, \mathcal{N})$, and the $U(\mathcal{N})$ R-symmetry generators $R^I{}_J$. Q and S satisfy

$$\{Q^I_\alpha, \overline{Q}_{\dot{\alpha} J}\} = 2P_{\alpha\dot{\alpha}} \delta^I{}_J, \tag{2.1.25}$$

$$\{\overline{S}^{\dot{\alpha} I}, S^\alpha_J\} = 2\tilde{K}^{\dot{\alpha}\alpha} \delta^I{}_J, \tag{2.1.26}$$

$$\{Q^I_\alpha, S^\beta_J\} = 4\delta^I{}_J (M_\alpha{}^\beta - \frac{i}{2}\delta_\alpha{}^\beta D) - 4\delta_\alpha{}^\beta R^I{}_J, \tag{2.1.27}$$

$$\{\overline{S}^{\dot{\alpha} I}, \overline{Q}_{\dot{\beta} J}\} = 4\delta^I{}_J (\overline{M}_{\dot{\alpha}}{}^{\dot{\beta}} + \frac{i}{2}\delta_{\dot{\alpha}}{}^{\dot{\beta}} D) - 4\delta_{\dot{\alpha}}{}^{\dot{\beta}} R^I{}_J, \tag{2.1.28}$$

where

$$P_{\alpha\dot{\alpha}} = (\sigma^\mu)_{\alpha\dot{\alpha}} P_\mu, \quad \tilde{K}^{\dot{\alpha}\alpha} = (\overline{\sigma}^\mu)^{\dot{\alpha}\alpha} K_\mu, \tag{2.1.29}$$

$$M_\alpha{}^\beta = -\frac{i}{4}(\sigma^\mu \overline{\sigma}^\nu)_\alpha{}^\beta M_{\mu\nu}, \quad \overline{M}^{\dot{\alpha}}{}_{\dot{\beta}} = -\frac{i}{4}(\overline{\sigma}^\mu \sigma^\nu)^{\dot{\alpha}}{}_{\dot{\beta}} M_{\mu\nu}, \tag{2.1.30}$$

$R^I{}_J$ satisfy

$$[R^I{}_J, R^K{}_L] = \delta^K{}_J R^I{}_L - \delta^I{}_L R^K{}_J. \tag{2.1.31}$$

The commutator of the supercharges and conformal generators are

$$[K_\mu, Q^I_\alpha] = -(\sigma_\mu)_{\alpha\dot{\alpha}} \overline{S}^{\dot{\alpha}I}, \quad [K_\mu, \overline{Q}_{\dot{\alpha}I}] = S^\alpha_I (\sigma_\mu)_{\alpha\dot{\alpha}}, \tag{2.1.32}$$

$$[P_\mu, \overline{S}^{\dot{\alpha}I}] = -(\overline{\sigma}_\mu)^{\dot{\alpha}\alpha} Q^I_\alpha, \quad [P_\mu, S^\alpha_I] = \overline{Q}_{\dot{\alpha}I} (\overline{\sigma}_\mu)^{\dot{\alpha}\alpha}, \tag{2.1.33}$$

$$[D, Q^I_\alpha] = \frac{i}{2} Q^I_\alpha, \quad [D, \overline{Q}_{\dot{\alpha}I}] = \frac{i}{2} \overline{Q}_{\dot{\alpha}I}, \tag{2.1.34}$$

$$[D, S^\alpha_I] = -\frac{i}{2} S^\alpha_I, \quad [D, \overline{S}^{\dot{\alpha}I}] = -\frac{i}{2} \overline{S}^{\dot{\alpha}I}. \tag{2.1.35}$$

The R-charges act on the supercharges according to

$$[R^I{}_J, Q^K_\alpha] = \delta^K{}_J Q^I_\alpha - \frac{1}{4} \delta^I{}_J Q^K_\alpha, \quad [R^I{}_J, \overline{Q}_{\dot{\alpha}K}] = -\delta^I{}_K \overline{Q}_{\dot{\alpha}J} + \frac{1}{4} \delta^I{}_J \overline{Q}_{\dot{\alpha}K}, \tag{2.1.36}$$

$$[R^I{}_J, S^\alpha_K] = -\delta^I{}_K S^\alpha_J + \frac{1}{4} \delta^I{}_J S^\alpha_K, \quad [R^I{}_J, \overline{S}^{\dot{\alpha}K}] = \delta^K{}_J \overline{S}^{\dot{\alpha}I} - \frac{1}{4} \delta^I{}_J \overline{S}^{\dot{\alpha}K}. \tag{2.1.37}$$

Note that when $\mathcal{N} = 4$ $R^I{}_I$ commutes with any generators, and we impose $R^I{}_I = 0$ so that the R-symmetry group is $SU(4)$.

We can construct representations of superconformal group in a similar way to that of conformal group. The commutation relations (2.1.34) and (2.1.35) tell us the supercharges Q^I_α and $\overline{Q}_{\dot{\alpha}I}$ raise the dimension by 1/2, and S^α_I and $\overline{S}^{\dot{\alpha}I}$ lower the dimension by 1/2. For this reason, we define a superconformal primary operator as an operator satisfying $[S^\alpha_I, \mathcal{O}(0)] = [\overline{S}^{\dot{\alpha}I}, \mathcal{O}(0)] = 0$ and we regard Q^I_α and $\overline{Q}_{\dot{\alpha}I}$ as raising operators.

In a CFT, primary operators are labeled by spins $(j, \overline{j})$ and conformal dimension Δ. In the case of SCFT, in addition to these, we need labels to specify the representation of the R-symmetry group. Thus we use the notation

$$[j, \overline{j}]^{R_{\text{rep}}}_\Delta \tag{2.1.38}$$

where R_{rep} is the representation of R-symmetry group, and we often use the Dynkin labels of $SU(\mathcal{N})$ and the charge of $U(1)$ as R_{rep}.

2.2 Deformation of SCFT

CFTs are realized at fixed points of renormalization group (RG) flow, where $\beta = 0$. Let us consider deformations of a CFT. We deform a CFT by changing coupling constants and shift the theory from the fixed point. We focus on a certain class of deformation which is realized by adding local operators to the Lagrangian as follows;

$$\delta\mathcal{L} = g\mathcal{O}, \tag{2.2.1}$$

where $\mathcal{O}$ is a local operator and g is an infinitesimal coupling constant corresponding to $\mathcal{O}$. We only consider the deformation $\delta\mathcal{L}$ preserving Lorentz symmetry. Thus we restrict the deformation operator $\mathcal{O}$ to be a scalar. The deformation $\delta\mathcal{L}$ can always be defined by using conformal perturbation theory, even if the original CFT is a non-Lagrangian theory. The deformation operators can be classified according to their dimension Δ into three types : relevant operators, irrelevant operators and marginal operators.

- $\Delta < 4$: a relevant deformation operator
- $\Delta > 4$: an irrelevant deformation operator
- $\Delta = 4$: a marginal deformation operator

Conformal symmetry is preserved only if deformation is a marginal deformation. In addition, we impose some conditions on deformation operators as follows :

- The operators should be conformal primaries. If $\mathcal{O}$ is a conformal descendant it is a total derivative of another operator, and $\delta\mathcal{L}$ does not modify the dynamics of the theory.
- To preserve the supersymmetry the operators must be a top (highest conformal dimension) component of a superconformal multiplet. If $\mathcal{O}$ is a top component of a superconformal multiplet, the supersymmetry transformation of $\delta\mathcal{L}$ is a total derivative of a fermionic operator. Therefore the deformed theory is still invariant under supersymmetry transformation.

2.2.1 Marginal deformation operator

Here we focus on the marginal deformation operators. If $\mathcal{O}$ is a marginal deformation operator, the corresponding coupling constant g is dimensionless, and (2.2.1) preserves the conformal invariance in the vicinity of the fixed point. If finite deformation by marginal deformation operator preserves conformal symmetry, such deformation is called exactly marginal deformation. We only consider the exactly marginal deformation below. By using exactly marginal deformation operators, we consider continuous deformations of CFTs. Namely, exactly marginal deformations take us from a given CFT to nearby CFT. Some interacting CFTs are constructed by continuous deformation from free field theories. The $\mathcal{N}=4$ SYM is an example of such theory. We can consider a free field theory of $\mathcal{N}=4$ vector multiplet [1]

$$\mathcal{L}=-\frac{1}{4}\mathrm{Tr}(F_{\mu\nu})^2-\frac{1}{2}\mathrm{Tr}(\partial_\mu\phi^{IJ})^2+\text{(fermionic part)}. \qquad (2.2.2)$$

By adding interaction term

$$\mathcal{L}_{\text{int}}=-2i\delta g_{\text{YM}}\mathrm{Tr}(\partial_\mu\phi^{IJ}[A^\mu,\phi^{IJ}])+\text{(fermionic term)} \qquad (2.2.3)$$

into (2.2.2), and by repeating this infinitesimal deformation, we obtain $\mathcal{N}=4$ $U(N)$ SYM. The interaction term (2.2.3) is an exactly marginal deformation.

2.2.2 Non-Lagrangian theories

As we have seen, if a theory has free parameters (coupling constants), a Lagrangian of the theory is given by perturbations of a free field theory. However, there is no general formulation of a construction of Lagrangian when a theory does not have a free parameter. In this case, such a theory is called non-Lagrangian theory. To be more precise, we define "non-Lagrangian theories" as CFTs having following property:

(♠) They cannot be obtained by continuous marginal deformations of any free field theories.

For example, let us consider a theory which contains electric and magnetic particles, and these electric particles and magnetic particles arise as the massless degrees of freedom. If this theory is a CFT, we must tune the value of the coupling constant e. One loop β-function of the theory is

$$\beta(e)\sim e^3-\frac{1}{e}. \qquad (2.2.4)$$

Then this theory is a CFT only if the coupling constant satisfies $\beta(e)=0$. In general, this type of CFTs are called isolated CFT. According to the above definition (♠), this theory is a non-Lagrangian theory. Actually, it is known that we cannot write local coupling of magnetic particles and gauge field. It is known that there are such kind of non-Lagrangian theories called Argyres-Douglas (AD) theories [3, 4]. AD theories were found at special point on the Coulomb branch of 4d $\mathcal{N}=2$ gauge

[1] For simplicity, we consider $U(N)$ SYM and change definition of covariant derivative from (1.2.1) as $D_\mu\phi=\partial_\mu\phi-ig_{\text{YM}}[A_\mu,\phi]$.

theories and they contain massless electric and magnetic particles. AD theories have been studied intensively in recent years.

We would like to consider a certain class of non-Lagrangian theories : genuine 4d $\mathcal{N}=3$ SCFT. For a long time, only $\mathcal{N}=1,2,4$ supersymmetric field theories had been known, and no one had known whether there are genuine $\mathcal{N}=3$ supersymmetric theories or not. One of the difficulties in the construction of $\mathcal{N}=3$ theory is that this theory cannot be free field theory. The only $\mathcal{N}=3$ multiplet of free field is the vector multiplet. The superconformal primary of the multiplet is scalar field ϕ^I $(I=1,2,3)$, and the multiplet is constructed by acting supercharges $Q^I_\alpha, \overline{Q}_{\dot{\alpha}I}$ on ϕ^I,

$$F^+_{\mu\nu} \overset{Q}{\leftarrow} \lambda_{\alpha I} \overset{Q}{\leftarrow} \phi^I \overset{\overline{Q}}{\rightarrow} \widetilde{\lambda}_{\dot{\alpha}}. \tag{2.2.5}$$

Because this multiplet is not CPT invariant, we have to combine it with the CPT conjugate multiplet

$$\overline{\widetilde{\lambda}}_\alpha \overset{Q}{\leftarrow} \overline{\phi}_I \overset{\overline{Q}}{\rightarrow} \overline{\lambda}^I_{\dot{\alpha}} \overset{\overline{Q}}{\rightarrow} F^-_{\mu\nu}. \tag{2.2.6}$$

The combination of these multiplets coincides with the $\mathcal{N}=4$ vector multiplet. Because this is the only free field multiplet of $\mathcal{N}=3$, the free $\mathcal{N}=3$ supersymmetric field theory automatically becomes $\mathcal{N}=4$ supersymmetric field theory. Conversely, if there are genuine 4d $\mathcal{N}=3$ supersymmetric theories, they must be strongly coupled and there are no weakly coupled descriptions.

2.3 Representations of superconformal algebra

2.3.1 $\mathcal{N}=2$ SCFT

An $\mathcal{N}=3$ SCFT can be regarded as a special $\mathcal{N}=2$ SCFT. First, we consider representations of $\mathcal{N}=2$ superconformal algebra[29] [2].

$\mathcal{N}=2$ superconformal group contains R-symmetry subgroup $SU(2)_R \times U(1)_r$. In order to denote a representation, we use $[j,\overline{j}]^{(R;r)}_E$ where R is the Dynkin label of $SU(2)_R$ and r is the $U(1)_r$ charge. The $\mathcal{N}=2$ supercharges $Q^I_\alpha, \overline{Q}_{\dot{\alpha}I}$ $(I=1,2)$ belong to the representations

$$Q \in [\frac{1}{2},0]^{(1;-1)}_{1/2}, \quad \overline{Q} \in [0,\frac{1}{2}]^{(1;1)}_{1/2}. \tag{2.3.1}$$

[2] We use notations of classifications appeared in [30]

An irreducible representation of the superconformal algebra is decomposed into irreducible representations of conformal algebra. For the construction of a superconformal representation it is enough to list up the conformal primaries for these conformal representations, and we can omit the conformal descendants obtained from them by P_μ action. Then we can treat Q and $\overline{Q}$ as if they are anti-commute. The states are treated as the tensor products of two sectors: the sector generated by Q and the sector generated by $\overline{Q}$.

The shortening conditions of $\mathcal{N}=2$ superconformal representations are classified in Tables 2.1 and 2.2[30]. Corresponding to two sectors associated with Q

Table 2.1: $\mathcal{N}=2$ SCFT Q shortening conditons

Name	Primary	Bound	Q Null state
L	$[j,\overline{j}]_\Delta^{(R;r)}$	$\Delta > 2+2j+R-\frac{1}{2}r$	
A_1	$[j,\overline{j}]_\Delta^{(R_1,R_2;r)},\ j\geq\frac{1}{2}$	$\Delta = 2+2j+R-\frac{1}{2}r$	$[j-\frac{1}{2},\overline{j}]_{\Delta+\frac{1}{2}}^{(R+1;r-1)}$
A_2	$[0,\overline{j}]_\Delta^{(R;r)}$	$\Delta = 2+R-\frac{1}{2}r$	$[0,\overline{j}]_{\Delta+1}^{(R+2;r-2)}$
B_1	$[0,\overline{j}]_\Delta^{(R;r)}$	$\Delta = R-\frac{1}{2}r$	$[\frac{1}{2},\overline{j}]_{\Delta+\frac{1}{2}}^{(R+1;r-1)}$

Table 2.2: $\mathcal{N}=2$ SCFT $\overline{Q}$ shortening conditons

Name	Primary	Bound	$\overline{Q}$ Null state
$\overline{L}$	$[j,\overline{j}]_\Delta^{(R;r)}$	$\Delta > 2+2\overline{j}+R+\frac{1}{2}r$	
$\overline{A}_1$	$[j,\overline{j}]_\Delta^{(R;r)},\ \overline{j}\geq\frac{1}{2}$	$\Delta = 2+2\overline{j}+R+\frac{1}{2}r$	$[j,\overline{j}-\frac{1}{2}]_{\Delta+\frac{1}{2}}^{(R+1;r+1)}$
$\overline{A}_2$	$[j,0]_\Delta^{(R;r)}$	$\Delta = 2+R+\frac{1}{2}r$	$[j,0]_{\Delta+1}^{(R+2;r+2)}$
$\overline{B}_1$	$[j,0]_\Delta^{(R;r)}$	$\Delta = R+\frac{1}{2}r$	$[j,\frac{1}{2}]_{\Delta+\frac{1}{2}}^{(R+1;r+1)}$

and $\overline{Q}$, there are two set of the shortening conditions. We obtain many types of representations by choosing one condition from each set.

We show some examples of representations.

- $\mathcal{N}=2$ vector mutiplet (Table 2.3)
  The representation $A_2\overline{B}_1[0;0]_1^{(0;2)}$ is the $\mathcal{N}=2$ vector multiplet, which contains a gauge field, two gauginos, and a complex scalar.

Table 2.3: $\mathcal{N}=2$ vector multiplet

	$R=-1$	$R=0$	$R=1$
$r=0$		$F_{\mu\nu}$	
$r=1$	λ^1		λ^2
$r=2$		ϕ	

- $\mathcal{N}=2$ hypermultiplet (Table 2.4)
  The representation $B_1\overline{B}_1[0,0]_1^{(1;0)}$ is the $\mathcal{N}=2$ hypermultiplet, which contains

Table 2.4: $\mathcal{N}=2$ hypermultiplet

	$R=-1$	$R=0$	$R=1$
$r=1$		$\widetilde{\psi}^\dagger$	
$r=0$	q		$\widetilde{q}^\dagger$
$r=-1$		ψ	

  two complex scalar fields and two Weyl spinor fields.

If a theory has non-trivial moduli space, it includes operators whose vacuum expectation values parametrize the moduli space. In $\mathcal{N}=2$ SCFT, it is known that the moduli space is parametrized by two types of operators.

- Higgs branch operators
  The Higgs branch operators are scalar operators annihilated by Q^1 and $\overline{Q}_2$. This conditions correspond to $B_1\overline{B}_1[0,0]_R^{(R;0)}$. Note that Higgs branch operators are $U(1)_r$ neutral and satisfy $\Delta=R$. Scalar fields in hypermultiplets are examples of Higgs branch operators. The subspace of the moduli space which is parametrized by Higgs branch operator is called the Higgs branch.

- Coulomb branch operators
  The Coulomb branch operators are scalar operators annihilated by $\overline{Q}_I$. This conditions correspond to $X\overline{B}_1[0,0]_{r/2}^{(0;r)}$, where $X=L,A_1,A_2,B_1$. Scalar fields in vector multiplets are examples of Coulomb branch operators. The subspace of the moduli space which is parametrized by Coulomb branch operators is called the Coulomb branch.

2.3.2 $\mathcal{N}=3$ SCFT

Let us consider representations of $\mathcal{N}=3$ superconformal symmetry. The R-symmetry subgroup of the $\mathcal{N}=3$ superconformal group is $SU(3)_R \times U(1)_r$. We use the notation

$$[j,\bar{j}]_{\Delta}^{(R_1,R_2;r)} \tag{2.3.2}$$

to write a representation. (R_1, R_2) are the Dynkin labels of $SU(3)_R$ and r is the $U(1)_r$ R-charge. 4d $\mathcal{N}=3$ supercharges $Q^I_\alpha, \overline{Q}_{\dot{\alpha}I}$ $(I=1,2,3)$ belong to

$$Q \in [\frac{1}{2},0]_{1/2}^{(1,0;-1)}, \quad \overline{Q} \in [0,\frac{1}{2}]_{1/2}^{(0,1;1)}. \tag{2.3.3}$$

The shortening conditions for $\mathcal{N}=3$ representations are classified in Tables 2.5 and 2.6.

Table 2.5: $\mathcal{N}=3$ SCFT Q shortening conditons

Name	Primary	Bound	Q Null state
L	$[j,\bar{j}]_{\Delta}^{(R_1,R_2;r)}$	$\Delta > 2+2j+\frac{2}{3}(2R_1+R_2)-\frac{1}{6}r$	
A_1	$[j,\bar{j}]_{\Delta}^{(R_1,R_2;r)},\ j\geq\frac{1}{2}$	$\Delta = 2+2j+\frac{2}{3}(2R_1+R_2)-\frac{1}{6}r$	$[j-\frac{1}{2},\bar{j}]_{\Delta+\frac{1}{2}}^{(R_1+1,R_2;r-1)}$
A_2	$[0,\bar{j}]_{\Delta}^{(R_1,R_2;r)}$	$\Delta = 2+\frac{2}{3}(2R_1+R_2)-\frac{1}{6}r$	$[0,\bar{j}]_{\Delta+1}^{(R_1+2,R_2;r-2)}$
B_1	$[0,\bar{j}]_{\Delta}^{(R_1,R_2;r)}$	$\Delta = \frac{2}{3}(2R_1+R_2)-\frac{1}{6}r$	$[\frac{1}{2},\bar{j}]_{\Delta+\frac{1}{2}}^{(R_1+1,R_2;r-1)}$

Table 2.6: $\mathcal{N}=3$ SCFT $\overline{Q}$ shortening conditons

Name	Primary	Bound	$\overline{Q}$ Null state
$\overline{L}$	$[j,\bar{j}]_{\Delta}^{(R_1,R_2;r)}$	$\Delta > 2+2\bar{j}+\frac{2}{3}(R_1+2R_2)+\frac{1}{6}r$	
$\overline{A}_1$	$[j,\bar{j}]_{\Delta}^{(R_1,R_2;r)},\ \bar{j}\geq\frac{1}{2}$	$\Delta = 2+2\bar{j}+\frac{2}{3}(R_1+2R_2)+\frac{1}{6}r$	$[j,\bar{j}-\frac{1}{2}]_{\Delta+\frac{1}{2}}^{(R_1,R_2+1;r+1)}$
$\overline{A}_2$	$[j,0]_{\Delta}^{(R_1,R_2;r)}$	$\Delta = 2+\frac{2}{3}(R_1+2R_2)+\frac{1}{6}r$	$[j,0]_{\Delta+1}^{(R_1,R_2+2;r+2)}$
$\overline{B}_1$	$[j,0]_{\Delta}^{(R_1,R_2;r)}$	$\Delta = \frac{2}{3}(R_1+2R_2)+\frac{1}{6}r$	$[j,\frac{1}{2}]_{\Delta+\frac{1}{2}}^{(R_1,R_2+1;r+1)}$

We introduce some multiplets that we use later.

Table 2.7: $\mathcal{N}=3$ vector multiplet $B_1\overline{B}_1[0,0]_1^{(1,0;2)}$

$\Delta\backslash r$	$r=0$	$r=1$	$r=2$	$r=3$
$\Delta=1$			$[0,0]_1^{(1,0;2)}$	
$\Delta=3/2$		$[\frac{1}{2},0]_{3/2}^{(0,1;1)}$		$[0,\frac{1}{2}]_{3/2}^{(0,0;3)}$
$\Delta=2$	$[1,0]_2^{(0,0;0)}$			

- $\mathcal{N}=3$ vector multiplet (Table 2.7)
  The representation $B_1\overline{B}_1[0,0]_1^{(1,0;2)}$ is the $\mathcal{N}=3$ vector multiplet, which contains (self-dual) gauge field $F^+_{\mu\nu}$, gauginos $\lambda_I, \widetilde{\lambda}$, scalars ϕ^I. This and its complex conjugate are the unique multiplets consisting of free fields.

- $\mathcal{N}=3$ stress tensor multiplet (Table 2.8)
  The representation $B_1\overline{B}_1[0,0]_2^{(1,1;0)}$ is the stress tensor multiplet, which contains the $SU(3)_R$ and $U(1)_r$ conserved currents $j^{\mu I}_J, j^\mu$, $\mathcal{N}=3$ supercurrents $J^{\mu I}_\alpha$, and the stress tensor $T_{\mu\nu}$.

Table 2.8: $\mathcal{N}=3$ tensor multiplet$B_1\overline{B}_1[0,0]_2^{(1,1;0)}$

$\Delta\backslash r$	$r=-3$	$r=-2$	$r=-1$	$r=0$	$r=1$	$r=2$	$r=3$
$\Delta=2$				$[0,0]_2^{(1,1;0)}$			
$\Delta=5/2$			$[\frac{1}{2},0]_{5/2}^{(1,0;-1)}$		$[0,\frac{1}{2}]_{5/2}^{(0,1;1)}$		
			$[\frac{1}{2},0]_{5/2}^{(0,2;-1)}$		$[0,\frac{1}{2}]_{5/2}^{(2,0;1)}$		
$\Delta=3$		$[1,0]_3^{(0,1;-2)}$		$[\frac{1}{2},\frac{1}{2}]_3^{(1,1;0)}$		$[0,1]_3^{(1,0;2)}$	
		$[0,0]_3^{(0,1;-2)}$		$[\frac{1}{2},\frac{1}{2}]_3^{(0,0;0)}$		$[0,0]_3^{(1,0;2)}$	
$\Delta=7/2$	$[\frac{1}{2},0]_{7/2}^{(0,0;-3)}$		$[1,\frac{1}{2}]_{7/2}^{(1,0;-1)}$		$[\frac{1}{2},1]_{7/2}^{(0,1;1)}$		$[0,\frac{1}{2}]_{7/2}^{(0,0;3)}$
$\Delta=4$				$[1,1]_4^{(0,0;0)}$			

- $B_1\overline{B}_1[0,0]^{(2,0;4)}$ (Table 2.9)
  The representation $B_1\overline{B}_1[0,0]^{(2,0;4)}$ contains marginal deformation operator $\mathcal{O}_{\rm MD}$ and an extra supercurrent $\mathcal{J}^\mu_\alpha$. This multiplet plays an important role to discuss properties of $\mathcal{N}=3$ SCFT.

2.3.3 $\mathcal{N}=4$ SCFT

We will consider a supersymmetry enhancement from $\mathcal{N}=3$ to $\mathcal{N}=4$. Here we explain $\mathcal{N}=4$ superconformal representations. The R-symmetry subgroup of the

Table 2.9: $B_1\overline{B}_1[0,0]_2^{(2,0;4)}$ multiplet contains marginal deformation and extra supercurrent

	$r=0$	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$
$\Delta=2$					$[0,0]_2^{(2,0;4)}$		
$\Delta=5/2$				$[\frac{1}{2},0]_{5/2}^{(1,1;3)}$		$[0,\frac{1}{2}]_{5/2}^{(1,0;5)}$	
$\Delta=3$			$[0,0]_3^{(0,2;2)}$ $[1,0]_3^{(1,0;2)}$		$[\frac{1}{2},\frac{1}{2}]_3^{(0,1;4)}$		$[0,0]_3^{(0,0;6)}$
$\Delta=7/2$		$[\frac{1}{2},0]_{7/2}^{(0,1;1)}$		$[1,\frac{1}{2}]_{7/2}^{(0,0;3)}$			
$\Delta=4$	$[0,0]_4^{(0,0;0)}$						

$\mathcal{N}=4$ superconformal group is $SU(4)_R$. We use the notation

$$[j,\overline{j}]_\Delta^{(R_1,R_2,R_3)} \tag{2.3.4}$$

to write a representation. (R_1,R_2,R_3) are the Dynkin labels of $SU(4)_R$. 4d $\mathcal{N}=4$ supercharges $Q_\alpha^I, \overline{Q}_{\dot{\alpha}I}$ $(I=1,\ldots,4)$ belong to

$$Q\in[\frac{1}{2},0]_{1/2}^{(1,0,0)},\quad \overline{Q}\in[0,\frac{1}{2}]_{1/2}^{(0,0,1)}. \tag{2.3.5}$$

The shortening conditions for $\mathcal{N}=4$ representations are classified in Tables 2.10 and 2.11.

Table 2.10: $\mathcal{N}=4$ SCFT Q shortening conditons

Name	Primary	Bound	Q Null state
L	$[j,\overline{j}]_\Delta^{(R_1,R_2,R_3)}$	$\Delta>2+2j+\frac{1}{2}(3R_1+2R_2+R_3)$	
A_1	$[j,\overline{j}]_\Delta^{(R_1,R_2,R_3)},\ j\geq\frac{1}{2}$	$\Delta=2+2j+\frac{1}{2}(3R_1+2R_2+R_3)$	$[j-\frac{1}{2},\overline{j}]_{\Delta+\frac{1}{2}}^{(R_1+1,R_2,R_3)}$
A_2	$[0,\overline{j}]_\Delta^{(R_1,R_2,R_3)}$	$\Delta=2+\frac{1}{2}(3R_1+2R_2+R_3)$	$[0,\overline{j}]_{\Delta+1}^{(R_1+2,R_2,R_3)}$
B_1	$[0,\overline{j}]_\Delta^{(R_1,R_2,R_3)}$	$\Delta=\frac{1}{2}(3R_1+2R_2+R_3)$	$[\frac{1}{2},\overline{j}]_{\Delta+\frac{1}{2}}^{(R_1+1,R_2,R_3)}$

We list some multiplets of $\mathcal{N}=4$ SCFT.

Table 2.11: $\mathcal{N}=4$ SCFT $\overline{Q}$ shortening conditons

Name	Primary	Bound	$\overline{Q}$ Null state
$\overline{L}$	$[j,\overline{j}]_\Delta^{(R_1,R_2,R_3)}$	$\Delta > 2+2\overline{j}+\frac{1}{2}(R_1+2R_2+3R_3)$	
$\overline{A}_1$	$[j,\overline{j}]_\Delta^{(R_1,R_2,R_3)}$, $\overline{j}\geq\frac{1}{2}$	$\Delta = 2+2\overline{j}+\frac{1}{2}(R_1+2R_2+3R_3)$	$[j,\overline{j}-\frac{1}{2}]_{\Delta+\frac{1}{2}}^{(R_1,R_2+1,R_3+1)}$
$\overline{A}_2$	$[j,0]_\Delta^{(R_1,R_2,R_3)}$	$\Delta = 2+\frac{1}{2}(R_1+2R_2+3R_3)$	$[j,0]_{\Delta+1}^{(R_1,R_2,R_3+2)}$
$\overline{B}_1$	$[j,0]_\Delta^{(R_1,R_2,R_3)}$	$\Delta = \frac{1}{2}(R_1+2R_2+3R_3)$	$[j,\frac{1}{2}]_{\Delta+\frac{1}{2}}^{(R_1,R_2,R_3+1)}$

- $\mathcal{N}=4$ vector multiplet
  The representation $B_1\overline{B}_1[0,0]_1^{(0,1,0)}$ is the $\mathcal{N}=4$ vector multiplet. This multiplet contains scalar fields ϕ^{IJ}, gauginos $\lambda_I, \overline{\lambda}^I$, and gauge field $F_{\mu\nu}$.

- $\mathcal{N}=4$ stress tensor multiplet (Table 2.12)
  The representation $B_1\overline{B}_1[0,0]_2^{(0,2,0)}$ is an $\mathcal{N}=4$ stress tensor multiplet. This

Table 2.12: $\mathcal{N}=4$ stress tensor multiplet$B_1\overline{B}_1[0,0]_2^{(0,2,0)}$

$\Delta=2$					$[0,0]_2^{(0,2,0)}$				
$\Delta=5/2$				$[\frac{1}{2},0]_{5/2}^{(0,1,1)}$		$[0,\frac{1}{2}]_{5/2}^{(1,1,0)}$			
$\Delta=3$			$[1,0]_3^{(0,1,0)}$		$[\frac{1}{2},\frac{1}{2}]_3^{(1,0,1)}$		$[0,1]_3^{(0,1,0)}$		
			$[0,0]_3^{(0,0,2)}$				$[0,0]_3^{(0,1,0)}$		
$\Delta=7/2$		$[\frac{1}{2},0]_{7/2}^{(0,0,1)}$		$[1,\frac{1}{2}]_{7/2}^{(1,0,0)}$		$[\frac{1}{2},1]_{7/2}^{(0,0,1)}$		$[0,\frac{1}{2}]_{7/2}^{(1,0,0)}$	
$\Delta=4$	$[0,0]_4^{(0,0,0)}$				$[1,1]_4^{(0,0;0)}$				$[0,0]_4^{(0,0,0)}$

multiplet contains $SU(4)_R$ current $[\frac{1}{2},\frac{1}{2}]_3^{(1,0,1)}$, $\mathcal{N}=4$ supercurrents $[1,\frac{1}{2}]_{7/2}^{(1,0,0)}$ and $[\frac{1}{2},1]_{7/2}^{(0,0,1)}$, two marginal deformation operators $[0,0]_4^{(0,0,0)}\times 2$, and the stress tensor $[1,1]_4^{(0,0,0)}$. In the context of $\mathcal{N}=3$ SCFT, the $\mathcal{N}=4$ stress tensor multiplet is decomposed into $\mathcal{N}=3$ stress tensor multiplet $B_1\overline{B}_1[0,0]_2^{(1,1;0)}$ and the representations which contain extra supercurrents $B_1\overline{B}_1[0,0]_2^{(2,0;4)}$ and $B_1\overline{B}_1[0,0]_2^{(0,2;-4)}$ as we list in Table 2.13.

2.4 Properties of $\mathcal{N}=3$ SCFT

In this section, we discuss properties of genuine $\mathcal{N}=3$ theories which do not have $\mathcal{N}=4$ supersymmetry[31].

Table 2.13: A decomposition of component fields of $\mathcal{N}=4$ into $\mathcal{N}=3$. Bold numbers denote dimensions of $SU(\mathcal{N})_R$ representations.

	$\mathcal{N}=4$ stress tensor multiplet	$B_1\overline{B}_1[0,0]_2^{(2,0;4)}$	$B_1\overline{B}_1[0,0]_2^{(1,1;0)}$	$B_1\overline{B}_1[0,0]_2^{(0,2;-4)}$
Bottom	$\mathbf{20}[0,0]_2^{(0,2,0)}$	$\mathbf{6}[0,0]_2^{(2,0;4)}$	$\mathbf{8}[0,0]_2^{(1,1;0)}$	$\overline{\mathbf{6}}[0,0]^{(0,2;-4)}$
R-current	$\mathbf{15}[\frac{1}{2},\frac{1}{2}]_3^{(1,0,1)}$	$\overline{\mathbf{3}}[\frac{1}{2},\frac{1}{2}]_3^{(0,1;4)}$	$\mathbf{8}[\frac{1}{2},\frac{1}{2}]_3^{(1,1;0)} \oplus \mathbf{1}[\frac{1}{2},\frac{1}{2}]_3^{(0,0;0)}$	$\mathbf{3}[\frac{1}{2},\frac{1}{2}]_3^{(1,0;-4)}$
Supercurrent	$\mathbf{4}[1,\frac{1}{2}]_{7/2}^{(1,0,0)}$	$\mathbf{1}[1,\frac{1}{2}]_{7/2}^{(0,0;3)}$	$\mathbf{3}[1,\frac{1}{2}]_{7/2}^{(1,0;-1)}$	
	$\overline{\mathbf{4}}[\frac{1}{2},1]_{7/2}^{(0,0,1)}$		$\overline{\mathbf{3}}[\frac{1}{2},1]_{7/2}^{(0,1;1)}$	$\mathbf{1}[\frac{1}{2},1]_{7/2}^{(0,0;-3)}$
Marginal def.	$\mathbf{1}[0,0]_4^{(0,0,0)}\times 2$	$\mathbf{1}[0,0]_4^{(0,0,0)}$		$\mathbf{1}[0,0]_4^{(0,0,0)}$
Stress tensor	$\mathbf{1}[1,1]_4^{(0,0,0)}$		$\mathbf{1}[1,1]_4^{(0,0,0)}$	

2.4.1 Embedding $\mathcal{N}=2$ R-symmetry group in $\mathcal{N}=3$ R-symmetry group

We regard $\mathcal{N}=3$ SCFT as a subclass of $\mathcal{N}=2$ SCFT. In this section, we denote the $\mathcal{N}=3$ supercharge by $Q'^{I'}$ $(I'=1,2,3)$ and the $\mathcal{N}=2$ supercharge by Q^I $(I=1,2)$. The representations of $Q'^{I'}$ and Q^I for the R-symmetry group $SU(3)_R\times U'(1)_r$ and $SU(2)_R\times U(1)_r$ are listed in Table 2.14 and Table 2.15. From the viewpoint of $\mathcal{N}=2$ theory, only the subgroup $SU(2)_R\times U(1)_r\times U(1)_F\subset SU(3)_R\times U'(1)_r$ of $\mathcal{N}=3$ R-symmetry is manifest. We use the following notation for Cartan generators:

Table 2.14: $SU(3)_R\times U'(1)_r$ charge of $\mathcal{N}=3$ supercharges

	$SU(3)_R$	$r^{\mathcal{N}=3}$
Q'	$\mathbf{3}$	-1
$\overline{Q}'$	$\overline{\mathbf{3}}$	$+1$

Table 2.15: $SU(2)_R\times U(1)_r$ charge of $\mathcal{N}=2$ supercharges

	$SU(2)_R$	$r^{\mathcal{N}=2}$
Q	$\mathbf{2}$	-1
$\overline{Q}$	$\overline{\mathbf{2}}$	$+1$

- $\mathcal{N}=2$ $\cdots$ $SU(2)_R : I_3$, $U(1)_r : r^{\mathcal{N}=2}$, $U(1)_F : F$
- $\mathcal{N}=3$ $\cdots$ $SU(3)_R : T_3, T_8$, $U(1)'_r : r^{\mathcal{N}=3}$

The explicit form of T_3 and T_8 are

$$T_3=\frac{1}{2}\lambda_3=\frac{1}{2}\left(\begin{array}{cc|c}1&&\\&-1&\\\hline&&0\end{array}\right),\quad T_8=\frac{1}{2}\lambda_8=\frac{1}{2\sqrt{3}}\left(\begin{array}{cc|c}1&&\\&1&\\\hline&&-2\end{array}\right). \tag{2.4.1}$$

We embed the $SU(2)_R$ in the top-left 2×2 block of the $SU(3)_R$, so that its Cartan generator I_3 is equal to T_3.

F and $r^{\mathcal{N}=2}$ are given by

$$r^{\mathcal{N}=2} = \frac{1}{3} r^{\mathcal{N}=3} - \frac{4}{\sqrt{3}} T_8, \tag{2.4.2}$$

$$F = r^{\mathcal{N}=3} + 2\sqrt{3} T_8. \tag{2.4.3}$$

These relations can be obtained from the following requirements.

- If we embed the $\mathcal{N}=2$ and $\mathcal{N}=3$ R-symmetry group in $\mathcal{N}=4$ R-symmetry group $SU(4)_R$, the action of $r^{\mathcal{N}=2}$ and $r^{\mathcal{N}=3}$ on the supercharges (Q^1, Q^2, Q^3, Q^4) must be

$$r^{\mathcal{N}=2} = \mathrm{diag}(-1,-1,1,1), \quad r^{\mathcal{N}=3} = \mathrm{diag}(-1,-1,-1,3). \tag{2.4.4}$$

- The $\mathcal{N}=2$ supercharges Q^I must be $U(1)_F$ neutral, while the third component of $\mathcal{N}=3$ supercharge Q'^3 has non-vanishing $U(1)_F$ charge.

2.4.2 Coulomb branch operators in $\mathcal{N}=3$ SCFT

As is explained in section 2.3.1 there are two branches in the moduli space, Higgs branch and Coulomb branch, in $\mathcal{N}=2$ supersymmetric theories. If $\mathcal{N}=3$ SCFT have a moduli space, since the only free field representation in $\mathcal{N}=3$ theories is the vector multiplet, the low-energy effective theory at a generic point on the moduli space should be described by vector multiplets. Such a moduli space should be called $\mathcal{N}=3$ Coulomb branch. The $\mathcal{N}=3$ free vector multiplet has six scalars, and then a rank r moduli space is parametrized by $6r$ scalars.

We can show that the operators parametrizing this moduli space belong to representations $B_1\overline{B}_1[0,0]^{(R_1,0;2R_1)}_{R_1}$. In terms of $\mathcal{N}=2$ SCFT, $\mathcal{N}=3$ vector multiplet is made of a vector multiplet and a hypermultiplet. The bottom components of the $\mathcal{N}=2$ vector multiplet are scalar fields which labels $2r$-dimensional Coulomb branch, and bottom components of the $\mathcal{N}=2$ hypermultiplet parametrize $4r$-dimensional Higgs branch. Because $\mathcal{N}=3$ SCFT is regarded as a special $\mathcal{N}=2$ SCFT, we can use the information of $\mathcal{N}=2$ Higgs and Coulomb branch operators.

First of all, we determine the irreducible decomposition of representations of $SU(3)_R \times U'(1)_r$ in the subgroup $SU(2)_R \times U(1)_r$. A state labeled by $SU(3)_R$ Dynkin labels $[R_1, R_2]$ has the Cartan charges $(T_3, T_8) = (\frac{R_1+R_2}{2}, \frac{R_1-R_2}{2\sqrt{3}})$. Since the $U(1)_r$ charge is given by (2.4.2), $[R_1, R_2]$ state has the $U(1)_r$ charge

$$r^{\mathcal{N}=2} = \frac{1}{3} r^{\mathcal{N}=3} - \frac{2}{3}(R_1 - R_2). \tag{2.4.5}$$

Let us consider which $\mathcal{N}=3$ representations correspond to the Coulomb branch operators. The $\mathcal{N}=3$ moduli space contains a subspace of $\mathcal{N}=2$ Coulomb branch and a subspace of $\mathcal{N}=2$ Higgs branch. These subspaces are parametrized by the bottom components of $X\overline{B}_1[0,0]^{(0;r)}_{r/2}$ and $B_1\overline{B}_1[0,0]^{(R;0)}_{R}$ (in terms of $\mathcal{N}=2$ SCFT), respectively. The $\mathcal{N}=2$ $B_1\overline{B}_1[0,0]^{(R;0)}_{R}$ representation should be in the $\mathcal{N}=3$ $B_1\overline{B}_1[0,0]^{(R_1,R_2;r^{\mathcal{N}=3})}_{\Delta}$ representation. Because of the $B_1\overline{B}_1$ condition, the $r^{\mathcal{N}=3}$ and Δ of this representation are given by

$$r^{\mathcal{N}=3} = 2(R_1 - R_2), \tag{2.4.6}$$

$$\Delta = R_1 + R_2. \tag{2.4.7}$$

It is known that if the $SU(3)_R \times U'(1)_r$ representation $(R_1, R_2; r^{\mathcal{N}=3})$ is decomposed to the irreducible representations of the subgroup $SU(2)_R \times U(1)_r$, the unique $SU(2)_R$ singlet representation is labeled by

$$(R; r^{\mathcal{N}=2}) = (0; \frac{1}{3}r^{\mathcal{N}=3} + \frac{4}{3}(R_1 - R_2)). \tag{2.4.8}$$

Thus we regard (2.4.8) as the $\mathcal{N}=2$ Coulomb branch operator. The $\mathcal{N}=2$ Coulomb branch operators must satisfy $\Delta = |r^{\mathcal{N}=2}|/2$. By substituting (2.4.6) to (2.4.8), we can see the Coulomb branch operator (2.4.8) has $\Delta = |R_1 - R_2|$, while the dimension of this operator should be (2.4.7). Therefore the $\mathcal{N}=3$ Coulomb branch operator must be labeled by

$$B_1\overline{B}_1[0,0]^{(R_1,0;2R_1)}_{R_1}, \tag{2.4.9}$$

or its complex conjugate.

However, some of $B_1\overline{B}_1[0,0]^{(R_1,0;2R_1)}_{R_1}$ representations are not allowed to exist. $B_1\overline{B}_1[0,0]^{(1,0;2)}_{1}$ is a free vector multiplet, and $B_1\overline{B}_1[0,0]^{(2,0;4)}_{2}$ contains extra supercurrent. Thus if these multiplets exist in a theory, the supersymmetry is enhanced to $\mathcal{N}=4$. Therefore in a genuine $\mathcal{N}=3$ SCFT, only $B_1\overline{B}_1[0,0]^{(R_1,0;2R_1)}_{R_1}$ $(R_1 \geq 3)$ are allowed to exist. This result was confirmed for rank-1 $\mathcal{N}=3$ theories in [32, 33].

2.4.3 Marginal operators

Another important property of genuine $\mathcal{N}=3$ SCFT is the absence of marginal deformation operators. The coupling constant of the theories must not be a free parameter. This is shown as follows.

For the existence of marginal deformation, there must be representations whose top component has quantum numbers $[0,0]^{(0,0;0)}_{4}$. This condition is restrictive and the only representations satisfying this condition are $B_1\overline{B}_1[0,0]_2^{(2,0;4)}$ and its complex conjugate (see Table 2.9). However, these multiplets contain extra supercurrents with quantum numbers $[1,\frac{1}{2}]_{7/2}^{(0,0;3)}$ or $[\frac{1}{2},1]_{7/2}^{(0,0;-3)}$ which cause supersymmetry enhancement from $\mathcal{N}=3$ to $\mathcal{N}=4$. Therefore a genuine $\mathcal{N}=3$ theory cannot have marginal deformations.

Deformations of a CFT by marginal operators make a family of CFTs labeled by continuous coupling constants. Because $\mathcal{N}=3$ SCFT do not have marginal operators, the coupling constants of the theory are fixed and there is no weak coupling region.

Summary

We summarize the properties of genuine $\mathcal{N}=3$ theories, which are important in the discussions in the next chapter.

- No marginal deformation operators

  Because marginal deformation operators and extra supercurrent belong to the same multiplet, an $\mathcal{N}=3$ theory containing marginal deformation operators should have $\mathcal{N}=4$ supersymmetry. Coupling constants of genuine $\mathcal{N}=3$ theories take a fixed values because of no marginal deformation.

- Dimensions of Coulomb branch operators must obey $\Delta \geq 3$

  If an $\mathcal{N}=3$ theory contain Coulomb branch operators whose dimensions are less than or equal to 2, the supersymmetry of the theory is enhanced to $\mathcal{N}=4$.

Chapter 3

4 dimensional $\mathcal{N} = 3$ supersymmetric field theories

An explicit construction of $\mathcal{N} = 3$ theories was suggested by García-Etxebarria and Regalado in [9]. They realize $\mathcal{N} = 3$ theories on a stack of N D3-branes by performing a $\mathbb{Z}_k$ orbifolding that project out one of four supercharges on the D3-branes. We call the orbifold "S-fold" following[10]. Aharony and Tachikawa [10] listed the dimensions of the Coulomb branch operators in these theories. In particular, they contain an operator with dimension N. Because genuine $\mathcal{N} = 3$ SCFT cannot have Coulomb branch operators with dimension 1 or 2, then, the theory with $N = 1$ or 2 must have hidden $\mathcal{N} = 4$ symmetry.

The purpose of this chapter is to obtain non-trivial evidence of this interesting phenomenon by analyzing the spectra of these theories. Because each theory is defined by a system of D3-branes in an S-fold, we can realize dyonic particles by string junctions [24, 25, 26, 27, 28] connecting D3-branes and the S-plane.

We compare the spectrum with the spectrum obtained by using another realization of the theory in which the $\mathcal{N} = 4$ symmetry is manifest. The $\mathcal{N} = 4$ theories with SU, SO, and Sp gauge groups have a simpler brane realization, in which $\mathcal{N} = 4$ supersymmetry and the associated central charges are realized perturbatively. We first compare the central charges Z that is realized perturbatively both in the S-fold and in the perturbative set-up. With this information, we establish relations between Coulomb moduli and dyonic charges in the S-fold and those in the other set up [12].

3.1 S-fold construction

In this section, we review how to construct four-dimensional $\mathcal{N} = 3$ field theory by using S-fold [9]. The S-fold is a $\mathbb{Z}_k$ ($k \in \mathbb{Z}$) orbifold such that only three supercharges on the D3-branes are preserved. Such an orbifold can be defined by combining $U(1)_R$ symmetry of type IIB supergravity [34, 35] and a rotation of the transverse space $\mathbb{R}^6$.

3.1.1 BPS bounds

Let us consider supersymmetry and BPS bounds in $\mathcal{N} = 4$ theories. Their anti-commutation relations are

$$\{Q^I_\alpha, \overline{Q}_{\dot\beta J}\} = (\sigma^\mu)_{\alpha\dot\beta}\delta^I_J P_\mu, \quad \{Q^I_\alpha, Q^J_\beta\} = \varepsilon_{\alpha\beta}\rho^{IJ}_m Z^{m+3}, \tag{3.1.1}$$

where Z^{m+3} ($m = 1, \dots, 6$) are central charges belonging to the vector representation of $SU(4)_R$, and ρ^{IJ}_m is an $SU(4)_R$ invariant tensor with two anti-symmetric fundamental indices and one vector index. The central charges Z^{m+3} form a complex vector in the $\mathbb{R}^6$ transverse to the D3-branes, and its real and imaginary part can be interpreted as the extension of fundamental strings and that of D-strings, respectively, in the brane realization of the $\mathcal{N} = 4$ SYM.

The BPS bound obtained from (3.1.1) is [36]

$$m^2 \geq |\operatorname{Re}\vec{Z}|^2 + |\operatorname{Im}\vec{Z}|^2 + |\operatorname{Re}\vec{Z}||\operatorname{Im}\vec{Z}|\sin\alpha, \tag{3.1.2}$$

where α is the angle between two vectors $\operatorname{Re}\vec{Z}$ and $\operatorname{Im}\vec{Z}$ in $\mathbb{R}^6$. For a junction to be BPS and to saturate this bound it must be planar [36, 37]. We consider planer junctions in 89 plane, and it is convenient to decompose the $SO(6)_{456789}$ fundamental representation into $SO(4)_{4567} \times SO(2)_{89}$ representations as

$$\mathbf{4} = (\mathbf{2}, \mathbf{1})_{+\frac{1}{2}} + (\mathbf{1}, \mathbf{2})_{-\frac{1}{2}}. \tag{3.1.3}$$

We denote the supercharges belonging to $(\mathbf{2}, \mathbf{1})_{+1/2}$ and $(\mathbf{1}, \mathbf{2})_{-1/2}$ by Q^a_α and $Q^{\dot a}_\alpha$, respectively, and their Hermitian conjugate by $\overline{Q}_{\dot\alpha a}$ and $\overline{Q}_{\dot\alpha\dot a}$, respectively. If only two components Z^8 and Z^9 out of the six are non-vanishing, Q^a_α and $Q^{\dot a}_\alpha$ anticommute to each other. Then the algebra (3.1.1) splits into two copies of $\mathcal{N} = 2$ algebra.

$$\begin{aligned}
&\{Q^a_\alpha, \overline{Q}_{\dot\beta b}\} = (\sigma^\mu)_{\alpha\dot\beta}\delta^a_b P_\mu, \quad \{Q^a_\alpha, Q^b_\beta\} = \varepsilon_{\alpha\beta}\varepsilon^{ab} Z, \\
&\{Q^{\dot a}_\alpha, \overline{Q}_{\dot\beta\dot b}\} = (\sigma^\mu)_{\alpha\dot\beta}\delta^{\dot a}_{\dot b} P_\mu, \quad \{Q^{\dot a}_\alpha, Q^{\dot b}_\beta\} = \varepsilon_{\alpha\beta}\varepsilon^{\dot a\dot b}\overline{Z}.
\end{aligned} \tag{3.1.4}$$

where $Z = Z^8 + iZ^9$ and $\overline{Z} = Z^8 - iZ^9$. Note that Z^8 and Z^9 are complex, and Z and $\overline{Z}$ are not conjugate to each other. Z and $\overline{Z}$ carry $U(1)_R \times SO(2)_{89}$ charges $(+1,+1)$ and $(+1,-1)$, respectively. Each $\mathcal{N}=2$ algebra in (3.1.4) gives the bound independently:

$$m \geq |Z|, \quad m \geq |\overline{Z}|. \tag{3.1.5}$$

3.1.2 S-fold

We consider N D3-branes in type IIB superstring theory, and 4d $\mathcal{N}=4$ $U(N)$ SYM is realized on the D3-branes. The $SU(4)_R \simeq SO(6)_R$ R-symmetry can be regarded as spacial rotation for the directions orthogonal to the branes. For the purpose of to construct an $\mathcal{N}=3$ theory, we use not only $SU(4)_R$ but also $U(1)_R$ R-symmetry in type IIB supergravity. Since a field strength $F_{\mu\nu}$ of $U(N)$ gauge field has $+1$ $U(1)_R$ charge, electric field and magnetic field are mixed by $U(1)_R$ transformation. Accordingly, electric and magnetic charges are also mixed. The $U(1)_R$ is broken by the quantization of (p,q)-string charges down to a discrete subgroup depending on the value of the complex field $\tau = C + ie^{-\phi}$.

However, since charges are quantized, the $U(1)_R$ symmetry brakes into a certain discrete subgroup. The discrete subgroup has to keep the charge lattice. We define complexified dyonic charge of (p,q)-string

$$Q \equiv p + q\tau. \tag{3.1.6}$$

Namely, τ is a complex structure of a charge lattice. For generic τ, the discrete subgroup is $\mathbb{Z}_2$. Under this $\mathbb{Z}_2$ transformation, the dyonic charge transforms as

$$\mathbb{Z}_2 : (p,q) \longrightarrow (-p,-q), \tag{3.1.7}$$

and then the orbifolded theory is still $\mathcal{N}=4$. There are special cases that discrete symmetry is enhanced when τ take a specific value. Concretely, in the case of $\tau = e^{2\pi i/k}(k=3,4,6)$ $U(1)_R$ breaks into $\mathbb{Z}_k$. The resulting orbifolds, which we call S-folds following [10], are labeled by two integers k and ℓ; k is the order of the orbifold group and ℓ is a divisor of k related to the discrete torsion. Therefore, the theory on the D3-branes are labeled by k, ℓ, and N, where N is the number of mobile D3-branes.

Let us consider an orbifolding by using $SU(4)_R$ and $U(1)_R$ symmetries. We denote $SU(4)_R$ Cartan generators by J_{45}, J_{67}, J_{89} and $U(1)_R$ charge by R. The quantum numbers of supercharges Q^a and $Q^{\dot{a}}$ are listed in Table 3.1. We introduce a new generator

$$\xi \equiv J_{45} - J_{67} + J_{89} - R, \tag{3.1.8}$$

Table 3.1: quantum numbers of $\mathcal{N} = 4$ supercharges

	Q^1	Q^2	$Q^{\dot{1}}$	$Q^{\dot{2}}$	Z	$\overline{Z}$
J_{45}	$+1/2$	$-1/2$	$+1/2$	$-1/2$	0	0
J_{67}	$+1/2$	$-1/2$	$-1/2$	$+1/2$	0	0
J_{89}	$+1/2$	$+1/2$	$-1/2$	$-1/2$	$+1$	-1
R	$+1/2$	$+1/2$	$+1/2$	$+1/2$	$+1$	$+1$
ξ	0	0	0	-2	0	-2

so that only one supercharge (here it is Q^4) is charged for ξ. An S-fold is defined by the projection which leaves states invariant under the $\mathbb{Z}_k$ action generated by

$$g = \gamma^{\xi} \quad (\gamma \equiv e^{2\pi i/k}). \tag{3.1.9}$$

The supercharges Q^a and $Q^{\dot{a}}$ and the central charges Z and $\overline{Z}$ are transformed as

$$(Q^1, Q^2, Q^{\dot{1}}, Q^{\dot{2}}) \to (Q^1, Q^2, Q^{\dot{1}}, \gamma^{-2}Q^{\dot{2}}), \quad Z \to Z, \quad \overline{Z} \to \gamma^{-2}\overline{Z}. \tag{3.1.10}$$

By performing the $\mathbb{Z}_k$ S-folding with $k \geq 3$, the supercharge Q^4 and $\overline{Z}$ are projected out and a $\mathcal{N} = 3$ theory is realized on the D3-branes. As same as an O-plane is a fixed point in an orientifold, we call the fixed point for the S-folding S-plane.

Note that the $\mathcal{N} = 3$ supercharges Q^A_α and $\overline{Q}_{\dot{\alpha}A}$ $(A = 1, 2, 3)$ satisfy the anti-commutation relations

$$\{Q^A_\alpha, \overline{Q}_{\dot{\beta}B}\} = (\sigma^\mu)_{\alpha\dot{\beta}}\delta^A_B P_\mu, \quad \{Q^A_\alpha, Q^B_\beta\} = \varepsilon_{\alpha\beta}Z^{AB}. \tag{3.1.11}$$

where Z^{AB} are $\mathcal{N} = 3$ central charges and they have anti-symmetric $SU(3)_R$ fundamental indices $Z^{AB} = -Z^{BA}$. Therefore, $\mathcal{N} = 3$ theories have three complex central charges. Although the $\mathcal{N} = 4$ theories have six central charges Z^{m+3} $(m = 1, \ldots, 6)$, the half of the central charges projected out by S-folding.

3.2 Junctions and BPS conditions

3.2.1 String realization of BPS state

For an open (p, q) string with the complex charge $Q = p + \tau q$ and the extension Δz on the 89 plane these central charges are given by

$$Z = Q\Delta z, \quad \overline{Z} = Q\Delta z^*. \tag{3.2.1}$$

(Note that Q and Δz carry $SO(2)_R \times SO(2)_{89}$ charges $(+1, 0)$ and $(0, +1)$, respectively.)

For a string junction consisting of more than one open string the central charges of the junction are given by

$$Z = \sum_i Q_i z_i, \quad \overline{Z} = \sum_i Q_i z_i^*, \tag{3.2.2}$$

where z_i are the positions of D3-branes and Q_i are complex charges of strings ending on them. If a junction contains more than two strings with different charges only one of the bounds in (3.1.5) can be saturated. To saturate the bound $m \geq |Z|$, all the constituent open strings should satisfy $m = |Z|$. This means that $\arg Z$ for the strings must be the same. Namely, the angles $\arg Q + \arg \Delta z$ are the same for all strings. We call such a junction "a holomorphic junction." On the other hand, for a junction saturating the bound $m \geq |\overline{Z}|$, the angles $\arg Q - \arg \Delta z$ are the same for all the constituent strings. We call such a junction "an anti-holomorphic junction." [1] Two bounds can be saturated at the same time only for a set of parallel strings with the same charge. Every state belongs to one of the following four types:

- $m = |Z| = |\overline{Z}|$: 1/2 BPS states
- $m = |Z| > |\overline{Z}|$: holomorphic 1/4 BPS states
- $m = |\overline{Z}| > |Z|$: anti-holomorphic 1/4 BPS states
- $m > |Z|$ and $m > |\overline{Z}|$: non-BPS states.

3.2.2 Junctions in S-folds

Let us first consider junctions in a system of n parallel D3-branes in the flat background without any S-folding. A junction $\boldsymbol{j}$ is specified by charges (p_a, q_a) $(a = 1, \ldots, n)$ of strings ending on the D3-branes. For strings ending on mobile D3-branes we always define string charges as incomming charges. We also use the complex charges $Q_a = p_a + \tau q_a$ for a string junction labeled by (p_a, q_a). The charge conservation requires Q_a to satisfy

$$\sum_{a=1}^{n} Q_a = 0. \tag{3.2.3}$$

[1] These two types of junctions are called class A and class B in [38].

When the positions of D3-branes are generic the low-energy effective theory of the parallel D3-branes is the $\mathcal{N} = 4$ $U(1)^n$ supersymmetric gauge theory. Let $U(1)_a$ be the gauge group corresponding to the a-th D3-brane. In the low-energy effective theory a junction is regarded as a particle with the $U(1)_a$ dyonic charge Q_a.

Let us consider junctions in $\mathbb{Z}_k$ S-fold. For $k \geq 3$ we set $\tau = \gamma \equiv e^{2\pi i/k}$, while for $k = 2$ we can take an arbitrary τ. Let Γ_k be the lattice on the complex plane spanned by 1 and τ. Note that g acts on the complex coordinate of the 89-plane z and the complex string charge Q as

$$z \xrightarrow{g} z' = \gamma z, \quad Q \xrightarrow{g} Q' = \gamma^{-1} Q. \tag{3.2.4}$$

We introduce N D3-branes on the S-fold. As in the case of orientifolds it may be possible to introduce D3-branes that are mirror to themselves. We introduce them later as a non-trivial discrete torsion. Here we assume the absence of such trapped D3-branes at the fixed point. Then in the covering space we have kN D3-branes. We label them by two indices $a = 1, \ldots, N$ and $i = 0, \ldots, k-1$, where a labels N independent branes and i labels k branes identified by $\mathbb{Z}_k$ action. Their positions $z_{i,a}$ are related by

$$z_{i,a} = \gamma^i z_a, \tag{3.2.5}$$

where $z_a \equiv z_{0,a}$. As in the flat background a junction is specified by the string charges $Q_{i,a} \in \Gamma_k$ on the D3-branes. We define $\mathbb{Z}_k$ action so that it rotates the coordinate z and the complex charge Q in the opposite direction by the same angle as given in (3.2.4). Due to the $\mathbb{Z}_k$ identification a string with charge $Q_{i,a}$ attached on the brane at $z_{i,a}$ is equivalent to a string with charge $\gamma^i Q_{i,a}$ attached on the brane at z_a. When we read off the $U(1)^N$ dyonic charges Q_a we should take account of this equivalence. We collect the endpoints on $z_{i,a}$ with $i = 0, \ldots, k-1$ to the brane with $i = 0$ by $\mathbb{Z}_k$ transformations and obtain

$$Q_a = \sum_{i=0}^{k-1} \gamma^i Q_{i,a}. \tag{3.2.6}$$

These charges are again elements of Γ_k. We have assumed trivial discrete torsion, and there are no D3-branes at the origin of the covering space. In this case a junction is attached on only mobile D3-branes and due to the charge conservation $Q_{i,a}$ must satisfy

$$\sum_{i,a} Q_{i,a} = 0. \tag{3.2.7}$$

Due to this constraint Q_a defined by (3.2.6) may not be all independent. The constraint is obtained as follows.

Let F_k be a $\mathbb{Z}_k$ invariant homomorphism from Γ_k onto some discrete group K. The $\mathbb{Z}_k$ invariance means $F_k(\gamma Q) = F_k(Q)$ for an arbitrary $Q \in \Gamma_k$[2]. By using (3.2.6), (3.2.7), and the $\mathbb{Z}_k$ invariance of F_k we can easily show

$$F_k(Q_0) = 0, \tag{3.2.8}$$

where Q_0 is the total charge

$$Q_0 = \sum_{a=1}^{N} Q_a. \tag{3.2.9}$$

If K is not trivial (3.2.8) gives a non-trivial constraint imposed on the set of the charges Q_a. On the S-fold, Q_0 can be regarded as the charge of a string attached on the S-plane. (For S-planes and O-planes we define the charge of a string ending on them as the outgoing charge.) Therefore, (3.2.8) gives the constraint on the string charge that can be attached on the S-plane.

The constraint (3.2.8) is a necessary condition. There may be some different choices of K, and for some of them (3.2.8) may not be a sufficient one. However, we can easily show that if we choose maximal K (3.2.8) gives the sufficient condition for the existence of junctions. The maximal K are in fact isomorphic to the discrete torsion group

$$\Gamma^{\rm tor} = H^3(\boldsymbol{S}^5/\mathbb{Z}_k, \widetilde{\mathbb{Z}+\mathbb{Z}}) \tag{3.2.10}$$

for the $\mathbb{Z}_k$ S-fold [10, 39, 40] (Table 3.2).

Up to here we consider S-folds with the trivial discrete torsion. Introduction of a non-vanishing discrete torsion changes the condition given above. To determine the modified condition it is convenient to realize such an S-plane by using wrapped fivebranes. (Similar realization of O-planes with non-trivial discrete torsions was given in [39, 41].) By the Poincare duality the discrete torsion can be also regarded as the two-cycle homology. Let us consider an S-fold with a torsion $t \in \Gamma^{\rm tor}$. We

[2] The group K is easily obtained as follows. Let us consider $k = 2$ case as an example. Then Γ_2 is the lattice generated by 1 and τ. Let a and b be two elements of K corresponding to them: $a = F_2(1)$ and $b = F_2(\tau)$. By the linearity $F_2(p + q\tau) = pa + qb$ $(p, q \in \mathbb{Z})$, and the $\mathbb{Z}_2$ invariance requires $2a = 2b = 0$. This means K contains at most four elements 0, a, b, and $a + b$. If these are all different we obtain the maximal K $(= \mathbb{Z}_2 + \mathbb{Z}_2)$. If one of a, b, and $a + b$ is zero $K = \mathbb{Z}_2$, and if $a = b = 0$ then $K = 0$.

Table 3.2: The discrete torsion groups for $\mathbb{Z}_k$ S-folds are shown.

k	2	3	4	6
$\Gamma^{\rm tor}$	$\mathbb{Z}_2 + \mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_2$	0

can realize such an S-fold from that with the trivial torsion by wrapping a fivebrane around the two-cycle specified by t. (Note that the coefficient group $\widetilde{\mathbb{Z}+\mathbb{Z}}$ is the sheaf of the (p,q) charges of fivebranes and an element of $\Gamma^{\rm tor}$ specifies not only the cycle wrapped by the fivebrane but also the fivebrane charges.) The existence of the wrapped (p,q) fivebrane allows (p,q) strings to end on it. Namely, string with charge Q_0 can be attached on the S-plane if

$$F(Q_0) \in \mathbb{Z}t. \tag{3.2.11}$$

The right hand side is not just t but $\mathbb{Z}t$ because we can attach an arbitrary number of strings.

3.3 Aharony and Tachikawa conjecture

In [10], Aharony and Tachikawa investigated the dimensions of the Coulomb branch operators in $\mathcal{N} = 3$ theories constructed by S-folding. S-folds are labelled by two integers k and ℓ; k is the order of the orbifold group and ℓ is a divisor of k related to the discrete torsion. Therefore, the theory on the D3-branes are labeled by k, ℓ, and N, where N is the number of mobile D3-branes.

The Coulomb branch operators contain an operator with dimension $N\ell$, which becomes 1 or 2 for some of the theories. As we mentioned the previous chapter, the dimensions of Coulomb branch operators in a genuine $\mathcal{N} = 3$ theory must be equal to or greater than 3. This fact implies that the $\mathcal{N} = 3$ theories with $N\ell = 1$ or 2 are in fact $\mathcal{N} = 4$ theories [10], which can be specified by giving the gauge group G. In [10] some BPS states were identified with 1/2 BPS W-bosons, and the gauge groups were determined(Table 3.3).

3.4 Matching of charges and moduli

Let us consider junctions in the $\mathbb{Z}_k$ S-fold with trivial discrete torsion. If we put two D3-branes the supersymmetry is expected to be enhanced from $\mathcal{N} = 3$ to $\mathcal{N} = 4$

Table 3.3: Theories in which $\mathcal{N}=3$ is enhanced to $\mathcal{N}=4$ are shown.

(k,ℓ,N)	G
$(k,1,1)$	$U(1)$
$(3,1,2)$	$SU(3)$
$(4,1,2)$	$SO(5)$
$(6,1,2)$	G_2

[10]. In this section we determine the both central charges Z and $\overline{Z}$ for junctions in the S-folds by using relations to perturbative realization of the same $\mathcal{N}=4$ theories.

Following [10] let us begin with the identification of W-bosons in the $\mathbb{Z}_k$ S-fold. We consider an open string connecting two D3-branes. We denote a string with complex charge Q that goes from a D3-brane at $z_{j,2}$ to another D3-brane at $z_{i,1}$ by

$$z_{i,1} \xleftarrow{Q} z_{j,2}, \quad Q \in \Gamma_k. \tag{3.4.1}$$

Let $Q_{i,a}^{(S)}$ be the charges of strings attached on D3-branes in the covering space. (We put the superscript "(S)" for distinction from the quantities in the dual set-up we will introduce later.) This is the junction with $Q_{i,1}^{(S)}=Q$, $Q_{j,2}^{(S)}=-Q$, and the other $Q_{i,a}^{(S)}$ vanishing. (3.2.6) gives

$$(Q_1^{(S)}, Q_2^{(S)}) = (\gamma^i Q, -\gamma^j Q). \tag{3.4.2}$$

Following [10] we impose the following electric condition:

$$Q_1^{(S)} = -Q_2^{(S)*}. \tag{3.4.3}$$

Then the string charge Q must satisfy

$$Q = \pm|Q|\gamma^{-\frac{i+j}{2}}, \tag{3.4.4}$$

and (3.4.2) becomes

$$(Q_1^{(S)}, Q_2^{(S)}) = \pm|Q|(\gamma^{\frac{i-j}{2}}, -\gamma^{\frac{j-i}{2}}). \tag{3.4.5}$$

Different $i-j$ gives different charges, and it is shown in [10] that these charges form the weight vectors in the adjoint representation of a rank-2 Lie group depending on k (Table 3.3).

In the following subsections, we give detailed analysis of the three models with supersymmetry enhancement.

3.4.1 $\mathbb{Z}_3$ S-fold

Let us first consider $\mathbb{Z}_3$ S-fold. In this case the modulous on the S-fold is $\tau^{(S)} = \omega = e^{2\pi i/3}$, and the complex string charges form the triangular lattice Γ_3 in Figure 3.1. (For $k = 3$ we use ω instead of γ.) The charges $Q_1^{(S)}$ and $Q_2^{(S)}$ are elements of Γ_3, and constrained by (3.2.8),

$$F_3(Q_1^{(S)} + Q_2^{(S)}) = 0, \tag{3.4.6}$$

where F_3 is the map from Γ_3 onto $\mathbb{Z}_3$. This defines the $\mathbb{Z}_3$ grading in the lattice shown in Figure 3.1. A string can end on the S-plane only when its charge is a red

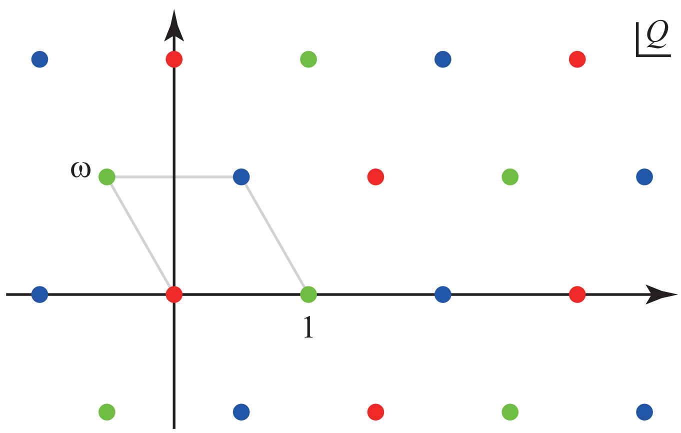

Figure 3.1: The $\mathbb{Z}_3$ grading of the lattice Γ_3 is shown. Red, green, and blue dots represent charges with $F(Q) = 0$, 1 and 2, respectively.

dot in Figure 3.1.

Different choices of j and i in (3.4.5) with the same $j - i$ give equivalent strings mapped to one another by $\mathbb{Z}_k$, and only $j - i$ is significant. Let $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ denote the strings corresponding to $j - i = 0$, 1, and 2. In the $\mathbb{Z}_3$-frame in which the string charge becomes $Q = 1$, these strings are

$$\boldsymbol{a} : z_1 \xleftarrow{1} z_2, \quad \boldsymbol{b} : \omega z_1 \xleftarrow{1} \omega^2 z_2, \quad \boldsymbol{c} : \omega^2 z_1 \xleftarrow{1} \omega z_2. \tag{3.4.7}$$

The dyonic charges and central charges of these strings are shown in Table 3.4. As is confirmed in [10] these are the same as those of W-bosons of $SU(3)$ SYM.

To see this more clearly, let us use another realization of the $\mathcal{N} = 4$ $SU(3)$ theory with parallel D3-branes in the flat background without any S-folding. We use y for

Table 3.4: The dyonic charges and the central charge of junctions in the S-fold and flat background are shown for three strings corresponding to W-bosons.

	$(Q_1^{(S)}, Q_2^{(S)})$	$Z^{(S)}$	$(Q_1^{(F)}, Q_2^{(F)})$	$Z^{(F)}$
$\boldsymbol{a}$	$(1, -1)$	$z_1 - z_2$	$(1, -1)$	$y_1 - y_2$
$\boldsymbol{b}$	$(\omega, -\omega^2)$	$\omega z_1 - \omega^2 z_2$	$(-1, 0)$	$-y_1$
$\boldsymbol{c}$	$(\omega^2, -\omega)$	$\omega^2 z_1 - \omega z_2$	$(0, 1)$	y_2

the coordinate of D3-branes. The center of mass motion is decoupled and we take the coordinate so that one of the D3-branes is located at $y = 0$, and denote the positions of the other two branes by y_1 and y_2. We denote the charges of the strings ending at y_1 and y_2 by $Q_1^{(F)}$ and $Q_2^{(F)}$, respectively. (The superscripts "(F)" indicate the flat background.) By using these charges we can determine the central charge $Z^{(F)}$ by the general formula (3.2.2).

Let us assume that the strings in (3.4.7) correspond to fundamental strings stretched between two D3-branes on the y-plane. The sum of the charges of the three strings in (3.4.7) vanishes, and we choose the corresponding strings so that they also carry charges with vanishing sum:

$$\boldsymbol{a} : y_1 \xleftarrow{1} y_2, \quad \boldsymbol{b} : 0 \xleftarrow{1} y_1, \quad \boldsymbol{c} : y_2 \xleftarrow{1} 0. \tag{3.4.8}$$

The central charge $Z^{(F)}$ and the dyonic charges $(Q_1^{(F)}, Q_2^{(F)})$ for these are shown in Table 3.4. Because all strings in (3.4.8) are fundamental ones, the central charge does not depend on the modulous $\tau^{(F)}$ on the flat background, which is not fixed yet.

We require $Z^{(S)} = Z^{(F)}$ for the W-boson states in Table 3.4. This means

$$Z = Q_1^{(F)} y_1 + Q_2^{(F)} y_2 = Q_1^{(S)} z_1 + Q_2^{(S)} z_2. \tag{3.4.9}$$

By requiring this equality for the W-bosons we obtain the relations among z_a and y_a:

$$z_1 = \frac{\omega y_1 + \omega^2 y_2}{\omega - \omega^2}, \quad z_2 = \frac{\omega^2 y_1 + \omega y_2}{\omega - \omega^2}. \tag{3.4.10}$$

$Z^{(S)}$ and $Z^{(F)}$ are holomorphic functions in the complex charges. This means that $Q_a^{(S)}$ and $Q_a^{(F)}$ should be related by holomorphic linear relations. The comparison of the complex charges of W-boson states in Table 3.4 determines the following relations.

$$Q_1^{(F)}=\frac{\omega Q_1^{(S)}+\omega^2 Q_2^{(S)}}{\omega-\omega^2},\quad Q_2^{(F)}=\frac{\omega^2 Q_1^{(S)}+\omega Q_2^{(S)}}{\omega-\omega^2}. \tag{3.4.11}$$

By taking account of $Q_a^{(S)}\in\Gamma_3$ and the constraint (3.4.6), we can show that $Q_a^{(F)}$ are also element of Γ_3 and no constraint is imposed. Furthermore, (3.4.11) is one-to-one. From the quantization of $Q_a^{(F)}$ we can fix the modulous on the flat background as $\tau^{(F)}=\omega$.

The other central charge $\overline{Z}$ can be directly obtained by the general formula (3.2.2) only in the flat background, and in the S-fold we cannot calculate it directly because it is generated non-perturbatively. However, once we obtain $\overline{Z}$ in the flat background, we can use the relations (3.4.10) and (3.4.11) to rewrite it as a function of the variables on the S-fold side. The result is

$$\overline{Z}=Q_1^{(F)}y_1^*+Q_2^{(F)}y_2^*=-Q_1^{(S)}z_2^*-Q_2^{(S)}z_1^*. \tag{3.4.12}$$

At the second equality we used (3.4.10) and (3.4.11). On the S-fold side this is non-local in the sense that each term contains the charge and the coordinate of different branes.

For the following discussions the equation

$$|Z|^2-|\overline{Z}|^2=(|Q_1^{(S)}|^2-|Q_2^{(S)}|^2)(|z_1|^2-|z_2|^2). \tag{3.4.13}$$

may be convenient. By using this we can easily check that for an electric junction $\boldsymbol{w}$ with $Q_2^{(S)}=-Q_1^{(S)*}$ the absolute values of two central charges coincides; $|\overline{Z}[\boldsymbol{w}]|=|Z[\boldsymbol{w}]|$.

As a consistency check let us compare singularities in the moduli spaces of the two set-ups.

When $z_1=z_2$, $z_1=\omega z_2$, or $z_1=\omega^2 z_2$, two D3-branes in the S-fold collide and the gauge symmetry is enhanced to $U(2)$. These three singular loci correspond to $y_1=y_2$, $y_1=0$, and $y_2=0$, respectively, in the flat background. On these loci two of three D3-branes coincide, and the gauge symmetry is enhanced to $U(2)$, just like on the S-fold side. This agreement of the loci is rather trivial because we determine the relations among moduli parameters by requiring the coincidence of the central charge Z of string connecting two independent D3-branes, which vanishes if two D3-branes collide.

On the S-fold side, we have another type of singularity. When one of z_1 and z_2 approaches the S-plane, a string connecting the D3-brane and one of its mirrors shrinks to zero length, and appearance of massless particles is naively expected. This limit, however, causes no singularity on the flat background. When $z_1 = 0$ or $z_2 = 0$, three D3-branes in the flat background form an equilateral triangle, and no junction shrinks to zero size. In Figure 3.2 (a) we show two strings

$$z_{1,a} \xleftarrow{1} z_{2,a}, \quad (a = 1, 2). \tag{3.4.14}$$

in a situation with $|z_1| \ll |z_2|$. The dyonic charges of these strings are

$$(Q_1^{(S)}, Q_2^{(S)}) = \begin{cases} (\omega - \omega^2, 0) & (a = 1) \\ (0, \omega - \omega^2) & (a = 2) \end{cases}. \tag{3.4.15}$$

By the relation (3.4.11) we obtain the charges of the corresponding junctions:

$$(Q_1^{(F)}, Q_2^{(F)}) = \begin{cases} (\omega, \omega^2) & (a = 1) \\ (\omega^2, \omega) & (a = 2) \end{cases}. \tag{3.4.16}$$

Junctions carrying these charges in the flat background are shown in Figure 3.2 (b). Although they have similar shapes the corresponding strings have completely

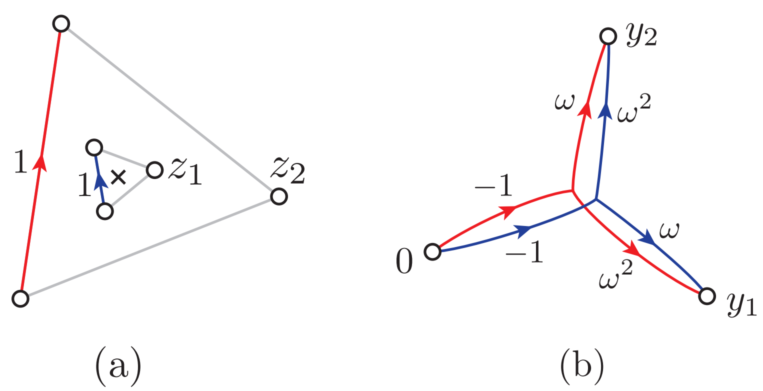

Figure 3.2: Examples of strings in the S-fold and corresponding junctions in the flat background.

different length in the S-fold. In particular, in the $z_1 = 0$ limit the junctions in (b) seem to have the same mass while in (a) the red one is massive and the blue one is massless.

One may think that all these strings/junctions are BPS, and naive estimation of masses should give the correct values. If it were the case, the above observation would mean inequivalence of theories given by two set-ups. This is, however, too naive.

Table 3.5: The discrete torsions and the gauge groups realized by the orientifolds with four kinds of O3-planes are shown. The first and the second component of t represent the torsions for R-R and NS-NS fluxes, respectively. N is the number of the mobile D3-branes.

	$\mathrm{O3}^-$	$\widetilde{\mathrm{O3}}^-$	$\mathrm{O3}^+$	$\widetilde{\mathrm{O3}}^+$
t	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
G	$SO(2N)$	$SO(2N+1)$	$Sp(N)$	$Sp(N)$

Remember that for planar junctions on the 89 plane in the $\mathcal{N} = 4$ theory there are two central charges Z and $\overline{Z}$, which gives independent BPS bounds. The two junctions in Figure 3.2 (b) are 1/4 BPS configurations saturating different bounds; the red one is a holomorphic junction saturating $m = |Z| > |\overline{Z}|$ while the blue one is an anti-holomorphic junction saturating $m = |\overline{Z}| > |Z|$. On the S-fold, however, only Z is realized perturbatively, and $\overline{Z}$ is generated by some unknown non-perturbative dynamics. From the length of strings in Figure 3.2 (a) we can determine only Z of two strings. Although they give the lower bounds for the masses, the saturation is not guaranteed. If $|\overline{Z}|$ is greater than $|Z|$ the bound $m \geq |Z|$ is never saturated, and this is the case for the blue string in Figure 3.2 (a). We emphasize that there is no reason that the perturbative estimation gives good approximation of the mass. We consider the situation in which z_1 is much smaller than the string scale and the string coupling is of order 1, and we cannot exclude large quantum corrections.

3.4.2 $\mathbb{Z}_4$ S-fold

Next let us consider $\mathbb{Z}_4$ case. The modulus is $\tau = i$ and the complex charge of strings in the S-fold takes value in the square lattice $\Gamma_4 = \{p + iq | p, q \in \mathbb{Z}\}$. If we put two D3-branes on this background it is expected to give the $\mathcal{N} = 4$ $SO(5)$ theory, which can also be realized by an orientifold. In this subsection we determine the spectrum of junctions in the S-fold and compare it with the spectrum on the orientifold side.

Before starting the analysis of the junction spectra let us briefly review some properties of O3-planes relevant to the analysis below. There are four types of O3-planes distinguished by the discrete torsion $t \in \mathbb{Z}_2 + \mathbb{Z}_2$ of R-R and NS-NS three-form fluxes [39]. If we put D3-branes in an orientifold $\mathcal{N} = 4$ theory with an orthogonal or symplectic gauge group is realized. The relation between the torsion and the gauge group is shown in Table 3.5. Among four types of O3-planes, the three but one

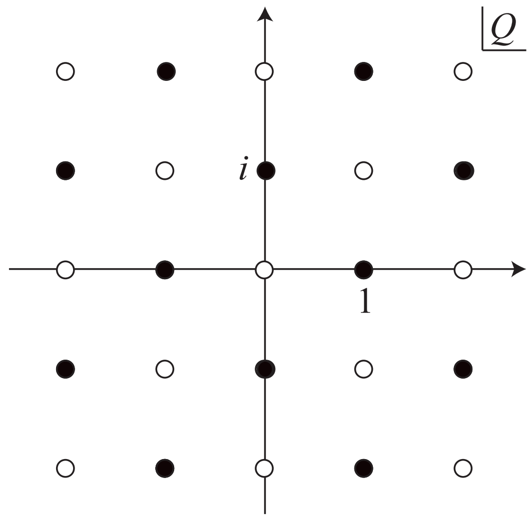

Figure 3.3: A $\mathbb{Z}_2$ grading of the lattice Γ_4 is shown. White and black dots represent charges with $F(Q) = 0$ and 1, respectively.

with $t = (0,0)$ are transformed among them by $SL(2,\mathbb{Z})$ symmetry of the type IIB string theory. Correspondingly, the gauge theories realized by them are equivalent via Montonen-Olive duality. In the case with two D3-branes, which we are interested in, the gauge groups $SO(5)$ and $Sp(2)$ are isomorphic, and the equivalence of these theories is trivial. However, the W-bosons are realized in different ways. We refer to these two realization of the theory as $SO(5)$ and $Sp(2)$ theories.

Let us first determine the junction spectrum in the $\mathbb{Z}_4$ S-fold with two D3-branes. The charges of strings $Q_a^{(S)}$ ($a = 1, 2$) ending on the D3-branes satisfy

$$F_4(Q_1^{(S)} + Q_2^{(S)}) = 0, \tag{3.4.17}$$

where F_4 is the homomorphism from Γ_4 onto $\mathbb{Z}_2$ which define the $\mathbb{Z}_2$ grading of Γ_4 shown in Figure 3.3. (3.4.17) shows that the charges $Q_1^{(S)}$ and $Q_2^{(S)}$ must have the same $\mathbb{Z}_2$ grading.

The strings whose charges satisfy the electric condition (3.4.3) are

$$\boldsymbol{a} : z_1 \xleftarrow{1} z_2, \quad \boldsymbol{b} : z_1 \xleftarrow{1-i} iz_2, \quad \boldsymbol{c} : z_1 \xleftarrow{i} -z_2, \quad \boldsymbol{d} : z_1 \xleftarrow{1+i} -iz_2, \tag{3.4.18}$$

and ones with opposite orientation. We would like to identify these strings with W-bosons of $SO(5)$ and $Sp(2)$ theories that are realized on the orientifolds.

Let us consider a system with an O3-plane at $y = 0$ and two D3-branes at $y = y_1, y_2$ (and their mirrors at $y = -y_1, -y_2$). The complex charge of (p, q)-string is $Q^{(O)} = p + q\tau^{(O)}$. (The superscripts "$(O)$" indicate orientifold.) We should notice that the modulus $\tau^{(O)}$ may not be the same as $\tau^{(S)} = i$. This should be determined so that the junction spectrum agrees with that of the S-fold. We denote complex charges of strings ending on the D3-branes by $Q_a^{(O)}$ $(a = 1, 2)$. Following (3.2.9) we also define

$$Q_0^{(O)} = Q_1^{(O)} + Q_2^{(O)}. \tag{3.4.19}$$

This is the charge of strings ending on the O3-plane. The general formula (3.2.11) says that a (p, q) string can end on an O3-plane with $t \neq 0$ only when the charges satisfy

$$(p, q) = 0 \text{ or } t \mod 2. \tag{3.4.20}$$

In particular, a fundamental string with charge $(1, 0)$ can end on $\widetilde{\text{O3}}^-$-plane, but not on O3^+- and $\widetilde{\text{O3}}^+$-planes.

The W-bosons of $SO(5)$ and $Sp(2)$ realizations are given in a consistent way to this constraint. For an $\widetilde{\text{O3}}^-$-plane $SO(5)$ W-bosons are given by open strings

$$y_2 \xleftarrow{1} y_1, \quad -y_1 \xleftarrow{1} y_2, \quad 0 \xleftarrow{1} y_1, \quad y_2 \xleftarrow{1} 0, \tag{3.4.21}$$

while for O3^+- and $\widetilde{\text{O3}}^+$-planes $Sp(2)$ W-bosons arise as strings

$$y_2 \xleftarrow{1} y_1, \quad -y_1 \xleftarrow{1} y_2, \quad -y_1 \xleftarrow{1} y_1, \quad y_2 \xleftarrow{1} -y_2. \tag{3.4.22}$$

We want to determine relations between strings (3.4.18) in the S-fold and (3.4.21) or (3.4.22) in the orientifold. The first two in (3.4.21) are the same with the first two in (3.4.22):

$$y_2 \xleftarrow{1} y_1, \quad -y_1 \xleftarrow{1} y_2. \tag{3.4.23}$$

Let us start with matching these two with two of the strings in (3.4.18). Because two strings in (3.4.23) are continuously deformed to each other by moving D3-branes, so are the corresponding strings. Among four strings in (3.4.18) $\boldsymbol{a}$ and $\boldsymbol{c}$ are deformed to each other, and so are $\boldsymbol{b}$ and $\boldsymbol{d}$, too. Therefore, we have only two essentially different choices.

First let us try matching strings in (3.4.23) with $\boldsymbol{d}$ and $\boldsymbol{b}$. Matching of these two is sufficient to determine the relation between the moduli parameters and that between dyonic charges. By identifying the central charges $Z^{(S)}$ and $Z^{(O)}$ we obtain the relations

$$z_1 = -\frac{1}{2}(y_1 + iy_2), \quad z_2 = \frac{1}{2}(y_1 - iy_2). \tag{3.4.24}$$

From the comparison of charges we obtain

$$Q_1^{(O)} = -\frac{1}{2}(Q_1^{(S)} - Q_2^{(S)}), \quad Q_2^{(O)} = -\frac{i}{2}(Q_1^{(S)} + Q_2^{(S)}). \tag{3.4.25}$$

These relations automatically fix the correspondents of $\boldsymbol{a}$ and $\boldsymbol{c}$ in (3.4.18), and we find that they correspond to the two remaining strings in (3.4.21). (See Table 3.6.) Namely, this gives the correspondence to the adjoint representation in the $SO(5)$ theory.

Table 3.6: The dyonic charges and the central charge of junctions in the S-fold and $\widetilde{\text{O3}}^-$-plane background are shown for four strings corresponding to W-bosons.

	$Z^{(S)}$	$(Q_1^{(S)}, Q_2^{(S)})$	$Z^{(O)}$	$(Q_1^{(O)}, Q_2^{(O)})$
$\boldsymbol{a}$	$z_1 - z_2$	$(1, -1)$	$-y_1$	$(-1, 0)$
$\boldsymbol{b}$	$(1-i)(z_1 - iz_2)$	$(1-i, -1-i)$	$-y_1 - y_2$	$(-1, -1)$
$\boldsymbol{c}$	$i(z_1 + z_2)$	(i, i)	y_2	$(0, 1)$
$\boldsymbol{d}$	$(1+i)(z_1 + iz_2)$	$(1+i, -1+i)$	$-(y_1 - y_2)$	$(-1, 1)$

$Q_a^{(S)}$ satisfying (3.4.17) can be given by

$$Q_a^{(S)} = p_a + iq_a, \quad p_1 + p_2 + q_1 + q_2 \in 2\mathbb{Z}. \tag{3.4.26}$$

Substituting this into (3.4.25) we obtain

$$Q_1^{(O)} = \frac{1}{2}[(p_1 - p_2) + i(q_1 - q_2)], \quad Q_2^{(O)} = \frac{1}{2}[(-q_1 - q_2) + i(p_1 + p_2)]. \tag{3.4.27}$$

(3.4.26) implies that these charges take values in the $\mathbb{Z}_2$ graded lattice in Figure 3.4, and $Q_1^{(O)}$ and $Q_2^{(O)}$ have the same grading. The latter statement means that $Q_0^{(O)}$, the charge of the string ending on the O3-plane, has grading 0.

From the quantization of each $Q_a^{(O)}$ we can fix the modulous in the orientifold to be

$$\tau^{(O)} = \frac{1}{2}(1 + i). \tag{3.4.28}$$

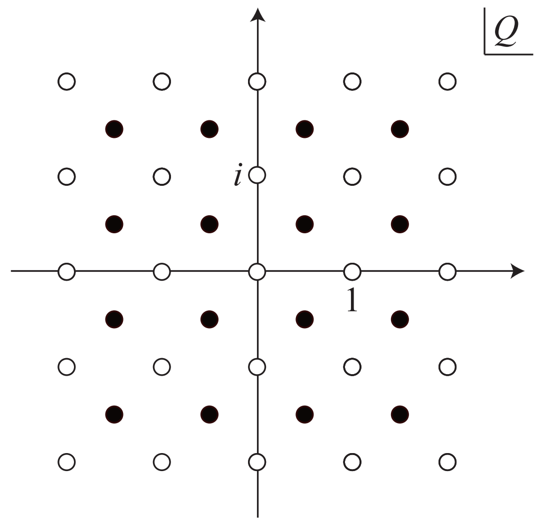

Figure 3.4: A $\mathbb{Z}_2$ grading of the lattice with $\tau = (1+i)/2$ is shown. White and black dots represent charges with 0 and 1 in $\mathbb{Z}_2$, respectively.

Then $Q_a^{(O)}$ are given in the form $p + q\tau$, and $Q_0^{(O)}$, which has trivial grading, can be expressed as

$$Q_0^{(O)} = p + 2q\tau^{(O)}, \quad p, q \in \mathbb{Z}. \tag{3.4.29}$$

This is nothing but the condition (3.2.11) for an $\widetilde{\text{O3}}^-$-plane, which gives the gauge group $SO(5)$.

Let us consider the other possibility: matching of two strings in (3.4.23) with $\boldsymbol{a}$ and $\boldsymbol{c}$ in (3.4.18). We obtain the relations

$$z_1 = -\frac{1}{2}(1-i)y_1 + \frac{1}{2}(1+i)y_2, \quad z_2 = \frac{1}{2}(1+i)y_1 - \frac{1}{2}(1-i)y_2, \tag{3.4.30}$$

and

$$Q_1^{(O)} = -\frac{1-i}{2}Q_1^{(S)} + \frac{1+i}{2}Q_2^{(S)}, \quad Q_2^{(O)} = \frac{1+i}{2}Q_1^{(S)} - \frac{1-i}{2}Q_2^{(S)}. \tag{3.4.31}$$

and the correspondents of $\boldsymbol{b}$ and $\boldsymbol{d}$ are the two remaining strings in (3.4.22). (See Table 3.7.)

Substituting (3.4.26) into (3.4.31) we obtain

$$Q_1^{(O)} = \frac{1+i}{2}[(p_2 - q_1) + i(p_1 + q_2)], \quad Q_2^{(O)} = \frac{1+i}{2}[(p_1 - q_2) + i(p_2 + q_1)]. \tag{3.4.32}$$

Table 3.7: The dyonic charges and the central charge of junctions in the S-fold and $O3^+$- and $\widetilde{O3}^+$-planes background are shown for four strings corresponding to W-bosons.

	$Z^{(S)}$	$(Q_1^{(S)}, Q_2^{(S)})$	$Z^{(O)}$	$(Q_1^{(O)}, Q_2^{(O)})$
$\boldsymbol{a}$	$z_1 - z_2$	$(1, -1)$	$-(y_1 - y_2)$	$(-1, 1)$
$\boldsymbol{b}$	$(1-i)(z_1 - iz_2)$	$(1-i, -1-i)$	$2y_2$	$(0, 2)$
$\boldsymbol{c}$	$i(z_1 + z_2)$	(i, i)	$-y_1 - y_2$	$(-1, -1)$
$\boldsymbol{d}$	$(1+i)(z_1 + iz_2)$	$(1+i, -1+i)$	$-2y_1$	$(-2, 0)$

These are similar to (3.4.27) and only difference up to unimportant signatures are the extra factors $1+i$. Therefore, the quantization and constraint for $Q_a^{(O)}$ can be obtained from the previous ones by simply rotate the lattice in Figure 3.4 by 45 degrees and expand it by the factor $\sqrt{2}$. The resulting lattice is the same as Figure 3.3. If we set $\tau^{(O)} = 1+i$ the charges $Q_a^{(O)}$ take the form $p + q\tau^{(O)}$, and $Q_0^{(O)}$ can be expressed as

$$Q_0^{(O)} = 2p + q\tau^{(O)}, \quad p, q \in \mathbb{Z}. \tag{3.4.33}$$

This is the constraint (3.2.11) for $O3^+$-plane. We can also set $\tau^{(O)} = i$. Then $Q_0^{(O)}$ can be expressed as

$$Q_0^{(O)} = 2p + q(1 + \tau^{(O)}), \quad p, q \in \mathbb{Z}. \tag{3.4.34}$$

This is the constraint (3.2.11) for $\widetilde{O3}^+$-plane. These two assignments of $\tau^{(O)}$ give two $Sp(2)$ theories with different θ-angle, which are dual to each other.

In both cases, the spectrum of the central charge Z in the S-fold matches completely with that of the orientifold.

Once we obtain these relations, we can rewrite $\overline{Z}$ as a function of the S-fold variables. For the $SO(5)$ matching with (3.4.24) and (3.4.25) and the $Sp(2)$ matching with (3.4.30) and (3.4.31) we obtain

$$SO(5) : \overline{Z} = Q_1^{(S)} z_2^* + Q_2^{(S)} z_1^*, \quad Sp(2) : \overline{Z} = -Q_1^{(S)} z_2^* - Q_2^{(S)} z_1^*. \tag{3.4.35}$$

The overall sign depends on the choice of string orientation and is the matter of conventions. This result is the same as (3.4.12) up to sign.

As in the $k = 3$ case let us illustrate the junctions corresponding to strings in S-fold. Strings connecting two independent D3-branes give W-bosons and they have been already analyzed above, and here we focus on strings connecting a D3-brane and one of its mirror branes. First we consider two strings

$$z_{0,a} \xleftarrow{1} z_{1,a}, \quad (a = 1, 2). \tag{3.4.36}$$

These are shown in Figure 3.5 (a) in a situation with $|z_1| \ll |z_2|$. The dyonic charges of these strings are

$$(Q_1^{(S)}, Q_2^{(S)}) = \begin{cases} (1-i, 0) & (a=1) \\ (0, 1-i) & (a=2) \end{cases}. \tag{3.4.37}$$

By the relation (3.4.25) we obtain the charges of the corresponding junctions:

$$(Q_1^{(O)}, Q_2^{(O)}) = \begin{cases} (\frac{-1+i}{2}, \frac{-1-i}{2}) & (a=1) \\ (\frac{1-i}{2}, \frac{-1-i}{2}) & (a=2) \end{cases}. \tag{3.4.38}$$

Junctions carrying these charges are shown in Figure 3.5 (b). As in Figure 3.2

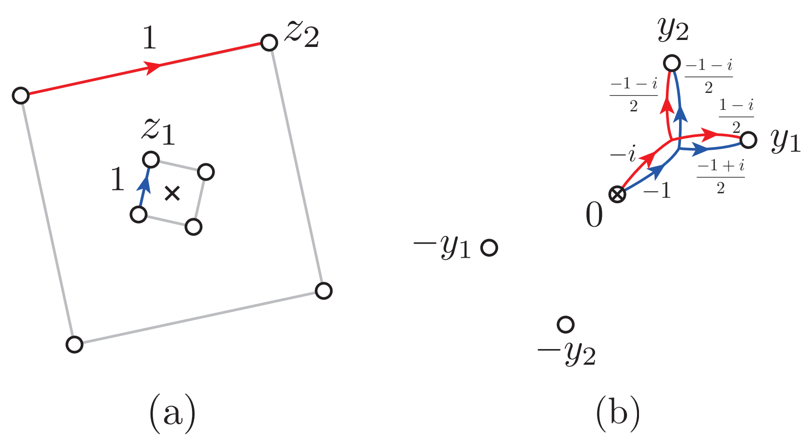

Figure 3.5: Examples of strings in the S-fold and corresponding junctions in the orientifold background.

the red and blue junctions in Figure 3.5 (b) are holomorphic and anti-holomorphic, respectively.

Another pair of two strings in S-fold is

$$z_{0,a} \xleftarrow{1} z_{2,a}, \quad (a = 1, 2). \tag{3.4.39}$$

We can easily check that these strings carry dyonic charges that are as $1+i$ times as those in (3.4.37). The corresponding junctions are ones in Figure 3.5 (b) with charges multiplied by $1+i$.

3.4.3 $\mathbb{Z}_6$ S-fold

For $k = 6$ we expect $\mathcal{N} = 4$ SYM with the gauge group G_2. The discrete torsion group for $k = 6$ is trivial, and an arbitrary pair of the dyonic charges $(Q_1^{(S)}, Q_2^{(S)}) \in \Gamma_6 + \Gamma_6$ is allowed. The central charge Z is given by

$$Z = Q_1^{(S)} z_1 + Q_2^{(S)} z_2. \tag{3.4.40}$$

Although no perturbative realization of G_2 theory is known in string theory and we cannot directly read off the other central charge $\overline{Z}$ from junctions, it is natural to guess from the results for $k \leq 4$ that

$$\overline{Z} = Q_1^{(S)} z_2^* + Q_2^{(S)} z_1^*, \tag{3.4.41}$$

up to overall phase. Indeed, if we assume

- Z is given by (3.4.40),
- $\overline{Z}$ is a holomorphic bilinear form of $Q_a^{(S)}$ and z_a^*. Namely, $\overline{Z}$ is given by $\overline{Z} = c_{ab} Q_a^{(S)} z_b^*$ with some coefficients c_{ab}, and
- $|Z| = |\overline{Z}|$ for charges satisfying the electric condition (3.4.3),

then (3.4.41) is the unique solution up to phase ambiguity.

Summary

We considered the S-fold theories realized by two D3-branes in the $\mathbb{Z}_k$ S-folds in order to confirm the supersymmetry enhancement. Note that the conjecture suggested that $\mathbb{Z}_3$ S-fold theory corresponds to $\mathcal{N} = 4$ SYM with gauge group $SU(3)$, and $\mathbb{Z}_4$ S-fold theory corresponds to $\mathcal{N} = 4$ SYM with gauge group $SO(5) \simeq Sp(2)$ when there are two D3-branes in these S-folds. We compared the S-fold theories with well-known brane realization of 4d $\mathcal{N} = 4$ SYM. By comparing the states realized by string junctions, we obtained following results :

- S-fold coordinates z_i correspond to flat space/orientifold coordinates y_i

  We found mobile D3-branes coordinates in $\mathbb{Z}_3$ ($\mathbb{Z}_4$) S-folds and these in flat space (orientifold) are in one-to-one correspondence. In usual brane construction of $\mathcal{N} = 4$ SYM, the coordinates of mobile D3-branes y_i are regarded as vacuum expectation values of $\mathcal{N} = 4$ Coulomb branch operators. Therefore, the correspondence between z_i and y_i suggest that the S-fold theories reproduce Coulomb branch operators of $\mathcal{N} = 4$ SYM.

- The dyonic charges of string junctions

  We considered Coulomb branch of both theories. At a generic point in the Coulomb branch, unbroken gauge group of these theories is $U(1)\times U(1)$, and we can consider charge lattices of electric and magnetic charges for the unbroken gauge group. The charge lattices of the S-fold theories and corresponding $\mathcal{N} = 4$ SYM are coincident. Namely, charge spectrum appearing in $SU(3)$ ($SO(5)$ and $Sp(2)$) $\mathcal{N} = 4$ SYM can be reproduced in $\mathbb{Z}_3$ ($\mathbb{Z}_4$) S-fold theory.

  $SO(5)$ and $Sp(2)$ are locally isomorphic, while brane constructions of $SO(5)$ and $Sp(2)$ SYM need different O3-plane. An $\widetilde{\text{O3}}^-$-plane realizes $SO(5)$ gauge group and O3^+- and $\widetilde{\text{O3}}^+$-plane realize $Sp(2)$ gauge group. It seems that the different O3-plane backgrounds make differences of dyonic charge spectrum. However, since it is known that these different O3-plane background theories are related by S-duality transformation, charge lattices of these theories are coincident. We found correspondences between the $\mathbb{Z}_4$ S-fold theory and any O3-plane background theories which are related by S-duality.

- $\mathbb{Z}_6$ S-fold theory and G_2 SYM

  We found formulae which give the central charges Z and $\overline{Z}$ from a string junction in $\mathbb{Z}_6$ S-fold. From Aharony and Tachikawa's conjecture this $\mathbb{Z}_6$ S-fold theory corresponds to $\mathcal{N} = 4$ SYM with a G_2 gauge group. Unfortunately, since no brane construction of the G_2 theory we cannot confirm the correctness of the Z and $\overline{Z}$ formulae. In principle, by studying BPS states of 4d $\mathcal{N} = 4$ G_2 SYM in the context of field theory, we can check the correspondence between $\mathbb{Z}_6$ S-fold theory and G_2 SYM.

There are remaining problems in our investigation as follows :

- Mass spectrum of BPS states

  We could not reproduce the mass spectrum of anti-holomorphic $\frac{1}{4}$-BPS states. Masses of these BPS states are bounded by $M \geq \overline{Z}$. We concluded that these mass spectrum cannot be obtained easily in S-fold theories because the central charge $\overline{Z}$ is projected out by S-folding. In order to understand the supersymmetry enhancement, we have to know how to appear the $\overline{Z}$ in S-fold theories.

- The marginal deformation parameter τ

  In $\mathcal{N} = 4$ supersymmetric theories, there are marginal deformation operators and we can always consider continuous deformation of $\mathcal{N} = 4$ theories. Thus these theories always contain continuous marginal deformation parameter τ. However, the supersymmetry enhanced S-fold theories only realized fixed values of τ because of $\mathbb{Z}_k$ invariance of $\mathbb{Z}_k$ S-fold. A mechanism which gives continuous parameter τ should exist in S-fold theories.

Chapter 4

Codimension-2 brane solutions in $D = 9, 8$ and 7 supergravities

In the last chapter, we investigated the particular class of $\mathcal{N} = 3$ SCFT. We saw that we can obtain new class of theories by using relatively simple orbifold backgrounds in string theory. This fact shows that the classification of background spacetime is important for realizations of a wide class of theories. In this chapter, we focus on codimension-2 branes. (The codimension is the number of the transverse directions.) The reason is as follows.

In the previous chapter, we investigated 4d $\mathcal{N} = 3$ supersymmetric field theories which are constructed by using D3-branes in S-fold. García-Etxebarria and Regalado suggested generalization of the S-fold construction and they realized another class of example of $\mathcal{N} = 3$ theories which cannot be constructed by S-fold[11]. They realize a 4d $\mathcal{N} = 3$ theory on 3-branes in 6d maximal supergravity. Notice that the codimension of 3-branes in 6d spacetime is two.

Not only 4d $\mathcal{N} = 3$ theories, codimension-2 branes often play an important role in the brane realization of field theories. An important feature of codimension 2-branes is that they can have non-trivial monodromies [42]. Namely, when we move charged objects around such branes they may get transformed to dual objects. The element of duality group specifying this duality transformation is called monodromy associated with the branes. In the context of brane realization of field theories such monodromy transformations are often interpreted as the electric-magnetic duality, and in some cases the existence of branes with non-trivial monodromies causes emergence of particles with mutually non-local charges. This is a common feature of some classes of non-Lagrangian theories such as Argyres-Douglas theories [3, 4] and 4d $\mathcal{N} = 3$ SCFT. This fact motivates us to investigate codimension 2-branes.

In section 4.1 we will review the codimension-2 brane construction of 4d $\mathcal{N} = 3$ theories, which is obtained by considering an S-fold in the context of M-theory. In section 4.2 we construct codimension-2 brane solutions in maximal supergravities. To obtain the solutions we solve the Killing spinor equations[13].

4.1 Codimension-2 brane construction of $\mathcal{N} = 3$ theory

In this section we review the new construction of $\mathcal{N} = 3$ theories proposed in [11]. It is obtained by combining the construction methods of two different field theories, one is the S-fold construction of 4d $\mathcal{N} = 3$ theories discussed in Chapter 3, and the other is 6d $\mathcal{N} = (2, 0)$ ADE type theories[43]. For the latter method it is convenient to re-formulate the S-fold in the context of M-theory.

We first explain the M-theory lift of S-folds and how to construct 6d ADE type theories. Then, we combine them to realize a new class of $\mathcal{N} = 3$ theories.

4.1.1 S-fold in M-theory

To reformulate the S-fold in M-theory we start with the setup in type IIB theory investigated in Chapter 3. We compactify two directions on S^1 which we denote by $\widetilde{S}^1_T$ and S^1_E. Due to the compactification on $\widetilde{T}^2 = \widetilde{S}^1_T \times S^1_E$ the rotational symmetry $SO(6)_R$ in the transverse space is broken to its subgroup $SO(4) \times \mathbb{Z}_2$ for generic $\widetilde{T}^2$. Because of the S-fold construction uses $\mathbb{Z}_k \subset SU(4)_R \simeq SO(6)_R$, to perform the S-fold projection, in addition the tuning of the complex scalar field τ, we also need to tune the complex structure $\widetilde{\tau}$ of $\widetilde{T}^2$ to the special values so that $\mathbb{Z}_2$ symmetry is enhanced to $\mathbb{Z}_k$ $(k = 3, 4, 6)$.

Let us take the following duality to move to M-theory.

$$
\begin{array}{ccc}
\text{IIB}/\widetilde{T}^2 \ N \ \text{D3} & \text{on} & \underline{\mathbb{R}^{1,3}} \times \widetilde{S}^1_{\text{T}} \times S^1_{\text{E}} \times \mathbb{C}^2 \\
\downarrow \text{T-dual } (S^1_{\text{T}}) & & \\
\text{IIA}/T^2 \ N \ \text{D4} & \text{on} & \underline{\mathbb{R}^{1,3} \times S^1_{\text{T}}} \times S^1_{\text{E}} \times \mathbb{C}^2 \\
\downarrow S^1_{\text{M}} \to 0 & & \\
\text{M}/T^3 \ N \ \text{M5} & \text{on} & \underline{\mathbb{R}^{1,3} \times S^1_{\text{M}} \times S^1_{\text{T}}} \times S^1_{\text{E}} \times \mathbb{C}^2
\end{array}
$$

The underline denotes the directions spanned by the branes.

By the T-duality for $\widetilde{S}^1_{\rm T}$ direction, the complex structure $\widetilde{\tau}$ of $\widetilde{S}^1_{\rm T}\times S^1_{\rm E}$ becomes Kähler parameter

$$\rho=\int_{T^2}B_2+i\sqrt{\det G_{T^2}}, \tag{4.1.1}$$

of $S^1_{\rm T}\times S^1_{\rm E}$ in type IIA, where G_{T^2} is the metric on the T^2. In this picture, $\mathbb{Z}_k$ action on ρ mixes B_2 and G_{T^2}, and is not geometrically manifest [1]. Moreover, we lift this configuration up to the M-theory. Then combining $T^2=S^1_{\rm T}\times S^1_{\rm E}$ and the eleven-th direction S^1_M, we obtain $T^3=S^1_{\rm M}\times S^1_{\rm T}\times S^1_{\rm E}$. The Kähler parameter ρ becomes M-theory parameter ρ_M given by

$$\rho_M\equiv\int_{T^3}A_3+i\sqrt{\det G_{T^3}}. \tag{4.1.2}$$

In the type IIB setup, the S-fold is the $\mathbb{Z}_k$ identification in the transverse space $\mathbb{R}^6$ accompanied by the S-duality transformation.

$$\text{IIB}/\widetilde{T}^2\quad N\text{ D3}\quad\text{on}\quad \underline{\mathbb{R}^{1,3}}\times(\widetilde{S}^1_{\rm T}\times S^1_{\rm E}\times\mathbb{C}^2)/\mathbb{Z}_k. \tag{4.1.3}$$

From the viewpoint of M-theory, the $SL(2,\mathbb{Z})$ of S-duality in type IIB is interpreted as the modular transformation of T^2 on which M5 are wrapped, and (4.1.3) corresponds to the configuration

$$N\text{ M5}\quad\text{on}\quad \underline{\mathbb{R}^{1,3}\times(S^1_{\rm M}\times S^1_{\rm T}}\times S^1_{\rm E}\times\mathbb{C}^2)/\mathbb{Z}_k,\quad \mathbb{Z}_k=\mathbb{Z}_k^{\tau}\cdot\mathbb{Z}_k^R\cdot\widetilde{\mathbb{Z}}_k^R, \tag{4.1.4}$$

in the M-theory picture, where $\mathbb{Z}_k^{\tau},\mathbb{Z}_k^R$, and $\widetilde{\mathbb{Z}}_k^R$ act on complex structure τ_M of $S^1_{\rm M}\times S^1_{\rm T}$, $\mathbb{C}^2$, and the M-theory parameter ρ_M of $S^1_{\rm M}\times S^1_{\rm T}\times S^1_{\rm E}$ respectively. $\mathbb{Z}_k^{\tau}\cdot\widetilde{\mathbb{Z}}_k^R$ is a subgroup of the duality group $SL(3,\mathbb{Z})\times SL(2,\mathbb{Z})_\rho$ of M-theory on T^3[44]. In order to construct 4d $\mathcal{N}=3$ theories, τ_M and ρ_M must be fixed so that we have the discrete symmetries $\mathbb{Z}_k$ $(k=3,4,6)$.

4.1.2 4d $\mathcal{N}=3$ theory based on E-type gauge group

We have reinterpreted S-fold construction in the context of M-theory. One of the advantages of the construction by using M-theory is that we can realize a new class of 4d $\mathcal{N}=3$ theories. In [11], they combine 6d $\mathcal{N}=(2,0)$ E-type theories from M-theory with S-fold construction.

[1] $\mathbb{Z}_k$ is not regarded as a subgroup of any spatial rotations.

First, we explain a realization of 6d $\mathcal{N} = (2, 0)$ ADE type theories briefly. Such theories appear as the low energy effective theories of ADE singularities in type IIB theory[43]. Consider type IIB theory on $\mathbb{R}^6 \times (\mathbb{C}^2/\Gamma)$, where Γ is a discrete subgroup of $SU(2)$. It is known that there is a correspondence between $\Gamma \subset SU(2)$ and simply laced Lie algebras. Namely, Γ is labeled by A_n, D_n or E_n. It is useful to regard $\mathbb{C}^2/\Gamma$ as a singular elliptic fibration over $\mathbb{C}$. Away from the singularity, such a space seems to be a $\mathbb{C} \times T^2$. At the singularity a cycle on the T^2 shrinks to zero size. When we go around a singularity, the complex structure of T^2 is transformed by the monodromy as

$$\tau \to \tau' = \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}). \tag{4.1.5}$$

The monodromies characterizes the singularities (Table 4.1).

Table 4.1: Monodromy of ADE singularities

Singularities	A_{N-1}	D_N	E_6	E_7	E_8
Monodromy	$\begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 4-N \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$

This type IIB configuration can also be transformed to M-theory by duality. Let us take the T-duality transformation along one of the directions of T^2. This procedure maps type IIB to type IIA and exchanges the complex structure τ and the Kähler parameter ρ.

Note that since ρ can be written as (4.1.1), the monodromy transformation can be regarded as what is induced by the existence of 5-branes in $\mathbb{R}^6$. For example, A_{N-1} type monodromy act on the Kähler parameter as $\rho \to \rho' = \rho + N$. The shift of the real part means ρ picks up the magnetic charge of B_2, and then the singularity corresponds to a stack of NS5-branes. The D_N type singularity is also interpreted as N NS5-branes and an ON5[45]. The brane interpretation corresponding to the E type singularities is not known. At the singularity, ρ is determined as a fixed point of the E type monodromy transformations: for E_6 and E_8, $\rho = e^{i\pi/3}$ and for E_7, $\rho = i$.

We can lift this up to M-theory as we have done in section4.1.1. From the viewpoint of M-theory, the monodromy transformation acts on the M-theory parameter ρ_M. For the E type singurality, ρ_M must be i or $e^{i\pi/3}$.

As we regarded the $\mathbb{C}^2/\Gamma$ as a singular T^2 fibration over $\mathbb{C}$ in type IIB setup, we can regard the T^3 fibration over $\mathbb{C}$ as $(\mathbb{C} \times T^3)/\mathbb{Z}_p$[11]. The $\mathbb{Z}_p$ is given by the combination of $\mathbb{Z}_p^{\mathbb{C}}$ and $\mathbb{Z}_p^{\rho}$ which acts on $\mathbb{C}$ as a rotation by $2\pi/k$ and on ρ_M via the monodromy, respectively. $p = 3, 4, 6$ depends on the choice of monodromy.

To combine the E type construction and S-fold construction, we consider the M-theory on a five-torus,

$$\mathbb{R}^4 \times S^1_a \times S^1_b \times S^1_c \times S^1_d \times S^1_e \times \mathbb{C}. \tag{4.1.6}$$

The S-fold quotient is given by

$$\mathbb{R}^4 \times (T^3_{abc} \times T^2_{de} \times \mathbb{C})/\mathbb{Z}^S_k, \quad \mathbb{Z}^S_k = \mathbb{Z}^\tau_k \cdot \mathbb{Z}^R_k \cdot \widetilde{\mathbb{Z}}^R_k, \tag{4.1.7}$$

and the E type quotient is given by

$$\mathbb{R}^4 \times T^2_{ab} \times (T^3_{cde} \times \mathbb{C})/\mathbb{Z}^E_p, \quad \mathbb{Z}^E_p = \mathbb{Z}^\rho_p \cdot \mathbb{Z}^{\mathbb{C}}_p, \tag{4.1.8}$$

where we denoted subtori by $T^2_{ab} = S^1_a \times S^1_b$, and so on. As a result, the 4d E type $\mathcal{N} = 3$ theories arise when we consider the M-theory on

$$\mathbb{R}^4 \times (\mathbb{C} \times T^5)/(\mathbb{Z}^S_k \times \mathbb{Z}^E_p). \tag{4.1.9}$$

The label p specifies one of $E_{6,7,8}$ appearing the 6d $\mathcal{N} = (2, 0)$ theories, and the label k specifies S-folds. The M-theory ρ parameter and complex structure of subtori are fixed according to the value of k and p.

The 4d theories are obtained by taking the small T^5 limit. Thus, we can interpret this 4d $\mathcal{N} = 3$ theories as world-volume theories of codimension-2 branes (3-branes) in the 6d maximal supergravity theory.

4.2 Codimension-2 brane solutions in maximal supergravities

The new class of 4d $\mathcal{N} = 3$ theories are given by the world volume theories of codimension-2 branes. In general, codimension-2 branes have non-trivial monodromies, and it is useful for field theory. For instance, a theory of 7-branes in type IIB superstring is a powerful tool for analyzing field theories, and it is called F-theory[46]. The 7-branes have already been classified[48], and some 4d field theories can be realized by proving D3-branes in 7-branes background[47]. Since the Argyres-Douglas theories also realized on a proved D3-brane in a certain 7-branes background, it seems that codimension-2 branes backgrounds are useful to construct non-Lagrangian field theories.

In this section, we consider codimension-2 brane solutions in various dimensions, and classify preserved supersymmetry on the branes. First, we explain maximal supergravity theories and how to construct codimension-2 brane solutions by reviewing the case of 10d type IIB supergravity. After that, we solve Killing spinor equations to obtain codimension-2 brane solutions in 9d, 8d, and 7d.

4.2.1 Maximal supergravities

In the context of string/M-theory, a maximal supergravity is obtained by torus compactification of ten- or eleven-dimensional theory. The U-duality group is generated by geometric coordinate changes of the torus and non-geometric duality transformations.

Maximal supergravities in various dimensions have common structure. The scalar manifolds of these theories have the form G/H, where G is the classical global symmetry and H is the local symmetry group, which is the maximal compact subgroup of G. See Table 4.2 for G and H in dimensions $D = 10, \ldots, 4$ [49]. The scalar fields

Table 4.2: The global symmetry group G, the duality group $G_{\mathbb{Z}}$, and the local symmetry group H in maximal supergravities

dim	G	$G_{\mathbb{Z}}$	H
10(A)	$SO(1,1)/\mathbb{Z}_2$	1	1
10(B)	$SL(2,\mathbb{R})$	$SL(2,\mathbb{Z})$	$SO(2)$
9	$SL(2,\mathbb{R}) \times O(1,1)$	$SL(2,\mathbb{Z}) \times \mathbb{Z}_2$	$SO(2)$
8	$SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$	$SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$	$SO(3) \times SO(2)$
7	$SL(5,\mathbb{R})$	$SL(5,\mathbb{Z})$	$SO(5)$
6	$O(5,5)$	$O(5,5;\mathbb{Z})$	$SO(5) \times SO(5)$
5	$E_{6(6)}$	$E_{6(6)}(\mathbb{Z})$	$USp(8)$
4	$E_{7(7)}$	$E_{7(7)}(\mathbb{Z})$	$SU(8)$

are coordinates of this manifold, and represented as a matrix $L \in G$ with left action of G and right action of H. The U-duality group $G_{\mathbb{Z}}$ is the integral form of G. For the theories in seven or higher dimensions G are all SL type and H are all SO type, and they can be dealt in similar ways.

A maximal supergravity contains

- the vielbein $e_M^{\widehat{M}}$
- scalar fields $L^\alpha{}_i$
- gravitino ψ_M
- dilatino λ_i

and anti-symmetric tensor fields of different ranks, which are not relevant to our analysis in this section. We use the following indices:

- $M, N, \ldots$: global coordinates
- $\widehat{M}, \widehat{N}, \ldots$: local Lorentz
- $\alpha, \beta, \ldots$: $SL(m)$ fundamental representation
- $i, j, \ldots$: $H = SO(n)$ vector.

The scalar fields appear in the action and the supersymmetry transformation laws through 1-form fields P and Q, which are defined as the symmetric and anti-symmetric parts of the Maurer-Cartan form:

$$P_{ij} = (L^{-1}dL)_{(ij)}, \quad Q_{ij} = (L^{-1}dL)_{[ij]}. \tag{4.2.1}$$

Under H transformation P transforms homogeneously as the symmetric matrix representation of H, while Q transforms inhomogeneously and plays the role of H-connection.

To obtain BPS solutions [2] we solve the Killing spinor equations for the gravitino ψ_M and dilatino λ_i.

4.2.2 $D = 10$

Let us consider BPS solutions in type IIB supergravity. Such solutions have been well investigated[42] and 7-branes are classified by Kodaira classification[50]. Various 4d $\mathcal{N} = 2$ supersymmetric theories are realized on D3-branes probing these solutions[51, 52, 53]. A purpose of this section is to review how we can obtain BPS solutions in ten dimensions by solving the Killing spinor equations. The derivations in lower dimensions are parallel.

The classical global symmetry of type IIB supergravity is $G = SL(2,\mathbb{R})$ and the local R-symmetry group is $H = SO(2)_R$. Namely, the scalar manifold is locally the two-dimensional homogeneous space $SL(2,\mathbb{R})/SO(2)_R$. When we discuss the global structure, we also need to take account of the duality group $G_{\mathbb{Z}} = SL(2,\mathbb{Z})$. Quantum numbers of scalar and spinor fields in type IIB supergravity [34] are summarized in Table 4.3. The gravitino field ψ_M belongs to the spinor representation of H. Namely, ψ_M has the spacetime vector index M and an $SO(2)_R$ spinor index which is implicit. The dilatino field λ_i has the $SO(2)_R$ vector index i and an implicit $SO(2)_R$ spinor index. It satisfies the ρ-traceless condition

$$\rho_i \lambda^i = 0, \tag{4.2.2}$$

[2] We would like to consider the codimension-2 branes which partially preserved supersymmetries.

Table 4.3: Quantum numbers of scalar and spinor fields in type IIB supergravity

	$G = SL(2,\mathbb{R})$	$H = SO(2)_R$
$L^\alpha{}_i$	2	± 1
ψ_M	1	$\pm\frac{1}{2}$
λ_i	1	$\pm\frac{3}{2}$
ϵ	1	$\pm\frac{1}{2}$

where ρ_i are Dirac matrices associated with the orthogonal group $H = SO(2)_R$. See appendix for our notation. This condition removes components carrying $SO(2)_R$ charge $\pm 1/2$ from λ^i, and the remaining components in λ^i carry $SO(2)_R$ charge $\pm 3/2$ as is shown in Table 4.3.

Due to the existence of the self-dual 4-form field it is difficult to write down the full Lagrangian of the type IIB supergravity. However, it is easy to give the Lagrangian of the subsector which is relevant to us. If we assume the vanishing anti-symmetric tensor fields the equations of motion for the remaining fields are obtained from the Lagrangian

$$\begin{aligned}\mathcal{L} = &\frac{e}{4}R + \frac{e}{2}(\psi_M \Gamma^{MNP} D_N \psi_P) \\ &- \frac{e}{4}(P_M{}^{ij})^2 + \frac{e}{2}(\lambda_i \Gamma^N D_N \lambda_i) + \frac{e}{2} P_M{}^{ij}(\psi_N \Gamma^M \Gamma^N \Gamma_i \lambda_j),\end{aligned} \tag{4.2.3}$$

up to higher order fermion terms.

The Killing spinor equations are

$$0 = \delta\psi_M = D_M \epsilon, \tag{4.2.4}$$

$$0 = \delta\lambda^i = P^{ij}_M \Gamma^M \rho_j \epsilon, \tag{4.2.5}$$

where D_M is the covariant derivative defined with the spin connection and the $SO(2)_R$ connection Q. Let us assume the solution has the eight dimensional Poincare invariance along the eight longitudinal directions. We take the ansatz

$$L^\alpha{}_i = L^\alpha{}_i(x^m), \quad e^{\widehat{\mu}} = f(x^m)\delta^{\widehat{\mu}}_\mu dx^\mu, \quad e^a = g(x^m)\delta^a_m dx^m, \quad \epsilon = \epsilon(x^m), \tag{4.2.6}$$

where we use x^μ ($\mu = 0, 1, \ldots, 7$) and x^m ($m = 8, 9$) for longitudinal and transverse coordinates. Because $L^\alpha{}_i$ is independent of the longitudinal coordinates x^μ, the longitudinal components of Q vanish. For the longitudinal components of the Killing spinor equation (4.2.4)

$$\delta\psi_\mu = D_\mu \epsilon = \left(\partial_\mu - \frac{1}{2g}(\partial_m f)\Gamma_{m\widehat{\mu}}\right)\epsilon = 0 \tag{4.2.7}$$

to have non-trivial solutions the function f must be constant, and without loss of generality we can set $f = 1$.

The covariant derivative in the transverse components of (4.2.4) include the connection of $SO(2)_{89}$, the rotation in the 8-9 plane, and that of $H = SO(2)_R$:

$$D_m\epsilon = \left(\partial_m + \frac{1}{2}\omega_{m89}\Gamma^{89} + \frac{1}{2}Q_{m12}\rho^{12}\right)\epsilon = 0. \tag{4.2.8}$$

For the existence of non-vanishing solutions, the action of two connections on some components of ϵ must be pure gauge. To study this condition, it is convenient to decompose the parameter ϵ into four parts $\epsilon_{s,r}$ according to $SO(2)_{89}$ and $SO(2)_R$ charges so that

$$\frac{1}{2}\Gamma_{89}\epsilon_{s,r} = is\epsilon_{s,r}, \quad \frac{1}{2}\rho_{12}\epsilon_{s,r} = ir\epsilon_{s,r}, \tag{4.2.9}$$

where both the indices s and r take values in $\{+\frac{1}{2}, -\frac{1}{2}\}$. We also decompose λ^i in the same way into λ^i_{sr}. For distinction we use $s = \{\uparrow, \downarrow\}$ for $SO(2)_{89}$ and $r = \{+, -\}$ for $SO(2)_R$. We also introduce $i = \{\oplus, \ominus\}$ for complex basis of $SO(2)_R$ vectors, which carry $SO(2)_R$ charge ± 1. See appendix for detail.

Let us require the solution to be half BPS. Without loosing generality we can assume that $\epsilon_{\uparrow +}$ and its Majorana conjugate $\epsilon_{\downarrow -}$ correspond to the unbroken supersymmetries. The other components are set to be zero: $\epsilon_{\downarrow +} = \epsilon_{\uparrow -} = 0$. Then, the non-vanishing components of $\delta\lambda$ are

$$\delta\lambda^i_{\downarrow -} = P^{i\oplus}_{z^*}\epsilon_{\uparrow +}, \tag{4.2.10}$$

and its complex conjugate.

Before proceeding, it would be instructive to check the consistency of the quantum numbers in (4.2.10). Let us first consider the $SO(2)_{89}$ quantum numbers. The left hand side has the lower index $\downarrow$. This means the component carries $SO(2)_{89}$ charge (spin) $-1/2$. On the right hand side, the parameter ϵ has lower index $\uparrow$ which means $SO(2)_{89}$ spin $+1/2$. In addition, P has lower index z^* and this component carries $SO(2)_{89}$ spin -1. Therefore, both left and right hand sides carry the same $SO(2)_{89}$ spin $-1/2$. The coincidence of the $SO(2)_R$ charge can be confirmed in a similar way. The index i is common for left and right hand sides and thus let us focus on other indices. On the left hand side we have lower $-$ index and this means it carries $SO(2)_R$ charge $-1/2$. On the right hand side there are the upper $\oplus$ index on P and the lower $+$ index on ϵ, which carry $SO(2)_R$ charges -1 and $+1/2$, respectively. Therefore, the left and right hand sides carry the same $SO(2)_R$ charge $-1/2$. The charge counting we have just explained is quite useful when we extract condition imposed on P from Killing spinor equations associated with dilatino fields in different dimensions.

The vanishing of (4.2.10) mean

$$P_{z^*}^{\oplus\oplus} = 0. \tag{4.2.11}$$

($P_{z^*}^{\ominus\oplus}$ is identically zero due to the traceless condition.) We want to solve this with respect to the scalar fields $L^\alpha{}_i$. For this purpose it is convenient to gauge fix the local $SO(2)_R$ symmetry so that the matrix L is given by

$$L = K(\tau), \tag{4.2.12}$$

where $K(\tau)$ for a complex number τ in the upper half plane is the following 2×2 matrix:

$$K(\tau) = \frac{1}{\sqrt{\tau_2}} \left(\begin{array}{cc} 1 & 0 \\ \tau_1 & \tau_2 \end{array} \right), \quad \tau \equiv \tau_1 + i\tau_2 \in H_+. \tag{4.2.13}$$

Then P and Q have the components

$$P^{ij} = \frac{1}{2\tau_2} \left(\begin{array}{cc} -d\tau_2 & d\tau_1 \\ d\tau_1 & d\tau_2 \end{array} \right), \quad Q^{ij} = \frac{d\tau_1}{2\tau_2} \left(\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right). \tag{4.2.14}$$

In this gauge the equation (4.2.11) gives

$$P_{z^*}^{\oplus\oplus} = \frac{i}{2\tau_2} \partial_{z^*} \tau = 0. \tag{4.2.15}$$

Namely, τ must be a holomorphic function of z.

Now let us turn to the equation $\delta\psi_m = D_m \epsilon = 0$. The components including the non-vanishing parameters $\epsilon_{\uparrow +}$ and $\epsilon_{\downarrow -}$ are

$$D_m \epsilon_{\uparrow +} = \left(\partial_m + \frac{i}{2} (\omega_{m89} + Q_{m12}) \right) \epsilon_{\uparrow +} = 0 \tag{4.2.16}$$

and its complex conjugation. For (4.2.16) to have solutions with $\epsilon_{\uparrow +} \neq 0$, the net connection $\omega_{89} + Q_{12}$ must be pure gauge, and we can take the gauge with $\omega_{89} + Q_{12} = 0$. The explicit form of the spin connection and the $SO(2)_R$ connection are

$$\omega_{89} = i \frac{\partial g}{g} dz - i \frac{\overline{\partial} g}{g} dz^*, \quad Q_{12} = -\frac{d\tau_1}{2\tau_2} = -i \frac{\partial \tau_2}{2\tau_2} dz + i \frac{\overline{\partial} \tau_2}{2\tau_2} dz^*. \tag{4.2.17}$$

where we used holomorphy of τ in the last equality. From $\omega_{89} + Q_{12} = 0$ we obtain

$$\frac{dg}{g} = \frac{d\tau_2}{2\tau_2} \tag{4.2.18}$$

and this is solved by

$$g = c\sqrt{\tau_2}, \tag{4.2.19}$$

where c is an arbitrary constant.

The solution is summarized as follows.

$$L^\alpha{}_i = K(\tau), \tag{4.2.20}$$
$$g = c\sqrt{\tau_2}, \tag{4.2.21}$$
$$\tau(z) = \tau_1 + i\tau_2, \quad \tau_2 > 0. \tag{4.2.22}$$

4.2.3 $D = 9$

Let us start the analysis in lower dimension following the prescription in the last subsection. The scalar and spinor fields in the nine-dimensional $\mathcal{N} = 2$ supergravity [54] are summarized in Table 4.4. The fields λ_i are subject to the gamma-traceless

Table 4.4: The quantum numbers of scalar and spinor fields in the nine-dimensional $\mathcal{N} = 2$ supergravity

	$SL(2)$	$SO(2)$
$L^\alpha{}_i$	2	± 1
φ	1	0
ψ_M	1	$\pm\frac{1}{2}$
λ_i	1	$\pm\frac{3}{2}$
$\widetilde{\lambda}$	1	$\pm\frac{1}{2}$
ϵ	1	$\pm\frac{1}{2}$

condition $\rho_i\lambda_i = 0$. The Lagrangian is

$$\begin{aligned}\mathcal{L} =& -\frac{e}{4}R - \frac{i}{2}e(\psi_L\Gamma^{LMN}D_M\psi_N)\\ &+\frac{e}{4}(P_{Mij})^2 + \frac{i}{2}e(\lambda_i\Gamma^M D_M\lambda_i) + \frac{i}{2}e(\psi_M\rho_i\Gamma^N\Gamma^M\lambda_j)P_{Nij}\\ &+\frac{e}{2}(\partial_M\varphi)^2 + \frac{i}{2}e(\widetilde{\lambda}\Gamma^M D_M\widetilde{\lambda}) + \frac{i}{\sqrt{2}}e(\psi_M\Gamma^N\Gamma^M\widetilde{\lambda}_j)\partial_N\varphi + \dots,\end{aligned} \tag{4.2.23}$$

where the dots represent terms with gauge fields and four-fermi terms, which play no role in the following analysis. The supersymmetry transformation rules for the spinor fields are

$$\delta\psi_M = D_M\epsilon, \tag{4.2.24}$$

$$\delta\lambda_i = \frac{1}{2}P_{Mij}\Gamma^M\rho_j\epsilon, \tag{4.2.25}$$

$$\delta\widetilde{\lambda} = \frac{1}{\sqrt{2}}D_M\varphi\Gamma^M\epsilon. \tag{4.2.26}$$

We want to obtain codimension 2 brane solutions by solving the Killing spinor equations. We take the ansatz

$$L^\alpha{}_i = L^\alpha{}_i(x^m), \quad \varphi = \varphi(x^m), \quad e^{\hat{\mu}} = \delta^{\hat{\mu}}_\mu dx^\mu, \quad e^a = g(x^m)\delta^a_m dx^m, \quad \epsilon = \epsilon(x^m) \tag{4.2.27}$$

In fact, the solution is almost the same as that of type IIB case. Although we have extra fields φ and $\widetilde{\lambda}$ compared to the ten-dimensional case, the condition $\delta\widetilde{\lambda} = 0$ forces φ to be constant;

$$0 = \delta\widetilde{\lambda} = \frac{1}{\sqrt{2}}\partial_m\varphi\Gamma^m\epsilon \quad \to \quad \partial_m\varphi = 0. \tag{4.2.28}$$

Therefore, we can forget about $\widetilde{\lambda}$ and φ, and remaining fields give the set of equations identical to the ten-dimensional case. The general solution is

$$\varphi = \text{const} \tag{4.2.29}$$

$$L^\alpha{}_i = K(\tau), \quad \tau = \tau_1 + i\tau_2 : \text{holomorphic function} \tag{4.2.30}$$

$$g = c\sqrt{\tau_2}, \quad c : \text{const} \tag{4.2.31}$$

4.2.4 $D = 8$

The scalar and fermion fields in 8d maximal supergravity [55] are shown in in Table 4.5. Classical p-brane solutions ($p = 0, 1, 3, 4$) are given in [56]. In the following we construct 5-brane solutions.

The scalar manifold of the eight dimensional maximal supergravity is the direct product of two homogeneous spaces: $SL(2,\mathbb{Z})/SO(2)_R \times SL(3,\mathbb{Z})/SO(3)_R$. Each factor can be interpreted geometrically in an appropriate duality frame. The $SL(2,\mathbb{Z})/SO(2)_R$ becomes manifest when we regard the theory as T^2 compactification of type IIB theory, while $SL(3,\mathbb{Z})/SO(3)_R$ can be regarded as the moduli space

Table 4.5: Quantum numbers of scalar and spinor fields in 8d maximal supergravity. The $SO(2)_R$ charge of each component of a spinor is proportional to the chirality.

	$SO(3)_R$	$SO(2)_R$
$\widetilde{L}^{\tilde{\alpha}}{}_{\tilde{i}}$	3	0
$L^{\alpha}{}_i$	1	± 1
ψ_M	2	$\frac{1}{2}\Gamma_9$
λ_i	2	$\frac{3}{2}\Gamma_9$
$\widetilde{\lambda}_{\tilde{i}}$	4	$-\frac{1}{2}\Gamma_9$

associated with T^3 compactification of M-theory. The S-duality group in the type IIB picture is a subgroup of $SL(3,\mathbb{Z})$.

For each factor of R-symmetry groups there is associated dilatino field. We denote fields associated with $SO(2)_R$ and $SO(3)_R$ by λ^i and $\widetilde{\lambda^{\tilde{i}}}$, respectively. All fermion fields have implicit spinor indices for all $SO(1,7)$, $SO(2)_R$, and $SO(3)_R$. In addition, λ and $\widetilde{\lambda}$ have $SO(2)$ and $SO(3)$ vector indices, respectively, and they satisfy the traceless conditions $\rho_i\lambda^i = 0$ and $\widetilde{\rho}_{\tilde{i}}\widetilde{\lambda^{\tilde{i}}} = 0$. Namely, λ and $\widetilde{\lambda}$ belong to $2_{\pm\frac{3}{2}}$ and $4_{\pm\frac{1}{2}}$, respectively, of $SO(2)_R \times SO(3)_R$.

The Lagrangian is

$$\begin{aligned}\mathcal{L} &= \frac{e}{4}R + \frac{e}{2}(\psi_M\Gamma^{MNP}D_N\psi_P)\\ &- \frac{e}{4}(P_M{}^{ij})^2 + \frac{e}{2}(\lambda_i\Gamma^N D_N\lambda_i) + \frac{e}{2}P_M{}^{ij}(\psi_N\Gamma^M\Gamma^N\rho_i\lambda_j)\\ &- \frac{e}{4}(\widetilde{P}_M{}^{\widetilde{ij}})^2 - \frac{e}{2}(\widetilde{\lambda}_{\tilde{i}}\Gamma^N D_N\widetilde{\lambda}_{\tilde{i}}) + i\frac{e}{2}\widetilde{P}_M{}^{\widetilde{ij}}(\psi_N\Gamma^M\Gamma^N\Gamma_{\tilde{i}}\widetilde{\lambda}_{\tilde{j}}) + \ldots \end{aligned} \tag{4.2.32}$$

where the dots represent four-fermion terms and terms with gauge fields. The supersymmetry transformation laws of fermions are

$$\delta\psi_M = D_M\epsilon, \tag{4.2.33}$$

$$\delta\lambda^i = \frac{1}{2}P_M{}^{ij}\Gamma^M\rho_j\epsilon, \tag{4.2.34}$$

$$\delta\widetilde{\lambda^{\tilde{i}}} = \frac{i}{2}\widetilde{P}_M{}^{\widetilde{ij}}\Gamma^M\widetilde{\rho}_{\tilde{j}}\epsilon. \tag{4.2.35}$$

We take the following ansatz:

$$L^{\alpha}{}_i = L^{\alpha}{}_i(x^m), \quad \widetilde{L}^{\tilde{\alpha}}{}_{\tilde{i}} = \widetilde{L}^{\tilde{\alpha}}{}_{\tilde{i}}(x^m), \quad e^{\hat{\mu}} = \delta^{\hat{\mu}}_{\mu}dx^{\mu}, \quad e^a = g(x^m)\delta^a_m dx^m, \quad \epsilon = \epsilon(x^m) \tag{4.2.36}$$

The covariant derivative $D_M\epsilon$ contains three connections ω, Q, and $\widetilde{Q}$, corresponding to $SO(2)_{67}$, $SO(2)_R$, and $SO(3)_R$, respectively. For the existence of non-trivial solution to $\delta\psi_m = 0$, the actions of three connections to some components of ϵ must be pure gauge. For this to be the case, non-vanishing components of $SO(3)_R$ connection $\widetilde{Q}$ should be in a certain $SO(2)$ subgroup of $SO(3)_R$. We can take the gauge such that it is rotation of $\widetilde{12}$ plane and

$$\widetilde{Q}^{\widetilde{i3}} = 0. \tag{4.2.37}$$

After taking this gauge, we have three $SO(2)$ connections ω_{67}, Q_{12} and $\widetilde{Q}_{\widetilde{12}}$. As in the 10 dimensional case it is convenient to divide the parameter ϵ into components $\epsilon_{sr\widetilde{r}}$ so that

$$\frac{1}{2}\Gamma_{67}\epsilon_{sr\widetilde{r}} = is\epsilon_{sr\widetilde{r}}, \quad \frac{1}{2}\rho_{12}\epsilon_{sr\widetilde{r}} = ir\epsilon_{sr\widetilde{r}}, \quad \frac{1}{2}\widetilde{\rho}_{12}\epsilon_{sr\widetilde{r}} = i\widetilde{r}\epsilon_{sr\widetilde{r}}, \tag{4.2.38}$$

where all of s, r, and $\widetilde{r}$ take values in $\{+\frac{1}{2}, -\frac{1}{2}\}$. For distinction we introduce the notation $s \in \{\uparrow, \downarrow\}$ for $SO(2)_{67}$, $r \in \{+, -\}$ for $SO(2)_R$, and $\widetilde{r} \in \{\widetilde{+}, \widetilde{-}\}$ for $SO(3)_R$. We also introduce $\{\oplus, \ominus\}$ for the complex basis of $SO(2)_R$ vector and $\{\widetilde{\oplus}, \widetilde{\ominus}, \widetilde{3}\}$ for the basis of $SO(3)_R$ vector that diagonalize $SO(2)_{\widetilde{12}}$.

The 6d chirality of $\epsilon_{sr\widetilde{r}}$ is given by s and $\widetilde{r}$ as

$$\gamma_7\epsilon_{sr\widetilde{r}} = \mathrm{sign}(s\widetilde{r})\epsilon_{sr\widetilde{r}}. \tag{4.2.39}$$

We want to consider solution in which some of $\epsilon_{sr\widetilde{r}}$ are preserved. Without loss of generality, we can suppose that $\epsilon_{\uparrow+\widetilde{+}}$ and its complex conjugate $\epsilon_{\downarrow-\widetilde{-}}$ are non-vanishing. Both of them have positive chirality, and they generate six-dimensional $\mathcal{N} = (1, 0)$ supersymmetry.

Let us consider the condition $\delta\lambda = 0$ first. The component of $\delta\lambda$ depending on $\epsilon_{\uparrow+\widetilde{+}}$ is

$$0 = \delta\lambda^{\oplus}_{\downarrow-\widetilde{+}} = P^{\oplus\oplus}_{z^*}\epsilon_{\uparrow+\widetilde{+}}. \tag{4.2.40}$$

For this to hold for $\epsilon_{\uparrow+\widetilde{+}} \neq 0$, $P^{\oplus\oplus}_{z^*}$ must vanish. This is the same as (4.2.11) in Section 4.2.2, and the solution is given by $L = K(\tau)$ with holomorphic $\tau(z)$.

We can also obtain similar condition for $\widetilde{L}$ form $\delta\widetilde{\lambda} = 0$. The components of $\delta\widetilde{\lambda}$ depending on $\epsilon_{\uparrow+\widetilde{+}}$ are $\delta\widetilde{\lambda}^{\widetilde{i}}_{\downarrow+\widetilde{\pm}}$, and we obtain the following Killing spinor equations.

$$0 = \delta\widetilde{\lambda}^{\widetilde{i}}_{\downarrow+\widetilde{+}} = \frac{i}{\sqrt{2}}\widetilde{P}^{\widetilde{i3}}_{z^*}\epsilon_{\uparrow+\widetilde{+}}, \tag{4.2.41}$$

$$0 = \delta\widetilde{\lambda}^{\widetilde{i}}_{\downarrow+\widetilde{-}} = i\widetilde{P}^{\widetilde{i\oplus}}_{z^*}\epsilon_{\uparrow+\widetilde{+}}. \tag{4.2.42}$$

The equation (4.2.41) require $\widetilde{P}^{\widetilde{i}\widetilde{3}} = 0$, and combining this with (4.2.37) we conclude that L is essentially $SL(2)$ element. Namely, in an appropriate choice of gauge it is given by

$$\widetilde{L}^{\widetilde{\alpha}}{}_{\widetilde{i}} = \widetilde{L}_0 \begin{pmatrix} K(\widetilde{\tau}) & 0 \\ 0 & 1 \end{pmatrix} \quad (\widetilde{\tau} \equiv \widetilde{\tau}_1 + i\widetilde{\tau}_2 \in H_+) \tag{4.2.43}$$

where $\widetilde{L}_0 \in SL(3,\mathbb{R})$ is a constant matrix. The condition $\widetilde{P}^{\widetilde{i}\widetilde{\oplus}}_{z^*} = 0$ obtained from (4.2.42) require the function $\widetilde{\tau}$ be a holomorphic function of z.

Finally, we can determine the function g by using $\delta\psi_m = D_m\epsilon = 0$. For this equation to hold for $\epsilon_{\uparrow+\widetilde{+}} \neq 0$, the sum of three connections ω, Q and $\widetilde{Q}$ must be pure gauge, and we can take the gauge in which

$$\omega_{m67} + Q_{m12} + \widetilde{Q}_{m\widetilde{1}\widetilde{2}} = 0. \tag{4.2.44}$$

This gives the differential equation

$$\frac{i}{g}\partial_z g = \frac{i}{2\tau_2}\partial_z\tau_2 + \frac{i}{2\widetilde{\tau}_2}\partial_z\widetilde{\tau}_2, \tag{4.2.45}$$

which is solved by

$$g = c\sqrt{\tau_2\widetilde{\tau}_2}. \tag{4.2.46}$$

The solution is summarized as follows.

$$L^{\alpha}{}_{i} = K(\tau), \quad \tau = \tau_1 + i\tau_2, \tag{4.2.47}$$

$$\widetilde{L}^{\widetilde{\alpha}}{}_{\widetilde{i}} = \widetilde{L}_0 \begin{pmatrix} K(\widetilde{\tau}) & 0 \\ 0 & 1 \end{pmatrix}, \quad \widetilde{L}_0 \in SL(3,\mathbb{R}), \quad \widetilde{\tau} = \widetilde{\tau}_1 + i\widetilde{\tau}_2. \tag{4.2.48}$$

$$g = c\sqrt{\tau_2\widetilde{\tau}_2}. \tag{4.2.49}$$

This is the general form of 1/4 BPS solutions.

1/2 BPS solutions are realized as special cases of this solution. Let us consider the case in which the supersymmetries associated with $\epsilon_{\uparrow-\widetilde{+}}$ and its conjugate $\epsilon_{\downarrow+\widetilde{-}}$ are also preserved. (4.2.39) shows that these components have negative chirality in six dimensions and we have $\mathcal{N} = (1,1)$ supersymmetry in this case. The Killing spinor equations including $\epsilon_{\uparrow-\widetilde{+}}$ are

$$0 = \delta\lambda^{\ominus}_{\downarrow+\widetilde{+}} = P^{\ominus\ominus}_{z^*}\epsilon_{\uparrow-\widetilde{+}}, \tag{4.2.50}$$

$$0 = \delta\widetilde{\lambda}^{\widetilde{i}}_{\downarrow-\widetilde{+}} = \frac{i}{\sqrt{2}}\widetilde{P}^{\widetilde{i}\widetilde{3}}_{z^*}\epsilon_{\uparrow-\widetilde{+}}, \tag{4.2.51}$$

$$0 = \delta\widetilde{\lambda}^{\widetilde{i}}_{\downarrow-\widetilde{-}} = i\widetilde{P}^{\widetilde{i}\widetilde{\oplus}}_{z^*}\epsilon_{\uparrow-\widetilde{+}}, \tag{4.2.52}$$

$$0 = \delta\psi_{m,\uparrow-\widetilde{+}} = D_m\epsilon_{\uparrow-\widetilde{+}}. \tag{4.2.53}$$

We have additional condition $P_{z^*}^{--} = 0$ from (4.2.50), and this require τ to be anti-holomorphic. This means τ must be a constant. Then the other equations hold.

There is another type of 1/2 BPS solutions with $\epsilon_{\uparrow+\tilde{-}}, \epsilon_{\downarrow-\tilde{+}} \neq 0$. (4.2.39) shows that these components have positive 6d chirality, and we obtain $\mathcal{N} = (2,0)$ supersymmetry in six dimensions. The Killing spinor equations including $\epsilon_{\uparrow+\tilde{-}}$ are

$$0 = \delta\lambda^{\oplus}_{\downarrow-\tilde{-}} = P^{\oplus\oplus}_{z^*}\epsilon_{\uparrow+\tilde{-}}, \tag{4.2.54}$$

$$0 = \delta\widetilde{\widetilde{\lambda}}^{i}_{\downarrow+\tilde{-}} = \frac{i}{\sqrt{2}}\widetilde{\widetilde{P}}^{i3}_{z^*}\epsilon_{\uparrow+\tilde{-}}, \tag{4.2.55}$$

$$0 = \delta\widetilde{\widetilde{\lambda}}^{i}_{\downarrow+\tilde{+}} = \widetilde{\widetilde{P}}^{i\widetilde{\ominus}}_{z^*}\epsilon_{\uparrow+\tilde{-}}, \tag{4.2.56}$$

$$0 = \delta\psi_{m,\uparrow+\tilde{-}} = D_m\epsilon_{\uparrow+\tilde{-}}. \tag{4.2.57}$$

(4.2.56) gives new condition $\widetilde{\widetilde{P}}^{i\widetilde{\ominus}}_{z^*} = 0$, and this means $\widetilde{\tau}$ is a constant. Then the other conditions are satisfied.

Finally, let us consider the case with $\epsilon_{\downarrow+\tilde{+}}$ and $\epsilon_{\uparrow-\tilde{-}}$ are non-vanishing. The Killing spinor equations including $\epsilon_{\downarrow+\tilde{+}}$ are

$$0 = \delta\lambda^{\oplus}_{\uparrow-\tilde{+}} = P^{\oplus\oplus}_{z}\epsilon_{\downarrow+\tilde{+}} = 0, \tag{4.2.58}$$

$$0 = \delta\widetilde{\widetilde{\lambda}}^{i}_{\uparrow+\tilde{+}} = \frac{i}{\sqrt{2}}\widetilde{\widetilde{P}}^{i3}_{z}\epsilon_{\downarrow+\tilde{+}} = 0, \tag{4.2.59}$$

$$0 = \delta\widetilde{\widetilde{\lambda}}^{i}_{\uparrow+\tilde{-}} = i\widetilde{\widetilde{P}}^{i\widetilde{\oplus}}_{z}\epsilon_{\downarrow+\tilde{+}} = 0, \tag{4.2.60}$$

$$0 = \delta\psi_m = D_m\epsilon_{\downarrow+\tilde{+}}. \tag{4.2.61}$$

The first gives the additional condition $P^{\oplus\oplus}_{z} = 0$, which require τ to be a constant, and the third gives $\widetilde{\widetilde{P}}^{i\widetilde{\oplus}}_{z} = 0$, and this means constant $\widetilde{\tau}$. Then, the solution becomes trivial flat solution, and all supersymmetries are preserved.

We summarize non-trivial BPS solutions in Table 4.6.

Table 4.6: The non-trivial 5-brane solutions in $D = 8$ supergravity and world volume supersymmetries.

	$\mathcal{N} = (1,0)$	$\mathcal{N} = (1,1)$	$\mathcal{N} = (2,0)$
τ	holomorphic	constant	holomorphic
$\widetilde{\tau}$	holomorphic	holomorphic	constant

The most general 1/4 BPS solution are embedded in $SL(2) \times SL(2) \subset SL(2) \times SL(3)$. These two $SL(2)$ factors are manifest in the type IIB frame. Namely, the

$SL(2)$ factor which is a subgroup of $SL(3)$ can be associated with the axio-dilaton field in type IIB theory, and the other $SL(2)$ is associated with the internal space T^2. From the viewpoint of F-theory the 1/4 BPS solution can be regarded as a compactification of the F-theory in a Calabi-Yau realized as T^4 fibration over $\mathbb{C}$.

4.2.5 $D = 7$

The 7-dimensional maximal supergravity has the field contents in Table 4.7 [57, 58].

Table 4.7: The field contents of 7-dimensional maximal supergravity are shown. Anti-symmetric tensor fields are omitted.

	$SO(5)_R$	
$e_M^{\widehat{M}}$	1	vielbein
$L^\alpha{}_i$	5	scalars
ψ_M	4	gravitino
λ_i	16	dilatino, $\rho_i\lambda_i = 0$

The scalar manifold of 7d maximal supergravity is $SL(5)/SO(5)$. There is no duality frame which manifests whole of the duality group $SL(5,\mathbb{Z})$ and the R-symmetry group $SO(5)_R$. When we regard the system as the T^4 compactification of M-theory $SL(4)/SO(4)$ becomes manifest, while T^3 compactification of type IIB theory manifests $SL(2)/SO(2) \times SL(2)/SO(2)$. Combining these we obtain the full symmetry.

The relevant part of the Lagrangian is

$$\begin{aligned}\mathcal{L} = &\frac{e}{2}R - \frac{e}{2}(\overline{\psi}_M\Gamma^{MNP}D_N\psi_P)\\ &- \frac{e}{2}P_{Mij}P^{Mij} - \frac{e}{2}(\overline{\lambda}^i\Gamma^M D_M\lambda_i) + \frac{e}{2}(\overline{\psi}_M\Gamma^N\Gamma^M\rho^i\lambda^j)P_{Nij},\end{aligned} \tag{4.2.62}$$

and the supersymmetry transformation rules for fermions are

$$\begin{aligned}\delta\psi_M &= D_M\epsilon,\\ \delta\lambda_i &= \frac{1}{2}P_{Mij}\Gamma^M\rho^j\epsilon.\end{aligned} \tag{4.2.63}$$

The transformation parameter ϵ belongs to the spinor representation of $H = SO(5)$, and the covariant derivative $D_M\epsilon$ includes the connection Q_{Mij}.

By assuming the Poincare invariance in the five dimensions parallel to the brane, we take the following ansatz.

$$L = L(x^m), \quad e^{\widehat{\mu}} = \delta^{\widehat{\mu}}_{\mu} dx^{\mu}, \quad e^a = g(x^m)\delta^a_m dx^m, \quad \epsilon = \epsilon(x^m). \tag{4.2.64}$$

Let us first consider the case with minimum number of unbroken supersymmetries. The supersymmetry parameter ϵ belongs to the **4** of $SO(5)_R$ symmetry, and in the minimum case we have only one non-vanishing component. Then the R-symmetry is broken to $SU(2) \times U(1) \subset SO(5)_R$.

It is convenient to consider the intermediate subgroup $SU(2)_l \times SU(2)_r \sim SO(4) \subset SO(5)_R$. The parameter ϵ is decomposed into four irreducible representation $(2,1)_{\pm\frac{1}{2}}$ and $(1,2)_{\pm\frac{1}{2}}$ of $SU(2)_l \times SU(2)_r \times SO(2)_{56}$. We denote them as undotted and dotted spinors.

$$\epsilon \to \{\epsilon_{s,a}, \epsilon_{s,\dot{a}}\}, \tag{4.2.65}$$

where $s = \uparrow, \downarrow$ represent the $SO(2)_{56}$ charges. The fields P and Q are decomposed as

$$Q_{ij} \to \{Q_{a\dot{a}}, Q_{(ab)}, Q_{(\dot{a}\dot{b})}\}, \quad P_{ij} \to \{P, P_{a\dot{b}}, P_{(ab)(\dot{a}\dot{b})}\}, \tag{4.2.66}$$

and the dilatino λ as

$$\lambda \to \{\lambda_{sa}, \lambda_{s\dot{a}}, \lambda_{s(ab)\dot{a}}, \lambda_{sa(\dot{a}\dot{b})}\}, \tag{4.2.67}$$

where a pair of indices in parenthesis are symmetric. If we choose $\epsilon_{\dot{1}}$ as the component for the unbroken supersymmetry, $SU(2)_r$ is broken to $U(1)_r$. The connection Q should take its value in $SU(2)_l \times U(1)_r$. Namely, $Q_{a\dot{a}} = Q_{\dot{1}\dot{1}} = Q_{\dot{2}\dot{2}} = 0$ and the only non-vanishing components are

$$Q_{(ab)}, \quad Q_{\dot{1}\dot{2}}. \tag{4.2.68}$$

$\epsilon_{\uparrow\dot{a}}$ appear in the supersymmetry transformation as

$$\delta\lambda_{\downarrow a} = P_{z^* a\dot{b}} \epsilon^{\dot{b}}_{\uparrow}, \tag{4.2.69}$$

$$\delta\lambda_{\downarrow\dot{a}} = P_{z^*} \epsilon_{\uparrow\dot{a}}, \tag{4.2.70}$$

$$\delta\lambda_{\downarrow(ab)\dot{a}} = P_{z^*(ab)(\dot{a}\dot{b})} \epsilon^{\dot{b}}_{\uparrow}, \tag{4.2.71}$$

$$\delta\lambda_{\downarrow a(\dot{a}\dot{b})} = P_{z^* a(\dot{a}} \epsilon_{\uparrow\dot{b})}. \tag{4.2.72}$$

(We omitted the numerical coefficients that are not important here.) If we require $\delta\lambda = 0$ for $\epsilon_{\uparrow\dot{1}} \neq 0$, we obtain $P_{z^*} = P_{z^* a\dot{a}} = P_{z^*(ab)(\dot{a}\dot{2})} = 0$ and the only non-vanishing components of P_{z^*} are

$$P_{z^*(ab)(\dot{1}\dot{1})}. \tag{4.2.73}$$

(We also have similar conditions for P_z from the equations containing $\epsilon_{\downarrow\dot{2}} \sim (\epsilon_{\uparrow\dot{1}})^*$.) (4.2.68) and (4.2.73) show that non-vanishing components of P and Q are associated with a subgroup $SO(4) \subset SO(5)_R$. As we mentioned above $SO(4)$ subgroup of $SO(5)_R$ can be realized geometrically if we regard the theory as T^4 compactification of M-theory.

We want to give the scalar fields L such that P and Q have only non-vanishing components (4.2.73) and (4.2.68). Unfortunately, we have not obtained the answer. To simplify the problem, let us consider a restricted case with $P_{(12)} = 0$. Then $F^{(Q)} = P \wedge P$ takes value in the Cartan part of $SU(2) \times U(1)$, and we can take the gauge such that $Q_{(11)} = Q_{(22)} = 0$, then non-vanishing components are

$$Q_{(12)}, \quad Q_{(\dot{1}\dot{2})}, \quad P_{(11)(\dot{1}\dot{1})}, \quad P_{(22)(\dot{1}\dot{1})}. \tag{4.2.74}$$

In this case, with an appropriate real basis, the 5×5 matrices P and Q are block diagonal matrices in the following form.

$$Q = \begin{pmatrix} Q' & & \\ & Q'' & \\ & & 0 \end{pmatrix}, \quad P = \begin{pmatrix} P' & & \\ & P'' & \\ & & 0 \end{pmatrix}. \tag{4.2.75}$$

Therefore, the solution reduces to the superposition of two copies of solutions for $SL(2,\mathbb{R})/SO(2)$ scalar manifold. In the same way as in higher dimensions, each $SL(2,\mathbb{R})$ part can be expressed in terms of a holomorphic function. Let the two holomorphic functions τ' and τ''. The solution is given by

$$L^{\alpha}{}_{i} = \begin{pmatrix} K(\tau') & & \\ & K(\tau'') & \\ & & 0 \end{pmatrix}, \tag{4.2.76}$$

$$g = c\sqrt{\tau'_2 \tau''_2}. \tag{4.2.77}$$

As a special case of this 1/4 BPS solution we can realize 1/2 BPS solution. Let us consider the cases there is another Killing spinor in addition to $\epsilon_{\dot{1}}$. There are two cases.

First, let us consider the case that ϵ_1 is also a Killing spinor. In this case, from

$$0 = \delta\lambda_{\downarrow(\dot{a}\dot{b})a} = P_{z^*(ab)(\dot{a}\dot{b})}\epsilon_{\uparrow}^{b} \tag{4.2.78}$$

we obtain $P_{z^*(a2)(\dot{a}\dot{b})} = 0$. Then the only non-vanishing component of P_{z^*} is $P_{z^*(11)(\dot{1}\dot{1})}$. In this case, just like the case of 1/2 BPS solution in 8d, we can show that one of τ' and τ'' must be z-independent constant.

If two Killing spinor have the same $SO(4)_R$ chirality, the equation (4.2.71) require $P_{z^*(ab)(\dot{a}\dot{b})} = 0$. Namely, all components of P vanish. Because $F^{(Q)} = P \wedge P = 0$ we can choose a gauge with $Q = 0$. Therefore, the solution is trivial.

Summary

We investigated codimension-2 brane solutions in 9d, $8d$, and $7d$ maximal supergravities. Because we want to BPS brane solutions, which preserve supersymmetry on the world volume, we solved Killing spinor equations $\delta\psi_M = \delta\lambda_i = 0$. The obtained solutions is as follows:

- $D = 9$

  There are general $\frac{1}{2}$-BPS solutions described by

$$\varphi = \text{const}, \quad L^\alpha{}_i = K(\tau), \tag{4.2.79}$$

$$ds^2_{9d} = \eta_{\mu\nu}dx^\mu dx^\nu + c^2\tau_2\delta_{mn}dx^m dx^n, \tag{4.2.80}$$

  where τ is a holomorphic function of $z = \frac{1}{\sqrt{2}}(x^7 + ix^8)$. Note that the $K(\tau)$ is a 2×2 matrix given by

$$K(\tau) = \frac{1}{\sqrt{\tau_2}} \begin{pmatrix} 1 & 0 \\ \tau_1 & \tau_2 \end{pmatrix}, \tag{4.2.81}$$

  and c is an arbitrary constant, which is also the same in below cases.

- $D = 8$

  We constructed the general $\frac{1}{2}$- and $\frac{1}{4}$-BPS solutions described by

$$L^\alpha{}_i = K(\tau), \tag{4.2.82}$$

$$\widetilde{L}^{\widetilde{\alpha}}{}_{\widetilde{i}} = \widetilde{L}_0 \begin{pmatrix} K(\widetilde{\tau}) & 0 \\ 0 & 1 \end{pmatrix}, \quad \widetilde{L}_0 \in SL(3,\mathbb{R}), \tag{4.2.83}$$

$$ds^2_{8d} = \eta_{\mu\nu}dx^\mu dx^\nu + c^2\tau_2\widetilde{\tau}_2\delta_{mn}dx^m dx^n, \tag{4.2.84}$$

  where τ and $\widetilde{\tau}$ are holomorphic functions of $z = \frac{1}{\sqrt{2}}(x^6 + ix^7)$. There are two kind of $\frac{1}{2}$-BPS solutions and a $\frac{1}{4}$-BPS solution, which are classified by whether holomorphic functions τ and $\widetilde{\tau}$ are constant or not (Table 4.8).

Table 4.8: The non-trivial 5-brane solutions in $D = 8$ supergravity and world volume supersymmetries.

	$\mathcal{N} = (1,0)$	$\mathcal{N} = (1,1)$	$\mathcal{N} = (2,0)$
τ	holomorphic	constant	holomorphic
$\widetilde{\tau}$	holomorphic	holomorphic	constant

- $D = 7$

  We construct the special $\frac{1}{4}$- and $\frac{1}{2}$-BPS solutions on which are imposed the restriction $P_{(12)} = 0$ given by

$$L^{\alpha}{}_{i} = \begin{pmatrix} K(\tau') & & \\ & K(\tau'') & \\ & & 0 \end{pmatrix}, \tag{4.2.85}$$

$$ds^2_{7d} = \eta_{\mu\nu} dx^{\mu} dx^{\nu} + c^2 \tau'_2 \tau''_2 \delta_{mn} dx^m dx^n, \tag{4.2.86}$$

  where τ' and τ'' are holomorphic functions of $z = \frac{1}{\sqrt{2}}(x^5 + ix^6)$. As the same in 8d, When one of the τ' and τ'' is a constant, the solution is a $\frac{1}{2}$-BPS state.

Chapter 5

Conclusions and discussions

In chapter 3, we investigated the phenomenon in which $\mathcal{N} = 3$ supersymmetry is enhanced to $\mathcal{N} = 4$ by using string junctions in S-folds. We succeeded in confirming some relations between a certain class of S-fold theories and 4d SYM with rank-2 gauge group.

The supersymmetry enhancement occurs when an S-fold theory is constructed by two D3-branes. We compared such S-fold theories and well-known brane constructions of 4d $\mathcal{N} = 4$ SYM with $SU(3)$ and $SO(5)$ gauge groups. Both the S-fold theories and the $\mathcal{N} = 4$ theories have two mobile D3-branes. The positions of the mobile D3-branes correspond to vacuum expectation values of Coulomb branch operators. We found one-to-one correspondence between positions of mobile D3-branes in the S-fold theories and those in the $\mathcal{N} = 4$ theories. Thus we confirmed that the S-fold theories can reproduce Coulomb branch operators appearing in $\mathcal{N} = 4$ theories. We also checked coincidence of charge spectra. At a generic point in the Coulomb branch, unbroken gauge group is $U(1) \times U(1)$ and charge spectrum of this $U(1) \times U(1)$ can be read off from string junctions. We found charge lattices of S-fold theories coincide with those of $\mathcal{N} = 4$ SYMs. For the $\mathbb{Z}_6$ S-fold theory, we could not compare this theory and $\mathcal{N} = 4$ SYM with G_2 gauge group since brane construction of the G_2 theory is not known excluding the $\mathbb{Z}_6$ S-fold theory. We considered formulae which give the central charges of G_2 theory in the context of the $\mathbb{Z}_6$ S-fold theory.

There are many unsolved problems. What is the most desired would be a direct determination of the non-perturbative central charges $\overline{Z}$. Due to the lack of the information of $\overline{Z}$ we could not directly determine the masses of BPS states. When we established the correspondence of the spectrum of junctions and $\mathcal{N} = 4$ dyonic particles, we use the information of one of the central charge Z, which can be seen perturbatively in the S-fold. However, to determine BPS saturating masses, we also need the other central charge $\overline{Z}$. Because this central charge is generated non-perturbatively together with the fourth supercharge in the supersymmetry enhancement, we cannot determine it by simply drawing the shapes of junctions.

There is another problem related to the marginal deformation. We showed that the spectrum of S-fold side gives dyonic spectrum of $\mathcal{N}=4$ theory with a particular value of the marginal deformation parameter τ. However, if $\mathcal{N}=4$ is realized, we should be able to freely change the deformation parameter. The change of the parameter affects the central charges, and we should have the corresponding parameter on the S-fold side. The identification of the parameter in the S-fold is a very important problem to understand the supersymmetry enhancement.

Even if we could obtain the central charges, it would be another problem to determine the BPS spectrum. The existence of BPS saturating states is a highly non-trivial problem and we need to perform quantum analysis of the junctions to determine it. As far as we know this is open problem even for orientifolds.

In chapter 4, instead of studying 4d non-Lagrangian theory directly, we study codimension-2 branes, which may play an important role in investigations of non-Lagrangian theories. Concretely, we studied codimension-2 BPS solutions in maximal supergravities in 9, 8, and 7 dimensions. The BPS solutions partially preserve supersymmetries. Therefore, effective theories on the branes are supersymmetric field theories, whose detail is not clear generically. In order to find the solutions, we solved Killing spinor equations, which give global supersymmetry transformation keeping background fermionic fields zero. Since we would like to construct codimension-2 brane solutions, we took the ansatz: all fields only depend on the codimension directions and directions where branes spread are Lorentz invariant. We only considered metric and scalar fields because the codimension-2 branes are magnetically charged objects for the scalar fields.

In maximal supergravities, the scalar fields take values in G/H, where G is the classical global symmetry and H is the R-symmetry. For 9d, 8d, and 7d theories, G are all SL type and H are all SO type. Thus to solve the Killing spinor equations of these theories can be dealt in similar ways.

The scalar manifold of 9d maximal supergravity is $(SL(2,\mathbb{R})/SO(2))\times\mathbb{R}$. In a BPS solution, the scalar field associated with the factor $\mathbb{R}$ must be constant and play no role. Therefore, the solutions are essentially the same as those of type IIB supergravity in 10d, and simply interpreted as the double dimensional reduction of type IIB 7-branes.

In 8d, the scalar manifold consists of two factors $SL(2,\mathbb{R})/SO(2)$ and $SL(3,\mathbb{R})/SO(3)$. Killing spinor equations associated with these factors decouple, and we can solve them one by one. From the $SL(2,\mathbb{R})/SO(2)$ part we obtain $\frac{1}{2}$-BPS branes on which six-dimensional $\mathcal{N}=(2,0)$ supersymmetry is realized while from the $SL(3,\mathbb{R})/SO(3)$ part we obtain $\frac{1}{2}$-BPS solutions on which six-dimensional $\mathcal{N}=(1,1)$ supersymmetry is realized. The latter is always embedded in $SL(2,\mathbb{R})/SO(2) \subset SL(3,\mathbb{R})/SO(3)$. These have essentially the same structure as the 7-brane solution in 10d. We also found $\frac{1}{4}$-BPS solutions, which are simple superposition of two types of 1/2 BPS solutions. If we regard the 8d supergravity as the T^2 compactification of type IIB theory, the two copies of $SL(2,\mathbb{R})/SO(2)$ are associated with the complex moduli of the compactification torus and the axio-dilaton field in type IIB theory, and both are geometrically realized in F-theory.

In 7d, a generic BPS solution is 1/4-BPS. We showed that such solution can be embedded in $SL(4,\mathbb{R})/SO(4) \subset SL(5,\mathbb{R})/SO(5)$. This means the solution can be realized as a geometric compactification of M-theory. We could not solve the Killing spinor solutions in the general situation. We introduced one additional restriction to simplify the problem, and then the solution is factorized into two copies of solutions associated with $SL(2,\mathbb{R})/SO(2)$. Again, similarly to the 8d case, the solution can be regarded as a simple superposition of $\frac{1}{2}$-BPS branes.

Although our original motivation for this work was to find essentially new BPS branes that cannot be regarded as a geometric compactification of higher dimensional theory all solutions we found have geometric realization in M or F-theory.

We investigated only $D = 9, 8$ and 7 maximal supergravities. However, the codimension-2 brane solutions in $D \leq 6$ maximal supergravities are also meaningful. Actually, 4d E type $\mathcal{N}=3$ theories should be given by a certain BPS solution in 6d maximal supergravity. Such codimension-2 brane solutions correspond to the solutions when the scalar fields are constant. Thus, if we construct more general BPS solutions, we expect that the solutions can be applied to 4d non-Lagrangian theories.

Appendix A

Notes on our notation

A.1 Notation of 4d field theory

This chapter specifies our notations and conventions in 4d field theories. Four-vector indices are represented by the Greek alphabet $\mu, \nu. \ldots = 0, 1, 2, 3$. The spacetime metric is

$$\eta_{\mu\nu} = \mathrm{diag}(-, +, +, +). \tag{A.1.1}$$

$$\tag{A.1.2}$$

Gamma matrices γ^μ satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}. \tag{A.1.3}$$

We usually use two-component Weyl spinor notation for fermions. It is convenient to use the following representations of gamma matrices:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix}, \tag{A.1.4}$$

where

$$\sigma^0 = \overline{\sigma}^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = -\overline{\sigma}^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = -\overline{\sigma}^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = -\overline{\sigma}^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.1.5}$$

In this representation, a four-component Dirac spinor is written in terms of two two-component Weyl spinor as

$$\Psi_D = \begin{pmatrix} \psi_\alpha \\ \xi^{\dagger\dot{\alpha}} \end{pmatrix}, \tag{A.1.6}$$

with left-handed spinor index $\alpha = 1, 2$ and right-handed spinor index $\dot{\alpha} = \dot{1}, \dot{2}$. The Hermitian conjugate of Weyl spinors are given by

$$(\psi^\alpha)^\dagger = \psi^{\dagger\dot{\alpha}}, \tag{A.1.7}$$
$$(\psi^\dagger_{\dot{\alpha}})^\dagger = \psi_\alpha. \tag{A.1.8}$$

Namely, the Hermitian conjugate relates a left-handed Weyl spinor to a right-handed Weyl spinor. The anti-symmetric epsilon symbol raise and lower the spinor indices as

$$\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta, \quad \psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta \tag{A.1.9}$$
$$\psi^\dagger_{\dot{\alpha}} = \varepsilon_{\dot{\alpha}\dot{\beta}}\psi^{\dagger\dot{\beta}}, \quad \psi^{\dagger\dot{\alpha}} = \varepsilon^{\dot{\alpha}\dot{\beta}}\psi^\dagger_{\dot{\beta}}, \tag{A.1.10}$$

and the anti-symmetric symbols are defined by

$$\varepsilon^{12} = 1, \quad \varepsilon_{12} = -1 \tag{A.1.11}$$
$$\varepsilon^{\dot{1}\dot{2}} = 1, \quad \varepsilon_{\dot{1}\dot{2}} = -1. \tag{A.1.12}$$

The sigma matrices σ^μ and $\overline{\sigma}^\mu$ have both spinor indices: $(\sigma^\mu)_{\alpha\dot{\beta}}, (\overline{\sigma}^\mu)^{\dot{\alpha}\beta}$.

It is convenient to define anti-symmetrized gamma and sigma matrices,

$$\gamma^{\mu\nu} \equiv \frac{1}{2}[\gamma^\mu, \gamma^\nu] = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \bar{\sigma}^{\mu\nu} \end{pmatrix}, \tag{A.1.13}$$
$$(\sigma^{\mu\nu})_\alpha{}^\beta = \frac{1}{2}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)_\alpha{}^\beta, \tag{A.1.14}$$
$$(\overline{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} = \frac{1}{2}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)^{\dot{\alpha}}{}_{\dot{\beta}}. \tag{A.1.15}$$

Spinor inner product is defined by contraction of spinor indices like

$$\psi\xi \equiv \psi^\alpha\xi_\alpha = \psi^\alpha\varepsilon_{\alpha\beta}\xi^\beta, \tag{A.1.16}$$
$$\psi^\dagger\xi^\dagger \equiv \psi^\dagger_{\dot{\alpha}}\xi^{\dagger\dot{\alpha}} = \psi^\dagger_{\dot{\alpha}}\epsilon^{\dot{\alpha}\dot{\beta}}\xi^\dagger_{\dot{\beta}}. \tag{A.1.17}$$

Note that $\psi^\dagger\xi^\dagger = (\psi\xi)^*$. Similarly, following relations are held,

$$\psi^\dagger\overline{\sigma}^\mu\xi = -\xi\sigma^\mu\psi^\dagger = (\xi^\dagger\overline{\sigma}^\mu\psi)^* = -(\psi\sigma^\mu\xi^\dagger)^*, \tag{A.1.18}$$
$$\psi\sigma^\mu\overline{\sigma}^\nu\xi = \xi\sigma^\nu\overline{\sigma}^\mu\psi = (\xi^\dagger\overline{\sigma}^\nu\sigma^\mu\psi^\dagger)^* = (\psi^\dagger\overline{\sigma}^\mu\sigma^\nu\xi^\dagger)^*. \tag{A.1.19}$$

A.2 Notation in Chapter 5

We denote the dirac matrices for $SO(2)_R$ group by ρ_i $(i = 1, 2)$. We use the representation

$$\rho_1 = \sigma_x, \quad \rho_2 = \sigma_y, \tag{A.2.1}$$

where $\sigma_{x,y,z}$ are the Pauli matrices. We use lower indices $a, b, \ldots$ for two-component spinors, and thus the matrices acting on them have lower and upper indices like $(\rho_i)_a{}^b$. For components of spinors we define the $SO(2)_R$ charge as eigenvalues of the generator $(-i/2)\rho_{12}$. This means the upper and the lower components of spinors carry the charges $+1/2$ and $-1/2$, respectively. To specify these components of a spinor χ we use the notation χ_+ and χ_-, respectively.

For the analysis of the Killing spinor equations it is convenient to the complex basis for vectors. For example, for a vector v^i $(i = 1, 2)$ we define $v^\oplus = v_\ominus = \frac{1}{\sqrt{2}}(v^1 + iv^2)$ and $v^\ominus = v_\oplus = \frac{1}{\sqrt{2}}(v^1 - iv^2)$. For ρ^i we have

$$\rho^\oplus = \rho_\ominus = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}, \quad \rho^\ominus = \rho_\oplus = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix}. \tag{A.2.2}$$

With this representation the lower index $\oplus$ and upper index $\ominus$ carry $SO(2)_R$ charge $+1$, while the lower $\ominus$ and upper $\oplus$ carry $SO(2)_R$ charge -1. This is checked by looking at the non-vanishing components of ρ^i. For example the non-vanishing component of $\rho_\oplus$ is $(\rho_\oplus)_-{}^+$, and the total charge of this component must be zero. The statement above about $SO(2)_R$ charge is consistent to this.

For the local rotation symmetry in the transverse space to branes we use up and down for the $SO(2)$ charge (spin) $\pm 1/2$. With the choice of the Dirac matrices, the lower indices z and z^* carry spin $+1$ and -1, respectively.

In 8d we also deal with $SO(3)_R$ symmetry. The notation is basically the same as the $SO(2)_R$ case except we put tildes on variables and indices for distinction from $SO(2)_R$ objects. We specify components of spinors by eigenvalues of the Cartan generator $(-i/2)\widetilde{\rho}_{\widetilde{12}}$. $\widetilde{\chi}_{\widetilde{\pm}}$ carry the charge $\pm 1/2$, and a vector $\widetilde{v}$ has three components $\widetilde{v}^{\widetilde{\ominus}} = \widetilde{v}_{\widetilde{\oplus}}$, $\widetilde{v}^{\widetilde{\oplus}} = \widetilde{v}_{\widetilde{\ominus}}$, and $\widetilde{v}^3 = \widetilde{v}_3$ that carry the Cartan charge $+1$, -1, and 0, respectively.

In the following we give relations of fields in this paper and those in references. We will not give detailed explanations for normalization of fields, spinor conventions, etc., because they are not important in our analysis of the Killing spinor equations. We focus on giving rough correspondence between fields used in this paper and those in references.

A.2.1 $D = 10$

Ten dimensional supergravity is given in [34]. The global symmetry $SL(2,\mathbb{R})$ is isomorphic to $SU(1,1)$. In [34] the scalar fields are expressed as the matrix $V^a_\pm$, which is defined with a complex basis natural for $SU(1,1)$. The real matrix $L^\alpha{}_i$ used in this paper is related with V by

$$\begin{pmatrix} V^1{}_- & V^1{}_+ \\ V^2{}_- & V^2{}_+ \end{pmatrix} = U \begin{pmatrix} L^1{}_1 & L^1{}_2 \\ L^2{}_1 & L^2{}_2 \end{pmatrix} U^\dagger, \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}. \tag{A.2.3}$$

The dilatino field λ in [34] is defined as the field with $U(1)_R$ charge $\pm 3/2$, while we denote this as a field with vector and spinor indices. They are related by

$$\lambda \sim \lambda^\ominus_+, \quad \overline{\lambda} \sim \lambda^\oplus_-. \tag{A.2.4}$$

Due to the ρ-traceless condition $\lambda^\oplus_+ = \lambda^\ominus_- = 0$.

A.2.2 $D = 9$

The 9-dimensional maximal supergravity is given in [54]. Two dilatino fields in [54] are renames as λ_i and $\widetilde{\lambda}$ to to match the fields in the other dimensions. $SO(2)$ Dirac matrices are denoted by τ_i in [54] while we use ρ_i for them.

A.2.3 $D = 8$

The 8-dimensional maximal supergravity is given in [55]. The dilatino field χ_i in [55] does not satisfy the ρ-traceless condition, and we decompose it into the traceless part $\widetilde{\lambda}_i$ and the trace part λ_I. To make the $SL(2,\mathbb{R})/SO(2)$ structure manifest we combine the scalar fields ϕ and B in [55] into the matrix $L^A{}_I$. With the gauge choice like (4.2.12) these are related by $L = K(\tau)$ with $\tau = -2B + ie^{2\phi}$.

A.2.4 $D = 7$

The 7-dimensional maximal supergravity is given in [58]. In the reference the scalar matrix is denoted by Π instead of L. Notation for other fields is similar to ours.

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