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Studies on Implications and Sequent Calculi for Quantum Logic

Tomoaki Kawano

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Contents

Chapter 1

Introduction

Quantum logic, a non-classical logic used to describe propositions of quantum physics, has been studied in the context of ortho and orthomodular lattices, and has developed through both logic and physics [4]. However, quantum logic is in an adolescent stage of development with many problems in basic concepts that still remain unsolved. In this paper, our discussion is centered on *implication* connective.

Although, quantum physics describes the behavior of small particles, it is quite different than classical mechanics because the particles behave in rather unpredictable ways. In most cases, these particles do not have precise values of physical quantities such as position and momentum. So this does not mean that we cannot determine known value, but it simply means the particles do not have precise values. A particle can be assigned only a probability distribution of values, even though particles on occasion could have specific physical values. According to the *Heisenberg uncertainty principle*, two specific identical physical values cannot have specific values at the same time. For example, when the momentum of a particle has a specific value, the position of this particle is completely unknown and vice versa. *Hilbert space* can be used to describe this odd behavior. The axis that passes through the origin of a Hilbert space represents the state of the particle. Each physical quantity can be assigned to an orthonormal basis in a Hilbert space, respectively. In order to express Heisenberg's uncertainty principle, both position and momentum are assigned to mutually unbiased bases. The probability distribution of any physical quantity in a state is determined by the component of the state by the basis to which the physical quantity is assigned. For example, if the particle state is consistent with any bases assigned to the momentum, then that particle has a deterministic momentum value. However, the particle position is completely random since this state has a base component equally assigned to the position. In quantum mechanics, the concept of *measurement* is also a bit odd but equally important.

In examining a physical quantity, since its distribution probability is determined by the given state, the value of the physical quantity can be determined by the ratio of the probability. After measurement, the state immediately moves to the eigenstate of the obtained value. In circumstances like those just described, the propositions of quantum mechanics can be roughly determined. In quantum logic, the proposition of quantum mechanics is defined as the proposition of the particle's physical quantity. For example, 'Momentum of a particle is 20 or 30' is a proposition of quantum logic. They correspond to the closed subspaces of a Hilbert space. 'Momentum of a particle is 20 or 30' is correspond to the closed subspace which is enclosed by the eigenstates 'Momentum = 20' and 'Momentum = 30' because if a state is included in this closed subspace, when we measure momentum of particle, 'Momentum = 20' or 'Momentum = 20' is always obtained.

The set of all closed subspace of one Hilbert space is known to be an *orthomodular lattice*. Therefore, quantum logic has been studied in the context of ortho and orthomodular lattices. An orthomodular lattice is derived from the ortho lattice and orthomodular law. The logic based on the ortho lattice is known as *orthologic* or *minimal quantum logic*, while the logic based on the orthomodular lattice is known as *orthomodular logic* or *quantum logic*. Both lattices have comparable Kripke semantics, namely an ortho-model (*O-model* [5] [19] [20]) and an orthomodular model (*OM-model* [5] [19]). In this paper, we use these models for semantics of quantum logic.

Both orthologic and orthomodular logic have some complete deduction system. In this paper, we adopt *sequent calculus* for deduction system. Sequent calculus, initially created to study the natural deduction [9] [10], has found utility in various logics. Sequent calculus plays an important role in the studies of logics. For example, *cut-elimination theorem* of a sequent calculus can be used to prove some natures of a logic such as *decidability*. The sequent calculi **GO**[19] and **GMQL**[20][21] are applied in orthologic, and **GOM**[19] is applied in orthomodular logic.

The relationships between quantum logic and other logics have been studied. In [5], the relationship between quantum logic and modal logic **B** is discussed. This relationship is similar to the relationship between intuitionistic logic and modal logic **S4**. That is, both quantum logic and **B** are based on the symmetric and reflexive frames, and both intuitionistic logic and **S4** are based on the transitive and reflexive frames. In [23], the symmetry of intuitionistic logic and quantum logic is discussed. This symmetry is discussed by using the formulas which are valid in one logic but not at the other logic, such as *excluded* *middle* or *distributive law*.

One recurring problem in quantum logic is the *implication problem*. In general, quantum logic includes only negation, conjunction and disjunction because orthomodular lattices can be treated only with these concepts. Although several implications in quantum logic arise [5], they all have difficulties for different reasons and result in a divided opinion on which implication best suits quantum logic. The main purpose of this paper is to construct and develop the sequent calculi for each implications. Furthermore, new frames and models for quantum logic with implications are proposed.

In Chapter 2, basic concepts of quantum logic such as ortho lattices, orthomodular lattices, O-models and OM-models are reviewed. Both O-models and OM-models are based on the orthogonality of closed subspace sets in Hilbert space. Also, in this chapter, the implications for quantum logic are reviewed. Since each implication has a unique set of characteristics, they are examined in detail. Especially, the *strict implication* plays an important role in orthologic, and the *Sasaki arrow* plays an important role in orthomodular logic.

In Chapter 3 and later, we discuss the results of the research by the author. The theorems in chapters 3, 4 and 5 are the result of the author unless otherwise noted.

In Chapter 3, new models called *dynamic orthomodular model* (*DOM-model*) is introduced, and some conditions on orthomodular lattices are discussed. Orthomodular lattices have been studied in terms of the detailed conditions required for Hilbert spaces. The *orthomodular law* is required to ensure the consistency of the dimension of Hilbert spaces. The *atomicity* expresses the need for each closed subspace of a Hilbert space to be expressed by a space enclosed by one-dimensional closed subspaces. The *covering law* expresses that there is no closed subspace between the n-dimensional subspace S and the $n+1$-dimensional subspace S' such that $S \subseteq S'$. However, if the model only has an orthogonal relationship, these conditions are difficult to apply. A solution to the problem can be found by adding the relation of *projections* to an OM-model.

In Chapter 4, we make sequent calculi including implications. We introduce sequent calculi for each implication. All calculi are based on **GO**, **GOM** or **GMQL**. However, these sequent calculi are incompatible with the cut-elimination theorem. The sequent calculi **GO** and **GOM** have a cut rule that cannot be removed. Conversely, **GMQL** does not have a (full) cut rule and it cannot be added. If a cut rule is added to **GMQL**, it collapses to classical logic [20].

In Chapter 5, we construct *labeled sequent calculi* for quantum logic to establish the cut-elimination theorem. A labeled sequent includes not only formulas but also relations based on a Kripke frame. These types of sequent calculi have been used to prove cut-elimination theorems [12] [17]. The details and the development of labeled sequent are summarized in [18]. In this chapter, three labeled sequent calculi **LGOI**, **LGOE** (for orthologic) and **LGOM** (for orthomodular logic) are constructed, and the cut-elimination theorem for these calculi are discussed. Each calculus contains an appropriate implication. We also prove that the validity problem of orthologic with strict implication is decidable, which was not known so far. **LGOM** includes the notion of DOM-models for simplification of the orthomodular law. However, this calculus still has some problems, and we cannot prove the completeness theorem and the cut-elimination theorem of **LGOM** at this point. So, we have to consider **LGOM** as a reference to improve labeled sequent calculi using DOM-models.

In Chapter 6, we discuss some unresolved issues.

Chapter 2

Preliminaries

In this chapter, some basics about quantum logic are discussed. For more details, see [5].

2.1 Basics

A language containing an infinite set of propositional variables, a propositional constant $\perp$, a unary connective $\neg$, and a binary connective $\wedge$ is used. Furthermore, in each chapter, necessary symbols will be added. Formulas are constructed using traditional method. The formulas $A \wedge B$ and $B \wedge A$ are not distinguished. The abbreviation $A \vee B$ represents $\neg(\neg A \wedge \neg B)$. The symbols $p, q, r, \ldots$ describe propositional variables while $A, B, C, \ldots$ denote formulas. The symbols $\mathit{\Gamma}, \mathit{\Delta}, \mathit{\Sigma}, \ldots$ denotes sets of formulas. A *sequent* is defined by $\mathit{\Gamma} \Rightarrow \mathit{\Delta}$ where $\mathit{\Gamma}$ and $\mathit{\Delta}$ are finite set of formulas.

2.2 Orthomodular lattice

An *ortho lattice* $(L, \leq, \sqcap, \sqcup, ', \mathbf{1}, \mathbf{0})$ is a lattice that satisfies the following conditions.

$\forall a \in L,\ \exists! a'$,

$a'' = a$,

If $a \leq b$, then $b' \leq a'$,

$a \sqcap a' = \mathbf{0}$,

$a \sqcup a' = \mathbf{1}$.

Given an ortho lattice $(L, \leq, \sqcap, \sqcup, ', \mathbf{1}, \mathbf{0})$, if all $a, b \in L$ satisfies the following condition called *orthomodular law*, $(L, \leq, \sqcap, \sqcup, ', \mathbf{1}, \mathbf{0})$ is referred as an *orthomodular lattice*.

$a \sqcap (a' \sqcup (a \sqcap b)) \leq b.$

The following conditions for an ortho lattice are equivalent to orthomodular law.

$a \sqcap (a' \sqcup (a \sqcap b)) = a \sqcap b.$

If $a \leq b$, then $b \leq a \sqcup (a' \sqcap b)$.

The most definitive feature of an ortho lattice and orthomodular lattice is that the *distributive law* (shown bellow) does not hold.

$a \sqcap (b \sqcup c) = (a \sqcap b) \sqcup (a \sqcap c)$

$a \sqcup (b \sqcap c) = (a \sqcup b) \sqcap (a \sqcup c)$

Examples of an ortho lattice and orthomodular lattice are shown in Figure 1. Lines in the figure represent the relation $\leq$. The lattice on the left is an ortho lattice but not an orthomodular lattice because $a \sqcap (a' \sqcup (a \sqcap b)) = a \not\leq b$. The lattice on the right is an orthomodular lattice.

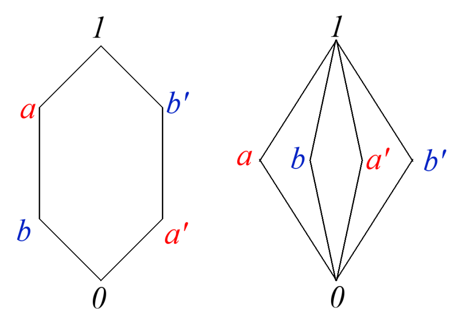

Figure 1

A *valuation* of a formula A in an ortho lattice (or orthomodular lattice) $(L, \leq, \sqcap, \sqcup, ', \mathbf{1}, \mathbf{0})$ is defined as follows. Valuation v is a function from formulas to L that satisfies the following conditions.

$v(A \wedge B) = v(A) \sqcap v(B)$

$v(\neg A) = v(A)'$

Formula A is *valid* in an ortho lattice (or orthomodular lattice) $(L, \leq, \sqcap, \sqcup, ', \mathbf{1}, \mathbf{0})$ if for all v, $v(A) = \mathbf{1}$.

2.3 Kripke frame

As mentioned earlier, in this paper, Kripke type semantics is used instead of lattices. We use one type of Kripke frame for orthologic (*O-frame*), and two types of Kripke frame for orthomodular logic (*OM-frame* and *TOM-frame*).

An *O-frame* is defined by a pair $(X, \perp)$ where X is a non-empty set, and $\perp$ is a binary relation on X that is irreflexive and symmetric. For technical reasons, the symbol $\perp$ is used both as a relation and as a formula. We write $x \perp Y$ if for all $y \in Y$, $x \perp y$. Given $Y \subseteq X$, the set $Y^{\perp}$ is defined by $\{x \in X | x \perp Y\}$. We say Y is $\perp$-*closed* if $Y^{\perp\perp} = Y$. We write $x \not\perp y$ if not $x \perp y$. Because $\perp$ is irreflexive and symmetric relation, $\not\perp$ can regarded as reflexive and symmetric relation on X.

Lemma 2.3.1 (Fundamental natures of an $\perp$-closed set). *Let $(X, \perp)$ be an O-frame. For any set $Y \subseteq X$, $Y^{\perp}$ is $\perp$-closed. For any set $Y \subseteq X$, $Y \subseteq Y^{\perp\perp}$. If $Y \subseteq X$ and $Z \subseteq X$ are $\perp$-closed, then $Y \cap Z$ is $\perp$-closed. If all $Y_i \subseteq X (i \in I)$ are$\perp$-closed, then $\bigcap_{i \in I} Y_i$ is $\perp$-closed.*

Proof. See [5]. □

$Y \sqcup Z$ is an abbreviation for $(Y^{\perp} \cap Z^{\perp})^{\perp}$. An *OM-frame* is an O-frame $(X, \perp)$ that satisfies a condition called *orthomodular law (for frames)*:

For all $\perp$-closed subsets Y, Z of X, $Y \cap (Y^{\perp} \sqcup (Y \cap Z)) \subseteq Z$.

An *O-model* is a triple $(X, \perp, V)$ where $(X, \perp)$ is an O-frame, and V is a function assigning each propositional variable p to an $\perp$-closed subset of X. The set $\|A\|$ is defined by induction on the composition of A.

$$\|p\| = V(p)$$

$$\|A \wedge B\| = \|A\| \cap \|B\|$$

$$\|\neg A\| = \|A\|^{\perp}$$

$$\|\perp\| = \emptyset$$

Formula A is *true* at $x \in X$ if $x \in \|A\|$ and write $x \models A$. Formula A is *false* at $x \in X$ if A is not true at x. Formula A is *valid* in an O-model $(X, \perp, V)$ if $\|A\| = X$.

An *OM-model* is an O-model $(X, \perp, V)$ in which $(X, \perp)$ is an OM-frame.

The expression $A \leq B$ represents $\|A\| \subseteq \|B\|$. The expression $A = B$ represents $\|A\| \subseteq \|B\|$ and $\|B\| \subseteq \|A\|$. A and B are *equivalent* if $A = B$ for every O-model or OM-model.

A *TOM-frame* (Traditional OM-frame), which is called OM-frame in [5] and [19], is defined as a triple $(X, \perp, \Psi)$ where $(X, \perp)$ is an O-frame, and Ψ is a nonempty collection of $\perp$-closed subsets of X that satisfies following conditions:

Ψ is closed under set intersection and the operation $^\perp$.

For all $Y, Z \in \Psi$, $Y \cap (Y^\perp \sqcup (Y \cap Z)) \subseteq Z$.

A *TOM-model* (Traditional OM-model), which is called OM-model in [5] and [19] is defined as a 4-tuple $(X, \perp, \Psi, V)$ where $(X, \perp, \Psi)$ is a TOM-frame and V is a function assigning each propositional variable p to an element of Ψ. In a TOM-model, only Ψ is required to be an orthomodular.

Lemma 2.3.2. *A formula A is valid in all ortho lattices iff A is valid in all O-models.*

Proof. See [5]. □

Lemma 2.3.3. *A formula A is valid in all orthomodular lattices iff A is valid in all TOM-models.*

Proof. See [5]. □

Whether or not the following assertion holds is an open question. *A formula A is valid in all OM-models iff A is valid in all TOM-models.*

2.4 Implications in quantum logic

In this section, the implications for quantum logic are reviewed. The classical implication $\neg A \vee B$ cannot be used in quantum logic because it does not satisfy modus ponens. Beside modus ponens, there are conditions to consider when constructing an implication. From both logic and philosophy, conditions that are satisfied by implications have been previously studied. To examine these detailed argument, see [5] and [11]. In this paper, the following conditions are adopted.

(E) If $A \leq B$, then $A \to B = T$

(MP) $A \wedge (A \to B) \leq B$

(MT) $\neg B \wedge (A \to B) \leq \neg A$

(NG) $A \wedge \neg B \leq \neg(A \to B)$

There are several implications - three "material implications," "strict implication," and "entailment," all of witch satisfy the above conditions. In the following, we explain each of them.

Firstly, we introduce the idea of *material implications*. They are implications composed by $\neg$ and $\wedge$. It is known that there are only three material implications that satisfy all the above conditions [5] [11].

1. *Sasaki arrow*
   $\|A \to_S B\| = \|\neg A \vee (A \wedge B)\|$
2. *Contrapositive Sasaki arrow*
   $\|A \to_C B\| = \|B \vee (\neg A \wedge \neg B)\|$
3. *Relevance arrow*
   $\|A \to_R B\| = \|(A \wedge B) \vee (\neg A \wedge B) \vee (\neg A \wedge \neg B)\|$

Sasaki arrow includes the notion of projections. Intuitively, $A \to_S B$ is considered as "When the state is projected to a closed subspace in which A is true, then B is also true." This condition is confirmed by Lemma 3.2.4.

Since the orthomodular law and the modus ponens of the Sasaki arrow are equivalent, the Sasaki arrow is unsuitable for orthologic because it does not satisfy the modus ponens in an O-model.

These arrows satisfy a useful lemma to make new rules. $A \to_C B$ and $A \to_R B$ can be defined by using Sasaki arrow.

Lemma 2.4.1. *In any OM-model and TOM-model, if $A \leq B$, then $B = A \vee (B \wedge \neg A)$.*

Proof. Use the fact about an orthomodular lattice. □

Lemma 2.4.2. *In any OM-model and TOM-model, $A \to_C B = \neg B \to_S \neg A$ and $A \to_R B = (A \to_S B) \wedge (A \to_C B)$.*

Proof. $A \to_C B = \neg B \to_S \neg A$ is deduced from the definition. From $A \wedge B \leq A \to_S B$ and $\neg A \wedge \neg B \leq \neg A \leq A \to_S B$, $(A \wedge B) \vee (\neg A \wedge \neg B) \leq A \to_S B$. Similarly, from $A \wedge B \leq B \leq A \to_C B$ and $\neg A \wedge \neg B \leq A \to_C B$, $(A \wedge B) \vee (\neg A \wedge \neg B) \leq A \to_C B$. Therefore, $(A \wedge B) \vee (\neg A \wedge \neg B) \leq (A \to_S B) \wedge (A \to_C B)$. Now we see $A \to_R B$ as $(\neg A \wedge B) \vee ((A \wedge B) \vee (\neg A \wedge \neg B))$. From above relations and

$\neg A \wedge B \leq (A \to_S B) \wedge (A \to_C B)$, $A \to_R B \leq (A \to_S B) \wedge (A \to_C B)$. Therefore, from Lemma 2.4.1, $(A \to_S B) \wedge (A \to_C B) = (A \to_R B) \vee (((A \to_S B) \wedge (A \to_C B)) \wedge \neg(A \to_R B))$. From Lemma 2.4.1, as $A \wedge B \leq A$ and $\neg A \wedge \neg B \leq \neg B$, $\neg(A \wedge B) \wedge (\neg A \vee (A \wedge B)) = \neg A$ and $\neg(\neg A \wedge \neg B) \wedge (B \vee (\neg A \wedge \neg B)) = B$ can be deduced. Therefore, $((A \to_S B) \wedge (A \to_C B)) \wedge \neg(A \to_R B) = (\neg A \vee (A \wedge B)) \wedge (B \vee (\neg A \wedge \neg B)) \wedge \neg(A \wedge B) \wedge \neg(\neg A \wedge \neg B) \wedge \neg(\neg A \wedge B) = (\neg A \wedge B) \wedge \neg(\neg A \wedge B) = \perp$. Then, we can conclude $((A \to_S B) \wedge (A \to_C B)) = (A \to_R B)$. $\square$

Secondly, *strict implication* is introduced. The symbol $\to$ denotes this implication. The definition of valuation for this implication is nearly identical to that of *intuitionistic logic*. In other words, $\|A \to B\|$ is defined in an O-model shown below.

$\|A \to B\| = \{x \in X |$ for all $y \in X$, if $x \not\perp y$ and $y \in \|A\|$, then $y \in \|B\|\}$.

In a Hilbert space, non-orthogonality is understood as the possible movement of the projection. Therefore, from a physical viewpoint, strict implication $A \to B$ can be interpreted as "After a measurement of any physical quantity, if A is true, then B is true."

Lemma 2.4.3. *In an O-model, $\|A \to B\|$ is $\perp$-closed.*

Proof. For all $x \in \|A \to B\|$, $x \perp \{y \in X \mid y \models A$ and $y \not\models B\}$. Then, $\{y \in X \mid y \models A$ and $y \not\models B\} \in \|A \to B\|^{\perp}$. Therefore, if $z \in \|A \to B\|^{\perp\perp}$, then $z \perp \{y \in X \mid y \models A$ and $y \not\models B\}$. This means $z \in \|A \to B\|$ and there is no point z that satisfies $z \notin \|A \to B\|$ and $z \in \|A \to B\|^{\perp\perp}$. Therefore, $\|A \to B\|$ is $\perp$-closed. $\square$

Strict implication is also useful for the following points.

- When orthomodular lattices are considered, material implications satisfies modus ponens. However, in cases involving ortho lattices, material implications do not satisfy modus ponens. On the other hand, strict implication satisfies modus ponens in both of lattices. Therefore, if ortho lattices are involved, strict implication is more suitable than material implications.

- From the definition, all of material implications are abbreviated formulas constructed by conjunction, disjunction and negation. On the other hand, strict implication cannot be constructed using these concepts. Therefore, when strict implication is added to quantum logic, description ability of the logic properly increases.

- The definition of strict implication in O-models is identical to the definition of implication of intuitionistic logic. The deduction rules of the strict implication is similar (or same) to those in the field of relevance logic. In this view, the relationship between quantum logic and other logics is analyzed. These analyses shed new light on the study of substructural logics.

Finally, *entailment* $\twoheadrightarrow$ is introduced. The valuation of entailment is defined as follows.

$$\|A \twoheadrightarrow B\| = \begin{cases} X & : \text{if } \ \|A\| \subseteq \|B\| \\ \phi & : \text{else} \end{cases} \tag{2.1}$$

This arrow only takes valuation of X or ϕ. In an orthomodular lattice, it can be only 1 or 0, and it is not very useful in the context of quantum physics. However, in mathematical logic, this arrow has been used to date [5].

Chapter 3

Advanced Kripke frame for quantum logic

In this chapter, conditions on O-frames are discussed. Furthermore, a new frame called *DO-frame* is introduced. For conditions on O-frames, we discuss the *orthomodular law*, *atomicity*, and *covering law*. The main results of this chapter is Theorem 3.2.5, which characterizes the orthomodular law, and Theorem 3.3.3, which characterizes the atomicity and covering law. The contents of this chapter is based on the author's paper [15].

3.1 Introduction

In this chapter, conditions on O-frames are discussed. Furthermore, a new frame called *DO-frame* is introduced. For conditions on O-frames, we discuss the *orthomodular law*, *atomicity*, and *covering law*. The orthomodular law is required to ensure the consistency of the dimension of Hilbert spaces. The atomicity expresses the need for each closed subspace of a Hilbert space to be expressed by a space enclosed by one-dimensional closed subspaces. The covering law expresses that there is no closed subspace between the n-dimensional subspace S and the $n+1$-dimensional subspace S' such that $S \subseteq S'$.

The OM-frame contains only orthogonal relation. With orthogonality alone, it is difficult to handle conditions for frames. We add the relations for projections, which are used in dynamic quantum logic [3], to the OM-frames and partially solve this problem. We call this new frame as *DO-frame*. The relations for projections represent the moves of states cased by projections in a Hilbert space. To introduce these new relations into an OM-frame, the Sasaki arrow and the formula called *Sasaki projection* are important.

3.2 Characterization of orthomodularity

In this section, we prove Theorem 3.2.5 which states the orthomodular law by the condition on a frame.

A *DO-frame* $(X, \perp, R)$ is derived from an O-frame $(X', \perp')$ below:

$X = X'$ and $\perp = \perp'$.

R is the set of all $R_{Y?}$. Y is any $\perp$-closed subset of X.

$R_{Y?}$ is the binary relation on X such that for any $x \in X$ and $y \in X$, $(x, y) \in R_{Y?}$ iff for any $\perp$-closed subset Z of X that includes x, $Y \cap (Y^\perp \sqcup Z)$ includes y.

A DO-frame $(X, \perp, R)$ is said to be a *DOM-frame* if $(X', \perp')$ is an OM-frame. We write $x(Y?)y$ for $(x, y) \in R_{Y?}$.

A *DO-model* $(X, \perp, R, V)$ is defined using the same definition of V in an O-model. A DO-model $(X, \perp, R, V)$ is said to be a *DOM-model* if $(X', \perp')$ is an OM-frame. We write $x(A?)y$ for $\|A\| = Y$ and $(x, y) \in R_{Y?}$.

The formula $A \wedge (\neg A \vee B)$ (or set $Y \cap (Y^\perp \sqcup Z)$) is called *Sasaki projection*. In a Hilbert space, this formula is true after a state in which B is true is projected to a subspace in which A is true. Therefore, intuitively, R expresses the change in state by the projection.

A DOM-model is essentially identical to an OM-model because $X = X'$, $\perp = \perp'$ and the definition of V is the same in an OM-model. Therefore, the same definition of truth and validity can be used for a formula, and the next lemma holds.

Lemma 3.2.1. *A formula A is valid in all DOM-models iff A is valid in all OM-models.*

We say a $\perp$-closed set Y is *self-adjoint* in a DO-frame $(X, \perp, R)$ if for every $x, y, z \in X$, if $x(Y?)y$ and $y \not\perp z$, then there exists $w \in X$ such that $z(Y?)w$ and $w \not\perp x$. This idea can be used like a frame condition for the orthomodular law. That is, a DO-frame $(X, \perp, R)$ is actually a DOM-frame iff all $\perp$-closed set Y are self-adjoint in $(X, \perp, R)$. To prove this lemma, the following lemmas are required.

Lemma 3.2.2. *In any OM-frame, for any $\perp$-closed subset Y, Z, W_i, if $(W_0 \sqcup Y^\perp) \cap (W_1 \sqcup Y^\perp) \cap \ldots \cap (W_n \sqcup Y^\perp) \cap Y \subseteq Z$, then $W_0 \cap W_1 \cap W_2 \cap \ldots \cap W_n \subseteq Y^\perp \sqcup (Y \cap Z)$.*

Proof. Suppose $Y \cap (\bigcap_i (W_i \sqcup Y^\perp)) \subseteq Z$. From$Y \cap (\bigcap_i (W_i \sqcup Y^\perp)) \subseteq Y$, $Y \cap (\bigcap_i (W_i \sqcup Y^\perp)) \subseteq Y \cap Z$. Therefore, $Y^\perp \sqcup (Y \cap (\bigcap_i (W_i \sqcup Y^\perp))) \subseteq Y^\perp \sqcup (Y \cap Z)$. From the (part of) distributive law and orthomodular law, $\bigcap_i W_i = (\bigsqcup_i W_i^\perp)^\perp \subseteq (Y \cap (Y^\perp \sqcup (Y \cap (\bigsqcup_i W_i^\perp))))^\perp \subseteq (Y \cap (Y^\perp \sqcup (\bigsqcup_i (Y \cap W_i^\perp))))^\perp = Y^\perp \sqcup (Y \cap (\bigcap_i (W_i \sqcup Y^\perp)))$. Therefore, $\bigcap_i W_i \subseteq Y^\perp \sqcup (Y \cap Z)$. □

Lemma 3.2.3. *In any DOM-frame $(X, \perp, R)$, if $x \notin Y$, then there exists some $y \in X$ such that $x(Y^\perp?)y$.*

Proof. Suppose $x \in Y^\perp$. Then, x is chosen as y, because $x(Y^\perp?)x$ if $x \in Y^\perp$.

Suppose $x \notin Y^\perp$. The expression $\{Z_0, Z_1, ...\}$ denotes the set of all $\perp$-closed subsets that include x. For the sake of contradiction, suppose that there are no $y \in X$ such that $y \in Y^\perp \cap (Y \sqcup Z)$ for all $Z \in \{Z_0, Z_1, ...\}$. That is, $\bigcap_i (Y^\perp \cap (Y \sqcup Z_i))$ is the empty set. From Lemma 3.2.2, $Z_0 \cap Z_1 \cap ... \subseteq Y$ because $Y^{\perp\perp} \sqcup (Y^\perp \cap \phi) = Y$. This contradicts $x \notin Y$. □

New sets $[Y?]Z$ are introduced as follows.

$$[Y?]Z = \{x \in X | \text{ for all } y \in X, \text{ if } x(Y?)y, \text{ then } y \in Z\}.$$

Furthermore, new formulas $[A?]B$ are introduced and $\|[A?]B\|$ is defined as $[\|A\|?]\|B\|$. In other words, for any formulas A and B, $[A?]B$ is a formula. This formula does not appear in the language of orthomodular logic because it is constructed for convenience in proving some lemmas. $[A?]B$ can be read as "When the state is projected to a closed subspace in which A is true, then B is also true."

Lemma 3.2.4 (Fundamental nature of the Sasaki arrow)**.** *In any DO-frame $(X, \perp, R)$, and any $\perp$-closed subset Y, if Y is self-adjoint, then $[Y?]Z = Y^\perp \sqcup (Y \cap Z)$ for any $\perp$-closed subset Z.*

Proof. Suppose $x \in X$ and $x \notin Y^\perp \sqcup (Y \cap Z)$. Then, there exists $y \in X$ such that $x \not\perp y$ and $y \in Y \cap (Y \cap Z)^\perp$. From $y(Y?)y$ and self-adjointness, there exists $z \in X$ such that $x(Y?)z$ and $z \not\perp y$. As $y \in (Y \cap Z)^\perp$, $z \notin (Y \cap Z)$. Therefore, $z \notin Z$ and $z \in Y$ because $x(Y?)z$. Therefore, $x \notin [Y?]Z$.

Suppose $x \in X$ and $x \notin [Y?]Z$. Then, there exists $y \in X$ such that $x(Y?)y$ and $y \notin Z$. Therefore, there exists $z \in X$ such that $y \not\perp z$ and $z \in Z^{\perp}$. From self-adjointness, there exists $w \in X$ such that $z(Y?)w$ and $w \not\perp x$. From $z \in Z^{\perp}$ and $z(Y?)w$, $w \in Y \cap (Y^{\perp} \sqcup Z^{\perp})$. Therefore, $x \notin (Y \cap (Y^{\perp} \sqcup Z^{\perp}))^{\perp} = Y^{\perp} \sqcup (Y \cap Z)$ because $w \not\perp x$. □

Although an OM-frame does not satisfy the *distributive law* $(Y \sqcup Z) \cap (Y \sqcup W) = Y \sqcup (Z \cap W)$, a part of the distributive law $Y \sqcup (Z \cap W) \subseteq (Y \sqcup Z) \cap (Y \sqcup W)$ is still valid in an OM-frame [5].

Theorem 3.2.5. *A DO-frame $(X, \perp, R)$ is a DOM-frame iff all $\perp$-closed subsets of X are self-adjoint.*

Proof. Suppose all $\perp$-closed subsets are self-adjoint in an DO-frame $(X, \perp, R)$. Then, from Lemma 3.2.4, $Y \cap Y^{\perp} \sqcup (Y \cap Z) = Y \cap [Y?]Z$. Therefore, if $x \in Y \cap Y^{\perp} \sqcup (Y \cap Z)$, then $x \in Z$ for any $x \in X$ because $x(Y?)x$ for any Y if $x \in Y$.

Suppose $(X, \perp, R)$ is a DOM-frame and suppose x, y and z satisfies $x, y, z \in X$, $x(Y?)y$ and $y \not\perp z$. From $y \in Y$, $z \notin Y^{\perp}$. From Lemma 3.2.3, there exists some $w \in X$ that satisfy $z(Y^{\perp\perp}?)w(= z(Y?)w)$. For the sake of contradiction, suppose there is no w such that $w \not\perp x$. Now $\{Z_0, Z_1, \ldots\}$ can be denoted as the set of all $\perp$-closed subset of X that includes z. Then, $x \in (Y \cap (Y^{\perp} \sqcup Z_0) \cap (Y^{\perp} \sqcup Z_1) \ldots)^{\perp}$ because x is orthogonal to all w that are included in $(Y \cap (Y^{\perp} \sqcup Z_0) \cap (Y^{\perp} \sqcup Z_1) \ldots)$. From the distributive law, $(Y \cap (Y^{\perp} \sqcup Z_0) \cap (Y^{\perp} \sqcup Z_1) \ldots)^{\perp} \subseteq (Y \cap (Y^{\perp} \sqcup (Z_0 \cap Z_1 \cap \ldots)))^{\perp} = Y^{\perp} \sqcup (Y \cap (Z_0 \cap Z_1 \cap \ldots)^{\perp})$. From $x(Y?)y$ and the orthomodular law, $y \in Y \cap (Y^{\perp} \sqcup (Y^{\perp} \sqcup (Y \cap (Z_0 \cap Z_1 \cap \ldots)^{\perp}))) = Y \cap (Y^{\perp} \sqcup (Y \cap (Z_0 \cap Z_1 \cap \ldots)^{\perp})) \subseteq (Z_0 \cap Z_1 \cap \ldots)^{\perp}$, which contradicts $y \not\perp z$. □

In [3], self-adjointness is introduced as a condition of frames for **QTS** (Quantum Transition Systems). In the case of a DOM-frame, self-adjointness is a theorem, not an axiom. **QTS** includes almost all complex dynamical notions in Hilbert spaces. Conversely, OM-frame does not include the notion of projections. Therefore, self-adjointness cannot be quickly confirmed in an OM-frame. From Theorem 3.2.5, it can be concluded that a DOM-frame is a natural expansion of an OM-frame.

3.3 Characterization of strong atomicity and the covering law

In this section, we prove Theorem 3.3.3 which states strong atomicity and the covering law by the condition on a frame.

In an orthomodular lattice $(L, \leq, \sqcap, \sqcup,', \mathbf{1}, \mathbf{0})$, we define the relation $a \lessdot b$ as follows.

$a \lessdot b \;\Leftrightarrow\; a < b$ and if $a \leq c \leq b$, then $c = a$ or $c = b$.

An *atomic element* of $(L, \leq, \sqcap, \sqcup,', \mathbf{1}, \mathbf{0})$ is defined as a such that $0 \lessdot a$. We say an orthomodular lattice satisfies the *atomicity* if the lattice satisfies following condition.

For all $a \in L$, $a = \bigsqcup\{b | b \leq a$ and b is an atomic element$\}$.

We say an orthomodular lattice satisfies the *covering law* if the lattice satisfies following condition.

For all $a \in L$ and all atomic elements $b \in L$, if $b \not\leq a$, then $a \lessdot a \sqcup b$.

Atomicity and covering law are defined also in OM-frames. We define $Y \prec Z$ for $\perp$-closed subsets Y, Z as follows.

$Y \prec Z \quad \Leftrightarrow \quad Y \subset Z$ and if $Y \subseteq W \subseteq Z$ and W is $\perp$-closed, then $W = Y$ or $W = Z$.

A $\perp$-closed subset Y is defined as an *atomic set* if $\emptyset \prec Y$. We say that an OM-frame has *atomicity* if the frame satisfies the following condition.

For all $\perp$-closed subset $Y \subseteq X$, $Y = \bigsqcup\{Z | Z \subseteq Y$ and Z is an atomic set$\}$.

We say that an OM-model satisfies the *covering law* if the frame satisfies the following condition.

For all $\perp$-closed subsets Y and all atomic sets Z, if $Z \not\subseteq Y$, then $Y \prec Y \sqcup Z$.

We now introduce the concept of *strong atomicity* because it is appropriate for DOM-frames. This change does not affect the essence of our argument. We say that an OM-model has strong atomicity if all singletons $\{x\}(x \in X)$ are $\perp$-closed. This notion is justified by the next lemma.

Lemma 3.3.1. *If an OM-frame $(X, \perp, R)$ has strong atomicity, then $(X, \perp, R)$ has atomicity.*

Proof. $\bigsqcup_{y \in Y} \{y\} = (\bigcap_{y \in Y} \{y\}^{\perp})^{\perp} = (\{z | \text{for all } y \in Y,\ y \perp z\})^{\perp} = (Y^{\perp})^{\perp} = Y$. □

The definition of $R_{Y?}$ for a DO-frame can be replaced by the following if the model has strong atomicity, because if $Z \subseteq Z'$, then $Y \cap (Y^{\perp} \sqcup Z) \subseteq Y \cap (Y^{\perp} \sqcup Z')$.

$R_{Y?}$ is the binary relation on X such that, for any $x \in X$ and $y \in X$, $(x, y) \in R_{Y?}$ iff $Y \cap (Y^{\perp} \sqcup \{x\})$ includes y.

Lemma 3.3.2. *In any DOM-frame $(X, \perp, R)$, if $x(Y?)y$, then $x \notin Y^{\perp}$.*

Proof. If $x(Y?)y$ and $x \in Y^{\perp}$, then $y \in Y \cap (Y^{\perp} \sqcup Y^{\perp}) = \emptyset$. This is a contradiction. □

Strong atomicity and the covering law can be represented by simple condition on a DOM-frame. We say a relation R' in a frame satisfies *partial functionality* if R' has the following nature.

For all $x, y, z \in X$, if $x(R')y$ and $x(R')z$, then $y = z$.

Theorem 3.3.3. *An OM-frame $(X, \perp)$ satisfies both strong atomicity and the covering law iff all relations in R for the DOM-frame derived from $(X, \perp)$ satisfy partial functionality.*

Proof. ($\Rightarrow$) For the sake of contradiction, suppose that the DOM-frame $(X, \perp, R)$ satisfies both strong atomicity and the covering law, but there exist some $x \in X$ and $W \subseteq X$ such that $2 \leq |\{y | x(W?)y\}|$. From the strong atomicity, the definition of $W?$, and Lemma 3.3.1, $\{y | x(W?)y\} = W \cap (W^{\perp} \sqcup \{x\})$. From Lemma 3.3.2 and $y(W?)y$, $x, y \notin W^{\perp}$. Then, from the covering law, $W^{\perp} \prec W^{\perp} \sqcup \{x\}$ and, for all $y \in W \cap (W^{\perp} \sqcup \{x\})$, $W^{\perp} \prec W^{\perp} \sqcup \{y\}$. From $\{y\} \subseteq W^{\perp} \sqcup \{x\}$ and $W^{\perp} \subseteq W^{\perp} \sqcup \{x\}$, $W^{\perp} \sqcup \{y\} \subseteq W^{\perp} \sqcup \{x\}$. Therefore, from $W^{\perp} \prec W^{\perp} \sqcup \{x\}$, $W^{\perp} = W^{\perp} \sqcup \{y\}$ or $W^{\perp} \sqcup \{y\} = W^{\perp} \sqcup \{x\}$. Suppose that $W^{\perp} = W^{\perp} \sqcup \{y\}$. Then, $W \cap W^{\perp} \neq \emptyset$ because $\{y\} \subseteq W \cap W^{\perp}$. This is a contradiction. Suppose that $W^{\perp} \sqcup \{y\} = W^{\perp} \sqcup \{x\}$. Then, $W \cap (W^{\perp} \sqcup \{x\}) = W \cap (W^{\perp} \sqcup \{y\})$. From the orthomodular law and $\{y\} \subseteq W$, $\{y\}^{\perp} \subseteq W^{\perp} \sqcup (W \cap \{y\}^{\perp}) = (W \cap (W^{\perp} \sqcup \{y\}))^{\perp}$. Therefore, $W \cap (W^{\perp} \sqcup \{x\}) = W \cap (W^{\perp} \sqcup \{y\}) \subseteq \{y\}$. This result contradicts $2 \leq |W \cap (W^{\perp} \sqcup \{x\})|$.

($\Leftarrow$) Suppose all relations in R for the DOM-frame $(X, \perp, R)$ satisfies partial functionality. From the definition of R and $X \cap (X^{\perp} \sqcup Y) =$

Y, $x(X?)y$ iff, for all Y such that $x \in Y$, $y \in Y$. Therefore, from $x(X?)x$ and the partial functionality, if $x \neq y$, there exists some $\perp$-closed subset Z such that $x \in Z$ and $y \notin Z$. The conjunction of all such subsets is $\{x\}$. Then, from Lemma 2.3.1, $\{x\}$ is $\perp$-closed. Therefore, $(X, \perp, R)$ has strong atomicity.

For the covering law, for the sake of contradiction, suppose W, U are $\perp$-closed, $W \subset U \subset W \sqcup \{x\}$, and $x \notin W$. From the nature of the least upper bound $\sqcup$, $W \sqcup \{x\} = U \sqcup \{x\}$. From $U^\perp \subseteq W^\perp$, $U^\perp \cap (U \sqcup \{x\}) = U^\perp \cap (W \sqcup \{x\}) \subseteq W^\perp \cap (W \sqcup \{x\})$. From the partial functionality, strong atomicity and Lemma 3.2.3, $W^\perp \cap (W \sqcup \{x\})$ is a singleton. Therefore, $U^\perp \cap (U \sqcup \{x\}) = \emptyset$ or $U^\perp \cap (U \sqcup \{x\}) = W^\perp \cap (W \sqcup \{x\})$. Suppose that $U^\perp \cap (U \sqcup \{x\}) = \emptyset$. Then, from the orthomodular law, $U^\perp \cap \{x\}^\perp = U^\perp \cap ((U^\perp \cap (U \sqcup \{x\}))^\perp) = U^\perp \cap \emptyset^\perp = U^\perp$. Therefore, $U = U \sqcup \{x\}$. This result contradicts $U \neq W \sqcup \{x\}$. Suppose that $U^\perp \cap (U \sqcup \{x\}) = W^\perp \cap (W \sqcup \{x\})$. Then, $W^\perp \cap (W \sqcup \{x\}) \subseteq U^\perp$. Therefore, from the orthomodular law and $(W \sqcup \{x\})^\perp \subseteq W^\perp$, $W^\perp \subseteq (W \sqcup \{x\})^\perp \sqcup (W^\perp \cap (W \sqcup \{x\})) \subseteq (W \sqcup \{x\})^\perp \sqcup U^\perp = U^\perp$. This result contradicts $W \subset U$. $\square$

Chapter 4

Sequent calculi for quantum logic with implications

In this chapter, we construct sequent calculi which include the implications introduced in Chapter 2, and prove the soundness and completeness theorems. Although some sequent calculi for quantum logic have been studied, they do not include an implication. Therefore, we construct new sequent calculi by adding the rules for the implications to these sequent calculi, which are already constructed. The sequent calculi in sections 4.2 - 4.6 are based on **GO** [19], and the sequent calculi in Section 4.7 are based on **GMQL** [20]. All the sequent calculi in this chapter are listed in Table 4.1. A part of the contents of this chapter is based on the author's paper [14].

Although almost all of these sequent calculi satisfy soundness and completeness theorems, they are incompatible with the cut-elimination theorem. We cannot remove the rule (cut) from the sequent calculi which are based on **GO**. Conversely, if we add the rule (cut) to **GMQL**, it will collapse to classical logic. Sequent calculi **GOL** [6] and **BS** [7] for quantum logic satisfy the cut-elimination theorem. However, the definitions of formulas or sequents in **GOL** and **BS** are somewhat complicated and they are not suitable for introducing rules for implications. Also, another sequent calculus for quantum (like) logic is discussed in [22], but this calculus has problems to prove the soundness and completeness theorems.

To sum up, there is difficulty in constructing a sequent calculus for quantum logic with implications that satisfies cut-elimination theorem. This difficulty will be solved in the next chapter.

	Name	Logic	Implication	Section
Known calculi by [19]	**GO**	O[1]		4.1
	GOM	OM[2]		4.1
New calculi based on **GO/GOM**	**GOI**	O	$\to$[3]	4.2
	GOI$_2$	O	$\to$	4.3
	GOMS	OM	$\to_S$[4]	4.4
	GOMC	OM	$\to_C$[5]	4.5
	GOMR	OM	$\to_R$[6]	4.5
	GOE	O	$\twoheadrightarrow$[7]	4.6
Known calculus by [20] [21]	**GMQL**	O		4.7
New calculi based on **GMQL**	**GMQLM**	OM		4.7
	GMQLMS	OM	$\to_S$	4.7

[1] orthologic
[2] orthomodular logic
[3] strict implication
[4] Sasaki arrow
[5] contrapositive Sasaki arrow
[6] relevance arrow
[7] entailment

Table 4.1: Sequent calculi in this chapter

4.1 Sequent calculi GO and GOM

In this section, the sequent calculi **GO** and **GOM**, which are introduced by Nishimura [19], are reviewed.

First, the sequent calculus **GO** is examined. **GO** is a calculus for O-models. This is the most basic sequent calculus for orthologic. Since an ortho lattice can be expressed using only negation and conjunction, **GO** includes only rules for $\neg$ and $\wedge$. The concept for rules of **GO** are nearly identical to the sequent calculus for classical logic **LK**. The major difference between **GO** and **LK** is the rule ($\neg$R). In **GO**, the context of the ($\neg$R) is restricted. Because of this restriction, the distributive law cannot be proven in **GO**.

GO

Axiom: $A \Rightarrow A$

Rules:

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\ \text{(cut)}$$

$$\frac{\Gamma \Rightarrow \Delta}{\Pi, \Gamma \Rightarrow \Delta, \Sigma}\ \text{(weakening)}$$

$$\frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ (\wedge\mathrm{L}) \quad \frac{B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ (\wedge\mathrm{L})$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\ (\wedge\mathrm{R})$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta}\ (\neg\mathrm{L}) \quad \frac{A \Rightarrow \Delta}{\neg \Delta \Rightarrow \neg A}\ (\neg\mathrm{R})$$

$$\frac{A, \Gamma \Rightarrow \Delta}{\neg\neg A, \Gamma \Rightarrow \Delta}\ (\neg\neg\mathrm{L}) \quad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \neg\neg A}\ (\neg\neg\mathrm{R})$$

In [19], Γ, Δ, Π and Σ can be infinite sets. In this paper, these sets are restricted to finite sets. This change does not affect the essence of the discussion.

Consider an O-model $(X, \perp, V)$. Sequent $\Gamma \Rightarrow \Delta$ is *false* at $x \in X$ if for all formulas A in Γ, $x \models A$, and for all formulas B in Δ, $x \not\models B$. If $\Gamma \Rightarrow \Delta$ is not false at x, then it is *true* at x. Sequent $\Gamma \Rightarrow \Delta$ is *valid* in an O-model $(X, \perp, V)$ if $\Gamma \Rightarrow \Delta$ is true at all $x \in X$.

In this system, we cannot regard $\Gamma \Rightarrow \Delta, A, B$ as $\Gamma \Rightarrow \Delta, A \vee B$. By definition, commas on the right side of the sequent means union of sets. However, $\|A\| \cup \|B\|$ is not always $\perp$-closed. $\|A\| \cup \|B\|$ and $\|A \vee B\|$ are different sets. For example, $\Rightarrow A, \neg A$ can not be proven in this system but $\Rightarrow A \vee \neg A$ ($= \Rightarrow \neg(\neg A \wedge \neg\neg A)$) can be proven.

Theorem 4.1.1 (Soundness and completeness theorem for **GO**). *$\Gamma \Rightarrow \Delta$ is provable in* **GO** *iff $\Gamma \Rightarrow \Delta$ is valid in all O-models.*

Proof. See [19]. □

Next, the sequent calculus **GOM** is examined. **GOM** is a calculus for TOM-models. This is the most basic sequent calculus for orthomodular logic. **GOM** is defined by adding the next rule to **GO**.

$$\frac{\neg B \Rightarrow \neg A \qquad \neg A, B \Rightarrow}{\neg A \Rightarrow \neg B}\ (\mathrm{OM})$$

The definitions of truth, falsity and validity of a sequent are identical to that in **GO** except that a model is a TOM-model. In [19], Nishimura adopted the rule (OM) as an orthomodular law. Another variant of an orthomodular law, $A \wedge (\neg A \vee (A \wedge B)) \Rightarrow B$, is provable in **GOM** [19].

Theorem 4.1.2 (Soundness and completeness theorem for **GOM**). *$\Gamma \Rightarrow \Delta$ is provable in* **GOM** *iff $\Gamma \Rightarrow \Delta$ is valid in all TOM-models.*

Proof. See [19]. □

4.2 Sequent calculus GOI

In this section, a sequent calculus that includes strict implication is established. In the next two sections, $\neg A$ is represented as an abbreviation of $A \to \bot$ because $\|A \to \bot\| = \|A\|^{\perp}$ can be easily checked. The sequent calculus **GOI** is defined as an expansion of **GO**. The rule ($\to$ R) and the axiom "$\bot \Rightarrow$" are added to **GO**. The rule ($\to$ R) is the transformation of the rule ($\to$) in [13]. As this rule ($\to$ R) is complex, it is better to use **GOI**$_2$ in the next chapter for main calculus of quantum logic with strict implication. However, ($\to$ R) is useful to prove completeness theorem. Therefore, first the details of **GOI** are shown. The definitions of truth, falsity and validity of a sequent are identical to that in **GO**.

$$\bot \Rightarrow \qquad (\bot)$$

$$\frac{\Gamma_1, A \Rightarrow B, \Delta_1, \Sigma \qquad \Gamma_2, A \Rightarrow B, \Delta_2, \Sigma \quad \ldots \qquad \Gamma_{2^n}, A \Rightarrow B, \Delta_{2^n}, \Sigma}{C_1 \to D_1, C_2 \to D_2, \ldots, C_n \to D_n, \Pi \Rightarrow A \to B, \Lambda}\ (\to\mathrm{R})$$

where, $0 \leq n$, $\Gamma_i = \{D_j | j \in \gamma(i)\}$, $\Delta_i = \{C_j | j \in \delta(i)\}$, $\langle \delta(i), \gamma(i) \rangle$ is i-th element of all partitions of $\{1, \ldots, n\}$. Π and Λ are sets of formulas. Σ is a set of all possible formulas of the shape $E \to F$ such that E is included in premise of the lower sequent, and F is included in the conclusion of the lower sequent, or $\bot$. i.e, $\Sigma = \{E \to F\ | E \in \{C_1 \to D_1, \ldots, C_n \to D_n, \Pi\}, F \in \{A \to B, \Lambda, \bot\}\}$.

For example, suppose $\Pi = \{I\}$, $\Lambda = \{J, K\}$. ($\to$R) in the case of $n = 0$, $n = 1$ and $n = 2$ are shown below.

$$\frac{A \Rightarrow B, I \to (A \to B), I \to J, I \to K, I \to \bot}{I \Rightarrow A \to B, J, K}$$

$$\frac{A \Rightarrow B, C_1, \Sigma \quad D_1, A \Rightarrow B, \Sigma}{C_1 \to D_1, I \Rightarrow A \to B, J, K}$$

where Σ is $\{(C_1 \to D_1) \to (A \to B), (C_1 \to D_1) \to J, (C_1 \to D_1) \to K, (C_1 \to D_1) \to \bot, I \to (A \to B), I \to J, I \to K, I \to \bot\}$.

$$\frac{A \Rightarrow B, C_1, C_2, \Sigma \qquad D_1, A \Rightarrow B, C_2, \Sigma \qquad D_2, A \Rightarrow B, C_1, \Sigma \qquad D_1, D_2, A \Rightarrow B, \Sigma}{C_1 \to D_1, C_2 \to D_2, I \Rightarrow A \to B, J, K}$$

where Σ is $\{(C_1 \to D_1) \to (A \to B), (C_1 \to D_1) \to J, (C_1 \to D_1) \to K, (C_1 \to D_1) \to \bot, (C_2 \to D_2) \to (A \to B), (C_2 \to D_2) \to J, (C_2 \to D_2) \to K, (C_2 \to D_2) \to \bot, I \to (A \to B), I \to J, I \to K, I \to \bot\}$.

In **GOI**, the rule $(\to \mathrm{L})$ is admissible.

$$\frac{\Gamma_1, \Rightarrow \Delta_1, A \qquad B, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, A \to B \Rightarrow \Delta_1, \Delta_2} \ (\to\mathrm{L})$$

Theorem 4.2.1 (Soundness theorem for **GOI**). *If $\Gamma \Rightarrow \Delta$ is provable in* **GOI***, $\Gamma \Rightarrow \Delta$ is valid in all O-models.*

Proof. Proven by induction on the construction of the proof of $\Gamma \Rightarrow \Delta$. Considering the rules in **GO**, the proof is identical to the proof in [19]. In the case of $(\to\mathrm{R})$, only the case where $n = 2$ is checked. All other cases are similar to this one. For the sake of contradiction, suppose all premises of the rule are valid, and there exist O-model $(X, \bot, V)$ and $x \in X$ such that the conclusion of the rule is false at x. Then, as $A \to B$ is not true at x, there exist $y \in X$ such that $x \not\perp y$, $y \models A$ and $y \not\models B$. Then, since it is assumed $y \models A$ and all premises are valid, from the first premise, B or C_1 or C_2 or one of the formulas in Σ is true at y. However, B is not true at y. Now suppose $E \to F \in \Sigma$. We have $x \models E$ and $x \not\models F$ by assumption and the definition of $(\to\mathrm{R})$. If $y \models E \to F$, from $y \not\perp x$ and $x \models E$, then $x \models F$. This result is a contradiction. Therefore, for all $E \to F \in \Sigma$, $y \not\models E \to F$. Therefore, C_1 or C_2 is true at y. In the former case, from $x \models C_1 \to D_1$ and $y \models C_1$, $y \models D_1$. Then, from the second premise, B or C_2 or one of the formulas in Σ is true at y. As previously shown, only possibility is C_2. Therefore, C_2 is true at y. Following this method to the end of premises, B and Σ are the only possibilities. This is a contradiction. The latter cases of the alternate possibilities lead to similar results using this method. $\square$

To prove the completeness theorem, the set Ω is defined as follows.

$\Omega(\Gamma \Rightarrow \Delta) = \{$ All subformulas in $\Gamma \cup \Delta\} \cup \{ \neg p \mid p$ appears in some formulas in $\Gamma \cup \Delta\} \cup \{\bot\}$.

For example, $\Omega(\neg(p \to q) \Rightarrow r \wedge q) = \{\bot, p, q, r, \neg p, \neg q, \neg r, p \to q, r \wedge q, \neg(p \to q)\}$. For each unprovable sequent $\Gamma \Rightarrow \Delta$, a *canonical model* $(X_c, \bot_c, V_c)$ of $\Gamma \Rightarrow \Delta$ is defined as shown below.

$$X_c = \{\Gamma' \Rightarrow \Delta' | \Gamma' \Rightarrow \Delta' \text{ is not provable and } \Gamma' \cup \Delta' = \Omega(\Gamma \Rightarrow \Delta)\}$$

$\perp_c = \{(\Gamma' \Rightarrow \Delta', \Gamma'' \Rightarrow \Delta'')|$ for some A and B, at least one of (1)(2) is true. (1) $A \to B \in \Gamma'$ and $A \in \Gamma''$ and $B \in \Delta''$. (2) $A \to B \in \Gamma''$ and $A \in \Gamma'$ and $B \in \Delta'\}$.

$V_c(p) = \{\Gamma' \Rightarrow \Delta' | p \notin \Delta'\}$.

Lemma 4.2.2. *$(X_c, \perp_c)$ is an O-frame. $V_c(p)$ is $\perp$-closed. Therefore, $(X_c, \perp_c, V_c)$ is an O-model.*

Proof. If $(\Gamma' \Rightarrow \Delta')\perp(\Gamma' \Rightarrow \Delta')$, there exists A and B such that $A \in \Gamma'$, $A \to B \in \Gamma'$ and $B \in \Delta'$. However, $A, A \to B \Rightarrow B$ is proven by using ($\to$L). Therefore, $\Gamma' \Rightarrow \Delta'$ can be proven. This is a contradiction. Therefore, for every $\Gamma' \Rightarrow \Delta' \in X_C$, $(\Gamma' \Rightarrow \Delta') \not\perp (\Gamma' \Rightarrow \Delta')$. Symmetry is obvious from the definition. If $p \notin \Omega(\Gamma \Rightarrow \Delta)$, then $V_c(p) = X_c$. This set is clearly $\perp$-closed. If $p \in \Omega(\Gamma \Rightarrow \Delta)$, then, for every $\Gamma' \Rightarrow \Delta' \in X_c$, $p \in \Gamma'$ or $p \in \Delta'$. Then, $(\Gamma' \Rightarrow \Delta') \models p$ iff $p \in \Gamma'$. Therefore, if the next statement can be proven, this lemma can be proven.

For all $(\Gamma' \Rightarrow \Delta') \in X_c$, if $p \in \Delta'$, then there exists $(\Gamma'' \Rightarrow \Delta'') \in X_C$ which satisfies $\neg p \in \Gamma''$ and $(\Gamma' \Rightarrow \Delta') \not\perp (\Gamma'' \Rightarrow \Delta'')$.

For convenience, this statement is proven after the next lemma. $\square$

Lemma 4.2.3. *For all canonical O-model, and for all formulas $A \in \Omega$, A is true at $(\Gamma' \Rightarrow \Delta')$ if $A \in \Gamma'$, and A is false at $(\Gamma' \Rightarrow \Delta')$ if $A \in \Delta'$.*

Proof. Proven by induction on the composition of A.

In the case of $A = p$, immediately from the definition of canonical O-model.

In the case of $A = B \wedge C$, the proof is the same as the proof in [19].

In the case of $A = \neg B$, the proof is included in the case of $A = B \to C$.

In the case of $A = B \to C$, suppose $B \to C \in \Gamma'$. Then, for all $(\Gamma'' \Rightarrow \Delta'')$ which satisfies $B \in \Gamma''$ and $C \in \Delta''$, $(\Gamma' \Rightarrow \Delta')\perp(\Gamma'' \Rightarrow \Delta'')$ from definition of canonical O-model. Then, by definition of $\to$ and induction hypothesis, $B \to C$ is true at $(\Gamma' \Rightarrow \Delta')$.

Suppose $B \to C \in \Delta'$. Since $\Gamma' \Rightarrow \Delta'$ cannot be proven, when we regard this sequent as the lower sequent of the rule ($\to$R), there exists an unprovable sequent $\Gamma'', B \Rightarrow C, \Delta''$ in the shape of one of the sequent in upper sequents of ($\to$R). Then, Γ'' and Δ'' distributes all formulas in the shape of $E \to F$ in Γ' regarded as one of a $C_i \to$

D_i. If there are some formulas in $\Gamma'', B \Rightarrow C, \Delta''$ not included in $\Omega(\Gamma \Rightarrow \Delta)$, they are deleted from $\Gamma'', B \Rightarrow C, \Delta''$ and a new sequent $\Gamma''', B \Rightarrow C, \Delta'''$ is formed. Then, $\Gamma''' \cup \{B, C\} \cup \Delta''' \subseteq \Omega(\Gamma \Rightarrow \Delta)$ and this sequent is still unprovable. This sequent can be expanded to the sequent $\Gamma'''' \Rightarrow \Delta'''' \in X_c$ because for all formulas G, at least one of a $\Gamma''', B \Rightarrow C, \Delta''', G$ or $G, \Gamma''', B \Rightarrow C, \Delta'''$ is unprovable because of the rule (cut) and $\Gamma''', B \Rightarrow C, \Delta'''$ is unprovable. Furthermore, $(\Gamma' \Rightarrow \Delta') \not\perp (\Gamma'''' \Rightarrow \Delta'''')$ is satisfied because all possibility of relation $\perp$ are removed when $\Gamma'', B \Rightarrow C, \Delta''$ is constructed. Therefore, by definition of $\to$ and induction hypothesis, $B \to C$ is false at $\Gamma' \Rightarrow \Delta'$. □

Now the statement in Lemma 4.2.2 can be proven using the method in the proof of Lemma 4.2.3. If $\Gamma' \Rightarrow \Delta', p$ $(\in X_c)$ is not provable, then, $\Gamma' \Rightarrow \Delta', p, \neg\neg p$ is also unprovable. $(p \to \bot) \to \bot$ is regarded as $B \to C$ in Lemma 4.2.3. Then, an identical argument in the case of $B \to C \in \Delta'$ in Lemma 4.2.3 can be applied. That is, we can find $(\Gamma'''' \Rightarrow \Delta'''') \in X_c$ which satisfies $(\neg p \in \Gamma'''')$, $(\bot \in \Delta'''')$ and $(\Gamma' \Rightarrow \Delta', p) \not\perp (\Gamma'''' \Rightarrow \Delta'''')$. If $\neg\neg p$ is included in $\Omega(\Gamma \Rightarrow \Delta)$, then $\Gamma' \Rightarrow \Delta', p, \neg\neg p$ is the same as $\Gamma' \Rightarrow \Delta', p$ and is included in X_c. If $\neg\neg p$ is not included in $\Omega(\Gamma \Rightarrow \Delta)$, sequent $\Gamma'''' \Rightarrow \Delta''''$ $(\neg p \in \Gamma'''')$ which is constructed from $\Gamma' \Rightarrow \Delta', p, \neg\neg p$ is included in X_c even if $\Gamma' \Rightarrow \Delta', p, \neg\neg p$ is not included in X_c because when $\Gamma''', \neg p \Rightarrow \bot, \Delta'''$ is constructed from $\Gamma' \Rightarrow \Delta', p, \neg\neg p$, all formulas which are not included in $\Omega(\Gamma \Rightarrow \Delta)$ are eliminated. Furthermore, they satisfy $(\Gamma' \Rightarrow \Delta', p) \not\perp (\Gamma'''' \Rightarrow \Delta'''')$ because $\Gamma' \Rightarrow \Delta', p$ is part of $\Gamma' \Rightarrow \Delta', p, \neg\neg p$.

Theorem 4.2.4 (Completeness theorem for **GOI**)**.** *If $\Gamma \Rightarrow \Delta$ is valid in all O-models, then $\Gamma \Rightarrow \Delta$ is provable in* **GOI**.

Proof. Suppose $\Gamma \Rightarrow \Delta$ is not provable. A canonical O-model of $\Gamma \Rightarrow \Delta$ can be made. Then, as (cut) is included in **GOI**, there exists $(\Gamma' \Rightarrow \Delta') \in X_c$ which is the expansion of $\Gamma \Rightarrow \Delta$. By Lemma 4.2.3, $\Gamma \Rightarrow \Delta$ is false at $(\Gamma' \Rightarrow \Delta')$. □

4.3 Sequent calculus $\mathbf{GOI}_2$

In this section, the sequent calculus $\mathbf{GOI}_2$ is established as the expansion of **GO**. This calculus is sound and complete with respect to an O-model the same as **GOI**. The axioms $(\to)$ and $(\bot)$, and the rule $(\to \mathrm{R})$' are added to **GO**. The rule $(\to \mathrm{R})$' is identical to the rule $(\to)$ in [13].

$$\perp \Rightarrow \qquad (\perp)$$

$$A \Rightarrow (A \rightarrow B) \rightarrow \perp, B \qquad (\rightarrow)$$

$$\frac{\mathit{\Gamma}_1, A \Rightarrow B, \mathit{\Delta}_1 \qquad \mathit{\Gamma}_2, A \Rightarrow B, \mathit{\Delta}_2 \quad \ldots \qquad \mathit{\Gamma}_{2^n}, A \Rightarrow B, \mathit{\Delta}_{2^n}}{C_1 \rightarrow D_1, C_2 \rightarrow D_2, \ldots, C_n \rightarrow D_n \Rightarrow A \rightarrow B} \ (\rightarrow\text{R})\text{'}$$

where, $0 \leq n$, $\mathit{\Gamma}_i = \{D_j | j \in \gamma(i)\}$, $\mathit{\Delta}_i = \{C_j | j \in \delta(i)\}$, $\langle \delta(i), \gamma(i) \rangle$ is i-th element of all partitions of $\{1, \ldots, n\}$.

The rule ($\rightarrow$ R)' is very natural expansion of the rule $(\neg R)$ in **GO**. In other words, if all of the D_j and B in ($\rightarrow$ R)' are $\perp$, it is identical to $(\neg R)$ in **GO** because $A \rightarrow \perp \equiv \neg A$.

Theorem 4.3.1 (Soundness and completeness theorem for **GOI**$_2$**).** *$\mathit{\Gamma} \Rightarrow \mathit{\Delta}$ is provable in* **GOI**$_2$ *iff $\mathit{\Gamma} \Rightarrow \mathit{\Delta}$ is valid in all O-models.*

Proof. All rules in **GOI** are constructed using rules in **GOI**$_2$, and vice versa. An explanation on creating ($\rightarrow$) by **GOI** and ($\rightarrow$R) by **GOI**$_2$ are shown below. The alternate cases are obvious.

$$\cfrac{\cfrac{A \rightarrow B \Rightarrow A \rightarrow B}{A \rightarrow B \Rightarrow \perp, A \rightarrow B, A \rightarrow ((A \rightarrow B) \rightarrow \perp), A \rightarrow \perp} \ (\text{weakening})}{A \Rightarrow (A \rightarrow B) \rightarrow \perp, B} \ (\rightarrow\text{R})$$

Suppose all sequents of upper sequents in ($\rightarrow$R) is provable. For example, suppose $n = 2$. Then,

$A \Rightarrow B, C_1, C_2, \mathit{\Sigma}$
$D_1, A \Rightarrow B, C_2, \mathit{\Sigma}$
$D_2, A \Rightarrow B, C_1, \mathit{\Sigma}$
$D_1, D_2, A \Rightarrow B, \mathit{\Sigma}$

are all provable. Now all formulas in $\mathit{\Sigma}$ are regarded as one of C_i $(n < i)$. For example, if $\mathit{\Sigma}$ has three elements, $\mathit{\Sigma}$ is regarded as $\{C_3, C_4, C_5\}$. Furthermore, all D_i $(n < i)$ is defined as $D_i = \perp$. Then,

$A \Rightarrow B, C_1, C_2, C_3, C_4, C_5$
$D_1, A \Rightarrow B, C_2, C_3, C_4, C_5$
$\ldots$
$D_1, D_2, D_3, D_4, D_5, A \Rightarrow B$

are all provable. If all formulas in $\Sigma = \{C_3, C_4, C_5\}$ are on the right side, it is obvious from assumption. If one of a $\{D_3, D_4, D_5\}$ is on the left side, then all of these are $\bot$, and this sequent is provable. Then, we can use all of these sequents and use (→R)'. Therefore, since $E \Rightarrow (E \to F) \to \bot, F$ is provable using (→), lower sequent of (→R) is provable by using (cut). □

Theorem 4.3.2. *In* **GOI** *and* $\mathbf{GOI_2}$*, the rule* ($\to$ *L) is admissible.*

Proof. By using (cut) and the fact that $A, A \to B \Rightarrow B$ is probable in these systems.

$$\dfrac{\dfrac{A \to B \Rightarrow A \to B}{A \to B, \Rightarrow ((A \to B) \to \bot) \to \bot} \quad \dfrac{A \Rightarrow (A \to B) \to \bot, B}{A, ((A \to B) \to \bot) \to \bot \Rightarrow B}}{A, A \to B \Rightarrow B}$$

□

While the rule of implication in **GOI** is a bit complex, the rule of implication in $\mathbf{GOI_2}$ is less complicated and is a natural expansion of the rule ($\neg R$). However, the axiom (→) must be added to $\mathbf{GOI_2}$. In both calculi, the cut-elimination theorem is not satisfied.

4.4 Sequent calculus GOMS

In this section, a sequent calculus including the Sasaki arrow is constructed. As was mentioned in Chapter 2, this arrow is not suitable for an O-model. Therefore, in this section, TOM-models are used for semantics.

GOMS

Axiom: $A \Rightarrow A \quad \bot \Rightarrow$

Rules:

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma} \text{(cut)}$$

$$\frac{\Gamma \Rightarrow \Delta}{\Pi, \Gamma \Rightarrow \Delta, \Sigma} \text{(weakening)}$$

$$\frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} (\wedge\text{L}) \quad \frac{B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} (\wedge\text{L})$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B} \ (\wedge \mathrm{R})$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta} \ (\neg \mathrm{L}) \quad \frac{A \Rightarrow \Delta}{\neg \Delta \Rightarrow \neg A} \ (\neg \mathrm{R})$$

$$\frac{A, \Gamma \Rightarrow \Delta}{\neg\neg A, \Gamma \Rightarrow \Delta} \ (\neg\neg \mathrm{L}) \quad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \neg\neg A} \ (\neg\neg \mathrm{R})$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{\Gamma, A \rightarrow_S B \Rightarrow \Delta} \ (\rightarrow_S \mathrm{L}) \quad \frac{\Gamma_{P(A)}, A \Rightarrow B}{\Gamma \Rightarrow A \rightarrow_S B} \ (\rightarrow_S \mathrm{R})$$

$$\frac{\neg B \Rightarrow \neg A \qquad \neg A, B \Rightarrow}{\neg A \Rightarrow \neg B} \ (\mathrm{OM})$$

where $\Gamma_{P(A)} = \{(\neg A \vee C) \wedge A \mid C \in \Gamma\}$.

This sequent calculus is the expansion of **GOM**.

The next lemma needs to be confirmed to prove the soundness theorem and completeness theorem.

Lemma 4.4.1. *In any TOM-models, for any formulas B_n and a formula A, $\bigvee_n (A \wedge B_n) \leq A \wedge \bigvee_n B_n$ is always satisfied.*

Theorem 4.4.2 (Soundness theorem for **GOMS**)**.** *If $\Gamma \Rightarrow \Delta$ is provable in* **GOMS***, then $\Gamma \Rightarrow \Delta$ is valid in all TOM-models.*

Proof. Proven by induction on the construction of a proof of the sequent $\Gamma \Rightarrow \Delta$. In cases of the rules for $\neg$ and $\wedge$, see [19]. For $(\rightarrow R)$, the situation where Γ has only two elements C and D is proven. Proofs for the other variants are similar. Suppose $\Gamma_A, A \Rightarrow B$ is valid. That is, in every TOM-model, $((\neg A \vee C) \wedge A) \wedge ((\neg A \vee D) \wedge A) \wedge A = A \wedge (\neg A \vee C) \wedge (\neg A \vee D) \leq B$. From this assumption and $A \wedge (\neg A \vee C) \wedge (\neg A \vee D) \leq A$, $A \wedge (\neg A \vee C) \wedge (\neg A \vee D) \leq A \wedge B$. Therefore, $\neg A \vee (A \wedge (\neg A \vee C) \wedge (\neg A \vee D)) \leq \neg A \vee (A \wedge B)$. Furthermore, from Lemma 4.4.1 and orthomodular law, $C \wedge D = \neg(\neg C \vee \neg D) \leq \neg(A \wedge (\neg A \vee (A \wedge (\neg C \vee \neg D)))) \leq \neg(A \wedge (\neg A \vee ((A \wedge \neg C) \vee (A \wedge \neg D)))) = \neg A \vee (A \wedge (\neg A \vee C) \wedge (\neg A \vee D))$. Therefore, $C \wedge D \leq \neg A \vee (A \wedge B)$.

In the case of rule $(\rightarrow \mathrm{L})$, proven by using the fact that orthomodular law implies that Sasaki arrow satisfies modus ponens. □

Lemma 4.4.3. $\neg A \vee (A \wedge B) \Rightarrow A \rightarrow_S B$ *and* $A \rightarrow_S B \Rightarrow \neg A \vee (A \wedge B)$ *are provable in* **GOMS**.

Proof.

$$
\dfrac{\dfrac{\dfrac{\vdots\,*}{(\neg A \vee (\neg A \vee (A \wedge B))) \wedge A \Rightarrow (\neg A \vee (A \wedge B)) \wedge A} \quad \dfrac{\vdots\,*}{(\neg A \vee (A \wedge B)) \wedge A \Rightarrow B}}{(\neg A \vee (\neg A \vee (A \wedge B))) \wedge A, A \Rightarrow B}}{\neg A \vee (A \wedge B) \Rightarrow A \rightarrow_S B}
$$

$$
\dfrac{\dfrac{\neg A \vee (A \wedge B) \Rightarrow A \rightarrow_S B}{\neg(A \wedge \neg(A \wedge B)) \Rightarrow \neg\neg(A \rightarrow_S B)} \quad \dfrac{\dfrac{\dfrac{A \Rightarrow A}{A \wedge \neg(A \wedge B) \Rightarrow A} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{B, A \Rightarrow A} \quad \dfrac{B \Rightarrow B}{B, A \Rightarrow B}}{B, A \Rightarrow A \wedge B}}{B, A, \neg(A \wedge B) \Rightarrow}}{B, A, A \wedge \neg(A \wedge B) \Rightarrow}}{B, A \wedge \neg(A \wedge B) \Rightarrow}}{A \rightarrow_S B, A \wedge \neg(A \wedge B) \Rightarrow}}{\neg\neg(A \rightarrow_S B), A \wedge \neg(A \wedge B) \Rightarrow}}{\neg\neg(A \rightarrow_S B) \Rightarrow \neg A \vee (A \wedge B)} \ (\mathrm{OM})
$$

$$
\vdots
$$

$$
\dfrac{A \rightarrow_S B \Rightarrow \neg\neg(A \rightarrow_S B) \quad \neg\neg(A \rightarrow_S B) \Rightarrow \neg A \vee (A \wedge B)}{A \rightarrow_S B \Rightarrow \neg A \vee (A \wedge B)} \ (\mathrm{cut})
$$

* Obvious from the completeness theorem.

□

The completeness theorem for **GOMS** is easily proven because of Lemma 4.4.3 and the facts that **GOM** is complete with respect to TOM-models.

Theorem 4.4.4 (Completeness theorem for **GOMS**). *If $\Gamma \Rightarrow \Delta$ is valid in all TOM-models, then $\Gamma \Rightarrow \Delta$ is provable in* **GOMS**.

Proof. The sequent $\Gamma' \Rightarrow \Delta'$ is constructed from $\Gamma \Rightarrow \Delta$ by changing all appearance of Sasaki arrow $A \rightarrow_S B$ to $\neg A \vee (A \wedge B)$ for every A and B. As $\Gamma \Rightarrow \Delta$ is valid, $\Gamma' \Rightarrow \Delta'$ is provable in **GOM** because of the completeness theorem for **GOM**. Suppose T is a proof of $\Gamma' \Rightarrow \Delta'$ in **GOM**. In **GOMS**, the formula $\neg A \vee (A \wedge B)$ can always be changed to $A \rightarrow_S B$ in T because of the rule (cut) and Lemma 4.4.3. Therefore, the proof of $\Gamma \Rightarrow \Delta$ in **GOMS** using T can be constructed. □

In general, a completeness theorem is proven using a canonical model $(X_c, \perp_c, V_c)$. Such a model is constructed using a class of sets of formulas or a set of sequent. In the case of **GOM**, Nishimura [19] used the set of unprovable sequent. In this model, A is true at $\Gamma \Rightarrow \Delta \in X_c$ iff $A \in \Gamma$. The Sasaki arrow $\neg A \vee (A \wedge B) \equiv \neg(A \wedge \neg(A \wedge B))$ contains a double layer of negation. Thus, when semantics are considered, not only the nearest elements in a model must be considered but also the second elements from a current element. This is awkward to create a canonical model by basic procedure. A DOM-model solves this problem because the notion of $\|[A?]B\|$ includes only the nearest elements in a model. From this analysis, we can see the features of the rule ($\to_S$R) as shown below. In a canonical model, if $(\Gamma \Rightarrow A \to_S B, \Delta) \in X_C$, then $\Gamma \Rightarrow A \to_S B, \Delta$ is not provable in **GOMS**. Thus, $\Gamma_{P(A)}, A \Rightarrow B$ is also not provable. Then, there is a sequent $(\Gamma', \Gamma_{P(A)}, A \Rightarrow B, \Delta') \in X_C$ that is not provable and a expansion of $\Gamma_{P(A)}, A \Rightarrow B$. therefore, $A \to_S B$ is not true at $(\Gamma \Rightarrow A \to_S B, \Delta) \in X_C$ because this element is related to $(\Gamma', \Gamma_{P(A)}, A \Rightarrow B, \Delta') \in X_C$ by $(A?)$ because of Γ and $\Gamma_{P(A)}$.

4.5 Sequent calculi GOMC and GOMR

In this section, sequent calculi including $\to_C$ and $\to_R$ are established. From lemma 2.4.2, the rules for $\to_C$ and $\to_R$ that are deformations of ($\to_S$L) and ($\to_S$R) in **GOMS** can be made. New rules ($\to_C$L) and ($\to_C$R) are added to **GOM** to create a new sequent calculus **GOMC**.

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{\Gamma, A \to_C B \Rightarrow \Delta}\ (\to_C \mathrm{L}) \qquad \frac{\Gamma_{P(\neg B)}, \neg B \Rightarrow \neg A}{\Gamma \Rightarrow A \to_C B}\ (\to_C \mathrm{R})$$

Theorem 4.5.1 (Soundness and completeness theorem for **GOMC**). *$\Gamma \Rightarrow \Delta$ is provable in* **GOMC** *iff $\Gamma \Rightarrow \Delta$ is valid in all TOM-models.*

Proof. By Lemma 2.4.2, (MP) for $\to_C$ and the completeness theorem for **GOMS**. □

The new rules ($\to_R$L) and ($\to_R$R) are added to **GOM** to create a new sequent calculus **GOMR**.

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{\Gamma, A \to_R B \Rightarrow \Delta}\ (\to_R \mathrm{L})$$

$$\frac{\Gamma_{P(A)}, A \Rightarrow B \quad \Gamma_{P(\neg B)}, \neg B \Rightarrow \neg A}{\Gamma \Rightarrow A \to_R B}\ (\to_R \mathrm{R})$$

Theorem 4.5.2 (Soundness and completeness theorem for **GOMR**). *$\Gamma \Rightarrow \Delta$ is provable in* **GOMR** *iff $\Gamma \Rightarrow \Delta$ is valid in all TOM-models.*

Proof. By Lemma 2.4.2, (MP) for $\to_R$ and the completeness theorem for **GOMS**. $\Box$

All of these calculi satisfy an important theorem called the disjunction theorem.

Theorem 4.5.3 (Disjunction theorem for **GOMS**, **GOMC** and **GOMR**). *If $\Gamma \Rightarrow \Delta$ is provable in* **GOMS** *(* **GOMC**, **GOMR***), there exist $A \in \Delta$ such that $\Gamma \Rightarrow A$ is provable in* **GOMS** *(* **GOMC**, **GOMR***).*

Proof. By induction on the construction of a proof of the sequent $\Gamma \Rightarrow \Delta$. This theorem has been established in the case of **GOM** [19]. So, we now only deal with the cases of the rules for implications. For the rules ($\to_S$R), ($\to_C$R) and ($\to_R$R), they are easily proven because the right side-of sequent in these rules include at most one formula. For the rules ($\to_S$L), ($\to_C$L) and ($\to_R$L), suppose $\Gamma \Rightarrow C$ is provable. Then, $\Gamma, A \to_{S(C,R)} B \Rightarrow C$ is provable by weakening. Next, suppose $\Gamma \Rightarrow A$ is provable and $B, \Gamma \Rightarrow C$ is provable. Then, $\Gamma, A \to_{S(C,R)} B \Rightarrow C$ is provable by ($\to_{S(C,R)}$L). $\Box$

Unfortunately, **GOM**, **GOMS**, **GOMC** and **GOMR** do not satisfy the cut-elimination theorem for the identical reason that **GOI** does not satisfy the cut-elimination theorem.

4.6 Sequent calculus GOE

In this section, the sequent calculus for entailment is introduced. An O-model is used for semantics. Sequent calculus **GOE** is defined by adding the following rules to **GO**.

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \twoheadrightarrow B, \Gamma \Rightarrow \Delta} (\twoheadrightarrow \mathrm{L}) \quad \frac{A \Rightarrow B}{\Rightarrow A \twoheadrightarrow B} (\twoheadrightarrow \mathrm{R})$$

Similar to other arrows, the left rule is identical to the rule for classical logic.

Theorem 4.6.1 (Soundness theorem for **GOE**). *If$\Gamma \Rightarrow \Delta$ is provable in* **GOE**, *then $\Gamma \Rightarrow \Delta$ is valid in all O-models.*

Proof. Cases where the last rule of the proof is ($\twoheadrightarrow$ R) are proven. Suppose $A \Rightarrow B$ is valid. Then, $A \leq B$ in any O-model. Therefore, $\Rightarrow A \twoheadrightarrow B$ is valid in any O-model.

$\Box$

Whether **GOE** satisfies the completeness theorem or not is still open. One difficulty of proving completeness is that when $\|A \twoheadrightarrow B\|$ is considered to prove completeness theorem, an obstacle arises where every element in a canonical model must be discussed. This issue can be resolved in conjunction with the cut-elimination theorem by introducing labeled sequent calculi in Chapter 5.

4.7 Sequent calculus GMQLMS

There is another type of sequent calculus for ortho logic called **GMQL** [20][21]. **GMQL** has similarities to the concepts that we can find in a lattice. In contrast to **GO**, in **GMQL**, $\Gamma \Rightarrow A, B$ can be regarded as $\Gamma \Rightarrow A \vee B$. Therefore, **GMQL** is symmetric because $\wedge$ and $\vee$ are symmetric in an ortho lattice. In this section, some rules for orthomodular law are added to **GMQL** to make new sequent calculi.

GMQL

Axiom: $A \Rightarrow A$

Rules:

$$\frac{\Gamma \Rightarrow \Delta}{\Pi, \Gamma \Rightarrow \Delta, \Sigma}\ (\text{weakening})$$

$$\frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ (\wedge\text{L}) \qquad \frac{B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ (\wedge\text{L}) \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\ (\wedge\text{R})$$

$$\frac{\Gamma \Rightarrow \Delta}{\neg\Delta, \Gamma \Rightarrow}\ (\neg\text{L}) \qquad \frac{\Gamma \Rightarrow \Delta}{\Rightarrow \Delta, \neg\Gamma}\ (\neg\text{R}) \qquad \frac{\Gamma \Rightarrow \Delta}{\neg\Delta \Rightarrow \neg\Gamma}\ (\neg\neg)$$

$$\frac{A, \Gamma \Rightarrow \Delta}{\neg\neg A, \Gamma \Rightarrow \Delta}\ (\neg\neg\text{L}) \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \neg\neg A}\ (\neg\neg\text{R})$$

$$\frac{\Gamma \Rightarrow \Delta, \neg A}{\Gamma \Rightarrow \Delta, \neg(A \wedge B)}\ (\neg\wedge\text{R}) \qquad \frac{\Gamma \Rightarrow \Delta, \neg B}{\Gamma \Rightarrow \Delta, \neg(A \wedge B)}\ (\neg\wedge\text{R})$$

$$\frac{\neg A \Rightarrow \Delta \quad \neg B \Rightarrow \Delta}{\neg(A \wedge B) \Rightarrow \Delta}\ (\neg\wedge\text{L}) \qquad \frac{\neg A \Rightarrow \Delta \quad \neg B \Rightarrow \Delta}{\Rightarrow \Delta, (A \wedge B)}\ (\neg\wedge\text{L})$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A \vee B}\ (\vee\text{R}) \qquad \frac{\Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \vee B}\ (\vee\text{R}) \qquad \frac{A \Rightarrow \Delta \quad B \Rightarrow \Delta}{A \vee B \Rightarrow \Delta}\ (\vee\text{L})$$

$$\frac{\neg A, \Gamma \Rightarrow \Delta}{\neg(A \vee B), \Gamma \Rightarrow \Delta}\ (\neg\vee\text{L}) \qquad \frac{\neg B, \Gamma \Rightarrow \Delta}{\neg(A \vee B), \Gamma \Rightarrow \Delta}\ (\neg\vee\text{L})$$

$$\frac{\Gamma \Rightarrow \neg A \quad \Gamma \Rightarrow \neg B}{\Gamma \Rightarrow \neg(A \vee B)}\ (\neg\vee\text{R}) \qquad \frac{\Gamma \Rightarrow \neg A \quad \Gamma \Rightarrow \neg B}{A \vee B, \Gamma \Rightarrow}\ (\neg\vee\text{R})$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad A \Rightarrow \Sigma}{\Gamma \Rightarrow \Delta, \Sigma}\ (\text{cut1}) \qquad \frac{\Gamma \Rightarrow A \quad A, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow, \Sigma}\ (\text{cut2})$$

The full cut rule is not admissible in this calculus. If we add full cut rule to the **GMQL**, it will subsequently collapse to classical logic [19]. In **GMQL**, only (cut1) and (cut2) are admissible. They are the restricted form of the full cut. The meaning of a sequent in this calculus is different from **GO**. A sequent $\Gamma \Rightarrow \Delta$ in **GO** implies $\bigcap \|\Gamma\| \subseteq \bigcup \|\Delta\|$ in an O-model. On the other hand, a sequent $\Gamma \Rightarrow \Delta$ in **GMQL** should be interpreted as $\| \bigwedge \Gamma\| \subseteq \| \bigvee \Delta\|$. (Note that $\bigcup \|\Delta\|$ is not always $\perp$-closed.) Therefore, for **GMQL**, the definition of truth for a sequent is altered. The altered definition states that $\Gamma \Rightarrow \Delta$ is *false* at $x \in X$ in an O-model $(X, \perp, V)$ if for all formulas $A \in \Gamma$, $x \in \|A\|$, and $x \notin (\bigcap_{B \in \Delta} \|B\|^{\perp})^{\perp}$. If $\Gamma \Rightarrow \Delta$ is not *false* at x, then it is *true* at x. The definition of validity of a sequent is identical to that in **GO**.

Theorem 4.7.1 (Soundness and completeness theorem for **GMQL**)**.** *$\Gamma \Rightarrow \Delta$ is provable in* **GMQL** *iff $\Gamma \Rightarrow \Delta$ is valid in all O-models.*

Proof. See [20]. □

GMQL is now transformed to **GMQLM** and **GMQLMS**. **GMQLM** is constructed by adding a rule for orthomodular law to **GMQL**. **GMQLMS** is constructed by adding the rules for Sasaki arrow to **GMQLM**. In these calculi, Γ, Δ, Π and Σ are finite sets. **GMQLM** is defined by adding following rule to **GMQL**.

$$\frac{(C \vee \neg A) \wedge A \Rightarrow B}{C \Rightarrow \neg A \vee (A \wedge B)}\ (\text{OM2})$$

There are several ways to represent the orthomodular law. We can also use the rule (OM3) below for orthomodular law instead of (OM) or (OM2). These rules can be deduced from each other. (OM2) is adopted for **GMQLM** because of the symmetry of the rule.

$$\frac{A \Rightarrow B}{B \Rightarrow A \vee (\neg A \wedge B)}\ (\text{OM3})$$

Lemma 4.7.2. *(OM), (OM2) and (OM3) are mutually derivable in* **GOM***,* **GOMS***,* **GOMC***,* **GOMR** *and* **GMQLM***.*

Proof. (OM) $\Rightarrow$ (OM2).

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\textit{The proof in}[19]\ \textit{using}\ (OM)}{A \wedge (\neg A \vee (A \wedge \neg C)) \Rightarrow \neg C}}{\vdots}}{C \Rightarrow \neg A \vee (A \wedge (\neg A \vee C))}
\quad
\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A \wedge (\neg A \vee C) \Rightarrow A} \quad A \wedge (\neg A \vee C) \Rightarrow B}{A \wedge (\neg A \vee C) \Rightarrow A \wedge B}}{A \wedge (\neg A \vee C) \Rightarrow \neg A \vee (A \wedge B)} \quad \dfrac{\neg A \Rightarrow \neg A}{\neg A \Rightarrow \neg A \vee (A \wedge B)}}{\neg A \vee (A \wedge (\neg A \vee C)) \Rightarrow \neg A \vee (A \wedge B)}
}{C \Rightarrow \neg A \vee (A \wedge B)}
$$

(OM2) $\Rightarrow$ (OM3).

$$
\dfrac{
\dfrac{\dfrac{\dfrac{B \Rightarrow B \quad \dfrac{A \Rightarrow B}{\neg\neg A \Rightarrow B}}{B \vee \neg\neg A \Rightarrow B}}{(B \vee \neg\neg A) \wedge \neg A \Rightarrow B}}{B \Rightarrow \neg\neg A \vee (\neg A \wedge B)}\ (\mathrm{OM2})
\quad
\dfrac{\vdots}{\neg\neg A \vee (\neg A \wedge B) \Rightarrow A \vee (\neg A \wedge B)}
}{B \Rightarrow A \vee (\neg A \wedge B)}
$$

(OM3) $\Rightarrow$ (OM).

$$
\dfrac{
\dfrac{\neg B \Rightarrow \neg A}{\neg A \Rightarrow \neg B \vee (\neg A \wedge \neg\neg B))}\ (\mathrm{OM3})
\quad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg A, B \Rightarrow}{\neg A, \neg\neg B \Rightarrow}}{\neg A \wedge \neg\neg B, \neg\neg B \Rightarrow}}{\neg A \wedge \neg\neg B \Rightarrow}}{\neg A \wedge \neg\neg B \Rightarrow \neg B} \quad \neg B \Rightarrow \neg B}{\neg B \vee (\neg A \wedge \neg\neg B)) \Rightarrow \neg B}
}{\neg A \Rightarrow \neg B}
$$

$\square$

Note that the rule for $\vee$ below is derivable in **GOM**.

$$
\frac{A \Rightarrow C \quad B \Rightarrow C}{A \vee B \Rightarrow C}
$$

The proof of orthomodular law $A \wedge (\neg A \vee (A \wedge B))) \Rightarrow B$ in [19] is easily changed to the proof constructed by only **GMQL** rules + (OM).

GMQLMS is constructed by adding the next two rules to **GMQLM**.

$$
\frac{\Gamma \Rightarrow A \quad B, \Gamma \Rightarrow C}{\Gamma, A \rightarrow_S B \Rightarrow C}\ (\rightarrow_S \text{L'}) \qquad \frac{\Gamma_{P(A)}, A \Rightarrow B}{\Gamma \Rightarrow A \rightarrow_S B}\ (\rightarrow_S \text{R})
$$

If Δ has only one element, the meaning of a sequent $\Gamma \Rightarrow \Delta$ is identical to that in **GOMS**. Therefore, the soundness of these new rules have been proven by Theorem 4.4.2 because the sequent in ($\rightarrow_S$L') includes only one element on the right-side of it and the shapes of ($\rightarrow_S$L') and ($\rightarrow_S$ R) are identical to those in **GOMS**.

For the completeness theorem for **GMQLM**, a canonical model needs to be constructed similar to that in a canonical model in [20].

$X_c = \{\Gamma \mid \Gamma$ is a finite set of formulas and $\Gamma \Rightarrow$ is not provable in **GMQLM**$\}$.

$\perp_c = \{(x, y) \mid x \in X_c, y \in X_c$, for some A, $x \Rightarrow A$ is provable and $y \Rightarrow \neg A$ is provable in **GMQLM**$\}$.

$V_c(p) = \{x \in X_c \mid x \Rightarrow p$ is provable in **GMQLM**$\}$.

It has been already proven that $(X_c, \perp_c, V_c)$ is an O-model [20].

Lemma 4.7.3. *For any $x \in X_c$ and A, A is true at x iff $x \Rightarrow A$ is provable in* **GMQLM**

Proof. See [20]. (We take Ω in [20] as the set of all formulas.) □

Lemma 4.7.4. $(X_c, \perp_c, V_c)$ *is an OM-model.*

Proof. Suppose $A \land (\neg A \lor (A \land B))$ is true at $\Gamma \in X_c$. By Lemma 4.7.3, $\Gamma \Rightarrow A \land (\neg A \lor (A \land B))$ is provable in **GMQLM**. It follows then that $\Gamma \Rightarrow B$ is provable in **GMQLM** because $A \land (\neg A \lor (A \land B)) \Rightarrow B$ is provable in **GMQLM** as shown below. Double negation can be eliminated by cut.

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg B \lor \neg A \Rightarrow \neg B \lor \neg A}{(\neg B \lor \neg A) \land A \Rightarrow \neg B \lor \neg A}}{\neg B \Rightarrow \neg A \lor (A \land (\neg B \lor \neg A))} \quad \dfrac{\vdots}{\neg A \lor (A \land (\neg B \lor \neg A)) \Rightarrow \neg(A \land (\neg A \lor (A \land B)))}}{\neg B \Rightarrow \neg(A \land (\neg A \lor (A \land B)))}}{\neg\neg(A \land (\neg A \lor (A \land B))) \Rightarrow \neg\neg B}$$

$$\dfrac{\vdots}{A \land (\neg A \lor (A \land B)) \Rightarrow B}$$

Therefore, from Lemma 4.7.3, B is true at Γ. □

Theorem 4.7.5 (Soundness and completeness theorem for **GMQLM**)**.** *$\Gamma \Rightarrow \Delta$ is valid in all OM-models iff $\Gamma \Rightarrow \Delta$ is provable in* **GMQLM**.

Proof. Soundness of the rules in **GMQL** are proven in [20]. Since (OM2) is a restricted form of ($\rightarrow_S$ R), the soundness of rule (OM2) can be proven in an identical way to Theorem 4.4.2. For the completeness theorem, suppose $\Gamma \Rightarrow \Delta$ is not provable. Then, $\Gamma \Rightarrow$ is not provable and $\Gamma \in X_c$. From Lemma 4.7.3, $\Gamma \not\models \neg(\bigwedge_{B\in\Delta} \neg B)$ because if $\Gamma \Rightarrow \neg(\bigwedge_{B\in\Delta} \neg B)$ is provable, then $\Gamma \Rightarrow \Delta$ is provable because $\neg(\bigwedge_{B\in\Delta} \neg B) \Rightarrow \Delta$ is provable in **GMQL**. Therefore, $\Gamma \Rightarrow \Delta$ is false at $\Gamma \in X_c$. □

Theorem 4.7.6 (Soundness and completeness theorem for **GMQLMS**). *$\Gamma \Rightarrow \Delta$ is valid in all OM-models iff $\Gamma \Rightarrow \Delta$ is provable in* **GMQLMS**.

Proof. By Theorem 4.7.5 and the fact that $\neg A \vee (A \wedge B) \Rightarrow A \rightarrow_S B$ and $A \rightarrow_S B \Rightarrow \neg A \vee (A \wedge B)$ are provable in **GMQLMS**. The proofs in Lemma 4.4.3 can be modified to the proofs of **GMQLMS** using Lemma 4.7.2. □

Chapter 5

Labeled sequent calculi for quantum logic

In this chapter, we introduce labeled sequent calculi for orthologic and orthomodular logic with implications, and prove the cut-elimination theorem for some of them. The calculi in this chapter are listed in Table 5.1. Furthermore, we show that the provability problem of **LGOI** is decidable; this implies that the validity problem of orthologic with strict implication is decidable. A part of the contents of this chapter is based on the author's paper [16].

5.1 Introduction

In this chapter, labeled sequent calculi for orthologic and orthomodular logic are discussed, because as mentioned earlier, normal sequent calculi of quantum logic are incompatible with the cut-elimination theorem. Intuitively, this incompatibility is identical to the normal sequent calculus for the modal logic **S5** which also does not satisfy

	Name	Logic	Implication	Section
New labeled sequent calculi	**LGOI**	O[1]	$\rightarrow$[2]	5.2
	LGOM	OM[3]	$\rightarrow_S$[4]	5.3
	LGOE	O	$\twoheadrightarrow$[5]	5.4

[1] orthologic
[2] strict implication
[3] orthomodular logic
[4] Sasaki arrow
[5] entailment

Table 5.1: Labeled sequent calculi in this chapter

the cut-elimination theorem. Both quantum logic and modal logic **S5** are based on symmetric Kripke frames. We could fall into an infinite loop when we try to prove the cut-elimination theorem semantically for a sequent calculus of a logic based on symmetric frames. In these cases, we have to deal with the unprovability of multiple sequents.

A few types of sequent calculi for quantum logic that satisfy the cut-elimination theorem have been proposed. However, it is difficult to add implications to these calculi. For example, Faggian and Sambin [7] treat $\neg p$ ($= p \to \bot$ in Section 5.2) as an atomic formula. It is difficult to add new implication rules into this system because we have to consider $p \to \bot$ and $A \to B$ separately.

The labeled sequent calculi solve these problems. A labeled sequent is a sequent which includes notion of relations in Kripke frames. Labeled sequent calculi have been studied to establish the cut-elimination theorem for several logics [12] [17]. An infinite loop will not appear in a proof of the completeness theorem if we use labeled sequent because we only have to deal with the unprovability of one sequent.

Implications suitable for orthologic and orthomodular logic are adopted. We adopt the strict implication for orthologic because a strict implication includes only the notion of orthogonality and not the notion of projections. We adopt the Sasaki arrow for orthomodular logic because we already confirmed that Sasaki arrow is suitable for orthomodular logic in Section 2.4.

We define the infinite set of *labels* which is denoted by $\{a,b,c,\ldots\}$. A *labeled formula* is defined by $a : A$ where a is a label and A is a formula. $\Gamma, \Delta, \Sigma, \ldots$ are also used to denote the sets of labeled formulas. The label a *appears* in Γ if $a : A$ is included in Γ for some A. A *relational expression* is defined by $a(R)b$ where a and b are labels and R is a symbol. R depends on the contexts. A *labeled sequent* is defined by $\{R\}\Gamma \Rightarrow \Delta$ where $\{R\}$ is a set of relational expressions, all $\{R\}$, Γ, and Δ are finite sets. In this paper, two kinds of binary relations are used on labels derived from the semantics. The first relation is one for non-orthogonality. We say that two labels a and b have a non-orthogonal relationship in $\{R\}\Gamma \Rightarrow \Delta$ if $a \not\perp b \in \{R\}$. The second relation deals with the projection of a formula. We say that a label a is related to b by relation for the projection of A if $a(A?)b \in \{R\}$.

$\{R\} \cup \{a \not\perp b\}$ is denoted as $\{R, a \not\perp b\}$. We say the label a appears in a set $\{R\}$ if $a \not\perp b$, $a(A?)b$ or $b(A?)a$ are included in $\{R\}$ for some b and A. A set $\{R\}_L$ is defined as a set of labels that appears in $\{R\}$.

An *Infinite labeled sequent* is defined by $\{R\}\Gamma \Rightarrow \Delta$, where $\{R\}$, Γ, or Δ is an infinite set, and the remaining conditions are identical to the labeled sequent. We say a labeled sequent $\{R'\}\Gamma' \Rightarrow \Delta'$ is a *subsequent* of $\{R\}\Gamma \Rightarrow \Delta$ if $\{R\}' \subseteq \{R\}, \Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$.

5.2 Labeled sequent calculus LGOI

In this section, a labeled sequent calculus for orthologic that includes strict implication is discussed. $\{R\}$ should include only the relations for non-orthogonality. As was considered in Section 4.2, $\neg A$ is regarded as an abbreviation of $A \to \bot$ in this section.

The labeled sequent calculus **LGOI** is defined as follows. $\{R\}$, Γ and Δ are finite sets.

LGOI

Axiom:

$$\{R\}a : A \Rightarrow a : A \qquad \{R\}a : \bot \Rightarrow$$

Rules:

$$\frac{\{R\}\Gamma \Rightarrow \Delta, a : A \quad \{R\}\ a : A, \Gamma \Rightarrow \Delta}{\{R\}\Gamma \Rightarrow \Delta}\ (\text{cut})$$

$$\frac{\{R\}\Gamma \Rightarrow \Delta}{\{R\}a : A, \Gamma \Rightarrow \Delta}\ (\text{wL}) \qquad \frac{\{R\}\Gamma \Rightarrow \Delta}{\{R\}\Gamma \Rightarrow \Delta, a : A}\ (\text{wR})$$

$$\frac{\{R\}a : A, a : B, \Gamma \Rightarrow \Delta}{\{R\}a : A \wedge B, \Gamma \Rightarrow \Delta}\ (\wedge\text{L})$$

$$\frac{\{R\}\Gamma \Rightarrow \Delta, a : A \quad \{R\}\Gamma \Rightarrow \Delta, a : B}{\{R\}\Gamma \Rightarrow \Delta, a : A \wedge B}\ (\wedge\text{R})$$

$$\frac{\{R, a \not\perp b\}\Gamma \Rightarrow \Delta, b : A \quad \{R, a \not\perp b\}\ b : B, \Gamma \Rightarrow \Delta}{\{R, a \not\perp b\}\ a : A \to B, \Gamma \Rightarrow \Delta}\ (\to \text{L})$$

$$\frac{\{R, a \not\perp b\}\ b : A, \Gamma \Rightarrow \Delta, b : B}{\{R\}\Gamma \Rightarrow \Delta, a : A \to B}\ (\to \text{R})\ ^{(1)}$$

$$\frac{\{R, a \not\perp b\}\ b : \neg p, \Gamma \Rightarrow \Delta}{\{R\}\ \Gamma \Rightarrow \Delta, a : p}\ (\neg)\ ^{(1)}$$

$$\frac{\{R, a \not\perp a\}\Gamma \Rightarrow \Delta}{\{R\}\Gamma \Rightarrow \Delta}\ (\text{Ref})^{(2)} \qquad \frac{\{R, a \not\perp b, b \not\perp a\}\Gamma \Rightarrow \Delta}{\{R, a \not\perp b\}\Gamma \Rightarrow \Delta}\ (\text{Sym})$$

(1) In the rules ($\to$ R) and ($\neg$), b must not appear in $\{R\}$, Γ and Δ.

(2) In the rule (Ref), a must appear in $\{R\}$, Γ, or Δ.

An *embedding* of a labeled sequent $\{R\}\Gamma \Rightarrow \Delta$ in an O-model $(X, \perp, V)$ is a function E from $(\{R\}\Gamma \Rightarrow \Delta)_L$ to X that satisfies the following condition:

If $a \not\perp b \in \{R\}$, $E(a) = x$, and $E(b) = y$, then $x \not\perp y$.

A labeled sequent $\{R\}\Gamma \Rightarrow \Delta$ is *true* in $(X, \perp, V)$ under an embedding E if, whenever A is true at $E(a)$ for all $a : A \in \Gamma$, B is true at $E(b)$ for some $b : B \in \Delta$. A labeled sequent $\{R\}\Gamma \Rightarrow \Delta$ is *false* in $(X, \perp, V)$ under an embedding E if $\{R\}\Gamma \Rightarrow \Delta$ is not true in $(X, \perp, V)$ under E. A labeled sequent $\{R\}\Gamma \Rightarrow \Delta$ is *valid* in an O-model, if for all E, $\{R\}\Gamma \Rightarrow \Delta$ is true under E. The definitions of embeddings are slightly different in Section 5.2 and Section 5.3 because $\{R\}$ includes the relations for projections in Section 5.3.

Before the proofs of the soundness and completeness theorems, we make remarks on rule $(\neg)$. Unfortunately, rule $(\neg)$ does not satisfy the subformula property, where the property means that each labeled formula $a : A$ in the premise of a rule is a subformula of a labeled formula in the conclusion of the rule. Without this rule, it is difficult to make complete labeled sequent calculus (using $\not\perp$ as relations) of orthologic that satisfies the cut-elimination theorem for reasons stated below.

If $(\neg)$ is removed from **LGOI**-$\{(\text{cut})\}$, we cannot prove $\{R\}a : \neg\neg(p \wedge q) \Rightarrow a : p$, which is provable in **LGOI**-$\{(\text{cut})\}$. From the shape of the formulas in $\{R\}a : \neg\neg(p \wedge q) \Rightarrow a : p$ and the subformula property of **LGOI**-$\{(\text{cut}), (\neg)\}$, if $\{R\}a : \neg\neg(p \wedge q) \Rightarrow a : p$ is provable, then the axioms of the proof must be $\{R'\}a : p \Rightarrow a : p$. However, we can check that regardless of how we reverse the deduction from $\{R\}a : \neg\neg(p \wedge q) \Rightarrow a : p$, we cannot reach $\{R'\}a : p \Rightarrow a : p$. Therefore, this sequent is not provable in **LGOI**-$\{(\text{cut}), (\neg)\}$. If we can use $(\neg)$, then $\{R\}a : \neg\neg(p \wedge q) \Rightarrow a : p$ is provable, as shown below.

$$\dfrac{\dfrac{\dfrac{\{a \not\perp b, b \not\perp c\}c : p \Rightarrow c : p}{\{a \not\perp b, b \not\perp c\}c : p, c : q \Rightarrow c : p}\,(\text{wL})}{\{a \not\perp b, b \not\perp c\}c : p \wedge q \Rightarrow c : p}\,(\wedge\text{L}) \qquad \dfrac{\{a \not\perp b, b \not\perp c\}c : \perp \Rightarrow}{\{a \not\perp b, b \not\perp c\}c : \perp, c : p \wedge q \Rightarrow}\,(\text{wL})}{\{a \not\perp b, b \not\perp c\}c : p \wedge q, b : \neg p \Rightarrow}\,(\to\text{L})$$

$$\vdots$$

$$\dfrac{\dfrac{\dfrac{}{\{a \not\perp b, b \not\perp c\}c : p \wedge q, b : \neg p \Rightarrow c : \perp}\,(\text{w R})}{\{a \not\perp b\}b : \neg p \Rightarrow b : \neg(p \wedge q)}\,(\to\text{R}) \qquad \dfrac{\{a \not\perp b\}b : \perp \Rightarrow}{\{a \not\perp b\}b : \perp, b : \neg p \Rightarrow}\,(\text{w L})}{\dfrac{\{a \not\perp b\}a : \neg\neg(p \wedge q), b : \neg p \Rightarrow}{\{\}a : \neg\neg(p \wedge q), \Rightarrow a : p}\,(\neg)}\,(\to\text{L})$$

Furthermore, $\{R\}a : \neg\neg(p \wedge q) \Rightarrow a : p$ has no proof in **LGOI**-$\{(\text{cut})\}$ that all deductions are subformulaic. Because we have to use rule $(\neg)$ to prove this sequent, $\neg p$ must appear somewhere in the proof. However, as $\neg p$ is not a subformula of $\neg\neg(p \wedge q)$ or p, there exist some deductions that are not subformulaic in the proof of $\{R\}a : \neg\neg(p \wedge q) \Rightarrow a : p$.

In the proof of the cut-free completeness theorem, we need rule $(\neg)$ to establish the falsity of p in a canonical model, as we will show in the proof of Lemma 5.2.2. The falsity of p depends on not only p but also $\neg p$ because of the nature of a $\perp$-closed set; that is, in an O-model $(X, \perp, V)$, p is false at $x \in X$ if and only if there exists $y \in X$ such that $y \in \|\neg p\|$ and $x \not\perp y$.

Moreover, we can show that rule $(\neg)$ corresponds to condition "$\perp$-closed" of O-models. We define a *pseudo-O-model* as $(X, \perp, V')$, where the definitions of X and $\perp$ are the same as those of an O-model, and V' is a function that assigns each propositional variable p to a subset of X; that is, $V'(p)$ does not need to be a $\perp$-closed set. Then, **LGOI**-$\{$*(cut), $(\neg)$*$\}$ *is sound and complete with respect to pseudo-O-models* (Theorem 5.2.5).

To conclude, rule $(\neg)$ is necessary to prove the cut-free completeness theorem; however, it destroys the subformula property. Thus, a natural question arises: *Does a labeled sequent calculus for orthologic that satisfies the subformula property exist*? This is an interesting open problem.

Now, we prove the soundness theorem.

Theorem 5.2.1 (Soundness theorem for **LGOI**). *If $\{R\}\Gamma \Rightarrow \Delta$ is provable in* **LGOI**, *then $\{R\}\Gamma \Rightarrow \Delta$ is valid in all O-models.*

Proof. We prove this by induction on the construction of the proof of sequent $\{R\}\Gamma \Rightarrow \Delta$. We only prove the cases in which the last rule used in the proof is $(\rightarrow \text{L})$, $(\rightarrow \text{R})$, (Ref), or (Sym). The proofs for the other cases are similar. The proof for $(\neg)$ is the similar to that for $(\rightarrow \text{R})$ because $\|p\| = \|\neg\neg p\| = \|\neg p \rightarrow \perp\|$.

First, we show the case in which the last rule is $(\rightarrow \text{L})$. Suppose that $\{R, a \not\perp b\}a : A \rightarrow B, \Gamma \Rightarrow \Delta$ is false in an O-model $(X, \perp, V)$ under embedding E. Then, A is false at $E(b)$ or B is true at $E(b)$. Therefore, $\{R, a \not\perp b\}\Gamma \Rightarrow \Delta, b : A$ is false under E or $\{R, a \not\perp b\}b : B, \Gamma \Rightarrow \Delta$ is false under E.

Next, we show the case in which the last rule is $(\rightarrow \text{R})$. For the sake of contradiction, suppose that $\{R, a \not\perp b\}b : A, \Gamma \Rightarrow \Delta, b : B$ is valid and that $\{R\}\Gamma \Rightarrow \Delta, a : A \rightarrow B$ is false in $(X, \perp, V)$ under E. Then there exists $x \in X$ such that $E(a) \not\perp x$ and A is true at x but B is not true at x. If we define E' as $E \cup \{(b, x)\}$, then

$\{R, a \not\perp b\} b : A, \Gamma \Rightarrow \Delta, b : B$ is false in $(X, \perp, V)$ under E'. This is a contradiction. Therefore, $\{R\}\Gamma \Rightarrow \Delta, a : A \to B$ is valid.

Next, we show the case in which the last rule is (Ref). Because relation $\not\perp$ in an O-model is reflexive and a appears in $\{R\}\Gamma \Rightarrow \Delta$, $\{R, a \not\perp a\}\Gamma \Rightarrow \Delta$ and $\{R\}\Gamma \Rightarrow \Delta$ have the same embeddings. Therefore, if $\{R, a \not\perp a\}\Gamma \Rightarrow \Delta$ is valid, then $\{R\}\Gamma \Rightarrow \Delta$ is also valid.

Next, we show the case in which the last rule is (Sym). Because the relation $\not\perp$ in an O-model is symmetric, $\{R, a \not\perp b, b \not\perp a\}\Gamma \Rightarrow \Delta$ and $\{R, a \not\perp b\}\Gamma \Rightarrow \Delta$ have the same embeddings. Therefore, if $\{R, a \not\perp b, b \not\perp a\}\Gamma \Rightarrow \Delta$ is valid, then $\{R, a \not\perp b\}\Gamma \Rightarrow \Delta$ is also valid. $\square$

For the completeness theorem, the contraposition of the theorem is proven. In other words, if the labeled sequent $\{R\}\Gamma \Rightarrow \Delta$ is not provable, then a model $(X, \perp, V)$ exists with an embedding E of $\{R\}\Gamma \Rightarrow \Delta$ to $(X, \perp, V)$ and $\{R\}\Gamma \Rightarrow \Delta$ is false in $(X, \perp, V)$ under E.

A canonical model $(X_c, \perp_c, V_c)$ is constructed from an unprovable sequent $\{R\}\Gamma \Rightarrow \Delta$ by creating new labels and relations while preserving unprovability. Changes in the sequent are denoted by $\{R\}_0\Gamma_0 \Rightarrow \Delta_0 (= \{R\}\Gamma \Rightarrow \Delta), \{R\}_1\Gamma_1 \Rightarrow \Delta_1, \ldots, \{R\}_i\Gamma_i \Rightarrow \Delta_i, \{R\}_{i+1}\Gamma_{i+1} \Rightarrow \Delta_{i+1}, \ldots$.

Suppose $\{R\}\Gamma \Rightarrow \Delta$ is not provable.

- Step 1:

  A new sequent is formed using an iterative procedure until the sequent remains unchanged.

  - If $a : (A \wedge B) \in \Gamma_i$, then we make $\Gamma_{i+1} = \Gamma_i \cup \{a : A, a : B\}$, $\Delta_{i+1} = \Delta_i$ and $\{R\}_{i+1} = \{R\}_i$. $\{R\}_{i+1}\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ is not provable because of the rule ($\wedge$L).
  - If $a : (A \wedge B) \in \Delta_i$, at least one of $\{R\}_i\Gamma_i \Rightarrow \Delta_i, a : A$ and $\{R\}_i\Gamma_i \Rightarrow \Delta_i, a : B$ is not provable because of the rule ($\wedge$R). Therefore, $\Delta_{i+1} = \Delta_i \cup \{a : A\}$ or $\Delta_i \cup \{a : B\}$ is made which preserves unprovability. In both cases, $\Gamma_{i+1} = \Gamma_i$ and $\{R\}_{i+1} = \{R\}_i$ are created.
  - If $a : (A \to B) \in \Gamma_i$ and $a \not\perp b \in \{R\}$, at least one of $\{R\}_i\Gamma_i \Rightarrow \Delta_i, b : A$ and $\{R\}_i\Gamma_i, b : B \Rightarrow \Delta_i$ is not provable because of ($\to$L). Therefore, we make $\Gamma_{i+1} = \Gamma_i \cup \{b : B | a \not\perp b \in \{R\}$ and $\{R\}_i\Gamma_i, b : B \Rightarrow \Delta_i$ is not provable$\}$ and $\Delta_{i+1} = \Delta_i \cup \{b : A | a \not\perp b \in \{R\}$ and $\{R\}_i\Gamma_i \Rightarrow \Delta_i, b : A$ is not provable $\}$. In each case, $\{R\}_{i+1} = \{R\}_i$ is created.

– If $a : (A \to B) \in \Delta_i$ and if this procedure is not executed to this formula, then a label b that does not appear in $\{R\}_i \Gamma_i \Rightarrow \Delta_i$ is selected and $\Gamma_{i+1} = \Gamma_i \cup \{b : A\}$, $\Delta_{i+1} = \Delta_i \cup \{b : B\}$ and $\{R\}_{i+1} = \{R, a \not\perp b, b \not\perp b\}_i$ are made. $\{R\}_{i+1}\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ is not provable because of the rule ($\to$R).

Note that only one label b for $a : (A \to B) \in \Delta_i$ can be added due to the second condition. Since all of the procedures decrease the complexity of the formulas, at some point, this step ends. Consequently, Step 2 is followed.

- Step 2:

  A new sequent is formed by applying an iterative procedure until the sequent remains unchanged.

  – If $a : p \in \Delta_i$ and if this procedure is not executed to this formula, then a label b is selected that does not appear in $\{R\}_i \Gamma_i \Rightarrow \Delta_i$ and we make $\Gamma_{i+1} = \Gamma_i \cup \{b : \neg p\}$, $\Delta_{i+1} = \Delta_i$ and $\{R\}_{i+1} = \{R, a \not\perp b, b \not\perp b\}_i$. $\{R\}_{i+1}\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ is not provable due to the rule ($\neg$).

  – If label a appears in $\{R\}_i \Gamma_i \Rightarrow \Delta_i$ and if $a \not\perp a$ is not included in $\{R\}_i$, then we define $\{R\}_{i+1} = \{R, a \not\perp a\}_i$, $\Gamma_{i+1} = \Gamma_i$ and $\Delta_{i+1} = \Delta_i$. Sequent $\{R\}_{i+1}\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ is not provable because of rule (Ref).

  – If $a \not\perp b$ is included in $\{R\}_i$ and if $b \not\perp a$ is not included in $\{R\}_i$, then we define $\{R\}_{i+1} = \{R, b \not\perp a\}_i$, $\Gamma_{i+1} = \Gamma_i$ and $\Delta_{i+1} = \Delta_i$. Sequent $\{R\}_{i+1}\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ is not provable because of rule (Sym).

  At some point, this step ends because there are a finite number of labels and propositional variables in $\{R\}_i \Gamma_i \Rightarrow \Delta_i$. Subsequently, Step 1 is revisited because new labels and relations are formed in Step 2.

We say an infinite labeled sequent is not provable if all finite subsequents are not provable. The infinite labeled sequent $\{R\}_c \Gamma_c \Rightarrow \Delta_c$ is formed by $\{R\}_c = \bigcup_{i=0}^{\infty} \{R\}_i$, $\Gamma_c = \bigcup_{i=0}^{\infty} \Gamma_i$ and $\Delta_c = \bigcup_{i=0}^{\infty} \Delta_i$. In other words, this sequent is the iterative limit of a repetitive process: Step 1 $\to$ Step 2 $\to$ Step 1 $\to$ Step 2 $\to \ldots$. The procedure described above confirms that this infinite labeled sequent is not provable.

The canonical model $(X_c, \perp_c, V_c)$ of $\{R\}\Gamma \Rightarrow \Delta$ is defined below.

$X_c = \{a \mid a \text{ appears in } \{R\}_c\}$

$\perp_c = \{a \perp b \mid a \not\perp b \notin \{R\}_c\}$

$V_c(p) = \{a \mid a : p \in \Gamma_c\}^{\perp\perp}$

From Lemma 2.3.1 and the fact that $\|A \to B\|$ is $\perp - closed$ [5], $\|A\|$ is $\perp$-closed for all A in $(X_c, \perp_c, V_c)$. From Step 2, $\perp_c$ is irreflexive and symmetric. Therefore, $(X_c, \perp_c, V_c)$ is actually an O-model. The embedding E_c of $\{R\}\Gamma \Rightarrow \Delta$ to $(X_c, \perp_c, V_c)$ is defined by $E_c(a) = a$.

Lemma 5.2.2. *If $a : A \in \Gamma_c$, then A is true at $a \in X_c$. If $a : A \in \Delta_c$, then A is false at $a \in X_c$. Therefore, $\{R\}\Gamma \Rightarrow \Delta$ is false in $(X_c, \perp_c, V_c)$ under E_c.*

Proof. Proven by induction on the construction of the labeled formulas in Γ and Δ. Suppose $a : p \in \Gamma$. From the definition of V_c and Lemma 2.3.1, $a \in \|p\|$. Suppose $a : p \in \Delta$. From the construction of $\{R\}_c\Gamma_c \Rightarrow \Delta_c$, there exists a label b such that $b : \neg p \in \Gamma_c$ and $a \not\perp b \in \{R\}_c$. If a label c exists that satisfies $b \not\perp c \in \{R\}_c$ and $c : p \in \Gamma_c$, then $\{R\}_i\Gamma_i \Rightarrow \Delta_i$ is provable in some i because $\{R, b \not\perp c\}\Gamma \cup \{b : \neg p, c : p\} \Rightarrow \Delta$ is provable. This is a contradiction. Consequently, if $b : \neg p \in \Gamma_c$, then $b \in \{a \mid a : p \in \Gamma_c\}^{\perp} = \{a \mid a : p \in \Gamma_c\}^{\perp\perp\perp} = \|\neg p\|$ by the definition of V_c and Lemma 2.3.1. Therefore, $a \notin \|p\|$.

In additional cases, easy confirmation occurs if $a : A \in \Gamma_c$, then A is true at $a \in X_c$ and if $a : A \in \Delta_c$, then A is false at $a \in X_c$ from Step 1 in the construction of $\{R\}_c\Gamma_c \Rightarrow \Delta_c$. Now consider an isolated case where $a : A \to B \in \Delta_c$. From Step1, a label b exists such that $a \not\perp b \in \{R\}_c$, $b : A \in \Gamma_c$ and $b : B \in \Delta_c$. From the inductive hypothesis, A is true at $b \in X_c$ and B is false at $b \in X_c$. Therefore, $A \to B$ is false at $a \in X_c$. □

Theorem 5.2.3 (Completeness theorem for **LGOI**). *If $\{R\}\Gamma \Rightarrow \Delta$ is valid in all O-models, then $\{R\}\Gamma \Rightarrow \Delta$ is provable in* **LGOI**.

Proof. From Lemma 5.2.2, if $\{R\}\Gamma \Rightarrow \Delta$ is not provable in **LGOI**, there exists an O-model $(X, \perp, V)$ and an embedding E such that $\{R\}\Gamma \Rightarrow \Delta$ is false in $(X, \perp, V)$ under E. □

Theorem 5.2.4 (Cut-elimination theorem for **LGOI**). *If $\{R\}\Gamma \Rightarrow \Delta$ is provable in* **LGOI**, *there exists a proof of $\{R\}\Gamma \Rightarrow \Delta$ that does note include the rule (cut).*

Proof. The completeness theorem was proven without the rule (cut). Therefore, the provability of a sequent is independent of the existence of (cut) in the calculus. □

The sequent $A, B \Rightarrow \neg(C \wedge \neg(A \wedge B))$ is not provable without (cut) in **GO** [19]. This sequent is now provable without (cut) in **LGOI** as shown below:

$$\dfrac{\{a \not\perp b, b \not\perp a\}a : A \Rightarrow a : A}{\{a \not\perp b, b \not\perp a\}a : A, a : B, b : C \Rightarrow a : A, b : \bot}\ (\text{wL}) \times 2,\ (\text{wR}) \times 1$$

$$\vdots$$

$$\dfrac{\vdots \qquad \dfrac{\{a \not\perp b, b \not\perp a\}a : B \Rightarrow a : B}{\{a \not\perp b, b \not\perp a\}a : A, a : B, b : C \Rightarrow a : B, b : \bot}\ (\text{wL}) \times 2,\ (\text{wR}) \times 1}{\{a \not\perp b, b \not\perp a\}a : A, a : B, b : C \Rightarrow a : A \wedge B, b : \bot}\ (\wedge\text{R})$$

$$\vdots$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots \qquad \dfrac{\{a \not\perp b, b \not\perp a\}a : \bot \Rightarrow}{\{a \not\perp b, b \not\perp a\}a : A, a : B, a : \bot, b : C \Rightarrow b : \bot}\ (\text{wL}) \times 3,\ (\text{wR}) \times 1}{\{a \not\perp b, b \not\perp a\}a : A, a : B, b : C, b : \neg(A \wedge B) \Rightarrow b : \bot}\ (\rightarrow\text{L})}{\{a \not\perp b, b \not\perp a\}a : A, a : B, b : C \wedge \neg(A \wedge B) \Rightarrow b : \bot}\ (\wedge\text{L})}{\{a \not\perp b\}a : A, a : B, b : C \wedge \neg(A \wedge B) \Rightarrow b : \bot}\ (\text{Sym})}{\{\}a : A, a : B \Rightarrow a : \neg(C \wedge \neg(A \wedge B))}\ (\rightarrow\text{R})$$

Theorem 5.2.5 (Soundness and completeness theorem for **LGOI**-$\{$(cut), $(\neg)\}$). *$\{R\}\Gamma \Rightarrow \Delta$ is provable in* **LGOI**-$\{$*(cut), $(\neg)$*$\}$ *iff $\{R\}\Gamma \Rightarrow \Delta$ is valid in all pseudo-O-models.*

Proof. For the soundness theorem, because we did not use the notion of a $\bot$-closed set in the proof of Theorem 5.2.1 except for rule $(\neg)$, we can use the same proof.

For the completeness theorem, we can make a canonical model for $\{R\}\Gamma \Rightarrow \Delta$ by applying all procedures in Step 1 and procedures for (Ref) and (Sym) in Step 2. We use $\{a \mid a : p \in \Gamma_c\}$ for $V'_c(p)$. Then, we can use almost the same proof as that of Lemma 5.2.2; however, it is much easier because we do not need to be concerned about $\bot$-closedness of sets. $\Box$

We can now prove that the validity problem for orthologic with strict implication is decidable using the completeness theorem. The provability of labeled sequent $\{\} \Rightarrow a : A$ can be determined by checking whether a canonical model for $\{\} \Rightarrow a : A$ exists. We can construct a canonical model that is essentially equivalent to $(X_c, \bot_c, V_c)$ in a finite number of steps. We execute the same procedure to construct $\{R\}_i\Gamma_i \Rightarrow \Delta_i$ but we stop the procedure at some i.

Suppose that label b is included in $\{R\}_i$ in the procedure for constructing a canonical model for $\{\} \Rightarrow a : A$. The *degree* of label b is $n(\geq 2)$ in $\{R\}_i$ if there exist labels $c_1, c_2, \ldots, c_{n-1}$ such that $a \not\perp c_1, c_1 \not\perp c_2, \ldots, c_{n-1} \not\perp b$ are included in $\{R\}_i$ and $a, b, c_1, c_2, \ldots, c_{n-1}$

are distinct labels. We define the degree of a as 0, and if $a \not\leq b \in \{R\}_i$, then we define the degree of $b(\neq a)$ as 1. The degree of a label is uniquely determined because we can confirm that there is no confluence (or loop) such as $d_0 \not\leq d_1, d_1 \not\leq d_2, \ldots, d_{m-1} \not\leq d, d_0 \not\leq e_1, e_1 \not\leq e_2, \ldots, e_{n-1} \not\leq d$ $(m \neq n)$ in $\{R\}_i$, where d_0,d_1,…,d_{m-1},d, e_1,…,e_{n-1} are distinct labels. Furthermore, the degree is well-defined for all labels included in $\{R\}_i$ because any new label added in Step 1 or Step 2 is related to a label already included in $\{R\}_i$.

The set $\Gamma(b)$ is defined as $\Gamma(b) = \{A|b : A \in \Gamma\}$. Because the number of subformulas of A is finite and the number of negations of propositional variables that appears in A is also finite, $|\Gamma_i(b) \cup \Delta_i(b)|$ is also finite. Furthermore, from the same reasoning, there exists a smallest k such that if $k \leq i$, then $|\Gamma_i(b) \cup \Delta_i(b)| = |\Gamma_k(b) \cup \Delta_k(b)|$. We say that b is already maximal in $\{R\}_i\Gamma_i \Rightarrow \Delta_i$ if i is larger than such k.

Theorem 5.2.6. *There is an effective procedure that determines whether formula A is valid in O-models.*

Proof. We show that we can stop the procedure to construct a canonical model for $\{\} \Rightarrow a : A$.

Suppose that the complexity of A is n. If the degree of b is greater than n in $\{R\}_i$, and if $b : B$ is included in Γ_i or Δ_i, then B is a propositional variable or the negation of a propositional variable because all procedures in Step 1 decrease the complexity of formulas, and Step 2 only creates the negation of propositional variables.

Suppose that the degree of b is n, $\Gamma_i(b) = \{p_1, p_2, \ldots, p_t, \neg q_1, \neg q_2, \ldots, \neg q_u\}$, $\Delta_i(b) = \{r_1, r_2, \ldots, r_v, \neg s_1, \neg s_2, \ldots, \neg s_w\}$, and b is already maximal. Note that all q_m are included in $\{r_1, r_2, \ldots, r_v\}$ because b is maximal. In Step 1, we have to add $b \not\leq c_m (1 \leq m \leq w)$ to $\{R\}_i$ for all m and add $c_m : s_m (1 \leq m \leq w)$ to Γ_i for all m. In Step 2, we have to add $b \not\leq d_m (1 \leq m \leq v)$ to $\{R\}_i$ for all m and add $d_m : \neg r_m (1 \leq m \leq v)$ to Γ_i for all m. All c_m and d_m are new labels. Furthermore, in Step 1, we have to add $q_m (1 \leq m \leq u)$ to Δ_i labeled by $c_m (1 \leq m \leq w)$ and $d_m (1 \leq m \leq v)$, and we also have to add $d_m : r_m (1 \leq m \leq v)$ to Δ_i. This new sequent is denoted by $\{R\}_j\Gamma_j \Rightarrow \Delta_j$.

We prove that we can stop the process for labels c_m and d_m. Although all $c_m : q_m \in \Delta_j$ and all $d_m : q_m \in \Delta_j$ request new labels in Step 2, we do not need new labels because label b already has its role; that is, because $b : \neg q_m \in \Gamma_j$ and $b \not\leq c_m \in \{R\}_j$, and because of the procedure for making symmetric relations in Step 2, the existence of b is sufficient to prove Lemma 5.2.2. Similarly, although $d_m : r_m \in \Delta_j$ requests a new label in Step 2, we do not need a new label because label d_m already has its role.

Therefore, a label with a degree greater than $n+2$ is not needed. Furthermore, the number of labels related to a given label is finite because, in the above procedure, one labeled formula can create at most one relation under the conditions of the procedure. Therefore, we can stop the procedure from constructing a canonical model in a finite number of steps without any loss of information.

Although there are multiple possibilities that depend on the choice of Step 1 for $A \wedge B \in \Delta_i$ and $A \to B \in \Gamma_i$, because the number of formulas appearing in $a : A$ is finite, the number of procedures is also finite.

If one of these finite procedures creates a canonical model, then we conclude that $\{\} \Rightarrow a : A$ is not provable and A is not valid. If these finite procedures does not create a canonical model, then we conclude that $\{\} \Rightarrow a : A$ is provable and A is valid. $\square$

For example, when we construct a canonical model for $\{\} \Rightarrow a : (q \to p)$, the procedure in this paper adds a number of $b_n : p$ in Δ_i (by Step 1), $b_n : \neg p$ in Γ_i (by Step 2), and $b_{n-1} \not\leq b_n$ in $\{R\}_i$ after some steps. We can easily check that more than four labels are redundant.

$\{\} \Rightarrow a : (q \to p)$

$\downarrow$ Step 1

$\{a \not\leq b_1\} b_1 : q \Rightarrow a : (q \to p), b_1 : p$

$\downarrow$ Step 2

$\{a \not\leq a, b_1 \not\leq b_1, b_2 \not\leq b_2, a \not\leq b_1, b_1 \not\leq a, b_1 \not\leq b_2, b_2 \not\leq b_1\} b_2 : \neg p, b_1 : q \Rightarrow a : (q \to p), b_1 : p$

$\downarrow$ Step 1 and Step 2 (Processing from here is superfluous).

$\{a \not\leq a, b_1 \not\leq b_1, b_2 \not\leq b_2, b_3 \not\leq b_3, a \not\leq b_1, b_1 \not\leq a, b_1 \not\leq b_2, b_2 \not\leq b_1, b_2 \not\leq b_3, b_3 \not\leq b_2\} b_3 : \neg p, b_2 : \neg p, b_1 : q \Rightarrow a : (q \to p), b_1 : p, b_2 : p$

$\downarrow$

$\ldots$

Theorem 5.2.7. LGOI *is decidable. That is, there is an effective procedure that determines whether labeled sequent* $\{R\}\Gamma \Rightarrow \Delta$ *is provable in* **LGOI**.

Proof. We can use almost the same method as that in the proof of Theorem 5.2.6; that is, we can stop the procedure from constructing a canonical model for $\{R\}\Gamma \Rightarrow \Delta$ in the same manner. We apply the notion of degree to all labels that appear in $\{R\}$. We say that the degree of label b for $a \in \{R\}_L$ is $n(\geq 2)$ in $\{R\}_i$ if there exist labels $c_1, c_2, \ldots, c_{n-1}$ such that $a \not\perp c_1, c_1 \not\perp c_2, \ldots, c_{n-1} \not\perp b$ are included in $\{R\}_i$, $a, b, c_1, c_2, \ldots, c_{n-1}$ are distinct labels, and $b, c_1, c_2, \ldots, c_{n-1}$ do not appear in $\{R\}$. We define the degree of $a \in \{R\}_L$ as 0, and if $a \not\perp b \in \{R\}_i$ and $b \notin \{R\}_L$, then we define the degree of b as 1.

Because the number of formulas that appear in $\Gamma \cup \Delta$ is finite and the number of labels that appear in $\{R\}$ is also finite, we can use the same argument as that in the proof of Theorem 5.2.6. Although there is a possibility that there is a confluence or loop in $\{R\}$, because the procedure does not make a confluence or loop, and because of the definition of degree, we can confirm that the proof is not affected by these conditions.

If one of these finite procedures creates a canonical model of $\{R\}\Gamma \Rightarrow \Delta$, then we conclude that $\{R\}\Gamma \Rightarrow \Delta$ is not provable. If these finite procedures do not create a canonical model of $\{R\}\Gamma \Rightarrow \Delta$, then we conclude that $\{R\}\Gamma \Rightarrow \Delta$ is provable. $\square$

5.3 Labeled sequent calculus LGOM

In this section, a labeled sequent calculus for orthomodular logic is discussed. The Sasaki arrow $A \rightarrow_S B$ is adopted for implication.

The easiest way to construct a labeled sequent calculus for orthomodular logic is to use the axiom of orthomodular law and regard $A \rightarrow_S B$ as an abbreviation of $\neg A \vee (A \wedge B)$. The labeled sequent calculus **qLGOM** is defined as **LGOI** + {the new axiom defied below }.

Axiom:

$$\{R\}a : A, a : \neg A \vee (A \wedge B) \Rightarrow a : B$$

The definition of a labeled sequent and an embedding are identical to the case of **LGOI**.

Theorem 5.3.1 (Completeness theorem for **qLGOM**)**.** *If $\Gamma \Rightarrow \Delta$ is provable in* **qLGOM***, then $\Gamma \Rightarrow \Delta$ is valid in all TOM-models.*

Proof. Suppose $\{R\}\Gamma \Rightarrow \Delta$ is not provable. By using the method in Section 5.2 and following step, canonical model for $\{R\}\Gamma \Rightarrow \Delta$ can be created.

For all A, if a appears in $\{R\}_i\Gamma_i \Rightarrow \Delta_i$, $a : A \notin \Gamma_i$ and $a : A \notin \Delta_i$, at least one of $\{R\}_i a : A, \Gamma_i \Rightarrow \Delta_i$ and $\{R\}_i\Gamma_i \Rightarrow \Delta_i, a : A$ is not provable because of the rule (cut). Therefore, $\{R\}_{i+1}\Gamma_{i+1} \Rightarrow \Delta_{i+1} = \{R\}_i\Gamma_i \cup a : A \Rightarrow \Delta_i$ or $\{R\}_{i+1}\Gamma_{i+1} \Rightarrow \Delta_{i+1} = \{R\}_i\Gamma_i \Rightarrow \Delta_i \cup a : A$ is created which preserves unprovability.

It is easy to confirm that this model satisfies orthomodular law because of the axiom. □

However, because of the axiom for orthomodular law, we cannot confirm that this calculus satisfy the cut-elimination theorem. Therefore, we adopt the concept of DOM-model and attempt to construct a new labeled sequent calculus for quantum logic. By using Theorem 3.2.5, we can express the orthomodular law with simple expression in $\{R\}$. As mentioned earlier, the definition of a labeled sequent and embedding is slightly modified. $\{R\}$ may include both non-orthogonal relations and the relations for projections. An embedding of a labeled sequent $\{R\}\Gamma \Rightarrow \Delta$ to a DOM-model $(X, \perp, R, V)$ is a function E from $\{R\}_L$ to X satisfying the following conditions:

If $a \not\perp b \in \{R\}$, $E(a) = x$, and $E(b) = y$, then $x \not\perp y$.

If $a(A?)b \in \{R\}$, $E(a) = x$, $E(b) = y$, and $\|A\| = Y$, then $x(Y?)y$.

The labeled sequent calculus **LGOM** is defined below. $\{R\}$, Γ, and Δ are finite sets.

LGOM

Axiom:

$$\{R\}a : A \Rightarrow a : A \qquad \{R\}a : \perp \Rightarrow$$

Rules:

$$\frac{\{R\}\Gamma \Rightarrow \Delta, a : A \quad \{R\}\ a : A, \Gamma \Rightarrow \Delta}{\{R\}\Gamma \Rightarrow \Delta}\ (\text{cut})$$

$$\frac{\{R\}\Gamma \Rightarrow \Delta}{\{R\}a : A, \Gamma \Rightarrow \Delta}\ (\text{wL}) \qquad \frac{\{R\}\Gamma \Rightarrow \Delta}{\{R\}\Gamma \Rightarrow \Delta, a : A}\ (\text{wR})$$

$$\frac{\{R\}a : A, a : B, \Gamma \Rightarrow \Delta}{\{R\}a : A\wedge B, \Gamma \Rightarrow \Delta}\ (\wedge\text{L})$$

$$\frac{\{R\}\Gamma \Rightarrow \Delta, a:A \quad \{R\}\Gamma \Rightarrow \Delta, a:B}{\{R\}\Gamma \Rightarrow \Delta, a:A \wedge B} \ (\wedge\mathrm{R})$$

$$\frac{\{R, a \not\perp b\}\ \Gamma \Rightarrow \Delta, a:A}{\{R, a \not\perp b\}\ b:\neg A, \Gamma \Rightarrow \Delta} \ (\neg\ \mathrm{L})$$

$$\frac{\{R, a \not\perp b\}\ b:A, \Gamma \Rightarrow \Delta}{\{R\}\ \Gamma \Rightarrow \Delta, a:\neg A} \ (\neg\ \mathrm{R})\ ^{(1)}$$

$$\frac{\{R, a(A?)b\}\ b:B, \Gamma \Rightarrow \Delta}{\{R, a(A?)b\}\ a:A \rightarrow_S B, \Gamma \Rightarrow \Delta} \ (\rightarrow\ \mathrm{L})$$

$$\frac{\{R, a(A?)b\}\ b:A, \Gamma \Rightarrow \Delta, b:B}{\{R\}\Gamma \Rightarrow \Delta, a:A \rightarrow_S B} \ (\rightarrow\ \mathrm{R})\ ^{(1)}$$

$$\frac{\{R, a \not\perp b\}\ b:\neg A, \Gamma \Rightarrow \Delta}{\{R\}\ \Gamma \Rightarrow \Delta, a:A} \ (\neg)\ ^{(1)}$$

$$\frac{\{R, a(A?)b, b \not\perp c, c(A?)d, d \not\perp a\}\Gamma \Rightarrow \Delta}{\{R, a(A?)b, b \not\perp c\}\ \Gamma \Rightarrow \Delta} \ (\mathrm{OM})^{(1)}$$

$$\frac{\{R, a \not\perp a\}\Gamma \Rightarrow \Delta}{\{R\}\Gamma \Rightarrow \Delta} \ (\mathrm{Ref})^{(2)} \qquad \frac{\{R, a \not\perp b, b \not\perp a\}\Gamma \Rightarrow \Delta}{\{R, a \not\perp b\}\Gamma \Rightarrow \Delta} \ (\mathrm{Sym})$$

(1) In the rules ($\neg$R), ($\rightarrow$R) and ($\neg$), b must not appear in the under sequent. In the rule (OM), d must not appear in the under sequent.

(2) In the rule (Ref), a must appear in $\{R\}$, Γ, or Δ.

$A \rightarrow_S B$ is not an abbreviation in this calculus. The rule (OM) expresses the orthomodular law using self-adjointness. as explained in the next theorem.

Theorem 5.3.2 (Soundness theorem for **LGOM**)**.** *If* $\{R\}\Gamma \Rightarrow \Delta$ *is provable in* **LGOM***, then* $\{R\}\Gamma \Rightarrow \Delta$ *is valid in all DOM-models.*

Proof. Proven by induction in proof construction of the sequent $\{R\}\Gamma \Rightarrow \Delta$. when the last rule of a proof is (OM), ($\rightarrow$L) or ($\rightarrow$R).

For (OM), suppose $\{R, a(A?)b, b \not\perp c, c(A?)d, d \not\perp a\}\Gamma \Rightarrow \Delta$ (d does not appear in $\{R\}, \Gamma$ and Δ) is valid. From Theorem 3.2.5, $\{R, a(A?)b, b \not\perp c, c(A?)d, d \not\perp a\}\Gamma \Rightarrow \Delta$ and $\{R, a(A?)b, b \not\perp c\}\Gamma \Rightarrow \Delta$ have same embeddings in the sense that the existence of x such that $E(c)(A?)x$ and $x \not\perp E(a)$ is guaranteed even if $c(A?)d$ and $d \not\perp a$ are not included in $\{R\}$. Therefore, $\{R, a(A?)b, b \not\perp c\}\Gamma \Rightarrow \Delta$ is also valid.

For (→L), suppose $\{R, a(A?)b\}b : B, \Gamma \Rightarrow \Delta$ is valid. We can use almost the same proof for (→L) in Theorem 5.2.1 because of Lemma 3.2.4 and the fact that $\{R, a(A?)b\}\Gamma \Rightarrow \Delta, b : A$ is valid. Therefore, $\{R, a(A?)b, \}a : A \to_S B, \Gamma \Rightarrow \Delta$ is valid.

For (→R), suppose $\{R, a(A?)b, b \not\perp b\}b : A, \Gamma \Rightarrow \Delta, b : B$ (b does not appear in $\{R\}, \Gamma$ and Δ) is valid. We can use almost the same proof for (→R) in Theorem 5.2.1 because of Lemma 3.2.4. Therefore, $\{R\}\Gamma \Rightarrow \Delta, a : A \to_S B$ is also valid.

Proofs for all other cases are similar to the cases that have been described. □

For the completeness theorem, a canonical model similar to the one for **LGOI** needs to be constructed. However, it is difficult to prove the completeness theorem only using the rule (OM). In a canonical model, it must be proven that *all* $\perp$-closed subsets are self-adjoint to prove a canonical model is a DOM-model. Therefore, we need additional rules for add some formulas when we construct a canonical model in order to represent all $\perp$-closed subset by formulas. Furthermore, to use the rule (OM) to construct a canonical model, underlying relations for projections are required since the relation $c(A?)d$ can be created when the relation $a(A?)b$ already exists.

Therefore, it is an open question whether **LGOM** satisfies the completeness theorem.

5.4 Labeled sequent calculus LGOE

In this section, the labeled sequent calculus for entailment is introduced. The labeled sequent calculus **LGOE** is defined below. $\{R\}$, Γ and Δ are finite sets. $\{R\}$ should include only the relations for non-orthogonality on labels. Different from the case of strict implication, entailment does not satisfy $\|A \twoheadrightarrow \perp\| = \|\neg A\|$. Therefore, we have to make rules for $\twoheadrightarrow$ and $\neg$ respectively.

LGOE

Axiom:

$$\{R\}a : A \Rightarrow a : A \qquad \{R\}a : \perp \Rightarrow$$

Rules:

$$\frac{\{R\}\Gamma \Rightarrow \Delta, a : A \quad \{R\}\ a : A, \Gamma \Rightarrow \Delta}{\{R\}\Gamma \Rightarrow \Delta}\ (\text{cut})$$

$$\frac{\{R\}\Gamma \Rightarrow \Delta}{\{R\}a : A, \Gamma \Rightarrow \Delta}\ (\text{wL}) \qquad \frac{\{R\}\Gamma \Rightarrow \Delta}{\{R\}\Gamma \Rightarrow \Delta, a : A}\ (\text{wR})$$

$$\frac{\{R\}a : A, a : B, \Gamma \Rightarrow \Delta}{\{R\}a : A \wedge B, \Gamma \Rightarrow \Delta}\ (\wedge\text{L})$$

$$\frac{\{R\}\Gamma \Rightarrow \Delta, a : A \quad \{R\}\Gamma \Rightarrow \Delta, a : B}{\{R\}\Gamma \Rightarrow \Delta, a : A \wedge B}\ (\wedge\text{R})$$

$$\frac{\{R, a \not\perp b\}\Gamma \Rightarrow \Delta, b : A}{\{R, a \not\perp b\}\ a : \neg A, \Gamma \Rightarrow \Delta}\ (\neg\ \text{L})$$

$$\frac{\{R, a \not\perp b\}\ b : A, \Gamma \Rightarrow \Delta}{\{R\}\Gamma \Rightarrow \Delta, a : \neg A}\ (\neg\ \text{R})\ ^{(1)}$$

$$\frac{\{R, a \not\perp b\}\ b : \neg p, \Gamma \Rightarrow \Delta}{\{R\}\ \Gamma \Rightarrow \Delta, a : p}\ (\neg)\ ^{(1)}$$

$$\frac{\{R\}\Gamma \Rightarrow \Delta, b : A \quad \{R\}\ b : B, \Gamma \Rightarrow \Delta}{\{R\}\ a : A \twoheadrightarrow B, \Gamma \Rightarrow \Delta}\ (\twoheadrightarrow\ \text{L})$$

$$\frac{\{R\}\ b : A, \Gamma \Rightarrow \Delta, b : B}{\{R\}\Gamma \Rightarrow \Delta, a : A \twoheadrightarrow B}\ (\twoheadrightarrow\ \text{R})\ ^{(1)}$$

$$\frac{\{R, a \not\perp a\}\Gamma \Rightarrow \Delta}{\{R\}\Gamma \Rightarrow \Delta}\ (\text{Ref})^{(2)} \qquad \frac{\{R, a \not\perp b, b \not\perp a\}\Gamma \Rightarrow \Delta}{\{R, a \not\perp b\}\Gamma \Rightarrow \Delta}\ (\text{Sym})$$

(1) In the rules ($\neg$R), ($\neg$) and ($\twoheadrightarrow$ R), b must not appear in $\{R\}$, Γ and Δ.

(2) In the rule (Ref), a must appear in $\{R\}$, Γ, or Δ.

The definition of embedding and truth for sequent is the same as in Section 5.2. The condition of ($\twoheadrightarrow$ R) means that b is not connected to any label by $\not\perp$.

Theorem 5.4.1 (Soundness theorem for **LGOE**)**.** *If $\{R\}\Gamma \Rightarrow \Delta$ is provable in* **LGOE***, then $\{R\}\Gamma \Rightarrow \Delta$ is valid in all O-models.*

Proof. For ($\twoheadrightarrow$ L), suppose $\{R\}\Gamma \Rightarrow \Delta, b : A$ and $\{R\}b : B, \Gamma \Rightarrow \Delta$ is valid. Then, in any embedding E, if all $c : C \in \Gamma$ are true at $E(c)$ and all $c : C \in \Delta$ are false at $E(c)$, A is true at $E(b)$ and B is false at $E(b)$. Therefore, in any element in a model, $A \twoheadrightarrow B$ is false. Therefore, $\{R\}a : A \twoheadrightarrow B, \Gamma \Rightarrow \Delta$ is valid.

For ($\twoheadrightarrow$ R), suppose $\{R\}b : A, \Gamma \Rightarrow \Delta, b : B$ is valid. Then, in any embedding E, if all $c : C \in \Gamma$ are true at $E(c)$ and all $c : C \in \Delta$ are false at $E(c)$, and if A is true at $E(b)$, then B is true at $E(b)$. Since

b does not appear in R, $E(b)$ could be any element of a model. This means that there are no elements x such that $x \models A$ and $x \not\models B$. Therefore, $\{R\}\Gamma \Rightarrow \Delta, a : A \twoheadrightarrow B$ is valid. □

For the completeness theorem, a canonical model $(X_c, \perp_c, V_c)$ is formed in a nearly identical way to that in the case of **LGOI**. We have to change the procedure in Step 1 for making a canonical model as follows.

- If $a : (A \twoheadrightarrow B) \in \Gamma_i$, for all b that appears in $\{R\}_i$, at least one of $\{R\}_i\Gamma_i \Rightarrow \Delta_i, b : A$ and $\{R\}_i\Gamma_i, b : B \Rightarrow \Delta_i$ is not provable because of the rule ($\twoheadrightarrow$L). Therefore, $\Gamma_{i+1} = \Gamma_i \cup \{b : B | b$ appears in $\{R\}$ and $\{R\}_i\Gamma_i, b : B \Rightarrow \Delta_i$ is not provable $\}$ and $\Delta_{i+1} = \Delta_i \cup \{b : A | b$ appears in $\{R\}$ and $\{R\}_i\Gamma_i \Rightarrow \Delta_i, b : A$ is not provable $\}$ are formed. In every case, $\{R\}_{i+1} = \{R\}_i$ is formed.
- If $a : (A \twoheadrightarrow B) \in \Delta_i$ and if this procedure is not executed to this formula, then a label b that does not appear in $\{R\}_i\Gamma_i \Rightarrow \Delta_i$ is selected and forms $\Gamma_{i+1} = \Gamma_i \cup \{b : A\}$, $\Delta_{i+1} = \Delta_i \cup \{b : B\}$ and $\{R\}_{i+1} = \{R\}_i$. $\{R\}_{i+1}\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ is not provable because of the rule ($\twoheadrightarrow$R).

The definition of an embedding E_c is identical to the case of **LGOI**.

Lemma 5.4.2. *If $a : A \in \Gamma_c$, then A is true at $a \in X_c$. If $a : A \in \Delta_c$, then A is false at $a \in X_c$. Therefore, $\{R\}\Gamma \Rightarrow \Delta$ is false in $(X_c, \perp_c, V_c)$ under E_c.*

Proof. This lemma is proven by induction on the construction of the labeled formulas included in Γ_c and Δ_c.

Suppose $a : A \twoheadrightarrow B \in \Gamma_c$. From Step 1, for all b that appears in $\{R\}_c$, $b : A \in \Delta_c$ or $b : B \in \Gamma_c$. Therefore, from the inductive hypothesis, for every $x \in X_c$, A is false at x or B is true at x. Therefore, $A \twoheadrightarrow B$ is true at $a \in X_c$.

Suppose $a : A \twoheadrightarrow B \in \Delta_c$. From Step1, there exists a label b such that $b : A \in \Gamma_c$ and $b : B \in \Delta_c$. From the inductive hypothesis, A is true at $b \in X_c$ and B is false at $b \in X_c$. Therefore, $A \twoheadrightarrow B$ is false at $a \in X_c$.

For additional cases, see Lemma 5.2.2. □

Theorem 5.4.3 (Completeness theorem for **LGOE**)**.** *If $\{R\}\Gamma \Rightarrow \Delta$ is valid in all O-models, then $\{R\}\Gamma \Rightarrow \Delta$ is provable in* **LGOE**.

Proof. From Lemma 5.4.2, if $\{R\}\Gamma \Rightarrow \Delta$ is not provable in **LGOE**, then an O-model $(X, \perp, V)$ and an embedding E exists such that $\{R\}\Gamma \Rightarrow \Delta$ is false in $(X, \perp, V)$ under E. □

Theorem 5.4.4 (Cut-elimination theorem for **LGOE**). *If* $\{R\}\Gamma \Rightarrow \Delta$ *is provable in* **LGOE**, *then there exists a proof of* $\{R\}\Gamma \Rightarrow \Delta$ *that does not include the rule (cut).*

Proof. The completeness theorem without the rule (cut) was proven. Therefore, provability of a sequent is independent of whether (cut) exist or not in the calculus. $\square$

Chapter 6

Conclusion and future work

In this chapter, some unresolved issues are discussed.

In Chapter 3, we introduced a new frame for quantum logic and showed the usefulness of it. However, this frame still has some problems. We have to discuss in frame level not formula level because the proof of Lemma 3.2.5 includes the notion of infinite conjunction which is not allowed in a formula. This problem is related to the problem that we could not prove the identity between OM-models and TOM-models, which is related to an infinite conjunctions. A solution is desired because this difference complicates the discussion especially on the completeness theorem.

In Chapter 4, we constructed sequent calculi for each implication. Although almost all of these sequent calculi satisfy soundness and completeness theorems, all of these calculi do not satisfy the cut-elimination theorem. Furthermore, some calculi have other problems. The rules for implication in **GOI** and $\mathbf{GOI}_2$ are complicated. For this, simplification is desired. We use the completeness theorem for **GOM** to prove the completeness theorem for **GOMS**, **GOMC** and **GOMR**. We could not make original canonical models for these calculi. This problem is related to that rule ($\to_S$ R), ($\to_C$ R), and ($\to_R$ R) do not satisfy the sub-formula property. In order to make a semantic analysis, we need a proof using a canonical model. It is open question that whether **GOE** satisfies the completeness theorem. This problem is caused by the nature of entailment, the valuation of which is influenced by the whole model.

In Chapter 5, we introduced sequent calculi for quantum logic which solve some problems of sequent calculi in Chapter 4. Some of these sequent calculi satisfy the cut-elimination theorem. Furthermore, we construct sequent calculus **LGOE** for entailment which satisfies the completeness theorem. However, some calculi still have problems. Although we proved the decidability of orthologic with strict implication by using **LGOI**, the rule ($\neg$) does not satisfy sub-formula property.

We can use following rule ($\neg\neg$ R) instead of ($\neg$), but sub-formula property is still broken.

$$\frac{\{R\}\Gamma \Rightarrow \Delta, a : \neg\neg p}{\{R\}\Gamma \Rightarrow \Delta, a : p}\ (\neg\neg\ \mathrm{R})$$

LGOM has several problems. At this point, whether or not **LGOM** satisfies the completeness theorem and cut-elimination theorem is open to debate. As mentioned earlier, in a canonical model, it must be proven that all $\perp$-closed subsets are self-adjoint. To do so, we have to add all formulas which are true in a canonical model to a labeled sequent when we make a canonical model. As one solution, following type of rule can be proposed.

$$\frac{\{R\}\ a : A, \Gamma \Rightarrow \Delta}{\{R\}\ \Gamma \Rightarrow \Delta}\ (\mathrm{O})$$

By using this rule, we can add true formulas to premise of a labeled sequent. We omit the details here but we can actually prove the completeness theorem and cut-elimination theorem by adding this rule and other complicated rules to **LGOM**. However, this rule has a random formula A similar to the rule (cut). Therefore, it is desirable to create a sequent calculus that satisfies completeness theorem without using such rules.

Bibliography

[1] Baltag, A., Smets, S.: The logic of quantum programs. Proceedings of the 2nd International Workshop on Quantum Programming Languages 2004. 39–56 (2004)

[2] Baltag, A., Smets, S.: Quantum logic as a dynamic logic. Synthese. 179, 285–306 (2011)

[3] Baltag, A., Smets, S.: The dynamic turn in quantum logic. Synthese. 186(3), 753–773 (2012)

[4] Birkhoff, G., Von Neumann, J.: The Logic of Quantum Mechanics. The Annals of Mathematics. 37(4), 823–843 (1936)

[5] Chiara, M. L. D., Giuntini, R.: Quantum Logics. Gabbay, D. M., Guenthner, F. (ed.): Handbook of Philosophical Logic 2nd Edition 6. Springer. 129–228 (2002)

[6] Egly, U., Tompits, H.: On Different Proof-Search Strategies for Orthologic. Studia Logica. 73(1), 131–152 (2003)

[7] Faggian, C., Sambin, G.: From Basic Logic to Quantum Logics with Cut-Elimination. International Journal of Theoretical Physics. 37(1), 31–37 (1998)

[8] Finch, P. D.: Quantum Logic as an Implication Algebra. Bulletin of the Australian Mathematical Society. 2(1), 101–106 (1970)

[9] Gentzen, G.: Untersuchungen ber das logische Schlieen. I. Mathematische Zeitschrift. 39 (2), 176–210 (1934)

[10] Gentzen, G.: Untersuchungen ber das logische Schlieen. II. Mathematische Zeitschrift. 39 (3), 405–431 (1935)

[11] Hardegree, G. M.: Material Implication in Orthomodular (and Boolean) Lattices. Notre Dame Journal of Formal Logic. 22(2), 163–182 (1981)

[12] Hou, Z., Tiu, A., Gore, R.: A Labelled Sequent Calculus for BBI: Proof Theory and Proof Search. TABLEAUX 2013, ed. D. Galmiche and D. Larchey-Wendling, Springer-Verlag Berlin Heidelberg, USA. 172–187 (2013)

[13] Ishii, K., Kashima, R., Kikuchi, K.: Sequent Calculi for Visser's Propositional Logics. Notre Dame Journal of Formal Logic. 42(1), 1–22 (2001).

[14] Kawano, T.: Sequent Calculus for Quantum Logic with Sasaki Arrow. Proceedings of Non-classical Logics, Theory and Applications. (2016)

[15] Kawano, T.: Advanced Kripke Frame for Quantum Logic. Proceedings of 25th Workshop on Logic, Language, Information and Computation. (2018)

[16] Kawano, T.: Labeled Sequent Calculus for Orthologic. To appear in Bulletin of the Section of Logic.

[17] Negri, S.: Proof Analysis in Modal Logic. Journal of Philosophical Logic. 34, 507–544 (2005)

[18] Negri, S.: Proof theory for modal logic. Philosophy Compass. 6(8), 523–538 (2011)

[19] Nishimura, H.: Sequential Method in Quantum Logic. The Journal of Symbolic Logic. 45(2), 339–352 (1980)

[20] Nishimura, H.: Proof Theory for Minimal Quantum Logic I. International Journal of Theoretical Physics. 33(1), 103–113 (1994)

[21] Nishimura, H.: Proof Theory for Minimal Quantum Logic II. International Journal of Theoretical Physics. 33(7), 1427–1443 (1994)

[22] Sambin, G., Battilotti, G., Faggian, C.: Basic logic: reflection, symmetry, visibility. The Journal of Symbolic Logic. 65(3), 979–1013 (2000)

[23] Takeuchi, G.: Linear Algebra and Quantum Mechanics (in Japanese). Shokabo. (1981)