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Construction of a third order Kusuoka scheme and its application to forward-backward stochastic differential equations

Solving numerical problems in mathematical finance

Summary verson

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Abstract

The main contributions of this work are the construction of a third order Kusuoka scheme and the application of Kusuoka scheme to forward-backward stochastic differential equations. Kusuoka scheme is a higher-order discretization framework for performing weak approximations of stochastic differential equations and the Ninomiya–Victoir and Ninomiya–Ninomiya methods are known as second order discretization methods that belong to the Kusuoka scheme class. The author constructs the third order discretization method based on polynomials of Gaussian random variables that approximate iterated Wiener integrals. Using these discretization methods, new higher order discretization methods of forward-backward stochastic differential equations are proposed. In addition, the author discusses the applications of the proposed methods to practically important financial problems, achieving the desired theoretical order and computational efficiency.

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Contents

Chapter 1

Introduction

1.1 Overview of thesis

1.1.1 Background

In applied science and engineering, solving stochastic differential equations (SDEs) numerically is crucial. Especially, the weak approximation of SDEs, i.e., the calculation of $E[f(X(T,x))]$, where $X(t,x)$ is the value at time t of an N-dimensional diffusion process and f is a function from $\mathbb{R}^N$ into $\mathbb{R}$, is important for the derivative pricing and risk management in the financial industry. A number of studies on this problem have been conducted [47, 39]. There are two approaches to solving this problem: using numerical methods for partial differential equations and performing simulations. It is important to use both approaches properly according to the problem. The former approach is effective for the law-dimensional problem, however it is hard to apply to more that several dimensional problem. On the other hand, the simulation approach can be applied to various problems including high-dimensional problems and enables to calculate the prices of the different derivative products in a unified way. In the simulation approach, the calculations are implemented in two steps. The first step is the discretization of SDEs, i.e., the construction of a set of random variables $\left\{X^{(n)}(t_i,x)\right\}_{i=0,1,\dots,n}(0=t_0<t_1\cdots<t_n=T)$ that approximates $\{X(t,x)\}_{0\leq t\leq T}$. Here, n represents the number of partitions of $[0,T]$. The Euler–

Maruyama (E–M) [58] and Milstein methods [60] are well-known discretization methods for SDEs. The second step is the calculation of $E[f\left(X^{(n)}(T,x)\right)]$, which involves the calculation of a finite-dimensional integration. $D(n)$ denotes the number of dimensions of the integration. One usually calculates the integral using the Monte Carlo(MC) or quasi-Monte Carlo method(QMC) [64, 70]. Two types of approximation errors are involved in the calculations, namely, discretization and integration errors. The discretization error is the difference between $E[f(X(T,x))]$ and $E[f\left(X^{(n)}(T,x)\right)]$, and an integration error is the error when one approximates the expectation with the sum of a random sample generated using the Monte Carlo method or the QMC. When the discretization error is $O(p)$, i.e., when there exists a positive constant C_p such that $\left|E[f(X(T,x))] - E[f\left(X^{(n)}(T,x)\right)]\right| < C_p/n^p$, it is said that the discretization method is of order p. Here, we define the order of the discretization method for weak approximation of SDEs; we discuss the order in this sense hereinafter. The E–M method is of order 1. Using higher-order discretization methods enables us to reduce n and $D(n)$. Consequently, $E[f\left(X^{(n)}(T,x)\right)]$ can be calculated through efficient use of the QMC; see Remark 6.2 of [66]. This work focuses on simulations, especially on the discretization method.

Kusuoka scheme [48, 50], which is also called K-scheme, KLNV-scheme, or Kusuoka approximation, is a higher-order discretization framework for performing weak approximations of SDEs. This framework was based on stochastic Taylor expansion, Malliavin calculus, free Lie algebra, and rough path theory. The Ninomiya–Victoir (N–V) [71] and Ninomiya–Ninomiya (N–N) [66] methods are practically feasible discretization methods that belong to the Kusuoka scheme. These are second-order weak discretization methods, and some extrapolations of them have been proposed.

1.1.2 Contribution

The main contributions of this work are the construction of a third-order Kusuoka scheme and the application of Kusuoka scheme to forward-backward stochastic differential equations (FBSDEs).

The first contribution is the construction of a third-order discretization method of SDEs, which lies in the framework of Kusuoka scheme. The method involves no extrapolation, therefore we can attain higher-order discretization with only positive weighted paths. Polynomials of Gaussian random variables that approximate iterated Wiener integrals play a key role in the new method. The author discusses the applications of the third-order methods to practically important financial models, achieving the desired theoretical order and computational efficiency. In fact, the proposed third-order method is 400 times faster than the E–M method in our example of pricing an Asian option under the SABR model.

The second contribution is the application of Kusuoka scheme to FBSDEs: second and third-order discretization methods for the weak approximations of FBSDEs are proposed using the Kusuoka scheme, the higher-order numerical integration methods, and the tree based branching algorithm technique (TBBA). In the proposed methods, the forward component is discretized using the Kusuoka scheme with discrete random variables, and the backward component using a higher-order numerical integration method suitable for the discretization method of the forward component. In addition, the author discusses the applications of the proposed methods to the XVA pricing, particularly to the credit valuation adjustment (CVA). The numerical results show that the expected theoretical order and computational efficiency could be achieved. Consequently, the proposed method enables us to solve the complicated program which can't be solve by the E–M method.

1.1.3 Outline

In following subsections of this chapter, the backgrounds and contributions of this work are presented for more detail. In Chapter 2, the notation and previous studies are introduce. Then, in Chapter 3, the construction of a third-order Kusuoka scheme is present with its numerical evidence. In Chapter 4, the application of Kusuoka scheme to FBSDEs is presented. Lastly, in Chapter 5, some concluding remarks are discussed.

1.2 Overview of Chapter 3

This chapter is based on the published papers [79, 80].

1.2.1 Background of Chapter 3

In this chapter, we focus on the following weak approximation problem of SDEs. Let $(\Omega, \mathcal{F}, P)$ be a probability space. For $t \in [0, \infty)$, let $B^0(t) = t$ and let $(B^1(t), \ldots, B^d(t))$ be a d-dimensional standard Brownian motion. $C_b^\infty(\mathbb{R}^N; \mathbb{R}^N)$ is the space of $\mathbb{R}^N$-valued smooth functions defined in $\mathbb{R}^N$ whose derivatives of any order are bounded. Let $X : \Omega \times [0, \infty) \times \mathbb{R}^N \to \mathbb{R}^N$ be a solution of the SDE written in Stratonovich form as

$$X(t, x) = x + \sum_{i=0}^{d} \int_0^t V_i \left(X\left(s, x\right)\right) \circ dB^i(s), \tag{1.1}$$

where $x \in \mathbb{R}^N$, $V_0, V_1, \ldots, V_d \in C_b^\infty(\mathbb{R}^N; \mathbb{R}^N)$. Our aim is the numerical evaluation of $E[f\left(X\left(T, x\right)\right)]$, where $f \in C_b^\infty(\mathbb{R}^N; \mathbb{R})$.

Kusuoka scheme is a higher-order discretization framework for performing weak approximations of SDEs, which is based on stochastic Taylor expansion, Malliavin calculus, free Lie algebra, and rough path theory [48, 50, 55]. Kusuoka introduces the notion of moment similar family and constructs a discretization method in [48]. A moment similar family is a family of random variables that mimics the family of iterated Wiener integrals in a specific sense. An example of a moment similar family for a discretization method of order 2 is presented in [48], and some discretization methods are proposed by applying it in [67, 68, 69, 65].

On the other hand, the N–V and N–N methods are practically feasible methods that belong to the Kusuoka scheme framework, which are constructed in accordance with other approaches. They are discretization methods of order 2, and some applications of them are studied in [3, 7]. These methods are executed by constructing $\left\{X^{(n)}\left(t_i, x\right)\right\}_{i=0,1,\ldots,n}$ as the solutions of ordinary differential equations (ODEs).

Techniques have been developed for attaining higher-order discretization. Romberg extrapolation, a general technique for increasing the orders of some methods, has been proven to be applicable to the E–M method in [84], the N–V and N–N methods in [51].

Other types of extrapolation methods for the N–V method, a kind of classical Richardson extrapolation, are studied in [35, 72, 27]. Note that these extrapolation methods involve negative weighted paths. Furthermore, the cubature method on Wiener space of higher-order is proposed in [41], and a discretization method of order 3 for a particular class of SDEs is studied in [3, 77]

1.2.2 Results of Chapter 3

In this chapter, we present a new discretization method of order 3 on the basis of a moment similar family. We describe the construction of a new example of a moment similar family for a discretization method of order 3 with polynomials of Gaussian random variables by focusing on the algebraic structure of iterated Wiener integrals. Then, we present the construction of an approximation operator for an SDE driven by 2-dimensional Brownian motion, applying the formula derived in [83] and changing the basis of the Lie series. When one uses the new method, it is required to solve the ODEs as in the N–V and N–N methods. Note that the new method enables us to attain higher-order discretization with only positive weighted paths.

To verify the numerical behavior of the new method, we applied it to financial problems of pricing some derivatives in practically important financial models: the Black–Scholes, Heston, and SABR models. This chapter presents the details of the implementation, specifically deriving the ODEs to be solved when one uses the new, N–V, and N–N methods. The performance of the new method is compared with other discretization methods. As the numerical results show, the desired theoretical order and computational efficiency can be achieved using the new method. In addition, the numerical examples demonstrate that the new method has some desirable properties, though no mathematical justification of these properties has yet been established.

1.3 Overview of Chapter 4

1.3.1 Background of Chapter 4

This chapter focuses on the discretization methods of FBSDEs. In particular, a solution (X_s, Y_s) for the following equations are discussed. Let $(\Omega, \mathcal{F}, P)$ be a probability space with a Brownian filtration $\{\mathcal{F}_u\}_{u\geq 0}$, and $B^0(u) = u$ and $(B^1(u), \ldots, B^d(u))$ be a d-dimensional standard Brownian motion for $u \in [0, \infty)$,

$$\begin{aligned} X_s &= x + \sum_{i=0}^{d} \int_t^s V_i\left(X_u\right) \circ dB^i(u), \\ Y_s^{t,x} &= \Phi(X_T^{t,x}) + \int_s^T f(X_u^{t,x}, Y_u^{t,x})\, du - \sum_{i=0}^{d} \int_s^T Z_u^{(i)}\, dB^i(u), \end{aligned} \tag{1.2}$$

where $x \in \mathbb{R}^N$, $V_0, V_1, \ldots, V_d \in C_b^\infty(\mathbb{R}^N; \mathbb{R}^N)$ and $\Phi : \mathbb{R}^N \to \mathbb{R}$, $f : \mathbb{R}^{N+1} \to \mathbb{R}$ are appropriately smooth functions. Note that we denote f a different function in Chapter 3 and Chapter 4 in order to be consistent with frequently used notation. Here, the forward component $X_s \in \mathbb{R}^N$ is written in the Stratonovich form and the backward component $Y_s^{t,x} \in \mathbb{R}$ in the Ito form. Our aim is the numerical evaluation of $Y_t^{t,x}$.

The framework of FBSDE was first introduced by Bismut [9], Pardoux and Peng proved the existence and uniqueness of FBSDE in [74] and established the Feynman–Kac formula in the general case in [73]. Since then, many researchers have investigated to this field from both the theoretical and application points of view. As for the theoretical developments, the smoothness of the solution of FBSDE was proved in [20], and the correspondence with fully nonlinear partial differential equations(PDEs) was proved in [19]. There are also substantial studies on financial applications of FBSDEs [25, 30, 16], refer to the monographs [31]. Recently, the XVA pricing i.e., various valuation adjustments for derivative instruments which is a current big issue in the financial industry, is formulated using FBSDE in a general and unified way[75, 15, 53, 11] . Although we can formulate various complicated problems using FBSDE, it is not easy to apply them to practical problems due to the difficulty of numerical treatments.

Many numerical methods have been proposed for FBSDE, they can be classified into three approaches. The first approach is the nonlinear PDE approach using the nonlinear Feynman–Kac formula which is also called the four-step scheme [56]. In this approach, the nonlinear Feynman–Kac formula and the numerical methods for solving nonlinear PDEs and forward SDEs are used. This approach has been extensively studied [28, 57]; in particular, higher-order discretization methods of forward SDEs were applied in [61]. The second approach is the simulation. As described below, in the simulation approach, the calculations consists of discretization and integration. Although some studies were conducted on the discretization of backward SDE [63], a discretization method for a general class of FBSDE was first introduced in [12, 87]. The third approach involves using various expansions [34, 82, 13, 43]. Some methods of this approach are practically feasible and can derive some meaningful calculation formulas. However, we cannot know the true value using this approach.

This study focuses on the second approach, i.e., the simulation approach. In the simulation approach, the calculations are implemented in two steps: discretization and integration. Discretization means the construction of a set of random variables $\{\hat{X}_{s_i}, \hat{Y}_{s_i}\}_{i=0,1,\ldots n}$ ($t = s_0 < s_1 \cdots < s_n = T$) that approximates the processes $(X_s, Y_s^{t,x})$. Then, since $\hat{Y}_t$ usually depends on the conditional expectations of $\{\hat{X}_{s_i}, \hat{Y}_{s_i}\}_{i=1,\ldots n}$, it is required to calculate these expectations by integration. Here, the integration involves hard calculations of the conditional expectations, which is significantly different from the simulation for SDEs; this makes practical implementations challenging. This problem is called "nested simulation" or "multiple Monte Carlo" problem. To overcome this issue, many methods have been proposed both for the discretization and integration, e.g., the quantization tree [6], the Malliavin integration by part formula [12, 87], the least square Monte-Carlo method [40], the Picard iteration [8], the sparse grid method [86], the multistep method [88, 17] the cubature method combined with the TBBA technique [22, 23], the Runge–Kutta method [18], and the Fourier method [78]. In [88, 23, 18], higher-order discretization methods of FBSDEs have been proposed. Here, the higher-order discretization methods of FBSDEs mean the constructions of sets of random variables $\{\hat{X}_{s_i}, \hat{Y}_{s_i}\}_{i=0,1,\ldots n}$ ($t = s_0 < s_1 \cdots < s_n = T$) such that $\left|Y_t^{t,x} - \hat{Y}_t\right| < C_p/n^p$ holds for $p \in \mathbb{N}$; these discretization methods are called order p.

Before the discretization methods of FBSDEs are studied, a large number of studies had been conducted for higher-order discretization methods of forward SDEs as described in Chapter 3. To use Kusuoka scheme, it is required to generate the ordinary differential equation(ODE) valued random variables for simulating a path. Certain discretization methods have been proposed using discrete random variables [55, 67]. When we use these methods iteratively along the time steps, the support of the intermediate measures grows exponentially as the number of partitions increases. To overcome this problem, the TBBA technique [21] was applied to these methods in [68, 69, 65, 24]. On the other hand, using the framework of Kusuoka scheme and the Gaussian random variables, the N–V and N–N methods have been proposed as practically feasible second-order methods, and the $Q_{(s)}^{(7,2)}$ method [79, 80] as a third-order method as described in Chapter 3.

1.3.2 Results of Chapter 4

In this chapter, we present second and third-order discretization methods of FBSDEs using the Kusuoka scheme, the higher-order numerical integration methods, and the TBBA technique. In our proposed methods, the forward component X_s is discretized using the Kusuoka scheme with discrete random variables, and the backward component Y_s using a higher-order numerical integration method suitable for the discretization method of the forward component; specifically, the N–V and N–N methods combined with the trapezoidal rule attain the second-order and the $Q_{(s)}^{(7,2)}$ method combined with the Simpson's the third-order. These methods construct random variables $\{\hat{X}_{s_i}, \hat{Y}_{s_i}\}_{i=0,1,\dots n}$ on a discrete probability space. We approximate the law of $\{\hat{X}_{s_i}, \hat{Y}_{s_i}\}_{i=0,1,\dots n}$ using the TBBA technique; in particular, we introduce an efficient implementation technique called the depth-first tree traversal. This technique is essential for practical implementation, because memory explosion might occur without this technique.

In addition, we discuss the applications of the proposed methods to the XVA pricing, particularly to the CVA. Using the proposed methods, we calculate the CVA prices formulated by FBSDEs under the Black–Scholes and Heston models, which are usually considered to be hard to solve numerically. Our numerical results show that the expected theoretical order and computational efficiency are achieved. Moreover, we could observe the effectiveness of higher-order methods rather than the case of SDEs.

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