

論文 / 著書情報
Article / Book Information

題目(和文)	3次楠岡近似の構成と前進後退確率微分方程式への応用 ~数理ファイナンスにおける数値計算問題への適用
Title(English)	Construction of a third-order Kusuoka scheme and its applications to forward-backward stochastic differential equations : Solving numerical problems in mathematical finance
著者(和文)	篠崎裕司
Author(English)	Yuji Shinozaki
出典(和文)	学位:博士(工学), 学位授与機関:東京工業大学, 報告番号:甲第10996号, 授与年月日:2018年9月20日, 学位の種別:課程博士, 審査員:二宮 祥一,尾形 わかは,水野 眞治,中丸 麻由子,中野 張
Citation(English)	Degree:Doctor (Engineering), Conferring organization: Tokyo Institute of Technology, Report number:甲第10996号, Conferred date:2018/9/20, Degree Type:Course doctor, Examiner:,,,,
学位種別(和文)	博士論文
Category(English)	Doctoral Thesis
種別(和文)	要約
Type(English)	Outline

Construction of a third order Kusuoka scheme and its application to forward-backward stochastic differential equations

Solving numerical problems in mathematical finance

Summary version

Yuji Shinozaki

A thesis submitted for the degree of
Doctor of Engineering

Tokyo Institute of Technology,
Japan

August 18, 2018

Abstract

The main contributions of this work are the construction of a third order Kusuoka scheme and the application of Kusuoka scheme to forward-backward stochastic differential equations. Kusuoka scheme is a higher-order discretization framework for performing weak approximations of stochastic differential equations and the Ninomiya–Victoir and Ninomiya–Ninomiya methods are known as second order discretization methods that belong to the Kusuoka scheme class. The author constructs the third order discretization method based on polynomials of Gaussian random variables that approximate iterated Wiener integrals. Using these discretization methods, new higher order discretization methods of forward-backward stochastic differential equations are proposed. In addition, the author discusses the applications of the proposed methods to practically important financial problems, achieving the desired theoretical order and computational efficiency.

Acknowledgements

I would like to express my deepest gratitude to my supervisor Prof. Syoiti Ninomiya for his continuous encouragement and valuable comments. I wish to give a special note of thanks to the member of the Ninomiya–Nakano laboratory for cheering me on. I also appreciate the members of the JAFEE derivative research group. Thanks are also extended to my colleagues in the MUFG.

Most importantly, I would like to thank my family and dear friends for their support and generosity in all respects.

Contents

1	Introduction	1
1.1	Overview of thesis	1
1.1.1	Background	1
1.1.2	Contribution	2
1.1.3	Outline	3
1.2	Overview of Chapter 3	4
1.2.1	Background of Chapter 3	4
1.2.2	Results of Chapter 3	5
1.3	Overview of Chapter 4	6
1.3.1	Background of Chapter 4	6
1.3.2	Results of Chapter 4	8
2	Preparation	10
2.1	Notation for free Lie algebras, vector fields, and iterated Wiener integrals	10
2.2	Previous study on Kusuoka scheme	13
2.2.1	Approximation theorem for Kusuoka scheme	13
2.2.2	Construction of approximation operators	14
2.3	Forward-backward SDE	16
3	Construction of a third order Kusuoka scheme and its application to financial models	18
3.1	Problem	18
3.2	Construction of a third order Kusuoka scheme	19

3.2.1	7-moment similar family	19
3.2.2	7-similar operator	25
3.3	Implementation and numerical results	28
3.3.1	Simulation method using $Q_{(s)}^{(7,2)}$	28
3.3.2	Black–Scholes model	30
3.3.3	Heston model	36
3.3.4	SABR model	43
3.4	Future work	50
4	Application of Kusuoka scheme to forward-backward SDEs	52
4.1	Problem	52
4.2	Higher-order discretization method of FBSDE	53
4.2.1	Discretization method of the forward component: Discrete Kusuoka scheme	53
4.2.2	Discretization method of the backward component	55
4.2.3	Estimation of the discretization error	56
4.3	TBBA	65
4.3.1	Algorithm	65
4.3.2	Efficient implementation	67
4.4	Numerical results: XVA pricing	67
4.4.1	XVA pricing	67
4.4.2	Black–Scholes model	69
4.4.3	Heston model	71
4.5	Future work	73
5	Concluding remarks	74
A	Maxima code to verify the algebraic condition of $Q_{(s)}^{(7,2)}$	76
B	Structured program for the implementation of the $Q_{(s)}^{(7,2)}$ method	81
C	Structured program for the implementation of the third order method for FBSDEs	83

Chapter 1

Introduction

1.1 Overview of thesis

1.1.1 Background

In applied science and engineering, solving stochastic differential equations (SDEs) numerically is crucial. Especially, the weak approximation of SDEs, i.e., the calculation of $E[f(X(T, x))]$, where $X(t, x)$ is the value at time t of an N -dimensional diffusion process and f is a function from $\mathbb{R}^N$ into $\mathbb{R}$, is important for the derivative pricing and risk management in the financial industry. A number of studies on this problem have been conducted [47, 39]. There are two approaches to solving this problem: using numerical methods for partial differential equations and performing simulations. It is important to use both approaches properly according to the problem. The former approach is effective for the low-dimensional problem, however it is hard to apply to more than several dimensional problem. On the other hand, the simulation approach can be applied to various problems including high-dimensional problems and enables to calculate the prices of the different derivative products in a unified way. In the simulation approach, the calculations are implemented in two steps. The first step is the discretization of SDEs, i.e., the construction of a set of random variables $\{X^{(n)}(t_i, x)\}_{i=0,1,\dots,n}$ ($0 = t_0 < t_1 < \dots < t_n = T$) that approximates $\{X(t, x)\}_{0 \leq t \leq T}$. Here, n represents the number of partitions of $[0, T]$. The Euler–

Maruyama (E–M) [58] and Milstein methods [60] are well-known discretization methods for SDEs. The second step is the calculation of $E[f(X^{(n)}(T, x))]$, which involves the calculation of a finite-dimensional integration. $D(n)$ denotes the number of dimensions of the integration. One usually calculates the integral using the Monte Carlo (MC) or quasi-Monte Carlo method (QMC) [64, 70]. Two types of approximation errors are involved in the calculations, namely, discretization and integration errors. The discretization error is the difference between $E[f(X(T, x))]$ and $E[f(X^{(n)}(T, x))]$, and an integration error is the error when one approximates the expectation with the sum of a random sample generated using the Monte Carlo method or the QMC. When the discretization error is $O(p)$, i.e., when there exists a positive constant C_p such that $|E[f(X(T, x))] - E[f(X^{(n)}(T, x))]| < C_p/n^p$, it is said that the discretization method is of order p . Here, we define the order of the discretization method for weak approximation of SDEs; we discuss the order in this sense hereinafter. The E–M method is of order 1. Using higher-order discretization methods enables us to reduce n and $D(n)$. Consequently, $E[f(X^{(n)}(T, x))]$ can be calculated through efficient use of the QMC; see Remark 6.2 of [66]. This work focuses on simulations, especially on the discretization method.

Kusuoka scheme [48, 50], which is also called K-scheme, KLVN-scheme, or Kusuoka approximation, is a higher-order discretization framework for performing weak approximations of SDEs. This framework was based on stochastic Taylor expansion, Malliavin calculus, free Lie algebra, and rough path theory. The Ninomiya–Victoir (N–V) [71] and Ninomiya–Ninomiya (N–N) [66] methods are practically feasible discretization methods that belong to the Kusuoka scheme. These are second-order weak discretization methods, and some extrapolations of them have been proposed.

1.1.2 Contribution

The main contributions of this work are the construction of a third-order Kusuoka scheme and the application of Kusuoka scheme to forward-backward stochastic differential equations (FBSDEs).

The first contribution is the construction of a third-order discretization method

of SDEs, which lies in the framework of Kusuoka scheme. The method involves no extrapolation, therefore we can attain higher-order discretization with only positive weighted paths. Polynomials of Gaussian random variables that approximate iterated Wiener integrals play a key role in the new method. The author discusses the applications of the third-order methods to practically important financial models, achieving the desired theoretical order and computational efficiency. In fact, the proposed third-order method is 400 times faster than the E–M method in our example of pricing an Asian option under the SABR model.

The second contribution is the application of Kusuoka scheme to FBSDEs: second and third-order discretization methods for the weak approximations of FBSDEs are proposed using the Kusuoka scheme, the higher-order numerical integration methods, and the tree based branching algorithm technique (TBBA). In the proposed methods, the forward component is discretized using the Kusuoka scheme with discrete random variables, and the backward component using a higher-order numerical integration method suitable for the discretization method of the forward component. In addition, the author discusses the applications of the proposed methods to the XVA pricing, particularly to the credit valuation adjustment (CVA). The numerical results show that the expected theoretical order and computational efficiency could be achieved. Consequently, the proposed method enables us to solve the complicated program which can't be solve by the E–M method.

1.1.3 Outline

In following subsections of this chapter, the backgrounds and contributions of this work are presented for more detail. In Chapter 2, the notation and previous studies are introduce. Then, in Chapter 3, the construction of a third-order Kusuoka scheme is present with its numerical evidence. In Chapter 4, the application of Kusuoka scheme to FBSDEs is presented. Lastly, in Chapter 5, some concluding remarks are discussed.

1.2 Overview of Chapter 3

This chapter is based on the published papers [79, 80].

1.2.1 Background of Chapter 3

In this chapter, we focus on the following weak approximation problem of SDEs. Let $(\Omega, \mathcal{F}, P)$ be a probability space. For $t \in [0, \infty)$, let $B^0(t) = t$ and let $(B^1(t), \dots, B^d(t))$ be a d -dimensional standard Brownian motion. $C_b^\infty(\mathbb{R}^N; \mathbb{R}^N)$ is the space of $\mathbb{R}^N$ -valued smooth functions defined in $\mathbb{R}^N$ whose derivatives of any order are bounded. Let $X : \Omega \times [0, \infty) \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ be a solution of the SDE written in Stratonovich form as

$$X(t, x) = x + \sum_{i=0}^d \int_0^t V_i(X(s, x)) \circ dB^i(s), \quad (1.1)$$

where $x \in \mathbb{R}^N$, $V_0, V_1, \dots, V_d \in C_b^\infty(\mathbb{R}^N; \mathbb{R}^N)$. Our aim is the numerical evaluation of $E[f(X(T, x))]$, where $f \in C_b^\infty(\mathbb{R}^N; \mathbb{R})$.

Kusuoka scheme is a higher-order discretization framework for performing weak approximations of SDEs, which is based on stochastic Taylor expansion, Malliavin calculus, free Lie algebra, and rough path theory [48, 50, 55]. Kusuoka introduces the notion of moment similar family and constructs a discretization method in [48]. A moment similar family is a family of random variables that mimics the family of iterated Wiener integrals in a specific sense. An example of a moment similar family for a discretization method of order 2 is presented in [48], and some discretization methods are proposed by applying it in [67, 68, 69, 65].

On the other hand, the N–V and N–N methods are practically feasible methods that belong to the Kusuoka scheme framework, which are constructed in accordance with other approaches. They are discretization methods of order 2, and some applications of them are studied in [3, 7]. These methods are executed by constructing $\{X^{(n)}(t_i, x)\}_{i=0,1,\dots,n}$ as the solutions of ordinary differential equations (ODEs).

Techniques have been developed for attaining higher-order discretization. Romberg extrapolation, a general technique for increasing the orders of some methods, has been proven to be applicable to the E–M method in [84], the N–V and N–N methods in [51].

Other types of extrapolation methods for the N–V method, a kind of classical Richardson extrapolation, are studied in [35, 72, 27]. Note that these extrapolation methods involve negative weighted paths. Furthermore, the cubature method on Wiener space of higher-order is proposed in [41], and a discretization method of order 3 for a particular class of SDEs is studied in [3, 77]

1.2.2 Results of Chapter 3

In this chapter, we present a new discretization method of order 3 on the basis of a moment similar family. We describe the construction of a new example of a moment similar family for a discretization method of order 3 with polynomials of Gaussian random variables by focusing on the algebraic structure of iterated Wiener integrals. Then, we present the construction of an approximation operator for an SDE driven by 2-dimensional Brownian motion, applying the formula derived in [83] and changing the basis of the Lie series. When one uses the new method, it is required to solve the ODEs as in the N–V and N–N methods. Note that the new method enables us to attain higher-order discretization with only positive weighted paths.

To verify the numerical behavior of the new method, we applied it to financial problems of pricing some derivatives in practically important financial models: the Black–Scholes, Heston, and SABR models. This chapter presents the details of the implementation, specifically deriving the ODEs to be solved when one uses the new, N–V, and N–N methods. The performance of the new method is compared with other discretization methods. As the numerical results show, the desired theoretical order and computational efficiency can be achieved using the new method. In addition, the numerical examples demonstrate that the new method has some desirable properties, though no mathematical justification of these properties has yet been established.

1.3 Overview of Chapter 4

1.3.1 Background of Chapter 4

This chapter focuses on the discretization methods of FBSDEs. In particular, a solution (X_s, Y_s) for the following equations are discussed. Let $(\Omega, \mathcal{F}, P)$ be a probability space with a Brownian filtration $\{\mathcal{F}_u\}_{u \geq 0}$, and $B^0(u) = u$ and $(B^1(u), \dots, B^d(u))$ be a d -dimensional standard Brownian motion for $u \in [0, \infty)$,

$$\begin{aligned} X_s &= x + \sum_{i=0}^d \int_t^s V_i(X_u) \circ dB^i(u), \\ Y_s^{t,x} &= \Phi(X_T^{t,x}) + \int_s^T f(X_u^{t,x}, Y_u^{t,x}) du - \sum_{i=0}^d \int_s^T Z_u^{(i)} dB^i(u), \end{aligned} \tag{1.2}$$

where $x \in \mathbb{R}^N$, $V_0, V_1, \dots, V_d \in C_b^\infty(\mathbb{R}^N; \mathbb{R})$ and $\Phi : \mathbb{R}^N \rightarrow \mathbb{R}$, $f : \mathbb{R}^{N+1} \rightarrow \mathbb{R}$ are appropriately smooth functions. Note that we denote f a different function in Chapter 3 and Chapter 4 in order to be consistent with frequently used notation. Here, the forward component $X_s \in \mathbb{R}^N$ is written in the Stratonovich form and the backward component $Y_s^{t,x} \in \mathbb{R}$ in the Ito form. Our aim is the numerical evaluation of $Y_t^{t,x}$.

The framework of FBSDE was first introduced by Bismut [9], Pardoux and Peng proved the existence and uniqueness of FBSDE in [74] and established the Feynman-Kac formula in the general case in [73]. Since then, many researchers have investigated to this field from both the theoretical and application points of view. As for the theoretical developments, the smoothness of the solution of FBSDE was proved in [20], and the correspondence with fully nonlinear partial differential equations(PDEs) was proved in [19]. There are also substantial studies on financial applications of FBSDEs [25, 30, 16], refer to the monographs [31]. Recently, the XVA pricing i.e., various valuation adjustments for derivative instruments which is a current big issue in the financial industry, is formulated using FBSDE in a general and unified way [75, 15, 53, 11]. Although we can formulate various complicated problems using FBSDE, it is not easy to apply them to practical problems due to the difficulty of numerical treatments.

Many numerical methods have been proposed for FBSDE, they can be classified into three approaches. The first approach is the nonlinear PDE approach using the nonlinear Feynman–Kac formula which is also called the four-step scheme [56]. In this approach, the nonlinear Feynman–Kac formula and the numerical methods for solving nonlinear PDEs and forward SDEs are used. This approach has been extensively studied [28, 57]; in particular, higher-order discretization methods of forward SDEs were applied in [61]. The second approach is the simulation. As described below, in the simulation approach, the calculations consists of discretization and integration. Although some studies were conducted on the discretization of backward SDE [63], a discretization method for a general class of FBSDE was first introduced in [12, 87]. The third approach involves using various expansions [34, 82, 13, 43]. Some methods of this approach are practically feasible and can derive some meaningful calculation formulas. However, we cannot know the true value using this approach.

This study focuses on the second approach, i.e., the simulation approach. In the simulation approach, the calculations are implemented in two steps: discretization and integration. Discretization means the construction of a set of random variables $\{\hat{X}_{s_i}, \hat{Y}_{s_i}\}_{i=0,1,\dots,n}$ ($t = s_0 < s_1 < \dots < s_n = T$) that approximates the processes $(X_s, Y_s^{t,x})$. Then, since $\hat{Y}_t$ usually depends on the conditional expectations of $\{\hat{X}_{s_i}, \hat{Y}_{s_i}\}_{i=1,\dots,n}$, it is required to calculate these expectations by integration. Here, the integration involves hard calculations of the conditional expectations, which is significantly different from the simulation for SDEs; this makes practical implementations challenging. This problem is called “nested simulation” or “multiple Monte Carlo” problem. To overcome this issue, many methods have been proposed both for the discretization and integration, e.g., the quantization tree [6], the Malliavin integration by part formula [12, 87], the least square Monte-Carlo method [40], the Picard iteration [8], the sparse grid method [86], the multistep method [88, 17] the cubature method combined with the TBBA technique [22, 23], the Runge–Kutta method [18], and the Fourier method [78]. In [88, 23, 18], higher-order discretization methods of FBSDEs have been proposed. Here, the higher-order discretization methods of FBSDEs mean the constructions of sets of random variables $\{\hat{X}_{s_i}, \hat{Y}_{s_i}\}_{i=0,1,\dots,n}$ ($t = s_0 < s_1 < \dots < s_n = T$) such that $\left|Y_t^{t,x} - \hat{Y}_t\right| < C_p/n^p$ holds for $p \in \mathbb{N}$; these

discretization methods are called order p .

Before the discretization methods of FBSDEs are studied, a large number of studies had been conducted for higher-order discretization methods of forward SDEs as described in Chapter 3. To use Kusuoka scheme, it is required to generate the ordinary differential equation(ODE) valued random variables for simulating a path. Certain discretization methods have been proposed using discrete random variables [55, 67]. When we use these methods iteratively along the time steps, the support of the intermediate measures grows exponentially as the number of partitions increases. To overcome this problem, the TBBA technique [21] was applied to these methods in [68, 69, 65, 24]. On the other hand, using the framework of Kusuoka scheme and the Gaussian random variables, the N–V and N–N methods have been proposed as practically feasible second-order methods, and the $Q_{(s)}^{(7,2)}$ method [79, 80] as a third-order method as described in Chapter 3.

1.3.2 Results of Chapter 4

In this chapter, we present second and third-order discretization methods of FBSDEs using the Kusuoka scheme, the higher-order numerical integration methods, and the TBBA technique. In our proposed methods, the forward component X_s is discretized using the Kusuoka scheme with discrete random variables, and the backward component Y_s using a higher-order numerical integration method suitable for the discretization method of the forward component; specifically, the N–V and N–N methods combined with the trapezoidal rule attain the second-order and the $Q_{(s)}^{(7,2)}$ method combined with the Simpson's the third-order. These methods construct random variables $\{\hat{X}_{s_i}, \hat{Y}_{s_i}\}_{i=0,1,\dots,n}$ on a discrete probability space. We approximate the law of $\{\hat{X}_{s_i}, \hat{Y}_{s_i}\}_{i=0,1,\dots,n}$ using the TBBA technique; in particular, we introduce an efficient implementation technique called the depth-first tree traversal. This technique is essential for practical implementation, because memory explosion might occur without this technique.

In addition, we discuss the applications of the proposed methods to the XVA pricing, particularly to the CVA. Using the proposed methods, we calculate the CVA

prices formulated by FBSDEs under the Black–Scholes and Heston models, which are usually considered to be hard to solve numerically. Our numerical results show that the expected theoretical order and computational efficiency are achieved. Moreover, we could observe the effectiveness of higher-order methods rather than the case of SDEs.

Bibliography

- [1] Anis Al Gerbi, Benjamin Jourdain, and Emmanuelle Clement. Ninomiya-Victoir scheme : strong convergence properties and discretization of the involved Ordinary Differential Equations. dec 2016.
- [2] Aurélien Alfonsi. On the discretization schemes for the CIR and Bessel squared processes. *Monte Carlo Methods and Applications*, 11(4):355–384, 2005.
- [3] Aurélien Alfonsi. High order discretization schemes for the CIR process: Application to affine term structure and Heston models. *Mathematics of Computation*, 79(269):209–237, 2010.
- [4] Aurélien Alfonsi. Strong order one convergence of a drift implicit Euler scheme: Application to the CIR process. *Statistics & Probability Letters*, 83(2):602–607, 2013.
- [5] Leif Andersen. Simple and efficient simulation of the Heston stochastic volatility model. *Journal of Computational Finance*, 11(3):1–42, 2008.
- [6] Vlad Bally, Gilles Pages, and Others. A quantization algorithm for solving multi-dimensional discrete-time optimal stopping problems. *Bernoulli*, 9(6):1003–1049, 2003.
- [7] Christian Bayer, Peter Friz, and Ronnie Loeffen. Semi-closed form cubature and applications to financial diffusion models. *Quantitative Finance*, 13(5):769–782, 2013.

- [8] Christian Bender and Robert Denk. A forward scheme for backward SDEs. *Stochastic processes and their applications*, 117(12):1793–1812, 2007.
- [9] Jean Michel Bismut. Conjugate convex functions in optimal stochastic control. *Journal of Mathematical Analysis and Applications*, 44(2):384–404, 1973.
- [10] Bela Bollobas. *Graph theory: an introductory course*. Graduate texts in mathematics. Springer Verlag, 1979.
- [11] Anastasia Borovykh, Andrea Pascucci, and Cornelis W Oosterlee. Efficient Computation of Various Valuation Adjustments Under Local Lévy Models. *SIAM Journal on Financial Mathematics*, 9(1):251–273, 2018.
- [12] Bruno Bouchard and Nizar Touzi. Discrete-time approximation and Monte-Carlo simulation of backward stochastic differential equations. *Stochastic Processes and their Applications*, 111(2):175–206, 2004.
- [13] Philippe Briand, Céline Labart, and Others. Simulation of BSDEs by Wiener chaos expansion. *The Annals of Applied Probability*, 24(3):1129–1171, 2014.
- [14] Damiano Brigo and Massimo Morini. Close-out convention tensions. *Risk Magazine*, pages 74–78, 2011.
- [15] Christoph Burgard and M Kjaer. PDE Representations of Derivatives with Bilateral Counterparty Risk and Funding Costs. *The journal of Credit Risk*, 7:75–93, 2011.
- [16] René Carmona. *Indifference pricing: theory and applications*. Princeton University Press, 2008.
- [17] Jean-François Chassagneux. Linear multistep schemes for BSDEs. *SIAM Journal on Numerical Analysis*, 52(6):2815–2836, 2014.
- [18] Jean-François Chassagneux, Dan Crisan, and Others. Runge–Kutta schemes for backward stochastic differential equations. *The Annals of Applied Probability*, 24(2):679–720, 2014.

- [19] Patrick Cheridito, H Mete Soner, Nizar Touzi, and Nicolas Victoir. Second-order backward stochastic differential equations and fully nonlinear parabolic PDEs. *Communications on Pure and Applied Mathematics*, 60(7):1081–1110, 2007.
- [20] Dan Crisan and François Delarue. Sharp derivative bounds for solutions of degenerate semi-linear partial differential equations. *Journal of Functional Analysis*, 263(10):3024–3101, 2012.
- [21] Dan Crisan and Terry Lyons. Minimal entropy approximations and optimal algorithms. *Monte Carlo Methods and Applications*, 8(4):343–356, 2002.
- [22] Dan Crisan and Konstantinos Manolarakis. Solving backward stochastic differential equations using the cubature method: Application to nonlinear pricing. *SIAM Journal on Financial Mathematics*, 3(1):534–571, 2012.
- [23] Dan Crisan and Konstantinos Manolarakis. Second order discretization of backward SDEs and simulation with the cubature method. *Annals of Applied Probability*, 24(2):652–678, 2014.
- [24] Dan Crisan and Salvador Ortiz-Latorre. A kusuoka–lyons–victoir particle filter. In *Proc. R. Soc. A*, volume 469, page 20130076. The Royal Society, 2013.
- [25] Jakša Cvitanić and Ioannis Karatzas. Hedging contingent claims with constrained portfolios. *The Annals of Applied Probability*, pages 652–681, 1993.
- [26] Steffen Dereich, Andreas Neuenkirch, and Lukasz Szpruch. An Euler-type method for the strong approximation of the Cox–Ingersoll–Ross process. In *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, volume 468, pages 1105–1115, 2011.
- [27] Philipp Doersek, Josef Teichmann, and Dejan Velušček. Cubature methods for stochastic (partial) differential equations in weighted spaces. *Stochastic Partial Differential Equations: Analysis and Computations*, 1(4):634–663, 2013.

- [28] Jim Douglas, Jin Ma, Philip Protter, and Others. Numerical methods for forward-backward stochastic differential equations. *The Annals of Applied Probability*, 6(3):940–968, 1996.
- [29] Darrell Duffie and Ming Huang. Swap rates and credit quality. *The Journal of Finance*, 51(3):921–949, 1996.
- [30] Nicole El Karoui, E Pardoux, and Marie Claire Quenez. Reflected backward SDEs and American options. *Numerical methods in finance*, 13:215–231, 1997.
- [31] Nicole El Karoui, Shige Peng, and Marie Claire Quenez. Backward stochastic differential equations in finance. *Mathematical finance*, 7(1):1–71, 1997.
- [32] William Feller. Two Singular Diffusion Problems. *Annals of Mathematics*, 54(1):173–182, 1951.
- [33] Masaaki Fujii and Akihiko Takahashi. Derivative pricing under asymmetric and imperfect collateralization and CVA. *Quantitative Finance*, 13(5):749–768, 2013.
- [34] Masaaki Fujii and Akihiko Takahashi. Perturbative expansion technique for non-linear FBSDEs with interacting particle method. *Asia-Pacific Financial Markets*, 22(3):283–304, 2015.
- [35] Takehiko Fujiwara. Sixth order methods of Kusuoka approximation. *PreprintSeries, Graduate School of Mathematical Sciences, Univ. Tokyo*, 2006.
- [36] J G Gaines. The algebra of iterated stochastic integrals. *Stochastics and Stochastic Reports*, 49(3-4):169–179, 1994.
- [37] J G Gaines. A basis for iterated stochastic integrals. *Mathematics and computers in simulation*, 38(1):7–11, 1995.
- [38] Michael Giles. Multilevel monte carlo path simulation. *Operations Research*, 56(3):607–617, 2008.
- [39] Paul Glasserman. *Monte Carlo methods in financial engineering*, volume 53. Springer Science & Business Media, 2013.

- [40] Emmanuel Gobet, Jean-Philippe Lemor, and Xavier Warin. A regression-based Monte Carlo method to solve backward stochastic differential equations. *The Annals of Applied Probability*, 15(3):2172–2202, 2005.
- [41] Lajos Gyurkó and Terry Lyons. Efficient and practical implementations of cubature on Wiener space. In *Stochastic analysis 2010*, pages 73–111. Springer, 2011.
- [42] Patrick Hagan, Deep Kumar, Andrew Lesniewski, and Diana Woodward. Managing smile risk. *The Best of Wilmott*, 1:249–296, 2002.
- [43] Pierre Henry-Labordère. Cutting CVA’s complexity. *Risk*, 25(7):67, 2012.
- [44] Steven Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of financial studies*, 6(2):327–343, 1993.
- [45] Stephen Joe and Frances Kuo. Remark on algorithm 659: Implementing {S}obol’s quasirandom sequence generator. *ACM Transactions on Mathematical Software (TOMS)*, 29(1):49–57, 2003.
- [46] Stephen Joe and Frances Kuo. Constructing {S}obol sequences with better two-dimensional projections. *SIAM Journal on Scientific Computing*, 30(5):2635–2654, 2008.
- [47] Peter Kloeden and Eckhard Platen. *Numerical Solution of Stochastic Differential Equations*. Springer, 1999.
- [48] Shigeo Kusuoka. Approximation of expectation of diffusion process and mathematical finance. In *Proceedings of Final Taniguchi Symposium, Nara*, volume 31 of *Advanced Studies in Pure Mathematics*, pages 147–165, 2001.
- [49] Shigeo Kusuoka. Malliavin calculus revisited. *J. Math. Sci. Univ. Tokyo*, 10(2):261–277, 2003.

- [50] Shigeo Kusuoka. Approximation of expectation of diffusion processes based on Lie algebra and Malliavin calculus. *Advances in Mathematical Economics*, 6:69–83, 2004.
- [51] Shigeo Kusuoka. Gaussian K-scheme: justification for KLVN method. *Advances in Mathematical Economics*, 17:71–120, 2013.
- [52] Shigeo Kusuoka, Mariko Ninomiya, and Syoiti Ninomiya. Application of the Kusuoka approximation to pricing barrier options. In *Proceedings of the ISCIE International Symposium on Stochastic Systems Theory and its Applications*, volume 2016, pages 171–180. The ISCIE Symposium on Stochastic Systems Theory and Its Applications, 2016.
- [53] Andrew Lesniewski and Anja Richter. Managing counterparty credit risk via BSDEs. 2016.
- [54] Christian Litterer, Terry Lyons, and Others. High order recombination and an application to cubature on Wiener space. *The Annals of Applied Probability*, 22(4):1301–1327, 2012.
- [55] Terry Lyons and Nicolas Victoir. Cubature on Wiener Space. In *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, volume 460, pages 169–198, 2004.
- [56] Jin Ma, Philip Protter, and Jiongmin Yong. Solving forward-backward stochastic differential equations explicitly a four step scheme. *Probability Theory and Related Fields*, 98(3):339–359, 1994.
- [57] Jin Ma, Jie Shen, and Yanhong Zhao. On numerical approximations of forward-backward stochastic differential equations. *SIAM Journal on Numerical Analysis*, 46(5):2636–2661, 2008.
- [58] Gisiro Maruyama. Continuous Markov processes and stochastic equations. *Rendiconti del Circolo Matematico di Palermo*, 4(1):48, 1955.

- [59] Guy Mélançon and Christophe Reutenauer. Lyndon words, free algebras and shuffles. *Canadian J. Math*, 41:577–591, 1989.
- [60] Milstein. Approximate integration of stochastic differential equations. *Theory of Probability & Its Applications*, 19(3):557–562, 1975.
- [61] Milstein and Tretyakov. Numerical Algorithms for Forward-Backward Stochastic Differential Equations. *SIAM Journal on Scientific Computing*, 28(2):561–582, 2006.
- [62] Yusuke Morimoto and Makiko Sasada. Algebraic structure of vector fields in financial diffusion models and its applications. *Quantitative Finance*, pages 1–13, 2017.
- [63] Toshiyuki Nakayama. Approximation of BSDE’s by Stochastic Difference Equation’s. *J. Math. Sci. Univ. Tokyo*, 9:257–277, 2002.
- [64] Harald Niederreiter. *Random number generation and quasi-Monte Carlo methods*. SIAM, Philadelphia, 1992.
- [65] Mariko Ninomiya. Application of the Kusuoka approximation with a tree-based branching algorithm to the pricing of interest-rate derivatives under the HJM model. *LMS Journal of Computation and Mathematics*, 13:208–221, 2010.
- [66] Mariko Ninomiya and Syoiti Ninomiya. A new higher-order weak approximation scheme for stochastic differential equations and the Runge–Kutta method. *Finance and Stochastics*, 13(3):415–443, 2009.
- [67] Syoiti Ninomiya. A new simulation scheme of diffusion processes: application of the Kusuoka approximation to finance problems. *Mathematics and Computers in Simulation*, 62(3):479–486, 2003.
- [68] Syoiti Ninomiya. A partial sampling method applied to the Kusuoka approximation. *Monte Carlo methods and Applications*, 9(1):27–38, 2003.

- [69] Syoiti Ninomiya and Makoto Mitsuzono. A partial sampling method for approximation of expectation of diffusion processes by using the exotic filters. *preprint*, 2004.
- [70] Syoiti Ninomiya and Shu Tezuka. Toward real-time pricing of complex financial derivatives. *Applied Mathematical Finance*, 3(1):1–20, 1996.
- [71] Syoiti Ninomiya and Nicolas Victoir. Weak approximation of stochastic differential equations and application to derivative pricing. *Applied Mathematical Finance*, 15(2):107–121, 2008.
- [72] Kojiro Oshima, Josef Teichmann, and Dejan Velušček. A new extrapolation method for weak approximation schemes with applications. *Annals of Applied Probability*, 22(3):1008–1045, 2012.
- [73] Etienne Pardoux and Shige Peng. Backward stochastic differential equations and quasilinear parabolic partial differential equations. In *Stochastic partial differential equations and their applications*, pages 200–217. Springer, 1992.
- [74] E Pardoux S Peng and Others. Adapted solution of backward stochastic differential equations. *System Control Lett*, 4:55–61, 1990.
- [75] Vladimir Piterbarg. Funding beyond discounting: collateral agreements and derivatives pricing. *Risk*, 10(2):97–102, 2010.
- [76] Christophe Reutenauer. *Free Lie Algebras*. LMS monographs. Clarendon Press, Oxford, 1993.
- [77] Clément Rey. Convergence in total variation distance for a third order scheme for one dimensional diffusion process. *to appear in Monte Carlo Methods and Applications*, 2017.
- [78] Maria J Ruijter and Cornelis W Oosterlee. Numerical Fourier method and second-order Taylor scheme for backward SDEs in finance. *Applied Numerical Mathematics*, 103:1–26, 2016.

- [79] Yuji Shinozaki. Higher order K-scheme and application to derivative pricing. In *Proceedings of the 47th ISCIE International Symposium on Stochastic Systems Theory and Its Applications (Dec. 2015, Honolulu)*, pages 137–143, 2016.
- [80] Yuji Shinozaki. Construction of a third-order K-scheme and its application to financial models. *SIAM Journal on Financial Mathematics*, 8(1):901–932, 2017.
- [81] Hector Sussmann. A product expansion for the Chen series. *Theory and applications of nonlinear control systems*, pages 323–335, 1986.
- [82] Akihiko Takahashi and Toshihiro Yamada. An Asymptotic Expansion for Forward–Backward SDEs: A Malliavin Calculus Approach. *Asia-Pacific Financial Markets*, 23(4):337–373, 2016.
- [83] Satoshi Takanobu. Multiple stochastic integrals appearing in the stochastic Taylor expansions. *Journal of the Mathematical Society of Japan*, 47(1):67–92, 1995.
- [84] Denis Talay and Luciano Tubaro. Expansion of the global error for numerical schemes solving stochastic differential equations. *Stochastic analysis and applications*, 8(4):483–509, 1990.
- [85] Haruo Yoshida. Construction of higher order symplectic integrators. *Physics Letters A*, 150(5):262–268, 1990.
- [86] Guannan Zhang, Max Gunzburger, and Weidong Zhao. A sparse-grid method for multi-dimensional backward stochastic differential equations. *Journal of Computational Mathematics*, 31(3):221–248, 2013.
- [87] Jianfeng Zhang. A numerical scheme for BSDEs. *Annals of Applied Probability*, 14(1):459–488, 2004.
- [88] Weidong Zhao, Guannan Zhang, and Lili Ju. A stable multistep scheme for solving backward stochastic differential equations. *SIAM Journal on Numerical Analysis*, 48(4):1369–1394, 2010.