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COMPARE THE RATIO OF SYMMETRIC POLYNOMIALS OF ODDS TO ONE AND STOP

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Abstract

In this paper, we deal with an optimal stopping problem whose objective is to maximize the probability of selecting k out of the last ℓ success, given a sequence of independent Bernoulli trials of length N, where k and ℓ are predetermined integers satisfying $1 \leq k \leq \ell < N$. This problem includes some odds problems as special cases, e.g., Bruss' odds problem, Bruss and Paindaveine's problem of selecting the last ℓ successes, and Tamaki's multiplicative odds problem for stopping at any of the last m successes. We show that an optimal stopping rule is obtained by a threshold strategy. We also present the tight lower bound and an asymptotic lower bound for the probability of win. Interestingly, our asymptotic lower bound is attained by using a variation of the well-known secretary problem, which is a special case of the odds problem. Our approach is based on the application of Newton's inequalities and optimization technique, which gives a unified view to the previous works.

Keywords: optimal stopping; odds problem; lower bound; secretary problem; Newton's inequality

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1. Introduction

In this paper, we discuss a variation of the odds problem, which is an extension of Bruss' odds problem discussed in [3] for the first time. Let $X_1, X_2, \ldots, X_N$ denote a sequence of independent Bernoulli random variables. If $X_j = 1$, we say that the outcome of random variable X_j is a *success*. Otherwise ($X_j = 0$), we say that the outcome of X_j is a *failure*. These random variables can be regarded as results of an underlying discrete stochastic process. For example, we can assume them to constitute the record process. This paper deals with an optimal stopping problem of maximizing the probability of selecting k out of the last ℓ successes where $1 \leq k \leq \ell < N$. More precisely, the problem may be stated as follows.

We consider a game in which a player is given the digits (realization of random variables) one by one and allowed to select the index of the variable when he observes a success. The number of selected indices of variables must be less than or equal to k. The player *wins* if he selected exactly k indices of variables contained in the set of last ℓ successes. For example, consider the case with $N = 8$, $k = 3$ and $\ell = 4$. When $(X_1, X_2, \ldots, X_8)$ has a vector of realized values $(0, 1, 1, 0, 0, 1, 1, 1)$, the player wins if he selected exactly 3 indices of variables in the set $\{X_3, X_6, X_7, X_8\}$. We deal with a problem of maximizing a probability of win. It is easy to see that the player wins if and only if the first selected variable is in $\{X_3, X_6\}$ by simply enumerating following $k = 3$ successes. Under this strategy, the player wins if the set of selected indices is corresponding to either $\{X_3, X_6, X_7\}$ or $\{X_6, X_7, X_8\}$. Thus, the player only need to observe the sequence with an objective to correctly

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predict the occurrence of the m-th last success satisfying $k \leq m \leq \ell$. From the above, the problem becomes a "single" stopping problem of maximizing the probability of stopping on a random variable X_m satisfying $X_m = 1$ and $\ell \geq X_m + X_{m+1} + \cdots + X_N \geq k$. We present an optimal stopping rule and an asymptotic lower bound for the probability of "*win*" (*i.e.*, obtaining m-th last success with $k \leq m \leq \ell$).

TABLE 1: Previous results and our results.

model	condition	lower bound		key inequality ($\star$)	
Bruss [3]	$\ell = k = 1$	e^{-1}	[4]	$\frac{r_i + r_{i+1} + \cdots + r_N}{1} = \frac{\boldsymbol{e}_1(\boldsymbol{r})}{\boldsymbol{e}_0(\boldsymbol{r})} < 1$	[3]
B&P(◇) [5]	$\ell = k \geq 1$	$\dfrac{\ell^\ell}{(\ell!)e^\ell}$	(*)	$\dfrac{\boldsymbol{e}_\ell(\boldsymbol{r})}{\boldsymbol{e}_{\ell-1}(\boldsymbol{r})} < 1$	[5]
Tamaki [13]	$\ell \geq k = 1$	$\exp\left(-(\ell!)^{\frac{1}{\ell}}\right) \sum_{m=1}^{\ell} \dfrac{(\ell!)^{\frac{m}{\ell}}}{m!}$	[10]	$\dfrac{\boldsymbol{e}_\ell(\boldsymbol{r})}{\boldsymbol{e}_0(\boldsymbol{r})} < 1$	[13]
this paper	$\ell \geq k \geq 1$	(‡) see below	(*)	$\dfrac{\boldsymbol{e}_\ell(\boldsymbol{r})}{\boldsymbol{e}_{k-1}(\boldsymbol{r})} < 1$	(*)

◇ Bruss and Paindaveine.

‡ $\exp\left(-\left(\frac{\ell!}{(k-1)!}\right)^{\frac{1}{\ell-k+1}}\right) \sum_{m=k}^{\ell} \left(\frac{1}{m!}\left(\frac{\ell!}{(k-1)!}\right)^{\frac{m}{\ell-k+1}}\right)$.

* Results obtained in this paper.

$\star$ An optimal stopping rule is attained by the threshold strategy defined by the minimum index i satisfying the key inequality in the last column (see (2) for detail) where $\boldsymbol{r} = (r_i, r_{i+1}, \ldots, r_N)$ and other notations are defined by (1).

When $\Pr[X_i = 1] = 1/i$, our problem gives a variation of the secretary problems. Especially, in the case that $\ell = k = 1$, the problem is equivalent to the classical secretary problem. One of the reason why the odds problems are popular in the optimal stopping theory is that it includes the secretary problem as a special case.

Although our problem setting looks artificial, it includes some odds problems as special cases (see Table 1). When $\ell = k = 1$, the problem is equivalent to the well-known Bruss' odds problem [3], which has an elegant and simple optimal stopping rule known as the *Odds theorem* or *Sum-the-Odds theorem.* A typical lower bound for an asymptotic optimal value (the probability of win), when N approaches infinity, has been shown to be e^{-1} by Bruss [4], which is equal to that for the classical secretary problem. If $\ell = k \geq 1$, Bruss and Paindaveine [5] showed that an optimal stopping rule is obtained by a threshold strategy. When $\ell \geq k = 1$, Tamaki [13] demonstrated the *Sum-the-Multiplicative-Odds theorem*, which gives an optimal stopping rule obtained using a threshold strategy. Recently, we discussed his model and showed a lower bound for the probability of win [10]. Bruss and Paindaveine [5] and Tamaki [13] also discussed the corresponding secretary problem and derived asymptotic optimal values. The related problem of the distribution of the rank of the accepted candidate has been studied by Bartoszyński [1] and of the last record rank before the last acceptance by Bruss [2].

In this paper, we describe an optimal stopping rule and derive the greatest lower bound for the probability of win for the the problem of selecting k out of the last ℓ success. The asymptotic value of our lower bound is equivalent to the asymptotic optimal value for the corresponding secretary problem appearing in Bruss [4], Bruss and Paindaveine [5], and Tamaki [13]. A special feature of our proof is the application of Newton's inequalities [11] and optimization technique to obtain our bound.

2. Elementary Symmetric Polynomials

For any pair of positive integers m, N satisfying $1 \leq m \leq N$ and a vector $\boldsymbol{r} \in \mathbb{R}^N$, $\boldsymbol{e}_m(\boldsymbol{r})$ denotes the m-th *elementary symmetric polynomial* (function) of $\boldsymbol{r} = (r_1, r_2, \dots, r_N)$ defined by

$$\boldsymbol{e}_m(\boldsymbol{r}) = \sum_{1 \le i_1 < i_2 < \cdots < i_m \le N} r_{i_1} r_{i_2} \cdots r_{i_m} = \sum_{\substack{B \subseteq \{1,2,\dots,N\} \\ \text{and } |B| = m}} \prod_{i \in B} r_i, \tag{1}$$

which is the sum of $\binom{N}{m}$ terms. We also define that $\boldsymbol{e}_0(\boldsymbol{r}) = 1$. The *m-th elementary symmetric mean* of $\boldsymbol{r}$ is defined by

$$S_m(\boldsymbol{r}) = \frac{\boldsymbol{e}_m(\boldsymbol{r})}{\binom{N}{m}} \quad (\forall m \in \{1, 2, \dots, N\}) \text{ and } S_0(\boldsymbol{r}) = 1.$$

We abbreviate $S_m(\boldsymbol{r})$ to S_m, when there is no ambiguity. The elementary symmetric polynomials satisfy the following inequalities shown by Newton.

Theorem 2.1. (Newton's inequalities [11]) *For every non-negative vector* $\boldsymbol{r} \in \mathbb{R}_+^N$ *and a positive integer* $1 \le m < N$,

$$S_m(\boldsymbol{r})^2 \ge S_{m-1}(\boldsymbol{r}) S_{m+1}(\boldsymbol{r}),$$

with equality exactly when all the r_i *are equal.*

Newton's inequalities directly imply the following.

Lemma 1. *For any positive vector* $\widetilde{\boldsymbol{r}} = (\widetilde{r}_1, \widetilde{r}_2, \dots, \widetilde{r}_N) > \mathbf{0}$ *and integers* (m, ℓ) *satisfying* $1 \le m \le \ell \le N$, *the inequality* $\dfrac{\boldsymbol{e}_{\ell-1}(\widetilde{\boldsymbol{r}})}{\boldsymbol{e}_{m-1}(\widetilde{\boldsymbol{r}})} \ge \dfrac{\boldsymbol{e}_\ell(\widetilde{\boldsymbol{r}})}{\boldsymbol{e}_m(\widetilde{\boldsymbol{r}})}$ *holds.*

Proof. The positivity of $\widetilde{\boldsymbol{r}}$ implies that $S_{m'}(\widetilde{\boldsymbol{r}}) > 0 \quad (0 \le \forall m' \le N)$. Newton's inequalities is equivalent to the midpoint log-concavity $\log(S_{m'}) \ge (1/2)(\log(S_{m'-1}) + \log(S_{m'+1}))$, which directly yields the concavity of the sequence $(\log(S_0), \log(S_1), \log(S_2), \dots, \log(S_N))$ and the following inequalities:

$$\begin{aligned}
\frac{\log(S_m) + \log(S_{\ell-1})}{2} &\ge \frac{\log(S_{m-1}) + \log(S_\ell)}{2}, \\
S_m S_{\ell-1} &\ge S_{m-1} S_\ell, \\
\frac{\binom{N}{m-1} \boldsymbol{e}_{\ell-1}(\widetilde{\boldsymbol{r}})}{\binom{N}{\ell-1} \boldsymbol{e}_{m-1}(\widetilde{\boldsymbol{r}})} = \frac{S_{\ell-1}}{S_{m-1}} &\ge \frac{S_\ell}{S_m} = \frac{\binom{N}{m} \boldsymbol{e}_\ell(\widetilde{\boldsymbol{r}})}{\binom{N}{\ell} \boldsymbol{e}_m(\widetilde{\boldsymbol{r}})}, \\
\frac{\boldsymbol{e}_{\ell-1}(\widetilde{\boldsymbol{r}})}{\boldsymbol{e}_{m-1}(\widetilde{\boldsymbol{r}})} &\ge \left(\frac{N-m+1}{N-\ell+1}\right) \left(\frac{\ell}{m}\right) \frac{\boldsymbol{e}_\ell(\widetilde{\boldsymbol{r}})}{\boldsymbol{e}_m(\widetilde{\boldsymbol{r}})} \ge \frac{\boldsymbol{e}_\ell(\widetilde{\boldsymbol{r}})}{\boldsymbol{e}_m(\widetilde{\boldsymbol{r}})}.
\end{aligned}$$

■

Lemma 2. *For any positive vector* $\widetilde{\boldsymbol{r}} = (\widetilde{r}_1, \widetilde{r}_2, \ldots, \widetilde{r}_N) > \mathbf{0}$ *and integers* (m, ℓ, N) *satisfying* $0 \leq m \leq \ell < N$ *and* $N \geq 2$*, the inequality* $\dfrac{\boldsymbol{e}_\ell(\widetilde{\boldsymbol{r}})}{\boldsymbol{e}_m(\widetilde{\boldsymbol{r}})} \geq \dfrac{\boldsymbol{e}_\ell(\widetilde{\boldsymbol{r}}_{-1})}{\boldsymbol{e}_m(\widetilde{\boldsymbol{r}}_{-1})}$ *holds, where* $\widetilde{\boldsymbol{r}}_{-1} = (\widetilde{r}_2, \ldots, \widetilde{r}_N)$.

Proof. When $m = 0$, it is obvious from the positivity of $\widetilde{\boldsymbol{r}}$. Let us consider cases that $m \geq 1$. If we apply Lemma 1 to the positive vector $\widetilde{\boldsymbol{r}}_{-1}$, then we obtain the inequality $\dfrac{\boldsymbol{e}_{\ell-1}(\widetilde{\boldsymbol{r}}_{-1})}{\boldsymbol{e}_{m-1}(\widetilde{\boldsymbol{r}}_{-1})} \geq \dfrac{\boldsymbol{e}_\ell(\widetilde{\boldsymbol{r}}_{-1})}{\boldsymbol{e}_m(\widetilde{\boldsymbol{r}}_{-1})}$, which directly implies that

$$\frac{\boldsymbol{e}_\ell(\widetilde{\boldsymbol{r}})}{\boldsymbol{e}_m(\widetilde{\boldsymbol{r}})} = \frac{\widetilde{r}_1\, \boldsymbol{e}_{\ell-1}(\widetilde{\boldsymbol{r}}_{-1}) + \boldsymbol{e}_\ell(\widetilde{\boldsymbol{r}}_{-1})}{\widetilde{r}_1\, \boldsymbol{e}_{m-1}(\widetilde{\boldsymbol{r}}_{-1}) + \boldsymbol{e}_m(\widetilde{\boldsymbol{r}}_{-1})} \geq \frac{\boldsymbol{e}_\ell(\widetilde{\boldsymbol{r}}_{-1})}{\boldsymbol{e}_m(\widetilde{\boldsymbol{r}}_{-1})}.$$ ■

3. Threshold Strategy

We deal with a sequence of independent 0/1 random variables $X_1, X_2, \ldots, X_N$, where N is a given positive integer and the distribution is $\Pr[X_i = 1] = p_i$, $\Pr[X_i = 0] = 1 - p_i = q_i$, $0 < p_i < 1$ for each i. We define $r_i = p_i/q_i$ for each i. The r_i's are called *odds*. Given a pair of integers (k, ℓ) satisfying $1 \leq k \leq \ell < N$, we discuss a problem to predict the m-th last success satisfying $k \leq m \leq \ell$, if any, with maximum probability at the time of its occurrence.

In the rest of this section, we denote the subvector $(r_i, r_{i+1}, \ldots, r_N)$ by $\boldsymbol{r}^{[i]}$ and introduce the notations:

$$W_i \stackrel{\text{def.}}{=} \Pr[k \leq X_{i+1} + \cdots + X_N \leq \ell] = \frac{\sum_{m=k}^{\ell} \boldsymbol{e}_m(\boldsymbol{r}^{[i+1]})}{\prod_{j=i+1}^{N}(1 + r_j)}, \text{ and}$$

$$V_i \stackrel{\text{def.}}{=} \Pr[k \leq X_i + \cdots + X_N \leq \ell \mid X_i = 1] = \frac{\sum_{m=k-1}^{\ell-1} \boldsymbol{e}_m(\boldsymbol{r}^{[i+1]})}{\prod_{j=i+1}^{N}(1 + r_j)}.$$

We define an index i_* by

$$i_* \stackrel{\text{def.}}{=} \min\left\{ i \,\middle|\, 1 \leq i \leq N - \ell \quad \text{and} \quad \frac{\boldsymbol{e}_\ell(\boldsymbol{r}^{[i+1]})}{\boldsymbol{e}_{k-1}(\boldsymbol{r}^{[i+1]})} < 1 \right\}. \tag{2}$$

When the minimum in (2) is taken on the empty set, we put $i_* \stackrel{\text{def.}}{=} N - \ell + 1$.

Now we give an optimal rule.

Theorem 3.1. *Let us consider the problem of stopping at m-th last success with $k \leq m \leq \ell$ defined on $X_1, X_2, \ldots, X_N$ satisfying $r_i > 0$ $(\forall i)$ and $1 \leq k \leq \ell < N$. An optimal stopping rule is obtained by stopping at the first success $X_i = 1$ with $i \geq i_*$, and the corresponding probability of win is equal to W_{i_*-1}.*

Proof. First, we show a property of the ratio W_i/V_i. The definition of i_* and Lemma 2 directly induce the following,

$$\frac{\boldsymbol{e}_\ell(\boldsymbol{r}^{[i+1]})}{\boldsymbol{e}_{k-1}(\boldsymbol{r}^{[i+1]})} \begin{cases} \geq 1 & (0 \leq \forall i \leq i_* - 1), \\ < 1 & (i_* \leq \forall i \leq N - \ell). \end{cases} \tag{3}$$

The definitions of W_i and V_i imply

$$\frac{W_i}{V_i} = \frac{\displaystyle\sum_{m=k}^{\ell} \boldsymbol{e}_m(\boldsymbol{r}^{[i+1]})}{\displaystyle\prod_{j=i+1}^{N} (1 + r_j)} \cdot \frac{\displaystyle\prod_{j=i+1}^{N} (1 + r_j)}{\displaystyle\sum_{m=k-1}^{\ell-1} \boldsymbol{e}_m(\boldsymbol{r}^{[i+1]})} = \frac{\displaystyle\sum_{m=k}^{\ell-1} \boldsymbol{e}_m(\boldsymbol{r}^{[i+1]}) + \boldsymbol{e}_\ell(\boldsymbol{r}^{[i+1]})}{\displaystyle\sum_{m=k}^{\ell-1} \boldsymbol{e}_m(\boldsymbol{r}^{[i+1]}) + \boldsymbol{e}_{k-1}(\boldsymbol{r}^{[i+1]})} \tag{4}$$

and thus we have the following inequality,

$$\frac{W_i}{V_i} \begin{cases} \geq 1 & (0 \leq \forall i \leq i_* - 1), \\ < 1 & (i_* \leq \forall i \leq N - \ell). \end{cases} \tag{5}$$

From property (5), our problem becomes a monotone stopping problem and the one-stage look-ahead strategy gives an optimal stopping rule (see [6, 7, 8, 12] for example). Thus, an optimal stopping rule is attained by the threshold strategy with the threshold value

$$\tau \stackrel{\text{def.}}{=} \min \left\{ i \,\middle|\, 1 \leq i \leq N - \ell \quad \text{and} \quad \frac{W_i}{V_i} < 1 \right\}.$$

When the minimum in the above definition is taken on the empty set, we put $\tau \stackrel{\text{def.}}{=} N - \ell + 1$. If we employ the optimal threshold strategy defined by τ, it is also known that the corresponding probability of win is equal to $W_{\tau-1}$.

From equality (4), we have the property,

$$\frac{W_i}{V_i} < 1 \quad \text{if and only if} \quad \frac{\boldsymbol{e}_\ell(\boldsymbol{r}^{[i+1]})}{\boldsymbol{e}_{k-1}(\boldsymbol{r}^{[i+1]})} < 1$$

and thus $i_* = \tau$. As a result, an optimal stopping rule is obtained by the threshold strategy with the threshold value i_*, which does not select any index less than i_* and selects the first variable $X_i = 1$ satisfying $i_* \leq i$. The corresponding probability of win is equal to $W_{\tau-1} = W_{i_*-1}$. ■

4. Lower Bound

In this section, we discuss a lower bound for the probability of win under the optimal stopping rule. First, we discuss a lemma which plays an important role in this section.

Lemma 3. *Every positive vector* $\widetilde{\boldsymbol{r}} = (\widetilde{r}_1, \widetilde{r}_2, \ldots, \widetilde{r}_N) > 0$ *satisfies that*

$$(S_{k-1}(\widetilde{\boldsymbol{r}}))^{\ell-m} \geq (S_m(\widetilde{\boldsymbol{r}}))^{\ell-k+1}(S_\ell(\widetilde{\boldsymbol{r}}))^{k-1-m}, \quad \forall m \in \{0,1,2,\ldots,k-1\}, \tag{6}$$

$$(S_m(\widetilde{\boldsymbol{r}}))^{\ell-k+1} \geq (S_{k-1}(\widetilde{\boldsymbol{r}}))^{\ell-m}(S_\ell(\widetilde{\boldsymbol{r}}))^{m-k+1}, \quad \forall m \in \{k,k+1,\ldots,\ell\}, \tag{7}$$

$$(S_\ell(\widetilde{\boldsymbol{r}}))^{m-k+1} \geq (S_{k-1}(\widetilde{\boldsymbol{r}}))^{m-\ell}(S_m(\widetilde{\boldsymbol{r}}))^{\ell-k+1}, \quad \forall m \in \{\ell+1,\ldots,N\}. \tag{8}$$

Proof. In the following, we abbreviate $S_m(\widetilde{\boldsymbol{r}})$ to S_m for simplicity. Newton's inequalities directly imply the concavity of the sequence $(\log(S_0), \ldots, \log(S_N))$ and thus we have the following inequalities:

$$\log(S_{k-1}) \geq \frac{(\ell-k+1)\log(S_m) + (k-1-m)\log(S_\ell)}{\ell-m}, \ \forall m \in \{0,1,\ldots,k-1\},$$
$$\log(S_m) \geq \frac{(\ell-m)\log(S_{k-1}) + (m-k+1)\log(S_\ell)}{\ell-k+1}, \ \forall m \in \{k,k+1,\ldots,\ell\},$$
$$\log(S_\ell) \geq \frac{(m-\ell)\log(S_{k-1}) + (\ell-k+1)\log(S_m)}{m-k+1}, \ \forall m \in \{\ell+1,\ell+2,\ldots,N\}.$$

Consequently, inequalities (6) (7) and (8) are obtained. ■

Theorem 4.1. *Let us consider the problem of stopping at m-th last success with* $k \leq m \leq \ell$ *defined on* $X_1, X_2, \ldots, X_N$ *satisfying* (1) $r_i > 0$ $(\forall i)$, (2) $1 \leq k \leq \ell < N$, (3) $1 > \frac{\boldsymbol{e}_\ell(\bar{\boldsymbol{r}})}{\boldsymbol{e}_{k-1}(\bar{\boldsymbol{r}})}$ *where* $\bar{\boldsymbol{r}} = (r_{N-\ell+1}, r_{N-\ell+2}, \ldots, r_N) \in \mathbb{R}^\ell$ *and*

(4) $\frac{\boldsymbol{e}_\ell(\boldsymbol{r})}{\boldsymbol{e}_{k-1}(\boldsymbol{r})} \geq 1$. *Under the optimal stopping rule, the greatest lower bound for the probability of win is equal to*

$$\frac{\sum_{m=k}^{\ell} \binom{N}{m} \theta^m}{(1+\theta)^N} \quad \textit{where} \quad \theta = \left(\frac{\binom{N}{k-1}}{\binom{N}{\ell}} \right)^{\frac{1}{\ell-k+1}}.$$

Proof. Since the optimal stopping rule defined by (2) is a threshold strategy, the truncation of the subsequence $X_1, X_2, \ldots, X_{i_*-1}$ does not affect the probability of win. Thus, we only need to consider a case where

$$\boldsymbol{e}_{k-1}(r_2, r_3, \ldots, r_N) - \boldsymbol{e}_\ell(r_2, r_3, \ldots, r_N) > 0 \quad \text{and} \tag{9}$$

$$\boldsymbol{e}_{k-1}(r_1, r_2, r_3 \ldots, r_N) - \boldsymbol{e}_\ell(r_1, r_2, r_3 \ldots, r_N) \leq 0. \tag{10}$$

Under assumptions (9) and (10), the optimal stopping rule is obtained by setting $i_* = 1$, and the probability of win is equal to

$$W_0 = \frac{\sum_{m=k}^{\ell} \boldsymbol{e}_m(\boldsymbol{r})}{(1+r_1)(1+r_2)\cdots(1+r_N)}.$$

Thus, the greatest lower bound for the probability of win under the optimal stopping rule is equal to the optimal value of an optimization problem,

$$\begin{aligned} \text{P1}: \quad & \text{min.} \quad && \text{P}_{\text{win}}(N) \stackrel{\text{def.}}{=} \frac{\sum_{m=k}^{\ell} \boldsymbol{e}_m(\boldsymbol{r})}{(1+r_1)(1+r_2)\cdots(1+r_N)} \\ & \text{s. t.} && 0 < r_i \quad (\forall i \in \{1, 2, \ldots, N\}), \\ &&& \boldsymbol{e}_{k-1}(\boldsymbol{r}_{-1}) - \boldsymbol{e}_\ell(\boldsymbol{r}_{-1}) > 0, \\ &&& \boldsymbol{e}_{k-1}(\boldsymbol{r}) - \boldsymbol{e}_\ell(\boldsymbol{r}) \leq 0, \end{aligned} \tag{11}$$

where $\boldsymbol{r}_{-1} = (r_2, r_3, \ldots, r_N)$.

We show that we only need to consider feasible solutions satisfying constraint (11) by equality. Let $\boldsymbol{r'}$ be a feasible solution of P1 satisfying $\boldsymbol{e}_{k-1}(\boldsymbol{r'}) - \boldsymbol{e}_\ell(\boldsymbol{r'}) < 0$. We introduce a function $f(r) : [0, r'_1] \to \mathbb{R}$ defined by

$$f(r) = \boldsymbol{e}_{k-1}(r, r'_2, r'_3, \ldots, r'_N) - \boldsymbol{e}_\ell(r, r'_2, r'_3, \ldots, r'_N),$$

which is obtained by fixing $N-1$ variables $\{r'_2, r'_3, \ldots, r'_N\}$. The assumption on $\boldsymbol{r'}$ directly implies that

$$f(r'_1) = \boldsymbol{e}_{k-1}(\boldsymbol{r'}) - \boldsymbol{e}_\ell(\boldsymbol{r'}) < 0 < \boldsymbol{e}_{k-1}(\boldsymbol{r'}_{-1}) - \boldsymbol{e}_\ell(\boldsymbol{r'}_{-1}) = f(0).$$

From the continuity of $f(r)$, the mean-value theorem implies the existence of a value $r'' \in (0, r'_1)$ satisfying $f(r'') = 0$. Obviously, $(r'', r'_2, r'_3, \ldots, r'_N)$ is feasible to P1. The objective function value $\mathrm{P}'_{\mathrm{win}}(N)$ corresponding to $\boldsymbol{r'}$ becomes

$$\begin{aligned}
\mathrm{P}'_{\mathrm{win}}(N) &= \frac{\sum_{m=k}^{\ell} \boldsymbol{e}_m(\boldsymbol{r'})}{(1+r'_1)(1+r'_2)\cdots(1+r'_N)} = \frac{\sum_{m=k}^{\ell}\Big(\boldsymbol{e}_m(\boldsymbol{r'}_{-1}) + r'_1 \boldsymbol{e}_{m-1}(\boldsymbol{r'}_{-1})\Big)}{(1+r'_1)(1+r'_2)\cdots(1+r'_N)} \\
&= \frac{\sum_{m=k}^{\ell}\Big(\boldsymbol{e}_m(\boldsymbol{r'}_{-1}) - \boldsymbol{e}_{m-1}(\boldsymbol{r'}_{-1}) + (1+r'_1)\boldsymbol{e}_{m-1}(\boldsymbol{r'}_{-1})\Big)}{(1+r'_1)(1+r'_2)\cdots(1+r'_N)} \\
&= \frac{(-1)\dfrac{\boldsymbol{e}_{k-1}(\boldsymbol{r'}_{-1}) - \boldsymbol{e}_\ell(\boldsymbol{r'}_{-1})}{1+r'_1} + \sum_{m=k}^{\ell} \boldsymbol{e}_{m-1}(\boldsymbol{r'}_{-1})}{(1+r'_2)\cdots(1+r'_N)}.
\end{aligned}$$

Since $\boldsymbol{e}_{k-1}(\boldsymbol{r'}_{-1}) - \boldsymbol{e}_\ell(\boldsymbol{r'}_{-1}) > 0$ and $r'' \in (0, r'_1)$, the objective function value of $(r'', r'_2, r'_3, \ldots, r'_N)$ is strictly less than that of $\boldsymbol{r'}$. As a result, we have shown that if a solution $\boldsymbol{r'}$ feasible to P1 satisfies $\boldsymbol{e}_{k-1}(\boldsymbol{r'}) - \boldsymbol{e}_\ell(\boldsymbol{r'}) < 0$, then there exists a feasible solution $\boldsymbol{r''}$ satisfying $\boldsymbol{e}_{k-1}(\boldsymbol{r''}) - \boldsymbol{e}_\ell(\boldsymbol{r''}) = 0$ with a strictly smaller objective value. Thus, we only need to consider a set of feasible solutions of P1 satisfying $\boldsymbol{e}_{k-1}(\boldsymbol{r}) - \boldsymbol{e}_\ell(\boldsymbol{r}) = 0$.

Let $\boldsymbol{r^*}$ be a feasible solution of P1 satisfying $\boldsymbol{e}_{k-1}(\boldsymbol{r^*}) - \boldsymbol{e}_\ell(\boldsymbol{r^*}) = 0$. Next, we derive an upper bound and/or a lower bound for $\boldsymbol{e}_m(\boldsymbol{r^*})$. We introduce the notations $\alpha \overset{\mathrm{def.}}{=} \left(\frac{(S_{k-1})^\ell}{(S_\ell)^{k-1}}\right)^{\frac{1}{\ell-k+1}}$ and $\theta \overset{\mathrm{def.}}{=} \left(\frac{S_\ell}{S_{k-1}}\right)^{\frac{1}{\ell-k+1}}$, for simplicity. The equality $\boldsymbol{e}_{k-1}(\boldsymbol{r}^*) - \boldsymbol{e}_\ell(\boldsymbol{r}^*) = 0$ directly implies

$$\theta = \left(\frac{S_\ell}{S_{k-1}}\right)^{\frac{1}{\ell-k+1}} = \left(\frac{\binom{N}{k-1}\boldsymbol{e}_\ell(\boldsymbol{r}^*)}{\binom{N}{\ell}\boldsymbol{e}_{k-1}(\boldsymbol{r}^*)}\right)^{\frac{1}{\ell-k+1}} = \left(\frac{\binom{N}{k-1}}{\binom{N}{\ell}}\right)^{\frac{1}{\ell-k+1}}.$$

(i) Inequalities (6) imply that for any $m \in \{0, 1, 2, \ldots, k-1\}$,

$$\begin{aligned}\boldsymbol{e}_m(\boldsymbol{r}^*) &= \binom{N}{m} S_m \leq \binom{N}{m} \left(\frac{(S_{k-1})^{\ell-m}}{(S_\ell)^{k-1-m}} \right)^{\frac{1}{\ell-k+1}} \\ &= \binom{N}{m} \left(\frac{(S_{k-1})^{\ell}}{(S_\ell)^{k-1}} \cdot \frac{(S_\ell)^m}{(S_{k-1})^m} \right)^{\frac{1}{\ell-k+1}} = \binom{N}{m} \alpha\, \theta^m.\end{aligned}$$

(ii) For each $m \in \{k, k+1, \ldots, \ell\}$, inequalities (7) give a lower bound (not upper bound):

$$\begin{aligned}\boldsymbol{e}_m(\boldsymbol{r}^*) &= \binom{N}{m} S_m \geq \binom{N}{m} \left((S_{k-1})^{\ell-m} (S_\ell)^{m-k+1} \right)^{\frac{1}{\ell-k+1}} \\ &= \binom{N}{m} \left(\frac{(S_{k-1})^{\ell}}{(S_\ell)^{k-1}} \cdot \frac{(S_\ell)^m}{(S_{k-1})^m} \right)^{\frac{1}{\ell-k+1}} = \binom{N}{m} \alpha\, \theta^m.\end{aligned}$$

(iii) Inequalities (8) imply that for any $m \in \{\ell+1, \ell+2, \ldots, N\}$,

$$\begin{aligned}\boldsymbol{e}_m(\boldsymbol{r}^*) &= \binom{N}{m} S_m \leq \binom{N}{m} \left(\frac{(S_\ell)^{m-k+1}}{(S_{k-1})^{m-\ell}} \right)^{\frac{1}{\ell-k+1}} \\ &= \binom{N}{m} \left(\frac{(S_{k-1})^{\ell}}{(S_\ell)^{k-1}} \cdot \frac{(S_\ell)^m}{(S_{k-1})^m} \right)^{\frac{1}{\ell-k+1}} = \binom{N}{m} \alpha\, \theta^m.\end{aligned}$$

Then the objective function value $\mathrm{P}^*_{\mathrm{win}}(N)$ corresponding to $\boldsymbol{r^*}$ satisfies:

$$\frac{1}{\mathrm{P}^*_{\mathrm{win}}(N)} = \frac{(1+r^*_1)(1+r^*_2)\cdots(1+r^*_N)}{\displaystyle\sum_{m=k}^{\ell}\boldsymbol{e}_m(\boldsymbol{r^*})} = \frac{\displaystyle\sum_{m=0}^{N}\boldsymbol{e}_m(\boldsymbol{r^*})}{\displaystyle\sum_{m=k}^{\ell}\boldsymbol{e}_m(\boldsymbol{r^*})}$$

$$= \frac{\displaystyle\sum_{m=0}^{k-1}\boldsymbol{e}_m(\boldsymbol{r^*})}{\displaystyle\sum_{m=k}^{\ell}\boldsymbol{e}_m(\boldsymbol{r^*})} + \frac{\displaystyle\sum_{m=k}^{\ell}\boldsymbol{e}_m(\boldsymbol{r^*})}{\displaystyle\sum_{m=k}^{\ell}\boldsymbol{e}_m(\boldsymbol{r^*})} + \frac{\displaystyle\sum_{m=\ell+1}^{N}\boldsymbol{e}_m(\boldsymbol{r^*})}{\displaystyle\sum_{m=k}^{\ell}\boldsymbol{e}_m(\boldsymbol{r^*})}$$

$$\leq \frac{\displaystyle\sum_{m=0}^{k-1}\binom{N}{m}\alpha\,\theta^m}{\displaystyle\sum_{m=k}^{\ell}\binom{N}{m}\alpha\,\theta^m} + 1 + \frac{\displaystyle\sum_{m=\ell+1}^{N}\binom{N}{m}\alpha\,\theta^m}{\displaystyle\sum_{m=k}^{\ell}\binom{N}{m}\alpha\,\theta^m} = \frac{\alpha\displaystyle\sum_{m=0}^{N}\binom{N}{m}\theta^m}{\alpha\displaystyle\sum_{m=k}^{\ell}\binom{N}{m}\theta^m} = \frac{(1+\theta)^N}{\displaystyle\sum_{m=k}^{\ell}\binom{N}{m}\theta^m}$$

and thus

$$\mathrm{P}^*_{\mathrm{win}}(N) \geq \frac{\displaystyle\sum_{m=k}^{\ell}\binom{N}{m}\theta^m}{(1+\theta)^N}. \tag{12}$$

Lastly, we discuss the tightness of the above lower bound. If we consider the case where $\widehat{r}_1 = \widehat{r}_2 = \cdots = \widehat{r}_N = \theta$, then we have that

$$\begin{aligned}
\boldsymbol{e}_{k-1}(\widehat{\boldsymbol{r}}) - \boldsymbol{e}_{\ell}(\widehat{\boldsymbol{r}}) &= \binom{N}{k-1}\theta^{k-1} - \binom{N}{\ell}\theta^{\ell} \\
&= \binom{N}{k-1}\left(\frac{\binom{N}{k-1}}{\binom{N}{\ell}}\right)^{\frac{k-1}{\ell-k+1}} - \binom{N}{\ell}\left(\frac{\binom{N}{k-1}}{\binom{N}{\ell}}\right)^{\frac{\ell}{\ell-k+1}} \\
&= \frac{\binom{N}{k-1}^{1+\frac{k-1}{\ell-k+1}}}{\binom{N}{\ell}^{\frac{k-1}{\ell-k+1}}} - \frac{\binom{N}{k-1}^{\frac{\ell}{\ell-k+1}}}{\binom{N}{\ell}^{-1+\frac{\ell}{\ell-k+1}}} = 0.
\end{aligned}$$

and

$$\begin{aligned}
\boldsymbol{e}_{k-1}(\widehat{\boldsymbol{r}}_{-1}) - \boldsymbol{e}_{\ell}(\widehat{\boldsymbol{r}}_{-1}) &= \binom{N-1}{k-1}\theta^{k-1} - \binom{N-1}{\ell}\theta^{\ell} \\
&= \left(\frac{N-k+1}{N}\right)\binom{N}{k-1}\theta^{k-1} - \left(\frac{N-\ell}{N}\right)\binom{N}{\ell}\theta^{\ell} \\
&= \left(\frac{N-k+1}{N}\right)\boldsymbol{e}_{k-1}(\widehat{\boldsymbol{r}}) - \left(\frac{N-\ell}{N}\right)\boldsymbol{e}_{\ell}(\widehat{\boldsymbol{r}}) \\
&= \left(\frac{N-k+1}{N}\right)\boldsymbol{e}_{k-1}(\widehat{\boldsymbol{r}}) - \left(\frac{N-\ell}{N}\right)\boldsymbol{e}_{k-1}(\widehat{\boldsymbol{r}}) = \frac{\ell-k+1}{N}\boldsymbol{e}_{k-1}(\widehat{\boldsymbol{r}}) > 0.
\end{aligned}$$

Thus, $\widehat{\boldsymbol{r}}$ is feasible for P1 and the corresponding probability of win (under the optimal stopping rule) attains the lower bound appearing in the right-hand side of (12). From the above, $\widehat{\boldsymbol{r}}$ is optimal for P1, which induces the tightness of our lower bound. ■

Finally, we consider an asymptotic lower bound that is independent of N. The greatest lower bound for the probability of win (under the optimal stopping rule) is non-increasing with respect to N. Thus we discuss the case that $N \to \infty$ and present a general lower bound.

Corollary 1. *Under the assumptions in Theorem 4.1, the probability of win is greater than*

$$\exp\left(-\left(\frac{\ell!}{(k-1)!}\right)^{\frac{1}{\ell-k+1}}\right)\sum_{m=k}^{\ell}\left(\frac{1}{m!}\left(\frac{\ell!}{(k-1)!}\right)^{\frac{m}{\ell-k+1}}\right).$$

Proof. It is easy to see that

$$\frac{\sum_{m=k}^{\ell}\binom{N}{m}\theta^m}{(1+\theta)^N} \geq e^{-N\theta}\sum_{m=k}^{\ell}\binom{N}{m}\theta^m = \exp\left(-\binom{N}{1}\theta\right)\sum_{m=k}^{\ell}\binom{N}{m}\theta^m.$$

For each $m \in \{0, 1, \ldots, N\}$, we can find an asymptotic value for $\binom{N}{m}\theta^m$:

$$
\begin{aligned}
\binom{N}{m}\theta^m &= \binom{N}{m}\left(\frac{\binom{N}{k-1}}{\binom{N}{\ell}}\right)^{\frac{m}{\ell-k+1}} \\
&= \frac{N!}{(N-m)!m!}\left(\frac{\ell!(N-\ell)!}{(k-1)!(N-k+1)!}\right)^{\frac{m}{\ell-k+1}} \\
&= \frac{1}{m!}\left(\frac{\ell!}{(k-1)!}\right)^{\frac{m}{\ell-k+1}} \frac{N!}{(N-m)!N^m}\left(\frac{(N-\ell)!N^{\ell-k+1}}{(N-k+1)!}\right)^{\frac{m}{\ell-k+1}} \\
&= \frac{1}{m!}\left(\frac{\ell!}{(k-1)!}\right)^{\frac{m}{\ell-k+1}} \frac{\left(1-\frac{0}{N}\right)\left(1-\frac{1}{N}\right)\cdots\left(1-\frac{m-1}{N}\right)}{\left(\left(1-\frac{k-1}{N}\right)\left(1-\frac{k}{N}\right)\cdots\left(1-\frac{\ell-1}{N}\right)\right)^{\frac{m}{\ell-k+1}}} \\
&\to \frac{1}{m!}\left(\frac{\ell!}{(k-1)!}\right)^{\frac{m}{\ell-k+1}}, \quad \text{as } N \to \infty.
\end{aligned}
$$

From the above discussion, we obtain the asymptotic lower bound

$$
\begin{aligned}
\lim_{N\to\infty} \frac{\sum_{m=k}^{\ell}\binom{N}{m}\theta^m}{(1+\theta)^N} &\geq \lim_{N\to\infty} \exp\left(-\binom{N}{1}\theta\right)\sum_{m=k}^{\ell}\binom{N}{m}\theta^m \\
&= \exp\left(-\left(\frac{\ell!}{(k-1)!}\right)^{\frac{1}{\ell-k+1}}\right)\sum_{m=k}^{\ell}\left(\frac{1}{m!}\left(\frac{\ell!}{(k-1)!}\right)^{\frac{m}{\ell-k+1}}\right). \qquad \blacksquare
\end{aligned}
$$

5. Conclusion

In this paper, we consider an optimal stopping problem of maximizing the probability of selecting k out of the last ℓ successes where $1 \leq k \leq \ell < N$. Our results thus cover quite a general class of odds problems which include the original Bruss' odds problem [3], as well as the results of Bruss and Paindaveine [5] and Tamaki [13]. We showed that an optimal stopping rule is given by a threshold strategy. We also gave a lower bound for the probability of win. Our proofs are based on Newton's inequalities and optimization technique.

Our general lower bound for the probability of win is attained by corresponding odds problems and/or secretary problems:

(1) e^{-1} (if $\ell = k = 1$), which is a well-known bound for the classical secretary problem and a lower bound for Bruss' odds problem shown by Bruss [4],

(2) $\dfrac{\ell^\ell}{(\ell!)e^\ell}$ (if $\ell = k \geq 1$) shown by Bruss and Paindaveine [5] for the secretary problem,

(3) $\exp\left(-(\ell!)^{\frac{1}{\ell}}\right) \sum_{m=1}^{\ell} \dfrac{(\ell!)^{\frac{m}{\ell}}}{m!}$ (if $\ell \geq k = 1$) shown by Tamaki [13] for the secretary problem, and by Matsui and Ano [10] for a variation of the odds problem proposed by Tamaki.

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