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著者(和文)	大久保隆史
Author(English)	Takafumi Okubo
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Doctoral Thesis

Quantum periods for $\mathcal{N} = 2$ Supersymmetric QCD at strong coupling

Takafumi Okubo

Department of Physics, Tokyo Institute of Technology
Tokyo, 152-8551, Japan

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Abstract

The purpose of this thesis is to study quantum periods in four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories with hypermultiplets defined in the Nekrasov-Shatashvili (NS) limit of the Ω-background. We first review the Seiberg-Witten theory. In $\mathcal{N} = 2$ supersymmetric gauge theories, their dynamics in the Coulomb moduli space is described by the Seiberg-Witten (SW) curve. The periods of the SW curve enable us to analyze the low-energy effective theory, including non-perturbative effects, at both weak and strong coupling. In the weak coupling region, one can investigate the non-perturbative instanton effects through the prepotential. At strong coupling, one obtains the periods around singularities in the Coulomb moduli space, where BPS particles become massless. The dual prepotentials around the massless monopole/dyon point are also determined from the SW curve and its periods. At a superconformal point, where mutually non-local BPS particles become massless, the curve degenerate and the theory becomes an interacting $\mathcal{N} = 2$ superconformal field theory, called the Argyres-Douglas (AD) theory. The AD theory has fractional scaling dimensional operators and no microscopic Lagrangian. The BPS spectrum of the AD theory is determined from the degenerated SW curve.

We next study the quantization of Seiberg-Witten curve for $\mathcal{N} = 2$ supersymmetric quantum chromodynamics (SQCD). The SW curve is quantized with help of the canonical quantization of the symplectic structure derived by the SW differential. The Planck constant $\hbar$ corresponds to the deformation parameter of the NS limit of the Ω-background. The quantum correction to the SW periods, obtained from the WKB solution, is given from the SW periods by acting some differential operator. In the weak coupling region, the quantum periods agree with those obtained from the NS limit of the Nekrasov partition function, where the gauge theory is defined in the Ω-background. We compute the quantum SW periods around the massless monopole point and the superconformal point up to the fourth order in $\hbar$. We then find the general formulas for the second and fourth order corrections to the SW periods in the $SU(N_c)$ SQCD and related Argyres-Douglas theory.

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Contents

Chapter 1

Introduction

1.1 Motivation

The standard model is the most successful quantum field theory which explains many experimental results of elementary particles. The standard model is based on a gauge theory which can do powerful predictions based on the perturbative method. However, it is difficult to study non-perturbative effects such as instanton effects and confinement of quarks, which are important to understand the low-energy dynamics of gauge theories. One can study such effects by introducing supersymmetry, which is a symmetry between bosonic and fermionic fields (particles). In particular, for four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories [1], we can explicitly analyze low-energy effective physics and the non-perturbative effects in both the weak and strong coupling region [1, 2]. By breaking $\mathcal{N} = 2$ to $\mathcal{N} = 1$, one has an explanation of confinement of electric charges via monopole condensations.

An exact solution to the low-energy effective theory of a four-dimensional $\mathcal{N} = 2$ theory was studied by Seiberg and Witten in 1994 [1, 2]. They proposed a procedure to analyze exactly the low-energy effective action for a $\mathcal{N} = 2$ $SU(2)$ gauge theory in both weak and strong coupling region [1, 2]. The low-energy effective theory is an interacting $\mathcal{N} = 2$ supersymmetric $U(1)$ gauge theory, where the low-energy effective action is obtained by integrating out the massive modes. In usual, it is difficult to perform the

[1]In general, the number of the supersymmetry refers to the number of conserved fermionic charges. Since we have four real components in a single fermion in four dimensions, for example, there are $4\mathcal{N}$ conserved fermionic charges or supercharges in $\mathcal{N}$ supersymmetric theory.

path integral directly and explicitly. Seiberg and Witten solved the low-energy effective action of the $\mathcal{N}=2$ supersymmetric gauge theory by using a Riemann surface, called the Seiberg-Witten curve, which characterizes the geometrical structure of the Coulomb moduli space. They discovered the Seiberg-Witten curve by using the holomorphy and the strong-weak coupling duality. Here the Coulomb moduli space is parameterized by the vacuum expectation value (vev) of the scalar fields in the $\mathcal{N}=2$ vector multiplets. The Seiberg-Witten curve and its periods enable us to understand both weak and strong coupling physics of the theory such as instanton effects, the BPS spectrum [1, 2] and in particular physics at nonlocal superconformal fixed point [3,4]. Their idea has been generalized to the theory based on various gauge group with or without hypermultiplets [5–10]. The Seiberg-Witten theory has been also extended to the higher dimensional theory [11].

The $\mathcal{N}=2$ supersymmetric gauge theory is constructed from superstring theory or M-theory. Superstring theory is the most promising candidate for the unified theory of particles and forces including gravity. Superstring theory is defined perturbatively and has five consistent theories. These five superstring theories are mutually related through duality transformations and can be unified by M-theory [12]. M-theory is expected to be the most fundamental theory of particle physics. However, we have no definitions of the M-theory since it is a strongly coupled theory. From the analysis of eleven-dimensional supergravity which is the low-energy effective theory of the M-theory, there are two fundamental objects, called M2-branes and M5-branes. In order to investigate M-theory, it is important to study physics of M2 and M5 branes. The model described multiple M2 branes on $\mathbf{C}^4/\mathbf{Z}_k$ is the ABJM model [13] which has been studied extensively. For M5 branes, the world volume theory is the six-dimensional $\mathcal{N}=(2,0)$ superconformal field theory (SCFT). Four-dimensional $\mathcal{N}=2$ gauge theories can be derived from compactifications of six-dimensional $\mathcal{N}=(2,0)$ SCFT on a Riemann surface with punctures [14–16]. It is useful to clarify four-dimensional $\mathcal{N}=2$ gauge theories from the viewpoint of M5 branes.

It is known that the $\mathcal{N}=2$ gauge theory are described by certain integrable models [17–19]. For example, for the $\mathcal{N}=2$ supersymmetric $SU(N_c)$ Yang-Mills theory, the SW curve is the same form as the spectral curve of a periodic A_{N_c-1} Toda lattice, which is a classical integrable model [18].

At strong coupling in Coulomb moduli space, there exist interacting $\mathcal{N}=2$ SCFTs, called Argyres-Douglas (AD) theories [3,4]. The AD theory arises at a superconformal RG fixed point where mutually non-local BPS particles become massless. There is no electric-magnetic duality transformation such that particles carry electric charge only. The AD theory is then a strongly coupled theory which has the particles with both electric and magnetic charges. This theory, therefore, has no local Lagrangian description. Moreover, the operators and their couplings have fractional scaling dimensions. The SW curve of the AD theory is obtained by taking the scaling limit around the superconformal point of corresponding $\mathcal{N}=2$ gauge theories [3, 4, 20–22]. Then the BPS spectrum of the AD theory can be studied by the degenerated SW curve. Recently, the dynamics of AD theories have been also studied from the viewpoint of the compactification of M5-branes on a punctured Riemann surface [23] and its relation to two-dimensional conformal field theories (CFT) [24–27]. By using the $\mathcal{N}=1$ deformation, the superconformal indices for the AD theory have been studied in [28–30].

It has been also found that a large class of the $\mathcal{N}=2$ SCFT corresponds to two-dimensional CFT with non-unitarity [24] by comparing central charges in both theories and analyzing the superconformal indices and the characters of two-dimensional chiral algebras. By applying the conformal bootstrap program [31], the correspondence between four-dimensional $\mathcal{N}=2$ SCFTs and two-dimensional CFTs has been also studied in [32, 33].

1.2 Omega background

We have considered the low-energy effective action of the $\mathcal{N}=2$ supersymmetric gauge theory in flat spacetime, obtained by using the Seiberg-Witten curve and its periods. In the weak coupling region, one can compute the partition function of $\mathcal{N}=2$ gauge theories based on the microscopic Lagrangian by introducing the Ω-background. The Ω-background deforms four-dimensional spacetime by the torus action with two parameters (ϵ_1,ϵ_2) [34,35]. The partition function, called the Nekrasov partition function, provides an exact formula of the effective Lagrangian including the non-perturbative instanton effects by taking the limit where $\epsilon_1,\epsilon_2\to 0$.

Recently it has been recognized that the $\mathcal{N} = 2$ supersymmetric gauge theories are related to various dimensional mathematical physics through the Nekrasov partition function. The Nekrasov partition functions are related to conformal blocks of two-dimensional CFTs [36,37], the partition function of topological strings [38,39] and the solutions of the Panlevé equations [40,41], where the Ω-deformation parameters enter into the formulas of the central charge and the string coupling.

The Nekrasov partition function in the weak coupling region can be computed with the help of localization technique. At strong coupling, however, we do not know the localization method to reproduce the dual effective action around singularities with massless BPS particles in the Coulomb moduli space. In particular, for the Argyres-Douglas theories, since we have no appropriate microscopic Lagrangian, one can not compute the partition function in a usual localization method. In the case of the self-dual Ω-background with $\epsilon_1 = -\epsilon_2$, the AD theories have been studied by using the holomorphic anomaly equation [38,42] and the E-strings [43].

1.3 Quantum Seiberg-Witten curve

The purpose of this thesis is to study the effects of the Ω-deformation at strong coupling. In particular, we consider the Nekrasov-Shatashvili (NS) limit [44] of the Ω background where one of the deformation parameters ϵ_2 goes to zero. In this limit, the SW curve becomes the so-called quantum Seiberg-Witten curve which is an ordinary differential equation. This differential equation is obtained by the canonical quantization procedure for the symplectic structure induced by the SW differential. Here the deformation parameter ϵ_1 plays a role of the Planck constant $\hbar$. The Ω-deformed SW periods in the NS limit, which are the main subjects of this thesis, are obtained from the WKB solution of the quantum SW curve.

In the weak coupling region, the validity of the quantum SW curve has been studied for various $\mathcal{N} = 2$ theories. For $SU(2)$ pure Yang-Mills theory, the quantum SW curve becomes the Schrödinger equation with the sine-Gordon potential [45] and the period computed from the WKB solution is shown to agree with that obtained from the Nekrasov partition function. For $\mathcal{N} = 2$ $SU(2)$ supersymmetric quantum chromodynamics (SQCD) with $N_f \leq 4$ hypermultiplets, the WKB solutions of the quantum SW curves have been studied in [46]. Generalization to other $\mathcal{N}=2$ theories and their relations to the Nekrasov partition functions have been investigated extensively [47–50]. The quantum SW curve is also derived from the analysis of the conformal block with the insertion of the surface operator [51–53]. From the classical limit of the conformal blocks of two-dimensional CFTs, the deformed prepotentials for the $SU(2)$ gauge theories with $N_f = 1, 2, 4$ hypermultiplets have been also obtained in [54–56]. In [57], the exact quantization condition for the $SU(N_c)$ pure Yang-Mills theory has been studied, which including non-perturbative effects in $\hbar$.

It is interesting to study perturbative and non-perturbative quantum corrections in the strong coupling region of the Coulomb moduli space, which might lead to the modification of strong coupling dynamics of the theory. The perturbative corrections around the massless monopole point in the $SU(2)$ pure Yang-Mills theory have been studied in [58]. In [59], the one-instanton correction in $\hbar$ to the dual prepotential has been calculated. The non-perturbative aspects of the $\hbar$ expansion in $\mathcal{N}=2$ theories have been studied in [60–63]. For the Argyres-Douglas theories, the quantum SW curve has been studied in [64] from the viewpoint of the ODE/IM correspondence (for a review of the ODE/IM correspondence see [65]).

In this thesis, we will study the perturbative corrections in $\hbar$ to the SW periods for $\mathcal{N}=2$ SQCD at the strong coupling, especially, around the massless monopole point and the superconformal point of the Coulomb moduli space. We will show that the higher order corrections in $\hbar$ to the SW periods can be expressed by acting the differential operators with respect to some parameters on the SW periods, such as the Coulomb moduli parameters and the mass parameters. We will then calculate the WKB solutions of the quantum SW curve and investigate the relation between the higher order corrections in $\hbar$ and the SW periods up to the fourth order in $\hbar$. Then we will compute the quantum corrections to the SW periods around the massless monopole point and the superconformal point up to the fourth order in $\hbar$. Around the massless monopole point of the $\mathcal{N}=2$ $SU(2)$ SQCD, we will calculate the the NS limit of the Ω-deformed dual prepotential by using the quantum SW periods and find the interesting phenomenon that the massless monopole point in the Coulomb moduli space is shifted by the quantum corrections up to the fourth order in $\hbar$. Around the superconformal point of the $\mathcal{N}=2$ $SU(2)$ SQCD, we will evaluate the quantum SW periods by applying the relation between the higher order correction in $\hbar$ and the SW periods up to the fourth order in $\hbar$. In AD theories realized from $\mathcal{N}=2$ $SU(N_c)$ SQCD, we will find general formulas for the second and fourth order corrections, which would be useful to explore higher order corrections.

1.4 Outline

This thesis is organized as follows:

In chapter 2, we will review the Seiberg-Witten theory. For $SU(2)$ pure Yang-Mills theory which is the simplest example of the SW theory, we will introduce the Seiberg-Witten curve and the SW differential. We will obtain the (dual) prepotential in both weak and strong coupling region. The construction of the SW solution for the pure $SU(2)$ theory can be generalized to the $SU(2)$ SQCD with $N_f(=1,2,3,4)$ hypermultiplets and the $SU(N_c)$ $(N_c \geq 3)$ SQCD with $N_f(< 2N_c)$ hypermultiplets.

In chapter 3, we will review the Argyres-Douglas theory. The SW curve for the AD theory is obtained from the degeneration of the SW curve for $\mathcal{N}=2$ gauge theories. The SW differentials for the AD theories take different forms for each N_f due to flavor symmetry. Then we will compute the period integrals around the superconformal point for the $SU(2)$ SQCD with $N_f = 1,2,3$ hypermultiplets. We will generalize to the case of the $SU(N_c)$ SQCD and derive the SW curve and the SW differential around the superconformal point.

In chapter 4, we will introduce the Ω-deformation of the four-dimensional spacetime. The Nekrasov partition function, which is computed with help of the Localization theorem, reproduces the prepotential in the weak coupling region. In the Nekrasov-Shatashvili (NS) limit, the low-energy effective theory is defined in the two-dimensional Ω-background with a deformation parameter ϵ_1. The supersymmetric vacua condition of the two-dimensional Ω-deformed theories derives that the SW periods satisfy the Bohr-Sommerfeld quantization condition which the Ω-deformation parameter ϵ_1 is a roll of the Plank constant.

In chapter 5, we will study the quantization of the SW curve for the $SU(2)$ gauge theory with $N_f(=0,\ldots,4)$ hypermultiplets. We will obtain the quantum corrections to the SW periods for the $SU(2)$ SQCD. We will show that the prepotentials obtained from the WKB solutions of the quantum SW curve agree with those obtained from the NS limit of the Nekrasov partition functions. Then we will compute the dual prepotential around the massless monopole point and discuss the modification of the strong coupling physics by the Ω-deformation. This chapter is based on the paper [66] of the author in collaboration with K. Ito and S. Kanno.

In chapter 6, we will study the quantum SW curve for the Argyres-Douglas theory. We will show the relation between the SW periods and the higher order corrections up to the fourth order in $\hbar$. We will calculate the quantum corrections to the SW periods near the superconformal point, which are expressed in terms of the hypergeometric function. Then we will extend to the AD theory associated with $SU(N_c)$ SQCD and show the higher order corrections can be also expressed by acting the differential operators on the SW periods up to fourth order in $\hbar$. This chapter is based on the paper [67] and the work [68].

In chapter 7, we summarize this research and discuss future works.

Chapter 2

Seiberg-Witten theory

This chapter is a review part of the Seiberg-Witten (SW) theory [1, 2]. The SW theory provides us a low-energy effective description for $\mathcal{N}=2$ gauge theories in both weak and strong coupling region. The basic facts of the supersymmetry are summarized in appendix A.

2.1 Effective action for $\mathcal{N}=2$ supersymmetric gauge theory

Let us consider the representation of the $\mathcal{N}=2$ supersymmetry. There are two types of multiplets, namely the vector multiplet and the hypermultiplet, which consist of the fields as follows:

$$
\begin{array}{lccc@{\qquad}lccc}
\mathcal{N}=2 \text{ vector multiplet :} & & & & \mathcal{N}=2 \text{ hypermultiplet :} & & & \\
\text{gauge field} & & A_\mu & & \text{Weyl spinor} & & \psi_q & \\
\text{Weyl spinor} & \lambda & & \psi & \text{complex scalar} & q & & \tilde{q}^\dagger \\
\text{complex scalar} & & \phi & & \text{Weyl spinor} & & \psi_{\tilde{q}}^\dagger &
\end{array}
\tag{2.1}
$$

where A_μ is a gauge field, (λ, ψ) are Weyl spinors and ϕ is a complex scalar filed in the vector multiplet. $(\psi_q, \psi_{\tilde{q}}^\dagger)$ are Weyl spinors and $(q, \tilde{q}^\dagger)$ are complex scalar fields in the hypermultiplet. The components fields in the $\mathcal{N}=2$ vector multiplet and the $\mathcal{N}=2$

hypermultiplet can be organized into the $\mathcal{N} = 1$ multiplets as follows:

$$\begin{array}{lllll} \mathcal{N}=2 \text{ vector multiplet :} & & \mathcal{N}=2 \text{ hypermultiplet :} & \\ \\ \mathcal{N}=1 \text{ vector multiplet :} & (A_\mu, \lambda), & \mathcal{N}=1 \text{ chiral multiplet :} & (q, \psi_q), \\ \mathcal{N}=1 \text{ chiral multiplet :} & (\phi, \psi), & \mathcal{N}=1 \text{ chiral multiplet :} & (\tilde{q}^\dagger, \psi_{\tilde{q}}^\dagger). \end{array} \tag{2.2}$$

The $\mathcal{N} = 2$ supersymmetry algebra contains the R-symmetry which rotates of supercharge by the $U(2)_R \simeq SU(2)_R \times U(1)_R$ group. The R-symmetry $U(1)_J \times U(1)_R$ acts on the component fields in the vector multiplet as

$$U(1)_J : \begin{pmatrix} & A_\mu & \\ \lambda & & \psi \\ & \phi & \end{pmatrix} \to \begin{pmatrix} & A_\mu & \\ e^{i\alpha}\lambda & & e^{-i\alpha}\psi \\ & \phi & \end{pmatrix}, \tag{2.3}$$

$$U(1)_R : \begin{pmatrix} & A_\mu & \\ \lambda & & \psi \\ & \phi & \end{pmatrix} \to \begin{pmatrix} & A_\mu & \\ e^{i\alpha}\lambda & & e^{i\alpha}\psi \\ & e^{2i\alpha}\phi & \end{pmatrix}, \tag{2.4}$$

where $U(1)_J$ denotes the diagonal subgroup of the $SU(2)_R$. Here we note (λ, ψ) are doublets under $SU(2)_R$. The fields belonging to the hypermultiplet transform as

$$U(1)_J : \begin{pmatrix} & \psi_q & \\ q & & \tilde{q}^\dagger \\ & \psi_{\tilde{q}^\dagger}^\dagger & \end{pmatrix} \to \begin{pmatrix} & \psi_q & \\ e^{i\alpha}q & & e^{-i\alpha}\tilde{q}^\dagger \\ & \psi_{\tilde{q}^\dagger}^\dagger & \end{pmatrix}, \tag{2.5}$$

$$U(1)_R : \begin{pmatrix} & \psi_q & \\ q & & \tilde{q}^\dagger \\ & \psi_{\tilde{q}^\dagger}^\dagger & \end{pmatrix} \to \begin{pmatrix} & e^{i\alpha}\psi_q & \\ q & & \tilde{q}^\dagger \\ & e^{-i\alpha}\psi_{\tilde{q}^\dagger}^\dagger & \end{pmatrix}, \tag{2.6}$$

under $U(1)_J \times U(1)_R$ where $(q, \tilde{q}^\dagger)$ are doublets under $SU(2)_R$. Note that the $U(1)_R$ symmetry is broken to a discrete subgroup at quantum level by the anomaly. For instance, in the $SU(N_c)$ gauge theory with N_f hypermultiplets, the chiral anomaly breaks $U(1)_R$ to the discrete subgroup $\mathbf{Z}_{4N_c - 2N_f}$.

Let us consider the low-energy effective theory for the $\mathcal{N} = 2$ pure Yang-Mills theory with gauge group G. The adjoint scalar field ϕ contained in the $\mathcal{N} = 2$ vector multiplet has the potential term:

$$\mathcal{V}(\phi) = \frac{1}{g^2}\mathrm{Tr}[\phi, \phi^\dagger]^2, \tag{2.7}$$

where g is the gauge coupling constant. In the classical vacuum defined by $\mathcal{V}=0$, the scalar field takes the vacuum expectation value (vev) as

$$\langle\phi\rangle = \sum_i a_i H^i, \tag{2.8}$$

up to gauge transformation where a_i is the complex parameter and H^i $(i=1,\cdots,r)$ belongs to the Cartan subalgebra for G. Here r is a rank of G. The classical vacua of this theory are degenerated continuously and parameterized by the gauge invariants $\left\langle \mathrm{Tr}\phi^k \right\rangle$ where k belongs to order of Casimir operators of G. The moduli space of vacua parameterized by $\left\langle \mathrm{Tr}\phi^k \right\rangle$ is called the Coulomb moduli space. In the generic classical vacua, the gauge group G of the theory is broken to the subgroup $U(1)^r$ by the vev of the scalar fields. Thus the low-energy effective theory for the $\mathcal{N}=2$ pure Yang-Mills theory becomes an interacting $\mathcal{N}=2$ $U(1)^r$ gauge theory. Integrating out the massive modes, we obtain the low-energy effective Lagrangian. The $\mathcal{N}=2$ supersymmetric effective Lagrangian can be written in terms of the $\mathcal{N}=1$ superfield by introducing the $\mathcal{N}=1$ field strength $W_\alpha = \sum_i W_{\alpha i} H^i$ and the $\mathcal{N}=1$ chiral superfield $\Phi = \sum_i \Phi_i H^i$. It takes the form:

$$\mathcal{L}_{\mathrm{eff}} = \frac{1}{4\pi}\mathrm{Im}\left[\int d^4\theta \Phi_D^i \bar{\Phi}_i + \frac{1}{2}\int d^2\theta \tau^{ij} W_i^\alpha W_{\alpha j}\right]. \tag{2.9}$$

Here the dual chiral superfield Φ_D^i and the complex effective coupling constant τ^{ij} are defined by

$$\Phi_D^i = \frac{\partial \mathcal{F}}{\partial \Phi_i}, \qquad \tau^{ij} = \frac{\partial^2 \mathcal{F}}{\partial \Phi_i \partial \Phi_j}, \tag{2.10}$$

respectively, where the holomorphic function $\mathcal{F}(\Phi)$ of Φ is a prepotential. For $SU(2)$ gauge theory, τ is the effective coupling constant, given by

$$\tau := \frac{\theta_{\mathrm{eff}}}{2\pi} + \frac{4\pi i}{g_{\mathrm{eff}}^2}, \tag{2.11}$$

where θ_{eff} is the effective theta angle and g_{eff} is the effective coupling constant. Classically, the prepotential is given by

$$\mathcal{F}_{\mathrm{cl}}(\Phi_i) = \frac{1}{2}\tau_{\mathrm{cl}} \Phi^i \Phi_i, \tag{2.12}$$

where $\tau_{\rm cl}$ is the bare coupling constant.

We determine the prepotential $\mathcal{F}(\Phi)$ to obtain the effective Lagrangian for the $\mathcal{N}=2$ gauge theory. The prepotential $\mathcal{F}(\Phi)$ includes not only the perturbative but also non-perturbative corrections.

2.2 Seiberg-Witten solution to $\mathcal{N}=2$ $SU(2)$ super Yang-Mills theory

2.2.1 Prepotential for $\mathcal{N}=2$ $SU(2)$ super Yang-Mills theory

Let us study the simplest example: $\mathcal{N}=2$ supersymmetric $SU(2)$ Yang-Mills theory. The prepotential for this theory has been obtained by Seiberg and Witten [1]. They gave a procedure to determine exactly the prepotential at both weak and strong couplings.

We first consider the vacuum structure of the $SU(2)$ gauge theory. The condition $\mathcal{V}(\phi)=0$ is satisfied when the scalar field takes the form $\langle\phi\rangle = a\sigma^3$ with $\sigma^3 = \mathrm{diag}(1,-1)$ where a is a complex parameter. The Coulomb moduli space has dimension one and is parameterized by

$$u := \left\langle \mathrm{Tr}\phi^2 \right\rangle = 2a^2. \tag{2.13}$$

For $u \neq 0$, the gauge group $SU(2)$ is broken to $U(1)$ by the Higgs mechanism and then the non-Abelian gauge bosons $A_\mu^\pm$ become massive. There is a singularity at $u=0$, since the gauge group $SU(2)$ is restored and the non-Abelian gauge bosons become massless. The quantum moduli space of vacua is also parameterized by the Coulomb moduli parameter u. In the quantum theory, $U(1)_R$ is broken to $\mathbf{Z}_8$ by the anomaly. The Coulomb moduli parameter u transforms as $u \to -u$ under this R-symmetry. The dynamics at the generic point of the quantum moduli space is described by the $U(1)$ gauge theory. The effective Lagrangian is given by (2.9) for $r=1$:

$$\mathcal{L}_{\rm eff} = \frac{1}{4\pi}\mathrm{Im}\left[\int d^4\theta \Phi_D \bar{\Phi} + \frac{1}{2}\int d^2\theta \tau W^\alpha W_\alpha\right], \tag{2.14}$$

where Φ_D and τ are defined by

$$\Phi_D = \frac{\partial \mathcal{F}}{\partial \Phi}, \qquad \tau = \frac{\partial^2 \mathcal{F}}{\partial \Phi^2}, \tag{2.15}$$

respectively. In the weak coupling region where $u \sim \infty$, the form of the full prepotential is determined as follows: Classically, the prepotential $\mathcal{F}_{\rm cl}(\Phi)$ is given by (2.12) for $r = 1$. From the non-renormalization theorem [69], we see that the perturbative correction has only the one-loop correction. Since the coupling constant is given by the second derivative of prepotential with respect to Φ (2.15), the one-loop correction to the prepotential is determined by the one-loop coupling constant. For the energy scale $\mu \geq |a|$, the one-loop coupling constant is given by

$$\tau_{\text{one-loop}}(\mu) = \frac{2i}{\pi} \log \frac{\mu}{\Lambda}, \tag{2.16}$$

where Λ is the cut-off of the high-energy renormalization scale. For $\mu < |a|$ the coupling remains constant since the gauge group $SU(2)$ is broken to $U(1)$ and then the field with the $U(1)$ charge decouple. The behavior of the running coupling is shown in Fig 2.1. In

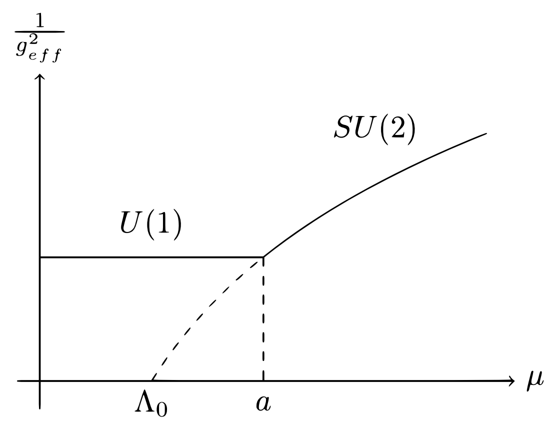

Figure 2.1: The one-loop coupling for the $SU(2)$ Yang-Mills theory

the vacuum breaking from $SU(2)$ to $U(1)$, the high-energy $SU(2)$ gauge field $A_\mu^{\rm SU(2)}$ is related to the low-energy $U(1)$ gauge field $A_\mu^{\rm U(1)}$ such as $A_\mu^{\rm SU(2)} = {\rm diag}(A_\mu^{\rm U(1)}, -A_\mu^{\rm U(1)})$. The relation between the coupling constant of the low-energy $U(1)$ gauge theory and that of the high-energy $SU(2)$ theory is given by

$$\tau^{U(1)} = 2\tau^{SU(2)}. \tag{2.17}$$

Thus in the low-energy limit, the effective coupling constant becomes

$$\tau(a) \sim \frac{4i}{\pi} \log \frac{a}{\Lambda_0}, \tag{2.18}$$

up to the perturbative correction where Λ_0 is the dynamically generated scale like the QCD scale parameter:

$$\Lambda_0^4 := \Lambda^4 e^{2\pi i \tau_{\rm cl}}. \tag{2.19}$$

Here subscript 0 of Λ_0 denotes the number of the hypermultiplets: $N_f = 0$. Integrating the effective coupling constant $\tau(a)$ over a twice, we see the one-loop correction to the prepotential given by

$$\mathcal{F}_{\text{one-loop}}(a) = \frac{i}{\pi} a^2 \log\left(\frac{a^2}{\Lambda_0^2}\right). \tag{2.20}$$

Although the prepotential is exact in the perturbation theory, the prepotential includes the non-perturbative correction due to instanton effect [69]. The instanton factor can be written as

$$e^{2\pi i \tau k} = \left(\frac{\Lambda_0}{a}\right)^{4k}, \tag{2.21}$$

by using the one loop effective coupling (2.18). The instanton factor is invariant under $\mathbf{Z}_8$ symmetry. Since the chiral superfield Φ has the charge 2 under the $U(1)_R$ symmetry, the prepotential transforms under the $U(1)_R$ symmetry as a field of charge 4 and then the non-perturbative correction to the prepotential is proportional to Φ^2. The full form of the prepotential, which receives the one-loop correction (2.20) and the instanton correction, takes

$$\mathcal{F}(\Phi) = \frac{i}{\pi}\Phi^2 \log\left(\frac{\Phi^2}{\Lambda_0^2}\right) + \frac{1}{2\pi i}\sum_{k=1}^{\infty} \mathcal{F}_k \Phi^2 \left(\frac{\Lambda_0}{\Phi}\right)^{4k}, \tag{2.22}$$

where the coefficients $\mathcal{F}_k$ can be calculated indirectly [1] as will be shown later. In the next subsection, we will discuss the duality between the weak and the strong coupling region for the $SU(2)$ Yang-Mills theory.

2.2.2 Duality

For the $\mathcal{N} = 2$ $SU(2)$ Yang-Mills theory in the Coulomb moduli space, there is the $SL(2, \mathbf{Z})$ duality transformation which acts on Φ and Φ_D as

$$\begin{pmatrix} \Phi_D \\ \Phi \end{pmatrix} \to M \begin{pmatrix} \Phi_D \\ \Phi \end{pmatrix}, \tag{2.23}$$

where $M \in SL(2,\mathbf{Z})$. Let us introduce the $SL(2,\mathbf{Z})$ generators by

$$T_k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.24}$$

where k is an integer. Under the T-transformation $M = T_k$, the effective Lagrangian (2.14) is modified such that the coupling constant is shifted by

$$\tau \to \tau + k, \tag{2.25}$$

but the path integral remains invariant.

We next consider the S-transformation $M = S$. The chiral superfield Φ and the dual chiral superfield Φ_D exchange each other under the S-transformation as

$$\begin{pmatrix} \Phi_D \\ \Phi \end{pmatrix} \to \begin{pmatrix} -\Phi \\ \Phi_D \end{pmatrix}. \tag{2.26}$$

This means that the theory should be described in terms of not (Φ, W_α) but $(\Phi_D, W_{D\alpha})$ where $W_{D\alpha}$ is the $\mathcal{N}=1$ dual field strength. By introducing the $\mathcal{N}=1$ dual vector superfield as the Lagrange multiplier and integrating out W_α, the dual effective Lagrangian can be written by

$$\mathcal{L}_{\mathrm{eff}D} = \frac{1}{4\pi}\mathrm{Im}\left[-\int d^4\theta \Phi \bar{\Phi}_D + \frac{1}{2}\int d^2\theta \tau_D W_D^\alpha W_{\alpha D}\right], \tag{2.27}$$

where Φ and τ_D are regarded as the function of Φ_D given by

$$\Phi = \frac{\partial \mathcal{F}_D}{\partial \Phi_D}, \qquad \tau_D = -\frac{\partial^2 \mathcal{F}_D}{\partial \Phi_D^2}, \tag{2.28}$$

with $\mathcal{F}_D(\Phi_D)$ being the dual prepotential. By using (2.28), we interpret τ_D as the dual coupling constant:

$$\tau_D = -\frac{1}{\tau}. \tag{2.29}$$

The duality under the S-transformation denotes the strong-weak coupling duality. From (2.25) and (2.29), we see the $SL(2,\mathbf{Z})$ group acts on τ as

$$\tau \to \frac{\alpha\tau + \beta}{\gamma\tau + \delta}, \tag{2.30}$$

where $\alpha\delta - \beta\gamma = 1$ and $\alpha, \beta, \gamma, \delta \in SL(2,\mathbf{Z})$.

The central charge Z of the $\mathcal{N}=2$ supersymmetry algebra is given by [70]

$$Z = n_e a + n_m a_D, \tag{2.31}$$

where n_e and n_m are the electric and magnetic charges, respectively. a_D corresponds to the vev of the dual scalar field including in the dual chiral superfield Φ_D, defined by

$$a_D := \frac{\partial \mathcal{F}}{\partial a}. \tag{2.32}$$

When (a, a_D) transform to $(-a_D, a)$ under the S-transformation, the central charge becomes $Z = -n_m a + n_e a_D$, so that $(-n_m, n_e)$ can be regarded as new electric and magnetic charges: $(\tilde{n}_e, \tilde{n}_m) := (-n_m, n_e)$. This is precisely the electric-magnetic duality transformation.

2.2.3 Structure of Coulomb moduli space

To study the geometrical structure of the Coulomb moduli space, we will discuss the monodromies around the singularities on the Coulomb moduli space. We will focus on the behavior of a and a_D around the singularities on u-plane. For large u, which corresponds to the weak coupling region, the theory is asymptotically free and the perturbative correction to the prepotential (2.20) gives a good approximation. By using (2.13) and (2.20), we obtain

$$a_D(u) \simeq \frac{2i}{\pi}\sqrt{\frac{u}{2}} \log \frac{u}{\Lambda_0^2}, \tag{2.33}$$

$$a(u) \simeq \sqrt{\frac{u}{2}}. \tag{2.34}$$

If the moduli parameter u circles as $u \to e^{2\pi i} u$, $a_D(u)$ and $a(u)$ become

$$\begin{pmatrix} a_D(u) \\ a(u) \end{pmatrix} \to M_\infty \begin{pmatrix} a_D(u) \\ a(u) \end{pmatrix}, \tag{2.35}$$

where M_∞ is the monodromy matrix around $u \sim \infty$:

$$M_\infty = \begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix}. \tag{2.36}$$

The existence of the non-trivial monodromy at large u implies that there must be other singular points on u-plane. Classically, since the non-Abelian gauge boson becomes massless, there is the singularity at the origin on the u-plane. If there is only two singularities

($u=\infty$ and $u=0$) on the quantum moduli space, the monodromy around $u=0$ should coincide with M_∞. This means that a^2 should be a good global complex coordinate. However, it is inconsistent with the positivity of the metric of the Coulomb moduli space, given by the holomorphic function $\mathrm{Im}\,\tau(a)$. We assume that there are three singularities including $u\sim\infty$ in the Coulomb moduli space [1]. The singularities arise at not the origin but $u=\pm\Lambda_0^2$ because of the $\mathbf{Z}_2$ symmetry on u-plane. What particles become massless at these singularities? The first guess would be that the non-Abelian gauge boson becomes massless. The existence of the massless gauge boson implies that the theory becomes the asymptotically conformal invariant theory in the IR limit. However, the point where gauge boson becomes massless is only $u=0$ due to the conformal invariance. Thus the massless particles belong to not the vector multiplet, but the hypermultiplet. There are no elementary particles belonging to the hypermultiplet in the pure $SU(2)$ gauge theory. Seiberg and Witten identified such particles as the monopole and dyon which are the BPS particles [1]. Here the BPS particles have no quantum correction and the mass of the BPS particle is given by

$$M=\sqrt{2}|Z|=\sqrt{2}|n_e a+n_m a_D|, \tag{2.37}$$

by using the $\mathcal{N}=2$ supersymmetry algebra and the formula for the central charge (2.31). We assume the monopole with $(n_e,n_m)=(0,1)$ becomes massless at $u=\Lambda_0^2$ and the dyon with $(n_e,n_m)=(-2,1)$ becomes massless at $u=-\Lambda_0^2$.

Let us consider the monodromy around $u=\Lambda_0^2$ where the theory is in the strong coupling region. At $u=\Lambda_0^2$, the theory becomes the magnetic $U(1)$ theory with the monopole coupling to the magnetic $U(1)$ gauge fields. It is convenient to describe by using the dual prepotential $\mathcal{F}_D(\Phi_D)$. Around $u=\Lambda_0^2$, the form of the full dual prepotential takes

$$\mathcal{F}_D(\Phi_D)=\frac{i}{4\pi}\Phi_D^2\log\left(\frac{\Phi_D}{\Lambda_0}\right)+\frac{1}{2\pi i}\Lambda_0^2\sum_{k=1}^{\infty}\mathcal{F}_{Dk}\left(\frac{i\Phi_D}{\Lambda_0}\right)^n. \tag{2.38}$$

[1] Actually the number of the singularities is determined three due to the consistency with the asymptotic forms of $a_D(u)$ and $a(u)$ at large u: (2.33) and (2.34) [71].

The asymptotic forms of $a_D(u)$ and $a(u)$ are given by

$$a_D(u) \simeq \frac{u-\Lambda_0^2}{\Lambda_0}, \tag{2.39}$$

$$a(u) \simeq \frac{i}{2\pi}\frac{u-\Lambda_0^2}{\Lambda_0}\log\left(\frac{u-\Lambda_0^2}{\Lambda_0^2}\right). \tag{2.40}$$

Under $u-\Lambda_0^2 \to e^{2\pi i}(u-\Lambda_0^2)$, $a_D(u)$ and $a(u)$ transform as

$$\begin{pmatrix} a_D(u) \\ a(u) \end{pmatrix} \to M_{\Lambda_0}\begin{pmatrix} a_D(u) \\ a(u) \end{pmatrix}, \tag{2.41}$$

where M_{Λ_0} is the monodromy matrix around $u=\Lambda_0$:

$$M_{\Lambda_0} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}. \tag{2.42}$$

If a_D and a in this case are written by

$$\begin{pmatrix} a_D \\ a \end{pmatrix} = \begin{pmatrix} n_m & n_e \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} \tilde{a}_D \\ \tilde{a} \end{pmatrix}, \tag{2.43}$$

under the $SL(2,\mathbf{Z})$ duality transformation where $n_m\delta - n_e\gamma = 1$ and $n_m, n_e, \gamma, \delta \in SL(2,\mathbf{Z})$, the mass of the monopole given by (2.37) with $(n_e, n_m) = (0,1)$ becomes that of the particle with (n_e, n_m):

$$M = \sqrt{2}|n_e\tilde{a} + n_m\tilde{a}_D|. \tag{2.44}$$

By using the monodromy transformation (2.41), we find that the monodromy matrix for ${}^t(\tilde{a}_D, \tilde{a})$ is of the form as

$$M_{(n_e,n_m)} = \begin{pmatrix} 1+n_en_m & n_e^2 \\ -n_m^2 & 1-n_en_m \end{pmatrix}. \tag{2.45}$$

This monodromy matrix is corresponds to that around a singularity where the BPS particle with charge (n_e, n_m) becomes massless. Indeed, by using (2.45) we obtain the monodromy $M_{-\Lambda_0}$ at $u=-\Lambda_0^2$ as

$$M_{-\Lambda_0} = M_{(-2,1)} = \begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix}. \tag{2.46}$$

The monodromy matrices M_∞ and $M_{\pm\Lambda_0}$ satisfy the condition of the monodromy on the u-plane:

$$M_\infty = M_{\Lambda_0}M_{-\Lambda_0}. \tag{2.47}$$

We note that under the $SL(2,\mathbf{Z})$ duality, the monodromies (2.42) and (2.46) can transform into those of other BPS particles with other charges (n_e, n_m). The monodromy matrices M_∞ and $M_{\pm\Lambda_0}$ generate the monodromy group $\Gamma_0(4)$, which is the subgroup of $SL(2,\mathbf{Z})$ and is defined by

$$\Gamma_0(4) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbf{Z}) \middle| \beta = 0 \bmod 4 \right\}. \tag{2.48}$$

The metric of the moduli space $\mathrm{Im}\,\tau(u)$ takes the positive due to unitarity. Thus the u-plane that we consider is viewed as the quotient of the upper half-plane H^+ by the monodromy group $\Gamma_0(4)$.

2.2.4 Seiberg-Witten curve for $\mathcal{N} = 2$ $SU(2)$ super Yang-Mills theory

So far, we find that the structure of the Coulomb moduli space corresponds to $H^+/\Gamma_0(4)$ when there are three singularities: $u = \infty$ and $u = \pm\Lambda_0^2$. The elliptic curve parameterized by $H^+/\Gamma_0(4)$ is described as the form [1, 2, 10, 72, 73]

$$y^2 = C(p)^2 - \Lambda_0^4 = (p^2 - u)^2 - \Lambda_0^4, \tag{2.49}$$

where

$$C(p) = p^2 - u. \tag{2.50}$$

This curve (2.49) is called the Seiberg-Witten curve for the pure $SU(2)$ gauge theory. The Riemann surface (2.49) is a torus. Since the metric of the Coulomb moduli space is proportional to $\mathrm{Im}\,\tau(u) > 0$ and the $SL(2,\mathbf{Z})$ duality act on $\tau(u)$ as (2.30), the coupling constant $\tau(u)$ is interpreted as the modulus of this torus and is defined by a ratio of the period integrals:

$$\tau(u) = \frac{\omega_D(u)}{\omega}, \tag{2.51}$$

where ω and ω_D is defined by the integration of a holomorphic differential $\frac{dp}{y}$ on the curve:

$$\omega(u) := \oint_\alpha \frac{2dp}{y}, \qquad \omega_D(u) := \oint_\beta \frac{2dp}{y}. \tag{2.52}$$

Here α and β are canonical one-cycles on the SW curve (2.49). From the relation of the coupling constant:

$$\tau(u) = \frac{\partial a_D}{\partial a} = \frac{\partial_u a_D(u)}{\partial_u a(u)}, \tag{2.53}$$

where $\partial_u = \frac{\partial}{\partial u}$, we find

$$\frac{\partial a(u)}{\partial u} = \omega(u), \quad \frac{\partial a_D(u)}{\partial u} = \omega_D(u). \tag{2.54}$$

Integrating them over u, $a(u)$ and $a_D(u)$ are given by the SW periods $\Pi := (a, a_D)$:

$$a(u) = \oint_\alpha \lambda_{\mathrm{SW}}, \quad a_D(u) = \oint_\beta \lambda_{\mathrm{SW}}, \tag{2.55}$$

where the meromorphic one form λ_{SW} is the SW differential

$$\lambda_{\mathrm{SW}} = p d \log \frac{C(p) - y}{C(p) + y}. \tag{2.56}$$

The u-derivative of the SW differential becomes the holomorphic differential:

$$\frac{\partial}{\partial u}\lambda_{\mathrm{SW}} = \frac{2p}{y} + d(*). \tag{2.57}$$

Hence the α cycle degenerates in the weak coupling region $u = \infty$ and the β cycle degenerates at the massless monopole point $u = \Lambda_0^2$. The massless dyon point $u = -\Lambda_0^2$ corresponds to the degeneration of the cycle $\beta - 2\alpha$. In figure 2.2, we show the schematic of the p-plane of the SW curve.

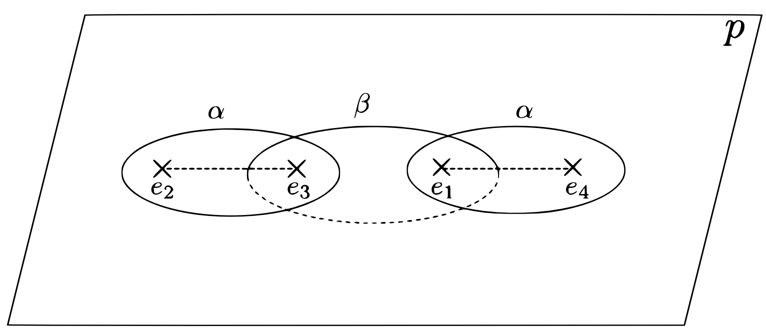

Figure 2.2: The p-plane of the SW curve for the $SU(2)$ theory.

We now compute the SW periods (2.55) and obtain the (dual) prepotential in both weak and strong coupling region. When we write the curve (2.49) in the form

$$y^2 = \prod_{i=1}^{4}(p - e_i), \tag{2.58}$$

where the weak coupling limit corresponds to $e_2 \to e_3$ and $e_1 \to e_4$, we can evaluate the period integrals $\partial_u \Pi$ (2.54), given by

$$\partial_u a = \oint_\alpha \frac{2dp}{y} = \frac{\sqrt{2}}{2\pi} \int_{e_2}^{e_3} \frac{dp}{[(p-e_1)(p-e_2)(p-e_3)(p-e_4)]^{\frac{1}{2}}}, \tag{2.59}$$

$$\partial_u a_D = \oint_\beta \frac{2dp}{y} = \frac{\sqrt{2}}{2\pi} \int_{e_1}^{e_3} \frac{dp}{[(p-e_1)(p-e_2)(p-e_3)(p-e_4)]^{\frac{1}{2}}}, \tag{2.60}$$

where the normalization is chosen such that the asymptotic forms around $u = \infty$ are compatible with those in the weak coupling region (2.33) and (2.34). In general, the period integrals $\partial_u \Pi$ can be represented in terms of the hypergeometric function:

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \qquad (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \tag{2.61}$$

after changing the variable and using the representation of the hypergeometric function:

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 dt\, t^{b-1}(1-t)^{c-b-1}(1-tz)^a. \tag{2.62}$$

We then obtain

$$\partial_u a = \frac{\sqrt{2}}{2}(e_1-e_2)^{-\frac{1}{2}}(e_3-e_4)^{-\frac{1}{2}} F\left(\frac{1}{2},\frac{1}{2};1;w'\right), \tag{2.63}$$

$$\partial_u a_D = \frac{\sqrt{2}}{2}(e_2-e_1)^{-\frac{1}{2}}(e_3-e_4)^{-\frac{1}{2}} F\left(\frac{1}{2},\frac{1}{2};1;1-w'\right), \tag{2.64}$$

where

$$w' = \frac{(e_3-e_2)(e_1-e_4)}{(e_1-e_2)(e_3-e_4)}. \tag{2.65}$$

Here the region around $w' = 0$ corresponds to the weak coupling region. The point $w' = 1$ corresponds to the massless monopole point and $w' = \infty$ is the massless dyon point. Since the variable of the hypergeometric function w' is complicated in general, we use the quadratic transformation [74, 75]

$$F\left(2a, 2b; a+b+\frac{1}{2}; z\right) = F\left(a, b; a+b+\frac{1}{2}; 4z(1-z)\right), \tag{2.66}$$

and the cubic transformation [74, 75]

$$F\left(3a, a+\frac{1}{6}; 4a+\frac{2}{3}; z\right) = \left(1-\frac{z}{4}\right) F\left(a, a+\frac{1}{3}; 2a+\frac{5}{6}; -\frac{27z^2}{(z-4)^3}\right), \tag{2.67}$$

such that the new variable becomes symmetric of the root e_i, which is given by

$$w = \frac{27w'^2(1-w')^2}{4(w'^2-w'+1)^3}. \tag{2.68}$$

From these transformation, we obtain the period integrals in the weak coupling region [75]:

$$\partial_u a = \frac{\sqrt{2}}{2}(-D)^{-\frac{1}{4}} F\left(\frac{1}{12}, \frac{5}{12}; 1; w\right), \tag{2.69}$$

$$\partial_u a_D = i\frac{\sqrt{2}}{2}(-D)^{-\frac{1}{4}}\left[\frac{3}{2\pi}\ln 12 F\left(\frac{1}{12}, \frac{5}{12}; 1; w\right) - \frac{1}{2\pi}F_*\left(\frac{1}{12}, \frac{5}{12}; 1; w\right)\right], \tag{2.70}$$

where

$$F_*(a,b;1;z) = F(a,b;1;z)\ln z + \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(n!)^2}\sum_{r=0}^{n-1}\left(\frac{1}{a+r}+\frac{1}{b+r}-\frac{2}{1+r}\right)z^n. \tag{2.71}$$

Here the variable (2.68) can be represented by

$$w = -\frac{27\Delta}{4D^3}, \tag{2.72}$$

where Δ is the discriminant of the curve (2.58):

$$\Delta = \prod_{i<j}(e_i - e_j)^2, \tag{2.73}$$

and D is given by

$$D = \sum_{i<j} e_i^2 e_j^2 - 6\prod_{i=1}^{4} e_i - \sum_{i<j<k}(e_i^2 e_j e_k + e_i e_j^2 e_k + e_i e_j e_k^2). \tag{2.74}$$

The integral $F = (-D)^{\frac{1}{4}} \oint \frac{dp}{y}$ obeys the hypergeometric differential equation:

$$w(1-w)\frac{d^2F}{dw^2} + (\gamma - (\alpha+\beta+1)w)\frac{dF}{dw} - \alpha\beta F = 0, \tag{2.75}$$

with $\alpha = \frac{1}{12}$, $\beta = \frac{5}{12}$ and $\gamma = 1$. Changing the variable from w to u, the hypergeometric differential equation (2.75) for $F(\frac{1}{12}, \frac{5}{12}; 1; w)$ leads to the differential equation with respect to u which takes the form:

$$\frac{\partial^3\Pi}{\partial u^3} + p_1\frac{\partial^2\Pi}{\partial u^2} + p_2\frac{\partial\Pi}{\partial u} = 0, \tag{2.76}$$

where p_1 and p_2 are given by

$$p_1 = \frac{\partial_u(-D)^{\frac{1}{4}}}{(-D)^{\frac{1}{4}}} - \frac{\partial_u^2 w}{\partial_u w} + \frac{\gamma - (1+\alpha+\beta)w}{w(1-w)}\partial_u w, \tag{2.77}$$

$$p_2 = \frac{\partial_u^2(-D)^{\frac{1}{4}}}{(-D)^{\frac{1}{4}}} + \frac{\partial_u(-D)^{\frac{1}{4}}}{(-D)^{\frac{1}{4}}}\left\{-\frac{\partial_u^2 w}{\partial_u w} + \frac{\gamma - (1+\alpha+\beta)w}{w(1-w)}\partial_u w\right\} - \frac{\alpha\beta}{w(1-w)}\left(\partial_u w\right)^2, \tag{2.78}$$

where $\alpha = \frac{1}{12}$, $\beta = \frac{5}{12}$ and $\gamma = 1$. This differential equation is called the Picard-Fuchs equation for $\partial_u \Pi$ [76, 72, 77–79, 75]. In the pure $SU(2)$ gauge theory, the Picard-Fuchs equation (2.76) turns out to be the second order differential equation for Π [76]

$$\frac{\partial^2 \Pi}{\partial u^2} - \frac{1}{4(\Lambda_0^4 - u)}\Pi = 0. \tag{2.79}$$

Let us calculate the prepotential in the weak coupling region for the pure $SU(2)$ gauge theory by using (2.69) and (2.70). The discriminant Δ and D for the pure $SU(2)$ gauge theory are given by

$$\Delta = 256\Lambda_0^8(u^2 - \Lambda_0^4), \qquad D = 12\Lambda_0^4 - 16u^2. \tag{2.80}$$

Substituting (2.80) into (2.69) and (2.70), we obtain the SW periods:

$$a(u) = \sqrt{\frac{u}{2}} - \frac{\Lambda_0}{2^4\sqrt{2}}\left(\frac{\Lambda_0^2}{u}\right)^{\frac{3}{2}} - \frac{15\Lambda_0}{2^{10}\sqrt{2}}\left(\frac{\Lambda_0^2}{u}\right)^{\frac{7}{2}} - \frac{105\Lambda_0}{2^{14}\sqrt{2}}\left(\frac{\Lambda_0^2}{u}\right)^{\frac{11}{2}} + \cdots, \tag{2.81}$$

$$a_D(u) = -\frac{i}{2\sqrt{2}\pi}\left(-4\sqrt{2}a(u)\log\frac{8u}{\Lambda_0^2} + 8\sqrt{u} - \frac{\Lambda_0}{4}\left(\frac{\Lambda_0^2}{u}\right)^{\frac{3}{2}} - \frac{47\Lambda_0}{2^9}\left(\frac{\Lambda_0^2}{u}\right)^{\frac{7}{2}} + \cdots\right), \tag{2.82}$$

by expanding (2.69) and (2.70) around $u = \infty$ and integrating over u. Solving u in terms of a in (2.81) and substituting it into a_D, a_D becomes a function of a. Then integrating it over a, we obtain the prepotential [72]

$$\mathcal{F}(a) = \frac{i}{2\pi}a^2\left(2\log\frac{a^2}{\Lambda_0^2} - 6 + 8\log 2 - \sum_{k=1}^{\infty}\mathcal{F}_k\left(\frac{\Lambda}{a}\right)^{4k}\right), \tag{2.83}$$

where the first several coefficients of $\mathcal{F}_k$ are listed in table 2.1. The first term of the

k	1	2	3	4	5
$\mathcal{F}_k$	$\frac{1}{2^5}$	$\frac{5}{2^{14}}$	$\frac{3}{2^{18}}$	$\frac{1469}{2^{31}}$	$\frac{4471}{2^{34}\cdot 5}$

Table 2.1: The coefficients of the prepotential (2.83) for the pure $SU(2)$ theory.

prepotential (2.83) is the perturbative correction to the prepotential and the last term is the non-perturbative instanton correction.

We can also study the SW periods in the strong coupling region. Around $u = \pm\Lambda_0^2$, solving the Picard-Fuchs equation (2.79) in terms of hypergeometric function, we can compute the SW periods [72]. For example, the SW periods around the massless monopole point $u = \Lambda_0$ are given by

$$a_D(\hat{u}) = i\Lambda_0 \left(\frac{1}{2}\frac{\hat{u}}{\Lambda_0^2} - \frac{1}{2^5}\left(\frac{\hat{u}}{\Lambda_0^2}\right)^2 + \frac{3}{2^9}\left(\frac{\hat{u}}{\Lambda_0^2}\right)^3 - \frac{5^2}{2^{14}}\left(\frac{\hat{u}}{\Lambda_0^2}\right)^4 + \cdots \right), \tag{2.84}$$

$$a(\hat{u}) = \frac{i}{2\pi}\left[a_D(u)\log\frac{\hat{u}}{2^5\Lambda_0^2} - i\Lambda_0\left(-\frac{1}{2}\left(\frac{\hat{u}}{\Lambda_0^2}\right) - \frac{3}{2^6}\left(\frac{\hat{u}}{\Lambda_0^2}\right)^2 + \frac{3}{2^8}\left(\frac{\hat{u}}{\Lambda_0^2}\right)^3 + \cdots \right)\right], \tag{2.85}$$

where $\hat{u} := u - \Lambda_0^2$. Inverting $a_D(\hat{u})$ in terms of $\hat{u}$ and inserting it into $a(\hat{u})$, we obtain a_D as a function of a. Then the dual prepotential around $u = \Lambda_0^2$ is given by the integration of a_D over a [72]:

$$\mathcal{F}_D(a_D) = \frac{i}{2\pi}\left(\frac{1}{2}a_D^2 \log\left(-\frac{ia_D}{16\Lambda_0}\right) - \sum_{k=1}^{\infty} \mathcal{F}_{Dk}\Lambda_0^2 \left(\frac{ia_D}{\Lambda_0}\right)^n \right), \tag{2.86}$$

where the first several coefficients of $\mathcal{F}_{Dk}$ are listed in the table 2.2. The first term of the dual prepotential (2.86) corresponds to the perturbative corrections to the dual prepotential (2.38). In the next section, we will extend the above discussion to the $SU(2)$ gauge theory with $N_f(= 1, 2, 3, 4)$ hypermultiplets.

2.3 Seiberg-Witten solution for $SU(2)$ SQCD

Let us discuss the low-energy effective theory for the $SU(2)$ gauge theory with $N_f(= 1\ldots 4)$ hypermultiplets [2]. In the previous section, we obtained the SW curve and the

k	1	2	3	4	5
$\mathcal{F}_{Dk}$	0	$\frac{3}{4}$	$\frac{1}{2^4}$	$\frac{5}{2^9}$	$\frac{11}{2^{12}\cdot 5}$

Table 2.2: The coefficients of the dual prepotential (2.86) for the pure $SU(2)$ theory.

SW differential for the $SU(2)$ pure Yang-Mills theory. In the next subsection, we will write down the superpotential for the hypermultiplets in terms of the $\mathcal{N}=1$ superfields and give the BPS mass formula including the mass of the squarks. In the next subsection, we will obtain the SW curve for the $SU(2)$ gauge theory with $N_f(=1\dots4)$ hypermultiplets.

2.3.1 Superpotential for hypermultiplet

In terms of the $\mathcal{N}=1$ superfields, the $\mathcal{N}=2$ hypermultiplets consist of two $\mathcal{N}=1$ chiral superfields Q_i^a and $\tilde{Q}_{ai}$ where i $(=1,2)$ is the color index and a $(=1,\cdots,N_f)$ is the flavor index. Here the chiral superfields Q_i^a and $\tilde{Q}_{ai}$ contain (ψ_q, q) and $(\psi_{\tilde{q}}^\dagger, \tilde{q})$, respectively, and belong to the fundamental representation of the $SU(2)$ gauge group. In the $\mathcal{N}=1$ language, the superpotential is given by

$$W=\sqrt{2}\tilde{Q}_a\Phi Q^a+\sum_{a=1}^{N_f} m_a\tilde{Q}_aQ^a, \tag{2.87}$$

where m_a is the bare mass for the N_f hypermultiplets and the color indices are suppressed. When $m_a=0$, the classical theory has the global symmetry which is the subgroup of $O(2N_f)\times SU(2)_R\times U(1)_R$. On the classical Coulomb moduli space, which implies that the vacuum condition is $\langle\phi\rangle = a\sigma^3$ and $\langle Q\rangle = \langle\tilde{Q}\rangle = 0$, the $SU(2)$ gauge group is spontaneously broken to $U(1)$. From the vev of the scalar field $\langle\phi\rangle$ and the superpotential (2.87), the squarks acquire mass. The squarks are the BPS particles and then the BPS mass formula including the squarks is give by [70, 2]

$$M=\sqrt{2}|Z|, \tag{2.88}$$

with

$$Z = n_e a + n_m a_D + \sum_{a=1}^{N_f} \frac{1}{\sqrt{2}} S_a m_a, \tag{2.89}$$

where S_a are the $U(1)$ charges corresponding to the additional symmetry, to which the global symmetry is broken by non-zero masses. One of the squarks has mass as $a \pm \frac{m_a}{\sqrt{2}}$. The effective Lagrangian also takes the form (2.14) by integrating out the massive modes, while the Coulomb moduli space for the $SU(2)$ SQCD is also parameterized by the Coulomb moduli parameter defined by (2.13). In the next subsection, we will consider the structure of the Coulomb moduli space and show the SW curve and the SW differential for the $SU(2)$ gauge theory with N_f hypermultiplets.

2.3.2 Seiberg-Witten curve for $SU(2)$ SQCD

As discussed in the subsection 2.2.4, by deriving the monodromy group generated by the monodromy around singularities, we can study the (singularity) structure of the quantum Coulomb moduli space. Furthermore, by taking the decoupling limit where

$$m_{N_f} \to \infty, \quad \Lambda_{N_f} \to 0, \quad \Lambda_{N_f-1}^{4-N_f+1} := m_{N_f} \Lambda_{N_f}^{4-N_f} : \text{fixed}, \tag{2.90}$$

with Λ_{N_f} being a QCD scale parameter for $N_f \leq 3$, the number of the hypermultiplets N_f are reduced to $N_f - 1$. For $N_f = 4$, the decoupling limit to the $N_f = 3$ is defined by

$$m_4 \to \infty, \quad q \to 0, \quad \Lambda_3 := m_4 q : \text{fixed}, \tag{2.91}$$

with $q = \exp(2\pi i \tau_{\text{UV}})$ where τ_{UV} denotes the UV coupling constant [80, 36]. Then we require that the SW curve and the SW differential for the N_f theory become those for the $N_f - 1$ theory in the decoupling limit (2.90) and (2.91). The SW curve for the $SU(2)$ gauge theory with $N_f(= 0 \ldots 4)$ hypermultiplets, which satisfies the above condition, is given by [10]

$$y^2 = C(p)^2 - \bar{\Lambda}^2 G(p), \tag{2.92}$$

where $\bar{\Lambda} = \Lambda_{N_f}^{2-\frac{N_f}{2}}$ for $N_f \leq 3$ and $\bar{\Lambda} = q^{\frac{1}{2}}$ for $N_f = 4$. $C(p)$ and $G(p)$ are given by

$$C(p) = \begin{cases} p^2 - u, & (N_f = 0, 1) \\ p^2 - u + \dfrac{\Lambda_2^2}{8} & (N_f = 2) \\ p^2 - u + \dfrac{\Lambda_3}{4}\left(p + \dfrac{m_1 + m_2 + m_3}{2}\right), & (N_f = 3) \\ \left(1 + \dfrac{q}{2}\right) p^2 - u + \dfrac{q}{4} p \sum_{i=1}^{4} m_i + \dfrac{q}{8} \sum_{i<j} m_i m_j, & (N_f = 4) \end{cases} \tag{2.93}$$

and

$$G(p) = \prod_{i=1}^{N_f} (p + m_i). \tag{2.94}$$

The SW differential is expressed as (2.56). The SW periods $\Pi := (a, a_D)$ are

$$a(u) = \oint_\alpha \lambda_{\mathrm{SW}}, \quad a_D = \oint_\beta \lambda_{\mathrm{SW}}, \tag{2.95}$$

where α and β are the canonical one-cycles on the SW curve. The SW curve (2.92) can be written into the form [10]

$$\frac{\bar{\Lambda}}{2}\left(G_+(p) z + \frac{G_-(p)}{z}\right) = C(p), \tag{2.96}$$

by introducing

$$y = \bar{\Lambda} G_+(p) z - C(p), \tag{2.97}$$

where

$$G_+(p) = \prod_{i=1}^{N_+} (p + m_i), \quad G_-(p) = \prod_{i=N_+ + 1}^{N_f} (p + m_i), \tag{2.98}$$

with N_+ being a fixed integer satisfying $1 \leq N_+ \leq N_f$. The SW differential becomes

$$\lambda_{\mathrm{SW}} = p\left(d \log \frac{G_-(p)}{G_+(p)} - 2d \log z\right). \tag{2.99}$$

The u-derivative of the SW differential becomes the holomorphic differential:

$$\frac{\partial \lambda_{\mathrm{SW}}}{\partial u} = \frac{2\partial_u z}{z} dp + d(*) = \frac{2dp}{y} + d(*), \tag{2.100}$$

where $\partial_u := \frac{\partial}{\partial u}$. Differentiating the SW periods Π with respect to u, one obtains the periods for the curve:

$$\partial_u a(u) = \oint_\alpha \frac{2\partial_u z}{z} dp = \oint_\alpha \frac{2dp}{y}, \quad \partial_u a_D(u) = \oint_\beta \frac{2\partial_u z}{z} dp = \oint_\beta \frac{2dp}{y}. \tag{2.101}$$

As discussed in the previous section, the period integrals (2.101) are given by (2.69) and (2.70) in the weak coupling region. We also find the periods $\partial_u \Pi$ obey the Picard-Fuchs equation (2.76). For the SW curve (2.96) with $N_f \leq 3$, the Picard-Fuchs equation (2.76) agrees with that in [78, 79]. Note that for the massless case, the Picard-Fuchs equation turns out to be the second order differential equation for Π [77].

To solve the Picard-Fuchs equation (2.76) for the N_f theories, we need to derive the discriminant Δ (2.73) and D (2.74) for each N_f. In the following Δ_{N_f}, D_{N_f} and w_{N_f} stand for Δ, D and w in (2.73), (2.74) and (2.72) for the N_f theory, respectively. In the case of the $N_f = 1$ theory, they are given by

$$\begin{aligned} \Delta_1 =& -\Lambda_1^6 (256u^3 - 256u^2 m_1^2 - 288 u m_1 \Lambda_1^3 + 256 m_1^3 \Lambda_1^3 + 27\Lambda_1^6), \\ D_1 =& -16u^2 + 12 m_1 \Lambda_1^3. \end{aligned} \tag{2.102}$$

For $N_f = 2$, Δ_2 and D_2 are obtained by

$$\begin{aligned} \Delta_2 =& \frac{\Lambda_2^{12}}{16} - 3\Lambda_2^{10} m_1 m_2 - \Lambda_2^8 \left(8u^2 - 36\left(m_1^2 + m_2^2\right)u + 27m_1^4 + 27m_2^4 + 6m_1^2 m_2^2\right) \\ &+ 256\Lambda_2^4 u^2 \left(u - m_1^2\right)\left(u - m_2^2\right) - 32\Lambda_2^6 m_1 m_2 \left(10u^2 - 9\left(m_1^2 + m_2^2\right)u + 8m_1^2 m_2^2\right), \\ D_2 =& -\frac{3}{4}\Lambda_2^4 + 12\Lambda_2^2 m_1 m_2 - 16u^2. \end{aligned} \tag{2.103}$$

We then consider the $N_f = 3$ and 4 theories, but Δ_{N_f} and D_{N_f} are rather complicated in the generic mass case. So we will write down them for these theories with the same mass $m := m_1 = m_2 = \cdots = m_{N_f}$. For $N_f = 3$, Δ_3 and D_3 become

$$\begin{aligned} \Delta_3 =& -\frac{\Lambda_3^2 \left(8m^2 + \Lambda_3 m - 8u\right)^3 \left(256\Lambda_3 \left(8m^3 - 3mu\right) + 8\Lambda_3^2 \left(3m^2 + u\right) + 3\Lambda_3^3 m - 2048u^2\right)}{4096}, \\ D_3 =& -\frac{\Lambda_3^4}{256} + 12\Lambda_3 m^3 + \Lambda_3^2 \left(u - \frac{9m^2}{4}\right) - 16u^2, \end{aligned} \tag{2.104}$$

in the same mass case. For $N_f = 4$, we have

$$\begin{aligned}\Delta_4 =&\frac{2^{24}q^2\left(m^2-u\right)^4\left(m^4(q-16)q+8m^2qu+16u^2\right)}{(q-4)^{10}},\\ D_4 =&\frac{16\left(-m^4q\left((q-12)^2q-192\right)-8m^2(q-8)q^2u-16((q-4)q+16)u^2\right)}{(q-4)^4}.\end{aligned} \tag{2.105}$$

We can also solve the Picard-Fuchs equation for general mass case based on Δ_{N_f} and D_{N_f} with $N_f = 3, 4$ but we do not show them here. It is shown that these formulas are consistent with the decoupling limit: (2.90) and (2.91).

In the weak coupling region, the periods $\partial_u a$ and $\partial_u a_D$ are found to be given by (2.69) and (2.70) with Δ_{N_f} and D_{N_f}. Expanding them around $u = \infty$ and integrating over u, we have the SW periods in the weak coupling region. The prepotential is derived from the SW periods at $u = \infty$. In chapter 5, we will compute the quantum SW periods for the $SU(2)$ gauge theory with $N_f(= 0, \cdots, 4)$ by solving the Picard-Fuchs equation.

We next consider the strong coupling region, where the monopole/ dyon particles become massless. We can also evaluate the SW periods by solving the Picard Fuchs equation (2.76) around the massless monopole/dyon point on the u-plane. Here we note that the singularities on the u-plane are obtained by the zero of the discriminant Δ_{N_f} at which the SW curve degenerates.

Let us consider the SW periods around the massless monopole point in general N_f, which the discriminant Δ_{N_f} becomes zero. Here we define u_0 as the massless monopole point. From the BPS mass formula (2.88) with only magnetic charge, we find the dual SW period $a_D(u)$ becomes zero at the massless monopole point $u = u_0$. By solving the Picard-Fuchs equation around $u = u_0$, the periods around the massless monopole point take the form as [75]

$$\partial_u a_D = \frac{\sqrt{2}i}{2}(-D_{N_f})^{-\frac{1}{4}}F\left(\frac{1}{12},\frac{5}{12};1;w_{N_f}\right), \tag{2.106}$$

$$\partial_u a = \frac{\sqrt{2}}{2}(-D_{N_f})^{-\frac{1}{4}}\left[\frac{3}{2\pi}\ln 12F\left(\frac{1}{12},\frac{5}{12};1;w_{N_f}\right)-\frac{1}{2\pi}F_*\left(\frac{1}{12},\frac{5}{12};1;w_{N_f}\right)\right]. \tag{2.107}$$

where w_{N_f} and D_{N_f} are given by (2.72) and (2.74), respectively. In general, w_{N_f} and $(-D_{N_f})^{\frac{1}{4}}$ are expanded around $u = u_0$ as follows:

$$w_{N_f} = \sum_{n=1}^{\infty} A_n \hat{u}^n, \qquad (-D_{N_f})^{-\frac{1}{4}} = \sum_{n=0}^{\infty} B_n \hat{u}^n, \tag{2.108}$$

where $\hat{u} = u - u_0$. Substituting (2.108) into (2.106) and (2.107) and integrating over u, the SW periods can be of the form as

$$a_D(\hat{u}) = \sum_{n=1}^{\infty} \mathcal{J}_n \hat{u}^n, \tag{2.109}$$

$$a(\hat{u}) = \frac{i}{2\pi}\left[l a_D(\hat{u}) \left\{ \log(A_l^{\frac{1}{l}} \hat{u}) - \frac{3}{l}\log 12 \right\} + \sum_{n=1}^{\infty} \mathcal{I}_n \hat{u}^n \right], \tag{2.110}$$

where a integral constant for a_D is determined such that the deal SW period satisfies the condition $a_D(0) = 0$. $a(\hat{u})$ is given up to independent constant of $\hat{u}$. We define the integer l as the smallest integer, giving nonzero A_n i.e. $A_n = 0$ $(n < l)$ and $A_l \neq 0$. $\mathcal{J}_n$ and $\mathcal{I}_n$ are given in terms of A_n and B_n. We show the first three terms of A_n and B_n as follows:

$$\begin{aligned}
\mathcal{J}_1 &= i\frac{B_0}{\sqrt{2}}, \\
\mathcal{J}_2 &= \frac{i}{2\sqrt{2}}\left(B_1 + B_0 A_1 f^{(1)}\right), \\
\mathcal{J}_3 &= \frac{i}{3\sqrt{2}}\left\{ B_2 + (B_0 A_2 + B_1 A_1) f^{(1)} + \frac{1}{2} B_0 A_1^2 f^{(2)} \right\},
\end{aligned} \tag{2.111}$$

$$\begin{aligned}
\mathcal{I}_1 &= -l B_1, \\
\mathcal{I}_2 &= -\frac{l}{2} B_2 + \frac{A_{l+1}}{A_l}\frac{1}{2} B_1 + \frac{i}{2\sqrt{2}} B_0 A_1 g^{(1)}, \\
\mathcal{I}_3 &= -\frac{l}{3} B_3 + \frac{A_{l+1}}{A_l}\frac{2}{3} B_2 + \left(\frac{A_{l+2}}{A_l} - \frac{A_{l+1}^2}{2A_l^2} \right)\frac{1}{3} B_1 + \frac{i}{3\sqrt{2}}\left\{ (B_0 A_2 + B_1 A_1) g^{(1)} + \frac{1}{2} B_0 A_1^2 g^{(2)} \right\},
\end{aligned} \tag{2.112}$$

where

$$\begin{aligned}
f^{(n)} &= \frac{(\alpha)_n (\beta)_n}{n!}, \\
g^{(n)} &= \frac{(\alpha)_n (\beta)_n}{(n!)^2} \sum_{r=0}^{n-1} \left(\frac{1}{\alpha + r} + \frac{1}{\beta + r} - \frac{2}{1 + r} \right).
\end{aligned} \tag{2.113}$$

with $\alpha = \frac{1}{12}$ and $\beta = \frac{5}{12}$. One can determine the higher order corrections in $\hat{u}$ in a similar way. In chapter 5, by using the above formulas, we will calculate the expansion of the quantum SW periods around the massless monopole point in some cases: the massless hypermultiplets and the massive hypermultiplets with the same mass. We note that the periods around the massless dyon point can be analyzed in the same manner.

2.4 Generalization to $SU(N_c)$ ($N_c \geq 3$) SQCD

In the previous sections, we had the SW curve and the SW differential for the $SU(2)$ gauge theory with $N_f(= 0, \cdots, 4)$ hypermultiplets. The construction of the curve for the $SU(2)$ SQCD can be generalized to the gauge theory with or without hypermultiplets [5–10]. In this section, we will consider the $SU(N_c)$ ($N_c \geq 3$) gauge theory with $N_f(< 2N_c)$ hypermultiplets [10].

For the $SU(N_c)$ gauge theory with $N_f(< 2N_c)$ hypermultiplets, the vev of the scalar fields (2.8) are given by

$$\langle \phi \rangle = \mathrm{diag}[a_1, \cdots, a_{N_c}], \tag{2.114}$$

where

$$\sum_{i=1}^{N_c} a_i = 0. \tag{2.115}$$

Then at generic point on the Coulomb moduli space, the $SU(N_c)$ gauge group is broken to $U(1)^{N_c-1}$ and the Coulomb moduli space is expressed as the $N_c - 1$ complex dimensional moduli space, parameterized by the gauge invariants:

$$u_k = \left\langle \mathrm{Tr}\phi^k \right\rangle = \sum_{i=1}^{N_c} a_i^k, \quad k = 2, \cdots N_c. \tag{2.116}$$

To construct the SW curve and the SW differential for $N_c \geq 3$, it is convenient to use the symmetric polynomial s_k of a_i:

$$s_k = (-1)^k \sum_{i_1 < \cdots < i_k} a_{i_1} \cdots a_{i_k}, \qquad k = 2, \cdots, N_c, \tag{2.117}$$

rather than the gauge invariants (2.116). The sets of s_k and u_k satisfy the Newton's formula [6]

$$k s_k + \sum_{i=1}^{k} s_{k-i} u_i = 0, \quad s_0 = 1, \quad s_1 = u_1 = 0. \tag{2.118}$$

This relation also satisfy at the quantum level.

The BPS mass formula (2.88) are generalized to the case of the $\mathcal{N}=2$ $SU(N_c)$ SQCD, given by [2, 10]

$$Z=\sum_{i=1}^{N_c}\left(n_e^i a_i+n_m^i a_{Di}\right)+\sum_{a=1}^{N_f}\frac{1}{\sqrt{2}}S_a m_a. \tag{2.119}$$

The electric n_e^i and magnetic n_m^i charges can be chosen to satisfy

$$\sum_{i=1}^{N_c} n_e^i=0,\quad \sum_{i=1}^{N_c} n_m^i=0. \tag{2.120}$$

Here a_{Di} $(i=1,\cdots,N_c)$ denotes the vev of the dual scalar fields satisfying $\sum_{i=1}^{N_c}a_{Di}=0$.

The SW curve for the $SU(N_c)$ $(N_c\geq 3)$ gauge theory with N_f hypermultiplets is given by

$$y^2=C(p)^2-\Lambda_{N_f}^{2N_c-N_f}G(p), \tag{2.121}$$

where $C(p)$ and $G(p)$ are given by

$$C(p)=p^{N_c}-\sum_{i=2}^{N_c}s_i p^{N_c-i}+\frac{\Lambda_{N_f}^{2N_c-N_f}}{4}\sum_{j=0}^{N_f-N_c}v_j p^{N_f-N_c-j}, \tag{2.122}$$

$$G(p)=\prod_{j=1}^{N_f}(p+m_j). \tag{2.123}$$

Here v_j $(j=1,\cdots,N_f)$ are some symmetric polynomial of m_j:

$$\sum_{j=0}^{N_f}v_j p^{N_f-j}:=\prod_{j=1}^{N_f}(p+m_j),\quad v_0=1. \tag{2.124}$$

The SW curve is a Riemann surface of the genus N_c-1. The SW periods $\Pi=(a_l,a_{Dl})$ $(l=1,\cdots,N_c-1)$ are given by

$$a_l=\oint_{\alpha_l}\lambda_{\mathrm{SW}},\qquad a_{Dl}=\oint_{\beta_l}\lambda_{\mathrm{SW}}, \tag{2.125}$$

where α_l and β_l are the canonical one-cycles on the curve. The SW differential λ_{SW} takes the same form as (2.56). The SW differential satisfies

$$\frac{\partial\lambda_{\mathrm{SW}}}{\partial s_i}=\omega_i+d(*), \tag{2.126}$$

where ω_i is the basis of the holomorphic 1-forms given by

$$\omega_i := \frac{2\partial_{s_i} C(p)}{y} dp, \tag{2.127}$$

with $\partial_{s_i} := \frac{\partial}{\partial s_i}$. In the decoupling limit which is given by

$$m_{N_f} \to \infty, \quad \Lambda_{N_f} \to 0, \quad \Lambda_{N_f-1}^{2N_c-N_f+1} = m_{N_c} \Lambda_{N_f}^{2N_c-N_f} : \text{fixed}, \tag{2.128}$$

we also find the SW curve and the SW differential for the N_f theory go to those for the $N_f - 1$ theory.

Summary

In this chapter, we explained the Seiberg-Witten theory for the $SU(N_c)$ gauge theory with N_f hypermultiplets. We constructed the SW curve and the SW differential for the $SU(2)$ pure Yang-Mills theory, for example. We then expressed the SW periods in terms of the hypergeometric function. We calculated the expansions of the SW periods and obtained the (dual) prepotential around $u = \infty$ and $u = \Lambda_0^2$. The construction in the case of the $SU(2)$ pure Yang-Mills theory was generalized to the case of $SU(N_c)$ SQCD.

Chapter 3

Argyres-Douglas theory

In this chapter, we will review the Argyres-Douglas (AD) theory [3,4]. The AD theory is an interacting $\mathcal{N} = 2$ superconformal field theory (SCFT) where mutually non-local BPS particles become massless. The BPS spectrum of the AD theory can be studied by the Seiberg-Witten curve, which is obtained from degeneration of the curve of $\mathcal{N} = 2$ gauge theory. We will discuss the general properties of an interacting $\mathcal{N} = 2$ superconformal field theory in the first section of this chapter. In the next section, we will discuss the SW curve for the AD theory, realized from the $SU(2)$ SQCD, by taking the scaling limit around the superconformal point on u-plane and then calculate the SW periods around the superconformal point. In the third section, we generalize the AD theories obtained from the $SU(N_c)$ SQCD.

3.1 Interacting $\mathcal{N} = 2$ SCFT

We begin by discussing four-dimensional interacting conformal field theories (CFTs) without supersymmetry. The generators of the conformal algebra satisfy the commutation relation as follows (see [81] for a textbook):

$$\begin{aligned}
[D, P_\mu] =& iP_\mu, \\
[D, K_\mu] =& -iK_\mu, \\
[K_\mu, P_\nu] =& 2i(\eta_{\mu\nu} D - L_{\mu\nu}), \\
[K_\mu, L_{\nu\rho}] =& i(\eta_{\mu\nu} K_\rho - \eta_{\mu\rho} K_\nu), \\
[P_\mu, L_{\nu\rho}] =& i(\eta_{\mu\nu} P_\rho - \eta_{\mu\rho} P_\nu), \\
[M_{\mu\nu}, L_{\rho\sigma}] =& i(\eta_{\nu\rho} L_{\mu\sigma} + \eta_{\mu\sigma} L_{\nu\rho} - \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\nu\sigma} L_{\mu\rho}),
\end{aligned} \tag{3.1}$$

with all other commutators vanishing where $L_{\mu\nu}$ is the Lorentz generator, P_μ is the translation generator, D is the dilation generator and K_μ is the special conformal transformation generator. All the fields in the CFT belong to the representations of the conformal algebra which are characterized by the scaling dimension and the $SU(2)_L \times SU(2)_R$ Lorentz spin (s_+, s_-). For example, we consider the field strength operator $F_{\mu\nu}$. It is convenient to introduce the two-form as $F := F_{\mu\nu} dx^\mu \wedge dx^\nu$. The field strength can be separated into the self and antiself dual parts.

$$F^\pm = F \pm {}^*F, \tag{3.2}$$

with the spins $(1, 0)$ and $(0, 1)$, respectively, where $*$ is the Hodge dual. $F^\pm$ is the conformal primary operator which is annihilated by K_μ. The descendants are created by acting P_μ on the primary states . The norm of the state of the conserved current $J^\pm = {}^*dF^\pm$ is given by

$$|\,|J^\pm\rangle\,|^2 = 2\left([F] - 2\right), \tag{3.3}$$

where $[F]$ denotes the scaling dimension of the field strength. From (3.3) and unitarity, we see the field strength satisfies $[F] \geq 2$. For $[F] = 2$, due to the Bianchi identity, we obtain the free equation of motion $dF^+ \pm dF^- = 0$ which implies that the theory becomes the free $U(1)$ theory. For $[F] > 2$, the theory has the non-zero conserved currents $J^\pm \neq 0$ which are the descendants of the primary operators $F^\pm$. Due to the presence of both the non-zero electric current $J_e := J^+ + J^-$ and the non-zero magnetic current $J_m := J^+ + J^-$, any interacting field strength must couple to both electrons and monopole in CFT. This statement is also valid in supersymmetric conformal field theories.

We next consider the $\mathcal{N} = 2$ SCFT. The superconformal algebra is generated by the superconformal charge $Q^I_\alpha, \bar{Q}_{\dot{\alpha}I},\ S^\alpha_I, \bar{S}^{\dot{\alpha}I}$ and the R charge $R^I{}_J$ in addition to the generators of the conformal algebra. The representations of the $\mathcal{N} = 2$ superconformal algebra are labeled by not only the scaling dimension and the Lorentz spins but also $U(1)_R$ charge R and $SU(2)_R$ spin I. This means that there is an anomaly free $U(1)_R$ symmetry, which is the subalgebra of the $\mathcal{N} = 2$ superconformal algebra, in $\mathcal{N} = 2$ SCFTs. As discussed in the previous chapter, in the $\mathcal{N} = 2$ gauge theory on the Coulomb moduli space, the classical $U(1)_R$ is broken by the anomaly. Thus there appears the accidental $U(1)_R$ symmetry in the IR, if the $\mathcal{N} = 2$ SCFTs appear at a certain point in the Coulomb moduli space.

In the Coulomb moduli space, the interacting $\mathcal{N} = 2$ SCFT arises at the special locus where mutually non-local BPS particles become massless. This $\mathcal{N} = 2$ SCFT is called the Argyres-Douglas theory [3, 4]. Here the term "mutually non-local" means there is no electric-magnetic duality transformation going to the frame such that the fields carry only electric charge. Since there are both the massless particles with electric charge and those with magnetic charge, the AD theory has no Lagrangian description.

The relevant operators $\tilde{u}_i$, which deform the superconformal point with their coupling $\tilde{M}_i$, are regarded as the chiral primary fields with $I = 0$ and $(s_+, s_-) = (0, 0)$ in the $\mathcal{N} = 2$ vector multiplets. For $\mathcal{N} = 2$ SCFTs, the scaling dimension of the chiral primary field is determined from the $U(1)_R$ charge R and has the bound:

$$[\tilde{u}_i] = \frac{1}{2} R \geq 1. \tag{3.4}$$

When the inequality is saturated, the fields satisfy the null state equations corresponding to the free Maxwell equations. At the superconformal point, there are no null states so that $[\tilde{u}_i] > 1$. Furthermore, the operators $\tilde{u}_i$ and their coupling $\tilde{M}_i$ satisfy the condition

$$[\tilde{u}_i] + [\tilde{M}_i] = 2. \tag{3.5}$$

From the above, one finds the bound of the scaling dimension of the relevant or marginal operators:

$$1 < [\tilde{u}_i] \leq 2. \tag{3.6}$$

If $[\tilde{u}_i] > 2$, the operator $\tilde{u}_i$ is irrelevant. Thus the operators for the AD theory have the fractional scaling dimension due to the accidental $U(1)_R$ symmetry. The scaling dimension of the operators can be determined from the SW curve of the theory and the fact the SW differential has the scaling dimension one. In the next section, we will focus on the $SU(2)$ SQCD and see that the SW curve and the SW periods for the AD theory are given by those for $SU(2)$ SQCD, taking the scaling limit.

3.2 AD theory realized from $SU(2)$ SQCD

In this section, we study the superconformal point in the Coulomb moduli space for $\mathcal{N} = 2$ $SU(2)$ SQCD where mutually non-local BPS particles become massless [3, 4].

The charges for two dyons to be the mutually non-local satisfy [3]

$$n_m^{(1)} n_e^{(2)} - n_m^{(2)} n_e^{(1)} \neq 0, \tag{3.7}$$

where $n_m^{(i)}$ and $n_e^{(i)}$ are the magnetic and electric charges of the i'th dyon.

For $SU(2)$ gauge theories with $N_f = 1, 2, 3$ hypermultiplets, there is a superconformal point on the u-plane, where the squark and monopole/dyon are both massless [4]. The superconformal point are given by choosing the Coulomb moduli parameter and the mass parameters as $u = u^*$ and $m_1 = \cdots = m_{N_f} = m^*$ where u^* and m^* are given in table 3.1. The SW curve (2.92) degenerates as

$$y^2 \sim (p - p^*)^3, \tag{3.8}$$

where p^* is the branch point of p given in table 3.1.

3.2.1 Seiberg-Witten curve at superconformal point

We study the SW curve and the SW differential around the superconformal point. The relevant operators and its corresponding couplings are identified by taking the scaling limit. One determines their scaling dimensions from the SW curve and the fact the differential has the scaling dimensions one.

Let us consider in the $N_f = 1$ theory at first. If we substitute the branch point $p = p^* = -\frac{\Lambda_1}{2}$ into the curve (2.96), the curve becomes zero at $z = \pm\frac{\Lambda_1^{\frac{1}{2}}}{2}$. We choose the

N_f	1	2	3
m^*	$\frac{3}{4}\Lambda_1$	$\frac{\Lambda_2}{2}$	$\frac{\Lambda_3}{8}$
u^*	$\frac{3}{4}\Lambda_1^2$	$\frac{3}{8}\Lambda_2^2$	$\frac{\Lambda_3^2}{32}$
p^*	$-\frac{1}{2}\Lambda_1$	$-\frac{\Lambda_2}{2}$	$-\frac{\Lambda_3}{8}$

Table 3.1: The superconformal point for the $SU(2)$ theory is given by tuning the moduli parameter and the mass parameter to $u = u^*$ and $m = m^*$, respectively. p^* is the branch point of p of the SW curve degenerated from the curve (2.92) at the superconformal point.

branch point $z = -\frac{\Lambda_1^{\frac{1}{2}}}{2}$ and introduce new variables as

$$\begin{aligned} p =& \epsilon\tilde{p} - \frac{\Lambda_1}{2}, \qquad z = \frac{i2^{\frac{1}{2}}\epsilon^{\frac{3}{2}}}{\Lambda_1}\tilde{z} - \frac{\epsilon^2\tilde{M}}{\Lambda_1^{\frac{1}{2}}} - \frac{\epsilon\tilde{p}}{\Lambda_1^{\frac{1}{2}}} - \frac{\Lambda_1^{\frac{1}{2}}}{2}, \\ u =& \epsilon^3\tilde{u} + \epsilon^2\tilde{M}\Lambda_1 + \frac{3}{4}\Lambda_1^2, \qquad m_1 = \epsilon^2\tilde{M} + \frac{3}{4}\Lambda_1. \end{aligned} \tag{3.9}$$

Expanding around $\epsilon = 0$ with keeping $\tilde{u}$ and $\tilde{M}$ finite, the leading order of the curve in ϵ corresponds to the curve for the AD theory of (A_1, A_2)-type:

$$\tilde{z}^2 = \tilde{p}^3 - \tilde{M}\Lambda_1\tilde{p} - \frac{\Lambda_1}{2}\tilde{u}. \tag{3.10}$$

The SW differential (2.99) becomes

$$\lambda_{SW} = \frac{i\epsilon^{\frac{5}{2}}}{2^{\frac{1}{2}}\Lambda_1^{\frac{1}{2}}}\tilde{\lambda}_{SW} + \dots, \tag{3.11}$$

$$\tilde{\lambda}_{SW} := -\frac{8}{\Lambda_1}\tilde{z}d\tilde{p}. \tag{3.12}$$

by using (3.9) and taking the scaling limit $\epsilon \to 0$. From the curve (3.10), the scaling dimensions of $\tilde{u}$ and $\tilde{M}$ are $\frac{6}{5}$ and $\frac{4}{5}$, respectively. Here $\tilde{u}$ is the relevant operator and $\tilde{M}$ is the corresponding coupling parameter.

For $N_f = 2$, we define the new variables

$$\begin{aligned}
p &= \epsilon\tilde{p} - \frac{\epsilon\tilde{M}}{3} - \frac{\Lambda_2}{2}, \qquad z = \frac{i2^{\frac{1}{2}}\epsilon^{\frac{3}{2}}}{\Lambda_2^{\frac{1}{2}}}\tilde{z} - \epsilon\tilde{p} - \frac{2\epsilon\tilde{M}}{3}, \\
u &= \epsilon^2\tilde{u} - \frac{(\epsilon\tilde{M})^2}{3} + \Lambda_2\epsilon\tilde{M} + \frac{3\Lambda_2^2}{8}, \\
m_1 &= \frac{\Lambda_2}{2} + \epsilon\tilde{M} + \epsilon^{\frac{3}{2}}\tilde{a}, \qquad m_2 = \frac{\Lambda_2}{2} + \epsilon\tilde{M} - \epsilon^{\frac{3}{2}}\tilde{a},
\end{aligned} \tag{3.13}$$

and consider the scaling limit $\epsilon \to 0$ of the curve (2.96). At leading order in ϵ the curve (2.96) becomes

$$\tilde{z}^2 = \tilde{p}^3 - \tilde{u}\tilde{p} - \frac{2}{3}\tilde{M}\tilde{u} + \frac{8}{27}\tilde{M}^3 - \frac{\tilde{C}_2\Lambda_2}{4}. \tag{3.14}$$

Here $\tilde{u}$, $\tilde{M}$ and $\tilde{C}_2 := 2\tilde{a}^2$ are the relevant operator, the corresponding coupling and the Casimir invariant of the $U(2)$ flavor symmetry, respectively. The curve (3.14) corresponds to that of the AD theory of (A_1, A_3)-type. In the scaling limit around the superconformal point, the SW differential (2.99) becomes

$$\lambda_{\mathrm{SW}} = \frac{i\epsilon^{\frac{3}{2}}}{2^{\frac{1}{2}}\Lambda_2^{\frac{1}{2}}}\tilde{\lambda}_{\mathrm{SW}} + \cdots, \tag{3.15}$$

up to the total derivatives where

$$\tilde{\lambda}_{\mathrm{SW}} = -4\tilde{z}\, d\log\left(\tilde{p} + \frac{2}{3}\tilde{M}\right). \tag{3.16}$$

The scaling dimensions of $\tilde{u}$, $\tilde{M}$ and $\tilde{C}_2$ are $\frac{4}{3}$, $\frac{2}{3}$ and 2, respectively.

For $N_f = 3$, introducing the scaling variables as

$$\begin{aligned}
p &= \epsilon^2\tilde{p} - \epsilon\tilde{M} + \frac{4\left((\epsilon\tilde{M})^2 + \epsilon^3\tilde{u}\right)}{3\Lambda_3} + \frac{16(\epsilon\tilde{M})^3}{9\Lambda_3^2} - \frac{\Lambda_3}{8}, \\
z &= \epsilon^3 i\tilde{z} - \frac{4(\epsilon\tilde{M})^3}{3\Lambda_3^{\frac{3}{2}}} - \frac{2(\epsilon\tilde{M})(\epsilon^2\tilde{p})}{\Lambda_3^{\frac{1}{2}}} - \frac{\epsilon^3\tilde{u}}{\Lambda_3^{\frac{1}{2}}}, \\
u &= \epsilon^3\tilde{u} - \frac{4(\epsilon\tilde{M})^3}{3\Lambda_3} + (\epsilon\tilde{M})^2 + \frac{3\Lambda_3\epsilon\tilde{M}}{8} + \frac{\Lambda_3^2}{32}, \\
m_1 &= \frac{\Lambda_3}{8} + \epsilon\tilde{M} + \epsilon^2\tilde{c}_1, \qquad m_2 = \frac{\Lambda_3}{8} + \epsilon\tilde{M} + \epsilon^2\tilde{c}_2, \qquad m_3 = \frac{\Lambda_3}{8} + \epsilon\tilde{M} - \epsilon^2(\tilde{c}_1 + \tilde{c}_2),
\end{aligned} \tag{3.17}$$

and then taking the scaling limit $\epsilon \to 0$ with fixed $\tilde{u}$, $\tilde{M}$, $\tilde{c}_1$ and $\tilde{c}_2$, we obtain the curve of the AD theory of (A_1, D_4)-type:

$$\tilde{z}^2 = \tilde{p}^3 - \tilde{p}\left(\frac{\tilde{C}_2}{2} + \frac{4\tilde{M}\tilde{u}}{\Lambda_3}\right) - \frac{\tilde{u}^2}{\Lambda_3} - \frac{8\tilde{M}^3\tilde{u}}{3\Lambda_3^2} + \frac{16\tilde{M}^6}{27\Lambda_3^3} - \frac{2\tilde{C}_2\tilde{M}^2}{3\Lambda_3} + \frac{\tilde{C}_3}{3}, \tag{3.18}$$

where

$$\tilde{C}_2 := 2\left(\tilde{c}_1^2 + \tilde{c}_1\tilde{c}_2 + \tilde{c}_2^2\right), \qquad \tilde{C}_3 := -3\left(\tilde{c}_1^2\tilde{c}_2 + \tilde{c}_1\tilde{c}_2^2\right). \tag{3.19}$$

Here $\tilde{u}$ and $\tilde{M}$ are the operator and the corresponding coupling, respectively. $\tilde{C}_2$ and $\tilde{C}_3$ refer to the Casimir invariants associated with the $U(3)$ flavor symmetry. Then the SW differential (2.99) around the superconformal point becomes

$$\lambda_{\text{SW}} = \frac{i\epsilon^2}{\Lambda_3^{\frac{1}{2}}}\tilde{\lambda}_{\text{SW}} + \cdots, \tag{3.20}$$

up to the total derivatives where

$$\tilde{\lambda}_{\text{SW}} = i\Lambda_3^{\frac{1}{2}}\left\{2\tilde{p}\, d\log\left(i\tilde{z} - \frac{2\tilde{M}\tilde{p}}{\Lambda_3^{\frac{1}{2}}} - \frac{4\tilde{M}^3}{3\Lambda_3^{\frac{3}{2}}} - \frac{\tilde{u}}{\Lambda_3^{\frac{1}{2}}}\right) - \sum_{i=1}^{3}\tilde{p}\, d\log(\tilde{p} + \tilde{m}_i)\right\}. \tag{3.21}$$

Here the parameters $\tilde{m}_i$ $(i = 1, \ldots, 3)$ are interpreted as the mass parameters at the superconformal point, which defined by

$$\tilde{m}_1 = \frac{4\tilde{M}^2}{3\Lambda_3} + \tilde{c}_1, \qquad \tilde{m}_2 = \frac{4\tilde{M}^2}{3\Lambda_3} + \tilde{c}_2, \qquad \tilde{m}_3 = \frac{4\tilde{M}^2}{3\Lambda_3} - (\tilde{c}_1 + \tilde{c}_2). \tag{3.22}$$

We see that the scaling dimensions of $\tilde{u}$, $\tilde{M}$, $\tilde{C}_2$ and $\tilde{C}_3$ are $\frac{3}{2}$, $\frac{1}{2}$, 2 and 3, respectively. We summarize the scaling dimensions of the operators for the AD theory realized from the $N_f = 1, 2, 3$ theories in table 3.2.

3.2.2 The period integrals around the superconformal point

We next discuss the SW periods for the AD theories realized from the $SU(2)$ gauge theory with N_f hypermultiplets. The SW curves for the N_f AD theory takes the form as

$$\tilde{z}^2 = \tilde{p}^3 - \rho_{N_f}\tilde{p} - \sigma_{N_f}, \tag{3.23}$$

N_f	AD theory	$[\tilde{u}]$	$[\tilde{M}]$	$[\tilde{C}_2]$	$[\tilde{C}_3]$
1	(A_1, A_2)	$\frac{6}{5}$	$\frac{4}{5}$	–	–
2	(A_1, A_3)	$\frac{4}{3}$	$\frac{2}{3}$	2	–
3	(A_1, D_4)	$\frac{3}{2}$	$\frac{1}{2}$	2	3

Table 3.2: The scaling dimension of the operators for the AD theory of (A_1, A_2), (A_1, A_3) and (A_1, D_4)-type. $[\mathcal{O}]$ denotes the scaling dimension of $\mathcal{O}$.

where ρ_{N_f} and σ_{N_f} are read off from (3.10), (3.14) and (3.18). The SW differentials $\tilde{\lambda}_{\rm SW}$ (3.12), (3.15) and (3.20) have been normalized such that

$$\frac{\partial}{\partial \tilde{u}} \tilde{\lambda}_{SW} = \frac{2 d\tilde{p}}{\tilde{z}}. \tag{3.24}$$

We then define the SW periods as

$$\tilde{\Pi} = (\tilde{a}, \tilde{a}_D) = \left(\int_{\tilde{\alpha}} \tilde{\lambda}_{SW}, \int_{\tilde{\beta}} \tilde{\lambda}_{SW} \right), \tag{3.25}$$

where $\tilde{\alpha}$ and $\tilde{\beta}$ are canonical 1-cycles on the curve (3.23). The $\tilde{u}$-derivative of the SW periods becomes the period integral of the holomorphic differential defined by

$$\omega = \int_{\tilde{\alpha}} \frac{d\tilde{p}}{\tilde{z}}, \quad \omega_D = \int_{\tilde{\beta}} \frac{d\tilde{p}}{\tilde{z}}. \tag{3.26}$$

As discussed in the $SU(2)$ SQCD, we express the period integral in terms of the hypergeometric functions of $\tilde{w}$ given by

$$\tilde{w}_{N_f} := -\frac{27\tilde{\Delta}_{N_f}}{4\tilde{D}_{N_f}^3} = 1 - \frac{27\sigma_{N_f}^2}{4\rho_{N_f}^3}. \tag{3.27}$$

Here $\tilde{\Delta}_{N_f}$ and $\tilde{D}_{N_f}$ correspond to Δ in (2.73) and D in (2.74), respectively, which are given by

$$\tilde{\Delta}_{N_f} = 4\rho_{N_f}^3 - 27\sigma_{N_f}^2, \tag{3.28}$$

$$\tilde{D}_{N_f} = -3\rho_{N_f}. \tag{3.29}$$

By using the quadratic (2.66) and cubic transformation (2.67) [74, 75], the periods (3.26) are obtained by

$$\omega^0(\tilde{w},\tilde{D}) = 2\pi\left(-\tilde{D}\right)^{-\frac{1}{4}} F\left(\frac{1}{12},\frac{5}{12};1;\tilde{w}\right), \tag{3.30}$$

$$\omega^0_D(\tilde{w},\tilde{D}) = -2i\pi\left(-\tilde{D}\right)^{-\frac{1}{4}}\left(\frac{3\log 12}{2\pi}F\left(\frac{1}{12},\frac{5}{12};1;\tilde{w}\right) - \frac{1}{2\pi}F_*\left(\frac{1}{12},\frac{5}{12};1;\tilde{w}\right)\right), \tag{3.31}$$

around the point $\tilde{w}=0$ where $F(\alpha,\beta;\gamma;z)$ and $F_*(\alpha,\beta;1;z)$ are defined by (2.61) and (2.71). Here the subscript N_f of $\tilde{w}$ and $\tilde{D}$ has been omitted for sake of simplicity. Since the dual period (3.31) has logarithmic behavior around $\tilde{w}=0$, it is not the expression around the superconformal point with the fractional dimensional $\tilde{u}$ and $\tilde{M}$.

From the analytic continuation of the solutions around $\tilde{w}=0$ to those of $\tilde{w}=\infty$, the periods (3.30) and (3.31) become

$$\begin{aligned}\omega^\infty(\tilde{w},\tilde{D}) =& 2\pi(-\tilde{D})^{-\frac{1}{4}}\left(\frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{5}{12}\right)\Gamma\left(\frac{11}{12}\right)}(1-\tilde{w})^{-\frac{1}{12}}F\left(\frac{1}{12},\frac{7}{12};\frac{2}{3};\frac{1}{1-\tilde{w}}\right)\right.\\ &\left.+\frac{\Gamma\left(-\frac{1}{3}\right)}{\Gamma\left(\frac{1}{12}\right)\Gamma\left(\frac{7}{12}\right)}(1-\tilde{w})^{-\frac{5}{12}}F\left(\frac{5}{12},\frac{11}{12};\frac{4}{3};\frac{1}{1-\tilde{w}}\right)\right),\end{aligned} \tag{3.32}$$

$$\begin{aligned}\omega^\infty_D(\tilde{w},\tilde{D}) =& 2i\pi(-\tilde{D})^{-\frac{1}{4}}\left(\frac{(-1)^{\frac{5}{6}}\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{5}{12}\right)\Gamma\left(\frac{11}{12}\right)}(1-\tilde{w})^{-\frac{1}{12}}F\left(\frac{1}{12},\frac{7}{12};\frac{2}{3};\frac{1}{1-\tilde{w}}\right)\right.\\ &\left.+\frac{(-1)^{\frac{1}{6}}\Gamma\left(-\frac{1}{3}\right)}{\Gamma\left(\frac{1}{12}\right)\Gamma\left(\frac{7}{12}\right)}(1-\tilde{w})^{-\frac{5}{12}}F\left(\frac{5}{12},\frac{11}{12};\frac{4}{3};\frac{1}{1-\tilde{w}}\right)\right),\end{aligned} \tag{3.33}$$

respectively. Here we use the connection formula [74]

$$\begin{aligned}F(\alpha,\beta;\gamma;z) =& \frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}(1-z)^{-\alpha}F\left(\alpha,\gamma-\beta;\alpha-\beta+1;\frac{1}{1-z}\right)\\ &+\frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)}(1-z)^{-\beta}F\left(\beta,\gamma-\alpha;-\alpha+\beta+1;\frac{1}{1-z}\right),\end{aligned} \tag{3.34}$$

where $|\arg(1-z)|<\pi$. Similarly, we use the connection formula

$$\begin{aligned}F(\alpha,\beta;\gamma;z) =& \frac{(1-z)^{-\alpha-\beta+\gamma}\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}F(\gamma-\alpha,\gamma-\beta;-\alpha-\beta+\gamma+1;1-z)\\ &+\frac{\Gamma(\gamma)\Gamma(-\alpha-\beta+\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}F(\alpha,\beta;\alpha+\beta-\gamma+1;1-z),\end{aligned} \tag{3.35}$$

in order to perform the analytic connection to the solutions around $\tilde{w}=1$. Thus we find the expansions around $\tilde{w}=1$:

$$\begin{aligned}\omega^1(\tilde{w},\tilde{D}) =&\pi^{-\frac{1}{2}}(-\tilde{D})^{-\frac{1}{4}}\left(6\Gamma\left(\frac{5}{12}\right)\Gamma\left(\frac{13}{12}\right)F\left(\frac{1}{12},\frac{5}{12};\frac{1}{2};1-\tilde{w}\right)\right.\\ &\left.-(1-\tilde{w})^{\frac{1}{2}}\Gamma\left(\frac{7}{12}\right)\Gamma\left(\frac{11}{12}\right)F\left(\frac{7}{12},\frac{11}{12};\frac{3}{2};1-\tilde{w}\right)\right),\end{aligned} \tag{3.36}$$

$$\begin{aligned}\omega_D^1(\tilde{w},\tilde{D}) =&-i\pi^{-\frac{1}{2}}(-\tilde{D})^{-\frac{1}{4}}\left(6\Gamma\left(\frac{5}{12}\right)\Gamma\left(\frac{13}{12}\right)F\left(\frac{1}{12},\frac{5}{12};\frac{1}{2};1-\tilde{w}\right)\right.\\ &\left.+(1-\tilde{w})^{\frac{1}{2}}\Gamma\left(\frac{7}{12}\right)\Gamma\left(\frac{11}{12}\right)F\left(\frac{7}{12},\frac{11}{12};\frac{3}{2};1-\tilde{w}\right)\right).\end{aligned} \tag{3.37}$$

By using these formulas, we will study the SW periods for the AD theories obtained from the $SU(2)$ theory with $N_f=1,2,3$ in the scaling limit as follows:

1. For $N_f=1$, $\tilde{w}_1$ and $\tilde{D}_1$ are given by

$$\tilde{w}_1 = 1-\frac{27\tilde{u}^2}{16\Lambda_1\tilde{M}^3}, \tag{3.38}$$

$$\tilde{D}_1 = -3\Lambda_1\tilde{M}. \tag{3.39}$$

The superconformal point corresponds to $\tilde{w}'_1 := \frac{1}{1-\tilde{w}_1} = 0$. Thus the expansions around the superconformal point are given by using Eqs. (3.32) and (3.33):

$$\frac{\partial\tilde{a}}{\partial\tilde{u}} = 2\omega^\infty(\tilde{w}_1,\tilde{D}_1), \quad \frac{\partial\tilde{a}_D}{\partial\tilde{u}} = 2\omega_D^\infty(\tilde{w}_1,\tilde{D}_1). \tag{3.40}$$

From the integration of these solution over $\tilde{u}$, one obtains the SW periods

$$\begin{aligned}\tilde{a} =&\frac{3^{\frac{1}{2}}\Lambda_1^{\frac{3}{2}}}{2^{\frac{1}{2}}\cdot 5\pi^{\frac{1}{2}}}\left(\frac{\tilde{u}}{\Lambda_1^2}\right)^{\frac{5}{6}}\left(2^{\frac{8}{3}}\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right)F\left(-\frac{5}{12},\frac{1}{12};\frac{2}{3};\tilde{w}'_1\right)\right.\\ &\left.+15\tilde{w}_1'^{\frac{1}{3}}\Gamma\left(-\frac{1}{6}\right)\Gamma\left(\frac{5}{3}\right)F\left(-\frac{1}{12},\frac{5}{12};\frac{4}{3};\tilde{w}'_1\right)\right),\end{aligned} \tag{3.41}$$

$$\begin{aligned}\tilde{a}_D =&\frac{3^{\frac{1}{2}}\Lambda_1^{\frac{3}{2}}}{2^{\frac{1}{2}}\cdot 5\pi^{\frac{1}{2}}}\left(\frac{\tilde{u}}{\Lambda_1^2}\right)^{\frac{5}{6}}\left(-2^{\frac{8}{3}}(-1)^{\frac{1}{3}}\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right)F\left(-\frac{5}{12},\frac{1}{12};\frac{2}{3};\tilde{w}'_1\right)\right.\\ &\left.+15(-1)^{\frac{2}{3}}\tilde{w}_1'^{\frac{1}{3}}\Gamma\left(-\frac{1}{6}\right)\Gamma\left(\frac{5}{3}\right)F\left(-\frac{1}{12},\frac{5}{12};\frac{4}{3};\tilde{w}'_1\right)\right).\end{aligned} \tag{3.42}$$

Note that the SW periods $\tilde{\Pi}$ satisfy the Picard-Fuchs equation [21]

$$(1-\tilde{w}'_1)\,\tilde{w}'_1\frac{\partial^2}{\partial\tilde{w}_1'^2}\tilde{\Pi}+\frac{2}{3}(1-\tilde{w}'_1)\frac{\partial}{\partial\tilde{w}'_1}\tilde{\Pi}+\frac{5}{144}\tilde{\Pi}=0. \tag{3.43}$$

From (3.41) and (3.42), we see that $\tilde{a} \sim \tilde{u}^{\frac{5}{6}}$ and $\tilde{a}_D \sim \tilde{u}^{\frac{5}{6}}$. We also find that the scaling dimension of $\tilde{u}$ and $\tilde{M}$ is given by $\frac{6}{5}$ and $\frac{4}{5}$, respectively, by using the fact the SW periods $\tilde{a}$ and $\tilde{a}_D$ have the scaling dimension one. From (3.41) and (3.42), we can compute the expansion of the coupling constant $\tau := \frac{\partial_{\tilde{u}}\tilde{a}_D}{\partial_{\tilde{u}}\tilde{a}}$ in $\tilde{w}_1'$. We then find it does not have logarithmic divergence, which implies that the theory is around the superconformal point. The SW periods (3.41) and (3.42) are interpreted as the expansions in the coupling $\tilde{M}$ with fixed $\tilde{u}$ in the scaling limit. We note that the present expansions for N_f theories are different from the results in the previous literature: In [75, 38], one have expanded in small $\tilde{u}$ without taking the scaling limit, after the coupling and the Casimir invariants have been chosen to be zero. In [82], the expansion of the SW periods without taking the scaling limit have been calculated.

2. For $N_f = 2$, $\tilde{w}_2$ and $\tilde{D}_2$ are obtained by

$$\tilde{w}_2 = 1 - \frac{(\frac{27}{2}\tilde{C}_2\Lambda_2 - 16\tilde{M}^3 + 36\tilde{M}\tilde{u})^2}{432\tilde{u}^3}, \tag{3.44}$$

$$\tilde{D}_2 = -3\tilde{u}. \tag{3.45}$$

The superconformal point corresponds to $\tilde{w}_2' := 1 - \tilde{w}_2 = 0$. Applying Eqs.(3.36) and (3.37), we obtain the expansion around the superconformal point:

$$\frac{\partial \tilde{a}}{\partial \tilde{u}} = 2\omega^1(\tilde{w}_2, \tilde{D}_2), \quad \frac{\partial \tilde{a}_D}{\partial \tilde{u}} = 2\omega_D^1(\tilde{w}_2, \tilde{D}_2). \tag{3.46}$$

After expanding them around $\tilde{w}_2' = 0$, where $\frac{\tilde{M}^2}{\tilde{u}} \ll 1$ and $\frac{\tilde{C}_2\Lambda_2}{\tilde{u}^{\frac{3}{2}}} \ll 1$, we integrate over $\tilde{u}$. Then the expansions of the SW periods around the superconformal point are given by

$$\begin{aligned}\tilde{a} =& \Lambda_2^{\frac{3}{2}} \left(\frac{\tilde{u}}{\Lambda_2^2}\right)^{\frac{3}{4}} \left(\frac{2^4\Gamma\left(\frac{5}{12}\right)\Gamma\left(\frac{13}{12}\right)}{3^{\frac{1}{4}}\pi^{\frac{1}{2}}} - \frac{2^3\cdot 3^{\frac{1}{4}}\Gamma\left(\frac{7}{12}\right)\Gamma\left(\frac{11}{12}\right)}{\pi^{\frac{1}{2}}} \left(\frac{\tilde{M}^2}{\tilde{u}}\right)^{\frac{1}{2}} \right. \\ & \left. - \frac{3^{\frac{1}{2}}\Gamma\left(\frac{7}{12}\right)\Gamma\left(\frac{11}{12}\right)}{\pi^{\frac{1}{2}}} \left(\frac{\tilde{C}_2^2\Lambda_2^2}{\tilde{u}^3}\right)^{\frac{1}{2}} + \cdots \right),\end{aligned} \tag{3.47}$$

$$\tilde{a}_D = \Lambda_2^{\frac{3}{2}} \left(\frac{\tilde{u}}{\Lambda_2^2}\right)^{\frac{3}{4}} \left(-\frac{2^4 i\Gamma\left(\frac{5}{12}\right)\Gamma\left(\frac{13}{12}\right)}{3^{\frac{1}{4}}\pi^{\frac{1}{2}}} - \frac{2^3\cdot 3^{\frac{1}{4}} i\Gamma\left(\frac{7}{12}\right)\Gamma\left(\frac{11}{12}\right)}{\pi^{\frac{1}{2}}}\left(\frac{\tilde{M}^2}{\tilde{u}}\right)^{\frac{1}{2}} - \frac{3^{\frac{1}{2}} i\Gamma\left(\frac{7}{12}\right)\Gamma\left(\frac{11}{12}\right)}{\pi^{\frac{1}{2}}}\left(\frac{\tilde{C}_2^2\Lambda_2^2}{\tilde{u}^3}\right)^{\frac{1}{2}} + \cdots\right). \tag{3.48}$$

We see again that the scaling dimensions of $\tilde{u}$, $\tilde{M}$ and $\tilde{C}_2$ are given by $\frac{4}{3}$, $\frac{2}{3}$ and 2, respectively. The expansions of the periods (3.47) and (3.48) do not contain logarithmic terms.

3. For $N_f = 3$, we have

$$\tilde{w}_3 = 1 - \frac{(-9\tilde{C}_3\Lambda_3^3 + 18\tilde{C}_2\Lambda_3^2\tilde{M}^2 - 16\tilde{M}^6 + 72\Lambda_3\tilde{M}^3\tilde{u} + 27\Lambda_3^2\tilde{u}^2)^2}{108\Lambda_3^6\left(\frac{\tilde{C}_2}{2} + 4\frac{\tilde{M}\tilde{u}}{\Lambda_3}\right)^3}, \tag{3.49}$$

$$\tilde{D}_3 = -3\left(\frac{\tilde{C}_2}{2} + \frac{4\tilde{M}\tilde{u}}{\Lambda_3}\right). \tag{3.50}$$

The superconformal point corresponds to $\tilde{w}_3 = \infty$ or $\tilde{w}'_3 := \frac{1}{1-\tilde{w}_3} = 0$. By using Eqs. (3.32) and (3.33), the periods around the superconformal point are given by

$$\frac{\partial\tilde{a}}{\partial\tilde{u}} = 2\omega^\infty(\tilde{w}_3, \tilde{D}_3), \quad \frac{\partial\tilde{a}_D}{\partial\tilde{u}} = 2\omega_D^\infty(\tilde{w}_3, \tilde{D}_3). \tag{3.51}$$

In a similar way as the case of the $N_f = 1$ and 2, we expand them around $\tilde{w}'_3 = 0$, where $\frac{\tilde{M}^3}{\tilde{u}\Lambda_3} \ll 1$, $\frac{\tilde{C}_2^3\Lambda_3^2}{\tilde{u}^4} \ll 1$ and $\frac{\tilde{C}_3\Lambda_3}{\tilde{u}^2} \ll 1$, and integrate over $\tilde{u}$. After that, we have the SW periods:

$$\tilde{a} = \Lambda_3^{\frac{3}{2}}(-1)^{\frac{5}{6}}\left(\frac{\tilde{u}}{\Lambda_3^2}\right)^{\frac{2}{3}}\left(\frac{5\Gamma\left(-\frac{5}{6}\right)\Gamma\left(\frac{1}{3}\right)}{2\cdot 3^{\frac{1}{2}}\pi^{\frac{1}{2}}} - \frac{2^3\Gamma\left(-\frac{1}{3}\right)\Gamma\left(\frac{5}{6}\right)}{3^{\frac{1}{2}}\pi^{\frac{1}{2}}}\left(\frac{\tilde{M}^3}{\tilde{u}\Lambda_3}\right)^{\frac{1}{3}} + \frac{\Gamma\left(-\frac{1}{3}\right)\Gamma\left(\frac{5}{6}\right)}{2\cdot\pi^{\frac{1}{2}}}\left(\frac{\tilde{C}_2^3\Lambda_3^2}{\tilde{u}^4}\right)^{\frac{1}{3}} + \frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right)}{2^2\cdot 3^{\frac{3}{2}}\pi^{\frac{1}{2}}}\left(\frac{\tilde{C}_3\Lambda_3}{\tilde{u}^2}\right) + \cdots\right), \tag{3.52}$$

$$\tilde{a}_D = \Lambda_3^{\frac{3}{2}}(-1)^{\frac{1}{6}}\left(\frac{\tilde{u}}{\Lambda_3^2}\right)^{\frac{2}{3}}\left(\frac{5\Gamma\left(-\frac{5}{6}\right)\Gamma\left(\frac{1}{3}\right)}{2\cdot 3^{\frac{1}{2}}\pi^{\frac{1}{2}}} - \frac{2^3\Gamma\left(-\frac{1}{3}\right)\Gamma\left(\frac{5}{6}\right)}{3^{\frac{1}{2}}\pi^{\frac{1}{2}}}\left(\frac{\tilde{M}^3}{\tilde{u}\Lambda_3}\right)^{\frac{1}{3}} + \frac{i\Gamma\left(-\frac{1}{3}\right)\Gamma\left(\frac{5}{6}\right)}{2\cdot\pi^{\frac{1}{2}}}\left(\frac{\tilde{C}_2^3\Lambda_3^2}{\tilde{u}^4}\right)^{\frac{1}{3}} + \frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right)}{2^2\cdot 3^{\frac{3}{2}}\pi^{\frac{1}{2}}}\left(\frac{\tilde{C}_3\Lambda_3}{\tilde{u}^2}\right) + \cdots\right). \tag{3.53}$$

The scaling dimensions of $\tilde{u}$, $\tilde{M}$, $\tilde{C}_2$ and $\tilde{C}_3$ are also read off $\frac{3}{2}$, $\frac{1}{2}$, 2 and 3, respectively. Furthermore, we see the expansion of the SW periods includes no logarithmic behavior.

In the next section, we will generalize the above discussion to the case of the $SU(N_c)$ SQCD and obtain the SW curve and the SW differential around the superconformal point.

3.3 AD theory realized from $SU(N_c)$ SQCD

We will consider the curve obtained from the degeneration of the SW curve (2.121) around the superconformal point of $\mathcal{N}=2$ $SU(N_c)$ SQCD. For the pure $SU(N_c)$ gauge theory, the SW curve around the superconformal point has been studied in [21]. The superconformal point for the $SU(N_c)$ gauge theory with N_f hypermultiplets has been studied in [20, 22]. For $SU(N_c)$ SQCD, the curve at the superconformal point are classified as follows:

1. For $N_f = 0$, the superconformal point is given by $s_i = s_i^*$ where

$$s_1^* = s_2^* = \cdots = s_{N_c-1}^* = 0, \quad s_{N_c}^* = \Lambda_0^{N_c}. \tag{3.54}$$

The SW curve (2.121) at the superconformal point becomes

$$y^2 \approx (p - p^*)^{N_c}, \tag{3.55}$$

where $p^* = 0$ is the branch point of p.

2. For $N_f = 1$, we can not distinguish squarks from other singularities, but we find the form of the curve takes

$$y^2 \approx (p - p^*)^{N_c+1}, \tag{3.56}$$

by counting the number of parameters, where p^* is the branch point of p. The theory described by the curve (3.56) belongs to the same universality class as the theory described by the curve (3.55).

3. For $N_f \geq 2$, we can find systematically the superconformal point [20]. We consider the case that all of the mass is same mass: $m := m_1 = \cdots = m_{N_f}$. In the case of

even hypermultiplets $N_f = 2n$, the SW curve (2.121) can be split by

$$y^2 = y_+(p)y_-(p),$$

where

$$y_\pm(p) = C(p) \pm \Lambda^{N_c-n}(p+m)^n.$$

The discriminant Δ of the curve (2.121) becomes

$$\Delta = \Delta_s\Delta_m, \tag{3.57}$$

where

$$\Delta_s = \mathrm{Res}(y_+, y_-)^2 = (C(p=-m))^{N_f}, \tag{3.58}$$

$$\Delta_m = \mathrm{Res}\left(y_+, \partial_p\, y_+\right) \mathrm{Res}\left(y_-, \partial_p\, y_-\right), \tag{3.59}$$

where $\mathrm{Res}(f,g)$ denotes the resultant of the two polynomials $f(p)$ and $g(p)$. In a similar way, we can treat the case of $N_f = 2n+1$ and find that the discriminant is of the form (3.57).

The point where $\Delta_s = 0$ relates to the massless squark point. When we fix $s_{N_c} = s^*_{N_c}$ where $s^*_{N_c}$ is the solution of $\Delta_s = 0$, then the function $C(p)$ and the discriminant Δ_m become

$$C(p) = (p+m)C_1(p), \qquad \Delta_m = \Delta_1\Delta_{2m}, \tag{3.60}$$

where $C_1(p)$ is a polynomial of order $N_c - 1$ and Δ_1 is (a power of) the resultant of $C_1(p)$ and $p+m$. The curve becomes

$$y^2 = (p+m)^2\left(C_1(p)^2 - \Lambda_{N_f}^{2N_c-N_f}(p+m)^{N_f-2}\right). \tag{3.61}$$

We next set $s_{N_c-1} = s^*_{N_c-1}$ which is the solution of the $\Delta_1 = 0$. Then $C_1(p)$ and Δ_{2m} are written as

$$C_1(p) = (p+m)C_2(x), \qquad \Delta_{2m} = \Delta_2\Delta_{4m}, \tag{3.62}$$

with a polynomial $C_2(p)$ of order $N_c - 2$. The curve becomes

$$y^2 = (p+m)^4\left(C_2(p)^2 - \Lambda_{N_f}^{2N_c-N_f}(p+m)^{N_f-4}\right). \tag{3.63}$$

Similarly we obtain the critical values $s^*_{N_c}, s^*_{N_c-1}, s^*_{N_c-2}, \cdots$ and m^* successively.

Eventually, we classify maximally superconformal point of $SU(N_c)$ theories with N_f hypermultiplets into four groups by degeneration of the curve

$$N_f = 0, 1, \qquad y^2 \approx (p - p^*)^l, \qquad \begin{cases} l = N_c, & \text{for } N_f = 0, \\ l = N_c + 1, & \text{for } N_f = 1, \end{cases} \tag{3.64}$$

$$N_f = 2, \qquad y^2 \approx (p + m^*)^{N_c+1}, \tag{3.65}$$

$$N_f = 2n + 1, \qquad y^2 \approx (p + m^*)^{N_f}, \tag{3.66}$$

$$N_f = 2n, \ (n \geq 2), \qquad y^2 \approx (p + m^*)^{N_c+n}, \tag{3.67}$$

where m^* is the superconformal point of m and p^* is the branch point of p for the superconformal point.

We study the SW curve for the AD theory obtained from the $SU(N_c)$ SQCD in the scaling limit. In order to obtain the SW curve around the superconformal point, we define the scaling variables from the moduli parameters s_i $(i = 1, \cdots, N_c)$, the mass parameters m_j $(j = 1, \cdots, N_f)$ and the coordinate p of the SW curve as

$$\begin{aligned} p =& p^* + \epsilon p_0 + \epsilon^2 \tilde{p}, \\ s_i =& s_i^* + \sum_{k=1}^{h_i - 1} \epsilon^k s_{0ik} + \epsilon^{h_i} \tilde{s}_i, \quad (i = 2, \cdots N_c), \\ m_j =& m^* + \sum_{k=1}^{h_M - 1} \epsilon^k m_{0jk} + \epsilon^{h_M} \tilde{M} + \epsilon^{h_c} \tilde{c}_j, \quad (j = 1, \cdots, N_f), \end{aligned} \tag{3.68}$$

where h_i, h_M and h_c are given by

$$h_i = \frac{2[\tilde{s}_i]}{[\tilde{p}]}, \quad h_M = \frac{2[\tilde{M}]}{[\tilde{p}]}, \quad h_c = \frac{2}{[\tilde{p}]}, \tag{3.69}$$

with $[\mathcal{O}]$ being the scaling dimension of $\mathcal{O}$. Here, $\tilde{c}_{N_f}$ is defined by

$$\tilde{c}_{N_f} := - \sum_{j=1}^{N_f - 1} \tilde{c}_j. \tag{3.70}$$

The parameters p_0, s_{0ij} and m_{0jk} are determined such that the leading term of the SW curve becomes

$$y^2 = \epsilon^{\frac{4[\tilde{y}]}{[\tilde{p}]}} \tilde{y}^2 + \cdots, \tag{3.71}$$

after expanding around $\epsilon = 0$ where $\tilde{y}$ corresponds to the SW curve for the AD theory realized from the $SU(N_c)$ gauge theory with N_f hypermultiplets in the scaling limit.

3.3.1 $N_f = 0$

For $N_f = 0$, we define $h_i = 2i$ in the scaling variables (3.68). Substituting (3.68) into (2.122) and (2.123), we find

$$C(p) = -\Lambda_0^{N_c} + \epsilon^{2N_c}\tilde{C}(\tilde{p}), \tag{3.72}$$

$$G(p) = 1, \tag{3.73}$$

where

$$\tilde{C}(\tilde{p}) = \tilde{p}^{N_c} - \sum_{i=2}^{N_c} \tilde{s}_i \tilde{p}^{N_c-i}. \tag{3.74}$$

The SW curve (2.121) and the SW differential (2.56) become

$$y^2 = \epsilon^{2N_c}\tilde{y}^2 + \epsilon^{4N_c}\tilde{C}(\tilde{p})^2, \tag{3.75}$$

$$\lambda_{\text{SW}} = \epsilon^{N_c+2}\tilde{\lambda}_{\text{SW}} + \cdots, \tag{3.76}$$

where the leading order term of the SW curve and the SW differential are

$$\tilde{y}^2 = -2\Lambda_0^{N_c}\tilde{C}(\tilde{p}), \tag{3.77}$$

$$\tilde{\lambda}_{\text{SW}} = -\frac{2}{\Lambda_0^{N_c}}\tilde{y}d\tilde{p}. \tag{3.78}$$

Since the scaling dimension of the SW differential is one, we find

$$[\tilde{y}] = \frac{N_c}{N_c+2}, \quad [\tilde{p}] = \frac{2}{N_c+2}, \quad [\tilde{s}_i] = \frac{2i}{N_c+2}. \tag{3.79}$$

3.3.2 $N_f = 1$

For the $SU(N_c)$ gauge theory with $N_f = 1$ hypermultiplet, we use the scaling parameters (3.68) with

$$h_i = 2i \ (i = 2 \cdots N_c - 1), \quad h_{N_c} = 2(N_c+1), \quad h_M = 2N_c. \tag{3.80}$$

Substituting (3.68) into (2.122) and (2.123), we find

$$C(p) = \pm\left(\Lambda_1^{2N_c-1}(p^*+m^*)\right)^{\frac{1}{2}} \pm \epsilon^2\frac{1}{2}\left(\frac{\Lambda_1^{2N_c-1}}{p^*+m^*}\right)^{\frac{1}{2}}\tilde{p} + \sum_{i=2}^{N_c}\epsilon^{2i}\hat{C}_i(\tilde{p}) - \epsilon^{2(N_c+1)}\tilde{s}_{N_c}, \tag{3.81}$$

$$\Lambda_1^{2N_c-1}G(p) = \Lambda_1^{2N_c-1}(p^*+m^*) + \epsilon^2\tilde{p} + \sum_{i=2}^{N_c-1}\epsilon^{2i}\alpha_i'\Lambda_1^{2N_c-i}\tilde{s}_i + \epsilon^{2N_c}\Lambda_1^{2N_c-1}\tilde{M}, \tag{3.82}$$

where α_i' $(i = 2, \cdots, N_c - 1)$ are some coefficients determined by the scaling variables (3.68). $\hat{C}_i(\tilde{p})$ $(i = 2, \cdots, N_c)$ are the functions of $\tilde{p}$ determined such that the leading terms of the SW curve (2.121) and the SW differential (2.56) become

$$y^2 = \epsilon^{2(N_c+1)} \tilde{y}^2 + \cdots , \tag{3.83}$$

$$\lambda_{\mathrm{SW}} = \epsilon^{N_c+3} \tilde{\lambda}_{\mathrm{SW}} + \cdots . \tag{3.84}$$

In the scaling limit $\epsilon \to 0$, we find the form of the curve $\tilde{y}$ and the SW differential $\tilde{\lambda}_{\mathrm{SW}}$ given by

$$\tilde{y}^2 = \tilde{u}_0 \tilde{p}^{N_c+1} + \sum_{i=2}^{N_c+1} \tilde{u}_i p^{N_c+1-i}, \tag{3.85}$$

$$\tilde{\lambda}_{\mathrm{SW}} = \frac{b_{N_c}}{\Lambda_1^{N_c}} \tilde{y} d\tilde{p}, \tag{3.86}$$

where $\tilde{u}_0$ is proportional to $\Lambda_1^{N_c-1}$ and $\tilde{u}_i := \tilde{u}_i(\tilde{s}_i, \tilde{M})$ $(i = 2, \cdots, N_c + 1)$ is the moduli parameter. The first several b_{N_c}'s are shown as follow:

$$b_2 = -4, \quad b_3 = 2^{\frac{9}{5}}, \quad b_4 = -\frac{4}{5^{\frac{1}{7}}}, \quad b_5 = \left(\frac{2^{17}}{7}\right)^{\frac{1}{9}}, \cdots . \tag{3.87}$$

The scaling dimensions of the parameters are read off from (3.85):

$$[\tilde{y}] = \frac{N_c + 1}{N_c + 3}, \qquad [\tilde{p}] = \frac{2}{N_c + 3}, \qquad [\tilde{u}_i] = \frac{2i}{N_c + 3}, \quad (i = 2, \cdots N_c + 1). \tag{3.88}$$

The SW curve and the SW differential correspond to those for the AD theory which obtained from the scaling limit of the pure $SU(N_c + 1)$ theory: (3.77) and (3.78).

3.3.3 $N_f = 2$

For $N_f = 2$, substituting the scaling variables (3.68) with

$$h_i = 2i, \qquad h_M = 2, \qquad h_c = N_c + 1, \tag{3.89}$$

into (2.122) and (2.123), we obtain

$$C(p) = -\epsilon^2 \Lambda_2^{N_c-1} (\tilde{p} + \tilde{M}) + \epsilon^{2N_c} \tilde{C}(\tilde{p}), \tag{3.90}$$

$$G(p) = \left(\epsilon^4 (\tilde{p} + \tilde{M})^2 + \epsilon^{2N_c+2} \tilde{C}_2\right), \tag{3.91}$$

where $\tilde{C}_2 = \tilde{c}_1 \tilde{c}_2$ and

$$\tilde{C}(\tilde{p}) = \tilde{p}^{N_c} - \sum_{i=2}^{N_c} \tilde{s}_i \tilde{p}^{N_c - i}. \tag{3.92}$$

By taking the scaling limit $\epsilon \to 0$, the SW curve (2.121) and the SW differential (2.56) become

$$y^2 = \epsilon^{2N_c+2} \tilde{y}^2 + \epsilon^{4N_c} \tilde{C}(\tilde{p})^2, \tag{3.93}$$

$$\lambda_{\rm SW} = \epsilon^{N_c - 1} \tilde{\lambda}_{\rm SW} + \cdots , \tag{3.94}$$

where

$$\tilde{y}^2 = -2\Lambda^{N_c-1} \left((\tilde{p} + \tilde{M}) \tilde{C}(\tilde{p}) + \frac{\Lambda_2^{N_c-1}}{2} \tilde{C}_2 \right), \tag{3.95}$$

$$\tilde{\lambda}_{\rm SW} = -\frac{2}{\Lambda_2^{N_f-1}} \tilde{y} d \log \left(\tilde{p} + \tilde{M} \right). \tag{3.96}$$

By using the fact that the SW differential has the scaling dimension one, we find

$$[\tilde{y}] = 1, \quad [\tilde{p}] = \frac{2}{N_c+1}, \quad [\tilde{s}_i] = \frac{2i}{N_c+1}, \quad [\tilde{M}] = \frac{2}{N_c+1}, \quad [\tilde{C}_2] = 2. \tag{3.97}$$

The AD theories associated with the $N_f = 2$ theories have the $U(2)$ flavor symmetry.

3.3.4 $N_f = 2n + 1$

In the $N_f = 2n+1$ $(n \geq 1)$ theory, using the scaling variables (3.68) with

$$h_i = \begin{cases} 1, & (2 \leq i \leq N_c - n), \\ 2(-N_c + n + i) + 1, & (N_c - n + 1 \leq i \leq N_c), \end{cases} \tag{3.98}$$

$$q = 1, \qquad h_c = 2, \tag{3.99}$$

then we obtain

$$C(p) = \epsilon^{N_f} \tilde{C}(\tilde{p}) + \cdots , \qquad G(p) = \epsilon^{2N_f} \tilde{G}(\tilde{p}), \tag{3.100}$$

in the scaling limit $\epsilon \to 0$ where $\tilde{C}(\tilde{p})$ and $\tilde{G}(\tilde{p})$ are given by

$$\tilde{C}(\tilde{p}) = \sum_{l=0}^{n} \tilde{u}_l \tilde{p}^{n-l}, \qquad \tilde{G}(\tilde{p}) = \prod_{j=1}^{N_f} (\tilde{p} + \tilde{c}_j), \tag{3.101}$$

with $\tilde{u}_l := \tilde{u}_l(\tilde{s}_i, \tilde{M})$ $(l = 0, \cdots, n)$. In the scaling limit $\epsilon \to 0$, the SW curve and the SW differential are given by

$$y^2 = \epsilon^{2N_f} \tilde{y}^2 + \cdots , \tag{3.102}$$

$$\lambda_{\rm SW} = \epsilon^2 \tilde{\lambda}_{\rm SW} + \cdots , \tag{3.103}$$

where

$$\tilde{y}^2 = \tilde{C}(\tilde{p})^2 - \Lambda_{N_f}^{2N_c - N_f} \tilde{G}(\tilde{p}), \tag{3.104}$$

$$\tilde{\lambda}_{\rm SW} = \tilde{p} d \log \frac{\tilde{C}(\tilde{p}) - \tilde{y}}{\tilde{C}(\tilde{p}) + \tilde{y}}. \tag{3.105}$$

The $N_f = 2n+1$ theories have the $U(N_f)$ flavor symmetry with $\tilde{c}_i$ being interpreted as the mass parameters where the scaling dimension is one. Since the power of $\tilde{C}(\tilde{p})^2$ is less than that of $\tilde{G}(\tilde{p})$ and the SW differential has the scaling dimension one, we find

$$[\tilde{y}] = \frac{N_f}{2}, \qquad [\tilde{p}] = 1, \qquad [\tilde{u}_l] = l + \frac{1}{2}, \qquad [\tilde{c}_i] = 1. \tag{3.106}$$

If we restrict $\tilde{u}_l$ to the relevant operator and its coupling, there exist only two parameters $\tilde{u}_0$ and $\tilde{u}_1$ where $[\tilde{u}_0] + [\tilde{u}_1] = 2$ and then $\tilde{C}(\tilde{p})$ becomes

$$\tilde{C}(\tilde{p}) = \tilde{u}_0 \tilde{p}^n + \tilde{u}_1 \tilde{p}^{n-1}. \tag{3.107}$$

Thus their theories are the SCFTs with the $U(N_f)$ flavor symmetry, which have a relevant operator and its coupling.

3.3.5 $N_f = 2n$ $(n \geq 2)$

For $N_f = 2n$ $(n \geq 2)$, there are two different subsectors obtained in different scaling limit where the scaling dimension of $\tilde{c}_i$ is one. These sectors have been studied in [22]. In order to study two sectors, we introduce two scales $\epsilon = \epsilon_A, \epsilon_B \ll 1$. In the A sector $(\epsilon = \epsilon_A)$, we define the scaling variables (3.68) with

$$h_i = 2(n - N_c + i) \ (\tilde{N} \leq i \leq N_c), \quad h_M = 2, \quad h_c = 2, \tag{3.108}$$

and $\tilde{s}_i$ $(2 \leq i < \tilde{N})$ setting to be zero where $\tilde{N} := N_c - n + 2$. In the B sector $(\epsilon = \epsilon_B)$, we use

$$h_i = 2i \ (2 \leq i \leq N_c), \qquad h_M = 2, \qquad h_c = \tilde{N} = N_c - n + 2, \tag{3.109}$$

of the scaling variables (3.68). Since $\tilde{c}_i$ has the scaling dimension one in both sectors, we derive the relation

$$\epsilon_A^2 = \epsilon_B^{\tilde{N}}, \qquad \epsilon_A \ll \epsilon_B. \tag{3.110}$$

It is possible to obtain the SW curve and the SW differential for the A and B subsectors by expanding around $\epsilon_A = 0$ and $\epsilon_B = 0$, respectively.

For $\epsilon = \epsilon_A$, substituting (3.68) with (3.108) into (2.122) and (2.123), we obtain

$$C(p) = \epsilon_A^{N_f} \tilde{C}(\tilde{p}) + \epsilon_A^{2N_f} \tilde{p}^{N_c}, \tag{3.111}$$

$$G(p) = \epsilon_A^{2N_f} \tilde{G}(\tilde{p}), \tag{3.112}$$

where

$$\tilde{C}(\tilde{p}) = -\Lambda_{N_f}^{N_c - n} \tilde{p}^n - \sum_{i=2}^{n} \tilde{s}_{\tilde{N}-2+i} \tilde{p}^{n-i}, \tag{3.113}$$

$$\tilde{G}(\tilde{p}) = \prod_{i=1}^{N_f} \left(\tilde{p} + \tilde{M} + \tilde{c}_i \right). \tag{3.114}$$

Then the SW curve (2.121) and the SW differential (2.56) become

$$y^2 = \epsilon_A^{2N_f} \tilde{y}^2 + \epsilon_A^{3N_f} \tilde{p}^{N_c} \tilde{C}(\tilde{p}) + \epsilon_A^{4N_f} \tilde{p}^{2N_c}, \tag{3.115}$$

$$\lambda_{\mathrm{SW}} = \epsilon_A^2 \tilde{\lambda}_{\mathrm{SW}} + \cdots, \tag{3.116}$$

where

$$\tilde{y} = \tilde{C}(\tilde{p})^2 - \Lambda_{N_f}^{2N_c - N_f} \tilde{G}(\tilde{p}), \tag{3.117}$$

$$\tilde{\lambda}_{\mathrm{SW}} = \tilde{p} d \log \frac{\tilde{C}(\tilde{p}) - \tilde{y}}{\tilde{C}(\tilde{p}) + \tilde{y}}. \tag{3.118}$$

By using the fact that the SW differential has scaling dimension one, we find

$$[\tilde{y}] = n, \quad [\tilde{p}] = 1, \quad [\tilde{s}_i] = -\tilde{N} + 2 + i, \quad [\tilde{M}] = 1, \quad [\tilde{c}_i] = 1. \tag{3.119}$$

For $i = \tilde{N}$, the parameter $\tilde{s}_{\tilde{N}}$ has the scaling dimension two: $[\tilde{s}_{\tilde{N}}] = 2$.

For $\epsilon = \epsilon_B$, by applying (3.68) with (3.109) to (2.122) and (2.123), we obtain

$$C(p) = -\epsilon_B^{2n} \Lambda_{N_f}^{N_c - n} \left(\tilde{p} + \tilde{M} \right)^n + \epsilon_B^{2N_c} \tilde{C}(\tilde{p}), \tag{3.120}$$

$$G(p) = \epsilon_B^{4n} \left(\tilde{p} + \tilde{M} \right)^{2n} + \sum_{j=2}^{2n} \epsilon_B^{j(N_c - n) + 4n} \left(\tilde{p} + \tilde{M} \right)^{2n-j} \tilde{C}_j, \tag{3.121}$$

where

$$\tilde{C}(\tilde{p}) = \tilde{p}^{N_c} - \sum_{i=2}^{N_c} \tilde{s}_i \tilde{p}^{N_c-i}, \tag{3.122}$$

and $\tilde{C}_j$ is the symmetric polynomial of $\tilde{c}_j$:

$$\sum_{j=0}^{N_f} \tilde{C}_j \tilde{p}^{N_f-j} := \prod_{j=1}^{N_f} (\tilde{p} + \tilde{c}_j), \quad \tilde{C}_0 := 1, \quad \tilde{C}_1 := 0. \tag{3.123}$$

Expanding in ϵ_B, the SW curve (2.121) and the SW differential (2.56) become

$$y^2 = \epsilon_B^{2N_c+2n} \tilde{y}^2 + \epsilon_B^{4N_c} \tilde{C}(\tilde{p})^2 - \sum_{j=3}^{2n} \epsilon_B^{j(N_c-n)+4n} \Lambda_{N_f}^{2n_c-2n} \left(\tilde{p} + \tilde{M}\right)^{2n-j} \tilde{C}_j, \tag{3.124}$$

$$\lambda_{\mathrm{SW}} = \epsilon_B^{N_c-n} \tilde{\lambda}_{\mathrm{SW}} + \cdots , \tag{3.125}$$

where

$$\tilde{y}^2 = -2\Lambda_{N_f}^{N_c-n} \left(\tilde{p} + \tilde{M}\right)^n \left(\tilde{C}(\tilde{p}) + \frac{\Lambda_{N_f}^{N_c-n}}{2} \left(\tilde{p} + \tilde{M}\right)^{n-2} \tilde{C}_2\right), \tag{3.126}$$

$$\tilde{\lambda}_{\mathrm{SW}} = -\frac{2}{\Lambda_{N_f}^{N_c-n}} \frac{\tilde{y}}{\left(\tilde{p} + \tilde{M}\right)^n} d\tilde{p}. \tag{3.127}$$

The scaling dimensions of the parameters are give by

$$\begin{aligned} [\tilde{y}] &= \frac{N_c + n}{N_c - n + 2}, \qquad [\tilde{p}] = \frac{2}{N_c - n + 2}, \\ [\tilde{s}_i] &= \frac{2i}{N_c - n + 2}, \qquad [\tilde{M}] = \frac{2}{N_c - n + 2}, \qquad [\tilde{C}_2] = 2. \end{aligned} \tag{3.128}$$

For $i = \tilde{N}$, the scaling dimension of $\tilde{s}_{\tilde{N}}$ is given by $[\tilde{s}_{\tilde{N}}] = 2$. We note the operators $\tilde{s}_j$ ($\tilde{N} + 1 < j \leq N_c$) are the irrelevant operators due to $[\tilde{s}_j] > 2$. If we assume that there are only the relevant operators and their couplings, $\tilde{C}(\tilde{p})$ becomes

$$\tilde{C}(\tilde{p}) = \tilde{p}^{N_c} - \sum_{i=2}^{\tilde{N}} \tilde{s}_i \tilde{p}^{N_c-i}. \tag{3.129}$$

These sectors A and B are coupled by an infrared-free $SU(2)$ [22]. For the A sector, the curve (3.117) has the operator which represents the squared mass parameter of the $SU(2)$ flavor symmetry. Similarly, the curve of the B sector (3.126) also has the squared mass parameter of the $SU(2)$ flavor symmetry.

Summary

In this chapter, we introduced the Seiberg-Witten curve and the Seiberg-Witten differential for a class of Argyres-Douglas theories obtained from $\mathcal{N}=2$ $SU(N_c)$ SQCD. For $SU(2)$ SQCD, the SW curves of the corresponding AD theories become a common cubic form but the SW differentials take a different form due to the flavor symmetry. The SW periods around the superconformal point is expressed in terms of the hypergeometric function and have no logarithmic divergence which implies the theory is around the superconformal point. The AD theories associated with $SU(N_c)$ SQCD are classified by the flavor symmetry as in table 3.3. We also find the SW differentials take a different form due to the flavor symmetry. The $N_f = 2n$ theory has two different subsectors in two different scaling limit and both sectors are coupled by $SU(2)$ flavor symmetry.

N_f	SW curve	SW differential
$N_f = 0, 1$	$\tilde{y}^2 \sim \sum_{i=0}^{N_c} \tilde{u}_i \tilde{p}^{N_c - i}$	$\tilde{\lambda}_{\text{SW}} \sim \tilde{y} d\tilde{p}$
$N_f = 2$	$\tilde{y}^2 \sim (\tilde{p} + \tilde{M})\tilde{C}(\tilde{p}) + \frac{\Lambda_2^{N_c-1}}{2}\tilde{C}_2$	$\tilde{\lambda}_{\text{SW}} \sim \tilde{y} d \log\left(\tilde{p} + \tilde{M}\right)$
$N_f = 2n+1$	$\tilde{y}^2 \sim \tilde{C}(\tilde{p})^2 - \Lambda_{N_f}^{2N_c - N_f}\tilde{G}(\tilde{p})$	$\tilde{\lambda}_{\text{SW}} \sim \tilde{p} d \log \frac{\tilde{C}(\tilde{p}) - \tilde{y}}{\tilde{C}(\tilde{p}) + \tilde{y}}$
$N_f = 2n\,(n \geq 2)\, A$	$\tilde{y}^2 \sim \tilde{C}(\tilde{p})^2 - \Lambda_{N_f}^{2N_c - N_f}\tilde{G}(\tilde{p})$	$\tilde{\lambda}_{\text{SW}} \sim \tilde{p} d \log \frac{\tilde{C}(\tilde{p}) - \tilde{y}}{\tilde{C}(\tilde{p}) + \tilde{y}}$
$N_f = 2n\,(n \geq 2)\, B$	$\tilde{y}^2 \sim \left(\tilde{p} + \tilde{M}\right)^n \left(\tilde{C}(\tilde{p}) + \frac{\Lambda_{N_f}^{N_c - n}}{2}\left(\tilde{p} + \tilde{M}\right)^{n-2}\tilde{C}_2\right)$	$\tilde{\lambda}_{\text{SW}} \sim \frac{\tilde{y}}{(\tilde{p} + \tilde{M})^n} d\tilde{p}$

Table 3.3: The SW curve and the SW differential for the AD theory obtained from the $\mathcal{N}=2$ $SU(N_c)$ SQCD.

Chapter 4

Ω-deformed $\mathcal{N}=2$ supersymmetric gauge theory

In this chapter, we will introduce the Ω-deformation for the $\mathcal{N}=2$ supersymmetric gauge theories. The four-dimensional theory in the Ω-background is constructed from the dimensional reduction of the six-dimensional theory in $\mathbf{R}^{1,3}\times\mathbf{T}^2$, where the metric is defined by [83, 35]

$$ds_6^2=\eta_{\mu\nu}\left(dx^\mu+\Omega_n{}^\mu{}_\rho x^\rho dx^n\right)\left(dx^\nu+\Omega_m{}^\nu{}_\rho x^\rho dx^m\right)-(dx^4)^2-(dx^5)^2, \tag{4.1}$$

with $\mu,\nu,\rho=0,1,2,3$ and $n,m=4,5$. x^4 and x^5 are the coordinates of the two-dimensional torus $\mathbf{T}^2$:

$$x^4\sim x^4+2\pi R_4,\quad x^5\sim x^5+2\pi R_5, \tag{4.2}$$

where R_4 and R_5 are the radii of compactification. Here $\Omega_n{}^\mu{}_\nu$ are the matrices of Lorentz rotations, where the complex linear combinations $\Omega^{\mu\nu}$ and $\bar{\Omega}^{\mu\nu}$ are defined by

$$\Omega^{\mu\nu}:=\frac{1}{\sqrt{2}}\left(\Omega_4{}^{\mu\nu}+i\Omega_5{}^{\mu\nu}\right)=\frac{1}{\sqrt{2}}\begin{pmatrix}0&\epsilon_1&0&0\\-\epsilon_1&0&0&0\\0&0&0&\epsilon_2\\0&0&-\epsilon_2&0\end{pmatrix}, \tag{4.3}$$

$$\bar{\Omega}^{\mu\nu}:=\frac{1}{\sqrt{2}}\left(\Omega_4{}^{\mu\nu}-i\Omega_5{}^{\mu\nu}\right)=\frac{1}{\sqrt{2}}\begin{pmatrix}0&\bar{\epsilon}_1&0&0\\-\bar{\epsilon}_1&0&0&0\\0&0&0&\bar{\epsilon}_2\\0&0&-\bar{\epsilon}_2&0\end{pmatrix}, \tag{4.4}$$

with (ϵ_1,ϵ_2) being the deformation parameters of the torus action. In the limit: $R_4,R_5\to 0$, the four-dimensional theory in the Ω-background is appeared. The Ω-background

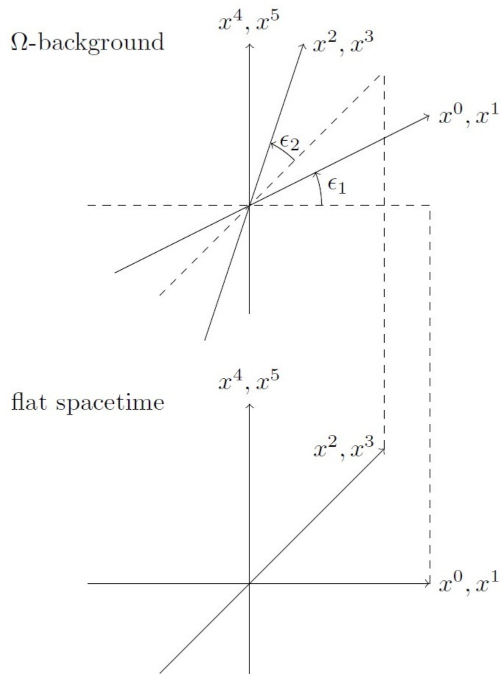

Figure 4.1: The deformation from the flat spacetime to the Ω-background.

deforms the four-dimensional spacetime by the torus action with ϵ_1 and ϵ_2. The schematic of the Ω-background is described as in figure 4.1.

In the weak coupling region of the four-dimensional $\mathcal{N}=2$ supersymmetric gauge theory, one can compute the Nekrasov partition function $Z_{\text{Nek}}(a;\epsilon_1,\epsilon_2)$ with help of the localization method [34, 35]. In the next section, we will write down the full form of the partition function for the $\mathcal{N}=2$ $SU(2)$ SQCD in the Ω-background. In the second section, we will take the Nekrasov-Shatashvili limit of the Ω-background and give the Ω-deformed prepotential in the NS limit.

4.1 Nekrasov partition function

The Nekrasov partition function for the $\mathcal{N}=2$ supersymmetric gauge theory can be written in terms of the prepotential $\mathcal{F}_{\text{Nek}}(a;\epsilon_1,\epsilon_2)$:

$$Z_{\text{Nek}}(a;\epsilon_1,\epsilon_2)=\exp\left(-\frac{1}{\epsilon_1\epsilon_2}\mathcal{F}_{\text{Nek}}(a;\epsilon_1,\epsilon_2)\right). \tag{4.5}$$

It is separated into the perturbative and non-perturbative contributions:

$$Z_{\text{Nek}}(a;\epsilon_1,\epsilon_2) := Z_{\text{pert}}(a;\epsilon_1,\epsilon_2) Z_{\text{inst}}(a;\epsilon_1,\epsilon_2), \tag{4.6}$$

where $Z_{\text{pert}}(a;\epsilon_1,\epsilon_2)$ is the perturbative part of the partition function and $Z_{\text{inst}}(a;\epsilon_1,\epsilon_2)$ is the instanton part.

In the $\mathcal{N}=2$ $U(2)$ gauge theory with N_f $(=0,\ldots,4)$ hypermultiplets, the perturbative and instanton parts of the Nekrasov partition function are written down as follows:

- The perturbative part $Z_{\text{pert}}(a;\epsilon_1,\epsilon_2)$ is given by [83, 35]

$$Z_{\text{pert}}(a;\epsilon_1,\epsilon_2) = \exp\left(\sum_{l,n=1}^{2} \gamma_{\epsilon_1,\epsilon_2}(a_l - a_n;\Lambda_{N_f}) - \sum_{l=1}^{2}\sum_{a=1}^{N_f} \gamma_{\epsilon_1,\epsilon_2}(a_l + m_a;\Lambda_{N_f})\right), \tag{4.7}$$

  where $a := a_1 = -a_2$ is the vev of the scalar field and

$$\gamma_{\epsilon_1,\epsilon_2}(x;\Lambda_{N_f}) = \frac{d}{ds}\bigg|_{s=0} \frac{\Lambda_{N_f}^s}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \frac{e^{-tx}}{(e^{\epsilon_1 t}-1)(e^{\epsilon_2 t}-1)}. \tag{4.8}$$

- The instanton partition function is given by [34–36]

$$Z_{\text{inst}}(a;\epsilon_1,\epsilon_2) = \sum_{\vec{Y}} q^{|\vec{Y}|} z_{\text{vector}}(\vec{a},\vec{Y}) \prod_{a=1}^{N_+} z_{\text{antifund}}(\vec{a},\vec{Y},m_a) \prod_{b=N_+ +1}^{N_f} z_{\text{fund}}(\vec{a},\vec{Y},m_b), \tag{4.9}$$

  where $\vec{a} := (a_1, a_2)$ and $q := e^{2\pi i \tau_{\text{UV}}}$ is the instanton factor. $\vec{Y} = (\mathbf{Y}_1, \mathbf{Y}_2)$ is the set of the Young tableaux where

$$\mathbf{Y}_l = (Y_{l,1}, Y_{l,2}, \cdots), \qquad Y_{l,1} \geq Y_{l,2} \geq \cdots \geq Y_{l,n} > 0 = Y_{l,n+1} = Y_{l,n+2} = \cdots, \tag{4.10}$$

  with $Y_{l,i}$ being the number of the boxes of the i-th column. The total number of the boxes $|\vec{Y}| := \sum_{l,i} Y_{l,i}$ refers to the instanton number. The contribution of the vector multiplet $z_{\text{vector}}(\vec{a},\vec{Y})$ is defined by

$$z_{\text{vector}}(\vec{a},\vec{Y}) = \left[\prod_{l,n=1}^{2} \prod_{(i,j)\in\mathbf{Y}_l} E\left(a_l - a_n, \mathbf{Y}_l, \mathbf{Y}_n; i,j\right) \prod_{(i,j)\in\mathbf{Y}_n} \left(\epsilon_1+\epsilon_2 - E\left(a_n - a_l, \mathbf{Y}_n, \mathbf{Y}_l; i,j\right)\right)\right]^{-1}, \tag{4.11}$$

where $E(a, \mathbf{Y}_1, \mathbf{Y}_2; i, j)$ is given by

$$E(a, \mathbf{Y}_1, \mathbf{Y}_2; i, j) = a - \epsilon_1 L_{\mathbf{Y}_2}(i, j) + \epsilon_2 (A_{\mathbf{Y}_1}(i, j) + 1). \tag{4.12}$$

$A_{\mathbf{Y}}(i, j)$ and $L_{\mathbf{Y}}(i, j)$ are the arm-length and leg-length for a box in the tableau $\mathbf{Y}$, defined by

$$A_{\mathbf{Y}}(i, j) = Y_i - j, \qquad L_{\mathbf{Y}}(i, j) = (Y^T)_j - i, \tag{4.13}$$

respectively, where (i, j) is the coordinate of the box and $(Y^T)_i$ denotes the number of the boxes of the i-th column for the transpose of $\mathbf{Y}$. The schematic of the Young tableau is shown in figure 4.2.

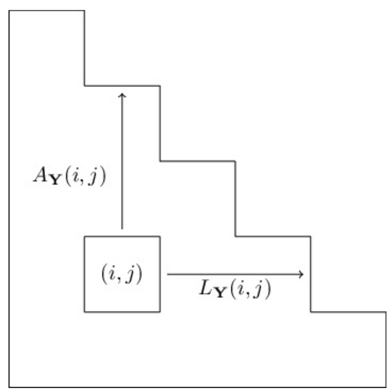

Figure 4.2: The arm-length and leg-length for a box in the Young tableau.

The contributions of the fundamental and anti-fundamental hypermultiplets $z_{\text{fund}}(\vec{a}, \vec{Y}, m)$ and $z_{\text{fund}}(\vec{a}, \vec{Y}, m)$ are defined by

$$z_{\text{fund}}(\vec{a}, \vec{Y}, m) = \prod_{l=1}^{2} \prod_{(i,j)\in\mathbf{Y}} (\phi(a; i, j) - m + \epsilon_1 + \epsilon_2), \tag{4.14}$$

$$z_{\text{antifund}}(\vec{a}, \vec{Y}, m) = z_{\text{fund}}(\vec{a}, \vec{Y}, \epsilon_1 + \epsilon_2 - m), \tag{4.15}$$

where $\phi(a; i, j)$ is given by

$$\phi(a; i, j) = a + \epsilon_1 (i - 1) + \epsilon_2 (j - 1). \tag{4.16}$$

The instanton partition function (4.9) includes the $U(1)$ gauge group contribution, given by the $U(1)$ factor:

$$Z^{U(1)} := (1 - q)^{2M_0(\epsilon_1 + \epsilon_2 - M_1)}, \tag{4.17}$$

where $M_0 := \frac{1}{2}(m_1 + m_2)$ and $M_1 := \frac{1}{2}(m_3 + m_4)$. Thus the instanton partition function for the $SU(2)$ SQCD is given by

$$Z_{\text{inst}}^{SU(2)} = Z_{\text{inst}}(a;\epsilon_1,\epsilon_2)/Z^{U(1)}. \tag{4.18}$$

In the limit $\epsilon_1, \epsilon_2 \to 0$, the Nekrasov partition function reproduces the Seiberg-Witten prepotential $\mathcal{F}_{\text{SW}}(a)$ in the weak coupling region [35]:

$$\mathcal{F}_{\text{SW}}(a) = \lim_{\epsilon_1,\epsilon_2\to 0} -\epsilon_1\epsilon_2 Z_{\text{Nek}}(a;\epsilon_1,\epsilon_2). \tag{4.19}$$

In the next section, we will take the special limit of the Ω-background, called the Nekrasov-Shatashvili limit [44].

4.2 Nekrasov-Shatashvili limit of Ω-background

In the Nekrasov-Shatashvili (NS) limit [44] of the Ω-background where one of the deformation parameters ϵ_2 is set to be zero, the prepotential at the weak coupling is derived from the Nekrasov partition function (4.5) as

$$\mathcal{F}_{\text{NS}}(a;\epsilon_1) = \lim_{\epsilon_2\to 0} -\epsilon_1\epsilon_2 \log Z_{\text{Nek}}(a;\epsilon_1,\epsilon_2). \tag{4.20}$$

The Ω-deformed prepotential in the NS limit consists of the perturbative and the instanton parts:

$$\mathcal{F}_{\text{NS}}(a;\epsilon_1) := \mathcal{F}_{\text{NS}}^{\text{pert}}(a;\epsilon_1) + \mathcal{F}_{\text{NS}}^{\text{inst}}(a;\epsilon_1), \tag{4.21}$$

where

$$\mathcal{F}_{\text{NS}}^{\text{pert}}(a;\epsilon_1) := \lim_{\epsilon_2\to 0} -\epsilon_1\epsilon_2 \log Z_{\text{pert}}(a;\epsilon_1,\epsilon_2), \tag{4.22}$$

$$\mathcal{F}_{\text{NS}}^{\text{inst}}(a;\epsilon_1) := \lim_{\epsilon_2\to 0} -\epsilon_1\epsilon_2 \log Z_{\text{inst}}(a;\epsilon_1,\epsilon_2). \tag{4.23}$$

We firstly consider the perturbative part of the Ω-deformed prepotential in the NS limit. In this limit, the function (4.8) becomes

$$\begin{aligned}\gamma_{\epsilon_1}(x,\Lambda_{N_f}) :=& \lim_{\epsilon_2\to 0} \epsilon_1\epsilon_2\gamma_{\epsilon_1,\epsilon_2}(x,\Lambda_{N_f}) \\ =&\epsilon_1^2 \left.\frac{d}{ds}\right|_{s=0} \frac{\Lambda_{N_f}^s}{\epsilon_1^s\Gamma(s)} \int_0^\infty dt\, t^{s-2} \frac{e^{-t\left(1+\frac{x}{\epsilon_1}\right)}}{(1-e^{-t})}.\end{aligned} \tag{4.24}$$

Differentiating it with respect to x, we find

$$\frac{\partial}{\partial x}\gamma_{\epsilon_1}(x;\Lambda_{N_f}) = \epsilon_1\left(\frac{1}{2}+\frac{x}{\epsilon_1}\right)\log\frac{\Lambda_{N_f}}{\epsilon_1} - \epsilon_1\log\Gamma\left(1+\frac{x}{\epsilon_1}\right)+\frac{\epsilon_1}{2}\log 2\pi. \tag{4.25}$$

From the above formula, the a-derivative of the perturbative part of the deformed prepotential in the NS limit becomes [46]

$$\begin{aligned}\frac{\partial}{\partial a}\mathcal{F}^{\text{pert}}_{\text{NS}}(a;\epsilon_1) =& -8a\left(\log\frac{2a}{\Lambda_{N_f}} - 1 + \sum_{k=1}^{\infty}\frac{B_{2k}}{2k(2k-1)}\left(\frac{\epsilon_1}{2a}\right)^{2k}\right) \\ &+\sum_{a=1}^{N_f}\left[2(a+m_a)\left(\log\frac{a+m_a}{\Lambda_{N_f}} - 1 + \sum_{k=1}^{\infty}\frac{(2^{1-2k}-1)B_{2k}}{2k(2k-1)}\left(\frac{\epsilon_1}{a+m_a}\right)^{2k}\right)\right],\end{aligned} \tag{4.26}$$

where B_k is the k-th Bernoulli number. Here we shift the mass parameter as $m_a \to m_a + \frac{\epsilon_1}{2}$. Integrating it over a, we obtain the perturbative part of the prepotential.

We next consider the instanton part of the Ω-deformed prepotential in the NS limit. For the instanton partition function (4.18), we shift the mass parameters: $m_a \to m_a + \frac{\epsilon_1+\epsilon_2}{2}$ for a fundamental hypermultiplet or $m_a \to \frac{\epsilon_1+\epsilon_2}{2} - m_a$ for an anti-fundamental hypermultiplet. In the $\mathcal{N} = 2$ $SU(2)$ SQCD, the instanton part of the prepotential in the NS limit is given by

$$\mathcal{F}^{\text{inst}}_{\text{NS}}(a;\epsilon_1) = \lim_{\epsilon_2\to 0} -\epsilon_1\epsilon_2\log Z^{SU(2)}_{\text{inst}}(a;\epsilon_1,\epsilon_2). \tag{4.27}$$

Then expanding it around $\epsilon_1 = 0$, we obtain the expansion of the instanton contribution of the prepotential.

The low-energy effective theory in the NS limit of the Ω-background arises in the two-dimensional Ω background with ϵ_1. The prepotential $\mathcal{F}_{\text{NS}}(a;\epsilon_1)$ leads to the twisted superpotential for the two-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory, which is expanded in ϵ_1 as

$$\mathcal{W}(a;\epsilon_1) \simeq \frac{1}{\epsilon_1}\mathcal{F}_{\text{NS}}(a;\epsilon_1) = \frac{1}{\epsilon_1}\mathcal{F}_{\text{SW}}(a) + \cdots, \tag{4.28}$$

where $\mathcal{F}_{\text{SW}}(a)$ is the SW prepotential. In the limit $\epsilon_1 \to 0$, the twisted superpotential $\mathcal{W}(a) := \lim_{\epsilon_1\to 0}\mathcal{W}(a;\epsilon_1)$ satisfies the two-dimensional supersymmetric vacua condition:

$$\frac{\partial\mathcal{W}(a)}{\partial a} = 2\pi i n \tag{4.29}$$

with n being an integer. Since the a-derivative of the SW prepotential is given by the integration of the SW differential, the deformed periods in the NS limit is found to satisfy the Bohr-Sommerfeld (BS) quantization condition [44] :

$$\oint \lambda_{\rm SW} = 2\pi i \left(n + \frac{1}{2} \right) \epsilon_1, \qquad (n \in \mathbf{Z}), \tag{4.30}$$

where the deformation parameter ϵ_1 is a roll of the Planck constant $\hbar$. Mironov and Morozov gave a different interpretation to the BS condition (4.30) [45]. The BS condition means that the SW curve is quantized by introducing the canonical quantization of the holomorphic symplectic structure induced by $d\lambda_{\rm SW}$. The quantum SW curve becomes the ordinary differential equation. Solving the differential equation, the quantum correction to the SW periods is obtained from the WKB solution. In the next chapter, we will check that the Ω-deformed prepotential for the $\mathcal{N} = 2$ $SU(2)$ SQCD in the NS limit agrees with that obtained from the quantum SW periods in the weak coupling region up to fourth order in ϵ_1. Then we will compute the quantum prepotential around the massless monopole point.

Summary

In this chapter, we reviewed the $\mathcal{N} = 2$ supersymmetric gauge theory in the Ω-background. In the weak coupling region, the Nekrasov partition function reproduces explicitly the full form of the prepotential including the instanton contribution. In the NS limit, the low-energy effective theory becomes a two-dimensional Ω-deformed theory with one deformation parameter ϵ_1. From the two-dimensional supersymmetric vacua condition, it is found that the SW periods in the NS limit satisfy the Bohr-Sommerfeld quantization condition.

Chapter 5

Quantum periods for $\mathcal{N} = 2$ $SU(2)$ SQCD

In this chapter, we will study the effect of the Ω-deformation at the massless monopole point for the $SU(2)$ gauge theory with $N_f(= 1, 2, 3, 4)$ hypermultiplets. In particular, we will take the Nekrasov-Shatashvili limit: $\epsilon_1 := \hbar$ and $\epsilon_2 \to 0$. In the NS limit, the SW periods satisfy the BS quantization condition (4.30) and the SW curve becomes a differential equation, obtained by the canonical quantization of the symplectic structure defined by $d\lambda_{\mathrm{SW}}$. The quantum corrections to the SW periods are obtained from the WKB solution of the quantum SW curve. Interestingly they can be represented by acting $\hat{\mathcal{O}}_k$ on the SW periods where $\hat{\mathcal{O}}_k$ denotes some differential operator with respect to the moduli parameter and the mass parameters. In the weak coupling region, the quantum prepotential obtained from the quantum SW periods agrees with that obtained from the Nekrasov partition function (4.20) [45].

In this chapter, we will construct $\hat{\mathcal{O}}_2$ and $\hat{\mathcal{O}}_4$ explicitly for $\mathcal{N} = 2$ $SU(2)$ SQCD and compute the second and fourth order corrections to the SW periods in $\hbar$ around $u \sim \infty$ and the massless monopole point $u \sim u_0$. In the following, $\Pi := (a, a_D)$ denotes the quantum SW periods while $\Pi^{(0)} := (a^{(0)}, a_D^{(0)})$ refers to the "undeformed" (or classical) SW periods defined by (2.95) as discussed previously.

5.1 Quantum SW curve

The SW curve and the SW differential for the $SU(2)$ gauge theory with the N_f hypermultiplets are given by (2.96) and (2.99), respectively. By introducing $z=e^{ix}$, the SW curve (2.92) is

$$C(p)-\frac{\bar{\Lambda}}{2}\left(G_+(p)e^{ix}+G_-(p)e^{-ix}\right)=0 \tag{5.1}$$

and the SW differential (2.99) becomes

$$\lambda_{\mathrm{SW}}=p\left(d\log\frac{G_-(p)}{G_+(p)}-2idx\right). \tag{5.2}$$

The differential gives a symplectic form $d\lambda_{\mathrm{SW}}=dp\wedge dx$ on the (p,x) space. By regarding the coordinate p as the differential operator $-i\hbar\frac{d}{dx}$, we obtain the quantum SW curve, taking the form of the differential equations

$$\left[C(-i\hbar\partial_x)-\frac{\bar{\Lambda}}{2}\left(e^{\frac{ix}{2}}G_+(-i\hbar\partial_x)e^{\frac{ix}{2}}+e^{-\frac{ix}{2}}G_-(-i\hbar\partial_x)e^{-\frac{ix}{2}}\right)\right]\Psi(x)=0, \tag{5.3}$$

where $\partial_x=\frac{\partial}{\partial x}$. $C(p)$ and $G_\pm(p)$ are given by (2.93) and (2.98), respectively. Here we take the ordering prescription of the differential operators as in [46]:

$$e^{itx}(-i\hbar\partial_x+m_i)e^{i(1-t)x}=(-i\hbar\partial_x+m_i-t\hbar)e^{ix}, \tag{5.4}$$

where $t=\frac{1}{2}$ in the second term in (5.3), for example.

We will choose the number of the hypermultiplet N_+ in $G_+(p)$ such that (5.3) becomes the second order differential equation:

$$\left(\partial_x^2+f(x)\partial_x+g(x)\right)\Psi(x)=0. \tag{5.5}$$

By introducing $\Psi(x)=\exp\left(-\frac{1}{2}\int f(x)dx\right)\psi(x)$, this equation becomes the Schrödinger type equation:

$$\left(-\hbar^2\partial_x^2+Q(x)\right)\psi(x)=0, \tag{5.6}$$

where $Q(x)=-\frac{1}{\hbar^2}(-\frac{1}{2}\partial_x f-\frac{1}{4}f^2+g)$. In the case of $SU(2)$ SQCD, the potential $Q(x)$ becomes the expansions in $\hbar$ as

$$Q(x)=Q_0(x)+\hbar^2 Q_2(x). \tag{5.7}$$

The quantum SW periods are defined by the WKB solution of the equation (5.6):

$$\psi(x) = \exp\left(\frac{i}{\hbar}\int^x \Phi(y)dy\right), \tag{5.8}$$

where

$$\Phi(y) = \sum_{n=0}^{\infty} \hbar^n \phi_n(y), \tag{5.9}$$

and $\phi_0(y) = p(y)$. Substituting the expansion (5.9) into (5.6), we obtain the recursion relation of $\phi_n(x)$'s:

$$Q_0(x) + \phi_0(x)^2 = 0, \tag{5.10}$$

$$Q_n(x) + 2\phi_0\phi_n + \sum_{l+k=n} \phi_l\phi_k - i\partial_x\phi_{n-1} = 0, \quad \text{for } n \geq 1, \tag{5.11}$$

where $Q_n(x) = 0$ for $n \neq 0, 2$ and $l, k \geq 1$. We separate $\Phi(x)$ into odd and even order corrections as

$$\Phi(x) = \Phi_{\text{odd}}(x) + \Phi_{\text{even}}(x), \tag{5.12}$$

where

$$\Phi_{\text{odd}}(x) = \sum_{j\geq 0} \hbar^{2j-1}\phi_{2j-1}(x), \qquad \Phi_{\text{even}}(x) = \sum_{j\geq 0} \hbar^{2j}\phi_{2j}(x). \tag{5.13}$$

We then find that Φ_{odd} becomes a total derivative:

$$\Phi_{\text{odd}}(x) = \frac{i}{2}\frac{\partial}{\partial x}\log \Phi_{\text{even}}. \tag{5.14}$$

There is only $\phi_{2n}(x)$ as the contribution to the period integrals. The first three ϕ_{2n}'s are given by

$$\phi_0(x) = i\sqrt{Q_0}, \tag{5.15}$$

$$\phi_2(x) = \frac{i}{2}\frac{Q_2}{\sqrt{Q_0}} + \frac{i}{48}\frac{\partial_x^2 Q_0}{Q_0^{\frac{3}{2}}}, \tag{5.16}$$

$$\phi_4(x) = -\frac{7i}{1536}\frac{(\partial_x^2 Q_0)^2}{Q_0^{\frac{7}{2}}} + \frac{i}{768}\frac{\partial_x^4 Q_0}{Q_0^{\frac{5}{2}}} - \frac{iQ_2\partial_x^2 Q_0}{32Q_0^{\frac{5}{2}}} + \frac{i\partial_x^2 Q_2}{48Q_0^{\frac{3}{2}}} - \frac{iQ_2^2}{8Q_0^{\frac{3}{2}}}, \tag{5.17}$$

up to total derivatives. The leading order term $\phi_0(x)$ leads to the SW periods $\Pi^{(0)} = \oint \phi_0(x)$. The quantum correction to the SW periods is given by the integration of $\phi_{2n}(x)$. Then one can expand the quantum periods $\Pi = \oint \Phi(x)dx = (a, a_D)$ in $\hbar$ as

$$\Pi = \Pi^{(0)} + \hbar^2\Pi^{(2)} + \hbar^4\Pi^{(4)} + \cdots , \tag{5.18}$$

where $\Pi^{(2n)} := \int \phi_{2n}(x)dx$.

The SW periods $\Pi^{(0)}$ can be evaluated by solving the Picard-Fuchs equation (2.76) as discussed previously. The higher correction $\Pi^{(k)}$ to the SW periods $\Pi^{(0)}$ can be computed by using the expressions as [38, 47, 58, 39]:

$$\Pi^{(k)} = \hat{\mathcal{O}}_k\Pi^{(0)}. \tag{5.19}$$

where $\hat{\mathcal{O}}_k$ is some differential operators, represented in various ways. For example, $\Pi^{(k)}$ can be expressed in terms of a basis $\partial_u\Pi^{(0)}$ and $\partial_u^2\Pi^{(0)}$:

$$\Pi^{(k)} = \left(X_k^1 \frac{\partial^2}{\partial u^2} + X_k^2 \frac{\partial}{\partial u} \right) \Pi^{(0)}. \tag{5.20}$$

Let us study the simplest example, the $N_f = 0$ theory. The quantum SW curve (5.6) becomes the Schrödinger type equation with the sine-Gordon potential:

$$Q(x) = -u - \frac{\Lambda_0^2}{2}(e^{ix} + e^{-ix}). \tag{5.21}$$

The SW periods $\Pi^{(0)}$ satisfy the Picard-Fuchs equation (2.79). By applying the WKB method we find the second and fourth order quantum corrections, given by [38, 45, 58]

$$\Pi^{(2)} = \left(\frac{1}{12} u \frac{\partial^2}{\partial u^2} + \frac{1}{24}\frac{\partial}{\partial u} \right) \Pi^{(0)}, \tag{5.22}$$

$$\Pi^{(4)} = \left(\frac{75\Lambda_0^8 - 4u^4 + 153\Lambda_0^4 u^2}{5760\,(u^2 - \Lambda_0^4)^{\,2}} \frac{\partial^2}{\partial u^2} - \frac{u^3 - 15\Lambda_0^4 u}{2880\,(u^2 - \Lambda_0^4)^{\,2}} \frac{\partial}{\partial u} \right) \Pi^{(0)}. \tag{5.23}$$

By using the Picard-Fuchs equation (2.79), it is found that a simpler formula for $\Pi^{(4)}$ is obtained by

$$\Pi^{(4)} = \left(\frac{7}{1440} u^2 \frac{\partial^4}{\partial u^4} + \frac{1}{48} u \frac{\partial^3}{\partial u^3} + \frac{5}{384}\frac{\partial^2}{\partial u^2} \right) \Pi^{(0)}. \tag{5.24}$$

In the weak coupling region where u is large, the expansions of the SW periods $(a^{(0)}, a_D^{(0)})$ are given by (2.81) and (2.82), respectively. Applying (5.22) and (5.24) on $(a(u), a_D(u))$, the expansions of the quantum SW periods around $u = \infty$ are given by

$$\begin{aligned}a(u) =& \left(\sqrt{\frac{u}{2}} - \frac{\Lambda_0}{16\sqrt{2}}\left(\frac{\Lambda_0^2}{u}\right)^{\frac{3}{2}} + \cdots\right) + \frac{\hbar^2}{\Lambda_0}\left(-\frac{1}{64\sqrt{2}}\left(\frac{\Lambda_0^2}{u}\right)^{\frac{5}{2}} - \frac{35}{2048\sqrt{2}}\left(\frac{\Lambda_0^2}{u}\right)^{\frac{9}{2}} + \cdots\right)\\&+\frac{\hbar^4}{\Lambda_0^3}\left(-\frac{1}{256\sqrt{2}}\left(\frac{\Lambda_0^2}{u}\right)^{\frac{7}{2}} - \frac{273}{16384\sqrt{2}}\left(\frac{\Lambda_0^2}{u}\right)^{\frac{11}{2}} + \cdots\right) + \cdots,\\a_D(u) =& -\frac{i}{2\sqrt{2}\pi}\left[-4\sqrt{2}a(u)\log\frac{8u}{\Lambda_0^2} + \left(8\sqrt{u} - \frac{\Lambda_0^4}{4u^{\frac{3}{2}}} + \cdots\right) + \frac{\hbar^2}{\Lambda_0}\left(-\frac{1}{6\sqrt{u}} - \frac{13}{96}\left(\frac{\Lambda_0^2}{u}\right)^{\frac{5}{2}} + \cdots\right)\right.\\&\left.+\frac{\hbar^4}{\Lambda_0^3}\left(\frac{1}{720u^{\frac{3}{2}}} - \frac{63}{1280}\left(\frac{\Lambda_0^2}{u}\right)^{\frac{7}{2}} + \cdots\right) + \cdots\right],\end{aligned} \tag{5.25}$$

up to the fourth order in $\hbar$. It has been showed that the prepotential obtained from them agrees with that obtained from the NS limit of the Nekrasov partition function [45, 58].

We can also study the quantum SW periods in the strong coupling region. We have the expansions of the SW periods around the massless monopole point $u = \Lambda_0^2$: (2.84) and (2.85). In order to analyze the quantum SW periods around the massless monopole point, it is convenient to use (5.24) rather than (5.23) since the coefficients in (5.23) have singularities at $u = \Lambda_0^2$. Then the expansions of the quantum SW periods around $u = \Lambda_0^2$ are given by [58]:

$$\begin{aligned}a_D(\hat{u}) =& i\left(\frac{\hat{u}}{2\Lambda_0} - \frac{\hat{u}^2}{32\Lambda_0^3} + \cdots\right) + \frac{i\hbar^2}{\Lambda_0}\left(\frac{1}{64} - \frac{5}{1024}\left(\frac{\hat{u}}{\Lambda_0^2}\right) + \cdots\right)\\&+\frac{i\hbar^4}{\Lambda_0^3}\left(-\frac{17}{65536} + \frac{721}{2097152}\left(\frac{\hat{u}}{\Lambda_0^2}\right) + \cdots\right) + \cdots,\\a(\hat{u}) =& \frac{i}{2\pi}\left[a_D(\hat{u})\log\frac{\hat{u}}{2^5\Lambda_0^2} + i\left(-\frac{\hat{u}}{2\Lambda_0} - \frac{3\hat{u}^2}{64\Lambda_0^3} + \cdots\right) + \frac{i\hbar^2}{\Lambda_0}\left(\frac{1}{24}\left(\frac{\hat{u}}{\Lambda_0^2}\right)^{-1} + \frac{5}{192} + \cdots\right)\right.\\&\left.+\frac{i\hbar^4}{\Lambda_0^3}\left(\frac{7}{1440}\left(\frac{\hat{u}}{\Lambda_0^2}\right)^{-3} - \frac{1}{2560}\left(\frac{\hat{u}}{\Lambda_0^2}\right)^{-2} + \cdots\right) + \cdots\right],\end{aligned} \tag{5.26}$$

where $\hat{u} := u - \Lambda_0^2$ up to fourth order in $\hbar$. In the following sections, we consider the quantum corrections to the SW periods at strong coupling for $N_f = 1, 2, 3, 4$ cases.

5.2 Quantum SW periods for $N_f \geq 1$

We discuss the quantum SW periods for the $SU(2)$ theory with $N_f \geq 1$ hypermultiplets. We will take N_+ of (2.98) such that the differential equation (5.3) becomes the second order differential equation. The quantum SW curve takes the form of the Schrödinger type equation (5.6) by the redefinition of the wave function. The higher order corrections to the SW periods are given by the integration of (5.16) and (5.17) over x along α and β cycles. The expressions are represented as $\hat{\mathcal{O}}_k \Pi^{(0)}$ with some differential operators $\hat{\mathcal{O}}_k$. In this section, we will find the second and fourth order corrections to the SW periods.

$N_f = 1$ theory

For $N_f = 1$, we can take $N_+ = 1$ in the SW curve (2.96) without loss of generality. We have the quantum curve as the Schrödinger type equation with the Tzitzéica-Bullough-Dodd type potential

$$Q(x) = -\frac{1}{2}\Lambda_1^{\frac{3}{2}} m_1 e^{ix} - u - \frac{1}{16}\Lambda_1^3 e^{2ix} - \frac{1}{2}\Lambda_1^{\frac{3}{2}} e^{-ix}, \tag{5.27}$$

where $Q_2(x) = 0$. The SW periods $\Pi^{(0)}$ satisfy not only the Picard-Fuchs equation (2.76) with (2.102) but also the differential equation:

$$\frac{\partial^2 \Pi^{(0)}}{\partial m_1 \partial u} = b_1 \frac{\partial^2 \Pi^{(0)}}{\partial u^2} + c_1 \frac{\partial \Pi^{(0)}}{\partial u}, \tag{5.28}$$

where

$$b_1 = -\frac{16 m_1 u - 9\Lambda_1^3}{8(4m_1^2 - 3u)}, \qquad c_1 = -\frac{m_1}{4m_1^2 - 3u}. \tag{5.29}$$

The second and fourth order corrections are given by the integration of (5.16) and (5.17) over x, respectively [39]. In terms of the basis $\partial_u \Pi^{(0)}$ and $\partial_u^2 \Pi^{(0)}$, we obtain the expressions of these corrections:

$$\Pi^{(2)} = \left(X_2^1 \frac{\partial^2}{\partial u^2} + X_2^2 \frac{\partial}{\partial u} \right) \Pi^{(0)}, \tag{5.30}$$

$$\Pi^{(4)} = \left(X_4^1 \frac{\partial^2}{\partial u^2} + X_4^2 \frac{\partial}{\partial u} \right) \Pi^{(0)}, \tag{5.31}$$

where the coefficients in (5.30) are given by

$$\begin{aligned} X_2^1 &= -\frac{-9\Lambda_1^3 m_1 - 16m_1^2 u + 24u^2}{48\,(4m_1^2 - 3u)}, \\ X_2^2 &= -\frac{3u - 2m_1^2}{12\,(4m_1^2 - 3u)}, \end{aligned} \tag{5.32}$$

and the coefficients in (5.31) are given by

$$\begin{aligned} X_4^1 =& \frac{\Lambda_1^{12}}{1440(4m_1^2 - 3u)\Delta_1^2}\big(-864\Lambda_1^9 m_1 \left(4350m_1^2 u + 1192m_1^4 + 441u^2\right) \\ &- 49152\Lambda_1^3 m_1 u^2 \left(-455m_1^2 u^2 + 609m_1^4 u - 204m_1^6 + 267u^3\right) \\ &+ 768\Lambda_1^6 \left(-19593m_1^2 u^3 + 42348m_1^4 u^2 - 22624m_1^6 u + 6400m_1^8 + 8235u^4\right) \\ &+ 131072u^4 \left(15m_1^2 u^2 + 6m_1^4 u - 2m_1^6 + 9u^3\right) - 729\Lambda_1^{12} \left(615u - 1792m_1^2\right)\big), \end{aligned} \tag{5.33}$$

$$\begin{aligned} X_4^2 =& \frac{\Lambda_1^{12}}{45(4m_1^2 - 3u)\Delta_1^2}\big(24\Lambda_1^6 \left(-1080m_1^2 u^2 + 4254m_1^4 u - 800m_1^6 + 1215u^3\right) \\ &- 768\Lambda_1^3 m_1 u \left(-185m_1^2 u^2 + 267m_1^4 u - 80m_1^6 + 159u^3\right) \\ &+ 2048u^3 \left(15m_1^2 u^2 + 6m_1^4 u - 2m_1^6 + 9u^3\right) - 81\Lambda_1^9 m_1 \left(235m_1^2 + 6u\right)\big). \end{aligned} \tag{5.34}$$

In the weak coupling region, the quantum SW periods reproduce the deformed prepotential obtained from the NS limit of the Nekrasov partition function as will show in the next section. From the above representation of the period integrals, it is possible to investigate the decoupling limit to the pure $SU(2)$ theory, which defined by (2.90) with $N_f = 1$. When we take the decoupling limit (2.90) with $N_f = 1$, the second and fourth order corrections (5.30) and (5.31) become (5.22) and (5.23).

In section 5.5, we will compute the deformed period integrals in the strong coupling region, where the monopole/dyon becomes massless. In this case, the coefficients in (5.30) and (5.31) become singular since the discriminant $\Delta_1 = 0$ at the massless BPS point. With help of the Picard-Fuchs equation (2.76) and the differential equation (5.28), the higher order corrections can be expressed such that all coefficients are regular with $\Delta_1 = 0$. We note that the coefficients of the differential operator for $\Pi^{(2)}$ can be written as

$$X_2^1 = \frac{1}{6}u + \frac{1}{6}m_1 b_1, \qquad X_2^2 = \frac{1}{12} + \frac{1}{6}m_1 c_1. \tag{5.35}$$

From the Picard-Fuchs equation (2.76) and the differential equation (5.28), the second and fourth order corrections to the SW periods can be expressed as

$$\Pi^{(2)} = \frac{1}{12}\left(2u\frac{\partial^2}{\partial u^2} + 2m_1\frac{\partial}{\partial m_1}\frac{\partial}{\partial u} + \frac{\partial}{\partial u}\right)\Pi^{(0)}, \tag{5.36}$$

$$\begin{aligned}\Pi^{(4)} = &\frac{1}{1440}\left(28u^2\frac{\partial^4}{\partial u^4} + 124u\frac{\partial^3}{\partial u^3} + 81\frac{\partial^2}{\partial u^2}\right.\\ &\left. + 56um_1\frac{\partial}{\partial m_1}\frac{\partial^3}{\partial u^3} + 28m_1^2\frac{\partial^2}{\partial m_1^2}\frac{\partial^2}{\partial u^2} + 132m_1\frac{\partial}{\partial m_1}\frac{\partial^2}{\partial u^2}\right)\Pi^{(0)}.\end{aligned} \tag{5.37}$$

We can easily analyze the quantum SW periods at the various strong coupling points in the Coulomb moduli space since all the coefficients have no singularities when $\Delta_1 = 0$.

$N_f = 2$ theory

In the case of $N_f = 2$, we can choose $N_+ = 1$ or $N_+ = 2$ in (2.98) for the SW curve (2.96). Although in either case, we have the quantum curves with the form of the Schrödinger type equations, they have apparently different $Q(x)$:

$$Q(x) = -u - \frac{\Lambda_2}{2}\left(m_1e^{ix} + m_2e^{-ix}\right) - \frac{\Lambda_2^2}{8}\cos 2x, \quad (N_+ = 1) \tag{5.38}$$

$$\begin{aligned}Q(x) = &-\frac{e^{ix}\Lambda_2^3 + \Lambda_2^2(e^{2ix}(m_1-m_2)^2 - 2) + 8\Lambda_2 e^{ix}(m_1m_2 - u) + 16u}{4(-2+e^{ix}\Lambda_2)^2}\\ &+\hbar^2\frac{e^{ix}\Lambda_2}{2(-2+e^{ix}\Lambda_2)^2}, \qquad (N_+ = 2)\end{aligned} \tag{5.39}$$

where for the $N_+ = 2$ case $Q(x)$ has the second order correction in $\hbar$. The quantum curves look quite different but it is shown that they give the same period integrals. One of the reasons is that the SW periods in both cases satisfy the same Picard-Fuchs equation with (2.103) and the differential equations

$$\frac{\partial^2\Pi^{(0)}}{\partial m_1\partial u} = \frac{1}{L_2}\left(b_2^{(1)}\frac{\partial^2\Pi^{(0)}}{\partial u^2} + c_2^{(1)}\frac{\partial\Pi^{(0)}}{\partial u}\right), \tag{5.40}$$

$$\frac{\partial^2\Pi^{(0)}}{\partial m_2\partial u} = \frac{1}{L_2}\left(b_2^{(2)}\frac{\partial^2\Pi^{(0)}}{\partial u^2} + c_2^{(2)}\frac{\partial\Pi^{(0)}}{\partial u}\right), \tag{5.41}$$

where

$$\begin{aligned}
L_2 =& -\Lambda_2^4 + 8m_1m_2\Lambda_2^2 + 32\big[4m_1^2m_2^2 - 3u(m_1^2+m_2^2) + 2u^2\big],\\
b_2^{(1)} =& 3\Lambda_2^4 m_1 - 4\Lambda_2^2 m_2(3m_1^2 - 9m_2^2 + 8u) - 64m_2u(m_1^2 - u),\\
c_2^{(1)} =& 4\Lambda_2^2 m_2 + 32m_1(m_2^2 - u),\\
b_2^{(2)} =& 3\Lambda_2^4 m_2 - 4\Lambda_2^2 m_1(3m_2^2 - 9m_1^2 + 8u) - 64m_1u(m_2^2 - u),\\
c_2^{(1)} =& 4\Lambda_2^2 m_1 + 32m_2(m_1^2 - u).
\end{aligned} \tag{5.42}$$

Since the SW periods are uniquely determined by solving the Picard-Fuchs equation around some singularities on the u-plane, we obtain the solutions of the SW periods which do not depend on the choice of N_+. The second reason is that one can also obtain the second and fourth order corrections with the independent of N_+, which are given by

$$\Pi^{(2)} = \frac{1}{6}\left(2u\frac{\partial^2}{\partial u^2} + \frac{3}{2}\left(m_1\frac{\partial}{\partial m_1}\frac{\partial}{\partial u} + m_2\frac{\partial}{\partial m_2}\frac{\partial}{\partial u}\right) + \frac{\partial}{\partial u}\right)\Pi^{(0)}, \tag{5.43}$$

$$\begin{aligned}
\Pi^{(4)} =& \frac{1}{360}\Bigg[28u^2\frac{\partial^4}{\partial u^4} + 120u\frac{\partial^3}{\partial u^3} + 75\frac{\partial^2}{\partial u^2}\\
&+ 42\left(um_1\frac{\partial}{\partial m_1}\frac{\partial^3}{\partial u^3} + um_2\frac{\partial}{\partial m_2}\frac{\partial^3}{\partial u^3}\right) + \frac{345}{4}\left(m_1\frac{\partial}{\partial m_1}\frac{\partial^2}{\partial u^2} + m_2\frac{\partial}{\partial m_2}\frac{\partial^2}{\partial u^2}\right)\\
&+ \frac{63}{4}\left(m_1^2\frac{\partial^2}{\partial m_1^2}\frac{\partial^2}{\partial u^2} + m_2^2\frac{\partial^2}{\partial m_2^2}\frac{\partial^2}{\partial u^2}\right) + \frac{126}{4}m_1m_2\frac{\partial}{\partial m_1}\frac{\partial}{\partial m_2}\frac{\partial^2}{\partial u^2}\Bigg]\Pi^{(0)}.
\end{aligned} \tag{5.44}$$

Here we required that all the coefficients of the expression do not have any singularity with $\Delta_2 = 0$. Thus we conclude that the quantum SW periods do not depend on the choice of N_+ at least up to the fourth order in $\hbar$. [46].

As explained in the previous sections, there are various way to represent the quantum corrections and the expressions (5.43) and (5.44) can be convert by using the Picard-Fuchs equation (2.76) and the differential equation (5.40). For example the expression of the second order correction (5.43) becomes that in terms of a basis $\partial_u^2\Pi^{(0)}$ and $\partial_u\Pi^{(0)}$ as

$$\Pi^{(2)} = \left[\left(\frac{1}{3}u + \frac{1}{4L_2}(m_1b_2^{(1)} + m_2b_2^{(2)})\right)\frac{\partial^2}{\partial u^2} + \left(\frac{1}{6} + \frac{1}{4L_2}(m_1c_2^{(1)} + m_2c_2^{(2)})\right)\frac{\partial}{\partial u}\right]\Pi^{(0)}, \tag{5.45}$$

where L_2, $b_2^{(1)}, \cdots c_2^{(2)}$ are given in (5.42). In the decoupling limit (2.90) with $N_f = 2$, the SW periods is reduced to that of the $N_f = 1$ theory. It can be also checked that the higher order corrections to the SW periods become those of the $N_f = 1$ theory up to fourth order in $\hbar$.

$N_f = 3$ theory

For $N_f = 3$, we should choose $N_+ = 1$ or 2 in (5.3) since the quantum SW curve becomes the third order differential equation if we take other N_+. We will take $N_+ = 2$ without loss of generality. The quantum SW curve becomes the form of the Schrödinger type equation (5.6) with

$$
\begin{aligned}
Q(x) =& \frac{e^{-2ix}}{16\left(-2+e^{ix}\Lambda_3^{\frac{1}{2}}\right)^2}\Big(-4\Lambda_3 - 4e^{3ix}\Lambda_3^{\frac{1}{2}}\left(m_3\Lambda_3+8m_1m_2-8u\right) - e^{2ix}\left(\Lambda_3^2 - 24m_3\Lambda_3+64u\right) \\
& - 4\left(m_1-m_2\right)^2 e^{4ix}\Lambda_3 + 4e^{ix}\Lambda_3^{\frac{1}{2}}\left(\Lambda_3 - 8m_3\right)\Big) + \hbar^2 \frac{e^{ix}\Lambda_3^{\frac{1}{2}}}{2\left(-2+e^{ix}\Lambda_3^{\frac{1}{2}}\right)^2}.
\end{aligned} \tag{5.46}
$$

where the potential $Q(x)$ has the second order corrections in $\hbar$. As explained in the $N_f = 1$ and 2 theories, the SW periods satisfy the Picard-Fuchs equation and the differential equations with respect to the mass parameter m_i $(i = 1, 2, 3)$ and the moduli parameter u. We consider the same mass $m := m_1 = m_2 = m_3$ for simplicity. The Picard-Fuchs equation is given by (2.76) with (2.104) and the differential equation takes the form

$$
\frac{\partial^2 \Pi^{(0)}}{\partial m \partial u} = b_3 \frac{\partial^2 \Pi^{(0)}}{\partial u^2} + c_3 \frac{\partial \Pi^{(0)}}{\partial u} \tag{5.47}
$$

where

$$
b_3 = \frac{3m\left(\Lambda_3^2 + 24\Lambda_3 m - 128u\right)}{16\left(16m^2 - \Lambda_3 m - 4u\right)}, \qquad c_3 = \frac{12m}{m\left(\Lambda_3 - 16m\right) + 4u}. \tag{5.48}
$$

In general mass case, it can be checked that the quantum corrections to the SW periods

$\Pi^{(0)}$ are expressed as

$$\Pi^{(2)} = \left[\left(\frac{5}{6}u - \frac{1}{384}\Lambda_3^2 \right) \frac{\partial^2}{\partial u^2} + \frac{1}{2}\sum_{i=1}^{3} m_i \frac{\partial}{\partial m_i}\frac{\partial}{\partial u} + \frac{5}{12}\frac{\partial}{\partial u} \right] \Pi^{(0)}, \tag{5.49}$$

$$\begin{aligned} \Pi^{(4)} =& \left[\frac{7}{10}\left(\frac{5}{6}u - \frac{1}{384}\Lambda_3^2 \right)^2 \frac{\partial^4}{\partial u^4} + \frac{47}{20}\left(\frac{241}{47}\frac{1}{6}u - \frac{1}{384}\Lambda_3^2 \right) \frac{\partial^3}{\partial u^3} + \frac{571}{480}\frac{\partial^2}{\partial u^2} \right. \\ & + \sum_{i=1}^{3} \left(\frac{7}{10}\left(\frac{5}{6}u - \frac{1}{384}\Lambda_3^2 \right) m_i \frac{\partial}{\partial m_i}\frac{\partial^3}{\partial u^3} + \frac{131}{120} m_i \frac{\partial}{\partial m_i}\frac{\partial^2}{\partial u^2} \right) \\ & \left. + \sum_{i=1}^{3}\sum_{j=1}^{3} \left(\frac{7}{40} m_i m_j \frac{\partial}{\partial m_i}\frac{\partial}{\partial m_j}\frac{\partial^2}{\partial u^2} \right) \right] \Pi^{(0)}. \end{aligned} \tag{5.50}$$

where all the coefficients are regular when $\Delta_3 = 0$. By using the Picard-Fuchs equation and the differential equation with respect to the mass parameters, the quantum SW periods (5.49) and (5.50) can be expressed in terms of a basis $\partial_u \Pi^{(0)}$ and $\partial_u^2 \Pi^{(0)}$. For the same mass case, we find that

$$\Pi^{(2)} = \left[\left(\frac{5}{6}u - \frac{1}{384}\Lambda_3^2 + \frac{1}{2}mb_3 \right) \frac{\partial^2}{\partial u^2} + \left(\frac{5}{12} + \frac{1}{2}mc_3 \right) \frac{\partial}{\partial u} \right] \Pi^{(0)}. \tag{5.51}$$

This representation is suitable to discuss the decoupling limit to the $N_f = 0$ theory, which is defined by $m \to \infty$ and $\Lambda_3 \to 0$ with $m^3 \Lambda_3 = \Lambda_0^4$ being fixed. By taking the decoupling limit, the SW periods for $N_f = 3$ theory are reduced to those for the $N_f = 0$ theory. Moreover, it can be shown that, in this limit, the second and fourth order corrections to the SW periods agree with those of the $N_f = 0$ theory.

$N_f = 4$ theory

In the case of $N_f = 4$, we will take $N_+ = 2$ in (5.3). Otherwise, the quantum SW curve becomes the third or fourth order differential equation. The quantum curve is the form of the Schrödinger type equation with

$$
\begin{aligned}
Q(x) =& \frac{e^{-2ix}}{4\left(-4\sqrt{q}\cos(x)+q+4\right)^2}\Big(4\sqrt{q}e^{3ix}\left(m_1^2 q + m_2^2 q - m_1 m_2(q+8) - m_3 m_4 q + 8u\right) \\
&+ 4\sqrt{q}e^{ix}\left(m_3^2 q + m_4^2 q - m_3 m_4(q+8) - m_1 m_2 q + 8u\right) \\
&- e^{2ix}\left(q\left(\left(m_1^2+m_2^2+m_3^2+m_4^2\right)q - 24\left(m_1 m_2 + m_3 m_4\right)\right) + 16(q+4)u\right) \\
&- 4qe^{4ix}\left(m_1-m_2\right)^2 - 4q\left(m_3-m_4\right)^2\Big) \\
&+ \hbar^2 \frac{\sqrt{q}e^{-ix}\left(qe^{2ix} - 8\sqrt{q}e^{ix} + q + 4e^{2ix} + 4\right)}{2\left(-4\sqrt{q}\cos(x)+q+4\right)^2}.
\end{aligned}
\tag{5.52}
$$

Due to complication in the general mass case, we consider the simpler case: massive hypermultiplets with the same mass: $m := m_1 = m_2 = m_3 = m_4$. In this case, the potential $Q(x)$ becomes

$$
\begin{aligned}
Q(x) =& -\frac{\left((m^2+u)\left(-16\sqrt{q}\cos(x)+(q-4)q+32\right) - (q-4)^2 u\right)}{\left(-4\sqrt{q}\cos(x)+q+4\right)^2} \\
&+ \hbar^2 \frac{\sqrt{q}e^{-ix}\left(qe^{2ix} - 8\sqrt{q}e^{ix} + q + 4e^{2ix} + 4\right)}{2\left(-4\sqrt{q}\cos(x)+q+4\right)^2}.
\end{aligned}
\tag{5.53}
$$

As discussed previously, the SW periods $\Pi^{(0)}$ satisfy the Picard-Fuchs equation (2.76) with (2.105). In terms of a basis $\partial_u \Pi^{(0)}$ and $\partial_u^2 \Pi^{(0)}$, the higher order corrections to the SW periods in $\hbar$ are given up to fourth order in $\hbar$ as follows: The second order correction is expressed as

$$
\Pi^{(2)} = \left(X_2^1 \frac{\partial^2}{\partial u^2} + X_2^2 \frac{\partial}{\partial u}\right)\Pi^{(0)}, \tag{5.54}
$$

where

$$
\begin{aligned}
X_2^1 &= -\frac{-18m^4 q + m^4 q^2 - 8m^2 u + 10m^2 qu + 24u^2}{96m^2}, \\
X_2^2 &= -\frac{-2m^2 + m^2 q + 6u}{48m^2}.
\end{aligned}
\tag{5.55}
$$

The fourth order correction is given by

$$
\Pi^{(4)} = \left(X_4^1 \frac{\partial^2}{\partial u^2} + X_4^2 \frac{\partial}{\partial u}\right)\Pi^{(0)}, \tag{5.56}
$$

where

$$\begin{aligned}X_4^1 =&\frac{1}{46080m^2\left(m^2-u\right)^2\left(m^2q-4m^2\sqrt{q}+4u\right)^2\left(m^2q+4m^2\sqrt{q}+4u\right)^2}\\&\times\Big(7m^{14}q^8-399m^{14}q^7+8484m^{14}q^6-80616m^{14}q^5+312480m^{14}q^4-284544m^{14}q^3\\&+153600m^{14}q^2+175m^{12}q^7u-7196m^{12}q^6u+96504m^{12}q^5u-436320m^{12}q^4u\\&+266496m^{12}q^3u-789504m^{12}q^2u+1848m^{10}q^6u^2-51624m^{10}q^5u^2+403488m^{10}q^4u^2\\&-896256m^{10}q^3u^2+2328576m^{10}q^2u^2+313344m^{10}qu^2+10648m^8q^5u^3\\&-190176m^8q^4u^3+820224m^8q^3u^3-1501184m^8q^2u^3-921600m^8qu^3+35968m^6q^4u^4\\&-377984m^6q^3u^4+881664m^6q^2u^4-26624m^6qu^4-8192m^6u^4+70656m^4q^3u^5\\&-344064m^4q^2u^5-325632m^4qu^5+24576m^4u^5+73728m^2q^2u^6+12288m^2qu^6\\&+319488m^2u^6+30720qu^7+122880u^7\Big),\end{aligned} \tag{5.57}$$

$$\begin{aligned}X_4^2 =&\frac{1}{23040m^2\left(m^2-u\right)^2\left(m^2q-4m^2\sqrt{q}+4u\right)^2\left(m^2q+4m^2\sqrt{q}+4u\right)^2}\\&\times\Big(7m^{12}q^7-287m^{12}q^6+3780m^{12}q^5-15816m^{12}q^4+1440m^{12}q^3-38400m^{12}q^2\\&+147m^{10}q^6u-4032m^{10}q^5u+29736m^{10}q^4u-55872m^{10}q^3u+225408m^{10}q^2u+30720m^{10}qu\\&+1260m^8q^5u^2-21768m^8q^4u^2+88704m^8q^3u^2-221952m^8q^2u^2-133632m^8qu^2\\&+5608m^6q^4u^3-56768m^6q^3u^3+147456m^6q^2u^3+7168m^6qu^3-2048m^6u^3\\&+13536m^4q^3u^4-64512m^4q^2u^4-58368m^4qu^4+6144m^4u^4+16512m^2q^2u^5+3072m^2qu^5\\&+79872m^2u^5+7680qu^6+30720u^6\Big).\end{aligned} \tag{5.58}$$

These expressions are useful to discuss the decoupling limit to $N_f = 0$: $m \to \infty$ and $q \to 0$ with $m^4 q = \Lambda_0^4$ being fixed. In the decoupling limit, the SW periods agree with those for the $N_f = 0$ theory. We can also show that the second and fourth order corrections of the quantum SW periods (5.54) and (5.56) are reduced to those for the $N_f = 0$ theory in this limit.

We then consider the massless limit. The Picard-Fuchs equation turns out a simple form:

$$\frac{\partial^2 \Pi^{(0)}}{\partial u^2} + \frac{1}{2u}\frac{\partial \Pi^{(0)}}{\partial u} = 0. \tag{5.59}$$

We note that the expressions (5.54) and (5.56) have the coefficients which become singular in $m \to 0$. In the massless case, it is found that the second and fourth order corrections to the SW periods convert into

$$\Pi^{(2)} = \left(-\frac{uq}{8}\frac{\partial^2}{\partial u^2} + \frac{(q-4)q}{16u}\frac{\partial}{\partial q}\right)\Pi^{(0)}, \tag{5.60}$$

$$\Pi^{(4)} = \left(\frac{-26q+11q^2}{2304}\frac{\partial^2}{\partial u^2} - \frac{(q-4)(-52q+35q^2)}{4608u^2}\frac{\partial}{\partial q} - \frac{(q-4)^2q^2}{288u^2}\frac{\partial^2}{\partial q^2}\right)\Pi^{(0)}, \tag{5.61}$$

with help of the Picard-Fuchs equation. We note that these formulas include the derivative with respect to q as well as u.

In the following sections, we will compute the quantum SW periods both in the weak and strong coupling regions and evaluate the deformed (dual) prepotentials.

5.3 Deformed SW periods in the weak coupling

In this section, we will study the expansion of the quantum SW periods in the weak coupling region for the completeness. We expand the quantum SW periods at $u = \infty$ and then obtain the deformed prepotential for the N_f theories [39,84]. Then we will check that obtained prepotential agrees with that obtained from the Nekrasov partition function in the NS limit [46]. The SW periods in the weak coupling region are given by integrating the expansions of the periods (2.69) and (2.70) over u [75]. The quantum SW periods are given by acting the differential operators on the SW periods $a^{(0)}$ and $a_D^{(0)}$.

5.3.1 $N_f \leq 3$

For $N_f = 1$, the discriminant Δ_1 and D_1 are given by (2.102). Substituting them into (2.69) and (2.70) and expanding around $u = \infty$, we obtain the expansions of the periods $\partial_u a^{(0)}$ and $\partial_u a_D^{(0)}$ in the weak coupling region. The expansions of the SW periods are obtained by the integration of them over u. Then applying (5.36) and (5.37) to the SW periods, the expansions of the quantum SW periods around $u = \infty$ are given by

$$\begin{aligned}a(u) =&\sqrt{\frac{u}{2}} - \frac{\Lambda_1^3 m_1 \left(\frac{1}{u}\right)^{\frac{3}{2}}}{2^4\sqrt{2}} + \frac{3\Lambda_1^6 \left(\frac{1}{u}\right)^{\frac{5}{2}}}{2^{10}\sqrt{2}} + \cdots \\ &+ \hbar^2 \left(-\frac{\Lambda_1^3 m_1 \left(\frac{1}{u}\right)^{\frac{5}{2}}}{2^6\sqrt{2}} + \frac{15\Lambda_1^6 \left(\frac{1}{u}\right)^{\frac{7}{2}}}{2^{12}\sqrt{2}} - \frac{35\Lambda_1^6 m_1^2 \left(\frac{1}{u}\right)^{\frac{9}{2}}}{2^{11}\sqrt{2}} + \cdots \right) \\ &+ \hbar^4 \left(-\frac{\Lambda_1^3 m \left(\frac{1}{u}\right)^{\frac{7}{2}}}{2^8\sqrt{2}} + \frac{63\Lambda_1^6 \left(\frac{1}{u}\right)^{\frac{9}{2}}}{2^{14}\sqrt{2}} - \frac{273\Lambda_1^6 m^2 \left(\frac{1}{u}\right)^{\frac{11}{2}}}{2^{14}\sqrt{2}} + \cdots \right) + \cdots ,\end{aligned} \tag{5.62}$$

$$\begin{aligned}a_D(u) =& -\frac{i}{2\sqrt{2}\pi}\Bigg[\sqrt{2}a(u)\left(i\pi - 3\log\frac{16u}{\Lambda_1^2}\right) + \left(6\sqrt{u} + \frac{m_1^2}{\sqrt{u}} + \frac{\frac{m_1^4}{6} - \frac{1}{4}\Lambda_1^3 m_1}{u^{\frac{3}{2}}} + \cdots \right) \\ &+ \hbar^2 \left(-\frac{1}{4\sqrt{u}} - \frac{m_1^2}{12u^{\frac{3}{2}}} + \frac{-\frac{9}{64}\Lambda_1^3 m_1 - \frac{m_1^4}{12}}{u^{\frac{5}{2}}} + \cdots \right) \\ &+ \hbar^4 \left(\frac{1}{160u^{\frac{3}{2}}} + \frac{7m_1^2}{240u^{\frac{5}{2}}} + \frac{\frac{7m_1^4}{96} - \frac{127\Lambda_1^3 m_1}{2560}}{u^{\frac{7}{2}}} + \cdots \right) + \cdots \Bigg].\end{aligned} \tag{5.63}$$

Inverting the series of $a(u)$ in (5.62), we obtain the expansion of u in terms of a. Substituting it into a_D, a_D becomes a function of a. The integration of a_D over a derives the deformed prepotential:

$$\mathcal{F}_1(a,\hbar) = \frac{1}{2\pi i}\left[\mathcal{F}_1^{\text{pert}}(a,\hbar) + \sum_{k=0}^{\infty}\sum_{n=1}^{\infty}\hbar^{2k}\mathcal{F}_1^{(2k,n)}\left(\frac{1}{a}\right)^{2n}\right], \tag{5.64}$$

where the first few coefficients of $\mathcal{F}_1^{(2k,n)}$ ($k = 0,1,2$) are listed in the table 5.1. The perturbative part $\mathcal{F}_1^{\text{pert}}(a,\hbar)$ of the prepotential takes the form

$$\begin{aligned}\mathcal{F}_1^{\text{pert}}(a,\hbar) =& -\frac{3}{2}a^2\log\frac{a^2}{\Lambda_1^2} + \frac{1}{2}\mathcal{F}_s^1 - a^2\log a - \frac{3m_1^2}{4} \\ &+ \hbar^2\left(-\frac{1}{12}\log a - \frac{1}{96}\frac{\partial^2\mathcal{F}_s^1}{\partial a^2} + \frac{1}{16}\right) + \hbar^4\left(-\frac{1}{5760a^2} + \frac{7}{2^{10}\cdot 3^2\cdot 5}\frac{\partial^4\mathcal{F}_s^1}{\partial a^4}\right) + \cdots ,\end{aligned} \tag{5.65}$$

where $\mathcal{F}_s^1$ is defined as [78]

$$\mathcal{F}_s^1 = \left(a + \frac{m_1}{\sqrt{2}}\right)^2\log\left(a + \frac{m_1}{\sqrt{2}}\right) + \left(a - \frac{m_1}{\sqrt{2}}\right)^2\log\left(a - \frac{m_1}{\sqrt{2}}\right). \tag{5.66}$$

k	$\mathcal{F}_1^{(2k,1)}$	$\mathcal{F}_1^{(2k,2)}$	$\mathcal{F}_1^{(2k,3)}$	$\mathcal{F}_1^{(2k,4)}$
0	$\frac{1}{32}\Lambda_1^3 m_1$	$-\frac{3\Lambda_1^6}{8192}$	$\frac{5\Lambda_1^6 m_1^2}{16384}$	$-\frac{7\Lambda_1^9 m_1}{393216}$
1	0	$\frac{1}{256}\Lambda_1^3 m_1$	$-\frac{15\Lambda_1^6}{65536}$	$\frac{21\Lambda_1^6 m_1^2}{65536}$
2	0	0	$\frac{\Lambda_1^3 m_1}{2048}$	$-\frac{63\Lambda_1^6}{524288}$

Table 5.1: The coefficients of the prepotential for the $N_f=1$ theory

Similarly, the expansions of the deformed prepotentials for $N_f=2$ and 3 theories are given by

$$\mathcal{F}_{N_f}(a,\hbar) = \frac{1}{2\pi i}\left[\mathcal{F}_{N_f}^{\text{pert}}(a,\hbar) + \sum_{k=0}^{\infty}\sum_{n=1}^{\infty}\hbar^{2k}\mathcal{F}_{N_f}^{(2k,n)}\left(\frac{1}{a}\right)^{2n}\right], \tag{5.67}$$

where some coefficients $\mathcal{F}_{N_f}^{(2k,n)}$ $(k=0,1,2)$ are shown in appendix B. The perturbative parts are expanded as

$$\begin{aligned}\mathcal{F}_2^{\text{pert}}(a,\hbar) =& -a^2\log\frac{a^2}{\Lambda_2^2} + \frac{1}{2}\mathcal{F}_s^2 - 2a^2\log a - \frac{3}{4}(m_1^2+m_2^2)\\ &+ \hbar^2\left(-\frac{1}{12}\log a - \frac{1}{96}\frac{\partial^2\mathcal{F}_s^2}{\partial a^2} + \frac{1}{8}\right) + \hbar^4\left(-\frac{1}{5760a^2} + \frac{7}{2^{10}\cdot 3^2\cdot 5}\frac{\partial^4\mathcal{F}_s^2}{\partial a^4}\right) + \cdots,\end{aligned} \tag{5.68}$$

$$\begin{aligned}\mathcal{F}_3^{\text{pert}}(a,\hbar) =& -\frac{1}{4}a^2\log\frac{a^2}{\Lambda_3^2} + \frac{1}{2}\mathcal{F}_s^3 - 3a^2\log a - \sum_{i=1}^{3}\frac{3}{4}m_i^2\\ &+ \hbar^2\left(-\frac{1}{12}\log a - \frac{1}{96}\frac{\partial^2\mathcal{F}_s^3}{\partial a^2} + \frac{3}{16}\right) + \hbar^4\left(-\frac{1}{5760a^2} + \frac{7}{2^{10}\cdot 3^2\cdot 5}\frac{\partial^4\mathcal{F}_s^3}{\partial a^4}\right) + \cdots,\end{aligned} \tag{5.69}$$

where $\mathcal{F}_s^{N_f}$ $(N_f=2,3)$ is defined as [79]

$$\mathcal{F}_s^{N_f} = \sum_{i=1}^{N_f}\left(\left(a+\frac{m_i}{\sqrt{2}}\right)^2\log\left(a+\frac{m_i}{\sqrt{2}}\right) + \left(a-\frac{m_i}{\sqrt{2}}\right)^2\log\left(a-\frac{m_i}{\sqrt{2}}\right)\right). \tag{5.70}$$

It can be shown that these deformed prepotentials are reduced to those for the theory with less number of the hypermultiplets in the decoupling limit.

We now compare the prepotentials for $N_f = 1, 2, 3$ theories, which obtained from the Nekrasov partition function in the NS limit. When we rescale the parameters $\hbar$, m_i ($i = 1, 2, 3$), and Λ_{N_f} as

$$2\pi i \mathcal{F}(a, \hbar) \to \mathcal{F}(a, \epsilon_1), \qquad \Lambda_{N_f} \to 2^{\frac{2}{(4-N_f)}} \sqrt{2} \Lambda_{N_f}, \qquad \hbar \to \sqrt{2}\epsilon_1, \qquad m_i \to \sqrt{2} m_i,$$

and then shift the mass parameters : $m_i \to m_i + \frac{\epsilon_1}{2}$ for a fundamental matter or $m_i \to \frac{\epsilon_1}{2} - m_i$ for an anti-fundamental matter, we find that the prepotential coincides with that with the NS limit of the Nekrasov partition [34], which is given in chapter 4.

5.3.2 $N_f = 4$

In the theory with the $N_f = 4$ hypermultiplets, we will rescale the coordinates y and p in the form of the SW curve (2.92) by a factor of $1 - \frac{q}{2}$, so that we can use the formulas (2.69) and (2.70). In the weak coupling region, the SW periods $a^{(0)}$ and $a_D^{(0)}$ are obtained by expanding the periods around $q = 0$ and integrating over u.

For simplicity, we consider the same mass case $m := m_1 = m_2 = m_3 = m_4$. The discriminant Δ_4 and D_4 are given in (2.105). The deformed prepotential takes the form:

$$\mathcal{F}_4 = \frac{1}{2\pi i} \left[\mathcal{F}_4^{\text{pert}}(a, \hbar) + \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \hbar^{2k} \mathcal{F}_4^{(2k,n)} q^n \right], \tag{5.71}$$

where the perturbative part is

$$\begin{aligned} \mathcal{F}_4^{\text{pert}}(a, \hbar) =& a^2 \log q + \frac{1}{2}\mathcal{F}_s^4 - 4a^2 \log a \\ &+ \hbar^2 \left(-\frac{1}{12}\log(a) - \frac{1}{96}\frac{\partial^2 \mathcal{F}_s^4}{\partial a^2} \right) + \hbar^4 \left(-\frac{1}{5760 a^2} + \frac{7}{2^{10}\cdot 3^2 \cdot 5}\frac{\partial^4 \mathcal{F}_s^4}{\partial a^4} \right) + \cdots , \end{aligned} \tag{5.72}$$

where

$$\mathcal{F}_s^4 = 4 \left(\left(a + \frac{m}{\sqrt{2}} \right)^2 \log \left(a + \frac{m}{\sqrt{2}} \right) + \left(a - \frac{m}{\sqrt{2}} \right)^2 \log \left(a - \frac{m}{\sqrt{2}} \right) \right). \tag{5.73}$$

The first several coefficients $\mathcal{F}_4^{(2k,n)}$ for $k = 0, 1, 2$ are given in appendix B.3. After rescaling the parameters $\hbar$, m and q as

$$2\pi i \mathcal{F}(a, \hbar) \to \mathcal{F}(a, \epsilon_1), \qquad q \to 4q, \qquad \hbar \to \sqrt{2}\epsilon_1, \qquad m \to \sqrt{2}m, \tag{5.74}$$

it can be checked that (5.71) coincides with the prepotential with the NS limit of the Nekrasov partition function of the theory with the same mass. Here we note that the mass parameter in the Nekrasov partition function are shifted as $m_i \to m_i + \frac{\epsilon_1}{2}$ for a fundamental matter or $m_i \to \frac{\epsilon_1}{2} - m_i$ for an anti-fundamental matter $(i = 1, \cdots 4)$.

For the massless case $m = 0$, the solution of the Picard-Fuchs equation (5.59) is given by

$$\Pi^{(0)} = f(q)u^{\frac{1}{2}}, \tag{5.75}$$

where

$$f(q) = \frac{\sqrt{2}}{((q-4)q+16)^{\frac{1}{4}}} F\left(\frac{1}{12};\frac{5}{12};1;\frac{108(q-4)^2q^2}{(q^2-4q+16)^3}\right). \tag{5.76}$$

Then, applying (5.60) and (5.61), the second and fourth order corrections to the SW periods are obtained by

$$\Pi^{(2)} = \frac{1}{32\sqrt{u}}\left(qf(q) + 2(q-4)\frac{\partial f(q)}{\partial q}\right), \tag{5.77}$$

$$\Pi^{(4)} = -\frac{q}{9216u^{\frac{3}{2}}}\left((11q-26)f(q) + 2(q-4)\left(16(q-4)q\frac{\partial^2 f(q)}{\partial q^2} + (35q-52)\frac{\partial f(q)}{\partial q}\right)\right). \tag{5.78}$$

After using (5.75), (5.77) and (5.78), we obtain the expansion of the prepotential around $q = 0$, which agree with (5.71) for $m = 0$. From the above discussions, it is found that the deformed periods explicitly coincide with those with the Nekrasov partition function in the NS limit up to the fourth order in $\hbar$.

5.4 Deformed effective coupling constant

The deformed effective coupling can be computed by using the relation (5.20) and the Picard-Fuchs equation (2.76). By applying the Picard-Fuchs equation (2.76), the u-derivative of the quantum corrections to the SW periods (5.20) is given by the form as

$$\frac{\partial}{\partial u}\Pi^{(2k)} = \left(Y^1_{2k}\frac{\partial^2}{\partial u^2} + Y^2_{2k}\frac{\partial}{\partial u}\right)\Pi^{(0)}, \tag{5.79}$$

where

$$Y_{2k}^1 := -\, p_1 X_{2k}^1 + \frac{\partial X_{2k}^1}{\partial u} + X_{2k}^2, \tag{5.80}$$

$$Y_{2k}^2 := -\, p_2 X_{2k}^1 + \frac{\partial X_{2k}^2}{\partial u}. \tag{5.81}$$

with p_1 and p_2 being the coefficients of the Picard-Fuchs equation (2.77) and (2.78). Then we have the u-derivative of the quantum SW period $\Pi = \sum_{k=0}^{\infty} \hbar^{2k} \Pi^{(2k)}$, which is of the form:

$$\frac{\partial}{\partial u}\Pi = \left(Y_1 \frac{\partial^2}{\partial u^2} + Y_2 \frac{\partial}{\partial u} \right) \Pi^{(0)}, \tag{5.82}$$

where

$$Y_1 = \sum_{n=1}^{\infty} \hbar^{2n} Y_{2n}^1, \qquad Y_2 = 1 + \sum_{n=1}^{\infty} \hbar^{2n} Y_{2n}^2. \tag{5.83}$$

The deformed effective coupling is defined by

$$\tau := \frac{\partial_u a_D}{\partial_u a}. \tag{5.84}$$

Substituting (5.82) into it and expanding in $\hbar$, we obtain the expansions of the deformed coupling constant, given by

$$\tau = \tau^{(0)} \left(1 + \hbar^2 Y_2^1 \partial_u \log \tau^{(0)} + \mathcal{O}(\hbar^4) \right). \tag{5.85}$$

up to second order in $\hbar$ where $\tau^{(0)} = \frac{\partial_u a_D^{(0)}}{\partial_u a^{(0)}}$. Since $\partial_u \log \tau^{(0)}$ is proportional to the beta functions at the weak coupling, the second order correction to the effective coupling constant in $\hbar$ is determined by a dimensionless constant Y_2^1 in (5.80).

We will compute the coefficient Y_2^1 for some simple cases, where all hypermultiplets have the same mass m. For $N_f = 0$, the coefficient of the Picard-Fuchs equation (2.77) is given by $p_1 = \frac{2u}{u^2 - \Lambda_0^4}$, while the coefficients X_2^1 and X_2^2 read off (5.22). From them, one finds

$$Y_2^1 = \frac{1}{8} - \frac{u^2}{6\left(u^2 - \Lambda_0^4\right)}. \tag{5.86}$$

In a similar way one can evaluate the coefficient Y_2^1 for $N_f \geq 1$. The results are the followings:

1. For $N_f = 1$, we have
$$Y_2^1 =\frac{1}{4}+\left(\frac{1}{2}m+\frac{3}{16}b_1\right)c_1-\frac{1}{6}\left(u+mb_1\right)\left(\frac{\partial_u\Delta_1}{\Delta_1}+\frac{3}{4m^2-3u}\right). \tag{5.87}$$

2. For $N_f = 2$, we have
$$Y_2^1 =\frac{1}{2}+\left(\frac{3m}{4}-2b_2\right)c_2-\left(\frac{1}{3}u+\frac{m}{4}b_2\right)\left(\frac{\partial_u\Delta_2}{\Delta_2}-\frac{8(3m^2-2u)}{8m^2-8u+\Lambda_2^2}\frac{c_2}{m}\right), \tag{5.88}$$
where
$$b_2=\frac{1}{L_2}(b_2^{(1)}+b_2^{(2)}),\qquad c_2=\frac{1}{L_2}(c_2^{(1)}+c_2^{(2)}). \tag{5.89}$$

3. For $N_f = 3$, we have
$$Y_2^1 = \frac{5}{4}+\left(\frac{3}{2}m-\frac{1}{6}b_3\right)-\left(\frac{5}{6}u-\frac{1}{384}\Lambda_3^2+\frac{1}{2}mb_3\right)\left(\frac{\partial_u\Delta_3}{\Delta_3}-\frac{24m^2+8u+m\Lambda_3}{-8m^2+8u-m\Lambda_3}\frac{c_3}{m}\right), \tag{5.90}$$
where b_3 and c_3 is given by (5.48).

4. For $N_f = 4$, we find
$$Y_2^1 =\frac{1-q}{8}-\frac{5u}{8m^2}-\frac{1}{96}\left(2(4-5q)u-m^2(q-18)q-\frac{24u^2}{m^2}\right)\left(\frac{\partial_u\Delta_4}{\Delta_4}+\frac{3}{m^2-u}\right). \tag{5.91}$$

The above formulas are consistent with the decoupling limit.

5.5 Deformed periods around massless monopole point

In this section, we consider the quantum SW periods at the strong coupling of the theories with $N_f = 1, 2, 3$ hypermultiplets, where a BPS monopole/dyon becomes massless. In particular, we will discuss the massless monopole point of the deformed theories where $a_D(u) = 0$. At the massless ("classical") monopole point where the dual SW period $a_D^{(0)}$ becomes zero, the discriminant Δ_{N_f} of the SW curve and also w_{N_f} become zero. In the following, we will compute the expansion of the quantum SW periods around the classical massless monopole point. We can analyze the periods around the dyon massless in the same way.

In general N_f, we gave the expansions of the SW periods around the massless monopole point $u = u_0$, which are expressed as (2.109) and (2.110). Once the SW periods around the massless monopole point are obtained, the quantum SW periods can be computed by applying the relations between the SW periods and the quantum corrections as discussed at the weak coupling. Since the coefficients of (2.109) and (2.110) are read off the series expansions of w_{N_f} and $(-D_{N_f})^{\frac{1}{4}}$ around $u = u_0$: (2.108), we should expand w_{N_f} and $(-D_{N_f})^{\frac{1}{4}}$ around $u = u_0$ which is one of the solution of $\Delta_{N_f} = 0$. In the following, we will only evaluate the expansion of the quantum SW periods in simpler cases; massless hypermultiplets and massive hypermultiplets with the same mass because the expression of u_0 is rather complicated for general mass parameters.

After analyzing the quantum SW periods around $u = u_0$, we find an interesting phenomenon by the quantum corrections. Although the undeformed SW period $a_D^{(0)}(u)$ is zero at $\hat{u} = 0$, the deformed SW period $a_D(u)$ does not become zero at same point. This means that the massless monopole point is shifted on the u-plane by the quantum correction. Indeed, the quantum correction to the SW periods around $u = u_0$ is expressed as

$$a_D^{(2k)} = \sum_{n=0}^{\infty} \mathcal{J}_n^{(2k)} \hat{u}^n. \tag{5.92}$$

Here $\mathcal{J}_n^{(0)} := \mathcal{J}_n$ in (2.109) with $\mathcal{J}_0^{(0)} = 0$ and $\mathcal{J}_1^{(0)}$, $\mathcal{J}_0^{(2)}$ and $\mathcal{J}_0^{(4)}$ take non-zero values by explicit calculation. Then we find the massless monopole point U_0 of the deformed theory is expanded as

$$U_0 = u_0 + \hbar^2 u_1 + \hbar^4 u_2 + \cdots , \tag{5.93}$$

where u_1 and u_2 are determined by

$$u_1 = -\frac{\mathcal{J}_0^{(2)}}{\mathcal{J}_1^{(0)}}, \tag{5.94}$$

$$u_2 = -\frac{\mathcal{J}_0^{(4)}}{\mathcal{J}_1^{(0)}} - \frac{\mathcal{J}_1^{(2)}}{\mathcal{J}_1^{(0)}} u_1 - \frac{\mathcal{J}_2^{(0)}}{\mathcal{J}_1^{(0)}} u_1^2. \tag{5.95}$$

We will explicitly calculate these corrections in the following examples.

5.5.1 Massless hypermultiplets

We consider the N_f theories with massless hypermultiplets. Since the Coulomb moduli space has some discrete symmetry, this case is a simple and interesting example. We will consider the massless monopole point of the u-plane. The SW periods around the massless monopole point u_0 have been studied in [77] by solving the Picard-Fuchs equation.

$N_f = 1$

For $N_f = 1$, the massless monopole point is given by $u_0 = -\frac{3\Lambda_1^2}{2^{\frac{8}{3}}}$. The expansions of w_1 and $(-D_1)^{-\frac{1}{4}}$ around $u = u_0$ are expressed as

$$w_1 = -\frac{2^{\frac{14}{3}}}{\Lambda_1^2}\hat{u} - \frac{2^{\frac{22}{3}}\cdot 5}{3\Lambda_1^4}\hat{u}^2 - \frac{47104}{27\Lambda_1^6}\hat{u}^3 + \cdots , \tag{5.96}$$

$$(-D_1)^{-\frac{1}{4}} = -i\left(\frac{2^{\frac{1}{3}}}{3^{\frac{1}{3}}\Lambda_1} + \frac{2^2}{3^{\frac{3}{2}}\Lambda_1^3}\hat{u} + \frac{2^{\frac{8}{3}}}{3^{\frac{3}{2}}\Lambda_1^5}\hat{u}^2 + \cdots\right). \tag{5.97}$$

Here the coefficients A_n and B_n in the expansions (2.108) can be read off from above expansions.

The SW periods around the massless monopole point are obtained by substituting these coefficients into (2.109) and (2.110). Applying the relations (5.36) and (5.37), the quantum SW periods around $\hat{u} = 0$ are expanded as

$$\begin{aligned} a_D(\hat{u}) =& \left(\frac{\hat{u}}{2^{\frac{1}{6}}\cdot 3^{\frac{1}{2}}\Lambda_1} + \frac{\hat{u}^2}{2^{\frac{1}{2}}\cdot 3^{\frac{5}{2}}\Lambda_1^3} + \frac{\hat{u}^3}{2^{\frac{5}{6}}\cdot 3^{\frac{11}{2}}\Lambda_1^5} + \cdots\right) \\ &+ \frac{\hbar^2}{\Lambda_1}\left(\frac{5}{2^{\frac{19}{6}}\cdot 3^{\frac{5}{2}}} + \frac{35}{2^{\frac{7}{2}}\cdot 3^{\frac{9}{2}}}\left(\frac{\hat{u}}{\Lambda_1^2}\right) + \frac{665}{2^{\frac{23}{6}}\cdot 3^{\frac{15}{2}}}\left(\frac{\hat{u}}{\Lambda_1^2}\right)^2 + \cdots\right) \\ &+ \frac{\hbar^4}{\Lambda_1^3}\left(\frac{2471}{6^{\frac{15}{2}}} + \frac{144347}{2^{\frac{53}{6}}\cdot 3^{\frac{19}{2}}}\left(\frac{\hat{u}}{\Lambda_1^2}\right) + \frac{1964347}{2^{\frac{55}{6}}\cdot 3^{\frac{23}{2}}}\left(\frac{\hat{u}}{\Lambda_1^2}\right)^2 + \cdots\right) + \cdots , \end{aligned} \tag{5.98}$$

k	$\mathcal{F}_{D1}^{(2k,1)}$	$\mathcal{F}_{D1}^{(2k,2)}$	$\mathcal{F}_{D1}^{(2k,3)}$	$\mathcal{F}_{D1}^{(2k,4)}$
0	0	-3	$-\frac{5}{12}\frac{1}{\tilde{c}(1)}$	$-\frac{515}{1152}\frac{1}{\tilde{c}(1)^2}$
1	$\frac{25}{96}\frac{1}{\tilde{c}(1)}$	$\frac{425}{4608}\frac{1}{\tilde{c}(1)^2}$	$-\frac{3275}{110592}\frac{1}{\tilde{c}(1)^3}$	$-\frac{50645}{294912}\frac{1}{\tilde{c}(1)^4}$
2	$\frac{104263}{5308416}\frac{1}{\tilde{c}(1)^3}$	$\frac{757333}{28311552}\frac{1}{\tilde{c}(1)^4}$	$-\frac{7173929}{1019215872}\frac{1}{\tilde{c}(1)^5}$	$-\frac{4749125675}{32614907904}\frac{1}{\tilde{c}(1)^6}$

Table 5.2: The coefficients of the dual prepotentials for the $N_f = 1$ theory, where $\tilde{c}(1) = -3^{\frac{3}{2}} \cdot 2^{\frac{-17}{6}}$ [77].

$$\begin{aligned} a(\hat{u}) =& \frac{i}{2\pi}\Bigg[a_D(\hat{u})\left(-i\pi + \log\frac{\hat{u}}{2^{\frac{4}{3}}\, 3^3\, \Lambda_1^2}\right) + i\left(-\frac{\hat{u}}{2^{\frac{1}{6}}\cdot 3^{\frac{1}{2}}\Lambda_1} - \frac{5\hat{u}^2}{2^{\frac{3}{2}}\cdot 3^{\frac{5}{2}}\Lambda_1^3} - \frac{298\hat{u}^3}{2^{\frac{5}{6}}\cdot 3^{\frac{13}{2}}\Lambda_1^5} + \cdots\right) \\ &+ \frac{i\hbar^2}{\Lambda_1}\left(-\frac{1}{2^{\frac{23}{6}}\cdot 3^{\frac{1}{2}}}\left(\frac{\hat{u}}{\Lambda_1^2}\right)^{-1} + \frac{13}{2^{\frac{19}{6}}\cdot 3^{\frac{7}{2}}} + \frac{101}{6^{\frac{9}{2}}}\left(\frac{\hat{u}}{\Lambda_1^2}\right) + \cdots\right) \\ &+ \frac{i\hbar^4}{\Lambda_1^3}\left(\frac{7}{2^{\frac{15}{2}}\cdot 3^{\frac{1}{2}}\cdot 5}\left(\frac{\hat{u}}{\Lambda_1^2}\right)^{-3} + \frac{29}{2^{\frac{47}{6}}\cdot 3^{\frac{5}{2}}\cdot 5}\left(\frac{\hat{u}}{\Lambda_1^2}\right)^{-2} + \frac{107}{2^{\frac{49}{6}}\cdot 3^{\frac{9}{2}}}\left(\frac{\hat{u}}{\Lambda_1^2}\right)^{-1} + \cdots\right)\Bigg]. \end{aligned} \tag{5.99}$$

Solving $\hat{u}$ in terms of a_D and substituting it into a, we have a as a function of a_D. Then the integration of a over a_D reproduces the dual prepotential:

$$\begin{aligned} \mathcal{F}_{D1}(a_D,\hbar) =& \frac{i}{8\pi}\Bigg[a_D^2\log\left(\frac{a_D}{\Lambda_1}\right)^2 - \frac{\hbar^2}{12}\log(a_D) - \frac{7\hbar^4}{5760a_D^2} + \cdots \\ &+ \sum_{k=0}^{\infty}\sum_{n=1}^{\infty}\Lambda_1^2\left(\frac{\hbar}{\Lambda_1}\right)^{2k}\mathcal{F}_{D1}^{(2k,n)}\left(\frac{a_D}{\Lambda_1}\right)^n\Bigg], \end{aligned} \tag{5.100}$$

where the first several coefficients $\mathcal{F}_{D1}^{(2k,n)}$ $(k = 0, 1, 2)$ are listed in the table 5.2.

$N_f = 2$

In the case of the $N_f = 2$ theory, the massless monopole point is given by $u_0 = \frac{\Lambda_2^2}{8}$. Then we have the expansions of w_2 and $(-D_2)^{-\frac{1}{4}}$ as

$$w_2 = \frac{108}{\Lambda_2^4}\hat{u}^2 - \frac{432}{\Lambda_2^6}\hat{u}^3 - \frac{3456}{\Lambda_2^8}\hat{u}^4 + \cdots, \tag{5.101}$$

$$(-D_2)^{-\frac{1}{4}} = \frac{1}{\Lambda_2} - \frac{\hat{u}}{\Lambda_2^3} - \frac{3\hat{u}^2}{2\Lambda_2^5} + \cdots. \tag{5.102}$$

The expansions of the quantum SW periods around $\hat{u}=0$ are given by

$$\begin{aligned} a_D(u) =& i\left(\frac{\hat{u}}{2^{\frac{1}{2}}\Lambda_2} - \frac{\hat{u}^2}{2^{\frac{3}{2}}\Lambda_2^3} + \frac{3\hat{u}^3}{2^{\frac{5}{2}}\Lambda_2^5} + \cdots\right) \\ &+ \frac{i\hbar^2}{\Lambda_2}\left(\frac{1}{2^{\frac{7}{2}}} - \frac{5}{2^{\frac{9}{2}}}\left(\frac{\hat{u}}{\Lambda_2^2}\right) + \frac{35}{2^{\frac{11}{2}}}\left(\frac{\hat{u}}{\Lambda_2^2}\right)^2 + \cdots\right) \\ &+ \frac{i\hbar^4}{\Lambda_2^3}\left(-\frac{17}{2^{\frac{17}{2}}} + \frac{721}{2^{\frac{21}{2}}}\left(\frac{\hat{u}}{\Lambda_2^2}\right) - \frac{10941}{2^{\frac{23}{2}}}\left(\frac{\hat{u}}{\Lambda_2^2}\right)^2 + \cdots\right) + \cdots, \end{aligned} \tag{5.103}$$

$$\begin{aligned} a(u) =& \frac{i}{2\pi}\left[2a_D(\hat{u})\log\frac{\hat{u}}{4\Lambda_2^2} + i\left(-\frac{2\hat{u}}{2^{\frac{1}{2}}\Lambda_2} - \frac{3\hat{u}^2}{2^{\frac{3}{2}}\Lambda_2^3} + \frac{12\hat{u}^3}{2^{\frac{5}{2}}\Lambda_2^5} + \cdots\right)\right. \\ &+ \frac{i\hbar^2}{\Lambda_2}\left(\frac{1}{2^{\frac{5}{2}}\cdot 3}\left(\frac{\hat{u}}{\Lambda_2^2}\right)^{-1} + \frac{10}{2^{\frac{7}{2}}\cdot 3} - \frac{77}{2^{\frac{9}{2}}\cdot 3}\left(\frac{\hat{u}}{\Lambda_2^2}\right) + \cdots\right) \\ &\left.+ \frac{i\hbar^4}{\Lambda_2^3}\left(\frac{7}{2^{\frac{11}{2}}\cdot 3^2\cdot 5}\left(\frac{\hat{u}}{\Lambda_2^2}\right)^{-3} - \frac{1}{2^{\frac{13}{2}}\cdot 5}\left(\frac{\hat{u}}{\Lambda_2^2}\right)^{-2} + \frac{53}{2^{\frac{15}{2}}\cdot 3\cdot 5}\left(\frac{\hat{u}}{\Lambda_2^2}\right)^{-1} + \cdots\right) + \cdots\right]. \end{aligned} \tag{5.104}$$

From the above formulas, one finds the deformed dual prepotential for the $N_f=2$ theory takes the forms as

$$\begin{aligned} \mathcal{F}_{D2}(a_D,\hbar) =& \frac{i}{8\pi}\left[2a_D^2\log\left(\frac{a_D}{\Lambda_2}\right)^2 + \frac{\hbar^2}{6}\log(a_D) - \frac{7\hbar^4}{2880a_D^2} + \cdots\right. \\ &\left.+ \sum_{k=0}^{\infty}\sum_{n=1}^{\infty}\Lambda_2^2\left(\frac{\hbar}{\Lambda_2}\right)^{2k}\mathcal{F}_{D2}^{(2k,n)}\left(\frac{a_D}{\Lambda_2}\right)^n\right] \end{aligned} \tag{5.105}$$

where the first several coefficients $\mathcal{F}_{D2}^{(2k,n)}$ are listed in the table 5.3.

$N_f=3$

For the $N_f=3$ theory, the massless monopole point is $u_0=0$. Then the expansions of w_3 and $(-D_3)^{-\frac{1}{4}}$ are given by

$$w_3 = \frac{2^{22}\cdot 3^3}{\Lambda_3^8}\hat{u}^4 + \frac{2^{31}\cdot 3^3}{\Lambda_3^{10}}\hat{u}^5 + \frac{2^{34}\cdot 3^5\cdot 5}{\Lambda_3^{12}}\hat{u}^6 + \cdots, \tag{5.106}$$

$$(-D_3)^{-\frac{1}{4}} = \frac{4}{\Lambda_3} + \frac{256}{\Lambda_3^3}\hat{u} + \frac{36864}{\Lambda_3^5}\hat{u}^2 + \cdots. \tag{5.107}$$

k	$\mathcal{F}_{D2}^{(2k,1)}$	$\mathcal{F}_{D2}^{(2k,2)}$	$\mathcal{F}_{D2}^{(2k,3)}$	$\mathcal{F}_{D2}^{(2k,4)}$
0	0	-6	$\frac{1}{2}\frac{1}{\tilde{c}(2)}$	$\frac{5}{64}\frac{1}{\tilde{c}(2)^2}$
1	$\frac{3}{16}\frac{1}{\tilde{c}(2)}$	$\frac{17}{256}\frac{1}{\tilde{c}(2)^2}$	$\frac{205}{6144}\frac{1}{\tilde{c}(2)^3}$	$\frac{315}{16384}\frac{1}{\tilde{c}(2)^4}$
2	$\frac{135}{32768}\frac{1}{\tilde{c}(2)^3}$	$\frac{2943}{524288}\frac{1}{\tilde{c}(2)^4}$	$\frac{69001}{10485760}\frac{1}{\tilde{c}(2)^5}$	$\frac{1422949}{201326592}\frac{1}{\tilde{c}(2)^6}$

Table 5.3: The coefficients of the dual prepotential for the $N_f = 2$ theory, where $\tilde{c}(2) = -i2^{-\frac{5}{2}}$ [77].

Then we obtain

$$\begin{aligned} a_D(u) =& i\left(\frac{2^{\frac{3}{2}}\hat{u}}{\Lambda_3} + \frac{2^{\frac{13}{2}}\hat{u}^2}{\Lambda_3^3} + \frac{2^{11}\cdot 3\hat{u}^3}{\Lambda_3^5} + \cdots\right) \\ &+ \frac{i\hbar^2}{\Lambda_3}\left(\frac{1}{2^{\frac{1}{2}}} + 2^{\frac{13}{2}}\left(\frac{\hat{u}}{\Lambda_3^2}\right) + 2^{19}\cdot 5^2\left(\frac{\hat{u}}{\Lambda_3^2}\right)^2 + \cdots\right) \\ &+ \frac{i\hbar^4}{\Lambda_3^3}\left(2^{\frac{5}{2}}\cdot 5 + 2^{\frac{17}{2}}\cdot 43\left(\frac{\hat{u}}{\Lambda_3^2}\right) + 2^{\frac{25}{2}}\cdot 1141\left(\frac{\hat{u}}{\Lambda_3^2}\right)^2 + \cdots\right), \end{aligned} \tag{5.108}$$

$$\begin{aligned} a(u) =& \frac{i}{2\pi}\left[4a_D(\hat{u})\log\frac{16\hat{u}}{\Lambda_3^2} + i\left(-\frac{2^{\frac{7}{2}}\hat{u}}{\Lambda_3} + \frac{2^{\frac{15}{2}}\cdot 3\hat{u}^2}{\Lambda_3^3} + \frac{2^{\frac{29}{2}}\cdot 3\hat{u}^3}{\Lambda_3^5} + \cdots\right)\right. \\ &+ \frac{i\hbar^2}{\Lambda_3}\left(-\frac{1}{2^{\frac{7}{2}}}\left(\frac{\hat{u}}{\Lambda_3^2}\right)^{-1} + \frac{2^{\frac{7}{2}}}{3} + \frac{2^{\frac{13}{2}}\cdot 29}{3}\left(\frac{\hat{u}}{\Lambda_3^2}\right) + \cdots\right) \\ &\left. + \frac{i\hbar^4}{\Lambda_3}\left(\frac{7}{2^{\frac{21}{2}}\cdot 3^2\cdot 5}\left(\frac{\hat{u}}{\Lambda_3^2}\right)^{-3} - \frac{1}{2^{\frac{9}{2}}\cdot 3\cdot 5}\left(\frac{\hat{u}}{\Lambda_3^2}\right)^{-2} + \frac{7}{2^{\frac{3}{2}}\cdot 5}\left(\frac{\hat{u}}{\Lambda_3^2}\right)^{-1} + \cdots\right)\right]. \end{aligned} \tag{5.109}$$

We then have the expansions of the dual prepotential around the massless monopole point:

$$\begin{aligned} \mathcal{F}_{D3}(a_D,\hbar) =& \frac{i}{8\pi}\left[4a_D^2\log\left(\frac{a_D}{\Lambda_3}\right)^2 + \frac{\hbar^2}{3}\log(a_D) - \frac{7\hbar^4}{1440a_D^2} + \cdots\right. \\ &\left. + \sum_{k=0}^{\infty}\sum_{n=1}^{\infty}\Lambda_3^2\left(\frac{\hbar}{\Lambda_3}\right)^{2k}\mathcal{F}_{D3}^{(2k,n)}\left(\frac{a_D}{\Lambda_3}\right)^n\right] \end{aligned} \tag{5.110}$$

where the first several coefficients of $\mathcal{F}_{D3}^{(2k,n)}$ are listed in the table 5.4.

k	$\mathcal{F}_{D3}^{(2k,1)}$	$\mathcal{F}_{D3}^{(2k,2)}$	$\mathcal{F}_{D3}^{(2k,3)}$	$\mathcal{F}_{D3}^{(2k,4)}$
0	0	-12	$\frac{1}{\tilde{c}(3)}$	$\frac{5}{32}\frac{1}{\tilde{c}(3)^2}$
1	$-\frac{1}{8}\frac{1}{\tilde{c}(3)}$	$-\frac{5}{128}\frac{1}{\tilde{c}(3)^2}$	$-\frac{19}{1024}\frac{1}{\tilde{c}(3)^3}$	$-\frac{85}{8192}\frac{1}{\tilde{c}(3)^4}$
2	$\frac{37}{49152}\frac{1}{\tilde{c}(3)^3}$	$\frac{239}{262144}\frac{1}{\tilde{c}(3)^4}$	$\frac{5221}{5242880}\frac{1}{\tilde{c}(3)^5}$	$\frac{102949}{100663296}\frac{1}{\tilde{c}(3)^6}$

Table 5.4: The coefficients of the dual prepotential for the $N_f=3$ theory, where $\tilde{c}(3)=i2^{-\frac{13}{2}}$ [77].

The dual prepotentials have the perturbative corrections as (5.65), (5.68) and (5.69) in the weak coupling region. These terms also arise in the $SU(2)$ pure Yang-Mills theory [58].

The deformed massless monopole point U_0 in the u-plane can be computed from the expansion of a_D. We then have

$$U_0=\begin{cases}\Lambda_0^2-\dfrac{1}{32}\hbar^2+\dfrac{9}{32768\Lambda_0^2}\hbar^4+\cdots, & N_f=0\\ -\dfrac{3\Lambda_1^2}{2^{\frac{8}{3}}}-\dfrac{5}{72}\hbar^2-\dfrac{1571}{2^{\frac{22}{3}}\,3^7\Lambda_1^2}\hbar^4+\cdots, & N_f=1\\ \dfrac{\Lambda_2^2}{8}-\dfrac{1}{8}\hbar^2+\dfrac{9}{256\Lambda_2^2}\hbar^4+\cdots, & N_f=2\\ -\dfrac{1}{4}\hbar^2-\dfrac{4}{\Lambda_3^2}\hbar^4+\cdots, & N_f=3.\end{cases} \tag{5.111}$$

In the next subsection, we will compute the expansion around the massless monopole point u_0 for the theory where all the hypermultiplets have the same mass.

5.5.2 Massive hypermultiplets with the same mass

For the same mass case $m := m_1 = \cdots = m_{N_f}$, one of the solutions of the discriminant $\Delta_{N_f} = 0$ corresponds to the classical massless monopole point u_0, given by

$$u_0 = \frac{-64m^4 - 216\Lambda_1^3 m + 8m^2 H_1^{\frac{1}{3}} - H_1^{\frac{2}{3}}}{24H_1^{\frac{1}{3}}}, \qquad \text{for } N_f = 1, \tag{5.112}$$

$$u_0 = -\frac{\Lambda_2^2}{8} + \Lambda_2 m, \qquad \text{for } N_f = 2, \tag{5.113}$$

$$u_0 = \frac{1}{512}\left(\Lambda_3^2 - 96\Lambda_3 m + \sqrt{\Lambda_3 (\Lambda_3 + 64m)^3}\right), \qquad \text{for } N_f = 3 \tag{5.114}$$

where

$$H_1 = 729\Lambda_1^6 - 512m^6 + 4320\Lambda_1^3 m^3 + 3\sqrt{3}\left(27\Lambda_1^4 - 64\Lambda_1 m^3\right)^{\frac{3}{2}}. \tag{5.115}$$

These points are consistent in the decoupling limit to the $N_f = 0$ theory: $m \to \infty$ and $\Lambda_{N_f} \to 0$ with $m^{N_f}\Lambda_{N_f}^{(4-N_f)} = \Lambda_0^4$ being fixed, where the massless monopole point for the $N_f = 0$ theory is Λ_0^2. In the massless limit $m \to 0$, the massless monopole points correspond to those for the massless N_f theories.

We start by discussing the $N_f = 1$ theory. Here we consider the small mass $|m| \ll \Lambda_1$, around which u_0 is expanded as [85]

$$u_0 = -\frac{3\Lambda_1^2}{2^{\frac{8}{3}}} - \frac{\Lambda_1 m}{2^{\frac{1}{3}}} + \frac{m^2}{3} + \cdots . \tag{5.116}$$

From (2.109), the expansion of the SW period $a_D^{(0)}$ around $u = u_0$ is given by

$$a_D^{(0)}(\hat{u}) = \hat{u}\left(\frac{1}{2^{\frac{1}{6}} \cdot 3^{\frac{1}{2}}\Lambda_1} - \frac{2^{\frac{3}{2}}m^2}{3^{\frac{7}{2}}\Lambda_1^3} + \cdots\right) + \hat{u}^2\left(\frac{1}{2^{\frac{1}{2}} \cdot 3^{\frac{5}{2}}\Lambda_1^3} + \frac{2^{\frac{17}{6}}m}{3^{\frac{7}{2}}\Lambda_1^4} + \cdots\right) + \cdots , \tag{5.117}$$

where $\hat{u} = u - u_0$. By using the relations (5.36) and (5.37), we obtain the second and fourth order corrections to the SW periods around $u = u_0$:

$$a_D^{(2)}(\hat{u}) = \left(\frac{5}{2^{\frac{13}{6}} \cdot 3^{\frac{5}{2}}\Lambda_1} - \frac{m}{2^{\frac{5}{6}} \cdot 3^{\frac{7}{2}}\Lambda_1^2} + \cdots\right) + \hat{u}\left(\frac{35}{2^{\frac{7}{2}} \cdot 3^{\frac{9}{2}}\Lambda_1^3} + \frac{5m}{2^{\frac{1}{6}} \cdot 3^{\frac{11}{2}}\Lambda_1^4} + \cdots\right) + \cdots , \tag{5.118}$$

$$a_D^{(4)}(\hat{u}) = \left(\frac{2471}{6^{\frac{15}{2}}\Lambda_1^3} - \frac{613m}{2^{\frac{31}{6}} \cdot 3^{\frac{15}{2}}\Lambda_1^4} + \cdots\right) + \hat{u}\left(\frac{144347}{2^{\frac{53}{6}} \cdot 3^{\frac{19}{2}}\Lambda_1^5} + \frac{26495m}{2^{\frac{9}{2}} \cdot 3^{\frac{21}{2}}\Lambda_1^6} + \cdots\right) + \cdots . \tag{5.119}$$

From above expansions, we have the quantum SW period $a_D = \sum \hbar^k a_D^{(k)}$ up to fourth order in $\hbar$. Then the monopole massless point U_0 is found to be (5.93) where

$$
\begin{aligned}
u_0 &= -\frac{3\Lambda_1^2}{2^{\frac{8}{3}}} - \frac{\Lambda_1 m}{2^{\frac{1}{3}}} + \frac{m^2}{3} + \cdots , \\
u_1 &= -\frac{5}{2^3 \cdot 3^2} + \frac{m}{2^{\frac{2}{3}} \cdot 3^3 \Lambda_1} + \frac{5m^2}{2^{\frac{1}{3}} \cdot 3^4 \Lambda_1^2} + \cdots , \\
u_2 &= -\frac{1571}{2^{\frac{22}{3}} \cdot 3^7 \Lambda_1^2} + \frac{613m}{2^5 \cdot 3^7 \Lambda_1^3} + \frac{11329m^2}{2^{\frac{11}{3}} \cdot 3^9 \Lambda_1^4} + \cdots ,
\end{aligned}
\tag{5.120}
$$

for the small mass.

In a similar way, we can obtain the massless monopole point U_0 in the deformed theory for $N_f = 2$ and 3. For $N_f = 2$, the massless monopole point U_0 is given by (5.93) where

$$
\begin{aligned}
u_0 &= -\frac{\Lambda_2^2}{8} + \Lambda_2 m, \\
u_1 &= -\frac{m - 2\Lambda_2}{32m - 16\Lambda_2}, \\
u_2 &= \frac{9\left(-8\Lambda_2^3 + m^3 - 2\Lambda_2 m^2 - 26\Lambda_2^2 m\right)}{2048\Lambda_2 \left(\Lambda_2 - 2m\right)^4}.
\end{aligned}
\tag{5.121}
$$

In the case of the small mass $|m| \ll \Lambda_2$, we find

$$
\begin{aligned}
u_0 &= -\frac{\Lambda_2^2}{8} + \Lambda_2 m, \\
u_1 &= -\frac{1}{8} - \frac{3m}{16\Lambda_2} - \frac{3m^2}{8\Lambda_2^2} + \cdots , \\
u_2 &= -\frac{9}{256\Lambda_2^2} - \frac{405m}{1024\Lambda_2^3} - \frac{2385m^2}{1024\Lambda_2^4} + \cdots .
\end{aligned}
\tag{5.122}
$$

For $N_f = 3$ with $|m| \ll \Lambda_3$, the massless monopole point U_0 is (5.93) where

$$
\begin{aligned}
u_0 &= -\frac{3\Lambda_3 m}{8} - 3m^2 + \cdots , \\
u_1 &= -\frac{1}{4} + \frac{6m}{\Lambda_3} - \frac{336m^2}{\Lambda_3^2} + \cdots , \\
u_2 &= -\frac{4}{\Lambda_3^2} + \frac{888m}{\Lambda_3^3} - \frac{131904m^2}{\Lambda_3^4} + \cdots .
\end{aligned}
\tag{5.123}
$$

Note that the first terms in the expansions of u_1 and u_2 agree with those in the massless limit.

In a similar calculation, we can obtain U_0 up to the fourth order in $\hbar$ for general m. We find that the massless monopole point is modified by the quantum correction in $\hbar$. In Fig. 5.1 , we have shown the graphs of the deformed massless monopole point as a function of $\frac{m}{\Lambda_{N_f}}$ with $\hbar = 1$. For $N_f = 2$, U_0 has singular at the superconformal point where $\frac{m}{\Lambda_2} = \frac{1}{2}$. This is because the ratios of $\mathcal{J}_n^{(k)}$ in (5.94) and (5.95) are divergent. However, for $N_f = 1$ and 3 their ratios remain finite. In order to study precisely the quantum SW periods around the superconformal point, we need to take the scaling limit of the Coulomb moduli and the mass parameters around the superconformal point. In the next chapter, we will discuss the quantum SW periods for the Argyres-Douglas theory.

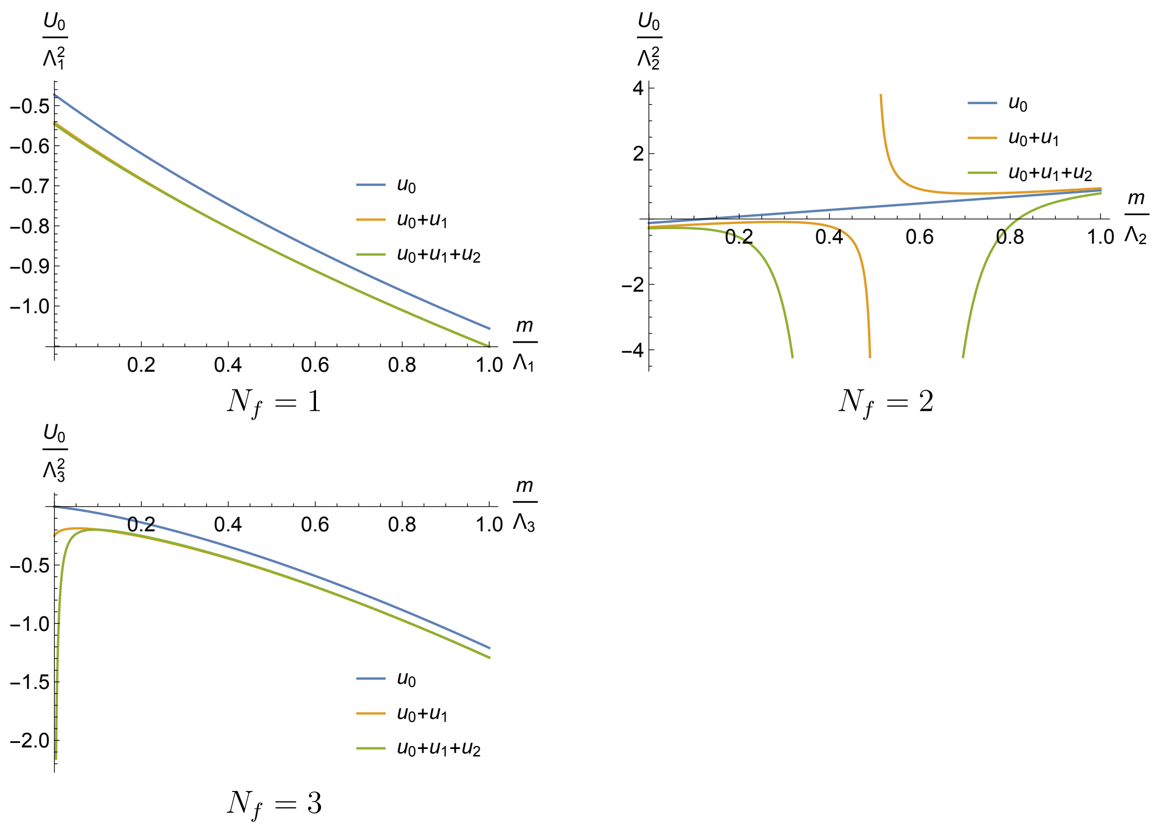

Figure 5.1: The graphs of u_0, $u_0 + \hbar^2 u_1$ and $u_0 + \hbar^2 u_1 + \hbar^4 u_2$ with respect to $\frac{m}{\Lambda_{N_f}}$ for $N_f = 1, 2$ and 3 where we choose $\hbar = 1$.

Summary

In this chapter, we argued the quantization of the SW curve for the $SU(2)$ gauge theory with $N_f(= 1, \dots 4)$ hypermultiplets. The quantum SW curve becomes the Schrödinger type equation. The quantum corrections to the SW periods are obtained by acting some differential operators on the SW periods. We calculated the expansion of the quantum SW periods in the weak coupling region and confirmed that the quantum prepotential agrees with that obtained from the NS limit of the Nekrasov partition function up to fourth order in $\hbar$. We also computed the expansion of the quantum SW periods around the massless monopole point up to fourth order in $\hbar$. We find that the massless monopole point is shifted by the quantum correction.

Chapter 6

Quantum periods for Argyres-Douglas theory

In the previous chapter, we have shown that the WKB solutions of the quantum SW curve at the weak coupling correspond to those for the Ω-deformed theory in the NS limit. We computed the quantum SW periods around the massless monopole point for $SU(2)$ SQCD. We then found the massless monopole point is shifted by the quantum correction. In this chapter, we will extend the above discussion to the Argyres-Douglas (AD) theory and obtain the quantum SW periods in terms of hypergeometric function up to the fourth order in $\hbar$.

6.1 $SU(2)$ SQCD around superconformal point

We will study the quantum SW periods around the superconformal point of the moduli space of $\mathcal{N}=2$ $SU(2)$ SQCD with $N_f = 1, 2, 3$ hypermultiplets. Around the superconformal point, the SW curve for N_f theories degenerates into a curve of a common cubic form (3.23), corresponding to the SW curve for the AD theory. From (3.12), (3.15) and (3.20), however, their SW differentials take different forms due to the flavor symmetry. This means that we need to introduce different quantization conditions for each N_f. In the following, we will quantize the SW curve for each N_f and construct the higher order corrections to the SW periods up to fourth order in $\hbar$.

6.1.1 $N_f = 1$

For $N_f = 1$, the differential (3.12) gives a symplectic form $d\tilde{\lambda}_{\mathrm{SW}} = d\tilde{z} \wedge d\tilde{p}$ on the $(\tilde{z}, \tilde{p})$ space. The quantization of the SW curve, given by (3.10), is performed by replacing the coordinate $\tilde{z}$ by the differential operator:

$$\tilde{z} = -i\hbar \frac{\partial}{\partial \tilde{p}}. \tag{6.1}$$

Then the quantum SW curve is given by the Schrödinger type equation:

$$\left(-\hbar^2 \frac{\partial^2}{\partial \tilde{p}^2} + Q(\tilde{p})\right) \Psi(\tilde{p}) = 0, \tag{6.2}$$

where

$$Q(\tilde{p}) = -\left(\tilde{p}^3 - \tilde{M}\Lambda_1 \tilde{p} - \frac{\Lambda_1}{2}\tilde{u}\right). \tag{6.3}$$

The WKB solution to the equation (6.2) takes the same form as (5.9) in terms of $\tilde{p}$. In a similar way to the previous chapter, we also find the second and fourth order corrections are of the same form as (5.16) and (5.17) with $Q_2 = 0$ up to total derivatives. The quantum SW periods are defined by

$$\tilde{\Pi} = (\tilde{a}, \tilde{a}_D) = \left(\oint_{\tilde{\alpha}} \Phi(\tilde{p}) d\tilde{p}, \int_{\tilde{\beta}} \Phi(\tilde{p}) d\tilde{p}\right), \tag{6.4}$$

with $\tilde{\alpha}$ and $\tilde{\beta}$ being the canonical 1-cycles. The periods are expanded in $\hbar$ as

$$\tilde{\Pi} = \tilde{\Pi}^{(0)} + \hbar^2 \tilde{\Pi}^{(2)} + \hbar^4 \tilde{\Pi}^{(4)} + \cdots, \tag{6.5}$$

where $\tilde{\Pi}^{(2n)} := \oint \phi_{2n}(\tilde{p}) d\tilde{p}$ with $\tilde{\Pi}^{(0)}$ denoting the classical SW period.

We will consider the differential equation which the quantum SW period obeys. It is found that the SW periods $\tilde{\Pi}^{(0)}$ satisfy not only the Picard-Fuchs equation (3.43) but also the differential equation with respect to $\tilde{M}$ and $\tilde{u}$:

$$\frac{\partial^2}{\partial \tilde{M} \partial \tilde{u}} \tilde{\Pi}^{(0)} = -\frac{3\tilde{u}}{2\tilde{M}} \frac{\partial^2}{\partial \tilde{u}^2} \tilde{\Pi}^{(0)} - \frac{1}{4\tilde{M}} \frac{\partial}{\partial \tilde{u}} \tilde{\Pi}^{(0)}. \tag{6.6}$$

Substituting (6.3) into (5.16) and (5.17) and integrating over $\tilde{p}$, we obtain the second and fourth order corrections to the SW periods in $\hbar$. We then find

$$\tilde{\Pi}^{(2)} = \frac{1}{\Lambda_1^2} \frac{\partial}{\partial \tilde{M}} \frac{\partial}{\partial \tilde{u}} \tilde{\Pi}^{(0)}, \tag{6.7}$$

$$\tilde{\Pi}^{(4)} = \frac{7}{10\Lambda_1^4} \frac{\partial^2}{\partial \tilde{M}^2} \frac{\partial^2}{\partial \tilde{u}^2} \tilde{\Pi}^{(0)}. \tag{6.8}$$

We note that the higher order corrections can be obtained from those for the $N_f = 1$ $SU(2)$ theory in the scaling limit around the superconformal point. The second and fourth order corrections to the SW periods for the $N_f = 1$ theory have been obtained by (5.36) and (5.37), respectively. It can be checked that, by taking the scaling limit (3.9), the leading orders in ϵ to the quantum SW periods (5.36) and (5.37) correspond to (6.7) and (6.8), respectively. Since the quantization conditions for the AD theories become different to those for the SQCD, it is nontrivial to check that the quantum SW periods of the AD theories agree with those of the SQCD in the scaling limit. In the subsection 6.1.4, we will compute the quantum SW periods around the superconformal point by applying the relations (6.7) and (6.8) up to fourth order.

6.1.2 $N_f = 2$

For $N_f = 2$, since the SW differential is given by (3.16), we need to introduce a new variable ξ by

$$\tilde{p} = e^{\xi} - \frac{2}{3}\tilde{M}, \tag{6.9}$$

such that the SW differential (3.15) takes a canonical form

$$\tilde{\lambda}_{\mathrm{SW}} = \tilde{z} d\xi. \tag{6.10}$$

Then the SW curve (3.14) becomes

$$\tilde{z}^2 - \left(e^{3\xi} - 2\tilde{M} e^{2\xi} + e^{\xi} \left(\frac{4\tilde{M}^2}{3} - \tilde{u} \right) - \frac{\Lambda_2 \tilde{C}_2}{4} \right) = 0. \tag{6.11}$$

The quantum SW curve is obtained by

$$\left(-\hbar^2 \frac{\partial^2}{\partial \xi^2} + Q(\xi) \right) \Psi(\xi) = 0, \tag{6.12}$$

where

$$Q(\xi) = -\left(e^{3\xi} - 2\tilde{M} e^{2\xi} + e^{\xi} \left(\frac{4\tilde{M}^2}{3} - \tilde{u} \right) - \frac{\Lambda_2 \tilde{C}_2}{4} \right). \tag{6.13}$$

by replacing $\tilde{z}$ by the differential operator

$$\tilde{z} = -i\hbar \frac{\partial}{\partial \xi}, \tag{6.14}$$

The WKB solution of the quantum SW curve takes (5.9) in terms of ξ, in which the leading term in $\hbar$ gives the classical SW periods $\tilde{\Pi}^{(0)} = \int \phi_0(\xi) d\xi$. The period integrals $(-\tilde{D}_2)^{\frac{1}{4}} \partial_{\tilde{u}} \tilde{\Pi}^{(0)}$ can be found to satisfy the Picard-Fuchs equation (2.75). $\tilde{\Pi}^{(0)}$ also obeys the differential equation

$$\frac{\partial^2}{\partial \tilde{M} \partial \tilde{u}} \tilde{\Pi}^{(0)} = L_2 \left(4\tilde{u} \frac{\partial^2}{\partial \tilde{u}^2} \tilde{\Pi}^{(0)} + \frac{\partial}{\partial \tilde{u}} \tilde{\Pi}^{(0)} \right), \tag{6.15}$$

where

$$L_2 := \frac{4\left(4\tilde{M}^2 - 3\tilde{u}\right)}{27\Lambda_2 \tilde{C}_2 + 24\tilde{M}\tilde{u} - 32\tilde{M}^3}. \tag{6.16}$$

By applying (5.16) and (5.17), the second and fourth order corrections are given by

$$\tilde{\Pi}^{(2)} = \left(\frac{1}{4} \frac{\partial}{\partial \tilde{M}} \frac{\partial}{\partial \tilde{u}} + \frac{\tilde{M}}{3} \frac{\partial^2}{\partial \tilde{u}^2} \right) \tilde{\Pi}^{(0)}, \tag{6.17}$$

$$\tilde{\Pi}^{(4)} = \left(\frac{7\tilde{M}^2}{90} \frac{\partial^4}{\partial \tilde{u}^4} + \frac{1}{20} \frac{\partial^3}{\partial \tilde{u}^3} + \frac{7}{160} \frac{\partial^2}{\partial \tilde{u}^2} \frac{\partial^2}{\partial \tilde{M}^2} + \frac{7\tilde{M}}{60} \frac{\partial^3}{\partial \tilde{u}^3} \frac{\partial}{\partial \tilde{M}} \right) \tilde{\Pi}^{(0)}. \tag{6.18}$$

Note that (6.17) and (6.18) are given up to the Picard-Fuchs equations. We also find that, after taking the scaling limit (3.13), the second and fourth order formulas of the $N_f = 2$ theory (5.43) and (5.44) agree with (6.17) and (6.18), respectively.

6.1.3 $N_f = 3$

Finally we discuss the quantum SW curve for the $N_f = 3$ theory. The SW differential (3.20) takes the canonical form

$$\tilde{\lambda}_{SW} = i\Lambda_3 \left(\tilde{p} d\tilde{\xi} + \sum_{i=1}^{3} \tilde{p} d \log(\tilde{p} + \tilde{m}_i) \right), \tag{6.19}$$

by introducing a new coordinate ξ:

$$\tilde{z} = -i \left(e^{\xi} + \frac{1}{2}(f_0 \tilde{p} + f_1) \right), \tag{6.20}$$

where we define

$$f_0 = \frac{4\tilde{M}}{\Lambda_3^{\frac{1}{2}}}, \qquad f_1 = \frac{8\tilde{M}^3}{3\Lambda_3^{\frac{3}{2}}} + \frac{2\tilde{u}}{\Lambda_3^{\frac{1}{2}}}, \qquad g(\tilde{p}) = \tilde{p}^3 - \rho_3 \tilde{p} - \sigma_3 + \left(\frac{2\tilde{M}\tilde{p}}{\Lambda_3^{\frac{1}{2}}} + \frac{4\tilde{M}^3}{3\Lambda_3^{\frac{3}{2}}} + \frac{\tilde{u}}{\Lambda_3^{\frac{1}{2}}} \right)^2. \tag{6.21}$$

Here the coefficients ρ_3 and σ_3 are read off from (3.18). Then the SW curve (3.18) becomes

$$e^{2\xi} + (f_0\tilde{p} + f_1)e^{\xi} + g(\tilde{p}) = 0. \tag{6.22}$$

Since the SW differential is defined a symplectic form $d\tilde{\lambda}_{\mathrm{SW}} \sim d\tilde{p} \wedge d\xi$, we quantize the SW curve by replacing the coordinate ξ as the differential operator

$$\xi = -i\hbar\frac{\partial}{\partial\tilde{p}}. \tag{6.23}$$

However, we need to consider the ordering of the operators, which defined by

$$t\tilde{p}e^{-i\hbar\partial_{\tilde{p}}}\Psi(\tilde{p}) + e^{-i\hbar\partial_{\tilde{p}}}\left((1-t)\tilde{p}\Psi(\tilde{p})\right) = (\tilde{p} - i(1-t)\hbar)e^{-i\hbar\partial_{\tilde{p}}}\Psi(\tilde{p}), \tag{6.24}$$

with the parameter t $(0 \leq t \leq 1)$. After taking the $t = \frac{1}{2}$ prescription as in the previous section [46], we obtain the quantum SW curve (6.22)

$$\left(\exp(-2i\hbar\partial_{\tilde{p}}) + \left(\frac{1}{2}f_0\tilde{p} + f_1\right)\exp(-i\hbar\partial_{\tilde{p}}) + \exp(-i\hbar\partial_{\tilde{p}})\frac{1}{2}f_0\tilde{p} + g(\tilde{p})\right)\Psi(\tilde{p}) = 0. \tag{6.25}$$

We consider the WKB solution to the quantum curve, where the leading term $\phi_0(\tilde{p}) := \xi(\tilde{p})$ leads to the SW periods. In order to obtain the higher order terms in $\hbar$, we convert the quantum SW curve (6.25) into

$$J(2) + \left(f_0\left(\tilde{p} - \frac{i}{2}\hbar\right) + f_1\right)J(1) + g(x) = 0, \tag{6.26}$$

by introducing

$$J(\alpha) := \exp\left(-\frac{i}{\hbar}\int^{\tilde{p}}\Phi(y)dy\right)\exp\left(-i\hbar\alpha\frac{\partial}{\partial\tilde{p}}\right)\exp\left(\frac{i}{\hbar}\int^{\tilde{p}}\Phi(y)dy\right). \tag{6.27}$$

After taking $\Phi(\tilde{p})$ in (6.26) as the form (5.9) in terms of $\tilde{p}$ we obtain the recursion relation of ϕ_n's. The leading correction $\phi_0(\tilde{p})$ is given by

$$\phi_0(\tilde{p}) = \log\left(\frac{1}{2}\left(-f_0\tilde{p} - f_1 + 2\tilde{y}\right)\right) \tag{6.28}$$

whose integration corresponds to the SW period. Here $\tilde{y}$ is defined by

$$\tilde{y}^2 = \frac{1}{4}(f_0\tilde{p} + f_1)^2 - g(\tilde{p}). \tag{6.29}$$

$\phi_1(\tilde{p})$ becomes the total derivative:

$$\phi_1(\tilde{p}) = \frac{\partial}{\partial \tilde{p}} \left(\frac{i}{2}\phi_0(\tilde{p}) + \frac{i}{4}\log 4\tilde{y} \right). \tag{6.30}$$

$\phi_3(\tilde{p})$ is also shown to be a total derivative. ϕ_2 and ϕ_4 are obtained by

$$\phi_2(\tilde{p}) = \frac{(-f_0\tilde{p} - f_1)\, g''(\tilde{p})}{96\tilde{y}^3} + \frac{f_0^2\,(f_0\tilde{p} + f_1)}{192\tilde{y}^3}, \tag{6.31}$$

$$\begin{aligned}\phi_4(\tilde{p}) =& g^{(4)}(\tilde{p}) \left(\frac{(f_0\tilde{p} + f_1)\, g(\tilde{p})}{1536\tilde{y}^5} + \frac{-f_0\tilde{p} - f_1}{5760\tilde{y}^3} \right) + g^{(3)}(\tilde{p}) \left(\frac{f_0 g(\tilde{p})}{480\tilde{y}^5} + \frac{f_0}{720\tilde{y}^3} \right) \\ &+ g''(\tilde{p}) \left(-\frac{7f_0^2\,(f_0\tilde{p} + f_1)\, g(\tilde{p})}{3072\tilde{y}^7} - \frac{7f_0^2\,(f_0\tilde{p} + f_1)}{7680\tilde{y}^5} \right) \\ &+ g''(\tilde{p})^2 \left(\frac{7\,(f_0\tilde{p} + f_1)\, g(\tilde{p})}{3072\tilde{y}^7} + \frac{7\,(f_0\tilde{p} + f_1)}{7680\tilde{y}^5} \right) + \frac{7f_0^4\,(f_0\tilde{p} + f_1)\, g(\tilde{p})}{12288\tilde{y}^7} + \frac{7f_0^4\,(f_0\tilde{p} + f_1)}{30720\tilde{y}^5},\end{aligned} \tag{6.32}$$

up to the total derivative.

For the classical SW periods $\tilde{\Pi}^{(0)}$, $(-\tilde{D}_3)^{\frac{1}{4}}\partial_{\tilde{u}}\tilde{\Pi}^{(0)}$ obeys the Picard-Fuchs equation (2.75). we also find $\tilde{\Pi}^{(0)}$ satisfies the differential equation with respect to $\tilde{M}$ and $\tilde{u}$:

$$\frac{\partial^2}{\partial \tilde{M} \partial \tilde{u}} \tilde{\Pi}^{(0)} = b_3 \frac{\partial^2}{\partial \tilde{u}^2} \tilde{\Pi}^{(0)} + c_3 \frac{\partial}{\partial \tilde{u}} \tilde{\Pi}^{(0)} \tag{6.33}$$

where

$$b_3 = \frac{4\tilde{M}\left(3\Lambda_3\tilde{M}\tilde{u} + 4\tilde{M}^4 - 3\Lambda_3^2\rho_3\right)\rho_3 + 27\Lambda_3^2\tilde{u}\sigma_3}{3\Lambda_3\left(9\Lambda_3\tilde{M}\sigma_3 - 4\tilde{M}^3\rho_3 - 3\Lambda_3\tilde{u}\rho_3\right)}, \tag{6.34}$$

$$c_3 = \frac{\left(4\tilde{M}^3 + 3\Lambda_3\tilde{u}\right)^2 - 12\Lambda_3^2\tilde{M}^2\rho_3}{3\Lambda_3\left(9\Lambda_3\tilde{M}\sigma_3 - 4\tilde{M}^3\rho_3 - 3\Lambda_3\tilde{u}\rho_3\right)}. \tag{6.35}$$

Substituting (6.21) into (6.31) and (6.32) we find that the second and fourth order corrections in $\hbar$ can be computed by applying the relations as

$$\tilde{\Pi}^{(2)} = \left(-\frac{\tilde{M}^2}{12}\frac{\partial^2}{\partial \tilde{u}^2} - \frac{\Lambda_3}{16}\frac{\partial}{\partial \tilde{u}}\frac{\partial}{\partial \tilde{M}} \right) \tilde{\Pi}^{(0)}, \tag{6.36}$$

$$\tilde{\Pi}^{(4)} = \left(\frac{7\tilde{M}^4}{1440}\frac{\partial^4}{\partial \tilde{u}^4} + \frac{\Lambda_3\tilde{M}}{192}\frac{\partial^3}{\partial \tilde{u}^3} + \frac{7\Lambda_3^2}{2560}\frac{\partial^2}{\partial \tilde{u}^2}\frac{\partial^2}{\partial \tilde{M}^2} + \frac{7\Lambda_3\tilde{M}^2}{960}\frac{\partial^3}{\partial \tilde{u}^3}\frac{\partial}{\partial \tilde{M}} \right) \tilde{\Pi}^{(0)}. \tag{6.37}$$

These formulas coincide with those of the scaling limit (3.17) for the $SU(2)$ gauge theory with $N_f = 3$ hypermultiplets: (5.49) and (5.50).

In the next subsection, we will compute the expressions of the quantum corrections to the SW periods around the superconformal point up to fourth order in $\hbar$.

6.1.4 Calculation of Quantum SW periods around superconformal point

In the previous subsection, we have investigated the quantum SW curves and the relation between the quantum SW periods and the classical SW periods in the AD theory. In this subsection, we will compute explicitly the quantum corrections to the SW periods around the superconformal point up to the fourth order in $\hbar$. We will study the expansion in the coupling constant of the relevant operator and the mass parameters for the AD theory.

$N_f = 1$ theory

We first discuss the $N_f = 1$ theory around the superconformal point. The SW periods around the superconformal point are expressed as (3.41) and (3.42). Substituting them into (6.7) and changing the variables $(\tilde{u}, \tilde{M})$ to $(\tilde{u}, \tilde{w}_1')$, we obtain the second order corrections to the SW periods in terms of hypergeometric function as

$$\tilde{a}^{(2)} = \frac{1}{2^{\frac{5}{2}} \cdot 3^{\frac{3}{2}} \pi^{\frac{1}{2}} \Lambda_1^{\frac{7}{2}}} \left(\frac{\tilde{u}}{\Lambda_1^2} \right)^{-\frac{5}{6}} \left(F_1^{(2)}(\tilde{w}_1') - F_2^{(2)}(\tilde{w}_1') \right), \tag{6.38}$$

$$\tilde{a}_D^{(2)} = \frac{1}{2^{\frac{5}{2}} \cdot 3^{\frac{3}{2}} \pi^{\frac{1}{2}} \Lambda_1^{\frac{7}{2}}} \left(\frac{\tilde{u}}{\Lambda_1^2} \right)^{-\frac{5}{6}} \left((-1)^{\frac{2}{3}} F_1^{(2)}(\tilde{w}_1') + (-1)^{\frac{1}{3}} F_2^{(2)}(\tilde{w}_1') \right), \tag{6.39}$$

where

$$F_1^{(2)}(\tilde{w}_1') = 2^{\frac{7}{3}} \cdot 3\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{5}{6}\right)\left(F\left(\frac{5}{12}, \frac{11}{12}; \frac{4}{3}; \tilde{w}_1'\right) - 5F\left(\frac{11}{12}, \frac{17}{12}; \frac{4}{3}; \tilde{w}_1'\right)\right), \tag{6.40}$$

$$F_2^{(2)}(\tilde{w}_1') = -7\tilde{w}_1'^{\frac{2}{3}}\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right)F\left(\frac{13}{12}, \frac{19}{12}; \frac{5}{3}; \tilde{w}_1'\right). \tag{6.41}$$

Similarly, applying (6.8) and changing the variables $(\tilde{u}, \tilde{M})$ to $(\tilde{u}, \tilde{w}_1')$, the fourth order corrections to the SW periods (6.5) are expressed as

$$\tilde{a}^{(4)} = -\frac{7}{2^{\frac{43}{6}}\cdot 3^{\frac{5}{2}}\cdot 5\pi^{\frac{1}{2}}\Lambda_1^{\frac{17}{2}}}\frac{\tilde{w}_1'^{\frac{1}{3}}}{(\tilde{w}_1'-1)}\left(\frac{\tilde{u}}{\Lambda_1^2}\right)^{-\frac{5}{2}}\left(-F_1^{(4)}(\tilde{w}_1')+F_2^{(4)}(\tilde{w}_1')\right), \tag{6.42}$$

$$\tilde{a}_D^{(4)} = -\frac{7}{2^{\frac{43}{6}}\cdot 3^{\frac{5}{2}}\cdot 5\pi^{\frac{1}{2}}\Lambda_1^{\frac{17}{2}}}\frac{\tilde{w}_1'^{\frac{1}{3}}}{(\tilde{w}_1'-1)}\left(\frac{\tilde{u}}{\Lambda_1^2}\right)^{-\frac{5}{2}}\left((-1)^{\frac{1}{3}}F_1^{(4)}(\tilde{w}_1')+(-1)^{\frac{2}{3}}F_2^{(4)}(\tilde{w}_1')\right), \tag{6.43}$$

where

$$F_1^{(4)}(\tilde{w}_1') = 2^3\cdot 7\cdot 13\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{7}{6}\right)\left((11\tilde{w}_1'+13)F\left(\frac{19}{12},\frac{25}{12};\frac{5}{3};\tilde{w}_1'\right)-5F\left(\frac{13}{12},\frac{19}{12};\frac{5}{3};\tilde{w}_1'\right)\right), \tag{6.44}$$

$$F_2^{(4)}(\tilde{w}_1') = 2^{\frac{1}{3}}\cdot 5\cdot 11\cdot 17\tilde{w}_1'^{\frac{1}{3}}\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{5}{6}\right)\left((7\tilde{w}_1'+17)F\left(\frac{23}{12},\frac{29}{12};\frac{7}{3};\tilde{w}_1'\right)-F\left(\frac{17}{12},\frac{23}{12};\frac{7}{3};\tilde{w}_1'\right)\right). \tag{6.45}$$

Expanding the quantum SW periods around $\tilde{w}_1' = 0$, we have the expansions of them as

$$\begin{aligned}\tilde{a} =& \Lambda_1^{\frac{3}{2}}\left(\frac{\tilde{u}}{\Lambda_1^2}\right)^{\frac{5}{6}}\left(-\frac{2^{\frac{7}{6}}\Gamma\left(-\frac{5}{6}\right)\Gamma\left(\frac{1}{3}\right)}{3^{\frac{1}{2}}\pi^{\frac{1}{2}}}-\frac{7\Gamma\left(-\frac{7}{6}\right)\Gamma\left(\frac{2}{3}\right)}{6^{\frac{1}{2}}\pi^{\frac{1}{2}}}\tilde{w}_1'^{\frac{1}{3}}+\cdots\right)\\ &+\frac{\hbar^2}{\Lambda_1^{\frac{7}{2}}}\left(\frac{\tilde{u}}{\Lambda_1^2}\right)^{-\frac{5}{6}}\left(-\frac{7\Gamma\left(-\frac{7}{6}\right)\Gamma\left(\frac{2}{3}\right)}{2^{\frac{1}{6}}\cdot 3^{\frac{5}{2}}\pi^{\frac{1}{2}}}+\cdots\right)\\ &+\frac{\hbar^4}{\Lambda_1^{\frac{17}{2}}}\left(\frac{\tilde{u}}{\Lambda_1^2}\right)^{-\frac{5}{2}}\left(\frac{7^2\cdot 13\Gamma\left(-\frac{5}{6}\right)\Gamma\left(\frac{1}{3}\right)}{2^{\frac{19}{6}}\cdot 3^{\frac{9}{2}}\pi^{\frac{1}{2}}}\tilde{w}_1'^{\frac{1}{3}}+\cdots\right)+\cdots,\end{aligned} \tag{6.46}$$

$$\begin{aligned}\tilde{a}_D =& \Lambda_1^{\frac{3}{2}}\left(\frac{\tilde{u}}{\Lambda_1^2}\right)^{\frac{5}{6}}\left(\frac{2^{\frac{7}{6}}(-1)^{\frac{1}{3}}\Gamma\left(-\frac{5}{6}\right)\Gamma\left(\frac{1}{3}\right)}{3^{\frac{1}{2}}\pi^{\frac{1}{2}}}-\frac{7(-1)^{\frac{2}{3}}\Gamma\left(-\frac{7}{6}\right)\Gamma\left(\frac{2}{3}\right)}{6^{\frac{1}{2}}\pi^{\frac{1}{2}}}\tilde{w}_1'^{\frac{1}{3}}+\cdots\right)\\ &+\frac{\hbar^2}{\Lambda_1^{\frac{7}{2}}}\left(\frac{\tilde{u}}{\Lambda_1^2}\right)^{-\frac{5}{6}}\left(-\frac{7(-1)^{\frac{2}{3}}\Gamma\left(-\frac{7}{6}\right)\Gamma\left(\frac{2}{3}\right)}{2^{\frac{1}{6}}\cdot 3^{\frac{5}{2}}\pi^{\frac{1}{2}}}+\cdots\right)\\ &+\frac{\hbar^4}{\Lambda_1^{\frac{17}{2}}}\left(\frac{\tilde{u}}{\Lambda_1^2}\right)^{-\frac{5}{2}}\left(-\frac{7^2\cdot 13(-1)^{\frac{1}{3}}\Gamma\left(-\frac{5}{6}\right)\Gamma\left(\frac{1}{3}\right)}{2^{\frac{19}{6}}\cdot 3^{\frac{9}{2}}\pi^{\frac{1}{2}}}\tilde{w}_1'^{\frac{1}{3}}+\cdots\right)+\cdots.\end{aligned} \tag{6.47}$$

Let us define the effective coupling constant[1] $\tilde{\tau}$ of the deformed theory by

$$\tilde{\tau} := \frac{\partial_{\tilde{u}}\tilde{a}_D}{\partial_{\tilde{u}}\tilde{a}}, \tag{6.48}$$

[1]Note that the present definition of the effective coupling constant is inverse of the one in [3].

where the expansion in $\hbar$ is given by

$$\tilde{\tau} = \tilde{\tau}^{(0)} + \hbar^2 \tilde{\tau}^{(2)} + \hbar^4 \tilde{\tau}^{(4)} + \cdots . \tag{6.49}$$

From (6.46) and (6.47), the deformed effective coupling constant (6.48) is expanded around the superconformal point as

$$\begin{aligned}\tilde{\tau} = &\left(-(-1)^{\frac{1}{3}} + \frac{3^{\frac{1}{2}} \cdot 7 i \pi^{\frac{1}{2}} \Gamma\left(-\frac{7}{6}\right)}{10 \Gamma\left(-\frac{5}{6}\right) \Gamma\left(\frac{1}{6}\right)} \tilde{w}_1'^{\frac{1}{3}} + \cdots \right) \\ &+ \frac{\hbar^2}{\Lambda_1^5} \left(-\frac{2^{\frac{4}{3}} \cdot 3^{\frac{1}{2}} i \pi^{\frac{1}{2}} \Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{1}{6}\right) \Gamma\left(-\frac{5}{6}\right)} \left(\frac{\tilde{u}}{\Lambda_1^2}\right)^{-\frac{5}{3}} + \cdots \right) \\ &+ \frac{\hbar^4}{\Lambda_1^{10}} \left(-\frac{2 \cdot 3^{\frac{3}{2}} i \pi^{\frac{1}{2}} \Gamma\left(\frac{5}{6}\right)^2 \Gamma\left(\frac{5}{3}\right)}{\Gamma\left(-\frac{5}{6}\right)^2 \Gamma\left(\frac{1}{6}\right) \Gamma\left(\frac{1}{3}\right)} \left(\frac{\tilde{u}}{\Lambda_1^2}\right)^{-\frac{10}{3}} + \cdots \right) + \cdots .\end{aligned} \tag{6.50}$$

$\tilde{\tau}$ can be expressed in terms of $\tilde{a}$ by inverting (6.46). We then obtain the free energy by integrating $\tilde{\tau}$ over $\tilde{a}$ twice. We find that the free energy at $\tilde{M} = 0$ coincides with that obtained from the E-string theory [86]. We note that the present expansions for N_f theories in the coupling parameter are different from those in the self-dual Ω-background [38], which are expanded in the operator have been done with the zero coupling and without taking the scaling limit.

$N_f = 2$ theory

In the case of the $N_f = 2$ theory, applying (6.17) to (3.46), we obtain the second order corrections in terms of the hypergeometric function as

$$\tilde{a}^{(2)} = -\frac{1}{2^4 \cdot 3^{\frac{15}{4}} \pi^{\frac{1}{2}} \Lambda_2^{\frac{3}{2}}} \left(\frac{\tilde{u}}{\Lambda_2^2}\right)^{-\frac{3}{4}} \left(F_1^{(2)}(\tilde{w}_2') - F_2^{(2)}(\tilde{w}_2') \right), \tag{6.51}$$

$$\tilde{a}_D^{(2)} = \frac{i}{2^4 \cdot 3^{\frac{15}{4}} \pi^{\frac{1}{2}} \Lambda_2^{\frac{3}{2}}} \left(\frac{\tilde{u}}{\Lambda_2^2}\right)^{-\frac{3}{4}} \left(F_1^{(2)}(\tilde{w}_2') + F_2^{(2)}(\tilde{w}_2') \right), \tag{6.52}$$

with $\tilde{w}_2' = 1 - \tilde{w}_2$. Here $F_1^{(2)}(\tilde{w}_2')$ and $F_2^{(2)}(\tilde{w}_2')$ are defined by

$$F_1^{(2)}(\tilde{w}_2') = 3^2 \Gamma\left(\frac{1}{12}\right) \Gamma\left(\frac{5}{12}\right) \left(2^2 \cdot 3^{\frac{1}{2}} \left(\frac{\tilde{M}^2}{\tilde{u}}\right)^{\frac{1}{2}} F\left(\frac{5}{12}, \frac{13}{12}; \frac{1}{2}; \tilde{w}_2'\right) - 5 \tilde{w}_2'^{\frac{1}{2}} F\left(\frac{13}{12}, \frac{17}{12}; \frac{3}{2}; \tilde{w}_2'\right) \right), \tag{6.53}$$

$$F_2^{(2)}(\tilde{w}_2') = 6^2 \Gamma\left(\frac{7}{12}\right) \Gamma\left(\frac{11}{12}\right) \left(3F\left(\frac{7}{12}, \frac{11}{12}; \frac{3}{2}; \tilde{w}_2'\right) + 7 X^{(2)} F\left(\frac{11}{12}, \frac{19}{12}; \frac{3}{2}; \tilde{w}_2'\right) \right), \tag{6.54}$$

where

$$X^{(2)} = -3 + 2 \cdot 3^{\frac{1}{2}} \tilde{w}_2'^{\frac{1}{2}} \left(\frac{\tilde{M}^2}{\tilde{u}} \right)^{\frac{1}{2}}. \tag{6.55}$$

The expansions of the second order terms in $\tilde{w}_2'$ are given by

$$\tilde{a}^{(2)} = \frac{1}{\Lambda_2^{\frac{3}{2}}} \left(\frac{\tilde{u}}{\Lambda_2^2} \right)^{-\frac{3}{4}} \left(-\frac{3^{\frac{1}{4}} \Gamma\left(\frac{7}{12}\right) \Gamma\left(\frac{11}{12}\right)}{2\pi^{\frac{1}{2}}} + \frac{\Gamma\left(\frac{1}{12}\right) \Gamma\left(\frac{5}{12}\right)}{2^4 \cdot 3^{\frac{5}{4}} \pi^{\frac{1}{2}}} \left(\frac{\tilde{M}^2}{\tilde{u}} \right)^{\frac{1}{2}} + \cdots \right), \tag{6.56}$$

$$\tilde{a}_D^{(2)} = \frac{1}{\Lambda_2^{\frac{3}{2}}} \left(\frac{\tilde{u}}{\Lambda_2^2} \right)^{-\frac{3}{4}} \left(-\frac{3^{\frac{1}{4}} i\Gamma\left(\frac{7}{12}\right) \Gamma\left(\frac{11}{12}\right)}{2\pi^{\frac{1}{2}}} - \frac{i\Gamma\left(\frac{1}{12}\right) \Gamma\left(\frac{5}{12}\right)}{2^4 \cdot 3^{\frac{5}{4}} \pi^{\frac{1}{2}}} \left(\frac{\tilde{M}^2}{\tilde{u}} \right)^{\frac{1}{2}} + \cdots \right). \tag{6.57}$$

Here we note that the expansion around $\tilde{w}_2' = 0$ corresponds to that in $\frac{\tilde{M}^2}{\tilde{u}} \ll 1$ and $\frac{\tilde{C}_2 \Lambda_2}{\tilde{u}^{\frac{3}{2}}} \ll 1$.

In a similar way, the fourth order corrections are given by

$$\tilde{a}^{(4)} = \frac{1}{2^9 \cdot 3^{\frac{11}{4}} \cdot 5\pi^{\frac{1}{2}} \Lambda_2^{\frac{9}{2}}} \frac{1}{\tilde{w}_2'^{\frac{1}{2}} (\tilde{w}_2' - 1)^2} \left(\frac{\tilde{u}}{\Lambda_2^2} \right)^{-\frac{9}{4}} \left(F_1^{(4)}(\tilde{w}_2') - F_2^{(4)}(\tilde{w}_2') \right), \tag{6.58}$$

$$\tilde{a}_D^{(4)} = -\frac{i}{2^9 \cdot 3^{\frac{11}{4}} \cdot 5\pi^{\frac{1}{2}} \Lambda_2^{\frac{9}{2}}} \frac{1}{\tilde{w}_2'^{\frac{1}{2}} (\tilde{w}_2' - 1)^2} \left(\frac{\tilde{u}}{\Lambda_2^2} \right)^{-\frac{9}{4}} \left(F_1^{(4)}(\tilde{w}_2') + F_2^{(4)}(\tilde{w}_2') \right), \tag{6.59}$$

where

$$F_1^{(4)}(\tilde{w}_2') = \Gamma\left(\frac{1}{12}\right) \Gamma\left(\frac{5}{12}\right) \left(-14 X_1^{(4)} F\left(\frac{1}{12}, \frac{5}{12}; \frac{1}{2}; \tilde{w}_2'\right) + X_2^{(4)} F\left(\frac{5}{12}, \frac{13}{12}; \frac{1}{2}; \tilde{w}_2'\right) \right), \tag{6.60}$$

$$F_2^{(4)}(\tilde{w}_2') = 14 \tilde{w}_2'^{\frac{1}{2}} \Gamma\left(\frac{7}{12}\right) \Gamma\left(\frac{11}{12}\right) \left(-2 X_1^{(4)} F\left(\frac{7}{12}, \frac{11}{12}; \frac{3}{2}; \tilde{w}_2'\right) + X_2^{(4)} F\left(\frac{11}{12}, \frac{19}{12}; \frac{3}{2}; \tilde{w}_2'\right) \right). \tag{6.61}$$

Here the coefficients X_1 and X_2 are defined by

$$\begin{aligned} X_1^{(4)} =& -2^2\cdot 3^{\frac{3}{2}}\tilde{w}_2'^{\frac{1}{2}}\left(10\tilde{w}_2'+11\right)+3\left(\frac{\tilde{M}^2}{\tilde{u}}\right)^{\frac{1}{2}}\left(377\tilde{w}_2'+127\right)\\ &-2^3\cdot 3^{\frac{1}{2}}\left(\frac{\tilde{M}^2}{\tilde{u}}\right)\tilde{w}_2'^{\frac{1}{2}}\left(13\tilde{w}_2'+113\right)+28\left(\frac{\tilde{M}^2}{\tilde{u}}\right)^{\frac{3}{2}}\left(13\tilde{w}_2'+11\right), \end{aligned} \tag{6.62}$$

$$\begin{aligned} X_2^{(4)} =& -3^{\frac{3}{2}}\tilde{w}_2'^{\frac{1}{2}}\left(1345\tilde{w}_2'+671\right)+6\left(\frac{\tilde{M}^2}{\tilde{u}}\right)^{\frac{1}{2}}\left(520\tilde{w}_2'^{2}+4639\tilde{w}_2'+889\right)\\ &-2^2\cdot 3^{\frac{3}{2}}\left(\frac{\tilde{M}^2}{\tilde{u}}\right)\tilde{w}_2'^{\frac{1}{2}}\left(593\tilde{w}_2'+1423\right)+56\left(\frac{\tilde{M}^2}{\tilde{u}}\right)^{\frac{3}{2}}\left(211\tilde{w}_2'+77\right). \end{aligned} \tag{6.63}$$

The fourth order corrections to the SW periods are expanded in $\tilde{w}_2'$ around $\tilde{w}_2'=0$ as

$$\tilde{a}^{(4)} =\frac{1}{\Lambda_2^{\frac{9}{2}}}\left(\frac{\tilde{u}}{\Lambda_2^2}\right)^{-\frac{9}{4}}\left(-\frac{11\Gamma\left(\frac{1}{12}\right)\Gamma\left(\frac{5}{12}\right)}{2^9\cdot 3^{\frac{5}{4}}\pi^{\frac{1}{2}}}-\frac{3^{\frac{1}{4}}\cdot 5\cdot 7\Gamma\left(\frac{7}{12}\right)\Gamma\left(\frac{11}{12}\right)}{2^8\pi^{\frac{1}{2}}}\left(\frac{\tilde{M}^2}{\tilde{u}}\right)^{\frac{1}{2}}+\cdots\right), \tag{6.64}$$

$$\tilde{a}_D^{(4)} =\frac{1}{\Lambda_2^{\frac{9}{2}}}\left(\frac{\tilde{u}}{\Lambda_2^2}\right)^{-\frac{9}{4}}\left(\frac{11i\Gamma\left(\frac{1}{12}\right)\Gamma\left(\frac{5}{12}\right)}{2^9\cdot 3^{\frac{5}{4}}\pi^{\frac{1}{2}}}-\frac{3^{\frac{1}{4}}\cdot 5\cdot 7i\Gamma\left(\frac{7}{12}\right)\Gamma\left(\frac{11}{12}\right)}{2^8\pi^{\frac{1}{2}}}\left(\frac{\tilde{M}^2}{\tilde{u}}\right)^{\frac{1}{2}}+\cdots\right). \tag{6.65}$$

We then expand the effective coupling constant $\tilde{\tau}$ in $\hbar$ as

$$\begin{aligned} \tilde{\tau} =& \left(-i-\frac{i\Gamma\left(\frac{7}{12}\right)\Gamma\left(\frac{11}{12}\right)}{3^{\frac{1}{2}}\Gamma\left(\frac{5}{12}\right)\Gamma\left(\frac{13}{12}\right)}\left(\frac{\tilde{M}^2}{\tilde{u}}\right)^{\frac{1}{2}}+\frac{i3^{\frac{1}{2}}\Gamma\left(\frac{7}{12}\right)\Gamma\left(\frac{11}{12}\right)}{2^3\Gamma\left(\frac{5}{12}\right)\Gamma\left(\frac{13}{12}\right)}\left(\frac{\tilde{C}_2^2\Lambda_2^2}{\tilde{u}^3}\right)^{\frac{1}{2}}+\cdots\right)\\ &+\frac{\hbar^2}{\Lambda_2^3}\left(\frac{\tilde{u}}{\Lambda_2^2}\right)^{-\frac{3}{2}}\left(\frac{3^{\frac{1}{2}}i\Gamma\left(\frac{7}{12}\right)\Gamma\left(\frac{11}{12}\right)}{2^4\Gamma\left(\frac{5}{12}\right)\Gamma\left(\frac{13}{12}\right)}+\frac{3^2i\Gamma\left(\frac{7}{12}\right)^2\Gamma\left(\frac{11}{12}\right)^2}{\Gamma\left(\frac{1}{12}\right)^2\Gamma\left(\frac{5}{12}\right)^2}\left(\frac{\tilde{M}^2}{\tilde{u}}\right)^{\frac{1}{2}}+\cdots\right)\\ &+\frac{\hbar^4}{\Lambda_2^6}\left(\frac{\tilde{u}}{\Lambda_2^2}\right)^{-3}\left(-\frac{3i\Gamma\left(\frac{7}{12}\right)^2\Gamma\left(\frac{11}{12}\right)^2}{2^9\Gamma\left(\frac{5}{12}\right)^2\Gamma\left(\frac{13}{12}\right)^2}\right.\\ &\left.-\frac{3^{\frac{1}{2}}i\left(3\Gamma\left(\frac{7}{12}\right)^3\Gamma\left(\frac{11}{12}\right)^3+19\pi^2\Gamma\left(\frac{5}{12}\right)\Gamma\left(\frac{13}{12}\right)\right)}{2^{10}\Gamma\left(\frac{5}{12}\right)^3\Gamma\left(\frac{13}{12}\right)^3}\left(\frac{\tilde{M}^2}{\tilde{u}}\right)^{\frac{1}{2}}+\cdots\right)+\cdots. \end{aligned} \tag{6.66}$$

It would be interesting to compare the free energy with that of the E-string theory, which is left for future work.

$N_f = 3$ theory

We now investigate the $N_f = 3$ case. Substituting (3.51) into (6.36), the second order corrections to the SW periods are obtained by

$$\tilde{a}^{(2)} = \frac{1}{2^{\frac{10}{3}} \cdot 3^{\frac{7}{2}} \pi^{\frac{1}{2}} \tilde{w}_3' \Lambda_3^3} \left(\frac{\tilde{u}}{\Lambda_3^2}\right) (-\sigma_3)^{-\frac{5}{6}} \left(1 + \frac{4}{3}\frac{\tilde{M}^3}{\tilde{u}\Lambda_3}\right) \left(F_1^{(2)}(\tilde{w}_3') + F_2^{(2)}(\tilde{w}_3')\right), \tag{6.67}$$

$$\tilde{a}_D^{(2)} = \frac{i}{2^{\frac{10}{3}} \cdot 3^{\frac{7}{2}} \pi^{\frac{1}{2}} \tilde{w}_3' \Lambda_3^3} \left(\frac{\tilde{u}}{\Lambda_3^2}\right) (-\sigma_3)^{-\frac{5}{6}} \left(1 + \frac{4}{3}\frac{\tilde{M}^3}{\tilde{u}\Lambda_3}\right) \left((-1)^{\frac{5}{6}} F_1^{(2)}(\tilde{w}_3') + (-1)^{\frac{1}{6}} F_2^{(2)}(\tilde{w}_3')\right), \tag{6.68}$$

where $\tilde{w}_3' := \frac{1}{1-\tilde{w}_3}$. $F_1^{(2)}(\tilde{w}_3')$ and $F_2^{(2)}(\tilde{w}_3')$ are given by

$$F_1^{(2)}(\tilde{w}_3') = 18\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right)\left(F\left(\frac{1}{12}, \frac{7}{12}; \frac{2}{3}; \tilde{w}_3'\right) - X^{(2)} F\left(\frac{7}{12}, \frac{13}{12}; \frac{2}{3}; \tilde{w}_3'\right)\right), \tag{6.69}$$

$$F_2^{(2)}(\tilde{w}_3') = -\frac{3\tilde{w}_3'}{2^{\frac{2}{3}}}\Gamma\left(-\frac{1}{6}\right)\Gamma\left(-\frac{1}{3}\right)\left(F\left(\frac{5}{12}, \frac{11}{12}; \frac{4}{3}; \tilde{w}_3'\right) - 5X^{(2)} F\left(\frac{11}{12}, \frac{17}{12}; \frac{4}{3}; \tilde{w}_3'\right)\right). \tag{6.70}$$

where

$$X^{(2)} = 1 + \frac{2^{\frac{2}{3}} \cdot 3\tilde{M}\Lambda_3}{(3\tilde{u}\Lambda_3 + 4\tilde{M}^3)}(-\sigma_3)^{\frac{1}{3}} \tilde{w}_3'^{\frac{2}{3}}. \tag{6.71}$$

We expand the second order corrections to the SW periods in $\tilde{w}_3'$, where $\frac{\tilde{M}^3}{\tilde{u}\Lambda_3} \ll 1$, $\frac{\tilde{C}_2^3\Lambda_3^2}{\tilde{u}^4} \ll 1$ and $\frac{\tilde{C}_3\Lambda_3}{\tilde{u}^2} \ll 1$. Then we have

$$\tilde{a}^{(2)} = \frac{1}{\Lambda_3^{\frac{1}{2}}}\left(\frac{\tilde{u}}{\Lambda_3^2}\right)^{-\frac{2}{3}}\left(-\frac{(-1)^{\frac{1}{6}}\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{5}{6}\right)}{2 \cdot 3^{\frac{1}{2}}\pi^{\frac{1}{2}}} + \frac{\left(1 + 19(-1)^{\frac{1}{3}}\right)\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right)}{3^4\pi^{\frac{1}{2}}}\left(\frac{\tilde{M}^3}{\tilde{u}\Lambda_3}\right)^{\frac{2}{3}} + \cdots\right), \tag{6.72}$$

$$\tilde{a}_D^{(2)} = \frac{1}{\Lambda_3^{\frac{1}{2}}}\left(\frac{\tilde{u}}{\Lambda_3^2}\right)^{-\frac{2}{3}}\left(-\frac{(-1)^{\frac{5}{6}}\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{5}{6}\right)}{2 \cdot 3^{\frac{1}{2}}\pi^{\frac{1}{2}}} + \frac{(-1)^{\frac{2}{3}}\left(1 + 19(-1)^{\frac{1}{3}}\right)\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right)}{3^4\pi^{\frac{1}{2}}}\left(\frac{\tilde{M}^3}{\tilde{u}\Lambda_3}\right)^{\frac{2}{3}} + \cdots\right), \tag{6.73}$$

It can be also found that the effective coupling constant is given by

$$\begin{aligned}\tilde{\tau} = &\left(-(-1)^{\frac{1}{3}} - \frac{2^4 i\pi^2}{\Gamma\left(\frac{1}{6}\right)^2\Gamma\left(\frac{1}{3}\right)^2}\left(\frac{\tilde{M}^3}{\tilde{u}\Lambda_3}\right)^{\frac{1}{3}} - \frac{2i\pi^2}{\Gamma\left(\frac{1}{6}\right)^2\Gamma\left(\frac{1}{3}\right)^2}\left(\frac{\tilde{C}_2^3\Lambda_3^2}{\tilde{u}^4}\right)^{\frac{1}{3}} + \cdots\right)\\ &+ \frac{\hbar^2}{\Lambda_3^2}(-1)^{\frac{5}{6}}\left(\frac{\tilde{u}}{\Lambda_3^2}\right)^{-\frac{4}{3}}\left(-\frac{3^{\frac{1}{2}}\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{5}{6}\right)}{5\Gamma\left(-\frac{5}{6}\right)\Gamma\left(\frac{1}{3}\right)} - \frac{2^{\frac{10}{3}}\cdot 3^{\frac{1}{2}}\Gamma\left(-\frac{1}{6}\right)\Gamma\left(\frac{5}{6}\right)^2}{5^2\Gamma\left(-\frac{5}{6}\right)^2\Gamma\left(\frac{1}{6}\right)}\left(\frac{\tilde{M}^3}{\tilde{u}\Lambda_3}\right)^{\frac{1}{3}} + \cdots\right)\\ &+ \cdots .\end{aligned} \tag{6.74}$$

The fourth order corrections to the effective coupling constant can be obtained in a similar manner as will shown in appendix C. The result is

$$\tilde{\tau}^{(4)} = \frac{(-1)^{\frac{1}{6}}}{\Lambda_3^4}\left(\frac{\tilde{u}}{\Lambda_3^2}\right)^{-\frac{8}{3}}\left(\frac{2^{\frac{4}{3}}\cdot 3^{\frac{1}{2}}\pi\Gamma\left(\frac{5}{6}\right)^2}{5^2\Gamma\left(-\frac{5}{6}\right)^2\Gamma\left(\frac{1}{6}\right)^2} + \frac{2^3\cdot 3^{\frac{3}{2}}\pi^{\frac{1}{2}}\Gamma\left(-\frac{1}{6}\right)\Gamma\left(\frac{5}{6}\right)^3}{5^3\Gamma\left(-\frac{5}{6}\right)^3\Gamma\left(\frac{1}{6}\right)^2}\left(\frac{\tilde{M}^3}{\tilde{u}\Lambda_3}\right)^{\frac{1}{3}} + \cdots\right). \tag{6.75}$$

In summary, for the AD theories of (A_1, A_2), (A_1, A_3) and (A_1, D_4)-types, we have the explicit form of the quantum corrections to the SW periods in terms of the hypergeometric functions up to the fourth orders in $\hbar$.

6.2 $SU(N_c)$ SQCD around superconformal point

In this section, we will study the quantum SW periods for the AD theory realized from the $SU(N_c)$ SQCD. The SW curve and the SW differential around the superconformal point were discussed in section 3.3. We have seen the SW differentials for each N_f take the different form due to flavor symmetry. Therefore we need to introduce different quantization condition in each N_f. In the following, we will discuss the quantum SW curve around the superconformal point and construct the differential operator on the SW periods to represent the quantum correction to the SW periods.

6.2.1 $N_f = 0, 1$

For $N_f = 0$ and $N_f = 1$, the SW differential (3.78) and (3.86) are defined by $d\tilde{\lambda}_{\rm SW} \sim d\tilde{y} \wedge d\tilde{p}$. The quantum SW curve for the $N_f = 0$ and 1 theories is given by

$$\left(-\hbar^2 \frac{\partial^2}{\partial \tilde{p}^2} + Q(\tilde{p})\right) \Psi(\tilde{p}) = 0. \tag{6.76}$$

The potential $Q(\tilde{p})$ is given by the SW curve (3.77) and (3.85):

$$Q(\tilde{p}) = -\tilde{y}^2. \tag{6.77}$$

where $Q_2 = 0$. The quantum SW periods can be obtained by the WKB solution of the quantum SW curve (6.76), which takes the form as (5.9) in terms of $\tilde{p}$. We note the quantum corrections to the SW periods agree with the leading term of those for the corresponding $\mathcal{N} = 2$ SQCD in the scaling limit up to total derivatives [2]. For $N_f = 0$, we find

$$\Pi^{(2)} = \epsilon^{-N_c+2} \left(-\frac{3\Lambda_0^{N_c}}{2} \tilde{\Pi}^{(2)}\right) + \cdots, \qquad \Pi^{(4)} = \epsilon^{-3N_c+2} \left(-\Lambda_0^{3N_c} \tilde{\Pi}^{(4)}\right) + \cdots. \tag{6.78}$$

For $N_f = 1$, we also find that the second and fourth order corrections correspond to those in the scaling limit up to total derivatives at least $N_c \leq 5$:

$$\Pi^{(2)} = \epsilon^{-(N_c+1)+2} b_{N_c}^{(2)} \Lambda_1^{N_c} \tilde{\Pi}^{(2)} + \cdots, \qquad \Pi^{(4)} = \epsilon^{-3(N_c+1)+2} b_{N_c}^{(4)} \Lambda_1^{3N_c} \tilde{\Pi}^{(4)} + \cdots, \tag{6.79}$$

where the first several coefficients $b_{N_c}^{(2)}$ and $b_{N_c}^{(4)}$ are given by

$$b_2^{(2)} = -\frac{3}{4}, \quad b_3^{(2)} = -\frac{3}{2^{\frac{9}{5}}}, \quad b_4^{(2)} = \frac{3 \cdot 5^{\frac{1}{7}}}{4}, \quad b_5^{(2)} = -\frac{3 \cdot 7^{\frac{1}{9}}}{2^{\frac{17}{9}}}, \cdots, \tag{6.80}$$

$$b_2^{(4)} = \frac{1}{8}, \quad b_3^{(4)} = \frac{1}{2^{\frac{12}{5}}}, \quad b_4^{(4)} = -\frac{5^{\frac{3}{7}}}{8}, \quad b_5^{(4)} = \frac{7^{\frac{1}{3}}}{2^{\frac{8}{3}}}, \cdots. \tag{6.81}$$

Since the SW curve for the AD theory realized from the pure $SU(N_c+1)$ gauge theory agrees with that obtained from the $SU(N_c)$ gauge theory with $N_f = 1$ hypermultiplet in the scaling limit, we obtain same quantum corrections from both cases. Thus we will choose the case of the $SU(N_c)$ gauge theory with $N_f = 1$ hypermultiplet and then obtain

[2] In appendix D, we perform the WKB approximation of the quantum SW curve for the $SU(N_c)$ SQCD. We then obtain the second and fourth order corrections to the SW periods.

the relation between the quantum corrections and the SW periods. The relation formulas between $\tilde{\Pi}^{(k)}$ $(k = 2, 4)$ and $\tilde{\Pi}^{(0)}$ are not unique and there are various ways to represent the differential operator $\hat{\mathcal{O}}_k$. For example, the second and fourth order corrections to the SW periods can be expressed as

$$\tilde{\Pi}^{(2)} = \frac{1}{12}\left(\sum_{i=0}^{N_c+1} U_{i2}\frac{\partial}{\partial \tilde{u}_{N_c+1}}\frac{\partial}{\partial \tilde{u}_i}\tilde{\Pi}^{(0)}\right), \tag{6.82}$$

$$\tilde{\Pi}^{(4)} = \frac{7}{1440}\left(\frac{5}{7}\sum_{i=0}^{N_c+1} U_{i4}\frac{\partial^2}{\partial \tilde{u}_{N_c+1}^2}\frac{\partial}{\partial \tilde{u}_i}\tilde{\Pi}^{(0)} + \sum_{i,j=0}^{N_c+1} U_{i2}U_{j2}\frac{\partial^2}{\partial \tilde{u}_{N_c+1}^2}\frac{\partial}{\partial \tilde{u}_i}\frac{\partial}{\partial \tilde{u}_j}\tilde{\Pi}^{(0)}\right), \tag{6.83}$$

where

$$U_{ij} := \frac{(N_c + 1 - i + j)!}{(N_c + 1 - i)!}\tilde{u}_{i-j}. \tag{6.84}$$

For the $SU(2)$ gauge theory with the $N_f = 1$ hypermultiplet, these relations are shown to agree with (6.7) and (6.8) up to the Picard-Fuchs equations.

6.2.2 $N_f = 2$

For $N_f = 2$, the SW curve (3.95) and the SW differential (3.96) become

$$\tilde{y}^2 = \sum_{l=0}^{N_c+1} \tilde{t}_l e^{(N_c+1-l)\xi}, \tag{6.85}$$

$$\tilde{\lambda}_{\mathrm{SW}} = -\frac{2}{\Lambda_2^{N_f-1}}\tilde{y}d\xi, \tag{6.86}$$

where $\tilde{t}_l = \tilde{t}_l(\tilde{M}, \tilde{s}_i)$ $(l = 0, \cdots, N_c + 1)$ by introducing $\tilde{p} = e^{\xi} - \tilde{M}$. The SW differential defines the symplectic one-form $d\tilde{\lambda}_{\mathrm{SW}} = d\tilde{y} \wedge d\xi$. The quantum SW curve takes the form as

$$\left(-\hbar^2\frac{\partial^2}{\partial \xi^2} + Q(\xi)\right)\Psi(\xi) = 0, \tag{6.87}$$

where

$$Q(\xi) = -\tilde{y}^2 = -\left(\sum_{l=0}^{N_c+1} \tilde{t}_l e^{(N_c+1-l)\xi}\right). \tag{6.88}$$

The quantum corrections to the SW periods are given by the WKB solutions of the quantum curve (6.87). Since the quantum SW curve (6.87) is the Schrödinger type equation, the second and fourth order terms are of the form as (5.16) and (5.17) in terms of ξ, respectively. The quantum corrections are shown to agree with those of the scaling limit of the $N_f = 2$ theory. When we change the variables ξ to $\tilde{p}$, then the second order correction to the SW periods becomes

$$\tilde{\Pi}^{(2)} = \oint \frac{\tilde{\phi}_2(\tilde{p})}{(\tilde{p}+\tilde{M})} d\tilde{p}, \tag{6.89}$$

where

$$\tilde{\phi}_2(\tilde{p}) = \frac{i(\tilde{p}+\tilde{M})}{48}\left(\frac{Q'(\tilde{p})}{Q(\tilde{p})^{\frac{3}{2}}} + \frac{(\tilde{p}+\tilde{M})Q''(\tilde{p})}{Q(\tilde{p})^{\frac{3}{2}}}\right). \tag{6.90}$$

This correction agrees with the second order correction to the SW periods for the original $N_f = 2$ theory by taking the scaling limit $\epsilon \to 0$. Similarly, we also find that the fourth order correction obtained by the quantum SW curve (6.87) agrees with that for the $N_f = 2$ theory in the scaling limit:

$$\Pi^{(2)} = \epsilon^{-(N_c+3)}\frac{3}{2}\Lambda_2^{(N_c-1)}\tilde{\Pi}^{(2)} + \cdots, \qquad \Pi^{(4)} = \epsilon^{-3(N_c-1)+2}\left(-\Lambda_2^{3(N_c-1)}\tilde{\Pi}^{(4)}\right) + \cdots. \tag{6.91}$$

The second and fourth order corrections to the SW periods are obtained by acting the differential operators on the SW periods as

$$\tilde{\Pi}^{(2)} = \frac{1}{12}\sum_{l=1}^{N_c+1}(N_c+2-l)^2\tilde{t}_{l-1}\frac{\partial}{\partial\tilde{t}_l}\frac{\partial}{\partial\tilde{t}_{N_c}}\tilde{\Pi}^{(0)}, \tag{6.92}$$

$$\tilde{\Pi}^{(4)} = \frac{7}{1440}\left(\frac{5}{7}\hat{\mathcal{O}}_A^{(4)}\tilde{\Pi}^{(0)} + \hat{\mathcal{O}}_B^{(4)}\tilde{\Pi}^{(0)}\right), \tag{6.93}$$

where

$$\hat{\mathcal{O}}_A^{(4)} = \sum_{l=1}^{N_c+1}(N_c+2-l)^4 t_{l-1}\frac{\partial}{\partial\tilde{t}_l}\frac{\partial}{\partial\tilde{t}_{N_c}}\frac{\partial}{\partial\tilde{t}_{N_c+1}}, \tag{6.94}$$

$$\hat{\mathcal{O}}_B^{(4)} = \sum_{l=1}^{N_c+1}\sum_{k=1}^{N_c+1}(N_c+2-l)^2(N_c+2-k)^2\tilde{t}_{l-1}\tilde{t}_{k-1}\frac{\partial}{\partial\tilde{t}_l}\frac{\partial}{\partial\tilde{t}_k}\frac{\partial^2}{\partial\tilde{t}_{N_c}^2}. \tag{6.95}$$

These formulas are consistent in the case of the $SU(2)$ gauge theory with $N_f = 2$ theory: (6.17) and (6.18) up to the Picard-Fuchs equation.

6.2.3 $N_f = 2n+1$

By introducing $\tilde{y} = \tilde{z} - \tilde{C}(\tilde{p})$, the SW curve (3.104) and the SW differential (3.105) become

$$\tilde{C}(\tilde{p}) - \frac{1}{2}\left(\tilde{z} + \frac{\Lambda_{N_f}^{2N_c-N_f}\tilde{G}(\tilde{p})}{\tilde{z}}\right) = 0, \tag{6.96}$$

$$\tilde{\lambda}_{\rm SW} = \tilde{p}\left(d\log\tilde{G}(\tilde{p}) - 2d\log\tilde{z}\right). \tag{6.97}$$

Rewriting the coordinates as

$$\tilde{z} = \exp\left(-i\hbar\frac{\partial}{\partial\tilde{p}}\right), \tag{6.98}$$

the quantum SW curve for the $N_f = 2n+1$ is given by

$$\left[\frac{1}{2}\left(\exp\left(-i\hbar\frac{\partial}{\partial\tilde{p}}\right) + \exp\left(i\frac{\hbar}{2}\frac{\partial}{\partial\tilde{p}}\right)\Lambda_{N_f}^{2N_c-N_f}\tilde{G}(\tilde{p})\exp\left(i\frac{\hbar}{2}\frac{\partial}{\partial\tilde{p}}\right)\right) - \tilde{C}(\tilde{p})\right]\Psi(\tilde{p}) = 0. \tag{6.99}$$

To discuss the higher order corrections to the SW periods, we rewrite the quantum SW curves as

$$\frac{1}{2}\left(J(1) + \Lambda_{N_f}^{2N_c-N_f}\tilde{G}\left(\tilde{p} + i\frac{\hbar}{2}\right)J(-1)\right) + \tilde{C}(\tilde{p}) = 0, \tag{6.100}$$

where $J(\alpha)$ is defined by (6.27). Note that we take the different ordering of the operators from the case of the $SU(2)$ gauge theory with $N_f = 3$ hypermultiplets as in the previous section. Although we look like quite different quantization, we can obtain the same results of quantum SW periods in each case. Expanding the quantum SW curve in $\hbar$, we obtain the second and fourth order corrections to the SW periods. Note the second and fourth order corrections to the SW periods for $N_f = 2n+1$ agree with the scaling limit of those for the $SU(N_c)$ gauge theory with $N_f = 2n+1$:

$$\Pi^{(2)} = \epsilon^2\tilde{\Pi}^{(2)} + \cdots, \qquad \Pi^{(4)} = \epsilon^2\tilde{\Pi}^{(4)} + \cdots. \tag{6.101}$$

Define

$$\tilde{U}_{ij} := \frac{(n-i+j)!}{(n-i)!}\tilde{u}_{i-j}, \qquad \tilde{V}_{ab} := \frac{(N_f-a+b)!}{(N_f-a)!}\tilde{C}_{a-b}, \tag{6.102}$$

where

$$\sum_{a=0}^{N_f} \tilde{C}_a \tilde{p}^{N_f-a} := \prod_{a=1}^{N_f} (\tilde{p} + \tilde{c}_a). \tag{6.103}$$

We also find the relations between the quantum corrections and the SW periods as

$$\tilde{\Pi}^{(2)} = \frac{1}{24} \left(\sum_{i,j=0}^{n} \left(\tilde{U}_{i0}\tilde{U}_{j2} + \tilde{U}_{i1}\tilde{U}_{j1} \right) \frac{\partial}{\partial \tilde{u}_i} \frac{\partial}{\partial \tilde{u}_j} \tilde{\Pi}^{(0)} + \sum_{i=0}^{n} \tilde{U}_{i2} \frac{\partial}{\partial \tilde{u}_i} \tilde{\Pi}^{(0)} + \sum_{i=0}^{n} \sum_{a=2}^{N_f} \tilde{U}_{i0} \tilde{V}_{a2} \frac{\partial}{\partial \tilde{u}_i} \frac{\partial}{\partial \tilde{C}_a} \tilde{\Pi}^{(0)} \right), \tag{6.104}$$

$$\begin{aligned}
\tilde{\Pi}^{(4)} =& \frac{1}{2^7 \cdot 3^2 \cdot 5} \left(\sum_{i,j,k,l=0}^{n} A^{(1)}_{ijkl} \frac{\partial}{\partial \tilde{u}_i} \frac{\partial}{\partial \tilde{u}_j} \frac{\partial}{\partial \tilde{u}_k} \frac{\partial}{\partial \tilde{u}_l} \tilde{\Pi}^{(0)} + \sum_{i,j,k=0}^{n} A^{(2)}_{ijk} \frac{\partial}{\partial \tilde{u}_i} \frac{\partial}{\partial \tilde{u}_j} \frac{\partial}{\partial \tilde{u}_k} \tilde{\Pi}^{(0)} \right. \\
& \left. + \sum_{i,j=0}^{n} A^{(3)}_{ij} \frac{\partial}{\partial \tilde{u}_i} \frac{\partial}{\partial \tilde{u}_j} \tilde{\Pi}^{(0)} + \sum_{i=0}^{n} A^{(4)}_{i} \frac{\partial}{\partial \tilde{u}_i} \tilde{\Pi}^{(0)} \right) \\
& + \frac{1}{2^7 \cdot 3^2 \cdot 5} \left(\sum_{i,j,k=0}^{n} \sum_{a=2}^{N_f} B^{(1)}_{ijka} \frac{\partial}{\partial \tilde{u}_i} \frac{\partial}{\partial \tilde{u}_j} \frac{\partial}{\partial \tilde{u}_k} \frac{\partial}{\partial \tilde{C}_a} \tilde{\Pi}^{(0)} + \sum_{i,j=0}^{n} \sum_{a,b=2}^{N_f} B^{(2)}_{ijab} \frac{\partial}{\partial \tilde{u}_i} \frac{\partial}{\partial \tilde{u}_j} \frac{\partial}{\partial \tilde{C}_a} \frac{\partial}{\partial \tilde{C}_b} \tilde{\Pi}^{(0)} \right. \\
& \left. + \sum_{i,j=0}^{n} \sum_{a=2}^{N_f} B^{(3)}_{ija} \frac{\partial}{\partial \tilde{u}_i} \frac{\partial}{\partial \tilde{u}_j} \frac{\partial}{\partial \tilde{C}_a} \tilde{\Pi}^{(0)} + \sum_{i=0}^{n} \sum_{a,b=2}^{N_f} B^{(4)}_{iab} \frac{\partial}{\partial \tilde{u}_i} \frac{\partial}{\partial \tilde{C}_a} \frac{\partial}{\partial \tilde{C}_b} \tilde{\Pi}^{(0)} + \sum_{i=0}^{n} \sum_{a=2}^{N_f} B^{(5)}_{ia} \frac{\partial}{\partial \tilde{u}_i} \frac{\partial}{\partial \tilde{C}_a} \tilde{\Pi}^{(0)} \right),
\end{aligned} \tag{6.105}$$

up to fourth order in $\hbar$ where

$$A^{(1)}_{ijkl} = 7 \left(\tilde{U}_{i0}\tilde{U}_{j0}\tilde{U}_{k2}\tilde{U}_{l2} + \tilde{U}_{i1}\tilde{U}_{j1}\tilde{U}_{k1}\tilde{U}_{l1} + 2\tilde{U}_{i0}\tilde{U}_{j1}\tilde{U}_{k1}\tilde{U}_{l2} \right), \tag{6.106}$$

$$A^{(2)}_{ijk} = 51\tilde{U}_{i0}\tilde{U}_{j2}\tilde{U}_{k2} + 84\tilde{U}_{i1}\tilde{U}_{j1}\tilde{U}_{k2} + 36\tilde{U}_{i0}\tilde{U}_{j1}\tilde{U}_{k3} + 5\tilde{U}_{i0}\tilde{U}_{j0}\tilde{U}_{k4}, \tag{6.107}$$

$$A^{(3)}_{ij} = 81\tilde{U}_{i2}\tilde{U}_{j2} + 78\tilde{U}_{i1}\tilde{U}_{j3} + 39\tilde{U}_{i0}\tilde{U}_{j4}, \tag{6.108}$$

$$A^{(4)}_{i} = 36\tilde{U}_{i4}, \tag{6.109}$$

$$B^{(1)}_{ijka} = 14\tilde{U}_{i0}\tilde{U}_{j1}\tilde{U}_{k1}\tilde{V}_{a2} + 14\tilde{U}_{i0}\tilde{U}_{j0}\tilde{U}_{k2}\tilde{V}_{a2}, \tag{6.110}$$

$$B^{(2)}_{ijab} = 7\tilde{U}_{i0}\tilde{U}_{j0}\tilde{V}_{a2}\tilde{V}_{b2}, \tag{6.111}$$

$$B^{(3)}_{ija} = 28\tilde{U}_{i1}\tilde{U}_{j1}\tilde{V}_{a2} + 64\tilde{U}_{i0}\tilde{U}_{j2}\tilde{V}_{a2} + 16\tilde{U}_{i0}\tilde{U}_{j1}\tilde{V}_{a3} + 5\tilde{U}_{i0}\tilde{U}_{j0}\tilde{V}_{a4}, \tag{6.112}$$

$$B^{(4)}_{iab} = 21\tilde{U}_{i0}\tilde{V}_{a2}\tilde{V}_{b2}, \tag{6.113}$$

$$B^{(5)}_{ia} = 36\tilde{U}_{i2}\tilde{V}_{a2} + 16\tilde{U}_{i1}\tilde{V}_{a3} + 19\tilde{U}_{i0}\tilde{V}_{a4}. \tag{6.114}$$

It can be shown that the above formulas are consistent with the case of the $SU(2)$ theory with $N_f = 3$ hypermultiplets up to the Picard-Fuchs equation. We note the second and fourth order corrections (6.104) and (6.105) include the differential operators with respect to the irrelevant operators of which the scaling dimensions are greater than 2. In order to obtain the second and fourth order correction for the AD theory associated with the $N_f = 2n + 1$ theory, we should use the relation (6.104) and (6.105) with keeping the irrelevant operators finite and then take the limit where these operators go to zero.

6.2.4 $N_f = 2n$ $(n \geq 2)$

For $N_f = 2n$ $(n \geq 2)$, we need to introduce the different quantization condition in each two sector $\epsilon = \epsilon_A$ and ϵ_B since the SW differential in the A sector (3.118) is different form from that in the B sector (3.127). We will first discuss the quantum SW periods in the A sector.

For $\epsilon = \epsilon_A$, the SW curve (3.117) and the SW differential (3.118) are same form as those for the $N_f = 2n + 1$ theory (3.104) and (3.105), respectively. The quantum SW curve is given by (6.99) where $\tilde{C}(\tilde{p})$ and $\tilde{G}(\tilde{p})$ are defined by (3.113) and (3.114). Here we shift the coordinate from $\tilde{p}$ to $\tilde{p}_1$ as

$$\tilde{p}_1 := \tilde{p} - \tilde{M}. \tag{6.115}$$

Then the functions (3.113) and (3.114) become

$$\tilde{C}(\tilde{p}) = \sum_{l=0}^{n} \tilde{u}_l \tilde{p}^{n-l}, \tag{6.116}$$

$$\tilde{G}(\tilde{p}) = \prod_{a=1}^{N_f} (\tilde{p}_1 + \tilde{c}_a) := \sum_{a=0}^{N_f} \tilde{C}_a \tilde{p}_1^{N_f - a}, \quad \tilde{C}_0 := 1, \quad \tilde{C}_1 := 0, \tag{6.117}$$

where $\tilde{u}_l = \tilde{u}_l(\tilde{M}, \tilde{s}_i)$ $(l = 0, \cdots, n)$ are the moduli parameters and $\tilde{C}_a$ $(a = 2, \cdots, N_f)$ are the Casimir invariants of $U(N_f)$ flavor symmetry. As discussed in the case of the $N_f = 2n + 1$ theory, by using the WKB method and solving the recursion relation of $\tilde{\phi}_k$ $(k = 0, 1, \cdots)$ on the $\tilde{p}_1$-plane, we find the relation between the SW periods and the quantum corrections to the SW periods are given by (6.104) and (6.105) up to forth order in $\hbar$. By taking the scaling limit $\epsilon_A \to 0$, we also find the leading order terms of the second and fourth order corrections for the $N_f = 2n$ theory agree with those obtained from the quantum SW curve for the A sector:

$$\Pi^{(2)} = \epsilon_A^2 \tilde{\Pi}^{(2)} + \cdots , \qquad \Pi^{(4)} = \epsilon_A^2 \tilde{\Pi}^{(4)} + \cdots . \tag{6.118}$$

For $\epsilon = \epsilon_B$, we introduce new variables ξ as

$$\tilde{p} = \xi^{-\frac{1}{n-1}} - \tilde{M}, \tag{6.119}$$

so that the SW curve (3.126) and the SW differential (3.127) become

$$\tilde{y}^2 = \sum_{l=0}^{N_c} \tilde{t}_l \xi^{-\frac{N_c - l + n}{n-1}}, \tag{6.120}$$

$$\lambda_{\mathrm{SW}} = -\frac{2}{\Lambda_{N_f}^{N_c - n}} \frac{1}{(-n+1)} \tilde{y} d\xi, \tag{6.121}$$

where $\tilde{t}_l = \tilde{t}_l(\tilde{M}, \tilde{s}_i)$ $(l = 0, \cdots , N_c)$. The SW differential defines the symplectic form $d\tilde{\lambda}_{\mathrm{SW}} \sim d\tilde{y} \wedge d\xi$. By introducing $\tilde{y} = -i\hbar \frac{\partial}{\partial \xi}$, we obtain the quantum SW curve:

$$\left(-\hbar^2 \frac{\partial^2}{\partial \xi^2} + Q(\xi) \right) \Psi(\xi) = 0, \tag{6.122}$$

with $Q(\xi) = Q_0(\xi) + \hbar^2 Q_2(\xi)$ where

$$Q_0(\xi) = -\left(\sum_{l=0}^{N_c} \tilde{t}_l \xi^{-\frac{N_c - l + n}{n-1}} \right), \qquad Q_2(\xi) = -\frac{n}{4(n-1)} \xi^{-2}. \tag{6.123}$$

The second and fourth order corrections are obtained by applying (5.16) and (5.17) in terms of ξ. These corrections agree with those for the $N_f = 2n$ $(n \geq 2)$ theory by taking the scaling limit $\epsilon_B \to 0$ up to total derivatives:

$$\Pi^{(2)} = \epsilon_B^{-(N_c - n)+2} (n-1) \Lambda_{N_f}^{N_c - n} \tilde{\Pi}^{(2)} + \cdots , \qquad \Pi^{(4)} = \epsilon_B^{-3(N_c - n)+2} (n-1)^3 \Lambda_{N_f}^{3(N_c - n)} \tilde{\Pi}^{(4)} + \cdots . \tag{6.124}$$

This is reason why we add Q_2 (6.123) in (6.122).

In order to obtain the relation between the higher order correction and the SW periods, it is convenient to introduce some functions as

$$\mu(i) := -\frac{N_c - i + n}{n-1}, \qquad \tilde{N} := N_c - n + 2, \qquad T_i^{(k)} := \frac{\Gamma(\mu(i) + 1)}{\Gamma(\mu(i) - k + 1)}. \tag{6.125}$$

Then we find the second and fourth order corrections are given by

$$\tilde{\Pi}^{(2)} = \frac{1}{12}\left(\sum_{i=1}^{N_c}\left(T^{(2)}_{i-1} - \frac{6n}{n-1}\right)\tilde{t}_{i-1}\frac{\partial}{\partial \tilde{t}_i}\frac{\partial}{\partial \tilde{t}_{\tilde{N}-1}}\tilde{\Pi}^{(0)} + \left(T^{(2)}_{N_c} - \frac{6n}{n-1}\right)\tilde{t}_{N_c}\frac{\partial}{\partial \tilde{t}_{N_c}}\frac{\partial}{\partial \tilde{t}_{\tilde{N}}}\tilde{\Pi}^{(0)}\right), \tag{6.126}$$

$$\tilde{\Pi}^{(4)} = \frac{7}{1440}\left(\frac{5}{7}\hat{\mathcal{O}}^{(4)}_A\tilde{\Pi}^{(0)} + \hat{\mathcal{O}}^{(4)}_B\tilde{\Pi}^{(0)} + \hat{\mathcal{O}}^{(4)}_C\tilde{\Pi}^{(0)} + \hat{\mathcal{O}}^{(4)}_D\tilde{\Pi}^{(0)}\right), \tag{6.127}$$

where

$$\hat{\mathcal{O}}^{(4)}_A = \sum_{i=1}^{N_c}\left(T^{(4)}_{i-1} + \frac{6n}{n-1}T^{(2)}_{i-1} + d_1(n)\right)\tilde{t}_{i-1}\frac{\partial}{\partial \tilde{t}_i}\frac{\partial}{\partial \tilde{t}_{\tilde{N}-1}}\frac{\partial}{\partial \tilde{t}_{\tilde{N}}}, \tag{6.128}$$

$$\hat{\mathcal{O}}^{(4)}_B = \sum_{i=1}^{N_c}\sum_{j=1}^{N_c}\left(T^{(2)}_{i-1}T^{(2)}_{j-1} + d_2(n)\right)\tilde{t}_{i-1}\tilde{t}_{j-1}\frac{\partial}{\partial \tilde{t}_i}\frac{\partial}{\partial \tilde{t}_j}\frac{\partial^2}{\partial \tilde{t}^2_{\tilde{N}-1}}, \tag{6.129}$$

$$\hat{\mathcal{O}}^{(4)}_C = \sum_{i=1}^{N_c}\left(\frac{2n(2n-1)}{(n-1)^2}T^{(2)}_{i-2} + d_3(n)\right)\tilde{t}_{i-2}\tilde{t}_{N_c}\frac{\partial}{\partial \tilde{t}_i}\frac{\partial}{\partial \tilde{t}_{N_c}}\frac{\partial^2}{\partial \tilde{t}^2_{\tilde{N}-1}}, \tag{6.130}$$

$$\hat{\mathcal{O}}^{(4)}_D = d_4(n)\tilde{t}_{N_c}\tilde{t}_{N_c}\frac{\partial^2}{\partial \tilde{t}^2_{N_c}}\frac{\partial^2}{\partial \tilde{t}^2_{\tilde{N}}}. \tag{6.131}$$

The coefficients $d_i(n)$ $(i = 1, \ldots, 4)$ are given by

$$d_1(n) := -7T^{(4)}_{N_c} + \left(T^{(2)}_{N_c-1}\right)^2 + \frac{4n(14n-23)}{(n-1)^2}T^{(2)}_{N_c}, \tag{6.132}$$

$$d_2(n) := \frac{1}{7}\left(-6T^{(4)}_{N_c} + \left(T^{(2)}_{N_c-1}\right)^2 + \frac{16n(3n-7)}{(n-1)^2}T^{(2)}_{N_c}\right), \tag{6.133}$$

$$d_3(n) := -\frac{4n(n+1)}{(n-1)^2}T^{(2)}_{N_c}, \tag{6.134}$$

$$d_4(n) := 6T^{(4)}_{N_c} - \left(T^{(2)}_{N_c-1}\right)^2 - \frac{n(62n-77)}{(n-1)^2}T^{(2)}_{N_c}. \tag{6.135}$$

As discussed in the case of the $N_f = 2n+1$ theory, we should use (6.126) and (6.127) with keeping the irrelevant operator finite since these formulas have the differential operator with respect to the irrelevant operator on the SW periods. After using (6.126) and (6.127), we obtain the quantum correction for the AD theory in the B sector by taking the limit where the irrelevant operators go to zero.

Summary

In this chapter, we studied the quantum SW periods for the Argyres-Douglas theories associated with $SU(N_c)$ SQCD. The quantum SW curve takes the different form for each N_f. The SW differential also takes the different form, which introduces the different quantization condition. In the case of $SU(2)$ SQCD, we obtained the relation between the quantum corrections and the classical SW periods. We computed the quantum SW periods around the superconformal point up to fourth order in $\hbar$. We then wrote down the explicit form of the quantum correction to the SW periods in terms of hypergeometric function up to the fourth order in $\hbar$. The quantum SW periods of the AD theory are shown to agree with those of the original SQCD by taking the scaling limit. We found the general formulas for the second and fourth order corrections in the AD theories realized from the $SU(N_c)$ SQCD, which are obtained from the SW periods by acting the differential operators.

Chapter 7

Conclusions and Discussions

In this thesis, we studied the low-energy effective theory of $SU(N_c)$ SQCD in the NS limit of the Ω-background. In chapter 2, we reviewed the basic idea of the Seiberg-Witten theory by adopting the $\mathcal{N}=2$ supersymmetric $SU(2)$ Yang-Mills theory as an example. We then introduced the Seiberg-Witten curve and the SW differential for the $SU(N_c)$ gauge theory with N_f hypermultiplets. In chapter 3, we obtained the SW curve for the AD theory by taking the scaling limit of the corresponding $\mathcal{N}=2$ gauge theory. For $SU(2)$ theory, the corresponding SW curves take the form of the cubic elliptic curve for all N_f, but the SW differentials take the different form. For $SU(N_c)$ SQCD, the AD theories are classified four groups by the number of the hypermultiplets.

In chapter 4, we introduced the Ω-deformed $\mathcal{N}=2$ supersymmetric gauge theories in the four-dimensional spacetime. In the weak coupling region, the Nekrasov partition function provides an exact formula of the prepotential including the instanton contribution. We then took the Nekrasov-Shatashvili limit of the Ω-background. In this limit, the low-energy effective theories appear in the two-dimensional Ω-background with one deformation parameter ϵ_1. The two-dimensional supersymmetric vacua condition is found to induce that the SW periods satisfy the Bohr-Sommerfeld quantization condition.

In chapter 5, we studied the low-energy effective theory of $SU(2)$ gauge theories with N_f hypermultiplets in the NS limit of the Ω-background. The deformed SW periods are given by the quantum SW curve, which is the ordinary differential equation and can be solved by the WKB method. By using the quantum SW curve and the Picard-Fuchs equation, it is possible to solve the series expansion with respect to the Coulomb moduli parameter and the deformation parameter $\hbar$. We found that the second and fourth order corrections to the SW periods are represented by simple formulas which are obtained by applying the differential operators on the SW periods. In the weak coupling region, we evaluated the quantum SW periods up to fourth order in $\hbar$. By using the quantum SW periods, we obtained the same prepotential as that given from the NS limit of the Nekrasov partition function. We then investigated the expansion of the quantum correction to the SW periods around the massless monopole point. The quantum corrections to the dual SW periods a_D are given by solving the Picard-Fuchs equation for the SW periods. Then we found the massless monopole points on the u-plane are shifted by the quantum corrections.

We also studied the quantum SW periods around the superconformal point in chapter 6. Since the SW differentials take the different form for each N_f, we need to introduce the different quantization condition. We also have the simple formulas to represent the second and fourth order corrections, which obtained from the classical periods by acting the differential operator with respect to operators and their corresponding coupling. They are shown to agree with the scaling limit of the formulas for the quantum SW periods of the original SQCD. For $SU(2)$ SQCD, we computed the quantum correction to the SW periods up to fourth order in $\hbar$ in terms of hypergeometric functions. Around the superconformal point, the SW periods and the effective coupling constant are expanded in the Coulomb moduli parameter with the fractional scaling dimension. We also find the general formulas for the second and fourth order corrections in AD theories associated with $\mathcal{N}=2$ $SU(N_c)$ SQCD.

It is interesting to explore the higher order corrections and how the structure of the moduli space is modified by the quantum corrections. In particular non-perturbative structure of the $\hbar$-corrections can be studied with the help of the resurgence method [60–63].

Although the SW differential for the AD theory associated with certain gauge theory takes the different form to that associated with other gauge theory, there are cases that both AD theories belong to same universality class [20, 23]. For example, the pure $SU(4)$ gauge theory associated with the AD theory of the (A_1, A_3)-type, which corresponds to the $SU(2)$ gauge theory with $N_f = 2$ hypermultiplets. Around the superconformal point of the pure $SO(8)$ gauge theory, the curve describes the same AD theory, which is (A_1, D_4) type, as the $SU(2)$ gauge theory with $N_f = 3$ hypermultiplets. It would be interesting to study the universality classes of $\mathcal{N} = 2$ SCFT in the NS limit of the Ω-background.

For AD theory, there are the singularities on the moduli space where one of the periods becomes the logarithmic behavior. It would be interesting to describe the theory around this point by the Nekrasov partition function.

The Ω-deformed theories in the NS limit are described by certain quantum integrable systems. The quantum SW curve yields the same data as the integrable systems. For the AD theory obtained from the $SU(N_c)$ Yang-Mills theory, the quantum curve takes the form of the Schrödinger equation with the polynomial potential. In [64], from the viewpoint of the ODE/IM correspondence (for a review see [65]), the exponential of the quantum period have been shown to be regarded as the Y-function of the quantum integrable model associated with the Yang-Lee edge singularity. It is interesting to investigate this relationship further by computing higher order corrections via the ODE/IM correspondence.

Appendix A

Short introduction to supersymmetry

A.1 Supersymmetry algebra

In this section, let us introduce the supersymmetry algebra in four-dimensional spacetime. The Poincaré symmetry is generated by the translations in $\mathbf{R}^{1,3}$ and the Lorentz transformations with the generators P_μ and $L_{\mu\nu}$, respectively (where the indices run over $\mu,\, \nu = 0, 1, 2, 3$). The Lorentz group $SO(3,1)$ is isomorphic to $SU(2)_L \times SU(2)_R$, labeled by two positive (or zero) spins (s_+, s_-) where $s_\pm \in \mathbf{Z}/2$. We show the representations with $s_\pm$ in the four-dimensional theory in the table A.1.

Representation	(s_+, s_-)	Representation	(s_+, s_-)
Scalar	$(0, 0)$	Left chiral fermion	$\left(\frac{1}{2}, 0\right)$
4-vector	$\left(\frac{1}{2}, \frac{1}{2}\right)$	Right chiral fermion	$\left(0, \frac{1}{2}\right)$
Symmetric tensors (rank 2)	(1,1)	Self-dual anti-symmetric tensor (rank 2)	$(1, 0)$
		Anti-selfdual anti-symmetric tensor (rank 2)	$(0, 1)$

Table A.1: The finite dimensional representations of $SL(2, \mathbf{C}) \sim SU(2)_L \times SU(2)_R$

The supersymmetry enlarges the Poincaré algebra by introducing the supercharge:

$$\begin{array}{lll} {Q_\alpha}^I & \text{left Weyl spinor} & (s_+, s_-) = \left(\frac{1}{2}, 0\right), \\ \bar{Q}_{\dot{\alpha} I} = \left({Q_\alpha}^I\right)^\dagger & \text{right Weyl spinor} & (s_+, s_-) = \left(0, \frac{1}{2}\right), \end{array} \tag{A.1}$$

$$\alpha,\, \dot{\alpha} = 1, 2, \qquad I = 1, 2, \cdots, \mathcal{N}, \tag{A.2}$$

where α and $\dot{\alpha}$ are the Weyl spinor indices and the label $I = 1, \cdots \mathcal{N}$ is the number of the independent supersymmetry. The supercharges transform as the Weyl spinors of $SO(3,1)$ and are translation invariant $[P_\mu, Q_\alpha{}^I] = 0$. The relevant relations of the supersymmetry algebra generators are given by

$$\{Q^I_\alpha, \bar{Q}_{\dot{\beta}J}\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^I_J, \tag{A.3}$$

$$\{Q^I_\alpha, Q^J_\beta\} = 2\sqrt{2}\epsilon_{\alpha\beta} Z^{IJ}, \tag{A.4}$$

$$\{\bar{Q}_{\dot{\alpha}I}, \bar{Q}_{\dot{\beta}J}\} = 2\sqrt{2}\epsilon_{\dot{\alpha}\dot{\beta}} Z^*_{IJ}, \tag{A.5}$$

where $\sigma^\mu_{\alpha\dot{\beta}}$ is the Pauli matrices and $\epsilon_{\alpha\beta}$ is the anti-symmetric tensor where $\epsilon_{12} = -\epsilon_{21} = -1$. The generators Z^{IJ} and Z^*_{IJ} are the anti-symmetric in the indices I and J and commute with all generators of the supersymmetric algebra, called central charges:

$$Z^{IJ} = -Z^{JI}, \qquad [Z^{IJ}, \text{anything}] = 0. \tag{A.6}$$

The central charge vanishes for the $\mathcal{N} = 1$ supersymmetry.

Let us study the representation of supersymmetry algebra. We firstly discuss the irreducible massless representation in which one can choose a Lorentz frame with $P^\mu = E(-1, 0, 0, 1)$. Then the supersymmetry algebra becomes

$$\{Q^I_\alpha, \bar{Q}_{\dot{\beta}J}\} = \begin{pmatrix} 4E & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot{\beta}} \delta^I_J. \tag{A.7}$$

The unitarity of the theory implies $\{Q^I_2, \bar{Q}_{\dot{2}J}\} = 0$ i.e. $Q^I_2 = 0$ and $Z^{IJ} = 0$. The remaining supercharges Q^I_1 and $\bar{Q}_{\dot{1}I}$ play a role of lowering and raising operators for helicity of the state by $\frac{1}{2}$. We define the Clifford vacuum $|\Omega_\lambda\rangle$ with the lowest helicity λ which satisfy $Q^I_1 |\Omega_\lambda\rangle = 0$. All the states in the massless representation can be constructed by acting the supercharge $\bar{Q}_{\dot{1}I}$ on $|\Omega_\lambda\rangle$. In CPT invariant theories, a fundamental multiplet contains the constructed state and its CPT conjugate. For $\mathcal{N} = 1$ and 2, the state in theories without gravity are listed in table A.2. For $\mathcal{N} = 1$, the multiplet with $\lambda = \frac{1}{2}$ is called a chiral multiplet while that with $\lambda = 0$ is a vector multiplet. For $\mathcal{N} = 2$, the multiplets with $\lambda = -\frac{1}{2}$ and 0 are called a vector multiplet and a hypermultiplet, respectively.

We next consider the irreducible massive representation. One can choose the rest frame with $P^\mu = (M, 0, 0, 0)$ so that the supersymmetry algebra (A.3) becomes

$$\{Q^I_\alpha, (Q^J_\beta)^\dagger\} = 2M\delta^\beta_\alpha \delta^I_J. \tag{A.8}$$

Helicity (≤ 1)	$\mathcal{N}=1$ $\lambda=\frac{1}{2}$	$\mathcal{N}=1$ $\lambda=0$	$\mathcal{N}=2$ $\lambda=0$	$\mathcal{N}=2$ $\lambda=-\frac{1}{2}$
1	1	0	1	0
$\frac{1}{2}$	1	1	2	2
0	0	$1+1$	$1+1$	4
$-\frac{1}{2}$	1	1	2	2
-1	1	0	1	0

Table A.2: Massless representations for $\mathcal{N}=1,2$ supersymmetry.

In this case, we have two sets of ladder operators for helicity. We consider the case of $\mathcal{N}=2$ supersymmetry. Under the unitary transformation, the central charge can be chosen the form:

$$Z^{IJ}=\epsilon^{IJ}Z, \tag{A.9}$$

with Z being the real values. We define the linear combinations of the supercharge, given by

$$\mathcal{Q}^1_\alpha=\frac{1}{2}\left(Q^1_\alpha+\epsilon_{\alpha\beta}(Q^2_\beta)^\dagger\right), \qquad \mathcal{Q}^2_\alpha=\frac{1}{2}\left(Q^1_\alpha-\epsilon_{\alpha\beta}(Q^2_\beta)^\dagger\right). \tag{A.10}$$

Using the supersymmetry algebra, they satisfy

$$\left\{\mathcal{Q}^1_\alpha,(\mathcal{Q}^1_\beta)^\dagger\right\}=\delta_{\alpha\beta}\left(M+\sqrt{2}Z\right), \qquad \left\{\mathcal{Q}^2_\alpha,(\mathcal{Q}^2_\beta)^\dagger\right\}=\delta_{\alpha\beta}\left(M-\sqrt{2}Z\right), \tag{A.11}$$

with all other anticommutators vanishing. Since all physical states should be the positive norm, we find a bound on the mass $M\geq\sqrt{2}|Z|$, called the BPS bound. For the saturation of the BPS bound, the states belong to a smaller representation of the supersymmetry algebra. The massive representation for the $\mathcal{N}=1$ and 2 is listed in table A.3.

A.2 $\mathcal{N}=1$ superfield

To study the $\mathcal{N}=1$ supersymmetric theories, it is convenient to introduce the Grassmann spinors θ^α and $\bar{\theta}_{\dot{\alpha}}$ in addition to the space-time coordinate x^μ. The Grassmann coordinates of superspace are defined by

$$\{\theta^\alpha,\theta^\beta\}=\{\bar{\theta}_{\dot{\alpha}},\bar{\theta}_{\dot{\beta}}\}=\{\theta^\alpha,\bar{\theta}_{\dot{\beta}}\}=0 \tag{A.12}$$

Spin (≤ 1)	$\mathcal{N}=1$ $\lambda=\frac{1}{2}$	$\mathcal{N}=1$ $\lambda=0$	$\mathcal{N}=2$ $\lambda=0$	$\mathcal{N}=2$ BPS $\lambda=0$	$\mathcal{N}=2$ BPS $\lambda=-\frac{1}{2}$
1	1	0	1	1	0
$\frac{1}{2}$	2	1	2	2	2
0	1	2	5	1	4

Table A.3: Massive representation for $\mathcal{N}=1$ and 2 supersymmetry. λ is the lowest spin of the Clifford vacuum.

In the following, we use the contraction conventions for the Grassmann spinors:

$$\theta\theta = \theta^\alpha\theta_\alpha, \quad \bar\theta\bar\theta = \bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha}, \tag{A.13}$$

$$\theta\sigma^\mu\bar\theta = \theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}, \quad \bar\theta\bar\sigma^\mu\theta = \bar\theta_{\dot\alpha}\bar\sigma^{\mu\dot\alpha\alpha}\theta_\alpha. \tag{A.14}$$

The integration over θ and $\bar\theta$ is defined by

$$\int d^2\theta\theta\theta = \int d^2\bar\theta\bar\theta\bar\theta = \int d^4\theta\theta\theta\bar\theta\bar\theta = 1. \tag{A.15}$$

The supercharges acting on the superspace are given by

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu, \qquad \bar Q_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu, \tag{A.16}$$

which satisfy

$$\{Q_\alpha, \bar Q_{\dot\beta}\} = 2i\sigma^\mu_{\alpha\dot\beta}\partial_\mu. \tag{A.17}$$

We also introduce the supercovariant derivatives:

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu, \qquad \bar D_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu, \tag{A.18}$$

which satisfy $\{D_\alpha, \bar D_{\dot\beta}\} = -2i\sigma^\mu_{\alpha\dot\beta}\partial_\mu$ and also anticommute with Q and $\bar Q$.

A superfield is defined by a function of x^μ, θ_α and $\bar\theta^{\dot\alpha}$. From the anticommutativity of the Grassmann coordinates, the superfield can be written as the finite series expansions in powers of the Grassmann variables θ and $\bar\theta$:

$$\begin{aligned} S(x,\theta,\bar\theta) =& \phi(x) + \theta\psi(x) + \bar\theta\bar\chi(x) + \theta\theta F(x) + \bar\theta\bar\theta G^*(x) \\ &+ \theta\sigma^\mu\bar\theta A_\mu(x) + \theta\theta\bar\theta\bar\lambda(x) + \bar\theta\bar\theta\theta\rho(x) + \theta\theta\bar\theta\bar\theta D(x). \end{aligned} \tag{A.19}$$

Under infinitesimal supersymmetry transformation, the superfield $S(x,\theta,\bar{\theta})$ transforms as

$$\delta_\xi S(x,\theta,\bar{\theta}) = (\xi^\alpha Q_\alpha + \bar{\xi}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}})S, \tag{A.20}$$

where the parameters ξ^α and $\bar{\xi}_{\dot{\alpha}}$ are anticommuting parameters. This can lead to the supersymmetric transformation of component fields including in the superfield. We then find that the variation of the top component $D(x)$ become a total derivative. The component fields belonging to the representation of the supersymmetry can be constructed from superfield $S(x,\theta,\bar{\theta})$, but it is highly reducible. The irreducible components can be derived by imposing the constraints on the superfield.

A chiral superfield Φ is defined by imposing the condition

$$\bar{D}_{\dot{\alpha}}\Phi = 0. \tag{A.21}$$

We introduce the coordinate $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$, satisfying

$$\bar{D}_{\dot{\alpha}}y^\mu = 0, \quad \bar{D}_{\dot{\alpha}}\theta^\beta = 0. \tag{A.22}$$

Then the chiral superfield can be written by the function (y,θ):

$$\Phi(y,\theta) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y). \tag{A.23}$$

If Φ is a scalar superfield, the components ϕ and ψ are the scalar and spinor fields respectively and F is an auxiliary field. Thus the $\mathcal{N}=1$ chiral multiplet can be represented by the chiral superfield. Similarly an antichiral superfield $\Phi^\dagger$ satisfies $D_\alpha\Phi^\dagger = 0$ and can be expanded as

$$\Phi^\dagger(y^\dagger,\bar{\theta}) = \phi^\dagger(y^\dagger) + \sqrt{2}\bar{\theta}\bar{\psi}(y^\dagger) + \bar{\theta}\bar{\theta}F^\dagger(y^\dagger), \tag{A.24}$$

where $y^{\dagger\mu} = x^\mu - i\theta\sigma^\mu\bar{\theta}$.

Here we consider the function constructed by chiral superfields Φ_i: $\mathcal{W}(\Phi_i)$. In general the function $\mathcal{W}(\Phi_i)$ is also a chiral superfield. The term with the highest power of θ of $\mathcal{W}(\Phi_i)$ is given by

$$\int d^2\theta \mathcal{W}(\Phi_i) = \frac{\partial\mathcal{W}}{\partial\phi_i}F_i - \frac{1}{2}\frac{\partial^2\mathcal{W}}{\partial\phi_i\partial\phi_j}\psi_i\psi_j. \tag{A.25}$$

Under infinitesimal supersymmetry transformation (A.20), this term becomes total derivatives. The function $\mathcal{W}(\Phi_i)$ is called the superpotential.

Any arbitrary function of Φ and $\Phi^\dagger$ is neither the chiral nor antichiral superfields. In the component fields, it is written by

$$\int d^4\theta K(\Phi_i, \Phi_j^\dagger) = \frac{1}{2}\frac{\partial^2 K}{\partial\phi_i\phi_j^\dagger}F_i F_j^\dagger - \frac{1}{2}\frac{\partial^3 K}{\partial\phi_i\partial\phi_j\partial\phi_k^\dagger}\psi_i\psi_j F_k^\dagger - \frac{1}{8}\frac{\partial^4 K}{\partial\phi_i\partial\phi_j\partial\phi_k^\dagger\partial\phi_l^\dagger}\psi_i\psi_j\bar{\psi}_k\bar{\psi}_l. \tag{A.26}$$

We also find that the variation of it becomes total derivatives. $K(\Phi_i, \Phi_j^\dagger)$ is referred as the Kähler potential.

A vector superfield satisfies the reality condition

$$V = V^\dagger. \tag{A.27}$$

In a similar way, the vector superfield is expanded as

$$\begin{aligned}
V(x,\theta,\bar{\theta}) =& C(x) + i\theta\chi(x) - i\bar{\theta}\bar{\chi}(x) \\
&+ \frac{i}{2}\theta\theta[M(x) + iN(x)] - \frac{i}{2}\bar{\theta}\bar{\theta}[M(x) - iN(x)] \\
&- \theta\sigma^\mu\bar{\theta}A_\mu(x) + i\theta\theta\bar{\theta}\left[\bar{\lambda}(x) + \frac{i}{2}\bar{\sigma}^\mu\partial_\mu\chi(x)\right] \\
&- i\bar{\theta}\bar{\theta}\theta\left[\lambda(x) + \frac{i}{2}\sigma^\mu\partial_\mu\bar{\chi}(x)\right] + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\left[D(x) + \frac{1}{2}\partial_\mu\partial^\mu C(x)\right],
\end{aligned} \tag{A.28}$$

where the component field C, D, M, N are the real fields and A_μ is regarded as the vector field.

We firstly consider in the case of the Abelian gauge theory. We find λ and D are invariant under the Abelian gauge transformation:

$$V \to V + \Phi_g + \Phi_g^\dagger, \tag{A.29}$$

where Φ_g and $\Phi_g^\dagger$ are the chiral and antichiral superfield. Thus λ and D are regarded as the gaugino and the auxiliary field, respectively. The Abelian field strength is defined by

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}D_\alpha V, \quad \bar{W}_\alpha = -\frac{1}{4}DD\bar{D}_{\dot{\alpha}}V, \tag{A.30}$$

so that W_α is a gauge invariant chiral superfield. Under the gauge transformation, we choose a special gauge fixing: $C=\chi=M=N=0$, called the Wess-Zumino gauge. In component fields, the Abelian field strength becomes

$$W_\alpha = -i\lambda_\alpha(y) + \theta_\alpha D - \frac{i}{2}(\sigma^\mu\bar{\sigma}^\nu\theta)_\alpha F_{\mu\nu} + \theta^2(\sigma^\mu\partial_\mu\bar{\lambda})_\alpha, \tag{A.31}$$

where $F_{\mu\nu}$ is the Abelian field strength. For the non-Abelian gauge, the vector superfield V belongs to the adjoint representation of the gauge group. The field strength is given by

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}\left(e^{-2V}D_\alpha e^{2V}\right), \qquad \bar{W}_{\dot\alpha} = -\frac{1}{4}DD\left(e^{2V}\bar{D}_\alpha e^{-2V}\right). \tag{A.32}$$

Under the gauge transformation:

$$e^V \to e^{-i\Phi_g^\dagger}e^{2V}e^{i\Phi_g}, \tag{A.33}$$

the field strength becomes $W_\alpha \to e^{-i\Phi_g}W_\alpha e^{i\Phi_g}$. In the Wess-Zumino gauge, the field strength is expanded as

$$W_\alpha = -i\lambda_\alpha(y) + \theta_\alpha D - \frac{i}{2}(\sigma^\mu\bar{\sigma}^\nu\theta)_\alpha F_{\mu\nu} + \theta^2(\sigma^\mu D_\mu\bar{\lambda})_\alpha, \tag{A.34}$$

where $F_{\mu\nu}$ and D_μ are the non-Abelian field strength and the covariant derivative.

Now we can write down the $\mathcal{N}=1$ supersymmetric Lagrangian in terms of the superfields. In the non-Abelian gauge theory with the chiral and vector multiplets, the renormalizable Lagrangian is given by

$$\mathcal{L} = \frac{1}{8\pi}\mathrm{Im}\left(\tau\mathrm{Tr}\int d^2\theta W^\alpha W_\alpha\right) + \int d^4\theta\Phi^\dagger e^{-2V}\Phi + \int d^2\theta\mathcal{W} + \int d^2\bar{\theta}\bar{\mathcal{W}}, \tag{A.35}$$

where τ is the complex gauge coupling: $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$.

A.3 $\mathcal{N}=2$ superfield

For $\mathcal{N}=2$ supersymmetric field theories, we introduce the coordinates of the $\mathcal{N}=2$ superspace as

$$x^\mu, \qquad \theta_{\alpha I} := (\theta_{\alpha 1}, \theta_{\alpha 2}), \qquad \bar{\theta}^{\dot\alpha J} := (\bar{\theta}^{\dot\alpha 1}, \bar{\theta}^{\dot\alpha 2}), \quad (I,J=1,2) \tag{A.36}$$

with $\{\theta, \theta\} = \{\bar{\theta}, \bar{\theta}\} = 0$. The generic superfield is defined as a function of $(x^\mu, \theta_I, \bar{\theta}^J)$, expanded as

$$S(x, \theta_I, \bar{\theta}^J) = \phi(x) + \theta_1\psi(x) + \bar{\theta}_1\bar{\chi}(x) + \cdots + \theta_1^2\theta_2^2\bar{\theta}_1^2\bar{\theta}_2^2 D(x). \tag{A.37}$$

As in the case of $\mathcal{N} = 1$, this is reducible. We need to impose the constraint on the superfields.

A $\mathcal{N} = 2$ chiral superfield is defined by imposing

$$\bar{D}_{\dot{\alpha}I}\Psi = 0, \tag{A.38}$$

where $\bar{D}_{\dot{\alpha}I}$ is the $\mathcal{N} = 2$ supercovariant derivative:

$$D_\alpha^I = \frac{\partial}{\partial\theta_I^\alpha} + i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}I}\partial_\mu, \qquad \bar{D}_\alpha^I = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}I}} + i\theta_I^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu. \tag{A.39}$$

Since the $\mathcal{N} = 2$ chiral superfield is reducible, we impose the constraint on the superfields, which given by

$$D^I D^J \Psi = \bar{D}^I \bar{D}^J \Psi = 0. \tag{A.40}$$

The expansion of Ψ in powers of θ_2 takes the form:

$$\Psi(y, \theta) = \Psi^{(1)}(y, \theta_1) + \sqrt{2}\theta_2^\alpha \Psi_\alpha^{(2)}(y, \theta_1) + \theta_2^\alpha\theta_{2\alpha}\Psi^{(3)}(y, \theta_1), \tag{A.41}$$

where $y^\mu = x^\mu + i\theta_1\sigma^\mu\bar{\theta}_1 + i\theta_2\sigma^\mu\bar{\theta}_2$. The $\mathcal{N} = 2$ multiplets can be expressed in terms of $\mathcal{N} = 1$ multiplets:

$$\Psi^{(1)} = \Phi(y, \theta_1), \quad \Psi^{(2)} = W_\alpha(y, \theta_1), \quad \Psi^{(3)} = \int d^2\bar{\theta}_1\, \Phi^\dagger(y - i\theta_1\sigma\bar{\theta}_1, \theta_1, \bar{\theta}_1)e^{-2V(y - i\theta_1\sigma\bar{\theta}_1, \theta_1, \bar{\theta}_1)}, \tag{A.42}$$

where Φ is the $\mathcal{N} = 1$ chiral superfield while W_α is the $\mathcal{N} = 1$ field strength. We then find that the component fields of Ψ are those of the $\mathcal{N} = 2$ vector multiplet.

By using the $\mathcal{N} = 2$ chiral superfield, the Lagrangian for the $\mathcal{N} = 2$ pure Yang-Mills theory can be written down as

$$\mathcal{L} = \frac{1}{4\pi}\mathrm{ImTr}\int d^2\theta_1 d^2\theta_2\, \frac{1}{2}\tau\Psi^2. \tag{A.43}$$

In terms of the $\mathcal{N}=1$ superfield, it takes the form

$$\mathcal{L} = \frac{1}{4\pi}\mathrm{ImTr}\left[\tau\left(\int d^4\theta\ \Phi^\dagger e^{-2V}\Phi + \frac{1}{2}\int d^2\theta\ W^\alpha W_\alpha\right)\right], \tag{A.44}$$

where $\theta := \theta_1$. We then introduce the function of Ψ as $\mathcal{F}(\Psi)$, which is also the $\mathcal{N}=2$ chiral superfield. For the $\mathcal{N}=2$ pure Yang-Mills theory, the generic Lagrangian is given by

$$\begin{aligned}\mathcal{L} =& \frac{1}{4\pi}\mathrm{ImTr}\int d^2\theta_1 d^2\theta_2\ \mathcal{F}(\Psi)\\ =& \frac{1}{4\pi}\mathrm{ImTr}\left[\int d^4\theta\ \Phi^\dagger e^{-2V}\frac{\partial\mathcal{F}(\Phi)}{\partial\Phi} + \frac{1}{2}\int d^2\theta\ \frac{\partial^2\mathcal{F}(\Phi)}{\partial\Phi^2}W^\alpha W_\alpha\right].\end{aligned} \tag{A.45}$$

where the function $\mathcal{F}(\Psi)$ is called the prepotential.

Appendix B

Coefficients $\mathcal{F}_{N_f}^{(2k,n)}$ for $N_f = 2, 3$ and 4 theories

In this appendix we explicitly write down some coefficients in the expansion of the prepotentials for $N_f = 2, 3, 4$ theories in the weak coupling region.

B.1 $N_f = 2$

For the $N_f = 2$ theory, the first four coefficients of the classical part of the prepotential in (5.67) are

$$\begin{aligned}
\mathcal{F}_2^{(0,1)} =& \frac{\Lambda_2^4}{4096} + \frac{1}{32}\Lambda_2^2 m_1 m_2, \\
\mathcal{F}_2^{(0,2)} =& -\frac{3\Lambda_2^4 m_1^2}{8192} - \frac{3\Lambda_2^4 m_2^2}{8192}, \\
\mathcal{F}_2^{(0,3)} =& \frac{5\Lambda_2^8}{134217728} + \frac{5\Lambda_2^4 m_1^2 m_2^2}{16384} + \frac{5\Lambda_2^6 m_1 m_2}{196608}, \\
\mathcal{F}_2^{(0,4)} =& -\frac{63\Lambda_2^8 m_1^2}{134217728} - \frac{63\Lambda_2^8 m_2^2}{134217728} - \frac{7\Lambda_2^6 m_1^3 m_2}{393216} - \frac{7\Lambda_2^6 m_1 m_2^3}{393216}.
\end{aligned} \tag{B.1}$$

The coefficients in the second order correction to the prepotential are

$$\begin{aligned}
\mathcal{F}_2^{(2,1)} =& 0, \\
\mathcal{F}_2^{(2,2)} =& \frac{\Lambda_2^4}{8192} + \frac{1}{256}\Lambda_2^2 m_1 m_2, \\
\mathcal{F}_2^{(2,3)} =& -\frac{15\Lambda_2^4 m_1^2}{65536} - \frac{15\Lambda_2^4 m_2^2}{65536},
\end{aligned}$$

$$\mathcal{F}_2^{(2,4)} = \frac{21\Lambda_2^8}{134217728} + \frac{21\Lambda_2^4 m_1^2 m_2^2}{65536} + \frac{35\Lambda_2^6 m_1 m_2}{786432}. \tag{B.2}$$

For the fourth order corrections, they are

$$\begin{aligned}
\mathcal{F}_2^{(4,1)} &= 0, \\
\mathcal{F}_2^{(4,2)} &= 0, \\
\mathcal{F}_2^{(4,3)} &= \frac{\Lambda_2^4}{16384} + \frac{\Lambda_2^2 m_1 m_2}{2048}, \\
\mathcal{F}_2^{(4,4)} &= -\frac{63\Lambda_2^4 m_1^2}{524288} - \frac{63\Lambda_2^4 m_2^2}{524288}.
\end{aligned} \tag{B.3}$$

B.2 $N_f = 3$

For $N_f = 3$, the coefficients of the prepotential in the expansion (5.67) are given by

$$\begin{aligned}
\mathcal{F}_3^{(0,1)} =& \frac{\Lambda_3^4}{33554432} + \sum_{i=1}^{3} \frac{\Lambda_3^2 m_i^2}{4096} + \frac{1}{32}\Lambda_3 m_1 m_2 m_3, \\
\mathcal{F}_3^{(0,2)} =& \sum_{i=1}^{3} -\frac{3\Lambda_3^4 m_i^2}{33554432} - \sum_{i<j} \frac{3\Lambda_3^2 m_i^2 m_j^2}{8192} - \frac{\Lambda_3^3 m_1 m_2 m_3}{32768}, \\
\mathcal{F}_3^{(0,3)} =& \frac{5\Lambda_3^8}{4503599627370496} + \sum_{i=1}^{3} \left(\frac{5\Lambda_3^6 m_i^2}{103079215104} + \frac{5\Lambda_3^4 m_i^4}{134217728} + \frac{5\Lambda_3^3 m_1 m_2 m_3 m_i^2}{196608} \right) \\
&+ \sum_{i<j} \frac{25\Lambda_3^4 m_i^2 m_j^2}{33554432} + \frac{5\Lambda_3^2 m_1^2 m_2^2 m_3^2}{16384} + \frac{7\Lambda_3^5 m_1 m_2 m_3}{268435456}, \\
\mathcal{F}_3^{(0,4)} =& \sum_{i=1}^{3} \left(-\frac{63\Lambda_3^8 m_i^2}{2251799813685248} - \frac{7\Lambda_3^6 m_i^4}{103079215104} - \frac{21\Lambda_3^5 m_i^2 m_1 m_2 m_3}{268435456} \right) + \sum_{i\neq j} -\frac{63\Lambda_3^4 m_i^4 m_j^2}{134217728} \\
&+ \sum_{i<j} \left(-\frac{35\Lambda_3^6 m_i^2 m_j^2}{34359738368} - \frac{7\Lambda_3^3 m_i^2 m_j^2 m_1 m_2 m_3}{393216} \right) - \frac{3\Lambda_3^7 m_1 m_2 m_3}{137438953472} - \frac{147\Lambda_3^4 m_1^2 m_2^2 m_3^2}{33554432},
\end{aligned} \tag{B.4}$$

for the classical part,

$$\begin{aligned}
\mathcal{F}_3^{(2,1)} =& -\frac{\Lambda_3^2}{16384}, \\
\mathcal{F}_3^{(2,2)} =& \frac{5\Lambda_3^4}{134217728} + \sum_{i=1}^{3} \frac{\Lambda_3^2 m_i^2}{8192} + \frac{1}{256}\Lambda_3 m_1 m_2 m_3,
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}_3^{(2,3)} =& -\frac{5\Lambda_3^6}{412316860416} - \sum_{i=1}^{3} \frac{65\Lambda_3^4 m_i^2}{268435456} - \sum_{i<j} \frac{15\Lambda_3^2 m_i^2 m_j^2}{65536} - \frac{35\Lambda_3^3 m_1 m_2 m_3}{786432}, \\
\mathcal{F}_3^{(2,4)} =& \frac{105\Lambda_3^8}{9007199254740992} + \sum_{i=1}^{3} \left(\frac{35\Lambda_3^6 m_i^2}{103079215104} + \frac{21\Lambda_3^4 m_i^4}{134217728} + \frac{35\Lambda_3^3 m_1 m_2 m_3 m_i^2}{786432} \right) \\
& + \sum_{i<j} \frac{147\Lambda_3^4 m_i^2 m_j^2}{67108864} + \frac{63\Lambda_3^5 m_1 m_2 m_3}{536870912} + \frac{21\Lambda_3^2 m_1^2 m_2^2 m_3^2}{65536}, \qquad \text{(B.5)}
\end{aligned}$$

for the second order in $\hbar$ and

$$\begin{aligned}
\mathcal{F}_3^{(4,1)} =& 0, \\
\mathcal{F}_3^{(4,2)} =& -\frac{\Lambda_3^2}{32768}, \\
\mathcal{F}_3^{(4,3)} =& \frac{141\Lambda_3^4}{2147483648} + \sum_{i=1}^{3} \frac{\Lambda_3^2 m_i^2}{16384} + \frac{\Lambda_3 m_1 m_2 m_3}{2048}, \\
\mathcal{F}_3^{(4,4)} =& -\frac{133\Lambda_3^6}{1649267441664} - \sum_{i=1}^{3} \frac{147\Lambda_3^4 m_i^2}{268435456} - \sum_{i<j} \frac{63\Lambda_3^2 m_i^2 m_j^2}{524288} - \frac{343\Lambda_3^3 m_1 m_2 m_3}{6291456}, \qquad \text{(B.6)}
\end{aligned}$$

for the fourth order in $\hbar$.

B.3 $N_f = 4$

For the $N_f = 4$ theory, the coefficients of the prepotential (5.71) are given by

$$\begin{aligned}
\mathcal{F}_4^{(0,1)} =& \frac{a^2}{8} + \frac{m^4}{32a^2}, \\
\mathcal{F}_4^{(0,2)} =& \frac{13a^2}{1024} + \frac{11m^4}{2048a^2} - \frac{3m^6}{2048a^4} + \frac{5m^8}{16384a^6}, \\
\mathcal{F}_4^{(0,3)} =& \frac{23a^2}{12288} + \frac{17m^4}{16384a^2} - \frac{m^6}{2048a^4} + \frac{15m^8}{65536a^6} - \frac{7m^{10}}{98304a^8} + \frac{3m^{12}}{262144a^{10}}, \\
\mathcal{F}_4^{(0,4)} =& \frac{2701a^2}{8388608} + \frac{1791m^4}{8388608a^2} - \frac{1125m^6}{8388608a^4} + \frac{6095m^8}{67108864a^6} - \frac{1673m^{10}}{33554432a^8} \\
& + \frac{2727m^{12}}{134217728a^{10}} - \frac{715m^{14}}{134217728a^{12}} + \frac{1469m^{16}}{2147483648a^{14}}, \qquad \text{(B.7)}
\end{aligned}$$

for the classical part,

$$\begin{aligned}
\mathcal{F}_4^{(2,1)} =& \frac{m^4}{256a^4}, \\
\mathcal{F}_4^{(2,2)} =& -\frac{m^2}{4096a^2} + \frac{5m^4}{4096a^4} - \frac{15m^6}{16384a^6} + \frac{21m^8}{65536a^8}, \\
\mathcal{F}_4^{(2,3)} =& -\frac{m^2}{16384a^2} + \frac{5m^4}{16384a^4} - \frac{5m^6}{12288a^6} + \frac{91m^8}{262144a^8} - \frac{43m^{10}}{262144a^{10}} + \frac{55m^{12}}{1572864a^{12}}, \\
\mathcal{F}_4^{(2,4)} =& -\frac{235m^2}{16777216a^2} + \frac{2487m^4}{33554432a^4} - \frac{8935m^6}{67108864a^6} + \frac{11235m^8}{67108864a^8} - \frac{38337m^{10}}{268435456a^{10}} \\
& + \frac{43505m^{12}}{536870912a^{12}} - \frac{29549m^{14}}{1073741824a^{14}} + \frac{18445m^{16}}{4294967296a^{16}}, \qquad \text{(B.8)}
\end{aligned}$$

for the second order in $\hbar$, and

$$\begin{aligned}
\mathcal{F}_4^{(4,1)} =& \frac{m^4}{2048a^6}, \\
\mathcal{F}_4^{(4,2)} =& \frac{1}{65536a^2} - \frac{m^2}{8192a^4} + \frac{7m^4}{16384a^6} - \frac{63m^6}{131072a^8} + \frac{219m^8}{1048576a^{10}}, \\
\mathcal{F}_4^{(4,3)} =& \frac{1}{262144a^2} - \frac{m^2}{32768a^4} + \frac{119m^4}{786432a^6} - \frac{133m^6}{393216a^8} + \frac{1689m^8}{4194304a^{10}} - \frac{253m^{10}}{1048576a^{12}} + \frac{1495m^{12}}{25165824a^{14}}, \\
\mathcal{F}_4^{(4,4)} =& \frac{235}{268435456a^2} - \frac{973m^2}{134217728a^4} + \frac{24571m^4}{536870912a^6} - \frac{9457m^6}{67108864a^8} + \frac{68835m^8}{268435456a^{10}} \\
& - \frac{625537m^{10}}{2147483648a^{12}} + \frac{1765673m^{12}}{8589934592a^{14}} - \frac{353325m^{14}}{4294967296a^{16}} + \frac{985949m^{16}}{68719476736a^{18}}, \qquad \text{(B.9)}
\end{aligned}$$

for the fourth order in $\hbar$.

Appendix C

Fourth order corrections in (A_1, D_4) theory

In this appendix, we will write down the fourth order corrections to the SW periods for AD theory of (A_1, D_4) type, associated with the $SU(2)$ gauge theory with $N_f = 3$ hypermultiplets. Using (6.37) and (3.51), we obtain the fourth order corrections to the SW periods, which are given by

$$\begin{aligned}\tilde{a}^{(4)} =& \frac{1}{2^{\frac{23}{3}} \cdot 3^{\frac{15}{2}} \cdot 5 \cdot 7\pi^{\frac{1}{2}} \tilde{w}_3'(\tilde{w}_3' - 1)^3} \frac{1}{\Lambda_3^{10}} \left(\frac{\tilde{u}}{\Lambda_3^2}\right)^3 (-\sigma_3)^{-\frac{5}{2}} \left(1 + \frac{4}{3}\frac{\tilde{M}^3}{\tilde{u}\Lambda_3}\right)^3 \\ & \times \left(F_1^{(4)}(\tilde{w}_3') + F_2^{(4)}(\tilde{w}_3')\right), \end{aligned} \tag{C.1}$$

$$\begin{aligned}\tilde{a}_D^{(4)} =& \frac{i}{2^{\frac{23}{3}} \cdot 3^{\frac{15}{2}} \cdot 5 \cdot 7\pi^{\frac{1}{2}} \tilde{w}_3'(\tilde{w}_3' - 1)^3} \frac{1}{\Lambda_3^{10}} \left(\frac{\tilde{u}}{\Lambda_3^2}\right)^3 (-\sigma_3)^{-\frac{5}{2}} \left(1 + \frac{4}{3}\frac{\tilde{M}^3}{\tilde{u}\Lambda_3}\right)^3 \\ & \times \left((-1)^{\frac{5}{6}} F_1^{(4)}(\tilde{w}_3') + (-1)^{\frac{1}{6}} F_2^{(4)}(\tilde{w}_3')\right), \end{aligned} \tag{C.2}$$

where

$$\begin{aligned}F_1^{(4)}(\tilde{w}_3') =& 2^{\frac{1}{3}} \cdot 21 \tilde{w}_3'^{\frac{1}{3}} \Gamma\left(\frac{1}{6}\right) \Gamma\left(\frac{1}{3}\right) \\ & \times \left(X_1^{(4)} F\left(-\frac{11}{12}, \frac{7}{12}; \frac{2}{3}; \tilde{w}_3'\right) + (\tilde{w}_3' - 1)\, X_2^{(4)} F\left(\frac{1}{12}, \frac{7}{12}; \frac{2}{3}; \tilde{w}_3'\right)\right), \end{aligned} \tag{C.3}$$

$$\begin{aligned}F_2^{(4)}(\tilde{w}_3') =& 2(\tilde{w}_3' - 1)\Gamma\left(-\frac{1}{6}\right) \Gamma\left(-\frac{1}{3}\right) \\ & \times \left(X_3^{(4)} F\left(-\frac{7}{12}, \frac{11}{12}; \frac{1}{3}; \tilde{w}_3'\right) + X_4^{(4)} F\left(\frac{5}{12}, \frac{11}{12}; \frac{1}{3}; \tilde{w}_3'\right)\right). \end{aligned} \tag{C.4}$$

Here the coefficients $X_i^{(4)}$ $(i = 1, 2, 3, 4)$ are defined as follows:

$$\begin{aligned}X_1^{(4)} =& -2^{\frac{2}{3}} \cdot 7^2 \tilde{w}_3'^{\frac{2}{3}}(77\tilde{w}_3' + 211) - \frac{126 \cdot 2^{\frac{1}{3}}\Lambda_3 \tilde{w}_3'^{\frac{1}{3}}(1423\tilde{w}_3' + 593)\tilde{M}}{4\tilde{M}^3 + 3\Lambda_3\tilde{u}}(-\sigma_3)^{\frac{1}{3}} \\ & \frac{252\Lambda_3^2(\tilde{w}_3'(889\tilde{w}_3' + 4639) + 520)\tilde{M}^2}{\left(4\tilde{M}^3 + 3\Lambda_3\tilde{u}\right)^2}(-\sigma_3)^{\frac{2}{3}} \\ & + \frac{756 \cdot 2^{\frac{2}{3}}\Lambda_3^3 \tilde{w}_3'^{\frac{2}{3}}\left(189\Lambda_3(\tilde{w}_3' - 1)\tilde{u} - (419\tilde{w}_3' + 1597)\tilde{M}^3\right)}{\left(4\tilde{M}^3 + 3\Lambda_3\tilde{u}\right)^3}(-\sigma_3) \\ & + \frac{1134 \cdot 2^{\frac{1}{3}}\Lambda_3^4(\tilde{w}_3' - 1)\tilde{w}_3'^{\frac{1}{3}}(95\tilde{w}_3' + 157)\tilde{M}}{\left(4\tilde{M}^3 + 3\Lambda_3\tilde{u}\right)^3}(-\sigma_3)^{\frac{4}{3}},\end{aligned} \tag{C.5}$$

$$\begin{aligned}X_2^{(4)} =& -3234 \cdot 2^{\frac{2}{3}}\tilde{w}_3'^{\frac{2}{3}}(\tilde{w}_3' + 3) - \frac{462 \cdot 2^{\frac{1}{3}}\Lambda_3 \tilde{w}_3'^{\frac{1}{3}}(347\tilde{w}_3' + 157)\tilde{M}}{4\tilde{M}^3 + 3\Lambda_3\tilde{u}}(-\sigma_3)^{\frac{1}{3}} \\ & - \frac{504\Lambda_3^2(\tilde{w}_3'(381\tilde{w}_3' + 2131) + 260)\tilde{M}^2}{\left(4\tilde{M}^3 + 3\Lambda_3\tilde{u}\right)^2}(-\sigma_3)^{\frac{2}{3}} \\ & + \frac{2079 \cdot 2^{\frac{2}{3}}\Lambda_3^3\tilde{w}_3'^{\frac{2}{3}}\left(63\Lambda_3(\tilde{w}_3' - 1)\tilde{u} - 32(4\tilde{w}_3' + 17)\tilde{M}^3\right)}{\left(4\tilde{M}^3 + 3\Lambda_3\tilde{u}\right)^3}(-\sigma_3) \\ & + \frac{486 \cdot 2^{\frac{1}{3}}\Lambda_3^4(\tilde{w}_3' - 1)\tilde{w}_3'^{\frac{1}{3}}(190\tilde{w}_3' + 349)\tilde{M}}{\left(4\tilde{M}^3 + 3\Lambda_3\tilde{u}\right)^3}(-\sigma_3)^{\frac{4}{3}},\end{aligned} \tag{C.6}$$

$$\begin{aligned}X_3^{(4)} =& 49 \cdot 2^{\frac{1}{3}}\tilde{w}_3'^{\frac{1}{3}}(11\tilde{w}_3' + 13) + \frac{168\Lambda_3(113\tilde{w}_3' + 13)\tilde{M}}{4\tilde{M}^3 + 3\Lambda_3\tilde{u}}(-\sigma_3)^{\frac{1}{3}} \\ & + \frac{126 \cdot 2^{\frac{2}{3}}\Lambda_3^2\tilde{w}_3'^{\frac{2}{3}}(127\tilde{w}_3' + 377)\tilde{M}^2}{\left(4\tilde{M}^3 + 3\Lambda_3\tilde{u}\right)^2}(-\sigma_3)^{\frac{2}{3}} \\ & + \frac{189 \cdot 2^{\frac{1}{3}}\Lambda_3^3\tilde{w}_3'^{\frac{1}{3}}\left(4(67\tilde{w}_3' + 101)\tilde{M}^3 - 63\Lambda_3(\tilde{w}_3' - 1)\tilde{u}\right)}{\left(4\tilde{M}^3 + 3\Lambda_3\tilde{u}\right)^3}(-\sigma_3) \\ & - \frac{162\Lambda_3^4(\tilde{w}_3' - 1)(95\tilde{w}_3' + 52)\tilde{M}}{\left(4\tilde{M}^3 + 3\Lambda_3\tilde{u}\right)^3}(-\sigma_3)^{\frac{4}{3}},\end{aligned} \tag{C.7}$$

$$X_4^{(4)} = -49 \cdot 2^{\frac{1}{3}} \tilde{w}_3'^{\frac{1}{3}} (11\tilde{w}_3'(6\tilde{w}_3' + 19) + 13) - \frac{42\Lambda_3(\tilde{w}_3'(3817\tilde{w}_3' + 2179) + 52)\tilde{M}}{4\tilde{M}^3 + 3\Lambda_3\tilde{u}}(-\sigma_3)^{\frac{1}{3}}$$
$$- \frac{378 \cdot 2^{\frac{2}{3}} \Lambda_3^2 \tilde{w}_3'^{\frac{2}{3}} (\tilde{w}_3'(254\tilde{w}_3' + 1463) + 299)\tilde{M}^2}{\left(4\tilde{M}^3 + 3\Lambda_3\tilde{u}\right)^2}(-\sigma_3)^{\frac{2}{3}}$$
$$- \frac{189 \cdot 2^{\frac{1}{3}} \Lambda_3^3 \tilde{w}_3'^{\frac{1}{3}} \left(4(\tilde{w}_3'(352\tilde{w}_3' + 1563) + 101)\tilde{M}^3 - 63\Lambda_3(\tilde{w}_3' - 1)(11\tilde{w}_3' + 1)\tilde{u}\right)}{\left(4\tilde{M}^3 + 3\Lambda_3\tilde{u}\right)^3}(-\sigma_3)$$
$$+ \frac{324\Lambda_3^4(\tilde{w}_3' - 1)(\tilde{w}_3'(285\tilde{w}_3' + 571) + 26)\tilde{M}}{\left(4\tilde{M}^3 + 3\Lambda_3\tilde{u}\right)^3}(-\sigma_3)^{\frac{4}{3}}. \tag{C.8}$$

Expanding (C.1) and (C.2) in $\frac{\tilde{M}^3}{\tilde{u}\Lambda_3}$, $\frac{\tilde{C}_2^2\Lambda_3^2}{\tilde{u}^4}$ and $\frac{\tilde{C}_3\Lambda_3}{\tilde{u}^2}$, the forth order corrections become

$$\tilde{a}^{(4)} = \frac{1}{\Lambda_3^{\frac{5}{2}}}\left(\frac{\tilde{u}}{\Lambda_3^2}\right)^{-2}\left(-\frac{\left(10 + 137(-1)^{\frac{2}{3}}\right)\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right)}{2^2 \cdot 3^5 \cdot 5\pi^{\frac{1}{2}}}\left(\frac{\tilde{M}^3}{\tilde{u}\Lambda_3}\right)^{\frac{1}{3}} - \frac{\left(1241 - 907(-1)^{\frac{1}{3}}\right)\Gamma\left(-\frac{1}{3}\right)\Gamma\left(\frac{5}{6}\right)}{2 \cdot 3^6\pi^{\frac{1}{2}}}\left(\frac{\tilde{M}^3}{\tilde{u}\Lambda_3}\right)^{\frac{2}{3}} + \cdots\right), \tag{C.9}$$

$$\tilde{a}_D^{(4)} = \frac{1}{\Lambda_3^{\frac{5}{2}}}\left(\frac{\tilde{u}}{\Lambda_3^2}\right)^{-2}\left(-\frac{(-1)^{\frac{1}{3}}\left(10 + 137(-1)^{\frac{2}{3}}\right)\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right)}{2^2 \cdot 3^5 \cdot 5\pi^{\frac{1}{2}}}\left(\frac{\tilde{M}^3}{\tilde{u}\Lambda_3}\right)^{\frac{1}{3}} - \frac{(-1)^{\frac{2}{3}}\left(1241 - 907(-1)^{\frac{1}{3}}\right)\Gamma\left(-\frac{1}{3}\right)\Gamma\left(\frac{5}{6}\right)}{2 \cdot 3^6\pi^{\frac{1}{2}}}\left(\frac{\tilde{M}^3}{\tilde{u}\Lambda_3}\right)^{\frac{2}{3}} + \cdots\right). \tag{C.10}$$

The $\hbar^4$ correction to the effective coupling constant is expanded as

$$\tilde{\tau}^{(4)} = \frac{(-1)^{\frac{1}{6}}}{\Lambda_3^4}\left(\frac{\tilde{u}}{\Lambda_3^2}\right)^{-\frac{8}{3}}\left(\frac{2^{\frac{4}{3}} \cdot 3^{\frac{1}{2}}\pi\Gamma\left(\frac{5}{6}\right)^2}{5^2\Gamma\left(-\frac{5}{6}\right)^2\Gamma\left(\frac{1}{6}\right)^2} + \frac{2^3 \cdot 3^{\frac{3}{2}}\pi^{\frac{1}{2}}\Gamma\left(-\frac{1}{6}\right)\Gamma\left(\frac{5}{6}\right)^3}{5^3\Gamma\left(-\frac{5}{6}\right)^3\Gamma\left(\frac{1}{6}\right)^2}\left(\frac{\tilde{M}^3}{\tilde{u}\Lambda_3}\right)^{\frac{1}{3}} + \cdots\right). \tag{C.11}$$

Appendix D

Quantum periods of $SU(N_c)$ SQCD

The SW curve (2.121) and the SW differential (2.56) become

$$C(p) - \frac{1}{2}\left(z + \frac{\Lambda_{N_f}^{2N_c - N_f} G(p)}{z}\right) = 0, \tag{D.1}$$

$$\lambda_{\mathrm{SW}} = p\left(d\log G(p) - 2d\log z\right), \tag{D.2}$$

by introducing

$$y = z - C(p). \tag{D.3}$$

The SW differential defines the holomorphic symplectic form $d\lambda_{\mathrm{SW}} = dp \wedge dx$ where $x \sim \log z$. Define

$$z = \exp\left(-i\hbar\frac{\partial}{\partial p}\right), \tag{D.4}$$

then we obtain the quantum SW curve

$$\left[\frac{1}{2}\left(\exp\left(-i\hbar\frac{\partial}{\partial p}\right) + \exp\left(-i\frac{\hbar}{2}\frac{\partial}{\partial p}\right)\Lambda_{N_f}^{2N_c-N_f} G(p)\exp\left(-i\frac{\hbar}{2}\frac{\partial}{\partial p}\right)\right) - C(p)\right]\Psi(p) = 0. \tag{D.5}$$

Here we take the ordering prescription of the differential operators as [46]. By using

$$J(\alpha) = \exp\left(-\frac{i}{\hbar}\int^p \sum_{k=0}^{\infty}\hbar^k \phi_k(p)dp\right)\exp\left(-i\hbar\alpha\frac{\partial}{\partial p}\right)\exp\left(\frac{i}{\hbar}\int^p \sum_{k=0}^{\infty}\hbar^k \phi_k(p)dp\right), \tag{D.6}$$

the quantum SW curve becomes

$$\frac{1}{2}\left(J(1)+\Lambda_{N_f}^{2N_c-N_f}G\left(p+i\frac{\hbar}{2}\right)J(-1)\right)+C(p)=0. \tag{D.7}$$

Expanding the quantum SW curve in $\hbar$, we obtain the recursion relation of ϕ_k. Solving the recursion relation, we find $\phi_{2n+1}(p)$ $(n=0,1)$ become the total derivatives. The first three ϕ_{2k}'s are given by

$$\phi_0(p) = \log\left(C(p)\pm y\right), \tag{D.8}$$

$$\phi_2(p) = \frac{C'(p)y'-C(p)y''}{16y^2}, \tag{D.9}$$

$$\begin{aligned}\phi_4(p) =& \frac{C'(p)^2C''(p)}{768y^3}-\frac{7C(p)C''(p)^2}{1536y^3}-\frac{C(p)C'(p)C^{(3)}(p)}{384y^3}\\ &+\left(-\frac{C(p)C'(p)^2}{256y^4}+\left(\frac{11C(p)^2}{512y^4}-\frac{11}{1536y^2}\right)C''(p)\right)y''+\left(\frac{17C(p)}{1536y^3}-\frac{5C(p)^3}{256y^5}\right)y''^2\\ &+\left(\frac{C(p)^2}{128y^4}-\frac{1}{384y^2}\right)C'(p)y^{(3)}+\left(\frac{11}{3072y}-\frac{3C(p)^2}{1024y^3}\right)C^{(4)}(p)+\left(\frac{3C(p)^3}{1024y^4}-\frac{5C(p)}{1024y^2}\right)y^{(4)},\end{aligned} \tag{D.10}$$

where $y' := \frac{\partial y}{\partial p}$. The integration of ϕ_k is interpreted as the quantum correction to the SW periods:

$$\Pi^{(k)} := \oint \phi_k(p)dp. \tag{D.11}$$

It can be checked that the second order correction to the SW periods agrees with that in [46] up to total derivatives. The second and fourth order corrections to the SW periods for the AD theory are obtained from not only the WKB solutions of the quantum SW curve for the AD theory, but also the scaling limit $\epsilon \to 0$ of the second and fourth order corrections for the corresponding $SU(N_c)$ SQCD: (D.9) and (D.10) .

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