論文 / 著書情報
Article / Book Information

題目(和文)	
Title(English)	Characteristic classes of fiber bundles and graph complexes
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出典(和文)	学位:博士(理学), 学位授与機関:東京工業大学, 報告番号:甲第11036号, 授与年月日:2019年3月26日, 学位の種別:課程博士, 審査員:遠藤 久顕,山田 光太郎,本多 宣博,服部 俊昭,野坂 武史,寺嶋 郁二
Citation(English)	Degree:Doctor (Science), Conferring organization: Tokyo Institute of Technology, Report number:甲第11036号, Conferred date:2019/3/26, Degree Type:Course doctor, Examiner:,,,,,
学位種別(和文)	博士論文
Type(English)	Doctoral Thesis

Characteristic classes of fiber bundles and graph complexes

February 22, 2019

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Contents

Introduction

The contents of this thesis are divided into these two parts roughly:

(i) cocycle-level constructions of characteristic classes of fiber bundles and their relations to cohomology of Lie algebras of derivations [32, 20, 31],
(ii) a relation between the dgl of symplectic derivations on a semi-free dgl and a certain graph complex [30].

Chapter 2 and 3 belong to (i), and Chapter 4 belongs to (ii).

Chapter 2. According to the result of K.T. Chen [5, 6], a Riemannian metric on a closed manifold X gives a formal homology connection on X via the Hodge decomposition defined by this metric. A notion of a formal homology connection on X is equivalent to a notion of a C_∞-algebra model of the de Rham complex of X [14]. This correspondence implies that a Riemannian metric of a fiber bundle $E \to B$ with fiber X gives a deformation of C_∞-algebra models of X, so it defines the map from the base space B to a certain moduli space of C_∞-algebra models of X. Under a certain condition, we can construct a flat connection on the moduli space. Using the Chern-Weil theory, we can obtain characteristic classes of a fiber bundle $E \to B$ satisfying a suitable condition as the image of the characteristic map from the Chevalley-Eilenberg complex of a Lie algebra of derivations on a Chen's model of X to the cohomology of the base space B through the map above. As an example of such characteristic classes, we have Morita-Miller-Mumford classes of surface bundles. These discussions are described in the papers [32, 20].

Chapter 3. The construction of Chapter 2 implies an existence of its simplicial enhancement. In this aspect, the moduli space $Q(X)$ defined in Chapter 2 is the set (space) of connected components of the simplicial set of C_∞-algebra models of X. In this chapter, we construct characteristic classes of a smooth fiber bundle $X \to E \to B$ by obstruction theory for a certain simplicial bundle $\mathcal{Q}_\bullet(E) \to S_\bullet(B)$ obtained from the original bundle. The base simplicial set $S_\bullet(B)$ of the simplicial bundle $\mathcal{Q}_\bullet(E) \to S_\bullet(B)$ is the simplicial set of singular simplices of B and the n-th set $\mathcal{Q}_n(E)_\sigma$ of the fiber over an n-simplex σ is the set of Chen's formal homology connections on σ^*E. A formal homology connection on a manifold X has rational homotopical information of X, which is equivalent to a minimal C_∞-algebra model $f : (H, m) \to A$ of the reduced de Rham complex A such that m is a minimal C_∞-algebra structure and the first term of f induces the identity map on cohomologies (see [14]). The fiber of the bundle is the simplicial set $Q_\bullet(X)$ of formal homology connections on $X \times \Delta^n$. This simplicial set is very close to the Maurer-Cartan simplicial set of the dgl $\hat{L}W \otimes A$, where $(\hat{L}W, \delta)$ is the dual of the bar-construction of the C_∞-algebra (H, m).

We introduce two versions of construction depending on whether the fiber $Q_\bullet(X)$ is connected or not. The homotopy group of the Muarer-Cartan simplicial set is known in [11, 1, 3]. So the homotopy groups of $Q_\bullet(X)$ can be also expressed as vector spaces by

$$\pi_n(Q_\bullet(X), \tau) = H_n(\mathrm{Der}(\hat{L}W), \delta)$$

for a formal homology connection $\tau = (\omega, \delta)$ on X.

In the case that $Q_\bullet(X)$ is connected, under certain conditions, an obstruction class of existence of a partial section over the n-skeleton of $\mathcal{Q}_\bullet(E) \to S_\bullet(B)$

$$\mathfrak{o}_n \in H^{n+1}(B; \Pi_n)$$

is obtained, where Π_n is the local system of the n-th homotopy groups of fibers of $\mathcal{Q}_\bullet(E) \to S_\bullet(B)$. Then we get the characteristic map

$$(\Lambda^p H_n(\mathrm{Der}(\hat{L}W), \delta)^*)^G \to H^{p(n+1)}(B; \mathbb{R})$$

for any $p \geq 1$. Here G is the structure group of $E \to B$. As an application, this yields the Euler class of a sphere bundle.

On the other hand, if $Q_\bullet(X)$ is not connected, the local system Π_0 of sets has a free and transitive action of a certain local system $\mathrm{QIAut}(E)$ of groups. Since this group has a natural filtration, we get the graded Lie algebra $\mathrm{gr}(\mathrm{QIAut}(E))$. The fiber of i-th part can be identified with a certain vector space $\mathrm{gr}_i(\mathrm{QDer}(\hat{L}W))$. Using this vector space in stead of the homotopy groups of $Q_\bullet(X)$, we can obtain the obstruction $\mathfrak{o}^{(i)} \in H^1(B; \mathrm{gr}_i(\mathrm{QIAut}(E)))$ and the characteristic map

$$(\Lambda^\bullet \mathrm{gr}_i(\mathrm{QDer}(\hat{L}W))^*)^G \to H^\bullet(B; \mathbb{R})$$

according to the stage i of extension of a partial section. Applying for a surface bundle, the obstruction class for $i = 0$ corresponds the twisted Morita-Miller-Mumford class and the characteristic map gives the Morita-Miller-Mumford classes.

Chapter 4. The Chevalley-Eilenberg complex of the limit of the Lie algebra of symmplectic derivations on (graded) free Lie algebras is isomorphic to the graph complex defined by the cyclic Lie operad (details in [25, 26, 8, 15]). In this paper, we introduce an extension of (the dual of) the construction to a Lie algebra of symmplectic derivations on free dgls. Let (W, ω) be a graded vector space with symmetric inner product of even degree N and δ a differential of degree -1 on the completed free Lie algebra $\hat{L}W$ satisfying the symplectic condition $\delta\omega = 0$. An important example is the case that $(\hat{L}W, \delta)$ is a Chen's dgl model of an even dimensional manifold and ω is its intersection form. We construct a W-labeled graph complex $C^{\bullet;\bullet}_{\mathrm{com}}(W)_+$ and a chain map

$$C^{\bullet;\bullet}_{\mathrm{com}}(W)_+ \to C^{\bullet;\bullet}_{CE}(\mathrm{Der}^+_\omega(\hat{L}W))$$

to the Chevalley-Eilenberg (double) complex $C^{\bullet;\bullet}_{CE}(\mathrm{Der}^+_\omega(\hat{L}W))$ of the differential graded Lie algebra $(\mathrm{Der}^+_\omega(\hat{L}W), \mathrm{ad}(\delta))$ of positive symplectic derivations on $\hat{L}W$. Furthermore the non-labeled part $C^{\bullet;\bullet}_{\mathrm{com}}(N, Z)_+$ of the graph complex, which depends on only the integer N and the set Z of degrees of W, we can obtain a chain map

$$C^{\bullet;\bullet}_{\mathrm{com}}(N, Z)_+ \subset C^{\bullet;\bullet}_{\mathrm{com}}(W)_+^{\mathrm{Sp}(W,\delta)} \to C^{\bullet;\bullet}_{CE}(\mathrm{Der}^+_\omega(\hat{L}W))^{\mathrm{Sp}(W,\delta)},$$

where $\mathrm{Sp}(W, \delta)$ is the group of graded linear isomorphisms of W preserving ω and δ. In the case of $N = 0$ and $Z = \{0\}$, the map corresponds to the Kontsevich's one [25, 26].

The construction above gives characteristic classes of fibrations. It is known that characteristic classes of simply-connected fibrations are related to Lie algebras of derivations [38, 42]. In non-simply connected cases, we got relations between characteristic classes and Lie algebras of derivations as in [32, 20]. In this paper, we consider the case that the boundary of a fiber is a sphere. For a simply-connected compact manifold X with $\partial X = S^{n-1}$, let $\mathrm{aut}_\partial(X)$ be the monoid of self-homotopy equivalences of X fixing the boundary pointwisely and $\mathrm{aut}_{\partial,0}(X)$ its connected component containing id_X. According to [2], the isomorphism

$$H^\bullet(B\,\mathrm{aut}_{\partial,0}(X); \mathbb{Q}) \simeq H^\bullet_{CE}(\mathrm{Der}^+_\omega(L_X))$$

is obtained. Here L_X is a cofbrant dgl model of X. The underlying Lie algebra of L_X is generated by the linear dual W of the suspension of the reduced cohomology of X. So the graph complex above gives the invariant part of the cohomology $H^{\bullet}_{CE}(\mathrm{Der}^+_{\omega}(L_X))$ with respect to the action of the group $\mathrm{Sp}(W,\delta)$ of automorphisms of W with intersection form preserving the differential δ of L_X. Using the Serre spectral sequence for the fibration

$$B\,\mathrm{aut}_{\partial,0}(X) \to B\,\mathrm{aut}_{\partial}(X) \to B\pi_0(\mathrm{aut}_{\partial}(X)),$$

the image of the natural map $H^{\bullet}(B\,\mathrm{aut}_{\partial}(X);\mathbb{Q}) \to H^{\bullet}(B\,\mathrm{aut}_{\partial,0}(X);\mathbb{Q})$ is included in the invariant part. We give a chain map

$$C^{\bullet;\bullet}_{\mathrm{com}}(N,Z)_+ \to C^{\bullet;\bullet}_{CE}(\mathrm{Der}^+_{\omega}(L_X))^{\mathrm{Sp}(W,\delta)}.$$

by the construction above. Considering W-labeled graphs, we can also obtain a W-labeled version $C^{\bullet;\bullet}_{\mathrm{com}}(W)_+$ and a chain map

$$C^{\bullet;\bullet}_{\mathrm{com}}(W)_+ \to C^{\bullet;\bullet}_{CE}(\mathrm{Der}^+_{\omega}(L_X)).$$

Acknowledgment. First, my deepest appreciation goes to my supervisor Prof. Y. Terashima form Tohoku University whose enormous support and insightful comments were invaluable during the course of my study. I would also like to show my appreciation to my another supervisor Prof. H.Endo from Tokyo Institute of Technology for support to get my degree and warm encouragements. I appreciate the feedback offered by Prof. A.Berglund from Stockholm University, Prof. H.Kajiura from Chiba University, Prof. K.Sakai from Shinshu University and Prof. T.Watanabe from Shimane University. Finally, I would also like to express my gratitude to my family for their moral and economic support.

A part of this work was supported by Grant-in-Aid for JSPS Research Fellow (No.17J01757).

Chapter 1. Derivations, homotopy algebras and related concepts

In this chapter, all vector spaces are over a field K with characteristic 0.

1. Permutations and sign notations

The symmetric group on r letters is denoted by $\mathfrak{S}_r$. For an integer $0 \le s \le r$, a permutation $\tau \in \mathfrak{S}_r$ satisfying

$$\tau^{-1}(1) < \cdots < \tau^{-1}(s), \quad \tau^{-1}(s+1) < \cdots < \tau^{-1}(r)$$

is called an $(s, r-s)$**-shuffle**. On the other hand, τ is called an $(s, r-s)$**-unshuffle** if τ^{-1} is $(s, r-s)$-shuffle. The set of $(s, r-s)$-shuffles is denoted by $\mathrm{Sh}(s, r-s)$ while the set of $(s, r-s)$-unshuffles is $\mathrm{Ush}(s, r-s)$.

We often denote by $|a|$ the degree of an element a. But we omit the symbol $|\cdot|$ of the degree when it appears in a power of -1. For example, $(-1)^{ab}$ means $(-1)^{|a||b|}$ for graded elements a, b.

Definition 1.1. We define the **Koszul sign** $\epsilon(\tau; x_1, \ldots, x_r)$ for a permutation $\tau \in \mathfrak{S}_r$ and letters $x_1, \ldots, x_r$ with degrees by the following axioms:

(i) $\epsilon(\tau; x_1, \ldots, x_r) \in \{\pm 1\}$ depends on only τ and the order of degrees of $x_1, \ldots, x_r$,

(ii) $\epsilon(1; x_1, \ldots, x_r) = 1$ and $\epsilon(\rho; x_1, \ldots, x_r) = (-1)^{x_i x_{i+1}}$ for a transposition $\rho = (i\ i+1)$,

(iii) $\epsilon(\tau\rho; x_1, \ldots, x_r) = \epsilon(\tau; x_{\rho(1)}, \ldots, x_{\rho(r)})\epsilon(\rho; x_1, \ldots, x_r)$.

The sign $\bar{\epsilon}(\tau; x_1, \ldots, x_r) = \mathrm{sgn}(\tau)\epsilon(\tau; x_1, \ldots, x_r)$ is called the **anti-Koszul sign**.

Example 1.2. For example,

$$\epsilon((1\ 2); x_1, x_2, x_3) = (-1)^{x_1 x_2}, \quad \bar{\epsilon}((1\ 2); x_1, x_2, x_3) = -(-1)^{x_1 x_2},$$
$$\epsilon((1\ 2\ 3); x_1, x_2, x_3) = \bar{\epsilon}((1\ 2\ 3); x_1, x_2, x_3) = (-1)^{x_1(x_2+x_3)},$$
$$\epsilon((1\ 3\ 2); x_1, x_2, x_3) = \bar{\epsilon}((1\ 3\ 2); x_1, x_2, x_3) = (-1)^{x_3(x_1+x_2)}.$$

Remark 1.3. The sign $\epsilon(\tau; x_1, \ldots, x_r)$ is a sign appearing in the equation

$$x_1 \cdots x_r = \epsilon(\tau; x_1, \ldots, x_r) x_{\tau(1)} \cdots x_{\tau(r)}$$

on the graded symmetric algebra generated by $x_1, \ldots, x_r$, while $\bar{\epsilon}(\tau; x_1, \ldots, x_r)$ is a sign appearing the same equation on the graded exterior algebra generated by $x_1, \ldots, x_r$.

Using the Koszul sign, we can describe the right $\mathfrak{S}_r$-action on $V^{\otimes r}$ by

$$(x_1 \otimes \cdots \otimes x_r)^{\tau} := \epsilon(\tau; x_1, \ldots, x_r) x_{\tau(1)} \otimes \cdots \otimes x_{\tau(r)}$$

for $x_1, \ldots, x_r \in V$ and $\tau \in \mathfrak{S}_r$.

2. Graded free algebras

2.1. Graded vector space.

Let V be a $\mathbb{Z}$-graded vector space.

2.1.1. *Grading.* We denote

- the subspace of elements of V of **cohomological degree** i by V^i, and
- the subspace of elements of V of **homological degree** i by $V_i = V^{-i}$.

Remark that the **linear dual** $V^* = \mathrm{Hom}(V, \mathbb{R})$ of V is graded by $(V^*)^i = \mathrm{Hom}(V_i, \mathbb{R})$.

2.1.2. *Suspension.* The p**-fold suspension** $V[p]$ of V for an integer p is defined by

$$V[p]^i := V^{i+p}$$

and elements of $V[p]^i$ are presented by the form σx or $x\sigma$ for $x \in V^{i+p}$ using the symbol σ of cohomological degree $-p$. In the case, we put $\sigma x = (-1)^{px} x\sigma$.

2.1.3. *Inner products.* Let $\alpha : V \otimes V \to K$ be a non-degenerate bilinear map of (cohomological) degree n. Out of the two conditions

(i) $\alpha(x, y) = (-1)^{xy}\alpha(y, x)$ for homogeneous elements $x, y \in V$,
(ii) $\alpha(x, y) = -(-1)^{xy}\alpha(y, x)$ for homogeneous elements $x, y \in V$,

the pair (V, α) is called **(graded) symmetric vector space** with degree n if satisfying (i), and **(graded) symplectic vector space** with degree n if satisfying (ii).

For a symmetric vector space (V, α), the desuspension $V[-1]$ has the canonical symplectic structure $\bar{\alpha}$ given by

$$\bar{\alpha}(\sigma a, \sigma b) = (-1)^a \alpha(a, b)$$

for $a, b \in V$. So $(V[-1], \bar{\alpha})$ is a symplectic vector space. The converse construction is also possible.

2.2. **Free algebra and free coalgebra.** We consider the tensor vector space

$$\bigoplus_{r=0}^{\infty} V^{\otimes r}.$$

An element $x_1 \otimes \cdots \otimes x_r$ is written by $x_1 \cdots x_r$ omitting the symbols $\otimes$ for simplicity, and r is called **the (tensor) rank** of $x_1 \cdots x_r$. This vector space has two bialgebra structures:

- the product ∇ and the coproduct Δ are defined by

$$\nabla(x_1 \cdots x_s, x_{s+1} \cdots x_r) = x_1 \cdots x_r,$$

$$\Delta(x_1 \cdots x_r) = \sum_{s=0}^{r} \sum_{\tau \in \mathrm{Ush}(s, r-s)} \epsilon \cdot (x_{\tau(1)} \cdots x_{\tau(s)}) \otimes (x_{\tau(s+1)} \cdots x_{\tau(r)})$$

  for homogeneous elements $x_1, \dots, x_r \in V$, where ϵ is the Koszul sign of the permutation $(x_1, \dots, x_r) \mapsto (x_{\tau(1)}, \dots, x_{\tau(r)})$. This bialgebra is denoted by TV and called **the tensor algebra**. We often use its completed version

$$\hat{T}V := \prod_{r=0}^{\infty} V^{\otimes r},$$

  which is called **the completed tensor algebra**. It can be also described by the completion of filtered algebra

$$\hat{T}V = \varprojlim_{r} TV/T_{r+1}V,$$

  where TV is filtered by $T_rV = \bigoplus_{n=r}^{\infty} V^{\otimes n}$.
- the product Δ^* and the coproduct ∇^* are defined by

$$\Delta^*(x_1 \cdots x_s, x_{s+1} \cdots x_r) = \sum_{\tau \in \mathrm{Sh}(s, r-s)} \epsilon \cdot x_{\sigma(1)} \cdots x_{\sigma(r)},$$

$$\nabla^*(x_1 \cdots x_r) = \sum_{s=0}^{r} (x_1 \cdots x_s) \otimes (x_{s+1} \cdots x_r)$$

for homogeneous elements $x_1, \ldots, x_r \in V$, where ϵ is the Koszul sign of the permutation $(x_1, \ldots, x_r) \mapsto (x_{\tau(1)}, \ldots, x_{\tau(r)})$. This bialgebra is denoted by T^cV and called **the tensor coalgebra**.

These structures are dual:

$$\hat{T}V^* = (T^cV)^*.$$

2.3. **Free Lie algebra.** The primitive part of TV is denoted by LV and called **the (graded) free Lie algebra** generated by V. Its rank r part is denoted by$L_rV = LV \cap V^{\otimes r}$. It is described by the unshuffle sum vanishing like the equation

$$L_rV = \left\{ A \in V^{\otimes r}; \sum_{\tau \in \mathrm{Ush}(s,r-s)} A^\tau = 0 \ (0 < s < r) \right\}.$$

Of course, it is a Lie algebra by the Lie bracket

$$[A, B] = AB - (-1)^{AB} BA$$

for $A, B \in LV$.

The free Lie algebra has a canonical filtration derived from rank. The lower central series $\{\Gamma_n\}_{n=0}^{\infty}$ is defined by

$$\Gamma_n = \bigoplus_{n=r}^{\infty} L_rV = LV \cap T_rV.$$

Then **the completed free Lie algebra** $\hat{L}V$, which is the primitive part of $\hat{T}V$, is described by the completion

$$\hat{L}V = \varprojlim_r LV/\Gamma_{r+1}.$$

The induced filtration $\{\hat{\Gamma}_n\}_{n=0}^{\infty}$ on $\hat{L}V$ is also the lower central series.

2.4. **Free symmetric algebra and free exterior algebra.**

- The **symmetric algebra** TV generated by V is the $\mathbb{Z}$-graded commutative algebra which is the quotient algebra obtained from the $\mathbb{Z}$-graded tensor algebra TV by introducing the relation

$$xy = (-1)^{xy} yx$$

for $x, y \in V$. The image of $V^{\otimes k}$ for an integer k in SV is denoted by S^kV. The algebra SV is isomorphic to the image of the symmetrization map $TV \to TV$

$$x_1 \cdots x_r \mapsto \sum_{\tau \in \mathfrak{S}_r} (x_1 \cdots x_r)^\tau.$$

- The **exterior algebra** ΛV generated by V is the $\mathbb{Z}$-graded anti-commutative algebra which is the quotient algebra obtained from the $\mathbb{Z}$-graded tensor algebra TV by introducing the relation

$$xy = -(-1)^{xy} yx$$

for $x, y \in V$. The image of $V^{\otimes k}$ for an integer k in ΛV is denoted by $\Lambda^k V$. The algebra ΛV is isomorphic to the image of the anti-symmetrization map $TV \to TV$

$$x_1 \cdots x_r \mapsto \sum_{\tau \in \mathfrak{S}_r} \operatorname{sgn}(\tau)(x_1 \cdots x_r)^{\tau}.$$

Note the canonical identification

$$\Lambda V \simeq S(V[1]).$$

3. Derivations

Definition 3.1. A derivation on each algebras is defined as follows.

- A **(algebra) derivation** on an algebra (A, ∇) is a linear map $D : A \to A$ satisfying
$$\nabla(D \otimes 1 + 1 \otimes D) = D\nabla.$$
- A **coderivation** on a coalgebra (A, Δ) is a linear map $D : A \to A$ satisfying
$$(D \otimes 1 + 1 \otimes D)\Delta = \Delta D.$$
- A **Hopf derivation** on a bialgebra (A, ∇, Δ) is a linear map $D : A \to A$ which is a derivation and a coderivation.
- A **Lie derivation** on a Lie algebra $(A, [\ ,\ ])$ is a linear map $D : A \to A$ satisfying
$$[\ ,\ ](D \otimes 1 + 1 \otimes D) = D[\ ,\ ].$$

The vector space of such derivations on A is denoted by $\operatorname{Der}(A)$. This is the graded Lie subalgebra of the graded Lie algebra $\operatorname{End}(A)$ of linear endomorphsims $A \to A$.

We mainly consider derivations on free algebras. Let V be a $\mathbb{Z}$-graded vector space. The Lie algebra of Hopf derivations on TV is isomorphic to the Lie algebra $\operatorname{Der}(LV)$ of Lie derivations on LV. So, in this paper, $\operatorname{Der}(TV)$ always means the Lie algebra of algebra derivations on TV. Note that $\operatorname{Der}(LV)$ is a Lie subalgebra of $\operatorname{Der}(TV)$. We also adapt the same notations $\operatorname{Der}(\hat{T}V)$ and $\operatorname{Der}(\hat{L}V)$ in the completed case.

3.1. The rank of derivations.

The Lie algebra $\operatorname{Der}(TV)$ has two gradings derived from two gradings of TV, the tensor rank and the grading of V. The degree of a derivation D with respect to the tensor rank is also called the rank of the derivation D, and the rank r part of $\operatorname{Der}(TV)$ is denoted by $\operatorname{Der}^r(TV)$. A derivation on TV is determined by only its evaluations on V. So we get the linear isomorphism

$$\operatorname{Der}^r(TV) \simeq \operatorname{Hom}(V, V^{\otimes(r+1)}).$$

For a linear map $f \in \operatorname{Hom}(V, V^{\otimes(r+1)})$, we denote the corresponding derivation by

$$\sum_{i=1}^{m} f(x^i) \frac{\partial}{\partial x^i},$$

where $x^1, \ldots, x^m$ is a basis of V. Using this grading, $\operatorname{Der}(TV)$ is filtered by

$$\operatorname{Der}^{\geq r}(TV) = \bigoplus_{n=r}^{\infty} \operatorname{Der}^n(TV)$$

and the completion with respect to the filtration is isomorphic to $\operatorname{Der}(\hat{T}V)$.

We can consider the corresponding grading of $\mathrm{Der}(LV)$. Putting

$$\mathrm{Hom}_{\mathrm{com}}(V, V^{\otimes(r+1)}) := \left\{ f \in \mathrm{Hom}(V, V^{\otimes(r+1)});\ \sum_{\tau \in \mathfrak{S}_{r+1}} f^{\tau} = 0 \right\},$$

we can describe the rank r part of $\mathrm{Der}(LV)$ by

$$\mathrm{Der}^r(LV) := \mathrm{Der}(LV) \cap \mathrm{Der}^r(TV) \simeq \mathrm{Hom}_{\mathrm{com}}(V, V^{\otimes(r+1)}).$$

The completion with respect to the filtration $\mathrm{Der}^{\geq r}(LV) := \mathrm{Der}^{\geq r}(TV) \cap \mathrm{Der}(LV)$ is isomorphic to $\mathrm{Der}(\hat{L}V)$ in the same way as $\mathrm{Der}(\hat{T}V)$.

3.2. **Symplectic derivations.** Let ω be a symplectic form on V with degree N. Using non-degeneracy of ω, we have the isomorphism $V \simeq V^*[N]$, so we can regard ω as an element of $V^{\otimes 2}$ through the isomorphism. This element ω is described explicitly by

$$\omega = \sum_{i,j} \omega_{ij} x^i x^j \ \in L_2 V,$$

where $x^1, \ldots, x^m$ is a basis of V and $(\omega_{ij})_{i,j}$ is the inverse matrix of $(\omega(x^i, x^j))_{i,j}$.

The Lie algebra of **symplectic derivations** on TV is defined by

$$\mathrm{Der}_\omega(TV) = \{D \in \mathrm{Der}(TV);\ D(\omega) = 0\}.$$

The Lie algebra also has the rank of elements and the filtration in the same way as $\mathrm{Der}(TV)$. Furthermore $\mathrm{Der}_\omega(LV)$ is also defined and the same structures exist.

Through the isomorphism $\mathrm{Der}^r(TV) \simeq \mathrm{Hom}(V, V^{\otimes(r+1)}) \simeq V^{\otimes(r+2)}[-N]$, we get the correspondence

$$\mathrm{Der}^r_\omega(TV) \simeq (V^{\otimes(r+2)})^{\mathbb{Z}/(r+2)\mathbb{Z}}[-N] =: V_{\mathrm{cyc}}(r+2)[-N].$$

Here $\mathbb{Z}/(r+2)\mathbb{Z}$ is the subgroup of cyclic permutations in $\mathfrak{S}_{r+2}$. Therefore putting

$$V_{\mathrm{Lcyc}}(r+1) := \left\{ x \in V_{\mathrm{cyc}}(r+1);\ \sum_{\tau \in \mathrm{Ush}(s, r-s)} x^{\tau} = 0 \ (0 < s < r) \right\},$$

where τ is regarded as a permutation on $r+1$ letters by the standard inclusion $\mathfrak{S}_r \subset \mathfrak{S}_{r+1}$, we have

$$\mathrm{Der}^r_\omega(LV) \simeq V_{\mathrm{Lcyc}}(r+2)[-N].$$

4. Homotopy algebra

4.1. **A_∞-algebras.** Let us review the notations on A_∞-algebra.

Definition 4.1 (A_∞-algebra [39, 40])**.** Let A be a $\mathbb{Z}$-graded vector space and $m = \{m_n : A^{\otimes n} \to A\}_{n \geq 1}$ be a family of linear maps with $\deg m_n = 2 - n$. The pair (A, m) satisfying the **A_∞-relations**

$$\sum_{k+l=n+1} \sum_{j=0}^{k-1} (-1)^{(j+1)(l+1)} m_k \circ (\mathrm{id}_A^{\otimes j} \otimes m_l \otimes \mathrm{id}_A^{\otimes(n-j-l)}) = 0$$

for $n \geq 1$ is called **A_∞-algebra**. Then m is called A_∞-structure on A.

The multilinear map m_k has degree $(2-k)$ indicates the degree of $m_k(a_1,\dots,a_k)$ is $|a_1|+\cdots+|a_k|+(2-k)$. The A_∞-relations implies $(m_1)^2=0$ for $n=1$, the Leibniz rule of the differential m_1 with respect to the product m_2 for $n=2$, and the associativity of m_2 up to homotopy for $n=3$. These facts further imply that the cohomology $H(A,m_1)$ has the structure of a (non-unital) algebra, where the product is induced from m_2.

Note that the product m_2 is strictly associative in A if m_3=0.

Definition 4.2. Let (A,m) be an A_∞-algebra.

- If higher products are all zero, i.e. $m_3=m_4=\cdots=0$, (A,m) is called **differential graded algebra (DGA)**.
- If $m_1=0$, (A,m) is called **minimal**.

Remark 4.3 (Bar construction of an A_∞-algebra)**.** Let (A,m) be an A_∞-algebra. The A_∞-structure corresponds to the codifferential $\mathfrak{m}$ on the coalgebra $BA:=T^c(A[1])$ as follows. Denote the suspension map by $s:A\to A[1]$. Defining the suspension of m_n by $\bar{m}_n:=s^{-1}\circ m_n\circ s^{\otimes n}$ for all $n\geq 1$, then the degree of $\bar{m}_n$ is 1 and the A_∞-relations are rewritten as the simpler equations

$$\sum_{k+l=n+1}\sum_{j=0}^{k-1}\bar{m}_k\circ(\mathrm{id}_{A[1]}^{\otimes j}\otimes\bar{m}_l\otimes\mathrm{id}_{A[1]}^{\otimes(n-j-l)})=0$$

(Getzler-Jones [13]). Then $\bar{m}_n:A[1]^{\otimes n}\to A[1]$ extents the unique coderivation $\mathfrak{m}_n:BA\to BA$ by the co-Leibniz rule $\Delta\circ\mathfrak{m}_n=(\mathfrak{m}_n\otimes\mathrm{id}+\mathrm{id}\otimes\mathfrak{m}_n)\circ\Delta$. Setting

$$\mathfrak{m}=\sum_{n=1}^{\infty}\mathfrak{m}_n\in\mathrm{Der}(BA),$$

then $\mathfrak{m}$ is a degree 1 codifferential, i.e. $\mathfrak{m}^2=0$, from the A_∞-relations of m. Thus an A_∞-algebra (A,m) is equivalent to a differential graded coalgebra (DGCA) $(BA,\mathfrak{m})$. The DGCA $(BA,\mathfrak{m})$ is called the **bar construction** of (A,m).

Definition 4.4 (A_∞-morphism)**.** Let (A,m) and (A',m') be A_∞-algebras. A family $f=\{f_n:A^{\otimes n}\to A'\}$ of linear maps with $\deg f_n=1-n$ satisfying the equations

$$\sum_{\substack{i\geq 1,\\ k_1+\cdots+k_i=n}}m'_i\circ(f_{k_1}\otimes\cdots\otimes f_{k_i})=\sum_{\substack{i+1+j=k,\\ i+l+j=n}}(-1)^{i+(n-i+1)l}f_k\circ(\mathrm{id}_A^{\otimes i}\otimes m_l\otimes\mathrm{id}_A^{\otimes(n-i-l)})$$

is called A_∞**-morphism** $f:(A,m)\to(A',m')$.

- If f_1 is a linear isomorphism, f is called A_∞**-isomorphism**.
- If $f_2=f_3=\cdots=0$, f is called **linear** A_∞-morphism.

The defining equation for A_∞-morphisms for $n=1$ implies that $f_1:A\to A'$ forms a chain map $f_1:(A,m_1)\to(A',m'_1)$. This together with the defining equation for $n=2$ implies that $f_1:A\to A'$ induces a (non-unital) algebra map from $H(A,m_1)$ to $H(A',m'_1)$. We denote it by $H(f_1):H(A,m_1)\to H(A',m'_1)$.

Definition 4.5. An A_∞-morphism $f:(A,m)\to(A',m')$ is called an A_∞**-quasi-isomorphism** if $f_1:(A,m_1)\to(A',m'_1)$ induces an isomorphism between the cohomologies of these two complexes.

Remark 4.6 (Bar construction of an A_∞-morphism)**.** Let $f:(A,m)\to(A',m')$ be an A_∞-morphism. Defining the suspension of f_n by $\bar{f}_n:=s\circ m_n\circ(s^{-1})^{\otimes n}$:

$A[1]^{\otimes n} \to A'[1]$ for all $n \geq 1$, then the degree of $\bar{f}_n$ is 0 and the relations for A_∞-morphism are rewritten as the equations

$$\sum_{\substack{i\geq 1,\\ k_1+\cdots+k_i=n}} \bar{m}'_i \circ (\bar{f}_{k_1} \otimes \cdots \otimes \bar{f}_{k_i}) = \sum_{\substack{i+1+j=k,\\ i+l+j=n}} \bar{f}_k \circ (\mathrm{id}_{A[1]}^{\otimes i} \otimes \bar{m}_l \otimes \mathrm{id}_{A[1]}^{\otimes(n-i-l)}).$$

Constructing the coalgebra map $BA \to BA'$

$$\mathfrak{f} = \sum_{n=1}^{\infty} \sum_{\substack{i\geq 1,\\ k_1+\cdots+k_i=n}} \bar{f}_{k_1} \otimes \cdots \otimes \bar{f}_{k_i}$$

from maps $\bar{f}_n$, then $\mathfrak{f}$ is a DGCA map $(BA, \mathfrak{m}) \to (BA', \mathfrak{m}')$ between bar constructions, i.e. $\mathfrak{f} \circ \mathfrak{m} = \mathfrak{m}' \circ \mathfrak{f}$ from the condition of A_∞-morphism.

The composition of A_∞-morphisms is defined by the composition of bar constructions of A_∞-morphisms. From the definition, any A_∞-isomorphism has its inverse A_∞-isomorphism uniquely.

On the other hand, it is easy to see that the composition of A_∞-quasi-isomorphisms is an A_∞-quasi-isomorphism. An A_∞-quasi-isomorphism has its inverse A_∞-quasi-isomorphism in a strict sence if and only if it is an A_∞-isomorphism, but always has its homotopy inverse as in Theorem 4.10. These facts imply that A_∞-quasi-isomorphisms define an equivalence relation between A_∞-algebras.

4.2. **Decomposition theorem of A_∞-algebras.** A pair of minimal A_∞-algebra (H, m^H) and an A_∞-quasi-isomorphism $(H, m^H) \to (A, m)$ is called **minimal model** of (A, m).

The following theorem was first mentioned in [24], and is called the decomposition theorem. A proof was given in [21] and was presented in [19]. See [7] for a filtered version.

Theorem 4.7. Any A_∞-algebra (A, m) is A_∞-isomorphic to the direct sum of a minimal A_∞-algebra M and a linear contractible A_∞-algebra C. Here, a **linear contractible A_∞-algebra** $C = (C, m^C)$ is an A_∞-algebra such that $m_2^C = m_3^C = \cdots = 0$ and the cohomology $H(C, m_1^C)$ is trivial.

Especially, we get the inclusion map $M \to (A, m)$ as a minimal model of (A, m).

Proof. We first choose a Hodge decomposition (H, ι, π, h) of the complex (A, m_1), that is, $H := H(A, m_1)$ is the cohomology, $\iota : H \to A$ and $\pi : A \to H$ are linear map of degree zero such that $\pi \circ \iota = id_H$, $h : A \to A$ is a linear map of degree minus one and they satisfy

$$m_1 h + h m_1 + P = \mathrm{id}_A, \qquad h^2 = 0$$

where $P := \iota \circ \pi$. This gives a Hodge decomposition of $(BA, \mathfrak{m}_1)$, as a complex of vector spaces, such that the cohomology is BH. Actually, ι and π extend to the (linear) coalgebra maps $\iota : BH \to BA$ and $\pi : BA \to BH$ and one can construct a chain homotopy $\mathfrak{h} : BA \to BA$ from $\bar{h}, \bar{P}$ and the identity map on $A[1]$.

We put $M := \operatorname{Im} P$ and $C := \operatorname{Im}(m_1 h + h m_1)$. Let us consider a coalgebra homomorphism $\mathfrak{f}^{(2)} : BA \to BA$ defined by $\bar{f}_1^{(2)} = id_{A[1]}$,

$$\bar{f}_2^{(2)} := \bar{h}\bar{m}_2 - \bar{P}\bar{m}_2\mathfrak{h},$$

and $\bar{f}_3^{(2)} = \cdots = 0$. This defines an A_∞-isomorphism $f^{(2)} : (A, m) \to (A, m^{(2)})$, where $\mathfrak{m}^{(2)} := \mathfrak{f}^{(2)} \circ \mathfrak{m} \circ (\mathfrak{f}^{(2)})^{-1}$. In particular, it turns out that $m_2^{(2)} = Pm_2(P \otimes P)$. Thus, $m_2^{(2)}$ defines a bilinear map on M. Inductively, assume now that $(A, m^{(n)})$ is an A_∞-algebra such that $m_2^{(n)}, \ldots, m_n^{(n)}$ defines a multilinear map on M. We set a coalgebra homomorphism $\mathfrak{f}^{(n+1)} : BA \to BA$ by $\bar{f}_1^{(n+1)} = id_{A[1]}$, $\bar{f}_2^{(n+1)} = \bar{f}_3^{(n+1)} = \cdots = \bar{f}_n^{(n+1)} = 0$,

$$\bar{f}_{n+1}^{(n+1)} := \bar{h}\bar{m}_{n+1}^{(n)} - \bar{P}\bar{m}_{n+1}^{(n)}\mathfrak{h},$$

and $\bar{f}_{n+2}^{(n+1)} = \bar{f}_{n+3}^{(n+2)} = \cdots = 0$. Then, one sees that $m_k^{(n+1)} = m_k^n$ for $k \leq n$ and $m_{n+1}^{(n+1)} = Pm_{n+1}^{(n)}(P \otimes \cdots \otimes P)$. Thus, the induction is completed. (For the details see [19, 21].) □

4.3. A_∞-homotopy.

Definition 4.8. Let (C, Δ), (C', Δ') be coalgebras, and $f : C \to C'$ be a coalgebra map. A linear map $D : C \to C'$ satisfying

$$\Delta' D = (f \otimes D + D \otimes f)\Delta$$

is a **coderivation over** f. For example, for a coderivation D on C', fD is a coderivation over f. If f is a coalgebra isomorphism, all coderivations over f are obtained in such way. Similarly for a coderivation D on C, Df is a coderivation over f and the parallel fact holds.

Definition 4.9 (A_∞-homotopy)**.** Two A_∞-morphisms $f, g : (A, m) \to (A', m')$ are A_∞**-homotopic** if there exists families of A_∞-morphisms $f(t) : (A, m) \to (A', m')$ and coderivations $\mathfrak{h}(t) : BA \to BA'$ over $\mathfrak{f}(t)$ parametrized piecewise algebraically by $t \in K$ such that

$$\frac{d\mathfrak{f}}{dt}(t) = \mathfrak{m}' \circ \mathfrak{h}(t) + \mathfrak{h}(t) \circ \mathfrak{m}.$$

Then we denote $f \sim g$, and $\{(f(t), \mathfrak{h}(t))\}_{t \in K}$ is called an A_∞**-homotopy from** f **to** g.

The decomposition theorem induces the following theorem along [21]. This theorem was first proved in [12] with a different method.

Theorem 4.10. Let (A, m) and (A', m') be A_∞-algebras. An A_∞-morphism $f : (A, m) \to (A', m')$ is an A_∞-quasi-isomorphism if and only if f is an A_∞**-homotopy equivalence**, i.e. there exists an A_∞-morphism $g : (A', m') \to (A, m)$ such that $g \circ f \sim \mathrm{id}_A$ and $f \circ g \sim \mathrm{id}_{A'}$.

Proof. Given a Hodge decomposition $(H = M, \iota, \pi, h)$ of (A, m_1), from the construction in Theorem 4.7 we have an A_∞-algebra structure on $M \oplus C$ and an A_∞-isomorphism $A \simeq M \oplus C$. Then, the pair (ι, π) extends to the pair of linear A_∞-quasi-isomorphisms

$$M \underset{\pi}{\overset{\iota}{\rightleftarrows}} M \oplus C.$$

Furthermore, the projection $P = \iota \circ \pi$ also extends to the linear $A\infty$-(quasi-iso)morphism $P : M \oplus C \to M \oplus C$ to M and it turns out to be A_∞-homotopic to the identity A_∞-(iso)morphism $id_{M \oplus C}$. In fact, setting $P_t := (1-t)\bar{P} + t\mathrm{id}_{A[1]} : A[1] \to A[1]$, by

$$(\mathrm{id}_{A[1]} - \bar{P})P_t = \mathrm{id}_{A[1]} - \bar{P}$$

and $m_1 P_t = P_t m_1$ we have

$$\begin{aligned}\frac{d}{dt}P_t^{\otimes} &= P_t^{\otimes} \otimes (\mathrm{id}_{A[1]} - \bar{P}) \otimes P_t^{\otimes} \\ &= P_t^{\otimes} \otimes (m_1\bar{h} + \bar{h}m_1) \otimes P_t^{\otimes} \\ &= [\mathfrak{m}, P_t^{\otimes} \otimes \bar{h} \otimes P_t^{\otimes}],\end{aligned}$$

where we express as $P_t^{\otimes}$ the coalgebra map corresponding to P_t. Thus, $id_{M\oplus C}$ and P is A_∞-homotopic to each other.

(In particular, the map

$$\mathfrak{h} := \int_0^1 (P_t^{\otimes} \otimes \bar{h} \otimes P_t^{\otimes})dt : BA \to BA$$

gives a chain homotopy from id_{BA} to $P^{\otimes}$. Namely, the Hodge decomposition of $(BA, \mathfrak{m}_1)$ is obtained.)

We also choose a Hodge decomposition $(H' = M', \iota', \pi', h')$ of (A', m_1'). Then we have the following diagram of A_∞-algebras and A_∞-(quasi-iso)morphisms

$$\begin{tikzcd} A \arrow[r, "\sim"] \arrow[d, "f"'] & M \oplus C \arrow[r, shift left, "\pi"] & M = H(A, m_1) \arrow[l, shift left, "\iota"] \arrow[d, "f_H"] \\ A' \arrow[r, "\sim"] & M' \oplus C' \arrow[r, shift left, "\pi'"] & M' = H(A', m_1'). \arrow[l, shift left, "\iota'"] \end{tikzcd}$$

and here we define f_H so that the diagram commutes. Since any composition of A_∞-quasi-isomorphisms is an A_∞-quasi-isomorphism, so is f_H. Furthermore, since M and M' are minimal A_∞-algebras, f_H is actually an A_∞-isomorphism. Thus, there exists the inverse A_∞-isomorphism $(f_H)^{-1}$. Then we define g by the commutative diagram

$$\begin{tikzcd} A \arrow[r, "\sim"] & M \oplus C \arrow[r, shift left] & M = H(A, m_1) \arrow[l, shift left] \\ A' \arrow[u, dotted, "g"] \arrow[r, "\sim"] & M' \oplus C' \arrow[r, shift left] & M' = H(A', m_1'). \arrow[l, shift left] \arrow[u, "(f_H)^{-1}"'] \end{tikzcd}$$

(Note that, in order to construct this g we need the decomposition theorem only, not the A_∞-homotopy.)

Now one can show $g \circ f \sim \mathrm{id}_A$ and $f \circ g \sim \mathrm{id}_{A'}$ since the correspond to $P \sim \mathrm{id}_{M\oplus C}$ on $M \oplus C$ and $P' \sim \mathrm{id}_{M'\oplus C'}$ on $M' \oplus C'$, respectively. □

From Theorem 4.10, an A_∞-quasi-isomorphism has its homotopy inverse.

4.4. **C_∞-algebra.** In this thesis, we use C_∞-algebras as generalization of differential graded commutative algebra rather than A_∞-algebras. For the concept of C_∞-algebra, we refer to [13].

Definition 4.11 (C_∞-algebra)**.** Let (A, m) be an A_∞-algebra. If

$$m_k \in \mathrm{Hom}_{\mathrm{com}}(A[1], A[1]^{\otimes(k+1)})$$

for any k, (A, m) is called a **C_∞-algebra**.

Definition 4.12 (C_∞-morphism)**.** Let $f : (A, m) \to (A', m')$ be an A_∞-morphism. If $f_k \in \mathrm{Hom}_{\mathrm{com}}(A[1], A'[1]^{\otimes(k+1)})$ for any k, f is called a **C_∞-morphism**.

Remark 4.13 (Bar construction of a C_∞-algebra)**.** Let (A, m) be a C_∞-algebra. By the definition of a C_∞-algebra, $\mathfrak{m}$ is a Hopf differential on the Hopf algebra BA. For any C_∞-morphism, its bar construction is a Hopf algebra morphism $BA \to BA'$.

The C_∞-versions of Theorem 4.7 and 4.10 also hold in the same way.

5. Chen's model of a manifold

Let X be an oriented manifold with finite-dimensional rational homology. Fix a base point $*$ of X. We denote the deRham complex on X by $A^\bullet(X)$, the reduced deRham complex and cohomology by

$$A = \tilde{A}^\bullet(X) := \{f \in A^0(X); f(*) = 0\} \oplus A^+(X), \quad H = \tilde{H}^\bullet_{DR}(X)$$

and the suspension of the reduced real homology by $W = \tilde{H}_\bullet(X; \mathbb{R})[-1]$.

5.1. Formal homology connection.

Definition 5.1 (Chen [5, 6])**.** A **formal homology connection** on X is a pair (ω, δ) satisfying the following conditions:

(i) an $\hat{L}W$-coefficient differential form $\omega \in A \otimes \hat{L}W$ with cohomological degree 1 is described by

$$\omega = \sum_{k=1}^{\infty} \sum_{i_1,\dots,i_k} \omega_{i_1\cdots i_k} x^{i_1} \cdots x^{i_k},$$

where $x^1, \dots, x^m$ is a homogeneous basis of W, such that

$$\int_{x_p} \omega_p = 1.$$

(ii) a linear map $\delta : \hat{L}W \to \hat{L}W$ is a differential with homological degree -1 of $\hat{L}W$ such that

$$\delta(W) \subset \hat{\Gamma}_2.$$

(iii) the form ω is a Maurer-Cartan element of $(A \otimes \hat{L}W, d + \delta)$, i.e., the flatness condition $\delta\omega + d\omega + \frac{1}{2}[\omega, \omega] = 0$ holds. (Though the sign notation may be different from Chen's original definition, they are equivalent.)

We call such a differential δ **Chen's differential** of X. If X is simply connected, we can replace the free Lie algebra LW and its derivation $\delta : LW \to LW$ with $\hat{L}W$ and $\delta : \hat{L}W \to \hat{L}W$ respectively.

It is well-known that, given a formal homology connection on X, we can compute the real cohomology of the loop space ΩX [5, 6].

Chen proved that a Riemannian metric gives a formal homology connection.

Theorem 5.2 (Chen [5, 6])**.** Given a Riemannian metric on X, we have uniquely a formal homology connection ω satisfying that ω_i is harmonic and $\omega_{i_1\dots i_k}$ for $k > 1$ is coexact with respect to the metric.

5.2. **C_∞-algebra and formal homology connection.** We shall mention the relation between a formal homology connection and a C_∞-algebra.

According to [14], a formal homology connection (ω, δ) on X is equivalent to a minimal C_∞-algebra model $f : (H, m) \to A$, i.e., a pair of a minimal C_∞-algebra structure on H and a C_∞-algebra morphism $f : (H, m) \to A$ such that the first term f_1 induces the identity on H. It is verified as follows: put

$$\omega = - \sum_{i_1,\dots,i_k} (-1)^\epsilon \sigma^{-1} \bar{f}_n(x^{i_1}, \dots, x^{i_k}) x_{i_1} \cdots x_{i_k},$$

$$\delta = \mathfrak{m}^*,$$

where

$$\epsilon = |x_{i_1}|(|x_{i_2}| + \cdots + |x_{i_k}|) + \cdots + |x_{i_{k-1}}||x_{i_k}|,$$

$\bar{f}_n = \sigma f_n(\sigma^{-1})^{\otimes n} : H[1]^{\otimes n} \to A[1]$, x^i is the dual basis of x_i, and $\mathfrak{m}$ is the bar-construction of m. Then the differential δ on the dual $(BH)^* = \hat{T}W$ of the bar-construction BH can be restricted on $\hat{L}W$ since δ is a coderivation. So the pair (ω, δ) is a formal homology connection on X. Conversely we can recover $f : (H, m) \to A$ from (ω, δ). Note that the condition that f is an A_∞-morphism corresponds to the flatness.

Given a Riemannian metric on X, we get a C_∞-minimal model associated to the Hodge decomposition from Theorem 4.7. This model corresponds the formal homology connection defined in Theorem 5.2.

5.3. **The simplicial set of formal homology connections.** The set of formal homology connections on X is denoted by $Q_0(X)$.

We define the simplical deRham dga $A_\bullet = \{A_n\}_{n=0}^\infty$ on X by

$$A_n := \tilde{A}^\bullet(X \times \Delta^n).$$

Its face maps and degeneracy maps are induced by the coface maps and codegeneracy maps of the cosimplicial space $\Delta^\bullet = \{\Delta^n\}_{n=0}^\infty$.

The family $Q_\bullet(X) = \{Q_n(X) := Q_0(X \times \Delta^n)\}_{n=0}^\infty$ of sets is a simplicial set by the induced structure by $A_\bullet$. Given a Chen's differential δ on X, the set of formal homology connections (ω, δ) on $X \times \Delta^n$ is denoted by $Q_n(X, \delta)$. Then $Q_\bullet(X, \delta)$ is also a simplicial set. We denote the set of Maurer-Cartan elements of $(A_n \otimes \hat{L}W, d + \delta)$ by $\mathrm{MC}_n(X, \delta)$. We obtain the simplicial set $\mathrm{MC}_\bullet(X, \delta)$, and then $Q_\bullet(X, \delta)$ is a subsimplicial set of $\mathrm{MC}_\bullet(X, \delta)$.

Lemma 5.3. For any n-th simplicial Muarer-Cartan element $\alpha \in \mathrm{MC}_n(X, \delta)$, if $\partial_i \alpha \in Q_{n-1}(X)$ for some $0 \leq i \leq n$, then $\alpha \in Q_n(X, \delta)$.

Proof. Regarding α as a C_∞-map $f : H \to A_n$, $f_1 : H \to H(A_n)$ is the identity map since ∂_i for any i gives the standard identification by $H^\bullet(X \times \Delta^n) \simeq H^\bullet(X \times \Delta^{n-1})$ and $\partial_i f_1 : H \to H(A_{n-1})$ is the identity map under the assumption.

□

Since the simplicial set $\mathrm{MC}(X, \delta)$ is a Kan complex (proved in Section 4 of [11]), the following lemma is obtained immediately from Lemma 5.3:

Lemma 5.4. The simplicial set $Q_\bullet(X)$ is a Kan complex. Furthermore the map induced by the inclusion

$$\pi_0(Q_\bullet(X, \delta)) \to \pi_0(\mathrm{MC}_\bullet(X, \delta))$$

is injective, and the map

$$\pi_n(Q_\bullet(X,\delta),\tau) \to \pi_n(\mathrm{MC}_\bullet(X,\delta),\tau)$$

for $\tau \in Q_0(X,\delta)$ and $n \geq 1$ is an isomorphism.

Theorem 5.5. The homotopy groups of the simplicial set $Q_\bullet(X)$ are described by

$$\pi_n(Q_\bullet(X),\tau) \simeq H_n(\mathrm{Der}(\hat{L}W),\delta)$$

for $n \geq 1$ and a formal homology connection $\tau = (\omega,\delta)$ on X, where $H_1(\mathrm{Der}(LW),\delta)$ is equipped with the Baker-Campell-Hausdorff product of $H_0(A \otimes \hat{L}W)$.

Proof. From Proposition 5.4 and Theorem 5.5 in [1], we have

$$\pi_n(Q_\bullet(X),\tau) \simeq \pi_n(\mathrm{MC}_\bullet(X,\delta),\tau) \simeq H_{n-1}(A \otimes \hat{L}W, d+\delta+[\omega,-]).$$

We shall prove the suspension of $(A \otimes \hat{L}W, d+\delta+[\omega,-])$ and the chain complex $\mathrm{Der}_F(BH,BA)$ of Hopf derivations over the bar-construction $F : BH \to BA$ of the C_∞-morphism corresponding to τ are isomorphic. Here the differential $\mathfrak{D}$ of the latter complex is defined by

$$\mathfrak{D}(D) = \mathfrak{m}^A \circ D - (-1)^D D \circ \mathfrak{m},$$

where $\mathfrak{m}^A$ and $\mathfrak{m}$ are the bar-constructions of C_∞-structures of A and (H,m) respectively.

Through the natural identification $\hat{T}W = (BH)^*$, consider the linear isomorphism $\Phi : A[1] \otimes \hat{L}W \to \mathrm{Der}_F(BH,BA) \subset \mathrm{Hom}(BH,A[1])$ defined by

$$\Phi(\alpha \otimes f)(x) = f(x)\alpha$$

for $x \in BH$. Here the differential on $A[1] \otimes \hat{L}W$ is equal to $\sigma(d+\delta+[\omega,-])\sigma^{-1}$. Then, using $F = \Phi(\sigma\omega)$, we have

$$\begin{aligned}
&\Phi(\sigma(d+\delta+[\omega,-])\sigma^{-1}(\alpha \otimes f))(x)\\
=&d\alpha f(x) + (-1)^{\alpha+1}\alpha\delta f(x) + \sigma[\omega,\sigma^{-1}\alpha \otimes f](x)\\
=&d\alpha f(x) - (-1)^{\alpha+f}\alpha f\mathfrak{m}(x) + \mathfrak{m}_2^A \circ (F \otimes \Phi(\alpha f))(x) + \mathfrak{m}_2^A \circ (\Phi(\alpha f) \otimes F)(x)\\
=&\mathfrak{D}\Phi(\alpha f)(x).
\end{aligned}$$

Thus the map Φ is a chain isomorphism.

On the other hand, the map

$$F \circ - : (\mathrm{Der}(\hat{L}W), \mathrm{ad}(\delta)) = (\mathrm{Der}(BH), \mathrm{ad}(\mathfrak{m})) \to (\mathrm{Der}_F(BH,BA), \mathfrak{D})$$

is a quasi-isomorphism because F is a quasi-isomorphism. So we get the isomorphism

$$H_{n-1}(A \otimes \hat{L}W, d+\delta+[\omega,-]) \simeq H_n(\mathrm{Der}(\hat{L}W), \mathrm{ad}(\delta)).$$

□

The set $Q(X) := \pi_0(Q_\bullet(X))$ of connected components can be identified with the set of homotopy classes of C_∞-morphisms $f : (H,m) \to A$ such that f_1 induces the identity map on H. The group $\mathrm{QIAut}(H,m)$ of homotopy classes of C_∞-automorphisms $f : (H,m) \to (H,m)$ such that $f_1 = \mathrm{id}_H$ acts on the right on $Q(X,\delta) := \pi_0(Q_\bullet(X,\delta))$ freely and transitively. We shall investigate the set $Q(X)$ as space in the next chapter.

Chapter 2. Characteristic classes through the cohomology of the moduli of homotopy algebra

Let X be an n-dimensional oriented closed manifold with base point $*$.

6. Moduli space of C_∞-minimal models

For a minimal C_∞-algebra structure m on H, **the moduli space $Q(X,m)$ of C_∞-quasi-isomorphisms over** m is the set of C_∞-homotopy classes of C_∞-quasi-isomorphisms $\iota : (H,m) \to A$ such that ι_1 induces the identity map on the their cohomology H.

The group $\mathrm{QIAut}(H,m)$ of homotopy classes of C_∞-automorphisms $f : (H,m) \to (H,m)$ such that $f_1 = \mathrm{id}_H$ is a inverse limit of finite-dimensional lie group:

$$\mathrm{QIAut}(H,m) = \varprojlim_n \mathrm{QIAut}(H,m)/\,\mathrm{QIAut}^{\geq n}(H,m),$$

where $\mathrm{QIAut}^{\geq n}(H,m)$ is the group consisting of classes of C_∞-automorphisms $f : (H,m) \to (H,m)$ such that $f_1 = \mathrm{id}_H, f_2 = 0, \ldots, f_{n-1} = 0$. Then each n-th quotient group is a finite-dimensional Lie group. The Lie ring of $\mathrm{QIAut}(H,m)$ is isomorphic to the Lie algebra $\mathrm{QDer}^+(H,m)$, which is the image of $\mathrm{Der}^{\geq 1}(BH)_0 \cap \mathrm{Ker}(\mathrm{ad}(\mathfrak{m}))$ in $H_0(\mathrm{Der}(BH), \mathrm{ad}(\mathfrak{m}))$.

The Lie group $\mathrm{QIAut}(H,m)$ acts on $Q(X,m)$ by

$$\iota \cdot f := \iota \circ f$$

for $\iota \in Q(X,m)$, $f \in \mathrm{QIAut}(H,m)$. This action is free and transitive since an C_∞-quasi-isomorphism has a homotopy inverse. So $Q(X,m)$ has the inverse limit of smooth manifolds, which is isomorphic to $\mathrm{QIAut}(H,m)$.

The set $C_\infty(X)$ of minimal C_∞-structures m on H such that $C_\infty(X,m) \neq \emptyset$ is parametrized by the space

$$\mathrm{IAut}(H,m)\backslash\,\mathrm{IAut}(BH).$$

So **the moduli space of C_∞-minimal models** of the reduced de Rham complex A of X

$$Q(X) := \coprod_{m \in C_\infty(X)} Q(X,m)$$

is parametrized by the space

$$Q(X,m) \times_{\mathrm{IAut}(H,m)} \mathrm{IAut}(BH)$$

fixing m. It is the space of C_∞-homotopy classes of C_∞-minimal models $\iota : (H,m) \to A$ such that ι_1 induces the identity map on the de Rham cohomology H.

The mapping class group of X

$$\mathcal{M}(X) := \mathrm{Diff}_+(X)/\,\mathrm{Diff}_0(X) = \pi_0(\mathrm{Diff}_+(X))$$

acts on $Q(X)$ as follows:

$$[\varphi] \cdot [\iota, m] := (\varphi \circ \iota \circ |\varphi|^{-1}, |\varphi| \circ m \circ |\varphi|^{-1})$$

for $[\iota, m] \in Q(X)$ and $[\varphi] \in \mathcal{M}(X)$. Here $|\varphi|$ is the map induced to H by φ. This action is well-defined since two isotopic diffeomorphisms φ_0, φ_1 of X induce C_∞-homotopic dga maps $A \to A$.

7. Construction

Let $E \to B$ be a smooth fiber bundle whose fiber is an oriented closed manifold X. For simplicity, we set

$$Q := Q(X),\ C_\infty := C_\infty(X),\ Q(m) := Q(X,m),\ \mathcal{M} := \mathcal{M}(X).$$

Choose a metric g of $E \to B$. The metric g_b on fiber E_b for $b \in B$ defines the Hodge composition on $A^\bullet(E_b)$. Since these Hodge decompositions give C_∞-minimal models of fibers, we can obtain the map $B \to \mathcal{S}\backslash Q$, where $\mathcal{S}$ is the image of the structure group of $E \to B$ in $\mathcal{M}$. Defining the de Rham complex of $\mathcal{S}\backslash Q$ by $A^\bullet(\mathcal{S}\backslash Q) := A^\bullet(Q)^{\mathcal{S}}$, we have the map $H^\bullet_{DR}(\mathcal{S}\backslash Q) \to H^\bullet_{DR}(B)$. Since any two metrics can be connected by a segment, the map is independent of a choice of a metric.

7.1. **Homologically trivial bundles.** We consider the case where the structure group of a fiber bundle acts trivially on the de Rham cohomology group of the fiber. In other words, suppose $\mathcal{S} = \mathcal{I} := \mathrm{Ker}(\mathcal{M} \to \mathrm{GL}(H))$. Then we have a map $q : B \to C_\infty$ by giving a metric of $E \to B$. Fix $m \in C_\infty$. Since the topological group $\mathrm{IAut}(H,m)$ is contractible, the pullback $q^*\,\mathrm{IAut}(BH) \to B$ of the principal $\mathrm{IAut}(H,m)$-bundle $\mathrm{IAut}(BH) \to C_\infty$ is trivial. Taking a trivialization of the principal bundle, we get the $\mathcal{I}$-equivalent map

$$s : q^*Q = Q(m) \times_{\mathrm{IAut}(H,m)} q^*\,\mathrm{IAut}(BH) \simeq Q(m) \times C_\infty \to Q(m).$$

Thus we can obtain the chain map

$$A^\bullet(Q(m))^{\mathcal{I}} \xrightarrow{s^*} A^\bullet(q^*Q)^{\mathcal{I}} \to A^\bullet(B).$$

Form the action of $\mathrm{QIAut}(H,m)$, the space $Q(m)$ has the Maurer-Cartan form $\eta \in A^1(Q(m); \mathrm{QDer}^+(H,m))$. Then we have the chain map

$$\Phi : C^\bullet_{CE}(\mathrm{QDer}^+(H,m)) \to A^\bullet(Q(m))^{\mathcal{I}}.$$

It is constructed as follows: for a cochain $c \in C^p_{CE}(\mathrm{QDer}^+(H,m))$, we define

$$\Phi(c) := c(\eta^p) = \sum_{i_1<\cdots<i_p} \eta_{i_1} \wedge \cdots \wedge \eta_{i_p} c(b_{i_1} \wedge \cdots \wedge b_{i_p}),$$

where we set

$$\eta = \sum_i \eta_i b_i$$

using a (topological) basis $\{b_i\}$ of $\mathrm{QDer}^+(H,m)$. The p-form $\Phi(c)$ is $\mathcal{I}$-invariant since $\mathcal{I}$ acts on H trivially. Then Φ is a chain map by the flatness of η

$$d\eta + \frac{1}{2}[\eta,\eta] = \sum_i d\eta_i b_i + \sum_{i<j} \eta_i \wedge \eta_j [b_i, b_j] = 0.$$

So we obtain the following:

Theorem 7.1. *Let $E \to B$ be a smooth fiber bundle with oriented closed fiber X whose structure group acts trivially on the real cohomology group of X. Then the chain map $C^\bullet_{CE}(\mathrm{QDer}^+(H,m)) \to A^\bullet(B)$ obtained by the construction above induces the map between cohomologies which is independent of a choice of a fiberwise metric.*

7.2. **Formal manifold bundles.** We consider the case where X is a formal manifold, i.e. C_∞ contains the algebra structure m of H, and there exists a decomposition of S-modules

$$\mathrm{Der}^+(BH)_0 = V \oplus \mathrm{Der}^+_m(BH)_0,$$

where S is the image of $\mathcal{S}$ in $\mathrm{GL}(H)$.

By the same discussion of Lemma 3.5 in [32], we can obtain the following:

Lemma 7.2. The S-equivariant principal $\mathrm{IAut}(H,m)$-bundle $\mathrm{IAut}(BH) \to C_\infty$ is S-equivariant trivial.

Then there exists an $\mathcal{S}$-equivariant diffeomorphism

$$Q = Q(m) \times_{\mathrm{IAut}(H,m)} \mathrm{IAut}(BH) \simeq Q(m) \times C_\infty.$$

Since the space C_∞ is also contractible, we can obtain the following:

Theorem 7.3. The space Q is $\mathcal{S}$-equivariant homotopic to $Q(m)$.

From the Maurer-Cartan form on $Q(m)$, we have the chain map

$$C^\bullet_{CE}(\mathrm{QDer}^+(H,m), S) \to A^\bullet(Q(m))^{\mathcal{S}}$$

in the same way as Subsection 7.1.

Theorem 7.4. Let $E \to B$ be a smooth fiber bundle with oriented closed formal fiber X. Suppose there exists a decomposition of S-modules

$$\mathrm{Der}^+(BH)_0 = V \oplus \mathrm{Der}^+_m(BH)_0,$$

where m is the algebra structure of H and S is the image of the structure group in $\mathrm{GL}(H)$. Then the chain map $C^\bullet_{CE}(\mathrm{QDer}^+(H,m), S) \to A^\bullet(B)$ obtained by the construction above induces the map between cohomologies which is independent of a choice of a fiberwise metric.

8. Relation to the construction using the fundamental group

For any $[m,\iota] \in Q$, we have the dual of the bar construction of ι

$$(BA)^* \to (BH)^* = \hat{T}W,$$

where $\hat{T}W$ means the completed tensor product generated by $W := H^*[-1]$. So composing the chain map $C_\bullet(\Omega X) \to (BA)^*$ obtained by iterated integrals from the cube chain complex of the loop space ΩX, we obtain the chain map

$$C_\bullet(\Omega X; \mathbb{R}) \to (\hat{T}W, \delta),$$

where $\delta := \mathfrak{m}^*$. The degree 0 part of (the completion of) map induced to homologies gives

$$\hat{\mathbb{R}}\pi_1 = \hat{H}_0(\Omega X; \mathbb{R}) \to H_0(\hat{T}W, \delta) = \hat{T}W_0/I_\delta,$$

where $\pi_1 := \pi_1(X)$, $W_0 := H_1(X;\mathbb{R})[-1]$ and $I_\delta := \delta(H_2(X;\mathbb{R})[-1])$. (This map given by the C_∞-minimal model defined by a metric g on X is the Chen expansion defined by g.) So we have the $\mathcal{M}$-equivalent map $\theta : Q \to \Theta(\pi_1)$. Here the definition of the space $\Theta(\pi_1)$ is obtained by replacing "Hopf algebra" with "algebra" from $\Theta(\pi_1)$ in [32].

Fixing m, we have the commutative diagram

$$\begin{tikzcd}
T_\iota Q(m) \arrow[r] \arrow[d, "\theta_*"] & T_1\,\mathrm{QIAut}(H,m) \arrow[r, equal] \arrow[d, "\theta_*"] & \mathrm{QDer}^+(H,m) \\
T_{\theta(\iota)}\Theta(\pi_1, I_\delta) \arrow[r] & T_1\,\mathrm{IAut}(\hat{T}W_0/I_\delta) \arrow[r, equal] & \mathrm{Der}^+(\hat{T}W_0/I_\delta).
\end{tikzcd}$$

So we obtain

$$\theta^*\eta = \theta_*\eta^\Theta,$$

where η^Θ is the Maurer-Cartan form on $\Theta(\pi_1)$ by the action of $\mathrm{IAut}(\hat{T}W_0/I_\delta)$.

Thus we obtain the following:

Theorem 8.1. We have the commutative diagram

$$\begin{tikzcd}
H^\bullet_{CE}(\mathrm{QDer}^+(H,m)) \arrow[r] & H^\bullet_{DR}(B) \\
H^\bullet_{CE}(\mathrm{Der}^+(\hat{T}W_0/I_\delta)) \arrow[u] \arrow[ur] &
\end{tikzcd}$$

under the assumption in Theorem 7.1 and

$$\begin{tikzcd}
H^\bullet_{CE}(\mathrm{QDer}^+(H,m), S) \arrow[r] & H^\bullet_{DR}(B) \\
H^\bullet_{CE}(\mathrm{Der}^+(\hat{T}W_0/I_\delta), S) \arrow[u] \arrow[ur] &
\end{tikzcd}$$

under the assumption in Theorem 7.4.

Chapter 3. Obstruction theoretic construction

9. The simplicial bundle of formal homology connections

Let $X \to E \to B$ be a smooth fiber bundle. In the section, we shall define the simplicial bundle of formal homology connections on fibers corresponding to a smooth bundle.

Definition 9.1. We define the simplicial bundle $\mathcal{Q}_\bullet(E) \to S_\bullet(B)$ over the simplicial set $S_\bullet(B)$ of singular simplices as follows:

- the fiber over an n-simplex $\sigma \in S_n(B)$ is $\mathcal{Q}_n(E)_\sigma := Q_0(\sigma^* E)$, and
- the face maps and the degeneracy maps are the induced maps $\mathcal{Q}_n(E)_\sigma \to \mathcal{Q}_{n-1}(E)_{\partial_i \sigma}$ and $\mathcal{Q}_n(E)_\sigma \to \mathcal{Q}_{n+1}(E)_{s_i \sigma}$ by the coface maps and the codegeneracy maps of $\Delta^\bullet$ respectively.

We can check that $\mathcal{Q}_\bullet(E) \to S_\bullet(B)$ is a bundle of simplicial sets in the sense of May [33].

Proposition 9.2. The simplicial map $\mathcal{Q}_\bullet(E) \to S_\bullet(B)$ is a simplicial bundle with fiber $Q_\bullet(X)$.

Proof. For an n-simplex $\sigma \in S_n(B)$ and a trivialization $\varphi_\sigma : \Delta^n \times X \simeq \sigma^* E$, we obtain the trivialization $\varphi_{\sigma,P} : \Delta^i \times X \simeq \sigma(P)^* E$ for $P \in \Delta[n]_i$ by the diagram

$$\begin{tikzcd} \Delta^n \times X \arrow[r, "\varphi_\sigma"] & \sigma^* E \\ \Delta^i \times X \arrow[u, "f_P \times \mathrm{id}_X"] \arrow[r, "\varphi_{\sigma,P}"] & \sigma(P)^* E \arrow[u] \end{tikzcd}$$

regarding σ as a simplicial map $\sigma : \Delta[n] \to S_\bullet(B)$. Here the map $f_P : \Delta^i \to \Delta^n$ is the induced map $P : \Delta[i] \to \Delta[n]$.

Then we obtain the simplicial trivialization

$$\hat{\varphi}_\sigma : \sigma^* \mathcal{Q}_\bullet(E) \simeq \Delta[n] \times Q_\bullet(X).$$

by $(P, \alpha) \mapsto (P, \varphi_{\sigma,P}^* \alpha)$, where

$$\sigma^* \mathcal{Q}_i(E) = \{(P, \alpha) \in \Delta[n]_i \times Q_0(\sigma(P)^* E)\}.$$

$\square$

We consider to fix a Chen's differential on fibers.

Definition 9.3. Fix a Chen's differential $\delta \in \mathrm{Der}(\hat{L}W)_{-1}$ of X is G-invariant with respect to the action of the homological structure group G on $\mathrm{Der}(\hat{L}W)$ (induced by the action on W). Then it gives the section $\hat{\delta}$ of the bundle

$$\mathcal{D}(E) \to B,$$

where $\mathcal{D}(E)_b := \{\text{Chen's differential of } E_b\}$ for $b \in B$. We call $\hat{\delta}$ a **section of Chen's differentials**. Given this, we can consider the simplicial bundle $\mathcal{Q}_\bullet(E, \hat{\delta}) \to S_\bullet(B)$ defined by

$$\mathcal{Q}_n(E, \hat{\delta})_\sigma := Q_0(\sigma^* E, \hat{\delta}(\sigma))$$

for $\sigma \in S_n(B)$. Here $\hat{\delta}(\sigma)$ is the Chen's differential of $\sigma^* E$ defined by $\hat{\delta}(\sigma_0)$ through the isomorphism $H(\sigma^* E) \simeq H(E_{\sigma_0})$. Here $\sigma_0 = \partial_1 \cdots \partial_n \sigma$ is the image of the base point of Δ^n.

For example, if X is formal, the differential δ corresponding to the cohomology ring structure of X is $\mathrm{Diff}(X)$-invariant.

10. Obstruction theory

Obstruction theory for simplicial sets is studied in [4, 9]. In Section 10.1 and 10.2, we shall review a part of them and rewrite obstruction theory as in Steenrod [41] for simplicial sets in order to fit our use briefly. In Section 4.2, we introduce obstruction classes to extend a section over the 0-skeleton stepwisely.

10.1. **Local system.**

We shall define cohomology with local coefficients briefly. We can see definitions in [4, 9].

Definition 10.1. Let $\mathcal{X}$ be a Kan complex. We define the **fundamental groupoid** $\Pi_1(\mathcal{X})$ of $\mathcal{X}$ such that the set of objects is $\mathcal{X}_0$ and the set of morphisms from x to y is the set of homotopy classes of $\gamma \in \mathcal{X}_1$ satisfying $\partial_0\gamma = x$ and $\partial_1\gamma = y$. A covariant functor $\Pi_1(\mathcal{X}) \to \mathcal{A}b$ is called a **local system** on $\mathcal{X}$. Here $\mathcal{A}b$ is the category of abelian groups.

Let $\mathcal{E} \to \mathcal{B}$ be a Kan simplicial bundle with n-simple fiber $\mathcal{X}$, i.e., $\mathcal{X}$ is a Kan complex and $\pi_1(\mathcal{X}, x)$ acts on $\pi_n(\mathcal{X}, x)$ trivially.

Definition 10.2. We define the local system $\Pi_n(\mathcal{E}/\mathcal{B})$ on $\mathcal{B}$ as follows: for a vertex $v \in \mathcal{B}_0$,

$$\Pi_n(\mathcal{E}/\mathcal{B})_v := \pi_n(v^*\mathcal{E}).$$

Note that we need not to choose a base point of $v^*\mathcal{E}$ because it is n-simple. For a path $\gamma \in \mathcal{B}_1$ such that $v_0 = \partial_1\gamma$ and $v_1 = \partial_0\gamma$, take a trivialization

$$\varphi_\gamma : \Delta[1] \times v_0^*\mathcal{E} \simeq \gamma^*\mathcal{E}$$

such that

$$\begin{tikzcd} \Delta[1] \times v_0^*\mathcal{E} \arrow[r, "\varphi_\gamma"] & \gamma^*\mathcal{E} \\ v_0^*\mathcal{E} \arrow[u, "\delta^1"] \arrow[r, equal] & v_0^*\mathcal{E}. \arrow[u, "\text{incl.}"'] \end{tikzcd}$$

Here $\delta^i : \Delta[0] \to \Delta[1]$ is the coface maps. Then we have the isomorphism $g_\gamma : v_0^*\mathcal{E} \to v_1^*\mathcal{E}$, which is called **holonomy** along γ, defined by

$$\begin{tikzcd} \Delta[1] \times v_0^*\mathcal{E} \arrow[r, "\varphi_\gamma"] & \gamma^*\mathcal{E} \\ v_0^*\mathcal{E} \arrow[u, "\delta^0"] \arrow[r, "g_\gamma"'] & v_1^*\mathcal{E}. \arrow[u, "\text{incl.}"'] \end{tikzcd}$$

So we put

$$\Pi_n(\mathcal{E}/\mathcal{B})(\gamma) := (g_\gamma^{-1})_* : \pi_n(v_1^*\mathcal{E}) \to \pi_n(v_0^*\mathcal{E}).$$

We can prove that it is depend on only the homotopy class of γ since $\mathcal{E} \to \mathcal{B}$ is Kan fibration. In fact, for another path γ' homotopic to γ by a homotopy $\sigma \in \mathcal{B}_2$, there exists a homotopy h satisfying the commutative diagram

$$\begin{tikzcd} \Lambda^2[2] \times v_0^*\mathcal{E} \arrow[r, "\varphi_\gamma \cup \varphi_{\gamma'}"] \arrow[d] & \sigma^*\mathcal{E} \arrow[d] \\ \Delta[2] \times v_0^*\mathcal{E} \arrow[r] \arrow[ur, dotted, "h"] & \Delta[2] \end{tikzcd}$$

by Theorem 7.8 in [33]. Here $\Lambda^2[2]$ is the $(2,2)$-horn.

The cochain complex and the cohomology with local coefficients are defined as follows.

Definition 10.3. Let X be a Kan complex, A a subsimplicial set of X, and $M : \Pi_1(X) \to \mathcal{A}b$ a local system on X. We define **the cochain complex with coefficient** M by

$$C^n(X, A; M) := \left\{ c : X_n \to \coprod_{v \in X_0} M(v);\ c(x) \in M(x_0),\ c|A = 0 \right\},$$

where $x_0 = \partial_1 \cdots \partial_n x$, and its normalized version by

$$N^n(X, A; M) := \bigcap_{i=0}^{n} \mathrm{Ker}(s_i^* : C^n(X, A; M) \to C^{n-1}(X, A; M)).$$

The differential $\delta : C^n(X, A; M) \to C^{n+1}(X, A; M)$ is defined by

$$\delta c(x) = M(x_{01})^{-1} c(\partial_0 x) - c(\partial_1 x) + \cdots + (-1)^{n+1} c(\partial_{n+1} x),$$

where $x_{01} = \partial_2 \cdots \partial_n x$. Its cohomology is denoted by $H^n(X, A; M)$.

10.2. **Obstruction cocycles and difference cochains.** Let $\mathcal{A}$ be a subsimplicial set of $\mathcal{B}$. We call a simplicial map s satisfying the following diagram an n**-partial section** relative to $\mathcal{A}$:

$$\begin{tikzcd} & \mathcal{E} \arrow[d] \\ \mathrm{sk}_n(\mathcal{B}) \cup \mathcal{A} \arrow[r] \arrow[ur, "s"] & \mathcal{B} \end{tikzcd}$$

Given an n-partial section $s : \mathrm{sk}_n(\mathcal{B}) \cup \mathcal{A} \to \mathcal{E}$ relative to $\mathcal{A}$, we shall construct the **obstruction cocycle** of s

$$c(s) \in N^{n+1}(\mathcal{B}, \mathcal{A}; \Pi_n(\mathcal{E}/\mathcal{B}))$$

to extend a partial section $\mathrm{sk}_{n+1}(\mathcal{B}) \cup \mathcal{A} \to \mathcal{E}$ as follows: for an $(n+1)$-simplex $\sigma \in \mathcal{B}_{n+1}$, we get the induced section s_σ such that

$$\begin{tikzcd} \sigma^*\mathcal{E} \arrow[r] & \mathcal{E} \\ \mathrm{sk}_n(\Delta[n+1]) \arrow[r, "\mathrm{sk}_n(\sigma)"] \arrow[u, dotted, "s_\sigma"] & \mathrm{sk}_n(\mathcal{B}). \arrow[u, "s"'] \end{tikzcd}$$

So we put

$$c(s)(\sigma) := g_\sigma^{-1}[s_\sigma] \in \pi_n(\sigma_0^*\mathcal{E}),$$

where $g_\sigma : \pi_n(\sigma_0^*\mathcal{E}) \to \pi_n(\sigma^*\mathcal{E})$ is an isomorphism induced by the inclusion $\sigma_0^*\mathcal{E} \to \sigma^*\mathcal{E}$.

Proposition 10.4. The cochain $c(s)$ is a cocycle.

Proof. For an $(n+2)$-simplex $\sigma \in \mathcal{B}_{n+2}$, we have

$$\begin{tikzcd} (\partial_i\sigma)^*\mathcal{E} \arrow[r] & \sigma^*\mathcal{E} \arrow[r] & \mathcal{E} \\ \mathrm{sk}_n(\Delta[n+1]) \arrow[u, "s_{\partial_i\sigma}"] \arrow[r, "\mathrm{sk}_n(\delta^i)"'] & \mathrm{sk}_n(\Delta[n+2]) \arrow[u, "s_\sigma"] \arrow[r, "\mathrm{sk}_n(\sigma)"'] & \mathrm{sk}_n(\mathcal{B}). \arrow[u, "s"] \end{tikzcd}$$

So the commutative diagrams for $i \neq 0$

$$\begin{tikzcd} \sigma_0^*\mathcal{E} \arrow[d] \arrow[dr] & \\ (\partial_i\sigma)^*\mathcal{E} \arrow[r] & \sigma^*\mathcal{E} \end{tikzcd} \qquad \begin{tikzcd} \sigma_1^*\mathcal{E} \arrow[d] & \sigma_0^*\mathcal{E} \arrow[l, "g_{\sigma_{01}}"'] \arrow[d] \\ (\partial_0\sigma)^*\mathcal{E} \arrow[r] & \sigma^*\mathcal{E} \end{tikzcd}$$

imply the equations

$$g_{\partial_i\sigma}^{-1}[s_{\partial_i\sigma}] = g_\sigma^{-1}(s_\sigma)_*[\mathrm{sk}_n(\delta^i)], \quad g_{\sigma_{01}}^{-1}g_{\partial_0\sigma}^{-1}[s_{\partial_0\sigma}] = g_\sigma^{-1}(s_\sigma)_*(\sigma_{01})_*[\mathrm{sk}_n(\delta^0)].$$

Here note that $[\mathrm{sk}_n(\delta^i)] \in \pi_n(\mathrm{sk}_n(\Delta[n+2]), 0)$ and $[\mathrm{sk}_n(\delta^0)] \in \pi_n(\mathrm{sk}_n(\Delta[n+2]), 1)$. Thus we obtain

$$(\delta c(s))(\sigma) = g_\sigma^{-1}(s_\sigma)_*\left((\sigma_{01})_*[\mathrm{sk}_n(\delta^0)] + \sum_{i\neq 0}(-1)^i[\mathrm{sk}_n(\delta^i)]\right) = 0,$$

using the relation $(\sigma_{01})_*[\mathrm{sk}_n(\delta^0)] + \sum_{i\neq 0}(-1)^i[\mathrm{sk}_n(\delta^i)] = 0$ in $\pi_n(\mathrm{sk}_n(\Delta[n+2]), 0)$. $\square$

We shall define the difference cochain for n-partial sections $s_0, s_1 : \mathrm{sk}_n(\mathcal{B}) \to \mathcal{E}$ and a fiberwise homotopy $h : \mathrm{sk}_{n-1}(\mathcal{B}) \times \Delta[1] \to \mathcal{E} \times \Delta[1]$ between their restriction on $\mathrm{sk}_{n-1}(\mathcal{B})$. Gluing these maps, we have the map

$$h^\square : (\mathrm{sk}_n(\mathcal{B}) \times \mathrm{sk}_0(\Delta[1])) \cup (\mathrm{sk}_{n-1}(\mathcal{B}) \times \Delta[1]) \to \mathcal{E} \times \Delta[1].$$

We consider the obstruction cocycle

$$c(h^\square) \in N^{n+1}(\mathrm{sk}_n(\mathcal{B}) \times \Delta[1], (\mathrm{sk}_n(\mathcal{B}) \times \mathrm{sk}_0(\Delta[1])) \cup (\mathrm{sk}_{n-1}(\mathcal{B}) \times \Delta[1]); \Pi_n^\square),$$

where $\Pi_n^\square = \Pi_n(\mathcal{E} \times \Delta[1]/\mathcal{B} \times \Delta[1])$. Note that faces of non-degenerate simplices of $\mathrm{sk}_n(\mathcal{B}) \times \Delta[1]$ are in $(\mathrm{sk}_n(\mathcal{B}) \times \mathrm{sk}_0(\Delta[1])) \cup (\mathrm{sk}_{n-1}(\mathcal{B}) \times \Delta[1])$. Through the Eilenberg-Zilber map

$$\times : N_n(\mathcal{B}) \otimes N_1(\Delta[1]) \to N_{n+1}(\mathrm{sk}_n(\mathcal{B}) \times \Delta[1], (\mathrm{sk}_n(\mathcal{B}) \times \mathrm{sk}_0(\Delta[1])) \cup (\mathrm{sk}_{n-1}(\mathcal{B}) \times \Delta[1])),$$

we can define the cochain $d(s_0, h, s_1) \in N^n(\mathcal{B}; \Pi_n(\mathcal{E}/\mathcal{B}))$ by

$$d(s_0, h, s_1)(\sigma) := (-1)^n c(h^\square)(\sigma \times I)$$

for $\sigma \in \mathcal{B}_n$. Here I is the unique non-degenerate simplex in $\Delta[1]_1$.

Proposition 10.5. The cochain $d(s_0, h, s_1)$ satisfies

$$\delta d(s_0, h, s_1) = c(s_1) - c(s_0).$$

Proof. It is proved by the equations

$$\begin{aligned}
\delta d(s_0,h,s_1)(\sigma) &= g_{\sigma_{01}}^{-1} d(s_0,h,s_1)(\partial_0\sigma) + \sum_{i\neq 0}(-1)^i d(s_0,h,s_1)(\partial_i\sigma)\\
&= (-1)^n g_{\sigma_{01}}^{-1} c(h^{\Box})(\partial_0\sigma\otimes I) + \sum_{i\neq 0}(-1)^{n+i} c(h^{\Box})(\partial_i\sigma\otimes I)\\
&= c(h^{\Box})(\sigma\otimes\partial I) - \delta c(h^{\Box})(\sigma\otimes I)\\
&= c(s_1) - c(s_0).
\end{aligned}$$

□

The next two propositions hold in the same way as in obstruction theory [41].

Proposition 10.6. An n-partial section $s : \mathrm{sk}_n(\mathcal{B}) \to \mathcal{E}$ extends to an $(n+1)$-partial section $\mathrm{sk}_{n+1}(\mathcal{B}) \to \mathcal{E}$ if and only if $c(s) = 0$.

Proposition 10.7. For n-partial sections $s, s' : \mathrm{sk}_n(\mathcal{B}) \to \mathcal{E}$, if obstruction cocycles $c(s)$ and $c(s')$ are cohomologue, there is a homotopy between $s|\,\mathrm{sk}_{n-1}(\mathcal{B})$ and $s'|\,\mathrm{sk}_{n-1}(\mathcal{B})$.

Suppose a fiber $\mathcal{X}$ of a Kan fiber bundle $\mathcal{E} \to \mathcal{B}$ is $(n-1)$-connected (and $\pi_1(\mathcal{X}, x)$ is abelian if $n = 1$). Then we can get an n-partial section $s : \mathrm{sk}_n(\mathcal{B}) \to \mathcal{E}$. If we get another n-partial section s', these is a homotopy between $s|\,\mathrm{sk}_{n-1}(\mathcal{B})$ and $s'|\,\mathrm{sk}_{n-1}(\mathcal{B})$. So we obtain an invariant

$$\mathfrak{o}_n(\mathcal{E}) := [c(s)] \in H^{n+1}(\mathcal{B}; \Pi_n(\mathcal{E}/\mathcal{B})).$$

It is called the **obstruction class** of $\mathcal{E} \to \mathcal{B}$.

10.3. **Obstruction for $n = 0$.** We consider an extension of a 0-partial section under the following situation: for a simplicial bundle $\mathcal{E} \to \mathcal{B}$, suppose that the local system $\Pi_0(\mathcal{E}/\mathcal{B})$ of sets has a free and transitive right action of a local system $\mathcal{G}$ of groups on $\mathcal{B}$.

At first, we define the non-abelian obstruction class of a 0-partial section. For that, we remark the definition of the non-abelian cohomology with values in a local system of non-abelian groups. Here "non-abelian cohomology" is in the sense of [10].

Definition 10.8. Let X be a simplicial set and $\mathcal{G}$ a local system of groups on X. Define the **(non-abelian) cochain complex** of X with coefficient $\mathcal{G}$

$$C^n(X;\mathcal{G}) := \left\{ c : X_n \to \coprod_{v\in X_0} \mathcal{G}(v);\ c(x) \in \mathcal{G}(x_0) \right\}$$

for $0 \le n \le 2$ and the following datum:

(i) the affine action φ of $C^0(X;\mathcal{G})$ on $C^1(X;\mathcal{G})$:

$$(\varphi(f)c)(\gamma) = f(\partial_1\gamma)c(\gamma)(\mathcal{G}(\gamma)^{-1} f(\partial_0\gamma)^{-1})$$

for $f \in C^0(X;\mathcal{G})$ and $c \in C^1(X;\mathcal{G})$,

(ii) the action ψ of $C^0(X;\mathcal{G})$ on $C^2(X;\mathcal{G})$:

$$(\psi(f)c)(\sigma) = \mathrm{Ad}(\mathcal{G}(\partial_2\sigma)^{-1} f(\partial_0\partial_2\sigma))(c(\sigma)),$$

(iii) the map $\delta : C^1(X;\mathcal{G}) \to C^2(X;\mathcal{G})$ satisfying $\delta(1) = 1$ and $\delta(\varphi(f)c) = \psi(f)c$ for $f \in C^0(X;\mathcal{G})$ and $c \in C^1(X;\mathcal{G})$:

$$\delta c(\sigma) = (\mathcal{G}(\partial_2\sigma)^{-1}c(\partial_0\sigma))c(\partial_1\sigma)^{-1}c(\partial_2\sigma)$$

for $c \in C^1(X;\mathcal{G})$ and $\sigma \in X_2$.

The we get the **0-th cohomology group**

$$H^0(X;\mathcal{G}) := \mathrm{Ker}(C^0(X;\mathcal{G}) \to \mathrm{Aut}(C^1(X;\mathcal{G})) \ltimes C^1(X;\mathcal{G}) \to C^1(X;\mathcal{G}))$$

and the **1-st cohomology set**

$$H^1(X;\mathcal{G}) := \delta^{-1}(1)/C^0(X;\mathcal{G}).$$

Given a 0-partial section $s : \mathrm{sk}_0(\mathcal{B}) \to \mathcal{E}$, put

$$c(s)(\gamma) = [s(\partial_1\gamma)]^{-1}(\Pi_0(\gamma)^{-1}[s(\partial_0\gamma)]) \in \mathcal{G}_{\gamma_0}$$

for $\gamma \in \mathcal{B}_1$, i.e., $c(s)(\gamma) \in \mathcal{G}_{\gamma_0}$ is the unique element satisfying

$$[s(\partial_1\gamma)]c(s)(\gamma) = \Pi_0(\gamma)^{-1}[s(\partial_0\gamma)].$$

By definition, $c(s) \in C^1(\mathcal{B};\mathcal{G})$ is a cocycle. For another section $s' : \mathrm{sk}_0(\mathcal{B}) \to \mathcal{E}$, if we can get $f \in C^0(\mathcal{B};\mathcal{G})$ uniquely such that

$$s'(x) = s(x)f(x)$$

for $x \in X_0$, then $c(s') = \varphi(f)c(s)$ holds. We denote f by $d(s,s')$ as in Section 10.2. Especially the cohomology class

$$\mathfrak{o}_0(\mathcal{E}) := [c(s)] \in H^1(\mathcal{B};\mathcal{G})$$

is independent of a choice of a 0-partial section $s : \mathrm{sk}_0(\mathcal{B}) \to \mathcal{E}$. As with usual obstructions, $\mathfrak{o}_0(\mathcal{E}) = 1$ if and only if there is a 1-partial section $\mathrm{sk}_1(\mathcal{B}) \to \mathcal{E}$. It follows from the following proposition:

Proposition 10.9. If $\mathfrak{o}_0(\mathcal{E}) = 1$, there exists a 0-partial section $s : \mathrm{sk}_0(\mathcal{B}) \to \mathcal{E}$ such that $c(s) = 1$.

Proof. If $[c(s)] = 1$, there exists $f \in C^0(\mathcal{B};\mathcal{G})$ such that $c(s) = \varphi(f)(1)$. So replacing s with sf^{-1}, we get the proposition. □

The non-abelian obstruction $\mathfrak{o}_0(\mathcal{E})$ is hard to deal with, we shall replace a certain abelian cocycle using a filtration $\{\mathcal{F}_i\mathcal{G}\}_{i=1}^{\infty}$ of $\mathcal{G}$ such that

$$\mathcal{G}_b = \mathcal{F}_1\mathcal{G}_b \rhd \mathcal{F}_2\mathcal{G}_b \rhd \cdots,$$

$$[\mathcal{F}_i\mathcal{G}_b, \mathcal{F}_j\mathcal{G}_b] \subset \mathcal{F}_{i+j}\mathcal{G}_b$$

for $b \in \mathcal{B}_0$, and the map $\mathcal{G}(\gamma)$ for $\gamma \in \mathcal{B}_1$ preserves the filtration. Given such a filtration, we can consider the local system of Lie algebras

$$\mathrm{gr}(\mathcal{G}) := \bigoplus_{i=1}^{\infty} \mathrm{gr}_i(\mathcal{G}) := \bigoplus_{i=1}^{\infty} \mathcal{F}_i\mathcal{G}/\mathcal{F}_{i+1}\mathcal{G}.$$

If the image of $c(s)$ to $C^1(\mathcal{B};\mathcal{G}/\mathcal{F}_i)$ is trivial, i.e., $c(s)(\gamma) \in \mathcal{F}_i\mathcal{G}_{\gamma_0}$ for $\gamma \in \mathcal{B}_1$, we get its image $c_i(s)$ to the (abelian) chain complex $C^1(\mathcal{B};\mathrm{gr}_i(\mathcal{G}))$. For another partial section $\mathrm{sk}_0(\mathcal{B}) \to \mathcal{E}$ satisfying the same condition, we can also get the image $d_i(s,s')$ of $d(s,s')$ to $C^1(\mathcal{B};\mathrm{gr}_i(\mathcal{G}))$. Then it satisfies the equation

$$c_i(s') - c_i(s) = \delta d_i(s,s').$$

It means $\mathfrak{o}^{(i)}(\mathcal{E}) := [c_i(s)] \in H^1(\mathcal{B};\mathrm{gr}_i(\mathcal{G}))$ is obtained uniquely.

Proposition 10.10. If $\mathfrak{o}^{(i)}(\mathcal{E})$ is defined and trivial, there exists a partial section $s : \mathrm{sk}_0(\mathcal{B}) \to \mathcal{E}$ such that $c(s)(\gamma) \in \mathcal{F}_{i+1}\mathcal{G}_{\gamma_0}$ for $\gamma \in \mathcal{B}_1$.

Proof. Supposing $\mathfrak{o}^{(i)}(\mathcal{E}) = [c_i(s)] = 1$, we have $1 = [c(s)] \in H^1(\mathcal{B}; \mathcal{G}/\mathcal{F}_{i+1}\mathcal{G})$. Then there exists a 0-partial section $s' : \mathrm{sk}_0(\mathcal{B}) \to \mathcal{E}$ such that $c(s') = 1 \in C^1(\mathcal{B}; \mathcal{G}/\mathcal{F}_{i+1}\mathcal{G})$. This section satisfies the required condition. □

11. Obstruction of the bundles of formal homology connections

Let $E \to B$ be a smooth fiber bundle with homological structure group G and fiber X. Fix a G-invariant Chen's differential δ on $\hat{L}W$, where $W = \tilde{H}_\bullet(X;\mathbb{R})[-1]$.

11.1. **Connected cases.** Suppose $\mathrm{QDer}^+(\hat{L}W, \delta) = 0$ and $H_i(\mathrm{Der}(\hat{L}W), \delta) = 0$ for $n > i > 0$. In addition, suppose, if $n = 1$, $H_1(\mathrm{Der}(\hat{L}W), \delta) \simeq H_0(\hat{L}W \otimes A, d + \delta + [\tau, -])$ is abelian with respect to the Baker-Campbell-Hausdorff product. Then we get the obstruction class of the simplicial bundle $\mathcal{Q}_\bullet(E, \hat{\delta}) \to S_\bullet(B)$

$$\mathfrak{o} = \mathfrak{o}_n(\mathcal{Q}_\bullet(E, \hat{\delta})) \in H^{n+1}(B; \Pi_n),$$

where $\Pi_n = \Pi_n(\mathcal{Q}_\bullet(E, \hat{\delta})/S_\bullet(B))$, and the characteristic maps of a fiber bundle $E \to B$

$$(\Lambda^p H_n(\mathrm{Der}(\hat{L}W), \delta)^*)^G \to H^{p(n+1)}(B; \mathbb{R})$$

by $\psi \mapsto \psi(\mathfrak{o}, \dots, \mathfrak{o})$ for $p \geq 1$.

11.2. **Example of a sphere bundle.** We consider the sphere bundle $S^2 \to E = S^3 \times_{S^1} S^2 \to S^2$ associated to the Hopf fibration $S^1 \to S^3 \to S^2$, where $U(1) = S^1$ acts on $S^2 = \mathbb{C} \cup \{\infty\}$ by rotations. Since the action of S^1 on S^2 has two fixed points 0 and ∞, this fiber bundle has a section $S^2 \to S^3 \times_{S^1} S^2$ defined by $[b] \mapsto [b, \infty]$. We fix the section.

Denote the volume form on the fiber $S^2 = \mathbb{C} \cup \{\infty\}$ by

$$v = \frac{\sqrt{-1}}{2\pi} \frac{dw d\bar{w}}{(1 + |w|^2)^2}$$

and the desuspension of the fundamental form by $x \in W = H_2(S^2)[-1]$. Then a dgl model of S^2 is given by

$$LW = L(x)\ (|x| = 1), \quad \delta x = 0$$

and its Lie algebra of derivations

$$\mathrm{Der}(LW) = \left\langle x \frac{\partial}{\partial x}, [x, x] \frac{\partial}{\partial x} \right\rangle.$$

Note that

$$H_1(\mathrm{Der}(LW), \delta) = \mathrm{Der}(LW)_1 = \left\langle [x, x] \frac{\partial}{\partial x} \right\rangle.$$

For simplicity, we restrict the bundle $\mathcal{Q}_\bullet(E) \to S_\bullet(S^2)$ to the Kan complex defined by

$$K_n = \{(\Delta^n, \mathrm{sk}_1 \Delta^n) \to (S^2, \infty)\} \subset S_n(S^2).$$

If $n \leq 1$, K_n is described by

$$K_0 = \{p_\infty\}, \quad K_1 = \{\gamma_\infty\},$$

where $p_\infty : \Delta^0 \to S^2$ and $\gamma_\infty : \Delta^1 \to S^2$ are constant maps to the point ∞. We put $\mathcal{Q}_\bullet := \mathcal{Q}_\bullet(E)|_{K_\bullet}$.

We use the map $\rho : D^2 \to S^2$ defined by

$$\rho(z) = \begin{cases} 2z/(1-|z|^2) & (|z|<1) \\ \infty & (|z|=1), \end{cases}$$

regarding $D^2 = \{z \in \mathbb{C}; |z| \le 1\} \subset \mathbb{C}$, and trivializations $\varphi_\rho : D^2 \times S^2 \to \rho^* E$ defined by

$$\varphi_\rho(z,w) = \left(z, \left[\left(\frac{2z}{1+|z|^2}, \frac{1-|z|^2}{1+|z|^2} \right), w \right] \right).$$

Choose an orientation-preserving diffeomorphism $h : \Delta^2/(\partial_1\Delta^2 \cup \partial_2\Delta^2) \simeq D^2$ such that

$$\Delta^1 \xrightarrow{\delta^0} \Delta^2/(\partial_1\Delta^2 \cup \partial_2\Delta^2) \xrightarrow{h} D^2$$

is given by $t \mapsto e^{2\pi\sqrt{-1}t}$. Then we get the 2-simplex in $K_\bullet$

$$\sigma : \Delta^2 \to \Delta^2/(\partial_1\Delta^2 \cup \partial_2\Delta^2) \xrightarrow{h} D^2 \xrightarrow{\rho} S^2$$

and the trivialization $\varphi_\sigma : \Delta^2 \times S^2 \simeq \sigma^* E$ induced by φ_ρ. The restriction $g : \Delta^1 \times S^2 \to \gamma_\infty^* E = \Delta^1 \times E_\infty \simeq \Delta^1 \times S^2$ of φ_σ on $\partial_0\Delta^2$ is described by

$$g(t,w) = (t, \varphi_0^{-1}([(e^{2\pi\sqrt{-1}t}, 0), w])) = (t, e^{-2\pi\sqrt{-1}t} w).$$

The partial section $s : \mathrm{sk}_1 K \to \mathcal{Q}$ is defined as follows:

$$s(p_\infty) := v_0 x \in \mathcal{Q}_0(E)_{p_\infty}, \quad s(\gamma_\infty) := v_1 x \in \mathcal{Q}_1(E)_{\gamma_\infty},$$

where $v_0 := (\varphi_0^{-1})^* v \in A^2(E_\infty)$ and $v_1 := (\varphi_0^{-1})^* v \in A^2(\gamma_\infty^* E)$ if the trivialization $\varphi_0 : S^2 \simeq p_\infty^* E = E_\infty$ and $\varphi_1 : \Delta^1 \times S^2 \simeq \gamma_\infty^* E = \Delta^1 \times E_\infty$ are defined by

$$\varphi_1(t,w) = (t, [(1,0), w]), \quad \varphi_0(w) = [(1,0), w].$$

Since $[s_\sigma] = [v_1 x] \in \pi_1(\sigma^* \mathcal{Q}_\bullet(E), v_0 x)$, we have

$$c(s)(\sigma) = g^*[s_\sigma] = g^*[v_1 x] = [g^*(v_1)x] \in \pi_1(Q_\bullet(S^2), vx)$$

under the identification $\varphi_0^* : \pi_1(Q_\bullet(E_\infty), v_0 x) \simeq \pi_1(Q_\bullet(S^2), vx)$. Calculating directly, we get

$$g^*(v) = v + \xi dt,$$

where

$$\xi = -\frac{\bar{w}dw + w d\bar{w}}{(1+|w|^2)^2} = df, \quad f(w) = \frac{1}{2}\frac{1}{1+|w|^2}.$$

Then putting

$$\Xi = t_1 \xi dt_2 - t_2 \xi dt_1 + 2 f dt_1 dt_2,$$

this satisfies the equation

$$(v+\Xi)^2 = 2v\Xi = 4fv dt_1 dt_2 = -4fv dt_0 dt_2 = -4d(fv(t_0 dt_2 - t_2 dt_0)).$$

So we obtain the formal homology connection $\alpha = (v+\Xi)x - 4fv(t_0 dt_2 - t_2 dt_0)[x,x] \in Q_2(S^2)$ satisfying

$$\partial_0 \alpha = (v + \xi dt_0)x, \quad \partial_1 \alpha = vx + 4fv dt_0 [x,x], \quad \partial_2 \alpha = vx.$$

Therefore the equation

$$[g^*(v_1)x] = [(v+\xi dt_0)x] = [vx + 4fv dt_0[x,x]] \in \pi_1(Q_\bullet(S^2), vx)$$

holds. Furthermore
$$\int_{S^2} 4fv = \int_{S^2} \frac{\sqrt{-1}}{\pi} \frac{dwd\bar{w}}{(1+|w|^2)^3} = \frac{1}{\pi}\int_0^\infty \frac{2rdr}{(1+r^2)^3}\int_0^{2\pi} d\theta = 2\int_0^\infty \frac{dx}{(1+x)^3} = 1$$
means that the deRham cohomology class $[4fv] \in H^2(S^2)$ is non-trivial. According to Theorem 4.10 of [1], we have $c(s)(\sigma) \neq 0$ and
$$\mathfrak{o} = [c(s)] \neq 0 \in H^2(K; H_1(\mathrm{Der}(LW))).$$

Finally evaluating the class with the dual basis ν of $[x,x]\partial/\partial x \in \mathrm{Der}(LW)_1$, we get the non-trivial characteristic class
$$\nu(\mathfrak{o}) \in H^2(K) = H^2(S^2),$$
which is the Euler class of the sphere bundle $E \to S^2$ (given in [35]).

11.3. **Non-connected cases.** If $\mathrm{QDer}^+(\hat{L}W, \delta) \neq 0$, we can apply the construction in Section 10.3. Putting $\Pi_0 = \Pi_0(\mathcal{Q}(E,\hat{\delta})/S_\bullet(B))$, we have the identification
$$\Pi_0(b) = \{C_\infty\text{-algebra map } (H(E_b), m_b) \to A(E_b) \text{ s.t. } (f_1)_* = \mathrm{id}_H\}/(C_\infty\text{-homotopic}),$$
where m_b is the C_∞-algebra structure on H corresponding to $\hat{\delta}(b)$. According to the homotopy theory of C_∞-algebras, the group $\mathrm{QIAut}(H(E_b), m_b)$ of homotopy classes of C_∞-automorphisms $(H(E_b), m_b) \to (H(E_b), m_b)$ such that $f_1 = \mathrm{id}_{H(E_b)}$ acts on the set $\Pi_0(b)$ on the right freely and transitively.

The local system $\mathrm{QIAut}(E)$ of groups is defined by
$$\mathrm{QIAut}(E)_b := \mathrm{QIAut}(H(E_b), m_b), \quad \gamma_*(f) := (g_\gamma^{-1})^* \circ f \circ (g_\gamma)^*$$
for $b \in B$, $\gamma \in S_1(B)$ and $f \in \mathrm{QIAut}(E_{\gamma(0)})$, where $g_\gamma : E_{\gamma(0)} \to E_{\gamma(1)}$ is the holonomy along γ. Then we get the non-abelian obstruction class
$$\mathfrak{o}_0 = \mathfrak{o}_0(\mathcal{Q}_\bullet(E)) \in H^1(B; \mathrm{QIAut}(E))$$
in Section 10.3.

Furthermore we have the filtration $\{\mathrm{QIAut}^{\geq i}(E)\}_{i=1}^\infty$ of $\mathrm{QIAut}(E)$ defined in Section **??**. By the observations in Section **??**, there exists the identification as local system of vector spaces
$$\mathrm{gr}_i(\mathrm{QIAut}(E)) \simeq \mathrm{gr}_i(\mathrm{QDer}^+(E)),$$
where the local system $\mathrm{QDer}^+(E)$ of Lie algebras is defined in the same way as $\mathrm{QIAut}(E)$. Here note that $\mathrm{gr}(\mathrm{QDer}^+(E))$ is defined similarly to $\mathrm{gr}(\mathrm{QIAut}(E))$ using its filtration.

Suppose we get the obstruction class $\mathfrak{o}_i \neq 0 \in H^1(B; \mathrm{gr}_i(\mathrm{QDer}^+(E)))$ with respect to the filtration. In the same way as in Section 10.2, the characteristic map
$$(\Lambda^\bullet \mathrm{gr}_i(\mathrm{QDer}^+(\hat{L}W, \delta))^*)^G \to H^\bullet(B; \mathbb{R})$$
is obtained.

Especially, if X is formal and δ corresponds to the product of the cohomology H of X, we obtain the characteristic map
$$(\Lambda^\bullet \mathrm{QDer}^i(\hat{L}W, \delta)^*)^G \to H^\bullet(B; \mathbb{R}).$$

We shall show a relation between the characteristic map constructed in [20] and the construction above. By discussions in [20], given a metric of the fiber bundle $E \to B$, we have the map $s : B \to \mathcal{Q}_0(E)$: for $b \in B$, the metric on E_b gives a Hodge decomposition of E_b, so let $s(b)$ be the C_∞-minimal model defined by the

Hodge decomposition. Composing the natural projection $\mathcal{Q}_0(E) \to \mathcal{D}(E)$ with s, we get a section of Chen's differential $\hat{\delta}$

Theorem 11.1. Let X be a pointed oriented closed manifold and $E \to B$ be a smooth bundle with section and metric. Suppose the metric gives a section $\hat{\delta}$ of Chen's differentials corresponding to a G-invariant Chen's differential δ of X. Then we have the commutative diagram of chain complexes

$$\begin{tikzcd} C^\bullet_{CE}(\mathrm{QDer}^+(\hat{L}W,\delta))^G \arrow[r, "\Phi"] \arrow[d] & A^\bullet(B) \arrow[d, "\int"] \\ (\Lambda^\bullet \mathrm{gr}_1(\mathrm{QDer}^+(\hat{L}W,\delta))^*)^G \arrow[r, "\Phi_1"] & C^\bullet(B;\mathbb{R}), \end{tikzcd}$$

where the first row map Φ is the characteristic map in [20], the second row Φ_1 is the characteristic map defined by

$$\Phi_1(\zeta)(\gamma) = \zeta(c_1(s)(\gamma), \dots, c_1(s)(\gamma))$$

for $\zeta \in (\Lambda^p \mathrm{gr}_1(\mathrm{QDer}^+(\hat{L}W,\delta))^*)^G$ and $\gamma \in S_1(B)$, the first column is the natural projection and the second column $\int$ is the deRham map.

Proof. Take a base point $*$ of B and put the universal covering of B

$$\tilde{B} = \{\gamma : [0,1] \to B; \gamma(0) = *\}/(\text{homotopy preserving boundary}).$$

We identify the fiber E_* on $*$ with the typical fiber X.

The smooth map $\mu : \tilde{B} \to Q(X,\delta)$ from the universal cover $\tilde{B}$ of B to the moduli space $Q(X,\delta) := \pi_0(Q_\bullet(X,\delta))$ of C_∞-algebra models of X is defined by

$$\mu([\gamma]) = g_\gamma^{-1} \cdot [s(\gamma(1))].$$

Here $g_\gamma : E_* \to E_{\gamma(1)}$ is the holonomy along γ. Pull-backing the right-invariant Maurer-Cartan form defined by the right-action of $\mathrm{QIAut}(\hat{L}W,\delta)$ on $Q(X,\delta)$

$$\eta \in A^1(Q(X,\delta); \mathrm{QDer}^+(\hat{L}W,\delta)),$$

we get the flat connection

$$\eta_\mu := \mu^*\eta \in A^1(\tilde{B}; \mathrm{QDer}^+(\hat{L}W,\delta)).$$

On the other hand, we can regard s as the 0-partial section $s : \mathrm{sk}_0(S_\bullet(B)) \to \mathcal{Q}_\bullet(E)$. Its non-abelian obstruction cocycle is described by

$$c(s)(\gamma) = [s(\gamma(0))]^{-1} g_\gamma^{-1} [s(\gamma(1))] = g_l(\mu([l])^{-1}\mu([\gamma l])),$$

where l is a path from $*$ to $\gamma(0)$ and a path $\tilde{\gamma} : [0,1] \to \tilde{B}$ is the lift of γ such that $\tilde{\gamma}(0) = [l]$. The map $\Psi : Q(X,\delta) \to Q(X,\delta)$ defined by $\Psi(\alpha) = \mu([l])^{-1}\alpha$ satisfies the differential equation $d\Psi = \Psi\eta$. Thus, solving the equation over the path $\mu\tilde{\gamma}$, we have

$$\mu([l])^{-1}\mu([\gamma l]) = \Psi(\mu\tilde{\gamma}(1)) = \sum \int_{\mu\tilde{\gamma}} \eta \cdots \eta.$$

Therefore we get the description using iterated integrals

$$c(s)(\gamma) = g_l \cdot \sum \int_{\mu\tilde{\gamma}} \eta \cdots \eta = g_l \cdot \sum \int_{\tilde{\gamma}} \eta_\mu \cdots \eta_\mu.$$

Its projection to $\mathrm{gr}_1(\mathrm{QDer}^+(\hat{L}W,\delta))$ is equal to $c_1(s)(\gamma) = g_l \int_{\tilde{\gamma}} \eta_\mu$ and

$$\int \Phi(\xi) = \int \xi(\eta_\mu, \dots, \eta_\mu) = \bar{\xi}\left(\int \eta_\mu, \dots, \int \eta_\mu\right) = \Phi_1(\bar{\xi}) \in C^p(\tilde{B})$$

for $\xi \in C^p_{CE}(\mathrm{QDer}^+(\hat{L}W,\delta))^G$, where $\bar{\xi}$ is the projection of ξ. Since the element is $\pi_1(B,*)$-invariant, we can regard it as element in $C^p(B)$.

□

Furthermore if $c_1(s) = \cdots = c_{i-1}(s) = 0$, we get the (cocycle-level) characteristic map $\Phi_i : (\Lambda^\bullet \mathrm{gr}_i(\mathrm{QDer}^+(\hat{L}W,\delta))^*)^G \to C^\bullet(B;\mathbb{R})$ defined by

$$\Phi_i(\zeta)(\gamma) = \zeta(c_i(s)(\gamma), \dots, c_i(s)(\gamma))$$

for $\zeta \in (\Lambda^p \mathrm{gr}_i(\mathrm{QDer}^+(\hat{L}W,\delta))^*)^G$ and $\gamma \in S_1(B)$ since $\eta_\mu \in A^1(B;\mathrm{QDer}^{\geq i}(\hat{L}W,\delta))$. Then the same commutative diagram holds. So the construction above using obstructions is the "leading term" of the characteristic map obtained in [20]

$$\Phi : C^\bullet_{CE}(\mathrm{QDer}^+(\hat{L}W,\delta))^G \to A^\bullet(B).$$

11.4. **Example of surface bundles.** We consider the case of $X = \Sigma_g$, which is the closed oriented surface with genus $g \geq 2$. This is a formal manifold, so we can put

$$\delta = \omega \frac{\partial}{\partial v},$$

where $v \in W_1$ is the fundamental form of Σ_g and $\omega \in [W_0, W_0]$ is the intersection form, i.e., $\omega = \sum_{i=1}^g [x^i, y^i]$ for a symmplectic basis $\{x^i, y^i\}$ of W_0 with respect to the intersection form of Σ_g.

11.4.1. *The first obstruction for surface bundles.* For a oriented surface bundle (with section), its homologically structure group is in the symplectic group $\mathrm{Sp}(W_0)$ of W_0.

Proposition 11.2. We have the identification as $\mathrm{Sp}(W_0)$-vector space

$$\mathrm{QDer}^1(\hat{L}W,\delta) \simeq \Lambda^3 W_0.$$

Proof. An element $D \in \mathrm{Der}^1(\hat{L}W)_0$ is described by the form

$$D = D_0 + [v,z]\frac{\partial}{\partial v}$$

for $D_0 \in \mathrm{Der}^1(LW_0)$ and $z \in W_0$. Then we can calculate the image by $\mathrm{ad}(\delta)$:

$$[\delta, D] = -D_0(\omega) + [\omega, z]\frac{\partial}{\partial v}.$$

So, D is in the kernel if and only if $D_0(\omega) \in (\omega)$, where (ω) is the Lie ideal in LW_0 generated by ω. This condition is equivalent to the condition: D_0 induces a derivation on $LW_0/(\omega)$

On the other hand, an element $P \in \mathrm{Der}^0(\hat{L}W)_1$ is described by

$$P = \sum b_i v \frac{\partial}{\partial x_i}$$

for $b_i \in \mathbb{R}$, where $\{x_i\}_{i=1}^{2g}$ is a basis of W_0. Its image of $\mathrm{ad}(\delta)$ is

$$[\delta, P] = \sum b_i \omega \frac{\partial}{\partial x_i} - P(\omega)\frac{\partial}{\partial v}.$$

Since we can prove $[v, W_0] = \{P(\omega); P \in \mathrm{Der}^0(\hat{L}W)_1\}$ by direct calculus, for any $D \in \mathrm{Der}^1(\hat{L}W)_0$, there exists $P \in \mathrm{Der}^0(\hat{L}W)_1$ such that

$$D_P := D + [\delta, P] \in \mathrm{Der}^1(LW_0).$$

Furthermore for another $P' \in \mathrm{Der}^0(\hat{L}W)_1$ such that $D_{P'} = D+[\delta, P'] \in \mathrm{Der}^1(LW_0)$, their difference $[\delta, P - P']$ is in $\mathrm{Hom}(W_0, \mathbb{R}\omega) \subset \mathrm{Der}^1(LW_0)$. So if D is in the kernel, D_P and $D_{P'}$ induce the same derivation on $LW_0/(\omega)$. Therefore we get the isomorphism

$$\mathrm{QDer}^1(\hat{L}W, \delta)_0 \simeq \mathrm{Der}^1(LW_0/(\omega)).$$

According to [36], we have the isomorphism $\mathrm{Der}^1(LW_0/(\omega)) \simeq \Lambda^3 W_0$. □

By the proposition above, for a oriented surface bundle $E \to B$ with section, we get the obstruction class

$$\mathfrak{o}^{(1)} = \mathfrak{o}^{(1)}(\mathcal{Q}(E, \hat{\delta})/S_\bullet(B)) \in H^1(B; \Lambda^3 W_0(E)).$$

Here $\Lambda^3 W_0(E)$ is the local system of vector spaces such that

$$\Lambda^3 W_0(E)(b) = \Lambda^3 \tilde{H}_1(E_b; \mathbb{R})[-1].$$

This local system is defined in the same way as $\mathrm{QIAut}(E)$ and $\mathrm{QDer}^+(E)$. Then we also get the characteristic map

$$(\Lambda^\bullet(\Lambda^3 W_0)^*)^{\mathrm{Sp}(W_0)} \to H^\bullet(B; \mathbb{R}).$$

11.4.2. *Twisted Morita-Miller-Mumford class.* We shall show that the obstruction $\mathfrak{o}^{(1)}$ can be regarded as one of the twisted Morita-Miller-Mumford classes. For the purpose, we use notations as follows:

- the mapping class group $\mathcal{M}_{g,*}$ of the oriented closed surface Σ_g with a base point,
- the space Met_g of Riemannian metrics which has constant curvature -1 on Σ_g,
- the Teichmüller space $\mathcal{T}_{g,*}$, which is the orbit space of Met_g by the action of the group $\mathrm{Diff}_0(\Sigma_g, *)$ of diffeomorphisms of $(\Sigma_g, *)$ isotopic to identity,
- the moduli space $\mathbb{M}_{g,*} = \mathcal{T}_{g,*}/\mathcal{M}_{g,*}$ of Riemann surfaces with a base point, and
- the universal family $\mathbb{C}_{g,*} = \mathrm{Met}_g \times_{\mathrm{Diff}(\Sigma_g,*)} \Sigma_g$ of Riemann surfaces with a base point.

Applying the construction in Section 11.4.1 for the "universal surface bundle" $\mathbb{C}_{g,*} \to \mathbb{M}_{g,*}$, we get the obstruction

$$\mathfrak{o}^{(1)} \in H^1(\mathbb{M}_{g,*}; \Lambda^3 W_0(\mathbb{C}_{g,*})).$$

Theorem 11.3. The obstruction class $\mathfrak{o}^{(1)}$ is equal to the minus of the twisted Morita-Miller-Mumford class

$$-m_{0,3} \in H^1(\mathcal{M}_{g,*}; \Lambda^3 W_0).$$

Proof. Take the canonical metric of $\mathbb{C}_{g,*} \to \mathbb{M}_{g,*}$. According to the proof of Theorem 11.1, we have $\mu : \mathcal{T}_{g,*} \to Q(\Sigma_g, \delta)$ and the cocycle

$$c_1(\gamma) = \int_{\tilde{\gamma}} \eta_\mu = \int_{\tilde{\gamma}} \eta_1,$$

where η_1 is the QDer^1-part of η_μ. So by the same discussion in [22], the cohomology class $\mathfrak{o}^{(1)} = [c_1(s)]$ is equal to the twisted Morita-Miller-Mumford class in $H^1(\mathcal{M}_{g,*}; \Lambda^3 W_0)$. (The discussion is also used in Section 4 of [32].)

□

So the obtained characteristic map

$$\Lambda^\bullet(\Lambda^3 W_0^*)^{\mathrm{Sp}(W_0)} \to H^\bullet(\mathbb{M}_{g,*}; \mathbb{R}) = H^\bullet(\mathcal{M}_{g,*}; \mathbb{R})$$

gives Morita-Miller-Mumford classes by the result of [23].

Chapter 4. Graph complex and characteristic classes of fibrations

12. Graph complex

12.1. **Orientation and ordering of graded sets.** The set of **orderings** on a set U is defined by

$$\mathrm{Ord}(U) := \{(u_1, \ldots, u_k) \in U^{\times k}; U = \{u_1, \ldots, u_k\}\},$$

where $k := \# U$.

Definition 12.1. Let U be a $\mathbb{Z}$-graded set, i.e. a finite set U given a map $|\cdot| : U \to \mathbb{Z}$.

- The graded vector space generated by U is denoted by $\mathbb{R}U$.
- The symmetric algebra generated by U is denoted by $SU := S(\mathbb{R}U)$.
- The exterior algebra generated by U is denoted by $\Lambda U := \Lambda(\mathbb{R}U)$.

For an element $(u_1, \ldots, u_k) \in \mathrm{Ord}(U)$, we denote the image of $u_1 \otimes \cdots \otimes u_k$ in ΛU by $[u_1, \ldots, u_k]$. The 1-dimensional vector space generated by this element is written by

$$O(U) := \langle [u_1, \ldots, u_k] \rangle \subset \Lambda U.$$

Definition 12.2. Let V be a $\mathbb{Z}$-graded vector space. We define the subspace $V_{\mathrm{cyc}}^{(k)}$ of **cyclic tensors** in $V^{\otimes k}$ by the image of the map $[-, \ldots, -]_{\mathrm{cyc}} : V^{\otimes k} \to V^{\otimes k}$ obtained by

$$x_1 \otimes \cdots \otimes x_k \mapsto \sum_{\tau \in \mathbb{Z}/k\mathbb{Z}} \epsilon \cdot x_{\tau(1)} \otimes \cdots \otimes x_{\tau(k)},$$

where $\mathbb{Z}/k\mathbb{Z}$ is identified with the group of cyclic permutations and ϵ is the Koszul sign of $(x_1, \ldots, x_k) \mapsto (x_{\tau(1)}, \ldots, x_{\tau(k)})$. For a $\mathbb{Z}$-graded set U, we denote

$$\mathrm{Cyc}(U) := \langle [u_1, \ldots, u_k]_{\mathrm{cyc}}; (u_1, \ldots, u_k) \in \mathrm{Ord}(U) \rangle \subset (\mathbb{R}U)_{\mathrm{cyc}}^{(k)}.$$

12.2. **Definition of graph complex.** Let W be a finite-dimensional symplectic vector space with form ω of degree N and suppose that N is even and $Z := \{a \in \mathbb{Z}; W_a \neq 0\} \subset \{0, \ldots, N\}$. Our labeled graph complex depends on (W, ω).

12.2.1. *Definition of graphs.*

Definition 12.3. An N**-graded graph** Γ consists of the following information:

- The set $H(\Gamma)$ of **half-edges**.
- The set $V(\Gamma)$ of **vertices**. It is a partition of the set $H(\Gamma)$, i.e.

  $$H(\Gamma) = \coprod_{v \in V(\Gamma)} v, \quad v \neq \emptyset \ (v \in V(\Gamma)).$$

  The number $\# v$ of elements of any $v \in V(\Gamma)$ is called the **valency** of v. A vertex with valency > 1 is called an **internal vertex** and one with valency 1 is called an **external vertex**. The set of internal (resp. external) vertices is denoted by $V_i(\Gamma)$ (resp. $V_e(\Gamma)$).
- The set $E(\Gamma)$ of **edges**. It is a partition of the set $H(\Gamma)$ such that the number of elements of any $e \in E(\Gamma)$ is two, i.e.

  $$H(\Gamma) = \coprod_{e \in E(\Gamma)} e, \quad \# e = 2 \ (e \in E(\Gamma)).$$

- The cohomological **degree of half-edges**. It is a map $|\cdot|: H(\Gamma) \to Z$ such that $|h_1| + |h_2| = N$ for an edge $e = \{h_1, h_2\} \in E(\Gamma)$. Then the cohomological degrees of vertices and edges are defined by
$$|v| := |h_1| + \cdots + |h_r| - N, \quad |e| := N$$
for $v = \{h_1, \dots, h_r\} \in V(\Gamma)$ and $e \in E(\Gamma)$.
- The division of the set $V_i(\Gamma)$ of internal vertices to two disjoint sets
$$V_i(\Gamma) = V_n(\Gamma) \amalg V_s(\Gamma)$$
such that all elements in $V_s(\Gamma)$ have cohomological degree -1 and the valency ≥ 3. An element of $V_n(\Gamma)$ is called **normal vertex**, and one of $V_s(\Gamma)$ is called **special vertex**.

The set of isomorphism classes of such graphs is denoted by $\mathcal{G}(N)$. Here an isomorphism between N-graded graphs is a bijection between the sets of half-edges preserving all information of N-graded graphs.

Example 12.4. In the case of $N = 4$ and $Z = \{0, 1, 2, 3, 4\}$, we can give examples of 4-graded graphs in Figure 1. In these figures,

- a black vertex $\bullet$ means a normal vertex, a white vertex $\circ$ a special vertex and a square vertex ■ a univalent vertex, and
- a number drawn beside a half-edge is its degrees.

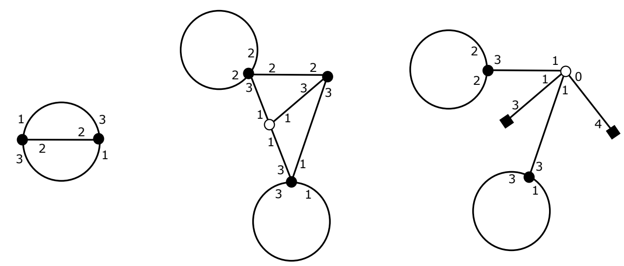


FIGURE 1. Examples of 4-graded graphs

12.2.2. *Decoration on vertices.* We shall give the relation equivalent to the dual of vertices defined by the cyclic Lie operad as in [8, 15, 28].

Definition 12.5. Let Γ be an N-graded graph.

- We introduce to $\mathrm{Cyc}(v)[N]$ for $v \in V_i(\Gamma)$ the **commutativity relation**
$$S_{v,h_r;s}(o) := \sum_{\tau \in \mathrm{Sh}(s, r-s-1)} o^{\tau^{(v,h_r)}} = 0,$$
$$o^{\tau^{(v,h_r)}} := \epsilon [h_{\tau(1)}, \dots, h_{\tau(r-1)}, h_r]_{\mathrm{cyc}} \sigma,$$
for $r - 1 > s > 0$ and $o = [h_1, \dots, h_r]_{\mathrm{cyc}} \sigma \in \mathrm{Cyc}(v)[N]$, where $\mathrm{Sh}(p, q)$ is the set of (p, q)-shuffles, σ is the symbol of the N-fold suspension, and ϵ is the Koszul sign. Then we denote the obtained space by $C(v) = \mathrm{Cyc}(v)[N]/(\text{com. rel.})$. (In the case of $r = 3$, it is the AS-relation for Jacobi diagrams.)

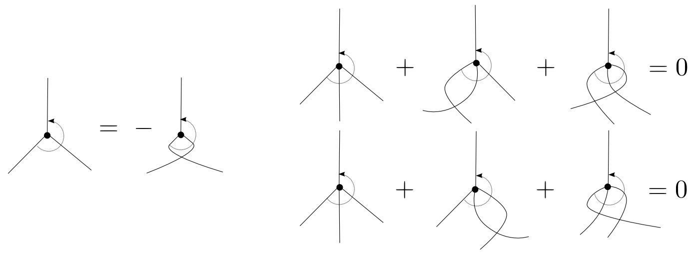


FIGURE 2. Commutativity ($r = 3, 4$). (Koszul signs are omitted in figures.)

12.2.3. *Decoration on N-graded graphs.* Set

$$\tilde{O}_{\mathrm{com}}(W,\Gamma) := \bigodot_{e\in E(\Gamma)} O(e) \otimes \bigodot_{u\in V_e(\Gamma)} W[-N]_{|u|} \otimes \bigwedge_{v^s\in V_s(\Gamma)} C(v^s) \otimes \bigwedge_{v\in V_n(\Gamma)} C(v),$$

where

$$\bigodot_{u\in U} V(u) := \left\{ v_{u_1}\cdots v_{u_k} \in S^k\left(\bigoplus_{u\in U} V(u)\right); v_{u_i}\in V(u_i),\ (u_1,\dots,u_k)\in \mathrm{Ord}(U)\right\},$$

$$\bigwedge_{u\in U} V(u) := \left\{ v_{u_1}\cdots v_{u_k} \in \Lambda^k\left(\bigoplus_{u\in U} V(u)\right); v_{u_i}\in V(u_i),\ (u_1,\dots,u_k)\in \mathrm{Ord}(U)\right\}$$

for a family $(V(u))_{u\in U}$ of $\mathbb{Z}$-graded vector spaces indexed by a finite set U. This tensor product consists of four factors: the first factor means directions of edges of Γ, the second factor W-labels of external vertices of Γ, the third factor (equivalence classes of) cyclic orderings on special vertices of Γ, and the fourth factor the same on normal vertices of Γ. Note that $W[-N]_{|u|} = W_{|h|}[-N]$ for an external vertex $u = \{h\}$.

We need to identify elements of $\tilde{O}_{\mathrm{com}}(W,\Gamma)$ by the symmetry of Γ. An automorphism α of an N-graded graph $\Gamma \in \mathcal{G}(N)$ induces the linear isomorphism $C(v) \to C(\alpha(v))$ for $v \in V_i(\Gamma)$ described by

$$[h_1,\dots,h_k]_{\mathrm{cyc}} \mapsto [\alpha(h_1),\dots,\alpha(h_k)]_{\mathrm{cyc}},$$

and the identity map $W[-N]_{|u|} \to W[-N]_{|\alpha(u)|} = W[-N]_{|u|}$ for $u \in V_e(\Gamma)$. Therefore the automorphism group of Γ acts on the vector space $\tilde{O}_{\mathrm{com}}(W,\Gamma)$ by the induced permutation of half-edges. Then the coinvariant vector space of $\tilde{O}_{\mathrm{com}}(W,\Gamma)$ by this action is denoted by $O_{\mathrm{com}}(W,\Gamma)$. We often consider an element o of $O_{\mathrm{com}}(W,\Gamma)$ described by the form

$$\begin{aligned} o &= [o_1,\dots,o_l; w_1,\dots,w_{k_e}; c^s_1,\dots,c^s_{k_s}; c_1,\dots,c_{k_n}]. \\ &:= (o_1\cdots o_l)\otimes(w_1\cdots w_{k_e})\otimes(c^s_1\cdots c^s_{k_s})\otimes(c_1\cdots c_{k_n}) \end{aligned}$$

where $w_i \in W[-N]_{|u_i|}$ and

$$o_i = [\hat{o}_i],\quad c^s_i = [\hat{c}^s_i]_{\mathrm{cyc}}\sigma,\quad c_i = [\hat{c}_i]_{\mathrm{cyc}}\sigma,$$

for $\hat{o}_i \in \mathrm{Ord}(e_i)$, $\hat{c}_i \in \mathrm{Ord}(v_i)$ and $\hat{c}^s_i \in \mathrm{Ord}(v^s_i)$. Such element o is called an **orientation** of Γ, a pair (Γ, o) is an **oriented graph**, and the information

$$\hat{o} = (\hat{o}_1,\dots,\hat{o}_l; w_1,\dots,w_{k_e}; \hat{c}^s_1,\dots,\hat{c}^s_{k_s}; \hat{c}_1,\dots,\hat{c}_{k_n})$$

is called a **lift** of an orientation $o = [\hat{o}]$ on Γ. The vector space $O_{\mathrm{com}}(W, \Gamma)$ is generated by orientations.

Example 12.6. In the case of $N = 4$ and $Z = \{0, 1, 2, 3, 4\}$, we can give examples of decorated 4-graded graphs in Figure 3 and 4. In these figures,

- an arrow on an edge means a direction, and
- an arc drawn around a vertex is an ordering of half-edges incident to this vertex.

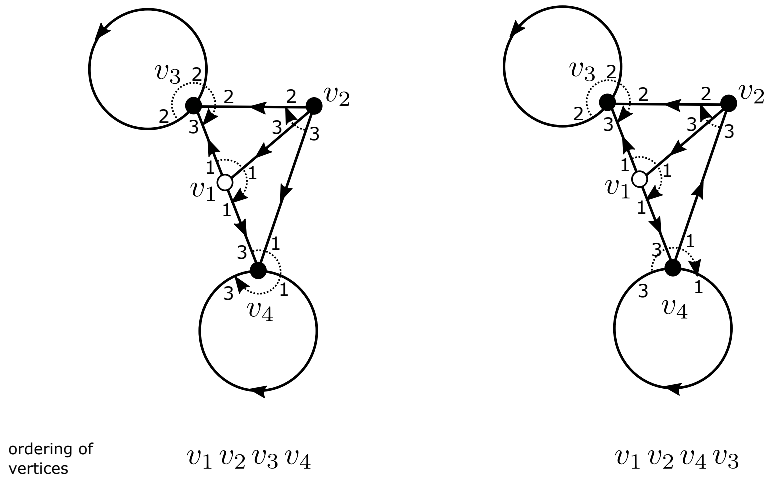


FIGURE 3. Non-labeled examples: the left (Γ, o_1) and the right (Γ, o_2)

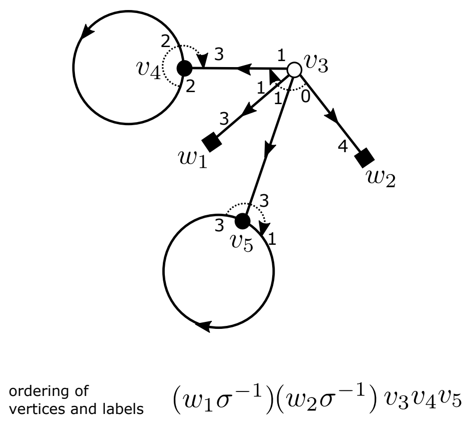


FIGURE 4. A labeled example

In Figure 3, the degrees of vertices are $v_1 = -1$, $v_2 = 4$, $v_3 = 5$, and $v_4 = 4$. In the space $O(\Gamma)$, we have

$$o_1 = (-1)^{5 \cdot 4+1}(-1)^{3 \cdot 1+1}(-1)^{3 \cdot (3+1+1)} o_2 = o_2,$$

where the signs $(-1)^{5\cdot 4+1}, (-1)^{3\cdot 1+1}, (-1)^{3\cdot(3+1+1)}$ are coming from changes of the ordering of vertices, the direction of the edge between v_2 and v_4 and the ordering of half-edges incident to v_4 respectively.

In Figure 4, elements $w_1 \in W_3$ and $w_2 \in W_4$ are labels of univalent vertices v_1, v_2 (their names v_1, v_2 of vertices are omitted in the figure). Note their degrees $|v_1| = |w_1\sigma^{-1}| = -1, |v_2| = |w_2\sigma^{-1}| = 0$.

12.2.4. *Definition of the bigraded vector space $\hat{C}^{\bullet,\bullet}_{com}(W)$.* The cohomological bidegree $(p,q) \in \mathbb{Z}\times\mathbb{Z}$ of $\Gamma \in \mathcal{G}(N)$ is defined by

$$p = \#V_n(\Gamma), \quad q = \sum_{v\in V_n(\Gamma)} |v| = \#V_s(\Gamma) + N(\#E(\Gamma) - \#V(\Gamma)) - \sum_{u\in V_e(\Gamma)} |u|,$$

and bidegree of elements in $O_{\mathrm{com}}(W,\Gamma)$ is defined by that of Γ. We define **the space of N-graded ribbon graphs** by

$$\hat{C}^{\bullet,\bullet}_{\mathrm{com}}(W) := \bigoplus_{\Gamma\in\mathcal{G}(N)} O_{\mathrm{com}}(W,\Gamma), \quad \hat{C}^{p,q}_{\mathrm{com}}(W) := \bigoplus_{\Gamma\in\mathcal{G}^{p,q}(W)} O_{\mathrm{com}}(W,\Gamma),$$

where $\mathcal{G}^{p,q}(W)$ is the subset of $\mathcal{G}(N)$ consisting N-graded graphs of degree (p,q). Then $\hat{C}^{\bullet,\bullet}_{\mathrm{com}}(W)$ can be regarded as bigraded vector space. We often denote an element in $\hat{C}^{\bullet,\bullet}_{\mathrm{com}}(W)$ corresponding to $o \in O_{\mathrm{com}}(W,\Gamma)$ for $\Gamma \in \mathcal{G}(N)$ by (Γ, o).

12.2.5. *Definition of the first differential d.* We define the linear map $d^{a,b}_{v;h^1,h^2} : O_{\mathrm{com}}(W,\Gamma) \to \hat{C}^{\bullet,\bullet}_{\mathrm{com}}(W)$ for an N-graded graph $\Gamma \in \mathcal{G}(N)$, a normal vertex $v \in V_n(\Gamma)$, two distinct half-edges h^1, h^2 incident to v, $a, b \in Z$ satisfying $a+b=N$. For an order $h_1,\ldots,h_r$ of half-edges incident to v such that $h^1 = h_r$ and $h^2 = h_i$, put

$$\begin{aligned} &d^{a,b}_{v;h^1,h^2}(\Gamma, [-;-;-;[h_1,\ldots,h_r]\sigma,-]) \\ &= (\Gamma^{a,b}_{v;h^1,h^2}, [-,[h',h''];-;-;[h_1,\ldots,h_i,h']\sigma,[h'',h_{i+1},\ldots,h_r]\sigma,-]). \end{aligned}$$

Here σ is the N-fold suspension, and the N-graded graph $\Gamma^{a,b}_{v;h^1,h^2}$ is defined by

$$H(\Gamma^{a,b}_{v;h^1,h^2}) = H(\Gamma) \amalg \{h',h''\}, \quad V(\Gamma^{a,b}_{v;h^1,h^2}) = (V(\Gamma)\setminus\{v\}) \amalg \{v',v''\},$$

$$V_s(\Gamma^{a,b}_{v;h^1,h^2}) = V_s(\Gamma), \quad E(\Gamma^{a,b}_{v;h^1,h^2}) = E(\Gamma) \amalg \{e_0\},$$

where $v' = \{h_1,\ldots,h_i,h'\}$, $v'' = \{h'',h_{i+1},\ldots,h_r\}$, $e_0 = \{h',h''\}$, $|h'| = a$ and $|h''| = b$. Note that the equation above is enough to define the operator $d^{a,b}_{v;h^1,h^2}$ and the operator is well-defined.

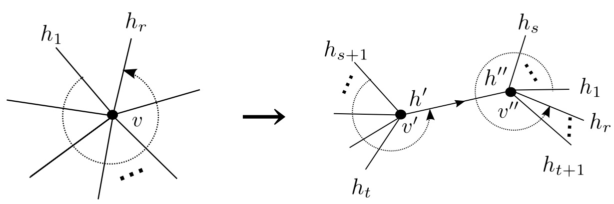


FIGURE 5. The operator $d^{a,b}_{v;h_s,h_t}$.

Then we obtain the linear map $d : \hat{C}^{\bullet;\bullet}_{\mathrm{com}}(W) \to \hat{C}^{\bullet;\bullet}_{\mathrm{com}}(W)$ by

$$d_v(\Gamma, o) := \frac{1}{2} \sum_{a+b=N} \sum_{h^1 \neq h^2 \in v} d^{a,b}_{v;h^1,h^2}(\Gamma, o), \quad d(\Gamma, o) := \sum_{v \in V_n(\Gamma)} d_v(\Gamma, o).$$

The map d can be also described by

$$d_v(\Gamma, o) = \sum_{a+b=N} \sum_{0 \leq s < t < r} d^{a,b}_{v;h_s,h_t}(\Gamma, o),$$

where $o = [-;-;-;[h_1, \dots, h_r]\sigma, -]$ and $v = \{h_1, \dots, h_r\}$. Remark the relation

$$d^{a,b}_{v;h^1,h^2}(\Gamma, o) = d^{b,a}_{v;h^2,h^1}(\Gamma, o)$$

for half-edges $h^1 \neq h^2 \in v$. Here well-definedness of d is proved by the relation with the commutativity relation:

Proposition 12.7. Using the notations above, $d_v S_{v,h_r;i}(\Gamma, o)$ is equal to zero under the commutativity relation.

Proof. For integers p, q, we define the linear ordered set $[p, q]$ by $\{p < p+1 < \cdots < q-1 < q\}$. If $p > q$, put $[p, q] = \emptyset$. For partial ordered sets P_1, P_2, we denote their direct sum by $P_1 + P_2$ (in the category of posets), and their ordinal sum by $P_1 \oplus P_2$. Then a (p, q)-shuffle is equivalent to the inverse of an order-preserving bijection $[1, p] + [p+1, p+q] \to [1, p+q]$.

Let $\tau^{-1} : [1, i] + [i+1, r-1] \to [1, r-1]$ be an $(i, r-i-1)$-shuffle and $0 \leq s < t < r$ integers. Put $L = \tau([s+1, t])$ and $l = t - s$.

If $\tau(s+1), \dots, \tau(t)$ are $\leq i$, then we have $\tau(s+m) = \tau(s+1) + (m-1)$ for $1 \leq m \leq t-s$ since $[1, i] \to \tau^{-1}([1, i])$ is an isomorphism between posets. Put $a = \tau(s+1) - 1$. Then we obtain the shuffle τ_2 by τ:

$$\begin{tikzcd}
{[1, i-l+1] + [i-l+2, r-l]} \arrow[r, "\tau_2^{-1}"] & {[1, r-l]} \\
{[1, a] \oplus \{*\} \oplus [a+l, i] + [i+1, r-1]} \arrow[u, "\text{canonical isom.}"] \arrow[r, "\text{bij.}"] & {[1, s] \oplus \{*\} \oplus [t+1, r-1]} \arrow[u, "\text{canonical isom.}"'] \\
{[1, i] + [i+1, r-1]} \arrow[u, two heads] \arrow[r, "\tau^{-1}"] & {[1, r-1]} \arrow[u, two heads]
\end{tikzcd}$$

The shuffle τ can recover from a pair (a, l, τ_2), where $\{a+1, \dots, a+l\} \subset [1, i]$ and an $(i-l+1, r-i-1)$-shuffle τ_2.

Similarly, if $\tau(s+1), \dots, \tau(t)$ are $\geq i+1$, we can obtain a triple (a, l, τ_2), where $\{a+1, \dots, a+l\} \subset [i+1, r-1]$ and an $(i-l+1, r-i-1)$-shuffle τ_2.

Otherwise, put $p = \#(L \cap [1, i])$. Then we obtain the shuffle τ_1 by restricting τ:

$$\begin{tikzcd}
{[1, p] + [p+1, l]} \arrow[r, "\tau_1^{-1}"] \arrow[d, "\text{canonical isom.}"'] & {[1, l]} \arrow[d, "\text{canonical isom.}"] \\
L \arrow[r, "\tau^{-1}"] \arrow[d, hook] & \tau^{-1}(L) \arrow[d, hook] \\
{[1, i] + [i+1, r-1]} \arrow[r, "\tau^{-1}"] & {[1, r-1]}
\end{tikzcd}$$

We consider $\bar{L} = ([1,i] + [i+1, r-1]) \setminus L$ and the order-preserving bijection $\rho^{-1} : \bar{L} \to [1,s] \oplus [t+1, r-1]$ defined by the restriction of τ^{-1}. The shuffle τ recovers from a pair (ρ, τ_1), where $\rho^{-1} : \bar{L} \to [1,s] \oplus [t+1, r-1]$ is an order-preserving bijection and τ_1 is a $(p, l-p)$-shuffle.

Thus we have

$$\begin{aligned} d_v S_{v,h_r;i}([h_1, \ldots, h_r]\sigma) &= \sum_{l=1}^{r-1} \left(\sum_{p=1}^{l-1} \sum_{\rho} \sum_{\tau_1} o_\rho^{\tau_1^{(v',h')}} + \sum_{a} \sum_{\tau_2} o_{a,l}^{\tau_2^{(v'',h_r)}} \right) \\ &= \sum_{l=1}^{r-1} \left(\sum_{p=1}^{l-1} \sum_{\rho} S_{v',h';p}(o_\rho) + \sum_{a} S_{v'',h_r;i-l+1}(o_{a,l}) \right), \end{aligned}$$

where $L = \{1, \ldots, r-1\} \setminus \bar{L} = \{u_1 < \cdots < u_p \text{ as integers}\}$,

$$\begin{aligned} o_\rho &= \epsilon[[h_{u_1}, \ldots, h_{u_p}, h']\sigma, [h_{\rho(1)}, \ldots, h_{\rho(s)}, h'', h_{\rho(t+1)}, \ldots, h_{\rho(r-1)}, h_r]\sigma], \\ o_{a,l} &= \epsilon'[[h_{a+1}, \ldots, h_{a+l}, h']\sigma, [h_1, \ldots, h_a, h'', h_{a+l+1}, \ldots, h_r]\sigma], \end{aligned}$$

and ϵ, ϵ' are appropriate Koszul signs. (In these equations, the subscriptions cyc are omitted.) $\square$

12.2.6. *Definition of the second differential L.* For $\Gamma \in \mathcal{G}(N)$, let $i_v(\Gamma)$ be the N-graded graph obtained by converting a normal vertex v of degree -1 to a special vertex. We define the linear map $i_v : O_{\text{com}}(W, \Gamma) \to O_{\text{com}}(W, i_v(\Gamma))$ for $o \in O_{\text{com}}(W, \Gamma)$ such that

$$i_v(\Gamma, [-;-;-;c,-]) = (i_v(\Gamma), [-;-;-,c;-])$$

for $c \in C(v)$ if v has degree -1 and valency ≥ 3, and $i_v(\Gamma, o) = 0$ if v does not. Since the relation

$$i_{v_1} S_{v_2,h_r;k}(\Gamma, o) = S_{v_2,h_r;k} i_{v_1}(\Gamma, o)$$

for $v_1, v_2 \in V_i(\Gamma)$ holds clearly, the map i_v is well-defined. Then the linear map $L : \hat{C}^{\bullet;\bullet}_{\text{com}}(W) \to \hat{C}^{\bullet;\bullet}_{\text{com}}(W)$ is defined by

$$L := id - di,$$

where the linear map $i : \hat{C}^{\bullet;\bullet}_{\text{com}}(W) \to \hat{C}^{\bullet;\bullet}_{\text{com}}(W)$ is obtained by

$$i(\Gamma, o) := \sum_{v \in V_n(\Gamma)} i_v(\Gamma, o).$$

The map L is also described by

$$L(\Gamma, o) = \sum_{v \in V_n(\Gamma)} (i_{v'} + i_{v''}) d_v(\Gamma, o)$$

since $i_u d_v = d_v i_u$ for normal vertices $u \neq v$.

Then d, i, and L have (cohomological) bidegree $(1,0)$, $(-1,1)$ and $(0,1)$ respectively.

12.2.7. *Definition of the underlying bigraded vector space $C^{\bullet;\bullet}_{com}(W)$.* The space $C^{\bullet;\bullet}_{\text{com}}(W)$ is the quotient space of $\hat{C}^{\bullet;\bullet}_{\text{com}}(W)$ by

- **(A_∞-relation)**

$$R_v(\Gamma, o) := i_{v'} i_{v''} d_v(\Gamma, o) = 0$$

for $\Gamma \in \mathcal{G}(N)$ and a normal vertex v (of degree -2).

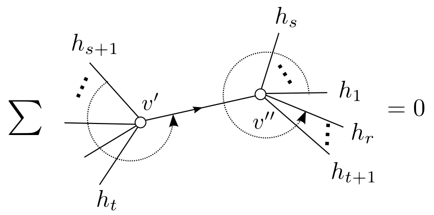


FIGURE 6. A_∞-relation.

- **(Cut-off relation)** For $\Gamma \in \mathcal{G}(N)$ and $e = \{h_1, h_2\} \in E(\Gamma)$, we define the N-graded graph Γ_e as follows:

$$\begin{aligned} H(\Gamma_e) &= H(\Gamma) \amalg \{\bar{h}_1, \bar{h}_2\}, \\ E(\Gamma_e) &= (E(\Gamma) \setminus \{e\}) \amalg \{\{h_1, \bar{h}_1\}, \{h_2, \bar{h}_2\}\}, \\ V(\Gamma_e) &= V(\Gamma) \amalg \{\{\bar{h}_1\}, \{\bar{h}_2\}\}, \\ |\bar{h}_1| = N - |h_1| =: a, &\quad |\bar{h}_2| = N - |h_2| =: b. \end{aligned}$$

Then

$$(\Gamma, [[h_1, h_2], -; -; -; -]) = \sum_{|x^i|=a, |x^j|=b} \omega_{ij} (\Gamma_e, [[h_1, \bar{h}_1], [\bar{h}_2, h_2], -; x^i\sigma^{-1}, x^j\sigma^{-1}, -; -; -]),$$

where $\{x^i\}$ is a homogeneous basis of W and (ω_{ij}) is the inverse matrix of $(\omega(x^i, x^j))$.

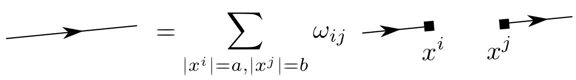


FIGURE 7. Cut-off relation.

Remark that $C^{\bullet;\bullet}_{\mathrm{com}}(W)$ is generated by W-labeled graphs with only one internal vertex by cut-off relation.

12.2.8. *On well-definedness of three operators d, i, L on $C^{\bullet;\bullet}_{com}(W)$.* The endomorphisms d, i and L of $\hat{C}^{\bullet;\bullet}_{\mathrm{com}}(W)$ induce endomorphisms of $C^{\bullet;\bullet}_{\mathrm{com}}(W)$ by the equations

$$dR_v(\Gamma, o) = \sum_{u \neq v} R_v d_u(\Gamma, o), \quad iR_v(\Gamma, o) = \sum_{u \neq v} R_v i_u(\Gamma, o)$$

for a normal vertex v of an N-graded graph Γ.

12.2.9. *On two differentials d, L on $C^{\bullet;\bullet}_{com}(W)$.*

Proposition 12.8. The bigraded vector space $C^{\bullet;\bullet}_{\mathrm{com}}(W)$ is a double complex with respect to differentials d and L. We call $C^{\bullet;\bullet}_{\mathrm{com}}(W)$ **double graph complex**.

Proof. First, we show the equation $d^2 = 0$. It is proved in the same way as Kontsevich's original graph complex. For a normal vertex v of an N-graded graph (Γ, o), let v', v'' be new vertices obtained by splitting at v. Then

$$d_{v'} d_v(\Gamma, o) = -d_{v''} d_v(\Gamma, o) \quad d_u d_v(\Gamma, o) = -d_v d_u(\Gamma, o)$$

for $u \neq v$ holds. The first equation is shown by Figure 8. In the figure, v' and v'' are defined such that the direction of the new edge is from v' to v'' in Figure 8, and

$(v')', (v')'', (v'')', (v'')''$ are also defined in the same way. So we obtain $d^2(\Gamma, o) = 0$ by cancellation.

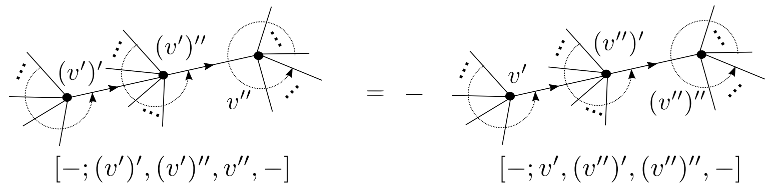


FIGURE 8. $d_{v'}d_v(\Gamma, o) = -d_{v''}d_v(\Gamma, o)$.

Next, we show $L^2 = 0$. From the equation in $\hat{C}^{\bullet;\bullet}_{\mathrm{com}}(W)$

$$\begin{aligned}(iL - Li)(\Gamma, o) &= \left(\sum_u i_u(i_{v'} + i_{v''})d_v - \sum_{u \neq v}(i_{v'} + i_{v''})d_v i_u\right)(\Gamma, o) \\ &= \sum_v (i_{v''}i_{v'} + i_{v'}i_{v''})d_v(\Gamma, o) \\ &= 2\sum_v R_v(\Gamma, o),\end{aligned}$$

we obtain the relation $iL - Li = 0$ in $C^{\bullet;\bullet}_{\mathrm{com}}(W)$. So the equations

$$L^2 = (id - di)L = idL - diL = idL - dLi = idid - didi,$$
$$L^2 = L(id - di) = Lid - Ldi = iLd - Ldi = -idid + didi$$

hold. Then we obtain $L^2 = 0$. Since $Ld + dL = -did + did = 0$ holds by definition of L, we get the proposition. $\square$

12.3. **Construction of the map to Chevalley-Eilenberg complexes.** Let (W, ω) and Z be as Section 12.2 and δ be a symplectic and quadratic differential of homological degree -1 on $\hat{L}W$. In this section, the Lie algebra $\mathrm{Der}_\omega(\hat{L}W)$ of symplectic derivations is denoted by $\mathcal{D}$. We construct a double chain map

$$C^{\bullet;\bullet}_{\mathrm{com}}(W) \to C^{\bullet;\bullet}_{CE}(\mathcal{D})$$

from the graph complex $C^{\bullet;\bullet}_{\mathrm{com}}(W)$ to the Chevalley-Eilenberg complex of the dgl $(\mathcal{D}, \mathrm{ad}(\delta))$.

Let (Γ, o) be an oriented graph and $\hat{o}$ be a lift of o. Put

$$k = \#V(\Gamma), \quad k_e = \#V_e(\Gamma), \quad k_s = \#V_s(\Gamma), \quad k_n = \#V_n(\Gamma),$$

$$\begin{aligned}(r_1, \ldots, r_k) &:= (\underbrace{1, \ldots, 1}_{k_e}, a_1, \ldots, a_{k_s + k_n}) \\ &:= (\underbrace{1, \ldots, 1}_{k_e}, \#v^s_1, \ldots, \#v^s_{k_s}, \#v_1, \ldots, \#v_{k_n})\end{aligned}$$

We denote by $\tau(\hat{o})$ the linear isomorphism (the permutation of factors of the tensor product)

$$W^{\otimes r_1} \otimes \cdots \otimes W^{\otimes r_k} \to W^{\otimes 2} \otimes \cdots \otimes W^{\otimes 2} = (W^{\otimes 2})^{\otimes l}$$

corresponding to the permutation of half-edges

$$(h_1, \ldots, h_{k_e}, \hat{c}^s_1, \ldots, \hat{c}^s_{k_s}, \hat{c}_1, \ldots, \hat{c}_{k_n}) \mapsto (\hat{o}_1, \ldots, \hat{o}_l).$$

Then we define the linear map $\alpha(\Gamma,\hat{o})$ of cohomological degree $(l-k)N$ by composing these maps

$$\alpha(\Gamma,\hat{o}) : W[-N]^{\otimes k_e}\otimes \mathrm{Der}_\omega(\hat{L}W)^{\otimes(k_s+k_n)} \overset{\text{proj.}}{\to} W[-N]^{\otimes k_e}\otimes \bigotimes_{i=1}^{k_s+k_n}\mathrm{Der}_\omega^{a_i+2}(\hat{L}W)$$

$$\overset{\Phi}{\simeq} W[-N]^{\otimes k_e}\otimes \bigotimes_{i=1}^{k_s+k_n} W(a_i)[-N] \subset \bigotimes_{i=1}^{k}(W^{\otimes r_i}[-N]) \overset{\sigma^{\otimes k}}{\to} \bigotimes_{i=1}^{k} W^{\otimes r_i} \overset{\tau(\hat{o})}{\to} (W^{\otimes 2})^{\otimes l} \overset{\omega_E}{\to} \mathbb{R},$$

where $\Phi := \mathrm{id}_{W[-N]}^{\otimes k_e}\otimes \Phi_\omega^{\otimes(k_s+k_n)}$, $\omega_E := \omega_{e_1}\otimes\cdots\otimes\omega_{e_l}$ and $\omega_{e_j} := \omega_{(|h_1^{e_j}|,|h_2^{e_j}|)}$ if $e_j = \{h_1^{e_j}, h_2^{e_j}\}$. Here we denote by $\omega_{(d_1,d_2)}$ for integers d_1,d_2 the composition of the projection $W\otimes W\to W_{d_1}\otimes W_{d_2}$ and the restriction of ω to $W_{d_1}\otimes W_{d_2}$. The map $\alpha(\Gamma,\hat{o})$ is independent of a choice of linear orders of half-edges representing cyclic orders, and compatible with the commutativity relation.

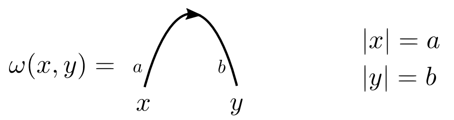
We define the map $\hat{\psi}(\Gamma,\hat{o}) : \mathcal{D}^{\otimes k_n}\to\mathbb{R}$ by

$$\hat{\psi}(\Gamma,\hat{o})(D_1,\dots,D_{k_n}) := \alpha(\Gamma,\hat{o})(w_1,\dots,w_{k_e},\underbrace{\delta,\dots,\delta}_{k_s},D_1,\dots,D_{k_n})$$

for $D_i\in\mathcal{D}$. Restricting the map[1] on the exterior algebra, we can get the map

$$\psi(\Gamma,o) = \hat{\psi}(\Gamma,\hat{o})\circ\mathrm{Alt}_{k_n} : \Lambda^{k_n}\mathcal{D}\to\mathbb{R}.$$

The map is independent of a representation $\hat{o}$ of o by the definition of an orientation. So we obtain the map $\psi : C^{\bullet,\bullet}_{\mathrm{com}}(W)\to C^{\bullet,\bullet}_{CE}(\mathcal{D})$.



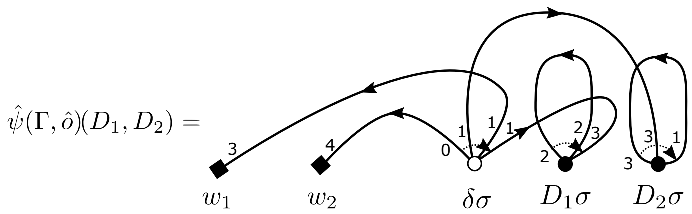


FIGURE 9. An example of $\hat{\psi}(\Gamma,\hat{o})(D_1,D_2)$ (Γ is the decorated graph in Figure 4.)

Well-definedness of ψ is proved by the correspondence through ψ between relations in the graph complex $C^{\bullet,\bullet}_{\mathrm{com}}(W)$ correspond to properties of derivations as the following table:

By definition, it is clear except for the A_∞-relation. The correspondence for the A_∞-relation is proved in the end of the proof of the following theorem.

---

[1]For a graded vector space V, the injective map $\mathrm{Alt}_n : \Lambda^n V\to V^{\otimes n}$ is defined by

$$\mathrm{Alt}_n(v_1\cdots v_n) = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\bar{\epsilon}(\sigma)v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(n)}$$

for $v_1,\dots,v_n\in V$, where $\bar{\epsilon}(\sigma)$ is the corresponding anti-Koszul sign.

graph complex	derivations
cyclicity	symplectic derivation
commutativity	Lie derivation
A_∞-relation	$\delta^2 = 0$
cut-off	symplectic form

Theorem 12.9. The map $\psi : C^{\bullet,\bullet}_{\mathrm{com}}(W) \to C^{\bullet,\bullet}_{CE}(\mathcal{D})$ is a double chain map.

Proof. First, we shall show that $d_{CE}\psi = \psi d$ on $\hat{C}^{\bullet,\bullet}_{\mathrm{com}}(W)$. To prove this, we need Lemma **??**.

For an oriented graph (Γ, o), we define the two lifts $\hat{o}^1$, $\hat{o}^2$ on $\Gamma^{s,t}_{v_i;h_\nu,h_\mu}$ as follows:

$$\hat{o}^1 = ((h', h''), -; -; -; v_1, \ldots, v_i', v_i'', \ldots, v_p),$$
$$\hat{o}^2 = ((h'', h'), -; -; -; v_1, \ldots, v_i'', v_i', \ldots, v_p),$$
$$v_i' = (h^i_{\nu+1}, \ldots, h^i_\mu, h'), \quad v_i'' = (h^i_1, \ldots, h^i_\nu, h'', h^i_{\mu+1}, \ldots, h^i_{r_i}),$$

where $r_i = \# v_i$. The signs ϵ_i defined by the equations

$$o^1 := \epsilon_1[\hat{o}^1], \quad o^2 := \epsilon_2[\hat{o}^2], \quad d^{s,t}_{v_i,h_\nu,h_\mu} o = (-1)^{i-1} o^1 = (-1)^{i-1} o^2.$$

So we obtain

$$\begin{aligned} d(\Gamma, o) &= \sum_{i=1}^k \sum_{\nu<\mu} \sum_{a+b=N} (-1)^{i-1} (\Gamma^{a,b}_{v_i;h_\nu,h_\mu}, o^1) \\ &= \sum_{i=1}^k \sum_{\nu<\mu} \sum_{a+b=N} (-1)^{i-1} (\Gamma^{a,b}_{v_i;h_\nu,h_\mu}, o^2). \end{aligned}$$

Note that

$$d_{CE}(\chi \circ \mathrm{Alt}_p) = \frac{1}{2} \sum_{s=1}^p (-1)^{s-1} \chi \circ (1^{\otimes s-1} \otimes [\ ,\ ] \otimes 1^{\otimes p-s}) \circ \mathrm{Alt}_{p+1}$$

for a linear map $\chi : W^{\otimes r_1}[-N] \otimes \cdots \otimes W^{\otimes r_p}[-N] \to \mathbb{R}$ and the anti-symmetrization Alt_p for p-components. So we should prove

$$\begin{aligned} &\hat{\psi}(\Gamma, \hat{o}) \circ (1^{\otimes i-1} \otimes [\ ,\ ] \otimes 1^{\otimes p-i-1}) \\ = &\sum_{\nu<\mu} \sum_{a+b=N} (\epsilon_1 \hat{\psi}(\Gamma^{a,b}_{v_i;h_\nu,h_\mu}, \hat{o}^1) + \epsilon_2 \hat{\psi}(\Gamma^{a,b}_{v_i;h_\nu,h_\mu}, \hat{o}^2) \circ \tau), \end{aligned}$$

where the map τ means the permutation

$$\begin{aligned} &X_1 \otimes \cdots \otimes (x_{\nu+1} \cdots x_\mu x') \otimes (x_1 \cdots x_\nu x'' x_{\mu+1} \cdots x_{r_i}) \otimes \cdots \otimes X_p \\ &\mapsto \epsilon \cdot X_1 \otimes \cdots \otimes (x_1 \cdots x_\nu x' x_{\mu+1} \cdots x_{r_i}) \otimes (x_{\nu+1} \cdots x_\mu x'') \otimes \cdots \otimes X_p \end{aligned}$$

and ϵ is the Koszul sign. It follows from the equations

$$\hat{\psi}(\Gamma, \hat{o}) \circ (1^{\otimes i-1} \otimes \sigma^{-1}(1 \otimes \omega_{(a,b)}) \pi^{r',r''}_{1;t} \sigma^{\otimes 2} \otimes 1^{\otimes p-i-1}) = \epsilon_1 \hat{\psi}(\Gamma^{a,b}_{v_i;h_\nu,h_\mu}, \hat{o}^1),$$

$$\hat{\psi}(\Gamma, \hat{o}) \circ (1^{\otimes i-1} \otimes \sigma^{-1}(1 \otimes \omega_{(a,b)}) \pi^{r',r''}_{2;t} \sigma^{\otimes 2} \otimes 1^{\otimes p-i-1}) = \epsilon_2 \hat{\psi}(\Gamma^{a,b}_{v_i;h_\nu,h_\mu}, \hat{o}^2) \circ \tau,$$

for $r' = \mu - \nu + 1$, $r'' = r - \mu + \nu + 1$, and $t = \nu + 1$. The first equation is verified as follows: we have by the definition of $\hat{\psi}$

$$\omega(x', x'') \hat{\psi}(\Gamma, \hat{o})(X_1, \ldots, X_p) = \epsilon_1 \hat{\psi}(\Gamma^{a,b}_{v_i;h_\nu,h_\mu}, \hat{o}^1)(X_1, \ldots, X_i', X_i'', \ldots, X_p)$$

for $X_s \in W^{\otimes r_s}$, $x' \in W_a$, and $x'' \in W_b$. Here we put $X_i' = x_{\nu+1}\cdots x_\mu x'\sigma^{-1}$ and $X_i'' = x_1\cdots x_\nu x'' x_{\mu+1}\cdots x_r\sigma^{-1}$ for $X_i = x_1\cdots x_r\sigma^{-1}$. So we obtain the first equation from

$$\epsilon_1 X_i\omega(x', x'') = \sigma^{-1}(1\otimes\omega)\pi_{1;t}^{r',r''}\otimes\sigma^{\otimes 2}(X_i'\otimes X_i'')$$

The second is also verified in the same way.

Next, we shall prove $i_\delta\psi = \psi i$ on $\hat{C}^{\bullet;\bullet}_{\mathrm{com}}(W)$. The ordering

$$\hat{o}_i := (-;-;-;v_i, v_1, \dots, \hat{v}_i, \dots, v_p)$$

is a lift of $\bar{\epsilon}_i\cdot o$, where $\bar{\epsilon}_i$ is the anti-Koszul sign of the permutation

$$(v_1,\dots,v_p)\mapsto(v_i,v_1,\dots,\hat{v}_i,\dots,v_p).$$

So we have

$$\begin{aligned}
&\psi i(\Gamma,o)(X_1,\dots,X_{p-1})\\
=&\sum_{s=1}^{j}\bar{\epsilon}_i\cdot\alpha(i_{v_i}(\Gamma),\hat{o}_i)(w_1,\dots,w_{k_e},\underbrace{\delta,\dots,\delta}_{k_s+1},\mathrm{Alt}_{p-1}(X_1,\dots,X_{p-1}))\\
=&\sum_{s=1}^{j}\sum_{\pi\in\mathfrak{S}_{p-1}}\bar{\epsilon}\cdot\alpha(\Gamma,\hat{o})(w_1,\dots,w_{k_e},\underbrace{\delta,\dots,\delta}_{k_s},X_{\pi(1)},\dots,\delta,\dots,X_{\pi(p-1)})\\
=&\alpha(\Gamma,\hat{o})(w_1,\dots,w_{k_e},\underbrace{\delta,\dots,\delta}_{k_s},\mathrm{Alt}_p(\delta,X_1,\dots,X_{p-1}))\\
=&i_\delta\psi(\Gamma,o)(X_1,\dots,X_{p-1})
\end{aligned}$$

where $\bar{\epsilon}$ is the anti-Koszul sign of

$$(\delta,X_1,\dots,X_{p-1})\mapsto(X_{\pi(1)},\dots,\delta,\dots,X_{\pi(p-1)}).$$

From the discussion above, the relation $\psi(R_v(\Gamma,o)) = 0$ follows from

$$\psi(R_v(\Gamma,o)) = \psi(i_{v'}i_{v''}d_v(\Gamma,o)) = \psi(\Gamma,o)([\delta,\delta],-) = 0.$$

Thus ψ induces the map $\psi : C^{\bullet;\bullet}_{\mathrm{com}}(W)\to C^{\bullet;\bullet}_{CE}(\mathcal{D})$. Furthermore, since ψ is commutative with d and i, so is L. So we complete the proof. $\square$

The group $\mathrm{Sp}(W,\delta)$ acts on $C^{\bullet;\bullet}_{\mathrm{com}}(W)$ by the action on the their labels. Then, the chain map $\psi : C^{\bullet;\bullet}_{\mathrm{com}}(W)\to C^{\bullet;\bullet}_{CE}(\mathcal{D})$ is $\mathrm{Sp}(W,\delta)$-equivariant clearly. Especially we can consider the $\mathrm{Sp}(W,\delta)$-invariant part $C^{\bullet;\bullet}_{\mathrm{com}}(W)^{\mathrm{Sp}(W,\delta)}$ of the complex $C^{\bullet;\bullet}_{\mathrm{com}}(W)$. It has the double subcomplex $C^{\bullet;\bullet}_{\mathrm{com}}(N,Z)$ consisting of N-graded graphs which have no external vertex. This complex $C^{\bullet;\bullet}_{\mathrm{com}}(N,Z)$ does not depend on the symplectic vector space W. It depends only a range Z of degrees and a degree N of a symplectic inner product.

Remark 12.10. We can define the associative version of $C^{\bullet;\bullet}_{\mathrm{com}}(W)$ as follows. Set

$$\tilde{O}_{\mathrm{ass}}(W,\Gamma) := \bigodot_{e\in E(\Gamma)} O(e)\otimes\bigodot_{u\in V_e(\Gamma)} W[-N]_{|u|}\otimes\bigwedge_{v^s\in V_s(\Gamma)}\mathrm{Cyc}(v^s)[N]\otimes\bigwedge_{v\in V_n(\Gamma)}\mathrm{Cyc}(v)[N],$$

$$C^{\bullet;\bullet}_{\mathrm{ass}}(W) := \bigoplus_{\Gamma\in\mathcal{G}(N)} O_{\mathrm{ass}}(W,\Gamma),\quad O_{\mathrm{ass}}(W,\Gamma) := \tilde{O}_{\mathrm{ass}}(W,\Gamma)_{\mathrm{Aut}(\Gamma)}.$$

Then $(C^{\bullet;\bullet}_{\mathrm{ass}}(W), d, L)$ is also a double $\mathrm{Sp}(W,\delta)$-chain complex and the chain map

$$C^{\bullet;\bullet}_{\mathrm{ass}}(W)\to C^{\bullet;\bullet}_{CE}(\mathrm{Der}_\omega(\hat{T}W))$$

can be defined in the same way. In this case, we can also consider the double subcomplex $C^{\bullet;\bullet}_{\mathrm{ass}}(N,Z)$ which consists of N-graded graphs without external vertices.

13. Applications and examples

Examples of relations between our chain map and a known notion are written in the following two Examples.

Example 13.1. For a cyclic minimal A_∞-algebra (H,I,m) with even degree, putting $W := H^*[-1]$, we have the map $C^{\bullet;\bullet}_{\mathrm{ass}}(W) \to C^{\bullet;\bullet}_{CE}(\mathrm{Der}_\omega(\hat{T}W))$. Here $\hat{T}W$ is the dual of the bar construction of (H,I,m). The map induced by the chain map

$$C^{0,\bullet}_{\mathrm{ass}}(N,Z) \to C^{0,\bullet}_{CE}(\mathrm{Der}_\omega(\hat{T}W)) = \mathbb{R}$$

is known as the Kontsevich cocycle ([25, 37, 17]) of a cyclic A_∞-algebra (H,I,m).

Example 13.2. In the case of $Z = \{0\}$ and $\delta = 0$, the chain map

$$C^{\bullet;0}_{\mathrm{ass}}(0,\{0\}) \to C^{\bullet;0}_{CE}(\mathrm{Der}_\omega(\hat{T}W))^{\mathrm{Sp}(W)}$$

is equal to Kontsevich's chain map [25, 26].

In the case that W is positively graded, we define a chain complex $C^{\bullet;\bullet}_{\mathrm{com}}(W)_+$ by

$$C^{\bullet;\bullet}_{\mathrm{com}}(W)_+ = C^{\bullet;\bullet}_{\mathrm{com}}(W)/(\text{positivity}),$$

where the positivity relation is as follows:

- **(positivity) (i)** a graph which has a normal vertex v satisfying $|v| \leq 0$ is zero, and **(ii)** $(i_{v'} + i_{v''})d_v(\Gamma, o) = 0$ for an oriented graph (Γ, o) and a normal vertex v of degree 0.

The differentials d, L are also defined on $C^{\bullet;\bullet}_{\mathrm{com}}(W)_+$, while i is not.

Proposition 13.3. The operators d, L induce the differentials on $C^{\bullet;\bullet}_{\mathrm{com}}(W)_+$.

Proof. It is clear that these operators are compatible with the former condition **(i)** of the positivity relation. Note that, to prove compatibility with L for a graph including a vertex with degree 0, we need to use **(ii)**.

We shall prove they are compatible with **(ii)**. First, we shall calculate the image of **(ii)** by the operator d. For $\Gamma \in \mathcal{G}(N)$ and a normal vertex v of degree 0, we have

$$\begin{aligned} d(i_{v'} + i_{v''})d_v &= d_{v''}i_{v'}d_v + d_{v'}i_{v''}d_v + \sum_{u \neq v', v''} d_u(i_{v'} + i_{v''})d_v \\ &= d_{v''}i_{v'}d_v + d_{v'}i_{v''}d_v - \sum_{u \neq v} (i_{v'} + i_{v''})d_v d_u. \end{aligned}$$

Here we used the equations in the proof of Theorem 12.8. For a splitting of v such that $|v'| = -1$, $d_{v''}i_{v'}d_v$ must have a non-positive vertex since $|v''| = 1$. In the same way, $d_{v'}i_{v''}d_v$ also have a non-positive vertex. So $d(i_{v'} + i_{v''})d_v$ is equal to zero under the positivity relation.

Next, we shall calculate the image of **(ii)** by the operator L:

$$\begin{aligned}L(i_{v'}+i_{v''})d_v=&\sum_u (i_{u'}+i_{u''})d_u(i_{v'}+i_{v''})d_v\\ =&(i_{(v'')'}+i_{(v'')''})d_{v''}i_{v'}d_v+(i_{(v')'}+i_{(v')''})d_{v'}i_{v''}d_v\\ &-\sum_{u\neq v}(i_{v'}+i_{v''})d_v(i_{u'}+i_{u''})d_u\\ =&(i_{(v'')'}+i_{(v'')''})i_{v'}d_{v''}d_v+(i_{(v')'}+i_{(v')''})i_{v''}d_{v'}d_v\\ &-\sum_{u\neq v}(i_{v'}+i_{v''})d_v(i_{u'}+i_{u''})d_u.\end{aligned}$$

By changing names of vertices like the proof of Theorem 12.8, we get

$$(i_{(v'')'}+i_{(v'')''})i_{v'}d_{v''}d_v=-(i_{(v')''}+i_{v''})i_{(v')'}d_{v'}d_v=-R_{v'}d_{v'}d_v-i_{(v')'}i_{v''}d_{v'}d_v,$$

and

$$\begin{aligned}&(i_{(v'')'}+i_{(v'')''})i_{v'}d_{v''}d_v+(i_{(v')'}+i_{(v')''})i_{v''}d_{v'}d_v\\ =&-R_{v'}d_{v'}d_v+i_{(v')''}i_{v''}d_{v'}d_v\\ =&-R_{v'}d_{v'}d_v-i_{(v'')'}i_{(v'')''}d_{v''}d_v\\ =&-R_{v'}d_{v'}d_v-R_{v''}d_{v''}d_v\end{aligned}$$

Using the A_∞-relation, $L(i_{v'}+i_{v''})d_v$ is equal to zero under the positivity relation. □

Then we can also get the chain map

$$\psi_+ : C^{\bullet,\bullet}_{\text{com}}(W)_+ \to C^\bullet(\mathrm{Der}^+_\omega(LW))$$

induced by ψ.

Example 13.4. Suppose $X=\#_g(S^n\times S^n)\backslash \mathrm{Int}\, D^{2n}$. Its Quillen model is described by:

$$L_X=L(u_1,v_1,\dots,u_g,v_g)\ (\deg u_i=\deg v_i=n-1),\quad \delta=0,$$
$$\omega(u_i,v_j)=\delta_{ij},\ \omega(u_i,u_j)=\omega(v_i,v_j)=0.$$

It means $N=2n-2$, $W=\langle u_1,v_1,\dots,u_g,v_g\rangle$ and $Z=\{n-1\}$. Then the dgl $(\mathrm{Der}^+_\omega(L_X),0)$ is a Quillen model of $B\,\mathrm{aut}_{\partial,0}(X)$ (which is proved in [2]). In the case, we can forget all special vertices in the graph complex sicne $\delta=0$. So we have the chain map

$$C^{\bullet,\bullet}_{\text{com}}(2n-2,\{n-1\})_+/(\text{special vertices})\to C^{\bullet,\bullet}_{CE}(\mathrm{Der}^+_\omega(L_X))^{\mathrm{Sp}(W)}.$$

This map is constructed by [2] and it is proved that the map is an isomorphism under the limit $g\to\infty$.

Example 13.5. Suppose $X=\mathbb{C}P^3\setminus \mathrm{Int}\, D^6$. Its Quillen model is described by:

$$L_X=L(u_1,u_2)\ (\deg u_i=2i-1),\quad \delta=\frac{1}{2}[u_1,u_1]\frac{\partial}{\partial u_2},$$
$$\omega(u_1,u_2)=\omega(u_2,u_1)=1.$$

It means $N=4$, $W=\langle u_1,u_2\rangle$ and $Z=\{1,3\}$. Then the dgl $(\mathrm{Der}^+_\omega(L_X),\delta)$ is a Quillen model of $B\,\mathrm{aut}_{\partial,0}(X)$. Since $\mathrm{Sp}(W,\delta)=1$, we have the chain map

$$C^{\bullet,\bullet}_{\text{com}}(W)_+\to C^{\bullet,\bullet}_{CE}(\mathrm{Der}^+_\omega(L_X))=C^{\bullet,\bullet}_{CE}(\mathrm{Der}^+_\omega(L_X))^{\mathrm{Sp}(W,\delta)}.$$

We shall define a certain sub dgl $\mathfrak{d}$ of $\mathrm{Der}_\omega(L_X)$. Put

$$A_1 = \frac{1}{2}[u_1, u_1]\frac{\partial}{\partial u_2}, \quad A_2 = \frac{1}{2}[u_2, u_2]\frac{\partial}{\partial u_1}$$

$$B_1 = \frac{1}{2}[u_1, u_1]\frac{\partial}{\partial u_1} + [u_1, u_2]\frac{\partial}{\partial u_2}, \quad B_2 = [u_1, u_2]\frac{\partial}{\partial u_1} + \frac{1}{2}[u_2, u_2]\frac{\partial}{\partial u_2}.$$

Then we have

$$\delta(A_1) = \delta(B_1) = \delta(B_2) = 0,$$

$$\delta(A_2) = \frac{1}{2}[[u_1, u_1], u_2]\frac{\partial}{\partial u_1} + \frac{1}{2}[[u_2, u_2], u_1]\frac{\partial}{\partial u_2} = [A_1, A_2] = -[B_1, B_2] =: C,$$

$$[A_i, B_j] = [A_i, A_i] = [B_j, B_j] = 0 \ (i, j = 1, 2),$$

$$\deg A_1 = -1, \ \deg A_2 = 5, \ \deg B_1 = 1, \ \deg B_2 = 3, \ \deg C = 4.$$

Here we put $\delta(Z) := [\delta, Z]$ for simplicity. By the relation above,

$$\mathfrak{d} := \langle A_1, A_2, B_1, B_2, C\rangle = \mathrm{Der}^1_\omega(L_X) \oplus \mathrm{Der}^2_\omega(L_X)$$

is a sub dgl of $\mathrm{Der}_\omega(L_X)$. Its positive truncation $\mathfrak{d}^+$ is described by

$$\mathfrak{d}^+ = \langle A_2, B_1, B_2, C\rangle\,,$$

$$\delta(A_2) = -[B_1, B_2] = C, \ \delta(B_1) = \delta(B_2) = \delta(C) = 0,$$

$$[A_2, B_i] = [A_2, A_2] = [B_i, B_i] = [A_2, C] = [B_i, C] = 0 \ (i = 1, 2).$$

Let x, y_1, y_2, z be the suspension of the dual basis of A_2, B_1, B_2, C. Then the Chevalley-Eilenberg complex of the dgl $\mathfrak{d}^+$ is written by

$$C^{\bullet;\bullet}_{CE}(\mathfrak{d}^+) = \Lambda(x, y_1, y_2, z) \ (\deg x = 6, \ \deg y_1 = 2, \ \deg y_2 = 4, \ \deg z = 5),$$

$$dx = dy_1 = dy_2 = 0, \ dz = x - y_1 y_2$$

and its total cohomology

$$H^\bullet_{CE}(\mathfrak{d}^+) = \Lambda(x, y_1, y_2)/(x - y_1 y_2).$$

Since $\mathfrak{d}^+$ is the rank ≤ 2 part of $\mathrm{Der}^+_\omega(L_X)$, the map $H^\bullet_{CE}(\mathrm{Der}^+_\omega(L_X)) \to H^\bullet_{CE}(\mathfrak{d}^+)$ induced by the inclusion has a section. So non-trivial classes in $H^\bullet_{CE}(\mathfrak{d}^+)$ gives non-trivial classes in $H^\bullet_{CE}(\mathrm{Der}^+_\omega(L_X))$.

The relation $dz = x - y_1 y_2$ in the Chevalley-Eilenberg complex is corresponding to the relation in the graph complex $C^{\bullet;\bullet}_{\mathrm{com}}(W)_+$ described in Figure 10. Here the classes x and $y_1 y_2$ corresponds to the first term and the sum of the second and third terms in the figure. Remark that y_1 and y_2 do not correspond to graphs without external vertices. According to the positivity relation, all the trivalent graphs appearing in the right hand side are cycles since the degrees of two half-edges incident to a permitted bivalent vertex in $C^{\bullet;\bullet}_{\mathrm{com}}(W)_+$ must be 3.

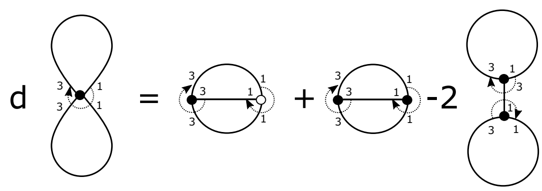


FIGURE 10. the relation of graphs (the orientations are omitted)

Example 13.6. Suppose $X = \mathbb{C}P^4 \setminus \operatorname{Int} D^8$. Its Quillen model is described by:

$$L_X = L(u_1, u_2, u_3) \ (\deg u_i = 2i-1), \quad \delta = \frac{1}{2}[u_1, u_1]\frac{\partial}{\partial u_2} + [u_1, u_2]\frac{\partial}{\partial u_3},$$
$$\omega(u_2, u_2) = \omega(u_1, u_3) = 1.$$

It means $N = 6$, $W = \langle u_1, u_2, u_3\rangle$ and $Z = \{1, 3, 5\}$. Then the dgl $(\mathrm{Der}^+_\omega(L_X), \delta)$ is a Quillen model of $B\operatorname{aut}_{\partial,0}(X)$. Defining the linear transformation τ by $\tau(u_1) = -u_1$, $\tau(u_2) = u_2$ and $\tau(u_3) = -u_3$, we have $\mathrm{Sp}(W, \delta) = \{1, \tau\}$. So $C^{\bullet;\bullet}_{\mathrm{com}}(W)^{\mathrm{Sp}(W,\delta)}$ is generated by graphs labeled by u_1, u_2, u_3 satisfying $\#\{u_1, u_3\text{-labeled vertex}\}$ is even. For simplicity, we put

$$[u_{i_1} \cdots u_{i_k}] := [u_{i_1}, \cdots, u_{i_k}]_{\mathrm{cyc}} = \sum_{s=1}^{k} (-1)^{s(k-s)} u_{i_{s+1}} \cdots u_{i_k} u_{i_1} \cdots u_{i_s} \in W^k_{\mathrm{cyc}}.$$

Using notations in Section 12.3, we can take a basis of $W(2)$

$$[u_i u_j] \ (\{i < j\} \subset \{1, 2, 3\}),$$

a basis of $W(3)$

$$\frac{1}{3}[u_i u_i u_i],\ [u_i u_j u_j],\ [u_i u_i u_j] \ (\{i < j\} \subset \{1, 2, 3\}),\ [u_1 u_2 u_3] + [u_1 u_3 u_2]$$

and a basis of $W(4)$

$$[u_i u_i u_j u_j] \ (i < j),$$
$$[u_1 u_1 u_2 u_3] + [u_1 u_1 u_3 u_2],\ [u_1 u_2 u_2 u_3] - [u_1 u_3 u_2 u_2],\ [u_1 u_2 u_3 u_3] - [u_1 u_3 u_3 u_2].$$

We put the corresponding rank 0, rank 1 and rank 2 basis of $\mathrm{Der}_\omega(L_X)$

$$P_{ij},\ A_{iii},\ A_{ijj},\ A_{iij},\ A_{123},\ B_{iijj},\ B_{1123},\ B_{1223},\ B_{1233},$$

and these dual basis p_{ij}, x_{ijk} and y_{ijkl} of P_{ij}, A_{ijk} and B_{ijkl}. Then by direct calculation we have the equations in $C^{\bullet;\bullet}_{CE}(\mathrm{Der}^+_\omega(L_X))$

$$dy_{1122} = x_{222} - 2x_{123} + x_{122}x_{113} - x_{122}x_{122},$$
$$dy_{2233} = x_{333}x_{122} + x_{233}x_{222} - x_{223}x_{223} - 2x_{123}x_{233} + x_{133}x_{223} + 2p_{23}y_{1233},$$
$$dy_{1133} = x_{233} - x_{133}x_{113} - x_{123}x_{123} - 2p_{23}y_{1123},$$
$$dy_{1123} = x_{223} - x_{133} - p_{23}y_{1122},$$
$$dy_{1223} = x_{233} + x_{223}x_{122} + x_{123}x_{123} - x_{223}x_{113} - x_{123}x_{222} - x_{133}x_{122} + p_{23}y_{1123},$$
$$dy_{1233} = x_{333} + x_{233}x_{122} - x_{123}x_{223} - x_{233}x_{113} - x_{123}x_{122} + p_{23}y_{1133}.$$

Here all terms appearing in the right-hand side of the equations are cocycles. For example, the fifth relation is corresponding to the relation in the graph complex $C^{\bullet;\bullet}_{\mathrm{com}}(W)_+$ described in Figure 11. In Figure 11, the image by ψ_+ of each graph appearing the last term of the right hand side is zero.

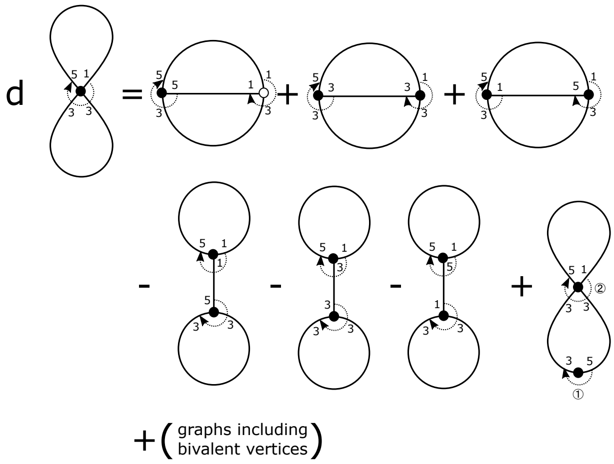


FIGURE 11. the relation of graphs (①, ② mean the orientation of vertices and the other orientations are omitted)

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