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Numerical method for combined stochastic control and its application for mathematical finance

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Abstract

This work contributes the numerical analysis of the Hamilton-Jacobi-Bellman quasi-variational inequality (HJBQVI) associated with the combined stochastic control and its applications to mathematical finance. Here the term "combined" means that we use both of the regular stochastic control and the impulse control. The regular stochastic control affects the drift and diffusion coefficients of the stochastic differential equation (SDE) describing the system, and the impulse control intervenes the value of the SDE directly.

We first sketch stochastic control theory by the classical approach in Chapter 1. The key points of this chapters are the dynamic programming equation and the verification theorem. We also briefly mention the notion of viscosity solution which is related to the discussions in the following chapters.

In Chapter 2, we construct an approximation scheme for the HJBQVI using the finite difference method. The key points of our scheme are: (i) we discretize the HJBQVI with the forward difference for the time grid; (ii) the impulse control part is kept in the same time step. We prove the monotonicity, the stability and the consistency of the scheme which are necessary for the convergence in the viscosity sense.

We propose an algorithm to solve the finite time horizon HJBQVI in Chapter 3. We focus on how to solve the discretized HJBQVI while we discuss its convergence in Chapter 2. The discretized HJBQVI is converted into an equivalent fixed point problem and is solved by the policy iteration. We provide a detailed procedure for the algorithm in matrix form for the sake of easy implementation. The optimal forest harvesting over finite time horizon is described as an example applied this algorithm.

An application of the combined stochastic control to the mathematical finance is discussed in Chapter 4. We investigate the optimal execution problem which seeks the optimal way to liquidate or acquire a large asset position. It is a trade-off between the fast trade facing the market impact risk and slow trade being exposed the risk that the price change due to the course of time. Using our novel algorithm, we seek the optimal execution strategy. Our execution strategy includes both of the limit and market orders and behaves dynamically reflecting on the price recovery from the market impact.

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Introduction

Stochastic control theory is a type of control theory and focuses on systems under noise in discrete and continuous time. We treat a continuous time problem and then the system is described by a stochastic differential equation (SDE) in which controls affect drift and diffusion coefficients of the SDE. In many cases, we measure the performance of the control by a scalar value. Our purpose then is to seek the value function that describes the performance that would be realized if we controlled the system optimally. A standard approach to analyzing the value function is to employ the dynamic programming principle pioneered by Bellman [Bel56]. This in turn leads the Hamilton-Jacobi-Bellman (HJB) equation, a non-linear partial differential equation that the value function satisfies. We refer to the literature: Fleming and Rishel [FR75], Krylov [Kry08], Fleming and Soner [FS06] and Pham [Pha09] for general topics regarding stochastic control theory; Björk [Bjö09] and Øksendal [Øks03] for readers interested in mathematical finance.

Mathematical finance is one of the most popular applications of stochastic control. A typical example is a portfolio optimization. The traditional mean-variance approach originated by Markowitz [Mar52] is a myopic method because it does not consider the rebalance after a fund has been launched. Hence, a continuous time approach employing a stochastic control is more suitable in the sense that it allows rebalancing of the portfolio until the fund is closed. Merton [Mer69] is a seminal work of continuous time portfolio optimization that suggests keeping a certain portfolio weight. For long-term investors such as pension funds, portfolio optimization in continuous time is a significant topic. Some examples of the literature treating managements of pension funds are as follows: Deelstra et al. [DGK03] and Giacinto et al. [DFG10] discuss the portfolio management for pension funds with a minimum guarantee; Menoncin and Scaillet [MS06] and Gerrard et al. [GHV12] deal with the pension scheme including the de-cumulation phase; Ieda et al. [IYN13] and [IYN14] investigate a liability tracking approach for pension fund management.

An impulse control is another way to control the system. Impulse control strategies consist of pairs of a stopping time and a random variable, that is, we are able to apply the control at the time which satisfies a certain condition. In contrast, we are able to execute a control continuously in the regular stochastic control. Another difference between impulse and stochastic controls is that an impulse control intervenes in the value of the process describing the system directly (recall that a stochastic control affects the coefficients of SDE). The main story of an impulse control theory is the same as that of a stochastic control theory: employing the dynamic programming principle, the value function is characterized by the Hamilton-Jacobi-Bellman quasi-variational inequality (HJBQVI). Bensoussan and Lions [BL84] and Øksendal and Sulem [ØS07] deal with the general theory of impulse control.

We present some of the literature which deal with the applications for the mathematical finance: Pliska and Suzuki [PS04], Palczewski and Zabczyk [PZ05] and Kharroubi and Pham [KP10] treat the portfolio optimization with transaction costs; Mundaca and Øksendal [MØ98] and Cadenillas and Zapatero [CZ00] study the control of the exchange rate by the central bank; Frey and Seydel [FS10] investigate the optimal loan securitization policy of a commercial bank and Korn [Kor99] provides an overview relating to the applications of impulse control.

Because both the HJB equation and HJBQVI are fully non-linear partial differential equations, analytical solutions are rarely available. Hence, numerical methods to solve them are necessary for applications in the real world. We enumerate popular numerical methods for the HJB equation: the finite difference method (see, e.g., Kushner and Dupuis [KD00] and Bonnans and Zidani [BZ03]); the Semi-Lagrangian method, a similar method to the finite element methods (e.g., Camilli and Falcone [CF95] and Debrabant and Jakobsen [DJ13]) probabilistic methods (see e.g., Pagès et al. [PPP04] and Fahim et al. [FTW11]); and meshfree methods (see e.g., Kansa [Kan90] and Nakano [Nak14])

Numerical methods for the HJBQVI are not as rich as the case of HJB equation. Bensoussan and Lions [BL84] approximate the impulse control problem using iterations of the optimal stopping problem, and hence the HJBQVI is translated to HJB variational inequalities (HJBVIs). Kharroubi et al. [Kha+10] and Elie and Kharroubi [EK10] take the approach using a penalty method involving the backward SDEs.

The finite difference approximation is also valid in the HJBQVI. Chancelier et al. [CMS06] solve the infinite time horizon HJBQVI by converting the discretized HJBQVI into an equivalent fixed point problem, although they do not prove the convergence of the discretized HJBQVI. For the finite horizon case, Chen and Forsyth [CF08] solves the finite time horizon HJBQVI associated with the pricing problem of the guaranteed minimum withdrawal benefit. However, there was no general method until Ieda [Ied15b].

Conributions of the present work

The first contribution of this work is the proposal of a novel numerical method solving the HJBQVI associated with the combined stochastic control over the finite time horizon. We discretize the HJBQVI by the finite difference method. The key points of the discretization are that (i) we discretize the HJBQVI with the forward difference for the time grid; (ii) the impulse control part is kept in the same time step. Hence, the present method is an implicit method. We prove the convergence of the numerical scheme by the method of Barles and Souganids [BS91]. If we show the monotonicity, stability, and consistency, the convergence is assured. The discretized HJBQVI is hard to solve in the original form. Therefore we convert the discretized HJBQVI to the equivalent fixed point problem. We solve the fixed point problem by the policy iteration.

The second contribution is the application of our novel algorithm to mathematical finance. We investigate the optimal execution problem under stochastic price recovery. The optimal execution problem, which seeks the optimal way to liquidate or acquire large asset positions, is a matter of concern in mathematical finance. It is a trade-off between fast trading facing market impact risk and slow trading being exposed to the risk that the price of the target asset will change over the course of time. We propose a new model that describes price recovery after the market impact based on the dynamics of the limit order book. We also enable the limit order strategy in addition to the market order strategy. Our execution problem is then formulated by the combined stochastic control problem. The analytical solution is not available for our problem, and thus we obtain the numerical solution using our novel algorithm. The main results of this study are: (i) we capture the dynamical behavior of the execution strategy reflecting on price recovery from the market impact; (ii) the limit order improves the execution performance.

Organization of the present dissertation

In Chapter 1, we sketch stochastic control theory by the classical approach. Section 1.1 reviews theory of regular stochastic control, and we mainly focus on the finite time horizon case. The key points are common results: the dynamic programming equation and the verification theorem. We mention the case of impulse control in Section 1.2. It follows in the same manner as Section 1.1; however, to the best of our knowledge, there are no well-known studies in which the HJBQVI is explicitly derived from the dynamic programming principle for the finite time horizon. Combined regular stochastic and impulse control is discussed in Section 1.3, and we briefly mention the notion of viscosity solution which is related to the discussion of the next chapter in Section 1.4.

The purposes of Chapter 2 are the construction of the finite difference method (FDM) for the HJB equations and its convergence proof. In Section 2.1 we mention the definition of the finite difference approximation and check accuracies of the approximation for several finite difference operators. Section 2.2 explains the Bales-Souganidis framework [BS91] which is a powerful tool to prove the convergence of approximation schemes for partial differential equations (PDEs). The monotonicity, stability, and consistency of numerical schemes play a central role in this framework. An illustrative example using the Bales-Souganidis framework is served in Section 2.3. We prove the monotonicity, stability, and consistency of the approximation scheme of the one dimensional Hamilton-Jacobi-Bellman equation. In Section 2.4 we investigate the convergence of the multi-dimensional Hamilton-Jacobi-Bellman quasi-variational inequality. The convergence of these approximation schemes tends to be treated as a well-known result. However, to the best of our knowledge, there are no well-known study that give the convergence proof of the implicit method for the one dimensional HJB equation. Furthermore, the convergence proof of the multi-dimensional HJBQVI is a completely novel result.

In Chapter 3, we focus on how to solve the discretized HJBQVI, while Chapter 2 gives its convergence proof. The majority of the contents of this chapter correspond to Ieda[Ied15b]. The heart of this chapter is Section 3.3. The discretized HJBQVI displayed in matrix form is translated into an equivalent fixed point problem. We solve the fixed point problem by the policy iteration proposed by Chancelier et al. [CMS06]

We also describe the procedure in detail from the viewpoint of the computational implementation. In Section 3.4, we apply the proposed method to the optimal forest harvesting problem. This is the problem that determines the optimal harvesting strategy for the ongoing forestry business. We employ the mathematical formulation of the problem proposed by Willassen [Wil98] based on the infinite time horizon impulse control framework for which analytical solutions of the value function and optimal strategy are provided. We introduce the terminal time, representing the time of exiting from the forest business and then the above problem becomes a finite time horizon problem. In this case, the analytical solution is unavailable, and thus we solve it numerically by the proposed algorithm.

The application of our novel algorithm to mathematical finance is discussed in Chapter 4. The majority of the contents of this chapter correspond to Ieda[Ied15a]. Section 4.2 introduces the mathematical formulation of our execution problem. The stochastic price recovery proposed in this section is a novel model. We use the combined stochastic control framework, and market and limit order strategies are formulated by the impulse and regular stochastic controls, respectively. The goal of this section is to derive the corresponding HJBQVI. In Section 4.3, we present a numerical method for solving the HJBQVI, following a similar procedure discussed in the previous chapter. The HJBQVI is discretized by a finite difference scheme and is converted to an equivalent fixed point problem. In Section 4.4, we present numerical results under several conditions, specifically the types of recovery intensity functions and either including or excluding the limit order.

1 Stochastic Control Problem

In this chapter, we sketch stochastic control theory from the classical approach. We first introduce a regular stochastic control, next mention an impulse control and finally describe combination of both controls. The key points of each topics are common results: the dynamic programming equation and the verification theorem. The description of the regular stochastic control is based on the literature von Handel [vHa07], Touzi [Tou12] and Pham [Pha09] . The impulse control strategy, especially finite time horizon case, is described in the original way inspired by the literature for von Handel [vHa07] and Økesendal and Sulem [ØS07] that deal with the elliptic case. To the best of our knowledge, there is no literature which derive explicitly the Hamilton-Jaboci-Bellman quasi variational inequality from the dynamic programming principle for the parabolic case. The last section briefly mentions the notion of the viscosity solution which is related to the discussion of the next chapter.

1.1 Regular Stochastic Control

Controlled diffusion process

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions and W_t be an m-dimensional $\mathcal{F}_t$-Wiener process. We consider a stochastic differential equation (SDE) with a control input as follow:

$$dX_t = \mu(t, X_t, u_t)dt + \sigma(t, X_t, u_t)dW_t, \tag{1.1}$$

where $u_t \in \mathbb{U}$ is a control input at time t, $\mathbb{U}$ is the set of values of control inputs, $\mu : \mathbb{T}\times\mathbb{R}^n\times\mathbb{U} \to \mathbb{R}^n$, $\sigma : \mathbb{T}\times\mathbb{R}^n\times\mathbb{U} \to \mathbb{R}^{n\times m}$ and $\mathbb{T}$ is the time interval. The time interval $\mathbb{T}$ is either finite or infinite but we mainly investigate the finite case in the following chapters. We often denote by $X_t^{s,x}$ the solution of the SDE (1.1) with initial condition $X_s = x$ to emphasize the initial condition.

We make the following assumptions:

Assumption 1.1.1
A control set $\mathbb{U}$ and a control strategy $u = \{u_t\}_{t\in\mathbb{T}}$ satisfy

- *$\mathbb{U}$ is a closed subset of R^q, $q \in \mathbb{N}$;*
- *$u_t(\omega) \in \mathbb{U}$ for every $(\omega, t) \in \Omega \times \mathbb{T}$;*
- *u_t is an $\mathcal{F}_t$-adapted process.*

We denote by $\mathcal{U}_0$ a collection of control strategies which satisfy the above conditions.

The coefficients μ and σ growth linearly in $\mathbb{R}^n$ and $\mathbb{U}$, and are Lipschitz continuous on $\mathbb{T}$ and $\mathbb{R}^n$: for all $s, t \in \mathbb{T}$, $x, y \in \mathbb{R}^n$ and $\alpha \in \mathbb{U}$,

$$
\begin{aligned}
&|\mu(t,x,\alpha)| \leq C(1 + |x| + |\alpha|),\\
&|\sigma(t,x,\alpha)| \leq C(1 + |x| + |\alpha|),\\
&|\mu(t,x,\alpha) - \mu(s,y,\alpha)| \leq C(|s-t| + |y-x|),\\
&|\sigma(t,x,\alpha) - \sigma(s,y,\alpha)| \leq C(|s-t| + |y-x|).
\end{aligned}
$$

where C is a positive constant.

Under the assumption 1.1.1, the SDE (1.1) has the unique strong solution.

Theorem 1.1.2
Suppose that the assumption 1.1.1 is satisfied and $\mathbb{T} = [0, T]$, $T \in (0, \infty)$. Then there exists a strong solution X_t of the SDE (1.1).

Theorem 1.1.3
The solution in the theorem 1.1.2 is unique $\mathbb{P}$-a.s.

These theorems are well known and we refer e.g. Øksendal [Øks03] for the proof.

Remark 1.1.4
The Lipschitz continuity of μ and σ on $\mathbb{T}$ does not need for the existence of the strong solution of the SDE (1.1). However in Chapter 2, we need this assumption to prove the convergence of numerical scheme.

Performance Criteria

To evaluate a given control $u \in \mathcal{U}_0$, we introduce a performance criterion $J^u : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}$. According to time horizons and domains, we consider several types of them defined as follows.

Finite time horizon and infinite domain

We first introduce a performance criterion over a finite time horizon and an infinite domain

$$
J^u(t,x) = \mathbb{E}\left[\int_t^T f(s, X_s, u_s)ds + g_T(X_T) \middle| X_t = x \right], \quad u \in \mathcal{U}(t,x),
$$

where $T \in (0, \infty)$ is a terminal time, $f : [0,T] \times \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}$ is a measurable function standing for running profit, $g_T : \mathbb{R}^n \to \mathbb{R}$ is a measurable function standing for a terminal profit and $\mathcal{U}(t,x)$ is a collection of control strategies $u \in \mathcal{U}_0$ such that

$$
\mathbb{E}\left[\int_t^T |f(s, X_s, u_s)|ds \right] < \infty.
$$

We see $J^u(t,x)$ is the expected profit using the control strategy u from t to T.

We remark that $J^u(t,x)$ is well-defined for all $u \in \mathcal{U}_0$ if both of f and g_T satisfy the quadratic growth condition, i.e., for all $(t,x,\alpha) \in [0,T] \times \mathbb{R}^n \times \mathbb{U}$,

$$|f(t,x,\alpha)| + |g_T(x)| < C(t + |x|^2 + |\alpha|^2),$$

where C is a positive constant.

Finite time horizon and finite domain

We next define a performance criterion over a finite time horizon and a finite domain. Let $\mathcal{S}$ be a finite open subset of $\mathbb{R}^n$ and $\partial\mathcal{S}$ be a boundary of $\mathcal{S}$. Then for $u \in \mathcal{U}(t,x)$,

$$J^u(t,x) = \mathbb{E}\left[\int_t^{T\wedge\tau_{\mathcal{S}}} f(s,X_s,u_s)ds + g_{\partial\mathcal{S}}(\tau_{\mathcal{S}}, X_{\tau_{\mathcal{S}}})\mathbf{1}_{\{\tau_{\mathcal{S}}<T\}} + g_T(X_T)\mathbf{1}_{\{T<\tau_{\mathcal{S}}\}}\middle| X_t = x\right], \tag{1.2}$$

where $\tau_{\mathcal{S}}$ is the first exit time that X_t exits from $\mathcal{S}$ and $g_{\partial\mathcal{S}} : (0,T) \times \partial\mathcal{S} \to \mathbb{R}$ is a bounded measurable function standing for a boundary profit. The difference from the infinite domain case is that when X_t exits from the domain $\mathcal{S}$ before the terminal time T we stop the control and obtain the boundary profit.

Infinite time horizon

Performance criteria over an infinite time horizon differ from ones over a finite time horizon because we have to ensure that the time integrals over $[0,\infty]$ in the criteria take finite values. Here we introduce two type of criteria:

- discounted criteria

  $$J_\lambda^u(x) = \mathbb{E}\left[\int_0^\infty e^{-\lambda t} f(X_t,u_t)dt \middle| X_0 = x\right],$$

  where $\lambda \in \mathbb{R}^+$ is a discount factor;

- time-average criteria

  $$J^u(x) = \limsup_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\int_0^\infty f(X_t,u_t)dt \middle| X_0 = x\right].$$

As same in the finite time horizon case, we are able to define performance criteria in a finite domain.

Admissibile Strategy and Markov Strategy

The control strategy u is said to be admissible if the corresponding performance criterion J^u or J_λ^u is well-defined. We also mention a special class of the admissible strategy, the Markov strategy:

Definition 1.1.5

An admissible control strategy u is called a Markov strategy if it is the form $u_t = \alpha(t,X_t)$ for some function $\alpha : \mathbb{T} \times \mathbb{R}^n \to \mathbb{U}$.

Dynamic Programming Principle

Here we concentrate the finite time horizon case with the finite domain (1.2) and define the value function V as follow:

$$V(t,x) = \sup_{u \in \mathcal{U}(t,x)} J^u(t,x), \quad (t,x) \in [0,T] \times \mathcal{S} \cup \partial\mathcal{S}. \tag{1.3}$$

Let us suppose that there exists the optimal control strategy $\hat{u}$, i.e.,

$$V(t,x) = J^{\hat{u}}(t,x), \quad (t,x) \in [0,T] \times \mathcal{S} \cup \partial\mathcal{S}.$$

Thus our goal is equivalent to determine the value function.

The dynamic programming principle is a fundamental tool in stochastic control theory. Roughly speaking, the dynamic programming principle allows us to find the value function by backward calculations from the terminal time.

Proposition 1.1.6
The value function V satisfies

$$\begin{aligned}
&V(t,x) = g_T(x), && x \in \mathcal{S} \cup \partial\mathcal{S};\\
&V(\tau_{\mathcal{S}},x) = g_{\partial\mathcal{S}}(x), && x \in \partial\mathcal{S};\\
&V(r,x) = \sup_{u \in \mathcal{U}(r,x)} \mathbb{E}\left[\int_r^t f(s,X_s,u_s)ds + V(t,x) \middle| X_t = x\right], && x \in \mathcal{S},
\end{aligned}$$

for $0 \le r \le t \le T \wedge \tau_{\mathcal{S}}$.

Proof. By the definition, the equation (1.2), it is clear that

$$V(t,x) = \sup_{u \in \mathcal{U}(T,x)} J^u(t,x) = g_T(x), \qquad x \in \mathcal{S} \cup \partial\mathcal{S}$$

and

$$V(\tau_{\mathcal{S}},x) = \sup_{u \in \mathcal{U}(\tau_{\mathcal{S}},x)} J^u(\tau_{\mathcal{S}},x) = g_{\partial\mathcal{S}}(x), \qquad x \in \partial\mathcal{S}.$$

Note that

$$\begin{aligned}
&J^u(t,X_t^{r,X_r}) - J^u(r,X_r)\\
&= \mathbb{E}\left[\int_t^{T\wedge\tau_{\mathcal{S}}} f(s,X_s,u_s)ds + g_{\partial\mathcal{S}}(\tau_{\mathcal{S}},X_{\tau_{\mathcal{S}}})\mathbf{1}_{\{\tau_{\mathcal{S}}<T\}} + g_T(X_T)\mathbf{1}_{\{T<\tau_{\mathcal{S}}\}} \middle| X_t^{r,X_r}\right]\\
&\quad - \mathbb{E}\left[\int_r^{T\wedge\tau_{\mathcal{S}}} f(s,X_s,u_s)ds + g_{\partial\mathcal{S}}(\tau_{\mathcal{S}},X_{\tau_{\mathcal{S}}})\mathbf{1}_{\{\tau_{\mathcal{S}}<T\}} + g_T(X_T)\mathbf{1}_{\{T<\tau_{\mathcal{S}}\}} \middle| X_r\right]
\end{aligned}$$

for $0 \leq r \leq t \leq T \wedge \tau_{\mathcal{S}}$. Taking the conditional expectation, we obtain

$$\begin{aligned}
&\mathbb{E}\left[J^u(t, X_t^{r,X_r}) \middle| X_r \right] - J^u(r, X_r) \\
&\quad = \mathbb{E}\left[\mathbb{E}\left[\int_t^{T\wedge\tau_{\mathcal{S}}} f(s, X_s, u_s)ds \middle| X_t^{r,X_r} \right] \middle| X_r \right] - \mathbb{E}\left[\int_r^{T\wedge\tau_{\mathcal{S}}} f(s, X_s, u_s)ds \middle| X_r \right] \\
&\qquad + \mathbb{E}\left[\mathbb{E}\left[g_{\partial\mathcal{S}}(\tau_{\mathcal{S}}, X_{\tau_{\mathcal{S}}})\mathbf{1}_{\{\tau_{\mathcal{S}}<T\}} \middle| X_t^{r,X_r} \right] \middle| X_r \right] - \mathbb{E}\left[\int_r^{T\wedge\tau_{\mathcal{S}}} g_{\partial\mathcal{S}}(\tau_{\mathcal{S}}, X_{\tau_{\mathcal{S}}})\mathbf{1}_{\{\tau_{\mathcal{S}}<T\}} + \middle| X_r \right] \\
&\qquad + \mathbb{E}\left[\mathbb{E}\left[g_T(X_T)\mathbf{1}_{\{T<\tau_{\mathcal{S}}\}} \middle| X_t^{r,X_r} \right] \middle| X_r \right] - \mathbb{E}\left[g_T(X_T)\mathbf{1}_{\{T<\tau_{\mathcal{S}}\}} \middle| X_r \right] .
\end{aligned}$$

The Markovian structure $X_s^{r,X_r} = X_s^{t,X_t^{r,X_r}}$, $r \leq t \leq s$, gives

$$\begin{aligned}
&\mathbb{E}\left[J^u(t, X_t^{r,X_r}) \middle| X_r \right] = \mathbb{E}\left[J^u(t, X_t) \middle| X_r \right], \\
&\mathbb{E}\left[\mathbb{E}\left[\int_t^{T\wedge\tau_{\mathcal{S}}} f(s, X_s, u_s)ds \middle| X_t^{r,X_r} \right] \middle| X_r \right] = \mathbb{E}\left[\int_t^{T\wedge\tau_{\mathcal{S}}} f(s, X_s, u_s)ds \middle| X_r \right], \\
&\mathbb{E}\left[\mathbb{E}\left[g_{\partial\mathcal{S}}(\tau_{\mathcal{S}}, X_{\tau_{\mathcal{S}}})\mathbf{1}_{\{\tau_{\mathcal{S}}<T\}} \middle| X_t^{r,X_r} \right] \middle| X_r \right] = \mathbb{E}\left[g_{\partial\mathcal{S}}(\tau_{\mathcal{S}}, X_{\tau_{\mathcal{S}}})\mathbf{1}_{\{\tau_{\mathcal{S}}<T\}} \middle| X_r \right] \\
&\mathbb{E}\left[\mathbb{E}\left[g_T(X_T)\mathbf{1}_{\{T<\tau_{\mathcal{S}}\}} \middle| X_t^{r,X_r} \right] \middle| X_r \right] = \mathbb{E}\left[g_T(X_T)\mathbf{1}_{\{T<\tau_{\mathcal{S}}\}} \middle| X_r \right] .
\end{aligned}$$

Thus we have

$$\mathbb{E}\left[J^u(t, X_t) \middle| X_r \right] - J^u(r, X_r) = -\mathbb{E}\left[\int_r^t f(s, X_s, u_s)ds \middle| X_r \right]$$

and hence

$$J^u(r, X_r) = \mathbb{E}\left[\int_r^t f(s, X_s, u_s)ds + J^u(t, X_t) \middle| X_r \right] .$$

Choose u' to be a strategy that coincides with u on $[0, t)$ and with $\hat{u}$ on $[t, T \wedge \tau_{\mathcal{S}}]$. Then

$$\begin{aligned}
V(r, X_r) \geq J^{u'}(r, X_r) &= \mathbb{E}\left[\int_r^t f(s, X_s, u'_s)ds + J_t^{u'}(X_t) \middle| X_r \right] \\
&= \mathbb{E}\left[\int_r^t f(s, X_s, u'_s)ds + V(t, X_t) \middle| X_r \right]
\end{aligned}$$

where we use the fact that $J_s^u(x)$ only depends on u on $[s, T \wedge \tau_{\mathcal{S}}]$. Thus we obtain

$$V(r, X_r) = \sup_{u \in \mathcal{U}(r,X_r)} \mathbb{E}\left[\int_r^t f(s, X_s, u_s)ds + V(t, X_t) \middle| X_r \right] .$$

□

Hamilton-Jacobi-Bellman equation

The Hamilton-Jabob-Bellman (HJB) equation is the infinitesimal version of the dynamic programming principle. We derive it formally under strong smoothness assumptions.

Before the derivation, we mention the notations. Denote by $\mathscr{L}_t^\alpha$ the infinitesimal generator of the process X_t:

$$\mathscr{L}_t^\alpha \phi(t,x) = \mu^\top(t,x,\alpha)\partial_x\phi(t,x) + \frac{1}{2}\mathrm{Tr}\left[\sigma\sigma^\top(t,x,\alpha)\partial_x^2\phi(t,x)\right],$$

for $\phi \in C^{1,2}([0,T]\times\mathbb{R}^n)$ and $\alpha \in \mathbb{U}$. We define the Hamiltonian $\mathscr{H} : [0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^{n\times n} \to \mathbb{R}$ as follow:

$$\mathscr{H}(t,x,p,M) = \sup_{\alpha\in\mathbb{U}}\left\{\mu^\top(t,x,\alpha)p + \frac{1}{2}\mathrm{Tr}\left[\sigma\sigma^\top(t,x,\alpha)M\right] + f(t,x,\alpha)\right\}.$$

Note that

$$\mathscr{H}(t,x,\partial_x V(t,x),\partial_x^2 V(t,x)) = \sup_{\alpha\in\mathbb{U}}\left\{\mathscr{L}_t^\alpha V(t,x) + f(t,x,\alpha)\right\}.$$

Suppose that $V(t,x)$ is C^1 in t and C^2 in x; then by the Ito's lemma we have

$$V(t,X_t) = V(r,X_r) + \int_r^t \left\{\partial_s V(s,X_s) + \mathscr{L}_s^{u_s} V(s,X_s)\right\} ds + \int_r^t \partial_x V(s,X_s)\sigma(s,X_s,u_s)dW_s$$

We additionally suppose that

$$\int_0^t \partial_x V(s,X_s)^\top \sigma(s,X_s,u_s)dW_s \text{ is a martingale for all } t. \tag{1.4}$$

Then taking the conditional expectation we obtain

$$V(r,X_r) = \mathbb{E}\left[\int_r^t -\left\{\partial_s V(s,X_s) + \mathscr{L}_s^{u_s} V(s,X_s)\right\} ds + V(t,X_t)\middle| X_r\right]$$

Using the dynamic programming principle, Proposition 1.1.6, we conclude that

$$\mathbb{E}\left[\int_r^t \left\{\partial_s V(s,X_s) + \mathscr{L}_s^{u_s} V(s,X_s)\right\} ds + f(s,X_s,u_s)\middle| X_r\right] \le 0,$$

for $0 \ge r \ge t \ge T\wedge\tau_{\mathcal{S}}$ and moreover the inequality becomes a equation if $u = \hat{u}$ on $[0,T\wedge\tau_{\mathcal{S}}]$. Thus, at least formally, we have

$$\sup_{u\in\mathcal{U}(r,X_r)}\left\{\partial_r V(r,X_r) + \mathscr{L}_r^{u_r} V(r,X_r) + f(r,X_r,u_r)\right\} = 0.$$

Using the pathwise nature of this equation and replacing r to t,

$$\sup_{\alpha\in\mathbb{U}}\left\{\partial_t V(t,x) + \mathscr{L}_t^\alpha V(t,X_t) + f(t,x,\alpha)\right\} = 0, \quad x\in\mathcal{S}.$$

and hence

$$\partial_t V(t,x) + \mathscr{H}(t,x,\partial_x V(t,x),\partial_x^2 V(t,x)) = 0, \quad x\in\mathcal{S}. \tag{1.5}$$

This is called the HJB equation.

Verification theorem

Using the HJB equation (1.5), we are able to find the value function by the methods of partial differential equations (PDEs). However a solution of the HJB equation is only a candidate of the value function. The verification theorem shows that the candidate is the value function under the proper assumptions. According to problem settings, there are various version of the verification theorem. Here we restrict the class of control strategies.

Theorem 1.1.7
Let $\phi(t,x)$ be a function which is C^1 in t and C^2 in x and $\mathcal{U}^M$ be a class of admissible strategy u such that

$$\int_0^t \partial_x\phi(s,X_s)^\top \sigma(s,X_s,u_s)dW_s \text{ is a martingale for all } t.$$

We set $\mathcal{U}(t,x) = \mathcal{U}^M$.

(i) Suppose that

$$\begin{aligned}
&\phi(t,x) \geq g_T(x), \qquad x \in \mathcal{S} \cup \partial\mathcal{S};\\
&\phi_{\tau_\mathcal{S}}(x) \geq g_{\partial\mathcal{S}}(x), \qquad x \in \partial\mathcal{S};\\
&-\partial_t\phi(t,x) - \mathscr{H}(t,x,\partial_x\phi(t,x) + \partial_x^2\phi(t,x)) \geq 0, \quad (t,x) \in [0,T)\times\mathcal{S}.
\end{aligned}$$

Then $\phi \geq V$ on $[0,T]\times\mathcal{S}\cup\partial\mathcal{S}$.

(ii) Suppose further that

- $\phi(t,x) = g_T(x), \qquad x \in \mathcal{S}\cup\partial\mathcal{S};$
- $\phi_{\tau_\mathcal{S}}(x) = g_{\partial\mathcal{S}}(x), \qquad x \in \partial\mathcal{S};$
- *there exists a function $\hat{\alpha}(t,x)$ which satisfies*

  $$-\partial_t\phi(t,x) - \mathscr{L}_t^{\hat{\alpha}(t,x)}V(t,x) - f(t,x,\hat{\alpha}(t,x)) = 0$$

  for all $(t,x) \in [0, T\wedge\tau_\mathcal{S})\times\mathcal{S}$;
- *the process $\hat{u} = \{\hat{\alpha}(t,x)\}_{t\in[0,T\wedge\tau_\mathcal{S})}$ is in $\mathcal{U}^M$.*

Then $\phi = V$ on $[0,T]\times\mathcal{S}\cup\partial\mathcal{S}$ and $\hat{u}$ is a optimal Markov strategy.

Proof. (i) Fix $u \in \mathcal{U}^M$. Applying the Ito's lemma to ϕ, we have

$$\begin{aligned}
&\phi(T\wedge\tau_\mathcal{S}, X_{T\wedge\tau_\mathcal{S}}) - \phi(t,X_t)\\
&\quad = \int_t^{T\wedge\tau_\mathcal{S}} \left(\partial_s\phi(s,X_s) + \mathscr{L}_s^{u_s}\phi(s,x)\right) ds + \int_t^{T\wedge\tau_\mathcal{S}} \partial_x\phi(s,X_s)\sigma(s,X_s,u_s)dW_s.
\end{aligned}$$

Taking the conditional expectation, we find

$$\begin{aligned}
&\phi(t,X_t)\\
&= \mathbb{E}\left[\int_t^{T\wedge\tau_\mathcal{S}} -\left(\partial_s\phi(s,X_s) + \mathscr{L}_s^{u_s}\phi(s,x)\right) ds \middle| X_t\right] + \mathbb{E}\left[\phi(T\wedge\tau_\mathcal{S}, X_{T\wedge\tau_\mathcal{S}})\middle| X_t\right],
\end{aligned}$$

where the stochastic integral term vanishes by the property of $\mathcal{U}^M$. By the assumptions in (i), we obtain

$$\begin{aligned}
&\phi(t, X_t) \\
&\geq \mathbb{E}\left[\int_t^{T\wedge\tau_{\mathcal{S}}} f(s, X_s, u_s)ds + g_{\partial\mathcal{S}}(\tau_{\mathcal{S}}, X_{\tau_{\mathcal{S}}})\mathbf{1}_{\{\tau_{\mathcal{S}}<T\}} + g_T(X_T)\mathbf{1}_{\{T<\tau_{\mathcal{S}}\}} \middle| X_t \right] \\
&= J^u(t, X_t).
\end{aligned}$$

The above discussion does not depend on u, and thus $\phi \geq V$.

(ii) Almost same as in (i). Fixing $u = \hat{u}$, applying the Ito's lemma to ϕ and taking the conditional expectation, we have

$$\begin{aligned}
&\phi(t, X_t) \\
= \quad & \mathbb{E}\left[\int_t^{T\wedge\tau_{\mathcal{S}}} -\left(\partial_s\phi(s, X_s) + \mathscr{L}_s^{\hat{\alpha}(s,X_s)}\phi(s, x)\right) ds \middle| X_t \right] + \mathbb{E}\left[\phi(T\wedge\tau_{\mathcal{S}}, X_{T\wedge\tau_{\mathcal{S}}}) \middle| X_t \right].
\end{aligned}$$

By the assumptions in (ii), we obtain

$$\begin{aligned}
&\phi(t, X_t) \\
&= \mathbb{E}\left[\int_t^{T\wedge\tau_{\mathcal{S}}} f(s, X_s, \hat{\alpha}(s, X_s))ds + g_{\partial\mathcal{S}}(\tau_{\mathcal{S}}, X_{\tau_{\mathcal{S}}})\mathbf{1}_{\{\tau_{\mathcal{S}}<T\}} + g_T(X_T)\mathbf{1}_{\{T<\tau_{\mathcal{S}}\}} \middle| X_t \right] \\
&= J^{\hat{u}}(t, X_t) \\
&= V(t, X_t).
\end{aligned}$$

□

Remark 1.1.8
The idea behind the restricted class $\mathcal{U}^M$ is the assumption (1.4). The advantage of this approach is that the derivation of HJB equation and the proof of verification theorem become quite simple.

Another approach which appears to be the most popular is imposing the quadratic growth condition to ϕ (instead of restricting $\mathcal{U}(t, x)$ to $\mathcal{U}^M$). This assumption is checked easier than that in theorem1.1.7. See e.g. Pham [Pha09] for the detail.

We summarize the way to obtain the optimal control strategy:

1. obtain candidates of the value function and optimal control from HJB eqation;

2. confirm that the candidates satisfy the assumptions in the verification theorem such as theorem1.1.7.

The approach using the verification theorem is called the classical approach.

1.2 Impulse control

In the regular stochastic control case, we control the drift and diffusion coefficients of the state process. We next consider an impulse control, a control which directly intervene the value of the state process. In this section, we discuss the impulse control problem on the finite time horizon and finite domain. The main story is almost the same for problems on other time horizons and domains.

Formulation

An impulse control strategy is consist of pairs of a stopping time and a random variable. Let $\{\tau_i\}_{i=0,1,\cdots,M+1}$ be a non-decreasing sequence of $\mathcal{F}_t$-stopping times, that is, $0 = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_{M+1}$ and $\{\zeta_i\}_{i=1,2,\cdots,M+1}$ be a sequence of $\mathcal{F}_{\tau_i}$-measurable $\mathcal{Z}$-valued random variables, where $M \in \mathbb{N}$ and $\mathcal{Z}$ is a compact subset of $\mathbb{R}^l$, $l \in \mathbb{N}$. We denote by v an impulse control strategy and it is defined by

$$v = (\tau_i, \zeta_i)_{i=1,\cdots,M+1}.$$

The controlled state process X_t is described by

$$\begin{aligned} dX_t &= \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \quad \tau_i < t < \tau_{i+1} \\ X_{\tau_i} &= \Gamma(X_{\tau_i^-}, \zeta_i) \end{aligned}$$

where $\Gamma : \mathcal{S} \cup \partial\mathcal{S} \times \mathcal{Z} \to \mathcal{S}$ is an intervention function. Here we assume that μ and σ satisfy the linear growth condition in $\mathcal{S}$, and are Lipschitz continuous on $[0, T]$ and $\mathcal{S}$. Then the controlled state process X_t exists and is unique. We futher assume that Γ is continuous on $\mathcal{S} \cup \partial\mathcal{S}$.

We evaluate a control strategy v by a scalar performance criterion $J^v(t, x)$:

$$\begin{aligned} J^v(t, x) =& \mathbb{E}\Bigg[\int_t^{T\wedge\tau_{\mathcal{S}}} f(s, X_s)ds + \sum_{i\in\{i:t<\tau_i<T\wedge\tau_{\mathcal{S}}\}} K(\tau_i^-, X_{\tau_i^-}, \zeta_i) \\ &+ g_{\partial\mathcal{S}}(\tau_{\mathcal{S}}, X_{\tau_{\mathcal{S}}})\mathbf{1}_{\{\tau_{\mathcal{S}}<T\}} + g_T(X_T)\mathbf{1}_{\{T<\tau_{\mathcal{S}}\}}\Big| X_t = x\Bigg] \end{aligned}$$

where $f : [0, T]\times\mathcal{S} \to \mathbb{R}$, $K : [0, T]\times\mathcal{S}\times\mathcal{Z} \to \mathbb{R}$, $g_T : \mathcal{S} \to \mathbb{R}$ and $g_{\partial\mathcal{S}} : (0, T)\times\partial\mathcal{S} \to \mathbb{R}$ are measurable functions which standing for a running profit, an intervention profit, terminal profit and a boundary profit respectively.

As same in the regular stochastic control case, we seek optimal strategy from admissible strategies. An impulse control strategy v is admissible if

- there exists an unique strong solution for the state process;
- the performance criterion is well-defined;
- $\tau_{M+1} > T$ a.s.

Remark 1.2.1
If $M = \infty$ the third condition of the above is modified to $\lim_{j\to\infty} \tau_j = \infty$. In Chapter 1 and Chapter 2, we consider the case that $M < \infty$.

We denote by $\mathcal{V}(t,x)$ a collection of impulse control strategies v such that

$$\mathbb{E}\left[\int_t^{T\wedge\tau_{\mathcal{S}}} |f(s,X_s)|ds \middle| X_t = x\right] < \infty,$$
$$\mathbb{E}\left[\sum_{i\in\{i:t<\tau_i<T\wedge\tau_{\mathcal{S}}\}} |K(\tau_i^-, X_{\tau_i^-}, \zeta_i)| \middle| X_t = x\right] < \infty,$$
$$\mathbb{E}\left[|g_T(X_T)| \middle| X_t = x\right] < \infty, \quad \mathbb{E}\left[|g_{\partial\mathcal{S}}(X_{\tau_{\mathcal{S}}})| \middle| X_t = x\right] < \infty.$$

The value function V is defined as follow:

$$V(t,x) = \sup_{v\in\mathcal{V}(t,x)} J^v(t,x).$$

Dynamic programming principle

As in the regular stochastic control case in previous section, the dynamic programming principle plays an important role. We suppose that there exists the optimal impulse control strategy $\hat{v}$, i.e.,

$$V(t,x) = J^{\hat{v}}(t,x), \quad (t,x) \in [0,T]\times\mathcal{S}\cup\partial\mathcal{S}.$$

Proposition 1.2.2
Let $\tilde{\tau}$ be a $\mathcal{F}_t$-stopping time on $[t, T\wedge\tau_{\mathcal{S}})$. Then the value function V satisfies

$$V(t,x) = g_T(x), \qquad x \in \mathcal{S}\cup\partial\mathcal{S};$$
$$V(\tau_{\mathcal{S}},x) = g_{\partial\mathcal{S}}(x), \qquad x \in \partial\mathcal{S};$$
$$V(t,x) = \sup_{v\in\mathcal{V}(t,x)} \mathbb{E}\left[\int_t^{\tilde{\tau}} f(s,X_s)ds + \sum_{i\in\{i|t<\tau_i\le\tilde{\tau}\}} K(\tau_i^-, X_{\tau_i^-}, \zeta_i) + V_{\tilde{\tau}}(x) \middle| X_t = x\right],$$
$$x \in \mathcal{S},\ 0 \le t \le \tilde{\tau}.$$

Proof. By the definition it is clear that

$$V(t,x) = \sup_{v\in\mathcal{V}(T,x)} J^v(t,x) = g_T(x), \qquad x \in \mathcal{S}\cup\partial\mathcal{S}$$

and

$$V(\tau_{\mathcal{S}},x) = \sup_{v\in\mathcal{V}(\tau_{\mathcal{S}},x)} J^v(\tau_{\mathcal{S}},x) = g_{\partial\mathcal{S}}(x), \qquad x \in \partial\mathcal{S}.$$

We see that

$$
\begin{aligned}
&J^v(\tilde{\tau}, X_{\tilde{\tau}}^{t,X_t}) - J^v(t, X_t) \\
&\quad = \mathbb{E}\Bigg[\int_{\tilde{\tau}}^{T\wedge\tau_{\mathcal{S}}} f(s, X_s)ds + \sum_{i\in\{i:\tilde{\tau}<\tau_i<T\wedge\tau_{\mathcal{S}}\}} K(\tau_i^-, X_{\tau_i^-}, \zeta_i) \\
&\qquad\qquad + g_{\partial\mathcal{S}}(\tau_{\mathcal{S}}, X_{\tau_{\mathcal{S}}})\mathbf{1}_{\{\tau_{\mathcal{S}}<T\}} + g_T(X_T)\mathbf{1}_{\{T<\tau_{\mathcal{S}}\}}\Big| X_{\tilde{\tau}}^{t,X_t}\Bigg] \\
&\quad\quad - \mathbb{E}\Bigg[\int_{t}^{T\wedge\tau_{\mathcal{S}}} f(s, X_s)ds + \sum_{i\in\{i:t<\tau_i<T\wedge\tau_{\mathcal{S}}\}} K(\tau_i^-, X_{\tau_i^-}, \zeta_i) \\
&\qquad\qquad + g_{\partial\mathcal{S}}(\tau_{\mathcal{S}}, X_{\tau_{\mathcal{S}}})\mathbf{1}_{\{\tau_{\mathcal{S}}<T\}} + g_T(X_T)\mathbf{1}_{\{T<\tau_{\mathcal{S}}\}}\Big| X_t\Bigg]
\end{aligned}
$$

for $0 \le t \le \tilde{\tau}$. Taking the conditional expectation and using the Markovian structure of X_t, we have

$$
\mathbb{E}\left[J^v(\tilde{\tau}, X_{\tilde{\tau}})\middle| X_t\right] - J^v(t, X_t) = \mathbb{E}\left[-\int_t^{\tilde{\tau}} f(s, X_s)ds - \sum_{i\in\{i:t<\tau_i\le\tilde{\tau}\}} K(\tau_i^-, X_{\tau_i^-}, \zeta_i)\middle| X_t\right]
$$

and thus

$$
J^v(t, X_t) = \mathbb{E}\left[\int_t^{\tilde{\tau}} f(s, X_s)ds + \sum_{i\in\{i:t<\tau_i\le\tilde{\tau}\}} K(\tau_i^-, X_{\tau_i^-}, \zeta_i) + J^v(\tilde{\tau}, X_{\tilde{\tau}})\middle| X_t\right].
$$

Choose v' to be a strategy that coincides with v on $[0, \tilde{\tau})$ and with $\hat{v}$ on $[\tilde{\tau}, T\wedge\tau_{\mathcal{S}}]$. Then

$$
\begin{aligned}
V(t, X_t) &\ge J^{v'}(t, X_t) \\
&= \mathbb{E}\left[\int_t^{\tilde{\tau}} f(s, X_s)ds + \sum_{i\in\{i:t<\tau_i\le\tilde{\tau}\}} K(\tau_i^-, X_{\tau_i^-}, \zeta_i) + J^{v'}(\tilde{\tau}, X_{\tilde{\tau}})\middle| X_t\right] \\
&= \mathbb{E}\left[\int_t^{\tilde{\tau}} f(s, X_s)ds + \sum_{i\in\{i:t<\tau_i\le\tilde{\tau}\}} K(\tau_i^-, X_{\tau_i^-}, \zeta_i) + V(\tilde{\tau}, X_{\tilde{\tau}})\middle| X_t\right],
\end{aligned}
$$

where we use that $J^v(s, x)$ only depends on v on $[0, T\wedge\tau_{\mathcal{S}}]$. Thus we obtain

$$
V(t, X_t) = \sup_{v\in\mathcal{V}(t,X_t)} \mathbb{E}\left[\int_t^{\tilde{\tau}} f(s, X_s)ds + \sum_{i\in\{i:t<\tau_i\le\tilde{\tau}\}} K(\tau_i^-, X_{\tau_i^-}, \zeta_i) + V(\tilde{\tau}, X_{\tilde{\tau}})\middle| X_t\right].
$$

$\square$

Hamilton-Jacobi-Bellman quasi-variational inequality

The infinitesimal version of the dynamic programming principle for the impulse control is called the Hamilton-Jacobi-Bellman quasi-variational inequality (HJBQVI). As in the regular control case, we derive it formally under strong smoothness assumptions.

Before the derivation, we introduce intervention operators $\mathscr{M}^z$, $z \in \mathcal{Z}$ and $\mathscr{M}$:

$$\mathscr{M}^z\phi(t,x) = \phi(t,\Gamma(x,z)) + K(t,x,z);$$
$$\mathscr{M}\phi(t,x) = \sup_{z\in\mathcal{Z}} \mathscr{M}^z\phi(t,x)$$

where $\phi : [0,T]\times\mathcal{S} \to \mathbb{R}$.

Suppose that $V(t,x)$ is C^1 in t and C^2 in x. Let us fix the strategy v and take $(\tilde{\tau},\tilde{\zeta}) = (\tau_i,\zeta_i)$ where $i = \min\{i|\tau_i \geq t,\ (\tau_i,\zeta_i)_{i=1,2,\cdots} = v\}$. By the dynamic programming principle, Proposition 1.2.2, we have

$$V(t,x) \geq \mathbb{E}\left[\int_t^{\tilde{\tau}} f(s,X_s)ds + K(\tilde{\tau}^-, X_{\tilde{\tau}^-},\tilde{\zeta}) + V(\tilde{\tau},X_{\tilde{\tau}})\middle| X_t = x\right]$$

The continuity of the value function gives

$$\begin{aligned} V(\tilde{\tau},X_{\tilde{\tau}}) &= V(\tilde{\tau},\Gamma(X_{\tilde{\tau}^-},\tilde{\zeta})) \\ &= V(\tilde{\tau}^-,\Gamma(X_{\tilde{\tau}^-},\tilde{\zeta})) \\ &= \mathscr{M}^{\tilde{\zeta}}V(\tilde{\tau}^-,X_{\tilde{\tau}^-}) - K(\tilde{\tau}^-,X_{\tilde{\tau}^-},\tilde{\zeta}) \end{aligned}$$

Hence we have

$$V(t,x) \geq \mathbb{E}\left[\int_t^{\tilde{\tau}} f(s,X_s)ds + \mathscr{M}^{\tilde{\zeta}}V(\tilde{\tau}^-,X_{\tilde{\tau}^-})\middle| X_t = x\right]. \tag{1.6}$$

This is true for arbitrary v and thus we obtain

$$V(t,x) = \sup_{(\tilde{\tau},\tilde{\zeta})}\mathbb{E}\left[\int_t^{\tilde{\tau}} f(s,X_s)ds + \mathscr{M}^{\tilde{\zeta}}V(\tilde{\tau}^-,X_{\tilde{\tau}^-})\middle| X_t = x\right].$$

On the other hand, applying the Ito's lemma to the value function and taking the conditional expectation, we find

$$\begin{aligned} V(t,X_t) =&\mathbb{E}\left[\int_t^{\tilde{\tau}^-} \{\partial_s V(s,X_s) + \mathscr{L}_s V(s,X_s)\}\, ds + V(\tilde{\tau}^-,X_{\tilde{\tau}^-})\middle| X_t\right] \\ &+ \mathbb{E}\left[\int_t^{\tilde{\tau}^-} \partial_x V(s,X_s)\sigma(s,X_s,u_s)dW_s\middle| X_t\right] \end{aligned}$$

As in the regular control case, if we additionally suppose that

$$\int_{\tau_i}^{\tau_{i+1}} \partial_x V(s,X_s)^\top\sigma(s,X_s,u_s)dW_s \text{ is a martingale for all } i.$$

then the second term of the above equation vanishes. Hence we have

$$V(t,X_t)=\mathbb{E}\left[\int_t^{\tilde{\tau}^-}-\{\partial_s V(s,X_s)+\mathscr{L}_s V(s,X_s)\}\,ds+V(\tilde{\tau}^-,X_{\tilde{\tau}^-})\middle|X_t\right] \tag{1.7}$$

Combining the inequality(1.6) and the equation(1.7), we find

$$\begin{aligned}0\leq&\mathbb{E}\left[\int_t^{\tilde{\tau}^-}-\left\{\partial_s V(s,X_s)+\mathscr{L}_s V(s,X_s)+f(s,X_s)\right\}ds\right.\\&\left.+V(\tilde{\tau}^-,X_{\tilde{\tau}^-})-\mathscr{M}^{\tilde{\zeta}}V(\tilde{\tau}^-,X_{\tilde{\tau}^-})\middle|X_t\right]\end{aligned}$$

Thus, at least formally, we have

$$\begin{aligned}-\partial_t V(t,X_t)-\mathscr{L}_t V(t,X_t)-f(t,X_t)&\geq 0,\quad t<\tilde{\tau},\\V(t,X_{t^-})-\mathscr{M}^{\tilde{\zeta}}V(t,X_{t^-})&\geq 0,\quad t=\tilde{\tau},\end{aligned}$$

where we use the continuity of the value function again: $V(t^-,\cdot)=V(t,\cdot)$. If $(\tilde{\tau},\tilde{\zeta})$ is optimal, the inequalities turn into equations. Hence using the pathwise nature of the above inequalities, we finally obtain the HJBQVI:

$$\min\left(-\partial_t V(t,x)-\mathscr{L}_t V(t,x)-f(t,x),V(t,x)-\mathscr{M}V(t,x)\right)=0. \tag{1.8}$$

Verification theorem

As the same for HJB equation, the HJBQVI(1.8) only gives a candidate of the value function. We here prove the verification theorem for impulse control with a restricted control strategy class.

Theorem 1.2.3
Let $\phi:[0,T]\times\mathcal{S}\cup\partial\mathcal{S}\to\mathbb{R}$ be a sufficiently smooth function to apply the Ito's lemma. We define $\mathcal{V}^M$ as a class of admissible impulse control strategy v such that

$$\int_{\tau_i}^{\tau_{i+1}}\partial_x\phi(s,X_s)^\top\sigma(s,X_s)dW_s \text{ is a martingale for all } i,$$

and set $\mathcal{V}(t,x)=\mathcal{V}^M$. Assume that f and K are bounded.

(i) Suppose that

$\phi(T,x)\geq g_T(x),\quad x\in\mathcal{S}\cup\partial\mathcal{S};$
$\phi(\tau_\mathcal{S},x)\geq g_{\partial\mathcal{S}}(x),\quad x\in\partial\mathcal{S};$
$\min\left(-\partial_t\phi(t,x)-\mathscr{L}_t\phi(t,x)-f(t,x),\phi(t,x)-\mathscr{M}\phi(t,x)\right)\geq 0,\quad (t,x)\in[0,T)\times\mathcal{S}.$

Then $\phi\geq V$ on $[0,T]\times\mathcal{S}\cup\partial\mathcal{S}$.

(ii) Suppose further that

- $\phi(T,x)\geq g_T(x),\quad x\in\mathcal{S}\cup\partial\mathcal{S},$

- $\phi(\tau_{\mathcal{S}}, x) \geq g_{\partial\mathcal{S}}(x), \quad x \in \partial\mathcal{S}$,
- *there exists a measurable function* $\hat{\zeta}(t,x)$ *which satisfies*
$$-\partial_t\phi(t,x) - \mathscr{L}_t\phi(t,x) - f(t,x) = 0 \quad x \in D_t$$
$$\phi(t,x) - \mathscr{M}^{\hat{\zeta}(t,x)}\phi(t,x) = 0 \quad x \in \mathcal{S} \setminus D_t,$$
where
$$D_t = \left\{ x \in \mathcal{S} | \phi(t,x) > \mathscr{M}^{\hat{\zeta}(t,x)}\phi(t,x) \right\}.$$
- *the strategy* $\hat{v}$ *defined below is in* $\mathcal{V}^M$:
$$\hat{v} = \left(\hat{\tau}_i, \hat{\zeta}(\tau_i^-, X_{\tau_i^-}) \right)_{i=1,2,\cdots}$$
where
$$\hat{\tau}_i = \inf\left\{ t > \hat{\tau}_{i-1} | X_t \notin D_t \right\}.$$

Then $\phi = V$ *on* $[0,T] \times \mathcal{S} \cup \partial\mathcal{S}$ *and* $\hat{v}$ *is an optimal impulse control strategy.*

Proof. By the boundedness of profit functions f and K, we are able to assume without loss of generality that f and K are both non-negative. [1]

(i) Fix a strategy $v \in \mathcal{V}^M$ let m be an index such that $m = \min\{m | \tau_m > t\}$, where $t < T \wedge \tau_{\mathcal{S}}$. Applying the Ito's lemma to ϕ, taking the conditional expectation and using the Markovian structure of X_t, we see

$$\begin{aligned}\mathbb{E}\left[\phi(\tau_m^-, X_{\tau_m^-}) \middle| X_t \right] - \phi(t, X_t) =& \mathbb{E}\left[\int_t^{\tau_m^-} \left\{ \partial_s\phi(s,X_s) + \mathscr{L}_s\phi(s,X_s) \right\} ds \middle| X_t \right] \\ &+ \mathbb{E}\left[\int_t^{\tau_m^-} \partial_x\phi(s,X_s)^\top \sigma(s,X_s)\phi(s,X_s) dW_s \middle| X_t \right]\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}\left[\phi(\tau_{i+1}^-, X_{\tau_{i+1}^-}) \middle| X_t \right] - \mathbb{E}\left[\phi(\tau_i, X_{\tau_i}) \middle| X_t \right] \\ &= \mathbb{E}\left[\int_{\tau_i}^{\tau_{i+1}^-} \left\{ \partial_s\phi(s,X_s) + \mathscr{L}_s\phi(s,X_s) \right\} ds \middle| X_t \right] \\ &\quad + \mathbb{E}\left[\int_{\tau_i}^{\tau_{i+1}^-} \partial_x\phi(s,X_s)^\top \sigma(s,X_s)\phi(s,X_s) dW_s \middle| X_t \right], \quad i \geq m.\end{aligned}$$

Here we implicitly suppose that $\tau_{i+1} < T \wedge \tau_{\mathcal{S}}$, $i \geq m$. By the definition of $\mathcal{V}^M$, the expectations for the stochastic integrals vanish. Summing up i from m to $j-1$, we have

$$\begin{aligned}&\mathbb{E}\left[\phi(\tau_j^-, X_{\tau_j^-}) \middle| X_t \right] + \mathbb{E}\left[\sum_{i=m+1}^{j-1} \phi(\tau_i^-, X_{\tau_i^-}) - \phi(\tau_i, X_{\tau_i}) \middle| X_t \right] - \mathbb{E}\left[\phi(\tau_m, X_{\tau_m}) \middle| X_t \right] \\ &= \mathbb{E}\left[\int_{\tau_m}^{\tau_j^-} \left\{ \partial_s\phi(s,X_s) + \mathscr{L}_s\phi(s,X_s) \right\} ds \middle| X_t \right].\end{aligned}$$

[1] We always able to accomplish this by shifting the profits by constants.

Thus we obtain

$$\begin{aligned}
&\mathbb{E}\left[\phi(\tau_j^-, X_{\tau_j^-})\middle| X_t\right] + \mathbb{E}\left[\sum_{i=m}^{j-1}\phi(\tau_i^-, X_{\tau_i^-}) - \phi(\tau_i, X_{\tau_i})\middle| X_t\right] - \phi(t, X_t)\\
&= \mathbb{E}\left[\int_t^{\tau_j^-}\left\{\partial_s\phi(s, X_s) + \mathscr{L}_s\phi(s, X_s)\right\} ds\middle| X_t\right],
\end{aligned}$$

and equivalently

$$\begin{aligned}
\phi(t, X_t) =&\mathbb{E}\left[\int_t^{\tau_j^-} - \left\{\partial_s\phi(s, X_s) + \mathscr{L}_s\phi(s, X_s)\right\} ds\right.\\
&\left. + \sum_{i=m}^{j-1}\left\{\phi(\tau_i^-, X_{\tau_i^-}) - \phi(\tau_i, X_{\tau_i})\right\} + \phi(\tau_j^-, X_{\tau_j^-})\middle| X_t\right]. \qquad (1.9)
\end{aligned}$$

On the other hand, assumptions in the statement give

$$-\partial_t\phi(t, x) - \mathscr{L}_t\phi(t, x) \geq f(t, x)$$

and

$$\begin{aligned}
\phi(\tau_i^-, X_{\tau_i^-}) - \phi(\tau_i, X_{\tau_i}) &= \phi(\tau_i^-, X_{\tau_i^-}) - \phi(\tau_i^-, \Gamma(X_{\tau_i^-}, \zeta_i))\\
&\geq K(\tau_i^-, X_{\tau_i^-}, \zeta_i)
\end{aligned}$$

where we use the continuity of ϕ for the first line of the second equation. Hence we have

$$\phi(t, X_t) \geq \mathbb{E}\left[\int_t^{\tau_j^-} f(s.X_s)ds + \sum_{i=m}^{j-1} K(\tau_i^-, X_{\tau_i^-}, \zeta_i) + \phi(\tau_j^-, X_{\tau_j^-})\middle| X_t\right].$$

Let us remove the assumption that $\tau_{i+1} < T \wedge \tau_{\mathcal{S}}$, $i \geq m$. Then the above inequality is rewritten by

$$\begin{aligned}
\phi(t, X_t) \geq&\mathbb{E}\left[\int_t^{\tau_j^- \wedge T\wedge\tau_{\mathcal{S}}} f(s.X_s)ds + \sum_{i\in\left\{i \middle| t<\tau_i<\tau_j^-\wedge T\wedge\tau_{\mathcal{S}}\right\}} K(\tau_i^-, X_{\tau_i^-}, \zeta_i)\right.\\
&\left. +\phi(\tau_j^- \wedge T \wedge \tau_{\mathcal{S}}, X_{\tau_j^-\wedge T\wedge\tau_{\mathcal{S}}})\middle| X_t\right].
\end{aligned}$$

The admissibility of the strategy v gives

$$\lim_{j\to\infty} \tau_j^- \wedge T \wedge \tau_{\mathcal{S}} = T \wedge \tau_{\mathcal{S}}.$$

We also find that

$$\lim_{j\to\infty}\int_t^{\tau_j^-\wedge T\wedge\tau_{\mathcal{S}}} f(s.X_s)ds = \int_t^{T\wedge\tau_{\mathcal{S}}} f(s.X_s)ds,$$

by the monotone convergence theorem. Hence taking the limit $j \to \infty$, we obtain

$$\begin{aligned}
\phi(t, X_t) &\geq \mathbb{E}\left[\int_t^{T\wedge\tau_{\mathcal{S}}} f(s.X_s)ds + \sum_{i\in\{i \,|t<\tau_i<T\wedge\tau_{\mathcal{S}}\}} K(\tau_i^-, X_{\tau_i^-}, \zeta_i) + \phi(T\wedge\tau_{\mathcal{S}}, X_{T\wedge\tau_{\mathcal{S}}}) \middle| X_t \right] \\
&\geq \mathbb{E}\left[\int_t^{T\wedge\tau_{\mathcal{S}}} f(s.X_s)ds + \sum_{i\in\{i \,|t<\tau_i<T\wedge\tau_{\mathcal{S}}\}} K(\tau_i^-, X_{\tau_i^-}, \zeta_i) \right. \\
&\qquad\qquad \left. + g_{\partial\mathcal{S}}(\tau_{\mathcal{S}}, X_{\tau_{\mathcal{S}}})\mathbf{1}_{\{\tau_{\mathcal{S}}<T\}} + g_T(X_T)\mathbf{1}_{\{T<\tau_{\mathcal{S}}\}} \middle| X_t \right] \\
&= J^v(t, X_t).
\end{aligned}$$

The above discussion does not depend on the specific v and hence $\phi \geq V$ on $[0, T] \times \mathcal{S} \cup \partial\mathcal{S}$.

(ii) Fix the strategy $\hat{v}$. The discussion is the same as in (i) until obtaining equation (1.9). Note that assumptions of the statement for (ii) gives

$$\begin{aligned}
&-\partial_t\phi(t, x) - \mathscr{L}_t\phi(t, x) = f(t, x), \\
&\phi(\hat{\tau}_i^-, X_{\hat{\tau}_i^-}) - \phi(\hat{\tau}_i, X_{\hat{\tau}_i}) = \phi(\hat{\tau}_i^-, X_{\hat{\tau}_i^-}) - \phi\left(\hat{\tau}_i^-, \Gamma(X_{\hat{\tau}_i^-}, \hat{\zeta}(\hat{\tau}_i^-, X_{\hat{\tau}_i^-}))\right) \\
&\qquad\qquad\qquad\qquad\qquad = K\left(\hat{\tau}_i^-, X_{\hat{\tau}_i^-}, \hat{\zeta}(\hat{\tau}_i^-, X_{\hat{\tau}_i^-})\right),
\end{aligned}$$

where we use the continuity of ϕ again. Thus we have

$$\phi(t, X_t) = \mathbb{E}\left[\int_t^{\tau_j^-} f(s.X_s)ds + \sum_{i=m}^{j-1} K\left(\hat{\tau}_i^-, X_{\hat{\tau}_i^-}, \hat{\zeta}(\hat{\tau}_i^-, X_{\hat{\tau}_i^-})\right) + \phi(\tau_j^-, X_{\tau_j^-}) \middle| X_t \right].$$

The following discussion is almost same as in (i). Removing the assumption $\tau_{i+1} < T\wedge\tau_{\mathcal{S}}$, $i \geq m$, the above equation is rewritten by

$$\begin{aligned}
\phi(t, X_t) =&\mathbb{E}\left[\int_t^{\tau_j^-\wedge T\wedge\tau_{\mathcal{S}}} f(s.X_s)ds + \sum_{i\in\left\{i\,\middle|t<\tau_i<\tau_j^-\wedge T\wedge\tau_{\mathcal{S}}\right\}} K\left(\hat{\tau}_i^-, X_{\hat{\tau}_i^-}, \hat{\zeta}(\hat{\tau}_i^-, X_{\hat{\tau}_i^-})\right) \right. \\
&\qquad \left. + \phi(\tau_j^- \wedge T\wedge\tau_{\mathcal{S}}, X_{\tau_j^-\wedge T\wedge\tau_{\mathcal{S}}}) \middle| X_t \right].
\end{aligned}$$

Using the admissibility of $\hat{v}$, monotone convergence theorem and assumptions on the statement and taking the limit $j \to \infty$, we obtain

$$
\begin{aligned}
\phi(t, X_t) &= \mathbb{E}\left[\int_t^{T\wedge\tau_{\mathcal{S}}} f(s.X_s)ds + \sum_{i\in\{i\,|t<\tau_i<T\wedge\tau_{\mathcal{S}}\}} K\left(\hat{\tau}_i^-, X_{\hat{\tau}_i^-}, \hat{\zeta}(\hat{\tau}_i^-, X_{\hat{\tau}_i^-})\right)\right. \\
&\qquad \left. +\phi(T\wedge\tau_{\mathcal{S}}, X_{T\wedge\tau_{\mathcal{S}}})\middle| X_t\right] \\
&= \mathbb{E}\left[\int_t^{T\wedge\tau_{\mathcal{S}}} f(s.X_s)ds + \sum_{i\in\{i\,|t<\tau_i<T\wedge\tau_{\mathcal{S}}\}} K\left(\hat{\tau}_i^-, X_{\hat{\tau}_i^-}, \hat{\zeta}(\hat{\tau}_i^-, X_{\hat{\tau}_i^-})\right)\right. \\
&\qquad \left. +g_{\partial\mathcal{S}}(\tau_{\mathcal{S}}, X_{\tau_{\mathcal{S}}})\mathbf{1}_{\{\tau_{\mathcal{S}}<T\}} + g_T(X_T)\mathbf{1}_{\{T<\tau_{\mathcal{S}}\}}\middle| X_t\right] \\
&= J^{\hat{v}}(t, X_t) \\
&= V(t, X_t).
\end{aligned}
$$

□

1.3 Combined stochastic control

We are able to combine regular stochastic control and impulse control. We denote by $w = (u, v)$ a combined control strategy where u is a regular control strategy and v is an impulse control strategy. We use the strategy u between the intervention times $\{\tau_i\}_{1\le i\le M}$, that is, we are not able to use both of regular and impulse control strategies at the same time. The controlled state process X_t is described by

$$
\begin{aligned}
dX_t &= \mu(t, X_t, u_t)dt + \sigma(t, X_t, u_t)dW_t, \quad \tau_i < t < \tau_{i+1} \\
X_{\tau_i} &= \Gamma(X_{\tau_i^-}, \zeta_i),
\end{aligned}
$$

where functions μ, σ and Γ satisfy the conditions imposed in the previous sections. We deal with the performance criterion $J^w(t, x)$ as follow:

$$
\begin{aligned}
J^w(t, x) =&\mathbb{E}\left[\int_t^{T\wedge\tau_{\mathcal{S}}} f(s, X_s, u_s)ds + \sum_{i\in\{i:t<\tau_i<T\wedge\tau_{\mathcal{S}}\}} K(\tau_i^-, X_{\tau_i^-}, \zeta_i)\right. \\
&\qquad \left. +g_{\partial\mathcal{S}}(\tau_{\mathcal{S}}, X_{\tau_{\mathcal{S}}})\mathbf{1}_{\{\tau_{\mathcal{S}}<T\}} + g_T(X_T)\mathbf{1}_{\{T<\tau_{\mathcal{S}}\}}\middle| X_t = x\right].
\end{aligned}
$$

The profit functions $f, K, g_{\partial\mathcal{S}}$ and g_T also satisfy the conditions imposed in the previous sections. We say the combined control w is admissible if

- there exists an unique strong solution for the state process;
- the performance criterion is well-defined;
- $\tau_{M+1} > T$ a.s.

We denote by $\mathcal{W}(t,x)$ a collection of combined control strategies $w = (u, v)$ such that

$$\mathbb{E}\left[\int_t^{T\wedge\tau_{\mathcal{S}}} |f(s, X_s, u_s)|ds \middle| X_t = x\right] < \infty, \quad u \in \mathcal{U}_0,$$
$$\mathbb{E}\left[\sum_{i\in\{i:t<\tau_i<T\wedge\tau_{\mathcal{S}}\}} |K(\tau_i^-, X_{\tau_i^-}, \zeta_i)| \middle| X_t = x\right] < \infty,$$
$$\mathbb{E}\left[|g_T(X_T)| \middle| X_t = x\right] < \infty, \quad \mathbb{E}\left[|g_{\partial\mathcal{S}}(X_{\tau_{\mathcal{S}}})| \middle| X_t = x\right] < \infty.$$

The value function V is defined as follow:

$$V(t,x) = \sup_{w\in\mathcal{W}(t,x)} J^w(t,x).$$

The dynamic programming principle for the combined stochastic control is proved in the same manner as the regular and impulse control cases. Suppose that there exists the optimal combined control strategy $\hat{w}$, i.e.,

$$V(t,x) = J^{\hat{w}}(t,x), \quad (t,x) \in [0,T]\times\mathcal{S}\cup\partial\mathcal{S}.$$

Proposition 1.3.1
Let $\tilde{\tau}$ be a $\mathcal{F}_t$-stopping time on $[t, T\wedge\tau_{\mathcal{S}})$. Then the value function V satisfies

$$V(t,x) = g_T(x), \qquad x \in \mathcal{S}\cup\partial\mathcal{S};$$
$$V(\tau_{\mathcal{S}},x) = g_{\partial\mathcal{S}}(x), \qquad x \in \partial\mathcal{S};$$
$$V(t,x) = \sup_{w\in\mathcal{W}(t,x)} \mathbb{E}\left[\int_t^{\tilde{\tau}} f(s, X_s, u_s)ds + \sum_{i\in\{i|t<\tau_i\le\tilde{\tau}\}} K(\tau_i^-, X_{\tau_i^-}, \zeta_i) + V_{\tilde{\tau}}(x) \middle| X_t = x\right],$$
$$x \in \mathcal{S},\ 0 \le t \le \tilde{\tau}.$$

Proof. It is proved in the same manner as Proposition 1.1.6 and Proposition 1.2.2. □

The corresponding HJBQVI is given by

$$\min\left(-\partial_t V(t,x) - \mathcal{H}(t,x,\partial_x V(t,x), \partial_x^2 V(t,x)), V(t,x) - \mathcal{M}V(t,x)\right) = 0, \tag{1.10}$$

where the Hamiltonian $\mathcal{H}$ is defined by

$$\mathcal{H}(t,x,p,A) = \sup_{u\in\mathbb{U}}\left[\mu(t,x,u)^\top p + \frac{1}{2}\mathrm{Tr}(a(t,x,u)A) + f(t,x,u)\right]$$

where $a = \sigma\sigma^\top$ and the intervention operator $\mathcal{M}$ is given by

$$\mathcal{M}\phi(t,x) = \sup_{z\in\mathcal{Z}} \phi(t,\Gamma(x,z)) + K(t,x,z)$$

for $\phi : [0,T]\times\bar{\mathcal{S}} \to \mathbb{R}$.

The verification theorem is also proved in the same manner as the regular and impulse control cases.

Theorem 1.3.2
Let $\phi : [0,T] \times \mathcal{S} \cup \partial\mathcal{S} \to \mathbb{R}$ be a sufficiently smooth function to apply the Ito's lemma. We define $\mathcal{W}^M$ as a class of admissible impulse control strategy v such that

$$\int_{\tau_i}^{\tau_{i+1}} \partial_x \phi(s, X_s, u_s)^\top \sigma(s, X_s) dW_s \text{ is a martingale for all } i,$$

and set $\mathcal{W}(t,x) = \mathcal{W}^M$. Assume that f and K are bounded.
(i) Suppose that

$$\begin{aligned}
&\phi(T,x) \geq g_T(x), \quad x \in \mathcal{S} \cup \partial\mathcal{S};\\
&\phi(\tau_{\mathcal{S}},x) \geq g_{\partial\mathcal{S}}(x), \quad x \in \partial\mathcal{S};\\
&\min\left(-\partial_t\phi(t,x) - \mathscr{L}_t\phi(t,x) - f(t,x), \phi(t,x) - \mathscr{M}\phi(t,x)\right) \geq 0, \quad (t,x) \in [0,T) \times \mathcal{S}.
\end{aligned}$$

Then $\phi \geq V$ on $[0,T] \times \mathcal{S} \cup \partial\mathcal{S}$.
(ii) Suppose further that

- $\phi(T,x) \geq g_T(x), \quad x \in \mathcal{S} \cup \partial\mathcal{S},$
- $\phi(\tau_{\mathcal{S}},x) \geq g_{\partial\mathcal{S}}(x), \quad x \in \partial\mathcal{S},$
- *there exists a measurable functions $\hat{\alpha}(t,x)$ $\hat{\zeta}(t,x)$ which satisfies*

$$\begin{aligned}
&-\partial_t\phi(t,x) - \mathscr{L}_t^{\hat{\alpha}(t,x)}\phi(t,x) - f(t,x) = 0 \quad x \in D_t\\
&\phi(t,x) - \mathscr{M}^{\hat{\zeta}(t,x)}\phi(t,x) = 0 \quad x \in \mathcal{S} \setminus D_t,
\end{aligned}$$

  where
$$D_t = \left\{x \in \mathcal{S} | \phi(t,x) > \mathscr{M}^{\hat{\zeta}(t,x)}\phi(t,x)\right\}.$$

- *the process $\hat{u} = \{\hat{\alpha}(t,x)\}_{t\in[0,T\wedge\tau_{\mathcal{S}})}$ is in $\mathcal{U}^M$,*
- *the strategy $\hat{v}$ defined below is in $\mathcal{V}^M$:*

$$\hat{v} = \left(\hat{\tau}_i, \hat{\zeta}(\tau_i^-, X_{\tau_i^-})\right)_{i=1,2,\cdots}$$

  where
$$\hat{\tau}_i = \inf\{t > \hat{\tau}_{i-1} | X_t \notin D_t\}.$$

Then $\phi = V$ on $[0,T] \times \mathcal{S} \cup \partial\mathcal{S}$ and $\hat{v}$ is an optimal impulse control strategy.

Proof. It is proved in the same manner as Theorem 1.1.7 and Theorem 1.2.3. □

1.4 Viscosity Solutions

The verification theorems in previous sections strongly depend on the smoothness assumption. However since both HJB equation and HJBQVI are fully non-linear PDE, smooth solutions are rarely available. An alternative approach instead of the classical verification approach is employing the notion of viscosity solutions. Viscosity solutions are interpreted as weak solutions of PDE.

Let us consider a second order PDE

$$F(x, V(x), \partial_x V(x), \partial_x^2 V(x)) = 0 \quad \text{for } x \in \mathcal{S} \tag{1.11}$$

where $F : \mathcal{S} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R}$ is a continuous map which satisfies the ellipticity condition

$$F(x, r, p, A) \leq F(x, r, p, B), \quad A \geq B. \tag{1.12}$$

A viscosity solution of (2.1) is defined as follow.

Definition 1.4.1
Let $V \in \bar{C}(\mathcal{S})$.
(i) V is a viscosity supersolution of (2.1) on $\mathcal{S}$ if

$$F(x_0, V(x_0), \partial_x \phi(x_0), \partial_x^2 \phi(x_0)) \geq 0$$

for all pairs $(x_0, \phi) \in \mathcal{S} \times C^2(\mathcal{S})$ such that x_0 is a minimizer of $V - \phi$ on $\mathcal{S}$.
(ii) V is a viscosity subsolution of (2.1) on $\mathcal{S}$ if

$$F(x_0, V(x_0), \partial_x \phi(x_0), \partial_x^2 \phi(x_0)) \leq 0$$

for all pairs $(x_0, \phi) \in \mathcal{S} \times C^2(\mathcal{S})$ such that x_0 is a maximizer of $V - \phi$ on $\mathcal{S}$.
(iii) V is a viscosity solution of (2.1) on $\mathcal{S}$ if it is both a viscosity supersolution and subsolution of (2.1).

We remark that both the HJB equation (1.5) and the HJBQVI (1.8) satisfy the condition (1.12). It is enough to show that the Hamiltonian satisfies the condition since the second order derivative is only contained in the regular control part of HJBQVI.

We introduce the definition of viscosity solutions because the numerical schemes discussed in the following chapters converge to viscosity solutions. However we dose not address the detail of viscosity solutions themselves. We refer Crandall et. al. [CIL92] for the general theory of viscosity solution and Pham [Pha09], Øksendal and Sulem [ØS07] and Fleming and Soner [FS06] for the relation of stochastic controls and viscosity solutions.

Connection between the classical and viscosity solutions

We mention why Definition1.4 is regarded as an extension of the classical solution of the PDE(2.1). Let us recall the definition of the classical solution of the PDE(2.1).

Definition 1.4.2
A function $V : \mathcal{S} \to \mathbb{R}$ is a classical supersolution of the PDE(2.1) if $V \in C^2(\mathcal{S})$ and

$$F(x, V(x), \partial_x V(x), \partial_x^2 V(x)) \geq 0, \quad x \in \mathcal{S}.$$

A function $V : \mathcal{S} \to \mathbb{R}$ is a classical subsolution of the PDE(2.1) if $V \in C^2(\mathcal{S})$ and

$$F(x, V(x), \partial_x V(x), \partial_x^2 V(x)) \leq 0, \quad x \in \mathcal{S}.$$

The notion of viscosity solutions is motivated by the following result.

Proposition 1.4.3
The following claims are equivalent.
(i) V is a classical supersolution (resp. subsolution) of the PDE(2.1).
(ii) Let $V \in C^2(\mathcal{S})$. For all pairs $(x_0, \phi) \in \mathcal{S} \times C^2(\mathcal{S})$ such that x_0 is a minimizer (resp. maximizer) of $V - \phi$ on $\mathcal{S}$, we have

$$F(x_0, V(x_0), \partial_x \phi(x_0), \partial_x^2 \phi(x_0)) \geq 0,$$
$$\left(\text{resp. } F(x_0, V(x_0), \partial_x \phi(x_0), \partial_x^2 \phi(x_0)) \leq 0\ \right).$$

Proof. Let us consider the supersolution. We first prove the statement (ii) from the statement (i). Since x_0 is a minimizer of $V - \phi$ we have

$$\partial_x (V - \phi)(x_0) = 0 \quad \text{and} \quad \partial_x^2 (V - \phi)(x_0) \geq 0$$

which are equivalent to

$$\partial_x V(x_0) = \partial_x \phi(x_0) \quad \text{and} \quad \partial_x^2 V(x_0) \geq \partial_x^2 \phi(x_0).$$

Hence the ellipticity condition of F implies that

$$F(x_0, V(x_0), \partial_x V(x_0), \partial_x^2 V(x_0)) \leq F(x_0, V(x_0), \partial_x \phi(x_0), \partial_x^2 \phi(x_0)).$$

Since V is classical supersolution, we obtain

$$0 \leq F(x_0, V(x_0), \partial_x \phi(x_0), \partial_x^2 \phi(x_0)).$$

which establishes the claim.

The proof of the statement (i) from the statement (ii) is almost obvious. Taking $\phi = V$ and the minimizer of $V - \phi$ is attained at any points in $\mathcal{S}$. Hence by the assumptions of the statement(ii) we have

$$0 \leq F(x_0, V(x_0), \partial_x \phi(x_0), \partial_x^2 \phi(x_0)) = F(x_0, V(x_0), \partial_x V(x_0), \partial_x^2 V(x_0)).$$

We are able to prove the subsolution case as in the same manner. □

The statement (ii) in Proposition1.4.3 is regarded as the another version of the definition of a classical solution. We find that if the smoothness assumption is weaken the statement coincides with the definition of the viscosity solution. Hence we are able to regard a viscosity solution as the extension of a classical solution.

2 Convergence Proof of Finite Difference Scheme for Hamilton-Jacobi-Bellman equations

The purposes of this chapter are construction of the finite difference method (FDM) for the Hamilton-Jacobi-Bellman equations and its convergence proof. In Section 2.1 we mention the definition of the finite difference approximation and check accuracies of the approximation for several finite difference operators. Section 2.2 explains the Bales-Souganidis framework which is a powerful tools to prove the convergence of approximation schemes for partial differential equations (PDEs). The monotonicity, stability and consistency of numerical schemes are play a central role in this framework. An illustrative example using the Bales-Souganidis framework is served in Section 2.3. We prove the monotonicity, stability and consistency of the approximation scheme of the 1-dimensional Hamilton-Jacobi-Bellman equation. In Section 2.4 we investigate the convergence of the multi dimensional Hamilton-Jacobi-Bellman quasi-variational inequality.

2.1 Formulation

Let us consider a sufficiently smooth function $f : \mathbb{R} \to \mathbb{R}$. Roughly speaking, FDM is an approximation method of the derivatives of f by the values of f on the finite grid points with grid size $\delta > 0$.

Definition 2.1.1
We define the first order finite difference operators Δ^+, Δ^-, Δ^c and the second order finite difference operator Δ^2 as follows:

$$\begin{aligned}
\Delta^+ f(x) &= \frac{f(x+\delta) - f(x)}{\delta}, \\
\Delta^- f(x) &= \frac{f(x) - f(x-\delta)}{\delta}, \\
\Delta^c f(x) &= \frac{f(x+\delta) - f(x-\delta)}{2\delta}, \\
\Delta^2 f(x) &= \frac{f(x+\delta) - 2f(x) + f(x-\delta)}{\delta^2}.
\end{aligned}$$

The differences $\Delta^+ f$, $\Delta^- f$ and $\Delta^c f$ are called the forward difference, backward difference and central difference, respectively.

It is easily checked that

$$\begin{aligned}\Delta^+ f(x) &= f'(x) + O(\delta),\\ \Delta^- f(x) &= f'(x) + O(\delta),\\ \Delta^c f(x) &= f'(x) + O(\delta^2),\\ \Delta^2 f(x) &= f''(x) + O(\delta^2),\end{aligned}$$

as $\delta \to 0$. For instance, by the Taylor expansion, we have

$$\begin{aligned}\Delta^c f(x) =& \frac{1}{2\delta}\left\{\left(f(x) + f'(x)\delta + \frac{1}{2}f''(x)\delta^2 + \frac{1}{6}f^{(3)}(x+\theta_+\delta)\delta^3\right)\right.\\ &\left. - \left(f(x) - f'(x)\delta + \frac{1}{2}f''(x)\delta^2 - \frac{1}{6}f^{(3)}(x-\theta_-\delta)\delta^3\right)\right\}\\ =& f'(x) + \frac{1}{12}\left(f^{(3)}(x+\theta_+\delta) + f^{(3)}(x+\theta_-\delta)\right)\delta^2\\ =& f'(x) + O(\delta^2),\end{aligned}$$

where $\theta_+, \theta_- \in (0,1)$.

We next consider the n-dimensional case: $f : \mathbb{R}^n \to \mathbb{R}$, $n \in \mathbb{N}$.

Definition 2.1.2
We define finite difference operators as follows:

$$\begin{aligned}\Delta_i^+ f(x) &= \frac{f(x+\delta_i e_i) - f(x)}{\delta_i},\\ \Delta_i^- f(x) &= \frac{f(x) - f(x-\delta_i e_i)}{\delta_i},\\ \Delta_i^c f(x) &= \frac{f(x+\delta_i e_i) - f(x-\delta_i e_i)}{2\delta_i},\\ \Delta_{ij}^+ f(x) &= \frac{1}{2\delta_i\delta_j}\left[2f(x) + f(x+\delta_i e_i + \delta_j e_j) + f(x-\delta_i e_i - \delta_j e_j)\right]\\ &\quad - \frac{1}{2\delta_i\delta_j}\left[f(x+\delta_i e_i) + f(x-\delta_i e_i) + f(x+\delta_j e_j) + f(x-\delta_j e_j)\right],\\ \Delta_{ij}^- f(x) &= \frac{1}{2\delta_i\delta_j}\left[f(x+\delta_i e_i) + f(x-\delta_i e_i) + f(x+\delta_j e_j) + f(x-\delta_j e_j)\right]\\ &\quad - \frac{1}{2\delta_i\delta_j}\left[2f(x) + f(x+\delta_i e_i - \delta_j e_j) + f(x-\delta_i e_i + \delta_j e_j)\right],\\ & i,j = 1,\cdots,n,\end{aligned}$$

where $\delta_i > 0$ is a step size for x_i and $\{e_1,\cdots,e_n\}$ is the standard basis for $\mathbb{R}^n$.

Remark 2.1.3
In the case of $i = j$, Δ^-_{ii} is preferable to Δ^+_{ii} when we solve second order PDEs. We need to the information of the 1 step distant points from boundary because Δ^+_{ii} contains the 2 step distant point from x:

$$\begin{aligned}\Delta^+_{ii}f(x) = &\frac{1}{2\delta_i\delta_j}\left[2f(x) + f(x+2\delta_i e_i) + f(x-2\delta_i e_i)\right]\\ &-\frac{1}{\delta_i^2}\left[f(x+\delta_i e_i) + f(x-\delta_i e_i)\right].\end{aligned}$$

On the other hand, Δ^-_{ii} is the same as the second order operator in the 1-dimensional case:

$$\Delta^-_{ij}f(x) = \frac{1}{\delta_i^2}\left[f(x+\delta_i e_i) - 2f(x) + f(x-\delta_i e_i)\right].$$

We check the approximation accuracy of $\Delta^+_{ij}f(x)$. For the sake of simply notation, let us set $n = 2$ and denote by

$$\begin{aligned}\Delta^+_{xy}f(x,y) = &\frac{1}{2\delta_x\delta_y}\left[2f(x,y) + f(x+\delta_x, y+\delta_y) + f(x-\delta_x, y-\delta_y)\right]\\ &-\frac{1}{2\delta_x\delta_y}\left[f(x+\delta_x,y) + f(x-\delta_x,y) + f(x,y+\delta_y) + f(x,y-\delta_y)\right].\end{aligned}$$

The Taylor expansion gives

$$\begin{aligned}&f(x+\delta_x, y+\delta_y) + f(x-\delta_x, y-\delta_y)\\ &\quad= 2f(x,y) + \frac{2}{2!}\left(\delta_x\partial_x + \delta_y\partial_y\right)^2 f(x,y)\\ &\qquad+\frac{2}{4!}\left(\delta_x\partial_x + \delta_y\partial_y\right)^4\left(f(x+\theta^{++}\delta_y, y+\theta^{++}\delta_y) + f(x+\theta^{--}\delta_y, y+\theta^{--}\delta_y)\right),\\ &f(x+\delta_x,y) + f(x-\delta_x,y)\\ &\quad= 2f(x,y) + \frac{2}{2!}\delta_x^2\partial_{xx}f(x,y) + \frac{2}{4!}\delta_x^4\partial_{xxxx}\left(f(x+\theta_x^+\delta_x,y) + f(x+\theta_x^-\delta_x,y)\right),\\ &f(x,y+\delta_y) + f(x,y-\delta_y)\\ &\quad= 2f(x,y) + \frac{2}{2!}\delta_y^2\partial_{yy}f(x,y) + \frac{2}{4!}\delta_y^4\partial_{yyyy}\Big(f(x,y+\theta_y^+\delta_y) + f(x,y+\theta_y^-\delta_y)\Big),\end{aligned}$$

where $\theta^{++}, \theta^{--}, \theta_x^+, \theta_x^-, \theta_y^+, \theta_y^- \in (0,1)$. Hence we obtain

$$\begin{aligned}\Delta^+_{xy}f(x,y) =&\partial_{xy}f(x,y)+\\ &+\frac{1}{4!\delta_x\delta_y}\Big(\left(\delta_x\partial_x + \delta_y\partial_y\right)^4\left(f(x+\theta^{++}\delta_x, y+\theta^{++}\delta_y) + f(x+\theta^{--}\delta_x, y+\theta^{--}\delta_y)\right)\\ &\qquad-\delta_x^4\partial_{xxxx}\left(f(x+\theta_x^+\delta_x,y) + f(x+\theta_x^-\delta_x,y)\right)\\ &\qquad-\delta_y^4\partial_{yyyy}\Big(f(x,y+\theta_y^+\delta_y) + f(x,y+\theta_y^-\delta_y)\Big)\Big)\end{aligned}$$

which implies

$$\Delta^+_{xy} f(x, y) = \partial_{xy} f(x, y) + O\left(\max(\delta_x^2, \delta_y^2)\right)$$

as $\delta = (\delta_x, \delta_y)^\top \downarrow 0$. From the similar computation, we have

$$\Delta^-_{xy} f(x, y) = \partial_{xy} f(x, y) + O\left(\max(\delta_x^2, \delta_y^2)\right)$$

as $\delta = (\delta_x, \delta_y)^\top \downarrow 0$.

Remark 2.1.4
Although we are able to define more higher order approximations, they are not tractable for the numerical schemes for second order PDEs. They tend to require the information of the outside of the boundary. It is the same situation in Remark 2.1.3.

2.2 Convergence to viscosity solution: the Barles-Souganidis framework

We introduce a powerful framework to prove the convergence of approximation schemes proposed by Barles and Souganidis [BS91]. Let us consider a second order parabolic PDE

$$F(x, V(x), \partial_x V(x), \partial_x^2 V(x)) = 0, \quad \text{for } x \in \mathcal{S} \cup \partial\mathcal{S}, \tag{2.1}$$

where $\mathcal{S} \cup \partial\mathcal{S} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R}$ is a continuous map which satisfies the ellipticity condition

$$F(x, r, p, A) \leq F(x, r, p, B), \quad A \geq B. \tag{2.2}$$

We define an abstract approximation scheme $\mathscr{F}_\delta$ for the PDE (2.1) as follow:

Definition 2.2.1 (Abstract approximation scheme)

$$\mathscr{F}_\delta(x, V^\delta(x), [V^\delta]_x) = 0 \tag{2.3}$$

where δ is the discretization parameter and $[V^\delta]_x$ represents the value of V^δ at other points than x.

Following properties of the scheme $\mathscr{F}_\delta$ are required for the convergence proof.

Definition 2.2.2 (Monotonicity)
The scheme $\mathscr{F}_\delta$ is said to be monotone if

$$\mathscr{F}_\delta(x, y, \phi) \leq \mathscr{F}_\delta(x, y, \psi),$$

for any $y \in \mathbb{R}^n$ and $\phi, \psi \in \mathcal{B}(\mathcal{S})$ such that $\phi \leq \psi$, where $\mathcal{B}(\mathcal{S})$ is the spcae of bounded functions defined on $\mathcal{S}$.

Definition 2.2.3 (Stability)
The scheme $\mathscr{F}_\delta$ is said to be stable if there exists a solution V^δ for the scheme (2.3) with a bound independent of δ.

Definition 2.2.4 (Consistency)
The scheme $\mathscr{F}_\delta$ is said to be consistent if for all $x \in \mathcal{S} \cup \partial\mathcal{S}$ and $\phi \in C^\infty(\mathcal{S} \cup \partial\mathcal{S})$

$$\lim_{\substack{y\to x,\ \xi\to 0\\ \delta\to 0}} \mathscr{F}_\delta(y, \phi(y) + \xi, [\phi]_y + \xi) = F\left(x, \phi(x), \partial_x\phi(x), \partial_x^2\phi(x)\right).$$

By Barles and Souganidis [BS91], the solution of (2.3) V^δ converges to the solution of (2.1) V in the viscosity sense If the scheme $\mathscr{F}_\delta$ satisfies the monotonicity, the stability and the consistency.

2.3 1-dimensional Hamilton-Jacobi-Bellman equation

In this section, we construct numerical schemes for a 1-dimensional HJB equation

$$\partial_t V(t,x) + \mathscr{H}(t,x,\partial_x V, \partial_{xx} V) = 0, \quad (t,x) \in [0,T) \times \mathcal{S}, \tag{2.4}$$

with terminal and boundary conditions

$$V(T,x) = g_T(x), \quad x \in \mathcal{S} \cup \partial\mathcal{S}, \tag{2.5}$$

$$V(t,x) = g_{\partial\mathcal{S}}(x), \quad (t,x) \in [0,T) \times \partial\mathcal{S}, \tag{2.6}$$

where $\mathcal{S} = (x_{\min}, x_{\max})$, $\partial\mathcal{S} = \{x_{\min}, x_{\max}\}$ and $x_{\min}, x_{\max} \in \mathbb{R}$ such that $x_{\min} < x_{\max}$. Here the Hamiltonian $\mathscr{H}$ is defined by

$$\mathscr{H}(t,x,p,A) = \sup_{u\in\mathbb{U}} \left[\mu(t,x,u)p + \frac{1}{2}a(t,x,u)A + f(t,x,u)\right]$$

where $a = \sigma^2$ and we additionally assume that the profit functions f, g_T and $g_{\partial\mathcal{S}}$ are continuous and satisfy

$$\begin{aligned}
|f(s,y,u) - f(t,x,u)| &\le C(1+|y|+|x|+|u|)(|s-t|+|y-x|)\\
|g_{\partial\mathcal{S}}(s,y) - g_{\partial\mathcal{S}}(t,x)| &\le C(1+|y|+|x|)(|s-t|+|y-x|)\\
|g_T(y) - g_T(x)| &\le C(1+|y|+|x|)|y-x|
\end{aligned}$$

for some constant C.

Let us prepare the notations for the discretization. We denote by $N^t, N^x \in \mathbb{N}$ the number of time and spatial grids and define the discretization parameters δ_t and δ_x and discretized domain $\mathcal{S}^\delta$ as follows:

- $\delta_t = \dfrac{T}{N^t}, \quad \delta_x = \dfrac{x_{\max} - x_{\min}}{N^x};$
- $\mathcal{S}^\delta = \{x_{\min} + i\delta_x \mid i = 1, \cdots N^x - 1\}$.

We introduce some symbols as follow:

- $t_k = k\delta_t, \quad x_i = i\delta_x + x_{\min},$
- $\tilde{\mathbb{T}}^\delta = [0, T - \frac{\delta_t}{2}), \quad \partial\tilde{\mathbb{T}}^\delta = [T - \frac{\delta_t}{2}, T],$
- $\tilde{\mathcal{S}}^\delta = (x_{\min} + \frac{\delta_x}{2}, x_{\max} - \frac{\delta_x}{2}), \quad \partial\tilde{\mathcal{S}}^\delta = [x_{\min}, x_{\min} + \frac{\delta_x}{2}] \cup [x_{\max} - \frac{\delta_x}{2}, x_{\max}].$

2.3.1 Explicit method

Let us approximate the HJB equation (2.4) using the backward difference Δ_t^- for t and the central difference Δ_x^c for x. Straightforward replacement gives

$$\begin{aligned}
&\partial_t V(t,x) + \mathscr{H}(t,x,\partial_x V, \partial_{xx} V) \\
&\quad \simeq \Delta_t^- V(t,x) + \mathscr{H}\left(t,x,\Delta_x^c V(t,x), \Delta_x^2 V(t,x)\right), \\
&\quad = \frac{V(t,x) - V(t-\delta_t,x)}{\delta_t} + \mathscr{H}\left(t,x,\Delta_x^c V(t,x), \Delta_x^2 V(t,x)\right),
\end{aligned}$$

for $(t,x) \in \left\{t_k \middle| \, k = 1,\cdots,N^t\right\} \times \mathcal{S}^\delta$. Shifting the time label, we define the explicit method $\mathscr{F}^{E,c}$ for the HJB equation (2.4) with central difference as follow.

Definition 2.3.1 (Explicit method for 1-dimensional HJB equation with central difference)

$$\begin{aligned}
&\mathscr{F}^{E,c}(t,x,V^\delta(t,x),[V^\delta]_{t,x}) \\
&= \begin{cases}
\dfrac{V^\delta(t+\delta_t,x) - V^\delta(t,x)}{\delta_t} + \mathscr{H}\left(t+\delta_t,x,\Delta_x^c V^\delta(t+\delta_t,x), \Delta_x^2 V^\delta(t+\delta_t,x)\right), & \\
\qquad\qquad\qquad\qquad\qquad\qquad (t,x) \in \left\{t_k \middle| \, k = 0,\cdots,N^t-1\right\} \times \mathcal{S}^\delta, & \\
g_T(x) - V^\delta(t,x), & (t,x) \in \partial\tilde{\mathbb{T}}^\delta \times (\mathcal{S} \cup \partial\mathcal{S}), \\
g_{\partial\mathcal{S}}(t,\varsigma(x)) - V^\delta(t,x), & (t,x) \in \tilde{\mathbb{T}}^\delta \times \partial\tilde{\mathcal{S}}^\delta, \\
\displaystyle\sum_{k=1}^{N^t-1}\sum_{i=1}^{N^x-1} V^\delta(t_k,x_i)\mathbf{1}_{\left\{t\in\left[t_k-\frac{\delta_t}{2},t_k+\frac{\delta_t}{2}\right),\, x\in\left[x_i-\frac{\delta_x}{2},x_i+\frac{\delta_x}{2}\right)\right\}} - V^\delta(t,x), & \text{otherwise.}
\end{cases}
\end{aligned}$$

where

$$\varsigma(x) = \underset{y\in\partial\mathcal{S}}{\operatorname{argmin}}\,|x-y|.$$

Remark 2.3.2
Definition 2.3.1 implies that

$$V^\delta(t,x) = V^\delta(t+\delta_t,x) + \mathscr{H}\left(t+\delta_t,x,\Delta_x^c V^\delta(t+\delta_t,x), \Delta_x^2 V^\delta(t+\delta_t,x)\right). \tag{2.7}$$

Since we are able to obtain the approximate solution $V^\delta(t,\cdot)$ explicitly by the backward induction, the scheme $\mathscr{F}^{E,c}$ is called the explicit method.

The numerical scheme using the backward or forward differences for the state variable x is not suitable to approximate the HJB equation, because the both schemes are, generally, not monotone. However the upwind scheme which is the combination of the backward and forward differences is quite useful. The (first order) upwind scheme switch the backward and forward differences depend on the coefficient of the first order derivative:

$$\mu(t,x,u)\partial_x V(t,x) \simeq \mu^+(t,x,u)\Delta_x^+ V(t,x) - \mu^-(t,x,u)\Delta_x^- V(t,x),$$

where μ^+ and μ^- is the absolute values of the positive and negative parts of the function μ:

$$\mu^+ = \max(\mu, 0),$$
$$\mu^- = \max(-\mu, 0).$$

To use the upwind difference, we introduce an operator $\tilde{\mathscr{H}}$ such that

$$\tilde{\mathscr{H}}(t,x,p^+,p^-,A) = \sup_{u\in\mathbb{U}}\left[\mu^+(t,x,u)p^+ - \mu^-(t,x,u)p^- + \frac{1}{2}a(t,x,u)A + f(t,x,u)\right].$$

We note that

$$\tilde{\mathscr{H}}(t,x,p,p,A) = \mathscr{H}(t,x,p,A).$$

and hence we approximate the HJB equation (2.4) by

$$\begin{aligned}
&\partial_t V(t,x) + \mathscr{H}(t,x,\partial_x V,\partial_{xx}V) \\
&\quad \simeq \Delta_t^- V(t,x) + \tilde{\mathscr{H}}\left(t,x,\Delta_x^+ V(t,x),\Delta_x^- V(t,x),\Delta_x^2 V(t,x)\right), \\
&\quad = \frac{V(t,x) - V(t-\delta_t,x)}{\delta_t} + \tilde{\mathscr{H}}\left(t,x,\Delta_x^+ V(t,x),\Delta_-^+ V(t,x),\Delta_x^2 V(t,x)\right),
\end{aligned}$$

for $(t,x) \in \left\{t_k \middle| \, k = 1,\cdots,N^t\right\} \times \mathcal{S}^\delta$. As in the case of central difference, we shift the time label and define the explicit method with the upwind scheme $\mathscr{F}^{E,u}$ as follow.

Definition 2.3.3 (Explicit method for 1-dimensional HJB equation with upwind difference)

$$\begin{aligned}
&\mathscr{F}^{E,u}(t,x,V^\delta(t,x),[V^\delta]_{t,x}) \\
&= \begin{cases}
\dfrac{V^\delta(t+\delta_t,x) - V^\delta(t,x)}{\delta} + \tilde{\mathscr{H}}\Big(t+\delta_t,x,\Delta_x^+ V^\delta(t+\delta_t,x),\Delta_x^- V^\delta(t+\delta_t,x),\Delta_x^2 V_t^\delta(t+\delta_t,x)\Big), \\
\qquad\qquad\qquad\qquad\qquad\qquad (t,x) \in \left\{t_k \middle| \, k = 0,\cdots,N^t-1\right\} \times \mathcal{S}^\delta, \\
g_T(x) - V^\delta(t,x), \qquad\qquad\qquad (t,x) \in \partial\tilde{\mathbb{T}}^\delta \times (\mathcal{S}\cup\partial\mathcal{S}), \\
g_{\partial\mathcal{S}}(t,\varsigma(x)) - V^\delta(t,x), \qquad\qquad (t,x) \in \tilde{\mathbb{T}}^\delta \times \partial\tilde{\mathcal{S}}^\delta, \\
\displaystyle\sum_{k=1}^{N^t-1}\sum_{i=1}^{N^x-1} V^\delta(t_k,x_i)\mathbf{1}_{\left\{t\in\left[t_k-\frac{\delta_t}{2},t_k+\frac{\delta_t}{2}\right),\, x\in\left[x_i-\frac{\delta_x}{2},x_i+\frac{\delta_x}{2}\right)\right\}} - V^\delta(t,x), \quad \text{otherwise.}
\end{cases}
\end{aligned}$$

Let us compare the schemes $\mathscr{F}^{E,c}$ and $\mathscr{F}^{E,u}$. Since the scheme $\mathscr{F}^{E,c}$ uses the central difference, the convergence speed of $\mathscr{F}^{E,c}$ is expected faster than that of $\mathscr{F}^{E,u}$. However, as shown in the latter subsection, the monotonicity of $\mathscr{F}^{E,c}$ is guaranteed under the condition

$$|\mu| \le \frac{a}{\delta_x},$$

while the scheme $\mathscr{F}^{E,u}$ dose not require such condition. Thus either method is superior to the other. We also remark that we have to determine the parameter δ_t and δ_x to satisfy the condition (like the CLF condition)

$$\frac{\delta_t}{\delta_x^2}a \le 1,$$

for both methods.

Remark 2.3.4
The convergence proof of the explicit methods is found in [FS06]. They use the probabilistic representation to prove the three properties. In this subsection, we prove them more directly.

We now prove the convergence of the approximation scheme $\mathscr{F}^{E,c}$. The case of $\mathscr{F}^{E,u}$ is proven in the almost same manner. All we need is to show the monotonicity, the stability and the consistency.

Lemma 2.3.5
Assume that

$$|\mu| \leq \frac{a}{\delta_x}, \qquad \frac{\delta_t}{\delta_x^2} a \leq 1.$$

Then the scheme $\mathscr{F}^{E,c}$ is monotone.

Proof. Let $y \in \mathbb{R}$ and $\phi, \psi \in \mathcal{B}(\mathcal{S})$ such that $\phi \leq \psi$. We see that

$$\mathscr{F}^{E,c}\left(t, x, y, [\psi]_{t,x}\right) - \mathscr{F}^{E,c}\left(t, x, y, [\phi]_{t,x}\right) = g_T(x) - y - (g_T(x) - y) = 0,$$

for $(t, x) \in \partial\tilde{\mathbb{T}}^\delta \times (\mathcal{S} \cup \partial\mathcal{S})$,

$$\mathscr{F}^{E,c}\left(t, x, y, [\psi]_{t,x}\right) - \mathscr{F}^{E,c}\left(t, x, y, [\phi]_{t,x}\right) = (g_{\partial\mathcal{S}}(t, \varsigma(x)) - y) - (g_{\partial\mathcal{S}}(t, x) - y) = 0$$

for $(t, x) \in \tilde{\mathbb{T}}^\delta \times \partial\tilde{\mathcal{S}}^\delta$, and

$$\begin{aligned}
&\mathscr{F}^{E,c}\left(t, x, y, [\psi]_{t,x}\right) - \mathscr{F}^{E,c}\left(t, x, y, [\phi]_{t,x}\right) \\
&= \sum_{k=1}^{N^t-1} \sum_{i=1}^{N^x-1} \psi(t_k, x_i) \mathbf{1}_{\left\{t \in \left[t_k - \frac{\delta_t}{2}, t_k + \frac{\delta_t}{2}\right),\, x \in \left[x_i - \frac{\delta_x}{2}, x_i + \frac{\delta_x}{2}\right)\right\}} - y \\
&\quad - \left(\sum_{k=1}^{N^t-1} \sum_{i=1}^{N^x-1} \phi(t_k, x_i) \mathbf{1}_{\left\{t \in \left[t_k - \frac{\delta_t}{2}, t_k + \frac{\delta_t}{2}\right),\, x \in \left[x_i - \frac{\delta_x}{2}, x_i + \frac{\delta_x}{2}\right)\right\}} - y \right) \\
&= \sum_{k=1}^{N^t-1} \sum_{i=1}^{N^x-1} \left(\psi(t_k, x_i) - \phi(t_k, x_i)\right) \mathbf{1}_{\left\{t \in \left[t_k - \frac{\delta_t}{2}, t_k + \frac{\delta_t}{2}\right),\, x \in \left[x_i - \frac{\delta_x}{2}, x_i + \frac{\delta_x}{2}\right)\right\}} \\
&\geq 0,
\end{aligned}$$

for $(t, x) \in [0, T] \times (\mathcal{S} \cup \partial\mathcal{S}) \setminus (\left\{t_k \middle|\, k = 0, \cdots, N^t - 1\right\} \times \mathcal{S}^\delta \cup \tilde{\mathbb{T}}^\delta \times (\mathcal{S} \cup \partial\mathcal{S}) \cup \tilde{\mathbb{T}}^\delta \times \partial\tilde{\mathcal{S}}^\delta)$.

The remaining is to show the case of $(t, x) \in \left\{t_k \middle|\, k = 0, \cdots, N^t - 1\right\} \times \mathcal{S}^\delta$. Straightforward computations give

$$
\begin{aligned}
&\mathscr{F}^{E,c}\left(t,x,y,[\psi]_{t,x}\right) - \mathscr{F}^{E,c}\left(t,x,y,[\phi]_{t,x}\right)\\
&\quad = \frac{\psi(t+\delta_t,x)-y}{\delta_t} + \sup_{u_1\in\mathbb{U}}\left[\mu(t+\delta_t,x,u_1)\frac{\psi(t+\delta_t,x+\delta_x)-\psi(t+\delta_t,x-\delta_x)}{2\delta_x}\right.\\
&\qquad\left. +\frac{1}{2}a(t+\delta_t,x,u_1)\frac{\psi(t+\delta_t,x+\delta_x)-2\psi(t+\delta_t,x)+\psi(t+\delta_t,x-\delta_x)}{\delta_x^2} + f(t+\delta_t,x,u_1)\right]\\
&\qquad - \frac{\phi(t+\delta_t,x)-y}{\delta_t} - \sup_{u_2\in\mathbb{U}}\left[\mu(t+\delta_t,x,u_2)\frac{\phi(t+\delta_t,x+\delta_x)-\phi(t+\delta_t,x-\delta_x)}{2\delta_x}\right.\\
&\qquad\left. +\frac{1}{2}a(t+\delta_t,x,u_2)\frac{\phi(t+\delta_t,x+\delta_x)-2\phi(t+\delta_t,x)+\phi(t+\delta_t,x-\delta_x)}{\delta_x^2} + f(t+\delta_t,x,u_2)\right]\\
&\quad = \sup_{u_1\in\mathbb{U}}\left[f(t+\delta_t,x,u_1) - \frac{1}{\delta_t}y + \left(\frac{1}{\delta_t} - \frac{a(t+\delta_t,x,u_1)}{\delta_x^2}\right)\psi(t+\delta_t,x)\right.\\
&\qquad\qquad + \frac{1}{2\delta_x}\left(\frac{a(t+\delta_t,x,u_1)}{\delta_x} + \mu(t+\delta_t,x,u_1)\right)\psi(t+\delta_t,x+\delta_x)\\
&\qquad\qquad \left. + \frac{1}{2\delta_x}\left(\frac{a(t+\delta_t,x,u_1)}{\delta_x} - \mu(t+\delta_t,x,u_1)\right)\psi(t+\delta_t,x-\delta_x)\right]\\
&\qquad - \sup_{u_2\in\mathbb{U}}\left[f(t+\delta_t,x,u_2) - \frac{1}{\delta_t}y + \left(\frac{1}{\delta_t} - \frac{a(t+\delta_t,x,u_2)}{\delta_x^2}\right)\phi(t+\delta_t,x)\right.\\
&\qquad\qquad + \frac{1}{2\delta}\left(\frac{a(t+\delta_t,x,u_2)}{\delta_x} + \mu(t+\delta_t,x,u_2)\right)\phi(t+\delta_t,x+\delta_x)\\
&\qquad\qquad \left. + \frac{1}{2\delta_x}\left(\frac{a(t+\delta_t,x,u_2)}{\delta_x} - \mu(t+\delta_t,x,u_2)\right)\phi(t+\delta_t,x-\delta_x)\right].
\end{aligned}
$$

The assumption in the statement implies that the coefficients of $\psi(t+\delta_t,x)$, $\psi(t+\delta_t,x+\delta_x)$

and $\psi(t+\delta_t, x-\delta_x)$ are non-negative. Hence we find that

$$\begin{aligned}
&\mathscr{F}^{E,c}\left(t,x,y,[\psi]_{t,x}\right) - \mathscr{F}^{E,c}\left(t,x,y,[\phi]_{t,x}\right) \\
&\geq \sup_{u_1\in\mathbb{U}}\left[f(t+\delta_t,x,u_1) - \frac{1}{\delta_t}y + \left(\frac{1}{\delta_t} - \frac{a(t+\delta_t,x,u_1)}{\delta_x^2}\right)\phi(t+\delta_t,x)\right. \\
&\qquad + \frac{1}{2\delta_x}\left(\frac{a(t+\delta_t,x,u_1)}{\delta_x} + \mu(t+\delta_t,x,u_1)\right)\phi(t+\delta_t,x+\delta_x) \\
&\qquad \left. + \frac{1}{2\delta_x}\left(\frac{a(t+\delta_t,x,u_1)}{\delta} - \mu(t+\delta_t,x,u_1)\right)\phi(t+\delta_t,x-\delta_x)\right] \\
&\quad - \sup_{u_2\in\mathbb{U}}\left[f(t+\delta_t,x,u_2) - \frac{1}{\delta_t}y + \left(\frac{1}{\delta_t} - \frac{a(t+\delta_t,x,u_2)}{\delta_x^2}\right)\phi(t+\delta_t,x)\right. \\
&\qquad + \frac{1}{2\delta_x}\left(\frac{a(t+\delta_t,x,u_2)}{\delta_x} + \mu(t+\delta_t,x,u_2)\right)\phi(t+\delta_t,x+\delta_x) \\
&\qquad \left. + \frac{1}{2\delta_x}\left(\frac{a(t+\delta_t,x,u_2)}{\delta} - \mu(t+\delta_t,x,u_2)\right)\phi(t+\delta_t,x-\delta_x)\right] \\
&= 0
\end{aligned}$$

which establishes the claim. □

Lemma 2.3.6
Assume that

$$|\mu| \leq \frac{a}{\delta_x}, \qquad \frac{\delta_t}{\delta_x^2}a \leq 1.$$

Then the scheme $\mathscr{F}^{E,c}$ is stable.

Proof. Let V^δ be a solution of $\mathscr{F}^{E,c}(t,x,V^\delta(t,x),[V^\delta]_{t,x}) = 0$. The existence of V^δ is obvious from equation (2.7) and thus we show that V^δ is bounded. It is enough to show the boundedness on $\{t_k = k\delta_t | k = 0,1,\cdots,N^t\} \times (\mathcal{S}^\delta \cup \partial\mathcal{S})$.

We see that

$$|V^\delta(t_{N^t},x)| = |V^\delta(T,x)| = |g_T(x)| \leq C_1, \quad x \in \mathcal{S}^\delta \cup \partial\mathcal{S},$$

and

$$|V^\delta(t_k,x)| = |g_{\partial\mathcal{S}}(t_k,x)| \leq C_2, \quad k = 0,1,\cdots,N^t-1,\ x \in \partial\mathcal{S}$$

where

$$\begin{aligned}
C_1 &= \sup_{y\in\mathcal{S}\cup\partial\mathcal{S}} |g_T(y)|; \\
C_2 &= \sup_{s\in[0,T),y\in\partial\mathcal{S}} |g_{\partial\mathcal{S}}(s,y)|.
\end{aligned}$$

Note that both of C_1 and C_2 are independent of δ_t and δ_x.

Next we consider the case that $(t_k, x) \in \{t_k | k = 0, 1, \cdots, N^t - 1\} \times \mathcal{S}^\delta$. Straightforward computations give

$$\begin{aligned}
&\frac{V^\delta(t_k+\delta_t,x)-V^\delta(t_k,x)}{\delta_t} + \sup_{u\in\mathbb{U}}\left[\mu(t+\delta_t,x,u)\frac{V^\delta(t_k+\delta_t,x+\delta_x)-V^\delta(t_k+\delta_t,x-\delta_x)}{2\delta_x}\right.\\
&\left.+\frac{1}{2}a(t_k+\delta_t,x,u)\frac{V^\delta(t_k+\delta_t,x+\delta_x)-2V^\delta(t_k+\delta_t,x)+V^\delta(t_k+\delta_t,x-\delta_x)}{\delta_x^2}+f(t+\delta_t,x,u)\right]\\
&=0.
\end{aligned}$$

This is equivalent to

$$\begin{aligned}
V^\delta(t_k,x) = \sup_{u\in\mathbb{U}}\Bigg[&\left(1-\frac{\delta_t}{\delta_x^2}a(t_{k+1},x,u)\right)V^\delta(t_{k+1},x)\\
&+\frac{\delta_t}{2\delta_x^2}\left(a(t_{k+1},x,u)+\delta\mu(t_{k+1},x,u)\right)V^\delta(t_{k+1},x+\delta)\\
&+\frac{\delta_t}{2\delta_x^2}\left(a(t_{k+1},x,u)-\delta\mu(t_{k+1},x,u)\right)V^\delta(t_{k+1},x-\delta)\\
&+\delta_t f(t_{k+1},x,u)\Bigg].
\end{aligned}$$

The assumption in the statement implies

$$\left(1-\frac{\delta_t}{\delta_x^2}a\right),\ \frac{\delta_t}{2\delta_x^2}(a\pm\delta\mu)\geq 0,$$

and hence we have

$$
\begin{aligned}
|V^\delta(t_k,x)| &= \Bigg|\sup_{u\in\mathbb{U}}\Bigg[\left(1-\frac{\delta_t}{\delta_x^2}a(t_{k+1},x,u)\right)V^\delta(t_{k+1},x)\\
&\qquad +\frac{\delta_t}{2\delta_x^2}\big(a(t_{k+1},x,u)+\delta\mu(t_{k+1},x,u)\big)\,V^\delta(t_{k+1},x+\delta)\\
&\qquad +\frac{\delta_t}{2\delta_x^2}\big(a(t_{k+1},x,u)-\delta\mu(t_{k+1},x,u)\big)\,V^\delta(t_{k+1},x-\delta)\\
&\qquad +\delta_t f(t_{k+1},x,u)\Bigg]\Bigg|\\
&\le \sup_{u\in\mathbb{U}}\Bigg[\left(1-\frac{\delta_t}{\delta_x^2}a(t_{k+1},x,u)\right)|V^\delta(t_{k+1},x)|\\
&\qquad +\frac{\delta_t}{2\delta_x^2}\big(a(t_{k+1},x,u)+\delta\mu(t_{k+1},x,u)\big)\,|V^\delta(t_{k+1},x+\delta)|\\
&\qquad +\frac{\delta_t}{2\delta_x^2}\big(a(t_{k+1},x,u)-\delta\mu(t_{k+1},x,u)\big)\,|V^\delta(t_{k+1},x-\delta)|\\
&\qquad +\delta_t|f(t_{k+1},x,u)|\Bigg]\\
&\le \sup_{u\in\mathbb{U}}\Bigg[\max_{x'\in\{x,x\pm\delta_x\}}\Bigg\{\left(1-\frac{\delta_t}{\delta_x^2}a(t_{k+1},x,u)\right)+\frac{\delta_t}{2\delta_x^2}\big(a(t_{k+1},x,u)+\delta\mu(t_{k+1},x,u)\big)\\
&\qquad +\frac{\delta_t}{2\delta_x^2}\big(a(t_{k+1},x,u)-\delta\mu(t_{k+1},x,u)\big)\Bigg\}|V^\delta(t_{k+1},x')|\\
&\qquad +\delta_t|f(t_{k+1},x,u)|\Bigg]\\
&= \sup_{u\in\mathbb{U}}\left[\max_{x'\in\{x,x\pm\delta_x\}}|V^\delta(t_{k+1},x')|+\delta_t|f(t_{k+1},x,u)|\right]\\
&= \max_{x'\in\{x,x\pm\delta_x\}}|V^\delta(t_{k+1},x')|+\delta_t\sup_{u\in\mathbb{U}}|f(t_{k+1},x,u)|\\
&\le \max\left[\max_{x'\in\mathcal{S}^\delta}|V^\delta(t_{k+1},x')|,\max_{x'\in\partial\mathcal{S}}|V^\delta(t_{k+1},x')|\right]+\delta_t C_3\\
&\le \max\left[\max_{x'\in\mathcal{S}^\delta}|V^\delta(t_{k+1},x')|,C_2\right]+\delta_t C_3,
\end{aligned}
$$

where $C_3=\sup_{s\in[0,T],\,y\in\mathcal{S},\,u\in\mathbb{U}}|f(s,y,u)|$. This evaluation implies that

$$|V^\delta(t_k,x)|\le\max[C1,C2]+(N^t-k)\delta_t C_3.$$

We are able to show the above by induction. For $k=N^t$,

$$|V^\delta(t_{N^t},x)|\le C_1\le\max[C1,C2]+(N^t-N^t)\delta_t C_3.$$

Assuming the claim is established for $k = k' + 1$,

$$\begin{aligned}
|V^\delta(t_{k'}, x)| &\leq \max\left[\max_{x' \in \mathcal{S}^\delta} |V^\delta(t_{k'+1}, x')|, C_2\right] + \delta_t C_3 \\
&\leq \max\left[\left\{\max[C1, C2] + (N^t - (k'+1))\delta_t C_3\right\}, C_2\right] + \delta_t C_3 \\
&\leq \max[C1, C2] + (N^t - (k'+1))\delta_t C_3 + \delta_t C_3 \\
&\leq \max[C1, C2] + (N^t - k')\delta_t C_3.
\end{aligned}$$

Note that $(N^t - k)\delta_t C_3 \leq TC_3$, $k = 0, 1, \cdots, N^t - 1$ and we obtain

$$|V^\delta(t_k, x)| \leq \max[C1, C2] + TC_3, \quad (t_k, x) \in \{t_k = k\delta_t | k = 0, 1, \cdots, N^t - 1\} \times \mathcal{S}^\delta,$$

which establishes the claim. □

Lemma 2.3.7
The scheme $\mathscr{F}^{E,c}$ is consistent.

Proof. Let $\phi \in C^\infty([0, T] \times (\mathcal{S}^\delta \cup \partial\mathcal{S}))$.

We first consider the case that $(t, x) \in [0, T) \times \mathcal{S}$. Suppose that δ_t and δ_x are small enough such that $(t, x) \in \tilde{\mathbb{T}}^\delta \times \tilde{\mathcal{S}}^\delta$. More precisely, we assume that δ_t and δ_x satisfy

$$\min(t, T - t) > \frac{1}{2}\delta_t, \quad \min_{x' \in \partial\mathcal{S}} |x - x'| > \frac{1}{2}\delta_x.$$

We choose (s, y) as follow:

$$s = \operatorname*{argmin}_{1 \leq k \leq N^t - 1} |t - t_k|, \quad y = \operatorname*{argmin}_{x' \in \mathcal{S}^\delta} |x - x'|.$$

Then we find

$$\begin{aligned}
&\left|\mathscr{F}^{E,c}(s, y, \phi(s, y) + \xi, \phi + \xi) - \left(\partial_t \phi(t, x) + \mathscr{H}\left(t, x, \partial_x \phi(t, x), \partial_x^2 \phi(t, x)\right)\right)\right| \\
&= \Bigg|\frac{\phi(s + \delta_t, y) + \xi - (\phi(s, y) + \xi)}{\delta_t} \\
&\quad + \sup_{u \in \mathbb{U}}\Bigg[f(s + \delta_t, y, u) + \mu(s + \delta_t, y, u)\frac{\phi(s + \delta_t, y + \delta_x) + \xi - (\phi(s + \delta_t, y - \delta_x) + \xi)}{2\delta_x} \\
&\quad + \frac{1}{2}a(s + \delta_t, y, u_1)\frac{\phi(s + \delta_t, y + \delta_x) + \xi - 2(\phi(s + \delta_t, y) + \xi) + \phi(s + \delta_t, y - \delta_x) + \xi}{\delta_x^2}\Bigg] \\
&\quad - \partial_t \phi(t, x) - \sup_{u \in \mathbb{U}}\left[\mu(t, x, u)\partial_x \phi(t, x) + \frac{1}{2}a(t, x, u)\partial_x^2 \phi(t, x) + f(t, x, u)\right]\Bigg|
\end{aligned}$$

where $\xi \in \mathbb{R}$. Note that for $A \subset \mathbb{R}^n$ and $f_1, f_2 : A \to \mathbb{R}$,

$$\sup_{x \in A} f_1(x) - \sup_{x \in A} f_2(x) \leq \sup_{x \in A}\left(f_1(x) - f_2(x)\right),$$

and we have

$$
\begin{aligned}
&\left|\mathscr{F}^{E,c}(s,y,\phi(s,y)+\xi,\phi+\xi)-\left(\partial_t\phi(t,x)+\mathscr{H}\left(t,x,\partial_x\phi(t,x),\partial_x^2\phi(t,x)\right)\right)\right|\\
&\leq\left|\frac{\phi(s+\delta_t,y)-\phi(s,y)}{\delta_t}-\partial_t\phi(t,x)\right.\\
&\quad+\sup_{u\in\mathbb{U}}\Big[f(s+\delta_t,y,u)-f(t,x,u)\\
&\quad\ +\mu(s+\delta_t,y,u)\frac{\phi(s+\delta_t,y+\delta_x)-\phi(s+\delta_t,y-\delta_x))}{2\delta_x}-\mu(t,x,u)\partial_x\phi(t,x)\\
&\quad\ +\frac{1}{2}a(s+\delta_t,y,u)\frac{\phi(s+\delta_t,y+\delta_x)+-2\phi(s+\delta_t,y)+\phi(s+\delta_t,y-\delta_x)}{\delta_x^2}\\
&\qquad\left.-\frac{1}{2}a(t,x,u)\partial_x^2\phi(t,x)\Big]\right|\\
&\leq\left|\frac{\phi(s+\delta_t,y)-\phi(s,y)}{\delta_t}-\partial_t\phi(t,x)\right|\\
&\quad+\sup_{u\in\mathbb{U}}|f(s+\delta_t,y,u)-f(t,x,u)|\\
&\quad\ +\sup_{u\in\mathbb{U}}\left|\mu(s+\delta_t,y,u)\frac{\phi(s+\delta_t,y+\delta_x)-\phi(s+\delta_t,y-\delta_x))}{2\delta_x}-\mu(t,x,u)\partial_x\phi(t,x)\right|\\
&\quad\ +\frac{1}{2}\sup_{u\in\mathbb{U}}\left|a(s+\delta_t,y,u)\frac{\phi(s+\delta_t,y+\delta_x)-2\phi(s+\delta_t,y)+\phi(s+\delta_t,y-\delta_x)}{\delta_x^2}\right.\\
&\qquad\quad\left.-a(t,x,u)\partial_x^2\phi(t,x)\right|.
\end{aligned}
$$

Denote by $\delta_{ts}=s-t$ and $\delta_{xy}=y-x$ and we evaluate each term.

We first evaluate the time derivative term. The Taylor expansion gives

$$
\begin{aligned}
&\left|\frac{\phi(s+\delta_t,y)-\phi(s,y)}{\delta_t}-\partial_t\phi(t,x)\right|\\
&=|\partial_t\phi(s,y)+\partial_t^2\phi(s+\theta_1\delta_t,y)\delta_t--\partial_t\phi(t,x)|\\
&=|\partial_t\phi(t+\delta_{ts},x+\delta_{xy})+\partial_t^2\phi(s+\theta_1\delta_t,y)\delta_t-\partial_t\phi(t,x)|\\
&=|\partial_t^2\phi(s+\theta_1\delta_t,y)\delta_t+\partial_t^2\phi(t+\theta_2\delta_t,x+\theta_2\delta_{xy})\delta_{ts}\\
&\qquad+\partial_x\partial_t\phi(t+\theta_2\delta_t,x+\theta_2\delta_{xy})\delta_{xy}|,
\end{aligned}
$$

where $\theta_1,\theta_2\in(0,1)$ and hence

$$
\left|\frac{\phi(s+\delta_t,y)-\phi(s,y)}{\delta_t}-\partial_t\phi(t,x)\right|\to 0\quad\text{as }\delta_t\to 0\text{ and }(s,y)\to(t,x).
$$

We next evaluate the terms related to the first derivative of x. Using the Taylor expansion, we find

$$\begin{aligned}
&\frac{\phi(s+\delta_t, y+\delta_x)-\phi(s+\delta_t, y-\delta_x)}{2\delta_x}\\
&= \partial_x\phi(s+\delta_t, y) + \frac{1}{12}\left(\partial_x^3\phi(s+\delta_t, y+\theta_3\delta_x)+\partial_x^3\phi(s+\delta_t, y-\theta_4\delta_x)\right)\delta_x^2\\
&= \partial_x\phi(t,x) + \frac{1}{12}\left(\partial_x^3\phi(s+\delta_t, y+\theta_3\delta_x)+\partial_x^3\phi(s+\delta_t, y-\theta_4\delta_x)\right)\delta_x^2\\
&\quad + \partial_t\partial_x\phi(t+\theta_5(\delta_{ts}+\delta_t), x+\theta_5\delta_{xy})(\delta_{ts}+\delta_t) + \partial_x^2\phi(t+\theta_5(\delta_{ts}+\delta_t), x+\theta_5\delta_{xy})\delta_{xy}.
\end{aligned}$$

where $\theta_i \in (0,1)$, $i = 3,4,5$. Thus we have

$$\begin{aligned}
&\sup_{u\in\mathbb{U}}\left|\mu(s+\delta_t, y, u)\frac{\phi(s+\delta_t, y+\delta_x)-\phi(s+\delta_t, y-\delta_x))}{2\delta_x} - \mu(t,x,u)\partial_x\phi(t,x)\right|\\
&\le \sup_{u\in\mathbb{U}}|\mu(s+\delta_t, y, u)-\mu(t,x,u)||\partial_x\phi(t,x)|\\
&\quad + \sup_{u\in\mathbb{U}}|\mu(s+\delta_t, y, u)|\left|\frac{1}{12}\left(\partial_x^3\phi(s+\delta_t, y+\theta_3\delta_x)+\partial_x^3\phi(s+\delta_t, y-\theta_4\delta_x)\right)\delta_x^2\right.\\
&\quad \left. + \partial_t\partial_x\phi(t+\theta_5(\delta_{ts}+\delta_t), x+\theta_5\delta_{xy})(\delta_{ts}+\delta_t) + \partial_x^2\phi(t+\theta_5(\delta_{ts}+\delta_t), x+\theta_5\delta_{xy})\delta_{xy}\right|.
\end{aligned}$$

The Lipschitz continuity and the linear growth condition of μ implies

$$\begin{aligned}
\sup_{u\in\mathbb{U}}|\mu(s+\delta_t, y, u)-\mu(t,x,u)||\partial_x\phi(t,x)| &\le \sup_{u\in\mathbb{U}} C(|s+\delta_t-t|+|y-x|)|\partial_x\phi(t,x)|\\
&= C(|\delta_t+\delta_{ts}|+|\delta_{xy}|)|\partial_x\phi(t,x)|
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{u\in\mathbb{U}}|\mu(s+\delta_t, y, u)|\left|\frac{1}{12}\left(\partial_x^3\phi(s+\delta_t, y+\theta_3\delta_x)+\partial_x^3\phi(s+\delta_t, y-\theta_4\delta_x)\right)\delta_x^2\right.\\
&\quad \left. + \partial_t\partial_x\phi(t+\theta_5(\delta_{ts}+\delta_t), x+\theta_5\delta_{xy})(\delta_{ts}+\delta_t) + \partial_x^2\phi(t+\theta_5(\delta_{ts}+\delta_t), x+\theta_5\delta_{xy})\delta_{xy}\right|\\
&\le C(1+|y|+\sup_{u\in\mathbb{U}}|u|)\left|\frac{1}{12}\left(\partial_x^3\phi(s+\delta_t, y+\theta_3\delta_x)+\partial_x^3\phi(s+\delta_t, y-\theta_4\delta_x)\right)\delta_x^2\right.\\
&\quad \left. + \partial_t\partial_x\phi(t+\theta_5(\delta_{ts}+\delta_t), x+\theta_5\delta_{xy})(\delta_{ts}+\delta_t) + \partial_x^2\phi(t+\theta_5(\delta_{ts}+\delta_t), x+\theta_5\delta_{xy})\delta_{xy}\right|.
\end{aligned}$$

Recall that $\mathbb{U}$ is compact and we obtain

$$\begin{aligned}
\sup_{u\in\mathbb{U}}\left|\mu(s+\delta_t, y, u)\frac{\phi(s+\delta_t, y+\delta_x)-\phi(s+\delta_t, y-\delta_x))}{2\delta_x} - \mu(t,x,u)\partial_x\phi(t,x)\right| &\to 0\\
\text{as } \delta_t \to 0,\ \delta_x \to 0 \text{ and } (s,y) \to (t,x).&
\end{aligned}$$

The terms related to the second derivative of x are evaluated in the same manner.

The Taylor expansion gives

$$
\begin{aligned}
&\frac{\phi(s+\delta_t,y+\delta_x)-2\phi(s+\delta_t,y)+\phi(s+\delta_t,y-\delta_x)}{\delta_x^2}\\
&=\partial_x\phi(s+\delta_t,y)+\frac{1}{24}\left(\partial_x^4\phi(s+\delta_t,y+\theta_6\delta_x)+\partial_x^4\phi(s+\delta_t,y-\theta_7\delta_x)\right)\delta_x^2\\
&=\partial_x\phi(t,x)+\frac{1}{24}\left(\partial_x^4\phi(s+\delta_t,y+\theta_6\delta_x)+\partial_x^4\phi(s+\delta_t,y-\theta_7\delta_x)\right)\delta_x^2\\
&\quad+\partial_t\partial_x\phi(t+\theta_5(\delta_{ts}+\delta_t),x+\theta_5\delta_{xy})(\delta_{ts}+\delta_t)+\partial_x^2\phi(t+\theta_5(\delta_{ts}+\delta_t),x+\theta_5\delta_{xy})\delta_{xy},
\end{aligned}
$$

where $\theta_i\in(0,1)$, $i=5,6,7$. Hence we see

$$
\begin{aligned}
&\sup_{u\in\mathbb{U}}\left|a(s+\delta_t,y,u)\frac{\phi(s+\delta_t,y+\delta_x)+-2\phi(s+\delta_t,y)+\phi(s+\delta_t,y-\delta_x)}{\delta_x^2}\right.\\
&\qquad\qquad\left.-a(t,x,u)\partial_x^2\phi(t,x)\right|\\
&\le\sup_{u\in\mathbb{U}}|a(s+\delta_t,y,u)-a(t,x,u)||\partial_x\phi(t,x)|\\
&\quad+\sup_{u\in\mathbb{U}}|a(s+\delta_t,y,u)|\left|\frac{1}{24}\left(\partial_x^4\phi(s+\delta_t,y+\theta_6\delta_x)+\partial_x^4\phi(s+\delta_t,y-\theta_7\delta_x)\right)\delta_x^2\right.\\
&\quad\left.+\partial_t\partial_x\phi(t+\theta_5(\delta_{ts}+\delta_t),x+\theta_5\delta_{xy})(\delta_{ts}+\delta_t)+\partial_x^2\phi(t+\theta_5(\delta_{ts}+\delta_t),x+\theta_5\delta_{xy})\delta_{xy}\right|.
\end{aligned}
$$

Using the Lipschitz continuity and the linear growth condition of σ, we have

$$
\begin{aligned}
|a(s+\delta_t,y,u)-a(t,x,u)|&=|\sigma^2(s+\delta_t,y,u)-\sigma^2(t,x,u)|\\
&\le\left(|\sigma(s+\delta_t,y,u)|+|\sigma(t,x,u)|\right)|\sigma(s+\delta_t,y,u)-\sigma(t,x,u)|\\
&\le C^2(2+|y|+|x|+2|u|)(|\delta_t+\delta_{ts}|+|\delta_{xy}|),
\end{aligned}
$$

and

$$|a(s+\delta_t,y,u)|\le C^2(1+|y|+|u|)^2.$$

Hence we obtain

$$
\begin{aligned}
&\sup_{u\in\mathbb{U}}|a(s+\delta_t,y,u)-a(t,x,u)||\partial_x\phi(t,x)|\\
&\qquad\le C^2(2+|y|+|x|+2\sup_{u\in\mathbb{U}}|u|)(|\delta_t+\delta_{ts}|+|\delta_{xy}|)|\partial_x\phi(t,x)|
\end{aligned}
$$

and

$$
\begin{aligned}
&\sup_{u\in\mathbb{U}}|a(s+\delta_t,y,u)|\left|\frac{1}{24}\left(\partial_x^4\phi(s+\delta_t,y+\theta_6\delta_x)+\partial_x^4\phi(s+\delta_t,y-\theta_7\delta_x)\right)\delta_x^2\right.\\
&\quad\left.+\partial_t\partial_x\phi(t+\theta_5(\delta_{ts}+\delta_t),x+\theta_5\delta_{xy})(\delta_{ts}+\delta_t)+\partial_x^2\phi(t+\theta_5(\delta_{ts}+\delta_t),x+\theta_5\delta_{xy})\delta_{xy}\right|\\
&\le\sup_{u\in\mathbb{U}}C^2(1+|y|+|u|)^2\left|\frac{1}{24}\left(\partial_x^4\phi(s+\delta_t,y+\theta_6\delta_x)+\partial_x^4\phi(s+\delta_t,y-\theta_7\delta_x)\right)\delta_x^2\right.\\
&\quad\left.+\partial_t\partial_x\phi(t+\theta_5(\delta_{ts}+\delta_t),x+\theta_5\delta_{xy})(\delta_{ts}+\delta_t)+\partial_x^2\phi(t+\theta_5(\delta_{ts}+\delta_t),x+\theta_5\delta_{xy})\delta_{xy}\right|.
\end{aligned}
$$

which imply

$$\sup_{u\in\mathbb{U}}\Big|a(s+\delta_t,y,u)\frac{\phi(s+\delta_t,y+\delta_x)+-2\phi(s+\delta_t,y)+\phi(s+\delta_t,y-\delta_x)}{\delta_x^2}$$
$$-a(t,x,u)\partial_x^2\phi(t,x)\Big|\to 0,\qquad \text{as } \delta_t\to 0,\ \delta_x\to 0 \text{ and } (s,y)\to(t,x).$$

By the assumption of f, the remaining evaluation is almost obvious:

$$\sup_{u\in\mathbb{U}}|f(s+\delta_t,y,u)-f(t,x,u)|\le C(1+|y|+|x|+\sup_{u\in\mathbb{U}}|u|)(|\delta_t+\delta_{ts}|+|\delta_{xy}|)$$

and thus

$$\sup_{u\in\mathbb{U}}|f(s+\delta_t,y,u)-f(t,x,u)|\to 0,\quad \text{as } \delta_t\to 0,\ \delta_x\to 0 \text{ and } (s,y)\to(t,x).$$

Therefore the claim is established on $(t,x)\in[0,T)\times\mathcal{S}$.

We next consider the case that $(t,x)\in[0,T)\times\partial\mathcal{S}$. Suppose that $(s,y)\in\mathbb{T}^\delta\times\partial\tilde{\mathcal{S}}^\delta$ and then

$$\begin{aligned}&\left|\mathscr{F}^{E,c}(s,y,\phi(s,y)+\xi,\phi+\xi)-\left(\partial_t\phi(t,x)+\mathscr{H}\left(t,x,\partial_x\phi(t,x),\partial_x^2\phi(t,x)\right)\right)\right|\\&=\left|g_{\partial\mathcal{S}}(s,\varsigma(y))-(\phi(s,y)+\xi)-(g_{\partial\mathcal{S}}(t,x)-\phi(t,x))\right|.\end{aligned}$$

The assumption imposed to $g_{\partial\mathcal{S}}$ gives

$$\begin{aligned}&\left|\mathscr{F}^{E,c}(s,y,\phi(s,y)+\xi,\phi+\xi)-\left(\partial_t\phi(t,x)+\mathscr{H}\left(t,x,\partial_x\phi(t,x),\partial_x^2\phi(t,x)\right)\right)\right|\\&\le C(1+|\varsigma(y)|+|x|)(|s-t|+|y-x|)+|\partial_t\phi(t+\theta_8\delta_{ts},x+\theta_8\delta_{xy})||\delta_{ts}|\\&\quad+|\partial_x\phi(t+\theta_8\delta_{ts},x+\theta_8\delta_{xy})||\delta_{xy}|+|\xi|\end{aligned}$$

where $\theta_8\in(0,1)$. Hence we obtain

$$\left|\mathscr{F}^{E,c}(s,y,\phi(s,y)+\xi,\phi+\xi)-\left(\partial_t\phi(t,x)+\mathscr{H}\left(t,x,\partial_x\phi(t,x),\partial_x^2\phi(t,x)\right)\right)\right|\to 0,$$
$$\text{as } \delta_t\to 0,\ \delta_x\to 0\ \xi\to 0 \text{ and } (s,y)\to(t,x).$$

which establishes the claim on $[0,T)\times\partial\mathcal{S}$.

Finally we consider the case that $(t,x)\in\{T\}\times(\mathcal{S}\cup\partial\mathcal{S})$. Suppose that $(s,y)\in\partial\mathbb{T}^\delta\times(\mathcal{S}\cup\partial\mathcal{S})$ and then

$$\begin{aligned}&\left|\mathscr{F}^{E,c}(s,y,\phi(s,y)+\xi,\phi+\xi)-\left(\partial_t\phi(t,x)+\mathscr{H}\left(t,x,\partial_x\phi(t,x),\partial_x^2\phi(t,x)\right)\right)\right|\\&=\left|g_T(y)-(\phi(s,y)+\xi)-(g_T(x)-\phi(T,x))\right|.\end{aligned}$$

The assumption imposed to g_T gives

$$\begin{aligned}&\left|\mathscr{F}^{E,c}(s,y,\phi(s,y)+\xi,\phi+\xi)-\left(\partial_t\phi(t,x)+\mathscr{H}\left(t,x,\partial_x\phi(t,x),\partial_x^2\phi(t,x)\right)\right)\right|\\&\le C(1+|y|+|x|)(|y-x|)+|\partial_t\phi(t+\theta_9\delta_{ts},x+\theta_9\delta_{xy})||\delta_{ts}|\\&\quad+|\partial_x\phi(t+\theta_9\delta_{ts},x+\theta_9\delta_{xy})||\delta_{xy}|+|\xi|\end{aligned}$$

where $\theta_9 \in (0,1)$. Hence we obtain

$$\left|\mathscr{F}^{E,c}(s,y,\phi(s,y)+\xi,\phi+\xi) - \left(\partial_t\phi(t,x) + \mathscr{H}\left(t,x,\partial_x\phi(t,x),\partial_x^2\phi(t,x)\right)\right)\right| \to 0,$$
$$\text{as } \delta_t \to 0,\ \delta_x \to 0\ \xi \to 0 \text{ and } (s,y) \to (t,x).$$

which establishes the claim on $\{T\} \times (\mathcal{S} \cup \partial\mathcal{S})$.

□

Lemma 2.3.5, Lemma 2.3.6, Lemma 2.3.7 and the convergence framework as in Barles and Souganidis [BS91] lead the following theorem.

Theorem 2.3.8
Denote by $V^\delta(t,x)$ the solution of the numerical scheme $\mathscr{F}^{E,c} = 0$. Then the limit

$$V(t,x) = \lim_{\delta_t,\delta_x \to 0,\ (s,y)\to(t,x)} V^\delta(s,y)$$

exists and is a unique viscosity solution if the comparison theorem is established.

Proof. Lemma 2.3.5, Lemma 2.3.6 and Lemma 2.3.7 establish the claim by Barles and Souganidis [BS91]. □

2.3.2 implicit method

We approximate the HJB equation (2.4) using the forward difference Δ_t^+ for t and the central difference Δ_x^c for x. Straightforward replacement gives

$$\begin{aligned}
&\partial_t V(t,x) + \mathscr{H}(t,x,\partial_x V,\partial_{xx} V) \\
&\quad \simeq \Delta_t^+ V(t,x) + \mathscr{H}\left(t,x,\Delta_x^c V(t,x),\Delta_x^2 V(t,x)\right), \\
&\quad = \frac{V(t+\delta_t,x) - V(t,x)}{\delta_t} + \mathscr{H}\left(t,x,\Delta_x^c V(t,x),\Delta_x^2 V(t,x)\right),
\end{aligned}$$

for $(t,x) \in \left\{t_k \middle| k = 0,\cdots,N^t-1\right\} \times \mathcal{S}^\delta$. We define the implicit method $\mathscr{F}^{I,c}$ for the HJB equation (2.4) with central difference as follow.

Definition 2.3.9 (Implicit method for 1-dimensional HJB equation with central difference)

$$\mathscr{F}^{I,c}(t,x,V^\delta(t,x),[V^\delta]_{t,x})$$
$$= \begin{cases}
\dfrac{V^\delta(t+\delta_t,x) - V^\delta(t,x)}{\delta_t} + \mathscr{H}\left(t,x,\Delta_x^c V^\delta(t,x),\Delta_x^2 V^\delta(t,x)\right), & \\
\qquad\qquad (t,x) \in \left\{t_k \middle| k = 0,\cdots,N^t-1\right\} \times \mathcal{S}^\delta, & \\
g_T(x) - V^\delta(t,x), & (t,x) \in \partial\tilde{\mathbb{T}}^\delta \times (\mathcal{S} \cup \partial\mathcal{S}), \\
g_{\partial\mathcal{S}}(t,\varsigma(x)) - V^\delta(t,x), & (t,x) \in \tilde{\mathbb{T}}^\delta \times \partial\tilde{\mathcal{S}}^\delta, \\
\displaystyle\sum_{k=1}^{N^t-1}\sum_{i=1}^{N^x-1} V^\delta(t_k,x_i)\mathbf{1}_{\left\{t\in\left[t_k-\frac{\delta_t}{2},t_k+\frac{\delta_t}{2}\right),\ x\in\left[x_i-\frac{\delta_x}{2},x_i+\frac{\delta_x}{2}\right)\right\}} - V^\delta(t,x), & \text{otherwise.}
\end{cases}$$

where

$$\varsigma(x) = \operatorname*{argmin}_{y \in \partial\mathcal{S}} |x - y|.$$

As in the explicit method, the upwind scheme is available.

Definition 2.3.10 (Implicit method for 1-dimensional HJB equation with upwind difference)

$$\mathscr{F}^{E,u}(t, x, V^\delta(t,x), [V^\delta]_{t,x})$$
$$= \begin{cases} \dfrac{V^\delta(t+\delta_t, x) - V^\delta(t,x)}{\delta} + \tilde{\mathscr{H}}\left(t, x, \Delta_x^+ V^\delta(t,x), \Delta_x^- V^\delta(t,x), \Delta_x^2 V_t^\delta(t,x)\right), \\ \qquad\qquad (t,x) \in \left\{t_k \middle| \, k = 0, \cdots, N^t - 1\right\} \times \mathcal{S}^\delta, \\ g_T(x) - V^\delta(t,x), \qquad (t,x) \in \partial\tilde{\mathbb{T}}^\delta \times (\mathcal{S} \cup \partial\mathcal{S}), \\ g_{\partial\mathcal{S}}(t, \varsigma(x)) - V^\delta(t,x), \qquad (t,x) \in \tilde{\mathbb{T}}^\delta \times \partial\tilde{\mathcal{S}}^\delta, \\ \displaystyle\sum_{k=1}^{N^t-1} \sum_{i=1}^{N^x-1} V^\delta(t_k, x_i) \mathbf{1}_{\left\{t \in \left[t_k - \frac{\delta_t}{2}, t_k + \frac{\delta_t}{2}\right),\, x \in \left[x_i - \frac{\delta_x}{2}, x_i + \frac{\delta_x}{2}\right)\right\}} - V^\delta(t,x), \quad \text{otherwise.} \end{cases}$$

where

$$\varsigma(x) = \operatorname*{argmin}_{y \in \partial\mathcal{S}} |x - y|,$$

and $\tilde{\mathscr{H}}$ is an operator such that

$$\tilde{\mathscr{H}}(t, x, p^+, p^-, A) = \sup_{u \in \mathbb{U}} \left[\mu^+(t,x,u)p^+ - \mu^-(t,x,u)p^- + \frac{1}{2}a(t,x,u)A + f(t,x,u)\right].$$

Remark 2.3.11
Comparing the explicit method, the scheme $\mathscr{F}^{I,\cdot}$ does not give the explicit formula for $V^\delta(t,\cdot)$. As shown in below, we need to solve the linear equation of $V^\delta(t,\cdot)$. Hence the scheme $\mathscr{F}^{I,\cdot}$ is called the implicit method.

Remark 2.3.12
Although the convergence of the implicit method is treated as the well-kown result, best of our knowledge, there is no literature which gives the convergence proof.

To prove the convergence of the approximation scheme $\mathscr{F}^{I,u}$, all we need is to show the monotonicity, the stability and the consistency.

Lemma 2.3.13
The scheme $\mathscr{F}^{I,u}$ is monotone.

Proof. Let $y \in \mathbb{R}$ and $\phi, \psi \in \mathcal{B}(\mathcal{S})$ such that $\phi \le \psi$. We see that

$$\mathscr{F}^{I,u}\left(t, x, y, [\psi]_{t,x}\right) - \mathscr{F}^{I,u}\left(t, x, y, [\phi]_{t,x}\right) = g_T(x) - y - (g_T(x) - y) = 0,$$

for $(t,x) \in \partial\tilde{\mathbb{T}}^\delta \times (\mathcal{S} \cup \partial\mathcal{S})$,

$$\mathscr{F}^{I,u}\left(t,x,y,[\psi]_{t,x}\right) - \mathscr{F}^{I,u}\left(t,x,y,[\phi]_{t,x}\right) = (g_{\partial\mathcal{S}}(t,\varsigma(x)) - y) - (g_{\partial\mathcal{S}}(t,x) - y) = 0$$

for $(t,x) \in \tilde{\mathbb{T}}^\delta \times \partial\tilde{\mathcal{S}}^\delta$ and

$$\begin{aligned}
&\mathscr{F}^{I,u}\left(t,x,y,[\psi]_{t,x}\right) - \mathscr{F}^{I,u}\left(t,x,y,[\phi]_{t,x}\right)\\
&= \sum_{k=1}^{N^t-1}\sum_{i=1}^{N^x-1} \psi(t_k,x_i)\mathbf{1}_{\left\{t\in\left[t_k-\frac{\delta_t}{2},t_k+\frac{\delta_t}{2}\right),\ x\in\left[x_i-\frac{\delta_x}{2},x_i+\frac{\delta_x}{2}\right)\right\}} - y\\
&\quad - \left(\sum_{k=1}^{N^t-1}\sum_{i=1}^{N^x-1} \phi(t_k,x_i)\mathbf{1}_{\left\{t\in\left[t_k-\frac{\delta_t}{2},t_k+\frac{\delta_t}{2}\right),\ x\in\left[x_i-\frac{\delta_x}{2},x_i+\frac{\delta_x}{2}\right)\right\}} - y\right)\\
&= \sum_{k=1}^{N^t-1}\sum_{i=1}^{N^x-1} \left(\psi(t_k,x_i) - \phi(t_k,x_i)\right)\mathbf{1}_{\left\{t\in\left[t_k-\frac{\delta_t}{2},t_k+\frac{\delta_t}{2}\right),\ x\in\left[x_i-\frac{\delta_x}{2},x_i+\frac{\delta_x}{2}\right)\right\}}\\
&\geq 0,
\end{aligned}$$

for $(t,x) \in [0,T] \times (\mathcal{S} \cup \partial\mathcal{S}) \setminus (\left\{t_k \middle|\, k = 0,\cdots,N^t-1\right\} \times \mathcal{S}^\delta \cup \tilde{\mathbb{T}}^\delta \times (\mathcal{S} \cup \partial\mathcal{S}) \cup \tilde{\mathbb{T}}^\delta \times \partial\tilde{\mathcal{S}}^\delta)$.

The remaining is to show the case of $(t,x) \in \left\{t_k \middle|\, k = 0,\cdots,N^t-1\right\} \times \mathcal{S}^\delta$. Straightforward computations give

$$\begin{aligned}
&\mathscr{F}^{I,u}\left(t,x,y,[\psi]_{t,x}\right) - \mathscr{F}^{I,u}\left(t,x,y,[\phi]_{t,x}\right)\\
&= \frac{\psi(t+\delta_t,x)-y}{\delta_t} + \sup_{u_1\in\mathbb{U}}\Bigg[\mu^+(t,x,u_1)\frac{\psi(t,x+\delta_x)-y}{\delta_x} - \mu^-(t,x,u_1)\frac{y-\psi(t,x-\delta_x)}{\delta_x}\\
&\qquad\qquad + \frac{1}{2}a(t,x,u_1)\frac{\psi(t,x+\delta_x)-2y+\psi(t,x-\delta_x)}{\delta_x^2} + f(t,x,u_1)\Bigg]\\
&\quad - \frac{\phi(t+\delta_t,x)-y}{\delta_t} + \sup_{u_2\in\mathbb{U}}\Bigg[\mu^+(t,x,u_2)\frac{\phi(t,x+\delta_x)-y}{\delta_x} - \mu^-(t,x,u_2)\frac{y-\phi(t,x-\delta_x)}{\delta_x}\\
&\qquad\qquad + \frac{1}{2}a(t,x,u_2)\frac{\phi(t,x+\delta_x)-2y+\phi(t,x-\delta_x)}{\delta_x^2} + f(t,x,u_2)\Bigg]\\
&= \sup_{u_1\in\mathbb{U}}\Bigg[f(t,x,u_1) + \frac{\psi(t+\delta_t,x)}{\delta_t} - \left(\frac{1}{\delta_t} + \frac{a(t,x,u_1)}{\delta_x^2} + \frac{\mu^+(t,x,u_1)+\mu^-(t,x,u_1)}{\delta_x}\right)y\\
&\qquad + \left(\frac{a(t,x,u_1)}{2\delta_x^2} + \frac{\mu^+(t,x,u_1)}{\delta_x}\right)\psi(t,x+\delta_x) + \left(\frac{a(t,x,u_1)}{2\delta_x^2} + \frac{\mu^-(t,x,u_1)}{\delta_x}\right)\psi(t,x-\delta_x)\Bigg]\\
&\quad - \sup_{u_2\in\mathbb{U}}\Bigg[f(t,x,u_2) + \frac{\phi(t+\delta_t,x)}{\delta_t} - \left(\frac{1}{\delta_t} + \frac{a(t,x,u_2)}{\delta_x^2} + \frac{\mu^+(t,x,u_2)+\mu^-(t,x,u_2)}{\delta_x}\right)y\\
&\qquad + \left(\frac{a(t,x,u_2)}{2\delta_x^2} + \frac{\mu^+(t,x,u_2)}{\delta_x}\right)\phi(t,x+\delta_x) + \left(\frac{a(t,x,u_2)}{2\delta_x^2} + \frac{\mu^-(t,x,u_2)}{\delta_x}\right)\phi(t,x-\delta_x)\Bigg]
\end{aligned}$$

Since the coefficients of $\psi(t+\delta_t,x)$, $\psi(t,x+\delta_x)$ and $\psi(t,x-\delta_x)$ are non-negative, we find that

$$\begin{aligned}
&\mathscr{F}^{I.c}\left(t,x,y,[\psi]_{t,x}\right)-\mathscr{F}^{I.c}\left(t,x,y,[\phi]_{t,x}\right)\\
&\geq \sup_{u_1\in\mathbb{U}}\Bigg[f(t,x,u_1)+\frac{\phi(t+\delta_t,x)}{\delta_t}-\left(\frac{1}{\delta_t}+\frac{a(t,x,u_1)}{\delta_x^2}+\frac{\mu^+(t,x,u_1)+\mu^-(t,x,u_1)}{\delta_x}\right)y\\
&\qquad+\left(\frac{a(t,x,u_1)}{2\delta_x^2}+\frac{\mu^+(t,x,u_1)}{\delta_x}\right)\phi(t,x+\delta_x)+\left(\frac{a(t,x,u_1)}{2\delta_x^2}+\frac{\mu^-(t,x,u_1)}{\delta_x}\right)\phi(t,x-\delta_x)\Bigg]\\
&\quad-\sup_{u_2\in\mathbb{U}}\Bigg[f(t,x,u_2)+\frac{\phi(t+\delta_t,x)}{\delta_t}-\left(\frac{1}{\delta_t}+\frac{a(t,x,u_2)}{\delta_x^2}+\frac{\mu^+(t,x,u_2)+\mu^-(t,x,u_2)}{\delta_x}\right)y\\
&\qquad+\left(\frac{a(t,x,u_2)}{2\delta_x^2}+\frac{\mu^+(t,x,u_2)}{\delta_x}\right)\phi(t,x+\delta_x)+\left(\frac{a(t,x,u_2)}{2\delta_x^2}+\frac{\mu^-(t,x,u_2)}{\delta_x}\right)\phi(t,x-\delta_x)\Bigg]\\
&\quad=0
\end{aligned}$$

which establishes the claim. □

Lemma 2.3.14
The scheme $\mathscr{F}^{I,u}$ is stable.

Proof. Let V^δ be a solution of $\mathscr{F}^{I,c}(t,x,V^\delta(t,x),[V^\delta]_{t,x})=0$. The existence and boundedness of V^δ are obvious on the boundary $\tilde{\mathbb{T}}^\delta\times\partial\tilde{\mathcal{S}}^\delta$ and the terminal $\partial\tilde{\mathbb{T}}^\delta\times(\mathcal{S}\cup\partial\mathcal{S})$. If we are able to show the existence and boundedness of V^δ on the grids $\left\{t_k|\ k=0,\cdots,N^t-1\right\}\times\mathcal{S}^\delta$, there exist bounded V^δ on the other region since it takes same value on the nearest grid.

We show the existence of V^δ on the grids $\left\{t_k|\ k=0,\cdots,N^t-1\right\}\times\mathcal{S}^\delta$. Note that

$$\begin{aligned}
&\mathscr{F}^{I,u}\left(t_k,x_i,V^\delta,[V^\delta]_{t_k,x_i}\right)\\
&=\frac{V^\delta(t_{k+1},x_i)-V^\delta(t_k,x_i)}{\delta_t}\\
&\quad+\sup_{u\in\mathbb{U}}\Bigg[\mu^+(t_k,x_i,u)\frac{V^\delta(t_k,x_{i+1})-V^\delta(t_k,x_i)}{\delta_x}-\mu^-(t_k,x_i,u)\frac{V^\delta(t_k,x_i)-V^\delta(t_k,x_{i-1})}{\delta_x}\\
&\qquad+\frac{1}{2}a(t_k,x_i,u)\frac{V^\delta(t_k,x_{i+1})-2V^\delta(t_k,x_i)+V^\delta(t_k,x_{i-1})}{\delta_x^2}+f(t_k,x_i,u)\Bigg]\\
&=\sup_{u\in\mathbb{U}}\Bigg[f(t_k,x_i,u)+\frac{V^\delta(t_{k+1},x_i)}{\delta_t}\\
&\qquad-\left(\frac{1}{\delta_t}+\frac{a(t_k,x_i,u)}{\delta_x^2}+\frac{\mu^+(t_k,x_i,u)+\mu^-(t_k,x_i,u)}{\delta_x}\right)V^\delta(t_k,x_i)\\
&\qquad+\left(\frac{a(t_k,x_i,u)}{2\delta_x^2}+\frac{\mu^+(t_k,x_i,u)}{\delta_x}\right)V^\delta(t,x_{i+1})\\
&\qquad+\left(\frac{a(t_k,x_i,u)}{2\delta_x^2}+\frac{\mu^-(t_k,x_i,u)}{\delta_x}\right)V^\delta(t,x_{i-1})\Bigg],\quad i=1,\cdots,N^x-1,
\end{aligned}$$

Define $\tilde{V} \in \mathbb{R}^{N^x+1}$ by $\tilde{V}_i^k = V^\delta(t_k, x_i)$, $i = 0, \cdots, N^x$ and the scheme $\mathscr{F}^{I,u}$ is displayed in the matrix form:

$$\sup_{u_1,\cdots,u_{N^x-1}\in\mathbb{U}} \left[L^{u,k}\tilde{V}^k + \tilde{f}^{u,k} + \tilde{V}^{k+1}\right] = 0, \quad k = 0, \cdots, N^t - 1,$$

where $L^{u,k} \in \mathbb{R}^{(N^x+1)\times(N^x+1)}$ is given by

$$L_{ij}^{u,k} = \begin{cases} 1, & i \in \{0, N^x\},\ j = i, \\ -1 - \dfrac{\delta_t}{\delta_x^2} a(t_k, x_i, u_i) - \dfrac{\delta_t}{\delta_x}\left(\mu^+(t_k, x_i, u_i) + \mu^-(t_k, x_i, u_i)\right), & i \notin \{0, N^x\},\ j = i, \\ \dfrac{\delta_t}{2\delta_x^2} a(t_k, x_i, u_i) + \dfrac{\delta_t}{\delta_x}\mu^+(t_k, x_i, u_i), & i \notin \{0, N^x\},\ j = i+1, \\ \dfrac{\delta_t}{2\delta_x^2} a(t_k, x_i, u_i) + \dfrac{\delta_t}{\delta_x}\mu^-(t_k, x_i, u_i), & i \notin \{0, N^x\},\ j = i-1, \\ 0, & \text{otherwise}, \end{cases}$$

and $\tilde{f}^{u,k}$ by

$$\tilde{f}_i^{u,k} = \begin{cases} g_{\partial\mathcal{S}}(t_k, x_i), & i = 0, \\ \delta_t f(t_k, x_i, u_i), & i = 1, \cdots, N^x - 1, \\ g_{\partial\mathcal{S}}(t_k, x_i), & i = N^x, \\ 0, & \text{otherwise}. \end{cases}$$

For any u, we find that

$$\begin{aligned} \sum_{j\neq i} |L_{ij}^{u,k}| &= \left|\frac{\delta_t}{2\delta_x^2} a(t_k, x_i, u_i) + \frac{\delta_t}{\delta_x}\mu^+(t_k, x_i, u_i)\right| + \left|\frac{\delta_t}{2\delta_x^2} a(t_k, x_i, u_i) + \frac{\delta_t}{\delta_x}\mu^-(t_k, x_i, u_i)\right| \\ &= \frac{\delta_t}{\delta_x^2} a(t_k, x_i, u_i) + \frac{\delta_t}{\delta_x}\left(\mu^+(t_k, x_i, u_i) + \mu^-(t_k, x_i, u_i)\right) \\ &< 1 + \frac{\delta_t}{\delta_x^2} a(t_k, x_i, u_i) + \frac{\delta_t}{\delta_x}\left(\mu^+(t_k, x_i, u_i) + \mu^-(t_k, x_i, u_i)\right) \\ &= |L_{ii}^{u,k}|, \qquad i = 1, \cdots, N^x - 1. \end{aligned}$$

Hence $L^{u,k}$ is strictly diagonally dominant matrix and thus is invertible (see e.g. [Var09]). Recall that $\tilde{V}_i^{N^t} = g_T(x_i)$, $i = 0, \cdots, N^x$ and we observe that $(L^{u,N^t-1})^{-1}(-\tilde{f}^{u,N^t-1} - \tilde{V}^{N^t})$ exists for any u. This implies the existence of $\tilde{V}^{N^t-1}$. For arbitrary $k \in \{0, \cdots, N^t - 2\}$, the existence of $\tilde{V}^k$ is obtained by the induction.

We next show the boundedness of $\tilde{V}^k$. Define $\phi^{u,k} \in \mathbb{R}^{N^x+1}$ as a solution of

$$L^{u,k}\phi^{u,k} + \tilde{f}^{u,k} + \tilde{V}^{k+1} = 0,$$

and

$$i^* = \underset{i}{\operatorname{argmax}}\, |\phi_i^{u,k}|.$$

If $i^* = 0$ or N^x, the it is trivial that

$$|\phi_{i^*}^{u,k}| = |g_{\partial\mathcal{S}}(t_k, x_{i^*})| \leq C_2,$$

where

$$C_2 = \sup_{s\in[0,T), y\in\partial\mathcal{S}} |g_{\partial\mathcal{S}}(s,y)|.$$

Let us consider the case that $i^* \in \{1, \cdots, N^x - 1\}$. Because $L_{ij}^{u,k} \geq 0$ for $i \neq j$ and $|\phi_{i^*}^{u,k}| \geq \mathrm{sign}(\phi_{i^*}^{u,k})\phi_j^{u,k}$, we see that

$$\sum_{j\neq i^*} L_{i^*j}^{u,k}|\phi_{i^*}^{u,k}| \geq \mathrm{sign}(\phi_{i^*}^{u,k}) \sum_{j\neq i^*} L_{i^*j}^{u,k}\phi_j^{u,k}.$$

Since $\sum_j L_{ij}^{u,k} = -1$ for $i \in \{1, \cdots, N^x - 1\}$, we have

$$\begin{aligned}
|\phi_{i^*}^{u,k}| &= -\sum_j L_{i^*j}^{u,k}|\phi_{i^*}^{u,k}| \\
&= -\sum_{j\neq i^*} L_{i^*j}^{u,k}|\phi_{i^*}^{u,k}| - L_{i^*i^*}^{u,k}\mathrm{sign}(\phi_{i^*}^{u,k})\phi_{i^*}^{u,k} \\
&\leq -\mathrm{sign}(\phi_{i^*}^{u,k})\sum_{j\neq i^*} L_{i^*j}^{u,k}\phi_j^{u,k} - \mathrm{sign}(\phi_{i^*}^{u,k})L_{i^*i^*}^{u,k}\phi_{i^*}^{u,k} \\
&= -\mathrm{sign}(\phi_{i^*}^{u,k})\sum_j L_{i^*j}^{u,k}\phi_j^{u,k} \\
&= \mathrm{sign}(\phi_{i^*}^{u,k})\left(\tilde{f}_{i^*}^{u,k} + \tilde{V}_{i^*}^{k+1}\right) \\
&\leq |\tilde{f}_{i^*}^{u,k}| + |\tilde{V}_{i^*}^{k+1}| \\
&\leq \delta_t C_3 + \max_i |\tilde{V}_i^{k+1}|,
\end{aligned}$$

where

$$C_3 = \sup_{s\in[0,T],\ y\in\mathcal{S},\ u\in\mathbb{U}} |f(s,y,u)|$$

The above discussion is valid for arbitrary u and hence we obtain

$$|\tilde{V}_i^k| \leq \max\left[C_2, \delta_t C_3 + \max_i |\tilde{V}_i^{k+1}|\right].$$

This evaluation implies that

$$|\tilde{V}_i^k| \leq \max[C1, C2] + (N^t - k)\delta_t C_3.$$

where

$$C_1 = \sup_{y\in\mathcal{S}\cup\partial\mathcal{S}} |g_T(y)|.$$

We are able to show the above by induction. For $k = N^t$,

$$|\tilde{V}_i^k| = |g_T(x_i)| \le C_1 \le \max[C1, C2] + (N^t - N^t)\delta_t C_3.$$

Assuming the claim is established for $k = k' + 1$,

$$\begin{aligned}
|\tilde{V}_i^{k'}| &\le \max\left[C_2, \delta_t C_3 + \max_i |\tilde{V}_i^{k'+1}|\right] \\
&\le \max\left[C_2, \delta_t C_3 + \max[C1, C2] + (N^t - (k'+1))\delta_t C_3\right] \\
&\le \max\left[C_2, \max[C1, C2] + (N^t - k')\delta_t C_3\right] \\
&\le \max[C1, C2] + (N^t - k')\delta_t C_3.
\end{aligned}$$

Note that $(N^t - k)\delta_t C_3 \le TC_3$, $k = 0, 1, \cdots, N^t - 1$ and we obtain

$$|\tilde{V}_i^{k'}| \le \max[C1, C2] + TC_3,$$

which establishes the claim. □

Lemma 2.3.15
The scheme $\mathscr{F}^{I,u}$ is consistent.

Proof. Let $\phi \in C^\infty([0, T] \times (\mathcal{S}^\delta \cup \partial\mathcal{S}))$.

We first consider the case that $(t, x) \in [0, T) \times \mathcal{S}$. Suppose that δ_t and δ_x are small enough such that $(t, x) \in \tilde{\mathbb{T}}^\delta \times \tilde{\mathcal{S}}^\delta$. More precisely, we assume that δ_t and δ_x satisfy

$$\min(t, T - t) > \frac{1}{2}\delta_t, \quad \min_{x' \in \partial\mathcal{S}} |x - x'| > \frac{1}{2}\delta_x.$$

We choose (s, y) as follow:

$$s = \operatorname*{argmin}_{1 \le k \le N^t - 1} |t - t_k|, \quad y = \operatorname*{argmin}_{x' \in \mathcal{S}^\delta} |x - x'|.$$

Then we find

$$\begin{aligned}
&\left|\mathscr{F}^{I,u}(s, y, \phi(s, y) + \xi, \phi + \xi) - \left(\partial_t \phi(t, x) + \mathscr{H}\left(t, x, \partial_x \phi(t, x), \partial_x^2 \phi(t, x)\right)\right)\right| \\
&= \Bigg| \frac{\phi(s + \delta_t, y) + \xi - (\phi(s, y) + \xi)}{\delta_t} \\
&\quad + \sup_{u \in \mathbb{U}} \Bigg[f(s, y, u) + \mu^+(s, y, u) \frac{\phi(s, y + \delta_x) + \xi - (\phi(s, y) + \xi)}{\delta_x} \\
&\qquad - \mu^-(s, y, u) \frac{\phi(s, y) + \xi - (\phi(s, y - \delta_x) + \xi)}{\delta_x} \\
&\qquad + \frac{1}{2} a(s, y, u) \frac{\phi(s, y + \delta_x) + \xi - 2(\phi(s, y) + \xi) + \phi(s, y - \delta_x) + \xi}{\delta_x^2} \Bigg] \\
&\quad - \partial_t \phi(t, x) - \sup_{u \in \mathbb{U}} \left[\mu(t, x, u) \partial_x \phi(t, x) + \frac{1}{2} a(t, x, u) \partial_x^2 \phi(t, x) + f(t, x, u) \right] \Bigg|
\end{aligned}$$

where $\xi \in \mathbb{R}$. Note that for $A \subset \mathbb{R}^n$ and $f_1, f_2 : A \to \mathbb{R}$,

$$\sup_{x\in A} f_1(x) - \sup_{x\in A} f_2(x) \leq \sup_{x\in A} \left(f_1(x) - f_2(x)\right) ,$$

and we have

$$\begin{aligned}
&\left|\mathscr{F}^{I,u}(s,y,\phi(s,y)+\xi,\phi+\xi) - \left(\partial_t\phi(t,x) + \mathscr{H}\left(t,x,\partial_x\phi(t,x),\partial_x^2\phi(t,x)\right)\right)\right| \\
&\leq \Bigg|\frac{\phi(s+\delta_t,y)-\phi(s,y)}{\delta_t} - \partial_t\phi(t,x) \\
&\quad + \sup_{u\in\mathbb{U}}\Big[f(s,y,u) - f(t,x,u) \\
&\qquad + \mu^+(s,y,u)\frac{\phi(s,y+\delta_x)-\phi(s,y)}{\delta_x} - \mu^-(s,y,u)\frac{\phi(s,y)-\phi(s,y-\delta_x)}{\delta_x} \\
&\qquad - \mu(t,x,u)\partial_x\phi(t,x) \\
&\qquad + \frac{1}{2}a(s,y,u)\frac{\phi(s,y+\delta_x)+-2\phi(s,y)+\phi(s,y-\delta_x)}{\delta_x^2} - \frac{1}{2}a(t,x,u)\partial_x^2\phi(t,x)\Big]\Bigg| \\
&\leq \left|\frac{\phi(s+\delta_t,y)-\phi(s,y)}{\delta_t} - \partial_t\phi(t,x)\right| + \sup_{u\in\mathbb{U}}|f(s,y,u)-f(t,x,u)| \\
&\quad + \sup_{u\in\mathbb{U}}\Bigg|\mu^+(s,y,u)\frac{\phi(s,y+\delta_x)-\phi(s,y)}{\delta_x} - \mu^-(s,y,u)\frac{\phi(s,y)-\phi(s,y-\delta_x)}{\delta_x} \\
&\qquad\qquad -\mu(t,x,u)\partial_x\phi(t,x)\Bigg| \\
&\quad + \frac{1}{2}\sup_{u\in\mathbb{U}}\left|a(s,y,u)\frac{\phi(s,y+\delta_x)+-2\phi(s,y)+\phi(s,y-\delta_x)}{\delta_x^2} - a(t,x,u)\partial_x^2\phi(t,x)\right|.
\end{aligned}$$

Denote by $\delta_{ts} = s - t$ and $\delta_{xy} = y - x$ and we evaluate each term.

We first evaluate the time derivative term. The Taylor expansion gives

$$\begin{aligned}
&\left|\frac{\phi(s+\delta_t,y)-\phi(s,y)}{\delta_t} - \partial_t\phi(t,x)\right| \\
&= |\partial_t\phi(s,y) + \partial_t^2\phi(s+\theta_1\delta_t,y)\delta_t - \partial_t\phi(t,x)| \\
&= |\partial_t\phi(t+\delta_{ts},x+\delta_{xy}) + \partial_t^2\phi(s+\theta_1\delta_t,y)\delta_t - \partial_t\phi(t,x)| \\
&= |\partial_t^2\phi(s+\theta_1\delta_t,y)\delta_t + \partial_t^2\phi(t+\theta_2\delta_t,x+\theta_2\delta_{xy})\delta_{ts} \\
&\qquad + \partial_x\partial_t\phi(t+\theta_2\delta_t,x+\theta_2\delta_{xy})\delta_{xy}|,
\end{aligned}$$

where $\theta_1, \theta_2 \in (0,1)$ and hence

$$\left|\frac{\phi(s+\delta_t,y)-\phi(s,y)}{\delta_t} - \partial_t\phi(t,x)\right| \to 0 \quad \text{as } \delta_t \to 0 \text{ and } (s,y) \to (t,x).$$

We next evaluate the terms related to the first derivative of x. Using Taylor expansion, we find

$$\begin{aligned}\frac{\phi(s,y+\delta_x)-\phi(s,y)}{\delta_x} &= \partial_x\phi(s,y)+\frac{1}{2}\partial_x^2\phi(s,y+\theta_3\delta_x)\delta_x \\ &= \partial_x\phi(t,x)+\frac{1}{2}\partial_x^2\phi(s,y+\theta_3\delta_x)\delta_x \\ &\quad+\partial_t\partial_x\phi(t+\theta_5\delta_{ts},x+\theta_5\delta_{xy})\delta_{ts}+\partial_x^2\phi(t+\theta_5\delta_{ts},x+\theta_5\delta_{xy})\delta_{xy}.\end{aligned}$$

and

$$\begin{aligned}\frac{\phi(s,y)-\phi(s,y-\delta_x)}{\delta_x} &= \partial_x\phi(s,y)-\frac{1}{2}\partial_x^2\phi(s,y-\theta_4\delta_x)\delta_x \\ &= \partial_x\phi(t,x)-\frac{1}{2}\partial_x^2\phi(s,y-\theta_4\delta_x)\delta_x \\ &\quad+\partial_t\partial_x\phi(t+\theta_5\delta_{ts},x+\theta_5\delta_{xy})\delta_{ts}+\partial_x^2\phi(t+\theta_5\delta_{ts},x+\theta_5\delta_{xy})\delta_{xy}.\end{aligned}$$

where $\theta_i\in(0,1)$, $i=3,4,5$. Thus we have

$$\begin{aligned}&\sup_{u\in\mathbb{U}}\Bigg|\mu^+(s,y,u)\frac{\phi(s,y+\delta_x)-\phi(s,y)}{\delta_x}-\mu^-(s,y,u)\frac{\phi(s,y)-\phi(s,y-\delta_x)}{\delta_x} \\ &\qquad -\mu(t,x,u)\partial_x\phi(t,x)\Bigg| \\ &\le \sup_{u\in\mathbb{U}}|\mu^+(s,y,u)-\mu^-(s,y,u)-\mu(t,x,u)||\partial_x\phi(t,x)| \\ &\quad+\sup_{u\in\mathbb{U}}|\mu^+(s,y,u)|\Bigg|\frac{1}{2}\partial_x^2\phi(s,y+\theta_3\delta_x)\delta_x+\partial_t\partial_x\phi(t+\theta_5\delta_{ts},x+\theta_5\delta_{xy})\delta_{ts} \\ &\qquad +\partial_x^2\phi(t+\theta_5\delta_{ts},x+\theta_5\delta_{xy})\delta_{xy}\Bigg| \\ &\quad+\sup_{u\in\mathbb{U}}|\mu^-(s,y,u)|\Bigg|\frac{1}{2}\partial_x^2\phi(s,y-\theta_4\delta_x)\delta_x+\partial_t\partial_x\phi(t+\theta_5\delta_{ts},x+\theta_5\delta_{xy})\delta_{ts} \\ &\qquad +\partial_x^2\phi(t+\theta_5\delta_{ts},x+\theta_5\delta_{xy})\delta_{xy}\Bigg|.\end{aligned}$$

The Lipschitz continuity and the linear growth condition of μ implies

$$\begin{aligned}&\sup_{u\in\mathbb{U}}|\mu^+(s,y,u)-\mu^-(s,y,u)-\mu(t,x,u)||\partial_x\phi(t,x)| \\ &\quad=\sup_{u\in\mathbb{U}}|\mu(s,y,u)-\mu(t,x,u)||\partial_x\phi(t,x)| \\ &\quad\le\sup_{u\in\mathbb{U}}C(|s-t|+|y-x|)|\partial_x\phi(t,x)| \\ &\quad=C(|\delta_{ts}|+|\delta_{xy}|)|\partial_x\phi(t,x)|,\end{aligned}$$

$$
\begin{aligned}
&\sup_{u\in\mathbb{U}} |\mu^+(s,y,u)| \bigg| \frac{1}{2}\partial_x^2\phi(s,y+\theta_3\delta_x)\delta_x + \partial_t\partial_x\phi(t+\theta_5\delta_{ts}, x+\theta_5\delta_{xy})\delta_{ts} \\
&\qquad\qquad\qquad + \partial_x^2\phi(t+\theta_5\delta_{ts}, x+\theta_5\delta_{xy})\delta_{xy} \bigg| \\
&\leq C(1+|y|+\sup_{u\in\mathbb{U}}|u|) \bigg| \frac{1}{2}\partial_x^2\phi(s,y+\theta_3\delta_x)\delta_x + \partial_t\partial_x\phi(t+\theta_5\delta_{ts}, x+\theta_5\delta_{xy})\delta_{ts} \\
&\qquad\qquad\qquad + \partial_x^2\phi(t+\theta_5\delta_{ts}, x+\theta_5\delta_{xy})\delta_{xy} \bigg|,
\end{aligned}
$$

and

$$
\begin{aligned}
&\sup_{u\in\mathbb{U}} |\mu^-(s,y,u)| \bigg| \frac{1}{2}\partial_x^2\phi(s,y-\theta_4\delta_x)\delta_x + \partial_t\partial_x\phi(t+\theta_5\delta_{ts}, x+\theta_5\delta_{xy})\delta_{ts} \\
&\qquad\qquad\qquad + \partial_x^2\phi(t+\theta_5\delta_{ts}, x+\theta_5\delta_{xy})\delta_{xy} \bigg| \\
&\leq C(1+|y|+\sup_{u\in\mathbb{U}}|u|) \bigg| \frac{1}{2}\partial_x^2\phi(s,y-\theta_4\delta_x)\delta_x + \partial_t\partial_x\phi(t+\theta_5\delta_{ts}, x+\theta_5\delta_{xy})\delta_{ts} \\
&\qquad\qquad\qquad + \partial_x^2\phi(t+\theta_5\delta_{ts}, x+\theta_5\delta_{xy})\delta_{xy} \bigg|.
\end{aligned}
$$

Recall that $\mathbb{U}$ is compact and we obtain

$$
\begin{aligned}
&\sup_{u\in\mathbb{U}} \bigg| \mu^+(s,y,u)\frac{\phi(s,y+\delta_x)-\phi(s,y)}{\delta_x} - \mu^-(s,y,u)\frac{\phi(s,y)-\phi(s,y-\delta_x)}{\delta_x} \\
&\qquad -\mu(t,x,u)\partial_x\phi(t,x) \bigg| \to 0, \quad \text{as } \delta_x \to 0 \text{ and } (s,y)\to(t,x)
\end{aligned}
$$

The terms related to the second derivative of x are evaluated in the same manner. Then, Taylor expansion gives

$$
\begin{aligned}
&\frac{\phi(s,y+\delta_x) - 2\phi(s,y) + \phi(s,y-\delta_x)}{\delta_x^2} \\
&= \partial_x\phi(s,y) + \frac{1}{12}\left(\partial_x^4\phi(s,y+\theta_6\delta_x) + \partial_x^4\phi(s,y-\theta_7\delta_x)\right)\delta_x^2 \\
&= \partial_x\phi(t,x) + \frac{1}{12}\left(\partial_x^4\phi(s,y+\theta_6\delta_x) + \partial_x^4\phi(s,y-\theta_7\delta_x)\right)\delta_x^2 \\
&\quad + \partial_t\partial_x\phi(t+\theta_5\delta_{ts}, x+\theta_5\delta_{xy})\delta_{ts} + \partial_x^2\phi(t+\theta_5\delta_{ts}, x+\theta_5\delta_{xy})\delta_{xy},
\end{aligned}
$$

where $\theta_i \in (0,1)$, $i = 5,6,7$. Hence we see

$$\begin{aligned}
&\sup_{u\in\mathbb{U}}\Big|a(s,y,u)\frac{\phi(s,y+\delta_x)+-2\phi(s,y)+\phi(s,y-\delta_x)}{\delta_x^2}\\
&\qquad -a(t,x,u)\partial_x^2\phi(t,x)\Big|\\
&\le \sup_{u\in\mathbb{U}}|a(s,y,u)-a(t,x,u)||\partial_x\phi(t,x)|\\
&\quad+\sup_{u\in\mathbb{U}}|a(s,y,u)|\Big|\frac{1}{12}\left(\partial_x^4\phi(s,y+\theta_6\delta_x)+\partial_x^4\phi(s,y-\theta_7\delta_x)\right)\delta_x^2\\
&\quad+\partial_t\partial_x\phi(t+\theta_5\delta_{ts},x+\theta_5\delta_{xy})\delta_{ts}+\partial_x^2\phi(t+\theta_5\delta_{ts},x+\theta_5\delta_{xy})\delta_{xy}\Big|.
\end{aligned}$$

Using the Lipschitz continuity and the linear growth condition of σ, we have

$$\begin{aligned}
|a(s,y,u)-a(t,x,u)| &= |\sigma^2(s,y,u)-\sigma^2(t,x,u)|\\
&\le \left(|\sigma(s,y,u)|+|\sigma(t,x,u)|\right)|\sigma(s,y,u)-\sigma(t,x,u)|\\
&\le C^2(2+|y|+|x|+2|u|)(|\delta_{ts}|+|\delta_{xy}|),
\end{aligned}$$

and

$$|a(s,y,u)| \le C^2(1+|y|+|u|)^2.$$

Hence we obtain

$$\begin{aligned}
&\sup_{u\in\mathbb{U}}|a(s,y,u)-a(t,x,u)||\partial_x\phi(t,x)|\\
&\qquad\le C^2(2+|y|+|x|+2\sup_{u\in\mathbb{U}}|u|)(|\delta_{ts}|+|\delta_{xy}|)|\partial_x\phi(t,x)|
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{u\in\mathbb{U}}|a(s,y,u)|\Big|\frac{1}{12}\left(\partial_x^4\phi(s,y+\theta_6\delta_x)+\partial_x^4\phi(s,y-\theta_7\delta_x)\right)\delta_x^2\\
&\quad+\partial_t\partial_x\phi(t+\theta_5\delta_{ts},x+\theta_5\delta_{xy})\delta_{ts}+\partial_x^2\phi(t+\theta_5\delta_{ts},x+\theta_5\delta_{xy})\delta_{xy}\Big|\\
&\le \sup_{u\in\mathbb{U}}C^2(1+|y|+|u|)^2\Big|\frac{1}{12}\left(\partial_x^4\phi(s,y+\theta_6\delta_x)+\partial_x^4\phi(s,y-\theta_7\delta_x)\right)\delta_x^2\\
&\quad+\partial_t\partial_x\phi(t+\theta_5\delta_{ts},x+\theta_5\delta_{xy})\delta_{ts}+\partial_x^2\phi(t+\theta_5\delta_{ts},x+\theta_5\delta_{xy})\delta_{xy}\Big|.
\end{aligned}$$

which imply

$$\begin{aligned}
&\sup_{u\in\mathbb{U}}\Big|a(s,y,u)\frac{\phi(s,y+\delta_x)+-2\phi(s,y)+\phi(s,y-\delta_x)}{\delta_x^2}\\
&\qquad -a(t,x,u)\partial_x^2\phi(t,x)\Big|\to 0, \qquad \text{as } \delta_x\to 0 \text{ and } (s,y)\to(t,x).
\end{aligned}$$

By the assumption of f, the remaining evaluation is almost obvious:

$$\sup_{u\in\mathbb{U}}|f(s,y,u)-f(t,x,u)| \le C(1+|y|+|x|+\sup_{u\in\mathbb{U}}|u|)(|\delta_{ts}|+|\delta_{xy}|)$$

and thus

$$\sup_{u\in\mathbb{U}} |f(s,y,u) - f(t,x,u)| \to 0, \quad \text{as } \delta_x \to 0 \text{ and } (s,y) \to (t,x).$$

Therefore the claim is established on $(t,x) \in [0,T) \times \mathcal{S}$.

We next consider the claim in the case that $(t,x) \in [0,T) \times \partial\mathcal{S}$. Suppose that $(s,y) \in \mathbb{T}^\delta \times \partial\tilde{\mathcal{S}}^\delta$ and then

$$\begin{aligned}
&\left|\mathscr{F}^{I,u}(s,y,\phi(s,y)+\xi,\phi+\xi) - \left(\partial_t\phi(t,x) + \mathscr{H}\left(t,x,\partial_x\phi(t,x),\partial_x^2\phi(t,x)\right)\right)\right| \\
&= \left|g_{\partial\mathcal{S}}(s,\varsigma(y)) - (\phi(s,y)+\xi) - (g_{\partial\mathcal{S}}(t,x) - \phi(t,x))\right|.
\end{aligned}$$

The assumption imposed to $g_{\partial\mathcal{S}}$ gives

$$\begin{aligned}
&\left|\mathscr{F}^{I,u}(s,y,\phi(s,y)+\xi,\phi+\xi) - \left(\partial_t\phi(t,x) + \mathscr{H}\left(t,x,\partial_x\phi(t,x),\partial_x^2\phi(t,x)\right)\right)\right| \\
&\le C(1+|\varsigma(y)|+|x|)(|s-t|+|y-x|) + |\partial_t\phi(t+\theta_8\delta_{ts}, x+\theta_8\delta_{xy})||\delta_{ts}| \\
&\quad + |\partial_x\phi(t+\theta_8\delta_{ts}, x+\theta_8\delta_{xy})||\delta_{xy}| + |\xi|
\end{aligned}$$

where $\theta_8 \in (0,1)$. Hence we obtain

$$\begin{aligned}
\left|\mathscr{F}^{I,u}(s,y,\phi(s,y)+\xi,\phi+\xi) - \left(\partial_t\phi(t,x) + \mathscr{H}\left(t,x,\partial_x\phi(t,x),\partial_x^2\phi(t,x)\right)\right)\right| &\to 0, \\
\text{as } \delta_t \to 0,\ \delta_x \to 0\ \xi \to 0 \text{ and } (s,y) \to (t,x)&.
\end{aligned}$$

which establishes the claim on $[0,T) \times \partial\mathcal{S}$.

Finally we consider the claim in the case that $(t,x) \in \{T\} \times (\mathcal{S} \cup \partial\mathcal{S})$. Suppose that $(s,y) \in \partial\mathbb{T}^\delta \times (\mathcal{S} \cup \partial\mathcal{S})$ and then

$$\begin{aligned}
&\left|\mathscr{F}^{I,u}(s,y,\phi(s,y)+\xi,\phi+\xi) - \left(\partial_t\phi(t,x) + \mathscr{H}\left(t,x,\partial_x\phi(t,x),\partial_x^2\phi(t,x)\right)\right)\right| \\
&= \left|g_T(y) - (\phi(s,y)+\xi) - (g_T(x) - \phi(T,x))\right|.
\end{aligned}$$

The assumption imposed on g_T gives

$$\begin{aligned}
&\left|\mathscr{F}^{I,u}(s,y,\phi(s,y)+\xi,\phi+\xi) - \left(\partial_t\phi(t,x) + \mathscr{H}\left(t,x,\partial_x\phi(t,x),\partial_x^2\phi(t,x)\right)\right)\right| \\
&\le C(1+|y|+|x|)(|y-x|) + |\partial_t\phi(t+\theta_9\delta_{ts}, x+\theta_9\delta_{xy})||\delta_{ts}| \\
&\quad + |\partial_x\phi(t+\theta_9\delta_{ts}, x+\theta_9\delta_{xy})||\delta_{xy}| + |\xi|
\end{aligned}$$

where $\theta_9 \in (0,1)$. Hence we obtain

$$\begin{aligned}
\left|\mathscr{F}^{I,u}(s,y,\phi(s,y)+\xi,\phi+\xi) - \left(\partial_t\phi(t,x) + \mathscr{H}\left(t,x,\partial_x\phi(t,x),\partial_x^2\phi(t,x)\right)\right)\right| &\to 0, \\
\text{as } \delta_t \to 0,\ \delta_x \to 0\ \xi \to 0 \text{ and } (s,y) \to (t,x)&.
\end{aligned}$$

which establishes the claim on $\{T\} \times (\mathcal{S} \cup \partial\mathcal{S})$.

□

Using Lemma 2.3.13, Lemma 2.3.14, Lemma 2.3.15 and the convergence framework as in Barles and Souganidis [BS91], we find the following result.

Theorem 2.3.16
Denote by $V^\delta(t,x)$ the solution of the numerical scheme $\mathscr{F}^{I,u}=0$. Then the limit

$$V(t,x)=\lim_{\delta_t,\delta_x\to 0,\ (s,y)\to(t,x)} V^\delta(s,y)$$

exists and is a unique viscosity solution if the comparison theorem is established.

Proof. Lemma 2.3.13, Lemma 2.3.14 and Lemma 2.3.15 establish the claim by Barles and Souganidis [BS91]. □

2.4 Multi dimensional Hamilton-Jacobi-Bellman quasi-variational inequality

Let us construct a numerical scheme for a n-dimensional HJBQVI

$$\max\left[\partial_t V(t,x)+\mathscr{H}(t,x,\partial_x V,\partial_x^2 V),\mathscr{M}V(t,x)-V(t,x)\right]=0,\quad (t,x)\in[0,T)\times\mathcal{S},\tag{2.8}$$

with terminal and boundary conditions

$$V(T,x)=g_T(x),\quad x\in\mathcal{S}\cup\partial\mathcal{S},\tag{2.9}$$
$$V(t,x)=g_{\partial\mathcal{S}}(x),\quad (t,x)\in[0,T)\times\partial\mathcal{S},\tag{2.10}$$

where $\mathcal{S}=\prod_{i=1}^n(x_i^{\min},x_i^{\max})$, $\partial\mathcal{S}=\bar{\mathcal{S}}\setminus\mathcal{S}$, $\bar{\mathcal{S}}$ is the closure of $\mathcal{S}$ and $x^{\min},x^{\max}\in\mathbb{R}^n$ such that $x^{\min}<x^{\max}$. Here the Hamiltonian $\mathscr{H}$ is defined by

$$\mathscr{H}(t,x,p,A)=\sup_{u\in\mathbb{U}}\left[\mu(t,x,u)^\top p+\frac{1}{2}\mathrm{Tr}(a(t,x,u)A)+f(t,x,u)\right]$$

where $a=\sigma\sigma^\top$ and the intervention operator $\mathscr{M}$ is given by

$$\mathscr{M}\phi(t,x)=\sup_{z\in\mathcal{Z}}\phi(t,\Gamma(x,z))+K(t,x,z)$$

for $\phi:[0,T]\times\bar{\mathcal{S}}\to\mathbb{R}$.

We assume that the profit functions f, g_T, $g_{\partial\mathcal{S}}$ and K are continuous and satisfy

$$\begin{aligned}
|f(t,x,u)|&\le C(1+|x|+|u|)\\
|f(s,y,u)-f(t,x,u)|&\le C(|s-t|+|y-x|),\\
|g_{\partial\mathcal{S}}(t,x)|&\le C(1+|x|),\\
|g_{\partial\mathcal{S}}(s,y)-g_{\partial\mathcal{S}}(t,x)|&\le C(|s-t|+|y-x|),\\
|g_T(x)|&\le C(1+|x|),\\
|g_T(y)-g_T(x)|&\le C|y-x|,\\
|K(t,x,z)|&\le C(1+|x|+|z|),\\
|K(s,y,z)-K(t,x,z)|&\le C(|s-t|+|y-x|),
\end{aligned}$$

for some constant C. We suppose that the control sets $\mathbb{U}$ and $\mathcal{Z}$ are compact and the intervention function Γ is continuous.

According to the verification theorem, we seek the optimal strategy from the strategies consist of the triplet (u, v, D) such that

$$\begin{aligned}
&\mathcal{D} = \{\mathcal{D}_t\}_{t\in[0,T)}, \quad \mathcal{D}_t \subseteq \mathcal{S},\\
&u = \{u(t,x)\}_{t\in[0,T),x\in\mathcal{S}},\\
&v = (\tau_i, \zeta(\tau_i, X_{\tau_i^-}))_{1\le i\le M+1},\\
&\tau_i = \inf\{s \ge \tau_{i-1} | X_s \notin \mathcal{D}_s\},
\end{aligned}$$

where $\zeta : [0,T)\times\mathcal{S} \to \mathcal{Z}$.

2.4.1 Discretization

We fdefine discretization parameters. Let $N_t, N_i \in \mathbb{N}$, $i = 1, \cdots, n$, be numbers of time grids and spacial grids of i-th axis. Then the grid sizes δ_t and $\delta_1, \cdots, \delta_n$ are given by

$$\delta_t = \frac{T}{N_t}, \quad \delta_i = \frac{x_i^{\max} - x_i^{\min}}{N_i}, \; i = 1, \cdots, n.$$

The spacial grid $\bar{\mathcal{S}}^\delta$ is defined by

$$\bar{\mathcal{S}}^\delta = \left\{ x^{\min} + \sum_{i=1}^n n_i\delta_i e_i \,\middle|\, n_i = 0, \cdots, N_i \right\}.$$

We split $\bar{\mathcal{S}}^\delta$ into the discretized domain $\mathcal{S}^\delta$ and the discretized boundary $\partial\mathcal{S}^\delta$:

$$\begin{aligned}
\mathcal{S}^\delta &= \left\{ x^{\min} + \sum_{i=1}^n n_i\delta_i e_i \,\middle|\, n_i = 1, \cdots, N_i - 1 \right\},\\
\partial\mathcal{S}^\delta &= \bar{\mathcal{S}}^\delta \setminus \mathcal{S}^\delta.
\end{aligned}$$

Denote by $N_{\bar{\mathcal{S}}}$, $N_{\mathcal{S}}$ and $N_{\partial\mathcal{S}}$, numbers of grid points in $\bar{\mathcal{S}}^\delta$, $\mathcal{S}^\delta$ and $\partial\mathcal{S}^\delta$. For latter proofs, we introduce some symbols as follows:

- $t_k = k\delta_t, \quad \tilde{\mathbb{T}}^\delta = [0, T - \frac{\delta_t}{2}), \quad \partial\tilde{\mathbb{T}}^\delta = [T - \frac{\delta_t}{2}, T]$,
- $\tilde{\mathcal{S}}^\delta = \prod_{i=1}^n (x_i^{\min} + \frac{\delta_i}{2}, x_i^{\max} - \frac{\delta_i}{2}), \quad \partial\tilde{\mathcal{S}}^\delta = \bar{\mathcal{S}} \setminus \tilde{\mathcal{S}}^\delta$,

We next approximate the regular control part. We use the forward difference for the time derivative and the central difference for the first spacial derivative:

$$\begin{aligned}
\partial_t V(t,x) &\simeq \Delta_t^+ V(t,x) = \frac{V(t+\delta_t, x) - V(t,x)}{\delta_t},\\
\partial_i V(t,x) &\simeq \Delta_i^+ V(t,x) = \frac{V(t, x+\delta_i e_i) - V(t, x-\delta_i e_i)}{2\delta_i}, \quad i = 1, \cdots, n.
\end{aligned}$$

For the second spacial derivative, both of Δ^+_{ij} and Δ^-_{ij} are invalid because the monotonicity and the stability are not guaranteed. To avoid the above issues, we employ the upwind scheme for the approximation of $\partial_{ij}V(t,x)$:

$$\begin{aligned}\mathrm{Tr}(a(t,x,u)\partial_x^2 V(t,x)) &= \sum_{i,j} a_{ij}(t,x,u)\partial_{ij}V(t,x)\\ \simeq \sum_i a_{ii}(t,x,u)\Delta^-_{ii}V(t,x) &+ \sum_{j\in\mathcal{J}(i)} a^+_{ij}(t,x,u)\Delta^+_{ij}V(t,x) - a^-_{ij}(t,x,u)\Delta^-_{ij}V(t,x),\end{aligned}$$

where $\mathcal{J}(i) = \{1,\cdots,n\}\setminus\{i\}$. Define the operator $\tilde{\mathscr{H}}$ by

$$\begin{aligned}&\tilde{\mathscr{H}}(t,x,p,A^+,A^-)\\ &= \sup_{u\in\mathbb{U}}\left[\mu(t,x,u)^\top p + \frac{1}{2}\sum_{i=1}^n \left\{a_{ii}(t,x,u)\Delta^-_{ii}A^-_{ii}\right.\right.\\ &\qquad\left.\left. + \sum_{j\in\mathcal{J}(i)} a^+_{ij}(t,x,u)A^+_{ij} - a^-_{ij}(t,x,u)A^-_{ij}\right\} + f(t,x,u)\right],\end{aligned}$$

and the approximation scheme for the regular control part is given by

$$\begin{aligned}&\partial_t V(t,x) + \mathscr{H}(t,x,\partial_x V,\partial_x^2 V)\\ &\simeq \Delta^+_t V(t,x) + \tilde{\mathscr{H}}(t,x,\Delta^c_x V(t,x),\Delta^{2+}_x V(t,x),\Delta^{2-}_x V(t,x)).\end{aligned}$$

Here $\Delta^c_x V(t,x)\in\mathbb{R}^n$ and $\Delta^{2+}_x V(t,x),\Delta^{2-}_x V(t,x)\in\mathbb{R}^{n\times n}$ are defined by

$$\begin{aligned}(\Delta^c_x V(t,x))_i &= \Delta^+_i V(t,x),\\ \left(\Delta^{2+}_x V(t,x)\right)_{ij} &= \Delta^+_{ij}V(t,x),\\ \left(\Delta^{2-}_x V(t,x)\right)_{ij} &= \Delta^-_{ij}V(t,x),\quad i,j=1,\cdots,n.\end{aligned}$$

The more explicit form of the above scheme is as follow:

$$\begin{aligned}&\partial_t V(t,x) + \mathscr{H}(t,x,\partial_x V,\partial_x^2 V)\\ &\simeq \frac{V(t+\delta_t,x)-V(t,x)}{\delta_t}\\ &\quad + \sup_{u\in\mathbb{U}}\left[\sum_{i=1}^n\left(\frac{-a_{ii}(t,x,u)}{\delta_i^2} + \sum_{j\in\mathcal{J}(i)}\frac{|a_{ij}(t,x,u)|}{2\delta_i\delta_j}\right)V(t,x)\right.\\ &\qquad + \frac{1}{2}\sum_{i=1}^n\sum_{\kappa=\pm1}\left(\frac{a_{ii}(t,x,u)}{\delta_i^2} - \sum_{j\in\mathcal{J}(i)}\frac{|a_{ij}(t,x,u)|}{\delta_i\delta_j} + \kappa\frac{\mu_i(t,x,u)}{\delta_i}\right)V(t,x+\kappa\delta_i e_i)\\ &\qquad \left. + \frac{1}{2}\sum_{i=1}^n\sum_{j\in\mathcal{J}(i)}\sum_{\kappa,\lambda=\pm1}\frac{a^{[\kappa\lambda]}_{ij}(t,x,u)}{2\delta_i\delta_j}V(t,x+\kappa\delta_i e_i+\lambda\delta_j e_j) + f(t,x,u)\right],\end{aligned}\tag{2.11}$$

where

$$a_{ij}^{[\kappa\lambda]} = a_{ij}^{+}\mathbf{1}_{\{\kappa\lambda=1\}} + a_{ij}^{-}\mathbf{1}_{\{\kappa\lambda=-1\}}, \quad \kappa, \lambda = \pm 1.$$

We impose the following condition to assure the monotonicity and stability of the scheme:

$$|\mu_i(t,x,u)| \le \frac{a_{ii}(t,x,u)}{\delta_i} + \sum_{j\in\mathcal{J}(i)} \frac{|a_{ij}(t,x,u)|}{\delta_j} \tag{2.12}$$

To approximate the impulse control part, we define a function $\Gamma^\delta : \mathcal{S}\times\mathcal{Z} \to \mathcal{S}^\delta$ as follow:

$$\Gamma^\delta(x,z) = \operatorname*{argmin}_{y\in\mathcal{S}^\delta} |\Gamma(x,z) - y|.$$

Then the impulse control part is approximated as follow:

$$\mathscr{M}V(t,x_i) \simeq \sup_{z\in\mathcal{Z}} V(t,\Gamma^\delta(x,z)) + K(t,x,z).$$

According the above discussion, we define the numerical scheme $\mathscr{F}^\delta$ for the HJB equation (2.8) as follow.

Definition 2.4.1 (Implicit method for HJBQVI)

$$\mathscr{F}^\delta(t,x,V^\delta(t,x),[V^\delta]_{t,x})$$
$$= \begin{cases} \max\Big[\Delta_t^+ V^\delta(t,x) + \tilde{\mathscr{H}}(t,x,\Delta_x^c V^\delta(t,x), \Delta_x^{2+}V^\delta(t,x), \Delta_x^{2-}V^\delta(t,x)), \\ \qquad + \sup_{z\in\mathcal{Z}} V^\delta(t,\Gamma^\delta(x,z)) + K(t,x,z)\Big], \quad (t,x)\in\{t_k \mid k=0,\cdots,N^t-1\}\times\mathcal{S}^\delta, \\ g_T(x) - V^\delta(t,x), \qquad (t,x)\in\partial\tilde{\mathbb{T}}^\delta\times(\mathcal{S}\cup\partial\mathcal{S}), \\ g_{\partial\mathcal{S}}(t,\varsigma_{\partial\mathcal{S}}(x)) - V^\delta(t,x), \qquad (t,x)\in\tilde{\mathbb{T}}^\delta\times\partial\tilde{\mathcal{S}}^\delta, \\ V^\delta(\varsigma_{\mathbb{T}}(t),\varsigma_{\mathcal{S}}(x)) - V^\delta(t,x), \quad \text{otherwise}, \end{cases}$$

where

$$\varsigma_{\mathbb{T}}(t) = \operatorname*{argmin}_{s\in\{t_0,\cdots,t_{N_t-1}\}} |t-s|,$$
$$\varsigma_{\mathcal{S}}(x) = \operatorname*{argmin}_{y\in\mathcal{S}^\delta} |x-y|,$$
$$\varsigma_{\partial\mathcal{S}}(x) = \operatorname*{argmin}_{y\in\partial\mathcal{S}} |x-y|.$$

Remark 2.4.2

Since the scheme $\mathscr{F}^\delta$ dose not gives a solution V^δ explicitly, the scheme $\mathscr{F}^\delta$ is an implicit method. We also remark that the condition (2.12) becomes milder if we use the upwind scheme to approximate the first spacial derivative, instead of the central difference.

2.4.2 Convergence proof

Monotonicity

Lemma 2.4.3
If the condition (2.12) is satisfied, the scheme $\mathscr{F}^\delta$ is monotone.

Proof. Let $y \in \mathbb{R}$ and $\phi, \psi \in \mathcal{B}(\bar{\mathcal{S}})$ such that $\phi \le \psi$. We see that

$$\mathscr{F}^\delta\left(t,x,y,[\psi]_{t,x}\right) - \mathscr{F}^\delta\left(t,x,y,[\phi]_{t,x}\right) = g_T(x) - y - (g_T(x) - y) = 0,$$

for $(t,x) \in \partial\tilde{\mathbb{T}}^\delta \times (\mathcal{S} \cup \partial\mathcal{S})$,

$$\mathscr{F}^\delta\left(t,x,y,[\psi]_{t,x}\right) - \mathscr{F}^\delta\left(t,x,y,[\phi]_{t,x}\right) = (g_{\partial\mathcal{S}}(t,\varsigma_{\partial\mathcal{S}}(x)) - y) - (g_{\partial\mathcal{S}}(t,x) - y) = 0$$

for $(t,x) \in \tilde{\mathbb{T}}^\delta \times \partial\tilde{\mathcal{S}}^\delta$ and

$$\begin{aligned}
&\mathscr{F}^\delta\left(t,x,y,[\psi]_{t,x}\right) - \mathscr{F}^\delta\left(t,x,y,[\phi]_{t,x}\right) \\
&\quad = \psi(\varsigma_{\mathbb{T}}(t),\varsigma_{\mathcal{S}}(x)) - y - \left(\phi(\varsigma_{\mathbb{T}}(t),\varsigma_{\mathcal{S}}(x)) - y\right) \\
&\quad \ge 0,
\end{aligned}$$

for $(t,x) \in [0,T] \times (\mathcal{S} \cup \partial\mathcal{S}) \setminus (\left\{t_k \middle|\ k = 0,\cdots,N^t-1\right\} \times \mathcal{S}^\delta \cup \tilde{\mathbb{T}}^\delta \times (\mathcal{S} \cup \partial\mathcal{S}) \cup \tilde{\mathbb{T}}^\delta \times \partial\tilde{\mathcal{S}}^\delta)$.

The remaining is to show the case of $(t,x_i) \in \left\{t_k \middle|\ k = 0,\cdots,N^t-1\right\} \times \mathcal{S}^\delta$. Recall that

$$a_1 \ge a_2,\ b_1 \ge b_2 \ \Rightarrow\ \max(a_1,b_1) \ge \max(a_2,b_2)$$

and we find that if we show

$$\begin{aligned}
&\Delta_t^+\psi(t,x)|_{\psi(t,x)=y} + \tilde{\mathscr{H}}(t,x,\Delta_x^c\psi(t,x),\Delta_x^{2+}\psi(t,x),\Delta_x^{2-}\psi(t,x))|_{\psi(t,x)=y} \\
&\quad \ge \Delta_t^+\phi(t,x)|_{\phi(t,x)=y} + \tilde{\mathscr{H}}(t,x,\Delta_x^c\phi(t,x),\Delta_x^{2+}\phi(t,x),\Delta_x^{2-}\phi(t,x_i))|_{\phi(t,x)=y},
\end{aligned} \tag{2.13}$$

and

$$\sup_{z\in\mathcal{Z}} \psi(t,\Gamma^\delta(x,z)) + K(t,x,z) \ge \sup_{z\in\mathcal{Z}} \phi(t,\Gamma^\delta(x,z)) + K(t,x,z) \tag{2.14}$$

for $(t,x_i) \in \left\{t_k \middle|\ k = 0,\cdots,N^t-1\right\} \times \mathcal{S}^\delta$, then the claim is established.

The inequality(2.14) is almost obvious:

$$\begin{aligned}
&\sup_{z\in\mathcal{Z}}\left\{\psi(t,\Gamma^\delta(x,z)) + K(t,x,z)\right\} - \sup_{z\in\mathcal{Z}}\left\{\phi(t,\Gamma^\delta(x,z)) + K(t,x,z)\right\} \\
&\ge \sup_{z\in\mathcal{Z}}\left\{\phi(t,\Gamma^\delta(x,z)) + K(t,x,z)\right\} - \sup_{z\in\mathcal{Z}}\left\{\phi(t,\Gamma^\delta(x,z)) + K(t,x,z)\right\} \\
&= 0.
\end{aligned}$$

Let show the inequality(2.13). By (2.11), we have

$$
\begin{aligned}
&\Delta_t^+\psi(t,x_i)|_{\psi(t,x_i)=y} + \tilde{\mathscr{H}}(t,x_i,\Delta_x^c\psi(t,x_i),\Delta_x^{2+}\psi(t,x_i),\Delta_x^{2-}\psi(t,x_i))|_{\psi(t,x_i)=y} \\
&\quad - \Delta_t^+\phi(t,x_i)|_{\phi(t,x_i)=y} - \tilde{\mathscr{H}}(t,x_i,\Delta_x^c\phi(t,x_i),\Delta_x^{2+}\phi(t,x_i),\Delta_x^{2-}\phi(t,x_i))|_{\phi(t,x_i)=y} \\
&= \frac{\psi(t+\delta_t,x)}{\delta_t} + \sup_{u\in\mathbb{U}}\left[\left\{\frac{1}{\delta_t} + \sum_{i=1}^n\left(\frac{-a_{ii}(t,x,u)}{\delta_i^2} + \sum_{j\in\mathcal{J}(i)}\frac{|a_{ij}(t,x,u)|}{2\delta_i\delta_j}\right)\right\}y\right. \\
&\qquad + \frac{1}{2}\sum_{i=1}^n\sum_{\kappa=\pm1}\left(\frac{a_{ii}(t,x,u)}{\delta_i^2} - \sum_{j\in\mathcal{J}(i)}\frac{|a_{ij}(t,x,u)|}{\delta_i\delta_j} + \kappa\frac{\mu_i(t,x,u)}{\delta_i}\right)\psi(t,x+\kappa\delta_i e_i) \\
&\qquad \left.+\frac{1}{2}\sum_{i=1}^n\sum_{j\in\mathcal{J}(i)}\sum_{\kappa,\lambda=\pm1}\frac{a_{ij}^{[\kappa\lambda]}(t,x,u)}{2\delta_i\delta_j}\psi(t,x+\kappa\delta_i e_i+\lambda\delta_j e_j) + f(t,x,u)\right] \\
&\quad - \frac{\phi(t+\delta_t,x)}{\delta_t} - \sup_{u\in\mathbb{U}}\left[\left\{\frac{1}{\delta_t} + \sum_{i=1}^n\left(\frac{-a_{ii}(t,x,u)}{\delta_i^2} + \sum_{j\in\mathcal{J}(i)}\frac{|a_{ij}(t,x,u)|}{2\delta_i\delta_j}\right)\right\}y\right. \\
&\qquad + \frac{1}{2}\sum_{i=1}^n\sum_{\kappa=\pm1}\left(\frac{a_{ii}(t,x,u)}{\delta_i^2} - \sum_{j\in\mathcal{J}(i)}\frac{|a_{ij}(t,x,u)|}{\delta_i\delta_j} + \kappa\frac{\mu_i(t,x,u)}{\delta_i}\right)\phi(t,x+\kappa\delta_i e_i) \\
&\qquad \left.+\frac{1}{2}\sum_{i=1}^n\sum_{j\in\mathcal{J}(i)}\sum_{\kappa,\lambda=\pm1}\frac{a_{ij}^{[\kappa\lambda]}(t,x,u)}{2\delta_i\delta_j}\phi(t,x+\kappa\delta_i e_i+\lambda\delta_j e_j) + f(t,x,u)\right]
\end{aligned}
$$

Since the coefficients of $\psi(t+\delta_t,x)$, $\psi(t,x+\kappa\delta_i e_i)$ and $\psi(t,x+\kappa\delta_i e_i+\lambda\delta_j e_j)$ are non-negative by the condition (2.12), we find that

$$
\begin{aligned}
&\Delta_t^+ \psi(t,x_i)|_{\psi(t,x_i)=y} + \tilde{\mathscr{H}}(t,x_i,\Delta_x^c \psi(t,x_i),\Delta_x^{2+}\psi(t,x_i),\Delta_x^{2-}\psi(t,x_i))|_{\psi(t,x_i)=y} \\
&\quad - \Delta_t^+ \phi(t,x_i)|_{\phi(t,x_i)=y} - \tilde{\mathscr{H}}(t,x_i,\Delta_x^c \phi(t,x_i),\Delta_x^{2+}\phi(t,x_i),\Delta_x^{2-}\phi(t,x_i))|_{\phi(t,x_i)=y} \\
&\le \frac{\phi(t+\delta_t,x)}{\delta_t} + \sup_{u\in\mathbb{U}}\left[\left\{\frac{1}{\delta_t} + \sum_{i=1}^{n}\left(\frac{-a_{ii}(t,x,u)}{\delta_i^2} + \sum_{j\in\mathcal{J}(i)}\frac{|a_{ij}(t,x,u)|}{2\delta_i\delta_j}\right)\right\} y\right. \\
&\qquad + \frac{1}{2}\sum_{i=1}^{n}\sum_{\kappa=\pm1}\left(\frac{a_{ii}(t,x,u)}{\delta_i^2} - \sum_{j\in\mathcal{J}(i)}\frac{|a_{ij}(t,x,u)|}{\delta_i\delta_j} + \kappa\frac{\mu_i(t,x,u)}{\delta_i}\right)\phi(t,x+\kappa\delta_i e_i) \\
&\qquad \left. +\frac{1}{2}\sum_{i=1}^{n}\sum_{j\in\mathcal{J}(i)}\sum_{\kappa,\lambda=\pm1}\frac{a_{ij}^{[\kappa\lambda]}(t,x,u)}{2\delta_i\delta_j}\phi(t,x+\kappa\delta_i e_i+\lambda\delta_j e_j) + f(t,x,u)\right] \\
&\quad - \frac{\phi(t+\delta_t,x)}{\delta_t} + \sup_{u\in\mathbb{U}}\left[\left\{\frac{1}{\delta_t} + \sum_{i=1}^{n}\left(\frac{-a_{ii}(t,x,u)}{\delta_i^2} + \sum_{j\in\mathcal{J}(i)}\frac{|a_{ij}(t,x,u)|}{2\delta_i\delta_j}\right)\right\} y\right. \\
&\qquad + \frac{1}{2}\sum_{i=1}^{n}\sum_{\kappa=\pm1}\left(\frac{a_{ii}(t,x,u)}{\delta_i^2} - \sum_{j\in\mathcal{J}(i)}\frac{|a_{ij}(t,x,u)|}{\delta_i\delta_j} + \kappa\frac{\mu_i(t,x,u)}{\delta_i}\right)\phi(t,x+\kappa\delta_i e_i) \\
&\qquad \left. +\frac{1}{2}\sum_{i=1}^{n}\sum_{j\in\mathcal{J}(i)}\sum_{\kappa,\lambda=\pm1}\frac{a_{ij}^{[\kappa\lambda]}(t,x,u)}{2\delta_i\delta_j}\phi(t,x+\kappa\delta_i e_i+\lambda\delta_j e_j) + f(t,x,u)\right] \\
&\quad = 0
\end{aligned}
$$

which establishes the inequality (2.13).

□

Stability

To prove the stability of the scheme $\mathscr{F}^\delta$, we investigate behaviour of the interventions at the same time. We first define the index function η.

Definition 2.4.4
$\eta : \{1,\cdots,N_{\mathcal{S}}\}\times\mathcal{Z}\to\{1,\cdots,N_{\mathcal{S}}\}$ such that

$$\eta(i,z) = \operatorname*{argmin}_{j\in\{1,\cdots,N_{\mathcal{S}}\}\setminus\{i\}} |\Gamma(x_i,z)-x_j|.$$

Fix an control $(u,v,\mathcal{D})$, and let us define

$$
\begin{aligned}
&\eta^{(0)}(t,i;\zeta) = i, \\
&\eta^{(m)}(t,i;\zeta) = \eta\left(\eta^{(m-1)}(t,i;\zeta),\zeta(t,x_{\eta^{(m-1)}(t,i;\zeta)})\right), \quad m\ge 1,
\end{aligned}
$$

and
$$m^c(t,i) = \operatorname*{argmin}_m \left\{ x_{\eta^{(m)}(t,i;\zeta)} \in \mathcal{D}_t \right\}.$$

The following lemma plays an important role to prove the existence of the scheme $\mathscr{F}^\delta$.

Lemma 2.4.5
If (u,v,D) is an admissible control strategy, for all $t \in [0,T)$ and $x_i \in \mathcal{S}^\delta \setminus \mathcal{D}_t$,

$$\eta^{(m)}(t,i;\zeta) \neq i, \quad 1 \leq m < m^c(t,i),$$

Proof. Suppose that there exist $t' \in [0,T)$ and $m' \in \{1,\cdots,m^c(t,i)-1\}$ such that
$$\eta^{(m')}(t',i;\zeta) = i.$$

Set $j^* = \operatorname*{argmax}_j \{\tau_j < t'\}$ and $X_{t'} = x_i$ and note that

$$\tau_{j^*+1} = \tau_{j^*+2} = \cdots = \tau_{j^*+m'} = t',$$

because $X_{\tau_{j^*+m}} = x_{\eta^{(m)}(t',i;\zeta)} \in \mathcal{S}^\delta \setminus \mathcal{D}_t,\ 1 \leq m \leq m'$. Moreover

$$X_{\tau_{j^*+m}} = x_{\eta^{(m)}(t',i;\zeta)} = x_{\eta^{(m-\lfloor \frac{m}{m'} \rfloor m')}(t',i;\zeta)} \in \mathcal{S}^\delta \setminus \mathcal{D}_t,$$

for $m > m'$. This implies that
$$\tau_{M+1} = t',$$
which contradicts the fact $\tau_{M+1} > T$ imposed by the admissibility. □

Let us define $\tilde{m}(t,i;\zeta)$ representing the maximum interventions number to reach the state x_i from x_j:
$$\tilde{m}(t,i;\zeta) = \max_{m,j} m\mathbf{1}_{\{i=\eta^{(m)}(t,j;\zeta)\}}$$

Lemma 2.4.6
We have
$$\tilde{m}(t,\eta(t,i;\zeta);\zeta) \geq \tilde{m}(t,i;\zeta)+1.$$

Proof. A straightforward computation gives

$$\begin{aligned}
\tilde{m}(t,\eta(t,i;\zeta);\zeta) &= \max_{m,j} m\mathbf{1}_{\{\eta(t,i;\zeta)=\eta^{(m)}(t,j;\zeta)\}} \\
&= \max_{j,k,m} m\mathbf{1}_{\{\eta(t,i;\zeta)=\eta(t,k;\zeta)\}}\mathbf{1}_{\{k=\eta^{(m-1)}(t,j;\zeta)\}} \\
&\geq \max_{j,m} m\mathbf{1}_{\{\eta(t,i;\zeta)=\eta(t,i;\zeta)\}}\mathbf{1}_{\{i=\eta^{(m-1)}(t,j;\zeta)\}} \\
&= \max_{j,m} m\mathbf{1}_{\{i=\eta^{(m-1)}(t,j;\zeta)\}}
\end{aligned}$$

Set $m' = m-1$, then

$$\begin{aligned}\max_{j,m} m\mathbf{1}_{\{i=\eta^{(m-1)}(t,j;\zeta)\}} &= \max_{j,m'}(m'+1)\mathbf{1}_{\{i=\eta^{(m')}(t,j;\zeta)\}}\\ &= \tilde{m}(t,i;\zeta)+1.\end{aligned}$$

Therefore the claim is established. □

Lemma 2.4.7
If the condition (2.12) is satisfied, the scheme $\mathscr{F}^\delta$ is stable.

Proof. Let V^δ be a solution of $\mathscr{F}^\delta(t,x,V^\delta(t,x),[V^\delta]_{t,x}) = 0$. The existence and boundedness of V^δ are obvious on the boundary $\tilde{\mathbb{T}}^\delta \times \partial\tilde{\mathcal{S}}^\delta$ and the terminal $\partial\tilde{\mathbb{T}}^\delta \times (\mathcal{S}\cup\partial\mathcal{S})$. If we are able to show the existence and boundedness of V^δ on the grids $\left\{t_k \middle| k = 0,\cdots,N^t-1\right\}\times\mathcal{S}^\delta$, there exist bounded V^δ on the other region since it takes the same value on the nearest grid.

We show the existence of V^δ on the grids $\left\{t_k \middle| k = 0,\cdots,N^t-1\right\}\times\mathcal{S}^\delta$. We display the scheme $\mathscr{F}^\delta$ on $\left\{t_k \middle| k = 0,\cdots,N^t-1\right\}\times\mathcal{S}^\delta$ in a matrix form. Fix an admissible control $w = (u,\zeta,\mathcal{D})$ and define $A^{w,k} \in \mathbb{R}^{N_{\bar{S}}\times N_{\bar{S}}}$ and $b^{w,k} \in \mathbb{R}^{N_{\bar{S}}}$ as follows:

$$A^{w,k}_{ij} = \begin{cases} L^{u,k}_{ij}, & x_i \in \mathcal{S}^\delta \cap \mathcal{D}_{t_k},\\ M^\zeta_{ij}, & x_i \in \mathcal{S}^\delta \cap (\mathcal{S}\setminus\mathcal{D}_{t_k}),\\ 1, & x_i \in \partial\mathcal{S}^\delta,\ i=j,\\ 0, & x_i \in \partial\mathcal{S}^\delta,\ i\neq j,\end{cases}$$

$$b^{w,k}_i = \begin{cases} f(t_k,x_i,u^k_i) + \frac{1}{\delta_t}V^\delta(t_{k+1},x_i) & x_i \in \mathcal{S}^\delta\cap\mathcal{D}_{t_k},\\ K(t_k,x_i,z^k_i), & x_i\in\mathcal{S}^\delta\cap(\mathcal{S}\setminus\mathcal{D}_{t_k}),\\ g_{\partial\mathcal{S}}(t_k,x_i), & x_i\in\partial\mathcal{S}^\delta.\end{cases}$$

where $u^k_i = u(t_k,x_i)$, $z^k_i = \zeta(t_k,x_i)$ and $L^{u,k}, M^\zeta \in \mathbb{R}^{N_{\bar{S}}\times N_{\bar{S}}}$ such that

$$L^{u,k}_{ij} = \begin{cases} 1, & \text{if } x_i\in\partial\mathcal{S}^\delta,\ i=j\\ -\dfrac{1}{\delta_t} + \displaystyle\sum_{i'=1}^{n}\frac{-a_{i'i'}(t_k,x_i,u^k_i)}{\delta^2_{i'}} + \sum_{j'\in\mathcal{J}(i')}\frac{|a_{i'j'}(t_k,x_i,u^k_i)|}{2\delta_{i'}\delta_{j'}}, & \text{if } x_i\in\partial\mathcal{S}^\delta,\ i=j\\ \dfrac{1}{2}\left\{\dfrac{a_{i'i'}(t_k,x_i,u^k_i)}{\delta^2_{i'}} - \displaystyle\sum_{j'\in\mathcal{J}(i')}\frac{|a_{i'j'}(t_k,x_i,u^k_i)|}{\delta_{i'}\delta_{j'}} + \kappa\frac{\mu_{i'}(t_k,x_i,u^k_i)}{\delta_{i'}}\right\}, & \\ \qquad \text{if } x_j = x_i+\kappa\delta_{i'}e_{i'},\ \kappa\pm1,\ i'\in\{1,\cdots,n\}, & \\ \dfrac{1}{4}\dfrac{a^{[\kappa\lambda]}_{i'j'}(t_k,x_i,u^k_i)}{\delta_{i'}\delta_{j'}}, & \text{if } x_i\in\partial\mathcal{S}^\delta, x_j = x_i+\kappa\delta_{i'}e_{i'}+\lambda\delta_{j'}e_{j'},\ \kappa,\lambda=\pm1,\\ & \qquad i'\in\{1,\cdots,n\},\ j'\in\mathcal{J}(i'),\\ 0, & \text{otherwize},\end{cases}$$

$M^\zeta_{ij} = -\mathbf{1}_{i=j} + \mathbf{1}_{i=\eta(i,z^k_i)}$.

Using $A^{w,k}$ and $b^{w,k}$, the scheme $\mathscr{F}^{\delta}$ is represented by

$$\sup_{w}\left\{A^{w,k}\tilde{V}^k + b^{w,k}\right\} = 0,$$

where $\tilde{V}^k \in \mathbb{R}^{N_{\bar{\mathcal{S}}}}$, $k = 0,\cdots,N_t$ such that $\tilde{V}^k_i = V(t_k,x_i)$, $i = 1,\cdots,N_{\bar{\mathcal{S}}}$. If $A^{w,k}$ is invertible, a linear equation

$$A^{w,k}\phi + b^{w,k} = 0,$$

has a solution $\phi = -(A^{w,k})^{-1}b^{w,k}$. This implies the existence of $\tilde{V}^k$. Therefore we need to show that $A^{w,k}$ is invertible.

We prove the case that $\mathcal{D}_{t_k} = \mathcal{S}$. In this case $A^{w,k} = L^{u,k}$. By the assumption (2.12), we find

$$\begin{aligned}
\sum_{j\neq i}|L^{u,k}_{ij}| &= \sum_{i'=1}^{n}\sum_{\kappa=\pm1}\frac{1}{2}\left\{\frac{a_{i'i'}(t_k,x_i,u^k_i)}{\delta^2_{i'}} - \sum_{j'\in\mathcal{J}(i')}\frac{|a_{i'j'}(t_k,x_i,u^k_i)|}{\delta_{i'}\delta_{j'}} + \kappa\frac{\mu_{i'}(t_k,x_i,u^k_i)}{\delta_{i'}}\right\}\\
&\quad + \sum_{i'=1}^{n}\sum_{j'\in\mathcal{J}(i')}\sum_{\kappa=\pm1}\frac{1}{4}\frac{a^{[\kappa\lambda]}_{i'j'}(t_k,x_i,u^k_i)}{\delta_{i'}\delta_{j'}}\\
&= \sum_{i'=1}^{n}\frac{a_{i'i'}(t_k,x_i,u^k_i)}{\delta^2_{i'}} - \sum_{j'\in\mathcal{J}(i')}\frac{|a_{i'j'}(t_k,x_i,u^k_i)|}{2\delta_{i'}\delta_{j'}}\\
&< \frac{1}{\delta_t} + \sum_{i'=1}^{n}\frac{a_{i'i'}(t_k,x_i,u^k_i)}{\delta^2_{i'}} - \sum_{j'\in\mathcal{J}(i')}\frac{|a_{i'j'}(t_k,x_i,u^k_i)|}{2\delta_{i'}\delta_{j'}}\\
&= |L^{u,k}_{ii}|, \qquad i\in\{l|x_l\in\mathcal{S}^{\delta}\}
\end{aligned}$$

Thus $L^{u,k}$ is a strictly diagonally dominant matrix. It is a well known result that a strictly diagonally dominant matrix is invertible. Hence $L^{u,k}$ is invertible.

Let us consider the claim in the case that $\mathcal{D}_{t_k} \neq \mathcal{S}$. In this case, $A^{w,k}$ is indeed a generalized diagonally dominant matrix which is also invertible (see e.g. [Var09] and [JR75]). We are able to define $c_A \in [0,1)$ satisfying

$$\max_{i\in\{l|x_l\in\mathcal{S}^{\delta}\cap\mathcal{D}_{t_k}\}}\frac{\sum_{j\in\mathcal{J}(i)}|L^{u,k}_{ij}|}{|L^{u,k}_{ii}|} < c_A^{\tilde{m}'+1}$$

where

$$\tilde{m}' = \max_{i\in\{l|x_l\in\mathcal{S}^{\delta}\cap(\mathcal{S}\setminus\mathcal{D}_{t_k})\}}\tilde{m}(t_k,i;\zeta).$$

Using a constant c_A, we define a diagonal matrix C_A by

$$(C_A)_{ii} = \begin{cases} c_A^{\tilde{m}(t_k,i;\zeta)}, & \text{if } x_i\in\mathcal{S}^{\delta}\cap(\mathcal{S}\setminus\mathcal{D}_{t_k}),\\ 1, & \text{otherwise.}\end{cases}$$

For $i \in \{l | x_l \in \mathcal{S}^\delta \cap \mathcal{D}_{t_k}\}$, we have

$$\sum_{j \neq i} |(A^{w,k} C_A)_{ij}| \leq \sum_{j \neq i} |A^{w,k}_{ij}| = \sum_{j \neq i} |L^{u,k}_{ij}| < c_A^{\tilde{m}'+1} |L^{u,k}_{ii}| \leq |(L^{u,k} C_A)_{ii}| = |(A^{w,k} C_A)_{ii}|,$$

because $(C_A)_{ij} \in [0, 1]$. By the lemma 2.4.5 and the lemma 2.4.6, we see

$$\begin{aligned}
\sum_{j \neq i} |(A^{w,k} C_A)_{ij}| &= \left| M^{\zeta}_{i,\eta(i,z_i^k)} \right| c_A^{\tilde{m}(t_k,\eta(i,z_i^k);\zeta)} \\
&\leq c_A^{\tilde{m}(t_k,i;\zeta)+1} \\
&< c_A^{\tilde{m}(t_k,i;\zeta)} \\
&= \left| M^{\zeta}_{ii} \right| c_A^{\tilde{m}(t_k,i;\zeta)} \\
&= |(A^{w,k} C_A)_{ii}|,
\end{aligned}$$

for $i \in \{l | x_l \in \mathcal{S}^\delta \cap (\mathcal{S} \setminus \mathcal{D}_{t_k})\}$. Therefore the matrix $A^{w,k} C_A$ is a strictly diagonally dominant matrix and $A^{w,k}$ is a generalized diagonally dominant matrix.

We next show the boundedness of $\tilde{V}^k$. Define $i^* \in \{1, \cdots, N_{\bar{\mathcal{S}}}\}$ such that

$$i^* = \operatorname*{argmax}_i |\phi_i|.$$

If $x_{i^*} \in \partial \mathcal{S}^\delta$, it is trivial that

$$|\phi_{i^*}| = |g_{\partial \mathcal{S}}(t_k, x_{i^*})| \leq C_2,$$

where

$$C_2 = \sup_{s \in [0,T), y \in \partial \mathcal{S}} |g_{\partial \mathcal{S}}(s, y)|.$$

Let us consider the case that $x_{i^*} \in \mathcal{S}^\delta$. We define $i^{*D} \in \{1, \cdots, N_{\bar{\mathcal{S}}}\}$ as follow:

$$i^{*D} = \operatorname*{argmax}_{i \in \{l | x_l \in \mathcal{S}^\delta \cap \mathcal{D}_{t_k}\}} |\phi_i|.$$

Because $A^{w,k}_{i^{*D} j} = L^{u,k}_{i^{*D} j} \geq 0$ and $|\phi_{i^{*D}}| \geq |\phi_j| \geq \operatorname{sign}(\phi_{i^{*D}}) \phi_j$ for $i^{*D} \neq j$, we see that

$$\sum_{j \neq i^*} A^{w,k}_{i^{*D} j} |\phi_{i^{*D}}| \geq \operatorname{sign}(\phi_{i^{*D}}) \sum_{j \neq i^{*D}} A^{w,k}_{i^{*D} j} \phi_j.$$

Since $\sum_j A^{w,k}_{i^{*D}j} = -\frac{1}{\delta_t}$, we have

$$\begin{aligned}
|\phi_{i^{*D}}| &= -\delta_t \sum_j A^{w,k}_{i^{*D}j} |\phi^{u,k}_{i^{*D}}| \\
&= -\delta_t \sum_{j \neq i^{*D}} A^{w,k}_{i^{*D}j} |\phi_{i^{*D}}| - \delta_t A^{w,k}_{i^{*D}i^{*D}} \mathrm{sign}(\phi_{i^{*D}}) \phi_{i^{*D}} \\
&\leq -\delta_t \mathrm{sign}(\phi_{i^{*D}}) \sum_{j \neq i^{*D}} A^{w,k}_{i^{*D}j} \phi_j - \delta_t \mathrm{sign}(\phi_{i^{*D}}) A^{w,k}_{i^{*D}i^{*D}} \phi_{i^{*D}} \\
&= -\delta_t \mathrm{sign}(\phi_{i^{*D}}) \sum_j A^{w,k}_{i^{*D}} \phi_j \\
&= \mathrm{sign}(\phi^{u,k}_{i^*}) \delta_t b^{w,k}_{i^{*D}} \\
&\leq |f(t_k, x_{i^{*D}}, u^k_i)| + |\tilde{V}^{k+1}_{i^{*D}}| \\
&\leq \delta_t C_3 + \max_i |\tilde{V}^{k+1}_i|,
\end{aligned}$$

where

$$C_3 = \sup_{s \in [0,T],\ y \in \mathcal{S},\ u \in \mathbb{U}} |f(s, y, u)|.$$

Hence if $x_{i^*} \in \mathcal{S}^\delta \cap \mathcal{D}_{t_k}$, we find that $|\phi_{i^*}| = |\phi_{i^{*D}}| \leq \delta_t C_3 + \max_i |V^{k+1}_i|$. If $x_{i^*} \in \mathcal{S}^\delta \cap (\mathcal{S} \setminus \mathcal{D}_{t_k})$, note that

$$\begin{aligned}
\phi_{i^*} &= \phi_{\eta(t_k, i^*; \zeta)} - K(t_k, x_{i^*}, z^k_i) \\
&= \phi_{m^c(t_k, i^*; \zeta)} - \sum_{m=0}^{m^c(t_k, i^*; \zeta)-1} K\left(t_k, x_{\eta^{(m)}(t_k, i^*; \zeta)}, z^k_{\eta^{(m)}(t_k, i^*; \zeta)}\right).
\end{aligned}$$

and thus

$$\begin{aligned}
|\phi_{i^*}| &\leq |\phi_{m^c(t_k, i^*; \zeta)}| + \sum_{m=0}^{m^c(t_k, i^*; \zeta)-1} \left| K\left(t_k, x_{\eta^{(m)}(t_k, i^*; \zeta)}, z^k_{\eta^{(m)}(t_k, i^*; \zeta)}\right) \right| \\
&\leq |\phi_{i^{*D}}| + C_4 \sum_{m=0}^{m^c(t_k, i^*; \zeta)-1} 1 \\
&\leq \delta_t C_3 + \max_i |\tilde{V}^{k+1}_i| + C_4 m^c(t_k, i^*; \zeta),
\end{aligned}$$

where

$$C_4 = \sup_{s \in [0,T],\ y \in \mathcal{S},\ z \in \mathcal{Z}} |K(s, y, z)|.$$

The above discussion is valid for arbitrary admissible w and hence we obtain

$$|\tilde{V}^k_i| \leq \max\left[C_2, \delta_t C_3 + C_4 m^c(t_k, i^{*k}; \zeta^*) + \max_i |\tilde{V}^{k+1}_i| \right],$$

here we denote

$$i^{*k} = \operatorname*{argmax}_i |\tilde{V}_i^k|.$$

This evaluation implies that

$$|\tilde{V}_i^k| \leq \max[C1, C2] + (N^t - k)\delta_t C_3 + C_4 \sum_{l=k}^{N_t-1} m^c(t_l, i^{*l}; \zeta^*).$$

where

$$C_1 = \sup_{y \in \mathcal{S} \cup \partial \mathcal{S}} |g_T(y)|.$$

We are able to show the above by induction. For $k = N^t$,

$$|\tilde{V}_i^k| = |g_T(x_i)| \leq C_1 \leq \max[C1, C2] + (N^t - N^t)\delta_t C_3 + C_4 \sum_{l=N_t}^{N_t-1} m^c(t_l, i^{*l}; \zeta^*).$$

Assuming the claim is established for $k = k' + 1$,

$$\begin{aligned}
|\tilde{V}_i^{k'}| &\leq \max\left[C_2, \delta_t C_3 + C_4 m^c(t_k, i^{*k'}; \zeta^*) + \max_i |\tilde{V}_i^{k'+1}|\right] \\
&\leq \max\Big[C_2, \delta_t C_3 + C_4 m^c(t_k, i^{*k'}; \zeta^*) \\
&\qquad\qquad + \max[C1, C2] + (N^t - (k'+1))\delta_t C_3 + C_4 \sum_{l=k'+1}^{N_t-1} m^c(t_l, i^{*l}; \zeta^*)\Big] \\
&\leq \max\left[C_2, \max[C1, C2] + (N^t - k')\delta_t C_3 + C_4 \sum_{l=k'}^{N_t-1} m^c(t_l, i^{*l}; \zeta^*)\right] \\
&\leq \max[C1, C2] + (N^t - k')\delta_t C_3 + C_4 \sum_{l=k'}^{N_t-1} m^c(t_l, i^{*l}; \zeta^*).
\end{aligned}$$

We see that $(N^t - k)\delta_t C_3 \leq TC_3, k = 0, 1, \cdots, N^t - 1$. Because the term $\sum_{l=k}^{N_t-1} m^c(t_l, i^{*l}; \zeta^*)$ means the count of interventions after t_k, we find $\sum_{l=k}^{N_t-1} m^c(t_l, i^{*l}; \zeta^*) \leq M$. Therefore we obtain

$$|\tilde{V}_i^{k'}| \leq \max[C1, C2] + TC_3 + MC_4,$$

which establishes the claim. □

Consistency

Lemma 2.4.8
The scheme $\mathscr{F}^\delta$ is consistent.

Proof. Let $\phi \in C^\infty([0,T] \times (\mathcal{S}^\delta \cup \partial\mathcal{S}))$ and $\delta_{ts} = s - t$ and $\delta^{xy} = y - x$.

We first consider the claim in the case that $(t,x) \in [0,T) \times \partial\mathcal{S}$. Suppose that $(s,y) \in \mathbb{T}^\delta \times \partial\tilde{\mathcal{S}}^\delta$ and then

$$\begin{aligned}
&\left|\mathscr{F}^\delta(s,y,\phi(s,y)+\xi,\phi+\xi) - \left(\partial_t\phi(t,x) + \mathscr{H}\left(t,x,\partial_x\phi(t,x),\partial_x^2\phi(t,x)\right)\right)\right| \\
&= \left|g_{\partial\mathcal{S}}(s,\varsigma(y)) - (\phi(s,y)+\xi) - (g_{\partial\mathcal{S}}(t,x) - \phi(t,x))\right|,
\end{aligned}$$

where $\xi \in \mathbb{R}$. The assumption imposed to $g_{\partial\mathcal{S}}$ gives

$$\begin{aligned}
&\left|\mathscr{F}^\delta(s,y,\phi(s,y)+\xi,\phi+\xi) - \left(\partial_t\phi(t,x) + \mathscr{H}\left(t,x,\partial_x\phi(t,x),\partial_x^2\phi(t,x)\right)\right)\right| \\
&\le C(|s-t| + |\varsigma(y)-x|) + \Bigg|\partial_t\phi(t+\theta_1\delta_{ts}, x+\theta_1\delta^{xy})\delta_{ts} \\
&\qquad\qquad + \sum_{i=1}^n \partial_i\phi(t+\theta_1\delta_{ts}, x+\theta_1\delta^{xy})\delta_i^{xy}\Bigg| + |\xi|
\end{aligned}$$

where $\theta_1 \in (0,1)$. Hence we obtain

$$\begin{aligned}
\left|\mathscr{F}^\delta(s,y,\phi(s,y)+\xi,\phi+\xi) - \left(\partial_t\phi(t,x) + \mathscr{H}\left(t,x,\partial_x\phi(t,x),\partial_x^2\phi(t,x)\right)\right)\right| &\to 0, \\
\text{as } \delta_t \to 0,\ \delta \to 0\ \xi \to 0 \text{ and } (s,y) \to (t,x).&
\end{aligned}$$

which establishes the claim on $[0,T) \times \partial\mathcal{S}$.

We consider the claim in the case that $(t,x) \in \{T\} \times (\mathcal{S} \cup \partial\mathcal{S})$. Suppose that $(s,y) \in \partial\mathbb{T}^\delta \times (\mathcal{S} \cup \partial\mathcal{S})$ and then

$$\begin{aligned}
&\left|\mathscr{F}^\delta(s,y,\phi(s,y)+\xi,\phi+\xi) - \left(\partial_t\phi(t,x) + \mathscr{H}\left(t,x,\partial_x\phi(t,x),\partial_x^2\phi(t,x)\right)\right)\right| \\
&= \left|g_T(y) - (\phi(s,y)+\xi) - (g_T(x) - \phi(T,x))\right|.
\end{aligned}$$

The assumption imposed to g_T gives

$$\begin{aligned}
&\left|\mathscr{F}^\delta(s,y,\phi(s,y)+\xi,\phi+\xi) - \left(\partial_t\phi(t,x) + \mathscr{H}\left(t,x,\partial_x\phi(t,x),\partial_x^2\phi(t,x)\right)\right)\right| \\
&\le C(|y-x|) + \Bigg|\partial_t\phi(T-\theta_2\delta_{sT}, x+\theta_2\delta^{xy})\delta_{sT} \\
&\qquad\qquad + \sum_{i=1}^n \partial_i\phi(T-\theta_2\delta_{sT}, x+\theta_2\delta^{xy})\delta_i^{xy}\Bigg| + |\xi|
\end{aligned}$$

where $\theta_2 \in (0,1)$. Hence we obtain

$$\begin{aligned}
\left|\mathscr{F}^\delta(s,y,\phi(s,y)+\xi,\phi+\xi) - \left(\partial_t\phi(t,x) + \mathscr{H}\left(t,x,\partial_x\phi(t,x),\partial_x^2\phi(t,x)\right)\right)\right| &\to 0, \\
\text{as } \delta_t \to 0,\ \delta \to 0\ \xi \to 0 \text{ and } (s,y) \to (t,x).&
\end{aligned}$$

which establishes the claim on $\{T\} \times (\mathcal{S} \cup \partial\mathcal{S})$.

The remaining task is to prove the case that $(t,x) \in [0,T) \times \mathcal{S}$. Suppose that δ_t and δ are small enough such that $(t,x) \in \tilde{\mathbb{T}}^\delta \times \tilde{\mathcal{S}}^\delta$. More precisely, we assume that δ_t and δ satisfy

$$\min(t, T-t) > \frac{1}{2}\delta_t, \quad |(x-x')_i| > \frac{1}{2}\delta_i, \quad x' = \operatorname*{argmin}_{x' \in \partial\mathcal{S}} |x - x'|, \quad i = 1, \cdots, n.$$

We choose (s, y) as follow:

$$s = \operatorname*{argmin}_{1\le k\le N^t-1} |t - t_k|, \quad y = \varsigma_{\mathcal{S}}(x).$$

Then we find

$$\begin{aligned}
&\left|\mathscr{F}^\delta(s, y, \phi(s, y) + \xi, \phi + \xi) - \left(\partial_t\phi(t, x) + \mathscr{H}\left(t, x, \partial_x\phi(t, x), \partial_x^2\phi(t, x)\right)\right)\right| \\
&= \Bigg|\max\Bigg[\Delta_t^+\phi(t, x) + \tilde{\mathscr{H}}(t, x, \Delta_x^c\phi(s, y), \Delta_x^{2+}\phi(s, y), \Delta_x^{2-}\phi(s, y)), \\
&\qquad\qquad + \sup_{z\in\mathcal{Z}}\left\{\phi(s, \Gamma^\delta(y, z)) + K(s, y, z)\right\} - \phi(s, y)\Bigg] \\
&\quad - \max\Bigg[\partial_t\phi(t, x) + \sup_{u\in\mathbb{U}}\left\{\mu(t, x, u)\partial_x\phi(t, x) + \frac{1}{2}a(t, x, u)\partial_x^2\phi(t, x) + f(t, x, u)\right\} \\
&\qquad\qquad + \sup_{z\in\mathcal{Z}}\left\{\phi(t, \Gamma(x, z)) + K(t, x, z)\right\} - \phi(t, x)\Bigg]\Bigg|.
\end{aligned}$$

Recall that

$$|a_1 - a_2| < \delta_a,\ |b_1 - b_2| < \delta_b \ \Rightarrow\ |\max(a_1, b_1) - \max(a_2, b_2)| < \max(\delta_a, \delta_b)$$

for $a_1, a_2, b_1, b_2 \in \mathbb{R}$ and thus it is enough to show that

$$\begin{aligned}
&\Bigg|\Delta_t^+\phi(t, x) + \tilde{\mathscr{H}}(t, x, \Delta_x^c\phi(s, y), \Delta_x^{2+}\phi(s, y), \Delta_x^{2-}\phi(s, y)) \\
&\quad -\partial_t\phi(t, x) - \sup_{u\in\mathbb{U}}\left\{\mu(t, x, u)\partial_x\phi(t, x) + \frac{1}{2}a(t, x, u)\partial_x^2\phi(t, x) + f(t, x, u)\right\}\Bigg| \to 0, \\
&\qquad\qquad \text{as } \delta_t \to 0,\ \delta \to 0 \text{ and } (s, y) \to (t, x),
\end{aligned} \tag{2.15}$$

and

$$\begin{aligned}
&\Bigg|\sup_{z\in\mathcal{Z}}\left\{\phi(t, \Gamma^\delta(y, z)) + K(s, y, z)\right\} - \phi(s, y) \\
&\quad - \sup_{z\in\mathcal{Z}}\left\{\phi(s, \Gamma(x, z)) + K(t, x, z)\right\} + \phi(t, x)\Bigg| \to 0, \\
&\qquad\qquad \text{as } \delta_t \to 0,\ \delta \to 0 \text{ and } (s, y) \to (t, x).
\end{aligned} \tag{2.16}$$

We prove the claim (2.16). Note that for $A \subset \mathbb{R}^n$ and $f_1, f_2 : A \to \mathbb{R}$,

$$\sup_{x\in A} f_1(x) - \sup_{x\in A} f_2(x) \le \sup_{x\in A}\left(f_1(x) - f_2(x)\right), \tag{2.17}$$

and we have

$$\begin{aligned}
&\left|\sup_{z\in\mathcal{Z}}\left\{\phi(s, \Gamma^\delta(y, z)) + K(s, y, z)\right\} - \phi(s, y) - \sup_{z\in\mathcal{Z}}\left\{\phi(t, \Gamma(x, z)) + K(t, x, z)\right\} + \phi(t, x)\right| \\
&\le \sup_{z\in\mathcal{Z}}\left|\phi(s, \Gamma^\delta(y, z)) - \phi(t, \Gamma(x, z))\right| + \sup_{z\in\mathcal{Z}}\left|K(s, y, z) - K(t, x, z)\right| \\
&\quad + |\phi(s, y) - \phi(t, x)|.
\end{aligned}$$

Let $\delta^{\Gamma(x,z)} = \Gamma^\delta(x,z) - \Gamma(x,z)$ and note that $|\delta_i^{\Gamma(x,z)}| \le \delta_i$, $i = 1, \cdots, n$, by the definition of Γ^δ. Hence we find

$$
\begin{aligned}
&|\phi(s,\Gamma^\delta(y,z)) - \phi(s,\Gamma(x,z))| \\
&= |\phi(s,\Gamma(y,z) + \delta^{\Gamma(y,z)}) - \phi(s,\Gamma(x,z))| \\
&= \left|\phi(s,\Gamma(y,z)) - \phi(s,\Gamma(x,z)) + \sum_{i=1}^{n} \partial_i\phi(s,\Gamma(y,z) + \theta_\Gamma\delta^{\Gamma(y,z)})\delta_i^{\Gamma(y,z)}\right| \\
&\le |\phi(s,\Gamma(y,z)) - \phi(s,\Gamma(x,z))| + \sum_{i=1}^{n} |\partial_i\phi(s,\Gamma(y,z) + \theta_\Gamma\delta^{\Gamma(y,z)})|\delta_i,
\end{aligned}
$$

where $\theta_\Gamma \in (0,1)$. Therefore the claim is established by the above estimation, the Lipschitz continuity of K and the continuity of ϕ.

We prove the claim (2.15). Using the inequality (2.17), we have

$$
\begin{aligned}
&\left|\mathscr{F}^\delta(s,y,\phi(s,y)+\xi,\phi+\xi) - \left(\partial_t\phi(t,x) + \mathscr{H}\left(t,x,\partial_x\phi(t,x),\partial_x^2\phi(t,x)\right)\right)\right| \\
&\le \left|\Delta_t^+\phi(s,y) - \partial_t\phi(t,x)\right| + \sup_{u\in\mathbb{U}} |f(s,y,u) - f(t,x,u)| \\
&\quad + \sum_{i=1}^{n} \sup_{u\in\mathbb{U}} \left|\mu_i(s,y,u)\Delta_i^c\phi(s,y) - \mu_i(t,x,u)\partial_i\phi(t,x)\right| \\
&\quad + \sum_{i=1}^{n} \sup_{u\in\mathbb{U}} \left|a_{ii}^-(s,y,u)\Delta_{ii}^-\phi(s,y) - a_{ii}(t,x,u)\partial_{ii}\phi(t,x)\right| \\
&\quad + \frac{1}{2}\sum_{i=1}^{n}\sum_{j\in\mathcal{J}(i)} \sup_{u\in\mathbb{U}} \left|a_{ij}^+(s,y,u)\Delta_{ij}^+\phi(s,y) - a_{ij}^-(s,y,u)\Delta_{ij}^-\phi(s,y) - a_{ij}(t,x,u)\partial_{ij}\phi(t,x)\right|.
\end{aligned}
$$

We first evaluate the time derivative term. The Taylor expansion gives

$$
\begin{aligned}
&\left|\Delta_t^+\phi(s,y) - \partial_t\phi(t,x)\right| \\
&= |\partial_t\phi(s,y) + \partial_t^2\phi(s+\theta_3\delta_t,y)\delta_t - \partial_t\phi(t,x)| \\
&= |\partial_t\phi(t+\delta_{ts},x+\delta^2xy) + \partial_t^2\phi(s+\theta_3\delta_t,y)\delta_t - \partial_t\phi(t,x)| \\
&= \Bigg|\partial_t^2\phi(s+\theta_3\delta_t,y)\delta_t + \partial_t^2\phi(t+\theta_4\delta_t,x+\theta_4\delta_{xy})\delta_{ts} \\
&\qquad + \sum_i \partial_i\partial_t\phi(t+\theta_4\delta_t,x+\theta_4\delta^{xy})\delta_i^{xy}\Bigg|,
\end{aligned}
$$

where $\theta_3, \theta_4 \in (0,1)$ and hence

$$
\left|\Delta_t^+\phi(s,y) - \partial_t\phi(t,x)\right| \to 0 \quad \text{as } \delta_t \to 0 \text{ and } (s,y) \to (t,x).
$$

We next evaluate the terms related to the first derivative of x. Using the Taylor expansion, we find

$$\begin{aligned}
\Delta_i^c \phi(s,y) &= \partial_i \phi(s,y) + \frac{1}{12}\sum_{\kappa=\pm 1} \phi(s, \theta_{5,\kappa,i}\kappa\delta_i e_i)\delta_i^2 \\
&= \partial_i \phi(t,x) + \frac{1}{12}\sum_{\kappa=\pm 1} \phi(s, \theta_{5,\kappa,i}\kappa\delta_i e_i)\delta_i^2 + \partial_t\partial_i\phi(t+\theta_{6,i}\delta_{ts}, x+\theta_{6,i}\delta^{xy})\delta_{ts} \\
&\quad + \sum_{j=1}^{n} \partial_j\partial_i\phi(t+\theta_{6,i}\delta_{ts}, x+\theta_{6,i}\delta^{xy})\delta_j^{xy}.
\end{aligned}$$

where $\theta_{5,\kappa,i}, \theta_{6,i} \in (0,1)$, $i = 1, \cdots, n$ and $\kappa = \pm 1$. Thus we have

$$\begin{aligned}
&\sup_{u\in\mathbb{U}} \left|\mu_i(s,y,u)\Delta_i^c\phi(s,y) - \mu_i(t,x,u)\partial_x\phi(t,x)\right| \\
&\le \sup_{u\in\mathbb{U}} |\mu_i(s,y,u) - \mu_i(t,x,u)||\partial_i\phi(t,x)| \\
&\quad + \sup_{u\in\mathbb{U}} |\mu_i(s,y,u)| \left|\frac{1}{12}\sum_{\kappa=\pm 1}\phi(s,\theta_{5,\kappa,i}\kappa\delta_i e_i)\delta_i^2 + \partial_t\partial_i\phi(t+\theta_{6,i}\delta_{ts}, x+\theta_{6,i}\delta^{xy})\delta_{ts}\right. \\
&\qquad\qquad \left. + \sum_{j=1}^{n}\partial_j\partial_i\phi(t+\theta_{6,i}\delta_{ts}, x+\theta_{6,i}\delta^{xy})\delta_j^{xy}\right|
\end{aligned}$$

The Lipschitz continuity and the linear growth condition of μ implies

$$\begin{aligned}
\sup_{u\in\mathbb{U}} |\mu_i(s,y,u) - \mu_i(t,x,u)| &\le \sup_{u\in\mathbb{U}} C(|s-t| + |y-x|)|\partial_i\phi(t,x)| \\
&= C(|\delta_{ts}| + |\delta^{xy}|),
\end{aligned}$$

and

$$\sup_{u\in\mathbb{U}} |\mu_i(s,y,u)| \le C(1 + |y| + \sup_{u\in\mathbb{U}} |u|).$$

Recall that $\mathbb{U}$ is compact and we obtain

$$\sup_{u\in\mathbb{U}} \left|\mu_i(s,y,u)\Delta_i^c\phi(s,y) - \mu_i(t,x,u)\partial_x\phi(t,x)\right| \to 0, \quad \text{as } \delta \to 0 \text{ and } (s,y) \to (t,x).$$

The terms related to the second derivative of x are evaluated in the same manner.

The Taylor expansion gives

$$
\begin{aligned}
&\Delta_{ij}^{+}\phi(s,y)\\
&= \partial_{ij}\phi(s,y)\\
&\quad + \frac{1}{24\delta_i\delta_j}\Big[-\delta_i^4\partial_{iiii}^4\left(\phi(s,y+\theta_{ij}^{+,i}\delta_i e_i)+\phi(s,y-\theta_{ij}^{-,i}\delta_i e_i)\right)\\
&\qquad\qquad - \delta_j^4\partial_{jjjj}^4\left(\phi(s,y+\theta_{ij}^{+,j}\delta_j e_j)+\phi(s,y-\theta_{ij}^{-,j}\delta_j e_j)\right)\\
&\qquad + \left(\delta_i\partial_i+\delta_j\partial_j\right)^4\left(\phi(s,y+\theta_{ij}^{++}\delta_i e_i+\theta_{ij}^{++}\delta_j e_j)+\phi(s,y-\theta_{ij}^{--}\delta_i e_i-\theta_{ij}^{--}\delta_j e_j)\right)\Big]\\
&= \partial_{ij}\phi(t,x)\\
&\quad + \partial_t\partial_{ij}\phi(t+\theta_{7,ij}\delta_{ts},x+\theta_{7,ij}\delta^{xy})\delta_{ts} + \sum_{l=1}^{n}\partial_l\partial_{ij}\phi(t+\theta_{7,ij}\delta_{ts},x+\theta_{7,ij}\delta^{xy})\delta_l^{xy}\\
&\quad + \frac{1}{24\delta_i\delta_j}\Big[-\delta_i^4\partial_{iiii}^4\left(\phi(s,y+\theta_{ij}^{+,i}\delta_i e_i)+\phi(s,y-\theta_{ij}^{-,i}\delta_i e_i)\right)\\
&\qquad\qquad - \delta_j^4\partial_{jjjj}^4\left(\phi(s,y+\theta_{ij}^{+,j}\delta_j e_j)+\phi(s,y-\theta_{ij}^{-,j}\delta_j e_j)\right)\\
&\qquad + \left(\delta_i\partial_i+\delta_j\partial_j\right)^4\left(\phi(s,y+\theta_{ij}^{++}\delta_i e_i+\theta_{ij}^{++}\delta_j e_j)+\phi(s,y-\theta_{ij}^{--}\delta_i e_i-\theta_{ij}^{--}\delta_j e_j)\right)\Big],
\end{aligned}
$$

and

$$
\begin{aligned}
&\Delta_{ij}^{-}\phi(s,y)\\
&= \partial_{ij}\phi(t,x)\\
&\quad + \partial_t\partial_{ij}\phi(t+\theta_{7,ij}\delta_{ts},x+\theta_{7,ij}\delta^{xy})\delta_{ts} + \sum_{l=1}^{n}\partial_l\partial_{ij}\phi(t+\theta_{7,ij}\delta_{ts},x+\theta_{7,ij}\delta^{xy})\delta_l^{xy}\\
&\quad + \frac{1}{24\delta_i\delta_j}\Big[\delta_i^4\partial_{iiii}^4\left(\phi(s,y+\theta_{ij}^{+,i}\delta_i e_i)+\phi(s,y-\theta_{ij}^{-,i}\delta_i e_i)\right)\\
&\qquad\qquad + \delta_j^4\partial_{jjjj}^4\left(\phi(s,y+\theta_{ij}^{+,j}\delta_j e_j)+\phi(s,y-\theta_{ij}^{-,j}\delta_j e_j)\right)\\
&\qquad - \left(\delta_i\partial_i-\delta_j\partial_j\right)^4\left(\phi(s,y+\theta_{ij}^{+-}\delta_i e_i+\theta_{ij}^{+-}\delta_j e_j)+\phi(s,y-\theta_{ij}^{-+}\delta_i e_i-\theta_{ij}^{-+}\delta_j e_j)\right)\Big],
\end{aligned}
$$

where $\theta_{ij}^{++},\theta_{ij}^{+-},\theta_{ij}^{-+},\theta_{ij}^{--},\theta_{ij}^{+,i},\theta_{ij}^{+,j},\theta_{ij}^{-,i},\theta_{ij}^{-,j},\theta_{7,ij}\in(0,1),\quad i,j=1\cdots,n.$ Hence we see

$$
\begin{aligned}
&\sup_{u\in\mathbb{U}}\Big|a_{ii}(s,y,u)\Delta_{ij}^{-}\phi(s,y)-a_{ii}(t,x,u)\partial_{ii}^{2}\phi(t,x)\Big|\\
&\le \sup_{u\in\mathbb{U}}|a_{ii}(s,y,u)-a_{ii}(t,x,u)||\partial_{ij}\phi(t,x)|\\
&\quad+\sup_{u\in\mathbb{U}}|a_{ii}(s,y,u)|\Big|\partial_t\partial_{ij}\phi(t+\theta_{5,ij}\delta_{ts},x+\theta_{5,ij}\delta^{xy})\delta_{ts}\\
&\qquad\qquad+\sum_{l=1}^{n}\partial_l\partial_{ij}\phi(t+\theta_{5,ij}\delta_{ts},x+\theta_{5,ij}\delta^{xy})\delta_l^{xy}\\
&\qquad\qquad+\frac{1}{12}\,\delta_i^4\partial_{iiii}^4\left(\phi(s,y+\theta_{ij}^{+,i}\delta_ie_i)+\phi(s,y-\theta_{ij}^{-,i}\delta_ie_i)\right)\Big|,
\end{aligned}
$$

and

$$
\begin{aligned}
&\sup_{u\in\mathbb{U}}\Big|a_{ij}^{+}(s,y,u)\Delta_{ij}^{+}\phi(s,y)-a_{ij}^{-}(s,y,u)\Delta_{ij}^{-}\phi(s,y)-a_{ij}(t,x,u)\partial_{ij}^{2}\phi(t,x)\Big|\\
&\le \sup_{u\in\mathbb{U}}|a_{ij}(s,y,u)-a_{ij}(t,x,u)||\partial_{ij}\phi(t,x)|\\
&\quad+\sup_{u\in\mathbb{U}}|a_{ij}(s,y,u)|\Big|\partial_t\partial_{ij}\phi(t+\theta_{7,ij}\delta_{ts},x+\theta_{7,ij}\delta^{xy})\delta_{ts}\\
&\qquad\qquad+\sum_{l=1}^{n}\partial_l\partial_{ij}\phi(t+\theta_{7,ij}\delta_{ts},x+\theta_{7,ij}\delta^{xy})\delta_l^{xy}\Big|\\
&\quad+\sup_{u\in\mathbb{U}}|a_{ij}^{+}(s,y,u)|\Big|\frac{1}{24\delta_i\delta_j}\Big[-\delta_i^4\partial_{iiii}^4\left(\phi(s,y+\theta_{ij}^{+,i}\delta_ie_i)+\phi(s,y-\theta_{ij}^{-,i}\delta_ie_i)\right)\\
&\qquad-\delta_j^4\partial_{jjjj}^4\left(\phi(s,y+\theta_{ij}^{+,j}\delta_je_j)+\phi(s,y-\theta_{ij}^{-,j}\delta_je_j)\right)\\
&\qquad\left(\delta_i\partial_i+\delta_j\partial_j\right)^4\left(\phi(s,y+\theta_{ij}^{++}\delta_ie_i+\theta_{ij}^{++}\delta_je_j)+\phi(s,y-\theta_{ij}^{--}\delta_ie_i-\theta_{ij}^{--}\delta_je_j)\right)\Big]\Big|\\
&\quad+\sup_{u\in\mathbb{U}}|a_{ij}^{-}(s,y,u)|\Big|\frac{1}{24\delta_i\delta_j}\Big[\delta_i^4\partial_{iiii}^4\left(\phi(s,y+\theta_{ij}^{+,i}\delta_ie_i)+\phi(s,y-\theta_{ij}^{-,i}\delta_ie_i)\right)\\
&\qquad+\delta_j^4\partial_{jjjj}^4\left(\phi(s,y+\theta_{ij}^{+,j}\delta_je_j)+\phi(s,y-\theta_{ij}^{-,j}\delta_je_j)\right)\\
&\qquad-\left(\delta_i\partial_i-\delta_j\partial_j\right)^4+\left(\phi(s,y+\theta_{ij}^{+-}\delta_ie_i+\theta_{ij}^{+-}\delta_je_j)+\phi(s,y-\theta_{ij}^{-+}\delta_ie_i-\theta_{ij}^{-+}\delta_je_j)\right)\Big]\Big|.
\end{aligned}
$$

Using the Lipschitz continuity and the linear growth condition of σ, we have

$$
\begin{aligned}
|a(s,y,u)-a(t,x,u)| &= |\sigma^2(s,y,u)-\sigma^2(t,x,u)|\\
&\le \left(|\sigma(s,y,u)|+|\sigma(t,x,u)|\right)|\sigma(s,y,u)-\sigma(t,x,u)|\\
&\le C^2(2+|y|+|x|+2|u|)(|\delta_{ts}|+|\delta^{xy}|),
\end{aligned}
$$

and

$$
|a(s,y,u)|\le C^2(1+|y|+|u|)^2.
$$

Recall that $\mathbb{U}$ is compact, and we obtain

$$\begin{aligned}
&\sup_{u\in\mathbb{U}}\left|a^{ii}(s,y,u)\Delta_{ii}^{-}\phi(s,y)-a_{ii}(t,x,u)\partial_{ii}^{2}\phi(t,x)\right|\to 0,\\
&\sup_{u\in\mathbb{U}}\left|a_{ij}^{+}(s,y,u)\Delta_{ij}^{+}\phi(s,y)-a_{ij}^{-}(s,y,u)\Delta_{ij}^{-}\phi(s,y)-a_{ij}(t,x,u)\partial_{ij}^{2}\phi(t,x)\right|\to 0,\\
&\hspace{20em}\text{as } \delta\to 0 \text{ and } (s,y)\to(t,x).
\end{aligned}$$

By the assumption of f, the remaining evaluation is almost obvious:

$$\sup_{u\in\mathbb{U}}|f(s,y,u)-f(t,x,u)|\le C(1+|y|+|x|+\sup_{u\in\mathbb{U}}|u|)(|\delta_{ts}|+|\delta_{xy}|)$$

and thus

$$\sup_{u\in\mathbb{U}}|f(s,y,u_{-}f(t,x,u)|\to 0,\quad \text{as } \delta_x\to 0 \text{ and } (s,y)\to(t,x).$$

Therefore the claim (2.15) is established.

□

By Lemma 2.4.3, Lemma 2.4.7, Lemma 2.4.8 and the convergence framework as in [BS91], we obtain the following theorem.

Theorem 2.4.9
Denote by $V^{\delta}(t,x)$ the solution of the numerical scheme $\mathscr{F}^{\delta}=0$. Then the limit

$$V(t,x)=\lim_{\delta_t,\delta\to 0,\ (s,y)\to(t,x)}V^{\delta}(s,y)$$

exists and is a unique viscosity solution if the comparison theorem is established.

Proof. Lemma 2.4.3, Lemma 2.4.7 and Lemma 2.4.8 establish the claim by Barles and Souganidis [BS91]. □

3 Implementation of Numerical Algorithm for Hamilton-Jacobi-Bellman Quasi-Variational Inequalities

In the last section of the previous chapter, we discuss the convergence of the implicit method of the HJBQVI. In this chapter, we provide a new algorithm to solve the discretized HJBQVI arising in the implicit method defined in the previous section. We also show an illustrative example, optimal forest harvesting problem in the finite time horizon. The majority of the contents of this chapter correspond to Ieda[Ied15b].

3.1 Introduction

Solving the Hamilton-Jacobi-bellman quasi-variational inequality (HJBQVI) is one of the most challenging issues in the stochastic optimal control problem. The HJBQVI is associated with the combined impulse and stochastic optimal control which is able to formulate the system which changes drastically by our control. The combined stochastic optimal control is a quite applicable framework. We present some of the literature which deal with the applications for the mathematical finance: Pliska and Suzuki [PS04], Palczewski and Zabczyk [PZ05] and Kharroubi and Pham [KP10] treat the portfolio optimization with transaction costs; Mundaca and Øksendal [MØ98] and Cadenillas and Zapatero [CZ00] study the control of the exchange rate by the Central Bank; Korn [Kor99] provides the overview relating to the applications of the impulse control. The applications to other areas, for instance, the management of electricity management, the problems of maintenance and quality control and information technology are found in Bensoussan and Lions [BL84] and the references therein.

The major approach to applying the impulse control framework has the aspect that problems or models are formulated to have an analytical solution. For the HJBQVI associated with the one-dimensional infinite horizon combined stochastic optimal control problem, the smooth-fit technique is an established method to obtain the solution. However this technique is not valid for solving general HJBQVIs and to our best knowledge, there is no established method for the general HJBQVI. Hence the development of the numerical method to solve the HJBQVI is significantly required.

The numerical approach to solve the HJBQVI is categorized into two types by the time horizon: the infinite or finite time horizon. We first mention the case of the infinite time horizon. Bensoussan and Lions [BL84] approximate the impulse control problem by iterations of the optimal stopping problem and hence HJBQVI is translated to the HJB variational inequalities (HJBVIs). The numerical method for the HJBVI is well studied due to the motivation for the pricing of American options in the mathematical finance. An alternative method is proposed in Chancelier et. al. [CMS06]. They provide the solving method for a fixed point problem which consists of the contractive operator and the non-expansive operator and the numerical algorithm for the infinite time horizon HJBQVI has appeared as an application. They discretize the HJBQVI by the finite difference scheme and lead an equivalent fixed point problem which is solvable by their algorithm.

In the case of the finite time horizon, we can employ the dynamic programming, the backward induction in a similar fashion as other optimal control problems which have the terminal condition. For instance, Chen and Forsyth [CF08] solves the finite time horizon HJBQVI associated with the annuity pricing problem with a guaranteed minimum withdrawal benefit. They construct the discretized equation which consistent with the original HJBQVI under the constraint that the time grid size is small enough to satisfy a certain condition depend on the space grid size. We note that this method is dependent on the model and thus it appears to be difficult to apply other problems directly.

In this chapter, we propose a new numerical algorithm solving the finite time horizon HJBQVI. Comparing the infinite time horizon case, we must treat the time stepping carefully. We provide an appropriate transform for the considered HJBQVI and lead the numerical scheme which does not need a relationship between the time and spacial stepping. The key points of the present study are that (i) we discretize the HJBQVI with the forward difference for the time grid; (ii) the impulse control part is kept in the same time step. Hence the present method is an implicit method.

The main advantage of the present work is providing a numerical scheme which is independent of the specific model formulation. The stability of the numerical scheme is especially sensitive in the finite time horizon case. The previous works such as [CF08] prove the stability for each concerned problems. The present work gives a proof of the stability in the general framework and hence our method is able to apply without the stability proof. To support the advantage of our method from the side of the implementation, we provide the detailed procedure of the algorithm in the matrix form.

This paper is organized as follows. We present the mathematical formulation of the HJBQVI in Section 3.2. The goal of this section is to display the discretized HJBQVI in the matrix form. Section 3.3 provides a detail of our algorithm. In Section 3.3.1, the matrix form HJBQVI obtained in Section 3.2 is translated into an equivalent fixed point problem. We describe the procedure in detail from the viewpoint of the computational implementation in Section 3.3.2. In Section 3.4, we apply the proposed method to the optimal forest harvesting problem. This problem that determines the optimal harvesting strategy for the ongoing forest. We employ the mathematical formulation of the problem proposed by Willassen [Wil98] based on the infinite time horizon impulse control framework having analytical solutions of the value function and optimal strategy are provided. We introduce the terminal time representing the time of exiting from the forest business and then the above problem turns into the finite time horizon one. In this case, the analytical solution is unavailable, and thus we solve it numerically by the proposed algorithm.

3.2 Mathematical formulation

3.2.1 Model

We consider the following combined stochastic and impulse control problems over a finite time horizon $[0,T]$ with performance criterion

$$J_t^w(x) = \mathbb{E}\left[\int_t^T f(s, X_s^w, u_s)dt + g(X_T^w) + \sum_{t<\tau_j<T} K(\tau_j^-, X_{\tau_j^-}^w, \zeta_j)\middle| X_t^w = x\right]. \tag{3.1}$$

The controlled process $\{X_t^w\}_{t\geq 0}$ is governed by

$$\begin{cases} dX_t^w = \mu(t, X_t^w, u_t)dt + \sigma(t, X_t^w, u_t)dW_t, & \tau_j \leq t < \tau_{j+1}, \\ X_{\tau_{j+1}} = \Gamma(X_{\tau_{j+1}^-}, \zeta_{j+1}), \quad j = 0, 1, 2, \cdots, \end{cases} \tag{3.2}$$

where W_t is a d-dimensional Brownian Motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, $\mu : [0,T]\times\mathbb{R}^n\times\mathbb{U} \to \mathbb{R}^n$, $\sigma : [0,T]\times\mathbb{R}^n\times\mathbb{U} \to \mathbb{R}^{n\times d}$, $\Gamma : \mathbb{R}^n\times\mathcal{Z} \to \mathbb{R}^n$, $\mathbb{U}\subset\mathbb{R}^l$, and $\mathcal{Z}\subset\mathbb{R}^l$. The combined control w consists of the Markov control strategy $u = \{u_t\}_{t\geq 0}$, an $\mathbb{U}$-valued stochastic process which is of the form $u_t = \alpha(t, X_t^w)$ for some function $\alpha : [0,T)\times\mathbb{R}^n \to \mathbb{U}$ and the impulse control strategy $v = \{(\tau_j, \zeta_j)\}_{j=1}^\infty$. Here $\tau_0 = 0$, $\tau_1 < \tau_2 < \cdots$ are $\mathcal{F}_t$-stopping times and $\zeta_j \in \mathcal{Z}$, $j \geq 1$, are $\mathcal{F}_{\tau_j}$-measurable random variables. The performance is measured by the following three functions: a profit rate function $f : [0,T]\times\mathbb{R}^n\times\mathbb{U} \to \mathbb{R}$, a bequest function $g : \mathbb{R}^n \to \mathbb{R}$, and an intervention profit function $K : [0,T), \mathbb{R}^n\times\mathcal{Z} \to \mathbb{R}$.

We denote by $\mathcal{W}$ the set of admissible combined controls, i.e., $w \in \mathcal{W}$ satisfies: (i) a unique strong solution of the SDE (3.2) with control w exists; (ii) $\lim_{j\to\infty}\tau_j = T$ a.s. We also assume that for $w \in \mathcal{W}$,

$$\mathbb{E}\left[\int_0^T \left|f(t, X_t^w, u_t)\right| dt\right] < \infty,\ \mathbb{E}\left[\left|g(X_T^w)\right|\right] < \infty,\ \mathbb{E}\left[\sum_{\tau_j<T}\left|K(\tau_j^-, X_{\tau_j^-}^w, \zeta_j)\right|\right] < \infty.$$

The value function corresponding to our problem (3.1) is defined by $V_t(x) = \sup_{w\in\mathcal{W}} J_t^w(x)$, $(t,x)\in[0,T]\times\mathbb{R}^n$. Hence the corresponding HJBQVI is given by (see e.g. [ØS07])

$$\max\left(\sup_{\alpha\in\mathbb{U}}\left\{\partial_t V_t(x) + \mathscr{L}_t^\alpha V_t(x) + f(t,x,\alpha)\right\}, \sup_{\zeta\in\mathcal{Z}}\left\{V_t(\Gamma(x,\zeta)) + K(t,x,\zeta) - V_t(x)\right\}\right) = 0, \tag{3.3}$$

with terminal condition $V_T(x) = g(x)$, $x\in\mathbb{R}^n$, where ∂_t is partial differential operator with respect to t, $\mathscr{L}_t^\alpha$ is the infinitesimal generator of the process X^w at time t:

$$\mathscr{L}_t^\alpha\Psi(x) = \mu(t,x,\alpha)^\top\partial_x\Psi(x) + \frac{1}{2}\mathrm{Tr}\left((\sigma\sigma^\top)(t,x,\alpha)\partial_x^2\Psi(x)\right)$$

for $\Psi \in C^2(\mathbb{R}^n)$. Here ∂_x^j is the j-th order partial differential operator with respect to x and the operator $\top$ means transposition. The continuation set $\mathcal{D}_t$ at time t is defined by

$$\mathcal{D}_t = \left\{ x \in \mathbb{R}^n \,\middle|\, \sup_{\zeta \in \mathcal{Z}} \left\{ V_t(\Gamma(x,\zeta)) + K(t,x,\zeta) - V_t(x) \right\} < 0 \right\}.$$

We impose the condition $\mathcal{D}_t \neq \emptyset$, $t \in [0,T)$ on the control set on, i.e., the control which intervenes at all region is not admissible. A simple way to satisfy this condition is that the intervention times $\{\tau_j\}_{j=1}^{\infty}$ meet the following statement: for each $t \in (0,T)$, there exists $x^R \in \mathcal{S}$ such that $X_t = x^R$ and $\tau_{j+1} \neq t$ where $j = \operatorname{argmax}_i \{\tau_i < t\}$. We are usually able to find such x^R from the problem formulation: an example of this case is seen in Section 3.4.1.

As similar to other numerical problems, we face the problem that the computer does not deal with the unbounded domain. Since our state process X_t^w leaves any bounded region until the termination time T with non-zero probability, this problem is inevitable. To cope with it, we restrict ourselves to the problem with the bounded domain. Let $\mathcal{S} \subset \mathbb{R}^n$ be the domain and $\partial\mathcal{S}$ be the boundary. We assume that (i) $\mathcal{D}_t \cap \mathcal{S} \neq \emptyset$, $t \in [0,T)$; (ii) the intervention function Γ satisfies $\Gamma : \mathcal{S} \to \mathcal{S}$. The boundary condition is given by the function $\psi : [0,T) \times \partial\mathcal{S} \to \mathbb{R}$.

3.2.2 Discretization

We discretize the QVI (3.3) using the standard finite difference scheme with the central difference. Let $\delta_t, \delta = (\delta_1, \cdots, \delta_n)^\top$ be the finite difference steps respect to t and x. We denote by $\mathcal{S}_\delta$ the spacial grid and then $\mathcal{S}_\delta = \mathcal{S} \cap \prod_{i=1}^n (\delta_i \mathbb{Z})$. The discretized boundary $\partial\mathcal{S}_\delta$ is also represented by $\partial\mathcal{S}_\delta = \partial\mathcal{S} \cap \prod_{i=1}^n (\delta_i \mathbb{Z})$. For the convenience we introduce symbols of the time grid points $\{t_i\}_{0 \le i \le N^t}$, $t_i \in [0,T]$ and the spacial grid points including boundary $\{x_i\}_{1 \le i \le N^x + \bar{N}^x}$, $x_i \in \mathcal{S}_\delta \cup \partial\mathcal{S}_\delta$, where N^t is the number of the time grid points, N^x is the number of the spacial grid points and $\bar{N}^x$ are the number of the discretized boundary points. We note that the subsequences $\{x_i\}_{1 \le i \le N^x}$ and $\{x_i\}_{N^x < i \le N^x + \bar{N}^x}$ represent the spacial (internal) grid points and discretized boundary points respectively. Furthermore the time grid points are defined sequentially: $t_0 = 0$, $t_1 = \delta_t, \cdots, t_{N^t} = N^t \delta_t = T$. Here we implicitly assume that δ_t is the number supporting the existence of $N^x \in \mathbb{N}$ s.t. $N^t \delta_t = T$. Let $\mathcal{Z}_\delta$ be a discretized set of the impulse control and we assume that: (i) $\Gamma : \mathcal{S}_\delta \times \mathcal{Z}_\delta \to \mathcal{S}_\delta$, (ii) we can take space indices which allow to define an integer function $\eta : \{1, \cdots, N^x\} \times \mathcal{Z}_\delta \to \{1, \cdots, N^x\}$ s.t.

$$\Gamma(x_i, \zeta) = x_{\eta(i,\zeta)}, \quad \eta(i,\zeta) < i, \quad \text{with} \quad i \in \{1, \cdots, N^x\},\ \zeta \in \mathcal{Z}_\delta, \tag{3.4}$$

and $x_1 \in \mathcal{D}_t$.

The discretized QVI of the QVI (3.3) is defined as follow:

$$\begin{cases} \max\left(\sup_{\alpha\in\mathbb{U}}\left\{\dfrac{\Phi^{k+1}(x)-\Phi^{k}(x)}{\delta_t}+\bar{\mathscr{L}}^{\alpha,k}\Phi^{k}(x)+f(t_k,x,\alpha)\right\},\right. \\ \qquad\qquad \left.\sup_{\zeta\in\mathcal{Z}_\delta}\left\{\Phi^{k}(\Gamma(x,\zeta))+K(t_k,x,\zeta)-\Phi^{k}(x)\right\}\right)=0, \quad x\in\mathcal{S}_\delta, \\ \Phi^{k}(x)=\psi(t_k,x), \quad x\in\partial\mathcal{S}_\delta, \end{cases} \tag{3.5}$$

for $k\in\{0,\cdots,N^t-1\}$, with terminal condition $\Phi^{N^t}(x)=g(x)$, $x\in\mathcal{S}_\delta\cup\partial\mathcal{S}_\delta$, where $\Phi^k:\mathcal{S}_\delta\cup\partial\mathcal{S}_\delta\to\mathbb{R}$ and $\bar{\mathscr{L}}^{\alpha,k}$ is the operator such that

$$\begin{aligned}\bar{\mathscr{L}}^{\alpha,k}\Psi(x)=&\Psi(x)\left\{\sum_{i=1}^{n}\frac{-(\sigma\sigma^\top)_{ii}}{\delta_i^2}+\sum_{j\in\mathcal{J}(i)}\frac{|(\sigma\sigma^\top)_{ij}|}{2\delta_i\delta_j}\right\}\\&+\frac{1}{2}\sum_{i=1}^{n}\sum_{\kappa=\pm1}\Psi(x+\kappa\delta_i e_i)\left\{\frac{(\sigma\sigma^\top)_{ii}}{\delta_i^2}-\sum_{j\in\mathcal{J}(i)}\frac{|(\sigma\sigma^\top)_{ij}|}{\delta_i\delta_j}+\kappa\frac{\mu_i}{\delta_i}\right\}\\&+\frac{1}{2}\sum_{i=1}\sum_{j\in\mathcal{J}(i)}\sum_{\kappa,\lambda=\pm1}\Psi(x+\kappa\delta_i e_i+\lambda\delta_j e_j)\frac{(\sigma\sigma^\top)_{ij}^{[\kappa\lambda]}}{\delta_i\delta_j}, \quad x\in\mathcal{S}_\delta,\end{aligned}$$

for $\Psi:\mathcal{S}_\delta\cup\partial\mathcal{S}_\delta\to\mathbb{R}$. Here we have used the notations

$$\begin{aligned}(\sigma\sigma^\top)_{ij}^{[\kappa\lambda]}=&\begin{cases}\max\left(0,(\sigma\sigma^\top)_{ij}\right), & \kappa\lambda=1,\\ -\min\left(0,(\sigma\sigma^\top)_{ij}\right), & \kappa\lambda=-1,\end{cases}\\ \mathcal{J}(i)=&\{1,\cdots,n\}\setminus\{i\},\end{aligned}$$

and have omitted the arguments of $(\sigma\sigma^\top)$ and μ, i.e, $(\sigma\sigma^\top)=(\sigma\sigma^\top)(t_k,x,\alpha)$ and $\mu=\mu(t_k,x,\alpha)$.

We have employed the central difference to obtain the discretized QVI (3.5). If we employ the one-sided difference the discretized operator $\bar{\mathscr{L}}^{\alpha,k}$ becomes

$$\begin{aligned}\bar{\mathscr{L}}^{\alpha,k}\Psi(x)=&\Psi(x)\left\{\sum_{i=1}^{n}\frac{-(\sigma\sigma^\top)_{ii}}{\delta_i^2}+\sum_{j\in\mathcal{J}(i)}\frac{|(\sigma\sigma^\top)_{ij}|}{2\delta_i\delta_j}-\frac{|\mu_i|}{\delta_i}\right\}\\&+\frac{1}{2}\sum_{i=1}^{n}\sum_{\kappa=\pm1}\Psi(x+\kappa\delta_i e_i)\left\{\frac{(\sigma\sigma^\top)_{ii}}{\delta_i^2}-\sum_{j\in\mathcal{J}(i)}\frac{|(\sigma\sigma^\top)_{ij}|}{\delta_i\delta_j}+\frac{\mu_i^{[\kappa]}}{\delta_i}\right\}\\&+\frac{1}{2}\sum_{i=1}\sum_{j\in\mathcal{J}(i)}\sum_{\kappa,\lambda=\pm1}\Psi(x+\kappa\delta_i e_i+\lambda\delta_j e_j)\frac{(\sigma\sigma^\top)_{ij}^{[\kappa\lambda]}}{\delta_i\delta_j}, \quad x\in\mathcal{S}_\delta,\end{aligned}$$

Finally we represent the discretized QVI employing the central difference (3.5) as the matrix QVI form:

$$\begin{cases} \max\left(\sup_{a \in \mathbb{U}^{N^x}} \left\{ \dfrac{\phi_i^{k+1} - \phi_i^k}{\delta_t} + \left(L^{a,k}\phi^k\right)_i + f_i^{a,k} \right\}, \right. \\ \qquad \left. \sup_{z \in \mathcal{Z}_\delta^{N^x}} \left\{ \left(M^z\phi^k\right)_i + K_i^{z,k} - \phi_i^k \right\} \right) = 0, \quad 1 \le i \le N^x, \\ \phi_i^k = \psi(t_k, x_i), \quad N^x < i \le N^x + \bar{N}^x, \end{cases} \tag{3.6}$$

for $k \in \{0, \cdots, N^t - 1\}$ with terminal condition $\phi_i^{N^t} = g(x_i)$, $i \in \{1, \cdots, N^x\}$, where ϕ^k is a $(N^x + \bar{N}^x)$-dimensional vector s.t. $\phi_i^k = \Phi^k(x_i)$, $a = \{a_i\}_{1 \le i \le N^x}$, $a_i \in \mathbb{U}$, $z = \{z_i\}_{1 \le i \le N^x}$, $z_i \in \mathcal{Z}_\delta$, $L^{\alpha_k,k}$ is a $N^x \times (N^x + \bar{N}^x)$ matrix such that for $i \in \{1, \cdots, N^x\}$,

$$L_{ij}^{a,k} = \begin{cases} \displaystyle\sum_{i'=1}^{n} \frac{-(\sigma\sigma^\top)_{i'i'}}{\delta_{i'}^2} + \sum_{j' \in \mathcal{J}(i')} \frac{|(\sigma\sigma^\top)_{i'j'}|}{2\delta_{i'}\delta_{j'}}, & \text{if } i = j, \\ \displaystyle\frac{1}{2}\left\{ \frac{(\sigma\sigma^\top)_{i'i'}}{\delta_{i'}^2} - \sum_{j' \in \mathcal{J}(i')} \frac{|(\sigma\sigma^\top)_{i'j'}|}{\delta_{i'}\delta_{j'}} + \kappa\frac{\mu_{i'}}{\delta_{i'}} \right\}, & \text{if } x_j = x_i + \kappa\delta_{i'}e_{i'}, \\ & \kappa \pm 1,\ i' \in \{1, \cdots, n\} \\ \displaystyle\frac{1}{2}\frac{(\sigma\sigma^\top)_{i'j'}^{[\kappa\lambda]}}{\delta_{i'}\delta_{j'}}, & \text{if } x_j = x_i + \kappa\delta_{i'}e_{i'} + \lambda\delta_{j'}e_{j'}, \\ & \kappa, \lambda = \pm 1, \\ & i' \in \{1, \cdots, n\}, \\ & j' \in \mathcal{J}(i'), \\ 0, & \text{otherwize}, \end{cases}$$

where $f^{a,k}$ is a N^x-dimensional vector s.t. $f_i^{a,k} = f(t_k, x_i, a_i)$, M^z is a $N^x \times (N^x + \bar{N}^x)$ matrix s.t. $M_{ij}^z = \mathbf{1}_{\{j = \eta(i, z_i)\}}$ and $K^{z,k}$ is a N^x-dimensional vector such that $K_i^{z,k} = K(t_k, x_i, z_i)$. We have omitted the arguments of $(\sigma\sigma^*)$ and μ again, i.e, $(\sigma\sigma^\top) = (\sigma\sigma^\top)(t_k, x_i, a_i)$ and $\mu = \mu(t_k, x_i, a_i)$.

Here we assume that the functions μ and σ satisfy the condition

$$|\mu_i(t, x, \alpha)| \le \frac{(\sigma\sigma^\top)_{ii}(t, x, \alpha)}{\delta_i} - \sum_{j \in \mathcal{J}(i)} \frac{|(\sigma\sigma^\top)_{ij}(t, x, \alpha)|}{\delta_j}, \quad (t, x, \alpha) \in [0, T) \times \mathcal{S}_\delta \times \mathbb{U}. \tag{3.7}$$

and then the equation (3.6) becomes stable. We remark that if we have employed the one-sided difference the condition (3.7) becomes the milder one:

$$0 \le \frac{(\sigma\sigma^\top)_{ii}(t, x, \alpha)}{\delta_i} - \sum_{j \in \mathcal{J}(i)} \frac{|(\sigma\sigma^\top)_{ij}(t, x, \alpha)|}{\delta_j}, \quad (t, x, \alpha) \in [0, T) \times \mathcal{S}_\delta \times \mathbb{U},$$

even though the convergence speed becomes slow.

3.3 Algorithm

3.3.1 The equivalent fixed point problem

We convert the matrix QVI (3.6) to the following equivalent fixed point problem:

$$\begin{cases} \phi_i = \max\left(\sup_{a\in\mathbb{U}^{N^x}}\left\{\left(\bar{L}^{a,k}\phi^k\right)_i + \bar{f}_i^{a,k}\right\}, \sup_{z\in\mathcal{Z}^{N^x}}\left\{\left(M^z\phi^k\right)_i + K_i^{z,k}\right\}\right), \quad 1 \le i \le N^x, \\ \phi_i^k = \psi(t_k, x_i), \quad N^x < i \le N^x + \bar{N}^x, \end{cases} \tag{3.8}$$

where $\bar{f}^{a,k}$ is a N^x-dimensional vector, $\bar{L}^{a,k}$ is a $N^x \times (N^x + \bar{N}^x)$ matrix and they are defined as follows:

$$\begin{cases} \bar{f}_i^{a,k} = \dfrac{h\delta_t}{h+\delta_t} f_i^{a,k} + \dfrac{h}{h+\delta_t}\phi_i^{k+1}, \\ \bar{L}^{a,k} = \dfrac{\delta_t}{h+\delta_t}(I + hL^{a,k}). \end{cases} \tag{3.9}$$

Here h is a positive number such that

$$h \le \inf_{a\in\mathbb{U}^{N^x}} \min_{1\le i\le N^x, 1\le k<N^t} \frac{1}{\left|L_{ii}^{a,k}\right|}$$

and I is a $N^x \times (N^x + \bar{N}^x)$ matrix such that $I_{ij} = \delta_{ij}$ where δ_{ij} is the Kronecker's delta. Then the fixed point problem (3.8) satisfies the conditions to apply the method proposed by Chancelier et. al. [CMS06]. The detail is discussed in the last subsection of this section.

3.3.2 Procedures

We are able to solve the problem (3.8) by the backward method. In the following steps we assume that we have already found the ϕ^{k+1}.

step1 set $\phi^k = \phi^{k+1}$,

step2 search the controls $\hat{a}^k$ and $\hat{z}^k$ such that

$$\hat{a}^k = \operatorname*{argmax}_{a\in\mathbb{U}^{N^x}}\left\{\bar{L}^{a,k}\phi^k + \bar{f}^{a,k}\right\}, \quad \hat{z}^k = \operatorname*{argmax}_{z\in\mathcal{Z}^{N^x}}\left\{M^z\phi^k + K^{z,k}\right\}$$

and define an index set $\mathcal{I}^k$ such that

$$\mathcal{I}^k = \left\{ i \in \{1, \cdots, N^x\} \,\middle|\, (\bar{L}^{\hat{a}^k,k}\phi^k)_i + \bar{f}_i^{\hat{a}^k,k} \ge (M^{\hat{z}^k}\phi^k)_i + K_i^{\hat{z}^k,k} \right\},$$

step3 determine a $(N^x + \bar{N}^x) \times (N^x + \bar{N}^x)$ matrix A and $(N^x + \bar{N}^x)$-dimensional vector b as follows:

$$A_{ij} = \begin{cases} \bar{L}_{ij}^{\hat{a}^k,k} & \text{if } i \in \mathcal{I}, \\ M_{ij}^{\hat{z}^k} & \text{if } i \in \{1, \cdots, N^x\} \backslash \mathcal{I}, \\ 0 & \text{if } i \in \{N^x+1, \cdots, N^x + \bar{N}^x\}, \end{cases}$$

$$b_i = \begin{cases} \bar{f}_i^{\hat{a}^k,k} & \text{if } i \in \mathcal{I}, \\ K_i^{\hat{z}^k,k} & \text{if } i \in \{1, \cdots, N^x\} \backslash \mathcal{I}, \\ \psi(t_k, x_i) & \text{if } i \in \{N^x+1, \cdots, N^x + \bar{N}^x\}, \end{cases}$$

step4 solve the linear equation $(I - A)\phi' = b$ where ϕ' is a $(N^x + \bar{N}^x)$-dimensional vector and I is a $(N^x + \bar{N}^x) \times (N^x + \bar{N}^x)$ identity matrix,

step5 if $\max|\phi'_i - \phi_i^k|$ exceeds the admissible error replace ϕ^k by ϕ' and back to step2, else also replace ϕ^k by ϕ' and go to step6,

step6 if $k \neq 1$ replace k by $k-1$ and go back to step1, else determine the Markov control $\hat{\alpha}$, the set $\hat{\mathcal{D}}_{t_k}$ and the impulse control $\hat{v} = (\hat{\tau}_i, \hat{\zeta}_i)_{i \geq 1}$ as follows:

$$\begin{aligned} &\hat{\alpha}(t_k, x_i) = \hat{a}_i^k, \quad \hat{\mathcal{D}}_{t_k} = \left\{ x_i \,\middle|\, i \in \mathcal{I}^k \right\}, \\ &\hat{\tau}_i = \min \left\{ t_k \in \{t_1, \cdots, t_{N^t}\} \,\middle|\, t_k > \hat{\tau}_{i-1}; X_{t_k}^{\hat{w}} \notin \hat{\mathcal{D}}_{t_k} \right\}, \\ &\hat{\zeta}_i(x_j) = \hat{z}_j^{k'}, \quad k' \in \{1, \cdots, N^t\} \text{ s.t. } t_{k'} = \hat{\tau}_i, \end{aligned}$$

where $\hat{\tau}_0 = 0$ and $\hat{w} = (\hat{\alpha}, \hat{v})$.

The control $\hat{w}$ obtained by the above procedure satisfies $J_{t_k}^{\hat{w}}(x_i) \geq J_{t_k}^{w}(x_i)$, $t_k \in \{t_0, \cdots, t_{N^t-1}\}$, $x_i \in \mathcal{S}_\delta$, $w \in \mathcal{W}$ and hence $\hat{w}$ is optimal control.

3.3.3 From the matrix QVI (3.6) to the fixed point problem (3.8)

We first verify that the matrix QVI (3.6) and the fixed point problem (3.8) are equivalent. The equations (3.9) imply that

$$\begin{aligned} f_i^{a,k} &= \left(\frac{1}{\delta_t} + \frac{1}{h} \right) \bar{f}_i^{a,k} - \frac{1}{\delta_t} \phi_i^{k+1}, \\ L^{a,k} &= \left(\frac{1}{\delta_t} + \frac{1}{h} \right) \bar{L}^{a,k} - \frac{1}{h} I. \end{aligned}$$

hence the matrix QVI (3.6) is rewritten in the following form:

$$\begin{cases} \max\left(\left(\dfrac{1}{\delta_t}+\dfrac{1}{h}\right)\sup\limits_{a\in\mathbb{U}^{N^x}}\left\{\left(\bar{L}^{a,k}\phi^k\right)_i+\bar{f}_i^{a,k}-\phi_i^k\right\}\right. \\ \qquad\qquad\left.,\ \sup\limits_{z\in\mathcal{Z}^{N^x}}\left\{\left(M^z\phi^k\right)_i+K_i^{z,k}-\phi_i^k\right\}\right)=0,\quad 1\le i\le N^x, \\ \phi_i^k=\psi(t_k,x_i),\quad N^x<i\le N^x+\bar{N}^x. \end{cases}$$

The parameters δ_t and h are positive numbers thus it is equivalent to [1]

$$\begin{cases} \max\left(\sup\limits_{a\in\mathbb{U}^{N^x}}\left\{\left(\bar{L}^{a,k}\phi^k\right)_i+\bar{f}_i^{a,k}\right\}-\phi_i^k\right. \\ \qquad\qquad\left.,\ \sup\limits_{z\in\mathcal{Z}^{N^x}}\left\{\left(M^z\phi^k\right)_i+K_i^{z,k}\right\}-\phi_i^k\right)=0,\quad 1\le i\le N^x, \\ \phi_i^k=\psi(t_k,x_i;\phi),\quad N^x<i\le N^x+\bar{N}^x. \end{cases}$$

Therefore we obtain the fixed point problem (3.8).

We next show that $\bar{L}^{a,k}$ is a contraction map displayed in the matrix form. By the definition of h, the condition (3.7) and the equations (3.9), we find that

$$0\le\bar{L}_{ij}^{a,k},<1\quad i\in 1,\cdots,N^x,\ j\in 1,\cdots,N^x+\bar{N}^x \tag{3.10}$$

$$\sum_{j=1}^{N^x+\bar{N}^x}\bar{L}_{ij}^{a,k}<1,\quad i\in 1,\cdots,N^x \tag{3.11}$$

Thus we obtain

$$\begin{aligned} \left\|\bar{L}^{a,k}\phi'-\bar{L}^{a,k}\phi\right\|_\infty &= \max_{1\le i\le N^x}\left|\sum_{j=1}^{N^x+\bar{N}^x}\bar{L}_{ij}^{a,k}\phi'_j-\bar{L}_{ij}^{a,k}\phi_j\right| \\ &\le \max_{1\le i\le N^x}\sum_{j=1}^{N^x+\bar{N}^x}\left|\bar{L}_{ij}^{a,k}\phi'_j-\bar{L}_{ij}^{a,k}\phi_j\right| \\ &\le \max_{1\le j'\le N^x}\left|\phi'_{j'}-\phi_{j'}\right|\max_{1\le i\le N^x}\sum_{j=1}^{N^x+\bar{N}^x}\left|\bar{L}_{ij}^{a,k}\right| \\ &< \max_{1\le j'\le N^x}\left|\phi'_{j'}-\phi_{j'}\right| \\ &= \left\|\phi'-\phi\right\|_\infty \end{aligned}$$

Hence $\bar{L}^{a,k}$ is a contraction map displayed in the matrix form.

[1] $\max[cf(x),g(x)]=0$ is equivalent to $\max[f(x),g(x)]=0$, for every $c>0$,

Finally we show that $\bar{L}^{a,k}$ satisfies the discrete maximum principle i.e.,

$$\bar{L}^{a,k}\phi' - \bar{L}^{a,k}\phi \leq \phi' - \phi \Rightarrow \phi' - \phi \geq 0.$$

We lead the contradiction of the above statement. Assume that $\phi'_n - \phi_n < 0$ and $\phi'_i - \phi_i \geq 0$, $i \neq k$. Then we have

$$\begin{aligned}
(\bar{L}^{a,k}\phi')_n - (\bar{L}^{a,k}\phi)_n &= \sum_j \bar{L}^{a,k}_{nj}(\phi'_j - \phi_j) \\
&= \sum_{j \neq n} \bar{L}^{a,k}_{nj}(\phi'_j - \phi_j) + \bar{L}^{a,k}_{nn}(\phi'_n - \phi_n) \\
&\geq \bar{L}^{a,k}_{nn}(\phi'_n - \phi_n) \\
&> (\phi'_n - \phi_n)
\end{aligned}$$

Hence we obtain

$$^{\exists}n \text{ s.t. } \phi'_n - \phi_n < 0 \;\Rightarrow \bar{L}^{a,k}\phi' - \bar{L}^{a,k}\phi \nleq \phi' - \phi$$

which establish the statement.

3.4 Numerical results

We apply our method to the finite horizon optimal forest harvesting problem based on the Willassen's formulation. The mathematical formulations of the original and extended problems are described in Section 3.4.1. The original work considers the infinite time horizon impulse control problem and gives the analytical solution in the unbounded domain. We first restrict the domain to the bounded one and add the boundary condition which gives the equivalent solution to the unbounded domain if the domain is large enough. In Section 3.4.2 we discuss the validity of our boundary condition using the numerical experiment. Our main target problem, the finite time horizon optimal forest harvesting problem which solution is not obtained analytically is discussed in Section 3.4.3.

3.4.1 Optimal forest harvesting problem

Let X_t be the biomass of a forest at time t and $\tau_1 < \tau_2 < \cdots$ be the tree harvesting times. We cut all trees in the forest at time τ_i and replant the biomass $\tilde{x} \in \mathbb{R}$. Suppose that the growth of the biomass follows the Geometric Brownian motion and then X_t is governed by

$$\begin{cases} dX^v_t = \mu X^v_t dt + \sigma X^v_t dW_t, \quad \tau_j < t < \tau_{j+1}, \\ X^v_{\tau_j} = \tilde{x}, \quad j = 1, 2, \cdots, \end{cases} \tag{3.12}$$

where μ and σ are positive constants. Furthermore we suppose that τ_i satisfies the conditions to be the intervention time, i.e., τ_j is an $\mathcal{F}_t$-stopping time and $\tau_j < \infty$ a.s.

Then X_t^v has a unique strong solution and $v := (\tau_1, \tau_2, \cdots)$ is the admissible impulse control strategy.

The original work formulated by Willassen [Wil98] is defined as the infinite time horizon optimal impulse control problem. Let $\beta \in (0,1)$ be the proportional harvesting cost and $Q > 0$ be the replanting cost of the biomass $\tilde{x}$. Then the performance criterion is

$$J^v(x) = \mathbb{E}\left[\sum_{j=1}^{\infty} e^{-\lambda \tau_j} \left((1-\beta) X_{\tau_j -} - Q \right) \middle| X_0^v = x \right], \tag{3.13}$$

where $\lambda > 0$ is the discounting factor. We impose $\tilde{x}$, β and Q on the condition $(1-\beta)\tilde{x} < Q$: if this is not the case, then the optimal strategy is that we harvest trees immediately after the replant which is a quite vacuity situation. The vacuity harvesting just after the replanting also suggests that $x^R = \tilde{x}$ for any $t \in (0, \infty)$.

The value function corresponding to the criterion (3.13) is defined by $V(x) = \sup_v J^v(x)$, $x \in \mathbb{R}$ and hence the corresponding HJBQVI is given by

$$\max\left(\frac{\sigma^2 x^2}{2} \partial_x^2 V(x) + \mu x \partial_x V(x) - \lambda V(x), V(\tilde{x}) + (1-\beta)x - Q - V(x) \right) = 0 \tag{3.14}$$

Willassen solved the HJBQVI (3.14) explicitly:

$$V(x) = \begin{cases} \Psi(x) & \text{for } x < y, \\ (1-\beta)x - Q + \Psi(\tilde{x}) & \text{for } x \geq y, \end{cases} \tag{3.15}$$

where

$$\Psi(x) = \frac{(1-\beta)y}{\gamma} \left(\frac{x}{y} \right)^{\gamma}, \quad \gamma = \frac{\sigma^2 - 2\mu + \sqrt{(\sigma^2 - 2\mu)^2 + 8\sigma^2 \lambda}}{2\sigma^2}$$

and $y > \tilde{x}$ is a solution of

$$y = \frac{\gamma Q - (1-\beta) y \left(\tilde{x}/y \right)^{\gamma}}{(1-\beta)(\gamma - 1)}.$$

We call y the strategy switch point. The key ideas to obtain the solution are as follows: (i) the condition for the cost suggests that we should wait for the harvesting until X_t exceeds a certain value y; (ii) since the partial differential equation $\frac{\sigma^2 x^2}{2} \partial_x^2 V(x) + \mu x \partial_x V(x) - \lambda V(x) = 0$ has the analytical general solution, we obtain the value function by connecting this analytical solution and $V(\tilde{x}) + (1-\beta)x - Q$ smoothly at y.

We extend the above problem by considering the case that the farmer exits the forest business at time T: he harvests all trees and does not replant at time T. Then the problem turns into the finite time horizon problem and the performance criterion is modified as follow:

$$J_t^v(x) = \mathbb{E}\left[\sum_{t < \tau_j < T} e^{-\lambda \tau_j} \left((1-\beta) X_{\tau_j^-}^v - Q \right) + e^{-\lambda T} (1-\beta) X_T^v \middle| X_t = x \right]. \tag{3.16}$$

The value function corresponding to this problem is defined by $V_t(x) = \sup_v J_t^v(x)$, $t \in [0, T)$, $x \in \mathbb{R}$ and thus the corresponding HJBQVI is given by

$$\max\Bigg(\partial_t V_t(x) + \frac{\sigma^2 x^2}{2}\partial_x^2 V_t(x) + \mu x \partial_x V_t(x), \\ V_t(\tilde{x}) + e^{-\lambda t}\left((1-\beta)x - Q\right) - V_t(x)\Bigg) = 0 \tag{3.17}$$

with terminal condition $V_T(x) = e^{-\lambda T}(1-\beta)x$.

In this case, the analytical solution is not available. However we can expect the behavior of the solution of the HJBQVI (3.17). Since the performance criterion J_0^v is equivalent to that of the infinite time horizon case (3.13) on the limit $T \to \infty$, the value function V_t and the optimal control $\hat{v}$ coincide with the infinite horizon ones if T is large enough and $t \ll T$.

3.4.2 Determination of the bounded domain and boundary condition

The idea introducing the strategy switch point y is significant for determination of the candidate finite domain and boundary condition. Let $x_{\max}$ be positive real value, $\mathcal{S} = (0, x_{\max})$ be the candidate bounded domain and boundary $\partial\mathcal{S} = \{0, x_{\max}\}$ be the candidate boundary. We first define ψ as the function giving the boundary condition of the infinite time horizon case such that

$$\begin{cases} \psi(0) = 0, \\ \psi(x_{\max}) = V(\tilde{x}) + (1-\beta)x_{\max} - Q. \end{cases} \tag{3.18}$$

Since we cannot expect the forest growth after the biomass reaches to 0, the value function should be 0 at $x = 0$. We are able to expect that this boundary condition gives the same solution in the infinite domain case if $x_{\max}$ is enough larger than y.

We examine this candidate domain and boundary condition by solving the HJBQVI (3.14) with them numerically. The method to solve the infinite horizon HJBQVI is proposed by Øksendal and Sulem [ØS07]. Since our state process is the 1-dimensional geometric Brownian motion, we are able to employ the method without extra assumptions. The HJBQVI is discretized by the standard finite difference scheme with central difference and we denote by δ_x be the finite difference step and N^x be the number of the grid points. We determine the parameters as table 3.1. In this situation, the value of strategy switch point y is 5.495503.

The numerical results obtained by the algorithm are as follows. Figure 3.1 shows the maximum error of the value function comparing the analytical solution (3.15) and the numerical solution. The order of the error is $O(\delta_x^2)$ which coincides with that implied by the finite difference scheme with the central difference. Figure 3.2 displaying the analytical strategy switch point with red line and the numerical one with blue line indicates that the numerical result of switch point is well accorded with the δ_x-order. Therefore we conclude that our candidate domain and boundary condition are valid.

Parameter	Description	Value
$x_{\max}$	right limit of the domain	10
$\tilde{x}$	initial biomass	1
β	harvesting cost rate	10%
Q	replanting cost	2
μ	expected growth rate	1
σ	volatility	1
λ	discount factor	2

Table 3.1: Parameters

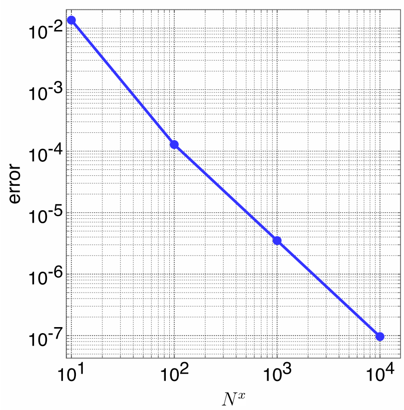

Figure 3.1: Maximum error.

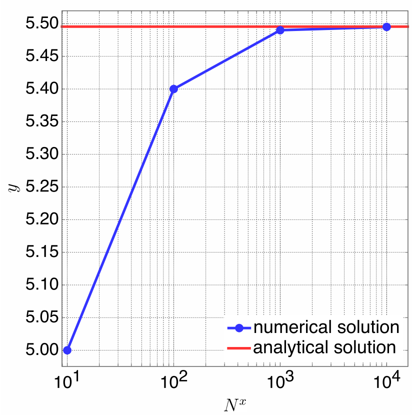

Figure 3.2: Strategy switch point.

We next define the boundary condition in the finite time horizon case as the slightly modified one from the infinite time horizon case:

$$\begin{cases} \psi(t,0) = 0, \\ \psi(t,x_{\max}) = V_t(\tilde{x}) + e^{-\lambda t}\left((1-\alpha)x_{\max} - Q\right). \end{cases}$$

This boundary condition works well under the case that $x_{\max}$ is large enough: the value of $x_{\max}$ should be larger than one of the infinite time horizon case. Because of the replanting cost Q we are able to expect that the optimal strategy close to the terminal time T is not cutting down the trees. However, our boundary condition has enforced the cutting down the trees at the boundary point $x_{\max}$. We can avoid this contradiction by taking the value of $x_{\max}$ large enough. The contribution of the cost Q to the value function $V_t(x)$ vanishes if the biomass x is large enough: the harvest profit which becomes larger with the growth of the biomass x; the cost Q does not depend on the biomass x. We discuss this issue again in the following section with the numerical results.

At the end of this subsection, we mention about the computational load to solve the infinite time horizon HJBQVI by this algorithm. Figure 3.3 describes the computational

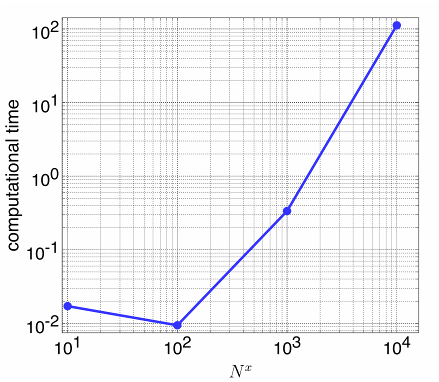

Figure 3.3: Computational time.

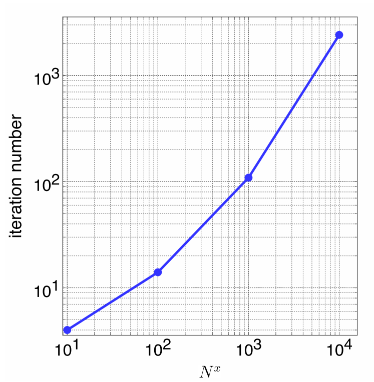

Figure 3.4: Iteration number.

time as a function of the number of the grid points [2]. We see that the computational time grows exponentially when δ_x becomes finer. The main load is the growth of size of the linear equations which is inevitable until we employ the finite difference scheme. The search of the optimal regular and impulse controls on the each grid points is the second load: the index i takes the larger range value when N^x becomes larger. There is the possibility that we can ease up this factor by the massively parallel computing such as the GPGPU or others. Figure 3.4 shows the inherent load of this algorithm. Since the finer finite difference step allows us to compute the more accurate value function, we need more iteration to reach the fixed point.

[2] The detail of our computational resources is as follows. The computer we use has Intel Core i5 650 @3.20GHz and 4GB RAMs. Our code is parallelized by the OpenMP, and we used the PARDISO [SG04], the linear equation solver.

3.4.3 Finite time horizon case

We now treat the main issue of this section. In this situation the matrix operators of our algorithm are defined as follows:

$$
\begin{aligned}
L_{ij}^{a,k} &= \begin{cases} -\dfrac{\sigma^2 x_i^2}{\delta_x^2} & i = j, \\ \dfrac{\sigma^2 x^2}{2\delta_x^2} \pm \dfrac{\mu x}{2\delta_x} & i = i \pm 1, \\ 0 & \text{otherwise}, \end{cases} \\
f^{a,k} &= 0, \quad \eta(i, z_i) = j \in \mathbb{N} \text{ s.t. } x_j = \tilde{x}, \\
K_i^{z,k} &= e^{-\lambda t_k} \left((1-\alpha)x_i - Q\right).
\end{aligned}
$$

We set $T = 3.0$ and $N^t = 3000$ and then $\delta_t = 0.001$. The other parameters are used the same value in table 3.1 except $x_{\max}$.

The first numerical result we discuss in this subsection is the behavior of $\tilde{y}_t$, the strategy switch point at time t. As mentioned in Section 3.4.1, we can expect that $\tilde{y}_t$ converges to y, the strategy switch point discussed in the previous section if t goes to 0 and T is large enough. We also remind that $\tilde{y}_t$ contains the error if $x_{\max}$ is not large enough. Hence we examine the various value of $x_{\max}$ and the result is shown in Figure 3.5.

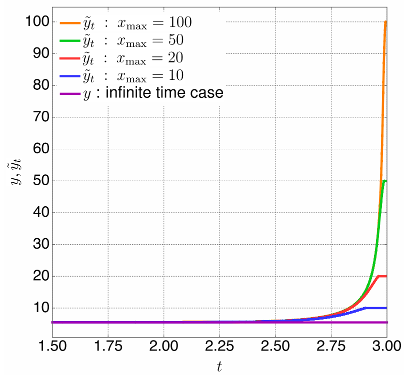

Figure 3.5: $\tilde{y}_t$ with various $x_{\max}$. The orange, green, red and blue lines indicate the strategy switch point obtained with the boundary $x_{\max}$ =100, 50, 20 and 10 respectively. The purple line indicates that of the infinite time horizon case.

The remarkable point we first mention is that $\tilde{y}_t$ converges to y regardless of the value of $x_{\max}$. It coincides with our expectation discussed in the previous subsection. We can understand the behavior of $\tilde{y}_t$ as follows. The behavior of $\tilde{y}_t$ close to T which takes the much higher value than $\tilde{y}$ suggests that we should keep the trees and this is quite consistent with the discussion in the previous subsection. If the time t disengages from

T, the merit of the waiting for the harvest caused by the replanting cost Q vanishes and hence the value of $\tilde{y}_t$ decreases.

We discuss the error caused by the improper boundary condition. In the previous subsection we hypothesize that the error vanishes The behavior of $\tilde{y}_t$ support our hypothesis. The evidence is the time interval that $\tilde{y}_t$ is fixed on $x_{\max}$. This time interval decreases with the increase of $x_{\max}$ and we cannot recognise the interval the case of $x_{\max} = 100$ in Figure 3.5. Another evidence is that $\tilde{y}_t$ with the smaller $x_{\max}$ consists with one with the larger $x_{\max}$ if t disengages from T. The time interval that they are consistent becomes longer if $x_{\max}$ become large.

We next discuss the computational load. We set $x_{\max} = 100$ and the results are displayed in Figure 3.6 and Figure 3.7. Figure 3.6 describes the computational time

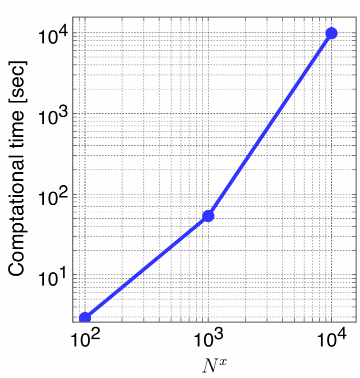

Figure 3.6: Computational time.

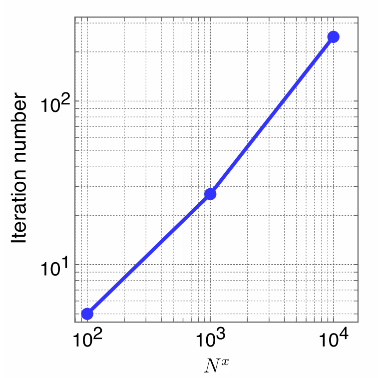

Figure 3.7: Iteration number.

whose transverse is the number of the grid points. The new load factor is obviously the time grid size N^t. The growth of size of the linear equation and the search cost for the optimal controls $\hat{a}^k$ and $\hat{z}^k$ are quite the same factors as the previous subsection. However, the iteration number for convergence to the fixed point is slightly different. Figure 3.7 which shows that the maximum iteration number of each time step suggests that we only need to the approximately 10% iteration comparing to the same grid size infinite time horizon case. This is due to the determination of the initial value of ϕ^k. Since δ_t is small enough the difference between ϕ^k and ϕ^{k+1} is expected to be small. Hence we are able to reduce the iteration number than the infinite case which is the case that we have no information about the solution.

3.5 Discussion

In this section, we check the robustness of our scheme using numerical results with various parameters. We use the same parameters in Table 3.1, except the right limit of the domain $x_{\max}$: we set $x_{\max} = 100$.

We first see the effect of the discount factor λ. Figure 3.8 shows the time evolution of the strategy switch points with various λ. The solid lines indicate the analytical optimal switch points y in the infinite time horizon case. The dashed lines are the numerical optimal switch points $\tilde{y}_t$ in the finite time horizon case. The behaviors of the numerical results are the same in Section 3.4.3: $\tilde{y}_t$ close to $x_{\max}$ if t near the terminal T; $\tilde{y}_t$ converge to y if $t \ll T$. If the discount factor takes the smaller value, we are able to expect that the switch point $\tilde{y}_t$ takes the larger value, since we weight the latter profits heavily. The numerical solutions capture this feature well.

We next mention how the environment parameters μ and σ. Figure 3.8 shows the time evolution of the strategy switch points with various μ and σ. Each panel displays the results with the fixed parameter μ indicated on the top of the panel. We observe that the switch point $\tilde{y}_t$ takes larger value if μ increases. Since the parameter μ stands for the average growth ratio of the forest, it is optimal to wait for the growth of the trees. Furtermore, the behaviors of the numerical results are also the same in Section 3.4.3.

The effect of the costs are evaluated in Figure 3.10 displaying the time evolution of the strategy switch points with various β and Q. We find that if the costs increases, the strategy switching point $\tilde{y}_t$ takes the larger value. It is rigorous because the difference of the profit between the terminal harvesting which does not require the replanting cost and the ongoing harvesting increases if the cost increases.

Finally, we remark that all of the numerical results in this section are calculated stably.

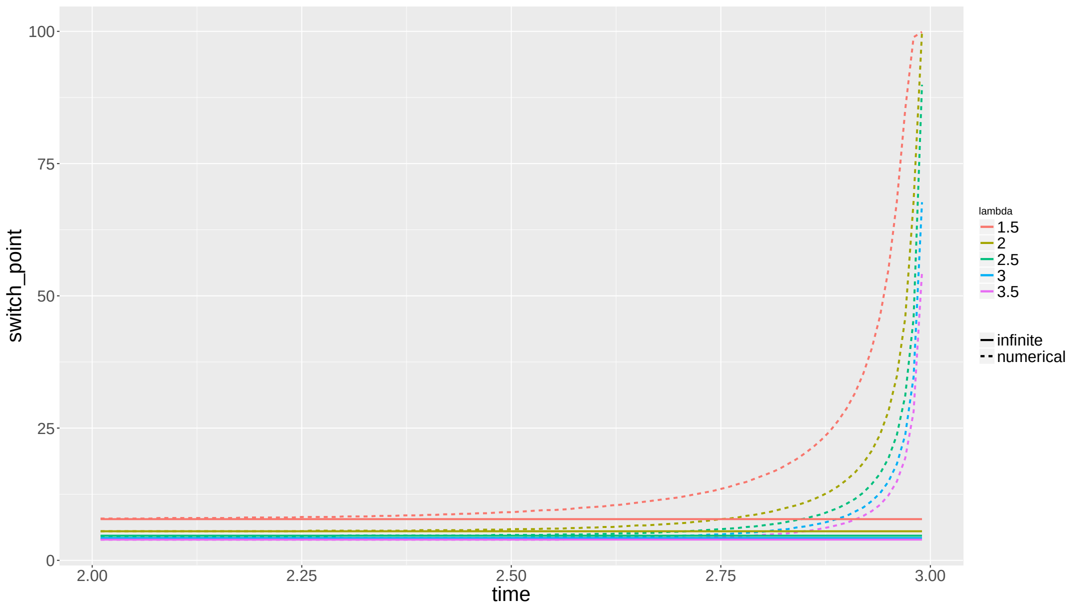

Figure 3.8: $\tilde{y}_t$ with various λ.

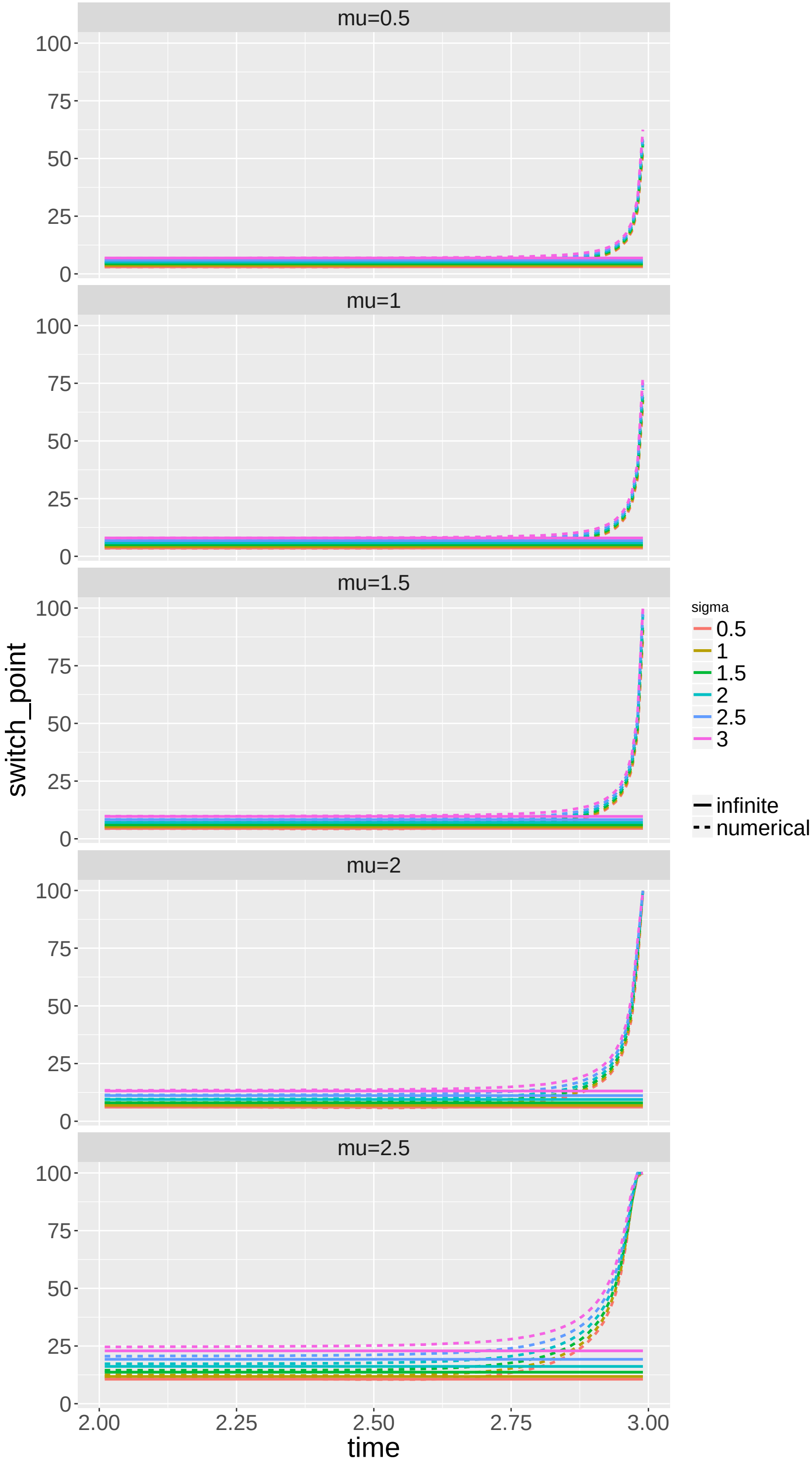

Figure 3.9: $\tilde{y}_t$ with various μ and σ.

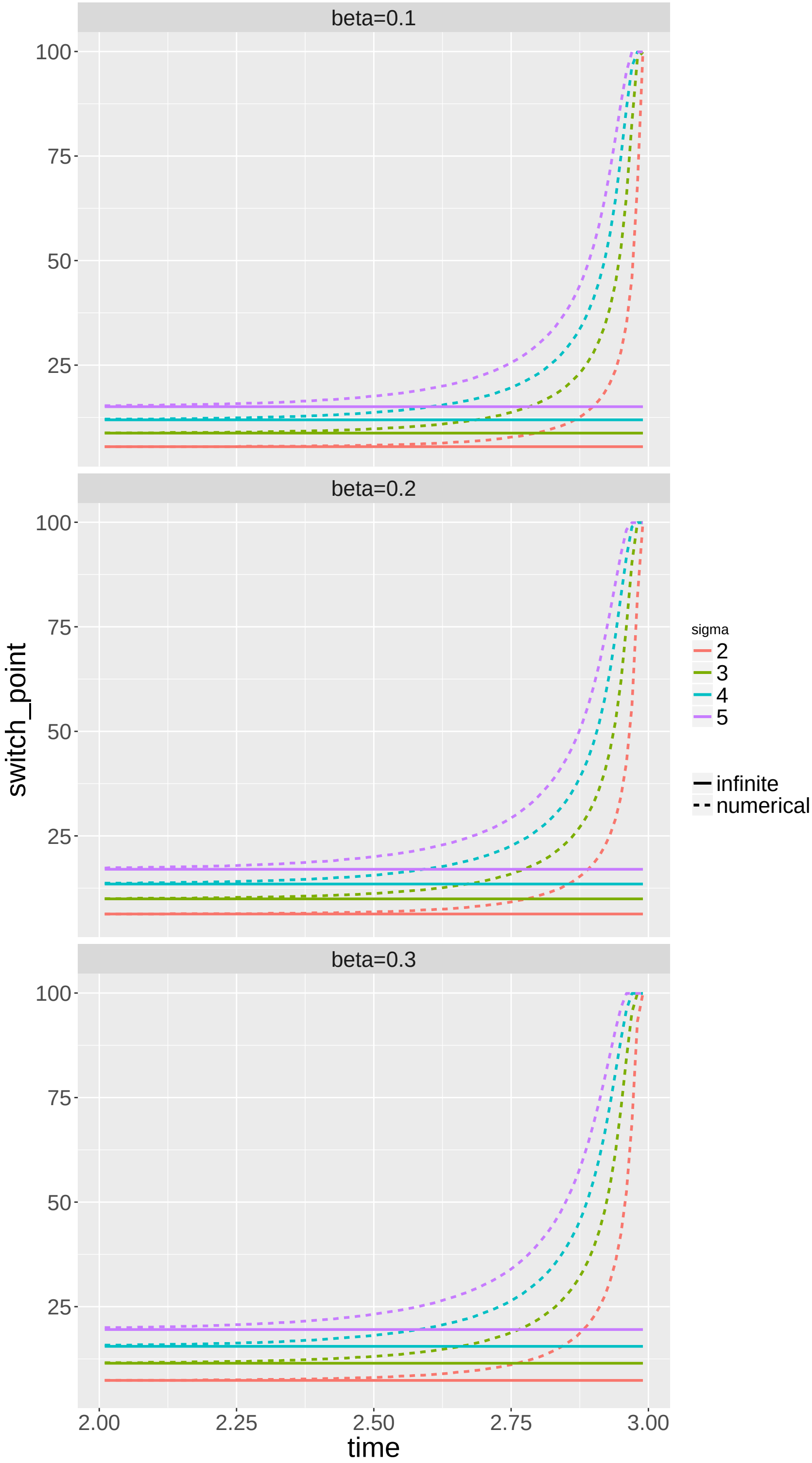

Figure 3.10: $\tilde{y}_t$ with various β and Q.

3.6 Summary

We have proposed a new numerical method to solve the HJBQVI associated with the combined impulse and stochastic optimal control problem over the finite time horizon.

The key points of our numerical scheme are that (i) we discretize the HJBQVI with the forward difference for the time grid; (ii) the impulse control part is kept in the same time step. Hence the present method is an implicit method. The main advantage of the present method is that it is independent of the specific model formulation. The present work gives a proof of the stability in the general framework and hence our method is able to apply without the stability proof. We have provided the detailed procedure of the algorithm displayed in the matrix form for the sake of the easy implementation.

We apply our method to the optimal forest harvesting problem. The original problem formulated by Willassen [Wil98] is defined as the infinite time horizon impulse control problem and has an analytical solution. We introduce the terminal time which represents the time of exiting from the forest business and then the above problem turns into the finite time horizon one. Since the original problem is defined on the unbounded domain, we have introduced the equivalent bounded domain and boundary condition and we have verified them by the numerical experiment.

The analytical solution of our finite time horizon problem is unavailable and hence we solve it by our algorithm. The behavior of the obtained optimal strategy is reasonable: the strategy coincides with the infinite time horizon one when the terminal time goes to infinity; the strategy switch point, the threshold of the biomass to harvest the trees is much higher than the infinite horizon case near the terminal time.

4 Optimal Execution Problem

In this chapter, we discuss an application of a combined stochastic control for mathematical finance.

We study the optimal execution problem under stochastic price recovery based on limit order book dynamics. We model price recovery after execution of a large order by accelerating the arrival of the refilling order,which is defined as a Cox process whose intensity increases by the degree of the market impact. We include not only the market order but also the limit order in our strategy in a restricted fashion. We formulate the problem as a combined stochastic control problem over a finite time horizon.

The corresponding Hamilton–Jacobi–Bellman quasi-variational inequality is solved numerically by the algorithm proposed in the previous chapter.

The majority of the contents of this chapter corresponds to Ieda[Ied15a].

4.1 Introduction

The optimal execution problem, which seeks the optimal way to liquidate or acquire a large asset position, is a matter of concern for both market practitioners and academic researchers. The pioneering work on the optimal execution problem, Bertsimas and Lo [BL98], provides an intuitive optimal strategy to minimize the purchasing cost under a discrete time grid and linear market impact, specifically dividing the amount of the target asset equally across the time grid. In seminal papers by Almgren and Chriss [AC01] and Almgren [Alm03], the performance criterion of the execution consists not only of the expected revenue but also a penalty for the uncertainty of the revenue. In these papers, the market impact is defined by two components: the temporary impact and the permanent impact. The former work considers a discrete time model and the latter a continuous time model. The approach using market impact consists of temporary and permanent components in continuous time with various risk-return criteria appears in papers such as Schied and Schöneborn [SS09], Forsyth [For11], Gatheral and Schied [GS11], Kato [Kat14] and Brigo and Graziano [BD14].

A recent trend in the study of the optimal execution problem is to include features of the limit order book (LOB). The shape of the LOB, or how the limit orders stack in the LOB, is a crucial aspect for modeling the market impact. The resilience of the LOB, which leads the price recovery found after the execution of large orders, is also a significant feature of the LOB. For studies of the empirical features of the LOB and modelling of the dynamics related to the LOB, we refer the reader to Almgren [Alm+05], Large [Lar07], Bouchaud et. al. [BFL09], Hall and Hautsch [HH07], Muke and Farmer [MF08], Toke [Tok11], Gould et. al. [Gou+13] and Cont et. al. [CST10]. Research reflecting

LOB information in optimal trade execution is addressed in Alfonsi and Schied [AS10], Predoiu et. al. [PSS11], Kharroubi and Pham [KP10], Guilbaud et. al. [GMP13], and Obizhaeva and Wang [OW13]. In particular, Obizhaeva and Wang [OW13] study linear market impact and exponential-type deterministic price recovery, finding an optimal strategy consisting of the initial large trade, pre-scheduled small trades during the period, and the terminal large trade.

In this chapter, we study the optimal execution problem under stochastic price recovery based on LOB dynamics. From the viewpoint of the LOB, the price recovery after execution of a large order is regarded as refilling the stacked limit order at the prior price. We model price recovery by accelerating the arrival of the refilling order. Following research on dynamics of the LOB such as [CST10] and [Lar07], we adopt the Cox process model to describe the arrival of the refilling order. The intensity of the Cox process is increased by the degree of the market impact. We define two types of intensity functions, linear and exponential, and call the former a weak recovery intensity and the latter a strong recovery intensity.

The price recovery possibly induces a waiting time to ease the market impact. To desterilize this waiting time, we include not only the market order but also the limit order in our execution strategy. In previous research dealing with the optimal execution problem, the limit order is rarely included in the optimal execution strategy as it is an intractable order method compared with the market order. Moreover, to the best of our knowledge, the dynamics of the market impact due to a large limit order remains uncertain. To avoid the complexity due to limit orders, including the market impact, we allow a restriction on the limit order. Our restricted limit order strategy is regarded as a quasi-iceberg strategy. We again use a Cox process model to describe the dynamics of the counterpart market order arrivals, which turns into a Poisson process under our restrictions.

The first main result of the present study is to provide a dynamic execution strategy depending on the state at each time step. Our new price recovery model, which is defined as a stochastic process, is the source of the dynamic behavior in the strategy. The key to understanding our strategy is tolerance for the market impact depending on the time remaining and the current state. The tolerance governs both the size and the timing of the trade. A quantitative demonstration of the usefulness of including the limit order in the strategy is the second main result. The proposed optimal strategy gives competitive results in numerical simulations, even though the limit order is allowed under conservative evaluation of the execution price.

This paper is organized as follows. Section 4.2 introduces the mathematical formulation of our execution problem. We use the combined stochastic control framework, and the market and limit order strategies are formulated by the impulse and regular stochastic controls respectively. The goal of this section is to derive the corresponding Hamilton–Jacobi–Bellman quasi-variational inequality (HJBQVI). In Section 4.3, we present a numerical method for solving the HJBQVI, following a similar procedure to that found in [Ied15b]. The HJBQVI is discretized by a finite difference scheme and is converted to an equivalent fixed point problem. In Section 4.4, we present numerical results under several conditions, specifically the types of recovery intensity functions and either including or excluding the limit order.

4.2 Model formulation

We consider the selling order execution problem: we must sell all share holdings before the terminal time. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space, x_0 be an initial number of share holdings and let T be the terminal time. We define our execution problem as the combined stochastic control problem over the finite time horizon T. We note that we call the number of remaining share holdings as the inventory.

4.2.1 Execution Strategy

We first formulate the market and limit order strategies using impulse and stochastic control strategies respectively. We note that we cannot use the both limit and market orders at the same time in this framework.

The market order strategy consists of the timing of each execution and the volume of each order. Let $\{\tau_j\}_{j>0}$ be a sequence of $\mathcal{F}_t$-stopping times, that is, the sequence of execution times of market orders, and let ζ_j be the market order volume at time τ_j defined as an $\mathcal{F}_{\tau_j}$-measurable $\mathcal{Z}_j$-valued random variable, where $\mathcal{Z}_j \subset \mathbb{R}^+\backslash\{0\}$. We prohibit short selling, that is, $\mathcal{Z}_j$ is bounded by the inventory at time τ_j. We denote $v = \{\tau_j, \zeta_j\}_{j>0}$, and note that v satisfies the definition of impulse control strategy..

We allow a limit order under the following restrictions: (i) the order volume is small enough; (ii) the execution price of the limit order is evaluated as a price that is slightly higher (by the fixed spread size s) than the best bid price, regardless of the quoted price (see equation (4.5) in Section 4.2.3). Then our limit order strategy consists of a single factor, the limit order volume. We denote the control set by $\mathbb{L} \subset \mathbb{R}^+$, which is the set of values we are allowed to choose as the limit order volume. We describe the arrival of the counterpart order to our limit order by a Poisson process (see, for example, [CST10][1]). We denote by $l_t \in \mathbb{L}$, N^L and λ^L the limit order volume at time t, the Poisson process describing the arrival of the counterpart order and the intensity of N^L respectively.

4.2.2 Price process and market impact

We next define processes related to the asset price. Let P_t be the unaffected part of the asset price process for the market sell order at time t, and suppose that P_t follows the stochastic differential equation

$$\begin{cases} dP_t = \sigma(t, P_t)dW_t, \\ P_0 = p_0 \in \mathbb{R}^+. \end{cases}$$

[1] In the literature, the arrival is modeled by a Cox process whose intensity depends on the spread size. Our situation coincides with a fixed spread size, and thus the intensity is a fixed value. Hence arrivals are indeed described by a Poisson process.

We note that P_t is the best bid price, and is not the mid-price. We denote by Ξ^v_t the degree of market impact at time t, which is governed by

$$\begin{cases} d\Xi^v_t = -\delta_\Xi dN^\Xi_t, \quad \tau_j \le t < \tau_{j+1}, \\ \Xi^v_{\tau_{j+1}} = \Xi^v_{\tau^-_{j+1}} + \Gamma(\zeta_{j+1}), \quad j = 0, 1, 2, \cdots, \\ \Xi^v_0 = 0, \end{cases} \tag{4.1}$$

where N^Ξ is the Cox process with intensity $\lambda^\Xi(\Xi^v)$. The market order increases the market impact through the function Γ, and the impact recovers following the process N^Ξ.

The process N^Ξ is the focus of the present research. Previous studies describe the dynamics of Ξ_t with a deterministic function. In contrast, our new model describes the dynamics more precisely in the sense that it includes features of the LOB in the dynamics. Price recovery is regarded as a random event corresponding to the arrival of the limit order in the LOB. This price recovery model is based on the view that (i) a large market order consumes the stacked limit order; (ii) the consumed orders at the prior price are refilled by new limit orders; (iii) the arrival of the limit order refilling the consumed order is more frequent than that of the usual limit order. The frequent arrival of the limit order is realized by increasing the intensity λ^Ξ in our model. Empirical studies such as [Lar07] take a similar approach.

With the perspective above, we define $\lambda^\Xi(\xi)$ to be an increasing function of ξ, and propose the following two candidates: a strong recovery intensity defined by the exponential function

$$\lambda^\Xi(\xi) = \bar\lambda_1 \left(e^{\bar\lambda_2 \xi} - 1 \right), \tag{4.2}$$

and a weak recovery intensity defined by the linear function

$$\lambda^\Xi(\xi) = \bar\lambda_1 \xi. \tag{4.3}$$

4.2.3 Inventory and cash holdings

Let X^w_t be the inventory, the number of remaining shares, at time t. Then X^w_t is governed by

$$\begin{cases} dX^w_t = -l_t dN^L_t, \quad \tau_j \le t < \tau_{j+1}, \\ X^w_{\tau_{j+1}} = X^w_{\tau^-_{j+1}} - \zeta_{j+1}, \quad j = 0, 1, 2, \cdots, \\ X^w_0 = x_0, \end{cases} \tag{4.4}$$

where N^L_t is a Poisson process describing the arrival of the counterpart order to our limit order.

We denote by Y^w_t the cash holdings at time t. The dynamics of Y^w_t are described by

$$\begin{cases} dY^w_t = l_t(P_t - \Xi_t + s) dN^L_t, \quad \tau_j \le t < \tau_{j+1}, \\ Y^w_{\tau_{j+1}} = Y^w_{\tau^-_{j+1}} - \zeta_{j+1}(P_{\tau_{j+1}} - \Xi_{\tau_{j+1}}), \quad j = 0, 1, 2, \cdots, \\ Y^w_0 = 0. \end{cases} \tag{4.5}$$

We remark that the profit from sell order execution is evaluated conservatively. As shown in the second line of equation (4.5), the execution price of the market order is assessed as the lowest price affected by the market order. The execution price of the limit order is appraised slightly higher than the best bid price $P_t + \Xi_t$ which is regarded as the minimum price that can be quoted. Hence the advantage of the limit order under the current setting is primarily avoidance of the market impact.

4.2.4 HJBQVI

We define J_t^w, the performance criterion at time t, as

$$J_t^w(x, y, p, \xi) = \mathbb{E}\left[g(X_T, Y_T, P_T, \Xi_T) \middle| X_t = x, Y_t = y, P_t = p, \Xi_t = \xi \right],$$

where $w = (l, v)$ is the combined stochastic control strategy and $g : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$. We remark that our performance criterion only depends on the terminal state. The value function V_t is defined by

$$V_t(x, y, p, \xi) := \sup_w J_t^w(x, y, p, \xi),$$

and the corresponding HJBQVI is given by (see e.g. [ØS07])

$$\max\left(\partial_t V_t(x, y, p, \xi) + \sup_{l \in \mathbb{L}} \left\{ \mathscr{L}^l V_t(x, y, p, \xi) \right\}, \right.$$
$$\left. \sup_{\zeta \in \mathcal{Z}(x)} \left\{ \mathscr{M}^\zeta V_t(x, y, p, \xi) \right\} - V_t(x, y, p, \xi) \right) = 0 \tag{4.6}$$

with terminal condition $V_T(x, y, p, \xi) = g(x, y, p, \xi)$, where $\mathscr{L}^l$ is the infinitesimal generator of the multi-dimensional stochastic process (X_t, Y_t, P_t, Ξ_t)

$$\begin{aligned} \mathscr{L}^l V_t(x, y, p, \xi) = {} & \frac{\sigma^2(t, p)}{2} \partial_p^2 V_t(x, y, p, \xi) \\ & + \lambda^L \left[V_t(x - l, y + (p - \xi + s) l, p, \xi) - V_t(x, y, p, \xi) \right] \\ & + \lambda^\Xi(\xi) \left[V_t(x, y, p, \xi - \delta_\Xi) - V_t(x, y, p, \xi) \right], \end{aligned} \tag{4.7}$$

λ^L is the intensity of N_t^L, and $\mathscr{M}^\zeta$ is the operator describing the intervention:

$$\mathscr{M}^\zeta V_t(x, y, p, \xi) := V_t\left(x - \zeta, y + \left\{ p - \xi - \Gamma(\zeta) \right\} \zeta, p, \xi - \Gamma(\zeta) \right). \tag{4.8}$$

4.3 Numerical method

4.3.1 Terminal wealth criterion and reduced form of the value function

To solve the HJBQVI numerically, we first specify the form of the function g. Following empirical studies of market impact such as [Alm+05], we use a power law function to describe the market impact:

$$\Gamma(x) = \theta_1 x^{\theta_2}, \tag{4.9}$$

where $\theta_1, \theta_2 \in \mathbb{R}^+$. We define the performance criterion as the terminal wealth including the terminal execution. The function g is determined by

$$g(x, y, p, \xi) = y + (p - \xi - \Gamma(x))x. \tag{4.10}$$

The second term of the above equation represents the terminal execution.

In the current setting, we are able to remove the arguments y and p from the value function by substituting the ansatz $V_t(x, y, p, \xi) = y + x(p - \xi) + \phi_t(x.\xi)$, so that the HJBQVI (4.6) becomes

$$\begin{aligned} \max\Bigg(&\partial_t\phi_t(x,\xi) + \sup_{l\in\mathbb{L}}\left\{\lambda^L\left[\phi_t(x-l,\xi) - \phi_t(x,\xi) + ls\right]\right\} \\ &\qquad + \lambda^{\Xi}(\xi)\left[\phi_t(x,\xi-\delta_\Xi) - \phi_t(x,\xi) + x\delta_\Xi\right], \\ &\sup_{\zeta\in\mathcal{Z}(x)}\left\{\phi_t(x-\zeta,\xi+\Gamma(\zeta)) - x\Gamma(\zeta) - \phi_t(x,\xi)\right\}\Bigg) = 0 \end{aligned} \tag{4.11}$$

with terminal condition $\phi_T(x, \xi) = -x\Gamma(x)$.

4.3.2 Discretization of HJBQVI

We discretize the HJBQVI (4.11) using a finite difference scheme with grid size $(\delta_t, \delta_x, \delta_\xi)$:

$$\begin{aligned} \max\Bigg(&\frac{\phi_{t_{k+1}}(x_{i_x},\xi_{i_\xi}) - \phi_k(x_{i_x},\xi_{i_\xi})}{\delta_t} + \sup_{l\in\mathbb{L}_\delta}\left\{\lambda^L\left[\phi_{t_k}(x_{i_x}-l,\xi_{i_\xi}) - \phi_{t_k}(x_{i_x},\xi_{i_\xi}) + ls\right]\right\} \\ &\qquad + \lambda^{\Xi}(\xi_{i_\xi})\left[\phi_{t_k}(x_{i_x},\xi_{i_\xi}-\delta_\Xi) - \phi_{t_k}(x_{i_x},\xi_{i_\xi}) + x_{i_x}\delta_\Xi\right], \\ &\sup_{\zeta\in\mathcal{Z}_\delta(x_{i_x})}\left\{\phi_{t_k}(x_{i_x}-\zeta,\xi_{i_\xi}+\Gamma(\zeta)) - x_{i_x}\Gamma(\zeta) - \phi_t(x_{i_x},\xi_{i_\xi})\right\}\Bigg) = 0, \end{aligned} \tag{4.12}$$

where $k, i_x, i_\xi \in \mathbb{N}$, $t_k = k\delta_t$, $x_{i_x} = i_x\delta_x$, $\xi_{i_\xi} = i_\xi\delta_\xi$, and $\mathbb{L}_\delta$ and $\mathcal{Z}_\delta(x)$ are the discretized sets of $\mathbb{L}$ and $\mathcal{Z}(x)$. The discretized terminal condition is represented by

$$\phi_{t_{N^t}}(x_{i_x}, \xi_{i_\xi}) = -x_{i_x}\Gamma(x_{i_x}),$$

where $N^t = \frac{T}{\delta_t}$, the other grid numbers are denoted by $N^x = \frac{x_0}{\delta_x}$ and $N^\xi = \frac{g(x_0)}{\delta_\xi}$. We have assumed that the grid sizes δ_t, δ_x and δ_ξ are chosen so that the grid numbers N^t, N^x and N^ξ are natural numbers.

We define the vector $\left(\phi^k_{(i_x,i_\xi)}\right)_{i_x\in\{0,\cdots,N^x\}, i_\xi\in\{0,\cdots,N^\xi\}}$ by $\phi^k_{(i_x,i_\xi)} = \phi_{t_k}(x_{i_x}, \xi_{i_\xi})$, from which we obtain the HJBQVI (4.12) in matrix form:

$$\max\left(\frac{\phi^{k+1}_{(i_x,i_\xi)}-\phi^{k}_{(i_x,i_\xi)}}{\delta_t}+\sup_{l\in\mathbb{L}_\delta}\left\{\lambda^L\left[\phi^k_{(i_x-i_l,i_\xi)}-\phi^k_{(i_x,i_\xi)}+ls\right]\right\}\right.$$
$$+\lambda^\Xi(\xi_{i_\xi})\left[\phi^k_{(i_x,i_\xi-1)}-\phi^k_{(i_x,i_\xi)}+x_{i_x}\delta_\Xi\right], \tag{4.13}$$
$$\left.\sup_{\zeta\in\mathcal{Z}_\delta(x_{i-x})}\left\{\phi^k_{(i_x-i^x_\zeta,i_\xi+i^\xi_\zeta)}-x_{i_x}\Gamma(\zeta)-\phi^k_{(i_x,i_\xi)}\right\}\right)=0,$$

where $i_l=\lceil\frac{l}{\delta_x}\rceil$, $i^x_\zeta=\lceil\frac{\zeta}{\delta_x}\rceil$ and $i^\xi_\zeta=\lceil\frac{\Gamma(\zeta)}{\delta_\Xi}\rceil$.

We now introduce the constant $h\in\mathbb{R}^+$ satisfying

$$h>\frac{1}{\delta_t}+2\left(\lambda^\Xi(g(x_0))+\lambda^L\right),$$

and convert the HJBQVI (4.13) into the equivalent fixed point problem

$$\phi^k=\max\left(\sup_{l\in\mathbb{L}_\delta}\left\{\bar{L}^l\phi^k+\bar{f}^{l,k}\right\},\sup_{\zeta\in\mathcal{Z}_\delta(x_{i_x})}\left\{M^\zeta\phi^k+K^\zeta\right\}\right), \tag{4.14}$$

where

$$\bar{L}^l_{ij}=\begin{cases}1-\dfrac{1}{h}\left(\dfrac{1}{\delta_t}+\lambda^\Xi(\xi_{i_\xi})+\lambda^L\right), & i=j=(i_x,i_\xi)\\ \dfrac{\lambda^\Xi(\xi_{i_\xi})}{h}, & i=(i_x,i_\xi),\, j=(i_x,i_\xi-1)\\ \dfrac{\lambda^L}{h}, & i=(i_x,i_\xi),\, j=(i_x-i_l,i_\xi)\\ 0, & \text{otherwise,}\end{cases}$$
$$\bar{f}^{l,k}_{(i_x,i_\xi)}=\frac{1}{h}\left(\frac{1}{\delta_t}\phi^{k+1}_{(ix,i_\xi)}+\lambda^\Xi(\xi_{i_\xi})x_{i_ix}\delta_\Xi+\lambda^L\right),$$
$$M^\zeta_{ij}=\mathbf{1}_{\left\{i=(ix,i_\xi),j=(i_x-i_l,i_\xi+i^\xi_\zeta)\right\}},$$
$$K^\zeta_{(i_x,i_\xi)}=x_{i_x}\Gamma(\zeta).$$

We find that, thanks to the factor h, $\bar{L}^l$ is a contraction map displayed in matrix form, and M^ζ is a non-expansive map. We refer to [Ied15b, Section 3.2] for the procedure to solve the problem in equation (4.14).

4.4 Numerical results

In this section, we investigate features of the optimal strategy obtained by the method described in the previous section. We divide this section into four subsections depending on the type of recovery intensity and the restrictions on limit orders. Throughout this section, we choose the parameters to obtain the optimal strategy as in Table 4.1.

Parameter	Description	Value
x_0	initial inventory	50
δ_x	minimum trading volume (grid size for X_t)	1
δ_t	size of time step	0.001
δ_Ξ	grid size for Ξ_t	1
s	spread size	1
θ_1	amplitude of market impact	2
θ_2	exponent of market impact	1
$\bar{\lambda}_1$	amplitude of price recovery	1
$\bar{\lambda}_2$	exponent of price recovery	1

Table 4.1: Parameters related to execution strategy

The key to understanding the optimal behavior is that the tolerance for the market impact depends on the time remaining and inventory. We are able to capture the intuitive ideas that (i) the incentive for quick liquidation using the market order becomes higher over the course of time, especially near the terminal time; (ii) a lower inventory and a larger market impact induce a wait for price to recover; and (iii) if the limit order is available, it is better to quote the limit order as compared with waiting for the price to recover.

4.4.1 Strong recovery intensity without limit order

We first consider the case with strong price recovery intensity as defined in equation (4.2), and set $T = 10$, $\lambda^L = 0$ and $l_{\max} = 0$. Note that we are not allowed to use the limit order strategy under the current setting.

Figure 4.1 shows the optimal strategy, where each panel displays a snapshot at the time indicated on the top of the panel. A red dot indicates that waiting for the price to recover is optimal, while a blue triangle indicates that selling one share, the minimum trading volume, by a market order is optimal. Because we prohibit short selling, the maximum value of Ξ_t is $g(x_0 - X_t)$. Hence, the optimal strategy is calculated in the triangular regions shown in Figure 4.1. We observe that the region filled by red dots is dominant except in the last panel, which describes snapshot point close to the terminal time. We also note that the blue triangles appear in the region where Ξ_t takes a small value. These results imply that if we observe the large market impact and enough time remains, then it is optimal to take no action because the price is expected to recover.

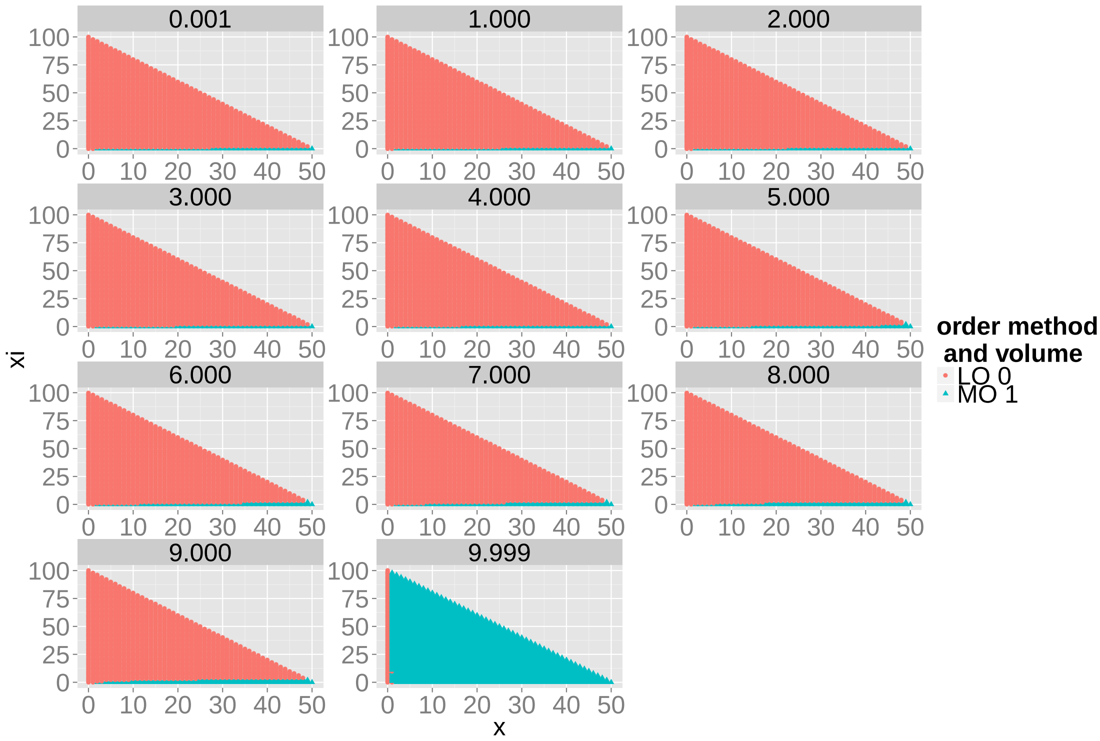

Figure 4.1: The optimal strategy under strong recover intensity without limit order when $T = 10$. Each panel displays a snapshot at the time indicated above the panel. A red dot indicates that waiting for the price to recover is optimal, while a blue triangle indicates that selling one share, the minimum trading volume, is optimal.

We next find that the region filled by blue triangles expands over the course of time. However, Figure 4.1 does not have enough resolution to discuss the time evolution of the optimal strategy. Therefore, we focus on the region where Ξ_t takes a lower value as shown in Figure 4.2. We find that the region filled by blue triangles expands from the bottom right corner to the top left corner over the course of time. We also observe that selling one share by a market order is optimal when $\Xi_t = 0$, that is, the market impact fully recovers.

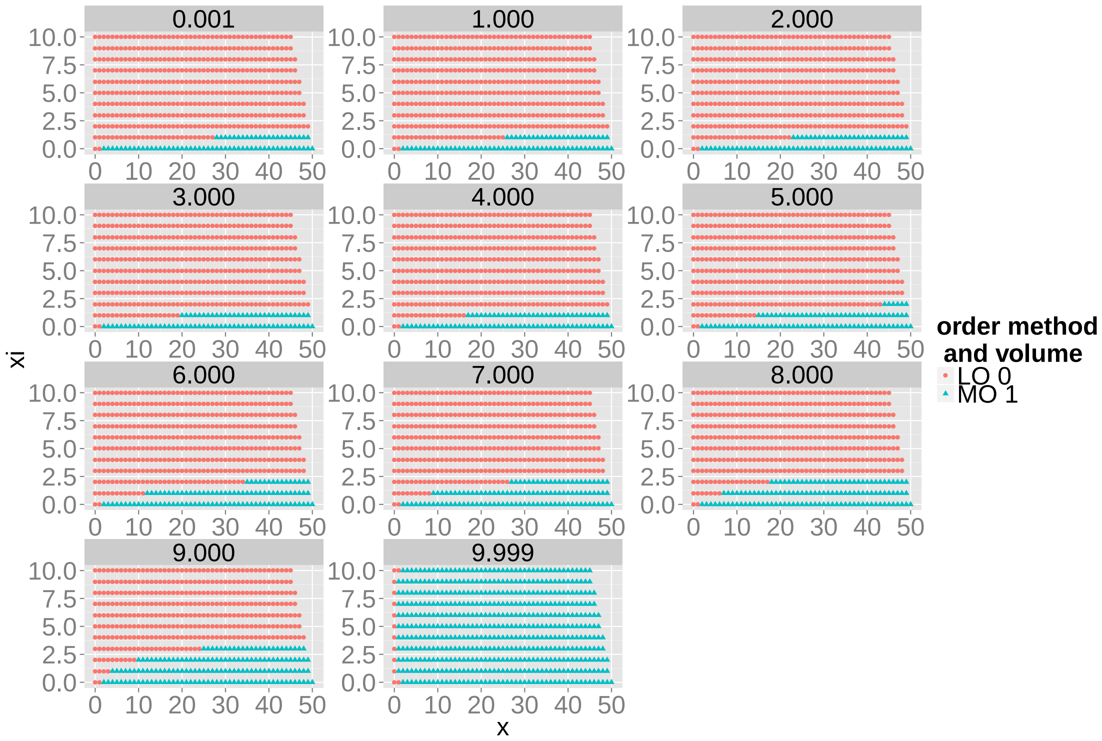

Figure 4.2: The optimal strategy, focusing on the region $\Xi_t \leq 10$ under strong recovery intensity without limit order when $T = 10$. A red dot indicates that waiting for the price to recover is optimal, while a blue triangle indicates that selling one share, the minimum trading volume, is optimal.

The key to understanding the dynamics described in Figure 4.2 is tolerance for the market impact. If the tolerance is high, the optimal strategy is to wait for the recovery of the market impact, whereas if the tolerance is low, it is optimal to liquidate by market orders. The region filled by red dots (respectively blue triangles) in Figure 4.2 is regarded as the high (resp. low) tolerance region. The manner of expanding the low tolerance region argues that tolerance for the market impact decreases with an increase in the inventory and over the course of time. This view is quite consistent with our intuition.

The interpretation using tolerance for the market impact is supported by Figure 4.3 and Figure 4.4. These figures display sample paths followed by the numerical simulation: the upper and lower panels describe the time evolution of the inventory X_t and the degree of market impact Ξ_t respectively. We observe that sell order execution by a market order with minimum trading volume occurs when Ξ_t takes a certain value. The value that triggers execution is $\Xi_t = 1$ before $t = 7$, after which different behaviors can be detected. From Figure 4.3 and Figure 4.4, we find that the triggers are $\Xi_t = 0$ and $\Xi_t = 2$ respectively We are able to explain this phenomenon by considering the remaining inventory. We observe that X_t in Figure 4.3 takes a lower value than in Figure 4.4 in the region after $t = 7$. Thus the tolerance increases in the sample path shown in Figure 4.3 and decreases in the path shown in Figure 4.4. Focusing on the lower panel in both figures, we find that Ξ_t decreases more frequently in Figure 4.3 than it does in 4.4. The natural behavior is to speed up execution when price recovery is fast, and so there are frequent arrivals of refill orders.

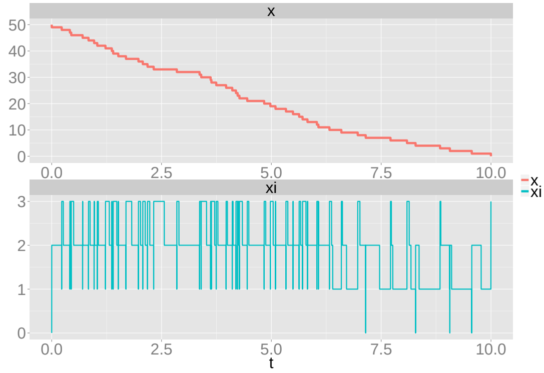

Figure 4.3: A sample path under strong recover intensity without limit orders when $T = 10$. This figure describes the fast price recovery case. The upper and lower panels display the time evolution of the inventory X_t and the degree of the market impact Ξ_t respectively.

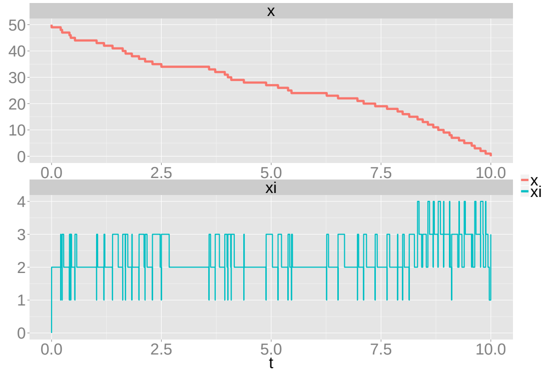

Figure 4.4: A sample path under strong recover intensity without limit orders when $T = 10$. This figure describes the slow price recovery case. The upper and lower panels display the time evolution of the inventory X_t and the degree of the market impact Ξ_t respectively.

From the point of view of tolerance for market impact, and the results when $T = 10$, we predict that (i) if we choose a larger value for T then the tolerance will increase; and (ii) as a result, the behavior tends to wait for full price recovery over the entire sample paths. A longer simulation gives results that support this. Figure 4.5 displays the optimal strategy when $T = 50$ in the same manner as in Figure 4.2. The high tolerance region is more dominant compared with that when $T = 10$. The sample paths vividly describe that the optimal strategy is to wait for full price recovery. We choose a sample path in the fast price recovery case, where price recovery is frequently found, and display it in Figure 4.6. A sample case of slow price recovery is displayed in Figure 4.7. Both imply that the optimal behavior coincides with our prediction: waiting for full price recovery is observed over the entire sample path.

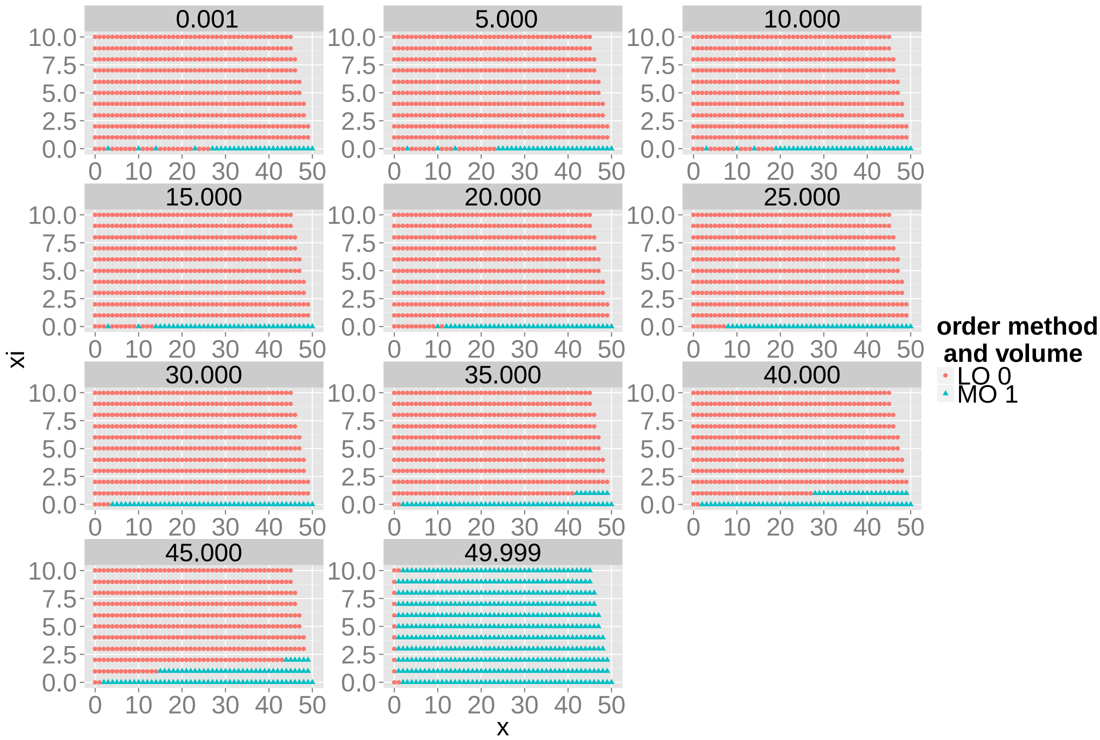

Figure 4.5: The optimal strategy focusing on the region $\Xi_t \le 10$ under strong recovery intensity without limit orders when $T = 50$. A red dot indicates that waiting for the price to recover is optimal, while a blue triangle indicates that selling one share, the minimum trading volume, is optimal.

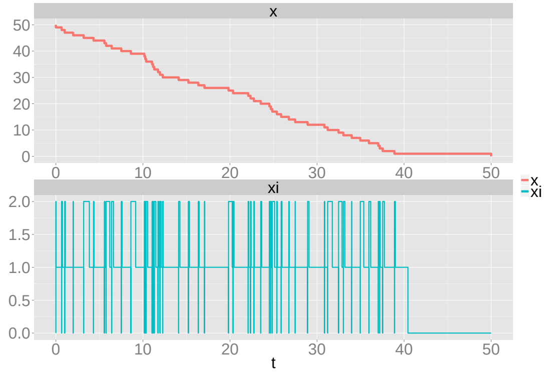

Figure 4.6: A sample path under strong recovery intensity without limit orders when $T = 50$. This figure describes the fast price recovery case. The upper and lower panels display the time evolution of the inventory X_t and the degree of the market impact Ξ_t respectively.

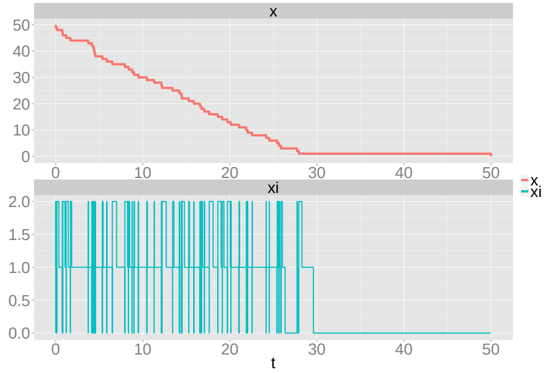

Figure 4.7: A sample path under strong recovery intensity without limit orders when $T = 50$. This figure describes the slow price recovery case. The upper and lower panels display the time evolution of the inventory X_t and the degree of the market impact Ξ_t, respectively.

4.4.2 Weak recover intensity without limit orders

In this subsection, we show the results using the weak recovery intensity defined in equation (4.3), again excluding limit orders.

Let us discuss the results for $T = 10$. Figure 4.8 describes the optimal strategy in the same manner as in Figure 4.1. It is obvious that the triangles representing point with low tolerance for market impact occupy a larger region than they do in Figure 4.2. This result indicates that tolerance for market impact is less than in the case studied in the previous subsection. The reason is clear: because the intensity in equation (4.3) implies that refill orders arrive less frequently compared with the intensity in equation (4.2), we do not expect the price to recover.

We also find that the time evolution of the tolerance follows the same pattern as in the previous subsection. Figure 4.9 shows a sample path for the current setting in the same manner as in Figure 4.3. We observe that the execution of a sell order occurs when Ξ_t takes a certain value, as we found previously. Decreased tolerance is also found in the sample path, as the value of Ξ_t that triggers the execution in Figure 4.9 is higher than the value in Figure 4.3.

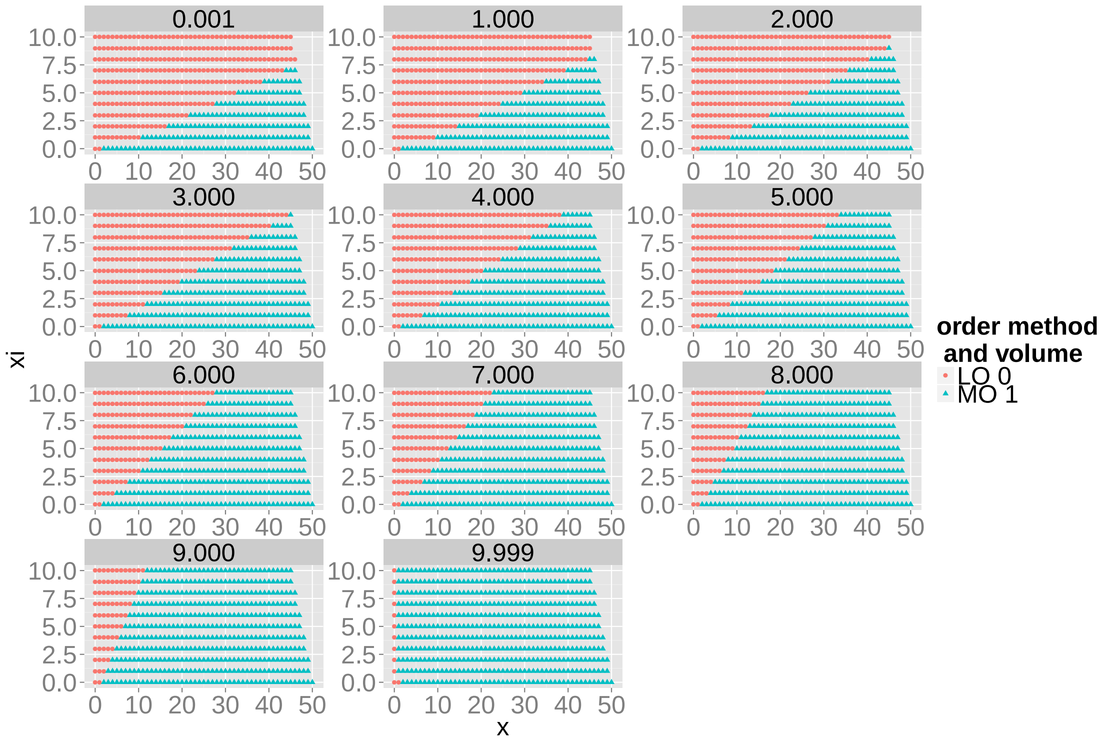

Figure 4.8: The optimal strategy focusing on the region $\Xi_t \leq 10$ under weak recovery intensity without limit orders when $T = 10$. A red dot indicates that waiting for the price to recover is optimal, while a blue triangle indicates that selling one share, the minimum trading volume, is optimal.

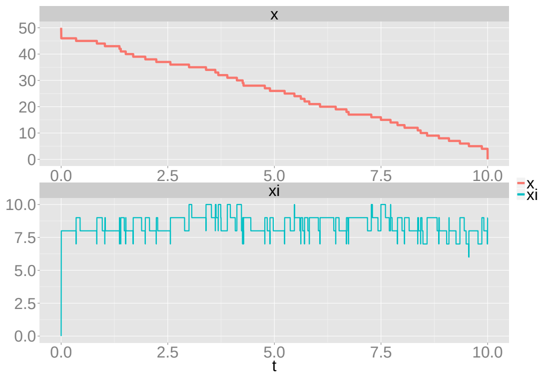

Figure 4.9: A sample path under weak recovery intensity without limit orders when $T = 10$. The upper and lower panels display the time evolution of the inventory X_t and the degree of the market impact Ξ_t, respectively.

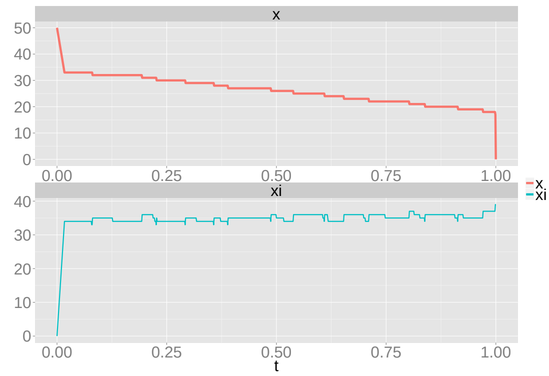

Figure 4.10: A sample path under weak recovery intensity without limit orders when $T = 1$. The upper and lower panels display the time evolution of the inventory X_t and the degree of the market impact Ξ_t, respectively.

The most remarkable finding in this subsection is observed in Figure 4.9, where we see large trades close to the initial and terminal times, unlike the previous case. The large trades are described with high resolution in Figure 4.10, which shows a sample path with the same parameters as in Figure 4.9 except that the terminal time is $T = 1$. These two large trades, however, each have a different structure.

Let us discuss the large trade near the initial time. We observe that it is constructed by a sequential market order with minimum trading volume. The sequential trades continue until Ξ_t hits the trigger value for execution in the short term. This is the result of a tradeoff between the following two desires: (i) reducing the inventory to a proper level, which we can describe using tolerance, as quick as possible; and (ii) avoiding an increase in the market impact Ξ_t as much as possible.

The large trade at the terminal time has different characteristics. Focusing on Figure 4.10, we see that (i) the slope of X_t at the terminal time is steeper than the slope near the initial time; and (ii) Ξ_t does not increase around the terminal time. These observations imply that the large trade at the terminal time is due to the terminal condition in equation (4.10). Because we are not concerned with what happens after the terminal time, this manner of terminal execution appears to be valid.

We summarize the features of our optimal strategy obtained from the results thus far. The strategy is made up of three components: (i) a large initial trade executed by sequential market orders with the minimum trade volume; (ii) small trades during the time period triggered by tolerance for the market impact; and (iii) a large terminal large trade due to the terminal condition. The large trades appearing around the initial and terminal times vanish if the execution period is long enough or if the price recovery is fast enough (see Section 4.4.1). The tolerance for market impact plays a central role in the interpretation of the optimal behavior.

Finally, we compare our optimal strategy with those investigated in previous research that models the price recovery as a deterministic process, such as Obizhaeva and Wang [OW13] . Both strategies consist of a large initial trade, small trades during the time period and a large terminal trade. The significant difference between them is found in the small trades. As mentioned above, the trigger for execution, which depends on the time remaining, the inventory and the market impact, is at the heart of our strategy. Because we model price recovery using stochastic processes as in equation (4.1), our optimal strategy behaves dynamically. The optimal strategies from previous studies are static in contrast. They describe price recovery using a deterministic function where we know the exact price recovery ex-ante and are then able to construct the execution strategy for the scheduled orders. Our model fits this view if we substitute Ξ_t into the expectation $\mathbb{E}[\Xi_t]$. Hence, the averaged version of our strategy corresponds to the previous strategy.

4.4.3 Enabling limit orders

The waiting times found in our strategy investigated in the previous subsections lead us to include limit orders in our strategy. Quoting the limit order appears to be efficient compared with simply waiting for the price to recover. The parts of our model related to the limit order are indeed motivated by this view. Recall that we include the limit order with small trade volume and a conservative evaluation of the execution price. The approach using the limit order with small trade volume to avoid the market impact is similar to the iceberg strategy used by market participants in the real world. [2] Hence we call our limit order strategy the quasi-iceberg strategy. In this subsection, we discuss how the limit order is included in the optimal strategy. Throughout this subsection we fix the parameters as follows: $\lambda^L = 0.1$, $l_{\max} = 3$ and $T = 30$.

Figure 4.11 describes the optimal strategy in a manner similar to Figure 4.2 although using different symbols and colors. We find that the dynamics of the market order part of the optimal strategy follows the same rule as it does without limit orders. We observe the new region occupied by the blue plus symbol which indicates that quoting a limit order with three shares is optimal. This behavior coincides with our expectations mentioned at the beginning of the present subsection. However, we still find that the region filled with red dots represents an optimal strategy of waiting for the price to recover. The execution price for the limit order is evaluated conservative and hence waiting is an optimal strategy when frequent price recovery is expected.

[2] For example, the NYSE offers the block reserve order which corresponds the iceberg strategy. Academic research about the iceberg strategy is found, for instance, in Esser and Mönch [EM07].

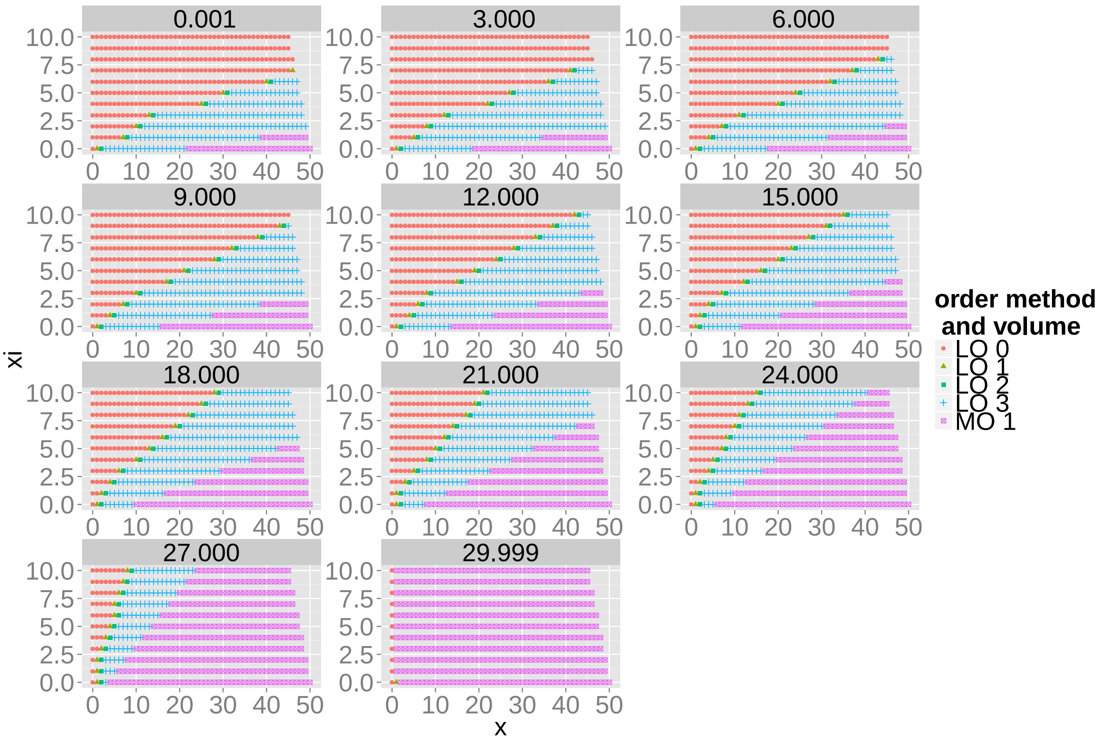

Figure 4.11: Optimal strategy with limit orders and intensity $\lambda^L = 0.1$ under weak recovery intensity when $T = 30$. The region shown is for $\Xi_t \leq 10$. Each panel displays a snapshot at the time indicated above the panel. A red dot indicates that waiting for the price to recover is optimal. A yellow triangle, a green square and a blue plus symbol indicate quoting a limit order with one, two or three shares is optimal, respectively. A purple square indicates that selling one share, the minimum traded volume, by a market order is optimal.

We note that the regions filled with yellow triangles and green squares only appear on the boundary between the regions occupied by red dots, which correspond to waiting for the price to recover, and by purple squares, which correspond to a quote of $l_{\max}$. Because there is no adverse effect due to the size of the limit order in our model, there is little incentive to choose trading volumes other than $l_{\max}$. From the technical perspective, the existence of these regions is interesting because it implies that the optimal strategy changes smoothly from waiting to quoting a limit order with $l_{\max}$ shares.

The tolerance for market impact continues to give a powerful interpretation in the current case. The optimal strategy has the following three phases. In the first phase, the tolerance is high and waiting is chosen as the optimal behavior. The tolerance then decreases and the optimal strategy changes to quoting a limit order with $l_{\max}$ shares. In the third phase, the tolerance hits the trigger discussed in the previous subsections, so immediately selling one share, the minimum trading volume, by a market order is optimal.

The strategy of waiting, however, does not appear in the investment simulation. Figure 4.12 displays a sample path under the current settings. In the lower right panel of the figure, which displays the optimal strategy over the time grid, we only observe immediately selling of one share by a market order and quoting a limit order with three shares. This observation implies that the market order part of the optimal strategy does not allow Ξ_t to exceed the trigger value, which changes the strategy to waiting. Figure 4.12 also suggests that the dynamics of the market order part of the optimal strategy is unchanged from the discussion above.

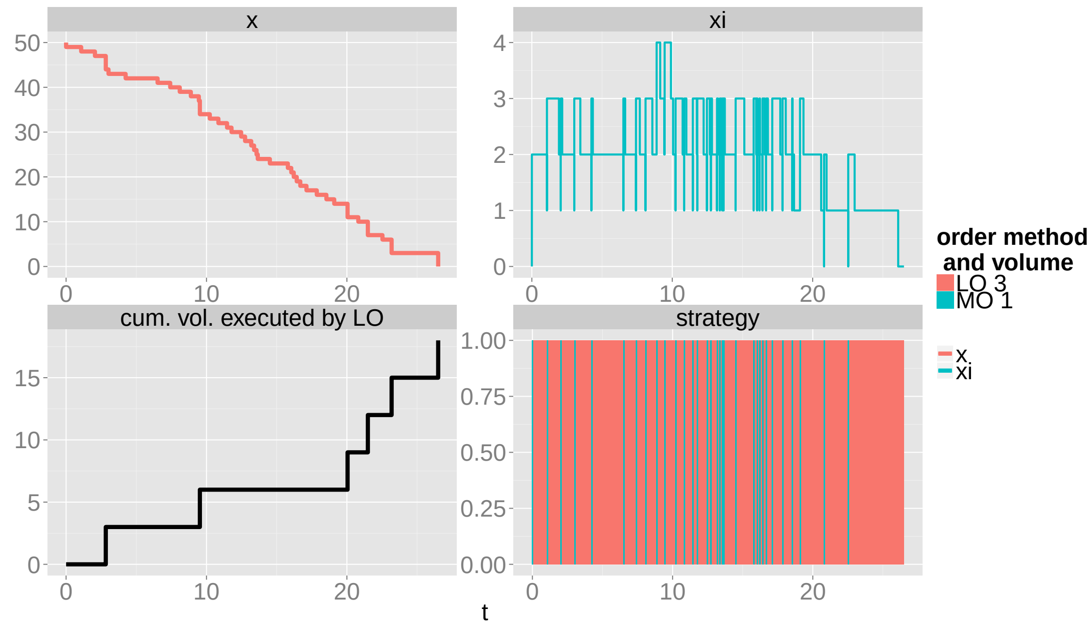

Figure 4.12: A sample path with limit orders and intensity $\lambda^L = 0.1$ under the weak recovery intensity when $T = 30$. The upper left and right panels display the time evolution of the inventory X_t and the degree of the market impact Ξ_t respectively. The lower left panel indicates the cumulative order volume executed by the limit orders. The lower right panel describes the optimal strategy: a blue bar indicates that immediately selling one share by a market order is optimal; a red bar indicates that quoting a limit order with three shares is optimal.

In this sample path, the quasi-iceberg strategy works well. The cumulative order volume executed by the limit orders is described in the lower panel of Figure 4.12. We observe that the limit order executions occur six times and that 18 shares are liquidated. This liquidation due to the limit orders leads to low tolerance for the market impact found near the terminal time in the upper right panel in Figure 4.12, which indicates the time evolution of Ξ_t. Another contribution of the quasi-iceberg strategy is demonstrated by the performance analysis discussed in the next subsection.

4.4.4 Performance analysis

In this subsection, we present a performance analysis of our optimal strategy. The results in the previous subsections suggest that we are able to reduce the market impact by increasing the terminal time T and taking a longer time span to liquidate the target asset. On the other hand, a larger value for T leads to large fluctuations from the initial price, as mentioned in the introduction. We quantify this tradeoff through a performance analysis using the liquidation rate R_T defined as follows:

$$R_T = \frac{Y_T - (P_T - \Xi_T - g(X_T))X_T}{x_0 p_0}.$$

The numerator of the above equation is the terminal wealth including the terminal execution and the denominator is the wealth obtained under perfect liquidity. Therefore, the liquidation rate is defined as the ratio between the total wealth with and without market impact. We run the simulation under weak recovery intensity and evaluate the expectation and the standard deviation of the liquidation rate, $\mathbb{E}[R_T]$ and $\mathrm{SD}(R_T)$, for each of the terminal times given in Table 4.2. We use geometric Brownian motion as the price process, that is, $\sigma(t, p) = \sigma p$, and produce 10^5 sample paths using the parameters in Table 4.2 as well as the parameters in Table 4.1.

Parameter	Description	Value
T	terminal time	1, 3, 5, 10, 20, 30, 50
σ	volatility	0.08
p_0	initial price	150
δ_p	grid size for P_t	1
–	number of sample paths	10^5

Table 4.2: Parameters related to numerical simulation

Let us first discuss the case that excludes the limit order. Figure 4.13 plots $\mathbb{E}[R_T]$ and $\mathrm{SD}(R_T)$ on the vertical and horizontal axes respectively. The points are colored with a lighter blue as T increases and are interpolated linearly. We observe that there is a monotonic increase for both $\mathbb{E}[R_T]$ and $\mathrm{SD}(R_T)$. The former and one of the features of our strategy imply that the improvement in performance relies on reducing the market impact. The latter is obviously caused by price fluctuation because of the large terminal time. Hence, using $\mathbb{E}[R_T]$ and $\mathrm{SD}(R_T)$ to quantify the tradeoff between avoidance of market impact and price fluctuation is somewhat primitive. The implication is obtained from the shape of the curve in Figure 4.13. Because the curve is concave, the amount of change in $\mathrm{SD}(R_T)$ required to improve $\mathbb{E}[R_T]$ a certain amount becomes larger as $\mathbb{E}[R_T]$ increases. This implies that, over a long time span, liquidating the asset is not profitable.

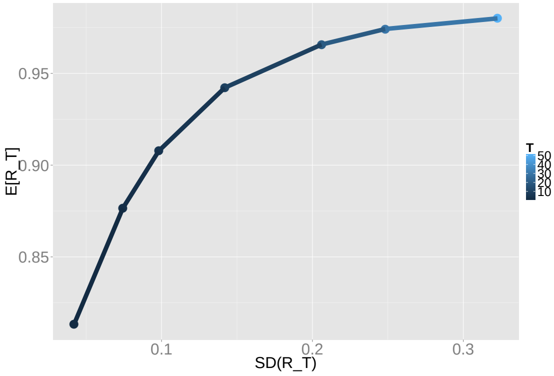

Figure 4.13: Performance graph when limit orders are prohibited. The expectation $\mathbb{E}[R_T]$ and the standard deviation $\mathrm{SD}(R_T)$ are plotted on the vertical and horizontal axes respectively. The results are plotted using points colored with a lighter blue as the terminal time T increases, and the points are interpolated linearly.

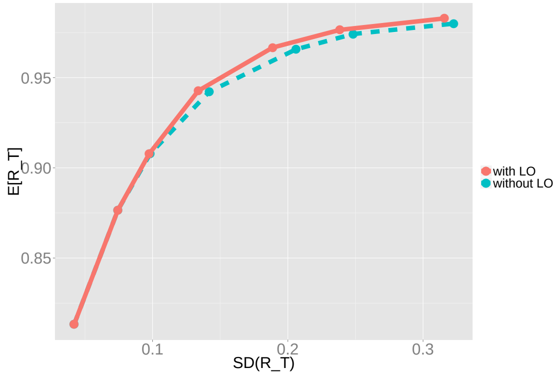

Figure 4.14: Performance graph when limit orders are permitted. The expectation $\mathbb{E}[R_T]$ and the standard deviation $\mathrm{SD}(R_T)$ are plotted on the vertical and horizontal axes respectively. The dashed blue line and the solid red line describe the performance with and without the limit order, respectively. The blue line is the same as that displayed in Figure 4.13.

We next consider the case where limit orders are permitted. Recall that the limit order in our optimal strategy replaces the strategy of waiting for the price to recover found in the optimal strategy without the limit order. Thus we expect an improvement in performance due to the limit order. Figure 4.14 plots $\mathbb{E}[R_T]$ and $\mathrm{SD}(R_T)$ in the same manner as in Figure 4.13 and supports this expectation. The dashed blue line and the solid red line in Figure 4.14 represent the results with and without the limit order respectively. We note that the former line is the exactly same as in Figure 4.13. We observe that the solid red line appears on the upper left side of the dashed blue line. This indicates that enabling limit orders leads to an improvement both in the risk and the return. We find that the improvement stands out when $T \geq 10$. The limit order is executed when the counterpart order arrives, and the number of limit order executions declines if the terminal time is a small value. Hence, the improvement due to the limit order is relatively small unless the terminal time is large enough. We also find that main contribution to the improvement comes from the decrease in $\mathrm{SD}(R_T)$. We impose a conservative evaluation of the execution value of the limit order, which appears to restrict the improvement in $\mathbb{E}[R_T]$. Therefore we expect further improvements in $\mathbb{E}[R_T]$ under a more sophisticated evaluation of the limit order execution.

4.5 Discussion

In this section, we discuss the relation between our approach and the practical trading.

We first introduce famous execution strategies which are used in real markets. Johnson [Joh10] categorizes algorithmic execution strategies into three types: impact-driven, cost-driven, and opportunistic strategies. Impact-driven strategies aim to minimize the market impact. The VWAP (volume weighted average price) strategy is one of the most famous strategies which are categorized in the impact-driven strategies. Cost-driven strategies stand the viewpoint of the reduction of the overall trading cost. A typical strategy categorized in the cost-driven strategies is the implementation shortfall strategy. Opportunistic strategies reflect on the market condition. The price inline strategy is a typical opportunistic strategy. We mention that we can categorize the algorithmic execution strategies in different fashions since there are vast strategies.

Since our optimal strategy reflects the degree of the market impact dynamically, our strategy is categorized in the opportunistic strategies mentioned above. Furthermore, the limit order in our strategy is regarded as the iceberg strategy. The iceberg strategy is the stream of small limit orders which are not posted until the previous order is executed. The purpose of the iceberg strategy is to avoid that other market participants detect our large trade position. We remark that the iceberg strategy is offered as an order method by some market maker, e.g., New York Stock Exchange [Exc] [3].

We next mention how to implement our strategy to real deals. In the practical case, the initial inventory x_0 and the execution span T are given. Recall that the key point of our strategy is the tolerance of the market impact and it depends on the intensity of the resilience and the remaining time. Since the execution span T is given, that is, remaining times are also given in each time steps, we can determine the optimal strategy if the resilience intensity is estimated. Although there is no dominant method to estimate the resilience, candidates are investigated in the empirical studies. We refer Aldridge [Ald13] and Large [Lar07] for the candidates of estimation methods. Using the estimated intensities, we calculate the execution strategy before the execution and post orders according to the observed market impact.

4.6 Concluding remarks

In this chapter, we study the optimal execution problem under stochastic price recovery based on LOB dynamics. We model the price recovery after execution of a large order by accelerating the arrival of the refilling order. The arrival of the refilling order is defined as a Cox process whose intensity is increased by the degree of the market impact. We include not only the market order but also the limit order in our execution strategy albeit in a restricted fashion. Our restricted limit order strategy is regarded as a quasi-iceberg strategy. We formulate our execution problem as a combined stochastic control problem over a finite time horizon. The corresponding HJBQVI is solved numerically using a

[3] They call the iceberg strategy as a reserve order.

scheme similar to that proposed by Ieda [Ied15b]. The performance criterion is used to maximize the terminal wealth including the terminal execution.

The optimal execution strategy without the limit order is made up of three components: (i) a large initial trade executed by sequential market orders with the minimum trading volume; (ii) unscheduled small trades during the time period; and (iii) a large terminal trade due to the terminal condition. The heart of our strategy is the tolerance for the market impact, which depends on the time remaining and inventory: if the tolerance is high, then the optimal strategy is to wait for the recovery of the market impact, while if the tolerance is low, then it is optimal to liquidate by market orders. The timing of the small trades, the second component of the strategy, is triggered by the tolerance. Since the tolerance reflects the state at the time, our optimal strategy is dynamic. The tolerance also governs the size of the large trades appearing around the initial and terminal times. These trades vanish if the execution period is long enough or if the price recovery is fast enough, which leads to a high tolerance situation.

The intervals between the small trades appearing in the above strategy motivate including limit orders in the strategy. Enabling limit orders under the restriction that a limit order does not affect the price dynamics, the strategy of waiting for the price to recover observed when limit orders are excluded is replaced by quoting the limit order. Although we impose a conservative evaluation on the limit order execution, our performance analysis based on the liquidation rate, the ratio between the total wealth with and without the market impact, demonstrates that the improvement is caused by the inclusion of the limit order.

We finally compare our result and previous studies from the viewpoint of practical applications. Bertsimas and Lo [BL98] is one of the most accepted results by practitioners, since the implication of their study, splitting equally a large order, is simple, intuitive and easy to implement. However, the real market is much more complex than the simplified market assumed in Bertsimas and Lo [BL98]. For example, because of the resilience of LOBs which is not considered in their work, practitioners empirically know that executing the sliced orders in the fixed intervals is not a good way. The timings executing orders seems to rely on the experience of traders. Hence, for their accountability, traders require academic studies which suggest when they should execute the order quantitatively. Obizhaeva and Wang [OW13] is an answer for the above issue. Their deterministic resilience captures the averaged behavior of the market. Our study dealing with the dynamic resilience is a more advanced answer. Since the market fluctuates volatile, our dynamic model seems to be preferable compared to the deterministic model in the sense that the dynamic behavior fits the experience of traders.

Conclusion

In this dissertation, we proposed a new numerical method to solve finite time horizon Hamilton-Jacobi-Bellman quasi-variational inequalities (HJBQVIs) and studied the optimal execution problem under stochastic price recovery using our novel method.

Chapter 1 served an introduction of stochastic and impulse controls. The main topics of this chapter were: (i) the derivation of Hamilton-Jacobi-Bellman (HJB) equations and HJBQVIs employing the dynamic programming principle and (ii) the verification theorem under smoothness assumption for value functions. We also briefly mentioned the notion of viscosity solutions as a preparation for subsequent chapters.

We constructed numerical schemes for HJB equations and HJBQVIs using the finite difference method in Chapter 2. To prove the convergence of numerical schemes, we employed the powerful framework proposed by Barles and Souganidis [BS91]: if a numerical scheme is monotone, stable and consistent, the solution of the scheme converges to the solution of the original partial differential equation in the viscosity sense.

As illustrative examples that how to apply the Barles-Souganidis framework, we dealt with both explicit and implicit methods for the one dimensional HJB equation. Both are regarded as well-known results, however, to the best of our knowledge, there is no well-known study in the literature that describes in detail the convergence proof of the implicit method.

The convergence proof of the numerical scheme for the multi-dimensional finite time horizon HJBQVI was the heart of this chapter. The most remarkable point of our method was that the discretization parameters of the time step and the spatial grids are independent. Hence, our method was regarded as a fully implicit method.

In Chapter 3, we proposed an algorithm to solve the finite time horizon HJBQVI. We focused on how to solve the discretized HJBQVI while we discussed its convergence in Chapter 2. The discretized HJBQVI was converted into an equivalent fixed point problem and was solved by the policy iteration. We provided a detailed procedure for the algorithm in matrix form for the sake of easy implementation.

We applied our method to the finite time horizon optimal forest harvesting problem, which considers exiting from the forestry business at a finite time. The original optimal forest harvesting problem is defined in the infinite time horizon and has an analytical solution. Because analytical solutions were not available in the finite time horizon, we sought the optimal strategy numerically by our method. The obtained optimal strategy coincided with our intuition: the strategy coincides with the optimal strategy in the infinite time horizon case when the terminal time goes to infinity; the strategy switch point, the threshold of the biomass to harvest the trees, is much higher than the infinite horizon case near the terminal time.

An application of the combined stochastic control for mathematical finance was discussed in Chapter 4. We investigate the optimal execution problem, which seeks the optimal way to liquidate or acquire a large asset position. It is a trade-off between fast trading facing market impact risk and slow trading being exposed to the risk that the price of the target asset will change over the course of time.

We proposed an execution mode with a price recovery based on the limit order book dynamics. We model price recovery after execution of a large order by accelerating the arrival of the refilling order, which is defined as a Cox process whose intensity increases by the degree of the market impact. We include not only the market order but also the limit order in our strategy in a restricted fashion. We formulate the problem as a combined stochastic control problem over a finite time horizon. The corresponding HJBQVI was solved numerically by the algorithm proposed in the previous chapter.

The optimal execution strategy without the limit order is made up of three components: (i) a large initial trade executed by sequential market orders with the minimum trading volume; (ii) unscheduled small trades during the time period, and (iii) a large terminal trade due to the terminal condition. The heart of our strategy is the tolerance for the market impact, which depends on the time remaining and inventory: if the tolerance is high, then the optimal strategy is to wait for the recovery from the market impact. If the tolerance is low, then it is optimal to liquidate by market orders. The timing of the small trades, the second component of the strategy, is triggered by the tolerance. Because the tolerance reflects the state at the time, our optimal strategy is dynamic. The tolerance also governs the size of the large trades appearing around the initial and terminal times. These trades vanish if the execution period is long enough or if the price recovery is fast enough, which leads to a high tolerance situation.

The intervals between the small trades appearing in the above strategy motivated the inclusion of limit orders in the strategy. With limit orders enabled under the restriction that a limit order does not affect the price dynamics, the strategy of waiting for the price to recover observed when limit orders are excluded is replaced by quoting the limit order. Although we impose a conservative evaluation on the limit order execution, our performance analysis based on the liquidation rate, the ratio between the total wealth with and without the market impact, demonstrates that the improvement is caused by the inclusion of the limit order.

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