論文 / 著書情報
Article / Book Information

題目(和文)	
Title(English)	Renormalization group equation for volume dependence of the O(N)sigma model
著者(和文)	CALLESergio
Author(English)	Sergio Calle
出典(和文)	学位:博士(理学), 学位授与機関:東京工業大学, 報告番号:甲第10712号, 授与年月日:2018年3月26日, 学位の種別:課程博士, 審査員:岡 眞,伊藤 克司,今村 洋介,山口 昌英,柴田 利明
Citation(English)	Degree:Doctor (Science), Conferring organization: Tokyo Institute of Technology, Report number:甲第10712号, Conferred date:2018/3/26, Degree Type:Course doctor, Examiner:,,,,
学位種別(和文)	博士論文
Type(English)	Doctoral Thesis

Renormalization group equation for volume dependence of the O(N) sigma model

Sergio Calle

Tokyo Institute of Technology

A thesis submitted for the degree of

Doctor of Science

Tokyo 2018

This thesis is dedicated
to my family and friends for
their support during years.

Acknowledgements

Firstly, I would like to express my sincere gratitude to my advisor Prof. Oka Makoto for the continuous support of my Ph.D study and related research, for his patience, motivation, and immense knowledge. His guidance helped me in all the time of research and writing of this thesis. I could not have imagined having a better advisor and mentor for my Ph.D study.

Besides my advisor, I would like to thank the rest of my thesis committee: Prof. Katsushi Ito, Prof. Yosuke Imamura, Prof Mashide Yamaguchi and Dr. Toshi-Aki Shibata, for their insightful comments and encouragement, but also for the hard question which stimule me to widen my research from various perspectives.

At the same time I am so grateful to Dr. Sasaki Kiyoshi who provided me an opportunity to have a grateful discussions and incredible guide for my research, it is difficult to express all my gratitude in just one sentence. Without his precious support it would not be possible to conduct this research.

Important mention too for the Kyoto UniversityA part of the numerical calculations was carried out on the super parallel computers, CRAY XC40 at YITP in Kyoto University.

I thank my fellow lab mates in Tokyo Institute of Technology, for the stimulating discussions, and interesting presentations during these years, and for all the fun we have had in the last four years. Also I thank my friends with who I shared a several moments during these years.

Important mention for the Kyoto University, where the numerical simulations were made the numerical calculations was carried out on the super parallel computers, CRAY XC40 at YITP.

At the same time, I am in debt with the Japanese Government Scholarship Students program (MEXT), for the economic support. Consequently to the Japanese society not only for to the economic support since they gave me the opportunity to live here in Japan and learn a lot of interesting tradition and culture.

Abstract

In this thesis, the O(N) sigma model is studied using the renormalization group, RG, equation, where the finite volume scheme is used. This RG equation is apply to expectation value of some operator to be able to get information of the operator.

Before entering the main issue, we summarize the basic properties on the 2-dimensional O(N) sigma model by historical order. In the middle '70s, the renormalizability, and the asymptotic freedom have been established. The asymptotic freedom is important to guarantee the applicability of the perturbative expansion in the high-energy region. The infrared (IR) divergence is regularized with a magnetic field in these studies. A vanishment of the magnetic field activates the IR divergence, again. However, the renormalization in minimal subtraction (MS) scheme requires only the cancellation of ultraviolet (UV) divergence. The IR divergence originates from the low dimensionality of the system.As Mermin-Wagner's theorem forbids the spontaneously symmetry breaking in 2 dimension, it is believed that the IR singularity generates a mass gap in the system, although there is no proof of this mechanism of the mass generation.

On the other hand, in the late '80s and the early '90s, some method were developed to extract information at infinite volume by positively using a finiteness of the box where the system is put. One of the extracted information is the scattering phase shift and another is the running coupling constant. The latter prescribes evolution of the renormalized coupling as a response to the change of the box size. The success of this method gives us a foresight of the existence of RG equation which describes the box-size dependence of the n-point Green function.

The goal of this thesis is to study and establish such a RG equation. For this purpose, as an example, we use the 2-dimensional O(N) sigma model. We define the RG equation where the box-size is considered as a renormalization variable. This RG equation in O(N) is based on study by Lüscher where a running coupling constant was defined. However, the wave function renormalization has not been discussed in the context where the system has been put in a finite box. Thus we present a calculation of the wave function renormalization in non-perturbative and perturbative calculations. We calculate the β and γ functions for the O(N),derive RG equation for two different operators, and subtract information on the composition of those operators.

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Chapter 1

O(N) Sigma model

In this chapter, we explain in detail the common knowledge for the sigma model. This knowledge is not lacking of controversy although the majority of the community accept the typical knowledge of O(N), which is predicted by perturbation theory. Because of the simple definition of this model and some important behaviours, the bibliography about this model is vast.
The main characteristics to explain are still quite unsettled, such us the mass gap, the asymptotic freedom, instanton contribution and so on. So this model is still an important model to study. This theory is defined by a field with N-components $S(x) = (S_1, S_2, ..., S_N)$, which satisfies the following condition,

$$S_1^2 + S_2^2 + ... + S_N^2 = 1. \tag{1.1}$$

This condition means that this field lives in the (N-1)-sphere, S^{N-1}. Its Lagrangian is given by:

$$\mathcal{L}_0 = \frac{1}{2g_0^2}(\partial_\mu S_i(x)\partial^\mu S_i(x)) \tag{1.2}$$

At first glance from the Lagrangian, one can think that this theory does not have any interaction term. But due to the condition (1.1), the Lagrangian is non-linear and contains non-trivial interaction.

1.1 Perturbative study on O(N)

First, we explore the common knowledge of this model which is derived from the perturbative analyses. Studies of perturbative sigma model were made by Brezin, Zinn-Justin and collaborators [58, 60, 7, 40]. The first step is to observe that this theory is renormalizable and it can be renormalized using two parameters, the coupling constant and the field strength.

1.1.1 Definition of O(N) sigma model perturbative calculation

In perturbative calculations we need to eliminate the condition (1.1). To be able to avoid this condition is an important step. We should understand it because when we have to parametrized this (N-1)-sphere, we can use different parametrization although in this section we will follow the natural parametrization

$$S(x) = (\pi^1(x), ..., \pi^{N-1}(x), \sigma(x)) \tag{1.3}$$

In this situation, it is clear that we break the symmetry because we take the vacuum at $S(x) = (0, ..., 0, 1)$, where the O(N) symmetry is reduced to O(N-1) symmetry. Although the O(N) symmetry is broken we have to mention that the coset of O(N-1) will give us a relation to infinitesimal transform,

$$\delta\pi^i = \omega_i(1 - \pi^i\pi^i)^{1/2}, \tag{1.4}$$

where ω_i is a constant parameter for the infinitesimal variation. This relation gives us the transformation of σ

$$\delta\sigma(x) \equiv \delta(1 - \pi^i(x)\pi^i(x))^{1/2} = -\omega^i\pi^i(x) \tag{1.5}$$

At the end, with this parametrization we obtain the Lagrangian

$$\mathcal{L}_0 = \frac{1}{2}G_{ij}(\pi)(\partial_\mu\pi_i(x)\partial_\mu\pi^j(x)) \tag{1.6}$$

$$G_{ij}(\pi) = \delta_{ij} + \frac{\pi_i\pi_j}{1 - \pi^k\pi^k} \tag{1.7}$$

1.1.2 Renormalization

As it is mentioned before this theory is renormalizable and two counter terms is enough to renormalize the theory. The first characteristic is quite easy to prove using power counting.

1.1.2.1 Power counting

From the free field Lagrangian, we can get the dimension of the field:

$$[\pi] = \frac{1}{2}(d-2) \tag{1.8}$$

From the Lagrangian, eq. (1.6), one sees all interaction terms in the Lagrangian satisfy:

$$[\partial_x^2 \pi^{2n}] = n(d-2) + 2 \tag{1.9}$$

Then as a consequence,
-for $d < 2$ the theory is super-renormalizable.
-for $d = 2$ it is just renormalizable.
-for $d > 2$ the theory is not renormalizable.
Here, we focus on dimension two or lower. It is important to mention that O(N) sigma model with dimension larger than two can be used as a quantum field theory, although we need new operators to regularize the theory, as was shown by Niedermayer, Hasenfratz and collaborators [67, 69, 11, 70, 71]. We find that this theory is renormalizable in dimension two. In the following we use the Ward-Takahashi indentity to get relations between renormalization constants.

1.1.2.2 Ward-Takahashi relation

To be able to obtain the Ward-Takahashi(WT) identity, we need to remember the relation of eq. (1.4), where the symmetry relation is kept. Using the WT relation plus the symmetry relation, we obtain

$$\int d^2x[\ J_i(x)\frac{\delta Z}{\delta H(x)} - H(x)\frac{\delta Z}{\delta J_i(x)}]\ = 0 \tag{1.10}$$

where $J_i(x)$ is the source of the field, $\pi^i(x)$, and $H(x)$ is a magnetization field.

$$\mathcal{S} = \int d^2x \Big(\mathcal{L} - J_i(x)\pi^i(x) - H(x)\sigma(x) \Big) \tag{1.11}$$

And the $Z(\vec{J}, H)$ is defined by:

$$Z(\vec{J}, H) = \int d\pi e^{-(\mathcal{S} - \int d^2x J_i(x)\pi^i(x))} \tag{1.12}$$

Using the Legendre transformation, with $W(J_i, H) = \log Z(J_i, H)$

$$\Gamma(\pi^i, H) = \int d^2x \pi^i(x) J_i(x) - W(J_i, H) \tag{1.13}$$

The we can translate the WT relation to

$$\int d^2x [\frac{\delta\Gamma}{\delta\pi_i(x)} \frac{\delta\Gamma}{\delta H(x)} - H(x)\pi_i(x)] \ = 0 \tag{1.14}$$

This functional can be expanded by perturbation in g^2, $\Gamma = \sum_{n=0}^{\infty} \Gamma^{(n)} g^{2n}$. Thanks to this expansion we can distinguish the divergence part from the non-divergence part. Γ^0 is the tree level contribution and non-divergent. Then the following relation will show up for the divergence part,

$$\int d^2x [\ \frac{\delta\Gamma^0}{\delta\pi_i(x)} \frac{\delta\Gamma^{(n)div}}{\delta H(x)} - \frac{\delta\Gamma^0}{\delta H(x)} \frac{\delta\Gamma^{(n)div}}{\delta\pi_i(x)}] \ = 0 \tag{1.15}$$

Due to this relation, we can solve the equation for any kind of operator of dimension 2 by inserting the following term to the bare Lagrangian to renormalize the theory,

$$\Gamma^{(n)div} = \int d^2x [B\pi + H(x)C\pi] \tag{1.16}$$

Then the divergent part in eq. (1.15) is going to be the Lagrangian itself, therefore for the renormalization we just need the redefinition of the constant term. For detail discussions we refer to the paper by Brezin [58].

1.1.3 Regularization

It is widely known that the quantum field theory needs to be regularized because of the divergence in the Feynmann diagrams. The divergences coming from the integration in the momemtum space are called ultraviolet divergences, UV, because they are coming from the high momemtum modes. Several different regularization method were invented but here we will only describe the main regularization features.
The O(N) sigma model is a massless theory in the bare Lagrangian and will generate the other kind of divergence than the UV divergence. Because the new divergence comes from low momentum modes, it is called infrared divergence, IR. This divergence is decoupled form the UV divergences, we treat each one separately.

1.1.3.1 UV divergence

This issue was well studied in the bibliography. The mostly used regularizations are the lattice regularization and the dimensional regularization. In our calculations in the later chapters, we show results of the dimensional regularization. Strictly speaking, we use the Z-function regularization, which in essence is the same as dimensional regularization. The difference is that the coordinate space is finite instead infinite.

Lattice regularization: First we need to go back to the definition of the Lagrangian and the parametrization made for the (N-1)-sphere. We need to discretize the derivative in lattice and different kinds of lattice definition can be made. A constraint is to get the same result in the continuum limit. As an example we mention the work made by Falcioni [59], where different definition was used.

$$\frac{1}{2g_0^2}(\partial_\mu S_i(x)\partial^\mu S_i(x)) \sim \frac{1}{2}[\ K_1 \sum_{n,\vec{\mu}} S_i(n)S_i(n+\vec{\mu}) +$$

$$K_2 \sum_n (S_i(n)S_i(n+\vec{x}+\vec{y}) + S_i(n)S_i(n+\vec{x}-\vec{y})) + K_3 \sum_{n,\vec{mu}} S_i(n)S_i(n+2\vec{mu})] \quad (1.17)$$

This is rewritten as:

$$\frac{1}{2}[\ \alpha_1 \sum_{n,\vec{\mu}} [\nabla_{\vec{\mu}} S_i(n)]^2 + \alpha_2 \sum_{n,\vec{\mu},\vec{\nu}} [\nabla_{\vec{\mu}} \nabla_{\vec{\nu}} S_i(n)]^2 + \alpha_2 \sum_{n,\vec{\mu}} [\nabla^2_{\vec{\mu}} S_i(n)]^2] \quad (1.18)$$

where

$$\nabla_{\vec{\mu}} F(n) = [F(n+\vec{mu}) - F(n)]/a$$

$$\alpha_1 = \frac{1}{2}K_1 + K_2 + 2K_3 \quad \alpha_2 = -\frac{1}{2}K_2 \quad \alpha_3 = \frac{1}{4}K_2 - \frac{1}{2}K_3 \tag{1.19}$$

The condition here is $\alpha_1 = 1$, because in the continuum limit the other terms are irrelevant. Another issue is the measure in the field $\int dS^{(N)}(x)$, because a term appears in the Lagrangian

$$\frac{1}{2}\sum_n log(1 - \pi^i(n)\pi^i(n)) \tag{1.20}$$

which comes from the parametrization of $S^N = (1 - \pi^i(n)\pi^i(n))^{1/2}$.

Dimensional regularization: For this regularization the problem of the derivative definition will not appear, so the unique issue to concern is the measure term. In this approach we start from the lattice definition and get the continuum limit,

$$\frac{1}{2}\sum_n \log(1 - \pi^i(n)\pi^i(n)) \to \frac{1}{2}\int d^2x\delta(0)\log(1 - \pi^i(n)\pi^i(n)). \tag{1.21}$$

So now the problem is how $\delta(0)$ is defined. When we use the dimensional regularization, as is shown in Appendix A.1, the value of $\delta(0)$ is zero. This calculation is made becuase in the usual dimensional regularization, infinite volume, the function to regularize is the Gamma function but in our case the function we have to regularize is the Riemann's Zeta function. So adding $\delta(0) = 0$ the Lagrangian is not modified by the measure term.

1.1.3.2 IR divergence

The IR divergence is coming from the propagator, $(p^2)^{-1}$ when the momentum is equal to zero.

Magnetic field: One way to avoid this IR divergence is straightforward. As it was already mentioned we insert a magnetic field coupled to the $\sigma(x)$ component,

$$\frac{1}{2}\int d^2xH\sigma(x) = \frac{1}{2}\int d^2xH(1 - \pi^i(x)\pi^i(x))^{1/2}. \tag{1.22}$$

This regularization using the magnetic field was already shown for the Ward-Takahashi relation in the previous section.

Finite volume: The main idea is to try to avoid the zero mode momentum in the internal momenta of the propagator. Using the finite volume method to avoid the IR divergence, there are several options. One, although it is not going to be used in our calculation, is to play with the boundary condition, for example by Floratos's calculation [50]. In his calculation, the boundary condition will break the O(N) symmetry and the divergence can be avoided because this boundary generates an expectation value which generates an effective mass gap. The following boundary is the one used by Floratos,

$$\vec{S}(0,t) = \vec{S}(L,t), \quad \forall t \tag{1.23}$$

$$\vec{S}(x,0) = \vec{\eta}, \quad \forall x \tag{1.24}$$

$$\vec{S}(y,0) = \vec{\xi}, \quad \forall y \tag{1.25}$$

with $\vec{\eta} = (1, 0, ..., 0)$ and $\vec{\xi} = (cos\theta_0, sin\theta_0, ..., 0)$. Then the expectation value will not be anymore the constant field,

$$\vec{S}_{cla} = \frac{1}{2g_0^2}\theta_0^2\frac{L}{T}, \tag{1.26}$$

but the Lagrangian is modified as:

$$S = S_{cl} + \frac{1}{2}\int d^2x(\partial_\mu\pi^i(x)\partial_\mu\pi^i(x) + \frac{\theta_0^2}{T^2}\pi^i(x)\pi^i(x)) + \frac{1}{2}\int d^2x\mathcal{L}_I. \tag{1.27}$$

Because here we are just considering the IR divergence, we will not describe the interaction Lagrangian. It is observed that this boundary will generate an effective mass. Another option is to consider a boundary condition which eliminates the zero mode. Two boundary conditions are possible Dirichlet boundary condition or anti-periodic boundary condition. In these cases the expectation value is not modified, but the zero mode momentum is avoided in the internal integration. The calculation is shown the later chapters. In [41, 26, 69, 70], the boundary condition is the Newmann boundary condition for perturbative calculation. This boundary condition is called free boundary condition in lattice calculation. We calculate our system in this boundary condition to be sure that we only describe the one excited state. Using this boundary, the zero momentum is not forbidden. Thus we need to avoid the zero modes. The main tool needed is an operator with O(N) symmetry that regularizes the IR divergence. This fact was shown for O(N) sigma model by David [23]. The next step is to integrate all the collective modes when the momentum is zero. The first paper

related with this was made by Hasenfratz [40] and later extended in [12]. The idea is to use the next relation,

$$1 = \int dm^i \prod_{i=1}^{N} \delta(m^i - \frac{1}{TL}\int d^2x S^i(x)). \tag{1.28}$$

Inserting the previous equation and assuming that the operator to study is O(N) symmetric we can integrate out the O(N) symmetry.

$$\vec{m} = |m|\vec{e}, \quad (\vec{e}) = 1, \quad dm^i = m^{N-1}dmde^i, \quad \int de^i = \kappa = \frac{2\pi^{N/2}}{\Gamma(N/2)} \tag{1.29}$$

Now the integration is made by a change of variable by $e^i = \Omega_{ij}(1, 0..., 0)$, where Ω_{ij} is a rotation matrix. So we define

$$S^i = \Omega_{ij}R^j. \tag{1.30}$$

The inserted equation, before it was transformed, is given by

$$\int d|m||m|^{N-1}\kappa \prod_{i=1}^{N} \delta(|m|(1, 0, ..., 0) - \frac{1}{TL}\int d^2x R^i(x)) =$$

$$= \kappa \prod_{i=1}^{N-1} \delta(\frac{1}{TL}\int d^2x R^i(x))(\frac{1}{TL}\int d^2x R^N(x))^{N-1}. \tag{1.31}$$

Then from this operator we reach two conclusions. We introduce a new operator in the Lagrangian.

$$(\frac{1}{TL}\int d^2x R^N(x))^{N-1} = e^{-[-(N-1)log(\frac{1}{TL}\int d^2x R^N(x))]} \tag{1.32}$$

Using the previous parametrization to the leading order, we obtain

$$e^{-[-(N-1)log(\frac{1}{TL}\int d^2x R^N(x))]} \approx e^{-[-(N-1)log(\frac{1}{TL}\int d^2x(1-\frac{1}{2}g_0^2\pi_i(x)\pi_i(x)]} \approx$$

$$\approx e^{-[-(N-1)\frac{1}{TL}\int d^2x \frac{1}{2}g_0^2\pi_i(x)\pi_i(x)]} \tag{1.33}$$

The zero mode for the π^i field is forbidden by

$$\prod_{i=1}^{N-1} \delta(\frac{1}{TL}\int d^2x \pi^i(x)) \tag{1.34}$$

This method can be used only when the zero mode is allowed and the vacuum expectation value is a constant. The boundary conditions which accomplish the previous characteristic are the Neumann and the periodic boundary conditions.

1.1.4 Renormalized factor in O(N) sigma model and γ and β functions

Previously we discuss the technical problems in O(N) sigma model. Now we can calculate the renormalized constants of the O(N) sigma model. In this section we show the renormalization factor. In this case all the calculations presented here are going to generate the Z factor which do not depend on the IR divergence. As it was told, this is possible because UV divergence and IR divergence are not coupled. This is important that because thanks to this IR independence these Z factors can be used in any kind of the IR regularization chosen. In the dimensional regularization the result upto the NNLO order is:

$$Z_{g^2}(d) = 1 + \frac{N-2}{\epsilon}\frac{g^2}{2\pi} + [(\frac{N-2}{\epsilon})^2 + \frac{N-2}{2\epsilon}](\frac{g^2}{2\pi})^2 +$$
$$+[(\frac{N-2}{\epsilon})^3 + \frac{7}{6}(\frac{N-2}{\epsilon})^2\frac{(N-2)(N+2)}{12\epsilon}](\frac{g^2}{2\pi})^3 \tag{1.35}$$
$$Z_\phi(d) = 1 + \frac{N-1}{\epsilon}\frac{g^2}{2\pi} + (\frac{(N-1)(N-\frac{3}{2})}{\epsilon^2}(\frac{g^2}{2\pi})^2 +$$
$$+[-\frac{N-1}{3\epsilon^3}(-3N^2 + \frac{19}{2}N - \frac{15}{2}) + \frac{(N-2)(N-1)}{3\epsilon^2}\frac{(N-2)(N-1)}{4\epsilon}](\frac{g^2}{2\pi})^3 \tag{1.36}$$

Those results was obtained by the calculation of the 2-point correlation functions and get the values for the next Lagrangian.

$$\int d^{2-\epsilon}x\mu^\epsilon[\frac{Z_\phi}{2}(\partial_\mu\pi_i(x)\partial^\mu\pi_i(x))$$
$$+Z_{g^2}g^2Z_\phi\frac{(\partial_\mu\pi_i(x)\partial^\mu\pi_i(x))}{1-Z_{g^2}g_\phi^2\pi_i(x)\pi_i(x)} + Z_{g^2} + \frac{H_R}{Z_\phi^{1/2}}g^2(1-Z_{g^2}g^2Z_\phi\pi_i(x)\pi_i(x))^{1/2}] \tag{1.37}$$

Where we have defined $g_0^2\mu^\epsilon = Z_{g^2}g_R^2$ and $\pi_0^i = Z_\phi\pi^i$ All the calculations are shown in the work made by Brezin and Zinn-Justin and collaborators [58, 60]. The next obvious step to follow is to get the β and γ function by the formula:

$$\beta(g^2) = \frac{-\epsilon g^2}{1 + g^2\frac{\partial}{\partial g^2}Z_{g^2}} \tag{1.38}$$

$$\gamma(g^2) = \beta(g^2)\frac{\partial}{\partial g^2}Z_\phi \tag{1.39}$$

Using the eq. (1.35) and eq. (1.36)

$$\beta_{\overline{MS}}(g^2) = -\epsilon g^2 - \frac{(N-2)}{2\pi}(g^2)^2 - \frac{(N-2)}{(2\pi)^2}(g^2)^3 - \frac{(N-2)(N+2)}{4(2\pi)^3}(g^2)^4 \tag{1.40}$$

$$\gamma_{\overline{MS}}(g^2) = \frac{(N-1)}{2\pi} g^2 + \frac{(N-2)(N-1)}{(2\pi)^3} (g^2)^3 \tag{1.41}$$

Now the next step is to calculate it in the other regularization, i.e., with lattice regularization. As we said before we should change the above lagrangian by the definition of the lattice derivative and insert the terms related with the measure term. So in this situation the Z-factor are not going to be presented, we just show β and γ functions to realize that these functions may depend on them. We are going to present the calculation made by Falcioni [59], where the calculation in detail can be checked. For other calculation, we recommend the paper by Caracciolo [56] for the sigma model and to learn how to compute perturbative lattice calculation [29]. We obtain

$$\beta_a(g^2) = -\frac{(N-2)}{2\pi}(g^2)^2 - \frac{(N-2)}{(2\pi)^2}(g^2)^3 - \frac{(N-2)[(N+2)h_1 + h_2]}{(2\pi)^3}(g^2)^4 \tag{1.42}$$

$$\gamma_a(g^2) = \frac{(N-1)}{2\pi} g^2 + \frac{(N-1)}{(2\pi)^2} k (g^2)^2. \tag{1.43}$$

This different values can be found, table 1.1.4.

	h_1	h_2	k
A_s	-0.09	0.52	1.57
A_{ti}	0.18	0.48	1.07
A_{bs}	0.22	-0.46	0.12

Table 1.1: Different β and γ functions, depending on the lattice derivative definition

The result is so revealing, first we see that the β-function for any kind of scheme is the same up to NLO. This is not trivial, that up to next leading order the beta function the coefficients are universal.

Another clear result is the differences between different Lagrangians, coming from the derivative definition. The difference remains even at the end of the calculation of the β-function when the regulator is neglected, as it is shown in the table . The situation is the same in the case of the γ-function. The difference is that the universality is just up to LO. The LO β-function is at $(g^2)^2$ order, while the LO γ-function is at g^2. For the next orders, γ-function varys with different Lagrangian definition.

1.2 O(N) controversy in perturbation theory

In the previous sections we were discussing the perturbative theory as correct tool to calculate QFT. But during long time there is not a real proof of perturbative calculation is the correct answer. This statement have been researched deeply by Patrascioiu and collaborators. We here give an idea of the most important points in this controversy.

Before starting the subject, we would like to point out that this controversy could change several important issues about the actual knowledge of quantum field theory and therefore the idea about QCD, the theory that describes the strong interaction in the nature. One of the change in our knowledge is the concept of asymptotic freedom, which is deeply believed for almost all the physics community.
This problem is related with one of the open problems in mathematical Physics, which is called "Soft phases in two dimensional O(N) models".

1.2.1 Different PT expansion

The initial point where all this controversy starts papers by Patrascioiu [47, 46]. In these papers, he questioned about the conventional perturbation theory and proposed new method to evaluate the partition function of the O(N) sigma model.
This new method (New PT) gave surprising results.
(a) New PT agrees with conventional PT only for $N = 2$ or the dimension, $d \to \infty$.
(b) New PT produces the correct assymptotic expansion of two point correlation function for any N in $d = 1$.
(c) For $d = 2$, the leading term of the β function computed with new PT has the same value as the conventional PT, but opposite sign.
This different sign of the β-function comes from the $d = 2$ correlation function in this method

$$\langle S^i(0,0)S^i(x,0)\rangle = 1 - \frac{N-1}{2\pi}g^2 log(x) + (\frac{N-1}{2(2\pi)^2} + \frac{(N-1)(N-2)}{2(2\pi)^2})(g^2 log^2(x))^2. \tag{1.44}$$

One sees the difference with the conventional PT in the last term which leads to

$$\beta(g^2) = \frac{(N-2)}{2\pi} g^2 \tag{1.45}$$

Until today there is no explanation why this method gives us a different result. As it was mentioned before, this new result will open new ideas in QFT, because if the β-function is not negative at the leading order, one may consider that there is some fixed point for the coupling, therefore a phase transition may occur.

1.2.2 Case of asymptotic freedom

The study by Patrascioui and collaborators has been widely known [48, 82, 80, 47] although a good summary of this problem can be found in [83] by Selier, where some evidence against the usual accepted concept of asymptotic freedom is called into question. There summarized the reasons for doubt on the O(N) sigma models for $N > 2$.
-The Mermin-Wagner theorem requires large fluctuations in any temperature and the fluctuations may destroy the logical basis of PT. That is why Patrascioiu presented a new boundary condition. The new boundary condition was called superinstanton and this leads us to different result.
-At the same time, the O(N) perturbation theory is believed to generate a mass gap dynamically. This finite mass gap generation should not depend on the boundary condition, however this statement is not correct.
-Another study shows that if we restrict one field component with $|S^N| < 1 - \epsilon$, the system will become massless at sufficiently low temperatures as Richard pointed out in [45]. This contradicts with the conventional PT because as it was presented before we only consider one north cascade in PT.

An important question which should arise is where in PT we have made a mistake! If we recheck the whole PT, the delicate problem is found where we expand the exponential form. This exponential expansion is correct in finite degree of freedom, but we should be aware that several theorems in mathematics do not hold for infinite systems. So some skeptical people doubts PT with lack of proof in QFT.
Anyway this problem is an interesting problem where several points of views can be discussed. Here we will present an interesting discussion between Patrascioiu and Niedermayer. The initial paper was made by Patrascioiu [82] "Superinstantons and

the Reliability of Perturbation Theory in Non-Abelian Models". In here they defined a vacuum expectation value

$$S_i = (\pi_i, \sqrt{1-\pi_i^2}\sin(\phi_i), \sqrt{1-\pi_i^2}\cos(\phi_i))$$

$$\phi(r,\theta) = c\log r/\log R, \quad \pi(r,\theta) = 0 \tag{1.46}$$

This configuration has an energy that goes to 0 at $R \to \infty$. This will generate a new propagator which is given by

$$G_{BC}^{SI}(x,y) = G_{BC}(x,y) - \frac{G_{BC}(x,y)G_{BC}(0,y)}{G_{BC}(0,0)} \tag{1.47}$$

With this definition it is straightforward to calculate some values. In this case, he calculated a value known in usual PT to compare them.

$$\langle S(0)\cdot S(1)\rangle_P = 1 - \frac{N-1}{2\beta}G(1,1) - \frac{1}{\beta^2}\{(N-1)^2[\frac{G(1,1)^2}{8} - \frac{t}{4}] + (N-1)[-\frac{G(1,1)^2}{4} + \frac{v}{2}]\} \tag{1.48}$$

From the Hasenfratz's paper, where usual periodic boundary condition is used:

$$\langle S(0)\cdot S(1)\rangle_H = 1 - \frac{N-1}{\beta}D(1) - \frac{N-1}{2\beta^2}D(1)^2, \quad D(1) = \frac{1}{2D}(1-\frac{1}{L^2}) \tag{1.49}$$

The Patrascioiu's result is not the same as Hasenfratz at large L limit. Although it is the same for the case of O(2) sigma model, commutative group.
This leads us to the next conclusion that at large L, different BC will give us different result, what contradict the Mermin-Wagner theorem.

In superinstanton BC,(SI BC) the coefficient $\frac{1}{\beta^2}$ goes $(N-1)^2$, while this term is absent in the standard BC. Since the propagator grows like $\log x$ at large distances and since the leading coefficient of the β function is obtained from the term $O(\log^2 x/\beta^2)$, we see that SI BC gives rise to a β function whose leading coefficient is different from the one found in the literature (obtained by using PT with periodic BC)

This generated some controversy, being answered by Niedermayer and others, [81]. This paper was a criticism about the Patrascioiu's work. The main point was as David pointed out in his comments on "Superinstantons and the Reliability of Perturbation Theory in Non-Abelian Models", that SI BC used by Patrascioiu is ill of IR divergences. Niedermayer took this idea and he calculated to third order PT using SI BC and he got the expected IR divergences as F. David pointed out. Then

Niedermayer opened some questions:
Q1:Is the asymptotic expansion of short-distance observables for fixed L and uniform in $L \to \infty$? Equivalently, can for such quantities the limits $L \to \infty$ and L fixed be exchanged in bare lattice PT?
Niedermayer's answer: It will depend on the BC. So for the case of SI BC the answer is NO. But in the case of BC without IF divergences this interchange is allowed. So if SI BC is discarded, in the cases where limits are commutative He opened new questions, to check the credibility of PT.
Q2: Which BC's α can be anticipated to give finite and coinciding answers for the $L \to \infty$ limits of their perturbative coefficients $c_r^\alpha(x_1, ..., x_n; L)$?
Q3: Do the infinite volume short-distance observables admit an asymptotic expansion in β^{-1}?
If "yes", can their coefficients $c_r^\infty(x_1, ..., x_n), r \geq 1$ be obtained from PT via

$$lim_{L\to\infty} c_r^\alpha(x_1, ..., x_n; L) = c_r^\infty(x_1, ..., x_n),$$

where α is one of the BC meeting the conditions in Q2?
His answer was YES if the BC is "well-behaved"

$$C^\alpha(\beta, L) = 1 - \sum_{r=1}^{k} c_r^\alpha(L)\beta^{-r} + R_k^\alpha(\beta, L)$$

A $c_r^\alpha(L)\beta^{-r}$ is bounded He claims that in the cases where the BC is "well-behaved" the correlation function will have the following relations

$$C_r^\alpha(x_1, ..., x_n; L) \leq C_r^\alpha(x_1, ..., x_n; L) \leq C_r^\infty(x_1, ..., x_n; L)$$

(This relation was just suggested but not rigorously proved). It was shown $C_r^\alpha(x_1, ..., x_n; L)$ is increasing at large L and $C_r^\infty(x_1, ..., x_n; L)$ is decreasing.

$$C_r^\alpha(x_1, ..., x_n; L) \leq C_r^\infty(x_1, ..., x_n) \leq C_r^\infty(x_1, ..., x_n; L) \tag{1.50}$$

After this paper, the Patrascioiu's answer was presented in [80] "Questionable Arguments for the Correctness of Perturbation Theory in Non-Abelian Models", which opposes the criticism made by Niedermayer.

The Niedermayer claims about rejecting SI BC because of IR divergence is not rigorous but Patrascioiu claims that this IR divergence can be canceled using OPE

and this may vary again our result of our conclusion. (But this problem was not tackle for anyone).

But even this IR divergence is solved that will happens at 3rd order, so at 2nd order we do not need it and we still get different β function.

The most important was about the concept of "well-defined" BC because that definition was made just to get BC to get the result was looking for

Patrascioiu explain that he did not claim anything against this relation.

$$C_r^{\alpha}(x_1, ..., x_n; L) \leq C_r^{\alpha}(x_1, ..., x_n; L) \leq C_r^{\infty}(x_1, ..., x_n; L) \tag{1.51}$$

But Patrascioiu mentioned this relation and the increasing and decreasing behaviour of $C_r^{\alpha}(x_1, ..., x_n; L), C_r^{\infty}(x_1, ..., x_n; L)$ respectively is not enough for the convergence of the PT.
As example:

$$C^{free}(0, 1, \beta, L) = 1 - \frac{1}{\beta}(1+1/L) \quad C^{Dir}(0, 1, \beta, L) = 1 - \frac{1}{\beta}(1+1/L) + \frac{1}{\beta^2}e^{-\frac{\beta}{L}} \tag{1.52}$$

So in the limit $L(\beta) \propto \beta$ we do not get the same results.

So the usual wisdom of QCD and O(N) about the theory is asymptotic free is not completely accept for all the physics community, maybe this is telling us it is needed a review for the concepts of QFT and the relation between non-perturbative and perturbative calculation.

1.2.3 Continuum limit extrapolation

This is anissue related to the previous one. First we describe the common knowledge about continuum extrapolation. The common continuum extrapolation is always based on the work made by Symanzik in [106] and [91]. There he calculated the Feynmann diagrams in lattice perturbation theory to get:

$$\mathcal{O}_a = \sum_{i=0} \sum_{k=0} K_i^{\mathcal{O}} \ {}_k a^{i+n} log^k a \tag{1.53}$$

where a is the lattice spacing. We observe that those terms $K_i^{\mathcal{O}}{}_k$ will depend on the operator we are calculating. The important conclusion to get from this calculation is that we will be able to describe the continuum limit using a polynomial in a. All of this calculation is a consequence of PT, so if we think that the PT in O(N) sigma model does not give the correct answer, why should we trust the conventional continuum limit?. This question was made from the other researchers too. The main idea of this suspicion is due to the bare correlation function is massless and may give us different result, in theory where the field has a mass the PT is believed that PT works well. Anyhow in [16] the study of O(2) was made. In this paper the continuum limit was reconsidered. It is important to mention that in O(2) there is no mass gap generation, this mass gap generation is when $N > 2$. Due to the lack of mass gap the O(2) theory will have a Kosterlitz-Thouless transition. In [16] it is clearly shown that the continuum limit can be taken without using a polynomial in a. In this case he used

$$\mathcal{O}_a = \sum_{k=0} K_k^{\mathcal{O}} \frac{1}{log^k a} \tag{1.54}$$

This kind of continuum limit worked well for O(2). So Patrascioi in [87] suggested to use this kind of continuum extrapolation to calculate the continuum extrapolation in Non-PT. He used the data used by Lüscher in [41] where the asymptotic freedom idea was reinforce, the final conclusion is using that this extrapolation different results for the β-function is found. The new β function is not clearly change. The calculation was made for a range of the coupling far from zero, so the behaviour in other range may change even more than it is expected. So this will encourage to be able to calculate better the continuum extrapolation, as we can observe this different extrapolation will give us the idea that there is not a mass gap generation and by dominos logic, fixed coupling point and not asymptotic freedom in O(N). This problem may have not been tackled because we should go to big lattice space to be able to get observe this $\frac{1}{log^k a}$ behaviour.

This section is not going to be important in our calculation, but it is important to understand how deep the change in physics can be made if all this controversy become real. In this case all the calculation in Non-PT calculation should have been checked. Becuase the most common and believed calculation of QFT using Non-PT is by using Lattice calculation where this continuum limit is all the time needed.

1.2.4 Open Problem in Mathematical Physics

Now we will present an open problem as it is defined in International Association of Mathematical Physics (IAMP) by A. Patrascioiu:

Soft phases in two-dimensional O(N) models.

Do the spin correlations in the classical Heisenberg model, and other O(N) models with $N > 2$, decay exponentially at all non-zero temperatures, or do the models have soft phases akin to that of the XY model?

It is generally believed by the statistical mechanics and quantum field communities that, unlike the xy-model, the classical Heisenberg O(N) model has exponentially decaying correlations at all temperatures. This belief is based mostly on formal perturbation theory (= low temperature expansion). While for the xy model it has been shown by Bricmont et al. that this expansion is indeed the correct asymptotic expansion in the thermodynamic limit, a proof for the classical Heisenberg model does not exist.

The conventional picture has been challenged by Patrascioiu and collaborators and they have proposed that the model (and indeed all O(N) models) has a transition to a soft phase with algebraic decay of correlations at low temperature. We gave arguments based on percolation theory and on the structure of low-lying excitations (super-instantons); we discussed the weaknesses of the conventional arguments.

The problem is of great significance for particle physics, because the classical Heisenberg model is a toy version of QCD, the theory of strong interactions. Existence of a soft phase also in QCD would most likely imply failure of the so-called asymptotic freedom.

1.Prove or disprove that the conventional perturbative expansion (low temperature expansion) gives the correct asymptotic expansion for correlation functions and thermodynamic quantities.

2.Prove or disprove that the model has a critical point at some nonzero temper-

ature.

3.If yes, determine the critical indices; in particular determine if they are temperature dependent like in the xy model.

1.3 Mass gap in O(N)

The last section in this chapter, but not least important, is devoted to the calculation of O(N) sigma model. This is an important open problem for QFT, until the point that the mass generation in Yang-Mills is one of the Millennium Prize Problems, this relation between O(N) sigma model is clearly said in the report on the status of the Yang-Mills Millenium Prize Problem by Michael R. Douglas:

"There are two classes of quantum field theories which are generally believed to bear a close similarity to four-dimensional Yang Mills theory. The first is the two-dimensional nonlinear sigma model with target space a group manifold G, or more generally a symmetric space M of positive Riemannian curvature. This is a theory whose fields are maps from two-dimensional space-time into M. Apparently, even these theories have not yet been constructed to the standards required in the problem description. On the other hand, the mass gap has been exhibited in a regulated version of the O(N) sigma model (C. Kopper, Comm. Math. Phys. 202, 89 (1999)). Physically, these theories are known to be integrable, a point we will return to below"

Nowadays there are several speculation of the origin of this mass gap generation. Although there is not any calculation using just the Wightman axioms, there is a calculation made by Hasenfratz and coll. in[18], where the mass gap was calculated but using some Bethe ansatz consideratation. Becuase of this Bethe ansatz the result is not consider as a rigorous mathematical result. We will explain the procedure for this calculation. The calculation is made in the sigma model where one conserved Noether charge is coupled to a chemical potencial, h.

$$Q^{\alpha\beta} = \int dx J_0^{\alpha\beta}, \quad J_\mu^{\alpha\beta} = n^\alpha \partial_\mu n^\beta - n^\beta \partial_\mu n^\alpha \tag{1.55}$$

Then free the sigma model energy can be calculated in two ways, using perturbative calculation and the calculation of with the Bethe ansatz solution. For the perturbative calculation we will consider $h >> m$ in this consideration.

$$f(h) - f(0) = -\frac{h^2}{4\pi}(log\frac{h}{e^{1/2}\Lambda_{\overline{MS}}} + log(log\frac{h}{\Lambda_{\overline{MS}}}) + \mathcal{O}(1)), \quad O(3) \tag{1.56}$$

$$f(h) - f(0) = -\frac{h^2}{8\pi}(log\frac{h}{2e^{1/2}\Lambda_{\overline{MS}}} + \frac{1}{2}log(log\frac{h}{\Lambda_{\overline{MS}}}) + \mathcal{O}(1)), \quad O(4) \tag{1.57}$$

For the calculation of the free energy variation with the chemical potential using Bethe ansatz, the method by Polyakov [20] is used in [18] where the numerical method can be used. The idea is to observe

$$f(h) - f(0) = -\frac{h^2}{4\pi}(log\frac{h}{m} + log(log\frac{h}{m}) + C + \mathcal{O}(\frac{log(log(h/m))}{log(h/m)})), \quad h >> m, \quad O(3) \tag{1.58}$$

In the numerical simulation where

$$f(h) - f(0) = -\frac{1}{2\pi}\int_{-B}^{B} mcosh\theta\epsilon(\theta)d\theta, \quad \epsilon(\pm B) = 0 \tag{1.59}$$

$$\epsilon(\theta) - \int_{-B}^{B} \frac{1}{\pi^2 + (\theta - \theta')^2}\epsilon(\theta')d\theta' = h - mcosh\theta \tag{1.60}$$

The calculation to get the behaviour expected we need to consider $h >> m$. In this case the value gotten is:

$$m = C\Lambda_{\overline{MS}}, \quad C = 0.58(1), \quad O(3) \tag{1.61}$$

Similar calculation can be made for any O(4). But in this case they used the Wiener-Hopf technique to transform the problem into a a form where the logarithms are explicitly separated.

As it was said above, the calculation was made numerically but in [19] the calculation was made analytically. So as in the previous calculation we need to calculate the free energy dependence on the chemical potential in the range where $h >> m$ in two

different technics. One will be perturbative calculation, in this scheme we get the next result

$$f(h)-f(0) = -(N-2)\frac{h^2}{4\pi}(log\frac{h}{e^{1/2}\Lambda_{\overline{MS}}}+\frac{1}{(N-2)}log(log\frac{h}{\Lambda_{\overline{MS}}})+\mathcal{O}(\frac{log(log(h/\Lambda_{\overline{MS}}))}{log(h/\Lambda_{\overline{MS}})})), \quad O(3) \tag{1.62}$$

The next technique is using similar procudure as before, where the starting point is the free energy calculated by [20] and the Wiener-Hopf technique used for any N in O(N). The final result is:

$$f(h)-f(0) = -(N-2)\frac{h^2}{4\pi}(log\frac{h}{m}+\frac{1}{(N-2)}log(log\frac{h}{m})+C_N+\mathcal{O}(\frac{log(log(h/m))}{log(h/m)})), \quad O(N) \tag{1.63}$$

Where

$$C_N = log[(\frac{8}{e})^{1/(N-2)}\frac{e^{-1/2}}{\Gamma(1+1/(N-2))}] \tag{1.64}$$

This lead us to

$$m = (\frac{8}{e})^{1/(N-2)}\frac{1}{\Gamma(1+1/(N-2))}\Lambda_{\overline{MS}} \tag{1.65}$$

It is important to remember that this result was calculated using the Bethe ansatz, due to this ansatz the S-matrices can be founded as in [21].

Other approach important was made in [37], where the instanton gas in the vacuum expectation value(VEV) is used. In here, if we consider the instanton VEV, a mass generation is speculated. The calculation in this approach is the azimutal green function,$G^{\phi}(x)$

$$G^{\phi}(x) = \langle e^{iargS_+(x)}e^{-iargS_+(0)}\rangle_{inst} \tag{1.66}$$

Becuase this quantity will have an exponential behaviour, so this will suggest the typical green function $\langle S(x)S(0)\rangle_{inst}$ will have it too, although they were not be able to calculated.

The approach mention in the Yang-Mills Millenium Prize Problem made by C. Kopper in [25]. We will not explain in detail, but we will mention the crucial points in his calculation. The initial point in his calculation is to relaxed the constrain and with a clever reparametrization

$$\int dS^j d\sigma S^i(x) S^i(0) e^{-\frac{1}{2g^2}\int (\delta S)^2 + i(S^2-1)\sigma + \frac{1}{K}\sigma^2} \tag{1.67}$$

Then this quantity is integrable in S^j, then we can get an action on just σ, the problem is that to be able to calcultated we needs to expand in large N. At the end with fancy maths the results the mass gap is obtained.

Before several approaches are been presented, although all of them will have some lacks for the real calculation of the mass gap. Now we will present important progress made by G. Dunne and coll. [72] [73],[74]. In those approach they use resurgences theories to be able to subtract the mass gap of the theory. The main idea of this approach starting long time ago with Dyson [75]. He showed that QFT in perturbative calculation will have divergence series. Then this divergences series can be calculated using the ideas of Borel summations, these Borel summations can lead us to a sitution where the result is not defined in some coupling value, then to avoid this the resurgents functions takes place. This resurgent functions are been considered as a non-perturbative effect because they do not appear in perturbation theory, getting in this way the desired non-perturbative effect which will make mass gap appears.

As we can see this opened problem is still an important issue in the physicist community, so the discover of this mechanism will be a milestone for the QFT.

1.4 Parametrization in O(N)

In all the previous sections, we have explained that controversy of PT and issues corresponding with non-perturbative calculation. In almost all the studies of PT one kind of parametrization is used. So in this section we define in O(N) sigma model in different parametrizations, to clarify if those different parametrization is one of the reason of this open problem in O(N) sigma model. Coming back to the initial point what we are doing is a function $f(M) \longrightarrow \mathbb{S}^{N-1}$, where M is the coordinate space. This coordinate space is not going to be the central study of this section. The unique condition for the coordinate space is that the (N-1)-cohomotopy group of M, $\pi^{N-1}(M)$, is trivial. As it was stated, in our calculation we are going to use free

boundary condition for the time component and periodic boundary condition for the spatial component. This leads us to:

$$M = [-\frac{T}{2}, -\frac{T}{2}] \otimes \mathbb{S}^1, \quad \pi^{N-1}(M) := [M \longrightarrow \mathbb{S}^{N-1}] = 0, \quad \forall N > 2 \tag{1.68}$$

This is really important because it affirms that using perturbative calculation is enough to describe all the posible functions in one topological sector. More about topology will be spoken in the section related with the mass gap.
Pointed out that we only have one the topological sector, we will discuss different parametrization of for the N-sphere, $\mathbb{S}^{N-1}$. First point to know is that parametrization of the sphere by a just one open set in a plane, chart, is imposible. And the perturbative calculation is based in one point in the image, which is consider as a vacuum expectation value. Therefore the perturbative calculation is always just considering one open set, chart, in each parametrization. The next parametrizations are consider as the most known or simple parametrizations.

1.4.1 Canonical parametrization, north pole

This parametrization is called canonical, because all the variables are defined in the same way except one coordinate, which is going to follow the constraint in terms of the other coordinates.

$$S^i(x) = g_0^2 \pi^i(x) \quad \forall i \in [1, N-1], \quad S^N(x) = (1 - g_0^2 \pi^i(x)\pi^i(x))^{1/2} \tag{1.69}$$

It is obvious that this parametrization only parametrizes upper hemisphere, as it can be seen in the figure 1.1
The values of the field are restrited to $|\pi(x)| < 1/g_0^2$, but in QFT the bares parameters go to infinite. Although this restriction does not affect the final results in the perturbative calculation. We might think that this parametrization is not correct for our propose in QFT but it will be shown that is not the case.
Then the final lagrangian to consider and the operator to do the calculation are

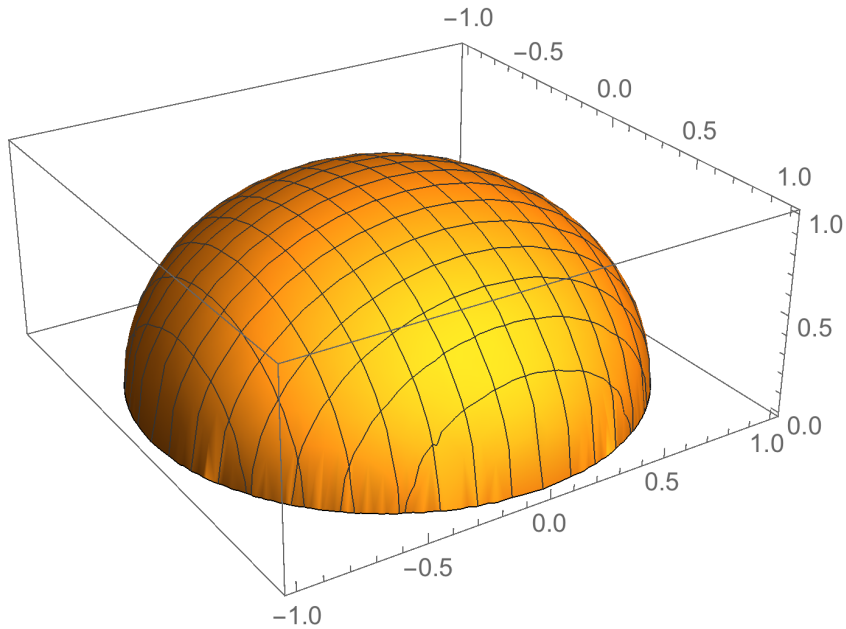

Figure 1.1: Upper hemisphere

$$\mathcal{L}_0 = \int (\sum_{i=1}^{N-1} \frac{1}{2}\partial_\mu \pi_i(x)\partial^\mu \pi_i(x) + g_0^2 \frac{1}{2} \sum_{i=1}^{N-1}\sum_{j=1}^{N-1} \pi_j(x)\partial_\mu \pi_j(x)\pi_i(x)\partial^\mu \pi_i(x) +$$

$$+ \sum_{i=1}^{N-1} \frac{1}{2TL}(N-1)g_0^2 \pi_i^2(x) + ...)dx \tag{1.70}$$

$$\langle S^i(x)S^i(y)\rangle = 1 + g_0^2(\langle \pi^i(x)\pi^i(y)\rangle - \frac{1}{2}\langle(\pi^i(x))^2\rangle - \frac{1}{2}\langle(\pi^i(y))^2\rangle) +$$

$$\frac{g_0^4}{8}(2\langle(\pi^i(x))^2(\pi^j(y))^2\rangle - \langle(\pi^i(y)^2(\pi^j(y))^2\rangle - \langle(\pi^i(x))^2(\pi^j(y)))^2\rangle) + \tag{1.71}$$

1.4.2 Stereographic projection

The stereographic projection is a famous parametrization for the N-sphere. In this parametrization one can parametrize all the sphere except for one point. The best explanation can be describe in the figure 1.2, for the case of $\mathbb{S}^2$.

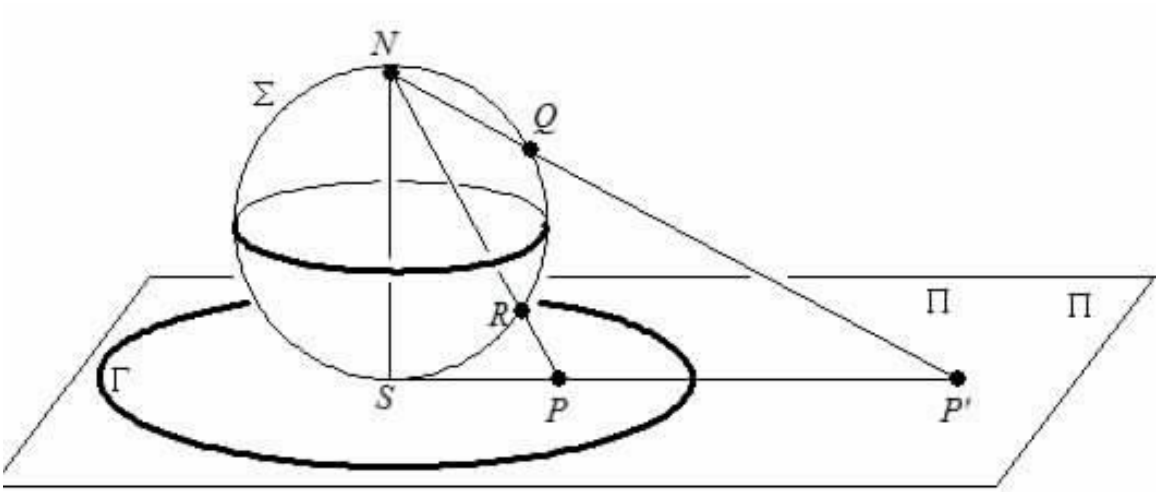

Figure 1.2: Stereographic projection of $\mathbb{S}^2$

In this case the definition of the new field, following [39], is:

$$S^i(x) = \frac{g_0\pi^i(x)}{1+\frac{1}{4}g_0^2\sum_{i=1}^{N-1}\pi^i(x)\pi^i(x)} \quad \forall i \in [i, N-1], \qquad S^N(x) = \frac{1-g_0^2\frac{1}{4}\sum_{i=1}^{N-1}\pi^i(x)\pi^i(x)}{1+g_0^2\frac{1}{4}\sum_{i=1}^{N-1}\pi^i(x)\pi^i(x)} \tag{1.72}$$

One may regard that in this parametrization we are considering all the points of the sphere less just one point. But the reality is far from this, as it is going to be explained soon.

The next step is to manipulate the lagrangian in the following form:

$$\mathcal{L}_0 = \int (\sum_{i=1}^{N-1}\frac{1}{2}\frac{\partial_\mu\pi_i(x)\partial^\mu\pi_i(x)}{1+g_0^2\frac{1}{4}\sum_{i=j}^{N-1}\pi_j(x)^2} =$$

$$= \int (\sum_{i=1}^{N-1}\frac{1}{2}\partial_\mu\pi_i(x)\partial^\mu\pi_i(x) - \frac{1}{8}g_0^2\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\partial_\mu\pi_i(x)\partial^\mu\pi_i(x)\pi_j^2(x) \tag{1.73}$$

The last step made is the perturbative expansion in g_0^2. This last step restricts the values of the field, because $\frac{1}{1+x^2} = 1 - x^2 + x^4 - ...$, if $|x| < 1$. So even in this parametrization, the field values are restricted. Anyway keeping this parametrization, where the last object to define is the 2-point correlation function with O(N) symmetry.

$$\langle S^i(x)S^i(y)\rangle = 1 + g_0^2(\langle\pi^i(x)\pi^i(y)\rangle - \frac{1}{2}\langle(\pi^i(x))^2\rangle - \frac{1}{2}\langle(\pi^i(y))^2\rangle)+$$

$$\frac{g_0^4}{8}(2\langle(\pi^i(x))^2(\pi^j(y))^2\rangle - \langle(\pi^i(y)^2(\pi^j(y))^2\rangle - \langle(\pi^i(x))^2(\pi^j(y)))^2\rangle-$$

$$2(\langle \pi^i(x)\pi^i(y))((\pi^i(x))^2 - (\pi^i(y))^2)\rangle)+ \tag{1.74}$$

After the observation of the lagrangian and the operator and as it was explained the enclosed problem appears. We get different operator expansion and different Lagrangian and operator, coming from the same theory. So an important question arises, are both parametrization equivalent? And if the answer is "no", which one is the correct answer? Those questions will be discuss later, where we present the calculation in each choice.

1.4.3 Spherical coordinates

This parametrization is the typical angular parametrization of the sphere.

$$S_1 = \sin(g_0\pi_1) \quad \pi_1 \in [-\frac{\pi}{2g_0}, \frac{\pi}{2g_0}]$$

$$S_2 = \cos(g_0\pi_1)\sin(g_0\pi_2) \quad \pi_2 \in [-\frac{\pi}{2g_0}, \frac{\pi}{2g_0}]$$

$$S_3 = \cos(g_0\pi_1)\cos(g_0\pi_2)\sin(g_0\pi_3) \quad \pi_3 \in [-\frac{\pi}{2g_0}, \frac{\pi}{2g_0}]$$

$$\vdots$$

$$S_{N-1} = \cos(g_0\pi_1)\cdots\cos(g_0\pi_{N-2})\sin(g_0\pi_{N-1}) \quad \pi_{N-1} \in [-\frac{\pi}{g_0}, \frac{\pi}{g_0}]$$

$$S_N = \cos(g_0\pi_1)\cdots\cos(g_0\pi_{N-2})\cos(g_0\pi_{N-1}) \tag{1.75}$$

For this parametrization, we are considering all the sphere, although other enclosed problem will show up too, as in the previous parametrization in QFT this constrain will not affect to the result. This parametrization were not investigated before so there is such us study in the bibliography.
Anyhow we will show here the lagrangian for this parametrization:

$$\mathcal{L}_0 = \int (\sum_{i=1}^{N-1} \frac{1}{2}\partial_\mu\pi_i(x)\partial^\mu\pi_i(x) + g_0^2(\frac{1}{2}\sum_{i=1}^{N-1}\sum_{j=1}^{i}\pi_j(x)\pi_j(x)\partial_\mu\pi_i(x)\partial^\mu\pi_i(x)+$$

$$+\sum_{i=1}^{N-1}\sum_{j=1}^{i}\pi_j(x)\partial_\mu\pi_j(x)\pi_i(x)\partial^\mu\pi_i(x)+\frac{1}{2}\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\pi_j(x)\partial_\mu\pi_j(x)\pi_i(x)\partial^\mu\pi_i(x)+$$

$$+\sum_{i=1}^{N-1}\frac{1}{2TL}(N-1)g_0^2\pi_i^2(x))+...dx \tag{1.76}$$

And the corresponding 2-point correlation function with O(N) symmetry:

$$\langle S^i(x)S^i(y)\rangle = 1+g_0^2(\langle\pi^i(x)\pi^i(y)\rangle-\frac{1}{2}\langle(\pi^i(x))^2\rangle-\frac{1}{2}\langle(\pi^i(y))^2\rangle)+$$

$$g_0^4(\frac{1}{6}(\sum_{i=1}^{N-1}\langle\pi^i(x)[\sum_{j=1}^{i-1}(3\pi^j(x))^2+(\pi^i(x))^2]\pi^i(y)\rangle+\langle\pi^i(x)[\sum_{j=1}^{i-1}(3\pi^j(y))^2+(\pi^i(y))^2]\pi^i(x)\rangle)+$$

$$\frac{1}{24}\sum_{i=1}^{N-1}(\langle(\pi^i(x))^2[\sum_{j=1,j\neq i}^{N}(6\pi^j(x))^2+(\pi^i(x))^2]\rangle+\langle(\pi^i(y))^2[\sum_{j=1,j\neq i}^{N}(6\pi^j(y))^2+(\pi^i(y))^2]\rangle))+... \tag{1.77}$$

We can realized that the Lagrangian and operator is quite troublesome, althought there is not any technical issue to consider.

1.4.4 Relaxed constrain

This parametrization has not been used often, although the best characteristic about this parametrization is the problem of the restricted field values is solved. We will follow the Cline's paper [43]. The central idea of this parametrization is that the constraint which can be understood as a Dirac's delta function is replaced by a limit of a gaussian function:

$$\delta(x)=\lim_{M\to\infty}\frac{M}{\sqrt{\pi}}e^{-M^2x^2} \tag{1.78}$$

In this case the constraint is $1=(S^i(x))^2$.
When we choose the vacuum expectation value, VEV, we will break the symmetry. As in the following parametrization

$$S^i(x)=g_0\pi^i(x)\quad\forall i\in[1,N-1],\quad S^N(x)=1-g_0\sigma(x) \tag{1.79}$$

The VEV is now (0,...0,1). In this case we will get the Lagrangian:

$$\mathcal{L}_0=\int(\sum_{i=1}^{N-1}\frac{1}{2}\partial_\mu\pi_i(x)\partial^\mu\pi_i(x)\frac{1}{2}\partial_\mu\sigma(x)\partial^\mu\sigma(x)+\frac{1}{2}M^2\sigma(x)\sigma(x)+$$

$$+M^2(\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\frac{1}{8}\pi_i^2(x)\pi_j^2(x)+\sum_{i=1}^{N-1}\frac{1}{2}\pi_i^2(x)\sigma(x)$$

$$+\sum_{i=1}^{N-1}\frac{1}{4}\pi_i^2(x)\sigma^2(x)+\sum_{i=1}^{N-1}\frac{1}{4}\pi_i^2(x)\sigma(x)+\frac{1}{2}\sigma^3(x)+\frac{1}{8}\sigma^4(x))dx \tag{1.80}$$

And the operator:

$$\langle S^i(x)S^i(y)\rangle = 1 - g_0\langle\sigma(x)+\sigma(y)\rangle + g_0^2(\langle\pi^i(x)\pi^i(y)\rangle + \langle\sigma(x)\sigma(y)\rangle)+ \tag{1.81}$$

Initially this new parametrization gives us a simple Lagrangian but to deal with it we should take care in some issue about the limit $M \to \infty$. As it is shown in [43] the interaction terms has M factors, however the propagator of the σ particle is proportional to the inverse to M. So we only get the correct answer after summing up all the diagrams and take the limit.

1.4.5 Summarize in all the parametrizations

In all the previous section different parametrizations are presented. All of them lead us to different Lagrangians and different operator equivalences. Then the obvious question arises.

-Are both parametrization equivalent?

-And if the answer is "no", which one is the correct answer?

The answer is that all these parametrizations give us the same result at NLO for the mass gap and the Z factor. For this calculation we have used tools which are going to be explained in the 3rd chapter. Here we would like to emphasize that this is not a result for this theory, the reason is linked with QFT. Before getting any conclusion we would like to summarize main difference in each parametrization.

-North pole and stereographic projection: In both we have just considered one cascade of the sphere, what implies the enclosed field but this constrain has not kept in mind for the calculation.

-Spherical coordinates: In this case whole sphere is parametrized. But as in the other calculations the fields are enclosed.

-Relaxed constrain: In this parametrization whole sphere is consider too but in here there is not constrain for the field values. So there is not a real reason to doubt about this parametrization.

The conclusion is quite clear: if the topology between the coordinate state and the field configuration is trivial. Any parametrization of the field in one open set in the field configuration, the perturbative calculation will be equivalent to any other. It is important to mention here the concept of perturbative calculation, because in the regime of non-perturbative calculation this equivalence may not occur. Anyway, this different parametrizations equivalence reforce the idea that the PT is a correct expansion.

Chapter 2

Finite volume effects

In this chapter we explain importance of volume dependence in quantum field theory. It is common to apply quantum field theory in the infinite volume, but for several reasons it is going to be important to be able to understand quantum field theory in finite volume. These reasons are not going to be just academic, because some of this volume dependence behaviour in nanotechnology with new materials should appear. At the same time, nowadays to be able to understand non-perturbative effect in quantum field theory there is no better tool than the computer calculation, lattice calculation, and in this case we have limited computer resources so that the infinite limit can not be reached. Normally the volume may be large enough but in other situations the volume is not sufficient so that we need to understand how to extrapolate to the infinite volume. This difference between the infinite volume result and the finite volume result is considered as a systematic error in lattice calculation. Those finite volume systematic errors in the actual calculations of lattice QCD are quite important to be able to get the real values to compare with experimental results of the particles. As it was explained in the previous chapter the finite volume can be used in theories where the field is massless, to avoid the infrared singularities. From all of these reasons we understand the importance of the volume dependence.

In this chapter, we present the most important ideas for finite volume calculations. We first show how we can go to finite volume from infinite volume regime with a simple recipe. Then in the next section we mention the most interesting works in finite volume regime. Many important studies in finite volume were made by Lüscher and collaborators and we describe the main ideas and conclusion in those studies. In

the second section, we show how we can treat the finite volume systematic error. In others words the deviation of the energy spectrum in finite volume from the infinite regime. In the third section, we explain how Lüscher gives a twist of the idea of finite volume effect. In this case the finite volume effect will be used to calculate the scattering matrix of two particles. In the next section, we explain how the renormalization group equation can be used to describe the volume effects too and in the last section we will describe a new renormalization group equation where the variable in the RGE is the volume itself. This will generate a new non-homogeneous term in the RGE.

2.1 From infinite volume to finite volume

It is a common knowledge how to treat QFT when the space is infinite. It is common to treat the interaction in the Feynman diagram scheme, where Feynman diagram is used. In the typical Feynman diagrams we use the momentum space to integrate the interaction terms. The needed information to be able to compute those interactions in each theory is the propagator of the particle in a free theory and the rest of the Lagrangian will be considered as interaction term and at the end we will integrate out the internal loops for each diagram. Obviously, if we want to calculate QFT in finite volume we need to observe how those informations change.

-Propagator in free theory

The way to obtain the propagator is using the concept of Green's function and to build the Green's function we need the eigenvalues and eigenfunctions of the Lagrangian. These eigenvalues and eigenfunctions will depend on the boundary condition employed in the finite volume. Obviously we have various boundary conditions, such as Periodic, Antiperiodic, Zero-Direchelt and Zero-Neumann, but now we just mention for simplicity the periodic boundary condition, PBC. One important issue to consider is when we get the infinite volume limit, in the situation where the field is massive, the infinite limit will give us the same values for any boundary, but when the field is not massive we should pay attention in the boundary condition because it will may change the calculation in infinite limit. This difference between the boundary conditions will emerge in our calculation. Coming back to the variation of the propagator, the usual propagator for a massive field in a d-dimension infinite volume is:

$$\int \langle \phi(x)\phi(y)\rangle_{Free} e^{-ip(x-y)} d^d x = \frac{1}{p^2+m^2} \quad p^2 = p_1^2 + ... + p_d^2 \tag{2.1}$$

Then if we want to change one dimension to finite volume with periodic boundary condition we get

$$\frac{1}{p^2 + (\frac{2\pi n}{L})^2 + m^2}, \quad n \in \mathcal{Z}, \quad p^2 = p_1^2 + ... + p_{d-1}^2. \tag{2.2}$$

In this case because the eigenfunction in PBC is the same as in infinite volume, the variation is quite simple. To check how this propagator varies in different boundary conditions we explain them in the Appendix A.2. We have presented the propagator in the momentum space, we can do the same in the coordinate space,

$$\sum_{m=-\infty}^{\infty} \frac{1}{-|p^2 + m^2|^{1/2}} e^{|p^2+m^2|^{1/2}|x-y+mL|}, \quad m \in \mathcal{Z}, \quad p^2 = p_1^2 + ... + p_{d-1}^2 \tag{2.3}$$

-Internal loops

From the Feynmann diagrams in the momentum space, we have to integrate in the internal loops so integration will change in a summation,

$$\int \frac{1}{2\pi} dp \to \frac{1}{L} \sum_{n=-\infty}^{\infty} \tag{2.4}$$

Where we use the propagator in the coordinate space, the change is

$$\int_{\infty}^{\infty} dx \to \int_{-L/2}^{L/2} dx \tag{2.5}$$

-Interaction term If in the interaction term has a derivative the form of the momentum will change accordingly.

2.2 Volume dependence of the energy spectrum in massive theories

After all the previous considerations, we are prepared to understand the work by Lüscher [91]. In [91], he explains how the mass shift, Δm, occurs in the finite volume system. So he starts from the usual definition of the Green's functions and amputates the Green's functions and he defines the variation of the propagator in finite volume, how it is shown in the figure 2.1

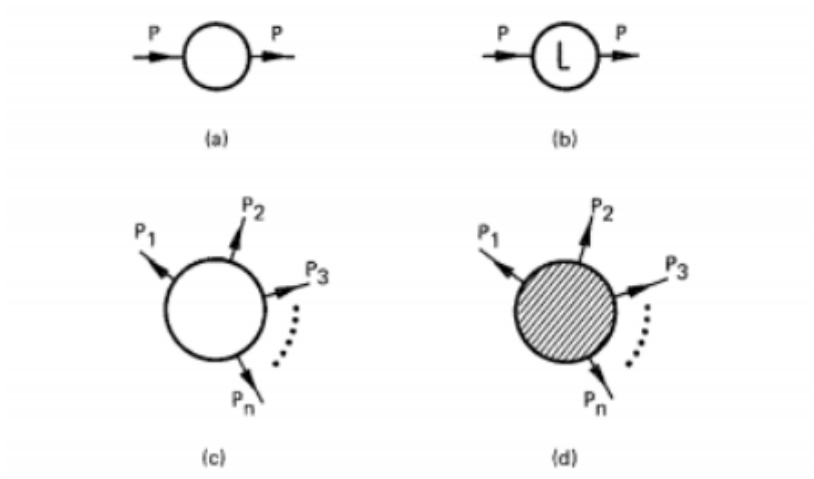

Figure 2.1: Interaction and propagator in infinite and finite volume

This variation of the propagator due to the finite volume is made by

$$G_L(p) = G(p)(\sum_{j=1}^{d} \cos(p_j L)) \tag{2.6}$$

Then from this Green's functions the decomposition of the 4-point function in 1-particle irreducible parts, as an example in the figure 2.2

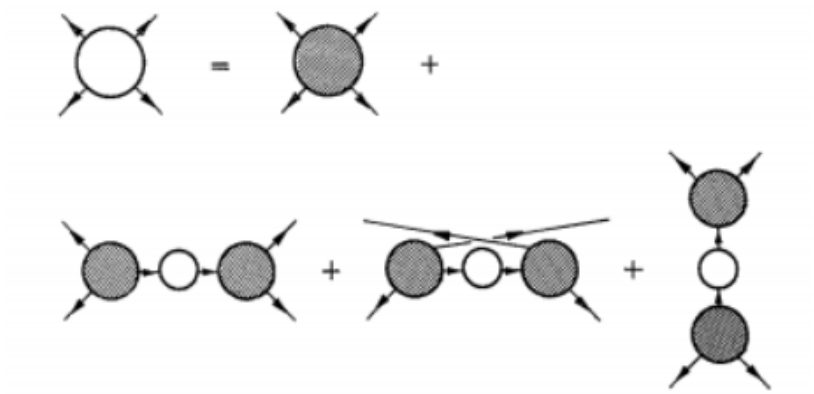

Figure 2.2: 4-points correlations functions

2.2.1 Volume dependence of the mass gap in simple scalar theories

Lüscher is interested in themass shift and so he only considers the selfenergy parts of the 2-point correlation function. To the leading order in the mass shift we obtain by

the following diagrams,

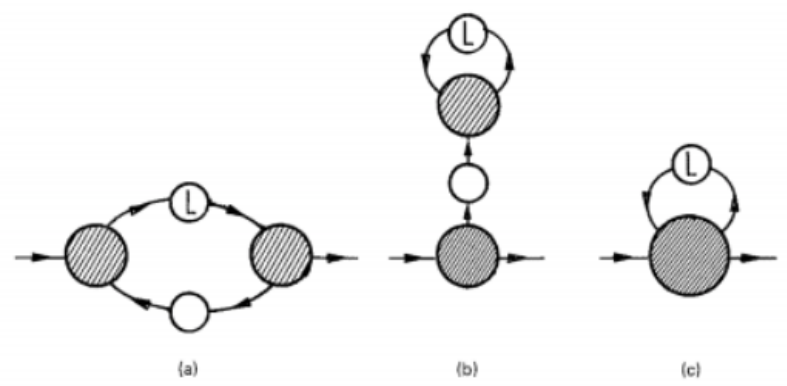

Figure 2.3: Self energy in finite volume

At the end, for the conclusion we need to define the forward amplitude

$$F(\nu) = T(p,q|p,q), \quad \nu \equiv (\omega(p)\omega(q) - pq)/m \tag{2.7}$$

where

$$\lim_{\nu \to \pm\frac{1}{2}m} (\nu^2 - \frac{1}{4}m^2)F(\nu) = \frac{1}{2}\lambda^2 \tag{2.8}$$

and the 3-point coupling constant λ is given by

$$\lambda = \Gamma(p,q,k), \quad p+q+k=0, \quad p^2 = q^2 = k^2 = -m^2 \tag{2.9}$$

Once we have defined the previous interaction terms, we find an asymptotic formula for the finite volume mass shift,

$$\Delta m = -\frac{3}{16\pi m^2 L}\{\lambda^2 e^{-\frac{\sqrt{3}}{2}mL} + \frac{m}{\pi}\int_{\infty}^{\infty} dy e^{-\sqrt{m^2+y^2}L} F(iy) + \mathcal{O}(e^{\sqrt{\frac{3}{2}}mL})\}. \tag{2.10}$$

2.2.2 Volume dependence of the bound state masses

At the same time in [91], Lüscher shows how we can calculate the mass shift Δm_B of the bound state. It is easy to check how the bound state may change with the previous calculation of the mass shift for the stable particle. First we define the bound state, ψ with a double line in the diagrams and how this bound state interact with the ϕ, particle with the same notation as before. Using the same notation we can build the Feynman diagrams, as in the figure 2.5.

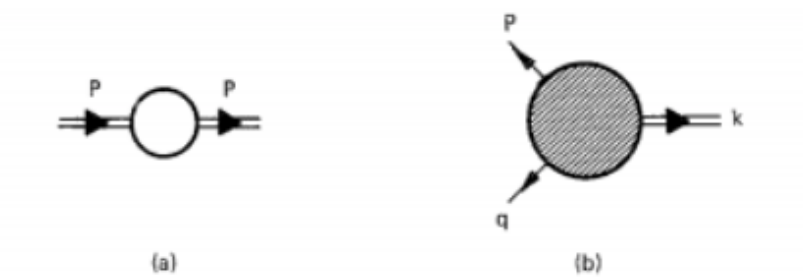

Figure 2.4: 4-points correlations functions

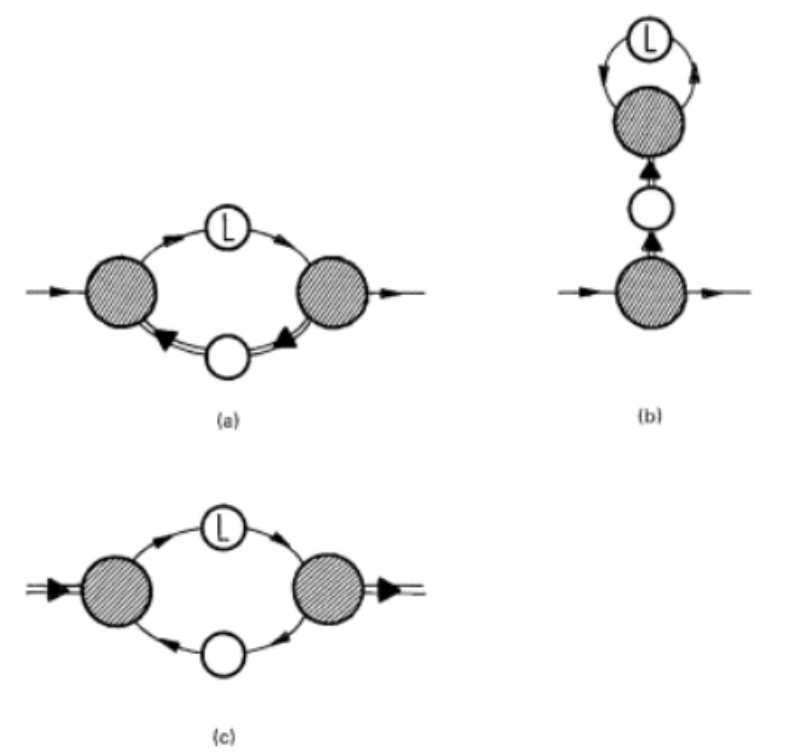

Figure 2.5: 4-points correlations functions

We observe in the c) diagram of the Fig. 2.5, how we can calculate the volume dependence of the bound state. Similarly the stable particle case, we proceed the calculation, where some new definitions are needed,

$$\nu = \pm\nu_B, \quad \nu_B = \frac{1}{2m}(m_B^2 - 2m^2) \tag{2.11}$$

where

$$\lim_{\nu\to\pm\nu_B}(\nu^2 - \nu_B^2)F(\nu) = -\frac{\nu_B}{m}g^2 \tag{2.12}$$

So we have

$$g = \Gamma(p, q; k), \quad p + q + k = 0, \quad p^2 = q^2 = -m^2, \quad k^2 = -m_B^2 \tag{2.13}$$

Now it is easy to get

$$\Delta m = -\frac{3g^2}{16\pi m_B^2 L}\{\lambda^2 e^{-\mu L} + \mathcal{O}(e^{\sqrt{\frac{3}{2}} m_B L})\}, \quad \mu = \sqrt{m^2 - \frac{1}{4}m_B^2} \tag{2.14}$$

For the modification of the mass shift of the stable particle, it can be calculated too with the help of the diagrams a) and b) in the Fig. 2.5. We do not showi this modification formula because it is given by Lüscher in [91]

2.2.3 Relation between volume dependence and scattering matrix

We will continue with the study made by Lüscher as he did not stop in the previous research but he kept going in the finite volume studies. The next study is [94] where we observe the relation between the energy eigenstate of two particles in a box of size L and the coefficient of the elastic scattering length. The definition of this scattering length is:

$$a_0 = \lim_{p \to 0} \frac{1}{2ip}(e^{2i\delta_0(p)} - 1) \tag{2.15}$$

where δ_0 is the $l = 0$ scattering phase shift as a function of relative momentum. This relation is given by

$$W = 2m - \frac{4\pi a_0}{mL^3}\{1 + c_1\frac{a_0}{L} + c_2\frac{a_0^2}{L^2}\} + \mathcal{O}(L^{-6}) \tag{2.16}$$

$$c_1 = -2.837, \quad c_2 = 6.375 \tag{2.17}$$

We are not trying to explain in detail the derivation but at least we will describe the leading order calculation. The definition of the scattering amplituted is

$$T_{nr} = -\frac{8\pi}{m}\sum_{l=0}^{\infty}(2l+1)P_l(cos\theta)t_l \tag{2.18}$$

$$t_l = \frac{1}{2ip}(e^{2i\delta_l} - 1) \tag{2.19}$$

where T_{nr} satisfies

$$T_{nr} = V(p',p) + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n} \int \frac{d^3k_1}{(2\pi)^3} ... \frac{d^3k_n}{(2\pi)^3} V(p',k_1) R_E(k_1) V(k_1,k_2) R_E(k_2) ... V(k_n,p) \tag{2.20}$$

$$V(k',k) = \int d^3z (e^{-i(k'-k)z} + e^{-i(k'+k)z}) V(z), \quad R_E(k) = 2\epsilon(k) - E - i\epsilon^{-1} \tag{2.21}$$

So if we are interested in the S-wave scaterring lenght a_0, we have

$$-\frac{8\pi}{m} a_0 = V(0,0) + ... \tag{2.22}$$

At the next to the leading order, the ingredients are that normalization of the states

$$\langle p', q' | p, q \rangle = L^6 \delta_{p',p} \delta_{q',q} + \delta_{p',q} \delta_{q',p} \tag{2.23}$$

$$\langle p', q' | V | p, q \rangle = L^3 \delta_{P',P} V(\frac{1}{2}(p'-q'), \frac{1}{2}(p-q)) \tag{2.24}$$

Collecting all, we will reach the eq. (2.16) at LO

$$W = 2m - \frac{4\pi a_0}{mL^3} + \mathcal{O}(L^{-3}) \tag{2.25}$$

For further calculation we should do the usual perturbative calculation. We find a relation between the energy of two particles in a box with the scattering length, at the momentum of the particles equal to zero. Furthermore, Lüscher derived on finite volume effect and reached an important relation in [95], where he relates the phase shift in any momentum value. As before we are not pretending to explain in detail Lüscher's study, just a brief explain to be able to understand the most important studies about this topic. The main idea was to observe the energy of a system of two particle in a box with the phase shift scattering. Different angular momentum, l, can be taken, so he could get those relations in different angular momentum. Other important issue is that the system is a box, so it is needed a relation between the cubic group and the angular momentum. As an example we will mention in the case when the state is $l = 0$, in this situation the we need the operator cubically invariant with zero total momentum ,(the A^+ sector). In this situation we can calculate the energy of the state and obtain the associated momemtum, k.

$$W = 2\sqrt{m^2 + k^2} \tag{2.26}$$

Where W is the state energy and m the mass of the particle.
Once we have calculated this associated momemtum, the relation with the phase shift is straigforward:

$$e^{2i\delta_0(k)} = \frac{\mathcal{Z}_{00}(1;q^2) + i\pi^{3/2}q}{\mathcal{Z}_{00}(1;q^2) - i\pi^{3/2}q} \tag{2.27}$$

Where

$$\mathcal{Z}_{00}(1;q^2) = \frac{1}{\sqrt{4\pi}} \sum_{n\in\mathcal{Z}^3} (n^2 - q^2)^{-s}, \quad q = \frac{kL}{2\pi} \tag{2.28}$$

The relations for other angular momentums, $l \neq 0$, we will refer to the Lüscher's paper. Although the relation looks simple but the derivation is complicated to explain here. This Lüscher's study was a milestone to calculate the phase shift. After [94], several extensions were made. For example, introduction 3particles in a box instead 2 and so on. Other important extension was made in [96]. In this study the Lüscher's relation is used but in this study the laboratory frame momentum is different than zero, because the frame is different than zero we are now able to study our system in different momentums. The key point is when the relation between the energy in the frame when the momentum is not zero, W_L and the center of mass frame, W_{CM}.

$$W_L^2 = P^2 + W_{CM}^2 \tag{2.29}$$

Where P^2 is the momentum in our frame and W_{CM} is going to have the relations in the Lüscher's paper, because his study was made in the center of mass. Maybe it is not clear why this extension is important, the reason is because we are using finite box so we have discretized possibility for the chosen momentum. Then with this extension we open more possibilities to study our system because the constrain in the momentum for the particles is vanished, now we do not need to consider just the sum of all particle momentum equal to zero. Now we can avoid this constrain and the possibility to get different associted momemtum, k, is increased.

The most important for our study and the reason why we explain those papers is that the finite volume effect in this case were not used to obtain how the energy changes in a box, in this study the volume dependence have been employed to get information in infinite volume regime. So this opens a new perspective if finite volume effect can be used to get information in infinite volume regime.

2.3 Renormalization group equation

This tool is widely employed in the literature, initially the idea is simple but inside has a lot of important QFT issues. The main idea is to calculate the variation of the operator that we study when we change some renomalized parameter in the theory. As we know from the regularization theory the bare parameters are not physical. To be able to subtract real information of the theory we need to regularize our theory . From this regularization we implement the relation of the renormalized parameters. As it is expected this relation between regularized parameter depends on which parameter we choose as the variable in our renormalization group equation. We have various renormalization group equations with different variables in the equation, such us, the lattice regularization parameter, the bare mass in the Lagrangian (Callan-Simanzik equation), minimal subtraction scheme renormalization group equation. In the previous comment, we check the variation of the renormalized parameters, which can be any kind of physical quantity, depending the range we are studying and which operator we study in terms of the those parameters, some equation will be more efficient than others.

We were speaking in general, so we proceed to the recipe for the renormalization group equation, RGE. First, in any kind of quantum field theory we need to regularize the theory. Our propose now it is not about regularize, each theory will need different kind of procedure for the regularization, although we define the cutoff parameter associated with the regularization, Λ, at the end of the calculation this Λ dependence dissapears. Then we choose the parameters which are going to be the variables for our renormalization group equation. It is so important to mention that we have to determine parameters as much as bare variables in the Lagrangian. Being (m) the number of bare variables:

$$g_R^{(1)} = \mathcal{O}^{(1)}(g_0^{(1)}, g_0^{(2)}..., g_0^{(m)}, \Lambda) \tag{2.30}$$

$$g_R^{(2)} = \mathcal{O}^{(2)}(g_0^{(1)}, g_0^{(2)}..., g_0^{(m)}, \Lambda) \tag{2.31}$$

$$g_R^{(m)} = \mathcal{O}^{(m)}(g_0^{(1)}, g_0^{(2)}..., g_0^{(m)}, \Lambda) \tag{2.32}$$

The wave function renormalization, strength field renormalization, is omitted but it has to be included. Once we have defined all the parameters, we have to determine the parameter in which we define our renormalization group equation. It is common to use the cut-off, lattice group equation, but in other cases we may use other parameter

like the μ parameter in the dimensional regularization or other parameters of the system such as the temperature or the volume. We stress again that the parameter in the RGE depends on the regularization of the system. It is called as the scheme of RGE. The final RGE, even if we use different regulator to regularize the theory, is the same when the scheme is the same. This lets us to compare different regulator results and show that the regulator is not physical because the results are not affected. At the end, we need to calculate the following parameters.

$$\beta^1(g_R^{(1)}, g_R^{(2)}..., g_R^{(m)}) = \mu_{RGE}\frac{dg_R^{(1)}}{d\mu_{RGE}} \tag{2.33}$$

$$\beta^2(g_R^{(1)}, g_R^{(2)}..., g_R^{(m)}) = \mu_{RGE}\frac{dg_R^{(2)}}{d\mu_{RGE}} \tag{2.34}$$

$$\beta^m(g_R^{(1)}, g_R^{(2)}..., g_R^{(m)}) = \mu_{RGE}\frac{dg_R^{(m)}}{d\mu_{RGE}} \tag{2.35}$$

where μ_{RGE} is the RGE parameter.
When the physical parameter is not dimensionless, such as the renormalized mass, the above equation are modified as

$$g_R^{(i)}\beta^i(g_R^{(1)}, g_R^{(2)}..., g_R^{(m)}) = \mu_{RGE}\frac{dg_R^{(i)}}{d\mu_{RGE}} \tag{2.36}$$

Now we have to add more definitions coming from the wave function renormalization, strength field renormalization,

$$\gamma^1(g_R^{(1)}, g_0^{(2)}..., g_0^{(m)}) = \mu_{RGE}\frac{d\log(Z_{\phi^{(1)}})}{d\mu_{RGE}} \tag{2.37}$$

$$\gamma^2(g_R^{(1)}, g_0^{(2)}..., g_0^{(m)}) = \mu_{RGE}\frac{d\log(Z_{\phi^{(2)}})}{d\mu_{RGE}} \tag{2.38}$$

$$\gamma^k(g_R^{(1)}, g_0^{(2)}..., g_0^{(m)}) = \mu_{RGE}\frac{d\log(Z_{\phi^{(k)}})}{d\mu_{RGE}} \tag{2.39}$$

With all the definitions for our renormalization group equation, we obtain:

$$[\mu_{RGE}\frac{\partial}{\partial\mu_{RGE}} + \beta^1 + ... + g_R^{(i)}\beta^i + ... + \beta^m + n_1\gamma_1 + n_k\gamma^k]\Gamma_R^{n_1,...,n_k}[p_1, ..., g_R^{(1)}, ..., g_R^{(m)}]$$
$$(\mu_{RGE}\frac{d}{d\mu_{RGE}}\Gamma_0^{n_1,...,n_k}[p_1, ..., g_R^{(1)}, ..., g_R^{(m)}])/(Z_{\phi_1}^{n_1}\cdot ... \cdot Z_{\phi_k}^{n_k}) \tag{2.40}$$

The way to obtain this equation is straightforward, we have defined

$$\Gamma_R^{n_1,...,n_k}[p_1, ..., g_R^{(1)}, ..., g_R^{(m)}] = (Z_{\phi^{(1)}})(Z_{\phi^{(2)}})...(Z_{\phi^{(k)}})\Gamma_R^{n_1,...,n_k}[p_1, ..., g_R^{(1)}, ..., g_R^{(m)}] \tag{2.41}$$

So the l.h.s of the eq. (2.40) is coming from the dependence of each variable by the RGE parameter and the r.h.s is the total derivative divide by Z factor which are coming to be able to get the log derivative of the Z factors. It seems that we are not earning much we this definition but as we will so soon this non-homogenous term is an important tool to be sure about the cutoff independence of the γ and β functions. The most used RGE is the one we define an external parameter, μ, for which this non-homogeneous term vanishes, this scheme is possible in dimensional regularization.

2.3.1 Callan-Symanzik RGE in $\lambda\phi^4$ theory

The propose of this equation is to obtain the variation of our theory by the change of the renormalized mass. To obtain this formula we will differentiate a contribution of each diagram by the renormalised mass and use the next expression

$$\frac{d}{d(m_0^2)}\langle\mathcal{O}\rangle = \langle\frac{1}{2}\int\phi(x)^2 dx\mathcal{O}\rangle_{connected} \tag{2.42}$$

Now we can obtain the well know Callan-Simanzik equation. which is non-homogeneous differential equation and the nonhomogeneous term comes from the previous equation, which will be explain in detail. The other terms will come from the derivation by the bare mass of the renormalised constant or, in other words, the constant defined in our scheme.

$$(m\frac{d}{dm} + \beta\frac{d}{dg} + \gamma)\langle\mathcal{O}\rangle_R = \alpha\langle\frac{1}{2}\int\phi(x)^2 dx\mathcal{O}\rangle_{R\quad connected} \tag{2.43}$$

The meaning of "connected" will be clarified later. In the left hand side of the equation, β, γ, α depends on the definition we impose for the RGE variable, for our actual propose this definition is not playing a crucial role because our goal is to generate the nonhomogeneous term, due to the derivation by the bare mass. Remembering, in the volume size equation, the other terms and functions are defined in the similar way, just changing the parameter of the scheme.
To prove the eq. (2.42), we will use two kind of approaches:

1. Perturbation theory formalism.

2. Path integral formalism

Each one gives us the knowledge to be applied in our propose of the renormalization equation by volume dependence.

2.3.1.1 Callan-Symanzik by perturbation theory formalism

We first take a look what the non-homogenous term means. The meaning is that we can decompose the interaction by Feynman diagrams, so if we obtain the same formula to all orders, that will mean it is preserved in non-perturbative regime. As an example we will draw one Feynman diagram in a $\lambda\phi^4$ scalar theory.

$$\int d^d x\Big((\partial_\mu\phi)^2 + m_0^2\phi^2 + \lambda\phi^4\Big) \tag{2.44}$$

So with the previous Lagrangian we can get the diagram of the figure 2.6.
The figure 2.6 gives the integral

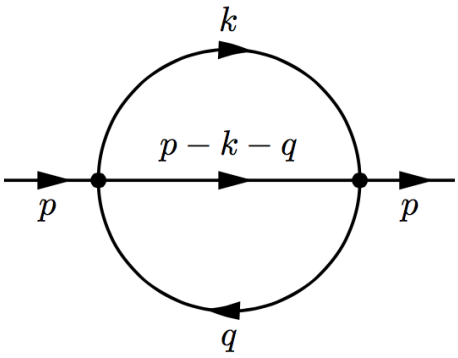

Figure 2.6: Feynmann diagram at second order of $\lambda\phi^4$ scalar theory

$$\frac{1}{p^2-m_0^2}\left(\int dk^4 dq^4 \frac{1}{(p-k-q)^2-m_0^2}\frac{1}{k^2-m_0^2}\frac{1}{q^2-m_0^2}\right)\frac{1}{p^2-m_0^2} \tag{2.45}$$

By deriving Eq (2.45) by m_0 we find that it depends on the derivative of each propagator. Then we find one operator which produces that effect in each propagator separately.

$$\frac{d}{d(m_0^2)}\frac{1}{p^2-m_0^2} = \frac{1}{p^2-m_0^2}\frac{1}{p^2-m_0^2} \tag{2.46}$$

Fortunately, the propagator which produces this effect is known and with a little calculation it is easy to obtain.

$$\int e^{-ipy}\langle\frac{1}{2}\int dx\phi(x)^2\phi(y)\phi(0)_{connected}\rangle dy =$$

$$\int e^{-ipy}\frac{e^{ik_1(x-y)}}{k_1^2-m_0^2}\frac{e^{ik_2(x)}}{k_2^2-m_0^2}dxdy =$$

$$\int e^{-ipy-ik_1y}\frac{1}{k_1^2-m_0^2}\frac{1}{k_1^2-m_0^2}dy = \frac{1}{p^2-m_0^2}\frac{1}{p^2-m_0^2} \tag{2.47}$$

Where we knew that

$$\int e^{-ipy}\langle\phi(y)\phi(0)_{connected}\rangle dy = \frac{1}{p^2-m_0^2} \tag{2.48}$$

Finally, we reach the previous formula announced before and needed for the Callan-Symanzik equation. The meaning of "connected" can be shown now.The relation

$$\langle\phi(x)^2\phi(y)\phi(z)\rangle = \langle\phi(x)\phi(y)\rangle\langle\phi(x)\phi(z)\rangle + \langle\phi(x)\phi(x)\rangle\langle\phi(y)\phi(z)\rangle \tag{2.49}$$

But we were used the expression:

$$\langle\phi(x)^2\phi(y)\phi(z)\rangle_{connected} = \langle\phi(x)\phi(y)\rangle\langle\phi(x)\phi(z)\rangle \tag{2.50}$$

We have to consider a general diagram and find the derivative of each propagator will give the same effect, for instance a diagram like the figure 2.8, We notice that the operator $\phi(x)^2$ always has to be connected to the original diagram and that is why we only take the connected diagrams.

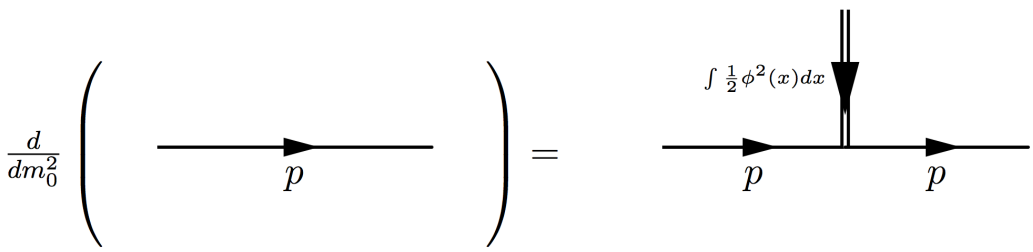

Figure 2.7: Derivation of the propagator by the bare mass

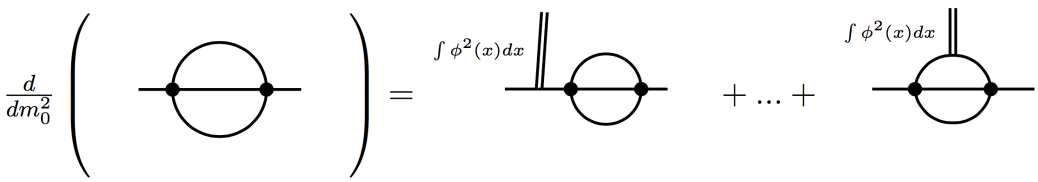

Figure 2.8: Derivation of one feynman diagram by the bare mass

Finally we obtain.

$$\frac{d}{d(m_0^2)}\langle\mathcal{O}\rangle = \langle\frac{1}{2}\int\phi(x)^2dx\mathcal{O}\rangle_{connected} \tag{2.51}$$

Where $\mathcal{O}$ can be any kind of operator.

2.3.1.2 Callan-Symanzik by the path integral formalism

In this case the previous formula will be obtained directly using the path integral approach. In this approach it is clearer the concept of "connected" diagram, presented above. The path integral formalism is based on the statistical physics. The way to obtain the vacuum expectation values of an operator is based on this formula:

$$\langle \mathcal{O} \rangle = \frac{\int D\phi \mathcal{O} e^{-S}}{\int D\phi e^{-S}} \tag{2.52}$$

$$S = \int \frac{1}{2}(\frac{d\phi(x)}{dx})^2 - \frac{1}{2}m_0^2\phi(x)^2 + \mathcal{L}_I dx \tag{2.53}$$

The unique term depending on mass in the action is the quadratic term. In this calculation we have to consider that integration terms does not depends on the mass. This consideration is obvious but as we will show later this consideration will be important for the other parameters.

$$\frac{d}{d(m_0^2)} S = \int \frac{1}{2}\phi(x)^2 dx \tag{2.54}$$

$$\begin{aligned} \tfrac{d}{d(m_0^2)}\langle \mathcal{O} \rangle &= \frac{\int D\phi \frac{dS}{d(m_0^2)} \mathcal{O} e^{-S}}{\int D\phi e^{-S}} - \frac{\int D\phi \frac{dS}{d(m_0^2)} e^{-S}}{\int D\phi e^{-S}} \frac{\int D\phi \mathcal{O} e^{-S}}{\int D\phi e^{-S}} = \\ &= \langle \tfrac{1}{2}\int \phi(x)^2 dx \mathcal{O} \rangle - \langle \tfrac{1}{2}\int \phi(x)^2 dx \rangle \langle \mathcal{O} \rangle \end{aligned} \tag{2.55}$$

At the first sight, we think the result is not the same, but actually it is the same. The difference resides in the second term but it cancels the non-connected diagrams which are taken into account in the first term.

2.3.1.3 Renormalization parameter definitions and conclusions

We mentioned the expression of the renormalization group equation, but we have not defined the parameters in the RGE. Before we have said that RGE was in terms of μ_{RGE}, but the bare mass, m_0, is not a real parameter in the theory, so by a simple changes of variables by the renormalized mass, m, we can get the desired RGE. So proceeding to the definition,:

$$\beta = m\frac{d}{dm}[\lambda(\lambda_0, \Lambda/m)]_{\Lambda,\lambda_0} \tag{2.56}$$

$$\gamma = -\frac{1}{2}\frac{d}{dm}[logZ_\phi(\lambda_0, \Lambda/m)]_{\Lambda,\lambda_0} \tag{2.57}$$

Then the the Calla-Symanzik equation is given by

$$(m\frac{d}{dm} + \beta\frac{d}{d\lambda} + \gamma_{\mathcal{O}})\langle\mathcal{O}\rangle_R = \alpha(m,\lambda)\langle\frac{1}{2}\int \phi(x)^2 dx\mathcal{O}\rangle_R \quad {\scriptstyle connected} \tag{2.58}$$

Transforming to the Γ functions:

$$(m\frac{d}{dm} + \beta\frac{d}{d\lambda} + \frac{n}{2}\gamma)\Gamma^n(p^2;\lambda,m) = \alpha(m,\lambda)\Gamma^n_{\phi^2}(p_{\phi^2} = 0, p^2;\lambda,m) \tag{2.59}$$

where $\alpha(m,g)$ is a function As it was explained before, we are not sure that the above equation is cutoff independence. That can be shown by using the fact that the renormalization scheme gives us the following renormalization conditions,

$$\Gamma^{(2)}_R(p^2 = 0;\lambda,m) = m^2 \tag{2.60}$$

$$\Gamma^2_{\phi^2}(p_{\phi^2} = 0, p^2 = 0;\lambda,m) = 1 \tag{2.61}$$

$$\frac{d\Gamma^2_R(p^2;\lambda,m)}{dp^2}\rfloor_{p^2=0} = -1 \tag{2.62}$$

These conditions and properties of Γ^n_R and $\Gamma^n_{\mathcal{O}}$ which are cutoff independent lead us to the condition that β and γ functions are cutoff independent. In this case, we consider a theory where the unique dimensional parameter is the mass m, the β and γ functions just depend on the coupling λ because they are dimensionless functions.

2.3.2 Renormalization group equation MS-scheme in a finite volume

This renormalization is based on the dimensional regularization. The reason why this RGE is widely study is because the r.h.s in the RGE, non-homogenous term, is zero. Thanks to non-homogenous term is zero the RGE can be integrated and a simple form in the equation is within easy reach. Departing from the RGE:

$$[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \frac{n}{2}\gamma(g)]\Gamma^{(n)}(p,g,L,\mu) = 0 \tag{2.63}$$

It is easy to solve the by characteristics method:

$$\Gamma^{(n)}(p, g, L, \mu) = F^n(g, s)\Gamma^{(n)}(p, g(s), L, s\mu) \tag{2.64}$$

Where

$$s\frac{d}{ds}g(s) = \beta(g), \quad g(1) = g^* \tag{2.65}$$

$$s\frac{d}{ds}\log(F(g, s)) = \gamma(g), \quad F(1) = 1 \tag{2.66}$$

Thanks of this RGE we can get information from the volume dependence as in [90]. Before we just wrote the simplest formula, but for more couplings or parameters in theory, an extension is possible. We will keep simple just to explain the most interesting points in this RGE. Now we can change the variables $sL\Lambda = 1$. By a simple dimension consideration we can say:

$$\Gamma^{(n)}(p, g, L, \Lambda) = \Lambda^{d-n(d-2)/2}\Gamma^{(n)}(p/\Lambda, g(s), L\Lambda, 1) \tag{2.67}$$

Then if we mix the relation for the amputed green function, we get:

$$\Gamma^{(n)}(p\Lambda, g, L, \mu) = \Lambda^{d-n(d-2)/2}e^{-n\int_0^s \gamma(g(g*,t))dt}F^n(g, s)\Gamma^{(n)}(p, g(g^*, s), Ls, \mu) \tag{2.68}$$

This is a general equation, although this expression can be used in regime where a phase transition occurs. The reason is because when the phase transtion happens the IR fixed point is reached. Then when we get this regime, we can observe that:

$$e^{-n\int_0^s \gamma(g(g,t))dt} \sim (s)^\eta \tag{2.69}$$

$$\Gamma^{(n)}(p, g_{IR}, L, 1) \sim L^{-d+n(d-2)/2+\eta}\Gamma^{(n)}(Lp, g_{IR}, 1, 1) \tag{2.70}$$

For more detail explanation we refer the readers to the Brezin [90]. It is of critical importance to mention that this RGE scheme is not going to be possible to use for us, because this RGE is not extrapolated to lattice regularization. Which is going to be our regularization for the non-perturbative calculation. That is why we have to define our RGE depending for other parameter, volume.

2.4 Renormalization group equation for volume dependence

So far we have explained the most common renormalization group equations. Now we present other kind of RGE. We use a new RGE because the previous one RGE will not fit in our propose. We want to define the RGE in the non-perturbative regime in a theory where there is no bare dimensional parameter. The reason of this propose is because in many interesting physics theories like, the sigma model, the Yangs-Mills theory and QCD in chiral limit, the parameters are dimensionless, so the Callan-Symanzik equation is not suitable. For non-perturbative calculation via Monte Carlo simulation, we use lattice regularization. We consider RGE which can be used in any scheme and we choose as RGE variable the volume where the system lives in. As it was mentioned in the previous sections the finite volume is commonly used in theories where there is no bare mass in the Lagrangian. It is used to avoid the IR divergences, but at the same time in our RGE the volume of the system is going to be used as a variable in the renormalization group equation.

2.4.1 Definition

The definition is straightforward by following the ideas we exposed at the beginning of the section. The RGE variable in this case is the volume of the system so:

$$\beta^{(i)}(g_R^{(1)}, g_R^{(2)} ..., g_{0R}^{(m)}) = -L\frac{dg_R^{(i)}}{dL} \tag{2.71}$$

$$\gamma^j(g_R^{(1)}, g_R^{(2)} ..., g_R^{(m)}) = -L\frac{d(log(Z^{(j)}))}{dL} \tag{2.72}$$

Where (i)'s are the different RGE variables, whose number of them correspond with the number of bare parameter. In the case of (j)'s is the number of different field in the theory.

This scheme for the β-function was first defined by Lüscher in [41], where the parameter g_R in the sigma model is related to the mass gap. Then it is easy now to get the equation

$$[-L\frac{\partial}{\partial L} + \beta^i + ... + n_j\gamma^j]\Gamma^{n_1,...,n_k}[p, g_R^{(i)}] = (-L\frac{d}{dL}\Gamma^{n_1,...,n_k}[p, g_R^{(i)}])/(\prod_j Z_{\phi_j}^{n_j}) \tag{2.73}$$

In the previous equation we have not defined the $g_R^{(i)}$ and $Z^{(i)}$. Those definitions can vary depending on the theory, but at the end the RGE will be the same, so we just need the r.h.s as in the case of Callan-Symanzik equation to get the RGE with volume dependence.

2.4.2 Non-homogenous term

To obtain the Eq. (2.73), we do derivative the same way to that in the CS equation. We consider the derivative with respect to the size of our system in this case. As Lüscher's procedure to obtain the non-perturbative renormalization scheme. We do not discuss about the β functions and so on, but we focus on the non-homogeneus term in RG equation.

As we showed before, the path integral formalism is quite good if the integration variable does not depend on the variable which we derive. But this is not the case, so we must use perturbation theory.

At the begining we have to introduce some concepts of finite size effects. There are some changes from the theory at infinite volume. The most important difference between infinite volume and finite volume is to replace

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dp^i = \sum_{n=-\infty}^{n=\infty} \frac{1}{L} \tag{2.74}$$

$$p = \frac{2\pi}{L} n \qquad n \in \mathbb{Z} \tag{2.75}$$

So if we want to calculate the Feynman diagram, we have to modify the previous prescriptions. As a example we will show the Feynman diagram in the figure 2.9. Which gives

$$\frac{1}{(p^0)^2+(\frac{2\pi n}{L})^2-m^2}\Big(\sum_m \frac{1}{L}\sum_l \frac{1}{L}\int dk^0 dq^0 \frac{1}{(p^0-k^0-q^0)^2+(\frac{2\pi n}{L}-\frac{2\pi l}{L}-\frac{2\pi m}{L})^2-m^2} \frac{1}{(q^0)^2+(\frac{2\pi m}{L})^2-m^2}\frac{1}{(k^0)^2+(\frac{2\pi l}{L})^2-m^2}\Big)\frac{1}{(p^0)^2+(\frac{2\pi n}{L})-m^2} \tag{2.76}$$

What we have made is just to use the previous prescriptions, notice that the momemtum is $p^2 = (p^0)^2 + (p^1)^2$ and just the second momentum is discretized becuase just the associated component x_1 was made finite. For the case of more dimensions: $p^2 = (p^0)^2 + (p^1)^2 + ... + (p^n)^2$, and we have to change to discretized momentum when its associated component is finite.

It is easy to realize that the volume dependence now is quite different from that in CS equation, mass dependence. we get this dependence from difference sources. It is split in three kinds of dependence:

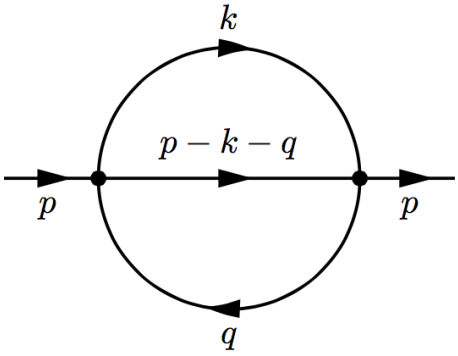

Figure 2.9: Feynmann diagram at second order of $\lambda\phi^4$ scalar theory

1. From the propagator
$$\frac{1}{(p^0)^2 + (\frac{2\pi n}{L})^2 - m^2} \tag{2.77}$$

2. From the summation in the loop
$$\sum \frac{1}{L} \tag{2.78}$$

3. If the coupling depends on the such us $\mathcal{L}_I = g(\frac{d\phi(x)}{dx})^k(\phi(x))^j$, from that factor
$$g(\frac{d}{dx})^k = g(\frac{2\pi n}{L})^k \tag{2.79}$$
(Notice that in previous diagram we do not have such interaction term but we take into account for our general case)

So when we operate $L\frac{d}{dL}$ we have to take account of all these contributions.

2.4.2.1 Propagator contribution

In this case the procedure is quite similar to the Callan-Simanzik equation
$$L\frac{d}{d(L)}\frac{1}{(p^0)^2 + (\frac{2\pi n}{L})^2 - m^2} = \frac{1}{(p^0)^2 + (\frac{2\pi n}{L})^2 - m^2}\frac{1}{(p^0)^2 + (\frac{2\pi n}{L})^2 - m^2}(\frac{2\pi n}{L})^2 \tag{2.80}$$
But in this time we have another term, which can be obtained by the Feynman diagram in the Figure 2.10.

Special mention for the $\sum_i \frac{d\phi(x)}{dx^i}$, is necessary that the summation indicates in the

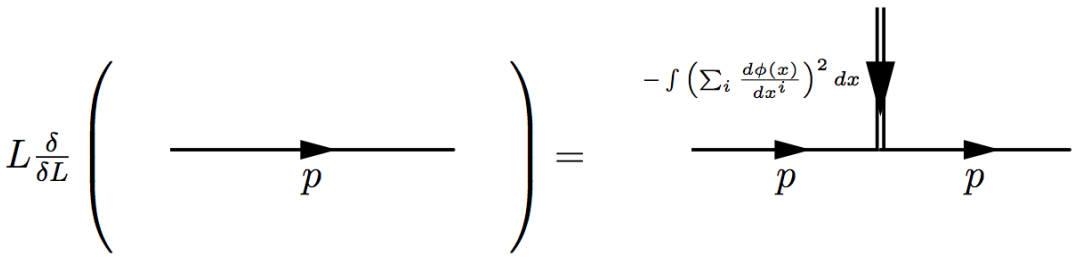

Figure 2.10: Derivation of the propagator by volume size

dimensions where the size is finite and we are doing the derivation.

$$L\frac{d}{dL}\langle\mathcal{O}\rangle = \langle\int\left(\frac{d\phi(x)}{dx^i}\right)^2 dx\mathcal{O}\rangle_{connected} \tag{2.81}$$

For any kind of operator, $\mathcal{O}$

2.4.2.2 Loop and interaction term with spatial derivative contribution

It is important to realize the sources of the dependence, which are from the loops, counting the dimensions that is finite, and the operator $\frac{d\phi(x)}{dx}$ in the Lagrangian interaction. So the factor which we have to analyze it is in the next way.

$$g^V\frac{1}{L^{dl+Vk}} \tag{2.82}$$

Notice that, d is the dimension where the size is finite. V, the number of vertex in the diagram. l, the number of loops in the diagram. k, t derivative in the components space in the interaction, $(\frac{d\phi(x)}{dx})^k$. To obtain the dependence we just need to derive,

$$L\frac{d}{dL}(g^V\frac{1}{L^{dl+Vk}}) = -(dl+Vk)(g^V\frac{1}{L^{dl+Vk}}) \tag{2.83}$$

Now in QFT, there is a relation between the loops and number of the vertex, V. This relation is kept if the external lines are the same. This condition "kept if the external lines are the same" is going to be important because thanks of this relation we can relate the L dependence with the coupling constant dependence, g, for any kind of diagram. In general case we will write the next relation.

$$l = mV + a \tag{2.84}$$

Where m depends on the vertex and a will depend on the external lines. Therefore we can change the volume derivative by the coupling

$$\begin{gathered} -((k+dm)g\tfrac{d}{dg} + da)(g^V \tfrac{1}{L^{dl+Vk}}) = -((k+dm)V + da)(g^V \tfrac{1}{L^{dl+Vk}}) = \\ -(dl+kV)(g^V \tfrac{1}{L^{dl+Vk}}) = L\tfrac{d}{dL}(g^V \tfrac{1}{L^{dl+Vk}}) \end{gathered} \tag{2.85}$$

We can see that this operator $-((k+dm)g\frac{d}{dg} + da)$ does not depend on the diagram we are calculating. We can say that operator is the same for all kind diagram in this theory with the same external link, getting the conclusion that the non-perturbative cases also obeys. Now we need to know what operator correspond to the derivative. We have to mention that this derivative is which respect to the bare coupling. For this purpose we examine the Callan-Simanzik equation in the path integral formalism.

2.4.2.3 Contributions from the derivative by the bare coupling

As we suspect, this calculation is going to be straightforward like in the CS equation. What we need to do is to derive of our Lagrangian as only this term depends on the coupling constant.

$$\begin{gathered} S = S_0 + S_I \qquad S_I = \int g\phi^n(x)(\tfrac{d\phi(x)}{dx})^m dx \\ g\tfrac{d}{dg}S = S_I \end{gathered} \tag{2.86}$$

$$\begin{gathered} g\tfrac{d}{dg}\langle\mathcal{O}\rangle = \tfrac{\int D\phi g\frac{dS}{dg}\mathcal{O}e^{-S}}{\int D\phi e^{-S}} - \tfrac{\int D\phi g\frac{dS}{dg}e^{-S}}{\int D\phi e^{-S}}\tfrac{\int D\phi\mathcal{O}e^{-S}}{\int D\phi e^{-S}} = \\ = \langle S_I\mathcal{O}\rangle - \langle S_I\rangle\langle\mathcal{O}\rangle = \\ = \langle S_I\mathcal{O}\rangle_{connected} \end{gathered} \tag{2.87}$$

We do not comment more about the meaning of connected but we mention the importance of this concept. Now we are prepared to combine all the information and get the final result.

2.4.3 RGE volume dependence with various couplings

We have already got all the tools to obtain the non-homogeneous term in our equation. Hence gathering all the results

$$L\frac{d}{dL}\langle\mathcal{O}\rangle = \langle(-\int(\frac{d\phi(x)}{dx})^2dx - (k+dm)S_I + da)\mathcal{O}\rangle_{connected} \tag{2.88}$$

We will try to obtain this result non perturbatively because there is no proof for non-perturbative calculations. For this propose the lattice field theory is used. Because the connected and disconnected diagrams are not distinguished so the next expression will be used in lattice.

$$L\frac{d}{dL}\langle\mathcal{O}\rangle = \langle(-\int(\frac{d\phi(x)}{dx})^2dx-(k+dm)S_I+da)\mathcal{O}\rangle-\langle(-\int(\frac{d\phi(x)}{dx})^2dx-(k+dm)S_I)\rangle\langle\mathcal{O}\rangle \tag{2.89}$$

The definition of each constant comes from the previous sections. We notice that this Lagrangian only have one coupling constant, but in the case where the Lagrangian has different couplings, we will need to know the relation between the loop and the vertex. As a general case we show here

$$l = mV + m'V' + m''V'' + ... + a \tag{2.90}$$

Then the equation will be modified as follows

$$\begin{aligned} L\tfrac{d}{dL}\langle\mathcal{O}\rangle = \langle(-\textstyle\int(\tfrac{d\phi(x)}{dx})^2dx - (k+dm)S_I - (k'+dm')S'_I - ... - da)\mathcal{O}\rangle- \\ \langle(-\textstyle\int(\tfrac{d\phi(x)}{dx_i})^2dx - (k+dm)S_I - (k'+dm')S'_I)\rangle\langle\mathcal{O}\rangle \end{aligned} \tag{2.91}$$

And this expression is written in terms of the Γ amputed functions:

$$(m\frac{d}{dm}+\beta^i\frac{d}{dg}+n\gamma^j)\Gamma^n(p^2;g_R^i,L) = \Gamma^n_{(\partial_{x_i}\phi)^2+(k_r+dm_r)S_I^r}(p*=0,p^2;g_R^i,L)-da\Gamma^n(p^2;g_R^i,L) \tag{2.92}$$

As it is expalined in the Callan-Simanzik equation, this relation will lead to the cutoff independence for this scheme. Then β and γ functions in this RGE have dependence only on the coupling if in out theory there is no bare parameter with dimensions in the Lagrangian. This is going to be so important because thanks of this condition, β and γ are L independent in the O(N) sigma model

2.4.4 RGE volume dependence in $\lambda\phi^4$ theory

The previous calculation was made in a general case, before use the final formula in the O(N) sigma model is convenient to use this renormalization group equation formalism for finite volume dependence in a more simple theory, like $\lambda\phi^4$. In the situation for O(N) sigma model we just consider the situation when the dimension was smaller than 2 because this theory is only regularizable for those dimensions. But in the case

of $\lambda\phi^4$ theory we can consider the dimension up to 4. That is why in this situation we will consider that our system live in a manifold, M, where $M = \mathbb{R}\times\mathbb{S}\times\mathbb{S}\times\mathbb{S}$. All the spheres of dimension 1 depends on L, being L the volume of each sphere. That is why our operator is modified by

$$\frac{1}{p_0^2 + (\frac{2\pi n_1}{L})^2 + (\frac{2\pi n_2}{L})^2 + (\frac{2\pi n_3}{L})^2 + m_0^2}, \quad n_1, n_2, n_3 \in \mathbb{Z}, \quad p_0 \in \mathbb{R} \tag{2.93}$$

And each loop integration

$$\int dp^1 dp^2 dp^3 = \frac{1}{L^3}\sum_{n_1,n_2,n_3} \tag{2.94}$$

So if we are interested on the calculation of 2-point correlation function, the relation between the quantity of loops and interaction terms is one to one. That is why

$$L\frac{d}{dL}\Big(\int dx e^{-ip\cdot x}\langle\phi(x)\phi(0)\rangle\Big) =$$

$$\langle\Big[-\int dx'\Big((\partial_1\phi(x'))^2+(\partial_2\phi(x'))^2+(\partial_3\phi(x'))^2\Big)dx'-3\lambda\phi^4(x')\Big]\Big(\int dx e^{-ip\cdot x}\phi(x)\phi(0)\Big)\rangle_{connected} \tag{2.95}$$

where the first volume dependence term is coming from the propagator dependence and the second term from the loop dependence. In this situation the calculation was straightforward because we have only one interaction term and we do not need to reparametrize the theory as O(N) sigma model, where the calculation is going to be more complicated.

From the equation we observe that to renormalize this insertion operator is enough to consider the renormalize factor 2-point correlation itself, because those insertions operators will not give us extra divergence. This is important because thanks of this,we can conclude that RGE for volume dependence is justified.

Chapter 3

Sigma model O(N) in a finite box with perturbation theory

Now we will proceed to the calculation of the O(N) sigma model by using perturbation theory (PT). To be able to do the calculation, first we need to explain the methodology to calculate the correlation function in the O(N) sigma model. In this methodology a delicate issue is related with the IR divergences because we have in this theory a massless field. As it was mentioned before we have several ways to avoid the IR divergences, although we will just use one kind regularization because one of our propose is to use our results in the finite volume scheme. So in this chapter, we start with the definition of the sigma model. After this, we will explain in detail some technical aspects in the methodology to get the correct results. Then we describe the correlation function and the vacuum energy density. Similar calculation was already performed [65], and extrapolated to 3 and 4 dimension in [69]. Those calculations were made in lattice regularization and transformed in dimensional regularization by the relation between both regularization. One present study in dimensional regularization was made using dimensional regularization in 2, 3 and 4 dimensions [70, 71]. But in all of those studies, only the mass gap was calculated therefore the β-function, so we proceed in the calculation of the Z factor to be able to calculate the γ-function in this finite volume scheme.

3.1 O(N) sigma model PT

We have explained before different parametrizations give us the same result for the perturbative calculation. The north pole parametrization gives the most simple Lagrangian and operators to calculate the propagator of 2-point correlation function. We use this parametrization in perturbative calculation. In other cases, the other kind parametrization may be more useful as we observe later.

3.1.1 Lagrangian in perturbation theory

The Lagrangian follows from the definition $S^i = g_0\pi$ when $i \neq N$ and $S^N = g_0^2\sqrt{1-g_0^2\pi^i\pi^i}$, generating the interaction Lagrangian:

$$\mathcal{L}_0 = \int (\sum_{i=1}^{N-1}\frac{1}{2}\partial_\mu\pi_i(x)\partial^\mu\pi_i(x) + g_0^2\frac{1}{2}\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\pi_j(x)\partial_\mu\pi_j(x)\pi_i(x)\partial^\mu\pi_i(x)+$$

$$(g_0^2)^2\frac{1}{2}\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\sum_{k=1}^{N-1}\pi_j(x)\partial_\mu\pi_j(x)\pi_i(x)\partial^\mu\pi_i(x)\pi_k^2(x) + ...)dx \tag{3.1}$$

Adding the interaction to avoid the zero mode by Hasenfratz's method [40], the IR divergences are avoided.

$$\mathcal{L}_{I(Zeromodes)} = \int (\sum_{i=1}^{N-1}\frac{1}{2TL}(N-1)g_0^2\pi_i^2(x)+\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\frac{1}{4TL}(N-1)(g_0^2)^2\pi_i^2(x)\pi_j^2(x)+...)dx \tag{3.2}$$

Another important interaction, which we should care, is the interaction coming from the integral measure and parametrization, which is written:

$$\mathcal{L}_{I(measure)} = -\delta^d(0)(\int (\sum_{i=1}^{N-1}\frac{1}{2}g_0^2\pi_i^2(x) + \sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\frac{1}{4}(g_0^2)^2\pi_i^2(x)\pi_j^2(x) + ...))dx \tag{3.3}$$

This measure term is coming from the field integration in the path integral formalism, $\int DS^N$. The value of $\delta^2(0) = \delta_{x_0}(0)\delta_{x_1}(0)$, depends on the regularization employed, in the case of the dimensional regularization. This value can be explained in infinite

volume by the usual regularization of Gamma distribution.

$$\delta^d(0) = \int_{\infty}^{\infty} d^d p \int_{\infty}^{\infty} d^d p \frac{p^2+m^2}{p^2+m^2} = \int_{\infty}^{\infty} d^d p (p^2+m^2) e^{-(p^2+m^2)} =$$

$$2^{-d}(m^2)^{d/2}\pi^{-d/2}\Big(\frac{d}{2}\Gamma[-d/2] + \Gamma[1-d/2]\Big) = 0 \tag{3.4}$$

Where $\Gamma(x)$ is the Gamma distribution. This regularization is based on the analytic continuation in the complex plane of this distribution. For our calculation we are using dimensional regularization, but for finite volume we have to regularize other function, Riemann zeta function, as in the appendix A.1 is calculated. In this case the Dirac delta function, the time component has a free boundary condition and in the spatial component PBC. In both boundaries, the results are the same, showing that the $\mathcal{L}_{measure}$ term is vanish in dimensional regularization for any scheme.

$$\delta^2(0) = \delta_{x_0}(0)\delta_{x_1}(0) = 0 \tag{3.5}$$

3.1.2 Operator in perturbation theory

The issue to consider now is related with the operator to calculate. For the mentioned parametrization, the operator we should study is presented below.

$$\langle S^i(x)S^i(y)\rangle = 1 + g_0^2(\langle \pi^i(x)\pi^i(y)\rangle - \frac{1}{2}\langle(\pi^i(x))^2\rangle - \frac{1}{2}\langle(\pi^i(y))^2\rangle)+$$

$$\frac{g_0^4}{8}(2\langle(\pi^i(x))^2(\pi^j(y))^2\rangle - \langle(\pi^i(y)^2(\pi^j(y))^2\rangle - \langle(\pi^i(x))^2(\pi^j(y)))^2\rangle)+$$

$$\frac{g_0^6}{16}(\langle(\pi^i(x))^2(\pi^k(x)^2(\pi^j(y))^2\rangle + \langle(\pi^i(x))^2(\pi^j(y))^2(\pi^k(y))^2\rangle-$$
$$\langle(\pi^i(x))^2(\pi^j(x))^2(\pi^k(x))^2\rangle - \langle(\pi^i(y))^2(\pi^j(y))^2(\pi^k(y))^2\rangle) \tag{3.6}$$

3.2 Technical aspects

The calculation of the previous operator is not going to be straight forward. As shortly commented before the temporal coordinate is going to be infinite in the end so one

can expect the result does not depend in the boundary condition in the temporal coordinate, but as it is mention in [65], [69], this conjectured is wrong. So one should use free boundary condition (in Lattice regularization), what it translates in continuum formalism (dimensional regularization) Newmann boundary condition equal to zero.
At this boundary condition, the zero mode is still present so the Hasenfratz method is required. The boundary condition are:

$$\frac{dS^i(-\frac{T}{2},x_1)}{dx_0} = \frac{dS^i(\frac{T}{2},x_1)}{dx_0} = 0, \quad S^i(x) = S^i(x+L), \quad x \in [-L/2, L/2] \times [-T/2, T/2] \tag{3.7}$$

The reason of this boundary condition in the time component is that if one uses the periodic boundary condition the higher isospin transitions also contribute in the perturbative calculation, being this method useless.

For the perturbative calculation, we are going to use dimensional regularization where the time component is non-integer and the spatial component is integer. We can calculate the other way around but with this scheme the function we get it is easier to calculate, regularize, although we know this choice will not change the final result [108].

3.2.1 Free propagator

The usual perturbation theory is made in the feymann diagrams in the momentum space, but due to the IR divergence the calculation is made in a mix between coordinate space and momentum space, as it is described in [70]. For this calculation, the next definitions are needed:

$$G(x, y, p_1 = 0) = \frac{1}{L}\left(\frac{T}{12} - \frac{|x_0 - y_0|}{2} + \frac{x_0^2 + y_0^2}{2T}\right) \tag{3.8}$$

And for the case $p_1 \neq 0$, the IR divergence does not appear so in this case we could think we take for the time coordinate infinite, but that it is not the case because in this Lagrangian the interaction terms have derivatives so we can not extrapolate to infinite time coordinate. In this situation we will get the next propagator:

$$G(x_0, x_1, p_1) = \frac{1}{L}\sum_m \frac{1}{2|p_1|}\left(e^{-|p_1||(x_0-y_0)+2mT|+ip(x_1-y_1)} + e^{-|p_1||(x_0+y_0)+(1+2m)T|+ip(x_1-y_1)}\right)$$

$$p_1 = \frac{2\pi n}{L} \quad n \in \mathbb{Z} - \{0\} \tag{3.9}$$

The derivation is shown in the appendix B, in case where we use more dimensions or different finite volume, we can modify the propagator.

3.2.2 Matching scheme

Now we should add to our scheme the concept of perturbative calculation for the Z factor and the mass gap. So the definition of our mass gap and Z factor is:

$$Ze^{-x_0 m} = (1 + Z_1 g_R^2 + Z_2 g_R^4 + Z_3 g_R^6) e^{-x_0(m_1 g_R^2 + m_2 g_R^4 + m_3 g_R^6)} = \tag{3.10}$$

To simplify our calculation we will consider the separation between the field will be 2τ instead just x_0. Then if we expand the previous result.

$$\begin{aligned} &1 + (-2m_1\tau + Z_1)g_R^2 + (-2m_2\tau + 2m_1^2\tau^2 - 2m_1\tau Z_1 + Z_2)g_R^4 \\ &+(-2m_3\tau + 4m_1 m_2\tau^2 - \frac{4m_1^3\tau^3}{3} - 2m_2\tau Z_1 + 2m_1^2\tau^2 Z_1 - 2m_1\tau Z_2 + Z_3)g_R^6 \end{aligned} \tag{3.11}$$

That is why by using typical perturbative expansion and using dimensional regularization the calculation can be made. In here we can mention the reason why we have chosen free boundary condition, as we observe in the previous equation from any order, the result to the second order is going to be related. For example at NLO has a term τ^2 this term is related with the order LO. This relation only happens when we study our operator with free boundary condition and the reason in the case of use other kind of boundary condition the propagator can not be described by only one exponential term. That is why we will need the FBC to be able to get just one exponential factor and get the mass gap value.

3.3 Correlation function

3.3.1 LO

The first step is to calculate at LO. The operator we should calculate at this order is:

$$1 + g_0^2 \frac{1}{L} \int_{-L/2}^{L/2} dx e^{-ip_1 \cdot x_1} (\langle \pi^i(-\tau, x_1)\pi^i(\tau, 0)\rangle - \frac{1}{2}\langle(\pi^i(-\tau, x_1))^2\rangle - \frac{1}{2}\langle(\pi^i(\tau, 0))^2\rangle) \tag{3.12}$$

In green function language, taking into account $G(-\tau,-\tau,p_1=0)=G(\tau,\tau,p_1=0)$ and $G(-\tau,-\tau,q_1=\frac{2\pi n}{L})=G(\tau,\tau,q_1=\frac{2\pi n}{L})=\frac{1}{2\pi}\int_{-\infty}^{\infty}dp_0G(p_0,q_1=\frac{2\pi n}{L})$ if $n\neq 0$. At the end of the calculation we consider the infinite time, so for all the calculation we will just show the values which do not vanish at infinite T,

$$1+g_0^2(G(-\tau,\tau,p_1=0)-G(\tau,\tau,p_1=0)-\sum_{n\neq 0}G(\tau,\tau,q_1=\frac{2\pi n}{L})). \tag{3.13}$$

The representations of the second and the third terms of these diagrams are in the figures 3.1 and 3.2.

Until now, we did not mention anything about the UV divergences, but the UV

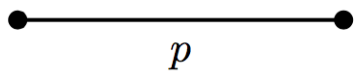

Figure 3.1: Feynmann diagram of $G(-\tau,\tau,p_1=0)$

Figure 3.2: Feynmann diagram of $\sum_{n\neq 0}(G(\tau,\tau,q_1=\frac{2\pi n}{L})$

regularization has to be made because the loop diagram of figure 3.2 is UV divergent. In our calculation, we will use the dimensional regularization, and the result to LO is

$$1+g_0^2(N-1)[\frac{1}{L}(\frac{T}{12}-\frac{|\tau-(-\tau)|}{2}+\frac{\tau^2+\tau^2}{2})-\frac{1}{L}(\frac{T}{12}+\frac{\tau^2+\tau^2}{2})-$$

$$-\frac{1}{L}\mu^{-\epsilon}(\frac{1}{\pi})^{(1-\epsilon)/2}\frac{1}{2^{1-\epsilon}}\mu^{\epsilon}\sum_{n_1\neq 0}\frac{1}{|\frac{2\pi n_1}{L}|^{1+\epsilon}}] =$$

$$= 1 + g_0^2(n-1)[-\frac{|\tau|}{L} - \frac{1}{L}\mu^{-\epsilon}(\frac{1}{\pi})^{(1-\epsilon)/2}\frac{1}{2^{1-\epsilon}}\mu^{\epsilon}\sum_{n_1\neq 0}\frac{1}{|\frac{2\pi n_1}{L}|^{1+\epsilon}}] \tag{3.14}$$

The last term is going to be calculated in detail, it has to be clear that the UV divergence at finite volume effect is going to be the same as in infinite volume, the previous integral gives:

$$A = \frac{1}{L}(\frac{1}{\pi})^{(1-\epsilon)/2}\frac{1}{2^{1-\epsilon}}\mu^{\epsilon}\sum_{n_1\neq 0}\frac{1}{|\frac{2\pi n_1}{L}|^{1+\epsilon}} =$$

$$= \frac{1}{2}(L)^{\epsilon}\pi^{-\frac{3}{2}-\frac{1}{2}\epsilon}\Gamma[\frac{1+\epsilon}{2}]Z_{Riemann}[1+\epsilon] \tag{3.15}$$

In the previous equation the definition of the Riemann's Zeta function ($Z_{Riemann}$) is used. The value of A is calculated by the series expansion of the $Z_{Riemann}$ and Γ function,

$$\Gamma(1/2 + x/2) = \sqrt{\pi}(1 + \frac{1}{2}(-\gamma_E - 2\log 2)x + \frac{1}{16}(\pi^2 + (-\gamma_E - 2\log 2))^2)x^2, \tag{3.16}$$

$$Z_{Riemann}[1+x] = \frac{1}{x} + \gamma_E - \gamma_{St1}x, \tag{3.17}$$

where γ_E is the Euler-Mascheroni constant and γ_{St1} is the 1-Stieltje's constant.

$$A = \frac{1}{2\pi}(\frac{1}{\epsilon} + \log(L\mu) + a + \epsilon(C_0 + (\frac{1}{2}(\log(L\mu))^2 + a)^2) + ...), \quad a = \frac{\gamma - log(4\pi)}{4\pi}, \tag{3.18}$$

$$a = \frac{\gamma - \log(4\pi)}{2}, \quad C_0 = \frac{-8\gamma_E^2 + \pi^2 - 16\gamma_{St1}}{16} \tag{3.19}$$

These values they are not important right now, but they will again appear for the calculation of the γ-function.

With the help of the eq. (3.11), the LO can be identified by $1 + (Z_1 - 2\tau m_1)g_0^2$, by direct relation, given by

$$m_1 = \frac{N-1}{2L} \tag{3.20}$$

$$Z_1 = -(N-1)A\mu^{-\epsilon} = -(N-1)(\frac{1}{2\pi\epsilon} + \frac{\gamma + 2log(L) - log(4\pi)}{4\pi})\mu^{-\epsilon}. \tag{3.21}$$

3.3.2 NLO

To do further calculations, it is needed to calculate at higher order the previous operator and the next order operator will appear, in other words the term g_0^4.

$$\langle \pi^i(x)\pi^i(y)\rangle_1 - \frac{1}{2}\langle(\pi^i(x))^2\rangle_1 - \frac{1}{2}\langle(\pi^i(y))^2\rangle_1 + \tag{3.22}$$

The term $_1$ means the order in perturbative calculation. And the operators at, $_0$, tree level.

$$\frac{1}{8}(2\langle(\pi^i(x))^2(\pi^j(y))^2\rangle_0 - \langle(\pi^i(y)^2(\pi^j(y))^2\rangle_0 - \langle(\pi^i(x))^2(\pi^j(y)))^2\rangle_0) \tag{3.23}$$

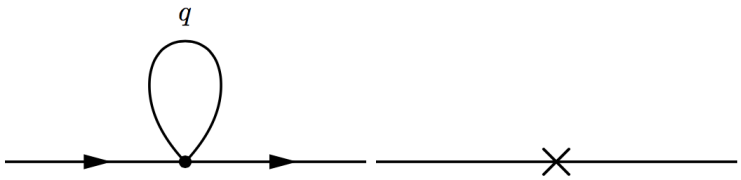

Figure 3.3: Feynmann diagram $\langle \pi^i(x)\pi^i(y)\rangle_1$

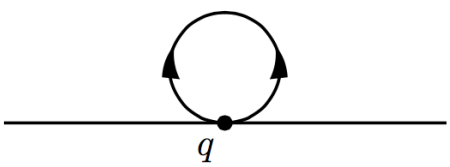

Figure 3.4: Feynmann diagram $\langle \pi^i(x)\pi^i(y)\rangle_1$

In the diagrams of 3.3 some propagators appear with one arrow, that means that propagator has a derivative, coming from the interaction term in the Lagrangian $L_I = \int dz \partial_\mu \pi_i(z)\pi_i(z)\partial_\mu \pi_i(z)\pi_i(z)$. Being left diagram of the figure 3.3:

$$\frac{1}{L}\int_{-L/2}^{L/2} dx e^{-ip_1\cdot x_1}\langle \pi^i(-\tau, x_1)\int dz \partial_\mu \pi^j(z_0,z_1)\pi^j(z_0,z_1)\pi^k(z_0,z_1)\partial_\mu \pi^k(z_0,z_1)\pi^i(\tau,0)\rangle =$$

$$= \frac{1}{L}\int dz(\partial_z G(z_0,\tau,p=0)\sum_n G(z_0,z_0,q_1=\frac{2\pi n}{L})\partial_z G(-\tau,z_0,p=0)+$$

$$= A(\frac{T}{12}-\tau+\frac{\tau^2}{T})\mu^{-\epsilon}+\frac{1}{L^2}[\frac{\tau^2}{12}+\frac{\tau^4}{2T^2}-\frac{\tau^3}{3T}-\frac{\tau T}{12}+\frac{13T^2}{1440}] \tag{3.24}$$

In the case of right diagram in the figure 3.3 the expression is calculated without the factor of the interaction Lagrangian.

$$\frac{1}{L}\int_{-L/2}^{L/2} dx e^{-ip_1\cdot x_1}\langle\pi^i(-\tau,x_1)\int dz\pi^j(z_0,z_1)\pi^j(z_0,z_1)\pi^i(\tau,0)\rangle =$$

$$= \int dz_0 G(\tau,z_0,p=0)G(-\tau,z_0,p=0) =$$

$$= \frac{1}{L}[-\frac{\tau^2 T}{3}-\frac{\tau^4}{3T}+\frac{2\tau^3}{3}+\frac{T^3}{720}] \tag{3.25}$$

For the figure 3.4 the calculation is not going to be straightforward. Before we will need calculation for $\partial_z\partial_{z'}G(z,z',p=0)$, this calculation is presented in the appendix A for the delta function. For the case $\partial_z\partial_{z'}G(z,z',p\neq 0)$ is calculated below.

$$(\partial_{y_0}\partial_{x_0}+\partial_{y_1}\partial_{x_1})G(x_0,x_1,y_0,y_1)\rfloor_{x=y} =$$

$$= \frac{1}{L}\sum_{p_1}\left(\frac{1}{L}\sum_m \frac{(|p_1|^2-|p_1|^2)}{2|p_1|}e^{-|p_1||(x-y)+(2m)T|+ip(x_1-y_1)}\right)\rfloor_{x=y}$$

$$+\frac{1}{L}\sum_{p_1}\left(\frac{1}{L}\sum_m \frac{(2|p_1|^2)}{2|p_1|}e^{-|p_1||(x+y)+(1+2m)T|+ip(x_1-y_1)}\right)\rfloor_{x=y} =$$

$$\frac{1}{L}\sum_{p_1}\left(\frac{1}{L}\sum_m \frac{(2|p_1|^2)}{2|p_1|}e^{-|p_1||2x+(1+2m)T|}\right), \quad p_1=\frac{2\pi n_1}{L} \quad | \quad n_1\in\mathcal{Z}/0, \quad m\in\mathcal{Z} \tag{3.26}$$

Is important to mention that this definition is coming from the direct definition of the green function with our boundary conditions, as it can be checked in the appendix D, where the images method was used to generate the green function. In this case this expression was used due to the simplicity of the calculations.

Inserting those calculations in the previous diagram calculation:

$$\frac{1}{L}\int_{-L/2}^{L/2} dx e^{-ip_1\cdot x_1}\langle\pi^i(-\tau,x_1)\int dz\pi^j(z_0,z_1)\partial_\mu\pi^j(z_0,z_1)\partial_\mu\pi^j(z_0,z_1)\pi^j(z_0,z_1)\pi^i(\tau,0)\rangle =$$

$$= \frac{1}{L}\int dz(\sum_n G(z_0,\tau,p=0)\partial_\mu\partial_{\mu'}G(z_0,z_0',p=\frac{2\pi n}{L})G(-\tau,z_0,p=0)+$$

$$= \frac{-2}{1152L^2}[-168\tau^2+T^2-\frac{2L^2\pi}{(2\pi)^2}] \tag{3.27}$$

It is important to note that other combinations for the derivative term are not going to be calculated because at $T\to\infty$ these contributions will be canceled at the end of the calculation. It is because the integration at infinite time component only the even integral in time component survives.

Now we will follow the same procedure for the next diagrams.

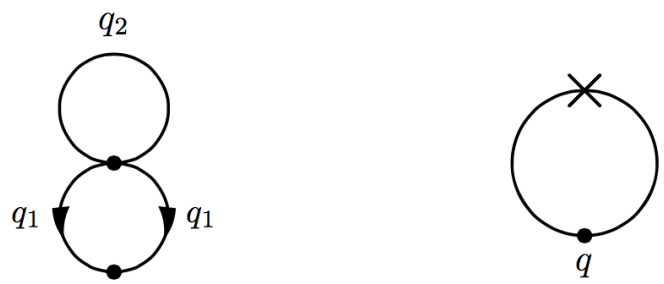

Figure 3.5: Feynmann diagram $\langle(\pi^i(x))^2\rangle^1$

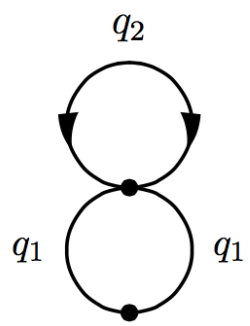

Figure 3.6: Feynmann diagram $\langle(\pi^i(x))^2\rangle^1$

The diagrams in figure 3.5 are given by:

-Left diagram:

$$\frac{1}{L}\int_{-L/2}^{L/2} dx e^{-ip_1\cdot x_1}\langle\pi^i(\tau,x_1)\int dz\partial_\mu\pi^j(z_0,z_1)\pi^j(z_0,z_1)\pi^k(z_0,z_1)\partial_\mu\pi^k(z_0,z_1)\pi^i(\tau,0)\rangle =$$

$$= A^2\mu^{-2\epsilon} + 2A(\frac{1}{L}(\frac{T}{12} + \frac{\tau^2}{T}))\mu^{-\epsilon} + \frac{1}{L^2}[\frac{\tau^2}{12} + \frac{\tau^4}{2T^2} + \frac{13T^2}{1440}], \tag{3.28}$$

-Right diagram:

$$\frac{1}{L}\int_{-L/2}^{L/2} dx e^{-ip_1\cdot x_1}\langle \pi^i(\tau, x_1)\int dz \pi^j(z_0, z_1)\pi^j(z_0, z_1)\pi^i(\tau, 0)\rangle =$$

$$= \frac{1}{L}[\frac{\tau^2 T}{6} - \frac{\tau^4}{3T} + \frac{2\tau^3}{3} + \frac{T^3}{720}]. \tag{3.29}$$

For the figure 3.6 the result is:

$$\frac{1}{L}\int_{-L/2}^{L/2} dx e^{-ip_1\cdot x_1}\langle \pi^i(\tau, x_1)\int dz \pi^j(z_0, z_1)\partial_\mu\pi^j(z_0, z_1)\partial_\mu\pi^j(z_0, z_1)\pi^j(z_0, z_1)\pi^i(\tau, 0)\rangle =$$

$$= \frac{1}{L^2}[-\frac{\tau^2}{6} + \frac{\tau^4}{3T^2} - \frac{2\tau^3}{3T} - \frac{T^2}{720}] \tag{3.30}$$

$$= \frac{-2}{1152L^2}[120\tau^2 + T^2 - \frac{2L^2\pi}{(2\pi)^2}]. \tag{3.31}$$

Figure 3.7: Feynmann diagram of $\langle(\pi^i(y)^2(\pi^j(y))^2\rangle_0$

The last diagrams to consider are given in the Fig. 3.7 and left diagram in the Fig. 3.8. They give the same results:

$$\frac{1}{L}\int_{-L/2}^{L/2} dx e^{-ip_1\cdot x_1}\langle \pi^i(-\tau, x_1)\pi^i(-\tau, x_1)\pi^j(\tau, 0)\pi^j(\tau, 0)\rangle =$$

$$= (\sum_n G(\tau, \tau, \frac{2\pi n}{L}))^2 =$$

$$= [A\mu^{-\epsilon} + \frac{1}{L}(\frac{T}{12} + \frac{\tau^2}{T})]^2. \tag{3.32}$$

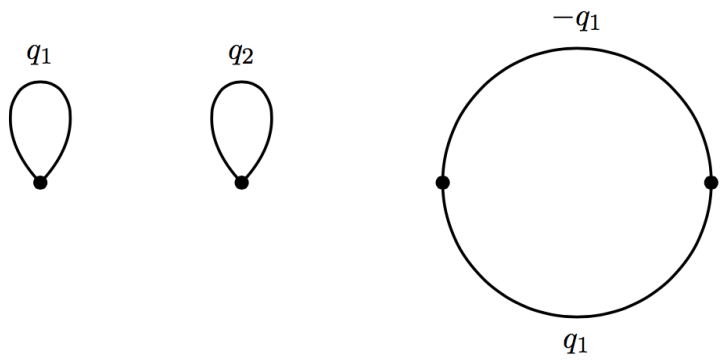

Figure 3.8: Feynmann diagram of $\langle(\pi^i(x))^2(\pi^j(y))^2\rangle$

For the diagram in the right side of the Fig 3.8, it is important to consider that $G(-\tau,\tau,\frac{2\pi n}{L}) \sim e^{-\tau(\frac{2\pi n}{L})}$ is at large τ, because this calculation is made in the δ-regime. The δ-regime is when the space we are studing is a large cylindric system, which agree with in our study.

$$\frac{1}{L}\int_{-L/2}^{L/2} dx e^{-ip_1\cdot x_1}\langle\pi^i(-\tau,x_1)\pi^j(\tau,0)\pi^i(-\tau,x_1)\pi^j(\tau,0)\rangle =$$

$$G(-\tau,\tau,p=0)^2 + \sum_{n\neq 0} G(-\tau,\tau,(\frac{2\pi n}{L})^2 \sim G(-\tau,\tau,p=0)^2 =$$

$$= [\frac{1}{L}(\frac{T}{12} - \tau + \frac{\tau^2}{T})]^2 \tag{3.33}$$

The last thing to point out is the statistical factor of each diagram. Now those statistical factor with the multiplication factor coming from the Lagrangian definition are presented in the table 3.1. When $x_0 = \tau$ and $y_0 = -\tau$, some diagrams has to be counted twice, such as the Fig. 3.7.

All the values of the Feynman diagrams are shown in the table 3.2 Collecting all the results and the counting in each diagram the result at NLO is given by:

$$\frac{1}{L}\sum_{x_1}\langle S(\tau,x_1)S(-\tau,0)\rangle \to NLO \to \frac{1}{2}(N-1)A^2 + A(N-1)\tau + (N-1)(N-1)\frac{\tau^2}{2L^2} \tag{3.34}$$

Then coming back to the equation 3.11 the result has to be equalize to:

$$(-2m_2\tau + 2m_1^2\tau^2 - 2m_1\tau Z_1 + Z_2) \tag{3.35}$$

Operator	Figure	Lagrangian	Statistical factor	Operator factor
$\langle \pi^i(x)\pi^i(y)\rangle_1$	3.3 left	$-\frac{1}{2}$	$2(N-1)$	1
	3.3 right	$-\frac{(N-1)}{2TL}$	$2(N-1)$	1
	3.4	$-\frac{1}{2}$	$2(N-1)$	1
$\langle(\pi^i(x))^2\rangle_1$	3.5 left	$-\frac{1}{2}$	$2(N-1)$	$-\frac{1}{2}(2)$
	3.5 right	$-\frac{(N-1)}{2TL}$	$2(N-1)$	$-\frac{1}{2}(2)$
	3.6	$-\frac{1}{2}$	$2(N-1)$	$-\frac{1}{2}(2)$
$\langle(\pi^i(y)^2(\pi^j(y))^2\rangle_0$	3.7	1	$[(N-1)^2+2(N-1)]$	$-\frac{1}{8}(2)$
$\langle(\pi^i(x)^2(\pi^j(y))^2\rangle_0$	3.8 left	1	$(N-1)^2$	$\frac{1}{4}$
	3.8 right	1	$2(N-1)$	$\frac{1}{4}$

Table 3.1: Multiplications factors

Operator	Figure	Feymann diagram value
$\langle \pi^i(x)\pi^i(y)\rangle_1$	3.3 left	$A\mu^{-\epsilon}(\frac{T}{12}-\tau+\frac{\tau^2}{T})+\frac{1}{L^2}[\frac{\tau^2}{12}+\frac{\tau^4}{2T^2}-\frac{\tau^3}{3T}-\frac{\tau T}{12}+\frac{13T^2}{1440}]$
	3.3 right	$\frac{1}{L}[-\frac{\tau^2 T}{3}-\frac{\tau^4}{3T}+\frac{2\tau^3}{3}+\frac{T^3}{720}]$
	3.4	$\frac{-2}{1152L^2}[-168\tau^2+T^2-\frac{2L^2\pi}{(2\pi)^2}]$
$\langle(\pi^i(x))^2\rangle_1$	3.5 left	$A^2\mu^{-2\epsilon}+2A\mu^{-\epsilon}(\frac{1}{L}(\frac{T}{12}+\frac{\tau^2}{T}))+\frac{1}{L^2}[\frac{\tau^2}{12}+\frac{\tau^4}{2T^2}+\frac{13T^2}{1440}]$
	3.5 right	$\frac{1}{L}[\frac{\tau^2 T}{6}-\frac{\tau^4}{3T}+\frac{2\tau^3}{3}+\frac{T^3}{720}]$
	3.6	$\frac{-2}{1152L^2}[120\tau^2+T^2-\frac{2L^2\pi}{(2\pi)^2}]$
$\langle(\pi^i(y)^2(\pi^j(y))^2\rangle_0$	3.7	$[A\mu^{-\epsilon}+\frac{1}{L}(\frac{T}{12}+\frac{\tau^2}{T})]^2$
$\langle(\pi^i(x)^2(\pi^j(y))^2\rangle_0$	3.8 left	$[A\mu^{-\epsilon}+\frac{1}{L}(\frac{T}{12}+\frac{\tau^2}{T})]^2$
	3.8 right	$[\frac{1}{L}(\frac{T}{12}-\tau+\frac{\tau^2}{T})]^2$

Table 3.2: Feynman diagrams calculation

$$m_2 = \frac{(N-1)}{2L}(N-2)A\mu^{-\epsilon} \tag{3.36}$$

$$Z_2 = \frac{1}{2}(N-1)A^2\mu^{-2\epsilon} \tag{3.37}$$

These results are in correspondence with those in [41] and [61]. The importance of the boundary condition emerges now because if the propagator was different the result in the corresponding to τ^2 term will not be equal to $2m_1^2\tau^2$. With other boundary condition in the time component other state will be included in the perturbative calculation, so more exponential states contributions may appear, doing useless this method to be unable to compare with an exponential function.

3.4 γ-function

As we have explained at the begining of the chapter the needed calculation is the γ-function at NNLO because the β-function and the mass gap was already calculated in other studies. The gamma function is defined by $\gamma(g_R^2) = -L\frac{d}{dL}(log(Z))$. We use the Z-factor and the previous calculation. The unique factor dependent on L is A. In this calculation, DR was used and it is important to mention the in this calculation the MS^*-scheme, $\mu^2 = \mu^2 4\pi e^{-\gamma_E}$, is going to be used. Using this MS^*-scheme, the bare coupling is replaced by the renormalized coupling:

$$g_0^2\mu^{-\epsilon} = g_R^2 + \frac{(N-2)}{(2\pi\epsilon)}(g_R^2)^2 + ((\frac{(N-2)}{(2\pi\epsilon)})^2 - \frac{(N-2)}{(2\pi\epsilon)2\pi}))(g_R^2)^3 \tag{3.38}$$

In the definition of our scheme, we are going to consider that renormalized coupling and μ do not depend on L. We can assume other dependence, although the L dependence relation between g_R and μ should be related in the way that g_0^2 is L independent by the relation $g_0^2 = g_R^2\mu^\epsilon$. This relation comes from the lattice regularization, because our scheme is to compare with lattice results where the bare parameter is going to be L independence. Where the relations between lattice regularization and dimensional regularization was study in [59].

We have chosen this μ as L independence to get simplest results.

3.4.1 NLO

Collecting the results the Z factor, the γ function can be straightforward calculated by:

$$1 - A(N-1)g_0^2\mu^{-\epsilon} + \frac{1}{2}(N-1)A^2(g_0^2)^2\mu^{-2\epsilon} \tag{3.39}$$

In all of this calculation we can redefine the value of μ by $\mu^* = \mu e^{-a}$

$$A^* = \frac{1}{2\pi\epsilon} + \frac{log(\mu^* L)}{2\pi} + \epsilon\frac{1}{2\pi}(\frac{1}{2}(\log\mu^* L)^2 + C_0) \tag{3.40}$$

$$C_0 = \frac{(-8\gamma_E^2 + \pi^2 - 16\gamma_{St1})}{16}) = 0.5230771586479023 \tag{3.41}$$

Then by simple substitution:

$$\log(Z) =$$
$$= -(N-1)(\frac{1}{2\pi\epsilon} + \frac{\log(\mu^* L)}{2\pi})g_R^2 + \frac{(N-1)(N-2)}{2}((\frac{1}{2\pi\epsilon})^2 - (\frac{\log(\mu^* L)}{2\pi})^2)g_R^4 \tag{3.42}$$

$$-L\frac{d}{dL}\log(Z) = \gamma(g_R^2) = \frac{(N-1)}{2\pi}g_R^2 + (\log(\mu^* L))\frac{(N-2)(N-1)}{(2\pi)^2}(g_R^2)^2 \tag{3.43}$$

The next step is to transform the gamma function in the coupling constant in the Lüscher's scheme, g_L^2:

$$g_L^2 = g_R^2 + (N-2)(\log(\mu^* L))g_R^4 \tag{3.44}$$

Then the final gamma function in the Lüscher's scheme is:

$$\gamma(g_L^2) = (N-1)\frac{1}{2\pi}g_L^2 + \mathcal{O}(g_L^4) \tag{3.45}$$

We find different results for different definitions of the coupling constant. The issue here is in the Lüscher's coupling definition the mass gap is used. This value can be calculated in lattice calculation, where there is no μ dependence. So at the end, this gamma function in terms of physical values is going to be μ independence, as in our case.

3.4.2 NNLO

To be able to calculate the gamma function at NNLO, we do not need to calculate the value of Z factor at NNLO. The information needed is the value of Z at NLO, the $\gamma_{\overline{MS}}(g_R^2) = \gamma_{MS^*}(g_R^2)$ and $\mu^*\frac{d\log Z}{d\mu^*} = 0$. The value $\gamma_{\overline{MS}}(g_R^2) = \gamma_{MS^*}(g_R^2)$ is well known. The calculation of $\gamma\overline{MS}(g_R^2)$ is made using dimensional regularization where the IR divergence is handled by linear magnetic field. Because the IR divergence is decoupled from the UV divergence we can relate the finite volume scheme with linear magnetic field. The other condition, $\mu^*\frac{d\log Z}{d\mu^*} = 0$, comes from the RGE. The RGE for the mass gap is given by

$$(\partial_\mu + \beta(g_R^2)\partial_{g_R^2})m = 0 \tag{3.46}$$

when we consider $Ze^{-|x_0|m}$). So if the RGE is preserved for the mass gap, $\frac{dZ}{d\mu} = 0$ has to be preserved too.

Now the definition of Z_2 is in general form:

$$Z_2 = \frac{(g_0^2)^3}{(2\pi)^3}\Big(\frac{P_{11}}{\epsilon^3} + \frac{P_{12}}{\epsilon^2} + \frac{P_{13}}{\epsilon} + P_{14} +$$

$$+log(L\mu^*)[\frac{P_{21}}{\epsilon^2} + \frac{P_{22}}{\epsilon} + P_{23}] + log^2(L\mu^*)[\frac{P_{31}}{\epsilon} + P_{32}] + log^3(L\mu^*)[P_{41}]\Big) \tag{3.47}$$

So the next step is to apply the previous conditions to fix the P_{XX}'s values:

$$\mu^* \frac{d \log Z}{d\mu^*} = 0. \tag{3.48}$$

$$\gamma_{MS^*}(g_R^2) = \mu^* \frac{d \log Z_{MS^*}}{d\mu^*} = \frac{(N-1)}{2\pi} g_R^2 + \frac{3(N-1)(N-2)}{4(2\pi)^3}(g_R^2)^3 + \mathcal{O}(g_R^8) \tag{3.49}$$

The constraint in Eq. (3.48) leads us to:

$$\begin{gathered} 0 = \mu^* \frac{d \log Z}{d\mu^*} = \\ \frac{1}{(2\pi)^3}\Big(\frac{-3P_{11}+P_{21}}{\epsilon^2} + \frac{-3P_{12}+P_{22}}{\epsilon} + \\ + \frac{-3P_{21}+2P_{31}}{\epsilon}\log(L\mu^*) + \frac{1}{4}(3-4N+N^2-12P_{31}+12P_{41})\log^2(L\mu^*) + \\ +(-3P_{22}+2P_{32})\log(L\mu^*) + \\ +\frac{1}{2}(3C_0 - 4C_0N + C_0N^2 - 6P_{13} + 2P_{23})\Big)g^6 \end{gathered} \tag{3.50}$$

For the 3.49 condition, we have to define Z_{MS^*}. So the definition of Z_{MS^*} is the same as Z but without the $\log(\mu^* L)$ terms and the finite parts, because we have to fix $\mu^* = 1/L$.

$$\begin{gathered} \mu^* \frac{dZ_{MS^*}}{d\mu^*} = \frac{(N-1)}{2\pi} g_R^2 + \\ + \frac{1}{(2\pi)^3}\Big(\frac{3-4N+N^2-6P_{11}}{2\epsilon^2} + \frac{-2+3N-N^2-6P_{12}}{2\epsilon} \\ +\frac{3}{2}(3C_0 - 4C_0N + C_0N^2 - 6C_0 - 2P_{13})\Big)g_R^6 \end{gathered} \tag{3.51}$$

So these two conditions fix the values of Z_2. They are:

$$P_{11} = \frac{1}{6}(3-4N+N^2), \quad P_{12} = \frac{1}{6}(-2+3N-N^2)$$

$$P_{13} = \frac{1}{4}(-2+3N-N^2+C_0(6-8N+2N^2)), \quad P_{21} = \frac{1}{2}(3-4N+N^2)$$

$$P_{22} = -\frac{1}{2}(-2+3N-N^2), \quad P_{23} = -\frac{1}{4}(-6+9N-3N^2+C_0(12-16N+4N^2))$$

$$P_{31} = -\frac{3}{4}(2-3N+N^2), \quad P_{32} = -\frac{3}{4}(2-3N+N^2), \quad P_{41} = \frac{2}{3}(3-4N+N^2) \quad (3.52)$$

The γ-function obtained at NNLO is:

$$\gamma(g_R^2) = \frac{(N-1)}{2\pi} g_R^2 + log(\mu^* L)\frac{(N-2)(N-1)}{(2\pi)^2}(g_R^2)^2 +$$
$$+(3+4log(\mu^* L)+4(N-2)log^2(\mu^* L))\frac{(N-2)(N-1)}{4(2\pi)^3}(g_R^2)^3 \quad (3.53)$$

As it was made above we need now to change the g_R in the Lüscher's definition of the coupling constant.

$$g_L^2 = g_R^2 + (N-2)^2 log(\mu^* L)\frac{g_R^4}{2\pi} + ((N-2)log^2(\mu^* L) + (N-2)log(\mu^* L) + \frac{3}{4}(N-2))\frac{g_R^6}{(2\pi)^2} \quad (3.54)$$

$$\gamma(g_L^2) = \frac{(N-1)}{2\pi} g_L^2 + \mathcal{O}(g_L^6) \quad (3.55)$$

By a quick look, we realize that this gamma function is μ independent as the renormalized coupling is in terms of physical values, which can be calculated in lattice regularization too.

3.5 Vacuum energy density

The expectation value of the vacuum energy density is calculated at LO.

We show that LO diagram is represented in the Fig. 3.9 and its calculation comes form the following integration:

$$\frac{1}{TL}\int dx \langle \partial_\mu \pi^i(x) \partial_\mu \pi^i(x) \rangle_1 \quad (3.56)$$

We do not calculate extra terms, because as it is going to be clear later, the calculation we should include all the terms because this value is in the non-perturbative regime. In the previous section we have done the perturbative calculation using FBC, the reason was because we just want one exponential form in the result because the mass gap is going to be compared with lattice calculation where the just one exponential form is taken. On the other hand for the case of vacuum energy density calculation, there is no problem to use PBC. So if we want to compare the result with the lattice

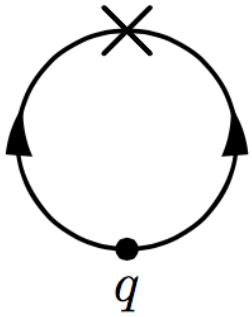

Figure 3.9: Feynmann diagram $\langle \partial_\mu \pi^i(x) \partial_\mu \pi^i(x) \rangle_1$

calculation we should use the same boundary condition. For this PBC the propagator has to be changed:

$$G(x, y, p_1 = 0) = \frac{1}{L}(\frac{T}{12} - \frac{|x_0 - y_0|}{2} + \frac{(x_0 - y_0)^2}{2T}) \tag{3.57}$$

$$G(x_0, x_1, p_1) = \frac{1}{L} \sum_m \frac{1}{|p_1|} e^{-|p_1||(x_0 - y_0) + 2mT| + ip(x_1 - y_1)}$$
$$p_1 = \frac{2\pi n}{L} \quad n \in \mathbb{Z} - \{0\} \tag{3.58}$$

We can see easily the difference with the FreeBC. In PBC the asymmetric part has been neglected. And the vacuum energy density is:

$$E^{(PBC)} = \frac{(N-1)^2}{12L^2} g_R^2 \tag{3.59}$$

From this calculation we may get wrong understanding. The reason is related to the non-perturbative regime. The vacuum energy density is calculated in lattice regularization [79], where the calculation was made by lattice perturbation theory in the infinite regime,

$$E_{Lattice} = \frac{N-1}{2} g_{aR}^2 a^2 \frac{(N-1)(N-3)}{8} g_{aR}^4 a^2 \tag{3.60}$$

Here we can see interesting behaviour comparing with dimensional regularization. If we go to infinite volume in the dimensional regularization we conclude that the vacuum energy density is zero but in the case of lattice regularization we know that when we go to smaller lattice we should resum all the terms to get the non-perturbative results [84] [85] and [86].
So if we calculate the mass gap using lattice regularization in finite volume [63], the lattice spacing, a, is always presented with a parameter in the system, volume N. At the end this parameter will absorb the lattice spacing to become a parameter in the continuum limit, $Na = L$. This issue is still unclear but we will explain in detail later where we compare perturbation results with Monte Carlo simulations.

Chapter 4

Sigma model O(3) non-perturbative calculations

In the previous chapter we have considered the O(N) sigma model for general N. In this chapter, we focus on $N = 3$ and analyze the O(3) sigma model in Euclidean 2-dimensional space using non-perturbative calculations of the path integral formalism. This non-perturbative calculation is performed by the Monte Carlo integration procedure. This Monte Carlo calculation is carried out on the discretized lattice space and this method is based on the random sampling generated by a Markov's chain mechanism. We use for this QFT in lattice regularization and at the end of the calculation this lattice regularization will be extrapolate to be able to get the continuum values. Frist, we give the definition of the O(3) sigma model in lattice regularization.

$$S = \frac{1}{2g_0^2} \sum_n \sum_d (S^i(x) - S^i(x + e_d)^2 \tag{4.1}$$

where

$$d = 2, \quad (S^1(x), S^2(x), S^3(x)) \in \mathbb{S}^2, \quad x \in [0, 1..., T] \times [0, 1..., L] \tag{4.2}$$

And e_d is the lattice unit vector for the space direction.

As it was mentioned in the first chapter, we have different ways to regularize this theory, using lattice regularization. In this case we are using the simplest regularization because we can compare with other studies.

Then from now in this chapter, we will explain the method to for the Monte Carlo simulation before showing the calculations. After those consideration we calculate straightaway the operator calculated in perturbation theory. Those results are considered bare results, non-processed, so we present and in the final section the process

to be able extrapolate to the continuum limit, therefore being ready to compare with perturbation theory.

4.1 Monte Carlo simulation

The Monte Carlo simulation is based on the random sampling, This random sample will be generated by a Markov's chain. Markov's chain is based in a probability distribution. This probability distribution as in the common QFT calculation is the exponential of the action, S. The reason is coming from path integral formalism.

$$\langle \mathcal{O} \rangle = \frac{\int D\phi \mathcal{O} e^{-S}}{\int D\phi e^{-S}} \tag{4.3}$$

where $\mathcal{O}$ is the operator we want to calculate. So it is clear that the probability distribution is the exponential of the action, e^{-S}.
Any kind of Markov's chain sample is related with the previous sample, this will affect the results. The relation of each sample is denominated by an autocorrelation function. Therefore to get rid of this problem the measure of operator is made when this autocorrelation vanishes. This is possible if the measurements of the operator are made after a large Markov's chain steps. This autocorrelation has to be studied for each operator. Another important issue in the Markov's chain simulation is the steps needed to reach the authentic distribution, because at the beginning we will have an initial distribution and we do not know if this distribution follow the probability distribution. This is independent of the operator we are studying, so we can ensure that we have reached the minimum of the action, we say that an authentic distribution is initial point independence.

4.1.1 Markov's chain

The Markov's chain used in our calculation can be explained easily. The idea is to be able to change the N-component field in all the sites in the space and this change follows the action probability, what is called the heat bath algorithm.
For the case of just change component in one site, in this case we can get an analytical distribution because we can ingrate out in the case of $N = 3$. But for higher values of N, we can generate a random point in the $\mathbb{S}^{N-1}$ and accept this field in that

position with the action distribution probability. To generate those random point in the sphere, we will mention an easy way to pick a random point on a sphere, one generates N Gaussian random variables ϕ_1, ϕ_2 and so one. Then the distribution of the vectors

$$S^i = \frac{\phi_i}{\sqrt{\phi_1^2 + ... + \phi_N^2}} \tag{4.4}$$

is uniform over the surface $\mathbb{S}^{N-1}$.

Until now we have just spoken about changing the field in one site of the space. The change for all the sites different procedure can be made.

Here, the procedure used is to divide our space in two areas. This division transform our space like a chess board, so we define the white position and black position. Due to this division we know that the points in each set are independent so we can randomly get the new distribution. Because this independence we are able to use MPI computations. Therefore after one set distribution is changed we will proceed to the next set. Another different Markov's chain has been developed we for the O(N) sigma model by Wolff [53], although for this study we just refer it.

4.1.2 Computational cost

As we are using computation calculation in our integration, so it is going to be limited by the computational power we can use. With a simple calculation we can know that the integration is an integration of several variables which this is encoded in the definition of the path formalism $\int DS^i(x)$. So if in general we want to calculate the O(N) sigma model in d-dimensions and each dimension is n_i times discretized, the integration is an integration of $N \cdot n_1 \cdot ... \cdot n_d$ variables. That is why is going to be important to consider spaces where the calculation is nowadays possible with our computational resources.

4.1.3 Boundary condition

Now we discuss about the boundary condition used in our calculations. In the case of perturbation theory we had to calculate our propagator using free boundary condition, the reason is because if we do not use free boundary condition other contributions are take into account in the calculation so it would be impossible to discern between different exponential forms to get the desired result.

In the case of Monte Carlo simulation this problem can be solved from the data without restricting the boundary condition. This is possible because when we get the results from Monte Carlo simulations the exponential form will be obtained so in large time separation only the state with smaller mass will survive. That is why in our calculation from Monte Carlo simulation to get the correlation function result the boundary condition in the time component will not affect the result at large time component. This issue is going to be clear when we extract the mass gap from the data. Just in case in our calculation we are going to use two different boundary condition for the time component, FBC and PBC, and for the spatial component PBC, as it was made in perturbative calculation.

4.2 2-point correlation function

In this case we do not need to do any expansion in the renormalized coupling, although we will have to take into account that our temporal coordinate is going to be finite too due to the computational cost. For the Monte Carlo simulations we will have the option to calculate it with different boundary conditions as it was mention before. It is expected because the theory has a mass gap and so different boundary conditions should give us the same results. The correlation function we are going to calculate is

$$\frac{1}{L}\sum_{x_s}\langle S(x_t,x_s)S(0,0)\rangle. \tag{4.5}$$

Anyhow, we will use two kinds of boundary conditions: free boundary condition, FBC, and periodic boundary condition, PBC.
For each BC we will have different issues to extrapole our data. For example in PBC, it is known that the operator will be given by cosh function. The reason is explained below

$$Ze^{-m\cdot x_0} = \frac{1}{2\pi}\int e^{ip_0\cdot x_0}\frac{2mZ}{p_0^2+m^2} \xrightarrow{Finite} = \frac{1}{T}\sum e^{ip_0\cdot x_0}\frac{2mZ}{p_0^2+m^2} = Z\frac{cosh(m(\frac{T}{2}-x_0))}{sinh(\frac{mT}{2})} \tag{4.6}$$

where we have changed the integration to a sum because of the discretized time. As it is easy to realize the $sinh(\frac{mT}{2})$ is going to be L independient, because m depends on L. So we should take this factor into to account in the PBC.
For the case of FBC the correlation function is proportional to $e^{-m|x_0|}$ so we will fit with the usual exponential form. If we check back in the definition of the propagator

in PT eq. (3.10), the definition should be the same. That is why the definition of the eq. (4.6) fits with the PT.

4.2.1 Raw parameters

Now we present the raw results from the calculations, where we distinguish the calculation using FBC and PBC

4.2.1.1 Periodic boundary condition

For this calculataion is performed on the $(T/a) \times (L/a)$, with T = 8L. For the thermalization we have used $3 \cdot 10^5$ sweeps. 10^5 measurements was made and between each measurement we have executed 10^4 sweeps. In this calculation the autocorrelation of the operator was studied in our sample and it was shown if the difference in each measure is 10^4, there is no autocorrelation between our data. In this case we need to calculate the 2-point correlation function where we can get the mass gap and the renormalization factor. As we are using computers we had finite resources, and finite time. Then we modify the exponential equation as it was shown in the equation 4.6.

$$\frac{1}{L}\int e^{-ip_1\cdot x_1}\langle S_i(x)S^j(0)\rangle dx\Big]_{p_1=0} = Z\frac{cosh(m(\frac{T}{2}-x_0))}{sinh(\frac{mT}{2})} \tag{4.7}$$

To be able to know if our propagator really corresponds to the previous form, it is natural to define the effective mass gap, which is defined by

$$m_{effective} = log\left\{\frac{\int e^{-ip_1\cdot x_1}\langle S_i(x_0+1,x_1)S^j(0)\rangle dx\Big]_{p_1=0}}{\int e^{-ip_1\cdot x_1}\langle S_i(x_0,x_1)S^j(0)\rangle dx\Big]_{p_1=0}}\right\} \tag{4.8}$$

In the Fig. 4.1 this effective mass is plotted. From Fig. 4.1 we observe two characteristic properties already mentioned. One is that at the initial the effective mass does not agree with the curve of the fitting, this is due to the excited states, so it makes sense to use only the large time values, because as it was explained this contribution will disappear at a large time separation. On the other hand we see how this cosh term will help us to get more accurate fitting with the figure. Because if we

just use an exponential form, only the regime where the effective mass has a plateau can be used for the fitting and we should reject important information.

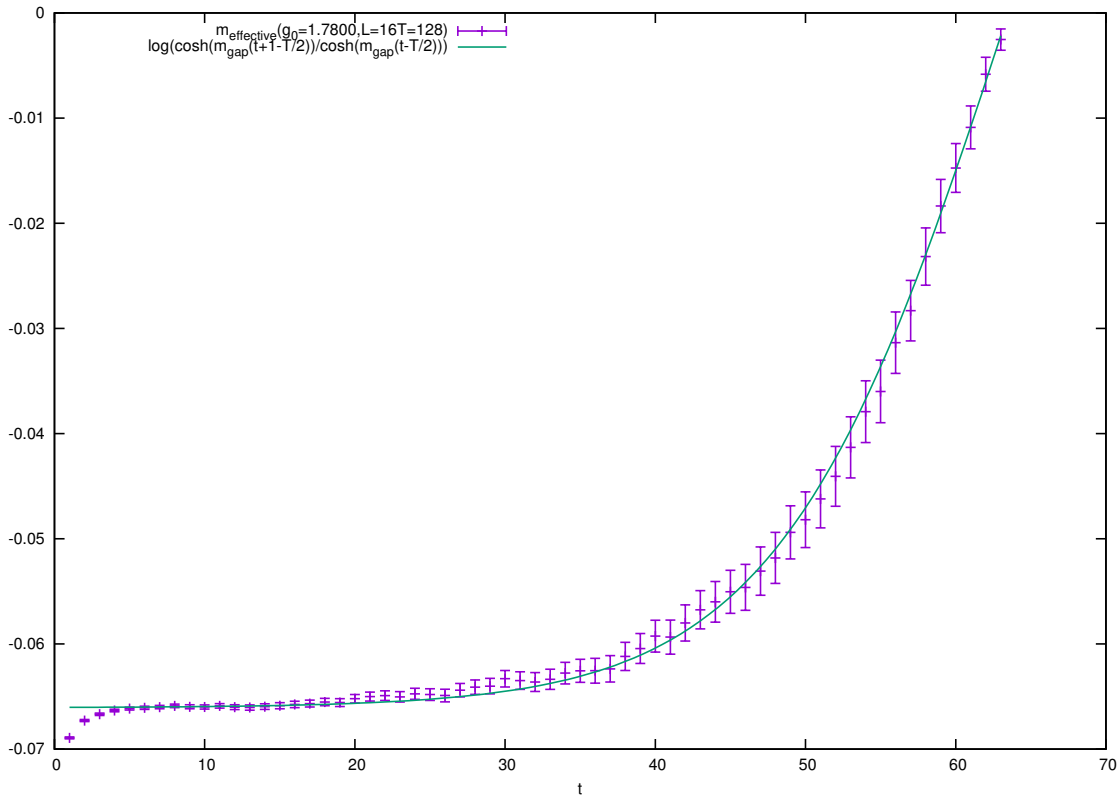

Figure 4.1: Effective mass

For different parameters we get the similar results, which we do not plot. we just summarize all the important values later on.

In the case of the renormalization factor, we define the effective factor, $Z_{effective}$, and check if this factor reaches a plateau in some temporal window.

$$Z_{effective} = \frac{\int e^{-ip_1 \cdot x_1} \langle S_i(x_0, x_1) S^j(0) \rangle dx \Big|_{p_1=0}}{e^{-m_{effective}|x_0|}} \tag{4.9}$$

The situation is exactly the same as in the effective mass. Again the cosh form is better than the exponentional form because it is hard to see a plateau as it is expected in the exponential form. At the same time we should remember that this effective mass is going to be modified by the formula:

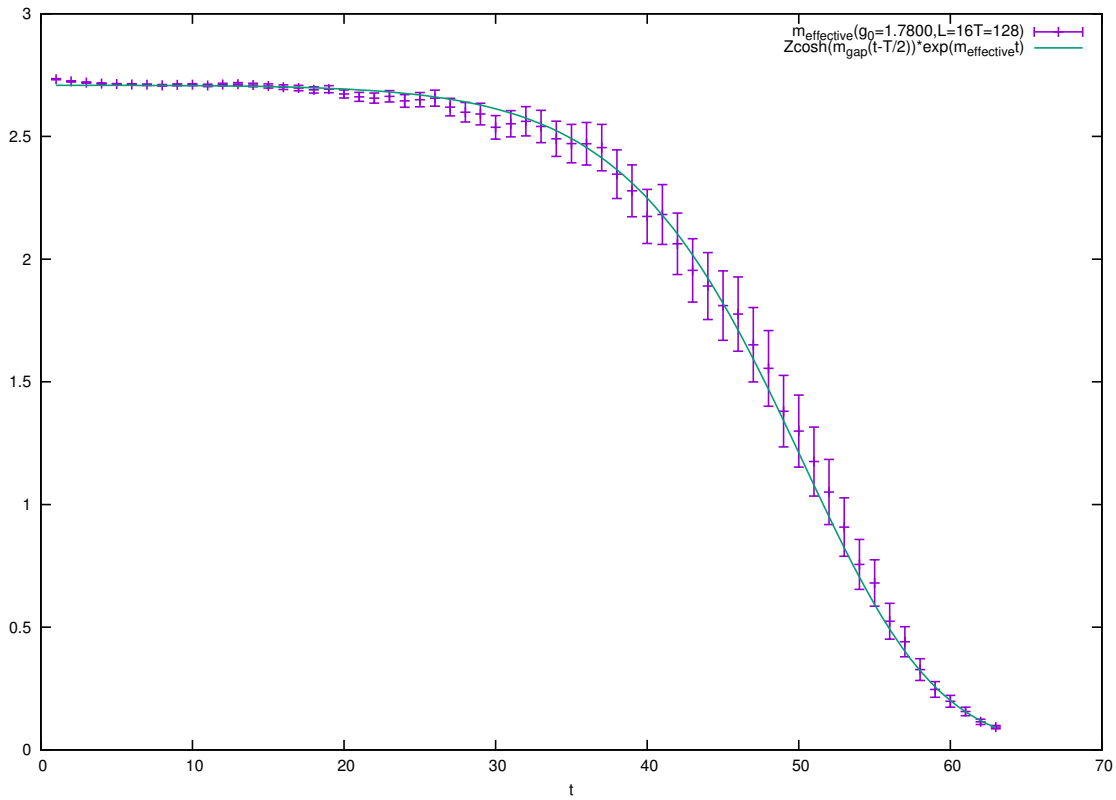

Figure 4.2: Effective Z factor

$$Z_{effective} = \frac{ZL}{sinh(\frac{mT}{2})} \tag{4.10}$$

So Figs 4.2 and 4.1 give us the idea that the equation 4.7 used in our fitting works very well. For this fitting the chi-squared is not shown, although it was calculated, and as it observes in the Fig .4.1, the chi squared analysis gives us the goodness of the fit. This procedure is made in several different parameters, presented in Table 4.1, where the results will be called raw values, which will compared with those modified later.

We use these values to check if our data agree with the Lüscher's data in [41] and the data agrees with each other.

4.2.1.2 Free boundary condition

As in the case of PBC the autocorrelation was studied and considered the results guarantees the correlation is negligible. In this case, more accurate calculation is

$1/g_0^2$	L/a	L'/a	$g^2(L)$	$g^2(L')$	$Z(L)$	$Z(L')$
2.1278	8	16	0.6770(9)	0.7388(9)	0.462(2)	0.397(2)
2.1403	10	20	0.6897(10)	0.7569(9)	0.442(2)	0.379(2)
2.1831	12	24	0.6852(9)	0.7481(8)	0.430(2)	0.368(2)
2.2403	16	32	0.6802(8)	0.7443(7)	0.410(2)	0.352(2)
2.0100	8	16	0.7405(10)	0.8181(10)	0.450(2)	0.381(2)
2.0489	10	12	0.7402(10)	0.8192(9)	0.432(2)	0.380(2)
2.0875	12	24	0.7363(10)	0.8120(8)	0.419(2)	0.354(2)
2.1260	16	32	0.7405(9)	0.8201(7)	0.397(2)	0.335(2)
1.8947	8	16	0.8176(12)	0.9180(12)	0.437(3)	0.361(2)
1.9319	10	20	0.8175(13)	0.9219(9)	0.417(3)	0.345(2)
1.9637	12	24	0.8181(12)	0.9188(8)	0.403(2)	0.333(2)
2.0100	16	32	0.8181(10)	0.9212(7)	0.380(2)	0.315(2)
1.7791	8	16	0.9173(15)	1.0593(15)	0.420(3)	0.338(3)
1.8171	10	20	0.9181(16)	1.0607(9)	0.399(3)	0.322(2)
1.8497	12	24	0.9162(13)	1.0551(8)	0.384(2)	0.310(2)
1.8965	16	32	0.9185 (12)	1.0595(7)	0.362(2)	0.292(2)
1.6589	8	16	1.0577(18)	1.2835(19)	0.398(4)	0.308(3)
1.6982	10	20	1.0591(19)	1.3073 (9)	0.377(4)	0.289(2)
1.7306	12	24	1.0598(16)	1.2768(8)	0.361(3)	0.280(2)
1.7800	16	32	1.0570(14)	1.2743(7)	0.338(3)	0.262(2)

Table 4.1: Raw values in PBC

made, 10^6 measurements. And this calculation the problem related with the autocorrelation was avoided with 10^3 steps between each sample. The numerical calculation is performed on the $(L/a)(T/a)$ lattice with $T = 5L$. For the 2-point function, the source point is set $t_{src} = L$. The fit is made by

$$G(t) = Ze^{-m(t-t_{src})} \tag{4.11}$$

The bare values for this periodic condition are shown in Table 4.2. In this case we did not vary the value of the Z factor because we are fitting our figure by the previous equation. As in the case of periodic boundary condition we could define the real expect operator for free boundary condition as it can be check in the appendix B. We have found that the variation can be neglected within the errors in our calculation. For all of this data the way to obtain the data we have used jackknife method.

$1/g_0^2$	L/a	L'/a	$g^2(L)$	$g^2(L')$	$Z(L)$	$Z(L')$
2.0786	6	12	0.67563(45)	0.73916(58)	1.47397(18)	1.26218(21)
2.1043	7	14	0.67546(52)	0.73939(53)	1.43340(24)	1.22751(16)
2.1275	8	16	0.67561(47)	0.73815(49)	1.40068(19)	1.19895(13)
2.1625	10	20	0.67531(50)	0.74181(50)	1.34583(20)	1.15280(12)
2.1954	12	24	0.67551(45)	0.73719(70)	1.30580(14)	1.11744(20)
2.2403	16	32	0.67522(66)	0.73888(70)	1.24228(27)	1.06337(17)
1.9637	6	12	0.73745(63)	0.81642(64)	1.43817(35)	1.21284(23)
1.9875	7	14	0.73928(57)	0.81902(59)	1.39553(26)	1.17638(18)
2.0100	8	16	0.74019(52)	0.81639(81)	1.36110(20)	1.14569(31)
2.0489	10	12	0.73762(65)	0.81689(70)	1.30494(30)	1.10022(20)
2.0794	12	24	0.73822(58)	0.81633(77)	1.26237(21)	1.06390(22)
2.1260	16	32	0.73847(57)	0.81893(78)	1.19807(17)	1.01021(18)
1.8439	6	12	0.81778(72)	0.92297(73)	1.39548(38)	1.15338(25)
1.8711	7	14	0.81674(79)	0.92205(67)	1.30026(31)	1.04648(21)
1.8947	8	16	0.81744(58)	0.91854(92)	1.31580(22)	1.08621(33)
1.9319	10	20	0.81597(73)	0.92028(80)	1.25618(33)	1.03871(22)
1.9637	12	24	0.81642(64)	0.91751(89)	1.21284(23)	1.00221(23)
2.0100	16	32	0.81639(81)	0.92035(89)	1.14569(31)	0.94769(19)
1.7276	6	12	0.91813(82)	1.06383(86)	1.34688(43)	1.08342(28)
1.7553	7	14	0.91733(72)	1.06329(78)	1.30026(31)	1.04648(21)
1.7791	8	16	0.91771(66)	1.05968(83)	1.26265(24)	1.01564(22)
1.8171	10	20	0.91628(83)	1.05876(93)	1.20088(36)	0.96696(24)
1.8497	12	24	0.91784(73)	1.0539(10)	1.15668(25)	0.93081(25)
1.8965	16	32	0.91762(92)	1.0560(10)	1.08754(33)	0.87618(21)
1.6050	6	12	1.05869(98)	1.2896(11)	1.28507(49)	0.99290(32)
1.6346	7	14	1.0574(11)	1.28754(96)	1.23520(58)	0.95627(24)
1.6589	8	16	1.06071(78)	1.2811(13)	1.19690(27)	0.92364(43)
1.6982	10	20	1.05953(82)	1.2814(11)	1.13264(27)	0.87625(27)
1.7306	12	24	1.05946(85)	1.2772(13)	1.08555(28)	0.84017(28)
1.7800	16	32	1.05791(83)	1.2709(13)	1.01609(22)	0.78761(23)

Table 4.2: Raw values in FreeBC

4.2.2 Continuum limit

Those fittings give the mass gap and the renormalization factor, but all of those calculations are made in a lattice field theory where the results suffer of the discretization error. To eradicate such errors we follow the procedure as [41], where the fixing pa-

rameter to get the continuum limit is the coupling, g^2, that is in the Lüscher scheme, $g^2 = mL$.

One of the main points of this studying is to get the step scaling function, SSF. The SSF is the ratio variation of a quantity with respect a variable, in this case the volume of the system. To get this SSF we use the Monte Carlo simulation. As we said before that to get the continuum limit is fixing g^2 in the smaller volume for each SSF. We need to adjust all the g^2 with the same value. We can do using the definition of the SSF using the β and γ functions. This relation between SSF and β and γ function will be explained in the next chapter. This variations is quite small, around 1%, we believe this variation will not give us relevant systematic errors. Therefore in Tables 4.3 and 4.4 we show the how the previous data are modified with this method.

L/a	L'/a	$g^2(L)$	$g^2(L')$	$\sigma_\phi = Z(L')/Z(L)$
8	16	0.6820	0.7449(15)	0.8581(55)
10	20	”	0.7475(16)	0.8592(55)
12	24	”	0.7442(14)	0.8584(49)
16	32	”	0.7464(13)	0.8587(46)
8	16	0.7400	0.8174(17)	0.8446(71)
10	12	”	0.8189(17)	0.8455(71)
12	24	”	0.8166(16)	0.8453(63)
16	32	”	0.8171(13)	0.8450(58)
8	16	0.8180	0.9185(20)	0.8267(80)
10	20	”	0.9225(20)	0.8288(77)
12	24	”	0.9186(17)	0.8280(69)
16	32	”	0.9211(16)	0.8285(62)
8	16	0.9170	1.0588(25)	0.8049(93)
10	20	”	1.0592(23)	0.8068(86)
12	24	”	1.0563(21)	0.8061(78)
16	32	”	1.0574(19)	0.8070(69)
8	16	1.057	1.2824(35)	0.773(11)
10	20	”	1.3038 (33)	0.770(10)
12	24	”	1.2724(27)	0.7751(90)
16	32	”	1.2742(25)	0.7762(79)

Table 4.3: Modified values in PBC

After this modification, we are now in the position to calculate the continuum limit. All the continuum extrapolations can be written in a single Table, so the

L/a	L'/a	$g^2(L)$	$g^2(L')$	$\sigma_\phi = Z(L')/Z(L)$
6	12	0.675	0.73840(79)	0.85645(28)
7	14	”	0.73883(82)	0.85646(32)
8	16	”	0.73742(75)	0.85610(28)
10	20	”	0.74143(79)	0.85664(30)
12	24	”	0.73658(88)	0.85586(29)
16	32	”	0.7386(10)	0.85603(41)
6	12	0.739	0.8183(10)	0.84298(40)
7	14	”	0.81867(92)	0.84302(34)
8	16	”	0.8150(10)	0.84198(36)
10	12	”	0.8186(11)	0.84282(40)
12	24	”	0.8173(11)	0.84262(36)
16	32	”	0.8196(11)	0.84308(34)
6	12	0.816	0.9206(11)	0.82691(45)
7	14	”	0.9210(12)	0.82695(49)
8	16	”	0.9167(11)	0.82582(40)
10	20	”	0.9203(12)	0.82687(45)
12	24	”	0.9169(12)	0.82642(39)
16	32	”	0.9198(14)	0.82726(48)
6	12	0.917	1.0622(14)	0.80465(50)
7	14	”	1.0628(13)	0.80490(42)
8	16	”	1.0587(12)	0.80453(38)
10	20	”	1.0598(15)	0.80504(50)
12	24	”	1.0528(14)	0.80491(43)
16	32	”	1.0552(13)	0.80579(53)
6	12	1.058	1.0622(14)	1.2885(19)
7	14	”	1.0628(13)	1.2884(20)
8	16	”	1.2871(35)	1.2769(18)
10	20	”	1.3086 (33)	1.2791(17)
12	24	”	1.2770(27)	1.2750(18)
16	32	”	1.2788(25)	1.2710(18)

Table 4.4: Modified values in FreeBC

following Tables show the calculation for each set of values. As an example, the Table 4.5 shows the values for PBC in time component for one set of values. This procedure is going to be used for all the set of the values for both the boundary conditions, at the same time we plot the continuum extrapolation for the case of the SSF for the coupling in the Fig (4.3) where the needed value is when $a = 0$, continuum

limit extrapolation. This continuum extrapolation was made using a polynomial in a^2 as it is made in [41]. This continuum extrapolation was confirm later in [68], where using Symanzik's effective action it shows $mathbbO(a^2)$ lattice corrections.

L/a	L'/a	$g^2(L)$	$g^2(L')$		$Z(L')/Z(L) = \sigma_\phi(2, g^2(L))$	
8	16	0.6820	0.7449(15)		0.8581(55)	
10	20	"	0.7475(16)		0.8592(55)	
12	24	"	0.7442(14)		0.8584(49)	
16	32	"	0.7464(13)		0.8587(46)	
Continuum limit		"	0.7451(70)		0.8558(5)	

Table 4.5: Continuum limit extrapolation for $g_L^2 = 0.6820$

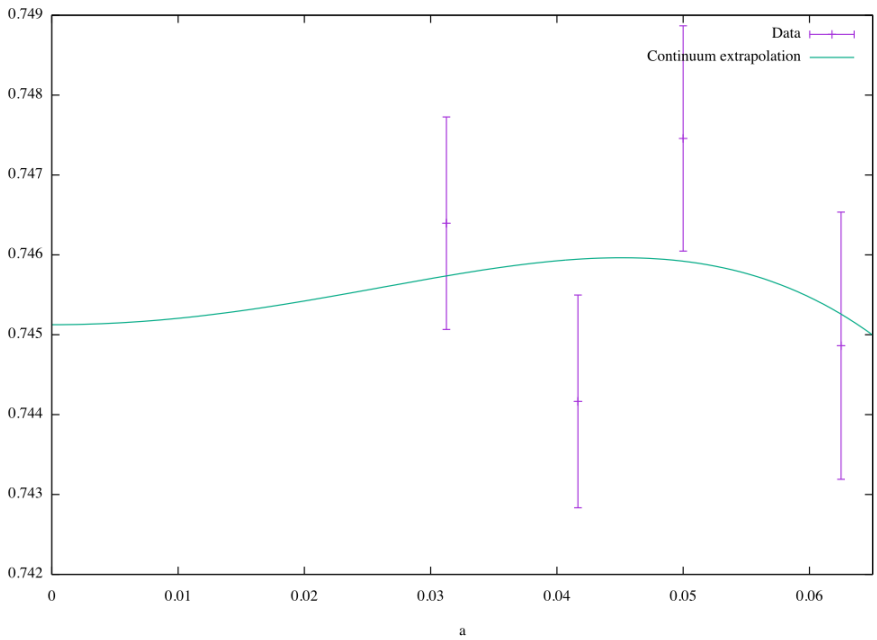

Figure 4.3: Continuum extrapolation for the SSF when $g^2 = 0.6820$ and $s = 2$

In Table 4.5 the important value is the result in the continuum limit. In each set of values we will proceed in the same way and Tables 4.7 and 4.6 summarize the data for each boundary condition.

Before going to the next step, we should mention that the continuum extrapolation is not a trivial issue. Actually there is a long discussion about it. For our continuum limit we use the same as Lüscher [41], in the other hand we mention another kind of continuum extrapolation was proposed by Patrascioiu [87]. This was based on the extrapolation in O(2) where this extrapolation is more accurate. We will not use the Patrascioiu's proposal because this O(3) and O(2) the difference are quite evidence, like the mass generation does not occur in O(2) sigma model.

$g^2(L)$	$g^2(L)$	$g^2(L')/g^2(L)$	$Z(L')/Z(L)$
0.6820	0.7451(70)	1.092(10)	0.8558(5)
0.7400	0.8150(37)	1.1013(50)	0.8436(6)
0.8180	0.9197(80)	1.1243(98)	0.8276(7)
0.9170	1.0564(46)	1.1521(49)	0.8060(5)
1.0570	1.244(47)	1.177(39)	0.7759(12)

Table 4.6: Continuum limit extrapolation in PBC

$g^2(L)$	$g^2(L')$	$g^2(L')/g^2(L)$	$Z(L')/Z(L)$
0.6820	0.7380(32)	1.0933(47)	0.8558(4)
0.7390	0.8209(26)	1.1109(35)	0.8434(7)
0.8180	0.9198(33)	1.1272(40)	0.8278(7)
0.9170	1.0509(39)	1.1460(42)	0.8060(5)
1.0600	1.2676(54)	1.1981(51)	0.7761(9)

Table 4.7: Continuum limit extrapolation in FreeBC

In Tables 4.7 and 4.6 we show the final result for the Monte Carlo simulation, i-e, non-perturbative calculation. These values are considered as the continuum limit and non-pertubative values of the SSF for O(3) sigma model. We are ready to compare the expected results from the perturbative calculation using the SSF tool.
Before going through the comparison some concepts should be understood. The difference between boundary conditions in time component in this calculation should not give any difference because we will treat the time component as an infinite component. This is possible because in lattice calculation, if the difference between the ground state and the first excited state is big enough, the ground state can be isolated, being this isolation impossible in perturbative calculations.
Anyhow, we can observe there is not much difference between the data from PBC and FBC, which confirms our previous speculation.
The difference observed is with in the error of the mass gap and the discrepancy in the last set of values for the PBC and FBC. The errors are smaller in FBC because our measured samples with this boundary condition is 10 times larger. On the other hand we observe the data in PBC in the last set, the final continuum extrapolation have errors around 3% while all the other error are around 0.5%. We suppose that this error is because we get the results in different discrete lattice which amplifies the error in the continuum limit. At the same time for this set of values we have the Lüscher's calculation [41], which agrees with our calculation with the FBC.

4.3 Another operators

In the previous part, we have calculated the operator which respect we have defined our scheme. So if we now want to evaluate the agreement for the RGE in finite volume scheme, we need to observe this RGE in other operators. Right now we consider the evaluation of two operators in the O(3) sigma model, vacuum energy density and magnetic susceptibility. These operators can be calculated in PT as we do for the vacuum energy density, although as we show in later section, the PT is not our goal because we want to be able to get the information in non-PT calculation. For those operators to get the continuum limit, as we discuss later on,this process varies with the previous method where the SSF was used.
At the same time for all of the computation we are using just PBC in our calculation and the values for the samples are the same as the measurements in the 2-point correlation function.

4.3.1 Vacuum energy density

The definition of this quantity is:

$$E = \frac{1}{TL}\int \langle \frac{1}{g_0^2}(\partial_{x_0}S^i(x)\partial_{x_0}S^i(x) + \partial_{x_1}S^i(x)\partial_{x_1}S^i(x))\rangle dx \tag{4.12}$$

In this case the definition is a direct value. So we do not need as the previous operators to fit the result to any function. In the Table 4.8 we present the obtained values.

At first glance, from the Table 4.8 we observe a surprising result if we compare with the calculation at LO in PT.

$$E_{(PT)}^{(PBC)} = \frac{(N-1)^2}{12L^2}g_R^2 \tag{4.13}$$

From this expression we should expect that the results doing SSF vary around 1/4. Because we have taken double volume, $L' = sL$, but that is not the case. This is telling us that the vacuum energy density have a clear non-perturbative function for the range we are considering. Using dimensional arguments we can get to the conclusion that

$$E_{(non-PT)}^{(PBC)} = C_E W(g_R^2)\Lambda_{MS}^2\mu^2 \tag{4.14}$$

β	L/a	L'/a	$g^2(L)$	$g^2(L')$	$E(L)$	$E(L')$
2.1278	8	16	0.6770(9)	0.7388(9)	1.0839(2)	1.0876(2)
2.1403	10	20	0.6897(10)	0.7569(9)	1.0850(2)	1.0872(1)
2.1831	12	24	0.6852(9)	0.7481(8)	1.0831(2)	1.0847(1)
2.2403	16	32	0.6802(8)	0.7443(7)	1.0806(2)	1.0816(1)
2.0100	8	16	0.7405(10)	0.8181(10)	1.0923(2)	1.0965(1)
2.0489	10	12	0.7402(10)	0.8192(9)	1.0916(2)	1.0938(1)
2.0875	12	24	0.7363(10)	0.8120(8)	1.0894(1)	1.0912(1)
2.1260	16	32	0.7405(9)	0.8201(7)	1.0877(1)	1.0888(1)
1.8947	8	16	0.8176(12)	0.9180(12)	1.1025(2)	1.10756(8)
1.9319	10	20	0.8175(13)	0.9219(9)	1.1013(1)	1.10442(7)
1.9637	12	24	0.8181(12)	0.9188(8)	1.0994(1)	1.10163(6)
2.0100	16	32	0.8181(10)	0.9212(7)	1.0965(1)	1.09786(4)
1.7791	8	16	0.9173(15)	1.0593(15)	1.11559(16)	1.12202(8)
1.8171	10	20	0.9181(16)	1.0607(9)	1.11373(13)	1.11777(7)
1.8497	12	24	0.9162(13)	1.0551(8)	1.11112(10)	1.11408(5)
1.8965	16	32	0.9185 (12)	1.0595(7)	1.10733(8)	1.10912(4)
1.6589	8	16	1.0577(18)	1.2835(19)	1.13333(16)	1.14294(8)
1.6982	10	20	1.0591(19)	1.3073 (9)	1.13067(13)	1.1385(7)
1.7306	12	24	1.0598(16)	1.2768(8)	1.12739(10)	1.13176(5)
1.7800	16	32	1.0570(14)	1.2743(7)	1.12189(8)	1.12453(4)

Table 4.8: Raw values of the vacuum energy density

Where C_E is a constant values, $W(g_R^2)$ an analytic function in g_R^2 and Λ_{MS} parameter in our theory. Because this Λ parameter is proportional to $e^{\frac{1}{\beta_0 g_R^2}}$ the previous expression is a genuine non-perturbative expression. Nowadays there is not any reliable way to compute this value using analytical result. Although as it was already pointed out in [86, 84], this vacuum energy density has genuine non-perturbative function. So this volume dependence confirms their claim too.

We are not going to process the values for the vacuum energy density to get the continuum limit as we proceed before. In this case we will mention a method using computational values in different values of bare parameter close to a sample already measure. The idea is to compute this variation of the bare coupling inside the operator computation.

$$\langle \mathcal{O} \rangle \rfloor_{g_0^2+\Delta g_0^2} = \langle \mathcal{O} e^{\frac{\Delta g_0^2}{(g_0^2)^2} S} \rangle \rfloor_{g_0^2} \tag{4.15}$$

Where S is the action. Thanks of this method we can compute the operator in a range of one sample, so we do not need the information of the SSF for this quantity. Anyway the most important idea of this analysis is that the vacuum energy density is a genuine non-perturbative effect. And the variation with the volume is absent.

4.3.2 Magnetic susceptibility

For the calculation of magnetic susceptibility, χ. Even we can think we did not calculate, but that is not the case. As an initial point, we should define this quantity:

$$\chi = \frac{1}{TL} \sum_{y_0, y_1} \int \langle S_i(y_0, y_1) S^i(x_0, x_1) \rangle \tag{4.16}$$

And we can remember the 2-point correlation function calculation in the case of PBC. So if we integrate the value of 2-point correlation function calculated before we get the value of the magnetic susceptibility.

$$\chi = \frac{1}{TL} \sum_{y_0, y_1} \langle S_i(y_0, y_1) S^i(x_0, x_1) \rangle = \frac{1}{T} \sum_{y_0} \frac{Z}{\sinh(\frac{mT}{2})} \cos(m(|y_0 - x_0 + T/2|)) = 2\frac{Z}{Tm} = \tag{4.17}$$

The previous equation can be modified if we use the relation used in the calculation $T = 8L$, So the value will follow $\chi \propto Z/g_L^2$. In the Table 4.9 the bare results are presented:

As in the case of the calculation of the vacuum energy density, we are not going to do the continuum limit because to be able to subtract information of this operator we will need to define some concepts of RGE in a finite box, considered in the next chapter.

β	L/a	L'/a	$g^2(L)$	$g^2(L')$	$\chi(L)$	$\chi(L')$
2.1278	8	16	0.6770(9)	0.7388(9)	$8.560(13)10^{-2}$	$6.734(11)10^{-2}$
2.1403	10	20	0.6897(10)	0.7569(9)	$8.031(12)10^{-2}$	$6.264(10)10^{-2}$
2.1831	12	24	0.6852(9)	0.7481(8)	$7.852(12)10^{-2}$	$6.167(10)10^{-2}$
2.2403	16	32	0.6802(8)	0.7443(7)	$7.542(11)10^{-2}$	$5.916(9)10^{-2}$
2.0100	8	16	0.7405(10)	0.8181(10)	$7.616(13)10^{-2}$	$5.838(10)10^{-2}$
2.0489	10	12	0.7402(10)	0.8192(9)	$7.306(12)10^{-2}$	$5.596(10)10^{-2}$
2.0875	12	24	0.7363(10)	0.8120(8)	$7.139(11)10^{-2}$	$5.471(10)10^{-2}$
2.1260	16	32	0.7405(9)	0.8201(7)	$6.699(10)10^{-2}$	$5.133(9)10^{-2}$
1.8947	8	16	0.8176(12)	0.9180(12)	$6.687(12)10^{-2}$	$4.930(9)10^{-2}$
1.9319	10	20	0.8175(13)	0.9219(9)	$6.402(11)10^{-2}$	$4.709(8)10^{-2}$
1.9637	12	24	0.8181(12)	0.9188(8)	$6.170(10)10^{-2}$	$4.541(8)10^{-2}$
2.0100	16	32	0.8181(10)	0.9212(7)	$5.839(10)10^{-2}$	$4.297(8)10^{-2}$
1.7791	8	16	0.9173(15)	1.0593(15)	$5.722(10)10^{-2}$	$3.999(8)10^{-2}$
1.8171	10	20	0.9181(16)	1.0607(9)	$5.451(10)10^{-2}$	$3.780(7)10^{-2}$
1.8497	12	24	0.9162(13)	1.0551(8)	$5.254(9)10^{-2}$	$3.685(7)10^{-2}$
1.8965	16	32	0.9185 (12)	1.0595(7)	$4.944(9)10^{-2}$	$3.456(7)10^{-2}$
1.6589	8	16	1.0577(18)	1.2835(19)	$4.713(10)10^{-2}$	$3.001(7)10^{-2}$
1.6982	10	20	1.0591(19)	1.3073 (9)	$4.464(9)10^{-2}$	$2.765(6)10^{-2}$
1.7306	12	24	1.0598(16)	1.2768(8)	$4.266(8)10^{-2}$	$2.749(6)10^{-2}$
1.7800	16	32	1.0570(14)	1.2743(7)	$3.999(8)10^{-2}$	$2.589(5)10^{-2}$

Table 4.9: Raw values of the magnetic susceptibility

Chapter 5

RGE for volume dependence of O(3) sigma model

In this chapter, the numerical values, non-perturbative results, are going to be compared with the perturbative calculations. To be able to compare those results, we need to understand the concept of step scaling function, SSF. The SSF is the key concept to compare the Monte Carlo simulations calculations we the expected values from perturbative calculation. Roughly speaking, the idea is that SSF is the integration of the RGE. So if we are able to integrate this RGE, we will get the information of this RGE.
The steps we are going to follow in this chapter are directed to generate this RGE for O(N) sigma model. After RGE definition, we will present the concept of SSF and how we can compare the expected SSF of PT with our non-PT calculation for the operator through the RGE was defined. Once we define all the tools of RGE in finite box, we will implement this RGE for two operators of which we can obtain information of their composition using RGE.

5.1 RGE in finite box

In the last section of the chapter 2 we have define the RGE in a finite box in a general way, but now we should define them for O(N) sigma model. From the previous sections the definition of the β and γ function was already made, so what we need to get is the r.h.s of the RGE in finite box. This is the non-homogeneous term in the equation, as it is shown in the chapter 2.

As initial point, we have developed the tools to obtain this non-homogeneous term, but those tools are based on Feymann diagrams ideas, therefore PT calculations. That is why we need to consider some issues about O(N) in perturbative calculations like instanton sectors and parametrization. Because the topology is trivial in a finite box, the PT in the trivial instanton sector is considered as a complete value. Now it is when the concept of different parametrization shows up, in PT calculation of O(N) sigma model was used the canonical parametrization, north pole parametrization, but for the propose of this section we are going to use other parametrization. We are going to use the relaxed constrain parametrization because for this propose we get simpler Lagrangian, due to the interaction terms in this parametrization are finite. We remember that the main idea of this parametrization is to interchange the sphere constriction for a limit of a gaussian function.

$$\delta(x) = \lim_{M\to\infty}(M/\sqrt{\pi})e^{(-M^2x^2)} \tag{5.1}$$

Translated for our constriction:

$$\delta(S^iS^i-1) = \lim_{M\to\infty}(M/(\sqrt{8\pi})e^{(-\frac{M^2}{8}(S^iS^i-1)^2)} \quad S^i \to (g\pi^i, 1+g\sigma) \tag{5.2}$$

This leads us to the next Lagrangian

$$\mathcal{L}_0 = \int(\sum_{i=1}^{N-1}\frac{1}{2}\partial_\mu\pi^i(x)\partial_\mu\pi^i(x)\frac{1}{2}\partial_\mu\sigma(x)\partial^\mu\sigma(x) + \frac{1}{2}M^2\sigma(x)\sigma(x)+$$

$$+M^2(\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\frac{1}{8}(\pi^i)^2(x)(\pi^j)^2(x) + \sum_{i=1}^{N-1}\frac{1}{2}(\pi^i)^2(x)\sigma(x)$$

$$+\sum_{i=1}^{N-1}\frac{1}{4}(\pi^i)^2(x)\sigma^2(x) + \sum_{i=1}^{N-1}\frac{1}{4}(\pi^i)^2(x)\sigma(x) + \frac{1}{2}\sigma^3(x) + \frac{1}{8}\sigma^4(x))dx \tag{5.3}$$

The reason why this parametrization is now evident, in this parametrization the number of interaction terms are finite and they do not have any derivative in this interaction terms, so the recipe for the non-homogeneous term is easier than for the others parametrizations, notice that in another parametrizations the interactions terms are always infinite. Calling the result of this r.h.s of RGE in finite volume

$$L\frac{d}{dL}\langle\mathcal{O}\rangle = \langle(-\int(\frac{d\phi(x)}{dx})^2dx-(k+dm)S_I+da)\mathcal{O}\rangle-\langle(-\int(\frac{d\phi(x)}{dx})^2dx-(k+dm)S_I)\rangle\langle\mathcal{O}\rangle \tag{5.4}$$

So if we fit the previous formula in one interaction term, for the 3-point interactions, $m_{3-p} = 1/2$, but for the 4-point interactions, $m_{4-p} = 1$. Due to the lack of derivative

terms in the interaction we get that k is always zero. At the same time, we only have one dimension which is finite and from which we perform the RGE, plus the operator in this parametrization

$$\langle S^i(x)S^i(y)\rangle = 1 - g_0\langle\sigma(x)+\sigma(y)\rangle + g_0^2(\langle\pi^i(x)\pi^i(y)\rangle + \langle\sigma(x)\sigma(y)\rangle). \tag{5.5}$$

On the other hand, the values for a are known too. Being for $\langle\sigma\rangle$, $a_{\langle\sigma\rangle} = 1/2$ and for $\langle\pi^2\rangle$ and $\langle\sigma^2\rangle$ $a_{\langle\pi^2\rangle} = a_{\langle\sigma^2\rangle} = 0$. Because a depends on the external field in the diagram, as it was explained before. Summing all this information, the non-homogenous insertion operator is coming from

$$\mathcal{O}_L = -\int \frac{1}{2}(\partial_{x_1}\pi^a\partial_{x_1}\pi^a + \partial_{x_1}\sigma\partial_{x_1}\sigma) +$$

$$M^2\int(\frac{1}{8}g^2(\pi^2+\sigma^2)^2 + \frac{1}{4}g\sigma(\pi^2+\sigma^2)) = \tag{5.6}$$

We did not get the final insertion operator yet, because we have to come back to the initial definition of the field $S^i = (g\pi^i, 1+g\sigma)$. That is why we have not considered $\langle\sigma\rangle$, because once we come back to the initial parametrization this value is vanish due to the symmetry, although if we keep it at the end of the calculation will be vanished too. From the eq. (5.6), we only describe the last term because the first term has a straightforward transformation. Then if we come back to the initial field definition:

$$M^2\int(\frac{1}{8g^2}(S^i\cdot S^i - 1)(S^i\cdot S^i - 1 + 2S^N) \tag{5.7}$$

This expression can be manipulated by some identities of Dirac's delta

$$\delta(x) = \lim_{M\to\infty}(M/\sqrt{\pi})e^{(-M^2x^2)}, \tag{5.8}$$

$$\delta'(x) = \lim_{M\to\infty}(-M^3/\sqrt{\pi})2xe^{(-M^2x^2)} \tag{5.9}$$

With the previous definitions the next expression can be absorb in $\delta'(x)$:

$$(M/\sqrt{\pi})\int dSM^2\int(\frac{1}{8g^2}(S^i\cdot S^i - 1)(S^i\cdot S^i - 1 + 2S^N)e^{-\mathcal{L}} =$$

$$= \frac{1}{2}\int\delta'(S_x^2 - 1)(S^i\cdot S^i + 1 - 2S^N)e^{-\mathcal{L}} =$$

$$= \int\delta(S_x^2 - 1)\int(1-\mathcal{L})e^{-\mathcal{L}} \tag{5.10}$$

Where in the last step, we have to define the N-component field in the spherical coordinate and the $\delta'(R_S^2 - 1)$, where R_S is the radius in the spherical coordinate of N-component field. Therefore the r.h.s in the RGE which is going to be called finite volume operator for O(N) sigma model:

$$\mathcal{O}_L = \int (\frac{1}{g^2}\partial_{x_1} S^i \partial_{x_1} S^i - \mathcal{L}) dx \tag{5.11}$$

This operator shows us important consequences. One is clear, this operator is O(N) invariant so as we could imagine that finite volume scheme does not break the O(N) symmetry.

$$[-L\frac{\partial}{\partial L} + \beta(g_L^2) + \frac{n}{2}\gamma(g_L^2)]G^n[p_1, ..., g_L^2, L] = G^n_{\mathcal{O}_\mathcal{L}}[p_1, ..., g_L^2, L] \tag{5.12}$$

Where $G^n_{\mathcal{O}_\mathcal{L}}[p_1, ..., g_L^2, L]$ is the n-correlation function of with the insertion of the finite volume operator of the O(N) sigma model. Now we can conclude importance consequence of this operator. The first one, and the most important, is that this insertion operator does not need and new renormalization operator because this insertion operator is defined in terms of the operators in the Lagrangian. Thanks of this we are able to speak about the reliability of the RGE for volume dependence because this will lead us that γ-function is finite and therefore regularization independence. As the same time, like Callan-Symanzik equation, this also lead us to the conclusion that γ and β functions are L independent in any order. Another important topic is that the operator in the eq. (5.12) is the operator of the non-homogenous term in the RGE equation. So we see the variation of any other operator is managed by this operator $\mathcal{O}_L$, where the volume dependence is encoded, and this operator is O(N) symmetric, so this volume finite box operator will not break the symmetry. Consequently there is no polarization in the system and the mass generation is a dynamic effect.

5.2 Step scaling function (SSF)

The concept of the step scaling function is quite straightforward, the idea is to choose one variable, in this case the volume, and observe the variation of other quantity, in our calculation the mass gap and the Z factor. In our procedure we will follow the SSF in [41] by Lüscher. Due to the this scheme for the γ and β functions, we can

relate the scaling function with the those functions if the variation of the quantity to study is the normalization factor and the renormalized coupling respectively.
The definition of the SSFs are:

$$g^2(sL) = \sigma_g(s, g^2(L)) \tag{5.13}$$

$$Z_\phi(sL) = \sigma_\phi(s, g^2(L)) \tag{5.14}$$

The relation with the γ and β functions is straightforward, becuase if we consider the variable s as the relation between the initial and final volume, $s = \frac{L'}{L}$.

$$\beta(g^2(sL)) = \beta(\sigma_g(s, g^2(L))) = -s\frac{\partial \sigma_g(s, g^2(L))}{s} \tag{5.15}$$

$$\gamma(g^2(sL)) = \gamma(\sigma_g(s, g^2(L))) = -s\frac{\partial \log[\sigma_\phi(s, g^2(L))]}{s} \tag{5.16}$$

Using the definition for the γ and β functions in the perturbative theory, the respective definition of the SSF are given by:

$$\sigma_g(s, u) = u + \sigma_{g0}(s)u^2 + \sigma_{g1}(s)u^3 + \sigma_{g2}(s)u^4 + ... \tag{5.17}$$

$$\sigma_\phi(s, u) = 1 + \sigma_{\phi 0}(s)u + \sigma_{\phi 1}(s)u^2 + \sigma_{\phi 2}(s)u^3 + ... \tag{5.18}$$

With the initial condition, $\sigma_{Xi}(s = 1) = 0$ for any physical quantity.
Within all of these considerations we finally express the SSF with the coefficients of γ and β functions.

$$\begin{aligned}
\sigma_{g0}(s) &= \beta_0 \log(s) \\
\sigma_{g1}(s) &= \beta_1 \log(s) + b_0^2 \log^2(s) \\
\sigma_{g2}(s) &= \beta_2 \log(s) + \frac{5}{2}\beta_0\beta_1 \log^2(s) + \beta_0^3 \log^3(s) \\
\sigma_{g3}(s) &= \beta_3 \log(s) + (\frac{3}{2}\beta_0\beta_2 + 3\beta_1^2) \log^2(s) + \frac{13}{3}(\beta_0^2\beta_1) \log^3(s) + \beta_0^4 \log^4(s)
\end{aligned} \tag{5.19}$$

$$\begin{aligned}
\sigma_{\phi 0}(s) &= \gamma_0 \log(s) \\
\sigma_{\phi 1}(s) &= \gamma_1 \log(s) + \frac{1}{2}(\beta_0 + \gamma_0)\gamma_0 \log^2(s) \\
\sigma_{\phi 2}(s) &= \gamma_2 \log(s) + (\frac{1}{2}\beta_1\gamma_0 + \beta_0\gamma_1 + \gamma_0\gamma_1) \log^2(s) + \frac{1}{3}(\beta_0 + \frac{1}{2}\gamma_0)(\beta_0 + \gamma_0)\gamma_0 \log^3(s)
\end{aligned} \tag{5.20}$$

For these results for the SSF for the coupling was made already by Lüscher up to NNLO. In here we present the SSF for the coupling at the NNNLO and the SSF for the strength field at NNLO. The values for the β's and γ's come from the perturbative calculation. Where β is up to calculated until NNNLO by [61] and γ for this scheme was calculated in the previous sections.

5.3 Comparison of non-perturbative calculation and perturbative of β and γ functions

With all the previous calculation, we now can evaluate if both formalisms give us similar results. we need to check if SSFs from the γ and β functions and the direct SSF obtained by direct calculation in non-perturbative lattice field theory. Then we will remember previous results and combine them.
We should start gathering the values of the γ and β functions in the scheme we are treating, volume dependence. We present the numerical values of the coefficients of those functions in the perturbation theory.

$$\beta_0 = 0.15915494, \quad \beta_1 = 0.02533029, \quad \beta_2 = 0.00806288, \quad \beta_3 = 0.00248074$$

$$\gamma_0 = 0.3183098, \quad \gamma_1 = 0.000, \quad \gamma_2 = 0.000 \tag{5.21}$$

The SSF calculation has to be compared between the Monte Carlo simulation, which was named as non-perturbative calculation, and the expect results coming from 1-loop, 2-loop and so on.

PBC results for the SSF of the coupling

$g^2(L)$	$\frac{\sigma_g^{1_{loop}}(2,g^2(L))}{g^2(L)}$	$\frac{\sigma_g^{2_{loop}}(2,g^2(L))}{g^2(L)}$	$\frac{\sigma_g^{3_{loop}}(2,g^2(L))}{g^2(L)}$	$\frac{\sigma_g^{4_{loop}}(2,g^2(L))}{g^2(L)}$	$\frac{g'^2(L'=2L)}{g^2(L)}$
0.682	1.0752	1.0891	1.0928	1.0940	1.092(10)
0.740	1.0816	1.0979	1.1027	1.1042	1.1013(50)
0.818	1.0902	1.1101	1.1168	1.1189	1.1243(98)
0.917	1.1012	1.1262	1.1352	1.1388	1.1521(49)
1.057	1.1166	1.1498	1.1637	1.1701	1.177(39)

PBC results for the SSF of the strength field

$g^2(L)$	$\sigma_\phi^{1_{loop}}(2,g^2(L))$	$\sigma_\phi^{2_{loop}}(2,g^2(L))$	$\sigma_\phi^{3_{loop}}(2,g^2(L))$	$\sigma_\phi(2,g^2(L))$
0.682	0.8495	0.8552	0.8546	0.8558(5)
0.740	0.8367	0.8434	0.8426	0.8436(6)
0.818	0.8195	0.8277	0.8266	0.8276(7)
0.917	0.7977	0.8079	0.8064	0.8060(5)
1.057	0.7668	0.7804	0.7781	0.7759(12)

FBC results for the SSF of the coupling

$g^2(L)$	$\frac{\sigma_g^{1loop}(2,g^2(L))}{g^2(L)}$	$\frac{\sigma_g^{2loop}(2,g^2(L))}{g^2(L)}$	$\frac{\sigma_g^{3loop}(2,g^2(L))}{g^2(L)}$	$\frac{\sigma_g^{4loop}(2,g^2(L))}{g^2(L)}$	$\frac{g'^2(L'=2L)}{g^2(L)}$
0.675	1.0745	1.0880	1.0916	1.0927	1.0933(47)
0.739	1.0815	1.0978	1.1025	1.1040	1.1109(35)
0.816	1.0900	1.1098	1.1162	1.1185	1.1272(40)
0.917	1.1011	1.1262	1.1352	1.1388	1.1460(42)
1.058	1.1167	1.1450	1.1639	1.1703	1.1981(51)

FBC results for the SSF of the strength field

$g^2(L)$	$\sigma_\phi^{1loop}(2,g^2(L))$	$\sigma_\phi^{2loop}(2,g^2(L))$	$\sigma_\phi^{3loop}(2,g^2(L))$	$\sigma_\phi(2,g^2(L))$
0.675	0.8511	0.8566	0.8560	0.8558(4)
0.739	0.8370	0.8436	0.8428	0.8434(7)
0.816	0.8200	0.8281	0.8270	0.8278(7)
0.917	0.7977	0.8079	0.8064	0.8060(5)
1.058	0.7666	0.7802	0.7779	0.7761(9)

We can easily calculate the SSF function using the Ep. (5.17), (5.18), (5.19) and (5.20). Because our propose is to compare results, the non-perturbative calculation is carried out in all the cases where the volume was twice bigger than the initial. So the renormalization coupling is the only free parameter to define in the SSF. Consequently the best way to compare all the results together is plotting SSF for different approaches and different couplings and keeping $s = 2$, as is shown in Fig. 5.1 and 5.2.

This way to plot the SSF is not the conventional way to represent, it is usual to compute all running coupling for the range of the coupling.But we consider this way it is more pragmatic because in the running coupling used to accumulate the error for each step while in our way, the error of each set of values not accumulates in other sets.

The way to look this kind of figures is to observe that for higher loop calculation the results get closer to the Monte Carlo simulation and with higher renormalized coupling the discrepancy increases. Those behaviour appears in Figs 5.1 and 5.2, although in Fig 5.1 is hardly observe of because the error bars are at the same order as the difference between the 2-loop and 3-loop calculations. Anyhow when the error is small this behaviour is easly observed. Actually the propose of this study is not this comparison of the β function because it was already made. So we are not aware to get results with smaller errors, although is important to show the results to compare with the previous papers and check there is not discrepancies with our results. In the other hand in the figure 5.2 the expected result is clearly obtained. In smaller coupling the errors for the SSF in the strength field is almost the same order as the difference

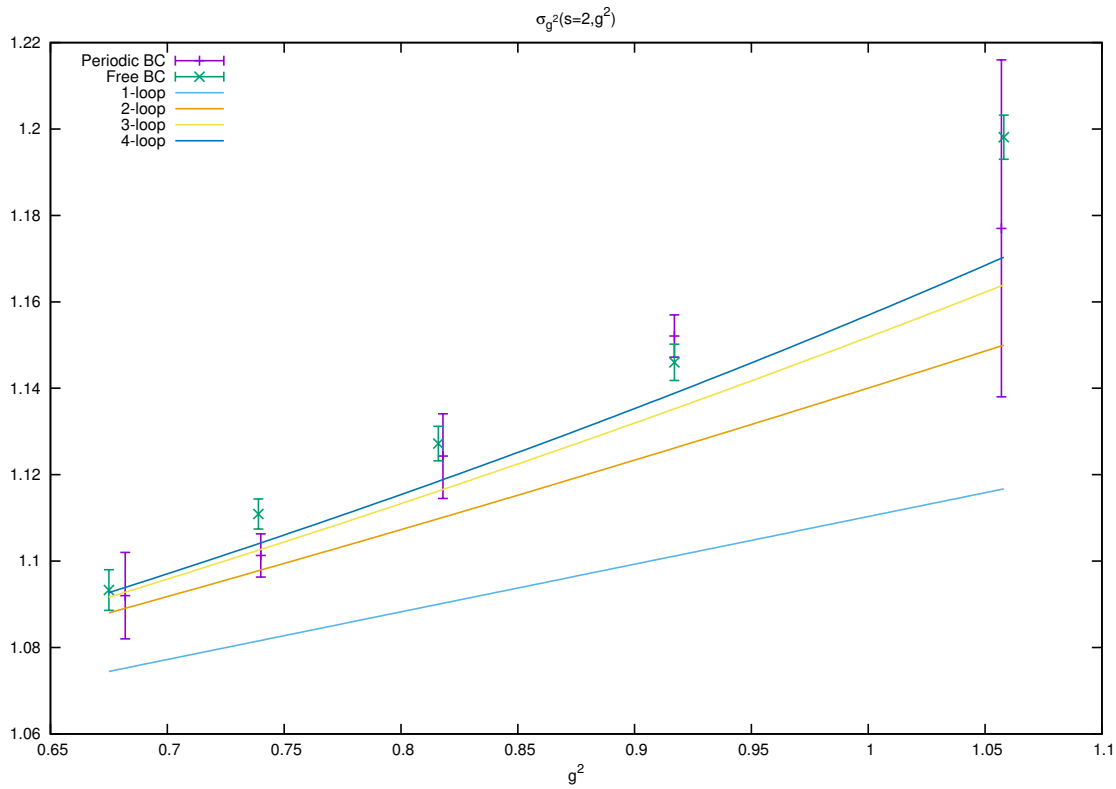

Figure 5.1: SSF of g_R^2.

between 2-loop and 3-loop, but when we are going to higher quantities it is easily observer that 3-loop has much better approximation than 2-loops approximation. We will not say easily this calculation confirm the perturbative calculation, but we can say that the perturbative calculation in this range of coupling is not far form the real non-perturbative calculation.

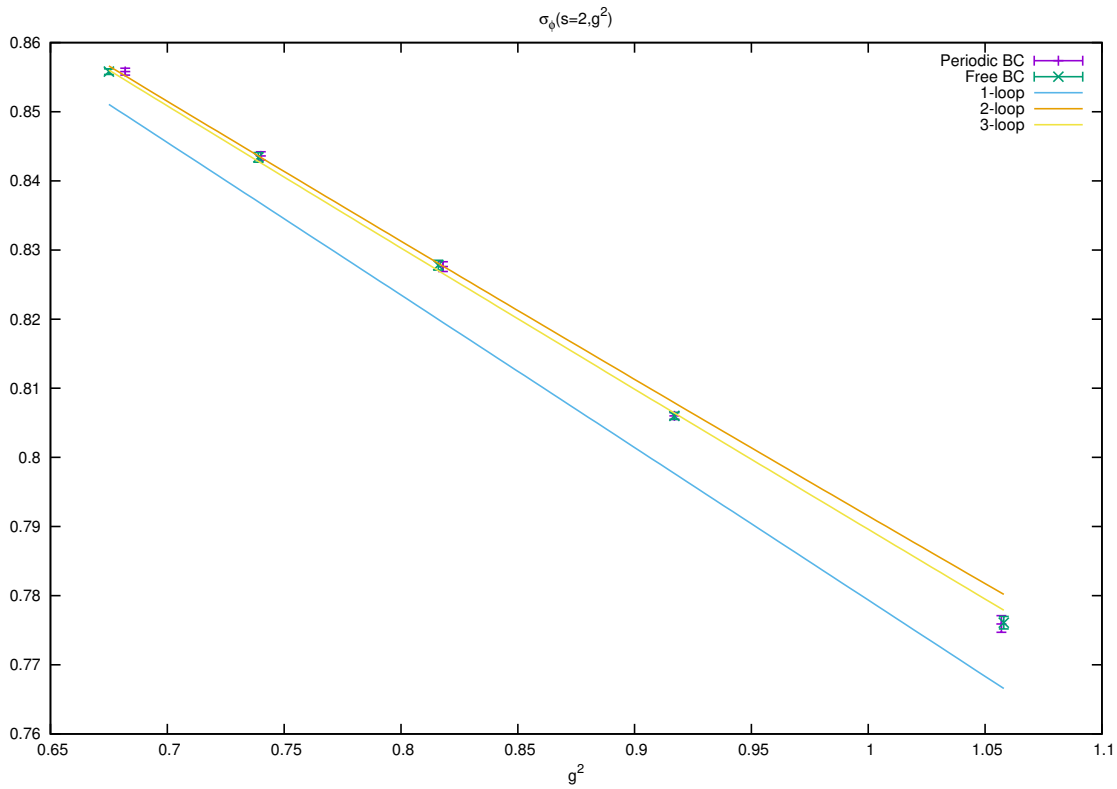

Figure 5.2: SSF of renormalizatoin factor,Z.

5.4 RGE in finite box application

Now all the tools of the RGE were defined and calculated for O(N) sigma model. So we can utilize this RGE for other operators different from the 2-point correlation function.

Summarizing, we have developed the RGE for volume dependence of the O(N) sigma model. This dependence can be described by an insertion operator which is defined from the Lagrangian operators. This leads us to the conclusion that we do not need any extra term to renormalize this insertion operator. Thanks of this result we can speak about the reliability of this RGE because of the cutoff independence of the γ-function. After that consideration we have proceeded to calculate our γ and β functions. So we are able to describe the volume variation of the operators. The operators which we consider are the vacuum energy density and magnetic susceptibility. In the Tables 4.8 and 4.9 we have shown the calculation of those operators, respectively, in different volumes. Before describing the volume dependence we first describe some definitions for the RGE, which help us to get information for those

operators

5.4.1 n-particle state operator

An important issue in QFT is the definition of number of particles in one operator. Actually this definition, if we use the perturbation theory, is quite straightforward, because we define 1-particle state for a theory in terms of how many fields, ϕ, we have in the operator. But this definition is no longer applied to the case of non-perturbative calculation. So we propose a well-defined definition for the concept of the 1-particle state and, as a result, the n-particle state.
The reason of the importance of this definition may be considered not for these theory, but in other theories like QCD, where the non-perturbative definition is needed. This definition helps us to understand the system we study better.
This idea resides in RGE for the volume dependence. Initially we have to consider the 2-point correlation function for our definitions. To be able to calculate the 2-point correlation function, we have used the simplest interpolator field and at the end of the calculation we have just taken the state with the smallest energy (large time separation).

$$\langle S(\tau,p_1)S(0,p_2)\rangle = A_1e^{-m_1\tau} + A_2e^{-m_2\tau} + A_3e^{-m_3\tau} + ... \xrightarrow{\tau\to\infty} A_1e^{-m_1\tau} \tag{5.22}$$

And we have defined Z factor for the of A_1 term, because m_1 is the smallest energy coupling with this interpolator.

$$\langle S(\tau,p_1)S(0,p_2)\rangle = \langle S(\tau,p_1)[|1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3| + ...]S(0,p_2)\rangle \xrightarrow{\tau\to\infty} A_1e^{-m_1\tau} \tag{5.23}$$

That is why we can consider this state as 1-particle state. It is important to realize that this state is not the same as the interpolation itself.

$$\langle S(\tau,p_1)S(0,p_2)\rangle = A_1\langle 1(t)|1(0)\rangle + A_2\langle 2(t)|2(0)\rangle + A_3\langle 3(t)|3(0)\rangle + ... \tag{5.24}$$

Therefore

$$\langle 1(t)|1(0)\rangle = B_1\langle S(\tau,p_1)S(0,p_2)\rangle + B_2\langle S(\tau,k_1)S(\tau,k_2)S(0,k_3)\rangle + ... \tag{5.25}$$

We saw this interpolator field has more excited states so to get rid of those excited state. This interpolator field should be modified by other composite operators, which

actually have more particles by our definition.
So if we are planning to check the volume dependence of any state, which we know has n particles, this operator follows the next RGE for volume dependence:

$$\mathcal{O}^{(n)}(g_L^2, L, \mu) \to L' = sL \to \mathcal{O}^{(n)}(\sigma_{g^2}(s, g_L^2), sL, \mu)(\sigma_\phi(s, g_L^2))^n \tag{5.26}$$

where the variable number of the particles,(n), is presented in this operator. So from Eq. (5.26), if we know this evolution, we can distinct how many particles in that operator by our denition.
As it was mentioned before, this definition is going to be important in the cases where the perturbative calculation is not enough for the operator in question. This definition is correct for the PT case and we can see that in the case of free theory this definition is correct too. In the next section we use this definition, so that we are able to get more information form the operator.
The data are going to be collected again for the operators we are going to study. In the case of the vacuum energy in the table 5.3 and the susceptibility in the 5.1.

5.4.2 Magnetic susceptibility

We start describing the magnetic susceptibility. We understood in the previous section that is expecting that susceptibility is proportional to Z/g_L^2. Then we can expect the behaviour of the ratio between the susceptibility of different volumes. This gives us the expected result which is the variation of these values by the volume dependence.

In the Table 5.2 we observe that the expected result fits to the results of calculations. This leads us to the conclusion that in this intermediate coupling regime the susceptibility is almost 100% coming from the 2-point correlation function.
Because the definiton of our Z_ϕ factor is made for the limit at $t \to \infty$ in the 2-point correlation function, we could have other contributions appear from other states, as we can expect in PBC and it is observed in the Figure 4.1. So we could have

$$\chi \propto x_1 \frac{Z}{g^2} + x_3 \frac{Z^3}{3g^2}, \tag{5.27}$$

where the first contribution is coming from the state calculated at $t \to \infty$, and the other contribution may come from the 3-particles state. When the time separation is not large the effective mass deviates from the expected effective mass, therefore this is telling us that contributions of other states are present in short time distance.

L/a	L'/a	$g^2(L)$	$g'^2(L)$	χ	χ'	χ'/χ
8	16	0.6770(9)	0.7388(9)	$8.560(13)10^{-2}$	$6.734(11)10^{-2}$	0.7867(18)
10	20	0.6897(10)	0.7569(9)	$8.031(12)10^{-2}$	$6.264(10)10^{-2}$	0.7800(18)
12	24	0.6852(9)	0.7481(8)	$7.852(12)10^{-2}$	$6.167(10)10^{-2}$	0.7854(18)
16	32	0.6802(8)	0.7443(7)	$7.542(11)10^{-2}$	$5.916(9)10^{-2}$	0.7844(18)
8	16	0.7405(10)	0.8181(10)	$7.616(13)10^{-2}$	$5.838(10)10^{-2}$	0.7666(18)
10	12	0.7402(10)	0.8192(9)	$7.306(12)10^{-2}$	$5.596(10)10^{-2}$	0.7658(18)
12	24	0.7363(10)	0.8120(8)	$7.139(11)10^{-2}$	$5.471(10)10^{-2}$	0.7662(18)
16	32	0.7405(9)	0.8201(7)	$6.699(10)10^{-2}$	$5.133(9)10^{-2}$	0.7661(18)
8	16	0.8176(12)	0.9180(12)	$6.687(12)10^{-2}$	$4.930(9)10^{-2}$	0.7373(18)
10	20	0.8175(13)	0.9219(9)	$6.402(11)10^{-2}$	$4.709(8)10^{-2}$	0.7355(18)
12	24	0.8181(12)	0.9188(8)	$6.170(10)10^{-2}$	$4.541(8)10^{-2}$	0.7360(18)
16	32	0.8181(10)	0.9212(7)	$5.839(10)10^{-2}$	$4.297(8)10^{-2}$	0.7359(18)
8	16	0.9173(15)	1.0593(15)	$5.722(10)10^{-2}$	$3.999(8)10^{-2}$	0.6988(18)
10	20	0.9181(16)	1.0607(9)	$5.451(10)10^{-2}$	$3.780(7)10^{-2}$	0.6970(18)
12	24	0.9162(13)	1.0551(8)	$5.254(9)10^{-2}$	$3.685(7)10^{-2}$	0.7014(18)
16	32	0.9185 (12)	1.0595(7)	$4.944(9)10^{-2}$	$3.456(7)10^{-2}$	0.6990(18)
8	16	1.0577(18)	1.2835(19)	$4.713(10)10^{-2}$	$3.001(7)10^{-2}$	0.6366(18)
10	20	1.0591(19)	1.3073 (9)	$4.464(9)10^{-2}$	$2.765(6)10^{-2}$	0.6194(18)
12	24	1.0598(16)	1.2768(8)	$4.266(8)10^{-2}$	$2.749(6)10^{-2}$	0.6445(18)
16	32	1.0570(14)	1.2743(7)	$3.999(8)10^{-2}$	$2.589(5)10^{-2}$	0.6474(18)

Table 5.1: Raw values of the magnetic susceptibility

$g^2(L)$	$g'^2(L)$	$g'^2(L)/g^2(L)$	Z'/Z	χ'/χ	Expected
0.6820	0.7451(70)	1.092(10)	0.8558(5)	0.7817(58)	0.7877(73)
0.7400	0.8150(37)	1.1013(50)	0.8436(6)	0.7659(3)	0.7536(50)
0.8180	0.9197(80)	1.1243(98)	0.8276(7)	0.7353(7)	0.7361(34)
0.9170	1.0564(46)	1.1521(49)	0.8060(5)	0.7000(22)	0.6995(30)
1.0570	1.244(47)	1.177(39)	0.7759(12)	0.649(15)	0.659(25)

Table 5.2: Continuum limit extrapolation in PBC

But the calculation of the susceptibility reveals that those states do not contribute in essence. That is why we stimulate the idea that this RGE study can show us the main state of the operator we are studying.

It is important to mention that we use the values of the non-perturbative calculation, and then these values are suitable to compare the results in other theories to subtract the information of the state that we want.

5.4.3 Vacuum energy density

Now we move to the study of the vacuum energy. For this operator and important problem arises, we do not have analytical function of this energy in terms of the renormalized parameter, so it is hard to get information from this operator.

β	L/a	L'/a	$g^2(L)$	$g'^2(L)$	E	E'
2.1278	8	16	0.6770(9)	0.7388(9)	1.0839(2)	1.0876(2)
2.1403	10	20	0.6897(10)	0.7569(9)	1.0850(2)	1.0872(1)
2.1831	12	24	0.6852(9)	0.7481(8)	1.0831(2)	1.0847(1)
2.2403	16	32	0.6802(8)	0.7443(7)	1.0806(2)	1.0816(1)
2.0100	8	16	0.7405(10)	0.8181(10)	1.0923(2)	1.0965(1)
2.0489	10	12	0.7402(10)	0.8192(9)	1.0916(2)	1.0938(1)
2.0875	12	24	0.7363(10)	0.8120(8)	1.0894(1)	1.0912(1)
2.1260	16	32	0.7405(9)	0.8201(7)	1.0877(1)	1.0888(1)
1.8947	8	16	0.8176(12)	0.9180(12)	1.1025(2)	1.10756(8)
1.9319	10	20	0.8175(13)	0.9219(9)	1.1013(1)	1.10442(7)
1.9637	12	24	0.8181(12)	0.9188(8)	1.0994(1)	1.10163(6)
2.0100	16	32	0.8181(10)	0.9212(7)	1.0965(1)	1.09786(4)
1.7791	8	16	0.9173(15)	1.0593(15)	1.11559(16)	1.12202(8)
1.8171	10	20	0.9181(16)	1.0607(9)	1.11373(13)	1.11777(7)
1.8497	12	24	0.9162(13)	1.0551(8)	1.11112(10)	1.11408(5)
1.8965	16	32	0.9185 (12)	1.0595(7)	1.10733(8)	1.10912(4)
1.6589	8	16	1.0577(18)	1.2835(19)	1.13333(16)	1.14294(8)
1.6982	10	20	1.0591(19)	1.3073 (9)	1.13067(13)	1.1385(7)
1.7306	12	24	1.0598(16)	1.2768(8)	1.12739(10)	1.13176(5)
1.7800	16	32	1.0570(14)	1.2743(7)	1.12189(8)	1.12453(4)

Table 5.3: Raw values of the energy

For the vacuum energy as it has been discussed before, there is no reliable expression for this quantity, although the data show the variation of the operator with the renormaized coupling variation. The main idea is as we said before that the vacuum energy is described by non-perturbative calculations,

$$E^{(PBC)}_{(non-PT)} = C_E W(g_R^2)\Lambda^2_{MS}. \tag{5.28}$$

Then we can say that there is not variation in bigger volume because when we increase the volume of the system g^2_{MS} is not varied as as we did in the calculation of the γ function. Then we can justify that the variation in our calculation only comes from the SSF of the strength field.

$$E^{(n)}(g_L^2, L, \mu) \to L' = sL \to E'^{(n)}(\sigma(s, g_L^2), L, \mu)(\sigma_\phi(s, g_L^2))^n = E^{(n)}(g_L^2, L, \mu)(\sigma_\phi(s, g_L^2))^n \tag{5.29}$$

So

$$\frac{E'^{(n)}}{E^{(n)}} = (\sigma_\phi(s, g_L^2))^n \tag{5.30}$$

Then all the variation will just depend on the quantity of fields in the state of the vaccum. And a quick look at the Table 5.4, we get the conclusion that n should be zero. This is an obvious result if we are studying the vacuum energy.

$g^2(L)$	$g'^2(L)$	$g'^2(L)/g^2(L)$	Z'/Z	E'/E
0.6820	0.7451(70)	1.092(10)	0.8558(5)	1.0001(1)
0.7400	0.8150(37)	1.1013(50)	0.8436(6)	1.0000(1)
0.8180	0.9197(80)	1.1243(98)	0.8276(7)	1.0001(1)
0.9170	1.0564(46)	1.1521(49)	0.8060(5)	1.0002(2)
1.0570	1.244(47)	1.177(39)	0.7759(12)	1.0002(7)

Table 5.4: Continuum limit extrapolation in PBC

Chapter 6

Conclusions

In this chapter we summarize the new ideas of this research.

Different parametrizations for O(N) sigma model:
We have explained the importance of the sigma model in the common knowledge of QFT and we show that there are several open problems concerning the sigma model. We described some important issues in the sigma model.
At the end of the first chapter, Different parametrizations in the sigma model are studied and it is shown that for the several standard parametrizations the results are the same. This can give reliability about the perturbative calculation, which is in contradiction with [14, 15, 48, 33]. We can reach to this conclusion because each parametrization suffers from different issues due to the impossibility to describe the $\mathbb{S}^{N-1}$-sphere by one plane.

Renormalization group equation for volume dependence:
We have proposed a renormalization group equation in a finite box. We have combined the techniques to obtain information at infinite volume from the finite volume scheme and used them for the definition of a new RGE definition which is an important to for QFT. We have define in general this renormalization group equation in finite volume, giving the recipe to generate it in any theory, where the critical point is the non-homogeous term in the theory. We could solve this problem because we found this non-homogenous term is an insertion operator of the operators defined in the Lagrangian, so this leads us to conclude that in this insertion operator calculation no new renormalization condition is needed. That is why our RGE for volume

dependence is consistent.
This recipe to calculate RGE for volume dependence is used in O(N) sigma model. One important issue is to calculate the insertion operator to describe the non-homogenous term and calculated in the chapter 5. This non-homogeneous term in the RGE for a volume dependence is O(N) symmetry, therefore the mass gap generation for finite volume is not due to the polarization of the system at finite volume, the mass generation is a dynamic mechanism in theory. This non-homogenous term is defined in terms of the Lagrangian operator so we can conclude that β and γ functions are volume independence functions.

Calculation of β and γ functions in O(N) sigma model: Non-perturbative and perturbative

After defining the RGE for a volume dependence, it is needed that β and γ are calculated. We have used two different approach to calculate them. The way to compare them is using the SSF and this comparison was a success. In this calculation we have relied on previous studies for the β function [41, 61]. This β function has been study in detail for other researchers, but for the case of the γ function, there are not studies for the both methods, PT and non-PT. We have explored both methods. For the PT calculations γ function is calculated until NNLO. For the non-PT we have used Monte Carlo simulations. As previously mention, the comparison was a success, so we can say now that all the tools for the RGE in finite box is made with precision.

Information from the renormalization group equation for a volume dependence, n-particle operator definition:

Apart from the calculation of β and γ functions, which are coming from the computation of 2-point correlation functions, we have calculated the vacuum energy and the magnetic susceptibility. We are able to get the information of the field composition of these operators by our definition of the n-particle state in the system. The definition of 1-particle state is coming from the definition of 2-point correlation function, where an elementary particle is considered as the first excited state of this 2-point correlation function.
After the definition of the concept of 1-particle, we are able to suggest the composition of other operators in terms of n-particles states. In the case of the vacuum energy, we got the expected results that the vacuum energy is in essence an operator with no particles, 0-particle state. In the other case, the susceptibility, we have checked this operator and found that the most important contribution comes from the propagator of one particle state.

The O(N) sigma model is a simple model where only one kind the particle definition has to be defined, so the idea is to extrapolate this method to other more complicated theories, such as QCD, where the definition of particles is not straightforward. We suggest that doing similar study of QCD will give us information of more complicated states.

One example is the case of $\Lambda(1405)$, where up to now there is no clear conclusion for the composition of this state [110, 111, 112]. There are several approaches to study its composition but all of these approaches rely on effective field theory ideas. That is why we suggest this method which can be applied directly in first principles, QCD, in our procedure we are counting the number of particles presented in the state related with the interpolator. So we suggest that using this method in particle states, where the composition is not understand yet, will clarify this states.

Appendix

A.1 Dirac's delta regularization in different boundary condition

The definition for the propagator is given in the chapter 3 to calculate the perturbative calculation. But in the calculations the derivative of the propagator is needed and so the Dirac delta function will appear in the calculation which is regularized. As it is mentioned before in the perturbative calculation, we will use DR in finite volume which is the same as the Zeta function regularization.

So starting from the initial point: the propagator is given by

$$G(x, y, p_1 = 0) = \frac{T}{12} - \frac{|x-y|}{2} + \frac{x^2+y^2}{2T} \tag{A.1}$$

This propagator is the one satisfying the free boundary condition, then we show later different boundary condition will bring us to a different propagator hence the regularization of the Dirac delta functino may be different.

It is important to realize that the derivative when $x \neq y$ has straightforward result

$$\partial_x G(x, y, p_1 = 0) = \begin{cases} -\frac{1}{2} + \frac{x}{T} & \text{for} \quad x > y \\ \# + \frac{x}{T} & \text{for} \quad x = y \\ \frac{1}{2} + \frac{x}{T} & \text{for} \quad x < y \end{cases} \tag{A.2}$$

And

$$\partial_y G(x, y, p_1 = 0) = \begin{cases} \frac{1}{2} + \frac{y}{T} & \text{for} \quad x > y \\ \# + \frac{y}{T} & \text{for} \quad x = y \\ -\frac{1}{2} + \frac{y}{T} & \text{for} \quad x < y \end{cases} \tag{A.3}$$

$$\partial_y\partial_x G(x,y,p_1=0) = \begin{cases} 0 & \text{for} \quad x>y \\ \delta(x-y)=\delta(0) & \text{for} \quad x=y \\ 0 & \text{for} \quad x<y \end{cases} \tag{A.4}$$

So to be able to calculate those quantities, we consider $\langle\pi(x)\pi(y)\rangle = G(x,y)$ when $x=y$

$$\partial_x(\langle\pi(x)\pi(x)\rangle) = \partial_x(G(x,x)) = \partial_x(\frac{T}{12}+\frac{x^2}{T}) \tag{A.5}$$

So this expression goes:

$$\langle\partial_x(\pi(x))\pi(x)\rangle + \langle\pi(x)\partial_x(\pi(x))\rangle = \frac{2x}{T} \tag{A.6}$$

Therefore

$$\langle\partial_x(\pi(x))\pi(x)\rangle = \partial_x(G(x,y))\rfloor_{x=y} = \frac{x}{T} \tag{A.7}$$

Giving the value the first derivate of the propagator , # = 0.
Now the last term to consider $\partial_y\partial_x G(x,y,p_1=0)$, we get the value of $\delta(x-y)=\delta(0)$. Using similar procedure as before one can reach

$$\langle\partial_x^2(\pi(x))\pi(x)\rangle + \langle\partial_x(\pi(x))\partial_x(\pi(x))\rangle = \frac{1}{T}. \tag{A.8}$$

With this information is not enough so it is needed to go to the definition of $G(x,y)$. The propagator is obtained by the usual eigenvalue procedure:

$$G(x,y) = \sum_{\lambda_1}(\frac{1}{\lambda_1^2}cos(\lambda_1 x)cos(\lambda_1 y)) + \sum_{\lambda_2}(\frac{1}{\lambda_2^2}sin(\lambda_2 x)sin(\lambda_2 y)) =$$

$$\sum_{\lambda_1}(\frac{1}{2\lambda_1^2}(cos(\lambda_1(x-y))+cos(\lambda_1(x+y)))) + \sum_{\lambda_2}(\frac{1}{2\lambda_2^2}(cos(\lambda_2(x-y))-cos(\lambda_2(x+y))))$$

$$\lambda_1 = \frac{2\pi n_1}{L}, \quad \lambda_2 = \frac{2\pi(n_2+1)}{L} \quad | \quad n_1 \in \mathcal{Z}/0, \quad n_2 \in \mathcal{Z} \tag{A.9}$$

Then $\partial_y\partial_x G(x,y)\rfloor_{x=y}$ and $\partial_x^2 G(x,y)\rfloor_{x=y}$ can be calculated

$$\partial_y\partial_x G(x,y)\rfloor_{x=y} = \sum_{\lambda_1}(\frac{1}{2}(1-cos(2\lambda_1 x))) + \sum_{\lambda_2}(\frac{1}{2}(1+cos(2\lambda_2 x))) = M+N \tag{A.10}$$

$$\partial_x^2 G(x,y)\rfloor_{x=y} = \sum_{\lambda_1}(\frac{1}{2}(-1-cos(2\lambda_1 x))) + \sum_{\lambda_2}(\frac{1}{2}(-1+cos(2\lambda_2 x))) = -M+N \tag{A.11}$$

Where

$$M = \sum_{\lambda_1} \frac{1}{2} + \sum_{\lambda_2} \frac{1}{2}, \quad N = \sum_{\lambda_1} (\frac{1}{2} cos(2\lambda_1 x)) - \sum_{\lambda_2} (\frac{1}{2} cos(2\lambda_2 x)) \tag{A.12}$$

The calculation of M is not difficult because we use the typical complex plane continuation of the zeta Riemann function, $Z_{Riemann}(0) = -1/2$. We have used now the regularization of the $Z_{Riemann}$ function, this kind of regularization is called Z regularization.

$$Z_{Riemann}(0) = \sum_n 1 \quad | \quad n \in \mathcal{Z}^+/0 \tag{A.13}$$

$$M = \frac{1}{2}(2Z_{Riemann}(0) + 2Z_{Riemann}(0) + 1) = -\frac{1}{2T} \tag{A.14}$$

Using the equation A.8:

$$M + N - M + N = \frac{1}{T}, \quad N = \frac{1}{2T} \tag{A.15}$$

The final solution using the equation A.8, one gets the value for FBC

$$\partial_x^2 G(x,y)\rfloor_{x=y} = \frac{1}{T}, \qquad \partial_y \partial_x G(x,y)\rfloor_{x=y} = 0, \quad \delta(0) = 0 \tag{A.16}$$

$$\partial_y \partial_x G(x,y,p_1 = 0) = \begin{cases} 0 & \text{for} \quad x > y \\ 0 & \text{for} \quad x = y \\ 0 & \text{for} \quad x < y \end{cases} \tag{A.17}$$

It is important to mention all the previous calculation was made in the case where the free boundary condition is applied. If one is interested in the case of periodic boundary condition, the procedure is the same, just some equation will change, because in this case the initial equation is:

$$G(x,y,p_1 = 0) = \frac{T}{12} - \frac{|x-y|}{2} + \frac{(x-y)^2}{2T} \tag{A.18}$$

Then

$$\partial_y \partial_x G(x,y,p_1 = 0) = \begin{cases} -\frac{1}{T} & \text{for} \quad x > y \\ \delta(0) - \frac{1}{T} & \text{for} \quad x = y \\ -\frac{1}{T} & \text{for} \quad x < y \end{cases} \tag{A.19}$$

And

$$\langle \partial_x^2(\pi(x))\pi(x)\rangle + \langle \partial_x(\pi(x))\partial_x(\pi(x))\rangle = 0 \tag{A.20}$$

Using the same procedure as before the new value for N in periodic boundary condition is $N = 0$. And in the case of M will change too because for peridic boundary condition, $\lambda_1 = \lambda_2, \quad \frac{2\pi n_1}{L} \quad | \quad n_1 \in \mathcal{Z}/0$.
Collecting all the results:

$$\partial_x^2 G(x,y)\rfloor_{x=y} = \frac{1}{T}, \qquad \partial_y \partial_x G(x,y)\rfloor_{x=y} = -\frac{1}{T}, \quad \delta(0) = 0 \tag{A.21}$$

$$\partial_y \partial_x G(x,y,p_1 = 0) = \begin{cases} -\frac{1}{T} & \text{for} \quad x > y \\ -\frac{1}{T} & \text{for} \quad x = y \\ -\frac{1}{T} & \text{for} \quad x < y \end{cases} \tag{A.22}$$

Then, this result confirm that for these boundary conditions the Dirac's delta it is independent of these boundary conditions.

A.2 Finite volume propagator representation

In this appendix we will derivate the equation 5.7-5.9 used in the Niedermayer's and Weisz's paper [70].
This representation is more suitable for finite volume calculation than the typical eigenvalue representation, which is commonly used in infinite volume, for the Feynman diagrams. The initial point is going to be the eigenvalue representation, momentum representation:

$$G(x_0, x_1, y_0, y_1) =$$
$$= \frac{1}{TL}\sum_{p_1}(\sum_{\lambda_1}\frac{1}{\lambda_1^2+p_1^2}cos(\lambda_1 x_0)cos(\lambda_1 y_0)+\sum_{\lambda_2}\frac{1}{\lambda_2^2+p_1^2}sin(\lambda_2 x_0)sin(\lambda_2 y_0))e^{ip_1(x_1-y_1)} = \tag{A.23}$$

$$\lambda_1 = \frac{2n\pi}{T}, \quad \lambda_2 = \frac{(2n+1)\pi}{T}, \quad p_1 = \frac{2n\pi}{T}, \quad n \in \mathbb{Z}, \quad (\lambda_1, p_1) \neq (0,0) \tag{A.24}$$

The eigenvalue values are coming from the boundary condition. With the trigonometry rule, we can already observe the periodic and anti-periodic regimes are split:

$$G(x_0, x_1, y_0, y_1) = \frac{1}{TL}\sum_{p_1}(\sum_{\lambda_1}\frac{1}{2(\lambda_1^2+p_1^2)}(cos(\lambda_1(x_0-y_0)) + cos(\lambda_1(x_0+y_0)))+$$
$$+\sum_{\lambda_2}\frac{1}{2(\lambda_2^2+p_1^2)}(cos(\lambda_2(x_0-y_0)) - cos(\lambda_2(x_0+y_0))))e^{ip_1(x_1-y_1)} \tag{A.25}$$

As we already know the case when $p_1 = 0$, we are not going to consider the full calculation. So we focus in the case where $p_1 \neq 0$. The term coming from $p_1 \neq 0$ is defined by $N_{p_1}(x, y)$. For all of $N_{p_1}(x, y)$, the calculation follows the same. Doing some algebra, as it is commonly follow by dimensional regularization scheme:

$$N_{p_1}(x_0, x_1, y_0, y_1) = \frac{1}{TL}\sum_{\lambda}\int dt\frac{1}{2}e^{-t(\lambda_1^2+p_1^2)}(e^{\lambda(x_0-y_0)} + e^{\lambda(x_0+y_0)+i\pi n})e^{ip_1(x_1-y_1)}$$
$$\lambda = \frac{n\pi}{T}, \quad n \in \mathbb{Z} \tag{A.26}$$

Now a critical point appears, the following used formula in the next step is called "Jacobi theta function relation" and one example can be found in [76]. This "Jacobi theta function relation" is important for any kind of finite volume effect as in Zeta function regularization. For our calculation we consider one dimension being finite. But for other cases where more dimensions can be consider finite, we must modify the following expression.
This "Jacobi theta function relation" is:

$$\sum_{\mathbf{n}\in\mathbb{Z}^p} exp(-\frac{1}{2}\mathbf{n}^T A\mathbf{n}+\mathbf{b}^T\mathbf{n}) = \frac{(2\pi)^{\frac{p}{2}}}{\sqrt{detA}}\sum_{\mathbf{m}\in\mathbb{Z}^p} exp[\frac{1}{2}(\mathbf{b}+2\pi i\mathbf{m})^T A^{-1}(\mathbf{b}+2\pi i\mathbf{m})] \quad \text{(A.27)}$$

where A is a matrix $p\times p$ and b a p-component vector.
Returning in the calculation of $N_{p_1}(x,y)$. For our case we have these values for the matrix A and vector b, where the dimension $p=1$:

$$A = 2(\frac{\pi}{T})^2 t, \quad b_1 = (\frac{2\pi}{T}(x-y)), \quad b_1 = (\frac{2\pi}{T}(x+y)+2\pi) \quad \text{(A.28)}$$

Now the algebra is straightforward:

$$N_{p_1}(x_0,x_1,y_0,y_1) =$$

$$= \frac{1}{L}\int_0^\infty \frac{1}{\sqrt{\pi t}}\sum_m dt\frac{1}{2}(e^{-tp_1^2-\frac{(x_0-y_0+2mT)^2}{4t}+ip_1(x_1-y_1)} + e^{-tp_1^2-\frac{(x_0+y_0+(1+2m)T)^2}{4t}+ip_1(x_1-y_1)}) \quad \text{(A.29)}$$

$$N_{p_1}(x_0,x_1,y_0,y_1) = \frac{1}{L}\sum_m \frac{1}{2|p_1|}(e^{-|p_1||x_0-y_0+2mT|+ip_1(x_1-y_1)}+e^{-|p_1||x_0+y_0+(1+2m)T|+ip_1(x_1-y_1)}) \quad \text{(A.30)}$$

This equation corresponds with Niedermayer's paper. As it was explained before, it may happen we need to know $N_{p_1}(x_0,x_1,y_0,y_1)\rfloor_{x_0=x_1,y_0=y_1}$ at large T, because in this situation we have to regularize this propagator. For this propose we are using dimensional regularization when the dimension of the time component is not natural number. Therefore we were using the "Theta jacobi relation" $d=1-\epsilon$. In the next calculation we keep the value of d in the equation.

$$N_{p_1}(x_0,x_1,y_0,y_1)\rfloor_{x_0=x_1,y_0=y_1} =$$

$$= \frac{1}{L}\int_0^\infty (\frac{(2\pi)}{2(\frac{\pi}{T})^2 t})^{d/2} \sum_m \frac{1}{2^d}(e^{-tp_1^2 - \frac{(2mT)^2}{4t}} + e^{-tp_1^2 - \frac{(2x_0+(1+2m)T)^2}{4t}})dt =$$

$$\xrightarrow{T\to\infty} = \frac{1}{L}\int_0^\infty (\frac{1}{\pi t})^{d/2}\frac{1}{2^d}e^{-tp_1^2}dt = \frac{1}{L}(\frac{1}{\pi})^{d/2}\frac{1}{2^d}|p_1|^{-2+d}\Gamma[1-\frac{d}{2}] \tag{A.31}$$

This value is used in our calculation. It is importance to notice that in this case we have use $t \to \infty$, because for this calculation the terms $m \neq 0$ vanish at $t \to \infty$. We can think this is natural because the momentum p_1 acts like a IR regulator, so if our calculation at the end we take $t \to \infty$, we do not have to be worry about the other terms. This assumption is not correct as we observe in 3.6 where this contribution have to take into account to get the correct result.

A.3 Dimensional regularization in not infinite space

This section is based on [108]. The usual dimensional regularization is made when the added space where the theory lives is $\mathbb{R}^\epsilon$. We can choose other kind of space in this regularization.

$$M' = M \times (E)^\epsilon \tag{A.32}$$

Where M is our initial space and E is the added space.

Because in PT we need to evaluate the Feymann diagrams using the free propagator $\langle\phi(x)\phi(y)\rangle\rfloor_{0,M'} = G(x,y)$. We have to know that the propagator in this new space M'. The way to obtain this propagator is using the Green's function definition

$$G(x,y) = \int e^{p\cdot x}\frac{1}{p_M^2 + p_E^2 + m^2}dp^{d_M+d_E}. \tag{A.33}$$

This expression can be modified by:

$$G(x,y) = \int dtK_t(x,y) \tag{A.34}$$

where this new function $\mathrm{K}_t(x,y)$ is called heat kernel.
Because the expression in A.33 is transformed to function where each momentum contribution is independent.

$$\frac{1}{p_M^2 + p_E^2 + m^2} = \int dte^{p_M^2+p_E^2+m^2} \tag{A.35}$$

This let us to separate each integration so the definition of each heat kernel is independent. This expression can be modified by:

$$K_t(x,y) = K_t^M(x,y)\cdot(K_t^E(x,y))^\epsilon \cdot e^{-tm^2} \tag{A.36}$$

Thanks of this notation we can know add any kind of space in the theory to use dimensional regularization. As an example we show the kernel of one circle, $\mathbb{S}^1$ when we $x = y$ and PBC.

$$K_t^{\mathbb{S}^1}(y,y) = \frac{1}{4\pi t}(1 + \frac{t}{3r^2} + \mathcal{O}(x^2)) \tag{A.37}$$

where r is the radius of the circle.
So if want to evaluate the now 1-loop integration in a plane space minus one circle ϵ times, $M' = \mathbb{R}^2 \times \mathbb{S}^1$

$$G(x,y) = \int dtK_t(x,y) = \int dtK_t(x,y)\cdot K_t(x,y)\cdot e^{-tm^2} \tag{A.38}$$

And this integration is finite. The important issue now is that if this freedom to choose the added space will modify the results, But the study in [108, 93] confirms that the final result will not depend on the choose of the space E made. Remembering In our calculation we have choose $E = \mathbb{S}^1$ in the temporal component at large radius and Free BC.

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