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Essays on Implementation Theory

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Abstract

This thesis consists of four essays on implementation theory. We investigate implementation problems under incomplete information; for the allocation of a social endowment of infinitely divisible resources; under the existence of partially honest agents; and under the existence of semi-socially-responsible agents.

First of all, in Chapter 1, we propose a brief history of implementation theory, and we shortly introduce four studies in this thesis.

In Chapter 2, we consider the implementation problem under incomplete information and private values. We investigate double implementability of social choice functions in dominant strategy equilibria and ex post equilibria. We show that the notion of ex post equilibrium is weaker than the notion of dominant strategy equilibrium. Then, this notion of double implementability is not trivial even under private values. We define a new strategic axiom that is stronger than "strategy-proofness," but weaker than "secure strategy-proofness." We call it "weak secure-strategy-proofness." We show that a social choice function is doubly implementable in dominant strategy equilibria and ex post equilibria if and only if it is *weakly securely-strategy-proof.*

In Chapter 3, we consider the allocation problem of infinitely divisible resources with at least three agents. For this problem, Thomson (2005) and Doğan (2016) propose simple but not procedurally fair mechanisms which implement the no-envy correspondence in Nash equilibria. By contrast, Galbiati (2008) constructs a procedurally fair but not simple mechanism which implements the no-envy correspondence in Nash equilibria. In this chapter, we design a both simple and procedurally fair mechanism which implements the no-envy correspondence in Nash equilibria.

In Chapter 4, we consider the implementation problem with at least three agents. We study double implementability of social choice correspondences in Nash equilibria and undominated Nash equilibria. We prove that "DZ-invariance," "weak no-veto-power," and "unanimity" together are sufficient for double implementability in Nash equilibria and undominated Nash equilibria. If there is at least one partially honest agent in the sense of Dutta and Sen (2012), then *weak no-veto-power* and *unanimity* together are sufficient for double implementability in Nash equilibria and undominated Nash equilibria. If there are at least two partially honest agents, then *unanimity* is sufficient for double implementability in Nash equilibria and undominated Nash equilibria. In addition, we show that if there is at least one partially honest agent and *unanimity* is satisfied, then "LY-condition" is necessary and sufficient for double implementability in Nash equilibria and undominated Nash equilibria. From these results, we obtain several positive corollaries such as in a bargaining problem (Nash, 1950).

In Chapter 5, we assume that each agent is "semi-socially-responsible" and we focus on social choice functions for double implementation in Nash equilibria and undominated Nash equilibria. We show that if there are at least three agents and each agent is semi-socially-responsible with respect to a *unanimous* social choice function, then a simple and procedurally fair mechanism doubly implements this social choice function in Nash equilibria and undominated Nash equilibria.

Finally, in Chapter 6, we conclude this thesis by summarizing its contributions and discussing three remaining issues in this thesis.

Contents

List of Tables

Chapter 1

Introduction

Consider a situation in which a social planner wants to choose an outcome to be socially desirable.[1] Examples of social desirability include the notions of *Pareto-efficiency*, *no-envy* (Foley, 1967), and others. For the planner to detect which outcome is socially desirable, she needs to know the agents' true preferences. However, she dose not have the information. If the planner simply asks each agent to reveal his preference and she selects an outcome to be socially desirable following the information revealed by the agents, some agent might strategically misrepresent it to realize an outcome which he prefers. The following example illustrates this problem.

Example 1.1. We consider the following voting problem. Suppose that there are three agents, there are three candidates, and each agent has two preferences. Let $N = \{1, 2, 3\}$ be the set of agents and $A = \{a, b, c\}$ be the set of candidates. For each $i \in N$, let $\mathcal{R}_i = \{R_i, R'_i\}$ be the set of preferences admissible for agent i. Let $\mathcal{R} = \times_{i \in N} \mathcal{R}_i$ be such that $a \; P_1 \; b \; I_1 \; c$, $b \; P_2 \; c \; P_2 \; a$, $c \; P_3 \; a \; P_3 \; b$, and for each $i \in N$, $a \; P'_i \; b \; P'_i \; c$.[2]

As an objective of a social planner, we consider the "Borda solution." To define this solution, we first propose a scoring rule. For each $i \in N$, each $R_i \in \mathcal{R}_i$, and each $d \in A$, if candidate d is the k-th most preferred outcome at R_i where $k \in \{1, 2, 3\}$, then let $B(R_i, d) \equiv k$.

[1] What we mean by an "outcome" will naturally depend on the context. As an example, for a government charged with delivering public goods, an outcome will consist of the provided quantities of public goods (e.g., national defense and security) together with the quantities of private goods (e.g., money).

[2] For each $i \in N$, $a \; R_i \; b \; R_i \; c$ means that by agent i, candidate a is at least preferred to candidate b and candidate b is at least preferred to candidate c.

Borda solution, $f^B : \mathcal{R} \to A$**:** For each $R \in \mathcal{R}$ and each $d \in A$, $\Sigma_{i\in N} B(R_i, f^B(R)) \leq \Sigma_{i\in N} B(R_i, d)$. If there is $e \in A$ such that $e \neq f^B(R)$ and $\Sigma_{i\in N} B(R_i, f^B(R)) = \Sigma_{i\in N} B(R_i, e)$, then $f^B(R)$ is selected in alphabetical order.[3]

Suppose that $(R_1, R_2, R_3) \in \mathcal{R}$ is the true preference profile. In this case, if the planner asks each agent to reveal his preference, each agent $i \in N$ simply reveals R_i, and the planner selects the candidate chosen by f^B for (R_1, R_2, R_3) i.e., c, then agent 1 can improve his preference by reporting R'_1 i.e., $f^B(R'_1, R_2, R_3) = a \; P_1 \; c = f^B(R_1, R_2, R_3)$.■

"Implementation theory" investigates the possibility of designing mechanisms by which a social planner implements her objective even when agents take strategic actions.[4,5] Formally, the objective of the planner is embodied by a "social choice correspondence (SCC)." An SCC is a set-valued mapping which, for each preference profile, selects a non-empty set of outcomes. Especially, if such a mapping is single-valued, then it is called a social choice function (SCF). Since the planner does not know the agents' true preferences, she must rely on the agents' strategic actions to indirectly cause the socially desirable outcome(s) to come about. Then, she specifies a message space for each agent and a single-value mapping which, for each possible message profile, chooses an outcome. The pair consisting of the list of the message spaces and the mapping is a "mechanism." As a special kind of mechanisms, in the direct mechanism associated with an SCF, the message space for each agent is the set of his possible preferences and the mapping is the SCF. A mechanism and the agents' preferences induce a "game." To capture the strategic actions of agents, the planner considers the equilibrium notion(s). She aims to design

[3]The Borda solution f^B is a single-valued "sub-solution" of the "Borda correspondence." The Borda correspondence F^B is defined as for each $R \in \mathcal{R}$, $F^B(R) = \{d \in A :$ for each $e \in A$, $\Sigma_{i\in N} B(R_i, d) \leq \Sigma_{i\in N} B(R_i, e)\}$. A single-valued sub-solution $\varphi_{F^B} : \mathcal{R} \to A$ of the Borda correspondence is a single-valued mapping such that for each $R \in \mathcal{R}$, $\varphi_{F^B}(R) \in F^B(R)$.

[4]In 1930s and 1940s, the problem of social decision making when information is decentralized is crystallized (von Mises, 1920; von Hayek, 1935, 1945; Lange, 1936, 1937; Lerner, 1936). This lengthy discussion is called "Hayek-Mises-Lange-Lerner debates" by Moore (1992). For surveys on these debates, see Brus (1990) and Kowalik (1990). After the contributions of Hurwicz (1960, 1972), mathematical analyses became possible.

[5]For surveys on implementation theory, see Moore (1992), Jackson (2001, 2014), Maskin and Sjöström (2002), Palfrey (2002), Serrano (2004), and Corchón (2015), For brief commentaries on implementation theory, see Maskin (2008, 2011).

a mechanism in which the set of outcomes chosen by an SCC coincides with the set of equilibrium outcomes of the game. Then, we say that the mechanism implements the SCC in the equilibrium notion(s). If there exists such a mechanism, then we say that the SCC is implementable.

Whether an SCC is implementable or not may depend on which game theoretic solution concept is invoked. The most demanding concept is "dominant strategy equilibrium." By definition, a dominant strategy of an agent is a best reply to any actions of the others. Thus, if there exists a dominant strategy equilibrium of a game, agents need not form any conjecture about the behavior of others in order to know what to do.

For an SCF to be dominant strategy implementable, the SCF must satisfy the property that in the direct mechanism associated with the SCF, truth-telling is a dominant strategy for each agent (Gibbard, 1973).[6] This property is known as *strategy-proofness*, and a number of papers propose *strategy-proof* SCFs in several environments (e.g., Vickrey, 1961; Moulin, 1980; Dubins and Freedman, 1981; Roth, 1982). To study whether *strategy-proof* SCFs work well in practice, a bunch of papers conducted laboratory experiments. Although *strategy-proof* SCFs are desirable from the theoretical viewpoint, laboratory experiments regarding such SCFs reported that in several games, some subjects did not select dominant strategies (e.g., Kagel, Harstad, and Levin, 1987; Attiyeh, Franciosi, and Isaac, 2000).[7]

These observations raise a concern for implementation theory. Although in experiments for the pivotal mechanism which is *strategy-proof*, some subjects did not adopt dominant strategies, they frequently selected a Nash equilibrium (e.g., Cason, Saijo, Sjöström, and Yamato, 2006). This observation led Saijo, Sjöström, and Yamato (2007) to formulate and investigate "secure implementation," namely double implementation in dominant strategy equilibria and Nash equilibria. If there is no dominant strategy for an agent, then any best reply of this agent depends on the choices of the other agents so

[6]This result is the so-called "revelation principle for dominant strategy implementability."

[7]For a summary of laboratory experiments regarding *strategy-proof* SCFs, see, e.g., Cason, Saijo, Sjöström, and Yamato (2006).

that it may be hard for this agent to predict the choices. For strategic uncertainty not to be important, Saijo, Sjöström, and Yamato (2007) study not only good Nash equilibria that induce socially desirable outcomes, but also good dominant strategy equilibria.

In the laboratory experiment in Cason, Saijo, Sjöström, and Yamato (2006), each subject knew only his own preference, so that incomplete information games were considered. Usually, to define the notion of Nash equilibrium, we investigate complete information games in which each agent knows the true preference profile.

In an attempt to explain the laboratory experiments in Cason, Saijo, Sjöström, and Yamato (2006), in Chapter 2, we study double implementability in dominant strategy equilibria and "ex post equilibria." By definition, an ex post equilibrium is a strategy profile in which, for each possible preference profile, the message profile for the preference profile is a Nash equilibrium. Every ex post equilibrium has the no regret property that no agent would have an incentive to change his message even if he were to be informed of the true preferences of the others. We would like to exclude bad Nash equilibria under incomplete information games.

As mentioned above, there exist *strategy-proof* SCFs in several environments such as under the set of single-peaked preference profiles. However, interesting SCFs satisfying *strategy-proofness* do not necessarily exist in other environments such as under the set of all preference profiles. (e.g., Hurwicz, 1972; Gibbard, 1973; Ledyard and Roberts, 1974; Satterthwaite, 1975).

If we drop the requirement of *strategy-proofness* and then we consider the notion of Nash equilibrium, the situation is much better. Nash implementation using mechanisms with general message spaces is studied by Maskin (1977, 1999).[8,9] He shows that "Maskin-invariance" is necessary for an SCC to be Nash implementable.[10] With at least three agents, *Maskin-invariance* together with "no-veto-power" is sufficient for Nash im-

[8] Maskin (1977) was later published as Maskin (1999).

[9] See also Groves and Ledyard (1977) for a class of economic environments and see Hurwicz and Schmeidler (1978) for the case of social choice from a finite set of alternatives. These two kinds of environments are included in the model of Maskin (1999).

[10] *Maskin-invariance* is also called "Maskin-monotonicity." For the terminology in this thesis, we follow Thomson (2018).

plementability by constructing the following mechanism (hereafter, the Maskin mechanism).[11] Each agent's strategy has three components, namely a preference profile, an outcome, and an integer. If each agent reports the same preference profile and the same outcome that is an element of the set of outcomes chosen by the SCC for the reported preference profile, then the reported outcome is chosen. If there exists only one agent reports a different preference profile or a different outcome, say a, from the above messages of the other agents, then the outcome proposed by the odd-man-out is chosen, provided that a is at least preferred to this outcome at his preference which is a component of the preference profile reported by the other agents; otherwise, a is chosen. In all other cases, the outcome is the one proposed by the agent with the highest index among those whose proposed integer is maximal.[12]

Regarding the result on sufficiency of Maskin (1999), it has been argued that there are the following two issues: (1) the Maskin mechanism is complex and it may not be procedurally fair i.e., some agent is not treated fairly (see, e.g., Thomson, 2005; Korpela, 2018); (2) if each agent does not select weakly dominated strategies, then the Maskin mechanism may not implement an SCC which is Nash implementable (see, e.g., Yamato, 1999).

In Chapter 3, we resolve the first issue for the allocation problem of infinitely divisible resources with at least three agents. For this problem, Thomson (2005) and Doğan (2016) propose simple but not procedurally fair mechanisms which implement the "no-envy" correspondence in Nash equilibria. By contrast, Galbiati (2008) constructs a procedurally fair but not simple mechanism which implements the no-envy correspondence in Nash equilibria. In this chapter, we design a both simple and procedurally fair mechanism which implements the no-envy correspondence in Nash equilibria.

In Chapter 4, we resolve the second issue. We study double implementability of SCCs in Nash equilibria and undominated Nash equilibria. We propose sufficient conditions

[11] The Maskin mechanism is also called a "canonical mechanism for Nash implementation."

[12] By means of a similar mechanism, Moore and Repullo (1990) propose a characterization of Nash implementability.

under each of three assumptions that the number of "partially honest" agent is 0, at least 1, or at least 2. In addition, we provide a necessary and sufficient condition under a minor qualification for double implementability in Nash equilibria and undominated Nash equilibria.

In Chapter 5, we resolve the both first and second issues for SCFs. We show that if there are at least three agents and each agent is "semi-socially-responsible" with respect to a *unanimous* SCF, then a simple and procedurally fair mechanism doubly implements this SCF in Nash equilibria and undominated Nash equilibria.

In Chapter 6, we conclude this thesis by summarizing its contributions and discussing the following three remaining issues in the thesis: (1) irrational choices; (2) repeated implementation; and (3) laboratory experiments.

Chapter 2

Double Implementation in Dominant Strategy Equilibria and Ex Post Equilibria with Private Values

2.1 Introduction

We investigate the implementation problem under incomplete information and private values. The objective of a social planner is embodied by a "social choice function (SCF)." Mathematically, an SCF is a single-valued mapping which, for each possible preference profile, specifies an outcome. The planner does not know the agents' preferences. Then, she specifies a message space for each agent and a single-value mapping which, for each possible message profile, chooses an outcome. The pair consisting of the list of agents' message spaces and a mapping is a "mechanism." In the direct mechanism associated with an SCF, the message space for each agent is the set of his possible preferences and the mapping is the function.

"Strategy-proofness" requires that in the direct mechanism associated with the SCF, truth-telling should be a dominant strategy for each agent. For each preference profile, the outcome chosen by the SCF is achieved at this dominant strategy equilibrium. An important point concerning a dominant strategy equilibrium is that each agent needs only information about his own preference. He need not care about the other agents' preferences nor strategies. However, laboratory experiments regarding *strategy-proof* SCFs reported that in several games, some subjects did not select dominant strategies.[1]

These observations raise a concern for implementation theory. Although in pivotal-mechanism experiments in which truth-telling is a dominant strategy for each agent, some subjects did not adopt dominant strategies, they frequently selected a Nash equilibrium (Cason, Saijo, Sjöström, and Yamato, 2006). There is an explanation for this observation. Suppose that there are only two subjects. If one of them, subject 1, finds a dominant strategy but the other, subject 2, does not, then as long as subject 2 chooses a best response to subject 1's strategy, a Nash equilibrium outcome is achieved. It should be easier to find a best response to subject 1's strategy than a dominant strategy. This observation led Saijo, Sjöström, and Yamato (2007) to formulate and investigate "secure implementation," namely double implementation in dominant strategy equilibria and Nash equilibria.[2] If there is no dominant strategy for an agent, then any best response of this agent depends on the choices of the other agents so that it may be hard for this agent to predict the choices. For strategic uncertainty not to be important, Saijo, Sjöström, and Yamato (2007) study not only good Nash equilibria that induce socially desirable outcomes, but also good dominant strategy equilibria.

In the laboratory experiments in Cason, Saijo, Sjöström, and Yamato (2006), each subject knew only his own preference, so that incomplete information games were considered.[3] Usually, to define the notion of Nash equilibrium, we investigate complete information games in which each agent knows the true preference profile.[4] Table 2.1

[1]For a summary of laboratory experiments regarding *strategy-proof* SCFs, see, e.g., Cason, Saijo, Sjöström, and Yamato (2006).

[2]Another study focuses on extensive form games. In an ascending auction and a second-price auction, subjects were substantially more likely to play truth-telling under the former than under the latter (Kagel, Harstad, and Levin, 1987). Inspired from this observation, "obvious" *strategy-proofness* is defined and characterized as a cognitively limited agent can recognize that truth-telling is a dominant strategy (Li, 2017).

[3]For other laboratory experiments under the incomplete information setting, see Attiyeh, Franciosi, and Isaac (2000) and Kawagoe and Mori (2001) for pivotal-mechanism experiments, and Kagel and Levin (1993) and Harstad (2000) for second-price-auction experiments.

[4]One justification of secure implementation as a theoretical prediction for the laboratory experiments in Cason, Saijo, Sjöström, and Yamato (2006) is that a Nash equilibrium can be interpreted as a rest point of the dynamic learning process (Cason, Saijo, Sjöström, and Yamato, 2006). However, secure implementation is a theoretical prediction in a one-shot game. Other justifications of secure implementation are characterizations by robust implementation notions (Adachi, 2014; Saijo, Sjöström, and Yamato, 2007). Even though these implementation notions are under the incomplete information setting, we might not explicitly study the observation of the experiments unlike secure implementation.

Table 2.1: The difference between Cason et al. (2006) and Saijo et al. (2007).

	Cason et al. (2006) (Laboratory experiments)	Saijo et al. (2007) (A theoretical prediction)
Information structure	Incomplete information	Complete information
Result	Subjects frequently selected a Nash equilibrium	Characterizations for secure implementability

illustrates this discussion.

In an attempt to explain the laboratory experiments in Cason, Saijo, Sjöström, and Yamato (2006), we study double implementability in dominant strategy equilibria and "ex post equilibria." By definition, an ex post equilibrium is a strategy profile in which, for each possible preference profile, the message profile for the preference profile is a Nash equilibrium. We would like to exclude bad Nash equilibria under incomplete information games. Every ex post equilibrium has the no regret property that no agent would have an incentive to change his message even if he were to be informed of the true preferences of the others. We would like to exclude bad Nash equilibria under incomplete information games.

Another possible way to explain the laboratory experiments in Cason, Saijo, Sjöström, and Yamato (2006) is to consider the notion of Bayesian Nash equilibrium, instead of the notion of ex post equilibrium. Wilson (1987) states that we should not rely on strong informational assumptions, such as the common prior assumption: there is common knowledge of a common prior on a set of preferences. In this sense, mechanisms for Bayesian Nash implementability are not practical, and it is difficult to impose the common prior assumption in the laboratory experiment in Cason, Saijo, Sjöström, and Yamato (2006). On the other hand, neither the notion of dominant strategy equilibrium nor the notion of ex post equilibrium refer to prior nor posterior probability distributions of the preferences. Then, these two notions are "belief free."[5] Especially, the notion of ex post

[5]If solution concepts do not depend on the beliefs and higher order beliefs of the agent, then we refer

equilibrium requires that for all beliefs and higher order beliefs, the strategies of the agents remain an "interim" equilibrium (Bergemann and Morris, 2007; Proposition 2).[6,7]

Bergemann and Morris (2008) claim that "*in an environment with private values, the notion of ex post equilibrium is equivalent to the notion of dominant strategy equilibrium*" (pp. 532). Our first result is that in general, the former is weaker than the latter (Fact 2.1, Example 2.1). Then, double implementability is not trivial even under private values.

For double implementability, we need to consider dominant strategy implementability. By the revelation principle for dominant strategy implementability, *strategy-proofness* is necessary (Gibbard, 1973). Based on this result, secure implementability is characterized by a stronger axiom, "secure strategy-proofness" (Saijo, Sjöström, and Yamato, 2007).[8] *Secure strategy-proofness* requires that the SCF should be *strategy-proof* and for each preference profile and each Nash equilibrium in the complete information game induced by the direct mechanism and the preference profile, the outcome at the equilibrium should be equal to the outcome chosen by the SCF for the preference profile.

We define a new strategic axiom, "weak secure-strategy-proofness." This axiom requires that the SCF should be *strategy-proof* and if a strategy profile is an ex post equilibrium in the incomplete information game induced by the direct mechanism and the set of preference profiles, then for each preference profile, the outcome at the equilibrium should be equal to the outcome chosen by the SCF for the preference profile. This axiom is weaker than *secure strategy-proofness* (Proposition 2.3, Example 2.2).

For the direct mechanism associated with an SCF, we show that dominant strategy implementability is weaker than ex post implementability (Lemma 2.1, Example 2.3).[9] In

to them as belief free solution concepts.

[6]For the definition of interim equilibrium, see Bergemann and Morris (2007).

[7]Under the model of Bergemann and Morris (2005), in the special case where the game is induced by a direct mechanism associated with an SCF and a social planner is trying to truthfully implement this function, the relationship between the notion of ex post equilibrium and the notion of interim equilibrium is provided (Bergemann and Morris, 2005; Propositions 1 and 3).

[8]In Saijo, Sjöström, and Yamato (2007), secure implementability is characterized by *strategy-proofness* and "rectangle property." For the definition of *rectangle property*, see Saijo, Sjöström, and Yamato (2007). It is easy to show that an SCF satisfies *strategy-proofness* and *rectangle property* if and only if it is *securely strategy-proof*.

[9]By this result, for the direct mechanism associated with an SCF, ex post "full" implementability is weaker than dominant strategy "full" implementability. Note that under private values, ex post "truthful" implementability is equivalent to dominant strategy "truthful" implementability by definition (see, e.g.,

addition, we show that an SCF is doubly implementable in dominant strategy equilibria and ex post equilibria if and only if it is *weakly securely-strategy-proof* (Theorem 2.1). The proof involves showing that any doubly implementable SCF in dominant strategy equilibria and ex post equilibria is implemented by its associated direct mechanism (Corollary 2.1). Hence, for double implementability, it suffices to focus on direct mechanisms.[10]

For secure implementation, negative results have been established for a number of interesting SCFs (e.g., Fujinaka and Wakayama, 2011). Even if an SCF is not securely implementable, it may be doubly implementable in dominant strategy equilibria and ex post equilibria (Corollary 2.2).[11] Are there such interesting SCFs? We provide one negative answer and one positive answer. In a school choice problem (Abdulkadiroğlu and Sönmez, 2003) under incomplete information, the tentative acceptance rule is not doubly implementable in dominant strategy equilibria and ex post equilibria (Example 2.4).[12] On the other hand, if the set of preference profiles is "large," then the rule may be doubly implemented in dominant strategy equilibria and ex post equilibria (Example 2.5). Identifying general conditions concerning the set of preference profiles for double implementability of the tentative acceptance rule in dominant strategy equilibria and ex post equilibria is an open question.

This chapter is organized as follows. Section 2.2 provides the notions of dominant strategy equilibrium and ex post equilibrium, and investigates the relationships. Section 2.3 provides the definitions of *strategy-proofness* and related properties, and establishes the relationships. Section 2.4 proposes the notions of dominant strategy implementability and ex post implementability, and reports our main results. Section 2.5 discusses several

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Bergemann and Morris, 2005). In other words, under private values, "ex post incentive compatibility" is equivalent to *strategy-proofness*.

[10]Saijo, Sjöström, and Yamato (2007) and Saran (2016) also provide, for other notions of implementability, revelation principles in which we can limit our attention to direct mechanisms.

[11]Note that this comparison is controversial and may not be precise, because secure implementability is under complete information, but double implementability in dominant strategy equilibria and ex post equilibria is under incomplete information. By this comparison, we just suggest that if an SCF is securely implementable under complete information, then this function is doubly implementable in dominant strategy equilibria and ex post equilibria if the complete information setting is changed into the incomplete information setting.

[12]The tentative acceptance rule is also called the deferred acceptance algorithm or the Gale-Shapley student optimal stable mechanism (Gale and Shapley, 1962).

applications.

2.2 Equilibrium Notions

Let $N = \{1, \cdots, n\}$ be the set of agents and A be the finite or infinite set of outcomes. For each $i \in N$, let $R_i \in \mathcal{R}_i$ be a preference for agent i, where $\mathcal{R}_i$ is the set of possible preferences for agent i over A. The asymmetric and symmetric components of $R_i \in \mathcal{R}_i$ are denoted by P_i and I_i, respectively. A preference profile is a list $R \equiv (R_1, \cdots, R_n) \in \mathcal{R}$, where $\mathcal{R} \equiv \times_{i \in N} \mathcal{R}_i$. For each $i \in N$ and each $R_i \in \mathcal{R}_i$, let $u_i : A \to \mathbb{R}$ be a utility representation for R_i such that for each pair $a, b \in A$, (1) $u_i(a) > u_i(b)$ if and only if $a \; P_i \; b$ and (2) $u_i(a) = u_i(b)$ if and only if $a \; I_i \; b$. Each agent's preferences do not depend on the other agents' preferences, so that we study private-values problems.[13]

A **social choice function** (**SCF**) $f : \mathcal{R} \to A$ is a single-valued mapping which, for each preference profile $R \in \mathcal{R}$, specifies an outcome $f(R) \in A$.

A **mechanism** Γ is a pair (M, g) such that $M = \times_{i \in N} M_i$, where for each $i \in N$, M_i is the message space for agent i, and $g : M \to A$ is the outcome mapping which, for each message profile $m \in M$, specifies an outcome $g(m) \in A$. Let $\Gamma^f = (\mathcal{R}, f)$ be the **direct mechanism associated with SCF** f.

Let $(\Gamma, \mathcal{R})$ be the **(incomplete information) game** induced by Γ and $\mathcal{R}$. A **(pure) strategy** $s_i : \mathcal{R}_i \to M_i$ **of** $(\Gamma, \mathcal{R})$ **for agent** $i \in N$ is a single-valued mapping which, for each preference $R_i \in \mathcal{R}_i$, specifies a message $s_i(R_i) \in M_i$. Let $s = (s_1, \ldots, s_n) \in S$ be a strategy profile, where S is a set of strategy profiles.

In game $(\Gamma, \mathcal{R})$, let us define the following two equilibrium notions which are central to our study.

Definition 2.1. A strategy profile $s \in S$ is a **dominant strategy equilibrium** of

[13] If each agent's preferences may depend on the other agents' preferences, then problems are under interdependent values. We can easily extend our results to interdependent-values problems. However, for double implementability, "dominant strategy incentive compatibility" (Bergemann and Morris, 2005) is necessary by the revelation principle for dominant strategy implementability. This axiom is stronger than *ex post incentive compatibility* and it is difficult to find interesting social choice functions satisfying this axiom under interdependent values.

$(\Gamma, \mathcal{R})$ if for each $i \in N$, each $R_i \in \mathcal{R}_i$, each $m_i \in M_i$, and each $m_{-i} \in M_{-i}$,

$$g(s_i(R_i), m_{-i}) \; R_i \; g(m_i, m_{-i}).$$

Let $DS(\Gamma, \mathcal{R}) \subseteq S$ be the **set of dominant strategy equilibria of** $(\Gamma, \mathcal{R})$.

Definition 2.2. A strategy profile $s \in S$ is an **ex post equilibrium** of $(\Gamma, \mathcal{R})$ if for each $R \in \mathcal{R}$, each $i \in N$, and each $m_i \in M_i$,

$$g(s_i(R_i), s_{-i}(R_{-i})) \; R_i \; g(m_i, s_{-i}(R_{-i})).$$

Let $EP(\Gamma, \mathcal{R}) \subseteq S$ be the **set of ex post equilibria of** $(\Gamma, \mathcal{R})$.

Bergemann and Morris (2008) claim that "*in an environment with private values, the notion of ex post equilibrium is equivalent to the notion of dominant strategy equilibrium*" (pp. 532). Our first result is that in general, the notion of ex post equilibrium is weaker than the notion of dominant strategy equilibrium (Fact 2.1, Example 2.1).

Fact 2.1. For each game $(\Gamma, \mathcal{R})$, $DS(\Gamma, \mathcal{R}) \subseteq EP(\Gamma, \mathcal{R})$.

Proof. Let $s \in DS(\Gamma, \mathcal{R})$. Suppose that $s \notin EP(\Gamma, \mathcal{R})$. Then, there are $R \in \mathcal{R}$, $i \in N$, and $m_i \in M_i$ such that $g(m_i, s_{-i}(R_{-i})) \; P_i \; g(s_i(R_i), s_{-i}(R_{-i}))$. Therefore, there are $i \in N$, $R_i \in \mathcal{R}_i$, $m_i \in M_i$, and $m_{-i} \equiv s_{-i}(R_{-i}) \in M_{-i}$ such that $g(m_i, m_{-i}) \; P_i \; g(s_i(R_i), m_{-i})$, which contradicts $s \in DS(\Gamma, \mathcal{R})$. □

The following example states that the converse of Fact 2.1 does not hold by showing that there is a game in which a strategy profile is an ex post equilibrium, but not a dominant strategy equilibrium.

Example 2.1: There is a game $(\Gamma, \mathcal{R})$ such that $DS(\Gamma, \mathcal{R}) \subsetneq EP(\Gamma, \mathcal{R})$.

Let $N = \{1, 2\}$, $A = \{a^1, a^2, a^3, a^4\}$, $\mathcal{R}_1 = \{R_1, R_1'\}$, $\mathcal{R}_2 = \{R_2, R_2'\}$, and $\mathcal{R} =$

$\times_{i \in N} \mathcal{R}_i$. Preferences are defined as follows:

R_1	R'_1		R_2	R'_2
a^1, a^2	a^2, a^3, a^4		a^1, a^2, a^3	a^2, a^4
a^3, a^4	a^1		a^4	a^1, a^3

Let (u_1, u_2) be a pair of utility representations for each preference profile such that for each agent, the utility of the most preferred outcome is 2 and the utility of the least preferred outcome is 1.

Let f be defined as follows:[14]

f	R_2	R'_2
R_1	a^1	a^2
R'_1	a^3	a^4

The game induced by Γ^f and $\mathcal{R}$ has the following utilities:

	true preference	R_2		R'_2	
true preference	message	R_2	R'_2	R_2	R'_2
R_1	R_1	$\underline{\mathbf{2}}, \underline{\mathbf{2}}$	$\underline{\mathbf{2}}, \underline{\mathbf{2}}$	$\underline{2}, 1$	$\underline{\mathbf{2}}, \underline{\mathbf{2}}$
	R'_1	$1, \underline{2}$	$1, 1$	$1, 1$	$1, \underline{2}$
R'_1	R_1	$1, \underline{2}$	$\underline{\mathbf{2}}, \underline{\mathbf{2}}$	$1, 1$	$\underline{\mathbf{2}}, \underline{\mathbf{2}}$
	R'_1	$\underline{\mathbf{2}}, \underline{\mathbf{2}}$	$\underline{2}, 1$	$\underline{2}, 1$	$\underline{\mathbf{2}}, \underline{\mathbf{2}}$

Let $(s_1, s_2) \equiv ((s_1(R_1), s_1(R'_1)), (s_2(R_2), s_2(R'_2)))$.[15] Then, $DS(\Gamma^f, \mathcal{R}) = \{((R_1, R'_1), (R_2, R'_2))\}$, and $EP(\Gamma^f, \mathcal{R}) = \{((R_1, R'_1), (R_2, R'_2))\ ,\ ((R_1, R_1), (R'_2, R'_2))\}$. Hence, the strategy profile $((R_1, R_1), (R'_2, R'_2))$ is an ex post equilibrium of $(\Gamma^f, \mathcal{R})$, but not a dominant strategy equilibrium of $(\Gamma^f, \mathcal{R})$. Then, $DS(\Gamma^f, \mathcal{R}) \subsetneq EP(\Gamma^f, \mathcal{R})$.■

[14] The SCF in Example 2.1 seems artificial. However, in a specific model, we can find an interesting SCF f such that $DS(\Gamma^f, \mathcal{R}) \subsetneq EP(\Gamma^f, \mathcal{R})$. See Example 2.4 in Section 2.5.

[15] Formally, let s_1 be the mapping such that for $R_1 \in \mathcal{R}_1$, agent 1 selects $s_1(R_1)$ and for $R'_1 \in \mathcal{R}_1$, agent 1 selects $s_1(R'_1)$, and let s_2 be the mapping such that for $R_2 \in \mathcal{R}_2$, agent 2 selects $s_2(R_2)$ and for $R'_2 \in \mathcal{R}_2$, agent 2 selects $s_2(R'_2)$.

2.3 Strategy-Proofness and Related Properties

The following axiom requires that in the direct mechanism associated with SCF f, truth-telling should be a dominant strategy for each agent.

Definition 2.3. An SCF f is **strategy-proof** if for each $R \in \mathcal{R}$, each $i \in N$, and each $R'_i \in \mathcal{R}_i$,

$$f(\overset{\text{truth}}{R_i}, R_{-i}) \overset{\text{truth}}{R_i} f(\overset{\text{lie}}{R'_i}, R_{-i}).$$

The following results are the revelation principles for dominant strategy implementability and ex post implementability.

Proposition 2.1. (1) (Gibbard, 1973) *If an SCF is dominant strategy implementable, it is* strategy-proof.
(2) (Bergemann and Morris, 2008) *If an SCF is ex post implementable, it is* strategy-proof.

The following axiom is a necessary and sufficient condition for secure implementation, namely double implementation in dominant strategy equilibria and Nash equilibria.

Definition 2.4. An SCF f is **securely strategy-proof** if (1) f is *strategy-proof*, and (2) for each pair $R, \tilde{R} \in \mathcal{R}$, if for each $i \in N$ and each $R'_i \in \mathcal{R}_i$, $f(\tilde{R}_i, \tilde{R}_{-i})\ R_i\ f(R'_i, \tilde{R}_{-i})$, then $f(\tilde{R}) = f(R)$.

To interpret this axiom, let us define the following notions. For each $R \in \mathcal{R}$, let (Γ, R) be the **complete information game** induced by Γ and R. A message profile $m \in M$ is a **dominant strategy equilibrium of** (Γ, R) if for each $i \in N$, each $m'_i \in M_i$, and each $m'_{-i} \in M_{-i}$, $g(m_i, m'_{-i})\ R_i\ g(m'_i, m'_{-i})$. Let $DS(\Gamma, R)$ be the **set of dominant strategy equilibria of** (Γ, R). A message profile $m \in M$ is a **Nash equilibrium of** (Γ, R) if for each $i \in N$ and each $m'_i \in M_i$, $g(m_i, m_{-i})\ R_i\ g(m'_i, m_{-i})$. Let $NE(\Gamma, R)$ be the **set of Nash equilibria of** (Γ, R).

An SCF f is **securely implementable** if there is a mechanism $\Gamma = (M, g)$ such that for each $R \in \mathcal{R}$, $\{f(R)\} = g(DS(\Gamma, R)) = g(NE(\Gamma, R))$.

Secure strategy-proofness requires that the SCF f should be *strategy-proof*, and for each preference profile $R \in \mathcal{R}$ and each Nash equilibrium of (Γ^f, R), the outcome at the Nash equilibrium should be equal to the outcome chosen by f for R. Secure implementability is characterized by this axiom.

Proposition 2.2. (Saijo, Sjöström, and Yamato, 2007). *An SCF is securely implementable if and only if it is* securely strategy-proof.

The following axiom is weaker than *secure strategy-proofness* as discussed in Saijo, Sjöström, and Yamato (2007).

Definition 2.5. An SCF f is **non-bossy (in welfare\outcome)** if for each $R \in \mathcal{R}$, each $i \in N$, and each $R'_i \in \mathcal{R}_i$, if $f(R_i, R_{-i})\ I_i\ f(R'_i, R_{-i})$, then $f(R_i, R_{-i}) = f(R'_i, R_{-i})$.

The following axiom of SCF f requires that f should be *strategy-proof*, and if a strategy profile is an ex post equilibrium in $(\Gamma^f, \mathcal{R})$, then for each preference profile, the outcome at the ex post equilibrium should be equal to the outcome chosen by f for the preference profile.

First, we define a notion. For each $i \in N$, a **deception** $d_i : \mathcal{R}_i \to \mathcal{R}_i$ **for agent** i is a single-valued mapping which, for each preference $R_i \in \mathcal{R}_i$, specifies a preference $d_i(R_i) \in \mathcal{R}_i$. We can interpret it as a strategy for agent i in the game induced by a mechanism in which for each agent $i \in N$, $M_i = \mathcal{R}_i$ and the set of preference profiles. Let $d = (d_i)_{i \in N} \in \mathcal{D}$ be a **deception profile**, where $\mathcal{D}$ is the **set of deception profiles**.

Definition 2.6. An SCF f is **weakly securely-strategy-proof** if (1) f is *strategy-proof*, and (2) for each $d \in \mathcal{D}$, if for each $R \in \mathcal{R}$, each $i \in N$, and each $R'_i \in \mathcal{R}_i$, $f(d_i(R_i), d_{-i}(R_{-i}))\ R_i\ f(R'_i, d_{-i}(R_{-i}))$, then $f \circ d = f$.

Weak secure-strategy-proofness is implied by *secure strategy-proofness* (Proposition 2.3), but the converse of this relationship does not hold (Example 2.2). Note that an ex post equilibrium $s \in S$ is a strategy profile in which, for each preference profile $R \in \mathcal{R}$, the message profile $s(R) \in M$ is a Nash equilibrium.

Proposition 2.3. *If an SCF is* securely strategy-proof, *then it is* weakly securely-strategy-proof.

Proof. Let f be a *securely strategy-proof* SCF. It suffices to show that it satisfies (2) of *weak secure-strategy-proofness.*

Let $d \in \mathcal{D}$. The proof is by contradiction. For each $R \in \mathcal{R}$, each $i \in N$, and each $R'_i \in \mathcal{R}_i$, suppose that $f(d_i(R_i), d_{-i}(R_{-i}))\ R_i\ f(R'_i, d_{-i}(R_{-i}))$. Suppose also that there is $R'' \in \mathcal{R}$ such that $f(d(R'')) \neq f(R'')$. Let $\tilde{R} = d(R'')$. We have that for each $i \in N$ and each $R'_i \in \mathcal{R}_i$, $f(\tilde{R}_i, \tilde{R}_{-i})\ R''_i\ f(R'_i, \tilde{R}_{-i})$, but $f(\tilde{R}) \neq f(R'')$, which contradicts (2) of *secure strategy-proofness.* □

The following example shows that the converse of Proposition 2.3 does not hold.

Example 2.2. An SCF is *weakly securely-strategy-proof*, but not *securely strategy-proof.*

Let $N = \{1, 2\}$, $A = \{a^1, a^2, a^3, a^4\}$, $\mathcal{R}_1 = \{R_1, R'_1\}$, $\mathcal{R}_2 = \{R_2, R'_2\}$, and $\mathcal{R} = \times_{i \in N} \mathcal{R}_i$. Preferences are defined as follows:

R_1	R'_1	R_2	R'_2
a^1, a^2	a^3, a^4	a^1, a^2, a^3	a^2, a^4
a^3, a^4	a^1, a^2	a^4	a^1, a^3

Let (u_1, u_2) be a pair of utility representations for each preference profile such that for each agent, the utility of the most preferred outcome is 2 and the utility of the least preferred outcome is 1.

Let f be defined as follows:[16]

f	R_2	R'_2
R_1	a^1	a^2
R'_1	a^3	a^4

[16]The SCF in Example 2.2 seems artificial. However, in a specific model, we can find an interesting SCF f that is *weakly securely-strategy-proof*, but not *securely strategy-proof.* See Example 2.5 in Section 2.5.

The game induced by Γ^f and $\mathcal{R}$ has the following utilities:

	true preference	R_2		R_2'	
true preference	message	R_2	R_2'	R_2	R_2'
R_1	R_1	$\underline{\mathbf{2}},\ \underline{\mathbf{2}}$	$\underline{\mathbf{2}},\ \underline{\mathbf{2}}$	$\underline{2}, 1$	$\underline{\mathbf{2}},\ \underline{\mathbf{2}}$
	R_1'	$1, \underline{2}$	$1, 1$	$1, 1$	$1, \underline{2}$
R_1'	R_1	$1, \underline{2}$	$1, \underline{2}$	$1, 1$	$1, \underline{2}$
	R_1'	$\underline{\mathbf{2}},\ \underline{\mathbf{2}}$	$\underline{2}, 1$	$\underline{2}, 1$	$\underline{\mathbf{2}},\ \underline{\mathbf{2}}$

Let $(d_1, d_2) \equiv ((d_1(R_1), d_1(R_1')), (d_2(R_2), d_2(R_2'))) = ((R_1, R_1'), (R_2, R_2'))$. Then, $DS(\Gamma^f, \mathcal{R}) = EP(\Gamma^f, \mathcal{R}) = \{(d_1, d_2)\}$ and $NE(\Gamma^f, (R_1, R_2)) = \{(d_1(R_1), d_2(R_2)), (R_1, R_2')\}$. The SCF f is *strategy-proof* and $f \circ d = f$. Therefore, it is *weakly securely-strategy-proof*. On the other hand, for $(R_1, R_2) \in \mathcal{R}$, $(R_1, R_2') \in NE(\Gamma^f, (R_1, R_2))$, but $f(R_1, R_2') \neq f(R_1, R_2)$. Hence, it is not *securely strategy-proof*. The SCF f does not satisfy *non-bossiness* either. For (R_1, R_2), agent 2, and $R_2' \in \mathcal{R}_2$, $f(R_1, R_2) = a^1\ I_2\ a^2 = f(R_1, R_2')$, but $a^1 \neq a^2$.■

2.4 Implementability Notions and Main Results

For SCF f, let us define the following two implementability notions which are central to our study.

Definition 2.7. An SCF f is **dominant strategy implementable** if there is a mechanism $\Gamma = (M, g)$ such that for each $s \in DS(\Gamma, \mathcal{R}) \neq \emptyset$, $g \circ s = f$.[17]

Definition 2.8. An SCF f is **ex post implementable** if there is a mechanism $\Gamma = (M, g)$ such that for each $s \in EP(\Gamma, \mathcal{R}) \neq \emptyset$, $g \circ s = f$.

For the direct mechanism associated with an SCF, dominant strategy implementability is weaker than ex post implementability (Lemma 2.1, Example 2.3).

[17] Given a mechanism $\Gamma = (M, g)$, a strategy profile $s \in S$ of $(\Gamma, \mathcal{R})$, and an SCF f, $g \circ s = f$ means that for each $R \in \mathcal{R}$, $g(s(R)) = f(R)$.

Lemma 2.1. *If an SCF f is implemented by Γ^f in ex post equilibria, then it is implemented by Γ^f in dominant strategy equilibria.*

Proof. Let f be an SCF that is implemented by Γ^f in ex post equilibria. Then, for each $s \in EP(\Gamma^f, \mathcal{R}) \neq \emptyset$, $f \circ s = f$. Since by Proposition 2.1 (2), f is *strategy-proof*, $DS(\Gamma^f, \mathcal{R}) \neq \emptyset$. Since f is implemented by Γ^f in ex post equilibria, by Fact 2.1, for each $s \in DS(\Gamma^f, \mathcal{R}) \subseteq EP(\Gamma^f, \mathcal{R})$, $f \circ s = f$. Therefore, f is implemented by Γ^f in dominant strategy equilibria. $\square$

The next example states that the converse of Lemma 2.1 does not hold by showing that the SCF in Example 2.1 is not ex post implementable. To prove this, we show that it does not satisfy the following axiom.

Definition 2.9. An SCF f is **ex post invariant**[18] if for each $d \in \mathcal{D}$ with $f \circ d \neq f$, there are $R \in \mathcal{R}$, $i \in N$, and $a \in A$ such that $a \ P_i \ f(d(R))$, and for each $R'_i \in \mathcal{R}_i$, $f(R'_i, d_{-i}(R_{-i})) \ R'_i \ a$.

The following result is applied in the next example.

Proposition 2.4. (Bergemann and Morris, 2008). *If an SCF is not* ex post invariant, *then it is not ex post implementable.*

In the following example, we consider the same setting as in Example 2.1.

Example 2.3. The SCF in Example 2.1 is not implementable in ex post equilibria.

Let $d \in \mathcal{D}$ be such that for each $\tilde{R}_1 \in \mathcal{R}_1$, $d(\tilde{R}_1) = R_1$ and for each $\tilde{R}_2 \in \mathcal{R}_2$, $d(\tilde{R}_2) = R'_2$. Then, $f \circ d \neq f$:

$f \circ d$	R_2	R'_2
R_1	a^2	a^2
R'_1	a^2	a^2

Then, for each $R \in \mathcal{R}$, $f(d(R)) = a^2$ and for each $i \in N$, each $R_i \in \mathcal{R}_i$, and each $a \in A$, $a^2 \ R_i \ a$. That is, for each $R \in \mathcal{R}$, each $i \in N$, and each $a \in A$, $f(d(R)) \ R_i \ a$. Therefore,

[18] *Ex post invariance* is called "ex post monotonicity" by Bergemann and Morris (2008).

f is not *ex post invariant.* By Proposition 2.4, f is not ex post implementable, although f is implemented by Γ^f in dominant strategy equilibria by the logic of Example 2.1.■

As discussed in Section 2.1, we would like to investigate double implementability in dominant strategy equilibria and ex post equilibria.

Definition 2.10. An SCF f is **doubly implementable** if there is a mechanism $\Gamma = (M, g)$ such that:

(1) for each $s \in DS(\Gamma, \mathcal{R}) \neq \emptyset$,

$$g \circ s = f,$$

(2) for each $s \in EP(\Gamma, \mathcal{R})$,

$$g \circ s = f.$$

Our main result is provided as follows:

Theorem 2.1. *An SCF is doubly implementable in dominant strategy equilibria and ex post equilibria if and only if it is* weakly securely-strategy-proof.

Proof. First, we consider the if part. Let f be a *weakly securely-strategy-proof* SCF. We show that $\Gamma^f = (\mathcal{R}, f)$ doubly implements f in dominant strategy equilibria and ex post equilibria. By (1) of *weak secure-strategy-proofness* and Fact 2.1, $\emptyset \neq DS(\Gamma^f, \mathcal{R}) \subseteq EP(\Gamma^f, \mathcal{R})$. By Lemma 2.1, it suffices to show that for each $s \in EP(\Gamma^f, \mathcal{R})$, $f \circ s = f$. Note that in $(\Gamma^f, \mathcal{R})$, for each $i \in N$, $s_i : \mathcal{R}_i \to \mathcal{R}_i$ so that $s \in \mathcal{D}$. By the definition of ex post equilibrium, for each $R \in \mathcal{R}$, each $i \in N$, and each $R'_i \in \mathcal{R}_i$, $f(s_i(R_i), s_{-i}(R_{-i}))\ R_i\ f(R'_i, s_{-i}(R_{-i}))$. By (2) of *weak secure-strategy-proofness*, $f \circ s = f$.

Next, we prove the only if part. Let f be a doubly implementable SCF in dominant strategy equilibria and ex post equilibria. Then, let $\Gamma = (M, g)$ be a mechanism which doubly implements f in dominant strategy equilibria and ex post equilibria. By Proposition 2.1 (1), f is *strategy-proof.* Therefore, it suffices to show that f satisfies (2) of *weak secure-strategy-proofness.*

Let $d \in \mathcal{D}$. Let the hypothesis of (2) be satisfied: for each $R \in \mathcal{R}$, each $i \in N$, and each $R'_i \in \mathcal{R}_i$, $f(d_i(R_i), d_{-i}(R_{-i}))\ R_i\ f(R'_i, d_{-i}(R_{-i}))$. We show that $f \circ d = f$.

Since Γ doubly implements f in dominant strategy equilibria and ex post equilibria, $DS(\Gamma, \mathcal{R}) \neq \emptyset$. Let $s \in DS(\Gamma, \mathcal{R})$. Since Γ implements f in dominant strategy equilibria, i.e., $g \circ s = f$, we have $g \circ s \circ d = f \circ d$. That is, for each $R \in \mathcal{R}$, $g(s(d(R))) = f(d(R))$. Similarly, since Γ implements f in dominant strategy equilibria, for each $i \in N$, each $R'_i \in \mathcal{R}_i$, and each $R_{-i} \in \mathcal{R}_{-i}$, $g(s_i(R'_i), s_{-i}(d(R_{-i}))) = f(R'_i, d(R_{-i}))$. Since $f(d_i(R_i), d_{-i}(R_{-i}))\ R_i\ f(R'_i, d_{-i}(R_{-i}))$ by the hypothesis of (2) in *weak secure-strategy-proofness*, $g(s(d(R))) = f(d(R))$, and $g(s_i(R'_i), s_{-i}(d_{-i}(R_{-i}))) = f(R'_i, d_{-i}(R_{-i}))$, we have $g(s_i(d_i(R_i)), s_{-i}(d_{-i}(R_{-i})))\ R_i\ g(s_i(R'_i), s_{-i}(d_{-i}(R_{-i})))$. When $R'_i = R_i$, since $s \in DS(\Gamma, \mathcal{R})$, for each $m_i \in M_i$, $g(s_i(R_i), s_{-i}(d_{-i}(R_{-i})))\ R_i\ g(m_i, s_{-i}(d_{-i}(R_{-i})))$. Therefore, for each $R \in \mathcal{R}$, each $i \in N$, and each $m_i \in M_i$, $g(s_i(d_i(R_i)), s_{-i}(d_{-i}(R_{-i})))\ R_i\ g(m_i, s_{-i}(d_{-i}(R_{-i})))$. Thus, $s \circ d$ is an ex post equilibrium. Since Γ implements f in ex post equilibria, $g \circ (s \circ d) = f$. Since $f \circ d = g \circ s \circ d$ and $g \circ s \circ d = f$, we have $f \circ d = f$. Therefore, f is *weakly securely-strategy-proof.* □

It involves showing that any doubly implementable SCF in dominant strategy equilibria and ex post equilibria is also doubly implemented in dominant strategy equilibria and ex post equilibria by the direct mechanism associated with it. Hence, for double implementability in dominant strategy equilibria and ex post equilibria, it suffices to focus on direct mechanisms.

Corollary 2.1. *An SCF f is doubly implementable in dominant strategy equilibria and ex post equilibria if and only if it is doubly implemented in dominant strategy equilibria and ex post equilibria by the direct mechanism associated with f.*[19]

By Proposition 2.3 and Theorem 2.1, *secure strategy-proofness* is sufficient for double implementation.

[19] The proof of Theorem 2.1 also involves showing that any ex post implementable SCF is also implemented in ex post equilibria by the direct mechanism associated with it.

Corollary 2.2. *If an SCF is* securely strategy-proof, *then it is doubly implementable in dominant strategy equilibria and ex post equilibria.*

2.5 Discussion

In the provision of a public good under a restricted domain, the Groves-Clarke rules are securely implementable (Saijo, Sjöström, and Yamato, 2003, 2007).[20] Also, in direct mechanisms, whether the Groves-Clarke rules work well in laboratory experiments has been investigated and one of the rules worked better than an SCF that is dominant strategy implementable, but not securely implementable (Cason, Saijo, Sjöström, and Yamato, 2006). By Corollary 2.2, the SCFs are also doubly implementable in dominant strategy equilibria and ex post equilibria.

For secure implementability, negative results have been established for a number of interesting SCFs (e.g., Fujinaka and Wakayama, 2011). Even if an SCF is not securely implementable, it may be doubly implementable in dominant strategy equilibria and ex post equilibria (Corollary 2.2). Are there such interesting SCFs? We provide one negative answer and one positive answer.

We consider the school choice problem (Abdulkadiroğlu and Sönmez, 2003) under incomplete information. Let N be a set of students, X be a set of schools, and ϕ means that for each student, he does not have any school and for each school, it gets an empty seat. Let $\mathcal{R}_i$ be the set of strict preferences over $X \cup \{\phi\}$ and $\mathcal{R} \equiv \times_{i \in N} \mathcal{R}_i$. Let $c \equiv (c_x)_{x \in X}$ be a capacity profile such that for each $x \in X$, $c_x \in \mathbb{N}$, where $\mathbb{N}$ is the set of positive integers.[21] A capacity for a school is the maximum number of students whom the school can accept. Let $\succsim \equiv (\succsim_x)_{x \in X}$ be a priority profile such that for each $x \in X$, $\succsim_x$ is a strict ordering over $N \cup \{\phi\}$. Let $(N, X, \mathcal{R}, c, \succsim)$ be a school choice problem under incomplete information.

Let $(a_1, \cdots, a_n) \in A \equiv (X \cup \{\phi\})^N$ be an outcome such that for each $x \in X$,

[20] For the definition of the Groves-Clarke rules, see, e.g., Saijo, Sjöström, and Yamato (2003).
[21] A capacity for a school is also called its "quota."

$|\{i \in N : a_i = x\}| \leq c_x$. Note that for each $i \in N$, each $R_i \in \mathcal{R}_i$, and each pair $a, b \in A$ such that $a = (a_1, \cdots, a_n)$ and $b = (b_1, \cdots, b_n)$, (1) $a_i \; P_i \; b_i$ if and only if $a \; P_i \; b$ and (2) $a_i \; I_i \; b_i$ if and only if $a \; I_i \; b$. Then, each agent's preferences over $X \cup \{\phi\}$ are extended to over A.

The following example is one negative result on double implementability.

Example 2.4. The tentative acceptance rule is not doubly implementable in dominant strategy equilibria and ex post equilibria.[22]

Let $(N, X, \mathcal{R}, c, \succsim)$ be such that $N = \{1, 2\}$, $X = \{a, b\}$, for each $i \in N$, $\mathcal{R}_i = \{R_i, R'_i\}$, and $\mathcal{R} = \times_{i \in N} \mathcal{R}_i$. Preferences and $(c, \succsim)$ are defined as follows: for each $i \in N$,

R_i	R'_i
a	b
b	a
ϕ	ϕ

$c_a = 1$	$c_b = 1$
$\succsim_a$	$\succsim_b$
2	1
1	2
ϕ	ϕ

Let (u_1, u_2) be a pair of utility representations for each preference profile such that for each agent, the utility of the most preferred school is 2, the utility of the second preferred school is 1, and the utility of the third preferred school is 0.

By computing the tentative acceptance rule, TA, for each preference profile, the outcome is chosen as follows:

TA	R_2	R'_2
R_1	(b, a)	(a, b)
R'_1	(b, a)	(b, a)

[22]For the definition of the tentative acceptance rule, see, e.g., Abdulkadiroğlu and Sönmez (2003).

The game induced by Γ^{TA} and $\mathcal{R}$ has the following utilities:

	true preference	R_2		R'_2	
true preference	message	R_2	R'_2	R_2	R'_2
R_1	R_1	$\mathbf{\underline{1}, \underline{2}}$	$\underline{2}, 1$	$\underline{1}, 1$	$\mathbf{\underline{2}, \underline{2}}$
	R'_1	$\mathbf{\underline{1}, \underline{2}}$	$1, \underline{2}$	$\mathbf{\underline{1}, \underline{1}}$	$1, \underline{1}$
R'_1	R_1	$\mathbf{\underline{2}, \underline{2}}$	$1, 1$	$\underline{2}, 1$	$1, \underline{2}$
	R'_1	$\mathbf{\underline{2}, \underline{2}}$	$\mathbf{\underline{2}, \underline{2}}$	$\mathbf{\underline{2}, \underline{1}}$	$\mathbf{\underline{2}, \underline{1}}$

Let $(d_1, d_2) \equiv ((R_1, R'_1), (R_2, R'_2))$. Then, $DS(\Gamma^{TA}, \mathcal{R}) = \{(d_1, d_2)\}$, and $EP(\Gamma^{TA}, \mathcal{R}) = \{(d_1, d_2), ((R'_1, R'_1), (R_2, R_2))\}$. Hence, for the preference profile $(R_1, R'_2) \in \mathcal{R}$, the ex post equilibrium $((R'_1, R'_1), (R_2, R_2))$ does not induce the outcome chosen by TA for (R_1, R'_2), although for each preference profile, the outcome at the dominant strategy equilibrium (s_1, s_2) is equal to the outcome chosen by TA for the preference profile. Therefore, the tentative acceptance rule does not satisfy (2) of *weak secure-strategy-proofness*, so that the rule cannot be doubly implemented in dominant strategy equilibria and ex post equilibria by the direct mechanism associated with TA. By Corollary 2.1, the tentative acceptance rule is not doubly implementable in dominant strategy equilibria and ex post equilibria.[23] ■

For the other models with restricted domains of preference profiles, some interesting rules are not doubly implementable in dominant strategy equilibria and ex post equilibria: (1) For the allocation problems of an indivisible good with money under quasi-linear preferences, the second-price auction is not doubly implementable in dominant strategy equilibria and ex post equilibria.[24] (2) In the location problem with single-peaked preferences, the median rule is not doubly implementable in dominant strategy equilibria and ex post equilibria.[25] (3) In the house reallocation problem, the top-trading-cycle rule is

[23] In the same example as Example 2.4, the top-trading-cycle rule is not doubly implementable in dominant strategy equilibria and ex post equilibria. For the definition of the top-trading-cycle rule, see, e.g., Abdulkadiroğlu and Sönmez (2003).

[24] For the definition of the second-price auction, see Vickrey (1961).

[25] For the definition of the median rule, see Moulin (1980).

not doubly implementable in dominant strategy equilibria and ex post equilibria.[26]

In contrast to Example 2.4, if the set of preferences for agent 1 includes a preference at which the ordering of school a is first, the ordering of ϕ is second, and the ordering of school b is third, then the tentative acceptance rule is doubly implementable in dominant strategy equilibria and ex post equilibria. Therefore, if the set of preference profiles is "large", then the rule may be doubly implemented in dominant strategy equilibria and ex post equilibria. Identifying general conditions on the set of preference profiles for double implementability of the tentative acceptance rule in dominant strategy equilibria and ex post equilibria is an open question.[27]

Example 2.5. The tentative acceptance rule is doubly implementable in dominant strategy equilibria and ex post equilibria under some condition on the set of preference profiles.

Let $(N, X, \mathcal{R}, c, \succsim)$ be the same setting as in Example 2.4 except for that $\mathcal{R}_1 = \{R_1, R_1', R_1''\}$. Preferences for agent 1 are defined as follows:

R_1	R_1'	R_1''
a	b	a
b	a	ϕ
ϕ	ϕ	b

Let (u_1, u_2) be a pair of utility representations for each preference profile such that for each agent, the utility of the most preferred school is 3, the utility of the second preferred school is 2, and the utility of the third preferred school is 1.

[26] For the definition of the top-trading-cycle rule for house reallocation problems, see Shapley and Scarf (1974).

[27] For laboratory experiments concerning the tentative acceptance rule, see, e.g., Chen and Sönmez (2006) and Chen, Liang, and Sönmez (2016). Although Chen, Liang, and Sönmez (2016) consider the complete information setting, Chen and Sönmez (2006) study the incomplete information setting. The two papers use relatively large sessions: there are 36 students and 36 school slots across seven schools. Then, in order to find whether in the two laboratory experiments, the tentative acceptance rule is doubly implementable in dominant strategy equilibria and ex post equilibria or not, we should investigate general conditions on the set of preference profiles for double implementability in dominant strategy equilibria and ex post equilibria of the rule.

By computing TA, for each preference profile, the outcome is chosen as follows:

TA	R_2	R'_2
R_1	(b, a)	(a, b)
R'_1	(b, a)	(b, a)
R''_1	(ϕ, a)	(a, b)

The game induced by Γ^{TA} and $\mathcal{R}$ has the following utilities:

	true preference	R_2		R'_2	
true preference	message	R_2	R'_2	R_2	R'_2
R_1	R_1	$\underline{\mathbf{2}}\mathbf{,}\ \underline{\mathbf{3}}$	$\underline{3}, 2$	$\underline{2}, 2$	$\underline{\mathbf{3}}\mathbf{,}\ \underline{\mathbf{3}}$
	R'_1	$\underline{\mathbf{2}}\mathbf{,}\ \underline{\mathbf{3}}$	$2, \underline{2}$	$\underline{\mathbf{2}}\mathbf{,}\ \underline{\mathbf{2}}$	$2, 2$
	R''_1	$1, \underline{3}$	$\underline{3}, 2$	$1, 2$	$\underline{\mathbf{3}}\mathbf{,}\ \underline{\mathbf{3}}$
R'_1	R_1	$\underline{\mathbf{3}}\mathbf{,}\ \underline{\mathbf{3}}$	$2, 2$	$\underline{3}, 2$	$2, \underline{3}$
	R'_1	$\underline{\mathbf{3}}\mathbf{,}\ \underline{\mathbf{3}}$	$\underline{\mathbf{3}}\mathbf{,}\ \underline{\mathbf{2}}$	$\underline{\mathbf{3}}\mathbf{,}\ \underline{\mathbf{2}}$	$\underline{\mathbf{3}}\mathbf{,}\ \underline{\mathbf{2}}$
	R''_1	$1, \underline{3}$	$2, 2$	$1, 2$	$2, \underline{3}$
R''_1	R_1	$1, \underline{3}$	$\underline{3}, 2$	$1, 2$	$\underline{\mathbf{3}}\mathbf{,}\ \underline{\mathbf{3}}$
	R'_1	$1, \underline{3}$	$1, \underline{2}$	$1, \underline{2}$	$1, \underline{2}$
	R''_1	$\underline{\mathbf{2}}\mathbf{,}\ \underline{\mathbf{3}}$	$\underline{3}, 2$	$\underline{2}, 2$	$\underline{\mathbf{3}}\mathbf{,}\ \underline{\mathbf{3}}$

Let $(d_1, d_2) \equiv ((R_1, R'_1, R''_1), (R_2, R'_2))$. Then, $DS(\Gamma^{TA}, \mathcal{R}) = EP(\Gamma^{TA}, \mathcal{R}) = \{(d_1, d_2)\}$. Hence, for each preference profile, the outcome at both the dominant strategy equilibrium and the ex post equilibrium is equal to the outcome chosen by the rule for the preference profile. Therefore, the rule is doubly implemented in dominant strategy equilibria and ex post equilibria by the direct mechanism associated with TA.■

Chapter 3

A Simple and Procedurally Fair Mechanism for Nash Implementation of the No-Envy Correspondence

3.1 Introduction

We consider the allocation problem of infinitely divisible resources with at least three agents. The objective of a social planner is embodied by a "social choice correspondence." Mathematically, a social choice correspondence is a set-valued mapping which, for each possible preference profile, specifies a non-empty set of outcomes. We assume that each agent knows the other agents' preferences, but the planner does not. Then, the planner specifies a message space for each agent and a mapping which, for each possible message profile, chooses an outcome. The pair consisting of the list of agents' message spaces and a mapping is a "mechanism."

In the allocation problem of infinitely divisible resources, the planner selects an allocation in which the summation of assignments for all agents is equal to a social endowment, i.e., the balance for the social endowment is satisfied, and she assigns a bundle of the allocation to each agent. We want this allocation to be "envy-free:" no agent prefers the bundle of a different agent over his own bundle (Foley, 1967). The "no-envy" correspondence selects the set of *envy-free* allocations for each preference profile.[1]

[1]One may ask the question that why does the planner want to implement the no-envy correspondence only. A social planner may want any *envy-free* allocation to be *Pareto-efficient*. Saijo, Tatamitani, and Yamato (1996) and Thomson (2005) suggest that for Nash implementability of the Pareto-efficiency solu-

We construct a mechanism for Nash implementation of the no-envy correspondence.[2] We call it "Choose-Two-Bundles-and-Transpose." In this mechanism, each agent announces two bundles each of which is a possible consumption bundle for the social endowment as well as the names of two agents. The two bundles are interpreted as the first bundle is for his own assignment and the second bundle is for his neighbor's assignment.[3] The outcome mapping is as follows: If the second bundle reported by each agent is the same as the first bundle reported by his neighbor and the list of bundles based on the announcements of agents is balanced for the social endowment, then each agent gets one bundle of the transposed allocation. If there is only one agent that reports a different bundle from the message reported by his neighbor and the list of bundles based on announcements of the other agents is balanced for the social endowment, then each agent gets one bundle of the transposed allocation. Therefore, a message regarding bundles reported by the odd-man-out will be ignored. Otherwise, each agent gets one bundle of the equal-division allocation. Note that this mechanism depends on the existence of the equal-division allocation.[4] By contrast, for the mechanisms of Thomson (2005) and Galbiati (2008), it is important that there exists a least preferred bundle by the assumption of *strict monotonicity* on preferences.

We show that in the allocation problem of infinitely divisible resources, if there are

tion, each of at least two agents reports a price vector, because the planner wants to obtain the common marginal rate of substitution at a *Pareto-efficient* allocation. Reporting a price vector may not be easy for the agents, so that this mechanism is not simple. We have already known that the no-envy solution is Nash implementable in the same model as ours (Doğan, 2016). However, for Nash implementability of the no-envy solution, no simple and procedurally fair mechanism has been constructed. Therefore, it is interesting to propose the possibility of Nash implementation of the no-envy solution by means of a simple and procedurally fair mechanism.

[2]We investigate "full implementation" of the no-envy correspondence, not partial implementation. Reasons for studying it are discussed in Thomson (1996).

[3]This interpretation is related to Saijo (1988) and Saijo, Tatamitani, and Yamato (1996). In the mechanism of Saijo (1988), each agent reports his own preference and his neighbor's. The idea of ordering the agents in a circular fashion and letting each of them report a message for the next agent in the circle is the same as in our mechanism. However, the message spaces to which this idea is applied in the mechanism of Saijo Saijo (1988) are different from those in our mechanism, and our mechanism is simpler than that of Saijo (1988). In the mechanism of Saijo, Tatamitani, and Yamato (1996), each agent reports only two bundles each of which is a possible consumption bundle for the social endowment. Although they apply the above same idea as in Saijo (1988) and this study, our mechanism is simpler than that of Saijo, Tatamitani, and Yamato (1996).

[4]Since Choose-Two-Bundles-and-Transpose depends on the existence of the equal-division allocation, this mechanism is not applicable to a model in which there is an indivisible good.

at least three agents, Choose-Two-Bundles-and-Transpose implements the no-envy correspondence in Nash equilibria (Theorem 3.1). Our result is applicable in, e.g., the cake division problem (Thomson, 2007) and the allocation problem of infinitely divisible resources with single-peaked preferences (Adachi, 2010; Morimoto, Serizawa, and Ching, 2013).[5]

This chapter is organized as follows. Section 3.2 provides the model for allocation of a social endowment of infinitely divisible resources and a mechanism for Nash implementation of the no-envy correspondence in the problem as well as our main result. Section 3.3 reports related literature. Section 3.4 proposes concluding remarks.

3.2 Allocation Problems of Infinitely Divisible Resources

Let $N = \{1, ..., n\}$ be a set of agents among whom a **social endowment** $\Omega \in \mathbb{R}^{\ell}_{++}$ of ℓ infinitely divisible resources has to be allocated. We assume that the resources cannot be disposed of. An **allocation for** $\Omega \in \mathbb{R}^{\ell}_{++}$ is a list $a = (a_1, \cdots, a_n) \in \mathbb{R}^{\ell n}_{+}$ such that $\Sigma_{i \in N} a_i = \Omega$. Let $A^{\Omega} = \{a \in \mathbb{R}^{\ell n}_{+} : \Sigma_{i \in N} a_i = \Omega\}$ be the **set of allocations for** $\Omega \in \mathbb{R}^{\ell}_{++}$. Let $X^{\Omega} = \{x \in \mathbb{R}^{\ell}_{+} : x \leq \Omega\}$ be the **set of possible consumption bundles for** $\Omega \in \mathbb{R}^{\ell}_{++}$.[6] Let $\tilde{a} \equiv \frac{\Omega}{|N|} \in A^{\Omega}$ be the **equal-division allocation for** $\Omega \in \mathbb{R}^{\ell}_{++}$. Let $R_i \in \mathcal{R}_i$ be a preference for agent $i \in N$ over X^{Ω}, where $\mathcal{R}_i$ is the set of preferences admissible for agent i. Let $R = (R_1, ..., R_n) \in \mathcal{R}$ be a preference profile, where $\mathcal{R} = \times_{i \in N} \mathcal{R}_i$. Note that for each $i \in N$, each $R_i \in \mathcal{R}_i$, and each pair $a, b \in A^{\Omega}$ such that $a = (a_1, \cdots, a_n)$ and $b = (b_1, \cdots, b_n)$, (1) $a_i \; P_i \; b_i$ if and only if $a \; P_i \; b$ and (2) $a_i \; I_i \; b_i$ if and only if $a \; I_i \; b$. Then, each agent's preferences over X^{Ω} are extended to over A^{Ω}. Note that in this chapter as well as in Doğan (2016), there is no monotonicity

[5] In the allocation problem of an infinitely divisible resource with single-peaked preferences (Sprumont, 1991), Thomson (2010) addresses Nash implementability of several social choice correspondences which do not satisfy "no-veto-power" in this model, in particular the no-envy correspondence. Since *no-veto-power* is one of sufficient conditions for Nash implementability of social choice correspondences (Saijo, 1988), the mechanism of Saijo (1988) is not applicable in this model. Then, Thomson (2010) shows that the no-envy correspondence is Nash implementable by the result of Yamato (1992). However, he does not propose any simple mechanism.

[6] Given $x, y \in \mathbb{R}^{\ell}_{+}$, $x \leq y$ means that for each $j \in \{1, \cdots, \ell\}$, $x_j \leq y_j$.

assumption on preferences, although Thomson (2005) and Galbiati (2008) impose the assumption of "strict monotonicity" on preferences.[7] Let (N, Ω, R) be the **allocation problem of infinitely divisible resources**. We fix N and Ω, so that a problem is represented by R.

An allocation $a \in A^{\Omega}$ is **envy-free for** $R \in \mathcal{R}$ if for each pair $i, j \in N$, $a_i \ R_i \ a_j$. The **no-envy correspondence** $F : \mathcal{R} \twoheadrightarrow A^{\Omega}$ is a set-valued mapping which, for each preference profile $R \in \mathcal{R}$, $F(R)$ is the set of *envy-free* allocations for R.

In order to design our mechanism to implement the no-envy correspondence in Nash equilibria, let us introduce a definition. Give a pair $i, j \in N$, let $T_j^i : A^{\Omega} \to A^{\Omega}$ be a **transposition mapping** which, for each allocation $a \in A^{\Omega}$, selects the allocation by transposing the bundles of agent i and agent j in a.

"Choose-Two-Bundles-and-Transpose" is a mechanism constructed for Nash implementation of the no-envy correspondence. In Choose-Two-Bundles-and-Transpose, each agent, $i \in N$, announces two bundles, $x^i, y^i \in X^{\Omega}$, and the names of two agents, $k^i, t^i \in N$. The outcome mapping is as follows: If the second bundle y^i reported by an agent, $i \in N$, is the same as the first bundle x^{i+1} reported by his neighbor, $i+1$, i.e., $y^i = x^{i+1} \equiv a_{i+1}$, and the list of bundles based on announcements of agents is balanced for the social endowment, i.e., $(a_1, \cdots, a_n) \in A^{\Omega}$, then each agent gets one bundle of the transposed allocation, i.e., $T_{t^n}^{k^n} \circ \cdots \circ T_{t^1}^{k^1}(a_1, \cdots, a_n)$. If there is only one agent, $j \in N$, reports a different bundle from the message reported by his neighbor, $j-1$ or $j+1$, and the list of bundles based on announcements of the other agents is balanced for the social endowment, i.e., $(a_1, \cdots, y^{j-1}, x^{j+1}, \cdots, a_n) \in A^{\Omega}$, then each agent gets one bundle of the transposed allocation, i.e., $T_{t^n}^{k^n} \circ \cdots \circ T_{t^1}^{k^1}(a_1, \cdots, y^{j-1}, x^{j+1}, \cdots, a_n)$. Therefore, a message regarding bundles reported by the odd-man-out will be ignored. Otherwise, each agent gets one bundle of the equal-division allocation.

Choose-Two-Bundles-and-Transpose, $\Gamma^{C2T} = (M, g)$: For each $i \in N$, $M_i = X^{\Omega} \times$

[7] In the same model as Galbiati (2008), Saijo, Tatamitani, and Yamato (1996) also construct mechanisms to implement the no-envy correspondence in Nash equilibria. However, the mechanism of Galbiati (2008) seems simpler than the mechanisms of Saijo, Tatamitani, and Yamato (1996).

$X^{\Omega} \times N \times N$. Given $m = (x^i, y^i, k^i, t^i)_{i \in N} \in \times_{i \in N} M_i \equiv M$, the outcome mapping $g : M \to A^{\Omega}$ is as follows:[8]

$$g(m) = \begin{cases} \text{Rule 1: } T_{t^n}^{k^n} \circ \cdots \circ T_{t^1}^{k^1}(a_1, \cdots, a_n) & \text{if } \begin{cases} \text{for each } i \in N, y^i = x^{i+1} \equiv a_{i+1}, \text{ and} \\ (a_1, \cdots, a_n) \in A^{\Omega} \end{cases} \\ \text{Rule 2: } T_{t^n}^{k^n} \circ \cdots \circ T_{t^1}^{k^1}(a_1, \cdots, y^{j-1}, x^{j+1}, \cdots, a_n) & \text{if } \begin{cases} \text{there is } j \in N \text{ such that for each } i \neq j, \\ y^i = x^{i+1} \equiv a_{i+1} \text{ and } [y^{j-1} \neq x^j \\ \text{ or } y^j \neq x^{j+1}], \text{ as well as} \\ (a_1, \cdots, y^{j-1}, x^{j+1}, \cdots, a_n) \in A^{\Omega} \end{cases} \\ \text{Rule 3: } \tilde{a} & \text{otherwise} \end{cases}$$

Given $R \in \mathcal{R}$, let (Γ^{C2T}, R) be the **game** induced by Γ^{C2T} and R. A message profile $m \in M$ is a **Nash equilibrium of** (Γ^{C2T}, R) if for each $i \in N$ and each $m'_i \in M_i$, $g(m_i, m_{-i})\ R_i\ g(m'_i, m_{-i})$. Let $NE(\Gamma^{C2T}, R)$ be the **set of Nash equilibria of** (Γ^{C2T}, R).

The mechanism Γ^{C2T} **implements the no-envy correspondence** F **in Nash equilibria** if for each $R \in \mathcal{R}$, $F(R) = g(NE(\Gamma^{C2T}, R))$.

The following is our main result.

Theorem 3.1. *Let* $n \geq 3$. *Choose-Two-Bundles-and-Transpose implements the no-envy correspondence in Nash equilibria.*

Proof. Let $R \in \mathcal{R}$. We prove it by two steps.

Step 1. $F(R) \subseteq g(NE(\Gamma^{C2T}, R))$.

Let $a = (a_1, \cdots, a_n) \in F(R)$ and $m = (a_i, a_{i+1}, i, i)_{i \in N}$. By Rule 1, $g(m) = a$. For each $i \in N$, let $m'_i \neq m_i$. By Rule 1 or 2, $g_i(m'_i, m_{-i}) \in \{a_1, \cdots, a_n\}$. Since $a \in F(R)$, $g_i(m_i, m_{-i}) = a_i\ R_i\ g_i(m'_i, m_{-i})$. Therefore, for each $i \in N$ and each $m'_i \in M_i$, $g_i(m_i, m_{-i})\ R_i\ g_i(m'_i, m_{-i})$. Hence, $m \in NE(\Gamma^{C2T}, R)$.

[8]Suppose that $a_{1-1} = a_n$ and $a_{n+1} = a_1$.

Step 2. $g(NE(\Gamma^{C2T}, R)) \subseteq F(R)$.

We show that if $g(m) \notin F(R)$, then $m \notin NE(\Gamma^{C2T}, R)$. Let $g(m) = a$ and $m = (x^i, y^i, k^i, t^i)_{i \in N}$. Since $a \notin F(R)$ and $\tilde{a} \in F(R)$, Rule 1 or 2 applies. Since $a \notin F(R)$, there is a pair $i, j \in N$ such that $a_j \ P_i \ a_i$. By selecting $m'_i = (x^i, y^i, k'^i, t'^i)$ appropriately, $g_i(m'_i, m_{-i}) = a_j$. Hence, $g_i(m'_i, m_{-i}) = a_j \ P_i \ a_i = g_i(m_i, m_{-i})$. Therefore, $m \notin NE(\Gamma^{C2T}, R)$. □

3.3 Related Literature

We first consider the allocation problem of infinitely divisible resources with "strictly monotonic" preferences. For each $i \in N$, a preference R_i is **strictly monotonic** if for each pair $x, y \in X^{\Omega}$, $x \geq y$ and $x \neq y$ imply $x \ P_i \ y$. For the problem with only two agents and a resource, a well-known mechanism for Nash implementation of the no-envy correspondence is "Divide-and-Choose." One agent divides the resource into two parts, and the other agent chooses one of them. Although Divide-and-Choose is simple, this mechanism works well only in the case of two agents.

For the allocation problem with strictly monotonic preferences and at least two agents, "Divide-and-Permute" implements the no-envy correspondence in Nash equilibria (Thomson, 2005). This mechanism resembles Divide-and-Choose. Although Divide-and-Permute is simple and works well with at least two agents, this mechanism is only applicable to models where the first and second agents always prefer any bundle to the bundle receiving nothing.

For the allocation problem in which such least-preferred bundles do not necessary exist, if there are at least three agents, "Divide-and-Transpose" implements the no-envy correspondence in Nash equilibria (Doğan, 2016).[9] This mechanism is a modification of Divide-and-Permute. Although Divide-and-Transpose is simple and applicable to models without a monotonic condition on preferences, this mechanism does not treat all agents

[9]Even if there is an indivisible good, Divide-and-Transpose works well. For example, the result regarding Nash implementation by means of this mechanism is applicable in the allocation problem of indivisible objects with monetary transfers (e.g., Svensson, 1983).

equally. Formally, this mechanism is not "ex ante fair" (Korpela, 2018)[10]: a mechanism $\Gamma = (S, h)$, where $h : S \to A^{\Omega}$, is *ex ante fair* if for each message profile $s \in S$ and each one-to-one function $\pi : N \to N$, there is another message profile $s' \in S$ such that $h(s') = \pi(h(s))$ and for each $i \in N$, $h(S_i, s'_{-i}) = \pi(h(S_{\pi(i)}, s_{-\pi(i)}))$.[11]

For the allocation problems with strictly monotonic preferences and at least two agents, "Galbiati's mechanism" implements the no-envy correspondence in Nash equilibria (Galbiati, 2008). This mechanism is another modification of Divide-and-Permute. In Galbiati's mechanism, each agent proposes an allocation, a one-to-one function from N to N, and the names of two agents. Although Galbiati's mechanism treats all agents equally so that it is *ex ante fair*, the message space for each agent is large. For example, suppose that there are ten agents and three types of resources. Each agent reports at least twenty-seven real-numbers for the other agents' assignments in addition to three real-numbers for his own assignments.

We designed a both simple and *ex ante fair* mechanism, Choose-Two-Bundles-and-Transpose, to implement the no-envy correspondence in Nash equilibria.

3.4 Concluding Remarks

For implementation theory, simple mechanisms are important.[12] If a mechanism is complicated, and an agent does not understand how to select outcomes, then even if he wants to achieve the best outcome, he may not choose a message that induces the best outcome for his preference over the set of attainable.

"Strategy-proofness" requires that in the direct mechanism associated with the single-

[10] As Korpela (2018) states that Divide-and-Permute is not *ex ante fair*, we also easily check for Divide-and-Transpose not being *ex ante fair*. In addition, the mechanisms of Saijo (1988), Yamato (1992), and Saijo, Tatamitani, and Yamato (1996) are not always *ex ante fair*.

[11] For each $a \in A^{\Omega}$, let $\pi(a) = (a_{\pi(1)}, \cdots, a_{\pi(n)})$. Given $s'_{-i} \in S_{-i}$, let $h(S_i, s'_{-i}) = \{h(s_i, s'_{-i}) : s_i \in S_i\}$. For each $A' \subseteq A^{\Omega}$, let $\pi(A') = \{\pi(a) : a \in A'\}$.

[12] In Thomson (2005), Doğan (2016), and this chapter, the precise definition of simplicity is not provided. When we say that a mechanism is simple, the message space for each agent is small and how to select an outcome for each message profile by the outcome mapping is natural as in Divide-and-Choose i.e., after dividing the social endowment as an allocation, agents exchange the components of this allocation.

valued correspondence, truth-telling should be a dominant strategy for each agent. Since the objective of the planner is achieved at a dominant strategy equilibrium, *strategy-proofness* is desirable. However, laboratory experiments regarding *strategy-proof* social choice functions reported that in several games, some subjects did not select dominant strategies.[13] For example, in several second-price-auction experiments, most bidders did not reveal true values (Kagel, Harstad, and Levin, 1987; Kagel and Levin, 1993; and Harstad, 2000). In an ascending auction and a second-price auction, subjects were substantially more likely to play truth-telling under the former than under the latter (Kagel, Harstad, and Levin, 1987). Inspired from these observations, "obvious" *strategy-proofness* is defined and characterized as a cognitively limited agent can recognize that truth-telling is a dominant strategy (Li, 2017). While second-price auctions are not *obviously strategy-proof*, ascending auctions are *obviously strategy-proof*. Therefore, even if a social choice function is *strategy-proof*, simpler mechanisms associated with the function work better.

Ex ante fairness should be also considered. A layman would say that he must have the same opportunities in the mechanism as others do. This suggests that procedural fairness can sometimes play out before the mechanism is actually executed as a participation constraint. *Ex ante fairness* guarantees that this cannot happen.[14]

[13]For a summary of laboratory experiments regarding *strategy-proof* social choice functions, see Cason, Saijo, Sjöström, and Yamato (2006).

[14]For other concepts of procedural fairness, see Gaspart (2003), Deb and Pai (2017), Azrieli and Jain (2018), and Korpela (2018). For the discussion regarding their concepts of procedural fairness, see Korpela (2018).

Chapter 4

Double Implementation in Nash Equilibria and Undominated Nash Equilibria without No-Veto-Power

4.1 Introduction

We consider the implementation problem with at least three agents. We study double implementability of social choice correspondences (hereafter, SCC) in Nash equilibria and undominated Nash equilibria. There are two reasons to investigate double implementability in Nash equilibria and undominated Nash equilibria. First, in laboratory experiments, subjects do not always adopt undominated strategies (see, e.g., Katok, Sefton, and Yavas, 2002; Cason, Saijo, Sjöström, and Yamato, 2006). Second, in several pivotal-mechanism experiments in which truth-telling is a dominant strategy for each agent, Nash equilibria have been frequently observed (Cason, Saijo, Sjöström, and Yamato, 2006). A possible explanation is that, even though some subjects could not identify undominated strategies, they were able to determine how to improve upon a strategy.

For Nash implementability, "Maskin-invariance" and "no-veto-power" together are sufficient (Maskin, 1999).[1] There are several SCCs that satisfy *Maskin-invariance* but violate *no-veto-power*.[2] In this case, to examine Nash implementability, we verify whether an SCC satisfies a necessary and sufficient condition, such as the one proposed by Sjöström

[1] *Maskin-invariance* is also called "Maskin-monotonicity."

[2] For these SCCs, see Section 4.5.

(1991). However, it is not easy to check the condition. Subsequently, to verify Nash implementability more easily, other sufficient conditions are proposed (Doghmi and Ziad, 2015): "DZ-invariance," "weak no-veto-power," and "unanimity."[3]

When there are at least three agents, if an SCC is Nash implementable, then it is doubly implementable in Nash equilibria and undominated Nash equilibria (Yamato, 1999). Then, *DZ-invariance*, *weak no-veto-power*, and *unanimity* together are sufficient for double implementability in Nash equilibria and undominated Nash equilibria (Yamato, 1999; Doghmi and Ziad, 2015). This result is provided indirectly by means of two mechanisms. We prove it directly by constructing another mechanism (Proposition 4.4). This mechanism is also applied in the proof of our first theorem.

We consider "partially honest" agents as defined by Dutta and Sen (2012).[4] A partially honest agent prefers reporting the true preference profile whenever a lie does not allow him to obtain an outcome that he prefers; otherwise, he prefers announcing a message inducing an outcome that he prefers.

For Nash implementability, if there are at least three agents out of which at least one agent is partially honest, then *no-veto-power* is sufficient (Dutta and Sen, 2012). We show that if there are at least three agents out of which at least one agent is partially honest, then *weak no-veto-power* and *unanimity* together are sufficient for double implementability in Nash equilibria and undominated Nash equilibria (Theorem 4.1). Each of *weak no-veto-power* and *unanimity* is weaker than *no-veto-power* (Remark 4.1).

For Nash implementability, if there are at least three agents out of which at least two agents are partially honest, then *unanimity* is sufficient (Kimya, 2015). We show that if there are at least three agents out of which at least two agents are partially honest, then

[3]The simplicity of the three conditions appears in several interesting applications, e.g., in allocation problems of an infinitely divisible resource with single-peaked preferences. See Doghmi and Ziad (2015) and Doghmi (2016).

[4]There are several papers on behavioral implementation theory. For implementation problems with "decent" agents, see Corchón and Herrero (2004). For implementation problems with "evidences," see Kartik and Tercieux (2012). For implementation problems with preferences for honesty, see Kartik, Tercieux, and Holden (2014), Lombardi and Yoshihara (2018, 2019), Mukherjee, Muto, and Ramaekers (2017), and Savva (2018). For implementation problems in exchange economies with "semi-responsible" agents, see Lombardi and Yoshihara (2017). For implementation problems with "semi-socially-responsible" agents, see Hagiwara, Yamamura, and Yamato (2018) and Hagiwara (2018).

Table 4.1: Previous results.

	Non-existence	Existence ≥1	Existence ≥2
Nash implementation	*Maskin-invariance* *No-veto-power* (Maskin, 1999) *DZ-invariance* *Weak no-veto-power* *Unanimity* (Doghmi and Ziad, 2015)	*No-veto-power* (Dutta and Sen, 2012)	*Unanimity* (Kimya, 2015)
Double implementation	Nash implementability $\Updownarrow$ Double implementability (Yamato, 1999)		

Table 4.2: Our results regarding sufficient conditions.

	Non-existence	Existence ≥1	Existence ≥2
Double implementation	*DZ-invariance* *Weak no-veto-power* *Unanimity* (Proposition 4.4, new mechanism)	*Weak no-veto-power* *Unanimity* (Theorem 4.1)	*Unanimity* (Theorem 4.2)

unanimity is sufficient for double implementability in Nash equilibria and undominated Nash equilibria (Theorem 4.2). Most of SCCs satisfy *unanimity*, so that those are doubly implementable in Nash equilibria and undominated Nash equilibria.

Table 4.1 illustrates previous results, and Table 4.2 summarizes our results regarding sufficient conditions.[5,6]

When at least one agent is partially honest, if some SCC does not satisfy *weak no-veto-*

[5] Non-existence is the assumption that there is no partially honest agent. Existence ≥1 is the assumption that there is at least one partially honest agent. Existence ≥2 is the assumption that there are at least two partially honest agents. For these three assumptions, see Section 4.2.

[6] In Tables 4.1 and 4.2, we simply refer to double implementation in Nash equilibria and undominated Nash equilibria as double implementation.

power, we cannot verify using Theorem 4.1 whether the SCC is doubly implementable in Nash equilibria and undominated Nash equilibria or not.[7] Then, it is important to provide a necessary and sufficient condition for double implementability in Nash equilibria and undominated Nash equilibria when at least one agent is partially honest.

Since most of SCCs satisfy *unanimity*, we focus on the SCCs that satisfy this condition. For Nash implementability, if there are at least three agents out of which at least one agent is partially honest and *unanimity* is satisfied, then *LY-condition* is necessary and sufficient (Lombardi and Yoshihara, 2019). We show that if there are at least three agents out of which at least one agent is partially honest and *unanimity* is satisfied, then *LY-condition* is necessary and sufficient for double implementability in Nash equilibria and undominated Nash equilibria (Theorem 4.3). Based on the result of Lombardi and Yoshihara (2019) and Theorem 4.3, double implementability in Nash equilibria and undominated Nash equilibria is equivalent to Nash implementability under a minor qualification (Corollary 4.1).

From our results, we obtain several positive corollaries in the allocation problem of an infinitely divisible resource with single-peaked preferences, with single-plateaued preferences, and the many-to-one matching problem. See Section 4.5. For several positive corollaries in other applications such as a coalitional problem, see Hagiwara (2017) and Lombardi and Yoshihara (2019).

This chapter is organized as follows: Section 4.2 presents the model. Section 4.3 reports sufficient conditions for double implementability in Nash equilibria and undominated Nash equilibria under the assumptions concerning the existence of partially honest agents when there are at least three agents. Section 4.4 provides a characterization for double implementability in Nash equilibria and undominated Nash equilibria when there are at least three agents out of which at least one partially honest agent exists and *unanimity* is satisfied. Section 4.5 presents the concluding remarks.

[7]By Observation 2 of Doghmi and Ziad (2015), *unanimity* is independent of *weak no-veto-power*. Then, there exists an SCC that satisfies *unanimity* but violates *weak no-veto-power*.

4.2 The Model

Let $N = \{1, ..., n\}$ be the set of agents and A be the set of outcomes. For each $i \in N$, let R_i be a preference for agent i over A, whose asymmetric and symmetric components are defined as P_i and I_i, respectively. For each $i \in N$, let $\mathcal{R}_i$ be the set of preferences admissible for agent i. Let $R = (R_1, ..., R_n) \in \mathcal{R}$ be a preference profile, where $\mathcal{R} \equiv \times_{i \in N} \mathcal{R}_i$.

For each $i \in N$ and each $R_i \in \mathcal{R}_i$, let $SL(a, R_i) \equiv \{b \in A : a \; P_i \; b\}$ be the **strictly lower contour set at** (a, R_i), $I(a, R_i) \equiv \{b \in A : a \; I_i \; b\}$ be the **indifferent set at** (a, R_i), and $L(a, R_i) \equiv SL(a, R_i) \cup I(a, R_i)$ be the **lower contour set at** (a, R_i).

A **social choice correspondence** (**SCC**) $F : \mathcal{R} \twoheadrightarrow A$ is a set-valued mapping which, for each preference profile $R \in \mathcal{R}$, specifies a non-empty set $F(R) \subseteq A$.

We focus on mechanisms in which as part of each agent's strategy, he reports the information regarding a preference profile. For each $i \in N$, let M_i be the **message space for agent** i, where $M_i = \mathcal{R} \times \mathcal{S}$ or $M_i = (\mathcal{R} \cup \Omega) \times \mathcal{S}$ in which Ω and $\mathcal{S}$ are the sets of supplemental messages. Let $M = \times_{i \in N} M_i$. The **outcome mapping** $g : M \to A$ is a single-valued mapping which, for each message profile $m \in M$, specifies an outcome $g(m) \in A$. A **mechanism** Γ consists of a pair (M, g).

We consider "partially honest" agents as defined by Dutta and Sen (2012). A partially honest agent prefers reporting the true preference profile whenever a lie does not allow him to obtain an outcome that he prefers; otherwise, he prefers announcing a message inducing an outcome that he prefers. Formally, preferences for partial honesty are defined as follows.

We "extend" an agent's preference over A to that over M. Given $R \in \mathcal{R}$ and $i \in N$, let $\succsim_i^R$ be **a preference for agent** i **over** M **at** R, whose asymmetric and symmetric components are denoted by $\succ_i^R$ and $\sim_i^R$, respectively:

Partial-honesty: For each pair $m_i, m'_i \in M_i$ such that $m_i = (m_1^i, s^i)$ and $m'_i = (m_1'^i, s'^i)$ and each $m_{-i} \in M_{-i}$,

(1) If $m_1^i = R$, $m_1'^i \neq R$, and $g(m_i, m_{-i})\ I_i\ g(m_i', m_{-i})$, then $(m_i, m_{-i}) \succ_i^R (m_i', m_{-i})$.

(2) In all other cases, $g(m_i, m_{-i})\ R_i\ g(m_i', m_{-i})$ if and only if $(m_i, m_{-i}) \succsim_i^R (m_i', m_{-i})$.

Agent $i \in N$ is **not partially honest** if for each $R \in \mathcal{R}$, each pair $m_i, m_i' \in M_i$, and each $m_{-i} \in M_{-i}$, $g(m_i, m_{-i})\ R_i\ g(m_i', m_{-i})$ if and only if $(m_i, m_{-i}) \succsim_i^R (m_i', m_{-i})$.[8]

We consider the following three assumptions regarding the existence of partially honest agents and the class of conceivable sets of partially honest agents:

Non-existence: There is no partially honest agent in N. i.e., $\mathcal{H}^0 \equiv \{\emptyset\}$ is the class of conceivable sets of partially honest agents.

Existence ≥1: There is at least one partially honest agent in N, and $\mathcal{H}^1 \equiv \{H \subseteq N : |H| \geq 1\}$ is the class of conceivable sets of partially honest agents.

Existence ≥2: There are at least two partially honest agents in N, and $\mathcal{H}^2 \equiv \{H \subseteq N : |H| \geq 2\}$ is the class of conceivable sets of partially honest agents.

Let $k \in \{0, 1, 2\}$. For each $R \in \mathcal{R}$ and each $H \in \mathcal{H}^k$, let $\succsim^{R,H} \equiv (\succsim_1^{R,H}, \cdots, \succsim_n^{R,H})$ be the **preference profile over** M such that for each $i \in H$, $\succsim_i^{R,H}$ is defined by partial honesty and for each $i \in N \backslash H$, $\succsim_i^{R,H}$ is defined by not-partial-honesty. Let $(\Gamma, \succsim^{R,H})$ be the **game** induced by Γ and $\succsim^{R,H}$. Note that under Non-existence, $(\Gamma, \succsim^{R,\emptyset})$ is equivalent to the game (Γ, R) induced by Γ and R.

A message profile $m \in M$ is a **Nash equilibrium of** $(\Gamma, \succsim^{R,H})$ if for each $i \in N$ and each $m_i' \in M_i$, $(m_i, m_{-i}) \succsim_i^{R,H} (m_i', m_{-i})$. Let $NE(\Gamma, \succsim^{R,H})$ be the **set of Nash equilibria of** $(\Gamma, \succsim^{R,H})$.

For each $i \in N$, agent i's message $m_i \in M_i$ is **weakly dominated by** $\tilde{m}_i \in M_i$ **at** $\succsim_i^{R,H}$ if for each $m_{-i} \in M_{-i}$, $(\tilde{m}_i, m_{-i}) \succsim_i^{R,H} (m_i, m_{-i})$ and for some $m_{-i} \in M_{-i}$, $(\tilde{m}_i, m_{-i}) \succ_i^{R,H} (m_i, m_{-i})$. Agent i's message $m_i \in M_i$ is **undominated at** $\succsim_i^{R,H}$ if it is not weakly dominated by any message in M_i at $\succsim_i^{R,H}$. A message profile $m \in M$

[8] Note that there is a pair consisting of a preference profile and a mechanism such that an agent is partially honest and "not" partially honest. Then, in general, the term of "not-partial-honesty" may not be precise. However, in our mechanisms constructed in this chapter, the two definitions of partial-honesty and not-partial-honesty are distinguished.

is an **undominated Nash equilibrium of** $(\Gamma, \succsim^{R,H})$ if for each $i \in N$, $m_i \in M_i$ is undominated at $\succsim_i^{R,H}$ and $m \in M$ is a Nash equilibrium of $(\Gamma, \succsim^{R,H})$. Let $UNE(\Gamma, \succsim^{R,H})$ be the **set of undominated Nash equilibria of** $(\Gamma, \succsim^{R,H})$. Note that $UNE(\Gamma, \succsim^{R,H}) \subseteq NE(\Gamma, \succsim^{R,H})$.

Let $k \in \{0, 1, 2\}$. An SCC F is **Nash implementable** if there is $\Gamma = (M, g)$ such that for each $R \in \mathcal{R}$ and each $H \in \mathcal{H}^k$, $F(R) = g(NE(\Gamma, \succsim^{R,H}))$. An SCC F is **doubly implementable in Nash equilibria and undominated Nash equilibria** if there is $\Gamma = (M, g)$ such that for each $R \in \mathcal{R}$ and each $H \in \mathcal{H}^k$, $F(R) = g(NE(\Gamma, \succsim^{R,H})) = g(UNE(\Gamma, \succsim^{R,H}))$.

4.3 Sufficient Conditions for Double Implementability in Nash Equilibria and Undominated Nash Equilibria

4.3.1 Under Non-Existence

First, we study double implementability of SCCs in Nash equilibria and undominated Nash equilibria under Non-existence with at least three agents.

The following are the two conditions which are studied by Maskin (1999):

Definition 4.1. An SCC F satisfies **Maskin-invariance** if for each pair $R, R' \in \mathcal{R}$ and each $a \in F(R')$, if for each $i \in N$, $L(a, R'_i) \subseteq L(a, R_i)$, then $a \in F(R)$.

Definition 4.2. An SCC F satisfies **no-veto-power** if for each $R \in \mathcal{R}$, each $i \in N$, and each $a \in A$, if for each $j \in N \backslash \{i\}$, $L(a, R_j) = A$, then $a \in F(R)$.

Proposition 4.1. (Maskin, 1999) *Let $n \geq 3$ and suppose that Non-existence holds. If an SCC satisfies* Maskin-invariance *and* no-veto-power, *then it is Nash implementable.*

As discussed in Section 4.1, we consider the following three conditions which are studied by Doghmi and Ziad (2015). These conditions are central to our study.

For each $i \in N$, each $R \in \mathcal{R}$, each $a \in F(R)$, and each $b \in I(a, R_i)\backslash\{a\}$, let $OI(a, b, R_i) \equiv \{c \in A\backslash\{a, b\} : a\ I_i\ b\ I_i\ c\}$ be the **set of other-indifferent outcomes for $\{a, b\}$ at R_i.**[9]

Definition 4.3. An SCC F satisfies **DZ-invariance**[10] if for each pair $R, R' \in \mathcal{R}$ and each $a \in F(R')$, if for each $i \in N$, $SL(a, R'_i) \cup OI(a, b, R'_i) \cup \{a\} \subseteq L(a, R_i)$ for some $b \in I(a, R'_i)\backslash\{a\}$, then $a \in F(R)$.

Definition 4.4. An SCC F satisfies **weak no-veto-power**[11] if for each $i \in N$, each pair $R, R' \in \mathcal{R}$, each $a \in F(R')$, and each $b \in A$, if for some $c \in I(a, R'_i)\backslash\{a\}$, $b \in SL(a, R'_i) \cup OI(a, c, R'_i) \subseteq L(b, R_i)$ and for each $j \in N\backslash\{i\}$, $L(b, R'_j) = A$, then $b \in F(R)$.

Definition 4.5. An SCC F satisfies **unanimity** if for each $R \in \mathcal{R}$ and each $a \in A$, if for each $i \in N$, $L(a, R_i) = A$, then $a \in F(R)$.

Remark 4.1. For each of *DZ-invariance*, *weak no-veto-power*, and *unanimity*, we have the following remark:

- *DZ-invariance* implies *Maskin-invariance* but the converse does not always hold (Doghmi and Ziad, 2015; Observation 1). If $I(a, R'_i)\backslash\{a\} = \emptyset$, then the hypothesis in *DZ-invariance* is $SL(a, R'_i) \cup \{a\} \subseteq L(a, R_i)$.

- *Weak no-veto-power* is implied by *no-veto-power* but the converse does not always hold (Doghmi and Ziad, 2015; Observation 2).

- *Unanimity* is independent of *weak no-veto-power* (Doghmi and Ziad, 2015; Observation 2), and *unanimity* is implied by *no-veto-power* but the converse does not always hold.$\Diamond$

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[9] The set of other-indifferent outcomes is called the "indifferent options subset" by Doghmi and Ziad (2015).

[10] *DZ-invariance* is called "I-monotonicity" by Doghmi and Ziad (2015).

[11] *Weak no-veto-power* is called "I-weak no-veto-power" by Doghmi and Ziad (2015).

The following is a previous result for Nash implementability under Non-existence with at least three agents.

Proposition 4.2. (Doghmi and Ziad, 2015) *Let $n \geq 3$ and suppose that Non-existence holds. If an SCC satisfies* DZ-invariance, weak no-veto-power, *and* unanimity, *then it is Nash implementable.*

From the following result, for double implementability in Nash equilibria and undominated Nash equilibria, it suffices to focus on Nash implementability under Non-existence with at least three agents.

Proposition 4.3. (Yamato, 1999) *Let $n \geq 3$ and suppose that Non-existence holds. An SCC is doubly implementable in Nash equilibria and undominated Nash equilibria if and only if it is Nash implementable.*

From Propositions 4.2 and 4.3, we obtain that with at least three agents and under Non-existence, *DZ-invariance*, *weak no-veto-power*, and *unanimity* together are sufficient for double implementability in Nash equilibria and undominated Nash equilibria. This result is provided indirectly by means of two mechanisms. We prove this result directly by constructing another mechanism. This mechanism is also applied in the proof of our first theorem.

Proposition 4.4. (Yamato, 1999; Doghmi and Ziad, 2015). *Let $n \geq 3$ and suppose that Non-existence holds. If an SCC satisfies* DZ-invariance, weak no-veto-power, *and* unanimity, *then it is doubly implementable in Nash equilibria and undominated Nash equilibria.*

Proof. Let F be an SCC satisfying *DZ-invariance*, *weak no-veto-power*, and *unanimity*. Let $\Gamma = (M, g)$ be the mechanism such that for each $i \in N$, $M_i = \mathcal{R} \times A \times A \times \{-n, ..., -1, 0, 1, ..., n\}$, with generic element $m_i = (R^i, a^i, b^i, k^i)$, and the outcome mapping $g : M \to A$ is defined by the following four rules.[12]

[12]The difference between our mechanism and Yamato's mechanism is in the second rule of the four

For each $i \in N$ and each $R_i \in \mathcal{R}_i$, $\bar{\text{b}}(R_i)$ and $\underline{\text{b}}(R_i)$ are defined as follows:

(1) If there is a pair $x, y \in A$ such that $x \; P_i \; y$, then let $\bar{\text{b}}(R_i) = x$ and $\underline{\text{b}}(R_i) = y$;

(2) Otherwise, pick any pair $x, y \in A$ with $x \neq y$, let $\bar{\text{b}}(R_i) = x$ and $\underline{\text{b}} \; (R_i) = y$.

Rule 1. If for each $i \in N$ and each $b^i \in A$, $m_i = (R, a, b^i, i)$ with $a \in F(R)$, then $g(m) = a$.

Rule 2. If there is $i \in N$ such that for each $j \in N \backslash \{i\}$ and each $b^j \in A$, $m_j = (R, a, b^j, j)$ with $a \in F(R)$ and for each $b^i \in A$, $m_i = (R^i, a^i, b^i, k^i)$ with $R^i \neq R$ or $a^i \neq a$ or $k^i \neq i$, then

$$g(m) = \begin{cases} b^i \text{ if for some } c \in I(a, R_i) \backslash \{a\}, b^i \in SL(a, R_i) \cup OI(a, c, R_i) \neq \emptyset, \\ a \text{ otherwise.} \end{cases}$$

Rule 3. If there is $i \in N$ such that for each $j \in N \backslash \{i\}$ and each $b^j \in A$, $m_j = (R, a, b^j, -i)$ with $a \in F(R)$, then

$$g(m) = \begin{cases} \bar{\text{b}}(R_i) \text{ if } m_i = (R, a, \bar{\text{b}}(R_i), i), \\ \underline{\text{b}}(R_i) \text{ if } m_i \neq (R, a, \bar{\text{b}}(R_i), i) \text{ with } k_i \leq 0 \text{ or } k_i = i. \end{cases}$$

Rule 4. In all other cases, $g(m) = b^{i^*}$, where $i^* = (\Sigma_{i \in N} \max\{0, k^i\})(mod \; n) + 1$.

Given $R \in \mathcal{R}$ and $a \in F(R)$, we first show that if for each $i \in N$, $m_i = (R, a, \bar{\text{b}}(R_i), i)$, then $m \in NE(\Gamma, R)$. By Rule 1, $g(m) = a$. Let $i \in N$ and let $m'_i = (R'^i, a'^i, b'^i, k'^i)$ be such that for some $c \in I(a, R_i) \backslash \{a\}, b'^i \in SL(a, R_i) \cup OI(a, c, R_i) \neq \emptyset$. By Rule 2, $g(m'_i, m_{-i}) = b'^i$. By the definitions of a strictly lower contour set, a lower contour set, and a set of other-indifferent outcomes, since $b'^i \in SL(a, R_i) \cup OI(a, c, R_i) \subset L(a, R_i)$, we

rules. In the second rule of Yamato's mechanism, the cases are divided whether the outcome reported by the odd-man-out is in the lower counter set based on a preference profile reported by all agents excluding the odd-man-out. By contrast, as in the mechanism of Doghmi and Ziad (2015), in the second rule of our mechanism, the cases are divided whether the outcome reported by the odd-man-out is in the union of the strictly lower counter set and the set of other-indifferent outcomes based on a preference profile reported by all agents excluding the odd-man-out.

have $g(m_i, m_{-i})\ R_i\ g(m'_i, m_{-i})$.

By the same proof as that for Lemma 2 of Yamato (1999), given $R \in \mathcal{R}$ and $a \in F(R)$, for each $i \in N$, $m_i = (R, a, \bar{b}(R_i), i)$ is undominated at R_i. Therefore, for each $R \in \mathcal{R}$, $F(R) \subseteq g(UNE(\Gamma, R))$.

Next, we show that for each $R \in \mathcal{R}$, $g(NE(\Gamma, R)) \subseteq F(R)$. Let $m \in NE(\Gamma, R)$. There are four cases concerning m.

Case 1. For each $R' \in \mathcal{R}$, each $a \in F(R')$, each $i \in N$, and each $b^i \in A$, $m_i = (R', a, b^i, i)$.

By Rule 1, $g(m) = a$. Let $i \in N$. For some $c \in I(a, R'_i)\backslash\{a\}$, let $d \in SL(a, R'_i) \cup OI(a, c, R'_i) \cup \{a\}$. Note that if $I(a, R'_i)\backslash\{a\} = \emptyset$, let $d \in SL(a, R'_i) \cup \{a\} \neq \emptyset$. Let $m'_i = (R'^i, a'^i, d, k'^i)$ with $R'^i \neq R'$ or $a'^i \neq a$ or $k^i \neq i$. By Rule 2, $g(m'_i, m_{-i}) = d$. Since $m \in NE(\Gamma, R)$, we have $a = g(m_i, m_{-i})\ R_i\ g(m'_i, m_{-i}) = d$. Then, $d \in L(a, R_i)$. Therefore, $SL(a, R'_i) \cup OI(a, c, R'_i) \cup \{a\} \subseteq L(a, R_i)$. By *DZ-invariance*, $a \in F(R)$.

Case 2. For each $R' \in \mathcal{R}$ and each $a \in F(R')$, there is $i \in N$ such that for each $j \in N\backslash\{i\}$ and each $b^j \in A$, $m_j = (R', a, b^j, j)$ and for each $b^i \in A$, $m_i = (R^i, a^i, b^i, k^i)$ with $R^i \neq R'$ or $a^i \neq a$ or $k^i \neq i$.

There are two subcases concerning $m_i = (R^i, a^i, b^i, k^i)$.

Subcase 2-1. For some $c \in I(a, R'_i)\backslash\{a\}$, $b^i \in SL(a, R'_i) \cup OI(a, c, R'_i) \neq \emptyset$.

By Rule 2, $g(m) = b^i$. Let $d \in SL(a, R'_i) \cup OI(a, c, R'_i) \neq \emptyset$ and $m'_i = (R'^i, a'^i, d, k'^i)$ with $R'^i \neq R'$ or $a'^i \neq a$ or $k'^i \neq i$. By Rule 2, $g(m'_i, m_{-i}) = d$. Since $m \in NE(\Gamma, R)$, we have $b^i = g(m_i, m_{-i})\ R_i\ g(m'_i, m_{-i}) = d$. Then, $d \in L(b^i, R_i)$. Therefore, $b^i \in SL(a, R'_i) \cup OI(a, c, R'_i) \subseteq L(b^i, R_i)$. For each $j \in N\backslash\{i\}$, since $m \in NE(\Gamma, R)$, we have $L(b^i, R_j) = A$. By *weak no-veto-power*, $b^i \in F(R)$.

Subcase 2-2. Otherwise.

By the same argument as Case 1, by *DZ-invariance*, $g(m) \in F(R)$.

Case 3. Rule 3 applies.

We show that if $g(m) \notin F(R)$, then $m \notin NE(\Gamma, R)$. Since $g(m) \notin F(R)$ and F satisfies *unanimity*, there are $\ell \in N$ and $b \in A$ such that $b\ P_\ell\ g(m)$. There are two subcases concerning agent ℓ.

Subcase 3-1. $\ell = i$.

If $i \neq 1$, let $m'_i = (R'^i, a'^i, b, i-1)$ and if $i = 1$, let $m'_i = (R'^i, a'^i, b, n)$. By Rule 4, $g(m'_i, m_{-i}) = b$. Then, $g(m'_i, m_{-i})\ P_i\ g(m_\ell, m_{-\ell})$. Hence, $m \notin NE(\Gamma, R)$.

Subcase 3-2. $\ell \neq i$.

If agent ℓ deviates to $m'_\ell = (R'^\ell, a'^\ell, b, k'^\ell) \neq m_\ell$ such that $(\Sigma_{m\neq\ell}k^m + k'^\ell)(mod\ n) + 1 = \ell$, then by Rule 4, $g(m'_\ell, m_{-\ell}) = b$. Then, $g(m'_\ell, m_{-\ell})\ P_\ell\ g(m_\ell, m_{-\ell})$. Hence, $m \notin NE(\Gamma, R)$.

Case 4. In all other cases, Rule 4 applies.

Suppose that $g(m) \notin F(R)$. Since F satisfies *unanimity*, there are $\ell \in N$ and $b \in A$ such that $b\ P_\ell\ g(m)$. Let $m'_\ell = (R'^\ell, a'^\ell, b, k'^\ell) \neq m_\ell$ be such that $(\Sigma_{m\neq\ell}k^m + k'^\ell)(mod\ n) + 1 = \ell$ and it induces Rule 4. By Rule 4, $g(m'_\ell, m_{-\ell}) = b$. Then, $g(m'_\ell, m_{-\ell})\ P_i\ g(m_\ell, m_{-\ell})$. Hence, $m \notin NE(\Gamma, R)$. □

4.3.2 Under Existence ≥1

The following is a previous result for Nash implementability under Existence ≥1 with at least three agents.

Proposition 4.5. (Dutta and Sen, 2012) *Let $n \geq 3$ and suppose that Existence ≥1 holds. If an SCC satisfies* no-veto-power, *then it is Nash implementable.*

We show that with at least three agents and under Existence ≥1, if an SCC satisfies *weak no-veto-power* and *unanimity*, then it is doubly implementable in Nash equilibria and undominated Nash equilibria. By Remark 4.1, each of *weak no-veto-power* and *unanimity* is weaker than *no-veto-power*.

Theorem 4.1. *Let $n \geq 3$ and suppose that Existence ≥1 holds. If an SCC satisfies* weak no-veto-power *and* unanimity, *then it is doubly implementable.*

Proof. Let F be an SCC satisfying *weak no-veto-power* and *unanimity.* Let $\Gamma = (M, g)$ be the same mechanism as in the proof of Proposition 4.4. By similar arguments to the proof of Lemma 4.1, for each $R \in \mathcal{R}$, each $H \in \mathcal{H}^1$, and each $a \in F(R)$, if for each $i \in N$, $m_i = (R, a, \bar{b}(R_i), i)$, then $m \in NE(\Gamma, \succsim^{R,H})$ with $g(m) = a$.

We show that given $R \in \mathcal{R}$, $H \in \mathcal{H}^1$, and $a \in F(R)$, for each $i \in N$, $m_i = (R, a, \bar{b}(R_i), i)$ is undominated at $\succsim_i^{R,H}$. Suppose that there is a pair $b, c \in A$ with $b\ P_i\ c$. Then, $\bar{b}(R_i)\ P_i\ \underline{b}(R_i)$. We show that for each $\tilde{m}_i \neq m_i$, there is $\tilde{m}_{-i} \in M_{-i}$ such that $(m_i, \tilde{m}_{-i}) \succ_i^{R,H} (\tilde{m}_i, \tilde{m}_{-i})$. There are two cases concerning $\tilde{m}_i \neq m_i$.

Case 1. $\tilde{k}^i \leq 0$ or $\tilde{k}^i = i$.

For each $j \in N \backslash \{i\}$ and each $b^j \in A$, let $\tilde{m}_j = (R, a, b^j, -i)$. By Rule 3, $g(m_i, \tilde{m}_{-i}) = \bar{b}(R_i)$ and $g(\tilde{m}_i, \tilde{m}_{-i}) = \underline{b}(R_i)$, so that $(m_i, \tilde{m}_{-i}) \succ_i^{R,H} (\tilde{m}_i, \tilde{m}_{-i})$.

Case 2. $\tilde{k}^i > 0$ and $\tilde{k}^i \neq i$.

Define $\tilde{m}_{-i} \in M_{-i}$ as follows: for some $j \in N \backslash \{i\}$, $\tilde{m}_j = (R', a', \underline{b}(R_i), j-1)$, for some $h \in N \backslash \{i, j\}$, $\tilde{m}_h = (R'', a'', \underline{b}(R_i), \tilde{k}^h)$, and for each $\ell \in N \backslash \{i, j.h\}$, each $R^\ell \in \mathcal{R}$, and each $b^\ell \in A$, $\tilde{m}_\ell = (R^\ell, b^\ell, \underline{b}(R_i), \tilde{k}^\ell)$, where $(R, a) \neq (R', a') \neq (R'', a'')$ and $(\Sigma_{p \neq i,j} \tilde{k}^p + i + (j-1))(mod\ n) + 1 = i$ with for each $q \in N \backslash \{i, j\}$, $\tilde{k}^q \geq 0$. By Rule 4, $g(m_i, \tilde{m}_{-i}) = \bar{b}(R_i)$ and $g(\tilde{m}_i, \tilde{m}_{-i}) = \underline{b}(R_i)$, so that $(m_i, \tilde{m}_{-i}) \succ_i^{R,H} (\tilde{m}_i, \tilde{m}_{-i})$.

Suppose that for each pair $b, c \in A$, $b\ I_i\ c$. Obviously, m_i is undominated at $\succsim_i^{R,H}$

To establish that for each $R \in \mathcal{R}$ and each $H \in \mathcal{H}^1$, $g(NE(\Gamma, \succsim^{R,H})) \subseteq F(R)$, the proof is the same as in Lemma 4.3, except for Case 1 in Lemma 4.3. We show that if $g(m) \notin F(R)$, then $m \notin NE(\Gamma, \succsim^{R,H})$. For each $R' \in \mathcal{R}$, each $a \in F(R') \backslash F(R)$, each $i \in N$, and each $b^i \in A$, let $m_i = (R', a, b^i, i)$. By Rule 1, $g(m) = a$. Under Existence $\geq$1, there is a partially honest agent $h \in H$. Let $m'_h = (R, a'^h, a, k'^h) \neq m_h$. By Rule 2, $g(m'_h, m_{-h}) = a$. Then, $g(m_h, m_{-h}) = g(m'_h, m_{-h})$. Since $h \in H$, $(m'_h, m_{-h}) \succ_h^{R,H} (m_h, m_{-h})$. Hence, $m \notin NE(\Gamma, \succsim^{R,H})$. □

4.3.3 Under Existence ≥2

The following is a previous result for Nash implementability under Existence ≥2 with at least three agents.

Proposition 4.6. (Kimya, 2015) *Let $n \geq 3$ and suppose that Existence ≥2 holds. If an SCC satisfies* unanimity, *then it is Nash implementable.*

We show that with at least three agents and under Existence ≥2, if an SCC satisfies *unanimity*, then it is doubly implementable in Nash equilibria and undominated Nash equilibria. Therefore, if the assumption concerning the existence of partially honest agents is changed from Existence ≥1 into Existence ≥2, a sufficient condition for double implementability in Nash equilibria and undominated Nash equilibria is weakened.

Theorem 4.2. *Let $n \geq 3$ and suppose that Existence ≥2 holds. If an SCC satisfies* unanimity, *then it is doubly implementable in Nash equilibria and undominated Nash equilibria.*

Proof. Let F be an SCC satisfying *unanimity*. Let $\Gamma = (M, g)$ be the mechanism such that for each $i \in N$, $M_i = \mathcal{R} \times A \times A \times \{-n, ..., -1, 0, 1, ..., n\}$ and the outcome mapping $g : M \to A$ is defined as follows:[13]

Rule 1: If there is $i \in N$ such that for each $j \in N \backslash \{i\}$ and each $b^j \in A$, $m_j = (R, a, b^j, j)$ with $a \in F(R)$, then $g(m) = a$.

Rule 2: If there is $i \in N$ such that for each $j \in N \backslash \{i\}$ and each $b^j \in A$, $m_j = (R, a, b^j, -i)$ with $a \in F(R)$, then

$$g(m) = \begin{cases} \bar{\mathrm{b}}(\tilde{R}_i) \text{ if } m_i = (R, a, \bar{b}(\tilde{R}_i), i) \\ \underline{\mathrm{b}}(\tilde{R}_i) \text{ if } m_i \neq (R, a, \bar{b}(\tilde{R}_i), i) \text{ with } k^i \leq 0 \text{ or } k^i = i. \end{cases}$$

[13] For the mechanism designed in Theorem 4.2, the first and second rules of the mechanism constructed in Proposition 4.4 is changed as in Dutta and Sen (2012). The reason is that in the case considered in the second rule of the mechanism constructed in Proposition 4.4, there is a partially honest agent in the set of all agents excluding the the odd-man-out by Existence ≥2.

Rule 3: In all other cases, $g(m) = b^{i^*}$, where $i^* = (\Sigma_{i \in N} \max\{0, k^i\})(mod\ n) + 1$.

Given $R \in \mathcal{R}$, $H \in \mathcal{H}^2$, and $a \in F(R)$, we first show that if for each $i \in N$, $m_i = (R, a, \bar{\text{b}}(R_i), i)$, then $m \in NE(\Gamma, \succsim^{R,H})$. By Rule 1, $g(m) = a$. No unilateral deviation can change the outcome and for each $i \in N$, $R^i = R$. Hence, $m \in NE(\Gamma, \succsim^{R,H})$.

By the same proof as in that for Theorem 4.1, given $R \in \mathcal{R}$, $H \in \mathcal{H}^2$, and $a \in F(R)$, for each $i \in N$, $m_i = (R, a, \bar{b}(R_i), i)$ is undominated at $\succsim_i^{R,H}$. Therefore, for each $R \in \mathcal{R}$ and each $H \in \mathcal{H}^2$, $F(R) \subseteq g(UNE(\Gamma, \succsim^{R,H}))$.

Next, we show that for each $R \in \mathcal{R}$ and each $H \in \mathcal{H}^2$, $g(NE(\Gamma, \succsim^{R,H})) \subseteq F(R)$. To show this, we prove that if $g(m) \notin F(R)$, then $m \notin NE(\Gamma, \succsim^{R,H})$. There are four cases concerning m.

Case 1. For each $R' \in \mathcal{R}$, each $i \in N$, each $a \in F(R')\backslash F(R)$, and each $b^i \in A$, $m_i = (R', a, b^i, i)$.

By Rule 1, $g(m) = a$. Under Existence ≥2, there is a partially honest agent $h \in H$. Let $m'_h = (R, a'^h, b'^h, k'^h)$. By Rule 1, $g(m'_h, m_{-h}) = a$. Then, $g(m'_h, m_{-h}) = g(m_h, m_{-h})$. Since $h \in H$, $(m'_h, m_{-h}) \succ_h^{R,H} (m_h, m_{-h})$. Hence, $m \notin NE(\Gamma, \succsim^{R,H})$.

Case 2. For each $R' \in \mathcal{R}$ and each $a \in F(R')\backslash F(R)$, there is $i \in N$ such that for each $j \in N\backslash\{i\}$ and each $b^j \in A$, $m_j = (R', a, b^j, j)$, and for each $b^i \in A$, $m_i \neq (R', a, b^i, i)$.

By Rule 1, $g(m) = a$. Under Existence ≥2, since $|H| \geq 2$, there is a partially honest agent $h \in H\backslash\{i\}$.[14] Without loss of generality, let $i = 1$ and $h = 2$. Let $m'_2 = (R, a'^2, b'^2, k'^2)$ be such that $(\Sigma_{j \neq 2} k^j + k'^2)(mod\ n) + 1 = 3$. By Rule 3, $g(m'_2, m_{-2}) = b^3 = a$. Then, $g(m'_2, m_{-2}) = g(m_2, m_{-2})$. Since $h = 2 \in H$, $(m'_2, m_{-2}) \succ_2^{R,H} (m_2, m_{-2})$. Hence, $m \notin NE(\Gamma, \succsim^{R,H})$.

Case 3. Rule 2 applies.

Case 4. In all other cases, Rule 3 applies.

The proofs of Case 3 and Case 4 are the same as the proof of Proposition 4.4. □

[14]Note that under Existence ≥1, when $|H| = 1$ and agent i is partially honest, there is no partially honest agent in $N\backslash\{i\}$.

4.4 A Characterization of Double Implementability in Nash Equilibria and Undominated Nash Equilibria

As discussed in Section 4.1, it is important to provide a necessary and sufficient condition for double implementability in Nash equilibria and undominated Nash equilibria under Existence ≥ 1 when there are at least three agents and *unanimity* is satisfied.

For each $a \in A$, each $i \in N$, each $R_i \in \mathcal{R}_i$, and each $B \subseteq A$, let $I(a, R_i, B) = \{b \in B : a \; I_i \; b\}$ be the **indifferent set at** (a, R_i) **restricted to** B. The following is the condition which is studied by Lombardi and Yoshihara (2019).

Definition 4.6. An SCC F satisfies **LY-condition** if there is $B \subseteq A$, and for each $R' \in \mathcal{R}$, each $i \in N$, and each $a \in B$ such that $a \in F(R')$, there is $C_i(a, R') \subseteq B$ with $a \in C_i(a, R') \subseteq L(a, R'_i)$ such that for each $R \in \mathcal{R}$ and each $H \in \mathcal{H}^1$, the following conditions are satisfied:

(1) There is $S_i(R', a; R) \neq \emptyset$ such that $S_i(R', a; R) \subseteq C_i(a, R')$,

(2) For each $h \in H$, if $a \notin S_h(R, a; R)$, then $S_h(R, a; R) \subseteq SL(a, R_h)$,

(3) If $b \in C_i(a, R') \subseteq L(a, R_i)$, $b \notin F(R)$, and for each $j \in N \backslash \{i\}$, $B \subseteq L(a, R_j)$,

(3-a) if $H = \{i\}$, then $S_i(R', a; R) \cap I(b, R_i, B) \neq \emptyset$ and $b \notin S_i(R', a; R)$, and

(3-b) if $i \notin H$, then there is $j \in H$ such that $a \notin S_j(R, a; R)$.

The following is the previous result of a characterization for Nash implementability under Existence ≥ 1 when there are at least three agents and *unanimity* is satisfied.

Proposition 4.7. (Lombardi and Yoshihara, 2019) *Let $n \geq 3$ and let F be an SCC satisfying* unanimity. *Suppose that Existence ≥ 1 holds. Then, F is Nash implementable if and only if it satisfies* LY-condition.

Note that "Assumption 2" of Lombardi and Yoshihara (2019), where the class of conceivable sets of partially honest agents has all non-empty subsets of N as elements, is included in Existence ≥ 1.

We obtain a characterization of double implementability in Nash equilibria and undominated Nash equilibria under Existence ≥ 1 when there are at least three agents and *unanimity* is satisfied.

Theorem 4.3. *Let $n \geq 3$ and let F be an SCC satisfying* unanimity. *Suppose that Existence ≥ 1 holds. Then, F is doubly implementable in Nash equilibria and undominated Nash equilibria if and only if it satisfies* LY-condition.

By Proposition 4.7 and Theorem 4.3, we obtain the following corollary.

Corollary 4.1. *Let $n \geq 3$ and let F be an SCC satisfying* unanimity. *Suppose that Existence ≥ 1 holds. Then, F is doubly implementable in Nash equilibria and undominated Nash equilibria if and only if it is Nash implementable.*

For Theorem 4.3, if an SCC is doubly implementable in Nash equilibria and undominated Nash equilibria, then it is Nash implementable. By Proposition 4.7, it satisfies *LY-condition*. Then, it suffices to show that if an SCC satisfies *LY-condition*, then it is doubly implementable in Nash equilibria and undominated Nash equilibria. The proof of this result is obtained by similar arguments of the proofs of Theorem 1 of Lombardi and Yoshihara (2019) and Theorem 4.1 in this chapter. Thus, we omit the formal proof. We only propose the mechanism to show that if an SCC satisfies *LY-condition*, then it is doubly implementable in Nash equilibria and undominated Nash equilibria.

Let $\Omega \neq \emptyset$ be an arbitrary set such that $\Omega \cap \mathcal{R} = \emptyset$ and that there is a bijection $\phi : \mathcal{R} \to \Omega$. Note that there is ϕ^{-1} such that for each $R \in \mathcal{R}$, $\phi^{-1} \circ \phi(R) = R$. For each $i \in N$, let $M_i = (\mathcal{R} \cup \Omega) \times A \times A \times \{-n, ..., -1, 0, 1, ..., n\}$, with generic element $m_i = (m_1^i, m_2^i, m_3^i, m_4^i) = (R^i, a^i, b^i, k^i)$. For each $R \in \mathcal{R}$, each $i \in N$, and each $a \in B$

with $a \in F(R)$, $\sigma_i(R,a)$ is defined as follows:

$$\sigma_i(R,a) = \begin{cases} \{\phi(R)\} \times \{a\} \times A \times \{-n,...,-1,0,1,...,n\} & \text{if there is } j \in N \backslash \{i\} \text{ such that} \\ & a \in S_j(R,a;R) \text{ and for each } \ell \in N\backslash\{j\}, \\ & a \notin S_\ell(R,a;R) \\ \{R\} \times \{a\} \times A \times \{-n,...,-1,0,1,...,n\} & \text{otherwise} \end{cases}$$

Let $\sigma(R,a) = (\sigma_i(R,a))_{i \in N}$, with generic element $((\sigma_1^i(R,a), \sigma_2^i(R,a), \sigma_3^i(R,a), \sigma_4^i(R,a)))_{i \in N}$. The outcome mapping $g : M \to A$ is defined as follows:

Rule 1: If for each $i \in N$, $m_i \in \sigma_i(R,a)$ and $m_4^i = i$, then $g(m) = a$.

Rule 2: If there is $i \in N$ such that $m_i \notin \sigma_i(R,a)$, and for each $j \in N\backslash\{i\}$, $m_j \in \sigma_j(R,a)$, and $m_4^j = j$, then there are three cases:

Rule 2-1: If $m_1^i = R^i = R$ or $m_1^i = \phi(R^i) = \phi(R)$, then $g(m) = a$.

Rule 2-2: If $m_1^i = R^i \neq R$ or $m_1^i = \phi(R^i) \neq \phi(R)$, then given $R = \phi^{-1} \circ \phi(R)$ and $R^i = \phi^{-1} \circ \phi(R^i)$, and for some $c \in S_i(R,a;R^i)$,

$$g(m) = \begin{cases} a^i & \text{if } a^i \in S_i(R,a;R^i) \\ a^i & \text{if } a^i \in C_i(a,R)\backslash S_i(R,a;R^i) \text{ and } S_i(R,a;R^i) \subseteq SL(a^i, R_i^i) \\ d & \text{if } a^i \in C_i(a,R)\backslash S_i(R,a;R^i) \text{ and } d \in S_i(R,a;R^i) \cap I(a^i, R_i^i, A) \\ c & \text{otherwise} \end{cases}$$

Rule 2-3: If $m_1^i = R^i = R \neq \sigma_1^i(R,a)$, then for some $c \in S_i(R,a;R)$,

$$g(m) = \begin{cases} a^i & \text{if } a^i \in S_i(R,a;R) \\ c & \text{otherwise} \end{cases}$$

Rule 3. If there is $i \in N$ such that for each $j \in N \setminus \{i\}$, $m_j \in \sigma_j(R, a)$ and $m_4^j = -i$, then given $R = \phi^{-1} \circ \phi(R)$,

$$g(m) = \begin{cases} \bar{\mathrm{b}}(R_i) \text{ if } m_i = (\sigma_1^i(R, a), \sigma_2^i(R, a), \bar{\mathrm{b}}(R_i), i), \\ \underline{\mathrm{b}}(R_i) \text{ if } m_i \neq (\sigma_1^i(R, a), \sigma_2^i(R, a), \bar{\mathrm{b}}(R_i), i) \text{ with } k^i \leq 0 \text{ or } k^i = i. \end{cases}$$

Rule 4: In all other cases, $g(m) = b^{i^*}$, where $i^* = (\Sigma_{i \in N} \max\{0, k^i\})(mod\ n) + 1$.

4.5 Concluding Remarks

If there are at least three agents and there is no partially honest agent, *DZ-invariance*, *weak no-veto-power*, and *unanimity* together are sufficient for double implementability in Nash equilibria and undominated Nash equilibria (Proposition 4.4). As an application, we consider the allocation problem of an infinitely divisible resource with single-peaked preferences. The no-envy correspondence satisfies *DZ-invariance*, *weak no-veto-power*, and *unanimity*, but violates *no-veto-power* (Doghmi and Ziad, 2008a, 2013, 2015; Thomson, 2010).[15,16] Thus, by Proposition 4.4, the no-envy correspondence is doubly implementable in Nash equilibria and undominated Nash equilibria.

If there are at least three agents out of which at least one agent is partially honest, then *weak no-veto-power* and *unanimity* together are sufficient for double implementability in Nash equilibria and undominated Nash equilibria (Theorem 4.1). As an application, we consider the allocation problem of an infinitely divisible resource with single-plateaued preferences. The Pareto correspondence satisfies *weak no-veto-power* and *unanimity*, but neither *DZ-invariance* nor *no-veto-power* (Doghmi and Ziad, 2013). By Theorem 4.1,

[15] For the other SCCs, see Thomson (2010) and Doghmi and Ziad (2013).

[16] By Lemma 1 of Doghmi and Ziad (2013), the no-envy correspondence satisfies *weak no-veto-power*. Moreover, by the previous results, the no-envy correspondence satisfies *DZ-invariance*: The no-envy correspondence satisfies *Maskin-invariance* but violates *no-veto-power* (see, e.g., Thomson, 2010). Doghmi and Ziad (2008a) show that in the allocation problem of an infinitely divisible resource with single-peaked preferences, *Maskin-invariance* is equivalent to its variant, which Doghmi and Ziad (2008b) called "strict monotonicity." In general, *DZ-invariance* is weaker than *strict monotonicity* but stronger than *Maskin-invariance* (Doghmi and Ziad, 2015, Observation 1). Then, in the problem, *Maskin-invariance* is equivalent to *DZ-invariance*. Therefore, the no-envy correspondence satisfies *DZ-invariance*.

the Pareto correspondence is doubly implementable in Nash equilibria and undominated Nash equilibria.[17] As another application, we consider the many-to-one matching problem (Gale and Shapley, 1962; Doghmi and Ziad, 2015). Any sub-solution of the stable matching correspondence satisfies *weak no-veto-power* and *unanimity* but may violate *no-veto-power* (Doghmi and Ziad, 2015). Thus, by Theorem 4.1, such SCCs are doubly implementable in Nash equilibria and undominated Nash equilibria.

If there are at least three agents out of which at least two agents are partially honest, then *unanimity* is sufficient for double implementability in Nash equilibria and undominated Nash equilibria (Theorem 4.2). From this result, we obtain several positive corollaries in several interesting applications such as a generalized one-to-one matching problem (Sönmez, 1996; Ehlers, 2004). See an earlier version of this chapter (Hagiwara, 2017).

If there are at least three agents out of which at least one agent is partially honest and *unanimity* is satisfied, then *LY-condition* is necessary and sufficient for double implementability in Nash equilibria and undominated Nash equilibria (Theorem 4.3). By Proposition 4.7 and Theorem 4.3, double implementability in Nash equilibria and undominated Nash equilibria is equivalent to Nash implementability (Corollary 4.1). Then, from Corollary 4.1 and the discussion of Lombardi and Yoshihara (2019) for Nash implementability in several interesting applications such as a bargaining problem (Nash, 1950), we obtain several positive corollaries regarding double implementability in Nash equilibria and undominated Nash equilibria. See Lombardi and Yoshihara (2019).

Although our results are positive, the mechanisms for these results are still complicated. Regarding a simple mechanism for double implementability of any *unanimous* single-valued SCC in Nash equilibria and undominated Nash equilibria, see Hagiwara (2018).

[17] In the allocation problem of an infinitely divisible resource with single-plateaued preferences, by Lemma 1 in Doghmi and Ziad (2013), *unanimity* is sufficient for double implementability in Nash equilibria and undominated Nash equilibria under Existence ≥ 1. We also obtain this result in allocation problems of an infinitely divisible resource with single-peaked preferences, and with single-dipped preferences. See Doghmi and Ziad (2013).

Chapter 5

A Simple and Procedurally Fair Mechanism for Double Implementation in Nash Equilibria and Undominated Nash Equilibria with Semi-Socially-Responsible Agents

5.1 Introduction

We investigate the possibility of double implementability of social choice functions in Nash equilibria and undominated Nash equilibria by means of a simple and procedurally fair mechanism when each agent is "semi-socially-responsible."[1]

We study the implementation problem under complete information. The objective of a social planner is embodied by a social choice function. Mathematically, a social choice function (SCF) is a single-valued mapping which, for each possible preference profile, specifies an outcome. Each agent knows the other agent preferences, but the planner does not know the agents' preferences. Then, she specifies a message space for each agent and a single-valued mapping which, for each possible message profile, chooses an outcome. The pair consisting of the list of agents' message spaces and a mapping is a mechanism.

Jackson, Palfrey, and Srivastava (1994), Tatamitani (1993), and Yamato (1999) have constructed mechanisms for double implementation in Nash equilibria and undominated Nash equilibria. In their mechanisms, each agent's strategy has at least three components,

[1]For reasons why we investigate double implementation in Nash equilibria and undominated Nash equilibria, see Chapter 4.

namely a preference profile, an outcome, and an integer, and the outcome mapping distinguishes between at least three types of message profiles. It has been argued that such mechanisms are complex.[2] In addition, their mechanisms are not always "ex ante fair" (Korpela, 2018).[3] The possibility of double implementation in Nash equilibria and undominated Nash equilibria by means of a simple and *ex ante fair* mechanism is remained open.

Traditional implementation theory usually assumes that each agent only cares about satisfying their own preferences. However, there are several experimental observations which suggest that some agents care about fairness, honesty, or reciprocity. For example, by Gneezy (2005) and Hurkens and Kartik (2009), agents are one of two kinds: either an agent will never lie, or an agent will lie whenever he prefers the outcome obtained by lying over the outcome obtained by telling the truth. In addition, the result of Gneezy (2005) suggests that the smaller the gains from lying are, the greater the number of subjects who tell the truth is (see Footnote 10 in the paper).[4] Following such experimental observations, a bunch of papers introduce behavioral implementation theory.[5]

For a mechanism in which a component of each agent's message space is the set of outcomes, an agent is "semi-socially-responsible" if he prefers reporting the socially desirable outcome at the true preference profile whenever announcing a socially undesirable outcome does not change the outcome to one that he prefers; otherwise, he reports a message inducing an outcome that he prefers.[6] Note that semi-social-responsibility is defined with respect to a particular SCF (Remark 5.1).

The existence of such agents is natural in a number of situations. Consider the

[2]Under some assumptions on environments, Kartik, Tercieux, and Holden (2014) and Yamato (1993) have constructed simpler mechanisms for double implementability in Nash equilibria and undominated Nash equilibria. For the definitions of those assumptions, see "separable punishment" in Kartik, Tercieux, and Holden (2014), and see (A1)-(A3) in Yamato (1993). We do not consider those assumptions. Note that Kartik, Tercieux, and Holden (2014) investigate the implementation notion in two rounds of iterated deletion of strictly dominated strategies, and this is also a study of double implementability in Nash equilibria and undominated Nash equilibria. I am grateful to Bhaskar Dutta for pointing out this fact.

[3]For the definition of *ex ante fairness*, see Chapter 3.

[4]In Hurkens and Kartik (2009), the same tendency as in Gneezy (2005) is observed, but there is no statistical significant unlike Gneezy.

[5]There are several papers on behavioral implementation theory. See Footnote 4 in Chapter 4.

[6]Semi-socially-responsible agents are called "socially responsible" agents by Hagiwara, Yamamura, and Yamato (2018).

following voting game: There are two candidates a and b. Each agent's message space is $\{a, b\}$. Given all agents' messages except for agent i's, candidate a is selected if agent i reports candidate a and candidate b is selected if he reports candidate b. Suppose that a preference profile for which an SCF selects candidate a is such that agent i is indifferent between the two candidates. In this case, since by reporting candidate a, he fulfills his social responsibility, we can imagine that agent i prefers reporting candidate a to candidate b.

Our point of departure for double implementability in Nash equilibria and undominated Nash equilibria by means of a simple and *ex ante fair* mechanism is the "HYY mechanism" (Hagiwara, Yamamura, and Yamato, 2018), which achieves Nash implementability with semi-socially-responsible agents.[7] This mechanism is simple and *ex ante fair*.[8] Indeed, in this mechanism, each agent reports an outcome and a positive integer between 1 and n where n is the number of agents. The outcome mapping is defined as follows: If at least $n - 1$ agents report the same outcome, then this outcome is chosen; otherwise, the outcome is chosen by a "modulo game."[9,10] There remains the issue of double implementability in Nash equilibria and undominated Nash equilibria by means of this mechanism.

We show that if there are at least three agents and each agent is semi-socially-responsible with respect to a "unanimous" SCF, then the HYY mechanism doubly implements this SCF in Nash equilibria and undominated Nash equilibria (Theorem 5.1). For this mechanism, if at least one agent is not semi-socially-responsible, then the set of undominated Nash equilibrium outcomes may be smaller than the set of Nash equilibrium outcomes (Example 5.1).

Although the basic structure of the HYY mechanism looks similar to that of Dutta

[7] The HYY mechanism is called the "outcome mechanism" by Hagiwara, Yamamura, and Yamato (2018).

[8] As discussed in Chapter 4, the precise definition of simplicity is not provided in this chapter either.

[9] In the modulo game in the HYY mechanism, the outcome reported by the agent whose identification matches the modulo of the sum of the integers reported by all agents plus 1 is chosen.

[10] For the results in Hagiwara, Yamamura, and Yamato (2018) and our result, we need to use a module game, not an "integer game." See Hagiwara, Yamamura, and Yamato (2018). The mechanism might be criticized because of the use of the modulo game which may lead to unwanted mixed strategy equilibria (see, e.g., Jackson, 1992).

and Sen's mechanism which is a variant of the canonical mechanisms for Nash implementability, the HYY mechanism has the following advantages over it.

First, in Dutta and Sen's mechanism, each agent reports the other agents' preferences as well as his own preference. On the other hand, in the HYY mechanism, each agent no longer needs to reveal any information on preferences.

The second advantage is that the complete information assumption can be weakened. In order to guarantee that which message profile is a Nash equilibria is common knowledge among agents in Dutta and Sen's mechanism, we usually need the assumption that all agents' preferences are common knowledge among agents.[11] By contrast, in the HYY mechanism, even when preferences are not common knowledge among agents, the set of Nash equilibria might be common knowledge. In Section 5.4, we illustrate how the complete information assumption can be weakened.

Our study is closely related to the following papers. Matsushima (2008) investigates the case in which some agent suffers a small utility loss from reporting socially undesirable outcomes. In his framework, the planner can impose small fines on agents. He shows that if there are at least three agents, every SCF is implementable in the iterative elimination of strictly dominated strategies. While Matsushima (2008) considers the probabilistic social choice problems with monetary transfers, we study deterministic social choice problems in which the planner cannot impose any small fines on agents. Moreover, Doğan (2013) examines allocation problems of indivisible goods with "responsible agents." A responsible agent wants to maximize the number of agents to whom socially optimal indivisible goods are allocated. In this setting, he shows that if the planner knows that there are at least three responsible agents, every SCF is implementable in Nash equilibria. While Doğan (2013) investigates allocation problems of indivisible goods in which each agent has a preference over the set of allocations, we consider general environments in which each agent has a preference over the set of message profiles.

This chapter is organized as follows: Section 5.2 presents the model. Section 5.3

[11]For the definition of common knowledge, see Aumann (1976).

reports our main result, a remark, and an example. Section 5.4 discusses informational requirements necessary for the HYY mechanism.

5.2 The Model

Let $N = \{1, \cdots, n\}$ be the set of agents and A be the set of outcomes. For each $i \in N$, let R_i be a preference of agent i over A, whose asymmetric and symmetric components are denoted by P_i and I_i, respectively. For each $i \in N$, let $\mathcal{R}_i$ be the set of possible preferences for agent i. Let $R = (R_1, ..., R_n) \in \mathcal{R}$ be a preference profile, where $\mathcal{R} = \times_{i \in N} \mathcal{R}_i$.

A **social choice function** (**SCF**) $f : \mathcal{R} \to A$ is a single-valued mapping which, for each preference profile $R \in \mathcal{R}$, specifies an outcome $f(R) \in A$, interpreted as the socially desirable outcome for R.

We investigate the possibility of double implementability in Nash equilibria and undominated Nash equilibria by means of a simple and *ex ante fair* mechanism. Our point of departure is the **HYY mechanism** (Hagiwara, Yamamura, and Yamato, 2018). For each $i \in N$, let $M_i = A \times N$ be the **message space of agent** i, with generic element $m_i = (a^i, k^i)$. Let $M = \times_{i \in N} M_i$. The **outcome mapping** $g : M \to A$ is defined as follows: Let $m = (m_i)_{i \in N} \in M$.

Rule 1: If there is $i \in N$ such that for each $j \in N \backslash \{i\}$, $m_j = (a, k^j)$, then $g(m) = a$.

Rule 2: In all other cases, $g(m) = a^{i^*}$, where $i^* = (\Sigma_{i \in N} k^i)(mod\ n) + 1$.

Let $\Gamma \equiv (M, g)$.

We "extend" an agent's preference over A to that over M. For the HYY mechanism, semi-socially-responsible preferences are defined as follows. Given an SCF f, $R \in \mathcal{R}$, and $i \in N$, let $\succsim_i^{f(R)}$ be the **preference of agent** i **over** M **at** $f(R)$, whose asymmetric and symmetric components are denoted by $\succ_i^{f(R)}$ and $\sim_i^{f(R)}$, respectively:

Semi-social-responsibility with respect to f:[12] For each pair $m_i, m_i' \in M_i$ such that

[12]In the model of Kartik and Tercieux (2012), "semi-socially-responsible preferences" can be defined

$m_i = (a^i, k^i)$ and $\tilde{m}_i = (\tilde{a}^i, \tilde{k}^i)$ and each $m_{-i} \in M_{-i}$,

(1) If $a^i = f(R)$, $\tilde{a}^i \neq f(R)$, and $g(m_i, m_{-i})\ I_i\ g(\tilde{m}_i, m_{-i})$, then $(m_i, m_{-i}) \succ_i^{f(R)} (\tilde{m}_i, m_{-i})$.

(2) In all other cases, $g(m_i, m_{-i})\ R_i\ g(\tilde{m}_i, m_{-i})$ if and only if $(m_i, m_{-i}) \succsim_i^{f(R)} (\tilde{m}_i, m_{-i})$.

Let $\succsim^{f(R)} \equiv (\succsim_1^{f(R)}, \cdots, \succsim_n^{f(R)})$.

Given $R \in \mathcal{R}$, let $(\Gamma, \succsim^{f(R)})$ be the **game** induced by Γ and $\succsim^{f(R)}$. A message profile $m \in M$ is a **Nash equilibrium of** $(\Gamma, \succsim^{f(R)})$ if for each $i \in N$ and each $\tilde{m}_i \in M_i$, $(m_i, m_{-i}) \succsim_i^{f(R)} (\tilde{m}_i, m_{-i})$. Let $NE(\Gamma, \succsim^{f(R)})$ be the **set of Nash equilibria of** $(\Gamma, \succsim^{f(R)})$.

The mechanism Γ **implements the SCF f in Nash equilibria** if for each $R \in \mathcal{R}$, $\{f(R)\} = g(NE(\Gamma, \succsim^{f(R)}))$.

For each $i \in N$, agent i's message $m_i \in M_i$ is **weakly dominated by** $\tilde{m}_i \in M_i$ **at** $\succsim_i^{f(R)}$ if for each $m_{-i} \in M_{-i}$, $(\tilde{m}_i, m_{-i}) \succsim_i^{f(R)} (m_i, m_{-i})$ and for some $m_{-i} \in M_{-i}$, $(\tilde{m}_i, m_{-i}) \succ_i^{f(R)} g(m_i, m_{-i})$. Agent i's message $m_i \in M_i$ is **undominated at** $\succsim_i^{f(R)}$ if it is not weakly dominated by any message in M_i at $\succsim_i^{f(R)}$. A message profile $m \in M$ is an **undominated Nash equilibrium of** $(\Gamma, \succsim^{f(R)})$ if for each $i \in N$, $m_i \in M_i$ is undominated at $\succsim_i^{f(R)}$ and $m \in M$ is a Nash equilibrium of $(\Gamma, \succsim^{f(R)})$. Let $UNE(\Gamma, \succsim^{f(R)})$ be the **set of undominated Nash equilibria of** $(\Gamma, \succsim^{f(R)})$.

The mechanism Γ **doubly implements the SCF f in Nash equilibria and undominated Nash equilibria** if for each $R \in \mathcal{R}$, $\{f(R)\} = g(NE(\Gamma, \succsim^{f(R)})) = g(UNE(\Gamma, \succsim^{f(R)}))$.

5.3 Results

Our main result is that the following axiom is sufficient for double implementability in Nash equilibria and undominated Nash equilibria by means of a simple and *ex ante fair*

similarly. However, their model is different from ours. In their model, each agent's strategy space is the product of his cheap-talk message space and his feasible set of "evidences." An evidence is a discriminatory signal about the true preference profile, as opposed to a cheap-talk message.

mechanism when each agent is semi-socially-responsible with respect to this SCF.

Definition 5.1. An SCF f is **unanimous** if for each $R \in \mathcal{R}$ and each $a \in A$, if for each $i \in N$ and each $b \in A$, $a \; R_i \; b$, then $a = f(R)$.

We apply the following result to our main result.

Proposition 5.1. (Hagiwara, Yamamura, and Yamato, 2018).[13] *Let $n \geq 3$ and f be a* unanimous *SCF. Suppose that at least two agents are semi-socially-responsible with respect to f.*[14] *The mechanism Γ implements f in Nash equilibria.*

The following is our main result.

Theorem 5.1. *Let $n \geq 3$ and f be a* unanimous *SCF. Suppose that each agent is semi-socially-responsible with respect to f. The mechanism Γ doubly implements f in Nash equilibria and undominated Nash equilibria.*

Proof. Let f be a *unanimous* SCF and $R \in \mathcal{R}$. Since $\{f(R)\} = g(NE(\Gamma, \succsim^{f(R)}))$ (Proposition 5.1) and $g(UNE(\Gamma, \succsim^{f(R)})) \subseteq g(NE(\Gamma, \succsim^{f(R)}))$, it suffices to show that $\{f(R)\} \subseteq g(UNE(\Gamma, \succsim^{f(R)}))$.

For each $i \in N$, let $m_i = (f(R), k^i)$. Note that $m \in NE(\Gamma, \succsim^{f(R)})$. We show that for each $\tilde{m}_i \in M_i$, [for each $\tilde{m}_{-i} \in M_{-i}$, $(m_i, \tilde{m}_{-i}) \sim_i^{f(R)} (\tilde{m}_i, \tilde{m}_{-i})$] or [there is $\tilde{m}_{-i} \in M_{-i}$ such that $(m_i, \tilde{m}_{-i}) \succ_i^{f(R)} (\tilde{m}_i, \tilde{m}_{-i})$].

We distinguish between two cases concerning the messages of agent i.

Case 1. For each $b \in A \backslash \{f(R)\}$ and each $\tilde{k}^i \in N$, $\tilde{m}_i = (b, \tilde{k}^i)$.

For each $j \in N \backslash \{i\}$, let $\tilde{m}_j = (b, \tilde{k}^j)$. By Rule 1, $g(m_i, \tilde{m}_{-i}) = g(\tilde{m}_i, \tilde{m}_{-i}) = b$. Since $a^i = f(R)$ and $b \neq f(R)$, then by semi-social-responsibility, $(m_i, \tilde{m}_{-i}) \succ_i^{f(R)} (\tilde{m}_i, \tilde{m}_{-i})$.

Case 2. For each $\tilde{k}^i \in N \backslash \{k^i\}$, $\tilde{m}_i = (f(R), \tilde{k}^i)$.

[13] Hagiwara, Yamamura, and Yamato (2018) propose the result of Proposition 5.1 concerning "social choice correspondences" in their Theorem (2). In the model with "partially honest" agents, similar results are established (Kimya, 2015; Lombardi and Yoshihara, 2019).

[14] The assumption that all agents are partially honest is also considered (see, e.g., Dutta and Sen, 2012; Kimya, 2015; Lombardi and Yoshihara, 2019).

When $|N| = 3$ and $|A| = 2$, for each message profile, Rule 1 applies. Then, for each $\tilde{m}_{-i} \in M_{-i}$, $g(m_i, \tilde{m}_{-i}) = g(\tilde{m}_i, \tilde{m}_{-i})$. Since $a^i = \tilde{a}^i = f(R)$, then for each $\tilde{m}_{-i} \in M_{-i}$, $(m_i, \tilde{m}_{-i}) \sim_i^{f(R)} (\tilde{m}_i, \tilde{m}_{-i})$.

In the other situations, we distinguish between two subcases concerning the preferences of agent i.

Subcase 2-1. There are $a, b \in A$ such that $a \; P_i \; b$.

There are $(\tilde{k}^1, \cdots, \tilde{k}^{i-1}, \tilde{k}^{i+1}, \cdots, \tilde{k}^n)$ and $\{\ell, \ell'\} \subset N$ such that agent $\ell \equiv (\Sigma_{j\in N\setminus\{i\}}\tilde{k}^j + k^i)(mod\ n) + 1$ reports a and agent $\tilde{\ell} \equiv (\Sigma_{j\in N\setminus\{i\}}\tilde{k}^j + \tilde{k}^i)(mod\ n) + 1(\neq \ell)$ reports b. Let $\tilde{m}_{-i} \in M_{-i}$ be messages satisfying these conditions and such that Rule 2 applies for both $(m_i, \tilde{m}_{-i})$ and $(\tilde{m}_i, \tilde{m}_{-i})$. By Rule 2, $g(m_i, \tilde{m}_{-i}) = a$ and $g(\tilde{m}_i, \tilde{m}_{-i}) = b$. Then, $(m_i, \tilde{m}_{-i}) \succ_i^{f(R)} (\tilde{m}_i, \tilde{m}_{-i})$.

Subcase 2-2. For each $a, b \in A$, $a \; I_i \; b$.

For each $\tilde{m}_{-i} \in M_{-i}$, $g(m_i, \tilde{m}_{-i}) \; I_i \; g(\tilde{m}_i, \tilde{m}_{-i})$. Since $a^i = \tilde{a}^i = f(R)$, then for each $\tilde{m}_{-i} \in M_{-i}$, $(m_i, \tilde{m}_{-i}) \sim_i^{f(R)} (\tilde{m}_i, \tilde{m}_{-i})$. □

There are the following two remarks:

Remark 5.1.[15] The HYY mechanism does not depend on the particular SCF f, but semi-socially-responsible preferences are defined with respect to f. On the other hand, in previous models (e.g., Jackson, Palfrey, and Srivastava, 1994; Kartik, Tercieux, and Holden, 2014; Tatamitani, 1993; Yamato, 1999), a mechanism doubly implementing the SCF f in Nash equilibria and undominated Nash equilibria depends on f, but preferences are defined independently of f.◊

Remark 5.2. The HYY mechanism is applicable to the case of at least four agents. In the case of only three agents, Hagiwara, Yamamura, and Yamato (2018) construct a **modified HYY mechanism**, $\Gamma^{mHYY} = (M, g)$. For each $i \in N$, the message space of agent i consists of $M_i = A \times \{0, 1\} \times N$, with generic element $m_i = (a^i, f^i, k^i)$. The

[15]I owe this remark to William Thomson.

outcome mapping $g : M \to A$ is defined as follows:

Rule 1: If there is $i \in N$ such that for each $j \neq i$, $m_j = (a, 0, k^j)$, then $g(m) = a$.

Rule 2: In all other cases, $g(m) = a^{i^*}$, where $i^* = (\Sigma_{i \in N} k^i)(mod\ n) + 1$. $\Diamond$

For the HYY mechanism, if at least one agent is not semi-socially-responsible, then the set of undominated Nash equilibrium outcomes may be smaller than the set of Nash equilibrium outcomes. This is illustrated by the following example.

Example 5.1. Let $N = \{1, 2, 3\}$, $A = \{a, b, c\}$, and $\mathcal{R} = \{R, R'\}$. Preferences are defined as follows:

R_1	R_2	R_3	R'_1	R'_2	R'_3
a	b	c	a	a	a
b, c	c	a	b	b	b
	a	b	c	c	c

Consider the following SCF. For each $i \in N$, each $R_i \in \mathcal{R}_i$, and each $a \in A$, let $B(R_i, a) \equiv k$ if outcome a is the k-th most preferred outcome at R_i.

Borda function, $f^{\mathbf{B}}$: For each $R \in \mathcal{R}$ and each $a \in A$,

$$\Sigma_{i \in N} B(R_i, f^{\mathbf{B}}(R)) < \Sigma_{i \in N} B(R_i, a).$$

We calculate that $f^{\mathbf{B}}(R) = c$ and $f^{\mathbf{B}}(R') = a$. Suppose that only agent 1 is not semi-socially-responsible: for each $\tilde{R} \in \mathcal{R}$, each pair $m_1, \tilde{m}_1 \in M_1$, and each $m_{-1} \in M_{-1}$, $g(m_1, m_{-1})\ \tilde{R}_1\ g(\tilde{m}_1, m_{-1})$ if and only if $(m_1, m_{-1}) \succsim_1^{f^{\mathbf{B}}(\tilde{R})} (\tilde{m}_1, m_{-1})$.

For each $i \in N$ and each $k^i \in N$, let $m_i = (c, k^i)$. Then, $(m_i)_{i \in N} \in NE(\Gamma, \succsim^{f^{\mathbf{B}}(R)})$. Actually, the message profiles at which each agent reports outcome c are only Nash equilibria of $(\Gamma, \succsim^{f^{\mathbf{B}}(R)})$. On the other hand, $m_1 = (c, k^1)$ is weakly dominated by $\tilde{m}_1 = (a, k^1)$ at $\succsim_1^{f^{\mathbf{B}}(R)}$. Therefore, $\{f^{\mathbf{B}}(R)\} = g(NE(\Gamma, \succsim^{f^{\mathbf{B}}(R)})) = \{c\}$, but $g(UNE(\Gamma, \succsim^{f^{\mathbf{B}}(R)})) = \emptyset$.[16]$\blacksquare$

[16]If we consider "social choice correspondences," instead of social choice functions, then our main result

5.4 Informational Requirements Necessary for the HYY Mechanism

In order to guarantee that which message profile is a Nash equilibrium is common knowledge among agents in the canonical mechanisms for Nash implementability, we usually need the assumption that all agents' preferences are common knowledge among agents. By contrast, in the HYY mechanism, even when preferences are not common knowledge among agents, the set of Nash equilibria might be common knowledge. The following example clarifies how the complete information assumption can be weakened.

Example 5.2.[17] Let $N = \{1, ..., n\}$ with $n \geq 3$, and let $A = \{a, b, c\}$. By abuse of notation, let abc denote the preference ordering $a\ P\ b\ P\ c$. For each $i \in \{1, 2\}$, let $\mathcal{R}_i = \{abc, acb, bca\}$, and for each $j \in N \backslash \{1, 2\}$, let $\mathcal{R}_j = \{abc\}$. By abuse of notation, an element of $\mathcal{R}$ is denoted by $(\alpha\beta\gamma, \delta\epsilon\varepsilon)$, because all agents other than agent 1 and agent 2 are essentially irrelevant. An SCF f is given as follows:

$R_1 \backslash R_2$	abc	acb	bca
abc	a	a	a
acb	a	a	a
bca	a	a	b

Note that f satisfies "no-veto-power."[18] An information partition is a partition of $\mathcal{R}$. The following left (resp. middle, right) table represents agent 1's (resp. agent 2's, for each $j \in N/\{1, 2\}$, agent j's) information partition $\mathcal{P}_1$ (resp. $\mathcal{P}_2$, $\mathcal{P}_j$) in terms of the most

does not hold. We investigate the same example except for $R_2 \in \mathcal{R}_2$ with $b\ P_2\ a\ I_2\ c$. We study the "Borda correspondence". For the definition of the Borda correspondence, see, e.g., Jackson, Palfrey, and Srivastava (1994). Then, even if all agents are semi-socially-responsible, the set of undominated Nash equilibrium outcomes is smaller than the set of Nash equilibrium outcomes.

[17] I owe this example to Hagiwara, Yamamura, and Yamato (2018).

[18] For the definition of *no-veto-power*, see Chapter 4.

preferred outcomes.[19]

$R_1\backslash R_2$	abc	acb	bca
abc	a	a	a
acb	a	a	a
bca	a	a	b

$R_1\backslash R_2$	abc	acb	bca
abc	a	a	a
acb	a	a	a
bca	a	a	b

$R_1\backslash R_2$	abc	acb	bca
abc	a	a	a
acb	a	a	a
bca	a	a	b

Let $E = \{(abc, abc), (abc, acb), (acb, abc), (acb, acb)\} \subset \mathcal{R}$ be an event. Suppose that the true preference profile is (abc, abc). The event E is common knowledge, although (abc, abc) is not common knowledge under these information partitions $\mathcal{P}_1, ..., \mathcal{P}_n$. We can easily check that for each pair $R, R' \in E$, $NE(\Gamma, \succsim^{f(R)})$ and $UNE(\Gamma, \succsim^{f(R)})$ are equal to $NE(\Gamma, \succsim^{f(R')})$ and $UNE(\Gamma, \succsim^{f(R')})$, respectively. Hence, as long as the event E is common knowledge, all agents can commonly know the set of Nash equilibria.■

In the HYY mechanism, the set of best response messages depends only on (1) the set of each agents' most preferred outcomes, (2) the set of socially desirable outcomes,[20] and (3) the set of semi-socially-responsible agents. Therefore, as long as (1) - (3) are common knowledge among agents, all agents deductively know which message profile is a Nash equilibrium. This advantage of the HYY mechanism can be a significant improvement if an SCF satisfies "tops-only."[21] In this case, if the set of the most preferred outcomes of each agent only is common knowledge, agents commonly deduce the socially desirable outcome, so that it can be also common knowledge. Especially, when there are a lot of

[19]Formally,

$$\mathcal{P}_1 = \{\{(abc, abc),(abc, acb)\}, \{(acb, abc),(acb, acb)\}, \{(bca, abc), (bca, acb)\}, \{(abc, bca)\}, \{(acb, bca)\}, \{(bca, bca)\}\},$$
$$\mathcal{P}_2 = \{\{(abc, abc),(acb, abc)\}, \{(abc, acb),(acb, acb)\}, \{(abc, bca), (acb, bca)\}, \{(bca, abc)\}, \{(bca, acb)\}, \{(bca, bca)\}\},$$

and for each $j \in N\backslash\{1, 2\}$,

$$\mathcal{P}_j = \{\{(abc, abc), (acb, abc), (abc, acb), (acb, acb)\}, \{(abc, bca), (acb, bca)\}, \{(bca, abc), (bca, acb)\}, \{(bca, bca)\}\}.$$

[20]If the event, $f^{-1}(f(R)) = \{R' \in \mathcal{R} : f(R) = f(R')\}$, is common knowledge among agents, then the socially desirable outcome $f(R)$ is also common knowledge among agents.

[21]Given $R_i \in \mathcal{R}_i$, let $t(R_i) = \{a \in A : aR_ib \text{ for each } b \in A\}$ be the set of top outcomes in A according to R_i. An SCF f satisfies *tops-only* if for each pair $R, R' \in \mathcal{R}$, if for each $i \in N$ $t(R_i) = t(R'_i)$, then $f(R) = f(R')$.

feasible outcomes, the use of the HYY mechanism can radically reduce the amount of information about other agents each agent has to know.

Chapter 6

Conclusion

We conclude this thesis by summarizing its contributions (Section 6.1) and discussing the following three remaining issues in the thesis (Section 6.2): (1) irrational choices; (2) repeated implementation; and (3) laboratory experiments.

6.1 Implications

In Chapter 2, we investigated the implementation problem under incomplete information and private values. We showed that an SCF is doubly implementable in dominant strategy equilibria and ex post equilibria if and only if it is *weakly securely-strategy-proof* (Theorem 2.1). This result involves showing that an SCF f is doubly implementable in dominant strategy equilibria and ex post equilibria if and only if it is doubly implemented in dominant strategy equilibria and ex post equilibria by the direct mechanism associated with f (Corollary 2.1). Therefore, for double implementability of an SCF in dominant strategy equilibria and ex post equilibria, it suffices to focus on one of simplest mechanisms, its associated direct mechanism. By our result, for a school choice problem (Abdulkadiroğlu and Sönmez, 2003) under incomplete information, the tentative acceptance rule is not doubly implementable in dominant strategy equilibria and ex post equilibria (Example 2.4). On the other hand, for a school choice problem under incomplete information and a condition on the set of preference profiles, the tentative acceptance rule is doubly implementable in dominant strategy equilibria and ex post equilibria (Example 2.5). These examples suggest that the "larger" the set of preference profiles is, the easier satisfying double implementability in dominant strategy equilibria and ex post equilibria is.

In Chapter 3, we considered the allocation problem of infinitely divisible resources with at least three agents. We showed that if there are at least three agents, Choose-Two-Bundles-and-Transpose implements the no-envy correspondence in Nash equilibria (Theorem 3.1). This result proposes the possibility of Nash implementation of the no-envy correspondence by means of a both simple and procedurally fair mechanism.

In Chapter 4, we considered the implementation problem with at least three agents and studied double implementability of SCCs in Nash equilibria and undominated Nash equilibria. We showed that if there are at least three agents out of which at least one agent is partially honest, then *weak no-veto-power* and *unanimity* together are sufficient for double implementability in Nash equilibria and undominated Nash equilibria (Theorem 4.1). In addition, we proved that if there are at least three agents out of which at least two agents are partially honest, then *unanimity* is sufficient for double implementability in Nash equilibria and undominated Nash equilibria (Theorem 4.2). Moreover, we showed that if there are at least three agents out of which at least one agent is partially honest and *unanimity* is satisfied, then *LY-condition* is necessary and sufficient for double implementability in Nash equilibria and undominated Nash equilibria (Theorem 4.3). By these results, we had several positive corollaries (see Section 4.4). Therefore, if the existence of some partially honest agent(s) is assumed, then much more SCCs are Nash implementable. However, for our results, complicated mechanisms are used. As in Hagiwara (2018) or Chapter 5, some simple mechanism(s) for double implementability in Nash equilibria and undominated Nash equilibria should be constructed.

In Chapter 5, we assumed that each agent is "semi-socially-responsible" and we investigated the possibility of double implementability of SCFs in Nash equilibria and undominated Nash equilibria by means of a simple and procedurally fair mechanism. We showed that if there are at least three agents and each agent is semi-socially-responsible with respect to a *unanimous* SCF, then the HYY mechanism doubly implements this SCF in Nash equilibria and undominated Nash equilibria (Theorem 5.1). For the HYY mechanism, if at least one agent is not semi-socially-responsible, then the set of undominated Nash equilibrium outcomes may be smaller than the set of Nash equilibrium outcomes (Example 5.1). This result suggests that if all agents want to report the socially desirable outcome to the planner as long as reporting a socially undesirable outcome induces an outcome which is at least less preferred, then most SCFs are doubly implementable in Nash equilibria and undominated Nash equilibria by means of a simple and procedurally fair mechanism, the HYY mechanism. However, for doubly implementability in Nash equilibria and undominated Nash equilibria, we may be able to construct a simple and procedurally fair mechanism which is not the HYY mechanism and to consider SCCs or the assumption that some agent is not semi-socially-responsible. These are open questions.

6.2 Further Research Topics

(1) Irrational choices:

In all chapters of this thesis, we assumed that each agent's choice is rational, in the sense of being consistent with the maximization of a context-independent preference. However, there is ample evidence on irrational choices in marketing, psychology, and behavioral economics. There are some classic examples such as temptation and self-control. In order to capture these examples, recent papers encode agents' choice correspondences instead of preferences. For this approach to irrationality, see Hurwicz (1986), Ray (2010), Korpela (2012), and de Clippel (2014). In addition, there are other ways to investigate implementation problems with irrational choices. Eliaz (2002) studies Nash implementation which is robust to the presence of "faulty" agents, where faulty agents may behave in any possible way.[1] Cabrales and Serrano (2011) investigate implementation problems under the behavioral assumption that agents adjust myopically their actions in the direction

[1]Ortner (2015) studies fault tolerant Nash implementability with "minimally honest" agents.

of better responses or best responses.[2] Renou and Schlag (2011) consider the problem of implementing SCCs in environments where agents have doubts about the rationality of their opponents.[3] Saran (2016) studies the implementation problem with bounded depths of rationality under complete information. de Clippel, Saran, and Serrano (2018) analyze the implementation problem with bounded depths of reasoning under incomplete information.[4]

de Clippel (2014) proposed a necessary condition and a sufficient condition for Nash implementation with irrational choices. However, a both necessary and sufficient condition has not been investigated. Therefore, providing a characterization is an open question.

(2) Repeated implementation:

In all chapters of this thesis, we considered one-shot implementation problems. However, a number of applications naturally fit several situations where the agents' preferences change over time in an uncertain way and the planner's objective is to repeatedly implement the same SCF for each possible preference profile. For example, in repeated auctions, the bidders' valuations over the objects could follow a stochastic process, and the planner wants to sell each object to the bidder with highest valuation. Lee and Sabourian (2011), Mezzetti and Renou (2017), and Āzacis and Vida (2019) investigate repeated implementation problems under complete information.[5] Lee and Sabourian (2011) provide a necessary and almost sufficient condition (*weak efficiency in the range of an SCF*) for infinitely repeated Nash implementation under some additional assumptions on the preferences. Mezzetti and Renou (2017) also propose a necessary and almost sufficient condition (*dynamic monotonicity*) for finitely or infinitely repeated Nash implementation.[6] Their results hold in both finite and infinite horizon problems irrespective

[2]For related literature, see Cabrales (1999), Mathevet (2010), Cabrales and Serrano (2012), Healy and Mathevet (2012), and Tumennasan (2013). For related experiments, see Chen and Plott (1996), Chen and Tang (1998), and Cabrales, Charness, and Corchón (2003).

[3]For implementation problems from the viewpoint of decision theory, see Bose and Renou (2014), Liu (2016), de Castro, Liu, and Yannelis (2017), and Guo and Yannelis (2017).

[4]For a related research, see de Clippel, Saran, and Serrano (2014).

[5]See also Chambers (2004), Kalai and Ledyard (1998), Lee and Sabourian (2015), and Renou and Tomala (2015).

[6]Mezzetti and Renou (2017) also investigate sufficiency of Nash repeated implementation when there

of the magnitude of the discount factor. Under the same model as Mezzetti and Renou (2017), Āzacis and Vida (2019) characterize *dynamic monotonicity* based on "Maskin monotonicity*." They also provide a characterization of SCFs that are repeatedly Nash implementable when there are at least three agents.

In contrast to the above three papers, Lee and Sabourian (2009, 2013) study repeated implementation problems under incomplete information. Lee and Sabourian (2009) provided sufficient conditions for repeated ex post implementation. However, they do not propose any necessary condition so that this is an open question.

For repeated implementation problems in the above papers, we assume that each agent is rational. Then, the repeated implementation problem when agents' choice may be irrational has not been investigated.

(3) Laboratory experiments:

In all chapters of this thesis, we studied implementation problems mainly from the theoretical viewpoint. However, even if an SCC is theoretically implemented by a mechanism, the mechanism may not work in practice. In order to investigate whether theoretical results are consistent with the observations in practical situations, a bunch of papers conducted experiments. We will see three points.

First, we usually assume that each agent maximizes his own preference whatever other agents get payoffs. Laboratory experiments observed that this assumption is not necessary satisfied. From the results of Gneezy (2005) and Hurkens and Kartik (2009), agents are one of two kinds: either an agent will never lie, or an agent will lie whenever he prefers the outcome obtained by lying over the outcome obtained by telling the truth.

Second, from the theoretical viewpoint, *strategy-proofness* is desirable as discussed in Chapters 1 and 2. However, laboratory experiments observed that this property is not necessarily enough to induce the socially desirable outcome. For pivotal-mechanism experiments, Attiyeh, Franciosi, and Isaac (2000) and Kawagoe and Mori (2001) observed that less than half of subjects did not adopt dominant strategies. For second-price-auction

are only two agents.

experiments, Kagel, Harstad, and Levin (1987), Kagel and Levin (1993), and Harstad (2000) observed that most bidders did not reveal true values.

Third, an appealing feature of iterative elimination of strictly dominated strategies is that it can be based on simple rationality assumptions. Abreu and Matsushima (1992b) propose a mechanism which virtually implements almost any SCF via iterative elimination of strictly dominated strategies. However, it has been argued that when many iterations of dominance are required, this solution concept is not compelling (Glazer and Rosenthal, 1992).[7] Then, Glazer and Rosenthal (1992) provide a modification of the Abreu-Matsushima mechanism for implementation via backward induction. Katok, Sefton, and Yavas (2002) report experimental results on the relative performance of mechanisms of Abreu and Matsushima (1992b) and Glazer and Rosenthal (1992).[8] Surprisingly, despite the above criticism, the performance of the mechanism of Abreu and Matsushima (1992b) is relatively better than that of Glazer and Rosenthal (1992) in their experiment.

The above three points suggest that we should not only study implementation theory but also conduct laboratory experiments to support the theoretical results. Even if the conducted experiments do not support the theoretical results, experimental results give us hints to modify the current theory of implementation, such as in Li (2017). The experiments for the mechanisms which are studied in this thesis should be conducted.

[7] For a response to Glazer and Rosenthal (1992), see Abreu and Matsushima (1992a).

[8] For another experiment on the Abreu-Matsushima mechanism, see Sefton and Yavas (1996).

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