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Authors	Kento Takagi, Ryo Kashima
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Kento Takagi and Ryo Kashima

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Department of
Mathematical and
Computing Sciences
Tokyo Institute of Technology

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On the Completeness and the Decidability of Strictly Monadic Second-Order Logic

Kento Takagi*and Ryo Kashima†

Department of Mathematical and Computing Science,
Tokyo Institute of Technology

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Abstract

Regarding monadic second-order logic, it is known that there are no deductive systems which are sound and complete w.r.t. standard semantics, although w.r.t. Henkin semantics, a standard second-order deductive system with full comprehension is sound and complete.

We show that as for strictly monadic second-order logic (SMSOL), which is the fragment of monadic second-order logic in which all predicate constants are unary and there are no function symbols, the standard deductive system is sound and complete w.r.t. standard semantics. This result is achieved by showing that in the case of SMSOL, the truth value of any formula in a faithful identity-standard Henkin structure is preserved when the structure is "standardized"; that is, the predicate domain is expanded into the set of all unary relations. In addition, we obtain a simpler proof of the decidability of SMSOL.

1 Introduction

For first-order logic, by Gödel's completeness theorem, there are deductive systems which are sound and complete with respect to standard semantics. It is also known that the validity problem for first-order logic is undecidable. However, for monadic first-order logic, the fragment of first-order logic in

*E-mail: takagi.k.am@m.titech.ac.jp
†E-mail: kashima@is.titech.ac.jp

which all predicate symbols are unary and no function symbols are allowed, the validity problem is decidable.

What about monadic second-order logic, the extension of first-order logic which allows quantification over unary (monadic) predicates? In the case of monadic second-order logic (or even worse full second-order logic), there are no deductive systems which are sound and complete with respect to standard semantics, although with respect to Henkin semantics, the standard deductive system, which is obtained from first-order predicate calculus by adding the rules for second-order quantification and full comprehension, is sound and complete. It is also known that the validity problem for monadic second-order logic w.r.t. standard semantics is undecidable. (For these results, see e.g. [2, 4].)

Then, what about the extension of monadic first-order logic which allows quantification over monadic predicates? Let us call this logic strictly monadic second-order logic, or SMSOL. It is known that the decidablity of the validity problem for SMSOL (w.r.t. standard semnatics) can be shown by the method of elimination of second-order quantifiers (see [1]). Thus, in the case of SMSOL, there is a deductive system which is sound and (weakly) complete w.r.t. standard sematics — we can obtain such a deductive system by choosing as axioms all the valid sentences. But then, there remains a natural question: is the standard decuctive system of SMSOL sound and complete w.r.t. standard semantics?

In this paper, we give a positive anser to this question: the standard deductive system of SMSOL is sound and (strongly) complete w.r.t. standard semantics. Note that, by Henkin completeness, this statement is equivalent to the one that in the case of SMSOL, standard semantics and Henkin semantics define the same logical consequence relation. In fact, we shall prove the latter of the two equivalent statements by showing that the truth values of all formulas are the same in a given faithful (identity-standard) Henkin structure and in the standard structure obtained from it by expanding the predicate domain into the set of all unary relations. Although it would also be possible to show the completeness by modifiying the above-mentioned proof of the decidability of SMSOL, our method provides a simpler way. And also, modifiying our method, we can obtain a new proof of the decidability of SMSOL.

This paper is organised as follows: In Section 2 and Section 3, we introduce the syntax and semantics of SMSOL respectively. (Those who want more detailed presentations for (not just strictly monadic but also full) second-order logic are referred to standard textbooks, like [4].) In Section 4, we give a proof of the completeness theorem along the lines mentioned above.

2 Syntax of SMSOL

A strictly monadic second-order language consists of the following symbols:

- Logical symbols: $\bot$, $\wedge$, $\vee$, $\rightarrow$, $\forall$ and $\exists$.
- Variables:
  - A countably infinite set of individual variables.
  - A countably infinite set of unary predicate variables.
- Constants:
  - A set $\mathrm{Const}^{\mathrm{i}}$ of individual constants.
  - A set $\mathrm{Const}^{\mathrm{p}}$ of unary predicate constants.

In what follows we will omit the adjective "unary" because we will treat only unary predicate variables and constants in this paper. We shall use x, y, u, and v as metavariables for individual variables; X, Y, and Z for predicate variables; c for individual constants; C for predicate constants.

A *term* is either an individual variable or an individual constant. (Note that our language have no function symbols.) We use t and s as metavariables for terms. An *atomic formula* is a logical constant $\bot$ or an expression of the form Xt or Ct. The *formulas* are defined by the following grammar:

$$\varphi ::= \alpha \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \forall x \varphi \mid \exists x \varphi \mid \forall X \varphi \mid \exists X \varphi\,,$$

where α is an atomic formula. We shall use $\equiv$ to stand for syntactical identity. The other logical connective are defined as usual: $\neg\varphi \equiv \varphi \rightarrow \bot$ and $\varphi \leftrightarrow \psi \equiv (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$.

The notion of free and bound variables are defined as usual. Let $\mathrm{FV}^{\mathrm{i}}(\varphi)$ (resp. $\mathrm{FV}^{\mathrm{p}}(\varphi)$) denotes the set of free individual (resp. predicate) variables of a formula φ. Similary, $\mathrm{Cn}^{\mathrm{i}}(\varphi)$ (resp. $\mathrm{Cn}^{\mathrm{p}}(\varphi)$) denotes the set of individual (resp. predicate) constants occuring in φ.

Here we shortly discuss a deduction system for strictly monadic second-order logic. As usual, it is obtained from the first-order predicate calculus by adding the rules for quantification over predicate variables and the *comprehension axioms*, that is, the formulas of the form $\exists X \forall x (Xx \leftrightarrow \varphi)$ where $X \notin \mathrm{FV}^{\mathrm{p}}(\varphi)$. There are several formulations of the system. If we choose the natural deduction style system, then the additional rules and axioms to first-order predicate calculus are described as follows:

$$\frac{}{\Gamma \vdash \exists X \forall x (Xx \leftrightarrow \varphi)} \ (X \notin \mathrm{FV}^{\mathrm{p}}(\varphi))$$

$$(\forall^2\mathrm{I}) \ \frac{\Gamma \vdash \varphi[X := Y]}{\Gamma \vdash \forall X \varphi} \ (Y \notin \mathrm{FV}^{\mathrm{p}}(\Gamma, \forall X \varphi))) \qquad (\forall^2\mathrm{E}) \ \frac{\Gamma \vdash \forall X \varphi}{\Gamma \vdash \varphi[X := T]}$$

$$(\exists^2\mathrm{E}) \ \frac{\Gamma \vdash \exists X \varphi \quad \Gamma, \varphi[X := Y] \vdash \psi}{\Gamma \vdash \psi} \ (Y \notin \mathrm{FV}^{\mathrm{p}}(\Gamma, \psi, \exists X \varphi)) \qquad (\exists^2\mathrm{I}) \ \frac{\Gamma \vdash \varphi[X := T]}{\Gamma \vdash \exists X \varphi}$$

Here T is any predicate variable or any predicate constant. $\varphi[X := T]$ means the capture-avoiding substitution of T for X in φ.

3 Semantics of SMSOL

3.1 Henkin Structures

Now we introduce *Henkin semantics* of strictly monadic second-order logic. A *Henkin structure* is a tuple $\mathcal{M} = \langle \mathcal{D}^{\mathrm{i}}, \mathcal{D}^{\mathrm{p}}, \mathcal{I} \rangle$ in which

- $\mathcal{D}^{\mathrm{i}}$ is a nonempty set, called *the individual domain* of $\mathcal{M}$;
- $\mathcal{D}^{\mathrm{p}}$ is a nonempty set of subsets of $\mathcal{D}^{\mathrm{i}}$, called *the predicate domain* of $\mathcal{M}$;
- $\mathcal{I}$ is a map that assigns an element of $\mathcal{D}^{\mathrm{i}}$ to each individual constant and a subset of $\mathcal{D}^{\mathrm{i}}$ to each predicate constant, called *the interpretation function* of $\mathcal{M}$.

An $\mathcal{M}$-*assignment* is a map that assigns an element of $\mathcal{D}^{\mathrm{i}}$ to each individual variable and an element of $\mathcal{D}^{\mathrm{p}}$ to each predicate variable. For an $\mathcal{M}$-assignment $\mathcal{V}$, individual variable x, and $a \in \mathcal{D}^{\mathrm{i}}$, we write $\mathcal{V}\frac{a}{x}$ for the assignment which maps x to a and is equal to $\mathcal{V}$ everywhere else. For an predicate variable X and $A \in \mathcal{D}^{\mathrm{i}}$, $\mathcal{V}\frac{A}{X}$ is defined similarly.

Let $\mathcal{M}$ be a Henkin structure and $\mathcal{V}$ an $\mathcal{M}$-assignment. Then, for each term t, the *denotation* of t under $\mathcal{M}$ with $\mathcal{V}$ (written $t^{\langle \mathcal{M}, \mathcal{V} \rangle}$) is defined as usual; namely, $t^{\langle \mathcal{M}, \mathcal{V} \rangle} = \mathcal{V}(t)$ if t is an individual variable and $t^{\langle \mathcal{M}, \mathcal{V} \rangle} = \mathcal{I}(t)$ if t is an individual constant.

The notion of a formula φ being *satisfied* (or *true*) in a Henkin structure $\mathcal{M}$ with an $\mathcal{M}$-assignment $\mathcal{V}$ (notation: $\mathcal{M}, \mathcal{V} \vDash \varphi$) is defined by induction as usual:

$$
\begin{array}{lll}
\mathcal{M}, \mathcal{V} \models Xt & \text{iff} & t^{\langle \mathcal{M}, \mathcal{V} \rangle} \in \mathcal{V}(X); \\
\mathcal{M}, \mathcal{V} \models Ct & \text{iff} & t^{\langle \mathcal{M}, \mathcal{V} \rangle} \in \mathcal{I}(C); \\
\mathcal{M}, \mathcal{V} \not\models \bot; & & \\
\mathcal{M}, \mathcal{V} \models \varphi \wedge \psi & \text{iff} & \mathcal{M}, \mathcal{V} \models \varphi \text{ and } \mathcal{M}, \mathcal{V} \models \psi; \\
\mathcal{M}, \mathcal{V} \models \varphi \vee \psi & \text{iff} & \mathcal{M}, \mathcal{V} \models \varphi \text{ or } \mathcal{M}, \mathcal{V} \models \psi; \\
\mathcal{M}, \mathcal{V} \models \varphi \to \psi & \text{iff} & \mathcal{M}, \mathcal{V} \not\models \varphi \text{ or } \mathcal{M}, \mathcal{V} \models \psi; \\
\mathcal{M}, \mathcal{V} \models \forall x \varphi & \text{iff} & \text{for all } a \in \mathcal{D}^{\mathrm{i}},\ \mathcal{M}, \mathcal{V}^{\frac{a}{x}} \models \varphi; \\
\mathcal{M}, \mathcal{V} \models \exists x \varphi & \text{iff} & \text{there exists some } a \in \mathcal{D}^{\mathrm{i}} \text{ such that } \mathcal{M}, \mathcal{V}^{\frac{a}{x}} \models \varphi; \\
\mathcal{M}, \mathcal{V} \models \forall X \varphi & \text{iff} & \text{for all } A \in \mathcal{D}^{\mathrm{p}},\ \mathcal{M}, \mathcal{V}^{\frac{A}{X}} \models \varphi; \\
\mathcal{M}, \mathcal{V} \models \exists X \psi & \text{iff} & \text{there exists some } A \in \mathcal{D}^{\mathrm{p}} \text{ such that } \mathcal{M}, \mathcal{V}^{\frac{A}{X}} \models \varphi.
\end{array}
$$

We say that a formula φ is *valid* in a Henkin structure $\mathcal{M}$ (notation: $\mathcal{M} \models \varphi$) if $\mathcal{M}, \mathcal{V} \models \varphi$ for all $\mathcal{M}$-assignments $\mathcal{V}$. For a set Γ of formulas, Γ is valid in $\mathcal{M}$ (notation: $\mathcal{M} \models \Gamma$) if every formula in Γ is valid in $\mathcal{M}$.

3.2 Definability and Comprehension Axioms

Let $\mathcal{M} = \langle \mathcal{D}^{\mathrm{i}}, \mathcal{D}^{\mathrm{p}}, \mathcal{I} \rangle$ be a Henkin structure. We say that a set A of individuals of $\mathcal{M}$ is *definable* in $\mathcal{M}$ if there are some formula φ, some individual variable x, and some $\mathcal{M}$-assignment $\mathcal{V}$ such that $A = \{a \in \mathcal{D}^{\mathrm{i}} \mid \mathcal{M}, \mathcal{V}^{\frac{a}{x}} \models \varphi\}$.

Let $\mathcal{M} = \langle \mathcal{D}^{\mathrm{i}}, \mathcal{D}^{\mathrm{p}}, \mathcal{I} \rangle$ be a Henkin structure and $\mathcal{V}$ an $\mathcal{M}$-assignment. Consider a comprehension axiom $\exists X \forall x (Xx \leftrightarrow \varphi)$ with $X \notin \mathrm{FV}(\varphi)$. We can see that $\mathcal{M}$ with $\mathcal{V}$ satisfies this comprehension axiom if and only if the set $\{a \in \mathcal{D}^{\mathrm{i}} \mid \mathcal{M}, \mathcal{V}^{\frac{a}{x}} \models \varphi\}$ is in $\mathcal{D}^{\mathrm{p}}$. As an easy consequence, all the comprehension axiom are valid in $\mathcal{M}$ if and only if $\mathcal{D}^{\mathrm{p}}$ is the set of all sets that are definable in $\mathcal{M}$. If either of these equivalent conditions holds, then we say that $\mathcal{M}$ is *faithful*. From this definition, the following proposition can be obtaned easily:

Proposition 3.1. *Let $\mathcal{D} = \langle \mathcal{D}^{\mathrm{i}}, \mathcal{D}^{\mathrm{p}}, \mathcal{I} \rangle$ be a faithful Henkin stucture. Then the followings hold.*

(i) $\emptyset \in \mathcal{D}^{\mathrm{p}}$ and $\mathcal{D}^{\mathrm{i}} \in \mathcal{D}^{\mathrm{p}}$.

(ii) If $A, B \in \mathcal{D}^{\mathrm{p}}$ then, $A \cup B$, $A \cap B$, and $A \setminus B \in \mathcal{D}^{\mathrm{p}}$.

The logical consequence relation over the class of faithful Henkin structures is denoted by $\models_{\mathrm{H}}$; that is, for a set Γ of formulas and a formula φ, we write $\Gamma \models_{\mathrm{H}} \varphi$ if $\mathcal{M}, \mathcal{V} \models \varphi$ for all faithful Henkin structures $\mathcal{M}$ and all $\mathcal{M}$-assigment $\mathcal{V}$ such that $\mathcal{M}, \mathcal{V} \models \gamma$ for all $\gamma \in \Gamma$.

With respect to the class of faithful Henkin structures, the deductive system defined in Section 2 is sound and (strongly) complete (see e.g. [4]).

Theorem 3.2. *For any set Γ of formulas and any formula φ, $\Gamma \vdash \varphi$ iff $\Gamma \vDash_{\mathrm{H}} \varphi$.*

A standard structure is a Henkin structure such that the predicate domain is the set of all subsets of the individual domain. Note that a standard structure is also a faithful Henkin strucuture. The logical consequence relation over the class of standard structures is denoted by $\vDash_{\mathrm{S}}$; that is, for a set Γ of formulas and a formula φ, we write $\Gamma \vDash_{\mathrm{S}} \varphi$ if $\mathcal{M}, \mathcal{V} \vDash \varphi$ for all standard structures $\mathcal{M}$ and all $\mathcal{M}$-assignment $\mathcal{V}$ such that $\mathcal{M}, \mathcal{V} \vDash \gamma$ for all $\gamma \in \Gamma$.

3.3 Equality

It is known that when defining the logical consequence relation over the class of faithful Henkin structures, namely $\vDash_{\mathrm{H}}$, we may actually consider only the class of faithful Henkin structures in which an equation $x = y$ can be defined by the formula $\forall X(Xx \to Xy)$, called *Leibniz equality.* (This fact is mentioned in [3, 4].) A faithful Henkin structure with this property is said to be identity-standard.

Here we shortly describe only the definitions and results concerning the notion of identity that will be needed in the later discussions. In Section A, a detailed explanation is described for those who are unfamiliar with these ideas.

Definition 3.3. Let $\mathcal{M} = \langle\langle \mathcal{D}^{\mathrm{i}}, \mathcal{D}^{\mathrm{p}}\rangle, \mathcal{I}\rangle$ be a faithful Henkin structure. We define the binary relation $\doteq_{\mathcal{M}}$ on $\mathcal{D}^{\mathrm{i}}$ as follows:

$$a \doteq_{\mathcal{M}} b \text{ if and only if for all } A \in \mathcal{D}^{\mathrm{p}},\ a \in A \text{ implies } b \in A\,.$$

Definition 3.4. Let $\mathcal{M} = \langle\langle \mathcal{D}^{\mathrm{i}}, \mathcal{D}^{\mathrm{p}}\rangle, \mathcal{I}\rangle$ be a faithful Henkin structure. We say that $\mathcal{M}$ is *identity-standard* if for all $a, b \in \mathcal{D}^{\mathrm{i}}$, $a \doteq_{\mathcal{M}} b$ implies $a = b$.

Note that if $\mathcal{M}$ is a standard structure, we can take $\{a\}$ as A in Definition 3.3. Thus, standard structures are identity-standard.

Theorem 3.5. *Let Γ be any set of formulas and φ any formula. Then, $\Gamma \vDash_{\mathrm{H}} \varphi$ holds if and only if $\mathcal{M}, \mathcal{V} \vDash \varphi$ for all faithful identity-standard Henkin structures $\mathcal{M}$ and $\mathcal{M}$-assignmets $\mathcal{V}$ such that $\mathcal{M}, \mathcal{V} \vDash \gamma$ for all $\gamma \in \Gamma$.*

For a faithful identity-standard Henkin strucutre $\mathcal{M} = \langle \mathcal{D}^{\mathrm{i}}, \mathcal{D}^{\mathrm{p}}, \mathcal{I}\rangle$, the singleton of an individual $b \in \mathcal{D}^{\mathrm{i}}$ can be defined by Leibniz equality; that is, for any $\mathcal{M}$-assignment $\mathcal{V}$ with $\mathcal{V}(y) = b$,

$$\{b\} = \{a \in \mathcal{D}^{\mathrm{i}} \mid \mathcal{M}, \mathcal{V}^{\frac{a}{x}} \vDash \forall X(Xx \to Xy)\}\,.$$

From this observation and Proposition 3.1, we obtain the following propositon. This is a key property to prove the main result.

Proposition 3.6. *For any faithful identity-standard Henkin structure $\mathcal{M} = \langle\langle\mathcal{D}^{\mathrm{i}}, \mathcal{D}^{\mathrm{p}}\rangle, \mathcal{I}\rangle$, any finite subset of $\mathcal{D}^{\mathrm{i}}$ belongs to $\mathcal{D}^{\mathrm{p}}$.*

4 Completeness of SMSOL

In this section, for simplicity, we only consider languages which have no constant symbols. But we can easily see that if a language has some constant symbols, everything in this section applies with trivial modifications. Note that, as far as languages have no constants, we may regard a Henkin structure as just a pair of an individual domain and a predicate domain.

4.1 Strategy

Our final goal is the completeness theorem with respect to standard semantics:

Theorem 4.1. *For any set Γ of formulas and any formula φ, $\Gamma \vdash \varphi$ iff $\Gamma \vDash_{\mathrm{S}} \varphi$.*

By theorem 3.2, this is equivalent to:

Theorem 4.2. *For any set Γ of formulas and any formula φ, $\Gamma \vDash_{\mathrm{H}} \varphi$ iff $\Gamma \vDash_{\mathrm{S}} \varphi$.*

We will prove the latter theorem. The left to right part is clear, since a standard structure is also a faithful Henkin structure. For the right to left part, by Theorem 3.5, we see that it suffices to show the following proposition.

Proposition 4.3. *Let $\mathcal{D} = \langle\mathcal{D}^{\mathrm{i}}, \mathcal{D}^{\mathrm{p}}\rangle$ be a faithful identity-standard Henkin structure and $\mathcal{W}$ be a $\mathcal{D}$-assignment. Let $\overline{\mathcal{D}}$ denote the standard structure $\langle\mathcal{D}^{\mathrm{i}}, \mathcal{P}(\mathcal{D}^{\mathrm{i}})\rangle$. Then, for all formulas φ, $\overline{\mathcal{D}}, \mathcal{W} \vDash \varphi$ iff $\mathcal{D}, \mathcal{W} \vDash \varphi$.*

We can observe that this proposition cannot be proved by simple induction on φ. For, when trying to prove $\mathcal{D}, \mathcal{W} \vDash \varphi$ from $\overline{\mathcal{D}}, \mathcal{W} \vDash \varphi$ in the case $\varphi \equiv \exists X \psi$, it may happen that the set $A \in \mathcal{P}(\mathcal{D}^{\mathrm{i}})$ which satisfies $\overline{\mathcal{D}}, \mathcal{W}^{\frac{A}{X}} \vDash \psi$ cannot be an element of $\mathcal{D}^{\mathrm{p}}$. (Of course, a similar difficulty arises in the case $\varphi \equiv \forall X \psi$.)

So, instead of proving this proposition, which mentions only the case in which the same assignment are given to $\overline{\mathcal{D}}$ and $\mathcal{D}$, we will propose a stronger claim, Lemma 4.6, which gives a sufficient condition for the truth value of φ in $\overline{\mathcal{D}}$ with $\mathcal{V}$ and the truth value of φ in $\mathcal{D}$ with $\mathcal{W}$ to be the same.

For the case that φ is only first-order, the condition is known, and this fact is used to prove the decidability of monadic first-order logic (see e.g. [2]). Indeed, our main lemma, Lemma 4.6, is an extension of the first-order version.

4.2 Notations

Now we prepare some notations for the main lemma. We will use the notation $|A|$ to mean the cardinality of a set A. For a set of predicate variables S, *S-index* is a mapping from S into $\{0,1\}$. Note that for disjoint sets S and T of predicate variables, the union of an S-index and a T-index is an $S \cup T$-index.

Let $\mathcal{D} = \langle \mathcal{D}^{\mathrm{i}}, \mathcal{D}^{\mathrm{p}} \rangle$ be a faithful Henkin structure and $\mathcal{V}$ a $\mathcal{D}$-assignment, and let S be a set of predicate variables and I an S-index. We define the *I-room* $\langle \mathcal{D}, \mathcal{V} \rangle_I$ of $\mathcal{D}$ with $\mathcal{V}$ as follows:

$$\langle \mathcal{D}, \mathcal{V} \rangle_I := \Big(\bigcap_{X \in I^{-1}(1)} \mathcal{V}(X) \Big) \cap \Big(\bigcap_{X \in I^{-1}(0)} \mathcal{D}^{\mathrm{i}} \setminus \mathcal{V}(X) \Big) .$$

In other words, $\langle \mathcal{D}, \mathcal{V} \rangle_I$ denotes the set of individuals a such that $\mathcal{D}, \mathcal{V}^{\frac{a}{x}} \models Xx$ if $I(X) = 1$, and $\mathcal{D}, \mathcal{V}^{\frac{a}{x}} \models \neg Xx$ if $I(X) = 0$.

The following lemma follows from Proposition 3.1.

Lemma 4.4. *If S is finite, then $\langle \mathcal{D}, \mathcal{V} \rangle_I \in \mathcal{D}^{\mathrm{p}}$.*

For each formula φ, we define $\mathrm{q}(\varphi)$ (resp. $\mathrm{Q}(\varphi)$) to be the maximum number of nested occurences of first-order (resp. second-order) quantifiers in φ. For example, $\mathrm{Q}(\varphi)$ is defined by

$$\begin{aligned} \mathrm{Q}(\alpha) &= 0 && \text{if } \alpha \text{ is atomic;} \\ \mathrm{Q}(\psi \circ \theta) &= \max(\mathrm{Q}(\psi), \mathrm{Q}(\theta)); \\ \mathrm{Q}(\triangledown x \psi) &= \mathrm{Q}(\psi); \\ \mathrm{Q}(\triangledown X \psi) &= 1 + \mathrm{Q}(\psi), \end{aligned}$$

where $\circ \in \{\wedge, \vee, \rightarrow\}$ and $\triangledown \in \{\forall, \exists\}$. Then we define $\mu(\varphi)$ as follows:

$$\mu(\varphi) = 2^{\mathrm{Q}(\varphi)} \cdot (|\,\mathrm{FV}^{\mathrm{i}}(\varphi)| + \mathrm{q}(\varphi)) .$$

Then immediately we have the following lemma:

Lemma 4.5. *If ψ is a subformula of φ, then it holds that $|\,\mathrm{FV}^{\mathrm{i}}(\psi)| + \mathrm{q}(\psi) \leq |\,\mathrm{FV}^{\mathrm{i}}(\varphi)| + \mathrm{q}(\varphi)$ and $\mu(\psi) \leq \mu(\varphi)$.*

4.3 Main Lemma

Now we are ready for the main lemma. We state the conditions for the truth value of φ in $\overline{\mathcal{D}}$ with $\mathcal{V}$ and that in $\mathcal{D}$ with $\mathcal{W}$ to be the same.

For a formula φ and two pairs $\langle \mathcal{E}, \mathcal{V} \rangle$ and $\langle \mathcal{D}, \mathcal{W} \rangle$ of faithful Henkin structures and assignments on them (we do not assume that $\mathcal{E}$ is necessarily $\overline{\mathcal{D}}$), we define three conditions $\mathfrak{P}_{\mathrm{pred}}$, $\mathfrak{P}_{=}$, and $\mathfrak{P}_{\mathrm{size}}$ as follows:

- $\mathfrak{P}_{\mathrm{pred}}(\varphi, \langle \mathcal{E}, \mathcal{V} \rangle, \langle \mathcal{D}, \mathcal{W} \rangle)$ holds if and only if for any $u \in \mathrm{FV}^{\mathrm{i}}(\varphi)$, the $\mathrm{FV}^{\mathrm{p}}(\varphi)$-index I with $\mathcal{V}(u) \in \langle \mathcal{E}, \mathcal{V} \rangle_I$ is the same as that with $\mathcal{W}(u) \in \langle \mathcal{D}, \mathcal{W} \rangle_I$. That is, $\mathfrak{P}_{\mathrm{pred}}(\varphi, \langle \mathcal{E}, \mathcal{V} \rangle, \langle \mathcal{D}, \mathcal{W} \rangle)$ holds if and only if $\mathcal{E}, \mathcal{V} \vDash Xu$ is equivalent to $\mathcal{D}, \mathcal{W} \vDash Xu$ for any $X \in \mathrm{FV}^{\mathrm{p}}(\varphi)$.

- $\mathfrak{P}_{=}(\varphi, \langle \mathcal{E}, \mathcal{V} \rangle, \langle \mathcal{D}, \mathcal{W} \rangle)$ holds if and only if for any $u, v \in \mathrm{FV}^{\mathrm{i}}(\varphi)$, $\mathcal{V}(u) = \mathcal{V}(v)$ iff $\mathcal{W}(u) = \mathcal{W}(v)$.

- $\mathfrak{P}_{\mathrm{size}}(\varphi, \langle \mathcal{E}, \mathcal{V} \rangle, \langle \mathcal{D}, \mathcal{W} \rangle)$ holds if and only if for any $\mathrm{FV}^{\mathrm{p}}(\varphi)$-index I, either (α) $|\langle \mathcal{E}, \mathcal{V} \rangle_I| = |\langle \mathcal{D}, \mathcal{W} \rangle_I|$ or (β) $|\langle \mathcal{E}, \mathcal{V} \rangle_I|, |\langle \mathcal{D}, \mathcal{W} \rangle_I| \geq \mu(\varphi)$.

Note that $\varphi, \langle \overline{\mathcal{D}}, \mathcal{W} \rangle, \langle \mathcal{D}, \mathcal{W} \rangle$ satisfies these three conditions. So the next lemma entails Proposition 4.3 and all we have to do is to show the next lemma. We will use the conjunction & like this: "$\mathfrak{P}_{\mathrm{pred}}$ & $\mathfrak{P}_{=}(\varphi, \langle \mathcal{E}, \mathcal{V} \rangle, \langle \mathcal{D}, \mathcal{W} \rangle)$ holds" means "$\mathfrak{P}_{\mathrm{pred}}(\varphi, \langle \mathcal{E}, \mathcal{V} \rangle, \langle \mathcal{D}, \mathcal{W} \rangle)$ holds and $\mathfrak{P}_{=}(\varphi, \langle \mathcal{E}, \mathcal{V} \rangle, \langle \mathcal{D}, \mathcal{W} \rangle)$ holds."

Lemma 4.6. *Let $\mathcal{D}$ be a faithful identity-standard Henkin structure. Then for any formula φ, any $\overline{\mathcal{D}}$-assignment $\mathcal{V}$, and any $\mathcal{D}$-assignment $\mathcal{W}$ such that $\mathfrak{P}_{\mathrm{pred}}$ & $\mathfrak{P}_{=}$ & $\mathfrak{P}_{\mathrm{size}}(\varphi, \langle \overline{\mathcal{D}}, \mathcal{V} \rangle, \langle \mathcal{D}, \mathcal{W} \rangle)$ holds, $\overline{\mathcal{D}}, \mathcal{V} \vDash \varphi$ if and only if $\mathcal{D}, \mathcal{W} \vDash \varphi$.*

Proof. In what follows we shall use Proposition 3.1 and Proposition 3.6 without references. Also sometimes we will use the notation $A \sqcup B$ to denote $A \cup B$ when A and B are disjoint and $\bigsqcup \mathcal{A}$ similarly. The proof is by induction on φ. The case that $\varphi \equiv Xx$ follows from the condition $\mathfrak{P}_{\mathrm{pred}}(\varphi, \langle \overline{\mathcal{D}}, \mathcal{V} \rangle, \langle \mathcal{D}, \mathcal{W} \rangle)$. The case $\varphi \equiv \bot$ is trivial.

[The case that $\varphi \equiv \psi_1 \circ \psi_2$, where $\circ \in \{\wedge, \vee, \rightarrow\}$]

By the induction hypothesis, it suffices to show that, for each $i \in \{1, 2\}$, $\mathfrak{P}_{\mathrm{pred}}$ & $\mathfrak{P}_{=}$ & $\mathfrak{P}_{\mathrm{size}}(\psi_i, \langle \overline{\mathcal{D}}, \mathcal{V} \rangle, \langle \mathcal{D}, \mathcal{W} \rangle)$ holds. Fix an $i \in \{1, 2\}$. The conditions $\mathfrak{P}_{\mathrm{pred}}(\psi_i, \langle \overline{\mathcal{D}}, \mathcal{V} \rangle, \langle \mathcal{D}, \mathcal{W} \rangle)$ and $\mathfrak{P}_{=}(\psi_i, \langle \overline{\mathcal{D}}, \mathcal{V} \rangle, \langle \mathcal{D}, \mathcal{W} \rangle)$ follow from the assumptions $\mathfrak{P}_{\mathrm{pred}}(\varphi, \langle \overline{\mathcal{D}}, \mathcal{V} \rangle, \langle \mathcal{D}, \mathcal{W} \rangle)$ and $\mathfrak{P}_{=}(\varphi, \langle \overline{\mathcal{D}}, \mathcal{V} \rangle, \langle \mathcal{D}, \mathcal{W} \rangle)$ respectively. We will consider the condition $\mathfrak{P}_{\mathrm{size}}(\psi_i, \langle \overline{\mathcal{D}}, \mathcal{V} \rangle, \langle \mathcal{D}, \mathcal{W} \rangle)$. Put $S = \mathrm{FV}^{\mathrm{p}}(\varphi) \setminus \mathrm{FV}^{\mathrm{p}}(\psi_i)$, and fix an arbitrary $\mathrm{FV}^{\mathrm{p}}(\psi_i)$-index J. Since the union of an $\mathrm{FV}^{\mathrm{p}}(\psi_i)$-index and an S-index is an $\mathrm{FV}^{\mathrm{p}}(\varphi)$-index, the following equations hold:

$$\langle \overline{\mathcal{D}}, \mathcal{V} \rangle_J = \bigsqcup_{K:\ S\text{-index}} \langle \overline{\mathcal{D}}, \mathcal{V} \rangle_{J \cup K}, \qquad \langle \mathcal{D}, \mathcal{W} \rangle_J = \bigsqcup_{K:\ S\text{-index}} \langle \mathcal{D}, \mathcal{W} \rangle_{J \cup K}\ .$$

On the other hand, by the assumption $\mathfrak{P}_{\mathrm{size}}(\varphi, \langle \overline{\mathcal{D}}, \mathcal{V} \rangle, \langle \mathcal{D}, \mathcal{W} \rangle)$, either of the following two holds for each S-index K: (α) $|\langle \overline{\mathcal{D}}, \mathcal{V} \rangle_{J \cup K}| = |\langle \mathcal{D}, \mathcal{W} \rangle_{J \cup K}|$; (β) $|\langle \overline{\mathcal{D}}, \mathcal{V} \rangle_{J \cup K}|, |\langle \mathcal{D}, \mathcal{W} \rangle_{J \cup K}| \geq \mu(\varphi)$. Hence, either

(i) for each S-index K, $|\langle\overline{\mathcal{D}},\mathcal{V}\rangle_{J\cup K}| = |\langle\mathcal{D},\mathcal{W}\rangle_{J\cup K}|$;

(ii) or there exists some S-index K_0 such that $|\langle\overline{\mathcal{D}},\mathcal{V}\rangle_{J\cup K_0}|, |\langle\mathcal{D},\mathcal{W}\rangle_{J\cup K_0}| \geq \mu(\varphi)$.

In the case (i),

$$|\langle\overline{\mathcal{D}},\mathcal{V}\rangle_J| = \sum_{K:\ S\text{-index}} |\langle\overline{\mathcal{D}},\mathcal{V}\rangle_{J\cup K}| = \sum_{K:\ S\text{-index}} |\langle\mathcal{D},\mathcal{W}\rangle_{J\cup K}| = |\langle\mathcal{D},\mathcal{W}\rangle_J|\,.$$

In the case (ii), by Lemma 4.5,

$$|\langle\overline{\mathcal{D}},\mathcal{V}\rangle_J| = \sum_{K:\ S\text{-index}} |\langle\overline{\mathcal{D}},\mathcal{V}\rangle_{J\cup K}| \geq |\langle\overline{\mathcal{D}},\mathcal{V}\rangle_{J\cup K_0}| \geq \mu(\varphi) \geq \mu(\psi_i)\,,$$

and

$$|\langle\mathcal{D},\mathcal{W}\rangle_J| = \sum_{K:\ S\text{-index}} |\langle\mathcal{D},\mathcal{W}\rangle_{J\cup K}| \geq |\langle\mathcal{D},\mathcal{W}\rangle_{J\cup K_0}| \geq \mu(\varphi) \geq \mu(\psi_i)\,.$$

Since J is an arbitrary $\mathrm{FV}^{\mathrm{p}}(\psi_i)$-index, $\mathfrak{P}_{\mathrm{size}}(\psi_i, \langle\overline{\mathcal{D}},\mathcal{V}\rangle, \langle\mathcal{D},\mathcal{W}\rangle)$ holds.

[The case that $\varphi \equiv \exists x\psi$]

Since the case in which $x \notin \mathrm{FV}^{\mathrm{i}}(\psi)$ is trivial, we only consider the case in which $x \in \mathrm{FV}^{\mathrm{i}}(\psi)$. By the induction hypothesis, in order to prove $\overline{\mathcal{D}},\mathcal{V} \vDash \varphi \implies \mathcal{D},\mathcal{W} \vDash \varphi$, it suffices to show that for any $a \in \mathcal{D}^{\mathrm{i}}$, there exists some $b \in \mathcal{D}^{\mathrm{i}}$ such that $\mathfrak{P}_{\mathrm{pred}}$ & $\mathfrak{P}_{=}$ & $\mathfrak{P}_{\mathrm{size}}(\psi, \langle\overline{\mathcal{D}},\mathcal{V}^{\frac{a}{x}}\rangle, \langle\mathcal{D},\mathcal{W}^{\frac{b}{x}}\rangle)$ holds. Since the proof of the right to left part is symmetric, we only describe the proof of the left to right part. Let a be an arbitrary element of $\mathcal{D}^{\mathrm{i}}$. We will show that there is a b with $\mathfrak{P}_{\mathrm{pred}}$ & $\mathfrak{P}_{=}$ & $\mathfrak{P}_{\mathrm{size}}(\psi, \langle\overline{\mathcal{D}},\mathcal{V}^{\frac{a}{x}}\rangle, \langle\mathcal{D},\mathcal{W}^{\frac{b}{x}}\rangle)$. The three conditions can be converted as follows:

- Since $\mathrm{FV}^{\mathrm{i}}(\psi) = \mathrm{FV}^{\mathrm{i}}(\varphi) \cup \{x\}$ and $\mathrm{FV}^{\mathrm{p}}(\psi) = \mathrm{FV}^{\mathrm{p}}(\varphi)$, $\mathfrak{P}_{\mathrm{pred}}(\psi, \langle\overline{\mathcal{D}},\mathcal{V}^{\frac{a}{x}}\rangle, \langle\mathcal{D},\mathcal{W}^{\frac{b}{x}}\rangle)$ holds if and only if $\mathfrak{P}_{\mathrm{pred}}(\varphi, \langle\overline{\mathcal{D}},\mathcal{V}\rangle, \langle\mathcal{D},\mathcal{W}\rangle)$ holds and the $\mathrm{FV}^{\mathrm{p}}(\varphi)$-index I such that $a \in \langle\overline{\mathcal{D}},\mathcal{V}\rangle_I$ is the same as the $\mathrm{FV}^{\mathrm{p}}(\varphi)$-index I such that $b \in \langle\mathcal{D},\mathcal{W}\rangle_I$.
- Since $\mathrm{FV}^{\mathrm{i}}(\psi) = \mathrm{FV}^{\mathrm{i}}(\varphi) \cup \{x\}$, $\mathfrak{P}_{=}(\psi, \langle\overline{\mathcal{D}},\mathcal{V}^{\frac{a}{x}}\rangle, \langle\mathcal{D},\mathcal{W}^{\frac{b}{x}}\rangle)$ holds if and only if $\mathfrak{P}_{=}(\varphi, \langle\overline{\mathcal{D}},\mathcal{V}\rangle, \langle\mathcal{D},\mathcal{W}\rangle)$ holds and $a = \mathcal{V}(u)$ iff $b = \mathcal{W}(u)$ for all $u \in \mathrm{FV}^{\mathrm{i}}(\varphi)$.
- Since $\mu(\varphi) \geq \mu(\psi)$ by Lemma 4.5 and $\mathrm{FV}^{\mathrm{p}}(\psi) = \mathrm{FV}^{\mathrm{p}}(\varphi)$, $\mathfrak{P}_{\mathrm{size}}(\psi, \langle\overline{\mathcal{D}},\mathcal{V}^{\frac{a}{x}}\rangle, \langle\mathcal{D},\mathcal{W}^{\frac{b}{x}}\rangle)$ holds if $\mathfrak{P}_{\mathrm{size}}(\varphi, \langle\overline{\mathcal{D}},\mathcal{V}\rangle, \langle\mathcal{D},\mathcal{W}\rangle)$ holds.

Since $\mathfrak{P}_{\rm pred}$ & $\mathfrak{P}_{=}$ & $\mathfrak{P}_{\rm size}(\varphi, \langle\overline{\mathcal{D}}, \mathcal{V}\rangle, \langle\mathcal{D}, \mathcal{W}\rangle)$ has been assumed, it suffices to show that there is some $b \in \mathcal{D}^{\rm i}$ such that

(i) the $\mathrm{FV}^{\rm p}(\varphi)$-index I with $a \in \langle\overline{\mathcal{D}}, \mathcal{V}\rangle_I$ coincides with the $\mathrm{FV}^{\rm p}(\varphi)$-index I with $b \in \langle\mathcal{D}, \mathcal{W}\rangle_I$,

(ii) and for any $u \in \mathrm{FV}^{\rm i}(\varphi)$, $a = \mathcal{V}(u)$ iff $b = \mathcal{W}(u)$.

Put $\mathrm{FV}^{\rm i}(\varphi) = \{x_1, \ldots, x_l\}$. We divide into two cases.

CASE (I): $a = \mathcal{V}(x_i)$ for some $i = 1, \ldots, l$. Take $b = \mathcal{W}(x_i)$. Then (i) and (ii) hold.

CASE (II): $a \notin \mathcal{V}(\{x_1, \ldots, x_l\})$. Let I_0 be the $\mathrm{FV}^{\rm p}(\varphi)$-index with $a \in \langle\overline{\mathcal{D}}, \mathcal{V}\rangle_{I_0}$. Then, by $\mathfrak{P}_{\rm pred}$ & $\mathfrak{P}_{=}(\varphi, \langle\overline{\mathcal{D}}, \mathcal{V}\rangle, \langle\mathcal{D}, \mathcal{W}\rangle)$,

$$|\langle\overline{\mathcal{D}}, \mathcal{V}\rangle_{I_0} \cap \mathcal{V}(\{x_1, \ldots, x_l\})| = |\langle\mathcal{D}, \mathcal{W}\rangle_{I_0} \cap \mathcal{W}(\{x_1, \ldots, x_l\})| \tag{1}$$

holds. By $\mathfrak{P}_{\rm size}(\varphi, \langle\overline{\mathcal{D}}, \mathcal{V}\rangle, \langle\mathcal{D}, \mathcal{W}\rangle)$, either ($\alpha$) $|\langle\overline{\mathcal{D}}, \mathcal{V}\rangle_{I_0}| = |\langle\mathcal{D}, \mathcal{W}\rangle_{I_0}|$ or (β) $|\langle\overline{\mathcal{D}}, \mathcal{V}\rangle_{I_0}|, |\langle\mathcal{D}, \mathcal{W}\rangle_{I_0}| \geq \mu(\varphi)$ holds. In the case (α), we have $|\langle\overline{\mathcal{D}}, \mathcal{V}\rangle_{I_0} \setminus \mathcal{V}(\{x_1, \ldots, x_l\})| = |\langle\mathcal{D}, \mathcal{W}\rangle_{I_0} \setminus \mathcal{W}(\{x_1, \ldots, x_l\})|$ by (1), and so $|\langle\mathcal{D}, \mathcal{W}\rangle_{I_0} \setminus \mathcal{W}(\{x_1, \ldots, x_l\})| \geq 1$ follows from $a \in \langle\overline{\mathcal{D}}, \mathcal{V}\rangle_{I_0} \setminus \mathcal{V}(\{x_1, \ldots, x_l\})$. In the case ($\beta$), since $\varphi \equiv \exists x\psi$,

$$\mu(\varphi) = 2^{\mathrm{Q}(\varphi)} \cdot (|\,\mathrm{FV}^{\rm i}(\varphi)| + \mathrm{q}(\varphi)) \geq |\,\mathrm{FV}^{\rm i}(\varphi)| + \mathrm{q}(\varphi) \geq |\,\mathrm{FV}^{\rm i}(\varphi)| + 1\,.$$

Hence,

$$|\langle\mathcal{D}, \mathcal{W}\rangle_{I_0} \setminus \mathcal{W}(\{x_1, \ldots, x_l\})| \geq \mu(\varphi) - |\,\mathrm{FV}^{\rm i}(\varphi)| \geq 1\,.$$

In both cases, we have $|\langle\mathcal{D}, \mathcal{W}\rangle_{I_0} \setminus \mathcal{W}(\{x_1, \ldots, x_l\})| \geq 1$, so that we can take b from $\langle\mathcal{D}, \mathcal{W}\rangle_{I_0} \setminus \mathcal{W}(\{x_1, \ldots, x_l\})$, and then (i) and (ii) hold.

[The case that $\varphi \equiv \forall x\psi$]

This case can be treated similarly to the case $\varphi \equiv \exists x\psi$.

[The case that $\varphi \equiv \exists X\psi$]

Since the case in which $X \notin \mathrm{FV}^{\rm p}(\psi)$ is trivial, we will consider only the case in which $X \in \mathrm{FV}^{\rm p}(\psi)$. In order to prove $\overline{\mathcal{D}}, \mathcal{V} \models \varphi \implies \mathcal{D}, \mathcal{W} \models \varphi$, it suffices to show that for an given $A \in \mathcal{P}(\mathcal{D}^{\rm i})$, there exists some $B \in \mathcal{D}^{\rm p}$ such that $\mathfrak{P}_{\rm pred}$ & $\mathfrak{P}_{=}$ & $\mathfrak{P}_{\rm size}(\psi, \langle\overline{\mathcal{D}}, \mathcal{V}\frac{A}{X}\rangle, \langle\mathcal{D}, \mathcal{W}\frac{B}{X}\rangle)$ holds. Similarly, in order to prove $\langle\overline{\mathcal{D}}, \mathcal{V}\rangle \models \varphi \impliedby \langle\mathcal{D}, \mathcal{W}\rangle \models \varphi$, it suffices to show that for an given $B \in \mathcal{D}^{\rm p}$, there exists some $A \in \mathcal{P}(\mathcal{D}^{\rm i})$ such that $\mathfrak{P}_{\rm pred}$ & $\mathfrak{P}_{=}$ & $\mathfrak{P}_{\rm size}(\psi, \langle\overline{\mathcal{D}}, \mathcal{V}\frac{A}{X}\rangle, \langle\mathcal{D}, \mathcal{W}\frac{B}{X}\rangle)$ holds. Contrary to the case that $\varphi \equiv \exists x\psi$, these two proofs are not symmetric. In the former case, B is required to belong to $\mathcal{D}^{\rm p}$, so that more careful consideration is needed.

We will describe a proof for the left to right part. The converse can be shown by a similar but simpler argument. (Of course, it could also be shown in exactly the same way as the former case, although then unnecessary arguments would be involved.)

Suppose an $A \in \mathcal{P}(\mathcal{D}^{\mathrm{i}})$ is given arbitrarily. We will show that there exists some $B \in \mathcal{D}^{\mathrm{p}}$ such that $\mathfrak{P}_{\mathrm{pred}}$ & $\mathfrak{P}_{=}$ & $\mathfrak{P}_{\mathrm{size}}(\psi, \langle \overline{\mathcal{D}}, \mathcal{V}\frac{A}{X}\rangle, \langle \mathcal{D}, \mathcal{W}\frac{B}{X}\rangle)$ holds. The three conditions can be converted as follows:

- Since $\mathrm{FV}^{\mathrm{p}}(\psi) = \mathrm{FV}^{\mathrm{p}}(\varphi) \cup \{X\}$ and $\mathrm{FV}^{\mathrm{i}}(\varphi) = \mathrm{FV}^{\mathrm{i}}(\psi)$, $\mathfrak{P}_{\mathrm{pred}}(\psi, \langle \overline{\mathcal{D}}, \mathcal{V}\frac{A}{X}\rangle, \langle \mathcal{D}, \mathcal{W}\frac{B}{X}\rangle)$ holds if and only if $\mathfrak{P}_{\mathrm{pred}}(\varphi, \langle \overline{\mathcal{D}}, \mathcal{V}\rangle, \langle \mathcal{D}, \mathcal{W}\rangle)$ holds and $\mathcal{V}(u) \in A$ iff $\mathcal{W}(u) \in B$ for all $u \in \mathrm{FV}^{\mathrm{i}}(\varphi)$.

- Since $\mathrm{FV}^{\mathrm{i}}(\psi) = \mathrm{FV}^{\mathrm{i}}(\varphi)$, $\mathfrak{P}_{=}(\psi, \langle \overline{\mathcal{D}}, \mathcal{V}\frac{A}{X}\rangle, \langle \mathcal{D}, \mathcal{W}\frac{B}{X}\rangle)$ holds if and only if $\mathfrak{P}_{=}(\varphi, \langle \overline{\mathcal{D}}, \mathcal{V}\rangle, \langle \mathcal{D}, \mathcal{W}\rangle)$ holds.

- Since $\mathrm{FV}^{\mathrm{p}}(\psi) = \mathrm{FV}^{\mathrm{p}}(\varphi) \cup \{X\}$, $\mathfrak{P}_{\mathrm{size}}(\psi, \langle \overline{\mathcal{D}}, \mathcal{V}\frac{A}{X}\rangle, \langle \mathcal{D}, \mathcal{W}\frac{B}{X}\rangle)$ holds if for any $\mathrm{FV}^{\mathrm{p}}(\varphi)$-index I, both of the following two hold:
  - $|\langle \overline{\mathcal{D}}, \mathcal{V}\rangle_I \cap A| = |\langle \mathcal{D}, \mathcal{W}\rangle_I \cap B|$ or $|\langle \overline{\mathcal{D}}, \mathcal{V}\rangle_I \cap A|, |\langle \mathcal{D}, \mathcal{W}\rangle_I \cap B| \geq \mu(\psi)$;
  - $|\langle \overline{\mathcal{D}}, \mathcal{V}\rangle_I \setminus A| = |\langle \mathcal{D}, \mathcal{W}\rangle_I \setminus B|$ or $|\langle \overline{\mathcal{D}}, \mathcal{V}\rangle_I \setminus A|, |\langle \mathcal{D}, \mathcal{W}\rangle_I \setminus B| \geq \mu(\psi)$.

Since $\mathfrak{P}_{\mathrm{pred}}$ & $\mathfrak{P}_{=}$ & $\mathfrak{P}_{\mathrm{size}}(\varphi, \langle \overline{\mathcal{D}}, \mathcal{V}\rangle, \langle \mathcal{D}, \mathcal{W}\rangle)$ has been assumed, it suffices to show that there is some $B \in \mathcal{D}^{\mathrm{p}}$ such that

(i) for any $u \in \mathrm{FV}^{\mathrm{i}}(\varphi)$, $\mathcal{V}(u) \in A$ iff $\mathcal{W}(u) \in B$,

(ii) and for any $\mathrm{FV}^{\mathrm{p}}(\varphi)$-index I, both of the following two hold:

- $|\langle \overline{\mathcal{D}}, \mathcal{V}\rangle_I \cap A| = |\langle \mathcal{D}, \mathcal{W}\rangle_I \cap B|$ or $|\langle \overline{\mathcal{D}}, \mathcal{V}\rangle_I \cap A|, |\langle \mathcal{D}, \mathcal{W}\rangle_I \cap B| \geq \mu(\psi)$;
- $|\langle \overline{\mathcal{D}}, \mathcal{V}\rangle_I \setminus A| = |\langle \mathcal{D}, \mathcal{W}\rangle_I \setminus B|$ or $|\langle \overline{\mathcal{D}}, \mathcal{V}\rangle_I \setminus A|, |\langle \mathcal{D}, \mathcal{W}\rangle_I \setminus B| \geq \mu(\psi)$.

For each $\mathrm{FV}^{\mathrm{p}}(\varphi)$-index I, let us denote $\langle \overline{\mathcal{D}}, \mathcal{V}\rangle_I \cap A$ by A_I. Our strategy is as follows: instead of trying to obtain $B \in \mathcal{D}^{\mathrm{p}}$ directly, we will try to make an appropriate B_I for each $\mathrm{FV}^{\mathrm{p}}(\varphi)$-index I, which is intended to be the I-room part of B, and finally put $B = \bigcup\{B_I \mid I\text{: } \mathrm{FV}^{\mathrm{p}}(\varphi)\text{-index}\}$. By $\mathfrak{P}_{\mathrm{pred}}(\varphi, \langle \overline{\mathcal{D}}, \mathcal{V}\rangle, \langle \mathcal{D}, \mathcal{W}\rangle)$,

$$\{v \in \mathrm{FV}^{\mathrm{i}}(\varphi) \mid \mathcal{V}(v) \in \langle \overline{\mathcal{D}}, \mathcal{V}\rangle_I\} = \{v \in \mathrm{FV}^{\mathrm{i}}(\varphi) \mid \mathcal{W}(v) \in \langle \mathcal{D}, \mathcal{W}\rangle_I\}$$

for each $\mathrm{FV}^{\mathrm{p}}(\varphi)$-index I. Denote this set by V_I for each $\mathrm{FV}^{\mathrm{p}}(\varphi)$-index I. Suppose a $B_I \subseteq \langle \mathcal{D}, \mathcal{W}\rangle_I$ is given for each $\mathrm{FV}^{\mathrm{p}}(\varphi)$-index I and all the B_I's satisfy the following two conditions:

$(\mathrm{i})_I$ $\mathcal{V}(u) \in A_I$ iff $\mathcal{W}(u) \in B_I$ for any $u \in V_I(\varphi)$;

$(\mathrm{ii})_I$ both of the following two hold:

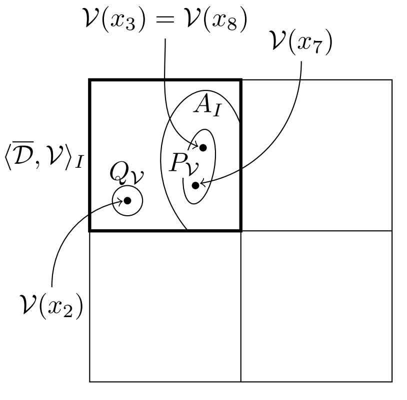
- $|A_I| = |B_I|$ or $|A_I|, |B_I| \geq \mu(\psi)$;
- $|\langle \overline{\mathcal{D}}, \mathcal{V} \rangle_I \setminus A_I| = |\langle \mathcal{D}, \mathcal{W} \rangle_I \setminus B_I|$ or $|\langle \overline{\mathcal{D}}, \mathcal{V} \rangle_I \setminus A_I|, |\langle \mathcal{D}, \mathcal{W} \rangle_I \setminus B_I| \geq \mu(\psi)$.

Then, when putting $B = \bigcup\{B_I \mid I\colon \mathrm{FV}^{\mathrm{p}}(\varphi)\text{-index}\}$, B satisfies (i) and (ii). In addition, if $B_I \in \mathcal{D}^{\mathrm{p}}$ for all I, then $B \in \mathcal{D}^{\mathrm{p}}$. Hence, if we can obtain $B_I \in \mathcal{D}^{\mathrm{p}}$ that satisfies $(\mathrm{i})_I$ and $(\mathrm{ii})_I$ for each $\mathrm{FV}^{\mathrm{p}}(\varphi)$-index I, then the proof is finished. Fix an $\mathrm{FV}^{\mathrm{p}}(\varphi)$-index I arbitrarily. We will prove that there exists some $B_I \subseteq \langle \mathcal{D}, \mathcal{W} \rangle_I$ which belongs to $\mathcal{D}^{\mathrm{p}}$ and satisfies $(\mathrm{i})_I$ and $(\mathrm{ii})_I$.

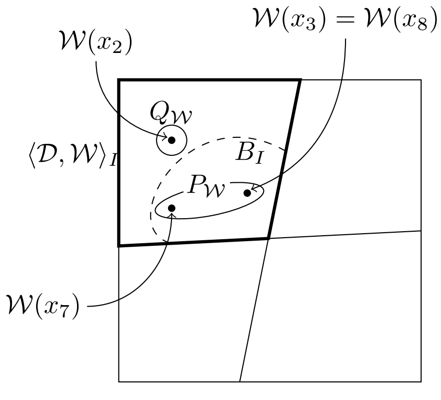
Before the proof, let us give names for certain parts of $\langle \overline{\mathcal{D}}, \mathcal{V} \rangle_I$ and $\langle \mathcal{D}, \mathcal{W} \rangle_I$ for clarity of the proof. We put

$$\begin{aligned} P_{\mathcal{V}} &:= \mathcal{V}(V_I \cap \mathcal{V}^{-1}(A_I)), & Q_{\mathcal{V}} &:= \mathcal{V}(V_I \setminus \mathcal{V}^{-1}(A_I)), & R_{\mathcal{V}} &:= \langle \overline{\mathcal{D}}, \mathcal{V} \rangle_I \setminus \mathcal{V}(V_I), \\ P_{\mathcal{W}} &:= \mathcal{W}(V_I \cap \mathcal{V}^{-1}(A_I)), & Q_{\mathcal{W}} &:= \mathcal{W}(V_I \setminus \mathcal{V}^{-1}(A_I)), & R_{\mathcal{W}} &:= \langle \mathcal{D}, \mathcal{W} \rangle_I \setminus \mathcal{W}(V_I). \end{aligned}$$

Note $\langle \overline{\mathcal{D}}, \mathcal{V} \rangle_I = P_{\mathcal{V}} \sqcup Q_{\mathcal{V}} \sqcup R_{\mathcal{V}}$ and $\langle \mathcal{D}, \mathcal{W} \rangle_I = P_{\mathcal{W}} \sqcup Q_{\mathcal{W}} \sqcup R_{\mathcal{W}}$. Look at the picture below to see what these sets represent. (The situation is as follows: both of the left and right outermost squares represent $\mathcal{D}^{\mathrm{i}}$; these left and right outermost squares are respectively partitioned by $\langle \overline{\mathcal{D}}, \mathcal{V} \rangle_J$'s and $\langle \mathcal{D}, \mathcal{W} \rangle_J$'s for $\mathrm{FV}^{\mathrm{p}}(\varphi)$-indices J; in particular, the rooms surrounded by thick lines at the left and right respectively represents $\langle \overline{\mathcal{D}}, \mathcal{V} \rangle_I$ and $\langle \mathcal{D}, \mathcal{W} \rangle_I$; V_I is $\{x_2, x_3, x_7, x_8\}$.)



$\mathcal{D}^{\mathrm{i}}$, partitioned by $\langle \overline{\mathcal{D}}, \mathcal{V} \rangle_J$'s



$\mathcal{D}^{\mathrm{i}}$, partitioned by $\langle \mathcal{D}, \mathcal{W} \rangle_J$'s

The picture also visualizes how we should make an appropriate B_I for the given A_I. Note that if we make B_I by adding some elements of $R_{\mathcal{W}}$ to $P_{\mathcal{W}}$, then the condition $(\mathrm{i})_I$ is satisfied. Indeed, in what follows, we will often make B_I such way.

Now we return to the proof of the existance of the desired B_I. Since $\mathfrak{P}_{\mathrm{size}}(\varphi, \langle\overline{\mathcal{D}}, \mathcal{V}\rangle, \langle\mathcal{D}, \mathcal{W}\rangle)$ is assumed, we can divide into two cases (I) $|\langle\overline{\mathcal{D}}, \mathcal{V}\rangle_I| = |\langle\mathcal{D}, \mathcal{W}\rangle_I|$ and (II) $|\langle\overline{\mathcal{D}}, \mathcal{V}\rangle_I|, |\langle\mathcal{D}, \mathcal{W}\rangle_I| \geq \mu(\varphi)$.

Case (I): $|\langle\overline{\mathcal{D}}, \mathcal{V}\rangle_I| = |\langle\mathcal{D}, \mathcal{W}\rangle_I|$. We divide into two subcases.

Subcase (I-1): A_I or $\langle\overline{\mathcal{D}}, \mathcal{V}\rangle_I \setminus A_I$ is finite. It can be seen that there exists a B_I with $(\mathrm{i})_I$ and $|B_I| = |A_I|$ as follows. By $\mathfrak{P}_=(\varphi, \langle\overline{\mathcal{D}}, \mathcal{V}\rangle, \langle\mathcal{D}, \mathcal{W}\rangle)$, we have $|P_\mathcal{V}| = |P_\mathcal{W}|$ and $|Q_\mathcal{V}| = |Q_\mathcal{W}|$, and hence $|R_\mathcal{V}| = |R_\mathcal{W}|$, too. Hence there are three bijections $f\colon P_\mathcal{V} \to P_\mathcal{W}$, $g\colon Q_\mathcal{V} \to Q_\mathcal{W}$, and $h\colon R_\mathcal{V} \to R_\mathcal{W}$. Put $e := f \sqcup g \sqcup h$. Then e is a bijection from $\langle\overline{\mathcal{D}}, \mathcal{V}\rangle_I$ onto $\langle\mathcal{D}, \mathcal{W}\rangle_I$. If we take $e(A_I)$ as B_I, then $(\mathrm{i})_I$ and $|B_I| = |A_I|$ are satisfied.

Since besides $|B_I| = |A_I|$ holds, $|\langle\overline{\mathcal{D}}, \mathcal{V}\rangle_I| = |\langle\mathcal{D}, \mathcal{W}\rangle_I|$ has been assumed, it follows that $|\langle\overline{\mathcal{D}}, \mathcal{V}\rangle_I \setminus A_I| = |\langle\mathcal{D}, W\rangle_I \setminus B_I|$. Thus we have $(\mathrm{ii})_I$.

The remaining task is to check if $B_I \in \mathcal{D}^{\mathrm{p}}$. When A_I is finite, then so is B_I, and hence $B_I \in \mathcal{D}^{\mathrm{p}}$. When $\langle\overline{\mathcal{D}}, \mathcal{V}\rangle_I \setminus A_I$ is finite, then so is $\langle\mathcal{D}, \mathcal{W}\rangle_I \setminus B_I$, and hence $\langle\mathcal{D}, \mathcal{W}\rangle_I \setminus B_I \in \mathcal{D}^{\mathrm{p}}$. In addition, since $\langle\mathcal{D}, \mathcal{W}\rangle_I \in \mathcal{D}^{\mathrm{p}}$ by lemma 4.4, it follows that $B_I = \langle\mathcal{D}, \mathcal{W}\rangle_I \setminus (\langle\mathcal{D}, \mathcal{W}\rangle_I \setminus B_I) \in \mathcal{D}^{\mathrm{p}}$. Thus $B_I \in \mathcal{D}^{\mathrm{p}}$ in both cases.

Sucase (I-2): A_I and $\langle\overline{\mathcal{D}}, \mathcal{V}\rangle_I \setminus A_I$ are infinite. Now $|\langle\overline{\mathcal{D}}, \mathcal{V}\rangle_I| = |\langle\mathcal{D}, \mathcal{W}\rangle_I|$ by the assumption of Case (I) and $\langle\overline{\mathcal{D}}, \mathcal{V}\rangle_I$ is infinite by the assumption of Case (I-2), so that $\langle\mathcal{D}, \mathcal{W}\rangle_I$ is infinite. Since, in addition, $\mathcal{W}(V_I)$ is finite, $R_\mathcal{W}$ is infinite. Take as B_I a set obtained by adding $\mu(\psi) - |P_\mathcal{W}|$ elements of $R_\mathcal{W}$ to $P_\mathcal{W}$. Then $(\mathrm{i})_I$ and $|B_I| = \mu(\psi)$ hold. Since A_I, $\langle\overline{\mathcal{D}}, \mathcal{V}\rangle_I \setminus A_I$, and $\langle\mathcal{D}, \mathcal{W}\rangle_I \setminus B_I$ are all infinite and $|B_I| = \mu(\psi)$, $(\mathrm{ii})_I$ holds. As B_I is finite, $B_I \in \mathcal{D}^{\mathrm{p}}$.

Case (II): $|\langle\overline{\mathcal{D}}, \mathcal{V}\rangle_I|, |\langle\mathcal{D}, \mathcal{W}\rangle_I| \geq \mu(\varphi)$. By Lemma 4.5, we have $|\,\mathrm{FV}^{\mathrm{i}}(\varphi)| + \mathrm{q}(\varphi) \geq |\,\mathrm{FV}^{\mathrm{i}}(\psi)| + \mathrm{q}(\psi)$. Thus we have

$$\begin{aligned}\mu(\varphi) &= 2^{\mathrm{Q}(\varphi)} \cdot (|\,\mathrm{FV}^{\mathrm{i}}(\varphi)| + \mathrm{q}(\varphi))\\ &= 2^{\mathrm{Q}(\psi)+1} \cdot (|\,\mathrm{FV}^{\mathrm{i}}(\varphi)| + \mathrm{q}(\varphi))\\ &\geq 2^{\mathrm{Q}(\psi)+1} \cdot (|\,\mathrm{FV}^{\mathrm{i}}(\psi) + \mathrm{q}(\psi)|)\\ &= 2\mu(\psi).\end{aligned}$$

Hence $|\langle\overline{\mathcal{D}},\mathcal{V}\rangle_I|, |\langle\mathcal{D},\mathcal{W}\rangle_I| \geq 2\mu(\psi)$. Thus it holds that

$$\begin{aligned} |R_{\mathcal{W}}| &= |\langle\mathcal{D},\mathcal{W}\rangle_I \setminus \mathcal{W}(V_I)| \\ &\geq 2\mu(\psi) - |\operatorname{FV}^{\mathrm{i}}(\varphi)| \\ &= 2\mu(\psi) - |\operatorname{FV}^{\mathrm{i}}(\psi)| \\ &\geq 2\mu(\psi) - \mu(\psi) \\ &= \mu(\psi). \end{aligned} \tag{2}$$

Since $\langle\overline{\mathcal{D}},\mathcal{V}\rangle_I = A_I \sqcup (\langle\overline{\mathcal{D}},\mathcal{V}\rangle_I \setminus A_I)$, either $|A_I| \geq \mu(\psi)$ or $|\langle\overline{\mathcal{D}},\mathcal{V}\rangle_I \setminus A_I| \geq \mu(\psi)$ holds. We divide into three subcases.

Subcase (II-1): $|A_I| < \mu(\psi)$ and $|\langle\overline{\mathcal{D}},\mathcal{V}\rangle_I \setminus A_I| \geq \mu(\psi)$. By (2) and the assumption $|A_I| < \mu(\psi)$, $R_{\mathcal{W}}$ has more than $|A_I|$ elements. Take as B_I a set obtained by adding $|A_I| - |P_{\mathcal{W}}|$ elements of $R_{\mathcal{W}}$ to $P_{\mathcal{W}}$. Then $(\mathrm{i})_I$ and $|B_I| = |A_I|$ hold. Since, besides $|B_I| = |A_I|$ and $|\langle\overline{\mathcal{D}},\mathcal{V}\rangle_I \setminus A_I| \geq \mu(\psi)$, it holds that

$$|\langle\mathcal{D},\mathcal{W}\rangle_I \setminus B_I| = |\langle\mathcal{D},\mathcal{W}\rangle_I| - |B_I| = |\langle\mathcal{D},\mathcal{W}\rangle_I| - |A_I| > 2\mu(\psi) - \mu(\psi) = \mu(\psi)\,,$$

$(\mathrm{ii})_I$ holds. By $|B_I| = |A_I| \leq \mu(\psi)$, $B_I \in \mathcal{D}^{\mathrm{p}}$.

Subcase (II-2): $|A_I| \geq \mu(\psi)$ and $|\langle\overline{\mathcal{D}},\mathcal{V}\rangle_I \setminus A_I| < \mu(\psi)$. By (2) and the assumption $|\langle\overline{\mathcal{D}},\mathcal{V}\rangle_I \setminus A_I| < \mu(\psi)$, $R_{\mathcal{W}}$ has more than $|\langle\overline{\mathcal{D}},\mathcal{V}\rangle_I \setminus A_I|$ elements. Let C_I be a set obtained by adding $|\langle\overline{\mathcal{D}},\mathcal{V}\rangle_I \setminus A_I| - |Q_{\mathcal{W}}|$ elements of R_W to $Q_{\mathcal{W}}$, and put $B_I = \langle\mathcal{D},\mathcal{W}\rangle_I \setminus C_I$. Then $(\mathrm{i})_I$ holds. Furthermore, we have $|\langle\mathcal{D},\mathcal{W}\rangle_I \setminus B_I| = |C_I| = |\langle\overline{\mathcal{D}},\mathcal{V}\rangle_I \setminus A_I|$, $A_I \geq \mu(\psi)$, and

$$|B_I| = |\langle\mathcal{D},\mathcal{W}\rangle_I| - |C_I| = |\langle\mathcal{D},\mathcal{W}\rangle_I| - |\langle\overline{\mathcal{D}},\mathcal{V}\rangle_I \setminus A_I| > 2\mu(\psi) - \mu(\psi) = \mu(\psi),$$

so that $(\mathrm{ii})_I$ holds.

Our final task is to check if $B_I \in \mathcal{D}^{\mathrm{p}}$. By $|C_I| = |\langle\overline{\mathcal{D}},\mathcal{V}\rangle_I \setminus A_I| < \mu(\psi)$, we have $C_I \in \mathcal{D}^{\mathrm{p}}$. Since, in addition, $\langle\mathcal{D},\mathcal{W}\rangle_I \in \mathcal{D}^{\mathrm{p}}$ holds by lemma 4.4, we have $B_I = \langle\mathcal{D},\mathcal{W}\rangle_I \setminus C_I \in \mathcal{D}^{\mathrm{p}}$.

Subcase (II-3): $|A_I| \geq \mu(\psi)$ and $|\langle\overline{\mathcal{D}},\mathcal{V}\rangle_I \setminus A_I| \geq \mu(\psi)$. From (2), we can take as B_I a set obtained by adding $\mu(\psi) - |P_{\mathcal{W}}|$ elements of $R_{\mathcal{W}}$ to $P_{\mathcal{W}}$. Then $(\mathrm{i})_I$ and $|B_I| = \mu(\psi)$ hold. By $|A_I| \geq \mu(\psi)$, $|B_I| = \mu(\psi)$, $|\langle\overline{\mathcal{D}},\mathcal{V}\rangle_I \setminus A_I| \geq \mu(\psi)$, and $|\langle\mathcal{D},\mathcal{W}\rangle_I \setminus B_I| \geq 2\mu(\psi) - \mu(\psi) = \mu(\psi)$, $(\mathrm{ii})_I$ holds. Since B_I is finite, $B_I \in \mathcal{D}^{\mathrm{p}}$.

[The case that $\varphi \equiv \forall X \psi$]

This case can be treated similarly to the case $\varphi \equiv \exists X \psi$. □

4.4 Decidability of SMSOL

It is known that the validity problem for SMSOL (w.r.t. standard semantics) is decidable by the method of elimination of second-order quantifiers (see [1]). In this subsection, we will see the arguments so far yield a simpler proof of the decidability. First, the following lemma can be shown in exactly the same way as Lemma 4.6. (The proof can even be simplified because now the both structures are standard.)

Lemma 4.7. *Let $\mathcal{D}$ and $\mathcal{E}$ be standard structures. Then for any formula φ, any $\mathcal{D}$-assignment $\mathcal{V}$, and any $\mathcal{E}$-assignment $\mathcal{W}$ such that $\mathfrak{P}_{\text{pred}}$ & $\mathfrak{P}_{=}$ & $\mathfrak{P}_{\text{size}}(\varphi, \langle\mathcal{D},\mathcal{V}\rangle, \langle\mathcal{E},\mathcal{W}\rangle)$ holds, $\mathcal{D},\mathcal{V} \vDash \varphi$ if and only if $\mathcal{E},\mathcal{W} \vDash \varphi$.*

Using this lemma, we can show the following theorem, from which the decidability of the satisfiability problem and hence that of the validity problem for SMSOL follows.

Theorem 4.8. *If a formula φ is satisfied by a standard structure, then φ is satisfied by a standard structure the cardinality of whose individual domain is less than or equal to $2^{|\mathrm{FV}^{\mathrm{p}}(\varphi)|} \cdot \mu(\varphi)$.*

Proof. Let $\mathcal{E}$ be a standard structure and $\mathcal{W}$ an $\mathcal{E}$-assgnment such that $\mathcal{E},\mathcal{W} \vDash \varphi$. From $\mathcal{E}$ and $\mathcal{V}$ we will make an standard structure $\mathcal{D}$ and a $\mathcal{D}$-assignment $\mathcal{V}$ such that the cardinality of the individual domain of $\mathcal{D}$ is less than or equal to $2^{|\mathrm{FV}^{\mathrm{p}}(\varphi)|} \cdot \mu(\varphi)$ and $\mathcal{D},\mathcal{V} \vDash \varphi$ holds. First, for each $\mathrm{FV}^{\mathrm{p}}(\varphi)$-index I, we take $D_I \subseteq \langle\mathcal{E},\mathcal{W}\rangle_I$ as follows:

Case (i): $|\langle\mathcal{E},\mathcal{W}\rangle_I| \le \mu(\varphi)$. Put $D_I := \langle\mathcal{E},\mathcal{W}\rangle_I$.

Case (ii): $|\langle\mathcal{E},\mathcal{W}\rangle_I| \ge \mu(\varphi)$. Since

$$|\langle\mathcal{E},\mathcal{W}\rangle_I \cap \mathcal{W}(\mathrm{FV}^{\mathrm{i}}(\varphi))| \le |\,\mathrm{FV}^{\mathrm{i}}(\varphi)| \le \mu(\varphi)\,,$$

we can take a set D_I of individuals the cardinality of which is $\mu(\varphi)$ and which satisfies

$$\langle\mathcal{E},\mathcal{W}\rangle_I \cap \mathcal{W}(\mathrm{FV}^{\mathrm{i}}(\varphi)) \subseteq D_I \subseteq \langle\mathcal{E},\mathcal{W}\rangle_I\,.$$

Put $D := \bigsqcup_I D_I$, and denote the standard structure $\langle D, \mathcal{P}(D)\rangle$ by $\mathcal{D}$. Take one individual $d_0 \in \mathcal{D}$, and define a $\mathcal{D}$-assignment $\mathcal{V}$ as follows:

$$\mathcal{V}(x) := \begin{cases} \mathcal{W}(x) & \text{if } x \in \mathrm{FV}^{\mathrm{i}}(\varphi), \\ d_0 & \text{o.w.} \end{cases} ; \qquad \mathcal{V}(X) := \mathcal{W}(X) \cap D\,.$$

Then $\langle\mathcal{D},\mathcal{V}\rangle_I = D_I$, and thus $\mathfrak{P}_{\text{pred}}$ & $\mathfrak{P}_{=}$ & $\mathfrak{P}_{\text{size}}(\varphi, \langle\mathcal{D},\mathcal{V}\rangle, \langle\mathcal{E},\mathcal{W}\rangle)$ holds. So, by Lemma 4.7, $\mathcal{D},\mathcal{V} \vDash \varphi$ follows from the assumption $\mathcal{E},\mathcal{W} \vDash \varphi$. Note that $|D| \le 2^{|\mathrm{FV}^{\mathrm{p}}(\varphi)|} \cdot \mu(\varphi)$ by definition.

□

5 Conclusion

We showed that the truth value of any formula of SMSOL in a faithful identity-standard Henkin structure is preserved when the structure is "standardized". This implies that the standard deductive system of SMSOL is sound and complete w.r.t. standard semantics. In addition, we obtained a simpler proof of the decidability of SMSOL.

On the other hand, it is known that the validity problem for monadic second-order logic w.r.t. standard semantics is no longer semi-decidable (and hence there are no sound and complete deductive systems) if as many constant symbols as the language of first-order arithmetic are allowed, because then the problem to decide whether a given first-order sentence is true or false in the standard structure of natural numbers can be reduced to the validity problem for monadic second-order logic (see e.g. [2]). There is an interesting question whether the standard deductive system of monadic second-order logic is complete or not w.r.t. standard semantics when a few constant symbols (of course, fewer than those of the language of first-order arithmetic) in addition to individual and unary predicate constants are allowed.

A Identity-standardization

In this section we give a proof of Theorem 3.5. The idea is as follows: Let $\mathcal{M} = \langle\langle\mathcal{D}^{\mathrm{i}}, \mathcal{D}^{\mathrm{p}}\rangle, \mathcal{I}\rangle$ be any faithful Henkin structure and $\mathcal{V}$ an $\mathcal{M}$-assingment. Then, although $\forall X(Xx \to Xy)$ may not define the equality relation on $\mathcal{D}^{\mathrm{i}}$, at least it defines an equivalence relation on $\mathcal{D}^{\mathrm{i}}$. Identifying the elements of $\mathcal{D}^{\mathrm{i}}$ up to this equivalence relation, we get a faithful identity-standard Henkin structure $\mathcal{M}_{\doteq}$ and an $\mathcal{M}_{\doteq}$-assignment $\mathcal{V}_{\doteq}$ such that the truth value of any formula φ in the structure $\mathcal{M}_{\doteq}$ with $\mathcal{V}_{\doteq}$ is the same as that in the structure $\mathcal{M}$ with $\mathcal{V}$.

Recall that for a faithful Hekin structure $\mathcal{M} = \langle\langle\mathcal{D}^{\mathrm{i}}, \mathcal{D}^{\mathrm{p}}\rangle, \mathcal{I}\rangle$, we defined the binary relation $\doteq_{\mathcal{M}}$ (c.f. Definition 3.3). This relation is an equivalence relation. For reflexivity and transitivity are trivial and symmetricity follows from the fact that $\mathcal{D}^{\mathrm{p}}$ is closed under set subtraction (c.f. Proposition 3.1).

Definition A.1. Let $\mathcal{M} = \langle\langle\mathcal{D}^{\mathrm{i}}, \mathcal{D}^{\mathrm{p}}\rangle, \mathcal{I}\rangle$ be a faithful Henkin structure. Let $[a]_{\doteq}$ denote the equivalence class of an individual a under $\doteq_{\mathcal{M}}$ and put $[A]_{\doteq} := \{[a]_{\doteq} \mid a \in A\}$ for $A \in \mathcal{D}^{\mathrm{p}}$. We define a Henkin structure $\mathcal{M}_{\doteq} = \langle\langle\mathcal{D}^{\mathrm{i}}_{\doteq}, \mathcal{D}^{\mathrm{p}}_{\doteq}\rangle, \mathcal{I}_{\doteq}\rangle$ as follows:

(i) $\mathcal{D}^{\mathrm{i}}_{\doteq} := \{[a]_{\doteq} \mid a \in \mathcal{D}^{\mathrm{i}}\}$;

(ii) $\mathcal{D}^{\mathrm{p}}_{\doteq} := \{[A]_{\doteq} \mid A \in \mathcal{D}^{\mathrm{p}}\} \subseteq \mathcal{P}(\mathcal{D}^{\mathrm{i}}_{\doteq})$;

(iii) $\mathcal{I}_{\doteq}(c) := [\mathcal{I}(c)]_{\doteq} \in \mathcal{D}^{\mathrm{i}}_{\doteq}$;

(iv) $\mathcal{I}_{\doteq}(C) := [\mathcal{I}(C)]_{\doteq} \in \mathcal{D}^{\mathrm{p}}_{\doteq}$.

Lemma A.2. *Let $\mathcal{M} = \langle\langle \mathcal{D}^{\mathrm{i}}, \mathcal{D}^{\mathrm{p}}\rangle, \mathcal{V}\rangle$ be a faithful Henkin structure. Then, for any $a \in \mathcal{D}^{\mathrm{i}}$ and any $A \in \mathcal{D}^{\mathrm{p}}$, $a \in A$ iff $[a]_{\doteq} \in [A]_{\doteq}$.*

Proof. The left to right part is trivial. For the right to left part, suppose $[a]_{\doteq} \in [A]_{\doteq}$. Then $[a]_{\doteq} = [b]_{\doteq}$ for some $b \in A$. From $a \doteq b$ and $A \in \mathcal{D}^{\mathrm{p}}$, $a \in A$ follows. □

Definition A.3. Let $\mathcal{M}$ be a faithful Henkin structure. For an $\mathcal{M}$-assignment $\mathcal{V}$, we define an $\mathcal{M}_{\doteq}$-assignment $\mathcal{V}_{\doteq}$ as follows:

(vi) $\mathcal{V}_{\doteq}(x) := [\mathcal{V}(x)]_{\doteq}$;

(vi) $\mathcal{V}_{\doteq}(X) := [\mathcal{V}(X)]_{\doteq}$.

Then we can easily get the following lemma.

Lemma A.4. *For a faithful Henkin structure $\mathcal{M}$ and an $\mathcal{M}$-assignment $\mathcal{V}$, the followings hold.*

(i) For any individual variable x and any $a \in \mathcal{D}^{\mathrm{i}}$, $(\mathcal{V}^{\frac{a}{x}})_{\doteq} = (\mathcal{V}_{\doteq})^{\frac{[a]_{\doteq}}{x}}$.

(ii) For any predicate variable X and any $A \in \mathcal{D}^{\mathrm{p}}$, $(\mathcal{V}^{\frac{A}{X}})_{\doteq} = (\mathcal{V}_{\doteq})^{\frac{[A]_{\doteq}}{X}}$.

Lemma A.5. *Let $\mathcal{M}$ be a faithful Henkin structure. Then, for any formula φ and any assignment $\mathcal{V}$ on $\mathcal{M}$, $\mathcal{M}, \mathcal{V} \vDash \varphi$ iff $\mathcal{M}_{\doteq}, \mathcal{V}_{\doteq} \vDash \varphi$.*

Proof. By induction on φ. When φ is of the form Xx, then the claim follows from Lemma A.2. When φ is a conjunction, disjunction, or implication, then use the induction hypothesis simply. When φ is of the form $\forall x\psi$, $\exists x\psi$, $\forall X\psi$, or $\exists X\psi$, then use Lemma A.4 and the induction hypothesis. □

Proposition A.6. *Let $\mathcal{M}$ be a faithful Henkin structure. Then the Henkin structure $\mathcal{M}_{\doteq}$ is faithful and identity-standard.*

Proof. For the faithfulness, we will prove that for any $\mathcal{M}_{\doteq}$-assignment $\mathcal{W}$ and any comprehension axiom φ, $\mathcal{M}_{\doteq}, \mathcal{W} \vDash \varphi$ holds. Take one assignment $\mathcal{V}$ on $\mathcal{M}$ such that $[\mathcal{V}(x)]_{\doteq} = \mathcal{W}(x)$ and $[\mathcal{V}(X)]_{\doteq} = \mathcal{W}(X)$. Then $\mathcal{V}_{\doteq} = \mathcal{W}$, and so, by Lemma A.5, $\mathcal{M}, \mathcal{V} \vDash \varphi$ iff $\mathcal{M}_{\doteq}, \mathcal{W} \vDash \varphi$. Note that $\mathcal{M}, \mathcal{V} \vDash \varphi$ holds since $\mathcal{M}$ is faithful.

It easily follows from Lemma A.2 that $\mathcal{M}_{\doteq}$ is identity-standard. □

Finally, we obtain Theorem 3.5 as an immediate consequence of Lemma A.5 and Lemma A.6.

References

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[2] George S. Boolos, John P. Burgess, and Richard C. Jeffrey. *Computability and logic.* Fifth edition, 2007.

[3] Colin Riba. A model theoretic proof of completeness of an axiomatization of monadic second-order logic on infinite words. In *IFIP International Conference on Theoretical Computer Science*, pp. 310–324. Springer, 2012. Full version available on HAL https://hal. archives-ouvertes.fr/hal-00692153.

[4] Stewart Shapiro. *Foundations without foundationalism: A case for second-order logic.* Oxford University Press, 1991.