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Cosmological Applications of Disformal Transformations

by

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Abstract

Since the discovery of the late-time accelerated expansion of the Universe, modified gravity models have been subjected to a renewed interest. A valuable tool in the study of such theories is the metric transformation, such as the conformal one, and its generalization, the disformal transformation. While the former amounts to a rescaling of the original gravitational metric, the latter includes not only a rescaling but also a stretch along the gradient of a scalar field.

In this work, we study the cosmological applications of disformal transformations. Our goal is two-fold: check whether physics changes under disformal transformations and how a disformal coupling between dark matter and dark energy affects the growth of the structure observed in our Universe.

We first study the frame dependence/independence of cosmological observables under disformal transformations, extending the previous results regarding conformal transformations, and provide the correspondence between Jordan-frame and Einstein-frame variables. The disformal transformation can be seen as a field redefinition, and as such, should not change the physical picture. To check this expectation and interpret eventual modifications, we explicitly compare quantities before and after a disformal transformation. We consider quantities such as the gravitational constant in the Newtonian limit, redshift, luminosity and angular diameter distances, as well as the distance-duality relation. Also, the Boltzmann equation, the observed specific intensity, and the adiabaticity condition are discussed. We show that the redshift is invariant and the distance-duality relation (between the luminosity distance and the angular diameter distance) still holds in general spacetimes even though the reciprocity relation (between two geometrical distances) is modified. In general, modifications can be interpreted as a change of units, albeit not as straightforwardly as in the conformal case.

We also investigate cosmological scenarios in which dark matter is non-minimally coupled to an extra scalar degree of freedom, the latter being responsible for the late-time acceleration of the Universe. The interaction is realized by means of conformal and disformal terms in the transformed gravitational metric. Considering linear perturbation theory, we show that the growth rate of dark matter differs from the uncoupled case and that the well-known Kaiser formula undergoes modification. As a result, redshift space distortion measurements cease to be a direct probe of the linear growth rate of total matter, since the distortion factor has an extra, coupling-dependent term. We study the effect of the coupling in three cosmological models, two conformally and one disformally coupled, and forecast the constraints on the coupling, and other cosmological parameters, from future galaxy surveys.

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Notations and Conventions

In this thesis, we use the metric signature convention $(-, +, +, +)$. Greek indices run over the four spacetime dimensions (0 to 3), whereas Latin indices run over the three spatial dimension (1 to 3). Unless otherwise stated, we use units such that $c = \hbar = 1$. G_N is Newton's gravitational constant and $M_{\text{Pl}}^2 = (8\pi G_N)^{-1}$ is the reduced Planck mass.

The line element with scalar linear perturbations in the Newtonian gauge around an FLRW background is defined as

$$ds^2 = -(1 + 2\Phi) dt^2 + a^2(t)(1 - 2\Psi)\gamma_{ij} dx^i dx^j,$$

where γ_{ij} is the metric of a maximally symmetric 3-space with Gaussian curvature K

$$d\sigma^2 = \gamma_{ij} dx^i dx^j = \frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

The four-dimensional covariant derivative is defined as ∇_μ and the d'Alembertian as $\square = g^{\mu\nu}\nabla_\mu\nabla_\nu$. Then, for the Laplacian we use the notation $\Delta = \gamma^{ij}\nabla_i\nabla_j$.

The shorthand notations $\phi_\mu = \partial_\mu\phi$ and $\phi_{\mu\nu} = \nabla_\mu\nabla_\nu\phi$ are often used to express the derivatives of the scalar field. Quantities with a subscript ϕ or X denote its derivative with respect to the scalar field ϕ or the kinetic term $X = -g^{\mu\nu}\phi_\mu\phi_\nu/2$, respectively:

$$A_\phi \equiv \frac{\partial A}{\partial \phi}, \quad A_X \equiv \frac{\partial A}{\partial X}.$$

Differentiation with respect to the cosmic time is denoted with a dot,

$$\dot{A} = \frac{dA}{dt}, \quad \dot{\tilde{A}} = \frac{d\tilde{A}}{d\tilde{t}},$$

where Jordan-frame quantities are usually denoted with a tilde.

Finally, parenthesis and square brackets around indices denote symmetrization and antisymmetrization, respectively:

$$A_{(\mu\nu)} = \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu}), \quad A_{[\mu\nu]} = \frac{1}{2}(A_{\mu\nu} - A_{\nu\mu})$$

Throughout this work, this notation will be restated whenever necessary.

List of Acronyms

BAO:	Baryon acoustic oscillations
BD:	Brans-Dicke
CDM:	Cold dark matter
CLASS:	Cosmic linear anisotropy solving system
CMB:	Cosmic microwave background
DE:	Dark energy
DHOST:	Degenerate higher-order scalar-tensor theory
DM:	Dark matter
EF:	Einstein frame
FLRW:	Friedmann-Lemaître-Robertson-Walker
GLPV:	Gleyzes-Langlois-Piazza-Vernizzi
GR:	General relativity
GUT:	Grand unified theory
JF:	Jordan frame
LSS:	Large scale structure
MaVaNs:	Mass varying neutrinos
MOND:	Modified Newtonian dynamics
QFT:	Quantum field theory
QS:	Quasi-static
RSD:	Redshift space distortions
TeVS:	Tensor-Vector-Scalar
WIMPs:	Weakly interacting massive particles

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1. Introduction

Einstein’s theory of General Relativity (GR) [1] is a remarkable accomplishment. Together with Quantum Field Theory (QFT) — used to describe, in a unified way, all non-gravitational interactions — it forms the basis of modern physics. Its elegant formulation based on differential geometry paved the way for advances in particle physics, such as gauge and string theories.

It also has an unarguable physical beauty: the identification of gravity with the curvature of spacetime itself has revolutionized our understanding of the Universe. Imposing requirements such as coordinate invariance, conservation laws, and agreement with the familiar Newtonian limit, Einstein came up with a theory, described by the action

$$S_{\text{GR}} = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} R + S_{\text{m}} , \quad (1.1)$$

that is not only elegant but also capable of changing the way we perceive space and time.

But even elegant theories need experimental validation and in that regard, GR passes with flying colors. Remaining virtually unchanged for over a century, it still provides the best description of gravitational physics on macroscopic scales. Einstein’s field equations,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G_N T_{\mu\nu} , \quad (1.2)$$

dictate the behavior of gravitational systems ranging from planets, stars, and black-holes all the way up to super-clusters of galaxies. It is thought to be inappropriate only in the microscopic realm of particles and high-energies, in which quantum effects become important. Nevertheless, GR remains the most accurate description of gravity, being tested on all other scales.

General Relativity also gave the theoretical foundation for the study of the cosmos, elevating cosmology from esoteric speculation to the status of legitimate science. The early works by Friedmann and Lemaître regarding expanding universes [2,3], combined with ideas about the geometry of homogeneous and isotropic spacetimes from Roberson [4] and Walker [5], then led to the development of the Friedmann-Lemaître-Robertson-Walker (FLRW) solution. This solution describing the background cosmological evolution was then extended with the theory of cosmological pertur-

bations, enabling us to accurately investigate and explain the formation of large-scale structures in our Universe.

Despite its current success, GR has not always enjoyed this status. Even with Einstein's explanation of the perihelion of Mercury and Eddington experiment with the deflection of sunlight, General Relativity remained outside the mainstream of theoretical physics until the 1960s and 1970s. It all came to change with both technological and theoretical advances: physicists began to better understand the intricate nuances of GR, whereas astronomers were able to build instruments capable of testing its predictions.

However, in the years and decades following Einstein's publication, theoreticians continued exploring other physical and mathematical possibilities. In general these alternative theories tried to extend GR, adopting its main ingredients but adding new features such as dynamical fields, different geometrical assumptions, or higher dimensions. Early attempts include Weyl's scale-independent theory [6], Eddington's theory of affine connections [7], and Kaluza and Klein's higher dimensional theory [8, 9].

Probably the most well-known of these early theories is the Brans-Dicke theory [10]. Based upon Dirac's ideas regarding the time variation of Newton's constant G_N , it was an attempt to construct a gravitational theory fully compatible with Mach's principle. This feature is implemented with the addition of a scalar field which directly couples to the curvature:

$$S_{\text{BD}} = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} \left(\phi R - \frac{\omega_{\text{BD}}}{\phi} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - U(\phi) \right) + S_{\text{m}}. \quad (1.3)$$

The magnitude of G_N would then be linked to the evolution of a cosmic field. The Brans-Dicke theory can be seen as the prototype scalar-tensor theory; it paved the way for the further development of a wide class of modified gravity theories.

From the theoretical point of view, there are at least two important reasons to try and extend General Relativity: its inability to describe physics at microscopic scales and the desire to unify all forces in nature into a single framework.

Albeit without much success, Einstein himself undertook the endeavor of creating a classical theory that could describe all fundamental forces within a single framework. More recently, with the development of the Standard Model of particle physics, theoretical physics saw a revival in attempts to build Grand Unified Theories (GUT). A strong GUT candidate is String Theory, which does not recover GR in the effective low-energy limit. Instead, one recovers the Brans-Dicke theory, with a dilaton field non-minimally coupled to the curvature.

Furthermore, it is not possible to quantize gravity using the standard QFT procedure. These attempts lead to a theory that is not renormalizable. General Relativity is then an effective field

theory, valid up to the Planck scale, indicating the need for a more complete description.

The limits of GR are again being tested with the emergence of the dark universe. Within the framework of Einstein's gravity, astrophysical and cosmological observations point to the existence of two new, mysterious components: dark energy and dark matter.

Since the early 1930s, measurements of the velocity dispersion and galaxy rotation curves indicated the existence of some invisible kind of matter that only interacts gravitationally with the visible stars, dubbed dark matter (DM) [11–13]. In the context of General Relativity, this new kind of matter behaves as pressureless (cold) dust, does not interact with electromagnetic radiation (hence, dark), and is responsible for roughly a quarter of the energy density of the Universe. Observational evidence for dark matter has accumulated over the years, from both astrophysical and cosmological observations [14–16].

Several extensions of gravity were proposed in an attempt to explain these observations without relying on dark matter. Examples include Milgrom's Modified Newtonian Dynamics (MOND) theory [17] and Bekenstein's Tensor-Vector-Scalar (TeVeS) gravity [18]. Despite showing some success, these theories still face serious challenges and a more popular approach is assuming that dark matter is a new particle, such as WIMPs and axions (see [19] for a comprehensive modern discussion).

A more compelling motivation for the search of modified theories of gravity was the discovery of the accelerated expansion of the Universe. In 1998, two independent groups studying the luminosity distance to type Ia Supernovae found evidence that our Universe is currently undergoing a phase of accelerated expansion [20, 21]. This fact has now been confirmed by several independent observations such as temperature anisotropies in the cosmic microwave background (CMB) [22, 23], inhomogeneities in the matter distribution [24–26], among others.

General Relativity, combined with the Standard Model of particle physics, cannot explain the accelerated expansion, requiring extra degrees of freedom. One can attribute these extra degrees of freedom to a new, exotic form of matter, generically called dark energy (DE): a diluted component with negative pressure that interacts only gravitationally and amounts to approximately 70% of the energy budget of the Universe.

The most natural DE candidate is Einstein's cosmological constant Λ . The observational evidence is consistent with Λ sourcing the accelerated expansion and cold dark matter (CDM) giving rise to the distribution of galaxies and clusters, the so-called Λ CDM model. Despite its simplicity and agreement with observations, the Λ CDM model presents severe theoretical shortcomings.

One issue, known as the cosmological constant problem [27], is that the observed value of Λ is orders of magnitude smaller than its estimate from particle physics. Another puzzle is the coincidence problem: since matter and dark energy evolve through cosmic history in very different

ways, why are their current values so similar?

One possible generalization of the Λ CDM model is to promote DE to a dynamical kind of matter. The simplest implementation of dynamical DE models consists of a scalar field, such as in quintessence models (a canonical scalar field slow-rolling down its potential [28–30]), or K-essence (where a non-canonical kinetic term fuels the acceleration [31, 32]). Indeed, these models can solve to some extent the cosmological constant and coincidence problems [33, 34].

A common feature of quintessence and K-essence is that the scalar field couples minimally to gravity, and it can be seen as a new kind of matter in the playground of General Relativity. However, modifying gravity itself is also perfectly justifiable: one could consider new theories generalizing the Einstein-Hilbert action

$$S_{\text{EH}} = \int d^4x \sqrt{-g} R, \quad (1.4)$$

For example, the so-called $f(R)$ theories [35–38], assumes that gravity is described by an arbitrary function of the Ricci scalar R .

In the same spirit of scalar field DE models, one could consider a new scalar degree of freedom non-minimally coupled to gravity; in fact, that is the case in the Brans-Dicke gravity. This class of models, generically called scalar-tensor theories, have received a lot of attention in recent years, but its origins predate the discovery of the accelerated expansion [10, 39–43].

One of the greatest developments in scalar-tensor theories was the construction of generic frameworks that allow the systematic study of many models in a unified way. For example, the Horndeski theory (also known as Generalized Galileon) [44–46], which is the most general 4-dimensional, local scalar-tensor theory that leads to second-order equations of motion. Further generalizations include the Beyond Horndeski (GLPV) [47] and DHOST theories [48].

The common feature of scalar-tensor theories is the inclusion of a scalar degree of freedom which mimics the role of the cosmological constant at late times. Their theoretical predictions, as well as observational constraints in the cosmological context, have been extensively investigated (see [49–52] for reviews).

One valuable tool in the study of modified gravity (and modern physics in general) is the conformal transformation [53]. In General Relativity, the gravitational degree of freedom is a rank-2 field that describes the geometry of spacetime. However, a theoretical possibility is that gravity requires two geometries: one describing gravity itself while the other specifying the matter dynamics [10, 54]. In such a case, the strong equivalence principle is violated, while the weak equivalence is preserved. The conformal transformation is defined as

$$\tilde{g}_{\mu\nu} = A g_{\mu\nu}, \quad (1.5)$$

where, in general, A is a function of the spacetime coordinates.

In recent years, the use of conformal transformation became even more widespread in attempts to generalize GR. The existence of two metrics can be understood as follows: in the matter frame, the scalar field is non-minimally coupled to the metric, whereas in the gravity frame, the scalar field is minimally coupled to the metric, but matter is non-minimally coupled to the scalar field [55, 56]. This duality allows different representations of the same theory. The representation in which the scalar field couples to gravity and matter is minimally-coupled is called Jordan frame, whereas the one in which the scalar field couples to matter and gravity is minimally-coupled is called Einstein frame.

The conformal transformation is the simplest way to relate two geometries, and one might wonder if it is possible to generalize it. Bekenstein introduced the disformal transformation while looking for a new form of metric that allows matter to couple non-trivially to a scalar field [54]. Defined as

$$\tilde{g}_{\mu\nu} = Ag_{\mu\nu} + B\partial_\mu\phi\partial_\nu\phi, \quad (1.6)$$

disformal transformations are a generalization of the conformal ones: while conformal transformations amount to a rescaling of the metric, disformal transformations also incorporate a stretching along the gradient of the scalar field.

Since physics is invariant under a redefinition of fields, it does not change under conformal transformations. The change can then be interpreted as a mere change of units [55, 57, 58]. On the other hand, it is not as straightforward to treat disformal transformations as a change of units, due to the introduction of derivative couplings [59, 60].

Scalar-tensor theories extended General Relativity with the introduction of a non-minimal coupling between the dark energy scalar field and the gravitational metric. After a conformal transformation, these theories are equivalent to GR in the sense that the scalar field couples minimally to gravity, but now dark energy couples universally to all matter fields [28, 56, 61]. The equivalence can be also extended to some classes of disformal transformation; in some simple cases, it is possible to recover the Einstein-Hilbert action after disformally transforming a scalar-tensor theory [62]. Hence, scalar-tensor theories are an example of interacting cosmology, motivating modified gravity models with a non-minimal coupling.

In these interacting dark energy scenarios, the scalar-mediated interaction is a fifth-force with a strength that is typically orders of magnitude weaker than the gravitational force [63, 64]. Nevertheless, the coupling can significantly change the cosmological evolution. The scalar field is then the mediator of cosmological interactions, the 'cosmon' particle as coined by Wetterich [28]. Effectively, the interaction acts as if the mass of all species (baryons, dark matter, neutrinos) was a function of the scalar field.

Tests of the equivalence principle and solar system constraints require the coupling between dark energy and baryons to be quite small [65]. Another important test is the variation of fundamental constants over cosmological time-scales [66, 67]. Moreover, since baryons make up a small fraction of the matter contained in the Universe, it is difficult to have strong cosmological effects due to coupling to baryons only.

The bounds on a coupling between dark energy and neutrinos, on the other hand, are set to be much more flexible. In fact, in scenarios such as MaVaNs [68] and growing neutrinos [69], the strength of the interaction could be comparable to the gravitational one.

Finally, an appealing possibility is the coupling between dark energy and dark matter. Such models were first introduced as an attempt to solve the cosmological constant problem [28] and the coincidence problem [70]. In principle, the coupling could be much stronger than the one with baryons since it is insensitive to tests on the equivalence principle or solar system constraints. This framework is generically called coupled quintessence and it was put to test with many choice of couplings [71–74] on several different contexts: background evolution [75], structure formation [76–79], and CMB observations [80–82].

In this work, we will apply the disformal transformation in two cosmological contexts. In the first, we investigate the relationship between cosmological observables defined in disformally related frames. Our goal is to check explicitly whether physics changes under disformal transformations. The second topic we intend to discuss is how a disformal coupling between dark matter and dark energy affects the growth of the structure observed in our Universe. In particular, we investigate the so-called Kaiser effect (the distortion in the observed galaxy distribution due to the peculiar motion of galaxies) to the and forecast the constraints on the coupling constant from future galaxy surveys. The results presented here were published in refs. [60, 78]

This thesis is organized as follows: In chapter 2 we briefly review General Relativity and the FLRW cosmology. The purpose is to remind the reader of essential concepts that are going to be further discussed on the remainder of the work, such as cosmological distance, cosmological perturbation theory, and redshift space distortions. In chapter 3, we discuss metric transformations. The discussion starts with a review of the conformal transformation, its basic properties, and applications in the study of modified gravity. After that, we introduce the idea of disformal transformations and discuss its applications on scalar-tensor theories, in particular in the definition of the Jordan and Einstein frames.

The disformal invariance is studied in chapter 4. Extending the previous results for the conformal case [58], we provide a comprehensive analysis regarding the transformation of observable quantities and relations of cosmological interest under disformal transformations. The results presented in this chapter were published in [60].

In chapters 5 and 6, we investigate redshift space distortions in the presence of non-minimally coupled dark matter, as published in [78]. We first extend the results of [74] and present the formalism of disformally coupled dark matter. Then, we show that, when the dark energy scalar field couples to dark matter, the Kaiser formula must be modified. In chapter 6, we explore the background and linear evolution of three concrete examples of coupled dark matter cosmologies and forecast the future constraints on the coupling parameter from the next generation of redshift surveys. Finally, on chapter 7 we conclude and summarize this work.

2. General Relativity and FLRW Cosmology

For over a century, Einstein's theory of General Relativity has proved itself to be robust, explaining fairly well observations ranging from our solar system to cosmological scales. On the other hand, the homogeneous and isotropic FLRW sets the framework to study the evolution of the Universe. The background cosmological evolution is then supplemented with cosmological perturbation theory to explain the observed cosmic structures. In this chapter we present a lightning review on General Relativity and, in particular, FLRW cosmology. The purpose is to introduce and review the basic topics that will be studied on later chapters.

2.1 Einstein's field equations

Let us consider the Einstein-Hilbert action with a cosmological constant and additional matter fields,

$$S = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} (R - 2\Lambda) + S_m, \quad (2.1)$$

where G_N is the Newton's gravitational constant, R is the Ricci scalar related to the metric $g_{\mu\nu}$, Λ is the cosmological constant, and S_m is the action for the matter fields. In standard General Relativity, matter is said to be minimally-coupled to the metric: it couples to $g_{\mu\nu}$ only via the volume element $\sqrt{-g}$ and eventual metric contractions. Then, the matter action can be written as

$$S_m = \int d^4x \sqrt{-g} \mathcal{L}_m[g_{\mu\nu}, \psi_m], \quad (2.2)$$

where $\mathcal{L}_m$ is the Lagrangian density describing the matter fields ψ_m .

To find the equations of motion, one needs to vary the action (2.1) with respect to the fields. In particular, the variation with respect to $g^{\mu\nu}$ yields the Einstein equations,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}. \quad (2.3)$$

The quantity $G_{\mu\nu}$ is the so-called Einstein tensor, which describes the geometry of the spacetime,

and can be expressed as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}, \quad (2.4)$$

in terms of the Ricci scalar R and the Ricci tensor $R_{\mu\nu}$. On the other hand, $T_{\mu\nu}$ is the energy-momentum tensor, which characterizes the material content in the spacetime. By construction, the field equations (2.3) and $T_{\mu\nu}$ are symmetric on its indices. This feature is naturally inherited from the symmetry of the metric tensor.

The Einstein tensor satisfies the Bianchi identities,

$$\nabla_\mu G^{\mu\nu} = \partial_\mu G^{\mu\nu} + \Gamma_{\mu\rho}^\mu G^{\rho\nu} + \Gamma_{\mu\rho}^\nu G^{\mu\rho} = 0, \quad (2.5)$$

where ∇_μ is the covariant derivative and $\Gamma_{\mu\nu}^\rho$ is the connection, both associated to the metric $g_{\mu\nu}$. As we will soon see, the Bianchi identity is closely related to an important feature of the field equations, namely, the energy-momentum conservation.

We now move on to the right-hand side of Einstein's field equations. Given an action, it is straightforward to compute the associated energy-momentum tensor. With upper indices, it is defined in terms of the Lagrangian density for matter fields as

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g_{\mu\nu}}. \quad (2.6)$$

However, it is often convenient to treat the material content in the Universe not in its representation as fundamental fields, but to model it as a generic fluid. As a result of its symmetry, $T_{\mu\nu}$, as measured by an observer moving with four-velocity u^μ , can be decomposed into parts parallel and orthogonal to u^μ ,

$$T_{\mu\nu} = \rho u_\mu u_\nu + q_\mu u_\nu + q_\nu u_\mu + Ph_{\mu\nu} + \pi_{\mu\nu}, \quad (2.7)$$

where, $h_{\mu\nu}$ is the induced metric on the hypersurface orthogonal to the four-velocity,

$$h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu, \quad (2.8)$$

and the velocity is normalized as $u^\mu u_\mu = -1$. With this split, the observer measures the relativistic energy density and pressure ρ and P , respectively; relativistic momentum density q_μ , which can also be interpreted as the energy flux relative to u^μ due to processes such as diffusion or heat conduction, and a relativistic, trace-free anisotropic stress tensor $\pi_{\mu\nu}$ due to viscosity effects or

magnetic fields. These quantities are given by the contractions

$$\rho = T_{\mu\nu} u^\mu u^\nu, \quad (2.9)$$

$$P = \frac{1}{3} h^{\mu\nu} T_{\mu\nu}, \quad (2.10)$$

$$q_\mu = -h^\nu{}_\mu T_{\nu\rho} u^\rho, \quad q_\mu u^\mu = 0, \quad (2.11)$$

$$\pi_{\mu\nu} = h^\rho{}_\mu h^\sigma{}_\nu T_{\rho\sigma} - \frac{1}{3} h^{\rho\sigma} T_{\rho\sigma} h_{\mu\nu}, \quad \pi_{\mu\nu} u^\mu = 0, \quad \pi^i{}_i = 0, \quad (2.12)$$

such that q_μ and $\pi_{\mu\nu}$ are orthogonal to u^μ . Due to its symmetry property, $T_{\mu\nu}$ has 10 independent components — two scalar quantities (ρ and P), the three components of the four-vector q_μ , and the five components of the tensor $\pi_{\mu\nu}$.

To find the equation of motion for the matter fields, we can either vary the action (2.1) with respect to ψ_m , like it was done for the metric field, or use the fact that its energy-momentum tensor is covariantly conserved. The latter procedure is also suitable when modelling matter as a generic fluid. Indeed, the Bianchi identity immediately implies the conservation of the total $T_{\mu\nu}$,

$$\nabla^\mu T_{\mu\nu} = 0. \quad (2.13)$$

For non-interacting fluids, the above equation holds individually for each one of them. However, in the presence of a non-minimal coupling to gravity or if two fluids or more are to interact to each other, this equation is modified. The modification will then reflect the flux of momentum and energy. This is one of the main topics that will be touched upon through out this work.

2.2 FLRW cosmology

The cosmological principle is at the basis of modern cosmology. It states that, on large scales, the Universe is homogeneous and isotropic. Observations of the cosmic microwave background radiation, along with the Copernican principle — there are no privileged positions in the Universe — indicate the homogeneity of the Universe.

The line element describing a 4-dimensional homogeneous and isotropic spacetime is

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t) d\sigma^2, \quad (2.14)$$

the so-called Friedmann-Lemaître-Robertson-Walker (FLRW) metric. The cosmic expansion is parametrized by the scale factor $a(t)$, where t is the cosmic time. Furthermore, $d\sigma^2$ is the line element of the time-independent, maximally symmetric 3-dimensional space with constant curvature

K . In spherical coordinates, it reads

$$d\sigma^2 = \gamma_{ij} dx^i dx^j = \frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (2.15)$$

where $K = +1, 0, -1$ corresponds to closed, flat, and open geometries.

In the cosmological setting, the dynamics of the whole Universe is determined by the properties of the matter in it. The homogeneity and isotropy of the FLRW spacetime restricts matter to take the form of a perfect fluid,

$$T^\mu{}_\nu = (\rho + P)u^\mu u_\nu + P\delta^\mu_\nu, \quad (2.16)$$

where u^μ is the four-velocity of an observer coming with the fluid. In that case, there is no momentum flux q_μ or anisotropic stress $\pi_{\mu\nu}$; the energy density ρ and pressure P are functions of time alone.

With the use of the metric (2.14) and energy-momentum tensor (2.16), the Einstein field equations (2.3) reduce to

$$H^2 = \frac{8\pi G_N}{3}\rho - \frac{K}{a^2}, \quad (2.17)$$

$$3H^2 + 2\dot{H} = -8\pi G_N P - \frac{K}{a^2}, \quad (2.18)$$

the Friedmann equations. Here $H = \dot{a}/a$ is the Hubble parameter and dots represent derivatives with respect to the cosmic time. The Friedmann equations determine how the spacetime in a homogeneous and isotropic cosmology — summarized by the function $a(t)$ — evolves given the energy content of the Universe.

The covariant conservation of the energy-momentum tensor (2.16) translates into,

$$\dot{\rho} + 3H(\rho + P) = 0, \quad (2.19)$$

the continuity equation. This expression can also be derived by differentiating eq. (2.17) and substituting the curvature term on eq. (2.18). The continuity equation represents the conservation law for a fluid in response to the Hubble expansion.

2.3 Cosmological distances

Before the recently inaugurated era of gravitational wave astronomy, all astronomical and cosmological observations relied on measurements of light coming from different sources. As such, the properties of light rays traveling on an FLRW universe are extremely important in making predictions about the cosmos. With measurements of the electromagnetic spectra of galaxies and

other sources, it is possible to estimate, among other things, the distance to these objects.

In an expanding universe, the wavelength of light λ is stretched, and we introduce the redshift to quantify this stretch in terms of the scale factor. Consider a signal emitted with λ_s and observed with a wavelength λ_0 . The redshift z is then defined as

$$z \equiv \frac{\lambda_0}{\lambda_s} = \frac{a_0}{a} - 1, \quad (2.20)$$

where subscript 0 corresponds to the present epoch, in which $z = 0$. It is customary to set a_0 to unity; unless otherwise stated, in the present work we follow this practice.

In expanding cosmologies, the definition of distance is not as straightforward as, for example, in the Minkowski spacetime. Let us now review some definitions of distance that will play an important role in the chapters that follow. Setting $r = \sin \chi$ for closed, $r = \chi$ for flat, and $r = \sinh \chi$ for open universes, the 3-sphere line element, eq. (2.15) can be written as

$$d\sigma^2 = d\chi^2 + f_K^2(\chi)(d\theta^2 + \sin^2 \theta d\phi^2), \quad (2.21)$$

where we have introduced the quantity $f_K(\chi)$, defined as

$$f_K(\chi) = \begin{cases} \sin \chi & (K = 1), \\ \chi & (K = 0), \\ \sinh \chi & (K = -1). \end{cases} \quad (2.22)$$

One can then define the comoving distance as the one traveled by light in the radial direction along null geodesics, $ds^2 = -dt^2 + a^2(t) d\chi^2 = 0$, between the emission point χ with $t = t_e$ and an observer at t_0 , with $\chi = 0$:

$$d_C \equiv \chi = \int_0^\chi d\chi = - \int_{t_0}^{t_e} \frac{dt}{a(t)} = \frac{1}{a_0 H_0} \int_0^z \frac{dz'}{E(z')}, \quad (2.23)$$

where $E(z) \equiv H(z)/H_0$ is the dimensionless Hubble parameter.

Another important way to define distance is by measuring the flux coming from an object with known luminosity. The luminosity distance d_L is defined in terms of the absolute luminosity L_s of a source and its observed flux F as

$$d_L^2 = \frac{L_s}{4\pi F}. \quad (2.24)$$

Due to the expansion of the Universe, the observed luminosity L_0 (detected at $\chi = 0$ and $z = 0$) is different from L_s (emitted at comoving distance χ and redshift z). The absolute luminosity is defined as $L_s = \Delta E_s / \Delta t_s$, where ΔE_s is the energy of light emitted in the interval

Δt_s . Analogously, the observed luminosity defined in terms of ΔE_0 , the energy of light observed in the interval Δt_0 , as $L_0 = \Delta E_0 / \Delta t_0$. Since the energy of photons is inversely proportional to its wavelength, $\Delta E_s / \Delta E_0 = \lambda_0 / \lambda_s = 1 + z$. On the other hand, the constancy of the speed of light implies $\Delta t_0 / \Delta t_1 = \lambda_0 / \lambda_s = 1 + z$. As a result,

$$\frac{L_s}{L_0} = (1 + z)^2. \quad (2.25)$$

The observed flux is defined as $F = L_0 / S$, where $S = 4\pi(a_0 f_K(\chi))^2$ is the area of a sphere centered at $z = 0$. Then the luminosity distance can finally be written as

$$d_L = a_0 f_K(\chi)(1 + z), \quad (2.26)$$

where χ , given in eq. (2.23), is the comoving distance.

In the low-redshift limit ($z \ll 1$), eq. (2.26) can be expanded around $z = 0$. Recovering the speed of light c , it leads to $d_L \simeq cz / H_0$, such that

$$v \simeq H_0 d_L. \quad (2.27)$$

In the previous equation we have used $z \simeq v/c$, which is valid as long as the recessional velocity v of the source is much smaller than the speed of light. This shows that in the low- z limit, the luminosity distance recovers Hubble's law.

The luminosity distance plays an important role in astronomy and cosmology. For instance, in the late 1990s, two independent teams reported the current accelerated expansion of the Universe after analyzing the luminosity of distant type Ia supernovae [20, 21].

Another important way to determine distances in astronomy is to measure the angle subtended by an object of known physical size. Consider an object with proper size Δx , orthogonal to the line of sight, subtending an angle $\Delta\theta$ in the sky. The angular diameter distance d_A is then defined as

$$d_A \equiv \frac{\Delta x}{\Delta\theta}. \quad (2.28)$$

If the source lies on the a sphere of radius χ , centered around the observer at $z = 0$, then its proper size at the time of emission t_e is given by

$$\Delta x = a(t_e) f_K(\chi) \Delta\theta. \quad (2.29)$$

Therefore, the angular diameter distance becomes

$$d_A = \frac{a_0 f_K(\chi)}{1+z}. \quad (2.30)$$

The angular diameter distance is often used when discussing the anisotropy of the cosmic microwave background (by means of the so-called distance priors [83]) or the acoustic features of the power spectrum (as, for example, in [84]).

Comparing eq. (2.30) with eq. (2.26), we see that the luminosity and angular diameter distances are related as

$$d_L = (1+z)^2 d_A. \quad (2.31)$$

This relation is called distance-duality or Etherington relation [85], and it is valid not only in FLRW spacetimes but it is a generic feature; as long as the flux is conserved, the distance-duality relation is satisfied. In chapter 4, we will revisit the distance-duality relation in the context of disformal transformations.

2.4 Perturbation theory

Linear perturbation theory is currently one of the main sources of cosmological information. The homogeneous and isotropic Universe is fairly well understood: despite some tensions, the Λ CDM model provides a good description of the background evolution of the Universe. In the past decades, after the initial measurements of the CMB and the large scale structure (LSS), a lot of effort has been put into getting accurate measurements of inhomogeneities; for that reason, it is often said we live in the era of precision cosmology. One of the main drives of such endeavor is to find stringent evidence in favor of modified gravity and pinpoint which, if any, extension of General Relativity actually describes our Universe.

With that in mind, we now turn to perturbation theory around the FLRW metric. In the present discussion, we restrict ourselves to scalar fluctuations. For a comprehensive discussion about cosmological perturbation theory, we refer the reader to [86–88]

At linear order in perturbation theory, the FLRW metric can then be written as

$$ds^2 = a^2 \{ -(1+2A) d\eta^2 + 2B_{;i} dx^i d\eta + [(1-2\psi)\gamma_{ij} + 2E_{;ij}] dx^i dx^j \}, \quad (2.32)$$

where A , B , ψ , and E are scalar perturbation variables, whereas $f_{;i} = \nabla_i f$, with ∇_i being the (background) covariant derivative compatible with the spatial metric γ_{ij} . We also introduced for

convenience the conformal time η , defined as

$$\eta = \int \frac{dt}{a}. \quad (2.33)$$

2.4.1 The choice of gauge

Since General Relativity is invariant under diffeomorphisms, the split between background and perturbed metric is not unique, which gives rise to a gauge freedom. For any variable Y , its perturbation δY may depend on the gauge choice. Since physics should not depend on a particular choice of gauge, δY is not necessarily an observable quantity. Indeed, for a coordinate transformation $\tilde{x}^\mu = x^\mu + \xi^\mu$, it is always possible to define a new variable $\delta\tilde{Y} = \delta Y + \mathcal{L}_\xi \bar{Y}$ using the Lie derivative along a vector field ξ^μ applied to the background value $\bar{Y}$. For scalar perturbations, one can define $\xi^\mu = (\xi^0, \nabla^i \xi)$, such that the components of the metric transform as [87]

$$\tilde{A} = A - \frac{a'}{a} \xi^0 - (\xi^0)', \quad (2.34)$$

$$\tilde{B} = B + \xi^0 - \xi', \quad (2.35)$$

$$\tilde{\psi} = \psi + \frac{a'}{a} \xi^0, \quad (2.36)$$

$$\tilde{E} = E - \xi, \quad (2.37)$$

where primes indicate derivatives with respect to the conformal time.

Given this set of four perturbation variables, two linear combinations can be removed. Common choices of gauge include: Newtonian gauge, corresponding to the choice $B = E = 0$; synchronous gauge, in which $A = B = 0$ (note, however, that this choice does not completely fix the gauge); and spatially flat gauge, where $\psi = E = 0$.

It is also possible to construct combinations of gauge-invariant quantities. Examples include the Bardeen potentials, $\hat{\Phi}$ and $\hat{\Psi}$, given by [89]

$$\hat{\Phi} = A - \frac{1}{a} [(B - E')a]', \quad \hat{\Psi} = \psi - \frac{a'}{a} (B - E'). \quad (2.38)$$

In the Newtonian gauge, we identify the gravitational potentials $\Phi = \hat{\Phi} = A$ and $\Psi = \hat{\Psi} = \psi$.

2.4.2 Evolution of perturbations

From now on, we focus on perturbations in the Newtonian gauge,

$$ds^2 = -[1 + 2\Phi(t, \mathbf{x})] dt^2 + a^2(t)[1 - 2\Psi(t, \mathbf{x})] d\mathbf{x}^2, \quad (2.39)$$

and reintroduce the cosmic time t . Up to the linear order, the energy-momentum tensor of a perfect fluid can be represented as

$$T^0_0 = -(\rho + \delta\rho), \quad (2.40)$$

$$T^0_i = -(\rho + P)\partial_i v, \quad (2.41)$$

$$T^i_j = (P + \delta P)\delta^i_j, \quad (2.42)$$

assuming that the fluid moves with a small velocity v , which is a perturbation of the same order as $\delta\rho$ and δP . Here we have made a simplifying assumption that the perturbed perfect fluid still is a perfect fluid. In general, that is not true: at linear order, one should also include an anisotropic stress. However, for the purpose of this review, $\pi_{\mu\nu}$ can safely be ignored.

When discussing linear cosmological perturbations, it is convenient to solve the linearized Einstein equations in Fourier space. In the Newtonian gauge, its $(0,0)$ and $(0,i)$ components, along with the trace and traceless part of (i,j) , become, respectively,

$$3H(\dot{\Psi} + H\Phi) + \frac{k^2}{a^2}\Psi = 4\pi G_N \delta T^0_0, \quad (2.43)$$

$$\dot{\Psi} + H\Phi = 4\pi G_N(\rho + P)v, \quad (2.44)$$

$$\ddot{\Psi} + H(\dot{\Phi} + 3\dot{\Psi}) + (2\dot{H} + 3H^2)\Phi = \frac{4\pi}{3}G_N \delta T^i_i, \quad (2.45)$$

$$\Psi - \Phi = 0. \quad (2.46)$$

Let $w = P/\rho$ be the equation of state of the fluid, and $\delta = \delta\rho/\rho$ its density contrast. Then, the conservation of the energy-momentum tensor, eq. (2.13), at linear order gives

$$\dot{\delta} = -3H\left(\frac{\delta P}{\delta\rho} - w\right)\delta - (1+w)\left(\frac{k^2}{a^2}v - 3\dot{\Psi}\right), \quad (2.47)$$

$$\dot{v} = \left(3wH - \frac{\dot{w}}{1+w}\right)v + \frac{\delta P/\delta\rho}{1+w}\delta + \Phi. \quad (2.48)$$

These are known as continuity and Euler equations, and are valid for a single, uncoupled fluid, or for the total fluid defined such that

$$\rho(1 + \delta) = \sum_i \rho_{(i)}(1 + \delta_{(i)}), \quad (2.49)$$

$$(\rho + P)v = \sum_i (\rho_{(i)} + P_{(i)})v_{(i)}, \quad (2.50)$$

where the index i runs over all particle species. If the individual components interact to each other, these equations must be modified. Such is the case before recombination, when baryons and photons are interacting via Thomson scattering. Another example will be studied on chapter 5,

where we will revisit the continuity and Euler equations when the dark matter component is coupled to a dark energy scalar field.

The continuity and Euler equations are appropriate to describe non-relativistic species, such as CDM and baryons (at the times of interest, after recombination). The cosmological evolution of relativistic species, such as photons or massless neutrinos, requires the Boltzmann equation formalism, in which the energy-momentum tensor is defined in terms of integrals over the full distribution function [88].

2.4.3 Pressure perturbations

Let us now briefly discuss pressure perturbations. For barotropic fluids — those whose pressure depends only on the energy density — one can express δP as

$$\delta P = c_s^2 \delta \rho, \quad c_s^2 = \frac{\dot{P}}{\dot{\rho}}, \quad (2.51)$$

where c_s^2 is the adiabatic sound speed squared. In general, however, there will also exist a dependency on the entropy such that δP can be split as

$$\delta P = c_s^2 \delta \rho + \dot{P} \Gamma, \quad (2.52)$$

where the first term is the adiabatic part, whereas the second is the entropic (non-adiabatic) one. The entropy perturbation Γ can be expressed as

$$\Gamma = \frac{\delta P}{\dot{P}} - \frac{\delta \rho}{\dot{\rho}}. \quad (2.53)$$

Defined in this way Γ is gauge invariant and represents the displacement of hypersurfaces of uniform pressure and uniform energy density. On chapter 4, we study the adiabaticity condition $\Gamma = 0$ in the context of diffeomorphisms.

2.4.4 Sub-horizon evolution

Current LSS surveys are restricted to scales much smaller than the Hubble horizon. These probes are only able to measure k -modes with $k \gg aH$. At late times, the contribution of relativistic species to the total energy density of the Universe becomes negligible. Then, setting $P = 0$, the continuity and Euler equations, (2.47) and (2.48), for non-relativistic matter in the sub-horizon

limit become

$$\dot{\delta}_{\text{m}} + \frac{k^2}{a^2} v_{\text{m}} = 0, \quad (2.54)$$

$$\dot{v}_{\text{m}} - \Phi = 0, \quad (2.55)$$

whereas from eq. (2.43) we find the Poisson equation

$$\frac{k^2}{a^2} \Phi = -4\pi G_N \rho_{\text{m}} \delta_{\text{m}}. \quad (2.56)$$

These equations can be combined, resulting in the evolution equation for the density contrast:

$$\ddot{\delta}_{\text{m}} + 2H\dot{\delta}_{\text{m}} - 4\pi G_N \rho_{\text{m}} \delta_{\text{m}} = 0. \quad (2.57)$$

From the previous expression, we see that the evolution of the density contrast is sourced by the gravitational potential and has a friction term due to the Hubble expansion. Furthermore, eq. (2.57) has two solutions: a growing mode and a decaying one.

Since the evolution equation for density contrast is independent of the wavenumber k , one can decompose the growing solution δ_{m} into a normalized, k -independent linear growth factor $D(t)$, and the initial density contrast δ_0 :

$$\delta_{\text{m}}(t, \mathbf{k}) = D(t) \delta_0(\mathbf{k}). \quad (2.58)$$

Defining the linear growth rate $f(t)$ as

$$f(t) = \frac{\text{d} \ln D}{\text{d} \ln a}, \quad (2.59)$$

the evolution equation (2.57) can be rewritten as

$$\frac{\text{d}f}{\text{d} \ln a} + \left(2 + \frac{\text{d} \ln H}{\text{d} \ln a}\right) f + f^2 - \frac{3}{2} \Omega_{\text{m}} = 0, \quad (2.60)$$

where $\Omega_{\text{m}} = 8\pi G_N \rho_{\text{m}} / 3H^2$ is the dimensionless matter density parameter. The growth rate, which quantifies the linear growth of matter perturbations in a Hubble time, can be parameterized as

$$f = \Omega_{\text{m}}^\gamma, \quad (2.61)$$

where γ is called the growth index. For General Relativity, assuming a constant dark energy

equation of state w , the fitting

$$\gamma = 0.55 + 0.05[1 + w(z = 1)] \quad (2.62)$$

describes the evolution of the growth rate extremely well [90].

The velocity potential can also be expressed in terms of the growth factor. From the continuity equation (2.54), it reads

$$v(t, \mathbf{k}) = -\frac{a^2 H}{k^2} f(t) \delta(t, \mathbf{k}). \quad (2.63)$$

The study of large-scale structures aims at recovering cosmological information from the analysis of matter clustering. The overdensity δ_m by itself does not provide such information. To quantify the clumpiness, we resort to the 2-point correlation function, or its Fourier-space version, the power spectrum.

2.4.5 Power spectrum

It is important to note that the density contrast $\delta(x)$ is not a deterministic quantity. In fact, it is a statistical random field with zero mean value, $\langle \delta \rangle = 0$ and variance

$$\langle \delta(\mathbf{k}) \delta^*(\mathbf{k}') \rangle = (2\pi)^3 P(\mathbf{k}) \delta_D(\mathbf{k} - \mathbf{k}'), \quad (2.64)$$

in Fourier space, where $P(\mathbf{k})$ is the power spectrum, and the Dirac delta function was introduced to make it explicit that each k -mode evolves independently.

Note that, from the Poisson equation (2.56), δ_m can be written in terms of the scalar potential Ψ . Indeed, the matter power spectrum can be expressed as

$$P_m(k, t) = D^2(t) M^2(k) P_\Psi(k), \quad (2.65)$$

where the growth of perturbations is parameterized using the growth function $D(t)$ and $M(k)$ is defined as

$$M(k) = \frac{2k^2 T(k)}{3\Omega_m^{(0)} H_0^2}. \quad (2.66)$$

In the previous expression $T(k)$ is the matter transfer function [91], which describes the evolution at the epoch of horizon crossing and matter-radiation transition, and $P_\Psi(k)$ is the primordial power spectrum of the curvature potential, set during inflation.

Primordial perturbations Ψ_p have zero mean and variance given by its power spectrum. In most inflationary models, the inflaton field follows a Gaussian distribution, such that the perturbations are completely characterized by the $P_\Psi(k)$. Current observations are consistent with purely

adiabatic curvature perturbations, as predicted by simple, single-field inflationary models [92], rendering the primordial power spectrum a function of k only,

$$P_{\Psi}(k) = \frac{2\pi^2}{k^3} A_s \left(\frac{k}{k_0} \right)^{n_s-1}, \quad (2.67)$$

where A_s is the scalar amplitude and n_s is the spectral index. Current constraint from the Planck mission [92] set $n_s = 0.9649 \pm 0.0042$ at 68% CL, such that $P_{\Psi}(k)$ is said to be nearly scale invariant.

The matter power spectrum contains a lot of information about structure formation. For instance, from the position of its peak one can infer the size of the cosmological horizon at the time of matter-radiation equality; the wiggles represent acoustic features imprinted by baryons due to the oscillation of the baryon-photon fluid before the recombination, a process known as baryon acoustic oscillation (BAO).

But the observed power spectrum obtained by LSS probes is in general slightly different from the matter power spectrum, which can be calculated for a given model. Its shape on small scales is distorted due to the small peculiar velocities of the galaxies used to trace the underlying matter distribution. This effect, known as redshift space distortion (RSD), can be used as a cosmological probe, and is the main topic of the next section (see also appendix C for a discussion about the galaxy power spectrum).

2.5 Redshift space distortions

Galaxy surveys infer the distance to galaxies from their redshift. However, galaxies tend to fall towards overdense regions in a coherent motion. Galaxy maps produced by estimating distances from observed radial velocities, which include components from both the Hubble flow and peculiar velocities driven by the clustering of matter, will then show an anisotropic galaxy distribution. Due to the peculiar velocities of galaxies, there is an error in the determination of distances. In other words, the positions are measured in the redshift space, whose coordinates are generally different from those in the real space.

The observed redshift has two contribution, the first coming from the Hubble expansion while the second from the peculiar velocities along the line of sight:

$$1 + z_{\text{obs}} \simeq (1 + z) \left(1 + \frac{v_{\parallel}(\mathbf{r})}{c} \right). \quad (2.68)$$

Hence, the inferred position $\mathbf{s}$ in redshift space is related to $\mathbf{r}$, the position in real space, as

$$\mathbf{s} = \mathbf{r} + \frac{v_{\parallel}(\mathbf{r})}{aH} \hat{\mathbf{r}}. \quad (2.69)$$

The number of galaxies in a particular region is the same in both real and redshift spaces $n(\mathbf{s}) d^3s = n(\mathbf{r}) d^3r$. Then, defining the density contrast

$$\delta(\mathbf{x}) = \frac{n(\mathbf{x})}{\bar{n}} - 1, \quad (2.70)$$

where $\bar{n}$ is the average number density, assumed to be the same in real and redshift space, imposing that the mapping conserves the number density we find

$$\delta(\mathbf{s}) = J(1 + \delta(\mathbf{r})) - 1, \quad (2.71)$$

where J is the Jacobian, given by

$$J = \left| \frac{d\mathbf{r}}{d\mathbf{s}} \right| = \left(1 + \frac{1}{aH} \frac{v_{\parallel}}{r} \right)^{-2} \left(1 + \frac{1}{aH} \frac{\partial v_{\parallel}}{\partial r} \right)^{-1}. \quad (2.72)$$

The first term is negligible, since

$$\frac{v_{\parallel}}{L} \ll k v_{\parallel}, \quad (2.73)$$

such that $kL \gg 1$ where L is the characteristic size of the survey and k is the Fourier mode of interest. The largest scales the survey can probe are $k \sim L^{-1}$. Hence, the modes that can effectively be measured are those with $kL \gg 1$. Therefore, the Jacobian can be expanded as

$$J \simeq \left(1 - \frac{1}{aH} \frac{\partial v_{\parallel}}{\partial r} \right) \quad (2.74)$$

and, from eq. (2.71), the density contrast in redshift space becomes

$$\delta(\mathbf{s}) = \delta(\mathbf{r}) - \frac{1}{aH} \frac{\partial v_{\parallel}(\mathbf{r})}{\partial r}. \quad (2.75)$$

Assuming that the angle being observed is not too wide and the distance between the observer and the galaxy's plane is much larger than the distance between galaxies, we can fix the direction of the line of sight, say $\hat{\mathbf{r}} \simeq \hat{\mathbf{z}}$. Then, in this distant observer approximation:

$$\delta(\mathbf{s}) = \delta(\mathbf{r}) - \frac{1}{aH} \frac{\partial v_z(\mathbf{r})}{\partial z}. \quad (2.76)$$

We now introduce the velocity potential v as

$$\vec{\mathbf{v}}(\mathbf{r}) = -\frac{1}{a} \vec{\nabla} v(\mathbf{r}), \quad (2.77)$$

such that, moving to the Fourier space, the density contrast given in eq. (2.76) becomes

$$\delta_s(\mathbf{k}) = \int d^3x \left(\delta(\mathbf{x}) - \frac{1}{aH} \frac{\partial v_z(\mathbf{x})}{\partial z} \right) e^{-i\mathbf{k} \cdot \mathbf{x}} \quad (2.78)$$

$$= \delta(\mathbf{k}) - \frac{1}{a^2 H} \int d^3x \left(\int \frac{d^3k'}{(2\pi)^3} v(\mathbf{k}') (k'_z)^2 e^{i\mathbf{k}' \cdot \mathbf{x}} \right) e^{-i\mathbf{k} \cdot \mathbf{x}} \quad (2.79)$$

$$= \delta(\mathbf{k}) - \frac{1}{a^2 H} k_z^2 v(\mathbf{k}). \quad (2.80)$$

Galaxy surveys measure the galaxy power spectra and the previous expression maps the galaxy density contrast from real to redshift space. But what can be calculated from cosmological models is the matter power spectrum. Hence, we need a relation between the distribution of galaxies and the fluctuations in the matter field. These are not exactly equal, but nevertheless, assuming that a given population of galaxies is a fair tracer of the underlying mass distribution, we introduce the k -independent linear galaxy bias $b_g(t)$ as

$$\delta_g(\mathbf{k}, t) = b_g(t) \delta_m(\mathbf{k}, t). \quad (2.81)$$

We also assume that, since galaxies move according to a common gravitational field, its velocities are not biased with respect to the matter velocity field, $v_g = v_m$; a valid assumption, at least in the Λ CDM model [93].

According to eq. (2.63), the matter velocity potential, in terms of the density contrast, is given by

$$v_m(\mathbf{k}) = -\frac{a^2 H}{k^2} f_m \delta_m(\mathbf{k}), \quad (2.82)$$

where $f_m(t)$ is the matter linear growth rate. Hence, eq. (2.80) becomes

$$\delta_g^s(\mathbf{k}) = (b_g + f_m \mu^2) \delta_m(\mathbf{k}) \quad (2.83)$$

where $\mu = (\mathbf{k} \cdot \hat{\mathbf{z}})/k$ is the angle between the wave vector and the line of sight. Note that we explicitly distinguish the redshift-space galaxy density contrast δ_g^s from the real-space matter density contrast δ_m . Thus, the galaxy power spectrum in redshift space in terms of the matter power spectrum in real space is given by

$$P_g^s(\mathbf{k}) = (b_g + f_m \mu^2)^2 P_m(\mathbf{k}), \quad (2.84)$$

the so-called Kaiser formula [94]. One can similarly define the relation between redshift space and

real space galaxy power spectra:

$$P_{\text{g}}^s(\mathbf{k}) = (1 + \beta\mu^2)^2 P_{\text{g}}(\mathbf{k}), \quad (2.85)$$

where $\beta = f_{\text{m}}/b_{\text{g}}$ is the redshift-space distortion parameter.

Due to the cosmological principle, the matter power spectrum usually depends only on the magnitude of each mode, $P_{\text{m}}(\mathbf{k}) = P_{\text{m}}(k)$. However, this is no longer true for the observed power spectrum. Due to the Kaiser effect, the redshift-space galaxy power spectrum will be distorted, having a clustering anisotropy induced by the peculiar velocities. On large scales, the clustering will be squashed along the line of sight while transverse modes are not modified. On the other hand, since galaxies are thought to be virialized at small scales, having random motion, the density field will appear to be stretched along the line of sight, the so-called Finger-of-God effect [95]

Redshift space distortions are an important probe of matter clustering since the velocity field is directly linked to the growth function [96]. Density perturbations grow in different ways depending on the gravitational theory. Thus, RSD lends itself to testing General Relativity and its extensions. In fact, measuring the Kaiser effect is one of the main drives of current and future galaxy redshift surveys such as PFS [97], DESI [98], Roman Space Telescope [99], EUCLID [100], and SKA [101].

3. Disformal Transformations

Metric transformations have played a significant role in the study and development of gravitational physics. Employing conformal transformations, one can represent a given theory in different ways. These different representations allow the study of several properties of the original theory, providing import physical insights. The natural generalization of the conformal transformation is the disformal one, and the inclusion of derivative couplings allows the exploration of a wider space of healthy gravitational theories.

In this chapter, we review the main features of conformal and disformal transformations, as well as their applications in modified gravity theories.

3.1 Conformal transformations and Brans-Dicke theory

In this section, we briefly review the basic properties of the conformal metric transformation and its applications in gravity theories. In particular, we will discuss the Brans-Dicke theory and explore how conformal transformations allow different representations of the same theory. For a deeper discussion about conformal transformations and its role in gravitational physics, the reader is referred to [53, 102].

3.1.1 Conformal transformations

The conformal transformation is a mathematical tool that proved to be remarkably useful in the study of General Relativity and its extensions. It amounts to a rescaling of the spacetime metric $g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu}$ such that geometrical quantities are also transformed. Applied at the action level, it is also used to put a given theory in different representations. When considering scalar-tensor theories, even though not strictly necessary, the transformation is usually accompanied by a field redefinition $\varphi \rightarrow \phi$, in order to recast the scalar field Lagrangian into its canonical form. The sets of dynamical variables $(\tilde{g}_{\mu\nu}, \varphi)$ and $(g_{\mu\nu}, \phi)$, together with their respective actions, describe the same theory but in different representations, or frames.

The conformal transformation is defined as

$$g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = A g_{\mu\nu}, \quad (3.1)$$

where the conformal factor A is a nowhere vanishing, regular function. The transformation can be understood as a rescaling: it changes the lengths of spacelike and timelike intervals, as well as the norms of spacelike and timelike vectors. On the other hand, null vectors and intervals are kept unchanged. As a result, light cones are invariant under conformal transformations, and the causal structure remains unaltered.

Geometrical quantities associated to the original metric are also modified. The inverse metrics and the determinants are related as:

$$\tilde{g}^{\mu\nu} = A^{-1} g^{\mu\nu}, \quad \tilde{g} = A^4 g. \quad (3.2)$$

The Christoffel symbol $\Gamma_{\mu\nu}^\rho$, defined in terms of the original metric $g_{\mu\nu}$,

$$\Gamma_{\mu\nu}^\rho = \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}), \quad (3.3)$$

transforms as

$$\tilde{\Gamma}_{\mu\nu}^\rho = \Gamma_{\mu\nu}^\rho + \frac{1}{2} (\delta_\mu^\rho \nabla_\nu \ln A + \delta_\nu^\rho \nabla_\mu \ln A - g_{\mu\nu} \nabla^\rho \ln A). \quad (3.4)$$

With the relation between the connections, one can readily determine the transformation of the covariant derivative [103]:

$$\tilde{\nabla}_\mu v^\nu = \nabla_\mu v^\nu + \frac{1}{2} (\delta_\mu^\nu \nabla_\rho \ln A + \delta_\rho^\nu \nabla_\mu \ln A - g_{\mu\rho} \nabla^\nu \ln A) v^\rho. \quad (3.5)$$

Finally, the transformation of the Ricci tensor is given by

$$\tilde{R}_{\mu\nu} = R_{\mu\nu} - \nabla_\mu \nabla_\nu \ln A - \frac{1}{2} g_{\mu\nu} \square \ln A + \frac{1}{2} \nabla_\mu \ln A \nabla_\nu \ln A - \frac{1}{2} g_{\mu\nu} \nabla^\rho \ln A \nabla_\rho \ln A, \quad (3.6)$$

whereas the Ricci scalar,

$$\tilde{R} = A^{-1} \left(R - 3 \square \ln A - \frac{3}{2} g^{\mu\nu} \nabla_\mu \ln A \nabla_\nu \ln A \right). \quad (3.7)$$

Naturally, the conformal transformation also modifies the components of the energy-momentum

tensor. The transformation is given by,

$$\tilde{T}^{\mu\nu} = \sqrt{\frac{\tilde{g}}{g}} \frac{\delta g_{\alpha\beta}}{\delta \tilde{g}_{\mu\nu}} T^{\alpha\beta}, \quad (3.8)$$

such that the following relations hold:

$$\tilde{T}^{\mu\nu} = A^{-3} T^{\mu\nu}, \quad \tilde{T}^\mu{}_\nu = A^{-2} T^\mu{}_\nu, \quad \tilde{T}_{\mu\nu} = A^{-1} T_{\mu\nu}, \quad \tilde{T} = A^{-2} T. \quad (3.9)$$

In general, tensor quantities are not conformally invariant, as just shown. Therefore, the physical interpretation of equations involving tensor will be modified accordingly. For example, let us consider the conservation of the energy momentum tensor:

$$\nabla^\mu T_{\mu\nu} = 0. \quad (3.10)$$

If this equation is satisfied, it is clear from eqs. (3.5) and (3.9) that $\tilde{T}_{\mu\nu}$ is usually not a conserved quantity. Indeed, it is straightforward to show that eq. (3.10) becomes

$$\tilde{\nabla}^\mu \tilde{T}_{\mu\nu} = -\frac{1}{2} \tilde{T} \tilde{\nabla}_\nu \ln A. \quad (3.11)$$

However, note that $\tilde{T}_{\mu\nu}$ is conserved only if the trace $\tilde{T}$ vanishes. This reflects the fact that null vectors are conformal invariant. Indeed, since the energy-momentum tensor for electromagnetism is trace-free, photons in the new frame still propagate along null geodesics, and the structure of light-cones is preserved.

When $\tilde{T}$ is non-vanishing, the geodesic equation describing the motion of particles in the new frame will be modified. These particles will experience a new force proportional to the gradient of the conformal factor.

3.1.2 Brans-Dicke and its representations

In several extensions of General Relativity, gravity is mediated by a scalar field in addition to the metric tensor field. Considering an additional scalar degree of freedom is arguably the simplest way in which one can build alternatives to GR. For that reason scalar-tensor theories have been exhaustively studied for quite some time, specially after discovery of the current accelerated expansion of the Universe. In fact, several attempts have been made to generalize the cosmological constant using scalar fields with the goal of solving, for instance, the fine-tuning and coincidence problems [72, 104–108].

Brans-Dicke (BD) gravity is the prototype scalar-tensor theory. Its motivation was to fully

incorporate Mach's principle into General Relativity. This principle states that forces observed in a local inertial frame may be interpreted as gravitational effects due to the average motion of distant astronomical objects [109]. General Relativity only partially incorporates this idea: although the mass distribution affects spatial geometries, it does not uniquely specify the geometrical structure. An example of a GR solution that violates Mach's principle is the Gödel universe [110]. One way to implement Mach's principle is to allow the gravitational coupling to vary from one point to the other to take into account the surrounding matter distribution.

In the Brans-Dicke theory [10], this feature is implemented using a scalar ϕ such that G_N is no longer constant, but linked to a cosmic field. The theory is described by the action

$$S(\tilde{g}_{\mu\nu}, \varphi, \psi_m) = \frac{1}{16\pi G_N} \int d^4x \sqrt{-\tilde{g}} \left(\varphi \tilde{R} - \frac{\tilde{\omega}_{\text{BD}}}{\varphi} \tilde{g}^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - U(\varphi) \right) + S_m(\tilde{g}_{\mu\nu}, \psi_m), \quad (3.12)$$

where the constant ω_{BD} is the Brans-Dicke parameter, and S_m is the action for the matter fields ψ_m . The coupling $\varphi \tilde{R}$ translates into a modification of the Newtonian constant. Indeed, using the parameterized post-Newtonian (PPN) formalism, one can show that in the weak field limit [111]

$$\tilde{G} = \frac{2\omega_{\text{BD}} + 4}{2\omega_{\text{BD}} + 3} \frac{1}{\varphi_0} G_N, \quad (3.13)$$

where φ_0 is the asymptotic value of the scalar field, far from the system being considered.

Upon a conformal transformation (3.1),

$$\tilde{g}_{\mu\nu} = A(\varphi) g_{\mu\nu} = \varphi^{-1} g_{\mu\nu}, \quad (3.14)$$

the action (3.12) can be recast as [55]

$$S(g_{\mu\nu}, \phi, \psi_m) = \int d^4x \sqrt{-g} \left(\frac{1}{16\pi G_N} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right) + S_m(A(\phi) g_{\mu\nu}, \psi_m), \quad (3.15)$$

where we employed the following redefinition of the scalar field:

$$\frac{1}{\varphi^2} \left(\frac{d\varphi}{d\phi} \right)^2 = \left(\frac{d \ln A}{d\phi} \right)^2 = \frac{16\pi G_N}{2\omega_{\text{BD}} + 3}, \quad (3.16)$$

$$V(\phi) = \frac{1}{16\pi G_N} \frac{U(\varphi)}{\varphi^2}. \quad (3.17)$$

The redefinition $\varphi \rightarrow \phi$ is done with the purpose of expressing the scalar Lagrangian in the canonical form. In this simple example of a scalar-tensor theory, the conformal transformation plays a crucial role: the actions (3.12) and (3.15) are said to be different representations of the same gravitational theory, namely, the Brans-Dicke gravity.

Expressed in terms of the variables $(\tilde{g}_{\mu\nu}, \varphi)$, the action is said to be in the Jordan frame (JF)

representation, in which the scalar field φ is a gravitational degree of freedom in addition to the metric $\tilde{g}_{\mu\nu}$. These two fields are non-minimally coupled to each other, such that even in the complete absence of matter, the gravitational dynamics differs from General Relativity. On the other hand, matter is minimally coupled to gravity through the metric $\tilde{g}_{\mu\nu}$.

In terms of $(g_{\mu\nu}, \phi)$, as in eq. (3.15), the BD theory is said to be in the Einstein frame (EF) representation. In this representation, the gravitational action consists of the Einstein-Hilbert term and a minimally coupled scalar field ϕ . However, matter is non-minimally coupled and all particles feel a new, universal force, mediated by the scalar ϕ in addition to the one due to the metric tensor. We already touched upon this feature when the transformation of the energy-momentum tensor, eq. (3.11), was discussed: the conservation of the EMT in one frame does not necessarily imply the conservation in the other.

The Jordan and Einstein frame representations are equivalent, at least in the classical level, and the conformal transformation can be interpreted as a change in local units of length [55, 57, 58, 61, 112].

The BD action (3.12) can be extended to encompass a larger class of scalar-tensor theories. This is done with the introduction of two new function that replace $\varphi\tilde{R}$ and the Brans-Dicke parameter $\tilde{\omega}_{\text{BD}}$:

$$S(\tilde{g}_{\mu\nu}, \varphi, \psi_{\text{m}}) = \int d^4x \sqrt{-\tilde{g}} \left(\frac{1}{2} f(\varphi, \tilde{R}) - \frac{1}{2} \zeta(\varphi) \tilde{g}^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \right) + S_{\text{m}}(\tilde{g}_{\mu\nu}, \psi_{\text{m}}). \quad (3.18)$$

Here f and ζ are generic functions and, for simplicity, we set $8\pi G_N = 1$. Theories such as $f(R)$ and dilaton gravity can be recovered from the action (3.18) with a suitable choice of f and ζ .

The $f(R)$ theory [113–115] corresponds to the choice $f(\varphi, \tilde{R}) = f(\tilde{R})$ and $\zeta = 0$. The effective theory coming from the low-energy limit of string theory known as dilaton gravity [116] corresponds to $f = 2e^{-\varphi}R - 2U(\varphi)$ and $\zeta = -2e^{-\varphi}$, where we have introduced the dilaton potential $U(\varphi)$. Naturally, both the Einstein-Hilbert and Brans-Dicke actions can also be recovered.

Even for this general class of gravitational theories it is possible to find their representation in the Einstein frame after applying a conformal transformation to its Jordan-frame representation, eq. (3.18). With the choice

$$A = \left(\frac{\partial f}{\partial \tilde{R}} \right)^{-1}. \quad (3.19)$$

the conformal transformation takes the action (3.18) to its Einstein frame representation [56].

3.2 Generalizing the conformal transformation

Given the usefulness of conformal transformations in the study of gravitational theories, a natural question is whether it can be generalized. This extension could, in principle, offer further insights about the structure and phenomenology of scalar-tensor theories. The fact the one can put the same theory in two different representations is interesting not only in practical applications (such as solving the equations of motion) but also provides insights about the theory itself.

Disformal transformations are the natural extension of the conformal ones. It was introduced by Bekenstein [54] as an attempt to generalize the concept of frame representation. The goal was finding the most general relation between two frames allowed by physics. To solve this problem, he assumed that the physical geometry, the one describing matter dynamics, is Finslerian, whereas the gravitational one is simply Riemannian. However, to preserve causality and the weak equivalence principle, Bekenstein showed that the Finsler geometry has to reduce the Riemannian one. The physical metric is then related to the gravitational one by a generalization of the conformal transformation, with the addition of a scalar field. This renders the disformal transformation the most general metric transformation involving a scalar field that respects both causality and the weak equivalence principle.

The dependence on the scalar field is introduced using derivative couplings. In the simplest case, the order of the derivatives is truncated to prevent ghost instabilities [54, 117]. Then, the disformal transformation from the gravitational-frame metric $g_{\mu\nu}$ to the matter-frame metric $\tilde{g}_{\mu\nu}$ is defined as [54]

$$g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = A(\phi, X)g_{\mu\nu} + B(\phi, X)\phi_\mu\phi_\nu, \quad (3.20)$$

where $X = -g^{\mu\nu}\phi_\mu\phi_\nu/2$, with $\phi_\mu = \partial_\mu\phi$. The factors $A(\phi, X)$ and $B(\phi, X)$ are arbitrary functions of ϕ and X , being respectively called conformal and disformal factors.

In contrast with the purely conformal case, disformal transformations correspond not only to a rescaling, but it also introduces a privileged direction. It changes the local units of length along the gradient of ϕ and perpendicular to it in different ways. For example, consider a null four-vector v^μ in the $g_{\mu\nu}$ frame, i.e. $g^{\mu\nu}v_\mu v_\nu = 0$. After the transformation, its length becomes

$$\tilde{g}_{\mu\nu}v^\mu v^\nu = B(v^\mu\phi_\mu)^2,$$

and if $B > 0$ the vector is now spacelike whereas for $B < 0$ it will be timelike.

3.2.1 Conditions for a well-defined transformation

Even though somewhat arbitrary, the choice of conformal and disformal functions has to satisfy a few conditions. These requirements must be enforced to guarantee that the metric transformation is well-defined. The transformation must preserve the Lorentz signature, causal structure, produce a non-singular volume element, and allow the existence and non-singularity of the inverse transformation. For our purposes, it is also important to ensure that the inverse transformation $\tilde{g}_{\mu\nu} \rightarrow g_{\mu\nu}$ is well-behaved.

To have a well-defined invariant integration volume, $\tilde{g}_{\mu\nu}$ must have a non-vanishing determinant. The determinant of a rank-2 tensor with Lorentzian signature in four dimensions is given by

$$-g = \frac{1}{4!} \epsilon^{\mu\nu\rho\sigma} \epsilon^{\alpha\beta\gamma\lambda} g_{\mu\alpha} g_{\nu\beta} g_{\rho\gamma} g_{\sigma\lambda}, \quad (3.21)$$

such that, for the disformal transformation (3.20), we find the requirement [18, 62]

$$\frac{\tilde{g}}{g} = A^3(A - 2BX) \geq 0. \quad (3.22)$$

In principle, this condition should be imposed for all values of X , i.e. $A > 0$ and $A > 2BX$ [118]. However, it was shown that, in cosmological settings, the second condition is dynamically satisfied. Whenever X approaches $A/(2B)$, the scalar field slows down, avoiding any instabilities. This behavior was observed for $B(\phi) > 0$ in both disformal couplings to matter [62, 119] and scalar field self couplings [120]. In chapter 6, we will introduce a model with disformal couplings to dark matter, with $A = 1$ and $B = B(X)$. The condition $A > 2BX$ will be then enforced for all possible values of X .

The condition of a non-vanishing metric determinant is also important in the definition of the inverse metric $\tilde{g}^{\mu\nu}$. Using $\tilde{g}^{\mu\rho}\tilde{g}_{\rho\nu} = \delta_\nu^\mu$, one can verify that the inverse metrics are related via

$$\tilde{g}^{\mu\nu} = \frac{1}{A} \left(g^{\mu\nu} - \frac{B}{A - 2BX} \phi^\mu \phi^\nu \right), \quad (3.23)$$

where $\phi^\mu = g^{\mu\nu} \phi_\nu$. Hence, whenever the condition (3.22) is satisfied, $\tilde{g}^{\mu\nu}$ will be well-defined.

We would like to extend the concept of Jordan and Einstein-frame representations to the case of disformal transformations. For that, we must guarantee that the inverse transformation $\tilde{g}_{\mu\nu} \rightarrow g_{\mu\nu}$ is also possible. For general disformal transformations, one can address this issue by evaluating the Jacobian determinant of the transformation [117]. The inverse function $g_{\mu\nu}(\tilde{g}_{\alpha\beta})$ exists around

any point for which

$$\left| \frac{\partial \tilde{g}_{\mu\nu}}{\partial g_{\alpha\beta}} \right| = \left| A \delta_\mu^\alpha \delta_\nu^\beta + \frac{1}{2} (A_X g_{\mu\nu} + B_X \phi_\mu \phi_\nu) \phi^\alpha \phi^\beta \right| \neq 0. \quad (3.24)$$

The determinant of the Jacobian can be found studying the set of eigenvalues and eigentensors of the Jacobian matrix. From this analysis, one finds that whenever

$$A = 0, \quad A - A_X X + 2B_X X^2 = 0, \quad (3.25)$$

it is not possible to find the inverse transformation. Note that, in the previous expression, we have defined $A_X = \partial A / \partial X$. The same notation was used for B_X .

The inverse map $g_{\mu\nu}(\tilde{g}_{\alpha\beta})$ can be found around any non-degenerate point. Assuming the ansatz

$$g_{\mu\nu} = \tilde{A}(\phi, \tilde{X}) \tilde{g}_{\mu\nu} + \tilde{B}(\phi, \tilde{X}) \phi_\mu \phi_\nu, \quad (3.26)$$

$$g^{\mu\nu} = \frac{1}{\tilde{A}(\phi, \tilde{X})} \left(\tilde{g}^{\mu\nu} - \frac{\tilde{B}(\phi, \tilde{X})}{\tilde{A}(\phi, \tilde{X}) - 2\tilde{B}(\phi, \tilde{X})\tilde{X}} \tilde{\phi}^\mu \tilde{\phi}^\nu \right), \quad (3.27)$$

$$\tilde{X}(\phi, \tilde{X}) = -\frac{1}{2} \tilde{g}^{\mu\nu} \phi_\mu \phi_\nu, \quad (3.28)$$

where $\tilde{\phi}^\mu = \tilde{g}^{\mu\nu} \phi_\nu$, we find the conditions

$$\tilde{A}(\phi, \tilde{X}) = \frac{1}{A(\phi, X)}, \quad \tilde{B}(\phi, \tilde{X}) = -\frac{B(\phi, X)}{A(\phi, X)}, \quad \tilde{X} = \frac{X}{A(\phi, X) - 2B(\phi, X)X}. \quad (3.29)$$

Besides its importance in determining the existence of the inverse transformation, the Jacobian is also important in the analysis of the energy-momentum tensor. As we saw in section 3.1, the transformation of the energy-momentum tensor is given by

$$\tilde{T}^{\mu\nu} = \sqrt{\frac{g}{\tilde{g}}} \frac{\delta g_{\alpha\beta}}{\delta \tilde{g}_{\mu\nu}} T^{\alpha\beta}, \quad (3.30)$$

which is relatively simple in the conformal case. In section 3.4, where we extend the analysis to disformal transformations, the Jacobian will be used to relate the energy fluxes as measured by observers in different frames.

3.2.2 Geometrical quantities

It is useful to show how geometrical quantities change under disformal transformations. The Christoffel symbol $\Gamma^\rho_{\mu\nu}$, defined in terms of the derivatives of the metric $g_{\mu\nu}$, can be written as

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}), \quad (3.31)$$

while $\tilde{\Gamma}^\rho_{\mu\nu}$ is similarly written in term of $\tilde{g}_{\mu\nu}$. Then, using eq. (3.20), it is straightforward to show that [121]

$$\begin{aligned} C^\rho_{\mu\nu} &\equiv \tilde{\Gamma}^\rho_{\mu\nu} - \Gamma^\rho_{\mu\nu} = \frac{1}{2}\tilde{g}^{\rho\sigma}(\nabla_\mu\tilde{g}_{\sigma\nu} + \nabla_\nu\tilde{g}_{\sigma\mu} - \nabla_\sigma\tilde{g}_{\mu\nu}) \\ &= \delta^\rho_{(\mu}\nabla_{\nu)}\ln A - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{g}^{\rho\sigma}\nabla_\sigma\ln A + \frac{B}{A-2XB}\phi^\rho\phi_{\mu\nu} \\ &\quad + \frac{B}{A-2XB}\phi^\rho\phi_{(\mu}\nabla_{\nu)}\ln(B/A) - \frac{B}{2}\phi_\mu\phi_\nu\tilde{g}^{\rho\sigma}\nabla_\sigma\ln(B/A), \end{aligned} \quad (3.32)$$

where we have introduced the notation $\phi_{\mu\nu} = \nabla_\mu\nabla_\nu\phi$. Also, note that the metric $\tilde{g}_{\mu\nu}$ and its inverse $\tilde{g}^{\mu\nu}$ are to be replaced using eqs. (3.20) and (3.23), respectively. Furthermore, since it is defined as the difference between two connections, the quantity $C^\rho_{\mu\nu}$ is in fact a tensor [103].

The transformation of other geometrical quantities is conveniently expressed in terms of the tensor $C^\rho_{\mu\nu}$. For example, with the transformations of the Christoffel symbol at hand, one can compute how the covariant derivative changes. Let $\tilde{\nabla}_\mu$ be the covariant derivative associated with $\tilde{g}_{\mu\nu}$ and ∇_μ the one related to $g_{\mu\nu}$. Then, the covariant derivative transforms as [103]

$$\tilde{\nabla}_\mu u^\nu = \nabla_\mu u^\nu + C^\nu_{\mu\rho} u^\rho. \quad (3.33)$$

We can use the definition of the Riemann tensor as the commutator of the covariant derivatives acting on a vector field,

$$2\nabla_{[\mu}\nabla_{\nu]}v^\alpha = R^\alpha_{\beta\mu\nu}v^\beta, \quad (3.34)$$

to compute its transformation:

$$\tilde{R}^\alpha_{\beta\mu\nu} = R^\alpha_{\beta\mu\nu} + 2C^\alpha_{\rho[\mu}C^\rho_{\nu]\beta} + 2\nabla_{[\mu}C^\alpha_{\nu]\beta}. \quad (3.35)$$

Then, the Ricci tensor, $R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}$ transforms under eq. (3.20) as [121]

$$\tilde{R}_{\mu\nu} = R_{\mu\nu} + \nabla_\rho C^\rho_{\mu\nu} - C^\rho_{\mu\sigma}C^\sigma_{\rho\nu} - \nabla_\mu\nabla_\nu\ln\sqrt{\frac{-\tilde{g}}{-g}} + C^\rho_{\mu\nu}\nabla_\rho\ln\sqrt{\frac{-\tilde{g}}{-g}}, \quad (3.36)$$

whereas, the Ricci scalar can be computed taking the contraction $\tilde{g}^{\mu\nu}\tilde{R}_{\mu\nu}$.

3.2.3 Extending the Jordan and Einstein frames

In the Brans-Dicke theory, we saw that the action in the Jordan frame can be recast in the Einstein frame by applying a conformal transformation. The resulting action is given by the Einstein-Hilbert term plus a minimally-coupled scalar field in the gravity sector, but matter is now non-minimally coupled to the gravitational degrees of freedom.

We would like to extend the concept of frames to the case of disformal transformations. To do so, we follow an approach similar to the one in the conformal case, but in the opposite direction: we first define the theory in the Einstein frame and then transform it into the Jordan-frame representation.

The action in the Einstein frame is then define as

$$S(g_{\mu\nu}, \phi, \psi_m) = \int d^4x \sqrt{-g} \left[\frac{R}{16\pi G_N} + K(\phi, X) \right] + S_m(Ag_{\mu\nu} + B\phi_\mu\phi_\nu, \psi_m), \quad (3.37)$$

where $K(\phi, X)$ represents a generic Lagrangian density for the scalar field. Note that since one gets to choose, in some sense, the function $K(\phi, X)$, it is not strictly necessary to perform a field redefinition. Thus, the scalar gravitational degree of freedom in both frames is identified with the same field ϕ . Using eq. (3.35), the Einstein-frame action becomes

$$S(\tilde{g}_{\mu\nu}, \phi, \psi_m) = \int d^4x \left(\sqrt{-\tilde{g}} \mathcal{L}_{\text{JF}} - \sqrt{-g} \nabla_\alpha \xi^\alpha \right) + S_m(\tilde{g}_{\mu\nu}, \psi_m), \quad (3.38)$$

where we have defined the Jordan frame Lagrangian ¹

$$\mathcal{L}_{\text{JF}} = \sqrt{\frac{g}{\tilde{g}}} \left[\frac{1}{16\pi G_N} g^{\mu\nu} \left(\tilde{R}_{\mu\nu} - 2C^\rho_{\sigma[\rho} C^\sigma_{\mu]\nu} \right) + K(\phi, X(\tilde{X})) \right], \quad (3.39)$$

and $\xi^\alpha = g^{\mu\nu} C^\alpha_{\mu\nu} - g^{\mu\alpha} C^\nu_{\mu\nu}$ appears as a total derivative, not contributing to the equations of motion. Hence, the disformal transformation does not introduce derivatives of the connection, thus keeping only second derivatives of the metric and scalar field in the Lagrangian.

3.3 Disformal transformations and scalar-tensor theories

As already discussed, scalar-tensor extensions are thought to be the simplest way one could modify General Relativity. Brans-Dicke theory, with its implementation of Mach's principle, motivates the search for more general scalar-tensor theories, which could solve current cosmological puzzles such as dark energy and the inflationary universe. Disformal transformations played an important role in this endeavor.

The construction of extended scalar-tensor theories is strongly limited by the Ostrogradsky theorem [122, 123]. It states that any non-degenerate theory with a Lagrangian containing second or higher derivatives with respect to time will have a Hamiltonian that is unbounded from below, such that the system is unstable — lacking a well-defined vacuum state — with unphysical degrees of freedom. General Relativity is special in the sense that Ostrogradsky theorem does not apply to it: even though the Einstein-Hilbert Lagrangian has second-order derivatives of the metric, GR

¹For the transformed Lagrangian in terms of the functions A and B , see [117, 121].

has the right degenerate structure, such that the equations of motion are still second order. In fact, it is the only four-dimensional, Lorentz-covariant local theory avoiding such instabilities [124].

Traditional scalar-tensor theories, such as Brans-Dicke and its extension, eq. (3.18), are described by Lagrangians containing only first derivatives of the fields, naturally avoiding Ostrogradsky instabilities. Generally speaking, one could allow higher-order terms, but it is often the case that these theories will have an extra ghost-like degree of freedom. The Horndeski Lagrangian describes the most general scalar-tensor theory whose equations of motion include up to second derivatives of the fields, despite having second derivative terms in the Lagrangian itself. It is invariant under special disformal transformations with $A(\phi)$, $B(\phi)$, but not under general ones; by invariant, we mean that upon a rearrangement of its terms, it will have the same functional form.

An important mark in the search for a generalized Lagrangian was the realization that disformal transformations do not introduce new degrees of freedom [121]. After that, it was discovered that the Jordan-frame representation of the Einstein-Hilbert action (3.38) contains terms that are not present in Horndeski's Lagrangian, which was thought to be the most general, ghost-free scalar-tensor theory. This realization indicates a loophole in Horndeski's theorem, motivating the creation of a new framework for healthy scalar-tensor theories. In this section, we summarize the recent developments in generalized scalar-tensor theories and the role played by disformal transformations. For a review on this topic, we refer the reader to [125].

Horndeski's theorem [44] states that the most general four-dimensional, Lorentz-covariant, local scalar-tensor theory with second-order equations of motion is given by

$$\mathcal{L}^{\text{H}} = \sum_{i=2}^5 \mathcal{L}_i^{\text{H}} \quad (3.40)$$

with

$$\mathcal{L}_2^{\text{H}} = G_2(\phi, X) \quad (3.41)$$

$$\mathcal{L}_3^{\text{H}} = G_3(\phi, X) \square \phi \quad (3.42)$$

$$\mathcal{L}_4^{\text{H}} = G_4(\phi, X) R + G_{4,X}(\phi, X) [(\square \phi)^2 - \phi^{\mu\nu} \phi_{\mu\nu}] \quad (3.43)$$

$$\mathcal{L}_5^{\text{H}} = G_5(\phi, X) G_{\mu\nu} \phi^{\mu\nu} - \frac{1}{6} G_{5,X}(\phi, X) [(\square \phi)^3 - 3(\square \phi) \phi_{\mu\nu} \phi^{\mu\nu} + 2\phi_{\mu\nu} \phi^{\mu\sigma} \phi^\nu{}_\sigma] \quad (3.44)$$

where $G_{i,X} = \partial G_i / \partial X$.

It is an extension of the galileon theory [126], which is formulated on the Minkowski spacetime, that includes curvature. The galileon Lagrangian cannot be generalized by simply replacing partial derivatives ∂_ν with covariant ones ∇_μ , since this modification leads to ghost instabilities. However, if one includes non-minimal couplings between the galileon scalar field and the curvature, the

extra degrees of freedom are removed [127, 128]. The resulting theory was named generalized galileon [45]. Furthermore, it can be shown that the Horndeski and generalized galileon theories are equivalent [46].

The Horndeski Lagrangian is useful in the study of scalar-tensor theories because it provides a framework that contains several other models, such as General Relativity, quintessence, K-essence, Brans-Dicke gravity, and dilaton. Given its generality, a natural question is whether it is possible to extend the concepts of Jordan and Einstein frame to the Horndeski Lagrangian. It turns out that it is for a model called DBI galileon [129]. With a simple disformal transformation in which both A and B are constants, it is possible to eliminate the non-minimal coupling to the Ricci scalar in this particular limit of the Horndeski theory [62].

The Horndeski Lagrangian is invariant under special disformal transformations $A(\phi)$, $B(\phi)$: with a redefinition of the functions $G_i(\phi, X)$ it is possible to recover the functional form of the Lagrangian (3.40–3.44) [130]. However, that is not the case for transformations with $A(\phi, X)$, $B(\phi, X)$, and the Horndeski Lagrangian is not invariant under general disformal transformations. But since the disformal transformation does not introduce higher-order terms [121], this indicates the possibility of constructing healthy theories beyond Horndeski [117].

It is, indeed, possible to come up with such Lagrangians. For instance, the GLPV theory (also known as Beyond Horndeski) suggests the inclusion of two terms absent in the original Horndeski theory [47]. Furthermore, one can consider even more general theories, in which the healthiness is not directly related to the order of the equations of motion. In the Degenerate Higher-Order Scalar-Tensor theory (DHOST), as long as the Lagrangian satisfies certain degeneracy condition, the theory will have no instabilities [48]. Both theories present enhanced stability under disformal transformations, with GLPV being invariant under $A(\phi)$, $B(\phi, X)$ and DHOST under $A(\phi, X)$, $B(\phi, X)$ [131, 132].

3.4 Energy-momentum tensor

Let us now focus on the transformation properties of the energy-momentum tensor. As in the conformal case, it is not invariant under disformal transformations. Therefore, components of the energy-momentum tensor measured in one frame will, in general, be different from those in the other.

The energy-momentum tensor in the Jordan frame, given by

$$\tilde{T}^{\mu\nu} = \frac{2}{\sqrt{-\tilde{g}}} \frac{\delta S_M}{\delta \tilde{g}_{\mu\nu}}, \quad (3.45)$$

is covariantly conserved, $\tilde{\nabla}_\mu \tilde{T}^{\mu\nu} = 0$, since matter is minimally coupled to the Jordan frame metric

$\tilde{g}_{\mu\nu}$. Furthermore, the energy-momentum tensor in the Einstein frame, $T^{\mu\nu}$, is related to $\tilde{T}^{\mu\nu}$ as

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g_{\mu\nu}} = \sqrt{\frac{-\tilde{g}}{-g}} \frac{\delta \tilde{g}_{\alpha\beta}}{\delta g_{\mu\nu}} \tilde{T}^{\alpha\beta}, \quad (3.46)$$

where the determinant transformation, eq. (3.22), and the Jacobian, eq. (3.24), are respectively given by

$$\tilde{g} = A^3(A - 2BX)g, \quad (3.47)$$

$$\frac{\delta \tilde{g}_{\alpha\beta}}{\delta g_{\mu\nu}} = A\delta_\alpha^\mu \delta_\beta^\nu + \frac{1}{2}(A_X g_{\alpha\beta} + B_X \phi_\alpha \phi_\beta) \phi^\mu \phi^\nu. \quad (3.48)$$

Hence, $T^{\mu\nu}$ is not covariantly conserved: $\nabla_\mu T^{\mu\nu} \neq 0$. In fact, the relation between the energy-momentum tensors in the two frames is

$$T^{\mu\nu} = A^{3/2}(A - 2BX)^{1/2} \left[A\tilde{T}^{\mu\nu} + \frac{1}{2}(A_X g_{\alpha\beta} + B_X \phi_\alpha \phi_\beta) \tilde{T}^{\alpha\beta} \phi^\mu \phi^\nu \right], \quad (3.49)$$

which coincides with the result obtained in [62] for a simpler case of $A = A(\phi)$, $B = B(\phi)$. Written in another form, separating Jordan-frame and Einstein-frame variables,

$$\tilde{T}^{\mu\nu} = A^{-5/2}(A - 2BX)^{-1/2} \left[T^{\mu\nu} - \frac{A_X g_{\alpha\beta} + B_X \phi_\alpha \phi_\beta}{2(A - A_X X + 2B_X X^2)} T^{\alpha\beta} \phi^\mu \phi^\nu \right]. \quad (3.50)$$

Let us now assume that matter in Jordan frame can be described by a perfect fluid, with the energy-momentum tensor, as measured by an observer with four-velocity $\tilde{u}^\mu = dx^\mu/d\tilde{\tau}$, being given by

$$\tilde{T}^{\mu\nu} = (\tilde{\rho} + \tilde{P})\tilde{u}^\mu \tilde{u}^\nu + \tilde{P}\tilde{g}^{\mu\nu}. \quad (3.51)$$

The four-velocities in the Jordan and Einstein frames are related as

$$\tilde{u}^\mu = \frac{dx^\mu}{d\tilde{\tau}} = \frac{d\tau}{d\tilde{\tau}} u^\mu, \quad (3.52)$$

such that, from eq. (3.49), the energy-momentum tensor in the Einstein frame can be expressed as

$$\begin{aligned} T^{\mu\nu} = A^{3/2}(A - 2BX)^{1/2} & \left[A \left(\frac{d\tau}{d\tilde{\tau}} \right)^2 (\tilde{\rho} + \tilde{P}) u^\mu u^\nu + \tilde{P} g^{\mu\nu} \right. \\ & + \left\{ \frac{\tilde{P}}{A - 2BX} \left[(2A - 3BX) \frac{A_X}{A} - B - B_X X \right] \right. \\ & \left. \left. - \frac{1}{2} \left(\frac{d\tau}{d\tilde{\tau}} \right)^2 (\tilde{\rho} + \tilde{P}) \left[A_X - B_X \left(\frac{d\phi}{d\tau} \right)^2 \right] \right\} \phi^\mu \phi^\nu \right]. \end{aligned} \quad (3.53)$$

From this equation, one can easily see that $T^{\mu\nu}$ does not necessarily take the form of a perfect fluid: it does so only when $u^\mu \propto \phi^\mu$, such as is the case of a homogeneous and isotropic universe.

Now, assume an FLRW metric in Einstein frame. The Jordan-frame metric is obtained by applying the disformal transformation,

$$\begin{aligned} d\tilde{s}^2 &= \tilde{g}_{\mu\nu} dx^\mu dx^\nu \\ &= (Ag_{\mu\nu} + B\phi_\mu\phi_\nu) dx^\mu dx^\nu \\ &= -(A - 2BX) dt^2 + Aa^2(t)\gamma_{ij} dx^i dx^j, \end{aligned} \quad (3.54)$$

where γ_{ij} is the spatially homogeneous and isotropic metric. It should be noticed that $X = \dot{\phi}(t)^2/2$, $A = A(\phi(t), X(t))$, and $B = B(\phi(t), X(t))$ represent the background values. Since the Einstein-frame metric is an FLRW one, the four-velocity of a comoving observer in the Einstein frame is $u^\mu = (1, \bar{0})$. Therefore the four-velocity of the comoving observer in Jordan frame is

$$\tilde{u}^\mu = \frac{u^\mu}{(A - 2BX)^{1/2}} = \frac{1}{(A - 2BX)^{1/2}}(1, \bar{0}), \quad (3.55)$$

and the components of the energy-momentum tensor in Jordan frame become

$$\tilde{T}^{00} = \frac{\tilde{\rho}}{A - 2BX}, \quad \tilde{T}^{0i} = \tilde{T}^{i0} = 0, \quad \tilde{T}^{ij} = \frac{\tilde{P}}{Aa^2}\gamma^{ij}. \quad (3.56)$$

Using eq. (3.49), we can calculate the components of the energy-momentum tensor in the Einstein frame. We find that the energy-momentum tensor remains diagonal in the Einstein frame and the energy density and the pressure of the perfect fluid in the Einstein frame are given by

$$\rho = \frac{A^{1/2}}{(A - 2BX)^{1/2}} \left[A(A - A_X X + 2B_X X^2)\tilde{\rho} + 3(A - 2BX)A_X X\tilde{P} \right], \quad (3.57)$$

$$P = A^{3/2}(A - 2BX)^{1/2}\tilde{P}. \quad (3.58)$$

Solving these relations inversely, we obtain

$$\tilde{\rho} = \frac{(A - 2BX)^{1/2}}{A^{5/2}(A - A_X X + 2B_X X^2)} (A\rho - 3A_X X P), \quad (3.59)$$

$$\tilde{P} = \frac{P}{A^{3/2}(A - 2BX)^{1/2}}. \quad (3.60)$$

If the perfect fluid is barotropic, then its equation of state parameter $\tilde{w} = \tilde{P}/\tilde{\rho}$ in Jordan frame and $w = P/\rho$ in the Einstein frame are related as

$$w = \frac{(A - 2BX)\tilde{w}}{A - (1 - 3\tilde{w})A_X X + 2B_X X^2 - 6\tilde{w}A_X BX^2/A}. \quad (3.61)$$

For a derivative conformal transformation $\tilde{g}_{\mu\nu} = A(\phi, X)g_{\mu\nu}$, one obtains the relation

$$w = \frac{\tilde{w}}{1 - (1 - 3\tilde{w})A_X X/A}. \quad (3.62)$$

Unless $\tilde{w} = 1/3$, i.e. the fluid is radiation, $w \neq \tilde{w}$. The fact that, for a generic conformal transformation, the observers in both frames perceive radiation fluid simultaneously is a reflex of the conformal symmetry of the electromagnetic Lagrangian. In particular, notice that for a simple field dependent conformal transformation, $\tilde{g}_{\mu\nu} = A(\phi)g_{\mu\nu}$, one has

$$\rho = \tilde{\rho}A^2, \quad P = \tilde{P}A^2, \quad w = \tilde{w}, \quad (3.63)$$

as expected.

3.5 Particle motion

Let us now see how the trajectories of test particles are modified when going from Jordan to Einstein frame.

The trajectory is parametrized by the proper time. For a test particle in the Jordan frame with four-velocity $\tilde{u}^\mu = dx^\mu/d\tilde{\tau}$, where $\tilde{\tau}$ is the proper time, the geodesic equation is given by

$$\tilde{u}^\nu \tilde{\nabla}_\nu \tilde{u}^\mu = 0. \quad (3.64)$$

Now, to compute the equations of motion in the EF, first note that the proper time in Jordan frame, $\tilde{\tau}$, is related to the proper time in the Einstein frame, τ , as

$$d\tilde{\tau}^2 = -\tilde{g}_{\mu\nu}dx^\mu dx^\nu = -(Ag_{\mu\nu} + B\phi_\mu\phi_\nu)dx^\mu dx^\nu = \alpha^2 d\tau^2, \quad (3.65)$$

where we have introduced the Einstein-frame four-velocity $u^\mu = dx^\mu/d\tau$ and the function

$$\alpha = \sqrt{A - B(\phi_\mu u^\mu)^2}. \quad (3.66)$$

Therefore, the four-velocity in Jordan frame is related to the one in Einstein frame as

$$\tilde{u}^\mu = \frac{dx^\mu}{d\tilde{\tau}} = \frac{1}{\alpha} u^\mu. \quad (3.67)$$

On the other hand, the relation between the covariant derivatives $\tilde{\nabla}_\mu$ and ∇_μ is given by the

connection (3.32):

$$\begin{aligned} C^\mu{}_{\nu\rho} &= \delta^\mu_{(\nu} \nabla_{\rho)} \ln A - \frac{1}{2} \tilde{g}_{\nu\rho} \tilde{g}^{\mu\sigma} \nabla_\sigma \ln A + \frac{B}{A - 2XB} \phi^\mu \phi_{\nu\rho} \\ &+ \frac{B}{A - 2XB} \phi^\mu \phi_{(\nu} \nabla_{\rho)} \ln(B/A) - \frac{B}{2} \phi_\nu \phi_\rho \tilde{g}^{\mu\sigma} \nabla_\sigma \ln(B/A). \end{aligned} \quad (3.68)$$

Combining these results, the geodesic equation (3.64) can be written as

$$u^\nu \nabla_\nu u^\mu = u^\mu u^\nu \nabla_\nu \ln \alpha - C^\mu{}_{\nu\rho} u^\nu u^\rho, \quad (3.69)$$

which is the equation of motion for a test particle in the Einstein frame. Since in the $g_{\mu\nu}$ frame matter fields are non-minimally coupled to gravity, test particles do not follow geodesics.

3.6 Geometric optics and propagation of photons

To derive the geodesic equation for photons, we consider the Maxwell equations in the geometric optics approximation [133]. From the action for electromagnetism in Jordan frame

$$S_{\text{EM}} = -\frac{1}{4} \int d^4x \sqrt{-\tilde{g}} \tilde{g}^{\alpha\mu} \tilde{g}^{\beta\nu} F_{\mu\nu} F_{\alpha\beta}, \quad (3.70)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, with A_μ being the electromagnetic gauge field, the source-free Maxwell equations, in the Lorentz gauge, are given by

$$\tilde{\nabla}^\nu \tilde{\nabla}_\nu A_\mu - \tilde{R}_\mu{}^\nu A_\nu = 0, \quad (3.71)$$

$$\tilde{\nabla}^\mu A_\mu = 0. \quad (3.72)$$

Under the geometric optics approximation, we seek a solution of the form

$$A_\mu = C_\mu e^{iS}, \quad (3.73)$$

where we assume that the phase S is large, varying rapidly compared to the amplitude C_μ , and that the wavelength of the electromagnetic wave is much smaller than the curvature radius. Substituting eq. (3.73) into eq. (3.71) yields, to the leading order,

$$\tilde{g}^{\mu\nu} k_\mu k_\nu = 0, \quad (3.74)$$

where $k_\mu = \partial_\mu S$ is the propagation vector of the electromagnetic wave, which is normal to the surfaces of constant phase. This shows that, in Jordan frame, the wave vector $\tilde{k}^\mu = \tilde{g}^{\mu\nu} k_\nu$ is a null

vector. The geodesic equation is derived by taking the gradient of eq. (3.74):

$$\tilde{k}^\nu \tilde{\nabla}_\nu k_\mu = 0, \quad (3.75)$$

which shows that light rays in the Jordan frame travel along null geodesics.

In the Einstein frame, the propagation vector $k_\mu = \partial_\mu S$ is the same, but its norm is different from zero: using the inverse metric (3.23), eq. (3.74) gives

$$g^{\mu\nu} k_\mu k_\nu = \frac{B}{A - 2BX} (\phi_\mu k^\mu)^2. \quad (3.76)$$

Then, by taking the gradient of eq. (3.76), the geodesic equation is given by

$$k^\nu \nabla_\nu k_\mu = \frac{1}{2} \nabla_\mu \left[\frac{B(\phi_\mu k^\mu)^2}{A - 2BX} \right], \quad (3.77)$$

where $k^\mu = g^{\mu\nu} k_\nu$.

The fact that the tangent vector k_μ is no longer null in the Einstein frame (although being null in the Jordan frame) implies that the speed of light in the $g_{\mu\nu}$ frame appears to be different from unity. In fact, for a flat spacetime in the Einstein frame, light rays obey

$$0 = d\tilde{s}^2 = A ds^2 + B \dot{\phi}^2 dt^2 = -\alpha^2 dt^2 + A d\mathbf{x}^2, \quad (3.78)$$

where we have assume $\phi(t, \mathbf{x}) = \phi(t)$ and $\alpha = \sqrt{A - B\dot{\phi}^2}$. Thus, the effective speed of light in the Einstein frame, c_E , is given by

$$c_E = \frac{\alpha}{A^{1/2}} = \sqrt{1 - \frac{B\dot{\phi}^2}{A}}. \quad (3.79)$$

Note that this apparent modification is a pure disformal effect: for $B = 0$, we recover $c_E = 1$, as expected.

4. Cosmological Observables and Disformal Transformations

An important application of conformal transformations is the possibility to define different representations of the same gravitational theory. Performing the transformation, takes the action of a scalar-tensor theory in the Jordan frame to that of Einstein gravity, but the metric is now non-minimally coupled to the matter sector. As a result, the issue of defining which frame actually describes physics naturally arises.

In previous attempts to tackle this problem, it was shown that cosmological observables — such as the redshift, luminosity distance, and temperature anisotropies — are frame-independent under conformal transformations [57, 58, 61]. Other topics studied include the adiabaticity condition and curvature perturbations [58, 134], as well as the frame dependence at the quantum level through one-loop corrections [135–137], and the construction of a frame-independent formulation [61, 138].

However, as discussed in the previous chapter, disformal transformations are a natural extension of the conformal ones. Indeed, it is the most general metric transformation involving a scalar field that respects causality and the weak equivalence principle [54]. The structure of light-cones seemingly changes under the disformal transformation, and the propagation of photons is modified. Since most astronomical and cosmological observations rely on measuring light, the frame-dependence of observables under disformal transformations is an issue that must be addressed. In other words, it is not clear whether cosmological observables are still frame-independent.

This begs the question: within a given gravitational theory, how to relate physical quantities that are defined in different frames, connected by disformal transformations? As in the conformal case, disformal transformation could, in principle, be interpreted as merely a change of variables or units, since physics should not change under a field redefinition. Therefore, one could expect that physical laws, and any measurable quantity, should not depend on a particular set of variables. The observables should be disformal-frame independent.

Previous studies regarding the disformal invariance include: distortions in the CMB spectrum and violations of the distance-duality relation [80]; curvature perturbations at linear [139, 140] and full non-linear order [141], and the invariance in the number of degrees of freedom [121]. In

particular, in reference [59] it was shown that at the action level, pure disformal transformations are equivalent to a rescaling of the time coordinate. The causal structure, propagation speed, cosmological perturbations were shown to be invariant.

In this chapter, we revisit this problem, extending the results for the conformal case [58]. We study several observable quantities and relations of cosmological interest. In section 4.1 we investigate the weak field limit and show that the disformal transformation leads to post-Newtonian corrections. The invariance of the redshift is shown in section 4.2 and the Sandage-Loeb effect (also called redshift drift) is studied in section 4.3. After that, we investigate the invariance of cosmological distances and, in particular, the condition to solve the horizon problem. Both the reciprocity and distance-duality relations in general spacetimes are studied in section 4.7. Then, we focus on the photon distribution function (section 4.8) and the black-body spectra (section 4.9). Finally, on section 4.10 we compute the adiabaticity condition, and on section 4.11 we provide a few comments on non-universal couplings. At each step we present the correspondence between Jordan-frame and Einstein-frame quantities, checking its frame dependence. The results presented in this chapter were published in [60].

4.1 Newtonian limit and gravitational constant

Let us start with the Newtonian limit (weak gravitational field and slow motion of test particles) on each frame. In standard General Relativity, the geodesic equation for minimally coupled particles can yield the Newtonian limit for a weak gravitational field only if

$$g_{00} = -(1 + 2\Phi), \quad g_{ij} = \delta_{ij}, \quad (4.1)$$

where Φ is the Newtonian potential. In accordance with the post-Newtonian bookkeeping system [111], we assign a bookkeeping label ϵ that keeps track of small quantities such that

$$\Phi \sim v^2 \sim \frac{P}{\rho} \sim \partial\phi \sim \mathcal{O}(\epsilon), \quad \frac{|\partial/\partial t|}{|\partial/\partial x|} \sim v, \quad (4.2)$$

where v represents the velocity of test particles. We naturally follow the same approach for quantities in the Jordan frame.

Note however that since the disformal factor B is always accompanied by factors $(\partial\phi)^2 \sim \mathcal{O}(\epsilon^2)$, these terms will lead to post-Newtonian corrections. Hence, as it will be shown, the Newtonian limit of disformally related theories is identical to the conformal case with $A = A(\phi)$.

Let us start with the Newtonian limit in the Einstein frame. For a canonical scalar field with the Lagrangian $K(\phi, X) = X - V(\phi)$, the Einstein frame action, eq. (3.37), leads to the equation

of motion

$$\square\phi - \frac{dV}{d\phi} = Q, \quad (4.3)$$

where Q represents the non-minimal coupling with matter fields, given by

$$Q = \nabla_\mu W^\mu - Z, \quad (4.4)$$

$$W^\mu = \frac{B}{A} T_{(m)}^{\mu\nu} \phi_\nu - \frac{C}{2A} (A - 2BX) \left(A_X T_{(m)} + B_X T_{(m)}^{\alpha\beta} \phi_\alpha \phi_\beta \right) \phi^\mu, \quad (4.5)$$

$$Z = \frac{1}{2A} \left([A_\phi + C A_X X (A_\phi - 2B_\phi X)] T_{(m)} + [B_\phi + C B_X X (A_\phi - 2B_\phi X)] T_{(m)}^{\mu\nu} \phi_\mu \phi_\nu \right). \quad (4.6)$$

In the equations above, $T_{(m)}^{\mu\nu}$ is the matter energy-momentum tensor, $T_{(m)}$ is its trace and for convenience we have defined

$$C = \frac{1}{A - A_X X + 2B_X X^2}. \quad (4.7)$$

Without loss of generality, in what follows we assume $V(\phi) = 0$. Then, keeping only terms of order $\mathcal{O}(\epsilon)$, the right-hand side of the equation of motion becomes

$$Q = \frac{1}{2} \frac{\partial \ln A}{\partial \phi} \rho, \quad (4.8)$$

such that, in the Newtonian limit, eq. (4.3) becomes

$$\Delta\phi = \frac{1}{2} \frac{\partial \ln A}{\partial \phi} \rho, \quad (4.9)$$

where $\Delta = \gamma^{ij} \nabla_i \nabla_j$ is the Laplacian operator. This equation admits the solution, up to order $\mathcal{O}(\epsilon)$,

$$\phi(\vec{x}) = \phi_0 - \frac{1}{8\pi} \frac{\partial \ln A}{\partial \phi} \bigg|_{\phi_0} \int d^3x' \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|}, \quad (4.10)$$

where the derivative of A is evaluated at ϕ_0 , the asymptotic value of the scalar field.

On the other hand, the evolution of the metric $g_{\mu\nu}$ is determined from eq. (3.37), leading to the usual Einstein's equation,

$$R_{\mu\nu} = 8\pi G_N \left[T_{\mu\nu}^{(m)} + T_{\mu\nu}^{(\phi)} - \frac{1}{2} g_{\mu\nu} \left(T^{(m)} + T^{(\phi)} \right) \right], \quad (4.11)$$

in which $T^{(\phi)}$ is the trace of $T_{\mu\nu}^{(\phi)}$, the energy-momentum tensor for the canonical scalar field, given by

$$T_{\mu\nu}^{(\phi)} = \phi_\mu \phi_\nu + g_{\mu\nu} X. \quad (4.12)$$

We note that $T_{\mu\nu}^{(\phi)}$ is a $\mathcal{O}(\epsilon^2)$ quantity and, in the Newtonian limit, the time-time component of

eq. (4.11) becomes

$$\Delta\Phi = 4\pi G_N \rho. \quad (4.13)$$

Similar to ϕ , the solution for the gravitation potential is

$$\Phi(\vec{x}) = \Phi_0 - G_N \int d^3x' \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|}. \quad (4.14)$$

To find the gravitational constant in Einstein frame, we use the fact that due to the non-minimal coupling to the scalar field, material particles experience a fifth-force. At the Newtonian order, the geodesic equation (3.69) becomes

$$\frac{d^2\vec{x}}{dt^2} = -\vec{\nabla}\Phi - \frac{1}{2} \frac{\partial \ln A}{\partial \phi} \vec{\nabla}\phi \quad (4.15)$$

$$= -G_N \left[1 + \frac{1}{16\pi G_N} \left(\frac{\partial \ln A}{\partial \phi} \right)^2 \Big|_{\phi_0} \right] \int d^3x' \frac{(\vec{x} - \vec{x}') \rho(\vec{x}')}{|\vec{x} - \vec{x}'|^3}, \quad (4.16)$$

where we have employed eqs. (4.10) and (4.14). The coefficient before the integral can be seen as the effective gravitational constant in Einstein frame,

$$G = G_N \left[1 + \frac{1}{16\pi G_N} \left(\frac{\partial \ln A}{\partial \phi} \Big|_{\phi_0} \right)^2 \right]. \quad (4.17)$$

Now we move on to Jordan frame. To calculate the effective gravitational constant $\tilde{G}$ we must first get the corresponding equations of motion, eqs. (4.9) and (4.11), in the Jordan frame.

Ignoring post-Newtonian terms, the connection relating the covariant derivatives in each frame, eq. (3.68), takes the form

$$C^\rho_{\mu\nu} = \frac{1}{2} \frac{\partial \ln A}{\partial \phi} (\delta^\rho_\mu \phi_\nu + \delta^\rho_\nu \phi_\mu - \tilde{g}_{\mu\nu} \tilde{g}^{\rho\sigma} \phi_\sigma), \quad (4.18)$$

being a $\mathcal{O}(\epsilon)$ quantity, and employing eq. (3.33) one finds $\square\phi = A\square\phi$. Thus, the equation of motion for ϕ , eq. (4.9), written in terms of Jordan frame quantities becomes

$$\Delta\phi = \frac{1}{2} \frac{\partial A}{\partial \phi} \tilde{\rho}, \quad (4.19)$$

where we have used the transformation of the energy-momentum tensor, eq. (3.49), to relate ρ to $\tilde{\rho}$.

Now we calculate the equation of motion for $\tilde{g}_{\mu\nu}$ from its Einstein-frame counterpart. Recalling

eq. (3.36), the Ricci tensor transforms as

$$R_{\mu\nu} = \tilde{R}_{\mu\nu} - \tilde{\nabla}_\rho C^\rho_{\mu\nu} - C^\rho_{\mu\sigma} C^\sigma_{\rho\nu} + \tilde{\nabla}_\mu \tilde{\nabla}_\nu \ln \sqrt{\frac{-\tilde{g}}{-g}} + C^\rho_{\mu\nu} \tilde{\nabla}_\rho \ln \sqrt{\frac{-\tilde{g}}{-g}}, \quad (4.20)$$

such that, neglecting terms $\sim \mathcal{O}(\epsilon^2)$, the field equations (4.11) become

$$\tilde{R}_{\mu\nu} = 8\pi G_N A \left(\tilde{T}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \tilde{T} \right) - \frac{\partial \ln A}{\partial \phi} \left(\tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi + \frac{1}{2} \tilde{g}_{\mu\nu} \square \phi \right), \quad (4.21)$$

where $\tilde{T}$ is the trace of $\tilde{T}_{\mu\nu}$. For the metric in the Newtonian limit, $\tilde{g}_{00} = -(1 + \tilde{\Phi})$, $\tilde{g}_{ij} = \delta_{ij}$, the time-time component of the above equation becomes

$$\Delta \tilde{\Phi} = 4\pi G_N A|_{\phi_0} \left[1 + \frac{1}{16\pi G_N} \left(\frac{\partial \ln A}{\partial \phi} \right)^2 \Big|_{\phi_0} \right] \tilde{\rho} \quad (4.22)$$

where we have used eq. (4.19). The Newtonian potential is related to $\tilde{\rho}$ via $\Delta \tilde{\Phi} = 4\pi \tilde{G} \tilde{\rho}$, so we identify the effective gravitational constant in Jordan frame:

$$\tilde{G} = A|_{\phi_0} G_N \left[1 + \frac{1}{16\pi G_N} \left(\frac{\partial \ln A}{\partial \phi} \right)^2 \Big|_{\phi_0} \right], \quad (4.23)$$

which could be measured, for instance, in a Cavendish-type experiment.

Hence, the effective gravitational constants are related as $\tilde{G} = A|_{\phi_0} G$, which is the same as in the conformal case [58]. As mentioned above, this result was expected since the disformal term leads to post-Newtonian corrections. This result is consistent with previous studies for the simple disformal transformation $A(\phi), B(\phi)$ using the full PPN analysis [142].

4.2 Redshift

As the first example of cosmological interest, let us examine the redshift and check whether it is invariant under disformal metric transformations. In accordance with eq. (2.14), the line elements of the FLRW spacetime in the Jordan and Einstein frames are given respectively by

$$d\tilde{s}^2 = -d\tilde{t}^2 + \tilde{a}(\tilde{t})^2 (d\chi^2 + f_K^2(\chi) d\Omega^2), \quad (4.24)$$

$$ds^2 = -dt^2 + a(t)^2 (d\chi^2 + f_K^2(\chi) d\Omega^2), \quad (4.25)$$

where $\tilde{t}$ is the cosmic time, $\tilde{a}$ is the scale factor, and $f_K(\chi) = \sin \chi, \chi, \sinh \chi$ for an closed, flat, or open universe, respectively. Since $d\tilde{s}^2 = A ds^2 + B \dot{\phi}^2 dt^2$, the corresponding cosmic time and scale

factor in the Einstein frame are given by

$$d\tilde{t} = \sqrt{A - B\dot{\phi}^2} dt = \alpha dt, \quad (4.26)$$

$$\tilde{a}(\tilde{t}) = A^{1/2} a(t). \quad (4.27)$$

First, we calculate the observed redshift in Jordan frame. Consider a light pulse emitted with a frequency $\tilde{\omega}_s$ by a source, and measured with a frequency $\tilde{\omega}_o$ by an observer. Here and in what follows, the subscripts s and o refer to quantities measured at the position of the source and observer, respectively. In Jordan frame, since photons are minimally coupled to the metric $\tilde{g}_{\mu\nu}$, the redshift is given by

$$1 + \tilde{z} = \frac{\tilde{\omega}_s}{\tilde{\omega}_o} = \frac{\tilde{a}_o}{\tilde{a}_s}. \quad (4.28)$$

Next, we turn to the redshift in Einstein frame. To do so, one must be careful about which units are being used to measure the frequency [58]. That is because the observation is made in the observer's reference frame, in which the units are, in general, different from those in the source's reference frame. In particular, according to eq. (4.26), for the same frequency $\tilde{\omega}$ in Jordan frame, the corresponding frequency in Einstein frame, as measured at a given position, would be $\omega_1 = \alpha_1 \tilde{\omega}$, whereas when measured at another, different position, would result in $\omega_2 = \alpha_2 \tilde{\omega}$. The relation between these measurements is

$$\frac{\omega_1}{\omega_2} = \frac{\alpha_1}{\alpha_2}, \quad (4.29)$$

corresponding to the change in the unit of time from one spacetime point to another.¹

Therefore, the observed redshift in Einstein frame is, more precisely, defined as the ratio between the frequency at the source, measured in units of the observer's reference frame, and the observed frequency. Taking into account the relation (4.29), it reads

$$1 + z = \frac{\alpha_o \omega_s}{\alpha_s \omega_o} = \frac{\tilde{\omega}_s}{\tilde{\omega}_o} = 1 + \tilde{z}. \quad (4.30)$$

Thus, the observed redshift is disformal-frame independent [59]. Furthermore, in agreement with eqs. (4.27) and (4.30), the relation between the redshift and scale factor in Einstein frame is given by

$$1 + z = \sqrt{\frac{A_o}{A_s}} \frac{a_o}{a_s}. \quad (4.31)$$

¹Using the effective speed of light eq. (3.79), given by $c_E = \alpha/A^{1/2}$, the corresponding relation between two wavelengths, λ_1 and λ_2 , is $\lambda_1/\lambda_2 = A_2^{1/2}/A_1^{1/2}$, which translates to a change in the unit of length.

4.3 Redshift drift

The redshift drift, or Sandage-Loeb effect [143, 144], corresponds to a change in the redshift of a distant source after some time interval. During said time, the expansion rate $H(z)$ changes by some amount and, consequently, the recession velocity and redshift of galaxies should change accordingly. Hence, if the Universe accelerates, the redshift should increase. The opposite also holds: if the Universe decelerates, the redshift must decrease. For that reason, the redshift drift is thought to be a direct probe of the cosmic acceleration [145].

To see this, let us consider light emitted from a source at the instants $\tilde{t}_s$ and $\tilde{t}_s + \Delta\tilde{t}_s$, reaching the observer at $\tilde{t}_o$ and at $\tilde{t}_o + \Delta\tilde{t}_o$. The redshift change is then given by

$$\Delta\tilde{z} = \frac{\tilde{a}(\tilde{t}_o + \Delta\tilde{t}_o)}{\tilde{a}(\tilde{t}_s + \Delta\tilde{t}_s)} - \frac{\tilde{a}(\tilde{t}_o)}{\tilde{a}(\tilde{t}_s)} \simeq \frac{\dot{\tilde{a}}(\tilde{t}_o) - \dot{\tilde{a}}(\tilde{t}_s)}{\tilde{a}(\tilde{t}_s)} \Delta\tilde{t}_o, \quad (4.32)$$

where a dot denotes the derivative with respect to $\tilde{t}$. Note that in the previous expression we have used the relation $\Delta\tilde{t}_o/\tilde{a}(\tilde{t}_o) = \Delta\tilde{t}_s/\tilde{a}(\tilde{t}_s)$ and assumed the time interval is small in the second equality. Therefore, $\Delta\tilde{z} > 0$ corresponds to $\dot{\tilde{a}}(\tilde{t}_o) - \dot{\tilde{a}}(\tilde{t}_s) > 0$, implying that the expansion of the Universe is accelerating. Conversely, if $\Delta\tilde{z} < 0$ we have $\dot{\tilde{a}}(\tilde{t}_o) - \dot{\tilde{a}}(\tilde{t}_s) < 0$ and the redshift drift suggests that the Universe is in a decelerated phase. Since the redshift drift is a kinematic relation, making no use of the Friedmann equations, the detection of a positive $\Delta\tilde{z}$ could be a direct proof of the acceleration of the cosmic expansion.

However, this statement is frame-dependent: according to eq. (4.27), the scale factor in Einstein frame involves A , such that the variation in the redshift Δz is given by

$$\Delta z = \Delta\tilde{z} = \frac{1}{A^{1/2}a(t)|_s} \left(\frac{1}{\alpha} \frac{d}{dt} [A^{1/2}a(t)] \Big|_o - \frac{1}{\alpha} \frac{d}{dt} [A^{1/2}a(t)] \Big|_s \right) \alpha \Delta t|_o. \quad (4.33)$$

Therefore, the sign of Δz in Einstein frame has no direct connection to the cosmic acceleration.

4.4 Luminosity distance and Hubble's law

The Jordan-frame luminosity distance $\tilde{d}_L$ is defined in terms of the source's (absolute) luminosity, $\tilde{L}_s$, and the observed flux, $\tilde{F}_o$, as

$$\tilde{d}_L = \sqrt{\frac{\tilde{L}_s}{4\pi\tilde{F}_o}}. \quad (4.34)$$

Due to the expansion of the Universe, $\tilde{L}_s$ is redshifted as $\tilde{L}_o = \tilde{L}_s(\tilde{\omega}_o/\tilde{\omega}_s)^2$, where $\tilde{L}_o$ is the apparent luminosity measured by the observer. On the other hand, the observed flux is defined as

$$\tilde{F}_o = \frac{\tilde{L}_o}{4\pi(a_o f_K(\chi))^2}, \quad (4.35)$$

where χ is the comoving distance to the source,

$$\chi = \int_{\tilde{t}_s}^{\tilde{t}_o} \frac{d\tilde{t}}{\tilde{a}} = \frac{1}{\tilde{a}_o \tilde{H}_0} \int_0^{\tilde{z}} \frac{d\tilde{z}'}{\tilde{E}(\tilde{z}')}, \quad (4.36)$$

$\tilde{H}_0 = \tilde{H}(0)$ is the present Hubble parameter, and $\tilde{E}(\tilde{z}) = \tilde{H}(\tilde{z})/\tilde{H}_0$. As a result, the luminosity distance in Jordan frame takes the usual form

$$\tilde{d}_L = \tilde{a}_o f_K(\chi)(1 + \tilde{z}). \quad (4.37)$$

Also, by taking the low- $\tilde{z}$ limit, we obtain Hubble's law,

$$\tilde{d}_L = \frac{\tilde{z}}{\tilde{H}_0}. \quad (4.38)$$

Now we calculate the luminosity distance d_L in Einstein frame. According to the discussion regarding the redshift, to define the luminosity distance one should consistently use the units in the observer's reference frame. Since the luminosity has units of energy divided by time, the source's absolute luminosity, when measured in the observer's units, becomes $L_s(\alpha_o/\alpha_s)^2$. Also, L_s is related to the apparent luminosity, at the observer's position, as $L_s = L_o(\omega_s/\omega_o)^2$. Therefore, the luminosity distance in Einstein frame is given by

$$d_L = \sqrt{\frac{L_s}{4\pi F_o} \left(\frac{\alpha_o}{\alpha_s}\right)^2} = a_o f_K(\chi)(1 + z). \quad (4.39)$$

Thus, as in the case of the conformal transformation [58], d_L differs from $\tilde{d}_L$ by the conformal factor: $d_L = \tilde{d}_L/A_o^{1/2}$. Note, however, that, due to the disformal transformation, χ is no longer determined by the null geodesics in the Einstein frame.

Once again, by taking the low- z limit, we obtain the Hubble law in Einstein frame,

$$d_L = \frac{z}{A_o^{1/2} \tilde{H}_0} = \frac{z}{H_{E0}}, \quad (4.40)$$

where we have defined the effective Hubble parameter,²

$$H_E = A^{1/2} \tilde{H} = \frac{A^{1/2}}{\alpha} \left(H + \frac{1}{2} \frac{d \ln A}{dt} \right), \quad (4.41)$$

and $H = \dot{a}/a$ is the usual one.

4.5 Horizon problem

We now discuss the condition for solving the horizon problem in both frames. For earlier attempts to solve cosmological puzzles employing disformal theories, we refer the reader to [146–148].

According to measurements of temperature fluctuations in the CMB, regions separated by distances larger than the horizon at the last scattering surface share the same temperature. In standard Big-Bang cosmology, this issue is known as the horizon problem.

Since the Universe became nearly homogeneous right after the decoupling epoch, the horizon problem can be solved if the distance light travels between two times (say, $\tilde{t}_i$ and $\tilde{t}_f$, with $\tilde{t}_f < \tilde{t}_{\text{dec}}$) is larger than the current size of the horizon. Using comoving coordinates, this condition in Jordan frame can be expressed as

$$\int_{\tilde{t}_i}^{\tilde{t}_f} \frac{d\tilde{t}}{\tilde{a}} = \int_{\tilde{a}_i}^{\tilde{a}_f} d \ln \tilde{a}' \frac{1}{\tilde{a}' \tilde{H}(\tilde{a}')} > \frac{1}{\tilde{a}_0 \tilde{H}_0}. \quad (4.42)$$

From $\tilde{a}_0 \tilde{H}_0 = a_0 H_{E0}$ and eqs. (4.26) and (4.27), this condition can be recast into

$$\int_{t_i}^{t_f} \frac{\alpha}{A^{1/2}} \frac{dt}{a} > \frac{1}{a_0 H_{E0}}. \quad (4.43)$$

Since the comoving Hubble distance $(\tilde{a} \tilde{H})^{-1}$ increases with time in the standard Big-Bang cosmology, it is manifest from eq. (4.42) that, in order to solve the horizon problem, there must be some period in the past, during which $(\tilde{a} \tilde{H})^{-1}$ decreased, that is,

$$\frac{d}{d\tilde{t}} (\tilde{a} \tilde{H}) = \frac{d^2 \tilde{a}}{d\tilde{t}^2} > 0. \quad (4.44)$$

Therefore, to solve the horizon problem the Universe must have experienced a phase of accelerated expansion in the early stages of its evolution. This condition is exactly the same as that of inflation in an expanding universe.

²Here we use z to define the effective Hubble parameter. If we instead use the recession velocity v , we need to multiply z by the effective speed of light $c_E = \alpha/A^{1/2}$ so that $v = c_E z = \alpha \tilde{H} d_L$, and the effective Hubble parameter is then $\alpha \tilde{H}$.

In Einstein frame, due to the relation $\tilde{a}\tilde{H} = aH_E$, this condition can be rewritten as

$$\frac{d}{dt}(aH_E) = \frac{d}{dt} \left[\frac{A^{1/2}}{\alpha} \left(\frac{da}{dt} + \frac{a}{2} \frac{d \ln A}{dt} \right) \right] > 0, \quad (4.45)$$

assuming $\alpha > 0$. Therefore, as in the case of the redshift drift, the cosmic acceleration does not have a direct connection to the solution of the horizon problem.

4.6 Angular diameter distance and distance-duality relation

Now we turn our attention to the angular diameter distance. Consider an object with proper diameter $\tilde{D}$, subtending an angle θ , as measured by an observer. In Jordan frame, the angular diameter distance to said object, $\tilde{d}_A$, is defined as

$$\tilde{d}_A \equiv \frac{\tilde{D}}{\theta} = \tilde{a}_s f_K(\chi) = \frac{\tilde{a}_o f_K(\chi)}{1 + \tilde{z}}. \quad (4.46)$$

As a result of the conservation of the number of photons [149], the angular diameter distance is related to luminosity distance, eq. (4.37), through the so-called distance-duality relation,

$$\tilde{d}_L = (1 + \tilde{z})^2 \tilde{d}_A. \quad (4.47)$$

On the other hand, the proper diameter D in Einstein frame, in units of the observer's position, becomes $D\sqrt{A_s/A_o}$, where the factor $\sqrt{A_s/A_o}$ comes from the change in the unit of length from the source's reference frame to the observer's reference frame. Hence, the angular diameter distance in Einstein frame is given by

$$d_A = \sqrt{\frac{A_s}{A_o}} \frac{D}{\theta} = \sqrt{\frac{A_s}{A_o}} a_s f_K(\chi) = \frac{a_o f_K(\chi)}{1 + z}, \quad (4.48)$$

where we have used eq. (4.31) to replace the scale factor a_s in favor of the redshift. Once again, as in the case of the luminosity distance, d_A differs from $\tilde{d}_A$ by the conformal factor, $d_A = \tilde{d}_A/A_o^{1/2}$, such that the distance-duality relation $d_L = (1 + z)^2 d_A$ still holds.³ This stems from the fact that the electromagnetic gauge 1-form is invariant under disformal transformation [59], and hence, the (modified) flux conservation law for photons still holds true in the Einstein frame. In the following subsection, we investigate the distance-duality relation in general spacetimes and show

³This result seems to be different from the one in ref. [80]. The difference comes from the definition of the redshift in Einstein frame, which in [80] is done without the unit conversion factor given in eq. (4.30), and hence, is not frame-independent quantity. When the unit conversion is taken into account, the result from [80] agrees with ours.

its frame-invariance.

4.7 Reciprocity relation in general spacetimes

The distance-duality relation — the relation between the luminosity distance and the angular diameter distance — in the Einstein frame can be derived for general spacetimes following [149, 150]. We first derive the (modified) reciprocity relation, the relation between two geometrical distances [149]. Before doing that, in order to introduce the notation, we first derive the reciprocity relation in Jordan frame.

Reciprocity relation in Jordan frame

In Jordan frame the second leading order of eq. (3.71) gives

$$\tilde{\nabla}_\mu(\tilde{C}^2 \tilde{k}^\mu) = 0, \quad (4.49)$$

where $\tilde{C}^2 = \tilde{g}^{\mu\nu} C_\mu C_\nu$. Consider a null geodesics congruence diverging from a source with cross-sectional area $d\tilde{S}$ perpendicular to the propagation vector k_μ . Then the rate of change of $d\tilde{S}$ along the null geodesics is determined by [133]

$$\tilde{k}^\mu \tilde{\nabla}_\mu d\tilde{S} = d\tilde{S} \tilde{\nabla}_\mu \tilde{k}^\mu. \quad (4.50)$$

Therefore, from eq. (4.49) and eq. (4.50),

$$\tilde{C}^2 d\tilde{S} = \text{constant} \quad (4.51)$$

along the geodesics. We note that the gauge condition eq. (3.72) implies $\tilde{g}^{\mu\nu} C_\mu k_\nu = 0$ which ensures that C_μ is spacelike.

The energy-momentum tensor of light-rays constructed out of the action eq. (3.70) under the geometric optics approximation is given by

$$\tilde{T}^{\mu\nu} = \tilde{C}^2 \tilde{k}^\mu \tilde{k}^\nu. \quad (4.52)$$

Then the energy flux measured in the rest frame of an observer with four-velocity $\tilde{u}^\mu$ is [149, 150]

$$\tilde{F}^\mu = -\tilde{h}^{\mu\nu} \tilde{T}_{\nu\rho} \tilde{u}^\rho = \tilde{F} \tilde{n}^\mu, \quad (4.53)$$

where $\tilde{n}^\mu$ is the unit vector orthogonal to the observer's four-velocity, $\tilde{n}^\mu \tilde{u}_\mu = 0$, and $\tilde{F}$ is the

observed flux:

$$\tilde{h}^{\mu\nu} = \tilde{g}^{\mu\nu} + \tilde{u}^\mu \tilde{u}^\nu, \quad (4.54a)$$

$$\tilde{k}^\mu = -\tilde{u}_\nu \tilde{k}^\nu (\tilde{u}^\mu + \tilde{n}^\mu) = \tilde{\omega}(\tilde{u}^\mu + \tilde{n}^\mu), \quad (4.54b)$$

$$\tilde{F} = \tilde{C}^2 \tilde{\omega}^2. \quad (4.54c)$$

Since the energy density measured by the observer is $\tilde{T}_{\mu\nu} \tilde{u}^\mu \tilde{u}^\nu = \tilde{C}^2 \tilde{\omega}^2$, eq. (4.49) and hence eq. (4.51) implies the conservation of the number of photons along the light-ray bundle.

The luminosity of the source $\tilde{L}_s$ is determined through the flux $\tilde{F}_s$ measured on a small sphere of radius $\Delta\tilde{r}$ around the source in the source rest frame,

$$\tilde{L}_s = 4\pi \Delta\tilde{r}^2 \tilde{F}_s = 4\pi \Delta\tilde{r}^2 \tilde{C}_s^2 \tilde{\omega}_s^2, \quad (4.55)$$

and the luminosity distance $\tilde{d}_L$ is defined as $\tilde{d}_L = (\tilde{L}_s/4\pi\tilde{F}_o)^{1/2}$.

Let us consider the situation where we observe the flux from the source along some bundle of geodesics which subtends a solid angle $d\Omega_s$ at the source and has cross-sectional area $d\tilde{S}_s$ at the observer. We then define the source area distance $\tilde{r}_s$ as [149]

$$d\tilde{S}_s = \tilde{r}_s^2 d\Omega_s. \quad (4.56)$$

Note that $\tilde{r}_s$ is not an observable quantity because $d\Omega_s$ cannot be directly measured. However, $\tilde{r}_s$ is related to the luminosity distance $\tilde{d}_L$. From the photon number conservation eq. (4.51), we have $\tilde{C}_s^2 \Delta\tilde{r}^2 d\Omega_s = \tilde{C}_o^2 d\tilde{S}_s$. Then

$$\tilde{d}_L^2 = \frac{\tilde{L}_s}{4\pi\tilde{F}_o} = \frac{\tilde{C}_s^2 \tilde{\omega}_s^2}{\tilde{C}_o^2 \tilde{\omega}_o^2} \Delta\tilde{r}^2 = (1+z)^2 \frac{d\tilde{S}_s}{d\Omega_s} = (1+z)^2 \tilde{r}_s^2. \quad (4.57)$$

Next consider the situation where we measure the solid angle $d\Omega_o$ subtended by some objects whose cross-sectional area is $d\tilde{S}_o$. Then, we define the observer area distance $\tilde{r}_o$ as [149]

$$d\tilde{S}_o = \tilde{r}_o^2 d\Omega_o. \quad (4.58)$$

From this definition, it is clear that $\tilde{r}_o$ is nothing but the angular diameter distance $\tilde{d}_A$. The relation between $\tilde{r}_s$ and $\tilde{r}_o$ is known as the reciprocity relation, which is written as

$$\tilde{r}_s = (1+z)\tilde{r}_o. \quad (4.59)$$

The reciprocity relation is derived from the null geodesic deviation equation [149, 150]. The rela-

tion between $\tilde{d}_L$ and $\tilde{d}_A$ is sometimes called the distance-duality relation, which is written using eq. (4.57) as

$$\tilde{d}_L = (1+z)^2 \tilde{d}_A, \quad (4.60)$$

and is a consequence of the photon number conservation.

Reciprocity relation in Einstein frame

To derive the reciprocity relation in the Einstein frame, we adopt the uniform- ϕ gauge, in which $\phi = \phi(t)$. In what follows, we also assume for simplicity that $g^{0i} = 0$, such that $g^{00} = (g_{00})^{-1}$. With these assumptions,

$$2X = -g^{\mu\nu} \phi_\mu \phi_\nu = (\phi_\mu u^\mu)^2 \quad (4.61)$$

for a comoving observer, such that

$$\alpha = \sqrt{A - B(\phi_\mu u^\mu)^2} = \sqrt{A - 2BX}. \quad (4.62)$$

Moreover, in the geometric optics approximation we assume $\phi(t)$ varies slowly compared to the phase S and only consider terms up to the first order in X .

First we note that, since the disformal transformation amounts to the change of units of time as $d\tilde{t} = \alpha dt$ and length as $d\tilde{x} = A^{1/2} dx$, eq. (4.51) now becomes

$$C^2 dS = \text{constant}, \quad (4.63)$$

where the scalar amplitude squared transforms as $C^2 = g^{\mu\nu} C_\mu C_\nu = A\tilde{C}^2$, and the cross section as $dS = A^{-1} d\tilde{S}$.

From eq. (3.49), the energy-momentum tensor of light rays in the Einstein frame under the geometric optics approximation up to the first order in X is

$$\begin{aligned} T_{\mu\nu} &= \alpha A^{1/2} \left[\tilde{T}_{\mu\nu} - B \left(\phi_\mu \phi_\alpha \tilde{T}^\alpha{}_\nu + \phi_\nu \phi_\alpha \tilde{T}^\alpha{}_\mu \right) \right] \\ &= \frac{\alpha}{A^{1/2}} C^2 \left[k_\mu k_\nu - \frac{B(k_\alpha \phi^\alpha)}{\alpha^2} (\phi_\mu k_\nu + \phi_\nu k_\mu) \right]. \end{aligned} \quad (4.64)$$

Thus, the measured energy flux corresponding to eq. (4.53) becomes

$$F^\mu = -h^{\mu\nu} T_{\nu\rho} u^\rho = F n^\mu, \quad (4.65)$$

where n^μ is the unit vector in the spatial direction of the propagation vector seen on the rest frame

of the observer, and F is the observed flux that are given by

$$h^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu, \quad (4.66a)$$

$$k^\mu = -u_\nu k^\nu \left(u^\mu + \frac{A^{1/2}}{\alpha} n^\mu \right) = \omega \left(u^\mu + \frac{A^{1/2}}{\alpha} n^\mu \right), \quad (4.66b)$$

$$F = \frac{A}{\alpha^2} C^2 \omega^2. \quad (4.66c)$$

However, care must be taken in defining the luminosity of the source L_s from the measured flux because the photon number is no longer conserved in the Einstein frame. This is seen from eq. (4.49) which implies the existence of the conserved current $\tilde{C}^2 \tilde{k}^\mu$ and the conserved photon number density $\tilde{C}^2 \tilde{\omega}$ in Jordan frame. On the other hand, eq. (4.49) implies that the conserved current in the Einstein frame is

$$J_E^\mu = \frac{\alpha}{A^{1/2}} C^2 \left(k^\mu - \frac{B(k_\alpha \phi^\alpha)}{\alpha^2} \phi^\mu \right), \quad (4.67)$$

and the conserved number density is

$$n_E = -u_\mu J_E^\mu = \frac{A^{1/2}}{\alpha} C^2 \omega. \quad (4.68)$$

This should be contrasted with the energy density measured by the observer which, up to in the first order in X , is given by

$$T_{\mu\nu} u^\mu u^\nu = \frac{A^{3/2}}{\alpha^3} C^2 \omega^2. \quad (4.69)$$

The bare number density $(A^{3/2}/\alpha^3) C^2 \omega = (A/\alpha^2) \times (A^{1/2}/\alpha) C^2 \omega$ is not a conserved quantity. Hence, to relate the measured flux to the source's luminosity we need to use the “modified” number density, eq. (4.68), and the modified flux F_E , defined as

$$F_E = \frac{\alpha^2}{A} F = C^2 \omega^2. \quad (4.70)$$

From eq. (4.63), this F_E obeys the flux conservation law: $F_E \propto dS^{-1}$. Thus, we define the source's luminosity L_s through the modified flux $F_{E,s}$ measured on a small sphere around the source with the radius Δr as

$$L_s = 4\pi \Delta r^2 F_{E,s} = 4\pi \Delta r^2 C_s^2 \omega_s^2, \quad (4.71)$$

and, in terms of the source's luminosity L_s measured in the observer's unit, the luminosity distance

d_L is defined as

$$d_L = \sqrt{\frac{L_s(\alpha_o/\alpha_s)^2}{4\pi F_{E,o}}}. \quad (4.72)$$

We define the source area distance r_s and the observer area distance r_o by

$$dS_s = r_s^2 d\Omega_s, \quad dS_o = r_o^2 d\Omega_o, \quad (4.73)$$

where dS_s is the cross-sectional area at the observer of the flux from the source and dS_o is the cross-sectional area of the source object. The relation between r_s and d_L is found as follows: from eq. (4.63), one can show that $C_s^2 \Delta r^2 d\Omega_s = C_o^2 dS_s$. Then, using eq. (4.30),

$$d_L^2 = \frac{L_s(\alpha_o/\alpha_s)^2}{4\pi F_o^E} = \frac{C_s^2 \omega_s^2 \alpha_o^2}{C_o^2 \omega_o^2 \alpha_s^2} \Delta r^2 = (1+z)^2 \frac{dS_s}{d\Omega_s} = (1+z)^2 r_s^2. \quad (4.74)$$

However, since the source area dS_o measured in the observer's unit is $(A_s/A_o) dS_o$, the angular diameter distance d_A is related to r_o as

$$d_A^2 = \frac{(A_s/A_o) dS_o}{d\Omega_o} = \frac{A_s}{A_o} r_o^2. \quad (4.75)$$

Next, following [149], we show that the following reciprocity relation in the Einstein frame, corresponding to eq. (4.59), holds although the geodesics deviation equation is modified :

$$r_s = (1+z) r_o \left(\frac{A_s}{A_o} \right)^{1/2}. \quad (4.76)$$

Consider a bundle of light rays diverging from S with a solid angle $d\Omega_s$, tangent vector k^μ , and affine parameter v . Let another bundle of light rays converging to O with a solid angle $d\Omega_o$ have tangent vector k'_μ and affine parameter v' . Hence, OS is the geodesic common to both bundles, such that $v = v'$ and $k^\mu = k'^\mu$ along it (figure 4.1). Let also η^μ, η'^μ be the Jacobi vectors for k^μ, k'^μ , respectively. Then, $k^\mu = \partial x^\mu / \partial v$ and from eq. (3.77)

$$\frac{Dk_\mu}{Dv} = k^\nu \nabla_\nu k_\mu = \frac{1}{2} \nabla_\mu \left(\frac{B(\phi^\mu k_\mu)^2}{A - 2BX} \right) \equiv \nabla_\mu K, \quad (4.77a)$$

$$\frac{D\eta_\mu}{Dv} = k^\nu \nabla_\nu \eta_\mu = \eta^\nu \nabla_\nu k_\mu. \quad (4.77b)$$

Similar equations hold for primed variables. From eqs. (4.77), we can derive a modified geodesic deviation equation

$$\frac{D^2 \eta_\mu}{Dv^2} = k^\rho \nabla_\rho (k^\nu \nabla_\nu \eta_\mu) = -R_{\mu\nu\rho\sigma} k^\nu \eta^\rho k^\sigma + \eta^\nu \nabla_\nu \nabla_\mu K, \quad (4.78)$$

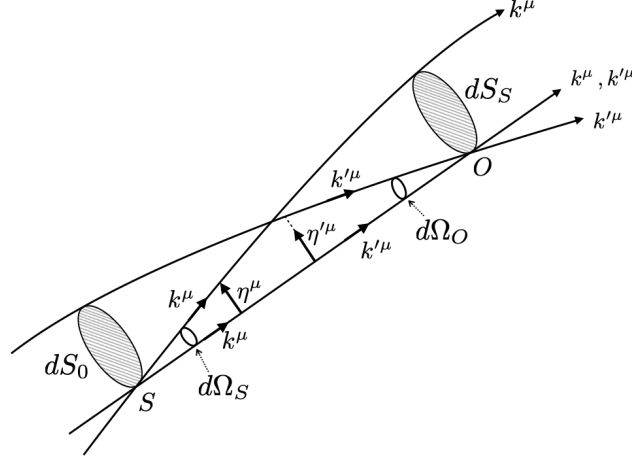


Figure 4.1: Schematic picture of congruences of photons expanding from a source S and converging to an observer O , with a common geodesic SO .

and similarly for primed variables.

From the geodesic deviation equation, we then have

$$\eta'^\mu \frac{d^2 \eta_\mu}{dv^2} - \eta^\mu \frac{d^2 \eta'_\mu}{dv^2} = 0 \quad (4.79)$$

along OS , which implies

$$\eta'^\mu \frac{d\eta_\mu}{dv} - \eta^\mu \frac{d\eta'_\mu}{dv} = 0. \quad (4.80)$$

Since $\eta^\mu = 0$ at S and $\eta'^\mu = 0$ at O , this gives

$$\eta'^\mu \left. \frac{d\eta_\mu}{dv} \right|_S = -\eta^\mu \left. \frac{d\eta'_\mu}{dv} \right|_O. \quad (4.81)$$

We assume that η^μ is orthogonal to the observer's four-velocity at O and η'^μ is orthogonal to the source's four-velocity at S . Then, as given in eq. (4.66c), k^μ and k'^μ can be decomposed into the temporal direction and the spatial direction in the rest frame of observer's four-velocity,

$$k^\mu = \omega \left(u^\mu + \frac{A^{1/2}}{\alpha} n^\mu \right), \quad (4.82)$$

such that an infinitesimal affine distance dv corresponds to the proper distance $d\ell$:

$$|d\ell| = \frac{\omega A^{1/2}}{\alpha} |dv|. \quad (4.83)$$

We choose a pair of Jacobi fields η_1^μ, η_2^μ such that

$$\left. \frac{d\eta_1^\mu}{dv} \frac{d\eta_2^\mu}{dv} \right|_S = \eta_1^\mu \eta_2^\mu \Big|_O = 0. \quad (4.84)$$

This suggests we may take $\eta_1'^\mu|_s \propto \eta_1^\mu|_s$ and $\eta_2'^\mu|_s \propto \eta_2^\mu|_s$, so we choose a pair of Jacobi fields $\eta_1'^\mu, \eta_2'^\mu$ determined by the condition

$$\eta_1'^\mu \frac{d\eta_{2\mu}}{dv} \Big|_s = \eta_2'^\mu \frac{d\eta_{1\mu}}{dv} \Big|_s = 0. \quad (4.85)$$

Then, from eq. (4.81), we have

$$\eta_1^\mu \frac{d\eta_{2\mu}'}{dv} \Big|_o = \eta_2^\mu \frac{d\eta_{1\mu}'}{dv} \Big|_o = 0, \quad (4.86)$$

which leads to

$$\eta_1'^\mu \eta_{2\mu}' \Big|_s = \frac{d\eta_1'^\mu}{dv} \frac{d\eta_{2\mu}'}{dv} \Big|_o = 0. \quad (4.87)$$

With this choice of Jacobi vectors, one finds the cross-sectional areas and solid angles

$$dS_s = \eta_1 \eta_2 \Big|_o, \quad dS_o = \eta_1' \eta_2' \Big|_s, \quad d\Omega_s = \frac{d\eta_1}{d\ell} \frac{d\eta_2}{d\ell} \Big|_s, \quad d\Omega_o = \frac{d\eta_1'}{d\ell} \frac{d\eta_2'}{d\ell} \Big|_o, \quad (4.88)$$

where η_1, η_1' are the norms of η_1^μ and $\eta_1'^\mu$, respectively. On the other hand, from eq. (4.81), we have

$$\eta_1 \eta_2 \frac{d\eta_1'}{dv} \frac{d\eta_2'}{dv} \Big|_o = \eta_1' \eta_2' \frac{d\eta_1}{dv} \frac{d\eta_2}{dv} \Big|_s. \quad (4.89)$$

Therefore, using eq. (4.83) and eq. (4.30), we finally obtain

$$dS_s d\Omega_o \tilde{\omega}_o^2 A_o = dS_o d\Omega_s \tilde{\omega}_s^2 A_s, \quad (4.90)$$

or, in terms of the area distances r_s and r_o ,

$$r_s = (1+z) r_o \left(\frac{A_s}{A_o} \right)^{1/2}, \quad (4.91)$$

as given in eq. (4.76).

Since r_s is not a measurable quantity, this modification has no observable effect and can be understood as a change in the units of length. However, in terms of observable distances d_L and d_A using eq. (4.74) and eq. (4.75), we find the distance-duality relation

$$d_L = (1+z)^2 d_A. \quad (4.92)$$

Therefore, the distance-duality relation is invariant under the disformal transformation although the reciprocity relation is modified.

4.8 Photon distribution function

We now consider observable quantities related to cosmic microwave background, starting from the frame-dependence of the photon distribution function.

First of all, note that in the geometric optics approximation, the photon four-momentum p_μ , conjugate to the comoving coordinate x^μ , is defined as $p_\mu = \hbar k_\mu$, where $k_\mu = \partial_\mu S$ is the propagation vector, invariant under disformal transformations. Therefore, the phase-space volume element $dx^1 dx^2 dx^3 dp_1 dp_2 dp_3 = d^3x d^3p$ is frame-invariant.

Moreover, in the geometric optic approximation, from eq. (3.49), the relationship between the Einstein frame and Jordan frame energy-momentum tensor is given by

$$T^{\mu\nu} = \alpha A^{5/2} \tilde{T}^{\mu\nu}. \quad (4.93)$$

These can be written in terms of integrals of the distribution functions as

$$T^{\mu\nu} = \frac{2}{(2\pi)^3} \int \frac{d^3p}{\sqrt{-g}} \frac{p^\mu p^\nu}{p^0} f, \quad \tilde{T}^{\mu\nu} = \frac{2}{(2\pi)^3} \int \frac{d^3p}{\sqrt{-\tilde{g}}} \frac{\tilde{p}^\mu \tilde{p}^\nu}{\tilde{p}^0} \tilde{f}, \quad (4.94)$$

where $p^\mu = dx^\mu / d\lambda$ and $\tilde{p}^\mu = dx^\mu / d\tilde{\lambda}$. Hence, the relation between the Jordan-frame and Einstein-frame distribution functions is given by [151]

$$\tilde{f} = \frac{1}{A} \frac{d\tilde{\lambda}}{d\lambda} f. \quad (4.95)$$

From the transformation of the measured frequency,

$$\tilde{\omega} = -\tilde{k}^\mu \tilde{u}_\mu = \frac{\omega}{\alpha} = -\frac{k^\mu u_\mu}{\alpha}, \quad (4.96)$$

we find $d\tilde{\lambda} = \alpha^2 d\lambda$, such that

$$\tilde{f} = \frac{\alpha^2}{A} f. \quad (4.97)$$

Hence, the number of photons in the JF phase-space, $d\tilde{N} = \tilde{f} d^3x d^3p$, is related to the one in the EF, $dN = f d^3x d^3p$, by $d\tilde{N} = (\alpha^2/A) dN$. Note that the factor α^2/A is consistent with the difference between the Jordan-frame and Einstein-frame photon number densities discussed in section 4.7.

In the Jordan frame, the rate of change of the distribution function along the photon world-line is given by the Boltzmann equation

$$\frac{D\tilde{f}}{D\tilde{\lambda}} = \frac{dx^\mu}{d\tilde{\lambda}} \frac{\partial \tilde{f}}{\partial x^\mu} + \frac{dp_\mu}{d\tilde{\lambda}} \frac{\partial \tilde{f}}{\partial p_\mu} = \tilde{\mathcal{C}}[\tilde{f}], \quad (4.98)$$

where $\tilde{\mathcal{C}}[\tilde{f}]$ represents the collision term. In the case of conformal transformations, since the electromagnetic Lagrangian, eq. (3.70), is conformal-invariant, the Boltzmann equation is also invariant under conformal transformations [58]. On the other hand, for the disformal transformation, the distribution function is no longer invariant. According to eq. (4.97), the Boltzmann equation is modified, becoming

$$\frac{Df}{D\lambda} = \frac{dx^\mu}{d\lambda} \frac{\partial f}{\partial x^\mu} + \frac{dp_\mu}{d\lambda} \frac{\partial f}{\partial p_\mu} = A\tilde{\mathcal{C}}\left[\left(\frac{\alpha^2}{A}\right)f\right] - \frac{A}{\alpha^2}f \frac{D}{D\lambda}\left(\frac{\alpha^2}{A}\right). \quad (4.99)$$

Despite the modification, we note that this does not immediately imply that the spectrum of temperature anisotropies is frame-dependent. We leave the detailed study of temperature anisotropies for a future work.

4.9 Specific flux and specific intensity

Next, we study the specific flux and specific intensity of CMB photons. Let us start with the analysis in the Jordan frame, in which the discussion follows closely the standard General Relativity case [149, 152]. The flux measured with frequencies between $\tilde{\omega}$ and $\tilde{\omega} + d\tilde{\omega}$ is given by

$$\tilde{F}_\omega d\tilde{\omega} = \frac{\tilde{L}_s}{4\pi \tilde{r}_s^2 (1 + \tilde{z})} \tilde{\mathcal{I}}[(1 + \tilde{z})\tilde{\omega}] d\tilde{\omega}, \quad (4.100)$$

where $\tilde{F}_\omega$ is the specific flux and $\tilde{\omega}$ is the observed frequency. Here we drop the subscript o to simplify the notation. The function $\tilde{\mathcal{I}}(\tilde{\omega})$ represents the source spectrum, such that $\tilde{L}_s \tilde{\mathcal{I}} d\tilde{\omega}$ is the source's emission rate in the frequency range $(\tilde{\omega}, \tilde{\omega} + d\tilde{\omega})$. Note that $\tilde{\omega}$ is the observed frequency, corresponding to a frequency $(1 + \tilde{z})\tilde{\omega}$ at emission. Then, the specific intensity $\tilde{I}_\omega$ — specific flux measured in a solid angle $d\Omega_o$ — can be written as

$$\tilde{I}_\omega d\tilde{\omega} = \frac{\tilde{F}_\omega d\tilde{\omega}}{d\Omega_o} = \frac{\tilde{L}_s}{4\pi d\tilde{S}_o} \left(\frac{\tilde{r}_o}{\tilde{r}_s}\right)^2 \frac{\tilde{\mathcal{I}}[(1 + \tilde{z})\tilde{\omega}] d\tilde{\omega}}{(1 + \tilde{z})} = \tilde{I}_s \frac{\tilde{\mathcal{I}}[(1 + \tilde{z})\tilde{\omega}] d\tilde{\omega}}{(1 + \tilde{z})^3}, \quad (4.101)$$

where $\tilde{I}_s = \tilde{L}_s/4\pi d\tilde{S}_o$ is the surface brightness, an intrinsic property of the source. In the equation above we have used the definition of the observer area, eq. (4.58), in the second equality and the reciprocity relation, eq. (4.59), in the third. We see that the observed specific intensity does not depend on the area distance of the source.

Now let us consider a source emitting black-body radiation, such as the CMB spectrum,

$$\tilde{I}_s \tilde{\mathcal{I}}(\tilde{\omega}_s) = \tilde{\omega}_s^3 \tilde{f}\left(\frac{\tilde{\omega}_s}{\tilde{T}_s}\right), \quad \tilde{f}\left(\frac{\tilde{\omega}}{\tilde{T}}\right) = \frac{4\pi\hbar}{c^2} \frac{1}{e^{\hbar\tilde{\omega}/k\tilde{T}} - 1}, \quad (4.102)$$

where $\tilde{f}(\tilde{\omega}/\tilde{T})$ is the Planck function for the black-body radiation with temperature $\tilde{T}$ and we have

restored $\hbar$ and c . Then, eqs. (4.101) and (4.58) lead to

$$\tilde{I}_\omega = \tilde{\omega}^3 \tilde{f}\left(\frac{\tilde{\omega}}{\tilde{T}(\tilde{z})}\right) \quad (4.103)$$

where $\tilde{T}(\tilde{z}) = \tilde{T}_s/(1 + \tilde{z})$ is the observed temperature, redshifted due to the expansion of the universe. Hence, the measured intensity does not deviate from the black-body spectrum.

In the Einstein frame, from eq. (4.74), the observed specific flux, calculated from the conserved flux F_E , is given by

$$F_\omega d\omega = \frac{\alpha_o L_s}{\alpha_s 4\pi r_s^2 (1+z)} \mathcal{I}\left[\frac{\alpha_s}{\alpha_o}(1+z)\omega\right] d\omega. \quad (4.104)$$

Accordingly, the specific intensity, $I_\omega d\omega = F_\omega d\omega/d\Omega_o$, then becomes

$$I_\omega d\omega = \frac{\alpha_o A_o}{\alpha_s A_s (1+z)^3} I_s \mathcal{I}\left[\frac{\alpha_s}{\alpha_o}(1+z)\omega\right] d\omega, \quad (4.105)$$

where $I_s = L_s/4\pi d S_o$ is the surface brightness. In the previous expression, we have used the definition of observer area distance in Einstein frame, eq. (4.58), and the reciprocity relation, eq. (4.76).

Once again, we consider the emission by a black-body source. Following the change in the distribution function, eq. (4.97), the source intensity is given by

$$I_s \mathcal{I}(\omega_s) = \omega_s^3 f\left(\frac{\omega_s}{T_s}\right), \quad f\left(\frac{\omega}{T}\right) = \frac{4\pi\hbar}{c_E^2} \frac{1}{e^{\hbar\omega/kT} - 1}, \quad (4.106)$$

where $c_E^2 = \alpha^2/A$ is the effective speed of light squared, eq. (3.79). By taking into account the difference between c_E at the source and that at the observer, the measured intensity I_ω is given by

$$I_\omega = \omega^3 f\left(\frac{\omega}{T(z)}\right), \quad (4.107)$$

where, we have defined the observed temperature

$$T(z) = \frac{\alpha_o}{\alpha_s} \frac{T_s}{(1+z)}. \quad (4.108)$$

With this definition, the observed temperature takes into account both the expansion of the universe and the change of units from the source's reference frame to the observer frame, consistent with eq. (4.30). We conclude that in the Einstein frame, I_ω represents the spectrum of black-body radiation with the temperature $T(z)$. Consequently, there is no spectral distortion.

4.10 Adiabaticity condition

Let us now consider the adiabaticity condition and non-adiabatic pressure perturbations. We first derive the adiabaticity condition for a generic disformal transformation, and later, by choosing the functions A and B , we discuss the particular cases.

The pressure perturbation of matter fields $\delta\tilde{P}$ can be split into two parts [86],

$$\delta\tilde{P} = \tilde{c}_s^2 \delta\tilde{\rho} + \frac{d\tilde{P}}{d\tilde{t}} \tilde{\Gamma}, \quad (4.109)$$

where the first term, proportional to the perturbation of the energy density $\delta\tilde{\rho}$, corresponds to the adiabatic part, and the second to the entropic (non-adiabatic) one. The sound speed, $\tilde{c}_s$, is defined as

$$\tilde{c}_s^2 = \frac{d\tilde{P}/d\tilde{t}}{d\tilde{\rho}/d\tilde{t}}, \quad (4.110)$$

whereas $\tilde{\Gamma}$ is the gauge-invariant non-adiabatic pressure perturbation, which measures the displacement between hypersurfaces of uniform (adiabatic) pressure and uniform energy density. In terms of Einstein-frame quantities, $\delta\rho$ and c_s^2 are similarly defined using the corresponding variables without tildes:

$$\delta P = c_s^2 \delta\rho + \frac{dP}{dt} \Gamma, \quad (4.111)$$

$$c_s^2 = \frac{dP/dt}{d\rho/dt}. \quad (4.112)$$

Before computing the relation between the entropy perturbations $\tilde{\Gamma}$ and Γ , let us further discuss the relationships between the energy densities and pressures in Einstein and Jordan frames (see also section 3.4). To get the components of the energy-momentum tensor, we must first split $T_{\mu\nu}$, as measured by an observer with four-velocity u^μ , into parts parallel and orthogonal to u^μ ,

$$T_{\mu\nu} = \rho u_\mu u_\nu + q_\mu u_\nu + q_\nu u_\mu + Ph_{\mu\nu} + \pi_{\mu\nu}, \quad (4.113)$$

where, its 10 components are

$$\rho = T_{\mu\nu} u^\mu u^\nu, \quad (4.114)$$

$$P = \frac{1}{3} h^{\mu\nu} T_{\mu\nu}, \quad (4.115)$$

$$q_\mu = -h^\nu{}_\mu T_{\nu\rho} u^\rho, \quad (4.116)$$

$$\pi_{\mu\nu} = h^\rho{}_\mu h^\sigma{}_\nu T_{\rho\sigma} - \frac{1}{3} h^{\rho\sigma} T_{\rho\sigma} h_{\mu\nu}, \quad (4.117)$$

the energy density, pressure, momentum density and anisotropic stress tensor, respectively, and

$$h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu \quad (4.118)$$

is the spatial metric relative to the observer. The decomposition of $\tilde{T}_{\mu\nu}$ is done in a similar way, with respect to tilde variables. Moreover, to determine the relationship between the respective Jordan-frame and Einstein-frame components, we need the relation between the observers' four-velocities, $\tilde{u}^\mu = u^\mu/\alpha$, as defined in eq. (3.67), where $\alpha = [A - B(\phi_\mu u^\mu)^2]^{1/2}$.

Since we are interested in cosmological applications, we consider linear perturbations around a homogeneous and isotropic background in both frames such that q_μ , $\pi_{\mu\nu}$, $\tilde{q}_\mu$, and $\tilde{\pi}_{\mu\nu}$ are first-order quantities. Furthermore, we take a comoving Jordan-frame observer, that is, $\tilde{u}^i = 0$. According to eq. (3.67), this choice implies $u^i = 0$ such that the equality $(\phi_\mu u^\mu)^2 = 2X$ is valid up to the linear order.

The components of the energy-momentum tensor measured in Jordan frame are then related to the ones in Einstein frame as

$$\tilde{\rho} = \xi_1 \rho + \xi_2 P, \quad (4.119a)$$

$$\tilde{P} = \frac{P}{\alpha A^{3/2}}, \quad (4.119b)$$

$$\tilde{q}_\mu = \frac{1}{A^{3/2}} [q_\mu - \zeta \phi_\rho u^\rho h^\nu{}_\mu \phi_\nu], \quad (4.119c)$$

$$\tilde{\pi}_{\mu\nu} = \frac{\pi_{\mu\nu}}{\alpha A^{1/2}}, \quad (4.119d)$$

where, for convenience, we have defined the functions

$$\xi_1 = \frac{(1 - 2BX/A)^{1/2}}{A(A - A_X X + 2B_X X^2)}, \quad (4.120)$$

$$\xi_2 = -\frac{3(1 - 2BX/A)^{1/2} A_X X}{A^2(A - A_X X + 2B_X X^2)}, \quad (4.121)$$

$$\zeta = \frac{BP}{A - 2BX} + \frac{(A_X - 2B_X X)\rho - 3A_X P}{2(A - A_X X + 2B_X X^2)}. \quad (4.122)$$

Once again, we point out that the above equations are valid up to linear order. Note that, if we allow the function A to depend on X , the energy density in Jordan frame will explicitly depend, not only on the Einstein-frame ρ , but also on its pressure P . This is a reflex of the fact that if $A = A(X)$, the metric transformation, eq. (3.20), is non-linear in $g_{\mu\nu}$, since X carries a hidden metric contraction.

A small comment is in order. We could also have chosen a Jordan-frame observer comoving with the fluid, such that $\tilde{u}^\mu$ is an eigenvector of $\tilde{T}^{\mu\nu}$ and $\tilde{q}_\mu$ vanishes. However, the corresponding Einstein-frame four-velocity, given by eq. (3.67), would not be an eigenvector of $T^{\mu\nu}$. This can be

seen from the transformations (4.119), which would still be valid to the linear order. In particular, eq. (4.119c) shows that $\tilde{q}_\mu = 0$ does not necessarily implies that q_μ vanishes. Nevertheless, it is still possible to choose observers comoving with the fluid in both Jordan and Einstein frames. In that case, $\tilde{q}^\mu = q^\mu = 0$, but the four-velocities will have a small spatial perturbation $\tilde{v}^i$ and v^i . These are related by

$$\tilde{u}^0 = \frac{u^0}{\alpha}, \quad \tilde{v}^i = \alpha^2 \left[\frac{(\rho + P)v^i - \zeta \dot{\phi} \delta\phi^i}{\alpha^2 \gamma + AP} \right], \quad \gamma = \frac{A\rho - 3A_X X P}{A - A_X X + 2B_X X^2}. \quad (4.123)$$

Still, the transformations of the energy density, pressure, and anisotropic stress in eq. (4.119) remain valid (at least up to the linear level).

The relation between the energy density and pressure perturbation in Jordan and Einstein frames can be readily computed from eqs. (4.119a) and (4.119b),

$$\delta\tilde{\rho} = \bar{\xi}_1 \delta\rho + \bar{\rho} \delta\xi_1 + \bar{\xi}_2 \delta P + \bar{P} \delta\xi_2, \quad (4.124)$$

$$\delta\tilde{P} = \frac{\bar{P}}{\bar{\alpha} \bar{A}^{3/2}} \left(\frac{\delta P}{\bar{P}} - \frac{\delta\alpha}{\bar{\alpha}} - \frac{3}{2} \frac{\delta A}{\bar{A}} \right), \quad (4.125)$$

where we explicitly discriminate with bars the background quantities. In what follows, we drop this notation. Also, note that since δA , $\delta\alpha$, $\delta\xi_1$, and $\delta\xi_2$ depend on X , the perturbations $\delta\tilde{\rho}$ and $\delta\tilde{P}$ implicitly include the scalar perturbations of $g_{\mu\nu}$, the Einstein frame metric.

Now, we turn our attention to the non-adiabatic pressure perturbations. First, we rewrite eq. (4.109) as

$$\tilde{\Gamma} = \frac{\delta\tilde{P}}{d\tilde{P}/d\tilde{t}} - \frac{\delta\tilde{\rho}}{d\tilde{\rho}/d\tilde{t}}. \quad (4.126)$$

Using the equations that describe the transformations of $\delta\tilde{\rho}$ and $\delta\tilde{P}$, eqs. (4.124) and (4.125), respectively, along with the derivatives with respect to the time $\tilde{t}$ of the energy density, eq. (4.119a), and pressure, eq. (4.119b), the expression above can be recast as

$$\begin{aligned} \frac{d \ln \tilde{P}}{d\tilde{t}} \tilde{\Gamma} &= \left(1 - w \frac{\tilde{c}_s^2}{\tilde{w}} \frac{\rho}{\tilde{\rho}} \xi_2 \right) \frac{c_s^2}{w} \frac{d \ln \rho}{dt} \Gamma \\ &+ \left[\frac{c_s^2}{w} - \frac{\tilde{c}_s^2}{\tilde{w}} \frac{\rho}{\tilde{\rho}} (\xi_1 + \xi_2 c_s^2) \right] \frac{\delta\rho}{\rho} - \frac{\tilde{c}_s^2}{\tilde{w}} \frac{\rho}{\tilde{\rho}} (\delta\xi_1 + w \delta\xi_2) - \frac{\delta\alpha}{\alpha} - \frac{3}{2} \frac{\delta A}{A}. \end{aligned} \quad (4.127)$$

Using the definitions of the sound speeds on Jordan and Einstein frames, eqs. (4.110) and (4.112), respectively, one can show that

$$\frac{\tilde{c}_s^2}{\tilde{w} \tilde{\rho}} = \frac{1}{\rho} \frac{\left(\frac{c_s^2}{w} \frac{\dot{\rho}}{\rho} - \frac{\dot{\alpha}}{\alpha} - \frac{3}{2} \frac{\dot{A}}{A} \right)}{\left[\xi_1 \left(\frac{\dot{\rho}}{\rho} + \frac{\dot{\xi}_1}{\xi_1} \right) + w \xi_2 \left(\frac{c_s^2}{w} \frac{\dot{\rho}}{\rho} + \frac{\dot{\xi}_2}{\xi_2} \right) \right]}. \quad (4.128)$$

In the above equation, the dots over Einstein frame quantities on the right-hand side denote the

derivative with respect to the time t .

Now, suppose that the adiabaticity condition, i.e. $\tilde{\Gamma} = 0$, holds in the Jordan frame. Then, according to eq. (4.127), it will also hold in Einstein frame ($\Gamma = 0$) if

$$\left[\frac{c_s^2}{w} - \frac{\tilde{c}_s^2}{\tilde{w}} \frac{\rho}{\tilde{\rho}} (\xi_1 + \xi_2 c_s^2) \right] \frac{\delta \rho}{\rho} = \frac{\tilde{c}_s^2}{\tilde{w}} \frac{\rho}{\tilde{\rho}} (\delta \xi_1 + w \delta \xi_2) + \frac{\delta \alpha}{\alpha} + \frac{3}{2} \frac{\delta A}{A}. \quad (4.129)$$

This expression represents the adiabaticity condition for a generic disformal transformation with $A(\phi, X)$ and $B(\phi, X)$.

In what follows, we make a simplification, assuming both A and B depend only on the scalar field, $A = A(\phi)$ and $B = B(\phi)$, and not on the kinetic term X . With this assumption, the Jordan-frame energy density, eq. (4.119a), will depend on ρ , without any explicit dependence on the Einstein-frame pressure P , and $\xi_1 = \alpha A^{-5/2}$. In this case, eq. (4.129) becomes

$$\begin{aligned} \left[\left(1 + \frac{c_s^2}{w} \right) \frac{\dot{\alpha}}{\alpha} + \left(\frac{3}{2} - \frac{5}{2} \frac{c_s^2}{w} \right) \frac{\dot{A}}{A} \right] \frac{\delta \rho}{\rho} &= \left[\left(1 + \frac{c_s^2}{w} \right) \frac{\dot{\rho}}{\rho} - 4 \frac{\dot{A}}{A} \right] \frac{\delta \alpha}{\alpha} \\ &+ \left[\left(\frac{3}{2} - \frac{5}{2} \frac{c_s^2}{w} \right) \frac{\dot{\rho}}{\rho} + 4 \frac{\dot{\alpha}}{\alpha} \right] \frac{\delta A}{A}. \end{aligned} \quad (4.130)$$

To proceed with the discussion, we identify two particular cases in which we fix either A or B .

Conformal case: For a conformal transformation $A = A(\phi)$ and $B = 0$, such that $\alpha^2 = A$, the condition eq. (4.130) translates into

$$\left(1 - \frac{c_s^2}{w} \right) \left(\frac{\dot{\alpha}}{\alpha} \frac{\delta \rho}{\rho} - \frac{\dot{\rho}}{\rho} \frac{\delta \alpha}{\alpha} \right) = 0. \quad (4.131)$$

This equation has two solutions [58],

$$c_s^2 = w, \text{ or } \frac{\delta \rho}{\dot{\rho}} = \frac{\delta \alpha}{\dot{\alpha}} = \frac{\delta \phi}{\dot{\phi}}, \quad (4.132)$$

the first corresponding to $w = \text{constant}$, whereas the second, to the situation in which the entropy perturbation between the matter field and the scalar field ϕ vanishes.

Purely disformal case: On the other hand, for a pure disformal transformation $A = 1$ and $B = B(\phi)$, one finds $\alpha^2 = 1 - 2BX$, as well as $\xi_1 = \sqrt{1 - 2BX}$ and $\xi_2 = 0$. The adiabaticity condition is then given by

$$\left(1 + \frac{c_s^2}{w} \right) \left(\frac{\dot{\alpha}}{\alpha} \frac{\delta \rho}{\rho} - \frac{\dot{\rho}}{\rho} \frac{\delta \alpha}{\alpha} \right) = 0, \quad (4.133)$$

which, similarly to the conformal case, also translates into two possibilities:

$$c_s^2 = -w, \text{ or } \frac{\delta\rho}{\dot{\rho}} = \frac{\delta\alpha}{\dot{\alpha}} = \frac{B_\phi X \delta\phi + B \delta X}{B_\phi X \dot{\phi} + B \dot{X}}. \quad (4.134)$$

The first possibility leads to

$$c_s^2 = \frac{dp}{d\rho} = -w = -\frac{P}{\rho}, \quad (4.135)$$

corresponding to the equation of state $P \propto \rho^{-1}$, which characterizes the Chaplygin gas [153]. According to eqs. (4.119a) and (4.119b), for $A = 1$ and $B = B(\phi)$, one has $\xi_1 = \alpha$ and $\xi_2 = 0$, which leads to the invariance of the quantity $\tilde{\rho}\tilde{P} = \rho P$ under disformal transformations. Therefore, a Chaplygin gas in the Jordan frame still behaves as a Chaplygin gas in Einstein frame, such that the adiabaticity condition is not violated.

On the other hand, the second possibility, when compared to the conformal case, now also has a X dependence coming from the disformal part. To ensure that the entropy perturbation between matter and the scalar field vanishes, one must also take into account the perturbations to the metric, hidden inside δX .

In conclusion, the notion of adiabaticity is generally not invariant under disformal transformations.

Comments on curvature perturbations

In ref. [58], it was shown that the uniform-density curvature perturbation $\tilde{\zeta}$ is conformally invariant when the adiabaticity condition eq. (4.132) is satisfied (in the case of constant equation of state, w has to be $1/3$). In Jordan frame, the conservation of energy and momentum leads to the conservation of the uniform-density curvature perturbation on super-horizon scales when the non-adiabatic pressure perturbation Γ is negligible [154]. However, since the matter energy-momentum tensor in the Einstein frame is not conserved, the condition $\Gamma = 0$ by itself does imply the conservation of ζ on large scales. Despite that, since $\tilde{\Gamma} = 0$ leads to the conservation of $\tilde{\zeta}$ on super-horizon scales, the adiabaticity condition ensures that ζ is also conserved on large enough scales. This relation between the adiabaticity and invariance of the uniform-density curvature perturbation cannot be easily extended to the disformal case, not even in the simple case $A = 1$, $B = B(\phi)$, illustrated in eq. (4.134) (the connection between the adiabaticity condition and invariance of curvature perturbations relies on the behavior of δA). In fact, in ref. [139], it was explicitly shown that $\Gamma = 0$ does not lead to the conservation of ζ on super-horizon scales for a disformal transformation with $A = A(\phi)$ and $B = B(\phi)$.

In the absence of matter ($\tilde{T}_{(m)}^{\mu\nu} = T_{(m)}^{\mu\nu} = 0$), the comoving curvature perturbation $\tilde{\mathcal{R}}$ is, not only conformally invariant [155–157], but also disformally invariant under the special transformation

$\tilde{g}_{\mu\nu} = A(\phi)g_{\mu\nu} + B(\phi)\phi_\mu\phi_\nu$ [139]. On the other hand, for general disformal transformations of the type $A(\phi, X)$, $B(\phi, X)$, that is no longer the case, and the difference between the comoving curvature perturbation in each frame can be written in terms of the perturbation of a scalar field [140]. However, it can be shown that this quantity vanishes on super-horizon scales at least in the framework of Horndeski theory such that the difference $\tilde{\mathcal{R}} - \mathcal{R}$ also vanishes [140].

4.11 Non-universal coupling

Throughout this chapter, we have assumed that the disformal coupling is universal: in Einstein frame, all species are non-minimally coupled to gravity via $Ag_{\mu\nu} + B\phi_\mu\phi_\nu$. In chapter 5, we will study in details cosmologies in which only dark matter is non-minimally coupled.⁴ In the meantime, let us make a few comments on the results of this chapter if the coupling was not universal.

To do so, we define a dark matter frame introducing the metric $g_{\mu\nu}^{(m)}$, in addition to the Jordan (photon) frame metric $\tilde{g}_{\mu\nu}$ and Einstein-frame metric $g_{\mu\nu}$:

$$g_{\mu\nu}^{(m)} = A_m g_{\mu\nu} + B_m \phi_\mu \phi_\nu, \quad (4.136)$$

$$\tilde{g}_{\mu\nu} = A g_{\mu\nu} + B \phi_\mu \phi_\nu. \quad (4.137)$$

Even in this case, the distance-duality relation is invariant under the disformal transformation $g_{\mu\nu}^{(m)} \rightarrow \tilde{g}_{\mu\nu}$, because $\tilde{g}_{\mu\nu}$ is related to $g_{\mu\nu}^{(m)}$ as

$$\tilde{g}_{\mu\nu} = \frac{A}{A_m} g_{\mu\nu}^{(m)} + \left(B - \frac{A}{A_m} B_m \right) \phi_\mu \phi_\nu. \quad (4.138)$$

Hence, we can similarly define

$$\alpha_m = \left[\frac{A}{A_m} - \left(B - \frac{A}{A_m} B_m \right) (\phi_\mu u_m^\mu)^2 \right]^{1/2}, \quad (4.139)$$

where $u_m^\mu = dx^\mu/d\tau_m$ with $d\tau_m^2 = -g_{\mu\nu}^{(m)} dx^\mu dx^\nu$. Therefore, the conclusions concerning photons in the previous sections still holds for non-universal coupling case as long as α_m is well-defined.

4.12 Summary

In this chapter, we discussed the frame dependence and independence of cosmological observables under disformal transformations. We provided the correspondence between several quantities of cosmological interest defined in the Jordan and Einstein frames.

⁴Disformally coupled dark matter theories were also investigated in [74, 79, 158, 159].

With the analysis of the Newtonian limit, we have found that the modification of the effective gravitational constant is the same as in the conformal case. We also investigated the redshift, the luminosity and angular diameter distances, and the observed specific intensity, finding that the redshift and the distance-duality relation are frame-independent and that the cosmic microwave background with initially Planck spectrum remains a Planck spectrum at the point of observation. In particular, we proved that, under the geometric optics approximation, the distance-duality relation holds for general spacetime geometries, even though the reciprocity relation is modified.

Generally speaking, dimensionful quantities (such as the Hubble parameter, the gravitational constant, and the distances) are frame-dependent, corresponding to a change of units, as in the case of the conformal transformation. Notably, the relation between the redshift drift and the acceleration of the universe is frame-dependent, which is also the case for the solution to the horizon problem. Moreover, unlike the case of conformal transformations, the distribution function of photons is frame-dependent which stems from the fact that the photon number is no longer conserved. It would be interesting to study the temperature anisotropies in both Einstein and Jordan frames.

We have also investigated the adiabaticity conditions and non-adiabatic pressure perturbations, extending the discussions presented in ref. [139] to a more generic case, and found that, in general, the adiabaticity condition is not invariant under disformal transformations.

In this chapter, we confine ourselves to classical theory. It would be interesting to extend our arguments to the quantum realm, as was already done in the conformal case [135–137].

5. RSD I: the Modified Kaiser Formula

A crucial question is whether and how matter (baryons and DM) is coupled to the dark energy scalar field. A robust way to parameterize the coupling is introducing a new metric $\tilde{g}_{\mu\nu}$, related to the original one, $g_{\mu\nu}$, by conformal and disformal factors, which in general depend on the scalar field and its kinetic term [54]. If both baryons and DM couple to the same $\tilde{g}_{\mu\nu}$, it is known that this non-minimally coupled theory can be mapped into the minimally coupled one, resulting in a scalar-tensor theory with the minimal coupling, converted from an original one through a disformal transformation [132, 160]. On the other hand, if baryons and DM couple to gravity with different metrics, this kind of mapping is unavailable.

Non-minimal couplings to baryons are strongly constrained by local tests of gravity both at solar-system and astrophysical scales [161, 162]. Modified gravity models usually employ screening mechanisms (e.g. chameleons [163, 164] and Vainshtein [165]) in order to explain these tight constraints [166]. Hence, it is quite natural to assume that baryons are minimally coupled, reducing the need for screening mechanisms. However, the situation changes when one considers non-minimally coupled DM, and an interaction between the scalar field and the DM particle cannot be excluded a priori (see [167] for a recent review and the references therein). The modifications introduced by the non-minimal coupling manifestly affect the cosmological scenario, and the resulting evolution of large-scale structures can be used to study said coupling.

In this chapter, we extend the analysis introduced in [74], computing explicitly the difference between the effective growth rate, inferred from the peculiar velocity, and the actual one. We consider Einstein's theory with an additional scalar degree of freedom non-minimally coupled to DM. The coupling is introduced using a new metric, consisting of a conformal part, proportional to the gravitational metric, and another term including derivatives of the scalar field, the disformal part.

In section 5.1 we define the framework for CDM conformally/disformally coupled to a K-essence scalar field, while in section 5.2 we present the relevant background and linear perturbation equations. We also derive the evolution equations for both baryons and DM in the quasi-static limit. In section 5.3 we study how the modifications to the DM continuity equation, due to the non-minimal coupling, motivate the definition of the effective growth rate and reformulate the

Kaiser formula. The discussion presented in this chapter was published in [78].

5.1 Conformally/Disformally coupled dark matter

Let us consider the action of Einstein's gravity with a generic scalar field ϕ and matter fields,

$$S = \int d^4x \sqrt{-g} \left[\frac{M_{\text{Pl}}^2}{2} R(g_{\mu\nu}) + \mathcal{L}_\phi(g_{\mu\nu}, \phi) \right] + S_{\text{m}}, \quad (5.1)$$

where $M_{\text{Pl}} = (8\pi G_N)^{-1/2}$ is the reduced Planck mass, R is the Ricci scalar related to the metric $g_{\mu\nu}$, $\mathcal{L}_\phi$ is the scalar field Lagrangian, and S_{m} is the action for the (non-relativistic and relativistic) matter sector. We assume that the Lagrangians for baryons and radiation are defined in the usual way, i.e., with these species coupled to the metric $g_{\mu\nu}$, while DM is coupled to the scalar field through the metric $\tilde{g}_{\mu\nu}$, such that S_{m} can be written as

$$S_{\text{m}} = \int d^4x \left[\sqrt{-g} \mathcal{L}_i(g_{\mu\nu}, \psi_i) + \sqrt{-\tilde{g}} \mathcal{L}_c(\tilde{g}_{\mu\nu}, \psi_c) \right], \quad (5.2)$$

where the index $i = \text{b, r}$ stands for baryons and radiation, respectively, whereas ψ_c denotes the (cold) dark matter field. The transformed metric is defined as

$$\tilde{g}_{\mu\nu} = A(\phi, X)g_{\mu\nu} + B(\phi, X)\phi_\mu\phi_\nu, \quad (5.3)$$

where $\phi_\mu = \partial_\mu\phi$. The functions A and B are called conformal and disformal factors, respectively, and, in principle, both depend on the scalar field ϕ and its kinetic term $X = -g^{\mu\nu}\phi_\mu\phi_\nu/2$. The case in which these couplings depend only on ϕ was investigated in [79, 158, 159]. For the scalar field, we consider the K-essence Lagrangian $\mathcal{L}_\phi = K(\phi, X)$, which does not change a gravitational sector in the sense that ϕ does not couple directly to R , the Ricci scalar.

The variation of the action with respect to $g^{\mu\nu}$ yields the Einstein equation,

$$M_{\text{Pl}}^2 G_{\mu\nu} = T_{\mu\nu}^{(\phi)} + T_{\mu\nu}^{(\text{m})} + T_{\mu\nu}^{(\text{r})}, \quad (5.4)$$

where $T_{\mu\nu}^{(\phi)}$, $T_{\mu\nu}^{(\text{m})} = T_{\mu\nu}^{(\text{b})} + T_{\mu\nu}^{(\text{c})}$, and $T_{\mu\nu}^{(\text{r})}$ are the scalar field, total matter, and radiation energy-momentum tensors, respectively, defined as

$$T_{(\phi)}^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_\phi)}{\delta g_{\mu\nu}}, \quad (5.5)$$

$$T_{(\text{i})}^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_i)}{\delta g_{\mu\nu}}, \quad (5.6)$$

$$T_{(\text{c})}^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-\tilde{g}}\mathcal{L}_c)}{\delta g_{\mu\nu}}. \quad (5.7)$$

The energy-momentum tensor of the scalar field can be directly calculated from the Lagrangian, yielding

$$T_{(\phi)}^{\mu\nu} = K g^{\mu\nu} + K_X \phi^\mu \phi^\nu, \quad (5.8)$$

where the subscript in K_X represents the derivative with respect to the kinetic term X . The Bianchi identity immediately implies the conservation of the total energy-momentum tensor,

$$\nabla^\mu T_{\mu\nu} = \nabla^\mu T_{\mu\nu}^{(\phi)} + \nabla^\mu T_{\mu\nu}^{(m)} + \nabla^\mu T_{\mu\nu}^{(r)} = 0. \quad (5.9)$$

Indeed, since baryons and relativistic matter are minimally coupled to the metric $g_{\mu\nu}$, their conservation equations take the usual form,

$$\nabla^\mu T_{\mu\nu}^{(b)} = 0, \quad \nabla^\mu T_{\mu\nu}^{(r)} = 0. \quad (5.10)$$

On the other hand, that does not hold anymore for the scalar field or DM, individually, but rather, the sum of these components is conserved:

$$\nabla^\mu T_{\mu\nu}^{(\phi)} + \nabla^\mu T_{\mu\nu}^{(c)} = 0. \quad (5.11)$$

In order to derive the equation of motion for ϕ , it is useful to first calculate the following quantities

$$\frac{\partial}{\partial \phi} (\sqrt{-\tilde{g}} \mathcal{L}_c) = \frac{\delta(\sqrt{-\tilde{g}} \mathcal{L}_c)}{\delta g_{\mu\nu}} \frac{\delta g_{\mu\nu}}{\delta \tilde{g}_{\alpha\beta}} \frac{\delta \tilde{g}_{\alpha\beta}}{\delta \phi} = \sqrt{-g} Z, \quad (5.12)$$

$$\frac{\partial}{\partial \phi_\mu} (\sqrt{-\tilde{g}} \mathcal{L}_c) = \frac{\delta(\sqrt{-\tilde{g}} \mathcal{L}_c)}{\delta g_{\mu\nu}} \frac{\delta g_{\mu\nu}}{\delta \tilde{g}_{\alpha\beta}} \frac{\delta \tilde{g}_{\alpha\beta}}{\delta \phi_\mu} = \sqrt{-g} W^\mu. \quad (5.13)$$

For the metric transformation (5.3), the functional derivatives become

$$\frac{\delta g_{\mu\nu}}{\delta \tilde{g}_{\alpha\beta}} = \frac{1}{A} \left[\delta_\mu^\alpha \delta_\nu^\beta - \frac{1}{2} C (A_X g_{\mu\nu} + B_X \phi_\mu \phi_\nu) \phi^\alpha \phi^\beta \right], \quad (5.14)$$

$$\frac{\delta \tilde{g}_{\alpha\beta}}{\delta \phi_\mu} = B (\delta_\alpha^\mu \phi_\beta + \delta_\beta^\mu \phi_\alpha) - A_X g_{\alpha\beta} \phi^\mu - B_X \phi_\alpha \phi_\beta \phi^\mu, \quad (5.15)$$

such that Z and W^μ can be written as

$$Z = \frac{1}{2A} \left([A_\phi + A_X C X (A_\phi - 2B_\phi X)] T_{(c)} + [B_\phi + B_X C X (A_\phi - 2B_\phi X)] T_{(c)}^{\mu\nu} \phi_\mu \phi_\nu \right), \quad (5.16)$$

$$W^\mu = \frac{B}{A} T_{(c)}^{\mu\nu} \phi_\nu - \frac{C}{2A} (A - 2BX) \left(A_X T_{(c)} + B_X T_{(c)}^{\alpha\beta} \phi_\alpha \phi_\beta \right) \phi^\mu. \quad (5.17)$$

In the above equations, $T_{(c)}$ is the trace of the DM energy-momentum tensor, while the function

C is defined as

$$C = \frac{1}{A - A_X X + 2B_X X^2}. \quad (5.18)$$

The subscripts ϕ and X represent the derivatives with respect to the scalar field and its kinetic term, respectively.

Using these results, the variation with respect to ϕ gives the equation of motion for the scalar field,

$$(K_X g^{\mu\nu} - K_{XX} \phi^\mu \phi^\nu) \phi_{\mu\nu} + K_\phi - 2K_{\phi X} X = Q, \quad (5.19)$$

where Q is the deviation from the minimally coupled DM,

$$Q = \nabla_\mu W^\mu - Z. \quad (5.20)$$

The combination of eqs. (5.11), (5.19) and (5.20) leads to

$$\nabla^\mu T_{\mu\nu}^{(c)} + Q \phi_\nu = 0. \quad (5.21)$$

Hence, the coupling function Q determines the exchange of energy and momentum between the scalar field and the DM component.

5.2 Cosmological evolution

Let us consider linear perturbations in the Newtonian gauge on a spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric,

$$ds^2 = -[1 + 2\Phi(t, \mathbf{x})] dt^2 + a^2(t)[1 - 2\Psi(t, \mathbf{x})] d\mathbf{x}^2, \quad (5.22)$$

and define the energy-momentum tensor for each fluid as

$$T^0_0 = -\rho(1 + \delta), \quad (5.23)$$

$$T^0_i = -(\rho + P) \partial_i v, \quad (5.24)$$

$$T^i_j = (P + \delta P) \delta^i_j + \pi^i_j, \quad \pi^i_i = 0. \quad (5.25)$$

Here ρ is the homogeneous energy density and $\delta = \delta\rho/\rho$ energy density contrast; P and δP are the background pressure and its perturbation, respectively; v is the velocity potential; and $\pi^i_j = T^i_j - \delta^i_j T^k_k/3$ is the anisotropic stress. We once again point out that this definition of the components of the energy-momentum tensor is valid for each species. In particular, for baryons

and cold dark matter, $P = \delta P = \pi_j^i = 0$.

From the above definition, the overdensity and velocity potential for the total matter fluid can be respectively expressed as

$$\delta_m = \omega_c \delta_c + \omega_b \delta_b, \quad (5.26)$$

$$v_m = \omega_c v_c + \omega_b v_b, \quad (5.27)$$

where $\omega_i = \rho_i/\rho_m$, $i = b, c$, is the fractional energy density. We also split the scalar field as $\phi(t, \mathbf{x}) \rightarrow \phi(t) + \delta\phi(t, \mathbf{x})$, and introduce the convenient notation $\delta Q(t, \mathbf{x}) = Q(t, \mathbf{x}) - Q_0(t)$, where Q_0 is a background value.

5.2.1 Background equations

At background level, Einstein equations become

$$H^2 = \frac{1}{3M_{\text{Pl}}^2}(\rho_\phi + \rho_b + \rho_c + \rho_r), \quad (5.28)$$

$$3H^2 + 2\dot{H} = -\frac{1}{M_{\text{Pl}}^2}\left(P_\phi + \frac{1}{3}\rho_r\right), \quad (5.29)$$

where $H = \dot{a}/a$, and the energy density and pressure for K-essence field are respectively given by

$$\rho_\phi = \dot{\phi}^2 K_X - K, \quad (5.30)$$

$$P_\phi = K. \quad (5.31)$$

The equation of motion for ϕ , eq. (5.19), yields

$$(K_X + K_{XX}\dot{\phi}^2)\ddot{\phi} + 3H\dot{\phi}K_X + K_{\phi X}\dot{\phi}^2 - K_\phi = -Q_0, \quad (5.32)$$

while the energy-momentum conservation for radiation, baryons and CDM leads to the background continuity equations

$$\dot{\rho}_r + 4H\rho_r = 0, \quad (5.33)$$

$$\dot{\rho}_b + 3H\rho_b = 0, \quad (5.34)$$

$$\dot{\rho}_c + 3H\rho_c = Q_0\dot{\phi}. \quad (5.35)$$

The background coupling Q_0 can be directly calculated from eq. (5.20), resulting in

$$Q_0 = \frac{\rho_c}{\dot{\phi}} \frac{d\Gamma}{dt}, \quad \Gamma = \ln \left(\frac{2A - A_X \dot{\phi}^2 + B_X \dot{\phi}^4}{2\sqrt{A - B\dot{\phi}^2}} \right). \quad (5.36)$$

In general, the above expression for Q_0 contains $\ddot{\phi}$ such that, in order to obtain the explicit expression of Q_0 , one needs to eliminate $\ddot{\phi}$ using the equation of motion for the scalar field, solving it for Q_0 .

Using the definitions of energy density and pressure for the scalar field, eqs. (5.30) and (5.31), respectively, the equation of motion (5.32) can be rewritten as

$$\dot{\rho}_\phi + 3H(1 + w_\phi)\rho_\phi = -Q_0\dot{\phi}, \quad (5.37)$$

which makes, according to the eq. (5.35), the conservation of the sum $\rho_c + \rho_\phi$ evident. The equation of state for the scalar field is defined in the usual way: $w_\phi = P_\phi/\rho_\phi$. Furthermore, the CDM energy density can be found from eq. (5.35), which yields the solution

$$\rho_c \propto a^{-3} e^{\Gamma(\phi, X)}. \quad (5.38)$$

At the classical level, the interaction can be interpreted as a variation in the mass of the CDM particle.

5.2.2 Linear perturbations

In the linear perturbation theory, the $(0, 0)$, $(0, i)$, along with the trace and traceless part of (i, j) , components of the Einstein equation in Fourier space read, respectively,

$$2M_{\text{Pl}}^2 \left[3H(H\Phi + \dot{\Psi}) + \frac{k^2}{a^2}\Psi \right] = -\delta\rho_\phi - \rho_m\delta_m - \rho_r\delta_r, \quad (5.39)$$

$$2M_{\text{Pl}}^2 \left[\dot{\Psi} + H\Phi \right] = K_X \dot{\phi}\delta\phi + \rho_m v_m + \frac{4}{3}\rho_r v_r, \quad (5.40)$$

$$2M_{\text{Pl}}^2 \left[\ddot{\Psi} + 3H\dot{\Psi} + H\dot{\Phi} + \left(2\dot{H} + 3H^2 \right)\Phi + \frac{1}{3}\frac{k^2}{a^2}(\Psi - \Phi) \right] = \delta P_\phi + \delta P_r, \quad (5.41)$$

$$2M_{\text{Pl}}^2 \frac{k^2}{a^2}(\Psi - \Phi) = 4\rho_r\sigma_r, \quad (5.42)$$

where $\delta\rho_\phi$ and δP_ϕ are the scalar field energy density and pressure perturbations, respectively, while the anisotropic shear σ is defined as

$$(\rho + P)\sigma \equiv -\left(\hat{k}_i \hat{k}_j - \frac{1}{3}\delta_{ij} \right) \pi^i_j, \quad (5.43)$$

with $\hat{k}_i = k_i/k$. From the energy-momentum tensor, eq. (5.8), combined with the definitions (5.23) and (5.24), we find

$$\delta\rho_\phi = -\left(K_\phi - \dot{\phi}^2 K_{\phi X}\right)\delta\phi + \left(K_X + \dot{\phi}^2 K_{XX}\right)\dot{\phi}\delta\dot{\phi} - \left(K_X + \dot{\phi}^2 K_{XX}\right)\dot{\phi}^2\Phi, \quad (5.44)$$

$$\delta P_\phi = K_\phi\delta\phi + \dot{\phi}K_X\left(\delta\dot{\phi} - \dot{\phi}\Phi\right). \quad (5.45)$$

Here, the subscripts ϕ and X in the right-hand side represent the derivatives with respect to the background values of the scalar field and its kinetic term, respectively. We can also compute the perturbed equation of motion for the scalar field,

$$-\mathcal{A}_1\frac{k^2}{a^2}\delta\phi - m_\phi^2\delta\phi + 3\mathcal{A}_1\dot{\phi}\dot{\Psi} + \mathcal{A}_2\dot{\phi}\dot{\Phi} + \mathcal{A}_3\Phi - \mathcal{A}_2\delta\ddot{\phi} + \mathcal{A}_4\delta\dot{\phi} = \delta Q, \quad (5.46)$$

where

$$\mathcal{A}_1 = K_X, \quad (5.47)$$

$$\mathcal{A}_2 = K_X + K_{XX}\dot{\phi}^2, \quad (5.48)$$

$$\mathcal{A}_3 = 6H\dot{\phi}K_X + \dot{\phi}^2 K_{\phi X} + 3H\dot{\phi}^3 K_{XX} + \dot{\phi}^4 K_{\phi XX} + \ddot{\phi}\left(2K_X + 5\dot{\phi}^2 K_{XX} + \dot{\phi}^4 K_{XXX}\right), \quad (5.49)$$

$$\mathcal{A}_4 = -3HK_X - 3H\dot{\phi}^2 K_{XX} - \ddot{\phi}\left(K_{XXX}\dot{\phi}^3 + 3K_{XX}\dot{\phi}\right) - \dot{\phi}K_{\phi X} - \dot{\phi}^3 K_{\phi XX}, \quad (5.50)$$

$$m_\phi^2 = -K_{\phi\phi} + 3K_{\phi X}H\dot{\phi} + K_{\phi\phi X}\dot{\phi}^2 + \ddot{\phi}\left(K_{\phi X} + K_{\phi XX}\dot{\phi}^2\right), \quad (5.51)$$

and the perturbation of Q is given by

$$\begin{aligned} \delta Q = & (R_1 + R_2)\dot{\phi}\dot{\delta}_c + Q_0\delta_c + R_1\dot{\phi}\frac{k^2}{a^2}v_c + R_2\frac{k^2}{a^2}\delta\phi - 3(R_1 + R_2)\dot{\phi}\dot{\Psi} \\ & - R_3\dot{\phi}\dot{\Phi} + R_4\dot{\phi}\dot{\Phi} + R_3\delta\ddot{\phi} + R_5\delta\dot{\phi} + R_6\delta\phi, \end{aligned} \quad (5.52)$$

in which, for convenience, we have defined the quantities

$$R_1 = \frac{B}{A}\rho_c, \quad (5.53)$$

$$R_2 = -\frac{C}{2A}(A - B\dot{\phi}^2)(A_X - B_X\dot{\phi}^2)\rho_c, \quad (5.54)$$

$$R_3 = \rho_c\dot{\phi}\omega_X, \quad (5.55)$$

$$R_4 = -Q_0(\omega + \omega_X\dot{\phi}^2) - \rho_c(\omega_\phi + 3\omega_X\ddot{\phi} + \dot{\omega}_X\dot{\phi} - \zeta_X\dot{\phi}), \quad (5.56)$$

$$R_5 = \omega_X Q_0\dot{\phi}^2 + \rho_c(\omega_\phi + \omega_X\ddot{\phi} + \dot{\omega}_X\dot{\phi} - \zeta_X\dot{\phi}), \quad (5.57)$$

$$R_6 = \omega_\phi Q_0\dot{\phi} + \rho_c(\dot{\omega}_\phi - \zeta_\phi), \quad (5.58)$$

with

$$\omega = (R_1 + R_2) \frac{\dot{\phi}}{\rho_c}, \quad (5.59)$$

$$\zeta = -\frac{1}{2A} \left[A_\phi + \frac{1}{2} C A_X \dot{\phi}^2 (A_\phi - B_\phi \dot{\phi}^2) \right] + \frac{\dot{\phi}^2}{2A} \left[B_\phi + \frac{1}{2} C B_X \dot{\phi}^2 (A_\phi - B_\phi \dot{\phi}^2) \right]. \quad (5.60)$$

The energy-momentum conservation for baryonic matter yields the standard equation,

$$\dot{\delta}_b - 3\dot{\Psi} + \frac{k^2}{a^2} v_b = 0, \quad (5.61)$$

$$\dot{v}_b - \Phi = 0, \quad (5.62)$$

while CDM follows

$$\dot{\delta}_c - 3\dot{\Psi} + \frac{k^2}{a^2} v_c = \frac{\dot{\phi}}{\rho_c} (\delta Q - Q_0 \delta_c) + \frac{Q_0}{\rho_c} \delta \dot{\phi}, \quad (5.63)$$

$$\dot{v}_c - \Phi = \frac{Q_0}{\rho_c} (\delta \phi - \dot{\phi} v_c). \quad (5.64)$$

For radiation, the respective equations do not change, remaining the same as in the standard Λ CDM case.

5.3 Effective growth rate and redshift space distortions

In this section, we derive the effective growth rate of matter perturbations for the disformally coupled CDM and show that it can significantly differ from the standard definition.

5.3.1 Quasi-static approximation

Currently, data coming from galaxy surveys are restricted to scales much smaller than the cosmological horizon. Moreover, for scales well inside the sound horizon of scalar field perturbations, one can employ the quasi-static approximation [168]. Effectively, this corresponds to dropping time derivatives, with respect to space derivatives, of metric and scalar field perturbations. Furthermore, since we are interested in the effects of the DE scalar field at late-times, we neglect the contributions from relativistic matter (radiation) here and hereafter.

In such an approximation, Einstein equations become constraint equations for the gravitational potentials Φ and Ψ ,

$$\frac{k^2}{a^2} \Psi = \frac{k^2}{a^2} \Phi = -\frac{1}{2M_{\text{Pl}}^2} \rho_m \delta_m, \quad (5.65)$$

while eqs. (5.46) and (5.52) give a constraint for the scalar perturbation,

$$-\mathcal{A} \frac{k^2}{a^2} \delta\phi \equiv -\left(\mathcal{A}_1 \frac{k^2}{a^2} + m_\phi^2\right) \delta\phi = \delta Q, \quad (5.66)$$

where δQ is given by

$$\delta Q = (R_1 + R_2) \dot{\phi} \dot{\delta}_c + Q_0 \delta_c + R_1 \dot{\phi} \frac{k^2}{a^2} v_c + R_2 \frac{k^2}{a^2} \delta\phi. \quad (5.67)$$

In eq. (5.66) we have defined $\mathcal{A} = \mathcal{A}(t, k)$ which in general depends on the wavenumber when the Compton wavelength of the scalar field is smaller than the Hubble horizon scale. When the scalar field consists of a canonical kinetic term and the scalar mass scale is irrelevant within sub-horizon scales (see appendix A), then we have $\mathcal{A} = 1$.

The continuity and Euler equations for baryon are given by

$$\dot{\delta}_b + \frac{k^2}{a^2} v_b = 0, \quad (5.68)$$

$$\dot{v}_b - \Phi = 0, \quad (5.69)$$

and those for CDM are modified as follows:

$$\dot{\delta}_c + \frac{k^2}{a^2} v_c = \frac{\dot{\phi}}{\rho_c} (\delta Q - Q_0 \delta_c), \quad (5.70)$$

$$\dot{v}_c - \Phi = \frac{Q_0}{\rho_c} (\delta\phi - \dot{\phi} v_c). \quad (5.71)$$

Using $\delta\phi$ and δQ given in eqs. (5.66) and (5.67), these become

$$(1 - \Upsilon_1) \left(\dot{\delta}_c + \frac{k^2}{a^2} v_c \right) = \Upsilon_2 (\dot{\delta}_c - \epsilon \delta_c), \quad (5.72)$$

$$(1 - \Upsilon_1) \frac{k^2}{a^2} (\dot{v}_c - \Phi) = \Upsilon_3 (\dot{\delta}_c - \epsilon \delta_c), \quad (5.73)$$

where, for convenience, we introduced the quantities

$$\Upsilon_1 = \frac{\dot{\phi}^2}{\rho_c} \frac{\mathcal{A} R_1}{\mathcal{A} + R_2}, \quad \Upsilon_2 = \frac{\dot{\phi}^2}{\rho_c} \frac{\mathcal{A} R_2}{\mathcal{A} + R_2}, \quad \Upsilon_3 = \frac{Q_0 \dot{\phi}}{\rho_c} (1 - \Upsilon_1 - \Upsilon_2) - \epsilon \Upsilon_2, \quad (5.74)$$

with $\epsilon = Q_0 / \mathcal{A} \dot{\phi}$. In the derivation of eq. (5.73) from eq. (5.71), we have eliminated v_c using eq. (5.72). Note that for purely conformal couplings, i.e. $A_X = B = 0$, the functions Υ_1 and Υ_2 vanish.

Taking the time derivative of the continuity equations, (5.68) and (5.70), and eliminating the metric perturbations, scalar field perturbations, and velocity terms, we obtain the following second-

order differential equations for baryons and CDM, respectively,

$$\ddot{\delta}_b + 2H\dot{\delta}_b - \frac{1}{2M_{\text{Pl}}^2}\rho_m\delta_m = 0, \quad (5.75)$$

$$(1 - \Upsilon_1 - \Upsilon_2)\ddot{\delta}_c + 2H(1 - \mathcal{E}_1)\dot{\delta}_c - \frac{1}{2M_{\text{Pl}}^2}[(1 - \mathcal{E}_2)\rho_c\delta_c + (1 - \Upsilon_1)\rho_b\delta_b] = 0, \quad (5.76)$$

where we have introduced two new functions of background quantities,

$$\mathcal{E}_1 = \Upsilon_1 + \frac{1}{2H} \left[\frac{1 - \Upsilon_1}{a^2} \frac{d}{dt} \left(\frac{a^2 \Upsilon_2}{1 - \Upsilon_1} \right) - \Upsilon_3 - \epsilon \Upsilon_2 \right], \quad (5.77)$$

$$\mathcal{E}_2 = \Upsilon_1 + \frac{2M_{\text{Pl}}^2}{\rho_c} \left[\frac{1 - \Upsilon_1}{a^2} \frac{d}{dt} \left(\frac{a^2 \epsilon \Upsilon_2}{1 - \Upsilon_1} \right) - \epsilon \Upsilon_3 \right]. \quad (5.78)$$

Hence, the CDM growth function experiences a different Hubble friction, H_{eff} , and feels a different gravitational pull, G_{eff} , depending on the couplings. As a result, the evolution of δ_c itself is different from the standard uncoupled case.

5.3.2 Linear growth rate

For sub-horizon scales, the system formed by the second-order differential eqs. (5.75) and (5.76) does not depend on the wavenumber k . Therefore, one can isolate the time dependence from the k dependence of the initial conditions and express the growing solutions for the baryon and CDM density contrasts as

$$\delta_i(t, \mathbf{k}) = D_i(t)\delta_0(\mathbf{k}). \quad (5.79)$$

Here, $D_i(t)$ is the k -independent growth factor while $\delta_0(\mathbf{k})$ represents the initial density contrasts.

Another quantity useful in describing the evolution of the perturbations is the so-called linear growth rate, defined as

$$f_i(t) \equiv \frac{d \ln D_i}{d \ln a}, \quad (5.80)$$

which quantifies the rate of growth in a Hubble time. The continuity equations, (5.68) and (5.70), can then be recast in terms of the respective growth factors, and the velocity potentials become

$$v_i(t, \mathbf{k}) = -\frac{a^2 H}{k^2} f_i^{\text{eff}}(t, k) \delta_i(t, \mathbf{k}). \quad (5.81)$$

For baryons, we immediately see that $f_b = f_b^{\text{eff}}$. On the other hand, eq. (5.70) implies that the effective linear growth rate for CDM, f_c^{eff} , can deviate significantly from the standard one:

$$f_c^{\text{eff}} = f_c - \frac{\Upsilon_2}{1 - \Upsilon_1} \left(f_c - \frac{Q_0}{\mathcal{A}H\dot{\phi}} \right) \equiv f_c + \Delta f_c. \quad (5.82)$$

Furthermore, employing eq. (5.27), the effective linear growth rate of the total matter becomes

$$f_{\text{m}}^{\text{eff}} = \frac{\omega_{\text{c}} D_{\text{c}} f_{\text{c}}^{\text{eff}} + \omega_{\text{b}} D_{\text{b}} f_{\text{b}}}{\omega_{\text{c}} D_{\text{c}} + \omega_{\text{b}} D_{\text{b}}} \equiv f_{\text{m}} + \Delta f_{\text{m}}, \quad (5.83)$$

$$\Delta f_{\text{m}} = \omega_{\text{c}} \frac{D_{\text{c}}}{D_{\text{m}}} \Delta f_{\text{c}} - \omega_{\text{b}} \frac{Q_0 \dot{\phi}}{H \rho_{\text{m}}} \frac{D_{\text{c}} - D_{\text{b}}}{D_{\text{m}}}, \quad (5.84)$$

where $D_{\text{m}} = \omega_{\text{c}} D_{\text{c}} + \omega_{\text{b}} D_{\text{b}}$ is the total matter growth function. The first term on the right-hand side of eq. (5.84) comes from the definition of the CDM effective growth rate, eq. (5.82), while the second is reflex of the non-minimal coupling on the actual growth rate (in the minimally coupled case, the density ratios ω_{b} and ω_{c} are constant, while the baryon and dark matter growth function are equal; naturally, that is no longer the case for when there is a non-minimal coupling).

Although $f_{\text{m}}^{\text{eff}}$ is simply given by the growth-factor-weighted average of the effective growth rates for CDM and baryons, the non-trivial terms in the CDM continuity equation and the background dynamics lead to a deviation from f_{m} , the actual growth rate.

5.3.3 Modified Kaiser formula

Spectroscopic surveys determine the distance to galaxies using their redshift. However, this measurement includes not only the Hubble flow but also a contribution from the galaxies' peculiar velocities, which will lead to errors in the determination of distances. On large scales, galaxies tend to follow the underlying matter distribution and fall into overdense regions. As a consequence, in real space, the clustering of galaxies tends to not have a preferred direction. In redshift space, however, the galaxy maps will show an anisotropy coming from the error in the determination of distances.

On linear scales, these anisotropies can be systematically taken into account. The peculiar velocity v_{g} can be related to the total matter fluid velocity v_{m} by imposing some reasonable physical condition, such as momentum-conservation law for each galaxy [79]. In the standard case, this leads to an expression for the galaxy power spectrum in redshift space, the so-called Kaiser formula [94]

$$P_{\text{g,s}}(\mathbf{k}, t) = [b_{\text{g}}(t) + f_{\text{m}}(t) \mu^2]^2 P_{\text{m}}(k, t), \quad (5.85)$$

where b_{g} is a linear, scale-independent bias relating the tracing population of galaxies and underlying matter field distribution, μ is the cosine of the angle between the wave vector and the line of sight, and P_{m} is the matter power spectrum in real space. According to eq. (5.79), the matter power spectrum can be written in terms of the growth factor as $P_{\text{m}}(k, t) = D_{\text{m}}^2(t) P_{\text{m},0}(k)$, where $P_{\text{m},0}$ is the initial matter power spectrum. In eq. (5.85) we assumed that, since galaxies move according to a common gravitational field, its velocities are not biased with respect to the matter

velocity field. This formula holds not only for Λ CDM, but also for quintessence and modified gravity models with a minimal coupling.

As previously seen, when there is an interaction in the dark sector, the velocity potential, eq. (5.81), is related not to the growth rate itself but to the effective growth rate. Hence, we extend eq. (5.85) to a modified Kaiser formula to take into account the coupling as

$$P_{g,s}(\mathbf{k}, t) = [b_g(t) + f_m^{\text{eff}}(t, k) \mu^2]^2 P_m(k, t), \quad (5.86)$$

with f_m^{eff} is given in eq. (5.83). For simplicity, as suggested by the Λ CDM case [93], we assume that the relation $v_g = v_m$ holds even when CDM is non-minimally coupled to the scalar field. In fact, as long as the galaxy peculiar velocity has some dependence on the dark matter velocity field (i.e. $v_g \neq v_b$), the coupling effect on RSD measurements of the growth rate will be present. The distribution of galaxies in redshift space depends on the effective growth rate and, in principle, an interaction between dark matter and the scalar field can be probed using clustering measurements.

One way to obtain the actual growth rate is to observe the time-evolution of large-scale structures using a direct probe, e.g. tomographic weak lensing. In such a case, the measurement of the gravitational potential gives clean information about the total matter density, via the Poisson equation (5.65). On the other hand, to probe the coupling between dark matter and the scalar field, the linear galaxy bias can be determined at each redshift by the cross-correlation between weak lensing and galaxy clustering [169] and one can then unambiguously measure the coupling using RSD measurements.

In terms of the real-space and redshift-space galaxy power spectra, $P_g \equiv b_g^2 P_m$ and $P_{g,s}$ respectively, the Kaiser formula, eq. (5.85), reads

$$P_{g,s}(\mathbf{k}, t) = [1 + \beta(t) \mu^2]^2 P_g(k, t), \quad (5.87)$$

where $\beta \equiv f_m/b_g$, which encodes the corrections due to redshift space distortions, is the redshift-distortion factor. In our setting, this quantity becomes

$$\beta^{\text{eff}} = \frac{f_m^{\text{eff}}}{b_g} = \beta + \Delta\beta, \quad (5.88)$$

with two contributions: the usual one, due to the peculiar velocity of galaxies, and also an additional term encompassing the modifications of the CDM continuity and Euler equations due to the non-minimal coupling. For a given choice of coupling, the latter can be characterized by

$$\frac{\Delta\beta}{\beta} = \frac{\Delta f_m}{f_m}, \quad (5.89)$$

where Δf_m was defined in eq. (5.83). For positive values of $\Delta f_m/f_m$, the difference between the real-space and redshift-space power spectra will be further increased, while for negative values, the coupling leads to a suppression of the redshift-distortion factor.

5.4 Summary

In this chapter, we extended the treatment of Einstein's gravity with a non-minimally coupled DM component recently presented in [74], generalizing it from the canonical scalar field Lagrangian to the general K-essence one. In this setup, the coupling between scalar field and DM is designed through the effective metric with conformal and disformal factors, whereas, to evade solar system constraints, baryon and radiation couple minimally to metric. Effectively, such dependence leads to an energy transfer between DM and DE.

At the homogeneous level, the interaction can be understood as if the mass of the DM particle depends on the value of the scalar field. As for linear perturbations, the interaction mediated by a scalar field through disformal and conformal transformations changes both the perturbed continuity and Euler equations for CDM, while baryons and radiation follow the standard equations. We showed that the modification is important even at sub-horizon scales (in the quasi-static limit) and the overall effect can be understood in terms of an effective Hubble friction and effective gravitational coupling. As a result, the growth of perturbation differs from the standard uncoupled case, and the difference can be, in principle, observationally tested.

In the presence of an interaction, since the modified Kaiser formula provides information only about the effective linear growth rate, determined using the galaxy peculiar velocity field, single-redshift RSD measurements are not able to measure the actual growth. However, multiple-redshift RSD measurements, or a combination of different types of observations, can help in breaking the degeneracy. Therefore, when considering extensions of the Λ CDM paradigm, one must carefully take such effects into account. These statements are true not only for our setup but also for any cosmological model including DM interacting with DE, as long as the continuity equation for the total matter is modified.

6. RSD II: Cosmological Examples and Forecast

One of the most reliable large-scale structure observable is the galaxy two-point correlation function (i.e. power spectrum), which can give information about the growth of density perturbations. Since spectroscopic surveys measure the redshift of galaxies, the observed position of each galaxy is different from the actual one, due to the combined effect of the peculiar velocity and the Hubble expansion. In other words, the power spectrum in real space does not coincide with the one in redshift space. This effect is called redshift space distortion (RSD) and it allows us to extract statistical information regarding the peculiar velocities of galaxies [94].

The next generation of spectroscopic surveys, such as PFS [170], eBOSS [171], Euclid ¹ [172], and WFIRST ² [173], combined with future HI galaxy surveys such as the Square Kilometre Array (SKA) ³ [174–176], will improve the measurements of peculiar velocities, enabling us to test various cosmological models at an extraordinary level. The recent measurements of the cosmic microwave background (CMB) by Planck satellite [23, 177] have greatly improved our knowledge regarding the universe at its early stages, and a similar gain is awaited from the forthcoming galaxy surveys, which are expected to constrain DE to an unrivaled precision.

As discussed on chapter 6, when only DM is non-minimally coupled to the DE scalar field, the matter linear growth rate inferred by measurements of galaxies peculiar velocities is no longer defined logarithmic derivative of the linear growth function, as in the standard Λ CDM case, but rather it has an additional dependence due to the modifications to the continuity equation [74, 78]. Therefore, the relation between the real space and redshift space power spectra, the so-called Kaiser formula, must be modified, and the non-minimal coupling leads to a different interpretation of the growth function for matter perturbations.

In this chapter, we dive deeper into this issue, studying three models parameterized by the conformal and disformal functions, and solve both the background and linear equations. Additionally, we forecast the constraints to the coupling parameter from two representative examples of future

¹<http://www.euclid-ec.org/>

²<https://wfirst.gsfc.nasa.gov/>

³<https://www.skatelescope.org/>

galaxy surveys, namely, Euclid and SKA.

Concrete cosmological examples are studied in section 6.1 choosing three different sets of conformal and disformal coupling functions. Using a modified version of the popular Boltzmann code CLASS, we numerically solve the dynamical equations and study how the interaction affects the cosmic evolution. Then, in section 6.2, we perform a forecast analysis and predict the constraining power from future galaxy surveys, assuming SKA2-like and Euclid-like specifications, to our choices of coupling functions. This chapter is based on [78].

6.1 Concrete examples

In this section we study three cosmological models, choosing the scalar field Lagrangian, along with the conformal and disformal factors, and discuss how the strength of the coupling changes the evolution of background and perturbed quantities. To simplify the notation, throughout this section we set the reduced Planck mass, M_{Pl} , to unity.

In what follows, we consider coupled quintessence models, with the canonical scalar field Lagrangian given by

$$K(\phi, X) = X - V(\phi), \quad (6.1)$$

with an inverse-power-law scalar potential [29, 30],

$$V(\phi) = M^2 \phi^{-n}, \quad (6.2)$$

where the slope of the potential n is a positive integer and M is a constant. The potential (6.2) belongs to the class of freezing models [178], in which the field rolls down the potential in the past, slowing down at the onset of the cosmic acceleration. In the uncoupled case, this model is known to give rise to tracking solutions [33, 34]: an attractor regime in which the energy density of the scalar field is subdominant, and both its equation of state and energy density parameter are approximately constant. For a wide range of initial conditions, the evolution of the scalar field will approach the previously mentioned attractor solution, with its equation of state, for the potential (6.2), being given by

$$w_\phi \approx \frac{nw_M - 2}{n + 2}, \quad (6.3)$$

where $w_M = P_M/\rho_M$ is the equation of state parameter of the dominant fluid (matter or radiation). Cosmological models that admit tracking solutions are interesting because such behavior lessens the dependence of the scalar field dynamics on the initial condition. In fact, a well-known result is that in the uncoupled case it is possible to solve the equation of motion for ϕ analytically (see, for instance, [179]).

As for the conformal and disformal factors, we consider the following one-parameter models:

Conformal model I:

$$A(\phi, X) = e^{-2\alpha\phi}, \quad B(\phi, X) = 0. \quad (6.4)$$

Conformal model II:

$$A(\phi, X) = e^{-\alpha\phi^2}, \quad B(\phi, X) = 0. \quad (6.5)$$

Disformal model III:

$$A(\phi, X) = 1, \quad B(\phi, X) = \alpha/X. \quad (6.6)$$

In the above expressions, α is the dimensionless coupling constant that controls the strength of the DE/CDM interaction and must be constrained by observations. For the conformal models I and II, the continuity equation in the quasi-static limit, eq. (5.72) is the standard one, with no explicit dependence on the coupling, whereas for model III the background coupling Q_0 vanishes. As will be discussed in the following sections, the most important difference between model I and model II is that, even for small values of the coupling constant, the former will deviate from the tracker solution, whilst the latter still hold, by construction, the tracker properties.

In the following subsections, we present the numerical solutions for the models considered in this work. The background and linear perturbation equations for each model were implemented in the publicly available CLASS code [180]. We assume that at early times the effect of the coupling is negligible and set the initial conditions for the scalar field using the analytic expression for the tracker solution. All plots were made using the cosmological parameters presented on table 6.1. Additionally, the value of M in eq. (6.2) is found using a shooting method such that we get the desired value of Ω_ϕ , the current value of the scalar field density parameter, whereas the slope parameter n is set to 0.5. The initial energy density of cold dark matter is also found employing the shooting method.

6.1.1 Background evolution

In order to study the background evolution, it is useful to rewrite the continuity equations for CDM and the scalar field, eqs. (5.35) and (5.37), respectively, as

$$\dot{\rho}_c + 3H(1 + w_c^{\text{eff}})\rho_c = 0, \quad (6.7)$$

$$\dot{\rho}_\phi + 3H(1 + w_\phi^{\text{eff}})\rho_\phi = 0, \quad (6.8)$$

where we have defined the expressions for the effective equation of state as

$$w_c^{\text{eff}} = -\frac{Q_0 \dot{\phi}}{3H\dot{\rho}_c}, \quad (6.9)$$

$$w_\phi^{\text{eff}} = w_\phi + \frac{Q_0 \dot{\phi}}{3H\dot{\rho}_\phi}. \quad (6.10)$$

As long as w_c^{eff} is small, much less than unity, it is expected that the evolution of the CDM energy density is not affected drastically by the interaction. Indeed, that is the case, for small values of the coupling constant, if the kinetic energy density of the scalar field is much smaller than the CDM energy density. Furthermore, when w_ϕ^{eff} is approximately constant the evolution of ϕ has a tracking behavior, even in the coupled case.

Additionally, for the scalar field Lagrangian (6.1), the equation of motion for ϕ at background level, eq. (5.32), becomes

$$\ddot{\phi} + 3H\dot{\phi} + V_\phi = -Q_0. \quad (6.11)$$

Conformal model I

The conformal model I, characterized by the coupling functions (6.4), is an example of purely conformal metric transformation. This kind of interaction was first proposed in the context of coupled quintessence [71] and has been further investigated in [69, 181, 182]. Imposing that both terms in the right hand side of eq. (6.10) are constant leads to a scaling behavior and constrains the potential to be the exponential one.

According to eq. (5.36), the coupling at background level is described by

$$Q_0 = -\alpha\rho_c, \quad \Gamma = -\alpha\phi, \quad (6.12)$$

such that, from eq. (5.38), the CDM energy density evolves as

$$\rho_c \propto a^{-3}e^{-\alpha\phi}, \quad (6.13)$$

while the equation of motion for the scalar field, eq. (6.11), gives

$$\ddot{\phi} + 3H\dot{\phi} + V_\phi = \alpha\rho_c. \quad (6.14)$$

When $\alpha > 0$, since ϕ is also positive, ρ_c dilutes faster than in the uncoupled case, and energy is transferred from dark matter to the scalar field, starting around the transition from radiation to matter epoch (figure 6.1a). As a result, for a given value of Ω_c today, the CDM energy density in the past will be higher in coupled models than it would be in the uncoupled case. This leads to

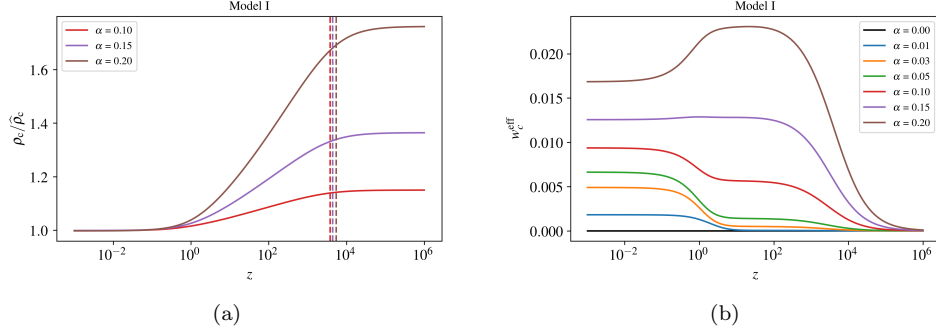


Figure 6.1: Left panel (a): evolution of the ratio between the coupled and uncoupled CDM energy density, ρ_c and $\hat{\rho}_c$, respectively. The dashed lines correspond to the redshift at matter-radiation equality. Right panel (b): evolution of the effective equation of state of dark matter, eq. (6.9). Both plots show the respective quantities as function of the redshift for the conformal model I, eq. (6.4), for different values of the coupling constant.

a shift in the time of matter-radiation equality, which will take place earlier, at a higher redshift. Also, for very small values of α (e.g. $\alpha \lesssim 0.15$ in figure 6.1b) the decay of ρ_c is more important during DE epoch, whereas for slightly larger values, the energy transfer is more salient during matter epoch.

In figure 6.2 we present the evolution of the scalar field equation of state as a function of the redshift. In the uncoupled case, ϕ follows the tracker solution with, according to eq. (6.3), $w_\phi \approx -0.73$ during radiation epoch, and $w_\phi \approx -0.8$ during matter epoch. Even though the initial conditions for ϕ and $\dot{\phi}$ in the coupled case were set using the tracker solution, the evolution of the scalar field during radiation epoch is such that its kinetic energy dominates over the potential and $w_\phi \approx 1$, departing from the tracker behavior. For small values of the coupling constant (e.g. $\alpha \lesssim 0.01$), the field catches up with the tracker solution around the time of the onset of the accelerated expansion. However, for larger values of α , the evolution can deviate significantly from the uncoupled tracker solution, getting asymptotically closer to the limit $w_\phi = -1$ at the present time.

Conformal model II

As previously discussed, cosmological scenarios that admit scalar field tracking solutions are interesting. In the uncoupled case, the equation of state of the scalar field follows the behavior of the dominant background fluid according to eq. (6.3). Hence, during the matter-dominated epoch, for small deviation from standard uncoupled quintessence, the equation of state of the background fluid $w_m^{\text{eff}} \simeq w_c^{\text{eff}}$ would still be constant, very close to zero. Then, it is expected that eq. (6.3) still holds, with $w_\phi^{\text{eff}} \simeq w_\phi$ also being approximately constant. With that in mind, by imposing a few conditions on the evolution of the scalar field, we can derive an interaction term that preserves the tracker behavior.

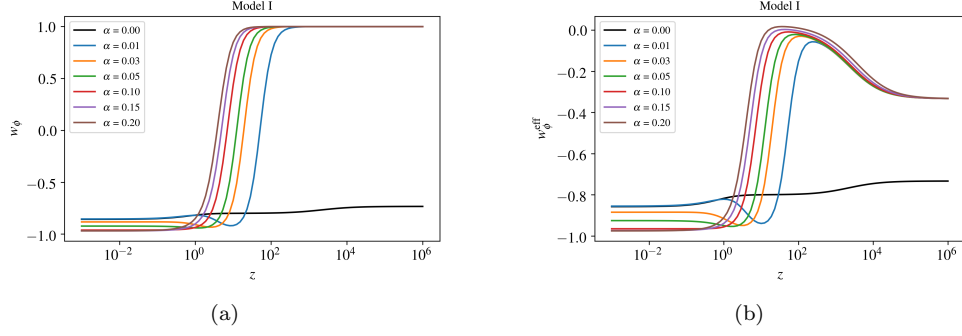


Figure 6.2: Left panel (a): evolution of the scalar field equation of state, $w_\phi = p_\phi/\rho_\phi$. Right panel (b): evolution of the effective equation of state, eq. (6.10). Both plots show the respective quantities as function of the redshift for the conformal model I, eq. (6.4), for different values of the coupling constant.

Suppose that, during the matter domination epoch, the scalar field not only admits a tracking solution, with both terms on the right-hand side of eq. (6.10) being constant, but also the ratio ρ_c/ρ_ϕ^p , for some integer $p > 1$, is constant. The exponent p guarantees that the energy density of the scalar field decreases slower than the CDM energy density, eventually becoming dominant. In such case, we have $\phi \propto \dot{\phi}^{1-p}$. Since constant w_ϕ implies that $V \propto \dot{\phi}^2$, the scalar potential must be $V \propto \phi^{2/(1-p)}$. Our parameterization corresponds to $p = 5$, namely $\rho_c \propto \rho_\phi^5$. Finally, if for simplicity we assume that the conformal function $A(\phi)$ depends only on ϕ and that the disformal one is $B = 0$, the requirement that the second term on the right-hand side of eq. (6.10) is constant, leads to the following expression for the background coupling:

$$Q_0 = -\alpha\rho_c\phi, \quad \Gamma = -\frac{\alpha}{2}\phi^2. \quad (6.15)$$

Then, from eq. (5.38), the CDM energy density evolves as,

$$\rho_c \propto a^{-3} \exp\left(-\frac{\alpha}{2}\phi^2\right), \quad (6.16)$$

and the equation of motion for the scalar field, eq. (6.11), becomes

$$\ddot{\phi} + 3H\dot{\phi} + V_\phi = \alpha\rho_c\phi. \quad (6.17)$$

Similarly to model I, for positive values of the coupling constant, ρ_c dilutes faster than in the uncoupled case, and there is an injection of energy from CDM to the scalar field. However, note that the conformal factor (6.5) has an additional ϕ dependence on the exponential term, when compared to model I. Typically, during the cosmic evolution, the dynamics are such that $0 < \phi(t) < 1$, in reduced Planck mass units. Hence, in comparison to model I, the interaction in

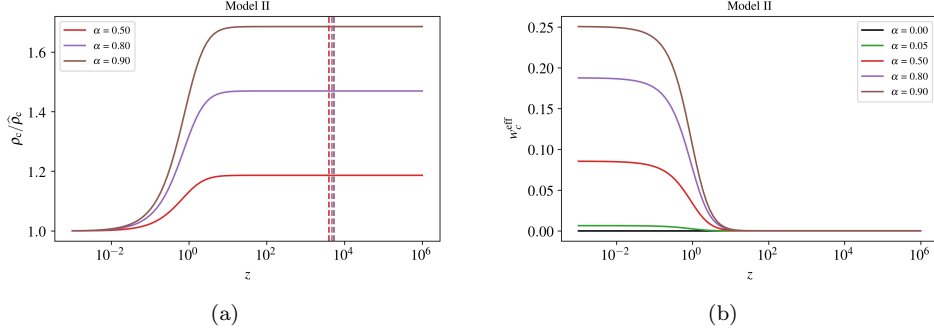


Figure 6.3: Left panel (a): evolution of the ratio between the coupled and uncoupled CDM energy density, ρ_c and $\hat{\rho}_c$, respectively. The dashed lines correspond to the redshift at matter-radiation equality. Right panel (b): evolution of the effective equation of state of dark matter, eq. (6.9). Both plots show the respective quantities as function of the redshift for the conformal model II, eq. (6.5), for different values of the coupling constant.

the dark sector for model II becomes relevant at later times, around the transition from matter to DE domination epochs, when the scalar field becomes large enough to flatten the potential (figure 6.3a). This stems from the fact that, by construction, model II effectively preserves the tracker properties up to the transition from matter to scalar field domination epochs. To get the same energy density at earlier times as in model I, the corresponding coupling constant must be significantly larger, and the ratio $\rho_c/\hat{\rho}_c$ increases steeply (as function of redshift) at late times. The effective equation of state for CDM remains close to zero during most part of the cosmic expansion (figure 6.3b) such that the evolution ρ_c does not deviate much from the uncoupled behavior, except at the low- z region.

In this model the evolution of the coupled scalar field indeed exhibits a tracker behavior up to matter domination epoch, when the scalar field equation of state is roughly constant, figure 6.4a, but the coupling slightly changes the value of w_ϕ . However, this change is compensated by the Q_0 term in eq. (6.10) such that the effective equation of state in the coupled case does not change with the interaction, remaining close to the uncoupled one, figure 6.4b. In contrast to model I, for larger values of the coupling constant, the kinetic energy of the scalar field becomes more important when compared to the scalar potential, and w_ϕ is less negative at late times.

Disformal model III

As an example of purely disformal coupling, we consider model III, characterized by the coupling functions (6.5). In this case, there is no interaction at the background level, $Q_0 = 0$. Hence, the energy density of cold dark matter evolves the usual way, $\rho_c \propto a^{-3}$, and the equation of motion for the scalar field is

$$\ddot{\phi} + 3H\dot{\phi} + V_\phi = 0. \quad (6.18)$$

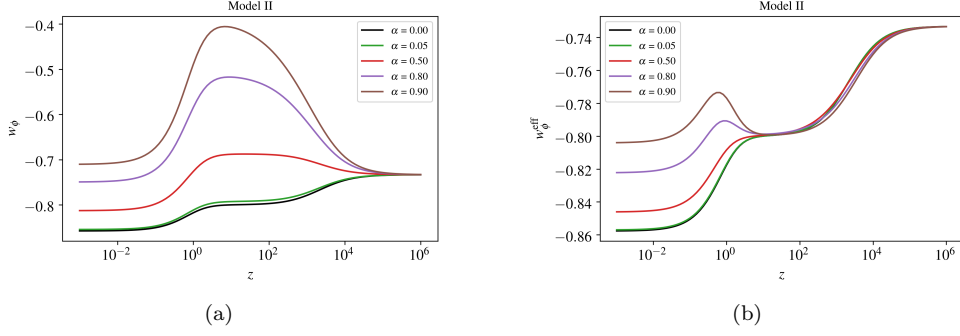


Figure 6.4: Left panel (a): evolution of the scalar field equation of state, $w_\phi = p_\phi/\rho_\phi$. Right panel (b): evolution of the effective equation of state, eq. (6.10). Both plots show the respective quantities as function of the redshift for the conformal model II, eq. (6.5), for different values of the coupling constant.

From the observational point of view, the lack of modifications at background level with respect to the standard quintessence scenario is interesting, given the current constraints on the properties of the homogeneous universe. The effects induced by the DE/CDM interaction are limited to change the evolution of linear perturbations only.

6.1.2 Perturbations

In this subsection, we present the evolution of linear perturbations and study each model in the quasi-static limit. Even though all the numerical results presented in this work were computed using the full set of k -dependent linear perturbation equations shown in section 5.2, it is elucidating to study the evolution in the quasi-static limit, since in said limit, one can write the equations of motion for the linear density in a simple way, allowing a straightforward discussion of the effects due to the coupling. Furthermore, as previously discussed, for cosmological scales probed by galaxy surveys, said approximation is valid and there is no loss of physical information. In what follow, we present the evolution of perturbations for modes with $k = 1h \text{ Mpc}^{-1}$. We numerically checked that these modes are sufficiently deep inside the sound horizon of scalar perturbation (see appendix B) and that the quasi-static approximation is indeed valid.

Conformal model I

In model I, since the coupling is purely conformal, both Υ_1 and Υ_2 , given by eq. (5.74), vanish and the equation of motion for the CDM growth function, eq. (5.70), becomes

$$\ddot{D}_c + 2H(1 - \mathcal{E}_1)\dot{D}_c - \frac{1}{2}[(1 - \mathcal{E}_2)\rho_c D_c + \rho_b D_b] = 0, \quad (6.19)$$

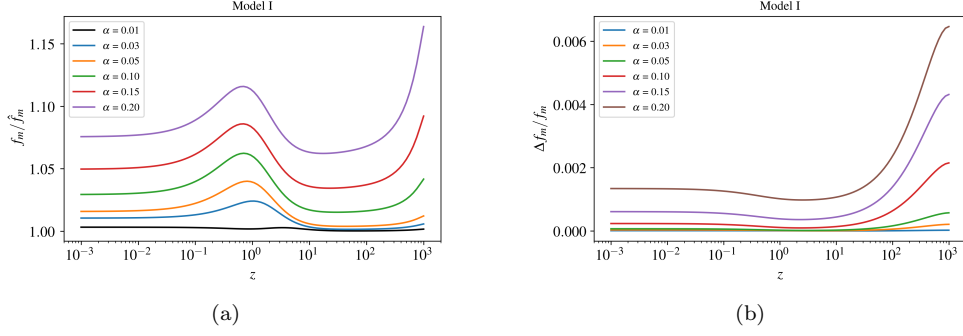


Figure 6.5: Right panel (a): ratio $f_m / \hat{f}_m$ where $\hat{f}_m$ is the growth rate in the uncoupled case and f_m is the one with non-zero coupling. Left panel (b): difference between the effective and actual growth rates $\Delta f_m / f_m$, eq. (5.83). Both figures are shown as function of the redshift for the conformal model I, eq. (6.4), for different values of the coupling constant.

with the the following contributions to the Hubble friction and gravitational constant terms, respectively:

$$\mathcal{E}_1 = \frac{\alpha}{2} \frac{\dot{\phi}}{H}, \quad \mathcal{E}_2 = -2\alpha^2. \quad (6.20)$$

First, by inspecting eqs. (6.10) and (6.15), note that $\mathcal{E}_1 = 3w_c^{\text{eff}}/2$, being proportional to the square root of the fractional kinetic energy of the scalar field such that, for small values of the coupling constant α , the usual Hubble friction term is recovered at early times. On the other hand, since $\mathcal{E}_2$ gives a constant contribution, the conformal coupling (6.4) changes the gravitational constant term by a factor $\sim \mathcal{O}(\alpha^2)$ at all epochs. For $\alpha > 0$, the interaction suppresses the friction term and enhances the gravitational pull, leading to more growth (figure 6.5a). This is a general feature of models with non-minimal couplings, where the interaction between matter particles is mediated by a scalar field, leading to an attractive fifth-force. As it will be shown, this behavior is also present in models II and III.

Besides, since Υ_2 vanishes, according to eq. (5.82) the effective and actual growth rates for CDM coincide. Even though $\Delta f_c = 0$, the effective linear growth rate for total matter, eq. (5.83), will differ from the actual growth rate, by a factor $\sim \mathcal{O}(\alpha^2)$, due to the combined effect of the background coupling Q_0 and the difference between the baryon and CDM growth functions (figure 6.5b). Hence, the non-minimal coupling reinforces the redshift-distortion factor.

Let us now turn to the matter power spectrum (figure 6.6). On small scales, i.e. in the quasi-static limit, since the scalar interaction induces more growth on the CDM linear density, the power spectrum on those scales will be equally augmented. Also, since larger α makes the matter-radiation equality to take place earlier, when the Hubble horizon was smaller, the wavelength of modes that entered the horizon at the equality will be smaller. Hence, the whole power spectrum will be shifted towards larger values of k . Another effect related to the background dynamics is that, since the ratio ρ_b / ρ_m at early times decreases for larger values of the coupling constant, the

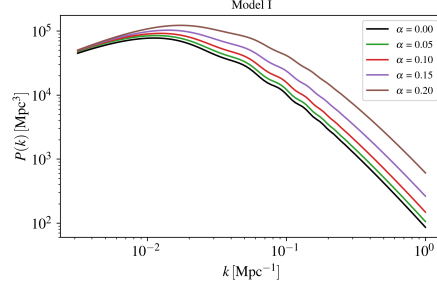


Figure 6.6: Linear matter power spectrum for conformal model I, eq. (6.4), for different values of the coupling constant.

amplitude of the baryon acoustic oscillations also decreases.

Conformal model II

In model II, once again, both Υ_1 and Υ_2 vanish and the modifications to the evolution of dark matter growth function appear through

$$\ddot{D}_c + 2H(1 - \mathcal{E}_1)\dot{D}_c - \frac{1}{2}[(1 - \mathcal{E}_2)\rho_c D_c + \rho_b D_b] = 0, \quad (6.21)$$

where

$$\mathcal{E}_1 = \frac{\alpha \phi \dot{\phi}}{2H}, \quad \mathcal{E}_2 = -2\alpha^2 \phi^2. \quad (6.22)$$

As in conformal model I, the modification to the Hubble friction term is related to the effective CDM equation of state by $\mathcal{E}_1 = 3w_c^{\text{eff}}/2$. Once again, the correction to the Hubble friction and gravitational constant terms are $\mathcal{O}(\alpha)$ and $\mathcal{O}(\alpha^2)$, respectively, and the scalar force leads to more clustering. Similarly to the behavior at background level, since both $\mathcal{E}_1$ and $\mathcal{E}_2$ are ϕ -suppressed, the growth function during matter domination epoch does not differ significantly from the uncoupled one during matter epoch, and all the coupling-induced growth happens at late times (figure 6.7b). Hence, in order to get a deviation from the uncoupled growth function comparable to the one in model I, the value of the coupling constant must be considerably larger (figure 6.8a).

As in model I, Δf_c is zero and the deviation between effective and actual matter growth rate, also by a factor $\sim \mathcal{O}(\alpha^2)$, is due to the background coupling and the difference between the baryon and CDM growth function (figure 6.8b). However, since in model II the quantity $\mathcal{E}_2$ is negligible at early times, the difference Δf_m is present only at later epochs. Since Δf_m is positive, on top of the enhancement of the actual growth rate due to the coupling, the effective growth will be further enhanced.

The behavior of the matter power spectrum of linear perturbations in model II (figure 6.9) resembles that of model I. For larger values of the coupling constant, because of the clustering induced by the coupling, it is enhanced at small scales, whereas the change in the time of matter-

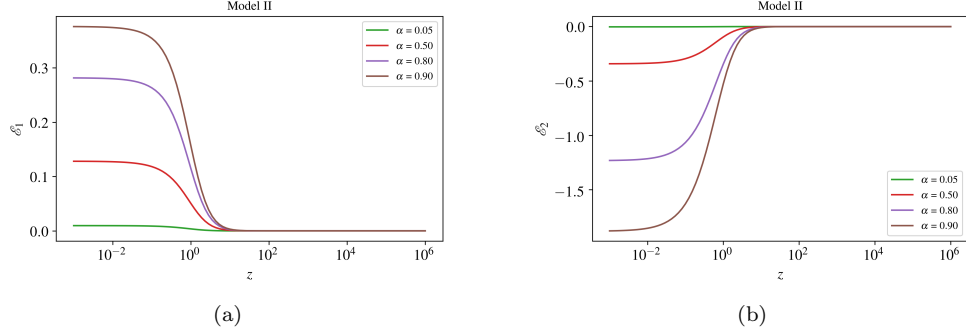


Figure 6.7: Evolution of the additional Hubble friction $\mathcal{E}_1$ (left panel (a)) and gravitational constant $\mathcal{E}_2$ (right panel (b)), eq. (6.22), as function of redshift for the conformal model II, eq. (6.5), for different values of the coupling constant.

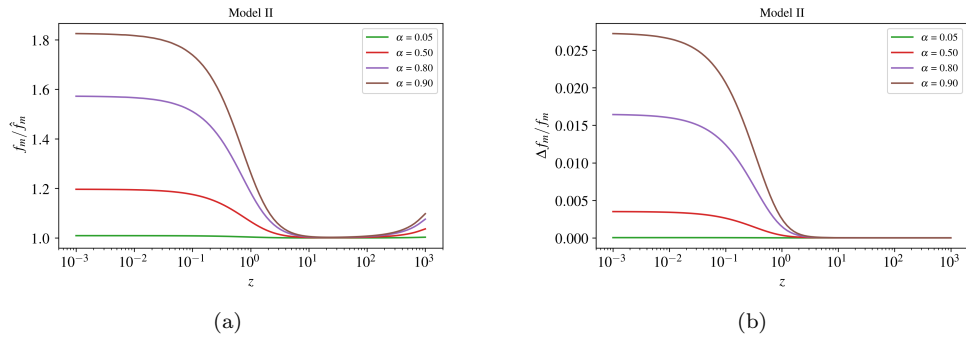


Figure 6.8: Left panel (a): ratio $f_m/\hat{f}_m$ where $\hat{f}_m$ is the growth rate in the uncoupled, case and f_m is the one with non-zero coupling. Right panel (b): difference between the effective and actual growth rates $\Delta f_m/\hat{f}_m$, eq. (5.83). Both figures are shown as function of the redshift for the conformal model II, eq. (6.5), for different values of the coupling constant.

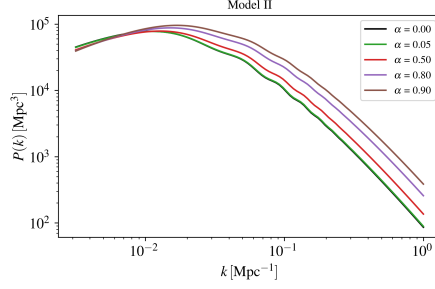


Figure 6.9: Linear matter power spectrum for conformal model II, eq. (6.5), for different values of the coupling constant.

radiation equality shifts the whole spectrum towards larger values of k . The amplitude of the baryonic feature also decreases. However, since the interaction is effective only during the scalar-field-dominated epoch, to get a modification of the same effect as in model I, the coupling constant must be roughly one order of magnitude larger.

Disformal model III

For model III, with conformal and disformal factors defined in eq. (6.6), one finds $\Upsilon_1 = -\Upsilon_2 = \mathcal{E}_2$. Hence, the equation of motion for the CDM growth function, eq. (5.70), becomes

$$\ddot{D}_c + 2H(1 - \mathcal{E}_1)\dot{D}_c - \frac{1}{2}(1 - \mathcal{E}_2)(\rho_c D_c + \rho_b D_b) = 0, \quad (6.23)$$

where,

$$\mathcal{E}_1 = -\frac{2\alpha^2 \rho_c \dot{\phi} (3H\dot{\phi} + 2V_\phi)}{H(\dot{\phi}^2 - 2\alpha\rho_c) \left[(1 - 2\alpha)\dot{\phi}^2 - 2\alpha\rho_c \right]}, \quad \mathcal{E}_2 = \frac{2\alpha\dot{\phi}^2}{\dot{\phi}^2 - 2\alpha\rho_c}. \quad (6.24)$$

Note that, in principle, when $\dot{\phi}^2 = 2\alpha\rho_c$ both $\mathcal{E}_1$ and $\mathcal{E}_2$ functions diverge, whereas when the evolution is such that $(1 - 2\alpha)\dot{\phi}^2 - 2\alpha\rho_c = 0$ only $\mathcal{E}_1$ has problems. The requirement that the disformally transformed metric $\tilde{g}_{\mu\nu}$ preserves the Lorentzian signature for this choice of the functions A and B translates into $2\alpha < 1$, which by itself does not guarantee that the functions $\mathcal{E}_1$ and $\mathcal{E}_2$ are well-defined. However, the disformal coupling modifies the sound speed of the scalar field, not being equal to unity as it is the case for standard quintessence. The requirement that $c_s^2 > 0$ leads to the constraint $\alpha < \dot{\phi}^2/(2\rho_c)$, the ratio between the scalar field kinetic energy and the CDM energy density (see appendix B). Since this relation must be satisfied throughout the cosmic evolution, we consider only negative values for the coupling constant. Similarly to model II, at earlier times the functions $\mathcal{E}_1$ and $\mathcal{E}_2$ are negligibly small, so the growth rate is insensitive to the interaction, and most of the effects due to the coupling become important during onset of the DE scalar field domination epoch (figure 6.10). That is a consequence of the fact that both functions depend not only on the coupling constant, but also on the ratio between the kinetic energy of the

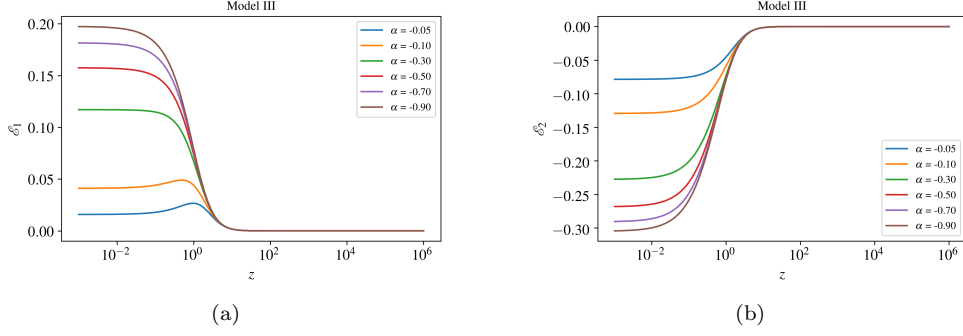


Figure 6.10: Evolution of the additional Hubble friction $\mathcal{E}_1$ (left panel (a)) and gravitational constant $\mathcal{E}_2$ (right panel (b)), eq. (6.24), as function of redshift for the disformal model III, eq. (6.6), for different values of the coupling constant.

scalar field, $\dot{\phi}^2/2$, and the CDM energy density. During matter domination epoch, this ratio is negligibly small, such that both $\mathcal{E}_1$ and $\mathcal{E}_2$ remain close to zero, regardless of the coupling constant value. However, as the ratio gets closer to unity at late times, the coupling becomes significant, with $\mathcal{E}_1$ and $\mathcal{E}_2$ being roughly proportional to α^2 and α , respectively.

Naturally, this choice of disformal factor also leads to more growth, since the Hubble friction is suppressed, while the gravitational constant term is enhanced (figure 6.11a). However, in contrast with the conformal models I and II, increasing the value of $|\alpha|$ indefinitely does not lead to more growth. Indeed, taking the limit $\alpha \rightarrow -\infty$, one finds

$$\mathcal{E}_1 \rightarrow -\frac{\dot{\phi}(3H\dot{\phi} + 2V_\phi)}{2H(\dot{\phi}^2 + \rho_c)}, \quad \mathcal{E}_2 \rightarrow -\frac{\dot{\phi}^2}{\rho_c}, \quad (6.25)$$

which means that the modifications due to the coupling saturate.

The lack of modification at the background level implies that the deviation of f_m^{eff} from f_m is due only to the term proportional to Δf_c in eq. (5.83). At late times, when $\dot{\phi}^2/(2\rho_c) \sim 1$, it becomes $\Delta f_c \approx 2\alpha f_c/(1-3\alpha)$. In contrast to the conformally coupled models, Δf_m for model III is negative (figure 6.11b), being roughly proportional to α for small values of the coupling constant. Therefore, the enhancement of the actual growth rate due to the coupling will be suppressed. However, as in the case of the growth function, $|\Delta f_m|$ does not increase indefinitely when one consider larger and larger values of $|\alpha|$.

Once again, because of the lack of modifications at the background level, the amplitude of the power spectrum simply increases, due to the coupling effect on the growth function, as shown in figure 6.12.

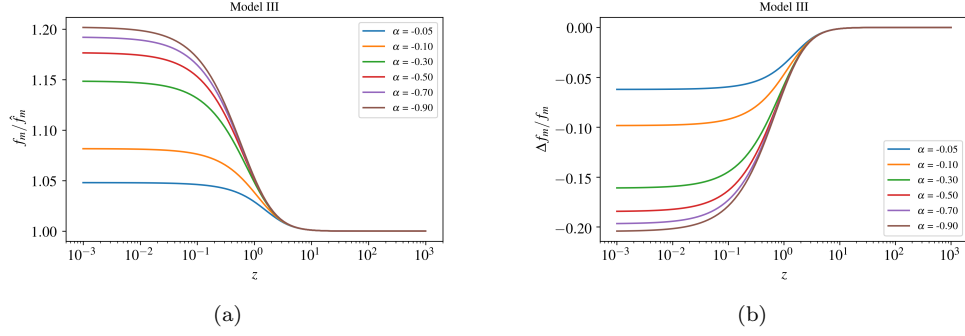


Figure 6.11: Left panel (a): ratio $f_m / \hat{f}_m$ where $\hat{f}_m$ is the growth rate in the uncoupled case and f_m is the one with non-zero coupling. Right panel (b): difference between the effective and actual growth rates $\Delta f_m / f_m$, eq. (5.83). Both figures are shown as function of the redshift for the disformal model III, eq. (6.6), for different values of the coupling constant.

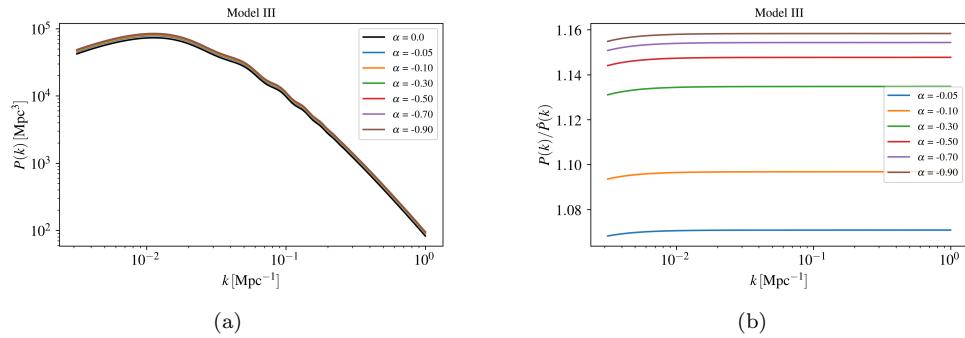


Figure 6.12: Left panel (a): linear matter power spectrum. Right panel (b): ratio $P(k) / \hat{P}(k)$ where $\hat{P}(k)$ is the power spectrum in the uncoupled case, and $P(k)$ is the one with non-zero coupling. Both spectra are computed at the redshift $z = 0$ for the disformal model III, eq. (6.6), for different values of the coupling constant.

6.2 Forecasts

In this section, we introduce the approach followed to obtain the constraining power of future galaxy surveys on cosmological parameters. Further details on the Fisher matrix analysis and the observed power spectrum are discussed in appendix C.

6.2.1 Fisher matrix analysis

In order to forecast how well the future galaxy surveys will be able to constraint the DE/CDM coupling, we employ the Fisher matrix analysis to the galaxy power spectrum sliced over several redshift bins. The technique consists of approximating the full likelihood for the cosmological parameters as a multivariate Gaussian distribution. After Taylor expanding the likelihood function $\mathcal{L}$, the Fisher matrix is defined by

$$F_{\alpha\beta} = - \left\langle \frac{\partial^2 \ln \mathcal{L}(\boldsymbol{\theta})}{\partial \theta^\alpha \partial \theta^\beta} \right\rangle \bigg|_{\hat{\boldsymbol{\theta}}}, \quad (6.26)$$

where $\hat{\boldsymbol{\theta}}$ is the vector containing the maximum likelihood estimators. The Fisher matrix estimates the best possible precision obtainable by a given experiment. Under the assumption that the parameters follow a Gaussian distribution, the Fisher matrix is the inverse of the covariance matrix. Generally, that is the case near the peak of the likelihood, and the Fisher matrix provides a good approximation to the uncertainties. The marginalized constraint on a given parameter θ^α is simply $(F^{-1})_{\alpha\alpha}^{1/2}$, while the unmarginalized one, when all the other parameters are fixed, is given by $(F_{\alpha\alpha})^{-1/2}$. As long as the parameters are not extremely degenerated, the Fisher matrix method is an efficient tool in estimating the best possible constraints.

When applied to galaxy power spectrum observations, sliced over several redshift bins, the Fisher matrix takes the form [183, 184],

$$F_{\alpha\beta} = \sum_{z_i} \int_{k_{\min}}^{k_{\max}} \frac{d^3 \mathbf{k}}{(2\pi)^3} V_{\text{eff}}(\mathbf{k}, z_i) \frac{\partial \ln P_{\text{obs}}(\mathbf{k}, z_i)}{\partial \theta^\alpha} \frac{\partial \ln P_{\text{obs}}(\mathbf{k}, z_i)}{\partial \theta^\beta}, \quad (6.27)$$

where the effective volume V_{eff} is defined as

$$V_{\text{eff}}(\mathbf{k}, z_i) = \left(\frac{n_g(z_i) P_{\text{obs}}(\mathbf{k}, z_i)}{n_g(z_i) P_{\text{obs}}(\mathbf{k}, z_i) + 1} \right)^2 V_{\text{survey}}(z_i), \quad (6.28)$$

which comes from the fact that, since galaxies follow a Poissonian distribution, the observed power spectrum has a shot-noise correction $P_{\text{shot}} = n_g^{-1}$, with $n_g(z_i)$ being the number density of galaxies in a redshift bin. When the sampling is good enough, $n_g P_{\text{obs}} \gg 1$ and both effective and actual survey volumes coincide. Note that the total Fisher matrix is given by the summation over all

redshift bins of the integrals over $\mathbf{k}$.

To compute the observed galaxy power spectrum, in addition to the modified Kaiser effect discussed in section (5.3.3), a few other effects must be taken into account, and the general expression reads [184]

$$P_{\text{obs}}(\mathbf{k}, z) = \mathcal{N}_{\text{AP}}(z) [b_g(z) + f_m^{\text{eff}}(z)\mu^2]^2 P_m(k, z) e^{-k^2 \mu^2 \sigma_{\text{NL}}^2}. \quad (6.29)$$

The Alcock-Packzynski factor [185], defined as

$$\mathcal{N}_{\text{AP}}(z) = \left(\frac{d_A^{\text{ref}}(z)}{d_A(z)} \right)^2 \frac{H(z)}{H^{\text{ref}}(z)}, \quad (6.30)$$

where d_A is the angular diameter distance and H is the Hubble function, is a correction due to the fact that when converting angles and redshifts into distances and wavenumbers one needs to, incorrectly, assume a reference cosmology. On the other hand, the exponential factor corresponds to errors in the observed redshift due to a line-of-sight smearing of the structure [186], which becomes relevant on small scales, and is characterized by the nuisance parameter σ_{NL} . Finally, we can compute the derivative of the observed galaxy power spectrum with respect to the cosmological parameters, resulting in [186]

$$\begin{aligned} \frac{d \ln P_{\text{obs}}}{d\theta^\alpha} &= \frac{\partial \ln P_m}{\partial \theta^\alpha} + \frac{2}{b_g + f_m^{\text{eff}} \mu^2} \left(\frac{\partial b_g}{\partial \theta^\alpha} + \mu^2 \frac{\partial f_m^{\text{eff}}}{\partial \theta^\alpha} \right) - k^2 \mu^2 \frac{\partial \sigma_{\text{NL}}^2}{\partial \theta^\alpha} \\ &+ \left[1 + \frac{4 f_m^{\text{eff}} \mu^2}{b_g + f_m^{\text{eff}} \mu^2} (1 - \mu^2) + \frac{\partial \ln P_m}{\partial \ln k} \mu^2 - 2 k^2 \mu^2 \sigma_{\text{NL}}^2 \right] \frac{d \ln H}{d\theta^\alpha} \\ &+ \left[-2 + \frac{4 f_m^{\text{eff}} \mu^2}{b_g + f_m^{\text{eff}} \mu^2} (1 - \mu^2) - \frac{\partial \ln P_m}{\partial \ln k} \right] \frac{d \ln D_A}{d\theta^\alpha}. \end{aligned} \quad (6.31)$$

In this work, we perform the forecast for the set of cosmological parameters

$$\boldsymbol{\theta} = \{\alpha, h, \Omega_b h^2, \Omega_\phi, 10^9 A_s, n_s\} + \{\sigma_{\text{NL}}, b_g(z_i)\}, \quad (6.32)$$

where α is the coupling constant for each model considered here, h is the dimensionless Hubble parameter, defined as $H_0 = 100h \text{ km/s/Mpc}$, Ω_b and Ω_ϕ are, respectively, the baryon and scalar field current density parameters, A_s is the amplitude of the primordial power spectrum at the scale $k = 0.05 \text{ Mpc}^{-1}$, and n_s is the spectral index of primordial perturbations. The error in the observed redshift, σ_{NL} , and the galaxy bias at each redshift bin, $b_g(z_i)$, are taken to be nuisance parameters and are marginalized over. Further, we consider a flat geometry.

For each model, the fiducial values for the coupling constant is chosen to be

- Conformal model I: $\alpha = 0.05$.
- Conformal model II: $\alpha = 0.1$.

Parameter	h	$\Omega_b h^2$	Ω_ϕ	$10^9 A_s$	n_s
Value	0.674	0.224	0.685	2.10	0.965

Table 6.1: Fiducial cosmological parameters consistent with Planck 2018 results for Λ CDM [23].

- Disformal model III: $\alpha = -0.05$.

This choice for model I is consistent with current constraints on coupled quintessence [187]. Even though we are interested in cosmological scenarios that differ from the standard one, we set the fiducial cosmology using current constraints on Λ CDM [23]. These values are shown in table 6.1. We choose the value $\sigma_{\text{NL}} = 7 \text{ Mpc}$, corresponding to a peculiar velocity dispersion of roughly 400 km/s [188, 189]. Moreover, we assume that the slope of the scalar field potential n is fixed, and set its value to $n = 0.5$, which is consistent with current constraints on quintessence [187].

Our goal is estimating the precision with which future galaxy redshift surveys will be able to measure the set of cosmological parameters (6.32), especially the coupling constant α . In this work, we consider experiments with Euclid-like [100] and SKA-like (Phase 2) [190] configurations. For the Euclid-like survey, we consider the range $0.7 < z < 2$, with a sky coverage of 15,000 squared degrees, and bias function $b_g = \sqrt{1+z}$, while for SKA2 we take $0.1 < z < 2$, a sky coverage of 30,000 squared degrees, and bias function $b_g = c_1 \exp(c_2 z)$, with constants c_1 and c_2 given in [190]. We adopt the predicted number density of galaxies as a function of redshift given in [100] and [190]. We divide the survey volume into redshift bins, with $\Delta z = 0.1$, and consider the expect galaxy number density $n_g(z_i)$ and galaxy bias $b_g(z_i)$ at each redshift slice z_i .

At each redshift bin, the limits of integration in eq. (6.27) are taken to be

$$k_{\min}(z_i) = \left(\frac{2\pi}{V_{\text{survey}}(z_i)} \right)^{1/3}, \quad k_{\max}(z_i) = 0.2h(1+z_i)^{2/(2+n_s)}, \quad (6.33)$$

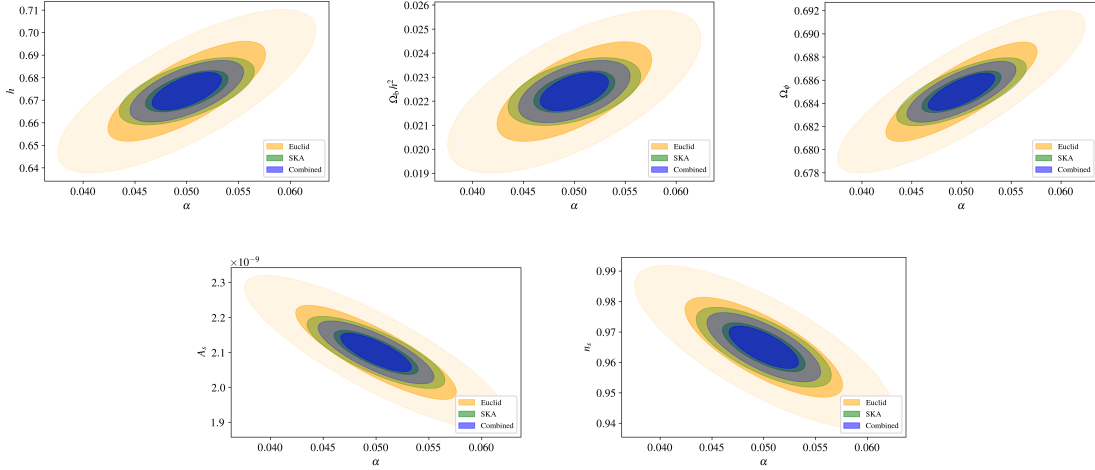
for both surveys. The maximum wavenumber $k_{\max}$ corresponds to a conservative cut-off of non-linear scales [191, 192]. The derivative with respect to a given cosmological parameter θ^α in the Fisher matrix are performed numerically, with a symmetric difference quotient, evaluating the observed power spectrum at $\theta^\alpha(1 \pm 10^{-2})$.

6.2.2 Results

On table 6.2 we present the marginalized 1σ C.L. errors on the fiducial parameters for all three models, while the two-dimensional contour plots for models I, II and III are presented in figures 6.13, 6.14 and 6.15, respectively. Note that, even though we did not include any prior to the value of α for the disformal model III, it must be kept in mind that the condition $\alpha < \dot{\phi}^2/(2\rho_c)$, discussed in section 6.1.2, must be satisfied. For each fiducial model, the forecasts for SKA and

Model	Survey	$10^2 \times \sigma(h)$	$10^3 \times \sigma(\Omega_b h^2)$	$10^3 \times \sigma(\Omega_\phi)$	$10^2 \times \sigma(\alpha)$	$10^2 \times \sigma(10^9 A_s)$	$10^3 \times \sigma(n_s)$
I	Euclid	1.46	1.36	2.38	0.50	8.86	1.08
	SKA2	0.60	0.57	1.18	0.26	4.14	0.53
	Combined	0.55	0.52	1.06	0.22	3.61	0.46
II	Euclid	0.84	0.88	9.30	10.46	4.79	0.65
	SKA2	0.53	0.57	5.35	5.62	2.88	0.45
	Combined	0.44	0.47	4.43	4.69	2.40	0.36
III	Euclid	0.78	0.85	2.41	24.93	3.83	0.72
	SKA2	0.47	0.50	1.24	6.99	2.59	0.41
	Combined	0.40	0.42	1.07	6.23	2.13	0.35

Table 6.2: Results for the Fisher analysis.

Figure 6.13: Confidence regions, 1σ and 2σ , for selected parameters vs α for the conformal model I. We present the independent SKA and Euclid results, and also the combined one. The contours are obtained marginalizing over all the parameters except the ones being plotted.

Euclid are combined, marginalizing over the biases and summing the resulting Fisher matrices.

Note that the forecasted constraints from the SKA-like survey are more stringent than those from Euclid-like one. The main driver of this result is the fact that the redshift window of SKA2 is larger than Euclid's, the former being able to measure galaxies at lower redshifts, and we assumed the same spacing for the redshift bins. Hence, the predicted errors measured by SKA are expected to be smaller.

Generally speaking, the effects of the non-minimal coupling on the observed power spectrum occur on the redshift-distortion factor, growth function, and the power spectrum (baryonic wiggles position and shape). In the previous sections, it was shown that in model I the interaction changes the evolution of both background and linear perturbations at all epochs, while in model II the modification arises only at late times. On the other hand, in model III, the coupling does not affect the background evolution but changes the behavior of linear perturbations at late times.

Since, at the background level, the main effect of the interaction is the energy transfer from CDM to the DE scalar field, the coupling causes a degeneracy between α and Ω_ϕ . Despite the degeneracy, both surveys would be able to constraint α and Ω_ϕ for model I better than for models II

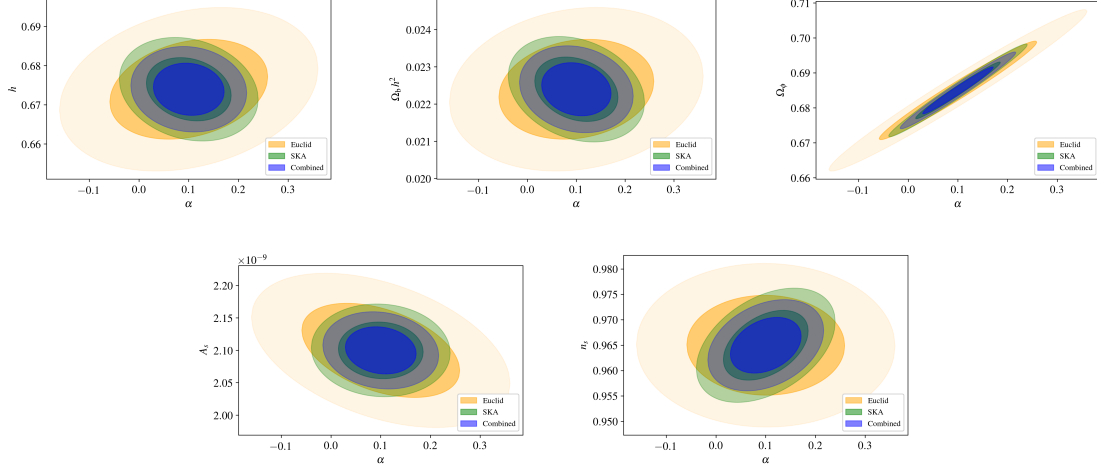


Figure 6.14: Confidence regions, 1σ and 2σ , for selected parameters vs α for the conformal model II. We present the independent SKA and Euclid results, and also the combined one. The contours are obtained marginalizing over all the parameters except the ones being plotted.

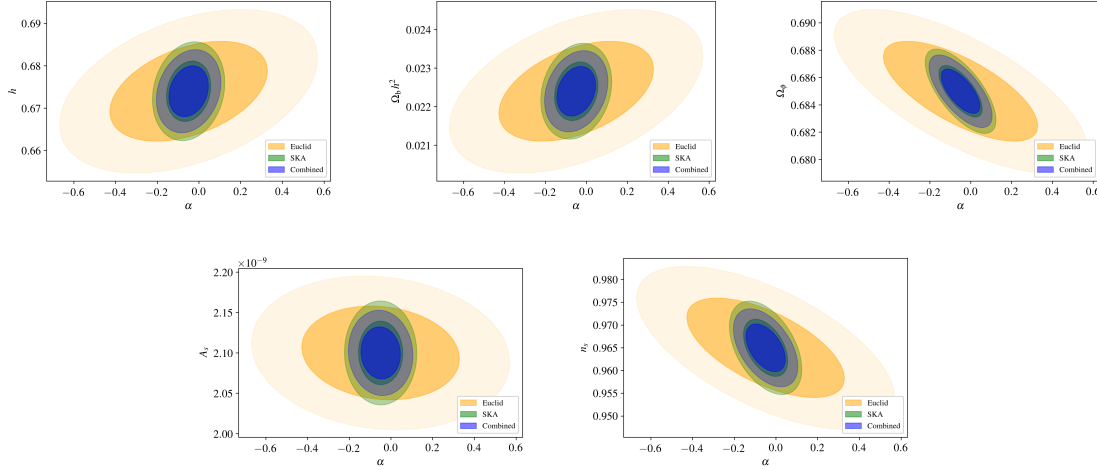


Figure 6.15: Confidence regions, 1σ and 2σ , for selected parameters vs α for the disformal model III. We present the independent SKA and Euclid results, and also the combined one. The contours are obtained marginalizing over all the parameters except the ones being plotted.

and III. The reason is that in model II, the coupling affects the evolution of background and perturbed quantities only at late times, effectively being less important than in model I. In model III the coupling only changes the growth function at late times, and also, since for a given α the negative Δf_m compensates the growth, the redshift-distortion factor β is not as sensitive to the non-minimal coupling as the conformal models. However, the precision with which all the other parameters could be measured by both surveys is higher for models II and III, since in model I the modification at higher redshift leads to indetermination on these parameters.

The constraints on all parameters except α and Ω_ϕ from SKA and Euclid are roughly of the same order of magnitude for all fiducial cosmologies considered here. In particular, for models II and III, the constraints from SKA and Euclid are of the same order of magnitude for all parameters except the coupling constant. The constraints from Euclid are much less stringent, since the modifications due to the coupling take place at low redshifts, outside its observational window. Note also that the error on Ω_ϕ for model III is comparable to the result for model I; even though the impact of the interaction is more important in the latter, the degeneracy between Ω_ϕ and α hinders the determination of Ω_ϕ .

The combined analysis helps in reducing the degeneracy and gives slightly better constraints. For all the fiducial cosmologies, the two-dimensional contour plots for (α, h) and $(\alpha, \Omega_b h^2)$ have roughly the same shape. For models II and III, since the modifications due to the non-minimal coupling are delayed in comparison to model I, these parameters show little to no correlation.

From the Fisher matrix analysis, we see that experiments with specifications similar to both surveys could determine if there is a non-minimal coupling in the dark sector for the conformal model I with our choice of fiducial cosmology. The coupling constant could be determined to the percentage level, significantly improving the current constraints [187]. For model II, the interaction could be probed to the 1σ level. On the other hand, for model III, the forecasted surveys do not have enough precision to determine the existence of the interaction, rendering these models observationally equivalent to the standard quintessence scenario. However, the combination of other cosmological probes, such as weak lensing, could help in breaking the degeneracy.

6.3 Summary

In this chapter, by considering three concrete cosmological models (two with purely conformal couplings and one with a disformal coupling), we studied the cosmological evolution of both background quantities and linear perturbations.

The well-known coupled quintessence model, here denoted by conformal model I, does not admit a tracker solution, and the presence of the coupling changes the evolution of matter linear

density even at early times due to the modification of the effective gravitational coupling. Then, we proposed an improved model, denoted by conformal model II, in which the tracker behavior is preserved even in the coupled case, and the effects of the coupling on the evolution of matter linear density become important only at late times.

For both conformal models I and II, the continuity equation for CDM follows the standard one, and thus the coupling effect in the effective linear growth rate is roughly the square of the coupling constant α .

We also proposed a purely disformal coupled model denoted by model III. Thanks to a particular choice of the form of the disformal factor in this model, the energy transfer at the background level does not occur, therefore the background cosmological evolution is identical to the one in quintessence model. Furthermore, an additional contribution due to the coupling in the effective gravitational constant is suppressed at early times, and the coupling effect on the effective linear growth rate, due to the modification of the continuity equation for CDM, is roughly of the same order as the coupling constant α .

In all three models explored, the DE-DM coupling leads to more clustering, because of the attractive nature of the scalar fifth-force. Considering the conformal models I and II, the difference between the effective and actual linear growth rates are positive, such that RSD measurements would indicate a higher growth rate than the true one. On the other hand, for the disformal model III, the effective linear growth rate is smaller than the actual one, suppressing the redshift-distortion effect. Hence, RSD measurements have less constraining power regarding the parameters of model III when compared to models I and II.

As for the prospect of future probes measuring the DE-DM coupling, by employing the Fisher matrix analysis, we showed that assuming either of the conformal models, the absence of coupling could be probed within 1σ , with the coupling constant being measured with a percent-level precision. On the other hand, for the disformal model III, RSD measurements alone would not be able to determine the coupling constant with sufficient accuracy. It is expected that a combination of RSD, weak lensing, and CMB measurements would give better constraints on the coupling constant, and we hope to come back to this issue in the future.

7. Summary and Conclusions

Even after 30 years of its discovery, the current accelerated expansion of the universe remains one of the major unsolved problems in physics. Even though General Relativity, combined with the Λ CDM model, can explain most observations, it still faces theoretical challenges, motivating the search for an alternative theory of gravity.

In this context, scalar-tensor theories present itself as a suitable candidate to replace GR, since it is arguably the simplest extension of Einstein’s gravity. Although several particular models have already been ruled out, the general class of scalar-tensor theories remains relevant and fruitful in the study of gravitational physics. The next generation of astronomical and cosmological surveys will allow us to further investigate the space of possible theories and maybe find a robust replacement for General Relativity.

A useful tool in the study of scalar-tensor theories is the disformal transformation. A natural generalization of the conformal transformation, introducing derivative coupling, the disformal transformation allows the investigation of existing theories as well as the construction of new models.

In this work, we studied disformal transformations in two contexts. One was the relationship between cosmological observables defined in disformally related frames. The other was how a disformal coupling between dark matter and dark energy affects the growth of the structure observed in our Universe.

To study the invariance of cosmology under disformal transformations, we first defined the Jordan and Einstein frames. In the former, the scalar field couples non-minimally to gravity and minimally to matter whereas in the latter it couples minimally to gravity but universally to matter.

Due to the new force mediated by the scalar field, Newton’s “constant” is no longer constant. As shown in section 4.1, since the effective gravitational constant is a dimensionful quantity, it is not disformal invariant. Furthermore, the disformal transformation introduces post-Newtonian corrections, such that the Newtonian limit is the same as in the conformal case.

The redshift, being a dimensionless quantity, does not change (section 4.2), but the relationship between the redshift and scalar factor does. Thus, it is possible to have an accelerated expansion in one frame but not in the other. As a result, concepts like the redshift drift (section 4.3) and

the solution to the horizon problem (sections 4.5) are frame-dependent. Also, we have shown that cosmological distances are frame-dependent (sections 4.4 and 4.6), although the distance-duality relation is still robust, not only for FLRW cosmologies but also in general spacetimes (section 4.7). In summary, dimensionless quantities are invariant, whereas dimensionful ones are not. The change from one frame to the other is then interpreted as a change in the units between different spacetime points in the Einstein-frame, which depends on the local value of the scalar field.

Now, contrary to the conformal case, the photon distribution function, and consequently the Boltzmann equation, also changes under disformal transformations (section 4.8). Nevertheless, the shape of the black-body CMB spectrum is frame-independent (section 4.9).

Finally, we also investigated the concept of adiabaticity (section 4.10) and showed that, in general, it is frame-dependent. Nevertheless, we found that for purely disformal transformations, the adiabaticity condition can be satisfied if the matter fluid behaves as a Chaplygin gas.

Another issue tackled in this work is how the cosmological evolution changes when the dark matter fluid and the dark energy scalar field are disformally coupled to each other. In the Einstein frame action, instead of a universal coupling to all species, ϕ couples only to the dark matter particle. Our interest lies in how to discriminate between coupled and uncoupled cosmologies. With that goal in mind, we showed that since the interaction modifies the continuity equation for dark matter, eq. (5.70), the relationship between the galaxies' peculiar velocities and the growth of the dark matter field is modified. The peculiar velocity is especially relevant when comparing the observed galaxy power spectrum with the theoretical matter distribution, due to the Kaiser effect. Hence, the interaction between dark matter and the scalar field motivates the introduction of the modified Kaiser formula, eq. (5.86), taking into account not only the redshift-space distortion but also the non-minimal coupling.

Since the galaxy power spectrum gives information about the effective matter growth rate, redshift space distortions are no longer able to measure the actual matter growth. Nevertheless, with the analysis of data spanning multiple redshifts, or combining observations targeting different physical phenomena (such as weak lensing), it is possible to separate the growth of structure from the effects of the coupling.

To supplement the qualitative discussion, in chapter 6 we studied concrete cosmological examples, dubbed models I, II, and III. The conformal Model I corresponds to the familiar coupled quintessence scenario. The effective gravitational coupling is non-negligible even in the early stages of the cosmic evolution, modifying the formation of structure during the matter epoch. Due to the coupling at the background level, it does not admit tracker solutions (useful in solving the coincidence problem), which motivates the introduction of model II. We proposed the conformal model II in an attempt to preserve the tracker behavior of the DE scalar field. Independently

of initial conditions, ϕ “tracks” the evolution of dominant component in the universe, which can alleviate the coincidence problem. The last possibility we studied was the disformal model III. By design, there is no interaction at the background level such that the background evolution is the same as in the standard quintessence scenario (in particular, the tracker behavior is preserved).

In all three models, the interaction enhances the matter clustering, a consequence of the attractive nature of the scalar fifth-force, and the growth rate is larger than in uncoupled cosmologies. However, while in conformal models I and II the effect of the interaction term on the effective growth rate increases the redshift-space distortion, in model III the disformal coupling has the opposite effect. It counterbalances the growth, suppressing the distortion. For that reason, a large dataset, combining different probes, is necessary to constraint the disformal coupling.

Current RSD data is not so abundant, but this is set to change with the new generation of galaxy and radio surveys. To get some insights about the predicted constraining power of future experiments on interacting cosmologies, we performed a Fisher matrix analysis. This procedure allows us to forecast how well the coupling constant could be constrained. For our choices of fiducial cosmologies, we found out that Euclid and SKA could, in principle, rule out or confirm the interaction within 1σ in both conformal cases. Unfortunately, the predicted constraint on the disformal model III is not tight enough, signaling the necessity of a combined analysis.

An interesting question that has not been fully explored yet is how the disformal coupling affects observables other than the galaxy power spectrum. Recently, a few steps were taken in this direction, with some studies on the effects of disformal coupling on the CMB [120, 151, 159] and N-body simulations [193]. However, these previous analyses were limited to simple disformal transformations $A(\phi)$, $B(\phi)$. Hence the space of possible disformally coupled dark matter models is still largely unexplored.

A. Scale Dependence of the Mass Term in the QS Limit

In eq. (5.66), we have defined the quantity $\mathcal{A}(t, k)$ as

$$-\mathcal{A} \frac{k^2}{a^2} \delta\phi \equiv -\left(\mathcal{A}_1 \frac{k^2}{a^2} + m_\phi^2\right) \delta\phi = \delta Q, \quad (\text{A.1})$$

coming from the equation of motion for the scalar field in the quasi-static limit. In general, it depends on the wavenumber when the Compton wavelength of the scalar field is smaller than the Hubble horizon scale. For the canonical scalar field, eq. (6.1), we have $\mathcal{A}_1 = 1$ and $m_\phi^2 = V_{\phi\phi}$, such that the scale dependence in the quasi-static limit appears as

$$\mathcal{A}(t, k) = \mathcal{A}_1 + \frac{a^2}{k^2} m_\phi^2 = 1 + \frac{a^2 H^2}{k^2} \frac{V_{\phi\phi}}{H^2}. \quad (\text{A.2})$$

For sub-horizon scales, $k \gg aH$, as long as the ratio $V_{\phi\phi}/H^2$ is not too large, we can safely ignore the scale dependence.

In figure A.1 we present the evolution of the ratio between the scalar mass, m_ϕ , and the Hubble parameter. In the uncoupled case, as well as for the conformal model II, the mass term is roughly the same order as the Hubble parameter during most part of the cosmic evolution. On the other hand, for model I the mass term is much smaller than H at early times, becoming of the same

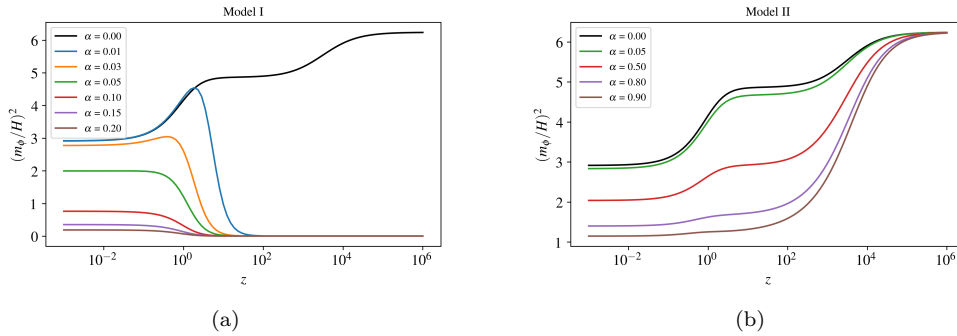


Figure A.1: Ratio between the m_ϕ^2 and H^2 in model I (left panel (a)) and model II (right panel (b)), as function of redshift for different values of the coupling constant.

other around the present epoch. Hence, In both cases, the mass term can be safely ignored when the quasi-static limit is considered.

B. Dispersion Relation

In this appendix, we investigate the dispersion relation of scalar perturbations. First of all, let us consider the conformal model I and II. In both cases, the coefficients (5.53) and (5.54) vanish, and therefore the gradient term in the perturbed equation for the scalar field (5.46) remains the same one as in quintessence case, implying the sound speed of the scalar perturbation is unity. Thus the sound horizon of scalar perturbation in both conformal models coincides with the cosmological horizon. On the other hand, the disformal model III has non-vanishing coefficients R_1 and R_2 , which modifies dispersion relation as we will see below.

The equation of motion for the perturbation of the canonical scalar field is

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} + \left(\frac{k^2}{a^2} + V_{\phi\phi}\right)\delta\phi - \dot{\phi}(\dot{\Phi} + 3\dot{\Psi}) + 2V_{\phi}\Phi = -2Q_0\Phi - \delta Q. \quad (\text{B.1})$$

In particular, for the disformal model III we have,

$$Q_0 = 0, \quad \delta Q = 2\alpha \frac{\rho_c}{\dot{\phi}^2} \frac{k^2}{a^2} (\dot{\phi}v_c - \delta\phi), \quad (\text{B.2})$$

and since there is no anisotropic stress, $\Phi = \Psi$. Hence, eq. (B.1) becomes

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} + \left[V_{\phi\phi} + \left(1 - 2\alpha \frac{\rho_c}{\dot{\phi}^2}\right) \frac{k^2}{a^2}\right]\delta\phi - 4\dot{\phi}\dot{\Phi} + 2V_{\phi}\Phi + 2\alpha \frac{\rho_c}{\dot{\phi}} \frac{k^2}{a^2} v_c = 0. \quad (\text{B.3})$$

To compute the dispersion relation we need to express v_c in terms of $\delta\phi$, Ψ , and its time derivatives. But first, note that for model III the Euler equations, (5.62) and (5.64), lead to $\dot{v}_b = \dot{v}_c$, which means that since there is no coupling at background level, the velocity potentials will evolve in the same way. They will differ at most by a constant, $v_b = v_c + \text{constant}$. In what follows, we set this constant to zero. With that in mind, we use the $(0, i)$ component of Einstein equation, eq. (5.40), to express the velocity as

$$(\rho_b + \rho_c)v_c = 2M_{\text{Pl}}^2(\dot{\Psi} + H\Psi) - \dot{\phi}\delta\phi, \quad (\text{B.4})$$

Plugging this result into eq. (B.3), we arrive at

$$\delta\ddot{\phi} + \beta_1\delta\dot{\phi} + \left(\beta_2 + \gamma_2\frac{k^2}{a^2}\right)\delta\phi + \left(\beta_3 + \gamma_3\frac{k^2}{a^2}\right)\dot{\Psi} + \left(\beta_4 + \gamma_4\frac{k^2}{a^2}\right)\Psi = 0, \quad (\text{B.5})$$

where,

$$\beta_1 = 3H, \quad \beta_2 = V_{\phi\phi}, \quad \beta_3 = -4\dot{\phi}, \quad \beta_4 = 2V_{\phi}, \quad (\text{B.6})$$

$$\gamma_2 = 1 - 2\alpha\left(\frac{\rho_c}{\dot{\phi}^2} + \frac{\rho_c}{\rho_b + \rho_c}\right), \quad \gamma_3 = 4\alpha\frac{M_{\text{Pl}}^2\rho_c}{\dot{\phi}(\rho_b + \rho_c)}, \quad \gamma_4 = 4\alpha\frac{M_{\text{Pl}}^2H\rho_c}{\dot{\phi}(\rho_b + \rho_c)}. \quad (\text{B.7})$$

To close the system, we use the (i, i) component of the Einstein equation, (5.41),

$$\ddot{\Psi} + \alpha_1\dot{\Psi} + \alpha_2\Psi + \alpha_3\delta\dot{\phi} + \alpha_4\delta\phi = 0, \quad (\text{B.8})$$

where the coefficients are given by

$$\alpha_1 = 4H, \quad \alpha_2 = 2\dot{H} + 3H^2 + \frac{\dot{\phi}^2}{2M_{\text{Pl}}^2}, \quad \alpha_3 = -\frac{\dot{\phi}}{2M_{\text{Pl}}^2}, \quad \alpha_4 = \frac{V_{\phi}}{2M_{\text{Pl}}^2}. \quad (\text{B.9})$$

Assuming the time dependence $\delta\phi = e^{i\omega t}\delta\phi_0$ and $\Psi = e^{i\omega t}\Psi_0$, the system formed by eqs. (B.5) and (B.8) admits solution if

$$\begin{vmatrix} -\omega^2 + i\alpha_1\omega + \alpha_2 & i\alpha_3\omega + \alpha_4 \\ i\left(\beta_3 + \gamma_3\frac{k^2}{a^2}\right)\omega + \beta_4 + \gamma_4\frac{k^2}{a^2} & -\omega^2 + i\beta_1\omega + \beta_2 + \gamma_2\frac{k^2}{a^2} \end{vmatrix} = 0. \quad (\text{B.10})$$

For sub-horizon scales, $k^{-1} \ll (aH)^{-1}$, the coefficients α_i , β_i and γ_i are roughly constant within one Hubble time, and the large- k limit of the above expression yields,

$$\omega^2 \simeq (\gamma_2 - \alpha_3\gamma_3)k^2. \quad (\text{B.11})$$

Hence, the sound velocity (group velocity) is given by

$$c_s^2 = \frac{\omega^2}{k^2} \simeq \gamma_2 - \alpha_3\gamma_3 = 1 - 2\alpha\frac{\rho_c}{\dot{\phi}^2}. \quad (\text{B.12})$$

As expected, the sound speed for the uncoupled canonical scalar field is equal to unity. However, that is changed by the disformal coupling between dark matter and dark energy. The fact that c_s^2 must be positive leads to an upper bound for the coupling constant,

$$\alpha < \frac{\dot{\phi}^2}{2\rho_c}. \quad (\text{B.13})$$

Furthermore, since during matter epoch $\Omega_c \gg \Omega_\phi$, we can approximate $\dot{\phi}^2/\rho_c \approx 0$, and a safer

bound against instabilities is $\alpha < 0$.

The quasi-static approximation is valid for modes inside the sound horizon of scalar perturbations. For uncoupled quintessence, since the sound speed is equal to unity, the quasi-static limit is valid for all modes well inside the Hubble horizon. In the general case, one can define a sound horizon scale $k_s^{-1} = c_s(aH)^{-1}$, such that the quasi-static approximation is valid for modes satisfying $k \gg k_s$. For instance, in model III we require

$$k \gg \frac{aH}{\sqrt{1 - 2\alpha \frac{\rho_c}{\phi^2}}} . \quad (\text{B.14})$$

C. Fisher Matrix for the Power Spectrum

In this appendix, we discuss in more details the observed power spectrum and how to use the Fisher matrix to forecast the constraining power of future galaxy surveys.

C.1 The power spectrum

The correlation function is a descriptive statistical tool used to condensate data into useful, compact information. The galaxy 2-point correlation function measures the deviation of the distribution of galaxies from a random distribution. In Fourier space, it becomes the power spectrum.

Since galaxies are point-like sources, its distribution is discrete. We are usually interested in studying continuous fields, like the matter distribution of the gravitational potential. In this sense, galaxies are tracers of the underlying field. Hence, to get information on the underlying field we need to study a discrete sample of galaxies, and in the simplest case, we assume that this sample is drawn from a Poisson distribution. We need a way to select galaxies, which is done with the introduction of a window function. In the end, the power spectrum will have two parts, one given by the true power spectrum and pure noise contribution, the so called shot-noise. The discussion presented here is largely based on [194].

C.1.1 Correlation function

Consider a survey volume V with N galaxies in it. The first descriptor of the galaxy distribution, the average number density $\bar{n} = N/V$ cannot describe any clustering of galaxies, so another statistic is necessary. Thus, consider a small volume dV taken at random from V , such that the average number of galaxies in it is $\bar{n} dV_i$.

The number of pairs separated by a distance $\mathbf{r}_{ab}$ is given by the product $N_a N_b$. If dN_{ab} is the average number of pairs, the 2-point correlation $\xi(\mathbf{r}_{ab})$ is then defined as

$$dN_{ab} = \langle N_a N_b \rangle = \bar{n}^2 dV_a dV_b [1 + \xi(\mathbf{r}_{ab})] \quad (\text{C.1})$$

If the particles are randomly distributed across the whole volume, then $N_i = \bar{n} dV$, and

the average number of pairs would simply be $\bar{n}^2 dV_a dV_b$. In this case, we say that distribution of galaxies is not correlated: $\xi(\mathbf{r}_{ab}) = 0$. On the other hand, if $\xi \neq 0$ it is said that the particles are correlated. Correlation can then understood as the deviation from a random distribution.

Defining the density contrast as

$$\delta(\mathbf{x}) = \frac{N(\mathbf{x})}{\bar{n} dV} - 1, \quad (\text{C.2})$$

the correlation function can be written as

$$\xi(r_{ab}) = \langle \delta(r_a) \delta(r_b) \rangle. \quad (\text{C.3})$$

Let $\mathbf{r} = \mathbf{x} - \mathbf{y}$ be the pairs separation. Then, the average is done over all possible positions:

$$\xi(\mathbf{r}) = \frac{1}{V} \int dV_y \delta(\mathbf{y}) \delta(\mathbf{r} + \mathbf{y}). \quad (\text{C.4})$$

C.1.2 Power spectrum

In Fourier space, any real quadratic function of perturbation variables is called power spectrum. One simple way to define it in terms of the density contrast is

$$P(\mathbf{k}) = V |\delta(\mathbf{k})|^2 = V \delta(\mathbf{k}) \delta^*(\mathbf{k}), \quad (\text{C.5})$$

where, in most situations, the volume V can be set to unit, and we have defined the Fourier transform of $\delta(\mathbf{x})$ as

$$\delta(\mathbf{k}) = \frac{1}{V} \int d^3x \delta(\mathbf{x}) e^{-i\mathbf{k} \cdot \mathbf{x}}. \quad (\text{C.6})$$

The power spectrum (C.5) can then be written as

$$P(\mathbf{k}) = \int d^3x \xi(\mathbf{r}) e^{-i\mathbf{k} \cdot \mathbf{x}}, \quad (\text{C.7})$$

which shows that it is the Fourier transform of the correlation function.

More generally, the power spectrum is defined as the ensemble average of the product of different k -modes:

$$V \langle \delta(\mathbf{k}) \delta^*(\mathbf{k}') \rangle = \frac{1}{V} \int d^3x d^3y \langle \delta(\mathbf{x}) \delta(\mathbf{y}) \rangle e^{-i\mathbf{k} \cdot \mathbf{x} + i\mathbf{k}' \cdot \mathbf{y}} = \frac{(2\pi)^3}{V} P(\mathbf{k}) \delta_D(\mathbf{k} - \mathbf{k}'), \quad (\text{C.8})$$

where $\delta_D(\mathbf{k} - \mathbf{k}')$ is the Dirac delta function. This show that, if the fields are statistically homogeneous, the different modes are uncorrelated.

We can think of galaxies as samples of the continue matter distribution. We actually do not know how to draw this sample, but a simple assumption is that it follows a Poisson distribution. Also, to take into account the discreteness of galaxies, we introduce a window function $W(\mathbf{x})$. We, again, chose the simplest possibility and assume that $W(\mathbf{x})$ is constant inside the survey volume and zero elsewhere (top-hat function). We choose to normalize it as

$$\int dV W(\mathbf{x}) = 1 \quad (\text{C.9})$$

such that

$$W(\mathbf{x}) = \begin{cases} 1/V, & \text{inside the volume,} \\ 0, & \text{elsewhere.} \end{cases} \quad (\text{C.10})$$

Now, the selected sample is given by

$$\delta_s(\mathbf{x}) = \delta(\mathbf{x}) V W(\mathbf{x}), \quad (\text{C.11})$$

where we must include the volume V because of our choice of normalization. The point-like nature of the density contrast can then be expressed as a sum of Dirac delta functions,

$$\delta(\mathbf{x}) = \frac{n(\mathbf{x})}{\bar{n}} - 1, \quad n(\mathbf{x}) = \sum_i^N \delta_D(\mathbf{x} - \mathbf{x}_i), \quad (\text{C.12})$$

where N is the total number of galaxies within V . Hence, the selected field becomes

$$\delta_s(\mathbf{x}) = \frac{V}{N} \sum_i w_i \delta_D(\mathbf{x} - \mathbf{x}_i) - V W(\mathbf{x}), \quad (\text{C.13})$$

where $w_i = V W(\mathbf{x}_i)$. In Fourier space, the previous expression reads

$$\delta_s(\mathbf{k}) = \frac{1}{N} \sum_i w_i e^{-i\mathbf{k} \cdot \mathbf{x}_i} - W(\mathbf{k}) \quad (\text{C.14})$$

where, for a spherical top-hat function of radius R ,

$$W(\mathbf{k}) = \int dV W(\mathbf{x}) e^{-i\mathbf{k} \cdot \mathbf{x}} = \frac{3(\sin kR - kR \cos kR)}{(kR)^3}. \quad (\text{C.15})$$

Finally, computing the variance of the density field $\delta_s(\mathbf{k})$, we find

$$\Delta^2(\mathbf{k}) = V \langle \delta(\mathbf{k}) \delta^*(\mathbf{k}) \rangle = P(\mathbf{k}) + P_{\text{shot}}, \quad (\text{C.16})$$

where we identify the true $P(\mathbf{k})$ and shot-noise P_{shot} power spectra:

$$P(\mathbf{k}) = \frac{V}{N^2} \sum_{i \neq j} \langle w_i w_j \rangle e^{-i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}_i)} - VW^2(\mathbf{k}), \quad (\text{C.17})$$

$$P_{\text{shot}} = \frac{V}{N}. \quad (\text{C.18})$$

The shot-noise corresponds to the power spectrum of a distribution without any intrinsic correlation, obtained by randomly selecting particles. Note that it becomes negligible for large densities N/V .

C.2 Fisher information matrix

A straightforward way to estimate the constraining power of a future experiment is to simulate data assuming a fiducial value for the parameters of interest $\hat{\theta}$. Repeating this procedure for all possible values of the fiducial parameter we can then find estimates for the accuracy with which a particular parameter can be measured. In practice, however, this procedure is computationally expensive. The Fisher matrix method provides a shortcut: approximating the likelihood for a future experiment, it is possible to find a lower bound to the errors on the parameters without the need of simulating data.

Assume that the distribution of the parameters around their best estimators can be approximated by a Gaussian distribution

$$L(\theta) \simeq N \exp\left(-\frac{1}{2}(\theta_i - \hat{\theta}_i)C_{ij}^{-1}(\theta_j - \hat{\theta}_j)\right) \quad (\text{C.19})$$

which can always be done near the peak of the likelihood. The values $\hat{\theta}_i$ are the maximum likelihood estimators, which are functions of the data, and C_{ij} is the correlation matrix. Expanding the log-likelihood around the best estimators $\hat{\theta}_i$,

$$\ln L(\theta_i) \simeq \ln L(\hat{\theta}_i) + \frac{1}{2} \frac{\partial^2 \ln L(\theta_i)}{\partial \theta_i \partial \theta_j} \bigg|_{\text{ML}} (\theta_i - \hat{\theta}_i)(\theta_j - \hat{\theta}_j), \quad (\text{C.20})$$

where the Hessian matrix is evaluated at the maximum likelihood estimators. Comparing, we find that the normalization is given by the fiducial parameters, whereas

$$C_{ij}^{-1} \simeq - \frac{\partial^2 \ln L(\theta)}{\partial \theta_i \partial \theta_j} \bigg|_{\text{ML}}. \quad (\text{C.21})$$

We define the Fisher information matrix as the expectation value of the Hessian,

$$F_{ij} \equiv - \left\langle \frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j} \right\rangle, \quad (\text{C.22})$$

such that if the distribution of the parameters follows a Gaussian, the Fisher matrix is identified as the inverse of the covariance matrix.

When used to produce a forecast of an experiment, the Fisher matrix gives information about the best accuracy with which a parameter could be determined. This estimate often times is very optimistic, assuming the best-case scenario. Nevertheless, the Fisher analysis is an important tool in the process of forecasting.

C.2.1 The Fisher matrix for the power spectrum

Suppose that a future redshift survey is set to measure the distribution of galaxies, providing us with the Fourier coefficients $\delta(\mathbf{k})$ and the corresponding power spectrum. The observations will be carried on different redshift bins providing, for each interval $(z, z + \Delta z)$, a set of m wavenumbers $\mathbf{k}_i$. Let us write the variance of the galaxy density contrast, eq. (C.16), as

$$\Delta(\mathbf{k}) = \langle \delta(\mathbf{k}) \delta^*(\mathbf{k}) \rangle = \langle \delta(\mathbf{k}) \delta(-\mathbf{k}) \rangle = P(\mathbf{k}, z; \theta) + \frac{1}{\bar{n}(z)}, \quad (\text{C.23})$$

where $\bar{n}(z)$ is the average galaxy density in a redshift bin. Recall that the last term is the shot-noise, which comes from the assumption that the sampling of galaxies is well described by a Poisson distribution. We can compute $P(\mathbf{k}, z)$ from our theory, but in any real survey the power spectrum will include the Poisson noise.

The average galaxy density is computed by the survey itself, $\bar{n} = \langle n(x) \rangle$. Since the density contrast is given by $\delta(x) = (n(x)/\bar{n}) - 1$, note that $\langle \delta(x) \rangle = \bar{\delta}(x) = 0$ by construction.

Let us assume that the galaxy distribution can be well approximated by a multivariate Gaussian and that the measures at every $\mathbf{k}_i$ are statistically independent, having no correlation. The likelihood is then

$$L = \frac{1}{(2\pi)^{m/2} \prod_i \Delta_i} \exp \left[-\frac{1}{2} \sum_i \frac{\delta_i^2}{\Delta_i^2} \right], \quad (\text{C.24})$$

where $\Delta_i = \Delta(\mathbf{k}_i)$, $\delta_i = \text{Re } \delta(\mathbf{k}_i)$, and the log-likelihood

$$\mathcal{L} = -\ln L = \frac{m}{2} \ln(2\pi) + \sum_i \ln \Delta_i + \sum_i \frac{\delta_i^2}{2\Delta_i^2}. \quad (\text{C.25})$$

Note that δ_i are given by the experiment, so it does not depend on the choice of fiducial parameters. The expectation value (and the true power spectrum), does. Computing the derivative,

we find the Fisher matrix,

$$F_{\alpha\beta} = \left\langle \frac{\partial^2 \mathcal{L}}{\partial \theta^\alpha \partial \theta^\beta} \right\rangle = \frac{1}{2} \sum_i \left(\frac{\bar{n} P_i}{\bar{n} P_i + 1} \right)^2 \frac{\partial \ln P_i}{\partial \theta^\alpha} \frac{\partial \ln P_i}{\partial \theta^\beta}. \quad (\text{C.26})$$

To proceed, we approximate the sum by an integral over all the wavenumbers $\mathbf{k}_i$. First, we need to count the number of modes in the Fourier volume $d^3k = 2\pi k dk d\mu$. However, the number of modes that can actually be seen is limited: modes with k^{-1} much larger than the characteristic scale of the survey cannot be measured, whereas short modes are sampled by very few galaxies, so its measurement is unreliable. Hence, we discretize the Fourier space into cells of volume $V_{\text{cell}} = (2\pi)^3/V_{\text{survey}}$ such that the number of modes in the survey volume can be written as

$$\frac{d^3k}{V_{\text{cell}}} = V_{\text{survey}} \frac{d^3k}{(2\pi)^3}. \quad (\text{C.27})$$

With the integral, the Fisher matrix then becomes [183]

$$F_{\alpha\beta} = \frac{1}{8\pi^2} \int_{-1}^1 d\mu \int_{k_{\min}}^{k_{\max}} k^2 dk \frac{\partial \ln P_i}{\partial \theta^\alpha} \frac{\partial \ln P_i}{\partial \theta^\beta} \left(\frac{n P_i}{n P_i + 1} \right)^2 V_{\text{survey}}. \quad (\text{C.28})$$

Note the the previous expression is the Fisher matrix for one redshift bin. If the survey's observations span over several redshift bins, the total FM is simply the sum of each $F_{\alpha\beta}(z_i)$.

C.3 Observed power spectrum

To compare the observed galaxy power spectrum with a given theory, we need not only the theoretical prediction for $P(\mathbf{k})$, but we should also take into account a few corrections. These include the Kaiser term due to redshift-space distortions, the Alcock-Packzynski (geometrical correction due to the wrong choice of fiducial cosmology), and eventual errors in the determination of the redshift. In this section, we further explain the last two.

C.3.1 Alcock-Packzynski effect

Galaxy surveys can measure angles and redshifts. To compute the power spectrum from the observed data, these angles and redshifts need to be converted into distances and wavenumbers, which requires a choice of reference cosmology. This leads to a difference between the power spectrum of reference and any other, due only to the wrong choice of cosmological model, the so-called Alcock-Packzynski effect. This begs the question: if we know the power spectrum for a given reference cosmology (say, the Λ CDM model), how to find the power spectrum for any other cosmology?

Several features of the power spectrum depend on the choice of cosmological model: the shape at $z = 0$, the amplitude at any other z (via the growth function); the separation between galaxies (therefore, wavenumbers), and the volume in which the power spectrum is computed, to name a few. Here we focus on volumes and wavenumbers.

Suppose an experiment measures an angle θ in the sky subtending a transverse comoving size λ_1 at z , where the subscript 1 expresses a given choice of cosmology. Then, the angular diameter distance is

$$d_A^{(1)}(z) = \frac{\lambda_1}{(1+z)\theta}. \quad (\text{C.29})$$

For any other cosmology, the scale has to change to keep the same subtended angle, and the angular diameter distance is

$$d_A^{(2)}(z) = \frac{\lambda_2}{(1+z)\theta}. \quad (\text{C.30})$$

Hence, the combination d_A/λ is independent of cosmology. Or, in terms of the transverse wavenumber $k_\perp$, the quantity $k_\perp d_A$ is the same in any cosmology. Thus, given a reference cosmology, the transverse wavenumber in any other cosmological model is

$$k_\perp = \frac{d_A^{\text{ref}}}{d_A} k_\perp^{\text{ref}}. \quad (\text{C.31})$$

Similarly, for comoving scales between z_1 and z_2 along the line of sight, $\lambda = \Delta z/H(z)$, where $\Delta z = z_2 - z_1$, the product $\lambda H(z)$ must remain constant. The longitudinal wavenumber is then

$$k_\parallel = \frac{H(z)}{H^{\text{ref}}(z)} k_\parallel^{\text{ref}}. \quad (\text{C.32})$$

With these results, we find that the modulus of $\mathbf{k}$ and the direction cosine $\mu = \mathbf{k} \cdot \hat{\mathbf{r}}/k$, where $\hat{\mathbf{r}}$ is the unit vector along the line of sight, are

$$k = \left(k_\parallel^2 + k_\perp^2\right)^{1/2} = R k^{\text{ref}}, \quad (\text{C.33})$$

$$\mu = \frac{k_\parallel}{\left(k_\parallel^2 + k_\perp^2\right)^{1/2}} = \frac{H \mu^{\text{ref}}}{H^{\text{ref}} R}, \quad (\text{C.34})$$

where we have defined

$$R = \frac{[H^2 d_A^2 \mu_{\text{ref}}^2 - H_{\text{ref}}^2 (d_A^{\text{ref}})^2 (\mu_{\text{ref}}^2 - 1)]^{1/2}}{H^{\text{ref}} d_A}. \quad (\text{C.35})$$

An immediate consequence is that if the reference power spectrum is isotropic, the one in any other cosmology will in general contain anisotropies.

Furthermore, the power spectrum is proportional to the volume in which the perturbations are measured, and the volume changes with the cosmology. To take this modification into account, consider a shell volume $V = r^2 d\Omega dr$, spanned by a solid angle $d\Omega$ and with thickness dr . Since $dr = dz/H(z)$ along the line of sight, and r is the observer area distance, i.e. angular diameter distance, the combination

$$d\Omega dz = \frac{H(z)}{d_A^2} V$$

is independent of the cosmology.

Therefore, the power spectrum assuming a reference cosmology — to convert wavenumbers and volumes — is related to the one in any other cosmology by

$$P(k^{\text{ref}}, \mu^{\text{ref}}) = \left(\frac{d_A^{\text{ref}}}{d_A} \right)^2 \frac{H(z)}{H^{\text{ref}}} P(k, \mu). \quad (\text{C.36})$$

C.3.2 Redshift error

The determination of the redshift itself is not free of uncertainties. An error σ_z in the measurement of the redshift propagates into the estimation of the radial distance: since $dr = dz/H(z)$, the error is propagated as $\sigma_r = \sigma_z/H(z)$.

Assuming the measured radial distance follows a Gaussian distribution around its true value r_0 ,

$$f(r; r_0, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(r-r_0)^2}{2\sigma^2}}, \quad (\text{C.37})$$

the “true” correlation function is given by the convolution

$$\xi(r_0, \sigma) = \int dr \xi(r) f(r; r_0, \sigma). \quad (\text{C.38})$$

Since in Fourier space the convolution becomes a product, the error in the distance translates into an uncertainty in the power spectrum given by

$$P(k) \rightarrow P(k) e^{-k^2 \mu^2 \sigma_r^2}. \quad (\text{C.39})$$

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