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Characterization of the Behavior of Dynamic Agents in Noncooperative Games

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Abstract

With the increasingly connected world, many systems can be modeled as games where dynamic agents interact noncooperatively with each other. Game theory is increasingly found to be useful to understand such systems. In general, there are two approaches in game theory: the descriptive approach, and the prescriptive approach. In the descriptive approach, the goal is to explain the behavior of agents in noncooperative situation. The descriptive approach can be found in economics or biology. In the prescriptive approach, the goal is to prescribe the behavior of agents in a noncooperative system such that the system behaves in a desirable manner. This approach can be found in engineering and computer science. In the first part of this thesis, using the prescriptive approach, we design Nash equilibrium seeking inputs for agents with multiple-order dynamics. We investigate the case of second-order dynamic agents and linear time-invariant dynamic agents in noncooperative games. In the second part, using the descriptive approach, we investigate the behavior of self-improving agents with quadratic payoff such that their payoffs are nondecreasing. We investigate the case where agents influence each other positively in the entire states, that is, the case of a game with totally positive externalities and the case where agents influence each other positively only in a certain set of positive externalities.

First, we design two Nash equilibrium seeking inputs for agents evolving under second-order dynamics. To develop the Nash equilibrium seeking inputs, we use tools from convex optimization and monotone operator. Inspired by the vanishing damping and Nesterov's gradient descent algorithms in convex optimization, we develop two Nash equilibrium seeking dynamics with similar forms to the corresponding optimization algorithms. Different from existing literatures that assume strong monotonicity on the pseudo-gradient, we prove convergence to the Nash equilibrium by assuming only cocoercivity condition on the pseudo-gradient. We note that the notion of cocoercivity is weaker than the notion of strong monotonicity. Furthermore, in the case of vanishing damping input, we are able to provide sufficient conditions for global uniform asymptotic stability for the Nash equilibrium. Then, we consider the problem of Nash equilibrium seeking for agents with linear time-invariant dynamics. In this

setting, first, we provide sufficient conditions for the existence and uniqueness of the Nash equilibrium of the game. Then, we use a passivation method to develop the Nash equilibrium seeking dynamics. By using the notion of incremental passivity, we construct a Lyapunov function to show the stability of the Nash equilibrium. Our approach can also accommodate a class of nonminimum phase linear time-invariant agents.

In the second part of the thesis, we begin by characterizing the behavior of self-improving agents in the game with totally positive externalities, that is the game where all agents influence each other positively everywhere in the state space. We show that if each agent evolves under affine dynamics and possesses quadratic payoff, then the class of pseudo-gradient agents coincide with the class of self-improving agents. In the case of self-improving agents with quadratic payoffs, we provide necessary and sufficient algebraic conditions for the game to have totally positive externalities. We also show that in such a game, there always exists a unique Nash equilibrium. Furthermore, we prove that if a certain condition relating the zeros of the pseudo-gradient and the equilibria of the dynamics holds, then the unique Nash equilibrium is asymptotically stable. Then, considering the pattern graph of the game, we show that if the pattern graph is connected, then the game by self-improving agents with totally positive externalities is a weighted identical interest game.

Then, we investigate the case of two pseudo-gradient agents with quadratic payoffs in noncooperative games. We introduced the set of positive externalities as the set where both agents influence each other positively. We show that the set of positive externalities is a union of convex polyhedra. Because the union of convex polyhedra is not necessarily connected, we provide sufficient conditions for connectedness of the set of positive externalities. When the set of positive externalities is a union of shifted polyhedral cones intersecting only at the unique Nash equilibrium, we show that this set is not a positively invariant set of the pseudo-gradient dynamics. However, we show that a certain subset of this set is a positively invariant set of the pseudo-gradient dynamics and provide sufficient conditions such that this subset of the set of positive externalities is positively invariant.

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Chapter 1

Introduction

1.1 Dynamics in Game Theory

In economics and evolutionary biology, dynamics in game theory arise as models of repeated interaction of agents. Following [1], depending on the complexities of strategies and beliefs of each agent, dynamics in game theory can be classified into three kinds: learning dynamics, adaptive heuristics, and evolutionary dynamics. Learning dynamics [2] models how agents learn Nash equilibrium and its refinements by updating their beliefs and acting optimally according to the beliefs. Adaptive heuristics [1] models agents adopting simple myopic "rules of thumbs" as their strategies. In contrast to learning dynamics and adaptive heuristics, evolutionary dynamics [3] models how strategies adopted by populations of agents change over time via selection and mutation. Despite differences in the assumptions made in these three kinds of dynamics, both the economics and the biologists use dynamics to *explain* and *describe* the behavior of agents in repeated interactions. For example, in [4], Kalai and Lehrer explained how agents in repeated interactions could *learn* Nash equilibria. Hart in [1] studied several simple *adaptive heuristics* that always lead to correlated equilibria. In biology [5, 6], two populations of individuals competing for a resource of fixed value were analyzed by using the replicator dynamics. In economics, Gintis in [7] attempted to explain why agents value goods they possess higher than the goods not in their possession via evolutionary approach. In all the aforementioned examples, the economists and the biologists attempt to explain the behavior of agents or populations of agents in repeated interactions. Following [8, 9], we call these three approaches as *descriptive*.

As early as [10], control engineers have also utilized game theoretic framework to model complex systems such as transportation networks [11–15], energy markets [16–18], cybersecurity [19–22], and communication networks [23–25]. These systems typically consist of multiple (possibly dispersed) physical components with various constraints and limitations interacting with each other. Game theory allows control engineers to mitigate the complexity of these systems by modeling them as collections of simpler interacting components (or agents) trying to improve their own payoffs. However, in contrast to the descriptive approach taken in economics and biology, the approach taken by the control engineers is *prescriptive* [8, 9]. In this approach, the goal is to *prescribe* the agents' behavior such that the whole system behaves in a desirable manner. In most cases, the control engineers would prescribe control law for each agent to converge to one of the Nash equilibria.

Since the inception of game theory, both economists and control engineers have designed various Nash equilibrium seeking dynamics and algorithms for different purposes. However, many of these dynamics and algorithms consider only agents that do not possess dynamics or agents evolving according to single-integrator dynamics. In economics, this restriction makes sense because the goal is to *explain* decision making in repeated interactions. However, in engineering, the agents often represent physical plants in a network such as mobile robots, electric vehicles, or smart grids. In this case, the agents might possess internal state that evolve according to some dynamics. Furthermore, existing Nash equilibrium seeking dynamics and algorithms do not take into account the case when the agents' payoffs do not depend on the full states of all the agents. For example, in the case of mobile robots, the agents' payoffs might depend only on the positions of the agents instead of both the positions and velocities. Thus, it is natural to ask how to design Nash equilibrium seeking dynamics for such systems. Taking the prescriptive approach, we consider this problem in the first part of this thesis.

One of the earliest dynamics considered as a model of agents learning a Nash equilibrium is the *pseudo-gradient* dynamics introduced in [26]. An important property of the pseudo-gradient dynamics is that an agent evolving under the pseudo-gradient dynamics (or a pseudo-gradient agent) is *self-improving*, that is, if the other agents' states are fixed, then the pseudo-gradient agent's payoff is nondecreasing. Thus, the pseudo-gradient dynamics roughly captures the notion that agents want to improve their own payoffs. However, if all the agents are pseudo-gradient agents (or more generally, all agents are self-improving agents), then it may not be the case that the agents' payoffs are improving. In fact, there are games by pseudo-gradient agents in which some agents' payoffs are *decreasing* before achieving the Nash equilibrium. It is then natural to ask the following question: *In what conditions would self-improving agents possess nondecreasing payoffs?* To answer this question, we introduce the concept of externality. In economics, externality is frequently defined as an agent's influence to another agent's payoff. If the influence is positive (resp. negative), then the externality are called positive (resp. negative) externality. Now, there are two conditions related to the notion of positive externality that lead to self-improving agents possessing nondecreasing payoffs:

- the condition of *totally positive externalities*, that is, the condition when all agents have positive externalities from each other everywhere in the state space, and
- when agents have positive externalities from each other in a certain set called the *set of positive externalities* and the agents' trajectories stay inside this set.

In the second part of this thesis, taking the descriptive approach, we derive conditions such that self-improving agents possess nondecreasing payoffs.

1.2 Overview

In the first part of this thesis, we develop Nash equilibrium seeking dynamics for multiple-order agents. Specifically, in Chapter 2, we design two Nash equilibrium seeking input functions for second-order dynamic agents. Inspired by the vanishing damping and the Nesterov's gradient descent algorithms from convex optimization, we design the corresponding vanishing damping and Nesterov's gradient inputs to seek the Nash equilibrium. In the case of vanishing damping input, we provide sufficient conditions for global asymptotic stability of the Nash equilbrium. In the case of Nesterov's gradient input, we provide sufficient conditions for convergence to the Nash equilibrium.

In Chapter 3, we consider the case of Nash equilibrium seeking for agents with linear time-invariant dynamics. We use a passivation method to asymptotically stabilize the Nash equilibrium. Our approach also works for a class of agents possesing nonminimum phase linear time-invariant dynamics

In Chapter 4, we study the game by self-improving agents with totally positive externalities. In the case of games by self-improving agents with quadratic payoffs, we provide algebraic necessary and sufficient conditions such that the game has totally positive externalities. We also show that the game by self-improving agents with totally positive externalities has a unique Nash equilibrium. If a certain condition regarding the zeros of the pseudo-gradient and the equilibria of the dynamics is assumed, then the unique Nash equilibrium is asymptotically stable. Furthermore, if the agents' payoffs depend on each other, then the game by self-improving agents with totally positive externalities is a weighted identical interest game.

In Chapter 5, we investigate the case where two pseudo-gradient agents with quadratic payoffs influence each other positively only in a certain set of positive externalities. We show that the set of positive externalities is the union of convex polyhedra. Furthermore, we also provide sufficient conditions for this set to be connected. We then study the case when the set of positive externalities is the union of shifted polyhedral cones that intersect only at the unique Nash equilibrium. In this case, we show that this set is not a positively invariant set of the pseudo-gradient dynamics. However, we provide sufficient conditions such that a certain subset of this set is positively invariant.

1.3 Notation

In this section, we provide the notation that we use throughout this thesis. Consider the finite set $\mathcal{N} = \{1, \ldots, N\}$ and let $\mathcal{X} \triangleq \prod_{i\in\mathcal{N}} \mathcal{X}_i \in \mathbb{R}^n$ be a product set, where $\mathcal{X}_i \subseteq \mathbb{R}^{n_i}$ and $n = \sum_{i\in\mathcal{N}} n_i$. If $x \in \mathcal{X}$, then we write $x = (x_i)_{i\in\mathcal{N}} = (x_i, x_{-i})$, where $x_i \in \mathcal{X}_i$ and $x_{-i} \triangleq (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N) \in \mathcal{X}_{-i} \triangleq \prod_{j\neq i} \mathcal{X}_j$. If we have $h_i : \mathcal{X}_i \to \mathbb{R}^p,\ i \in \mathcal{N}$, then we define $h : \mathcal{X} \to \mathbb{R}^{Np}$ by $h(x) = (h_i(x_i))_{i\in\mathcal{N}}$.

Given a vector $x \in \mathbb{R}^n$, its transpose is denoted by x^{T}. For $x, x' \in \mathbb{R}^n$, $\langle x, x'\rangle \triangleq x^{\mathrm{T}}x'$ denotes the inner product and $\|x\|$ denotes the Euclidean vector norm. A vector $x \in \mathbb{R}^n$ with nonnegative coordinates is denoted by $x \geq 0$. Furthermore, let $f : Y \to \mathbb{R}^n$ and $h : Y \to \mathbb{R}$ be n-times continuously differentiable functions, where Y is an open connected subset of $\mathbb{R}^n$. Then, the Lie derivative of h along f is written as $L_f h(x) = \frac{\partial h}{\partial x}(x) f(x)$. For $2 \leq k \leq n$, the kth Lie derivative of h along f is written as $L_f^k h(x) = L_f L_f^{k-1} h(x) = \frac{\partial L_f^{k-1} h(x)}{\partial x} f(x)$.

Given a matrix $A \in \mathbb{R}^{n\times n}$, $A^{ij}, i, j \in \{1, \ldots, n\}$, denotes the element of matrix A in the ith row and jth column. The ith row of a matrix A is denoted by $\mathrm{row}_i(A)$. The identity matrix in $\mathbb{R}^{n\times n}$ is denoted by I_n. We denote the set of $n \times n$ diagonal matrices by $\mathbb{D}^n$. A symmetric positive-definite matrix $P \in \mathbb{R}^{n\times n}$ is denoted by $P \succ 0$. A square matrix $M \in \mathbb{R}^{n\times n}$ is a *Metzler matrix* if $M^{ij} \geq 0,\ i \neq j,\ i, j = 1, \ldots, n$. A nonnegative matrix $K \in \mathbb{R}^{m\times n}$ is denoted by $K \geq 0$.

Given a set $S \subseteq \mathbb{R}^n$, we denote its interior by $\mathrm{Int}(S)$. The set $-S$ is defined as $-S \triangleq \{y \in \mathbb{R}^n : y = -x,\ x \in S\}$.

Chapter 2

Nash Equilibrium Seeking in a Game by Second-Order Dynamic Agents

2.1 Introduction

In this chapter, we study the problem of Nash equilibrium seeking where each agent has second-order dynamics. We study the game by second-order dynamic agents because this game is the simplest nontrivial game by higher-order dynamic agents and many physical systems can be modeled by second-order differential equations. In deriving the Nash equilibrium seeking dynamics for second-order agents, we are mainly interested in whether there are dynamics with similar properties to the pseudo-gradient dynamics introduced in [26]. Recall that the pseudo-gradient dynamics is defined in such a way that for an agent i, when the states of all the other agents are fixed, the payoff of agent i is nondecreasing along her trajectory. As a consequence, for each agent i, when all the other agents' states are fixed, the pseudo-gradient dynamics is the weighted gradient dynamics. Thus, for an agent i, if the all the other agents' states are fixed, then she is *optimizing* her own payoff. Motivated by this observation, it is reasonable to reformulate the two most famous second-order optimization dynamics, that is, the heavy ball method (or more generally the gradient ascent with vanishing damping ([27, 28])) and Nesterov's accelerated gradient ascent [29], in the context of game by second-order dynamic agents.

In the control literature, as far as we know, the earliest work dealing with multiple-order agents are the work of Arslan and Shamma in [30, 31]. In [30, 31], Arslan and Shamma considered the *derivative-action learning* dynamics and showed that derivative-action learning enables convergence of trajectories to the mixed-strategy Nash equilibria even in games where no first-order smooth dynamics can bring the state to mixed-strategy Nash equilibria. In [32–34], Frihauf et al. considered games where the game itself evolves according to continuous-time nonlinear dynamics. Specifically, averaging and singular perturbation technique are used to semi-globally practically asymptotically stabilize the Nash equilibrium of the game. Our work is most closely related to several recent attempts to design Nash equilibrium seeking dynamics for multiple-order dynamic agents in [35–39]. Our results are different from the aforementioned results mainly because we do not deal with *distributed* or *decentralized* dynamics, that is, we assume that each agent can observe all the other agents' outputs. Although this assumption on the agents' interaction is a strong assumption, as a tradeoff, we can relax the assumption on the structure of the game. Instead of assuming strong monotonicity of the pseudo-gradient as in [35–37, 39], we derive the Nash equilibrium seeking dynamics only with the assumption of cocoercivity. Note that strong monotonicity and Lipschitz continuity of the pseudo-gradient implies cocoercivity but cocoercivity only implies Lipschitz continuity of the pseudo-gradient. In contrast to [38], we do not need to assume that the game is a potential game. Furthermore, as mentioned earlier, our Nash equilibrium seeking dynamics has a clear link to the corresponding second-order optimization algorithms. When the states of all the other agents are fixed, each agent is *merely optimizing* her own payoff with second-order optimization dynamics such as the vanishing damping dynamics or the Nesterov's accelerated gradient ascent dynamics. Furthermore, in the case of there is only one agent or the agents are in a potential game, our Nash equilibrium seeking dynamics is exactly the usual second-order optimization dynamics (vanishing damping dynamics and Nesterov's gradient dynamics).

In the game theory literature, as far as we know, Flam and Morgan in [40] are the first to consider multiple-order dynamics. In [40], Flam and Morgan proposed the *heavy-ball with friction* dynamics to asymptotically stabilize the Nash equilibrium of a class of games including the potential game and games with affine pseudo-gradient. Laraki and Mertikopoulos in [41] derived the *higher-order replicator dynamics* and showed that in addition to convergence to a Nash equilibrium, both dominated strategies and weakly-dominated strategies are eliminated in the long-run. This is different from the first-order replicator dynamics where weakly-dominated strategies may survive. In [42], the same authors introduced the *inertial game dynamics* and showed that for potential games, Nash equilibria that are potential maximizers are asymptotically stable. In contrast to [40, 42], we allow for time varying coefficient in the dynamics of each agent. Furthermore, we establish uniform global asymptotic stability for the case of vanishing damping input in addition to convergence to the Nash equilibrium.

The contents of this chapter are as follows. In Section 2.2, we introduce the notion of a game by second-order dynamic agents and characterize the Nash equilibrium of this game. In Section 2.3, we we use tools from convex optimization [28, 43] to derive the inputs that guarantee convergence to the Nash equilibrium for this game. Although we are indebted to [28] to derive our vanishing damping input, because we work in Euclidean space instead of real Hilbert space, we are able to show uniform global asymptotic stability of the Nash equilibrium by explicitly constructing a time-varying Lyapunov function. To comply with the requirement that each agent only knows her own payoff function but not the other agents' payoffs, we modify the Nesterov's gradient method in [43] in deriving our Nesterov's gradient input. In Section 2.4, we provide numerical examples to illustrate our results. Finally, we summarize this chapter in Section 2.5.

2.2 Game by Second-Order Dynamic Agents

Consider the finite set $\mathcal{N}$ of N agents. Let agent $i \in \mathcal{N}$ have state $x_i \in \mathbb{R}^2$ and evolve under the continuous-time dynamics

$$\dot{x}_i(t) = f_i(x_i(t)) + g_i(x_i(t))u_i(t), \quad x_i(t_0) = x_{i0}, \quad t \geq t_0, \tag{2.1}$$

$$y_i(t) = h_i(x_i(t)), \tag{2.2}$$

where $u_i(t) \in \mathbb{R}^2$ is the input of agent i, $y_i(t) \in \mathbb{R}$ is the output of agent i, $f_i : \mathbb{R}^2 \to \mathbb{R}^2$, $g_i : \mathbb{R}^2 \to \mathbb{R}^2$ with $g_i(x_i) \neq 0$ for all $x_i \in \mathbb{R}^2$, $i \in \mathcal{N}$, and t_0 is the initial time. Furthermore, each agent i's dynamics have *uniform relative degree* 2 from u_i to y_i for all $i \in \mathcal{N}$ [44, Chapter 9]. In the following, we develop time-varying inputs so that the initial time is not restricted to 0.

We assume that the input of agent i is given as a feedback of agent i's state x_i as well as the other agents' *output* $y_{-i} \triangleq (y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_N)$ described by

$$u_i(t) = \phi_i(x_i(t), y_{-i}(t)), \quad i \in \mathcal{N}, \tag{2.3}$$

where $\phi_i : \mathbb{R}^2 \times \mathbb{R}^{N-1} \to \mathbb{R}$. Furthermore, each agent i has a payoff function $J_i : \mathbb{R}^N \to \mathbb{R}$ and wants to increase her own payoff. We assume that agent i knows her own payoff function, but not the other agents' payoff functions. In other words, the input of each agent i is assumed to be *uncoupled* [45]. Furthermore, the payoff functions $J_i(y_i, y_{-i}),\ i \in \mathcal{N}$, satisfy the following concavity and differentiability conditions.

Assumption 2.1. *The payoff functions* $J_i(y_i, y_{-i})$, $i \in \mathcal{N}$, *are continuously differentiable in* $\mathbb{R}^N$ *and concave in* $y_i \in \mathbb{R}$ *for every fixed* $y_{-i} \in \mathbb{R}^{N-1}$.

Now, we define the Nash equilibrium of this game with multi-dimensional continuous-time dynamics for each agent.

Definition 2.1. *Let* $x^\star \in \mathbb{R}^{2N}$ *and* $y^\star \triangleq h(x^\star)$. *Then,* $x^\star$ *is a* Nash equilibrium *of the closed-loop system* (2.1)–(2.3) *along with the payoff functions* $J_i(y)$, $i \in \mathcal{N}$, *if*

1. $x^\star$ *satisfies*

$$J_i(y_i^\star, y_{-i}^\star) \geq J_i(h_i(x_i), y_{-i}^\star), \quad x_i \in \mathbb{R}^2, \quad i \in \mathcal{N}, \tag{2.4}$$

and

2. $x^\star$ *is a closed-loop equilibrium, that is,*

$$f_i(x_i^\star) + g_i(x_i^\star)\phi_i(x_i^\star, y_{-i}^\star) = 0, \quad i \in \mathcal{N}. \tag{2.5}$$

The condition (2.4) is standard in defining Nash equilibria in the conventional game theory literature. However, since we consider the case where the payoff function $J_i(y_i, y_{-i})$ depends only on $y \in \mathbb{R}^N$ instead of $x \in \mathbb{R}^{2N}$, the equilibrium state should be additionally characterized over the entire state x as given by (2.5). It is important to note that the notion of stability in the sense of Lyapunov around the Nash equilibrium does not make sense unless underlying dynamics such as (2.1) are defined.

Assumption 2.1 define a game $\mathcal{G}(J)$, where $J = (J_i)_{i\in\mathcal{N}}$, with second-order dynamic agents. Our goal is to find the input (2.3) such that if $x^\star = (x_i^\star)_{i\in\mathcal{N}}$ is an unknown Nash equilibrium of the closed-loop system (2.1)–(2.3), then $\lim_{t\to\infty} x(t) = x^\star$.

2.2.1 Characterization of Nash equilibrium of $\mathcal{G}(J)$

Define the pseudo-gradient [26] $F : \mathbb{R}^N \to \mathbb{R}^N$ as

$$F(y) \triangleq \left[\frac{\partial J_1(y)}{\partial y_1}, \ldots, \frac{\partial J_N(y)}{\partial y_N}\right]^{\mathrm{T}}. \tag{2.6}$$

Then, we have the following.

Proposition 2.2. *Consider the game* $\mathcal{G}(J)$ *given by* (2.1)–(2.3). *Then,* $x^\star \in \mathbb{R}^{2N}$ *is a Nash equilibrium of* $\mathcal{G}(J)$ *if and only if*

$$F(y^\star) = 0, \tag{2.7}$$

and (2.5) *holds, where* $y^\star \triangleq h(x^\star)$.

Proof. From [46, Proposition 1.4.2] and Assumption 2.1, inequality (2.4) holds if and only if (2.7) holds. The result follows. □

By [46, Proposition 2.2.7], if $G(y) \triangleq -F(y)$ is continuous and satisfies

$$\lim_{\|y\|\to\infty} \frac{\langle y, G(y)\rangle}{\|y\|} = \infty, \tag{2.8}$$

then there exists a solution $y^\star$ to (2.7). If $G(y)$ also satisfies

$$\langle y - y', G(y) - G(y')\rangle \geq 0, \quad y, y' \in \mathbb{R}^N, \tag{2.9}$$

with strict inequality whenever $y \neq y'$, then the solution to (2.7) is unique [46, Theorem 2.3.3]. If, in addition, there exists a unique $x^\star \in \mathbb{R}^{2N}$ satisfying (2.5), then it is the unique Nash equilibrium of the closed-loop system (2.1)–(2.3). Henceforth, we have the following assumption.

Assumption 2.2. *There exists a unique Nash equilibrium of the closed-loop system* (2.1)–(2.3) *of the game* $\mathcal{G}(J)$.

2.2.2 Normal form representation for agents' dynamics

Recall that agent i's dynamics have uniform relative degree 2. Hence, by [44, Proposition 9.1.1], there are diffeomorphisms $\Phi_i,\ i \in \mathcal{N}$, each of which are defined in $\mathbb{R}^2$, such that the coordinate transformation $z_i = \Phi_i(x_i)$ yields

$$\dot{z}_i^1(t) = z_i^2(t), \quad z_i(t_0) = z_{i0} \triangleq \Phi_i(x_{i0}), \quad t \geq t_0, \tag{2.10}$$

$$\dot{z}_i^2(t) = b_i(z_i(t)) + a_i(z_i(t))u_i(t), \tag{2.11}$$

$$y_i(t) = z_i^1(t), \tag{2.12}$$

where $a_i : \mathbb{R}^2 \to \mathbb{R}$ with $a_i(z_i) \neq 0,\ z_i \in \mathbb{R}^2$, and $b_i : \mathbb{R}^2 \to \mathbb{R}$ for all $i \in \mathcal{N}$.

Note that $z^\star \triangleq (\Phi_i(x_i^\star))_{i\in\mathcal{N}}$ is an equilibrium of the closed-loop system (2.3), (2.10)–(2.12) if and only if

$$z_i^{\star 2} = 0, \qquad i \in \mathcal{N}, \tag{2.13}$$

$$b_i(z_i^\star) + a_i(z_i^\star)\phi_i(\Phi_i^{-1}(z_i^\star), y_{-i}^\star) = 0, \qquad i \in \mathcal{N}, \tag{2.14}$$

where $y^\star_{-i} = \left[z_1^{\star 1}, \ldots, z_{i-1}^{\star 1}, z_{i+1}^{\star 1}, \ldots, z_N^{\star 1}\right]^{\mathrm{T}} \in \mathbb{R}^{N-1}$. Proposition 2.3 below shows that the set of equilibria of the closed-loop system (2.1)–(2.3) in the original coordinate frame is equivalent to the set of equilibria of the closed-loop system (2.3), (2.10)–(2.12) in the transformed coordinate frame.

Proposition 2.3. *Consider the game $\mathcal{G}(J)$ given by* (2.1)–(2.3). *Then, $x^\star$ is an equilibrium of the closed-loop system* (2.1)–(2.3) *if and only if $z^\star$ is an equilibrium of the closed-loop system* (2.3), (2.10)–(2.12).

Proof. Let $x_0 \triangleq \left[x_{10}^{\mathrm{T}}, \ldots, x_{N0}^{\mathrm{T}}\right]^{\mathrm{T}}$ and $z_0 \triangleq \left[z_{10}^{\mathrm{T}}, \ldots, z_{N0}^{\mathrm{T}}\right]^{\mathrm{T}}$ and let $\tau_1(t, x_0)$ and $\tau_2(t, z_0)$ be the flows of the closed-loop systems (2.1)–(2.3) and (2.3), (2.10)–(2.12), respectively. Then, we have $\tau_2(t, z_0) = \Phi(\tau_1(t, \Phi^{-1}(z_0)))$ and $\tau_1(t, x_0) = \Phi^{-1}(\tau_2(t, \Phi(x_0)))$, $t \geq t_0$.

Assuming that $x^\star$ is an equilibrium of the closed-loop system (2.1)–(2.3), we have $\tau_2(t, z^\star) = \Phi(\tau_1(t, \Phi^{-1}(z^\star))) = \Phi(\tau_1(t, x^\star)) = \Phi(x^\star) = z^\star$ for all $t \geq t_0$. Hence, $z^\star$ is an equilibrium of the closed-loop system (2.3), (2.10)–(2.12). By the same reasoning, the converse also holds. Hence, the result follows. □

2.3 Nash Equilibrium Seeking

Using the results from [28] and [43], in this section we derive inputs that guarantee the convergence of the state profile of all agents to the unique Nash equilibrium for the game $\mathcal{G}(J)$ satisfying Assumptions 2.1–2.2 with λ-cocoercivity hypothesis on $-F$.

Definition 2.4. *A function $G : \mathbb{R}^n \to \mathbb{R}^n$ is λ-cocoercive if there exists a constant $\lambda > 0$ such that for all $y, y' \in \mathbb{R}^n$,*

$$\langle y - y', G(y) - G(y') \rangle \geq \lambda \|G(y) - G(y')\|^2. \tag{2.15}$$

Remark 2.5. *Note that* (2.15) *implies* (2.9). *Furthermore, if $G(y)$ is λ-cocoercive, then it is Lipschitz continuous with the Lipschitz constant $\frac{1}{\lambda}$ [47, Corollary 23.10]. However, if $G(y)$ only satisfies* (2.9), *then it is not necessarily the case that $G(y)$ is Lipschitz continuous.*

2.3.1 Vanishing damping input

In [28], Bot and Csetnek introduced the *second-order dynamics with vanishing damping* in the real Hilbert space $\mathcal{H}$ given by

$$\ddot{x}(t) + \alpha(t)\dot{x}(t) + \beta(t)B(x(t)) = 0, x(t_0) = x_0, \quad \dot{x}(t_0) = v_0, \quad t \geq t_0, \tag{2.16}$$

where $B : \mathcal{H} \to \mathcal{H}$ is a λ-cocoercive operator in the real Hilbert space $\mathcal{H}$ and the functions $\alpha, \beta : [t_0, +\infty) \to (0, +\infty)$ are Lebesgue measurable and continuously differentiable.

Remark 2.6. *If we set $\alpha(t) \equiv \gamma > 0$ and $\beta(t) \equiv 1$, then we recover the* heavy-ball with friction dynamics *[48].*

The following fact from [28, Lemma 7] about the dynamics (2.16) is needed to derive our result.

Lemma 2.7. *[28, Lemma 7 and Theorem 8] Consider the dynamics* (2.16) *with $B(\cdot)$ being λ-cocoercive. Suppose that there exists $p \in \mathcal{H}$ such that $B(p) = 0$. Let $\alpha(t), \beta(t)$ be chosen such that*

$$\frac{\alpha^2(t)}{\beta(t)} \geq \frac{1+\theta}{\lambda}, \quad \dot{\alpha}(t) \leq 0 \leq \dot{\beta}(t), \tag{2.17}$$

with $\theta > 0$ for $t \geq t_0$ and let $h_p : \mathbb{R} \to \mathbb{R}$ be a function given by

$$h_p(t) \triangleq \frac{1}{2}\alpha(t)\|x(t) - p\|^2 + \lambda\frac{\alpha(t)}{\beta(t)}\|\dot{x}(t)\|^2 + \langle x(t) - p, \dot{x}(t)\rangle, \tag{2.18}$$

where $x(t)$ is the solution to (2.16)*. Then, it follows that*

1. $\dot{h}_p(t) \leq 0,\ t \geq t_0$,
2. $\lim_{t\to\infty} \dot{x}(t) = \lim_{t\to\infty} \ddot{x}(t) = \lim_{t\to\infty} B(x(t)) = 0$,
3. *$x(t)$ weakly converges to $p \in \mathcal{H}$.*

As mentioned in [28], by choosing

$$\alpha(t) = ae^{-\rho t} + b, \quad \beta(t) = \frac{1}{ce^{-\delta t} + d}, \tag{2.19}$$

where $a, b, c, d, \rho, \delta > 0$ with $b^2 d > \frac{1}{\lambda}$, (2.17) is satisfied. From (2.17), because $\dot{\alpha}(t) \leq 0 \leq \dot{\beta}(t)$, it follows that $\alpha_{\max} \triangleq \sup_{t\geq t_0} \alpha(t) = \alpha(t_0) < \infty$, $\beta_{\min} \triangleq \inf_{t\geq t_0} \beta(t) = \beta(t_0) > 0$ and thus

$$\frac{\alpha^2_{\max}}{\beta_{\min}} \geq \frac{\alpha^2(t)}{\beta(t)} \geq \frac{1+\theta}{\lambda}, \quad t \geq t_0. \tag{2.20}$$

Hence, it follows that $\beta(t) \leq \frac{\lambda}{1+\theta}\frac{\alpha^2_{\max}}{\beta_{\min}} < \infty$ and $\alpha(t) \geq \frac{1+\theta}{\lambda}\beta_{\min} > 0$ for $t \geq t_0$. As $\alpha(t)$ is nonincreasing and $\beta(t)$ is nondecreasing, it follows that $\alpha_{\min} \triangleq \lim_{t\to\infty} \alpha(t) = \inf_{t\geq t_0} \alpha(t) > 0$ and $\beta_{\max} \triangleq \lim_{t\to\infty} \beta(t) = \sup_{t\geq t_0} \beta(t) < \infty$. Hence from (2.17), it follows that

$$\frac{\alpha^2(t)}{\beta(t)} \geq \frac{\alpha^2_{\min}}{\beta_{\max}} \geq \frac{1+\theta}{\lambda}, \quad t \geq t_0. \tag{2.21}$$

Now we state our result inspired by the properties above of the second-order dynamics with vanishing damping.

Theorem 2.8. *Consider the game $\mathcal{G}(J)$ given by (2.1)–(2.3) and suppose that $G(y) = -F(y)$ is λ-cocoercive. If the input for each agent is given by*

$$\phi_i(t) = \frac{1}{L_{g_i} L_{f_i} h_i(x_i(t))} (-L_{f_i}^2 h_i(x_i(t)) + v_i(t)), \tag{2.22}$$

with

$$v_i(t) = -\alpha(t) L_{f_i} h_i(x_i(t)) + \beta(t) \frac{\partial J_i(y(t))}{\partial y_i}, \tag{2.23}$$

where $\alpha(t), \beta(t)$ are chosen to satisfy (2.17), then the solution $x(t) \equiv x^\star$ of the dynamics (2.1)–(2.3) with input (2.22), (2.23) is globally asymptotically stable.

Proof. In normal form representation, since $L_{f_i} h_i(x_i) = z_i^2$, the closed-loop dynamics of agent i are given by

$$\dot{z}_i^1(t) = z_i^2(t), \quad z(t_0) = z_0, \quad t \geq t_0, \tag{2.24}$$

$$\dot{z}_i^2(t) = -\alpha(t) z_i^2(t) + \beta(t) \frac{\partial J_i(z^1(t))}{\partial y_i}, \tag{2.25}$$

where $z^1 \triangleq [z_1^1, \ldots, z_N^1]^{\mathrm{T}}$. In other words, letting $z^2 \triangleq [z_1^2, \ldots, z_N^2]^{\mathrm{T}}$, the closed-loop dynamics of all agents can be equivalently written as

$$\dot{z}^1(t) = z^2(t), \quad z(t_0) = z_0, \quad t \geq t_0, \tag{2.26}$$

$$\dot{z}^2(t) = -\alpha(t) z^2(t) - \beta(t) G(z^1(t)). \tag{2.27}$$

Because $\dot{z}^1 = z^2$, the above dynamics can be rewritten as

$$\ddot{z}^1(t) + \alpha(t)\dot{z}^1(t) + \beta(t) G(z^1(t)) = 0, \quad z(t_0) = z_0, \quad t \geq t_0. \tag{2.28}$$

Note that (2.28) is a particular case of (2.16) since $G(z^1)$ is λ-cocoercive. Furthermore $z^\star \triangleq (z^{\star 1}, 0) \in \mathbb{R}^{2N}$ such that $G(z^{\star 1}) = 0$ is an equilibrium of the dynamics (2.28).

First, we show that the solution $z(t) \triangleq \left[z^{1^{\mathrm{T}}}(t), z^{2^{\mathrm{T}}}(t)\right]^{\mathrm{T}} \equiv z^\star$ of the dynamics (2.28) is Lyapunov stable. Consider the time-varying Lyapunov function candidate

$$V(t, z) \triangleq \tilde{z}^{\mathrm{T}} \begin{bmatrix} \frac{1}{2}\alpha(t) I_N & \frac{1}{2} I_N \\ \frac{1}{2} I_N & \lambda \frac{\alpha(t)}{\beta(t)} I_N \end{bmatrix} \tilde{z},$$

where I_N denotes the identity matrix in $\mathbb{R}^{N\times N}$ and $\tilde{z} \triangleq \left[(z^1 - z^{\star 1})^{\mathrm{T}}, z^{2\mathrm{T}}\right]^{\mathrm{T}}$. Note that $V(t, z^\star) = 0,\ t \geq t_0$. Because $\alpha_{\min} \leq \alpha(t)$ and $\beta(t) \leq \beta_{\max}$, it follows that

$$\tilde{z}^{\mathrm{T}}(W_{\min} \otimes I_N)\tilde{z} \leq V(t, z), \quad t \geq t_0,$$

where $W_{\min} \triangleq \begin{bmatrix} \frac{1}{2}\alpha_{\min} & \frac{1}{2} \\ \frac{1}{2} & \lambda\frac{\alpha_{\min}}{\beta_{\max}} \end{bmatrix}$ and $\otimes$ denotes the Kronecker product. Note that $W_{\min}$ is a symmetric matrix with positive trace. Furthermore, from (2.21), it follows that $\det(W_{\min}) = \frac{\lambda}{2}\frac{\alpha_{\min}^2}{\beta_{\max}} - \frac{1}{4} > 0$. Thus, it follows that $W_{\min}$ is a positive-definite matrix. As the eigenvalues of the Kronecker product $W_{\min} \otimes I_N$ are the multiplications of the eigenvalues of $W_{\min}$ and I_N [49, Theorem 4.2.12], it follows that the eigenvalues of $W_{\min} \otimes I_N$ are the same as the eigenvalues of $W_{\min}$. Hence, it follows from Rayleigh's theorem [50, Theorem 4.2.2] that

$$\gamma_{\min}(W_{\min})\|\tilde{z}\|^2 \leq V(t, z), \quad t \geq t_0,$$

where $\gamma_{\min}(W_{\min}) > 0$ is the smallest eigenvalue of the matrix $W_{\min}$. Furthermore, note that $V(t, z(t)) = h_{z^{\star 1}}(t)$ by (2.18) and thus from Lemma 2.7, it follows that $\dot{V}(t, z(t)) = \dot{h}_{z^{\star 1}}(t) \leq 0$. We conclude from [51, Theorem 4.6] that the solution $z^\star$ of the dynamics (2.28) is Lyapunov stable.

Now we show that $\lim_{t\to\infty} x(t) = x^\star$. First, in $\mathbb{R}^N$, the notion of weak convergence is the same as the notion of convergence in standard topology [52, Theorem 3.6]. Hence, it follows from Lemma 2.7 that $\lim_{t\to\infty} z^1(t) = z^{\star 1}$ and $\lim_{t\to\infty} z^2(t) = 0$, where $G(z^{\star 1}) = -F(z^{\star 1}) = 0$.

Because $z^\star$ is an equilibrium of the closed-loop system (2.3), (2.10)–(2.12) with input (2.22), (2.23), it follows from Proposition 2.3 that the state $q \in \mathbb{R}^{2N}$ such that $z^\star = (\Phi_i(q_i))_{i\in\mathcal{N}}$ is an equilibrium of the closed-loop system (2.1)–(2.3) with input (2.22), (2.23). Furthermore, because $F(z^{\star 1}) = 0$, by Proposition 2.2, q is a Nash equilibrium of the game $\mathcal{G}(J)$. By Assumption 2.2, there exists a unique Nash equilibrium of the game $\mathcal{G}(J)$ and hence $q = x^\star$. Thus, we conclude that $\lim_{t\to\infty} x(t) = x^\star$. Because x_0 and t_0 are arbitrary, it follows that the solution $x(t) \equiv x^\star$ is globally asymptotically stable. □

2.3.2 Nesterov's gradient input

In [53], Su, Boyd, and Candès analyzed the continuous-time version of Nesterov's accelerated gradient method [29] for convex optimization. In [43], Attouch and Peypouquet generalized the continuous-time Nesterov's dynamics for *maximal monotone operator* $B : \mathcal{H} \to \mathcal{H}$ in the real Hilbert space $\mathcal{H}$ by considering the dynamics

$$\ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + B_{\hat{\lambda}(t)}(x(t)) = 0, x(t_0) = x_0, \quad \dot{x}(t_0) = v_0, \quad t \geq t_0 > 0, \tag{2.29}$$

where $B_w(x),\ w > 0$, is the Yosida regularization of the operator $B(x)$ given by

$$B_w(x) = \frac{1}{w}\big(x - (I + wB)^{-1}(x)\big),$$

$I : \mathcal{H} \to \mathcal{H}$ is the identity operator in the real Hilbert space $\mathcal{H}$, and $\hat{\lambda}(t) : [t_0, \infty) \to (0, \infty)$ is a continuous function.

For our purpose, we consider a slightly modified version of (2.29) in order to comply with the assumption that the input of each agent is uncoupled. Specifically, let $B : \mathcal{H} \to \mathcal{H}$ be a λ-*cocoercive function* in the real Hilbert space $\mathcal{H}$ and consider the dynamics given by

$$\ddot{x}(t) + \frac{\alpha}{t}\dot{x}(t) + \mu(t)B(x(t)) = 0, x(t_0) = x_0, \quad \dot{x}(t_0) = v_0, \quad t \geq t_0 > 0, \tag{2.30}$$

where $\mu : [t_0, \infty) \to (0, \infty)$ is a continuous function. Following the results given in [43], below are several properties of dynamics (2.30) that we need.

Lemma 2.9. *Consider the dynamics* (2.30) *with* $t_0 > 0$ *and let* $p \in \mathcal{H}$ *such that* $B(p) = 0$. *Define* $h_p : [t_0, \infty) \to [0, \infty)$ *by*

$$h_p(t) \triangleq \frac{1}{2}\|x(t) - p\|^2, \tag{2.31}$$

where $x(t)$ *is the solution to* (2.30). *Then, for all* $t \geq t_0$*, it follows that*

$$\ddot{h}_p(t) + \frac{\alpha}{t}\dot{h}_p(t) + \mu(t)\lambda\|B(x(t))\|^2 \leq \|\dot{x}(t)\|^2. \tag{2.32}$$

Proof. The proof is similar to [43, Lemma 2.2]. □

Lemma 2.10. *Consider the dynamics* (2.30) *with* $t_0 > 0$. *Letting*

$$\alpha > 2, \quad \mu(t) = C\frac{1}{1+\epsilon}\frac{\alpha^2}{t^2}, \tag{2.33}$$

where

$$\epsilon > \frac{2}{\alpha - 2}, \quad \lambda > C, \tag{2.34}$$

it follows that

1. *there exists* $D_1 > 0$ *such that* $\|x(t)\| \leq D_1 < \infty$ *for all* $t \geq t_0$*,*

2. *there exists* $D_2 > 0$ *such that* $\|\dot{x}(t)\| \leq \frac{D_2}{t}$ *for all* $t \geq t_0$*,*

3. *there exists* $D_3 > 0$ *such that*

$$\int_{t_0}^{\infty} t\|\dot{x}(t)\|^2 \mathrm{d}t \leq D_3. \tag{2.35}$$

Proof. The proof is similar to [43, Proposition 2.4]. □

Now we state our result using the above results concerning Nesterov's gradient dynamics.

Theorem 2.11. *Consider the game* $\mathcal{G}(J)$ *given by* (2.1)–(2.3) *and suppose that* $G(y) = -F(y)$ *is* λ*-cocoercive. If the input for each agent is given by* (2.22) *with*

$$v_i(t) = -\frac{\alpha}{t} L_{f_i} h_i(x_i(t)) + \mu(t) \frac{\partial J_i(y(t))}{\partial y_i}, \tag{2.36}$$

(2.33)*, and* (2.34)*, then* $\lim_{t\to\infty} x(t) = x^\star$*.*

Proof. With the same reasoning as in the beginning of the proof of Theorem 2.8, the closed-loop dynamics of all agents can be written as

$$\ddot{z}^1(t) + \frac{\alpha}{t}\dot{z}^1(t) + \mu(t) G(z^1(t)) = 0, \quad z(t_0) = z_0, \quad t \geq t_0. \tag{2.37}$$

Note that (2.37) is a particular case of (2.30) since $G(z^1)$ is λ-cocoercive. Hence, it follows from Lemma 2.10 that $\lim_{t\to\infty} \dot{z}^1(t) = \lim_{t\to\infty} z^2(t) = 0$.

Now, for $p \in \mathbb{R}^N$ such that $G(p) = 0$, we show that $\lim_{t\to\infty} z^1(t) = p$. We follow the proof of [43, Theorem 2.1] with some modifications. Define $h_p(t)$ as in (2.31). Then, by Lemma 2.9,

$$t\ddot{h}_p(t) + \alpha \dot{h}_p(t) + t\mu(t)\lambda \left\|G(z^1(t))\right\|^2 \leq t\left\|\dot{z}^1(t)\right\|^2. \tag{2.38}$$

First, note that $h_p(t)$ is bounded below by 0. Furthermore, by Lemma 2.10, the integral $\int_{t_0}^{\infty} t\|\dot{z}^1(t)\|^2 \mathrm{d}t < \infty$. Thus, we can conclude from [43, Lemma A6] that $\lim_{t\to\infty} h_p(t)$ exists and that

$$\int_{t_0}^{\infty} t\mu(t)\lambda \left\|G(z^1(t))\right\|^2 \mathrm{d}t = K \int_{t_0}^{\infty} \frac{1}{t} \left\|G(z^1(t))\right\|^2 \mathrm{d}t < \infty, \tag{2.39}$$

where $K = \frac{C\lambda\alpha^2}{1+\epsilon}$.

We now show that (2.39) implies $\lim_{t\to\infty} \|G(z^1(t))\|^2 = 0$. First, because $G(y)$ is λ-cocoercive, it is Lipschitz continuous with Lipschitz constant $\frac{1}{\lambda}$ (Remark 2.5). Furthermore, $z^1(t)$ is also Lipschitz continuous because $\dot{z}^1(t)$ is bounded (Lemma 2.10). Thus, $G(z^1(t))$ is Lipschitz continuous as a function of t. Hence, by Rademacher's theorem [54, Theorem 3.2], $G(z^1(t))$ is differentiable for a.e. $t \geq t_0$. Note that for $t \geq t_0$ such that $\dot{G}(z^1(t))$ exists, $\dot{G}(z^1(t)) = \lim_{s\to t} \frac{G(z^1(s))-G(z^1(t))}{s-t}$. Now by Lipschitz continuity of $G(y)$, we have

$$\left\|G(z^1(t)) - G(z^1(s))\right\| \leq \frac{1}{\lambda}\left\|z^1(t) - z^1(s)\right\|, \quad t, s \geq t_0. \tag{2.40}$$

Dividing both sides of (2.40) by $\|t-s\|$ with $t \neq s$ and letting $s \to t$, we arrive at $\left\|\dot{G}(z^1(t))\right\| \leq \frac{1}{\lambda}\|\dot{z}^1(t)\|$ for a.e. $t \geq t_0$ such that $\dot{G}(z^1(t))$ exists. By Lemma 2.10, we have $\|\dot{z}^1(t)\| \leq \frac{D_2}{t}$ for some $D_2 > 0$ so that $\left\|\dot{G}(z^1(t))\right\| \leq \frac{1}{\lambda}\frac{D_2}{t}$ for a.e. $t \geq t_0$.

Let $w(t) \triangleq \|G(z^1(t))\|^2$. First, we show that $w(t)$ is differentiable for a.e. $t \geq t_0$. We have for $t, s \geq t_0$,

$$\begin{aligned}
|w(t) - w(s)| &= \left|\left\|G(z^1(t))\right\|^2 - \left\|G(z^1(s))\right\|^2\right| \\
&= \left|\left(\left\|G(z^1(t))\right\| - \left\|G(z^1(s))\right\|\right)\right| \cdot \left|\left(\left\|G(z^1(t))\right\| + \left\|G(z^1(s))\right\|\right)\right| \\
&\leq \left\|G(z^1(t)) - G(z^1(s))\right\| \cdot \left|\left(\left\|G(z^1(t)) - G(p)\right\| + \left\|G(z^1(s)) - G(p)\right\|\right)\right| \\
&\leq \frac{K_2}{\lambda}|t-s|\left(\left\|z^1(t) - p\right\| + \left\|z^1(s) - p\right\|\right) \\
&\leq \frac{2K_2K_3}{\lambda}|t-s|,
\end{aligned}$$

where K_2 is the Lipschitz constant of $G(z^1(t))$ as a function of t, K_3 is an upper bound of $h_p(t)$, and p satisfies $G(p) = 0$. Thus, $w(t)$ is Lipschitz continuous and hence again by Rademacher's theorem, it is differentiable for a.e. $t \geq t_0$. Note that for $t \geq t_0$ such that the derivative $\dot{w}(t)$ exists, $\dot{w}(t) = 2\left\langle G(z^1(t)), \dot{G}(z^1(t))\right\rangle$ and hence

$$\begin{aligned}
|\dot{w}(t)| &= 2\left|\left\langle G(z^1(t)), \dot{G}(z^1(t))\right\rangle\right| \\
&\leq 2\left\|G(z^1(t))\right\|\left\|\dot{G}(z^1(t))\right\| \\
&\leq \frac{2}{\lambda}\left\|z^1(t) - p\right\|\left\|\dot{G}(z^1(t))\right\| \\
&\leq \frac{2}{\lambda^2}\frac{D_1D_2}{t} \triangleq \eta(t), \quad \text{a.e. } t \geq t_0.
\end{aligned} \tag{2.41}$$

Because $w(t), \eta(t)$ are nonnegative functions that are differentiable for a.e. $t \geq t_0$, $\eta(t) \notin L^1(t_0, \infty)$, and inequalities (2.39) and (2.41) hold, it follows from [43, Lemma A5] that $\lim_{t\to\infty} w(t) = \lim_{t\to\infty} \|G(z^1(t))\|^2 = 0$.

Now, because $\sup_{t\geq t_0} z^1(t) < \infty$, there exists a sequence $\{t_n\}, t_n \to \infty$, such that $\lim_{n\to\infty} z^1(t_n) = q$ for some $q \in \mathbb{R}^N$. Note that continuity of $G(y)$ necessarily implies $G(q) = 0$ and because the Nash equilibrium is unique (Assumption 2.2), it follows that $q = p$. Thus, by (2.31), it follows that $\lim_{n\to\infty} h_p(t_n) = 0$. Because $h_p(t)$ has a limit, we deduce that $\lim_{t\to\infty} h_p(t) = 0$. Consequently, $\lim_{t\to\infty} z^1(t) = p$. Then, by the same reasoning as the last paragraph of the proof of Theorem 2.8, we conclude that $\lim_{t\to\infty} x(t) = x^\star$. □

2.4 Numerical Example

To illustrate Theorems 2.8 and 2.11, we consider the following game by two agents with second-order dynamics satisfying the followings:

1. agent 1's state evolves according to the dynamics

$$\begin{aligned}\dot{x}_1^1(t) &= x_1^2(t),\\ \dot{x}_1^2(t) &= \sinh(x_1^1(t)) + u_1(t),\end{aligned}$$

   with $y_1(t) = x_1^1(t)$ and the payoff function given by

$$J_1(y) \triangleq -y^{\mathrm{T}} \begin{bmatrix} 1 & -1/2 \\ -1/2 & 2/3 \end{bmatrix} y + y^{\mathrm{T}} \begin{bmatrix} 4 \\ 3 \end{bmatrix};$$

2. agent 2's state evolves according to the dynamics

$$\begin{aligned}\dot{x}_2^1(t) &= x_2^2(t),\\ \dot{x}_2^2(t) &= 3(x_2^1(t))^3 - u_2(t),\end{aligned}$$

   with $y_2(t) = x_2^1(t)$ and the payoff function given by

$$J_2(y) \triangleq -y^{\mathrm{T}} \begin{bmatrix} 4 & -1/2 \\ -1/2 & 1 \end{bmatrix} y + y^{\mathrm{T}} \begin{bmatrix} 2 \\ -5 \end{bmatrix}.$$

In this setting, it can be shown that Assumption 2.1 is satisfied. Moreover, the pseudo-gradient is given by

$$F(y) = -\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} y + \begin{bmatrix} 4 \\ -5 \end{bmatrix},$$

so that $-F(y)$ is λ-cocoercive with $\lambda = \frac{1}{3}$. Furthermore, $x^\star = [1, 0, -2, 0]^{\mathrm{T}}$ is the unique Nash equilibrium for the closed-loop system (2.1)–(2.3) with the inputs (2.22), (2.23) and (2.22), (2.36).

2.4.1 Heavy ball with friction input

The heavy ball with friction input is the vanishing damping input with $\alpha(t) \equiv \gamma$ and $\beta(t) \equiv 1$ satisfying (2.17), i.e., $\gamma^2\lambda > 1$. In this case, it follows from Theorem 2.8 that the solution $x(t) \equiv x^\star$ of the closed-loop system (2.1)–(2.3), (2.22), (2.23) is asymptotically stable. Figure 2.1 shows the trajectories of both agents $x_1(t)$ and $x_2(t)$.

2.4.2 Vanishing damping input

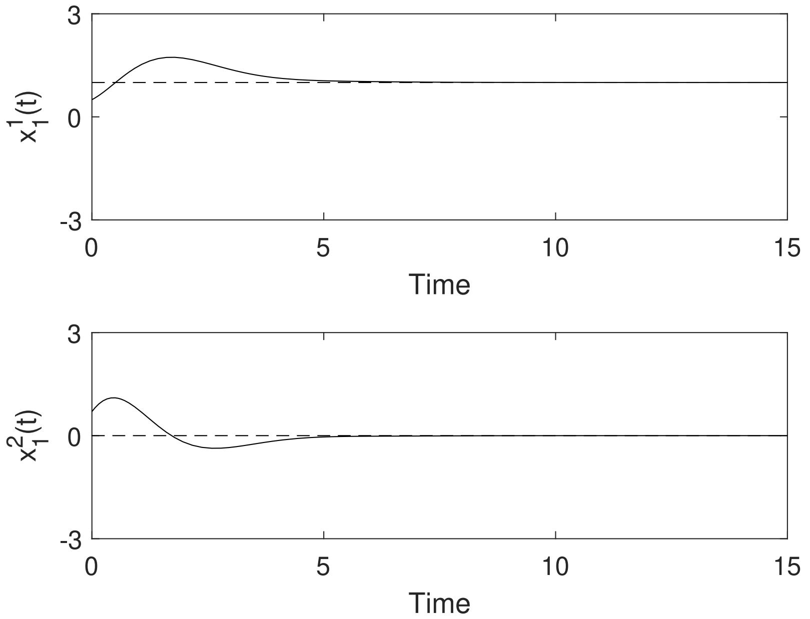
For vanishing damping input, we use the functions $\alpha(t)$ and $\beta(t)$ in (2.19) with parameters $a = 3,\ b = 2,\ c = 2,\ d = \frac{7}{4},\ \rho =\ 5$, and $\delta = 5$ satisfying (2.17). Hence, it follows from Theorem 2.8 that the the solution $x(t) \equiv x^\star$ of the closed-loop system (2.1)–(2.3), (2.22), (2.23) is asymptotically stable. Figure 2.2 shows the trajectories of both agents $x_1(t)$ and $x_2(t)$.

2.4.3 Nesterov's gradient input

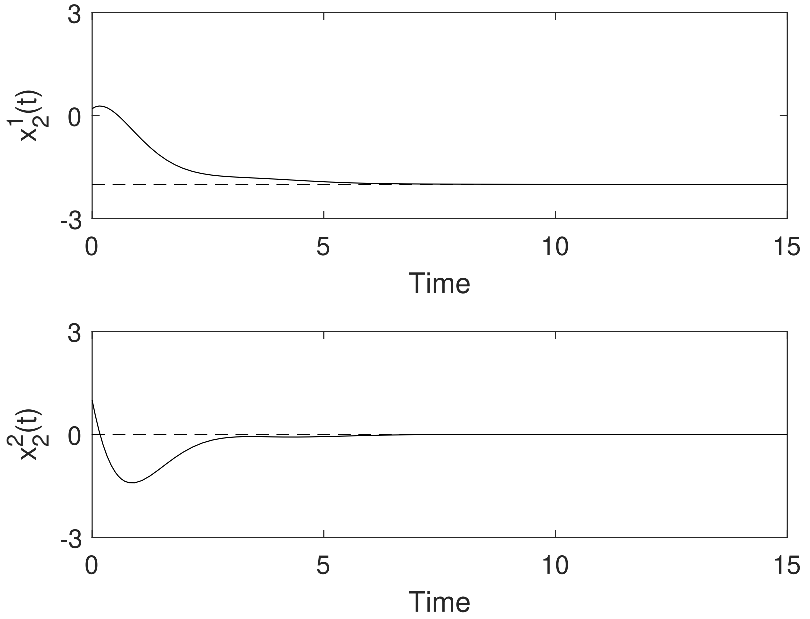
For Nesterov's gradient input, we start the dynamics from $t_0 = 1$ with parameters $\alpha = 8, C = \frac{4}{15}$, and $\epsilon = \frac{4}{3}$ satisfying (2.33) and (2.34). Thus, it follows from Theorem 2.11 that the trajectory converges to the Nash equilibrium of the closed-loop system (2.1)–(2.3), (2.22), (2.36). Figure 2.3 shows the trajectories of both agents $x_1(t)$ and $x_2(t)$.

2.5 Chapter Summary

We explored a game with second-order dynamic agents where each agent has a concave and continuously differentiable payoff function. From the results in convex optimization in the literature, we proposed several inputs that depend only on the knowledge of

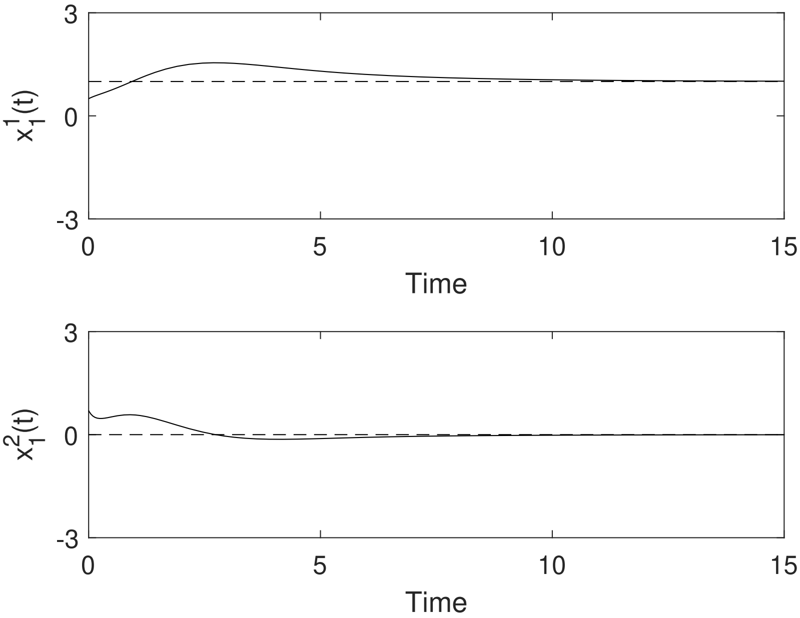


(a) State of agent 1 using heavy ball input

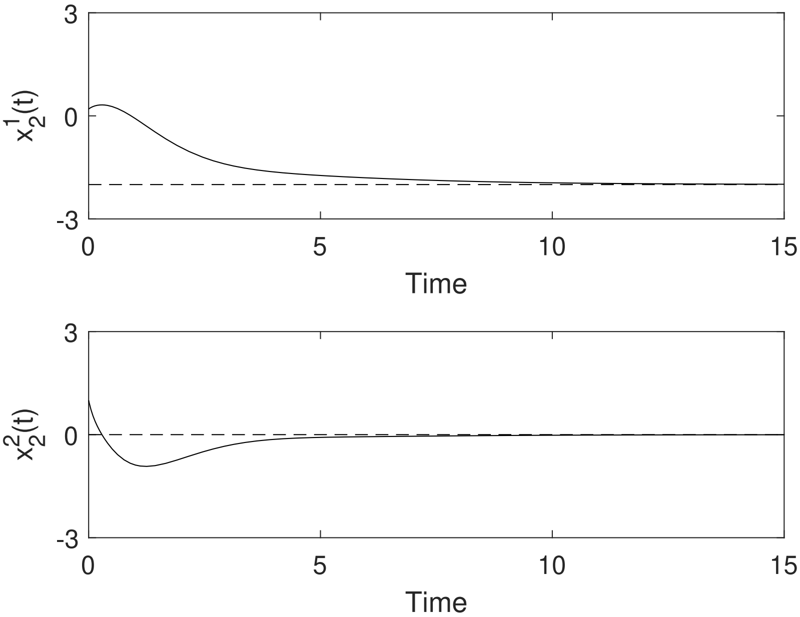


(b) State of agent 2 using heavy ball input

Fig. 2.1 The trajectories of the agents' using heavy ball input with the Nash equilibrium indicated by the dashed line

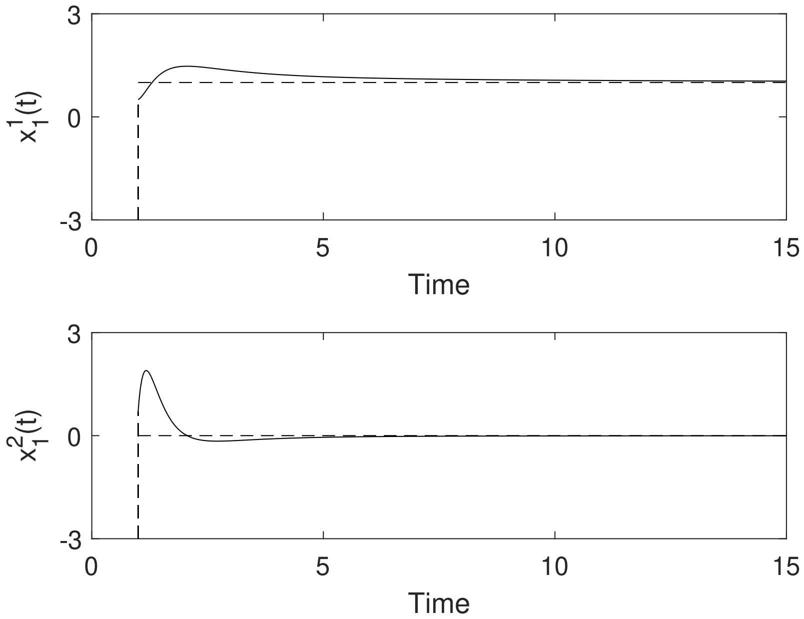


(a) State of agent 1 using vanishing damping input

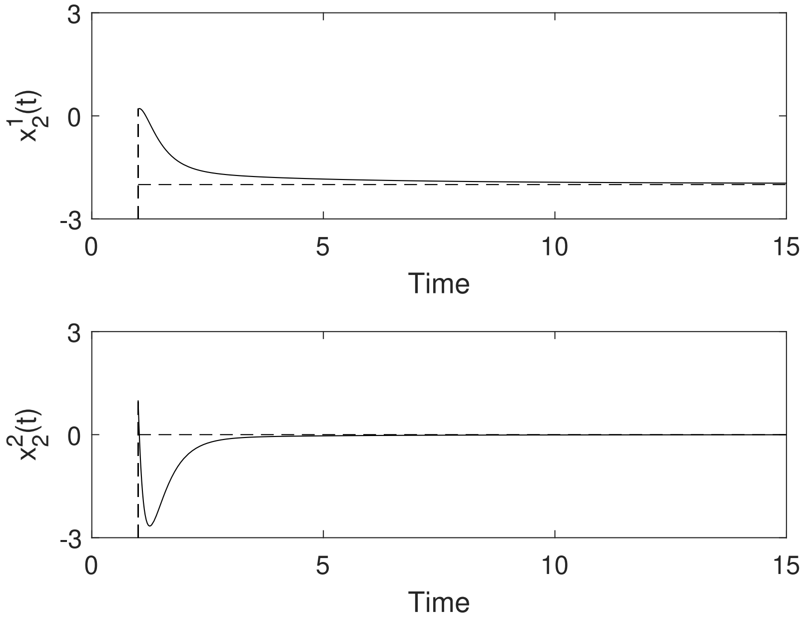


(b) State of agent 2 using vanishing damping input

Fig. 2.2 The trajectories of the agents using vanishing damping input with the Nash equilibrium indicated by the dashed line



(a) State of agent 1 using Nesterov's gradient input



(b) State of agent 2 using Nesterov's gradient input

Fig. 2.3 The trajectories of the agents using Nesterov's gradient input with the Nash equilibrium indicated by the dashed line and $t_0 = 1$

each agent's partial derivative of payoff function. We showed that the proposed input schemes guarantee convergence to the Nash equlibrium of the game with second-order dynamic agents.

Chapter 3

Nash Equilibrium Seeking in a Game by Linear Time-Invariant Dynamic Agents

3.1 Introduction

Building the work developed in Chapter 2, in this chapter, we consider the problem of Nash equilibrium seeking dynamics where each agent evolves under linear time-invariant dynamics. As in Chapter 2, we do not deal with distributed or decentralized dynamics. However, because in this chapter we deal with agents evolving under linear time-invariant dynamics, we need to assume strong monotonicity of the pseudo-gradient. Note that the dynamics considered in [35–37, 39, 38] are higher-order integrator dynamics and hence a subclass of linear time-invariant dynamics. In this chapter, we derive sufficient conditions for the existence and uniqueness of the Nash equilibrium for game with linear dynamic agents. This condition includes a class of nonminimum phase linear time-invariant system. Thus, our method can handle a class of dynamics that can not be handled by the approach of [55] that only deals with passive dynamics. Following [56], by designing new output such that each agent is strictly passive with respect to this new output, we derive input that asymptotically stabilize the Nash equilibrium of the game. We also illustrate our result with numerical example. Furthermore, we show that if a linear time-invariant system can be made passive with dynamic feedback with respect to the original output, then it is necessarily minimum phase and has relative degree 1 from the original input to the original output. This result complements the result of Johansson and Robertsson [57] that established Kalman-Yakubovich-Popov lemma for linear time-invariant system with dynamic feedback.

The contents of this chapter are as follows. In Section 3.2, we review the notion of passivity and incremental passivity for linear time-invariant system. In Section 3.3, we introduce the notion of a game by linear dynamic agents and characterize the Nash equilibrium of game with linear dynamic agents. In Section 3.4, we derive the input that guarantee convergence to the Nash equilibrium for game with linear dynamic agents. In Section 3.5, we give numerical example to illustrate our result. In Section 3.6, we show that if it is possible to make a linear time-invariant system passive with respect to its output with dynamic feedback, then necessarily the system is minimum phase and has relative degree 1 from its input to its output. Finally, we summarize this chapter in Section 3.7.

3.2 Mathematical Preliminaries

3.2.1 Strictly passive linear system

Consider the linear time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) \triangleq x_0, \quad t \geq 0 \tag{3.1}$$

$$y(t) = Cx(t), \tag{3.2}$$

where $x(t) \in \mathbb{R}^n, u(t), y(t) \in \mathbb{R}$, (A, B) is controllable and (C, A) is observable. The transfer function of the linear time-invariant system (3.1), (3.2) is

$$H(s) \triangleq C(sI_n - A)^{-1}B = \frac{\alpha_{n-1}s^{n-1} + \cdots + \alpha_0}{s^n + \beta_{n-1}s^{n-1} + \cdots + \beta_0}. \tag{3.3}$$

The linear time-invariant system (3.1), (3.2) is *strictly passive* [58, Definition 6.3] if there exists a continuously differentiable positive semidefinite function $V(x)$ such that $\frac{\partial V(x)}{\partial x}(Ax + Bu) \leq uy - \psi(x)$ for $x \in \mathbb{R}^n, u \in \mathbb{R}$, and $y \in \mathbb{R}$, where $\psi(x)$ is a positive definite function. A necessary and sufficient condition for strict passivity of linear time-invariant system is provided by the Kalman-Yakubovich-Popov Lemma [58, Lemma 6.3 and Lemma 6.4]:

Lemma 3.1 (KYP Lemma)**.** *The linear time-invariant system* (3.1), (3.2) *is strictly passive if there exist symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$, a matrix $L \in \mathbb{R}^{n\times n}$,* *and* $\epsilon > 0$ *such that*

$$A^{\mathrm{T}}P + PA = -L^{\mathrm{T}}L - \epsilon P, \tag{3.4}$$
$$B^{\mathrm{T}}P = C. \tag{3.5}$$

The linear time-invariant system (3.1), (3.2) is *strictly incrementally passive* if there exists a continuously differentiable function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that

$$V(x_1, x_2) > 0, \quad x_1 \neq x_2, \tag{3.6}$$
$$V(x, x) = 0, \tag{3.7}$$
$$\begin{aligned}\frac{\partial V(x_1,x_2)}{\partial x_1}(Ax_1 + Bu_1) + \frac{\partial V(x_1,x_2)}{\partial x_2}(Ax_2 + Bu_2)&\\ \leq (u_1 - u_2)(y_1 - y_2) - \psi(x_1, x_2),&\end{aligned} \tag{3.8}$$

where $\psi(x_1, x_2)$ satisfies $\psi(x_1, x_2) > 0,\ x_1 \neq x_2,\ x_1, x_2 \in \mathbb{R}^n$ and $\psi(x, x) = 0,\ x \in \mathbb{R}^n$. For linear time-invariant systems, (strict) passivity and (strict) incremental passivity are equivalent [59].

3.2.2 Relative degree and transmission zeros of linear system

In this subsection, we follow the terminology of [60, Chapter 1]. Consider the linear time-invariant system (3.1), (3.2). The *relative degree* $r > 0$ is the least integer such that $CA^{r-1}B \neq 0$. The *transmission zero* is defined as the zeros of the polynomial

$$T(s) \triangleq \det \begin{bmatrix} A - sI_n & B \\ C & 0 \end{bmatrix}.$$

The linear time-invariant system (3.1), (3.2) is *minimum-phase* if all of its transmission zeros have *negative* real part. If the linear time-invariant system (3.1), (3.2) is assumed to have relative degree 1, the following theorem is useful for checking whether the linear time-invariant system (3.1) has minimum phase.

Lemma 3.2 ([61, Theorem 1])**.** *Suppose that the linear time-invariant system* (3.1), (3.2) *has relative degree 1. Let*

$$A_\star \triangleq \big(\Psi - \Psi B(CB)^{-1}C\big)A\Psi^{\mathrm{T}}, \tag{3.9}$$

where $\Psi \in \mathbb{R}^{(n-1)\times n}$ satisfies $C\Psi^{\mathrm{T}} = 0$ and $\Psi\Psi^{\mathrm{T}} = I_{n-1}$. The linear time-invariant system (3.1)*,* (3.2) *is minimum phase if and only if all eigenvalues of $A_\star$ have negative real parts.*

3.3 Game by Linear Time-Invariant Dynamic Agents

Consider the finite set $\mathcal{N}$ of N agents where each agent $i \in \mathcal{N}$ has a state $x_i \in \mathbb{R}^n$. Note that the total state space of all agents is $\mathbb{R}^{Nn}$ and we call $x \in \mathbb{R}^{Nn}$ the state profile of all agents. Let the agents' state evolve under the linear time-invariant continuous-time dynamics with scalar input and scalar output

$$\dot{x}_i(t) = A_i x_i(t) + B_i u_i(t), \quad x_i(0) \triangleq x_{i0}, \quad t \geq 0, \quad i \in \mathcal{N}, \tag{3.10}$$

$$y_i(t) = C_i x_i(t), \quad i \in \mathcal{N}, \tag{3.11}$$

where $A_i \in \mathbb{R}^{n\times n}, B_i \in \mathbb{R}^n, C_i \in \mathbb{R}^{1\times n}$, $u_i(t) \in \mathbb{R}$ is the input of agent i, and $y_i(t) \in \mathbb{R}$ is the output of agent i. We assume that (A_i, B_i) is controllable and (C_i, A_i) is observable for $i \in \mathcal{N}$. Furthermore, following [62], we assume that the $(n + 1) \times (n + 1)$-dimensional matrix

$$\mathcal{A}_i \triangleq \begin{bmatrix} A_i & B_i \\ C_i & 0 \end{bmatrix}, \quad i \in \mathcal{N}, \tag{3.12}$$

is full-rank and hence invertible.

Suppose that each agent i has a payoff function $J_i : \mathbb{R}^N \to \mathbb{R}$, $(y_i, y_{-i}) \mapsto J_i(y_i, y_{-i})$, and wants to increase her own payoff. We assume that each agent knows her own payoff function, but not the other agents' payoff functions. Furthermore, the payoff functions $J_i(y_i, y_{-i}),\ i \in \mathcal{N}$, satisfy the following concavity and differentiability conditions.

Assumption 3.1. *The payoff functions $J_i(y_i, y_{-i}),\ i \in \mathcal{N}$, are continuously differentiable in $\mathbb{R}^N$ and concave in y_i for every fixed $y_{-i} \in \mathbb{R}^{N-1}$.*

We call the game described by (3.10), (3.11) and satisfying Assumption 3.1 as *game with linear dynamic agents*. We denote this game by $\mathcal{G}(J)$, where $J = (J_i)_{i\in\mathcal{N}}$.

We assume that each agent i is allowed to broadcast a scalar $w_i \in \mathbb{R}$ to other agents. It is important to note that even though agents are allowed to broadcast a scalar, the game is still a noncooperative game. In game theory, the distinction between noncooperative game and cooperative game is in whether agents are allowed to make agreement between them [63, Chapter 15–20]. In noncooperative game theory, even though agents communicate or receive some signals or recommendations, agents are in principle *free* to act or not to act on the information. In contrast, in cooperative game, if the agents are in agreement, their actions are *restricted by the agreement*.

The input of agent i is given by

$$u_i(t) = r\frac{\partial J_i(w_i(t), w_{-i}(t))}{\partial y_i} + q_i w_i(t) - K_i x_i(t), \tag{3.13}$$

$$w_i(t) = B_i^{\mathrm{T}} P_i x_i(t), \tag{3.14}$$

where $K_i, P_i \in \mathbb{R}^{n\times n}$, $q_i, r \in \mathbb{R}$, and $\frac{\partial J_i(w_i,w_{-i})}{\partial y_i} \triangleq \frac{\partial J_i(y_i,y_{-i})}{\partial y_i}|_{y_i=w_i, y_{-i}=w_{-i}}$.

Remark 3.3. *If the agents are not allowed to communicate with each other, an alternative information structure is as follows. Suppose there exists a central operator that is accessible to all agents $i \in \mathcal{N}$. At each time t, each agent i submits $w_i(t) \in \mathbb{R}$ to the central operator and then the central operator broadcasts $w(t) \triangleq [w_1(t), \ldots, w_N(t)]^{\mathrm{T}} \in \mathbb{R}^N$ to all agents. This information structure can be found in many contexts, for examples: i) in Nash equilibrium seeking for network aggregative games [64]; ii) in game theoretic method for frequency control in power system [65]; iii) in dynamic congestion control [66]; iv) in charging control of electric vehicle [67]; v) in rate allocation in wireless networks [68], or v) in autonomous demand-side management scheduling for smart grid [17].*

Now, we define the Nash equilibrium of this game with linear time-invariant continuous-time dynamics for each agent.

Definition 3.4. *Let $x^\star \in \mathbb{R}^{Nn}$ and $y^\star \triangleq (C_i x_i^\star)_{i\in\mathcal{N}}$. Then, $x^\star$ is a* Nash equilibrium *of the closed-loop system* (3.10), (3.11) *with input* (3.13), (3.14) *along with the payoff functions $J_i(y)$, $i \in \mathcal{N}$, if*

1. *$x^\star$ satisfies the inequalities*

$$J_i(y_i^\star, y_{-i}^\star) \geq J_i(C_i x_i, y_{-i}^\star), \quad x_i \in \mathbb{R}^n, \quad i \in \mathcal{N}, \tag{3.15}$$

and

2. *$x^\star$ is a closed-loop equilibrium, that is,*

$$A_i x_i^\star + rB_i\frac{\partial J_i(w_i^\star, w_{-i}^\star)}{\partial y_i} + q_i B_i w_i^\star - B_i K_i x_i^\star = 0, \quad i \in \mathcal{N}, \tag{3.16}$$

$$w_i^\star - B_i^{\mathrm{T}} P_i x_i^\star = 0, \quad i \in \mathcal{N}, \tag{3.17}$$

$$y_i^\star = w_i^\star, \quad i \in \mathcal{N}. \tag{3.18}$$

From Assumption 3.1, we have the following characterization of the Nash equilibrium of this game.

Proposition 3.5. *Consider the game $\mathcal{G}(J)$ described by* (3.10), (3.11) *with input* (3.13), (3.14) *satisfying Assumption 3.1. Let $x^\star \in \mathbb{R}^{Nn}$ and $y^\star = (C_i x_i^\star)_{i\in\mathcal{N}}$. Then $x^\star \in \mathbb{R}^{Nn}$ is a Nash equilibrium if and only if*

$$\frac{\partial J_i(y_i^\star, y_{-i}^\star)}{\partial y_i} = 0, \quad i \in \mathcal{N}, \tag{3.19}$$

$$A_i x_i^\star + q_i B_i y_i^\star - B_i K_i x_i^\star = 0, \quad i \in \mathcal{N}, \tag{3.20}$$

$$C_i x_i^\star - B_i^{\mathrm{T}} P_i x_i^\star = 0, \quad i \in \mathcal{N}. \tag{3.21}$$

Proof. From [46, Proposition 1.4.2], inequality (3.15) holds if and only if (3.19) holds. Conditions (3.20)–(3.21) are restatement of conditions (3.16)–(3.18). □

Remark 3.6. *Condition* (3.21) *does not necessarily imply $C_i = B_i^{\mathrm{T}} P_i$, $i \in \mathcal{N}$, because* (3.21) *only needs to hold for $x^\star \in \mathbb{R}^{Nn}$. For example, for $N = 1$, if $x^\star = [1\ 1]^{\mathrm{T}}$ and $C = [0\ 2]$, then with $B^{\mathrm{T}} P = [1\ 1]$,* (3.21) *holds, but $C \neq B^{\mathrm{T}} P$.*

3.3.1 Nash Equilibrium of the Game $\mathcal{G}(J)$

Proposition 3.5 provides a characterization for a Nash equilibrium of the game $\mathcal{G}(J)$. Suppose that there exists $y^\star \triangleq [y_1^\star, \ldots, y_N^\star]^{\mathrm{T}} \in \mathbb{R}^N$ such that (3.19) holds. The following lemma gives a sufficient condition for the existence and uniqueness of $x^\star \in \mathbb{R}^{Nn}$ such that (3.20), (3.21) hold.

Lemma 3.7. *Consider the game $\mathcal{G}(J)$ described by* (3.10), (3.11) *with input* (3.13), (3.14) *satisfying Assumption 3.1. Suppose there exists $y^\star \in \mathbb{R}^N$ such that* (3.19) *holds. If K_i, P_i, $i \in \mathcal{N}$, are chosen such that*

$$\hat{A}_i \triangleq A_i - B_i K_i \text{ is invertible}, \quad i \in \mathcal{N}, \tag{3.22}$$

$$C_i \hat{A}_i^{-1} B_i \neq 0, \quad i \in \mathcal{N}, \tag{3.23}$$

$$B_i^{\mathrm{T}} P_i \hat{A}_i^{-1} B_i = C_i \hat{A}_i^{-1} B_i, \quad i \in \mathcal{N}, \tag{3.24}$$

and

$$q_i \triangleq -\frac{1}{C_i \hat{A}_i^{-1} B_i}, \quad i \in \mathcal{N}, \tag{3.25}$$

then there exists a unique $x^\star \triangleq \left[x_1^{\star\mathrm{T}}, \ldots, x_N^{\star\mathrm{T}}\right]^{\mathrm{T}} \in \mathbb{R}^{Nn}$ *such that* (3.20), (3.21) *hold. Furthermore,*

$$x_i^\star = -q_i \hat{A}_i^{-1} B_i y_i^\star, \quad i \in \mathcal{N}. \tag{3.26}$$

Proof. We start by showing the existence of the Nash equilibrium. Because $\hat{A}_i,\ i \in \mathcal{N}$, are invertible, (3.20) holds if and only if (3.26) holds. Substituting $x_i^\star,\ i \in \mathcal{N}$, to (3.20) and (3.21), it follows that

$$y_i^\star \begin{bmatrix} 1 \\ 1 \end{bmatrix} = y_i^\star q_i \begin{bmatrix} -C_i \hat{A}_i^{-1} B_i \\ -B_i^{\mathrm{T}} P_i \hat{A}_i^{-1} B_i \end{bmatrix}, \quad i \in \mathcal{N}. \tag{3.27}$$

Thus, if (3.27) holds, $x^\star$ is a Nash equilibrium. If $P_i,\ i \in \mathcal{N}$, are chosen such that (3.23), (3.24) hold and $q_i,\ i \in \mathcal{N}$, are as defined in (3.25), then, by direct calculation, (3.27) holds. Hence $x^\star$ is a Nash equilibrium.

To show uniqueness, let $v_i^\star \triangleq q_i y_i^\star,\ i \in \mathcal{N}$. Then, (3.20) and (3.21) can be written as

$$\begin{bmatrix} 0 \\ y_i^\star \\ y_i^\star \end{bmatrix} = \hat{\mathcal{A}}_i \begin{bmatrix} x_i^\star \\ v_i^\star \end{bmatrix} \triangleq \begin{bmatrix} \hat{A}_i & B_i \\ C_i & 0 \\ B_i^{\mathrm{T}} P_i & 0 \end{bmatrix} \begin{bmatrix} x_i^\star \\ v_i^\star \end{bmatrix}, \quad i \in \mathcal{N}. \tag{3.28}$$

Because we assume that for $i \in \mathcal{N}$, rank $\mathcal{A}_i = (n+1)$, it follows that

$$\begin{aligned} \det \hat{\mathcal{A}}_i^1 &\triangleq \det \begin{bmatrix} \hat{A}_i & B_i \\ C_i & 0 \end{bmatrix} \\ &= \det \begin{bmatrix} A_i - B_i K_i & B_i \\ C_i & 0 \end{bmatrix} \\ &= \det \left(\begin{bmatrix} A_i & B_i \\ C_i & 0 \end{bmatrix} \begin{bmatrix} I_n & 0 \\ -K_i & 1 \end{bmatrix} \right) \\ &= \det \begin{bmatrix} A_i & B_i \\ C_i & 0 \end{bmatrix} \det \begin{bmatrix} I_n & 0 \\ -K_i & 1 \end{bmatrix} \\ &= \det \begin{bmatrix} A_i & B_i \\ C_i & 0 \end{bmatrix} \\ &\neq 0, \end{aligned}$$

for $i \in \mathcal{N}$, where the second to last equality follows because

$$\det \begin{bmatrix} I_n & 0 \\ -K_i & 1 \end{bmatrix} = 1, \quad i \in \mathcal{N}.$$

Hence, rank $\hat{\mathcal{A}}_i^1 = (n+1)$. Because $\begin{bmatrix} B_i^{\mathrm{T}} P_i & 0 \end{bmatrix} \in \mathbb{R}^{1\times(n+1)}$ and rank $\hat{\mathcal{A}}_i^1 = (n+1)$, it follows that $\begin{bmatrix} B_i^{\mathrm{T}} P_i & 0 \end{bmatrix}$ is in the rowspace of $\hat{\mathcal{A}}_i^1$. Thus, rank $\hat{\mathcal{A}}_i =$ rank $\hat{\mathcal{A}}_i^1 = n+1$, and hence the matrix $\hat{\mathcal{A}}_i$ is full column rank and necessarily injective. Because $[x_i^\star \;\; v_i^\star]$, $i \in \mathcal{N}$, satisfy (3.28) and $\hat{\mathcal{A}}_i$, $i \in \mathcal{N}$, are injective, $[x_i^\star \;\; v_i^\star]$, $i \in \mathcal{N}$, are necessarily unique. □

In Lemma 3.7, we can always choose K_i, $i \in \mathcal{N}$, such that (3.22) holds because (A_i, B_i), $i \in \mathcal{N}$, are controllable. In fact, we can always choose K_i, $i \in \mathcal{N}$, such that $\hat{A}_i$, $i \in \mathcal{N}$, in (3.22) are Hurwitz. However, it is not clear when conditions (3.23) and (3.24) hold. Furthermore, because we want to stabilize Nash equilibrium, our choice of P_i, $i \in \mathcal{N}$, is limited to symmetric positive-definite matrices that satisfy the Lyapunov inequalities $\hat{A}_i^{\mathrm{T}} P_i + P_i \hat{A}_i \prec 0$, $i \in \mathcal{N}$, where $\hat{A}_i$, $i \in \mathcal{N}$, are defined in (3.22). We need the following technical lemmas:

Lemma 3.8. *Consider the linear time-invariant system* (3.1), (3.2) *with transfer function $H(s)$* (3.3). *Suppose A is Hurwitz. Then, the following statements hold:*

1. $\beta_0 > 0$;
2. $CA^{-1}B = -\frac{\alpha_0}{\beta_0}$, *and*
3. *if P is symmetric positive-definite matrix such that*

$$A^{\mathrm{T}} P + PA \prec 0, \tag{3.29}$$

then $B^{\mathrm{T}} P A^{-1} B < 0$.

Proof. Item 1) is immediate. To see 2), because A is Hurwitz, $(sI_n - A)$ is invertible for any complex number s with nonnegative real part. Hence, $CA^{-1}B = -C(sI_n - A)^{-1}B|_{s=0} = -H(0) = -\frac{\alpha_0}{\beta_0}$. To see 3), let P be a symmetric positive-definite matrix such that (3.29) holds. Right multiplying by A^{-1} and left multiplying by A^{-T} on both sides of (3.29), it follows that $PA^{-1} + A^{-T}P \prec 0$. Because $B \in \mathbb{R}^n$, necessarily $B^{\mathrm{T}}PA^{-1}B + B^{\mathrm{T}}A^{-T}PB = 2B^{\mathrm{T}}PA^{-1}B < 0$ and hence $B^{\mathrm{T}}PA^{-1}B < 0$. □

From Lemma 3.8, we have the following:

Lemma 3.9. *Consider the linear time-invariant system* (3.1), (3.2) *with transfer function $H(s)$ given by* (3.3). *Suppose A is Hurwitz. Then, a symmetric positive-definite matrix P satisfying* (3.29) *and*

$$CA^{-1}B \neq 0, \tag{3.30}$$

$$B^{\mathrm{T}}PA^{-1}B = CA^{-1}B, \tag{3.31}$$

exists if and only if $\alpha_0 > 0$.

Proof. To show sufficiency, suppose that $\alpha_0 > 0$. Because A is Hurwitz, from Lemma 3.8, it follows that $CA^{-1}B < 0$. Furthermore, because A is Hurwitz, there exists a symmetric positive-definite matrix P such that (3.29) holds. Thus, from Lemma 3.8, it follows that $B^{\mathrm{T}}PA^{-1}B < 0$. Let $\hat{P} \triangleq \frac{CA^{-1}B}{B^{\mathrm{T}}PA^{-1}B}P$. Then, because $\frac{CA^{-1}B}{B^{\mathrm{T}}PA^{-1}B} > 0$, it follows that $A^{\mathrm{T}}\hat{P} + \hat{P}A = \frac{CA^{-1}B}{B^{\mathrm{T}}PA^{-1}B}\big(A^{\mathrm{T}}P + PA\big) \prec 0$. Furthermore, $B^{\mathrm{T}}\hat{P}A^{-1}B = \frac{CA^{-1}B}{B^{\mathrm{T}}PA^{-1}B}B^{\mathrm{T}}PA^{-1}B = CA^{-1}B$. Thus, $\hat{P}$ satisfies (3.29)–(3.31) simultaneously.

To show necessity, suppose there exists a symmetric positive-definite matrix P such that (3.29)–(3.31) hold. Then, because A is Hurwitz, from Lemma 3.8, it follows that $B^{\mathrm{T}}PA^{-1}B < 0$ and hence $CA^{-1}B < 0$. Furthermore, again from Lemma 3.8, it follows that $\alpha_0 = -\beta_0 CA^{-1}B$. Because from Lemma 3.8 we also have that $\beta_0 > 0$, it follows that $\alpha_0 > 0$. □

From Lemma 3.9, we have the following:

Proposition 3.10. *Consider the game* $\mathcal{G}(J)$ *described by* (3.10), (3.11) *with input* (3.13), (3.14) *satisfying Assumption 3.1. Suppose there exists a unique* $y^\star \in \mathbb{R}^N$ *such that* (3.20) *holds. Let*

$$H_i(s) \triangleq C_i(sI_n - A_i)^{-1}B_i = \frac{\alpha_{i(n-1)}s^{n-1} + \cdots + \alpha_{i0}}{s^n + \beta_{i(n-1)}s^{i(n-1)} + \cdots + \beta_{i0}}, \quad i \in \mathcal{N}, \tag{3.32}$$

be the agents' transfer functions and suppose that $\alpha_{i0} > 0$, $i \in \mathcal{N}$. *If* K_i, $i \in \mathcal{N}$, *are chosen such that* $\hat{A}_i$, $i \in \mathcal{N}$, *defined in* (3.22) *are Hurwitz and* q_i, $i \in \mathcal{N}$, *are defined as in* (3.25), *then* $x^\star \in \mathbb{R}^{Nn}$ *as defined in* (3.26) *is the unique Nash equilibrium of the game* $\mathcal{G}(J)$.

Proof. Let $v_i \triangleq r\frac{\partial J_i(w_i, w_{-i})}{\partial y_i} + q_i w_i$, $i \in \mathcal{N}$. Then, the interconnected system (3.10), (3.11), (3.13), (3.14) can be written as

$$\dot{x}_i(t) = (A_i - B_iK_i)x_i(t) + B_iv_i(t), \quad i \in \mathcal{N}, \tag{3.33}$$

$$w_i(t) = B_i^{\mathrm{T}}P_ix_i(t), \quad i \in \mathcal{N}, \tag{3.34}$$

$$y_i(t) = C_ix_i(t), \quad i \in \mathcal{N}. \tag{3.35}$$

Let

$$\begin{aligned}\hat{H}_i(s) &\triangleq C_i(sI_n - (A_i - B_iK_i))^{-1}B_i \\ &= \frac{\hat{\alpha}_{i(n-1)}s^{n-1} + \cdots + \hat{\alpha}_{i0}}{s^n + \hat{\beta}_{i(n-1)}s^{i(n-1)} + \cdots + \hat{\beta}_{i0}}, \quad i \in \mathcal{N},\end{aligned} \tag{3.36}$$

be the transfer function of the interconnected system (3.33)–(3.35) from v_i to y_i for $i \in \mathcal{N}$. If $\hat{\alpha}_{i0} > 0,\ i \in \mathcal{N}$, it follows from Lemmas 3.7 and 3.9 that $x^\star \in \mathbb{R}^{Nn}$ is the unique Nash equilibrium of the game $\mathcal{G}(J)$. Thus, it is sufficient to show that $\hat{\alpha}_{i0} > 0,\ i \in \mathcal{N}$.

To see that $\hat{\alpha}_{i0} > 0,\ i \in \mathcal{N}$, from [60, equation (2.11)], it follows that the numerators of $\hat{H}_i(s),\ i \in \mathcal{N}$, are

$$\begin{aligned}\hat{N}_i(s) &= \det \begin{bmatrix} A_i - B_iK_i - sI_n & B_i \\ C_i & 0 \end{bmatrix} \\ &= \det \left(\begin{bmatrix} A_i - sI_n & B_i \\ C_i & 0 \end{bmatrix} \begin{bmatrix} I_n & 0 \\ -K_i & 1 \end{bmatrix} \right) \\ &= \det \begin{bmatrix} A_i - sI_n & B_i \\ C_i & 0 \end{bmatrix} \det \begin{bmatrix} I_n & 0 \\ -K_i & 1 \end{bmatrix} \\ &= \det \begin{bmatrix} A_i - sI_n & B_i \\ C_i & 0 \end{bmatrix} \\ &= N_i(s), \quad i \in \mathcal{N},\end{aligned} \tag{3.37}$$

where $N_i(s),\ i \in \mathcal{N}$, are the numerator polynomials of the agents' transfer functions $H_i(s),\ i \in \mathcal{N}$, and the second to last inequality holds because

$$\det \begin{bmatrix} I_n & 0 \\ -K_i & 1 \end{bmatrix} = 1.$$

From (3.37) we have $\hat{N}_i(s) = N_i(s),\ i \in \mathcal{N}$, and hence $\hat{\alpha}_{ij} = \alpha_{ij},\ j = 1, \ldots, n-1,\ i \in \mathcal{N}$. In particular, it follows that $\hat{\alpha}_{i0} = \alpha_{i0} > 0,\ i \in \mathcal{N}$. Thus, from Lemmas 3.7 and 3.9, it follows that $x^\star$ is the unique Nash equilibrium of the game $\mathcal{G}(J)$. $\square$

3.4 Nash Equilibrium Seeking

In this section, we derive inputs that guarantee the convergence of the state profile of all agents to a Nash equilibrium for the game $\mathcal{G}(J)$ described by (3.10), (3.11), with input (3.13), (3.14) satisfying Assumption 3.1. We need the following proposition from [56]:

Proposition 3.11 ([56, Proposition 3.1]). *Consider the linear time-invariant system* (3.1), (3.2). *Let* $u \triangleq v - Kx$ *and* $w \triangleq B^{\mathrm{T}}Px$, *where* K *is chosen such that* $A - BK$ *is Hurwitz and* P *is a symmetric positive-definite matrix such that* $(A-BK)^{\mathrm{T}}P + P(A-BK) \prec 0$. *Then, the linear-time invariant system* (3.1), (3.2) *with input* u *is strictly incrementally passive from* v *to* w.

Before we state our main result, recall that the *pseudo-gradient* [26] $F : \mathbb{R}^N \to \mathbb{R}^N$ of the game $\mathcal{G}(J)$ is described by $F(y) \triangleq \left[\frac{\partial J_1(y)}{\partial y_1}, \ldots, \frac{\partial J_N(y)}{\partial y_N}\right]^{\mathrm{T}}$. We assume the following holds for the pseudo-gradient $F(y)$ of the game $\mathcal{G}$.

Assumption 3.2. *The pseudo-gradient* $F(y)$ *is strongly antimonotone [46, Definition 2.3.1], that is,* $\langle (F(y) - F(y')), y - y' \rangle \leq -\lambda \|y - y'\|^2$, *where* $\lambda > 0$ *is the strong antimonotonicity constant.*

Now we state our main result.

Theorem 3.12. *Consider the game* $\mathcal{G}(J)$ *described by* (3.10), (3.11) *with input* (3.13), (3.14) *satisfying Assumptions 3.1 and 3.2. Let* $H_i(s)$, $i \in \mathcal{N}$, *be the agents' transfer functions from* u_i *to* y_i *described by* (3.32) *and suppose that* $\alpha_{i0} > 0$, $i \in \mathcal{N}$. *Furthermore, suppose that the matrices* K_i, $i \in \mathcal{N}$, *are chosen such that* $A_i - B_iK_i$, $i \in \mathcal{N}$, *are Hurwitz, the symmetric positive-definite matrices* P_i, $i \in \mathcal{N}$, *are chosen such that* (3.23), (3.24) *hold, and* q_i, $i \in \mathcal{N}$, *are defined as in* (3.25). *If* $r > \max_{i \in \mathcal{N}} \frac{|q_i|}{\lambda}$, *then* $\lim_{t\to\infty} x(t) = x^\star$, *where* $x_i^\star \triangleq -q_i\hat{A}_i^{-1}B_iy_i^\star$, $i \in \mathcal{N}$, *is the unique Nash equilibrium of the game* $\mathcal{G}$.

Proof. Let $v_i \triangleq r\frac{\partial J_i(w_i, w_{-i})}{\partial y_i}$, $i \in \mathcal{N}$. Then, by Proposition 3.11, each agent i is strictly incrementally passive from v_i to w_i. Consider the Lyapunov function candidate $W(x) \triangleq \sum_{i\in\mathcal{N}} (x_i - x_i^\star)^{\mathrm{T}} P_i (x_i - x_i^\star)$. Then, because each agent is strictly incrementally passive, it follows that

$$
\begin{aligned}
\frac{\partial W(x)}{\partial x} f(x) &\leq \sum_{i\in\mathcal{N}} (v_i - v_i^\star)(w_i - w_i^\star) \\
&= \sum_{i\in\mathcal{N}} \left(r\frac{\partial J_i(w_i, w_{-i})}{\partial y_i} + q_i w_i - q_i w_i^\star \right)(w_i - w_i^\star) \\
&= \sum_{i\in\mathcal{N}} \left(r\frac{\partial J_i(w_i, w_{-i})}{\partial y_i} - r\frac{\partial J_i(y_i^\star, y_{-i}^\star)}{\partial y_i} \right)(w_i - w_i^\star) + \sum_{i\in\mathcal{N}} q_i (w_i - w_i^\star)^2 \\
&= r\langle w - w^\star, F(w) - F(w^\star)\rangle + \sum_{i\in\mathcal{N}} q_i (w_i - w_i^\star)^2 \\
&\leq -r\lambda \|w - w^\star\|^2 + \max_i |q_i| \|w - w^\star\|^2 < 0, \quad t \geq 0. \qquad (3.38)
\end{aligned}
$$

Hence $x^\star$ is Lyapunov stable.

To show asymptotic stability, from inequality (3.38), it follows that $\frac{\partial W(x)}{\partial x} f(x) = 0$ implies $w = w^\star = y^\star$. From [46, Theorem 2.3.3], by strong antimonotonicity of the pseudo-gradient $F(y)$ (Assumption 3.2), there exists a unique $y^\star$ such that (3.19) holds. Then, by Proposition 3.10, $x^\star$ is the unique equilibrium of the game $\mathcal{G}$. Hence, by Lasalle's invariance principle [58, Theorem 4.4], we conclude that $\lim_{t\to\infty} x(t) = x^\star$. $\square$

In Proposition 3.10, the existence and uniqueness of the Nash equilibrium of the game $\mathcal{G}(J)$ is related to the positivity of α_{i0}, $i \in \mathcal{N}$. Corollary 3.13 below provides sufficient condition for the existence and uniqueness of the Nash equilibrium in terms of the high-frequency gain and the transmission zeros of each agent.

Corollary 3.13. *Consider the game $\mathcal{G}(J)$ described by* (3.10), (3.11) *with input* (3.13), (3.14) *satisfying Assumption 3.1. Let $r_i > 0$ be the relative degree and $G_i \triangleq C_i A_i^{r_i-1} B_i$ be the high-frequency gain of agent $i \in \mathcal{N}$. Suppose that for each agent $i \in \mathcal{N}$, one of the following conditions holds.*

1. *$G_i > 0$, and agent i's dynamics has even number of transmission zeros, 0 is not a transmission zero, and there are even number of transmission zeros with positive real parts, or*

2. *$G_i < 0$ and agent i's dynamics has odd number of transmission zeros, 0 is not a transmission zero, and there are odd number of transmission zeros with positive real parts.*

Suppose also that there exists a unique $y^\star$ such that (3.19) *holds. If K_i, $i \in \mathcal{N}$, are chosen such that $\hat{A}_i$, $i \in \mathcal{N}$, defined in* (3.22) *are Hurwitz and q_i, $i \in \mathcal{N}$, are defined* *as in* (3.25)*, then* $x^\star \in \mathbb{R}^{Nn}$ *defined in* (3.26) *is the unique Nash equilibrium of the game* $\mathcal{G}(J)$*.*

Proof. Both conditions ensure that $\alpha_{i0} > 0,\ i \in \mathcal{N}$. Then, the result follows directly from Proposition 3.10. □

From Corollary 3.13, we see that if all agents' dynamics are minimum phase with positive high-frequency gain, then there exists a unique Nash equilibrium. However, from Corollary 3.13, we also see that the set of agents' dynamics that can be driven to the Nash equilibrium also include some classes of agents' dynamics that are nonminimum phase.

3.5 Numerical Example

To illustrate Theorem 3.12, we consider the following game with 2 linear dynamic agents. Suppose that the transfer function of agent 1 is given by

$$H_1(s) = \frac{(s-3)(s-6)}{(s-2)(s-1)(s+4)(s-5)}.$$

Note that the transfer function $H_1(s)$ has two zeros in the right hand plane and relative degree 2, so agent 1 is not passive and is nonminimum phase. A controllable and observable realization of $H_1(s)$ is given by, for example,

$$A_1 = \begin{bmatrix} 4 & 15 & -58 & 40 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} 0 & 1 & -9 & 18 \end{bmatrix}.$$

Suppose that the payoff of agent 1 is given by

$$J_1(y) = -y^{\mathrm{T}} \begin{bmatrix} 2 & 0.5 \\ 0.5 & 1 \end{bmatrix} y + \begin{bmatrix} 5 \\ 1 \end{bmatrix}^{\mathrm{T}} y.$$

Suppose that the transfer function of agent 2 is given by

$$H_2(s) = \frac{(s-4)(s-1)}{(s-3)(s-2)(s-5)(s-7)}.$$

The transfer function $H_2(s)$ has two zeros in the right hand plane and relative degree 2, so agent 2 is also not passive and is nonminimum phase. A controllable and observable realization of $H_2(s)$ is given by, for example,

$$
\begin{aligned}
A_2 &= \begin{bmatrix} 17 & -101 & 247 & -210 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \\
C_2 &= \begin{bmatrix} 0 & 1 & -4 & 4 \end{bmatrix}.
\end{aligned}
$$

Suppose that the payoff of agent 2 is given by

$$
J_2(y) = -y^{\mathrm{T}} \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1.5 \end{bmatrix} y + \begin{bmatrix} 4 \\ 10 \end{bmatrix}^{\mathrm{T}} y.
$$

We note that both matrices $\mathcal{A}_i,\ i = 1, 2$, as defined in (3.12) are full rank and $\alpha_{i0} > 0,\ i = 1, 2$. The pseudo-gradient is

$$
F(y) = -\begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} y + \begin{bmatrix} 5 \\ 10 \end{bmatrix}.
$$

It can be shown that $F(y)$ is strongly antimonotone with $\lambda = 0.117$. It can be also shown that $y^\star = [0.4545, 3.1818]^{\mathrm{T}}$ satisfies (3.19).

The matrices $K_i,\ i = 1, 2$, are chosen such that $\hat{A}_i \triangleq A_i - B_i K_i,\ i = 1, 2$, are Hurwitz. The positive definite matrices $P_i,\ i = 1, 2$, are chosen by solving the following optimization problem:

$$
\begin{aligned}
&\min\ \mathrm{Trace}(P_i) \\
&\text{subject to } -\hat{A}_i^{\mathrm{T}} P_i - P_i \hat{A}_i \succ 0, \\
&\qquad B_i^{\mathrm{T}} P_i \hat{A}_i^{-1} B_i = C_i \hat{A}_i^{-1} B_i,
\end{aligned}
$$

for $i = 1, 2$. From Lemma 3.7, there always exists a feasible solution to the above optimization problem. Hence, there always exists $P_i,\ i = 1, 2$, such that (3.30), (3.31) hold. We set $q_i,\ i = 1, 2$, as in (3.25) and $r \triangleq 1 + \max_{i \in \mathcal{N}} \frac{q_i}{\lambda}$. We note that there are other ways to choose $K_i, P_i,\ i \in \mathcal{N}$. For example, we can follow the procedure outlined in [56] instead of solving the above optimization problem.

From Proposition 3.10, it can be shown that $x_1^\star = [0, 0, 0, 0.0253]^\mathrm{T}$ and $x_2^\star = [0, 0, 0, 0.7955]^\mathrm{T}$ is the unique Nash equilibrium of this game. Fig. 3.1 shows that the trajectory of both agents $x_1(t)$ (shown in Fig. 3.1(a)) and $x_2(t)$ (shown in Fig. 3.1(b)) converge to the Nash equilibrium (shown in dashed line in the figures). Fig. 3.1(c) shows the outputs of both agents, and we can see that the output of both agents converge to $y^\star$.

3.6 Further Discussion

Following [56, 69–71], our main idea is to design input such that each agent's interconnected dynamics (3.33)–(3.35) are strictly passive from v_i to a new output w_i for $i \in \mathcal{N}$. Because the input of each agent i depends on w_{-i}, it is necessary for each agent i to broadcast w_i to all the other agents. Then, it is natural to ask whether it is possible to design input u_i such that each agent's dynamics are passive with respect to the original outputs $y_i,\ i \in \mathcal{N}$, instead of the new outputs $w_i,\ i \in \mathcal{N}$. If it were possible, then each agent i can stabilize the Nash equilibrium without having to broadcast w_i to all the other agents.

In the static feedback input case, Byrnes, Isidori and Willems [72] showed that nonlinear system

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + g(x(t))u(t), \quad x(0) \triangleq x_0, \quad t \geq 0, \\ y(t) &= h(x(t)), \end{aligned}$$

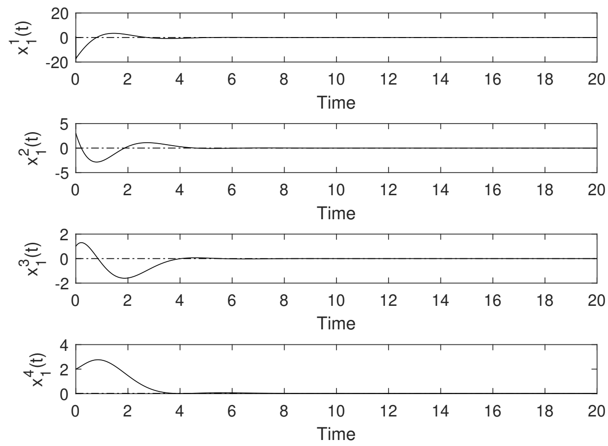
that can be made passive by static feedback of the form $u = \alpha(x) + \beta(x)v$ necessarily have relative degree 1 and minimum phase from u to y in the neighborhood of the origin.

The following proposition shows that if there exists *dynamic feedback* input in the form of

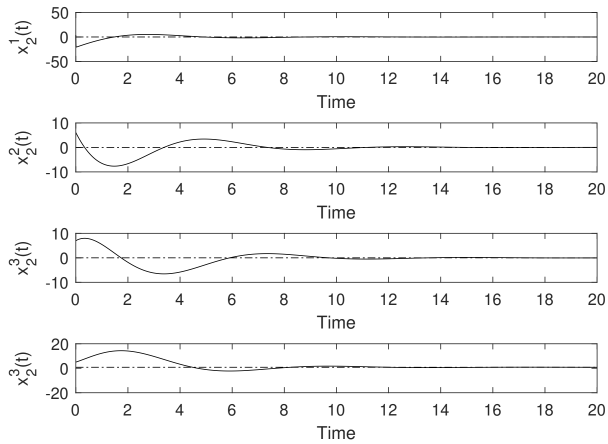
$$u(t) = Nz(t) + v(t), \tag{3.39}$$

$$\dot{z}(t) = Lz(t) + Mx(t) + Kv(t), \tag{3.40}$$

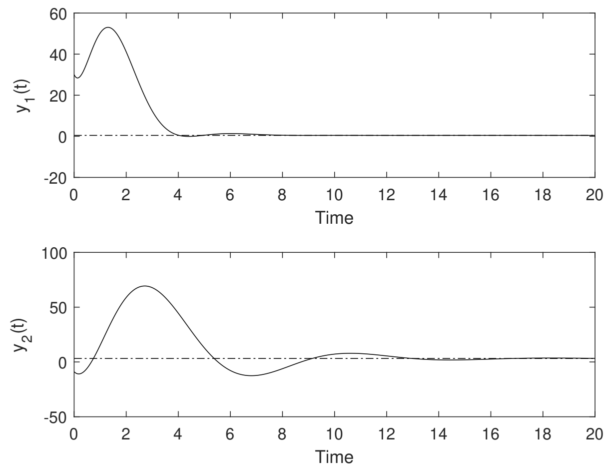
for the linear-time invariant system (3.1), (3.2) such that the interconnected system (3.1), (3.2), (3.39), (3.40) is passive from v to y, then the original linear time-invariant system (3.1), (3.2) is minimum phase and has relative degree 1 from u to y. In other words, if the original linear time-invariant system has relative degree larger than 1



(a) States of Agent 1



(b) States of Agent 2



(c) The Agents' Outputs

Fig. 3.1 The trajectories and outputs of the agents from u to y or if it is nonminimum phase, then passivation requires new output. The dynamic feedback inputs considered in [69, 70, 56] can be written in the form of (3.39), (3.40) for some matrices K, L, M, N.

Proposition 3.14. *Consider the linear time-invariant system* (3.1), (3.2) *with dynamic feedback input* (3.39), (3.40)*. If the interconnected system* (3.1), (3.2), (3.39), (3.40) *is passive from v to y, then the original linear time-invariant system* (3.1), (3.2) *is minimum phase and it has relative degree 1 from u to y.*

Proof. First, we show that the original linear time-invariant system (3.1), (3.2) has relative degree 1 from u to y. Let $\xi \triangleq [x \quad z]^{\mathrm{T}}$. Then, the interconnected system (3.1), (3.2), (3.39), (3.40) is given by

$$\begin{aligned} \dot{\xi}(t) &= \begin{bmatrix} \dot{x}(t) \\ \dot{z}(t) \end{bmatrix} = \tilde{A}\xi(t) + \tilde{B}v(t) \\ &\triangleq \begin{bmatrix} A & BN \\ M & L \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} B \\ K \end{bmatrix} v(t), \\ y(t) &= \tilde{C}\xi(t) \triangleq \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}. \end{aligned}$$

Taking the derivative of $y(t)$ with respect to time, it follows that $\dot{y} = C\dot{x} = CAx + CBNz + CBv$. Because the interconnected system (3.1), (3.2), (3.39), (3.40) is passive from v to y, by [72], it has relative degree 1 from v to y and thus, $CB \neq 0$. However, $CB \neq 0$ implies that the original linear time-invariant system (3.1), (3.2) has relative degree 1 from u to y.

To show that the original linear-time invariant system (3.1), (3.2) is minimum phase, we use Lemma 3.2. Let

$$\tilde{\Psi} \triangleq \begin{bmatrix} \Psi & 0 \\ 0 & I_n \end{bmatrix},$$

where $\Psi \in \mathbb{R}^{(n-1)\times n}$ satisfies $C\Psi^{\mathrm{T}} = 0$ and $\Psi\Psi^{\mathrm{T}} = I_{n-1}$. Then, it follows that

$$\begin{aligned} \tilde{A}_\star &= (\tilde{\Psi} - \tilde{\Psi}\tilde{B}(\tilde{C}\tilde{B})^{1}\tilde{C})\tilde{A}\tilde{\Psi}^{\mathrm{T}} \\ &= \begin{bmatrix} (\Psi - \Psi B(CB)^{-1}C)A\Psi^{\mathrm{T}} & 0 \\ M - K(CB)^{-1}A & L - KN \end{bmatrix}. \end{aligned}$$

Because the interconnected system (3.1), (3.2), (3.39), (3.40) is passive from v to y, by [72], it is minimum phase. From Lemma 3.2, it follows that all eigenvalues of $\tilde{A}_\star$ have negative real parts. Because $\tilde{A}_\star$ is a lower triangular block matrix, the eigenvalues of $\tilde{A}_\star$ are the eigenvalues of $A_\star \triangleq (\Psi - \Psi B(CB)^{-1}C)A\Psi^{\mathrm{T}}$ and the eigenvalues of $L - KN$. Hence, all the eigenvalues of $A_\star$ have negative real parts. From Lemma 3.2 again, it follows that the original linear time-invariant system (3.1), (3.2) is minimum phase. $\square$

Recently in [71], Sassano and Astolfi considers a different dynamic feedback input described by

$$u(t) = Pz(t), \tag{3.41}$$

$$\dot{z}(t) = Qz(t) + Rx(t) + Sv(t), \tag{3.42}$$

$$w(t) = y(t) + Tz(t), \tag{3.43}$$

for some matrices P, Q, R, S, T, such that the interconnected system (3.1), (3.2), (3.41)–(3.43)

$$\begin{aligned}
\begin{bmatrix} \dot{x}(t) \\ \dot{z}(t) \end{bmatrix} &= \begin{bmatrix} A & BP \\ R & S \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} 0 \\ S \end{bmatrix} v(t), \\
y(t) &= \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}, \\
w(t) &= \begin{bmatrix} C & T \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix},
\end{aligned}$$

is passive from v to w. By taking the derivative of $y(t)$ with respect to time, it follows that $\dot{y}(t) = C\dot{x}(t) = CAx(t) + CBPz(t)$, and hence the interconnected system (3.1), (3.2), (3.41)–(3.43) has relative degree *strictly greater* than 1 from v to y. Hence, it is not possible to use dynamic feedback input (3.41), (3.42) to make the linear time-invariant system (3.1), (3.2) passive from v to y.

3.7 Chapter Summary

We explored a game with linear time-invariant dynamic agents where each agent has a concave and continuously differentiable payoff function. By using passivation, we proposed control law that asymptotically stabilize the Nash equlibrium of the game.

Chapter 4

Characterization of a Game by Self-Improving Agents with Totally Positive Externalities

4.1 Introduction

In the previous part, we take the prescriptive approach in designing a Nash equilibrium seeking dynamics in games by second-order and linear dynamic agents. In the this chapter and the next, instead of taking the prescriptive approach, we take the descriptive approach. We investigate the qualitative behavior of a dynamic agent in interaction with other agents. Without knowing anything about the other agents' payoff function, it is reasonable to assume that a dynamic agent will try to improve her own payoff. We call a dynamic agent with such behavior as a *self-improving* agent. Specifically, an agent is self-improving if she changes her state in such a way that her payoff is nondecreasing when the other agents' states are fixed. Formally, the notion of a self-improving agent is similar to the notion of weak compatibility [73], myopic adjustment [74], and positive correlation [75] in the context of population games and evolutionary dynamics. An agent evolving under the *pseudo-gradient* dynamics introduced in [26] is an example of a self-improving agent.

Although all the agents might be self-improving, it is not the case that their payoffs are improving. Because an agent's payoff depend on the other agents' states, we need to consider the effect of the other agents' states to an agent's payoff. In economics, the effect of other agents' actions to an agent's payoff is called externality [76]. If the effect of the other agents' action is positive (resp., negative), then the externality is called positive (resp., negative) externality. We call the game where all agents influence each other positively as the *game with totally positive externalities.* An example of the game with totally positive externalities is the game by self-improving agents with identical interest [77]. In the game with identical interest, because all the agents have the same payoff, if an agent improve her own payoff, then the the other agents' payoffs are also improving. Hence it is not hard to see that if all the agents are self-improving, then the agents influence each other positively. I A game with totally positive externalities where each agent is self-improving is called a *game by self-improving agents with totally positive externalities.*

In this chapter, we investigate the qualitative behavior of agents in the simplest setting, that is, the case where each agent has quadratic payoff. We provide a necessary and sufficient condition for such a game to be a game by self-improving agents with totally positive externalities. We also found that a game by self-improving agents with totally positive externalities is a weighted identical interest game. Informally, our result shows that in noncooperative systems, if all the agents influence each other positively everywhere in the total state space, then all the agents have a common interest. In the context of potential games, our results highlight the connection of qualitative behavior of each agent with the structure of the agents' payoffs. Thus, our work differs from the work in [78] and [79] that focus on characterizing the set of Nash equilibria of the potential game and the work of [80] whose main result is characterizing the evolutionary stability of equilibria of the potential game.

The contents of this chapter are as follows. In Section 4.2, we review some notions from graph theory and linear algebra that we need to derive our results. In Section 4.3, we introduce the notion of a game by self-improving agents with totally positive externalities and review the notion of a potential game by dynamic agents. In Section 4.4, after formulating our problem setting, we characterize the game by self-improving agents with totally positive externalities. We also analyze the existence, uniqueness, and stability of Nash equilibrium of such game. Furthermore, we also show the relation between the game by self-improving agents with totally positive externalities and the weighted identical interest game. In Section 4.5, we give an example to illustrate our result. Finally, we summarize this chapter in Section 4.6.

4.2 Mathematical Preliminaries

4.2.1 Graph theory

Graph theory

Let a *graph* $G = (V, E, \omega)$ consist of the set of n nodes $V = \{1, \ldots, n\}$, the set of m edges $E \subseteq V \times V$, and the weights $\omega(i, j) \neq 0,\ (i, j) \in E$. An *undirected path* from node $j \in V$ to node $l \in V$ in the graph G is a sequence $(i_1, \ldots, i_k),\ k \geq 1$, such that $i_1, \ldots, i_{k-1}$ are pairwise distinct, $i_1 = j, i_k = l$, and either $(i_q, i_{q+1}) \in E$ or $(i_{q+1}, i_q) \in E$ for $q = 1, \ldots, k-1$. The graph G is *weakly connected* if there exists an undirected path from any node $i \in V$ to any node $j \in V$. A *weakly connected component* of the graph G is a set $C \subseteq V$ such that there exists an undirected path from node i to j for $i, j \in C$.

Pattern Graph

For $A \in \mathbb{R}^{n \times n}$, the *pattern graph* [81] is the graph $G(A) = (V_G, E_G, \omega_G)$ with the node set $V_G \triangleq \{1, \ldots, n\}$, the edge set $E_G \triangleq \{(i, j) : A^{ij} \neq 0,\ i, j \in V_G\}$, and the weights $\omega_G(i, j) = A^{ij},\ (i, j) \in E_G$.

We state the result we need in Section 4.4. This results follow from [81].

Lemma 4.1. *Let $A \in \mathbb{R}^{n \times n}$ and $X_1, X_2, \in \mathbb{D}^n$ and suppose that $X_i,\ i = 1, 2,$ are positive-definite. Then, the pattern graph $G(A)$ is weakly connected if and only if*

$$X_1 A X_1^{-1} = X_2 A X_2^{-1}$$

implies $X_2 = \lambda X_1$ for some $\lambda > 0$.

4.2.2 Linear algebra

Lemma 4.2. *Let $f : \mathbb{R}^N \to \mathbb{R}, g : \mathbb{R}^N \to \mathbb{R}$ be affine functions described by*

$$f(x) = a^{\mathrm{T}} x + b, \text{ and } g(x) = c^{\mathrm{T}} x + d,$$

where $a, c \in \mathbb{R}^N$ are non-zero vectors and $b, d \in \mathbb{R}$. Then $f(x)g(x) \geq 0,\ x \in \mathbb{R}^N$ if and only if $g(x) = \lambda f(x),\ \lambda > 0$.

Proof. The proof for sufficiency is straightforward. To show necessity, suppose that $g(x) \neq \lambda f(x),\ \lambda > 0$. We show that there exists $\bar{x} \in \mathbb{R}^N$ such that $f(\bar{x})g(\bar{x}) < 0$ and hence $f(x)g(x) \ngeq 0,\ x \in \mathbb{R}^N$.

First, if $g(x) = \lambda f(x),\ \lambda < 0$ then $f(x)g(x) \leq 0,\ x \in \mathbb{R}^N$. Furthermore, if $\lambda = 0$, then $g(x) \equiv 0$ contrary to our assumption that $c \neq 0$. Thus, without loss of generality, we assume that $g(x) \neq \lambda f(x),\ \lambda \in \mathbb{R}$. There are two cases: (i) a, c is linearly independent, or (ii) a, c are linearly dependent. We start with case (i). Because a, c is linearly independent, the matrix $F \triangleq \begin{bmatrix} a & c \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{2\times N}$ has full row rank and is surjective. Let $e, f \in \mathbb{R}$ be such that $(e-b)(f-d) < 0$, $g \triangleq \begin{bmatrix} b & d \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^2$, and $h \triangleq \begin{bmatrix} e & f \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^2$. Then, since F is surjective, there exists a solution $\bar{x} \in \mathbb{R}^N$ to the equation $Fx + g = h$. Thus, it follows from the definition of g and h that $f(\bar{x})g(\bar{x}) < 0$.

In case (ii), because a, c is linearly dependent and $a, c \neq 0$, it follows that $c = \lambda a,\ \lambda \neq 0$. Since we assume that $g(x) \neq \lambda f(x)$, it follows that $d \neq \lambda b$. Letting $h(u) \triangleq \lambda u^2 + (d + \lambda b)u + bd$, it follows that $f(x)g(x) = \lambda(a^{\mathrm{T}}x)^2 + (d+\lambda b)(a^{\mathrm{T}}x) + bd = h(a^{\mathrm{T}}x)$. Note that the discriminant of $h(u)$ is given by $(d+\lambda b)^2 - 4\lambda bd = (d - \lambda b)^2 \geq 0$. Because $d \neq \lambda b$, $(d-\lambda b)^2 > 0$. Thus, $h(u)$ always has two roots and hence there exists $\bar{u}$ such that $h(\bar{u}) < 0$. Letting $\bar{x} = \frac{a}{\|a\|^2}\bar{u}$, it follows from calculation that $f(\bar{x})g(\bar{x}) = h(a^{\mathrm{T}}\bar{x}) = h(\bar{u}) < 0$. $\square$

4.3 Game by Dynamic Agents

Consider a finite set $\mathcal{N} = \{1, \ldots, N\}$ of N agents. Each agent $i \in \mathcal{N}$ controls her state $x_i \in \mathbb{R}^{n_i}$. The total state space of all agents is $\mathbb{R}^n$, where $n = \sum_{i\in\mathcal{N}} n_i$. If $x \in \mathbb{R}^n$, then we write $x = (x_i)_{i\in\mathcal{N}} = (x_i, x_{-i})$, where $x_{-i} \in \mathbb{R}^{n-n_i}$. The agents' states evolve according to the dynamics

$$\dot{x}_i(t) = f_i(x(t)), \quad i \in \mathcal{N}, \quad x_i(0) \triangleq x_{i0}, \quad t \geq 0, \tag{4.1}$$

where $f_i : \mathbb{R}^n \to \mathbb{R}^{n_i}$ is continuously differentiable. Note that each agent's dynamics depend on all the agents' states.

Suppose that each agent $i \in \mathcal{N}$ has a payoff function $J_i : \mathbb{R}^n \to \mathbb{R}$, $(x_i, x_{-i}) \mapsto J_i(x_i, x_{-i})$ and wants to increase her own payoff. We assume that each agent knows her own payoff function, but not the other agents'. Furthermore, the payoff functions $J_i(x_i, x_{-i}),\ i \in \mathcal{N}$, are continuously differentiable concave in $x_i \in \mathbb{R}^{n_i}$ for every fixed $x_{-i} \in \mathbb{R}^{n-n_i}$. We denote noncooperative dynamical system $\mathcal{G}(J)$ with $J \triangleq \{J_i\}_{i\in\mathcal{N}}$ and the dynamics (4.1). We denote the agent i with state $x_i \in \mathbb{R}^{n_i}$ that evolves acccording to (4.1) with $f_i(x) = \lambda_i \frac{\partial J_i(x)}{\partial x_i}^{\mathrm{T}}$, $\lambda_i > 0$, as a *pseudo-gradient* agent. The *externality* of agent $i \in \mathcal{N}$ from agent $j \in \mathcal{N} \setminus \{i\}$ is given by $\frac{\partial J_i(x)}{\partial x_j} f_j(x)$. Agent $i \in \mathcal{N}$ has *positive* *externality* (resp., negative externality) from agent $j \in \mathcal{N} \setminus \{i\}$ if

$$\frac{\partial J_i(x)}{\partial x_j} f_j(x) \geq 0, \text{ (resp., } \frac{\partial J_i(x)}{\partial x_j} f_j(x) \leq 0). \tag{4.2}$$

Note that the posivity (resp., negativity) of externality may depend on the state x.

Now we define the Nash equilibrium of the game $\mathcal{G}(J)$.

Definition 4.3. *Consider the game $\mathcal{G}(J)$ with* (4.1). *The state $x^\star \in \mathbb{R}^n$ is a* Nash equilibrium *of the game $\mathcal{G}(J)$ if $x^\star$ is an equilibrium of the dynamics* (4.1)*, that is,*

$$f_i(x^\star) = 0, \quad i \in \mathcal{N}. \tag{4.3}$$

and satisfies

$$J_i(x_i^\star, x_{-i}^\star) \geq J_i(x_i, x_{-i}^\star), \quad x_i \in \mathbb{R}^{n_i}, \quad i \in \mathcal{N}. \tag{4.4}$$

It follows from Definition 4.3 and [46, Proposition 1.4.2] that $x^\star \in \mathbb{R}^n$ is a Nash equilibrium if and only if (4.3) holds and

$$\frac{\partial J_i(x_i^\star, x_{-i}^\star)}{\partial x_i} = 0, \quad i \in \mathcal{N}. \tag{4.5}$$

Thus, if all agents are pseudo-gradient agents, then $x^\star \in \mathbb{R}^n$ is a Nash equilibrium if and only if 4.5 holds.

Now, we formally define the notions of a self-improving agent and the game with totally positive externalities. Consider the game $\mathcal{G}(J)$ with (4.1). We say that agent $i \in \mathcal{N}$ is *self-improving* if

$$\frac{\partial J_i(x)}{\partial x_i} f_i(x) \geq 0, \quad x \in \mathbb{R}^n. \tag{4.6}$$

Note that if agent i is self-improving, then for every fixed $x_{-i} \in \mathbb{R}^{n-n_i}$, $\dot{J}_i(x_i(t), x_{-i}) \triangleq \frac{\mathrm{d}}{\mathrm{d}t} J_i(x_i(t), x_{-i}) \geq 0$ along the trajectory of (4.1). If each agent $i \in \mathcal{N}$ is self-improving, then we say that the game $\mathcal{G}(J)$ is a *game by self-improving agents*. We denote the set of games by self-improving agents by $\mathcal{G}^{\mathrm{SI}}$. Note that Nash equilibria of a game in the class of $\mathcal{G}^{\mathrm{SI}}$ are not necessarily stable.

The game $\mathcal{G}(J)$ is a game with *totally positive externalities* if (4.2) holds for all $i, j \in \mathcal{N}$ and $x \in \mathbb{R}^n$ such that $i \neq j$, that is, each agent $i \in \mathcal{N}$ has positive externality from all the other agents $j \in \mathcal{N} \setminus \{i\}$ everywhere in the state space. We denote the set of games by agents with totally positive externalities under the dynamics (4.1) by $\mathcal{G}^{\mathrm{TPE}}$. The set of games by self-improving agents with totally positive externalities is denoted by $\mathcal{G}^{\mathrm{SI\&TPE}} \triangleq \mathcal{G}^{\mathrm{SI}} \cap \mathcal{G}^{\mathrm{TPE}}$. It follows directly from (4.2) and (4.6) that if $\mathcal{G}(J) \in \mathcal{G}^{\mathrm{SI\&TPE}}$, then $\dot{J}_i(x(t)) = \sum_{j\in\mathcal{N}} \frac{\partial J_i(x(t))}{\partial x_j} f_j(x(t)) \geq 0,\ i \in \mathcal{N}$, along the trajectory of (4.1). Thus, when $\mathcal{G}(J) \in \mathcal{G}^{\mathrm{SI\&TPE}}$, each agent's payoff is nondecreasing for all trajectories $x(t) \in \mathbb{R}^n$.

4.3.1 Potential game by dynamic agents

Consider the game $\mathcal{G}(J)$ with (4.1). The game $\mathcal{G}(J)$ is called a *weighted potential game* [78, Section 2] if there exists a function $\Phi : \mathbb{R}^n \to \mathbb{R}$ and weights $w^i > 0,\ i \in \mathcal{N}$, such that $J_i(x_i, x_{-i}) - J_i(y_i, x_{-i}) = w^i\left(\Phi(x_i, x_{-i}) - \Phi(y_i, x_{-i})\right)$ for $x_i, y_i \in \mathbb{R}^{n_i},\ x_{-i} \in \mathbb{R}^{n-n_i},\ i \in \mathcal{N}$. If $w^i = 1,\ i \in \mathcal{N}$, then the game $\mathcal{G}(J)$ is called an *exact potential game*. We denote the set of all weighted potential games by $\mathcal{G}^{\mathrm{WP}}$ and the set of all exact potential games by $\mathcal{G}^{\mathrm{P}}$. By definition, the inclusion $\mathcal{G}^{\mathrm{P}} \subsetneq \mathcal{G}^{\mathrm{WP}}$ holds.

One particularly relevant class of weighted potential game is weighted identical interest game. The game $\mathcal{G}(J)$ is called a *weighted identical interest game* [77, Section 12.1] if there exist a function $\Phi : \mathbb{R}^n \to \mathbb{R}$, weights $w^i > 0,\ i \in \mathcal{N}$, and constants $c_i \in \mathbb{R},\ i \in \mathcal{N}$, such that

$$J_i(x) = w^i\Phi(x) + c_i, \quad x \in \mathbb{R}^n, \quad i \in \mathcal{N}. \tag{4.7}$$

If $w^i = 1,\ i \in \mathcal{N}$, then the game $\mathcal{G}(J)$ is called an *identical interest game*. We denote the set of all weighted identical interest games by $\mathcal{G}^{\mathrm{WII}}$ and the set of all identical interest games by $\mathcal{G}^{\mathrm{II}}$. By definition, the relations $\mathcal{G}^{\mathrm{II}} \subsetneq \mathcal{G}^{\mathrm{WII}} \subsetneq \mathcal{G}^{\mathrm{WP}}$ and $\mathcal{G}^{\mathrm{II}} \subsetneq \mathcal{G}^{\mathrm{P}} \subsetneq \mathcal{G}^{\mathrm{WP}}$ hold. Since it follows from (4.2), (4.6), and (4.7) that $\frac{\partial J_i(x)}{\partial x_j} = w^i \frac{\partial \Phi(x)}{\partial x_j} f_j(x) = \frac{w^i}{w}^j \frac{\partial J_j(x)}{\partial x_j} f_j(x) \geq 0$, for $i, j \in \mathcal{N}$ such that $i \neq j$ and $x \in \mathbb{R}^n$, the relation $\mathcal{G}^{\mathrm{WII}} \cap \mathcal{G}^{\mathrm{SI}} \subseteq \mathcal{G}^{\mathrm{SI\&TPE}}$ holds.

4.4 Main Results

we analyze the simplest nontrivial game by N dynamic agents, that is, we analyze the game $\mathcal{G}(J)$ with (4.1) where the state of each agent is scalar so that $n = N$ and the payoff function of each agent is quadratic.

Assumption 4.1. *Each agent $i \in \mathcal{N}$ has a scalar state $x_i \in \mathbb{R}$ that evolves according to the dynamics (4.1). Furthermore, the payoff function $J_i(x)$ of agent $i \in \mathcal{N}$ is given in the form*

$$J_i(x) = -\frac{1}{2}x^{\mathrm{T}} P_i x + q_i^{\mathrm{T}} x + r_i, \tag{4.8}$$

where $P_i \succ 0,\ q_i \in \mathbb{R}^N,$ *and* $r_i \in \mathbb{R}$.

Note that we keep $r_i,\ i \in \mathcal{N}$, in (4.8) for formality but their actual values do not matter in the results because $J_i(x), i \in \mathcal{N}$, appear along with their partial derivatives in the following characterizations.

4.4.1 Characterization of the game by self-improving agents with totally positive externalities

We begin by an interesting result concerning the game by self-improving agents.

Theorem 4.4. *Consider the game* $\mathcal{G}(J)$ *with* (4.1) *and Assumption 4.1. Suppose that each agent's state evolves according to the dynamics* (4.1) *with* $f(x)$ *in the form*

$$f_i(x) = a_i^{\mathrm{T}} x + b_i, \tag{4.9}$$

where $a_i \in \mathbb{R}^N \setminus \{0\}$ *and* $b_i \in \mathbb{R}$. *Then,* $\mathcal{G}(J) \in \mathcal{G}^{\mathrm{SI}}$ *if and only if each agent* $i \in \mathcal{N}$ *is a pseudo-gradient agent.*

Proof. The proof of sufficiency is straightforward. Necessity follows directly from (4.9) and Lemma 4.2. □

Now, let $P \in \mathbb{R}^{N \times N}$ be the (not necessarily symmetric) matrix described by

$$P \triangleq \left[\mathrm{row}_1(P_1)^{\mathrm{T}}, \ldots, \mathrm{row}_N(P_N)^{\mathrm{T}}\right]^{\mathrm{T}}, \tag{4.10}$$

where $P_i \in \mathbb{R}^{N \times N}$ denotes the symmetric positive-definite matrix in the payoff function (4.8) of agent $i \in \mathcal{N}$ and let

$$q \triangleq \left[q_1^1, \ldots, q_N^N\right]^{\mathrm{T}} \in \mathbb{R}^N, \tag{4.11}$$

where $q_i \in \mathbb{R}^N$ is the vector in the payoff function (4.8) of agent $i \in \mathcal{N}$. We have the following characterization of the game by self-improving agents with totally positive externalities.

Theorem 4.5. *Consider the game* $\mathcal{G}(J)$ *with* (4.1) *and Assumption 4.1. Suppose that each agent* i*'s vector field satisfies the following condition:*

$$f_i(x) = 0 \iff \frac{\partial J_i(x)}{\partial x_i} = 0, \quad x \in \mathbb{R}^N. \tag{4.12}$$

Then, $\mathcal{G}(J) \in \mathcal{G}^{\mathrm{SI\&TPE}}$ *if and only if for each* $i \in \mathcal{N}$*, there exists a unique diagonal matrix*

$$D_i \triangleq \mathrm{diag}\big[d_i^1, \ldots, d_i^{i-1}, 1, d_i^{i+1}, \ldots, d_i^N\big] \in \mathbb{D}^N, \tag{4.13}$$

where $d_i^j > 0,\ i \in \mathcal{N},\ j \in \mathcal{N} \setminus \{i\}$*, such that*

$$D_i P = P_i, \quad i \in \mathcal{N}, \tag{4.14}$$

$$D_i q = q_i, \quad i \in \mathcal{N}. \tag{4.15}$$

Proof. First, we show that $\mathcal{G} \in \mathcal{G}^{\mathrm{SI\&TPE}}$ (i.e., (4.6) and 4.2 hold) if and only if there exists $d_i^j > 0$ such that

$$\frac{\partial J_i(x)}{\partial x_j} = d_i^j \frac{\partial J_j(x)}{\partial x_j}, \quad x \in \mathbb{R}^N, \quad i, j \in \mathcal{N}. \tag{4.16}$$

Let $i, j \in \mathcal{N}$ such that $i \neq j$. We need to consider two cases: (i) when both (4.2) and (4.6) hold with strict inequality, and (ii) when (4.2) and (4.6) hold with either (4.2) or (4.6) holds with equality. We start with case (i). Note that (4.2) holds with strict inequality if and only if $\mathrm{sign}\Big(\frac{\partial J_i(x)}{\partial x_j}\Big) = \mathrm{sign}(f_j(x)),\ i, j \in \mathcal{N},\ i \neq j$, where for $a \in \mathbb{R}$, $\mathrm{sign}(a) = 1$ if $a > 0$, $\mathrm{sign}(a) = -1$ if $a < 0$, and $\mathrm{sign}(a) = 0$ otherwise. Also note that (4.6) holds with strict inequality if and only if $\mathrm{sign}\Big(\frac{\partial J_i(x)}{\partial x_i}\Big) = \mathrm{sign}(f_i(x)),\ i \in \mathcal{N}$. Consequently, (4.2) and (4.6) hold with strict inequality if and only if $\mathrm{sign}\Big(\frac{\partial J_i(x)}{\partial x_j}\Big) = \mathrm{sign}\Big(\frac{\partial J_j(x)}{\partial x_j}\Big)$ for all $i, j \in \mathcal{N}$. Thus, (4.2) and (4.6) both hold with strict inequality if and only if $\frac{\partial J_i(x)}{\partial x_j}\frac{\partial J_j(x)}{\partial x_j} > 0$. Now we consider case (ii). We show that (4.2) and (4.6) hold with either (4.2) or (4.6) holds with equality. First, note that (4.2) holds with equality if and only if either $\frac{\partial J_i(x)}{\partial x_j} = 0$ or $f_j(x) = 0$. When $\frac{\partial J_i(x)}{\partial x_j} = 0$, it follows that $\frac{\partial J_i(x)}{\partial x_j}\frac{\partial J_j(x)}{\partial x_j} = 0$. If $f_j(x) = 0$, it follows from (4.12) that $\frac{\partial J_j(x)}{\partial x_j} = 0$. Thus, in this case, it also follows that $\frac{\partial J_i(x)}{\partial x_j}\frac{\partial J_j(x)}{\partial x_j} = 0$. On the other hand, $\frac{\partial J_i(x)}{\partial x_j}\frac{\partial J_j(x)}{\partial x_j} = 0$ implies either $\frac{\partial J_j(x)}{\partial x_j} = 0$ or $\frac{\partial J_i(x)}{\partial x_j} = 0$. If $\frac{\partial J_j(x)}{\partial x_j} = 0$, then it follows from (4.12) that (4.2) holds with equality. On the other hand, it follows trivially from $\frac{\partial J_i(x)}{\partial x_j} = 0$ that (4.2) holds with equality. Thus, (4.2) holds with equality if and only if $\frac{\partial J_i(x)}{\partial x_j}\frac{\partial J_j(x)}{\partial x_j} = 0$. By similar reasoning, it can be shown that (4.6) holds with equality if and only if $\frac{\partial J_i(x)}{\partial x_j}\frac{\partial J_j(x)}{\partial x_j} = 0$. Hence, it follows that (4.2) and (4.6) hold if and only if $\frac{\partial J_i(x)}{\partial x_j}\frac{\partial J_j(x)}{\partial x_j} \geq 0$ for all $i, j \in \mathcal{N}$. Furthermore, note that $\frac{\partial J_i(x)}{\partial x_j} = -\mathrm{row}_j(P_i)x + q_i^j$ is an affine function of x for $i, j \in \mathcal{N}$. Thus, it follows from Lemma 4.2 that $\frac{\partial J_i(x)}{\partial x_j}\frac{\partial J_j(x)}{\partial x_j} \geq 0$ for all $i, j \in \mathcal{N}$ (and hence (4.6) and (4.2) hold) if and only if there exists $d_i^j > 0$ such that (4.16) holds. Now, noting that

$$\left[\frac{\partial J_1(x)}{\partial x_1}, \dots, \frac{\partial J_N(x)}{\partial x_N}\right]^{\mathrm{T}} = -Px + q, \tag{4.17}$$

and writing (4.16) in vector notation, it follows that $\mathcal{G}(J) \in \mathcal{G}^{\mathrm{SI\&TPE}}$ if and only if

$$\begin{aligned}\frac{\partial J_i(x)}{\partial x}^{\mathrm{T}} &= \operatorname{diag}\big[d_i^1, \dots, d_i^N\big]\left[\frac{\partial J_1(x)}{\partial x_1}, \dots, \frac{\partial J_N(x)}{\partial x_N}\right]^{\mathrm{T}} \\ &= D_i(-Px + q), \quad i \in \mathcal{N}.\end{aligned} \tag{4.18}$$

Noting that $\frac{\partial J_i(x)}{\partial x}^{\mathrm{T}} = -P_i x + q_i,\ i \in \mathcal{N}$ (Assumption 4.1), the result follows.

To show uniqueness, suppose that $D_{i,1}, D_{i,2} \in \mathbb{D}^N$ both satisfy (4.14) for all $i \in \mathcal{N}$. Then, it follows that

$$D_{i,1}P = P_i = D_{i,2}P, \quad i \in \mathcal{N}. \tag{4.19}$$

Because $D_{i,1}, D_{i,2}$, and P_i are invertible, the matrix P is also invertible. Consequently, right multiplying (4.19) by P^{-1}, it follows that $D_{i,1} = D_{i,2}$. □

4.4.2 Nash equilibrium and stability analysis

In this section, we show that the game by self improving agents with totally positive externalities has a unique Nash equilibrium that is asymptotically stable.

Theorem 4.6. *Consider the game $\mathcal{G}(J)$ with* (4.1) *and Assumption 4.1. If $\mathcal{G}(J) \in \mathcal{G}^{\mathrm{SI\&TPE}}$, then there exists a unique Nash equilibrium $x^\star \in \mathbb{R}^N$ of the game $\mathcal{G}(J)$. If, in addition,*

$$\frac{\partial J_i(x)}{\partial x_i} f_i(x) = 0,\ i \in \mathcal{N},\ \text{if and only if } x = x^\star, \tag{4.20}$$

then the unique Nash equilibrium $x^\star$ is globally asymptotically stable.

Proof. First, it follows from (4.5) that a Nash equilibrium $x^\star$ satisfies $-Px^\star + q = 0$. Furthermore, since $\mathcal{G}(J) \in \mathcal{G}^{\mathrm{SI\&TPE}}$, it follows from Theorem 4.5 that there exist positive-definite diagonal matrices $D_i,\ i \in \mathcal{N}$, such that (4.14) holds. Because the matrices $P_i,\ i \in \mathcal{N}$, and $D_i,\ i \in \mathcal{N}$, are invertible, $P \in \mathbb{R}^{N\times N}$ is invertible. Thus, the unique Nash equilibrium is given by $x^\star = P^{-1}q$.

Next, we show that the unique Nash equilibrium $x^\star \in \mathbb{R}^N$ is asymptotically stable. Consider the Lyapunov function candidate

$$V(x) \triangleq \sum_{i\in\mathcal{N}} J_i(x^\star) - J_i(x). \tag{4.21}$$

Because $\mathcal{G}(J) \in \mathcal{G}^{\text{SI\&TPE}}$, it follows from (4.14) and (4.15) that (4.21) can be rewritten as

$$\begin{aligned}
V(x) &= \sum_{i\in\mathcal{N}} \frac{1}{2}(x^{\mathrm{T}} P_i x - x^{\star\mathrm{T}} P_i x^\star) + q_i^{\mathrm{T}}(x^\star - x) \\
&= \sum_{i\in\mathcal{N}} \frac{1}{2}(x^{\mathrm{T}} P_i x - x^{\star\mathrm{T}} P_i x^\star) + q^{\mathrm{T}} D_i(x^\star - x) \\
&= \sum_{i\in\mathcal{N}} \frac{1}{2}(x^{\mathrm{T}} P_i x - x^{\star\mathrm{T}} P_i x^\star) + x^{\star\mathrm{T}} P^{\mathrm{T}} D_i(x^\star - x) \\
&= \sum_{i\in\mathcal{N}} \frac{1}{2}(x^{\mathrm{T}} P_i x - x^{\star\mathrm{T}} P_i x^\star) + x^{\star\mathrm{T}} P_i(x^\star - x) \\
&= \frac{1}{2}\sum_{i\in\mathcal{N}} (x - x^\star) P_i (x - x^\star),
\end{aligned} \tag{4.22}$$

where the third equality follows from the fact that $Px^\star = q$. Since $x^\star$ is the unique Nash equlibrium, it follows from (4.22) that $V(x) \geq 0$ and $V(x) = 0$ if and only if $x = x^\star$. Because the matrices P_i, $i \in \mathcal{N}$, are positive-definite, $V(x)$ is radially unbounded. Furthermore, because $\mathcal{G}(J) \in \mathcal{G}^{\text{SI\&TPE}}$, it follows from (4.6), (4.2), and (4.21) that

$$\frac{\partial V(x)}{\partial x} f(x) = \sum_{i\in\mathcal{N}} -\frac{\partial J_i(x)}{\partial x} f(x) = -\sum_{i\in\mathcal{N}}\sum_{k\in\mathcal{N}} \frac{\partial J_i(x)}{\partial x_k} f_k(x) \leq 0, \tag{4.23}$$

for $x \in \mathbb{R}^N$ and hence the solution $x(t) \equiv x^\star$ is Lyapunov stable. Furthermore, the equality in (4.23) holds if and only if $\frac{\partial J_i(x)}{\partial x_k} f_k(x) = 0,\ i, k \in \mathcal{N}$. Now, consider the set $\mathcal{R} \triangleq \left\{x \in \mathbb{R}^N : \frac{\partial J_i(x)}{\partial x_k} f_k(x) = 0,\ i, k \in \mathcal{N}\right\}$. We show that $\mathcal{M} = \{x^\star\}$ is the largest invariant set contained in $\mathcal{R}$. First, it follows from (4.3) that $\{x^\star\}$ is an invariant set contained in $\mathcal{R}$. Furthermore, note that $\frac{\partial J_i(x)}{\partial x_i} f_i(x) = 0,\ x \in \mathcal{R},\ i \in \mathcal{N}$. As a result, it follows from (4.20) that $\mathcal{R} = \{x^\star\}$ and hence $\mathcal{M} = \{x^\star\}$. Thus, we can conclude using Krasovskii-Lasalle invariance principle that the unique Nash equilibrium is globally asymptotically stable. $\square$

Note that if each agent is a pseudo-gradient agent, then (4.20) holds. Consequently, it follows from Theorem 4.6 that if each agent is a pseudo-gradient agent, then the unique Nash equilibrium $x^\star$ is globally asymptotically stable.

4.4.3 Connection with potential games

In this section, we show that the class of games by self-improving agents with totally positive externalities is a *subset* of the class of *weighted identical interest games*.

Consider the matrix P defined in (4.10) and its pattern graph $G(P)$. Here, we deal with the case where $G(P)$ is weakly connected. The case where $G(P)$ is not weakly connected can be handled by analyzing each weakly connected component separately.

Theorem 4.7. *Consider the game $\mathcal{G}(J)$ with* (4.1) *and Assumption 4.1. Suppose that $\mathcal{G}(J) \in \mathcal{G}^{\mathrm{SI\&TPE}}$. Let $P \in \mathbb{R}^{N\times N}$ be the matrix defined in* (4.10)*. If the pattern graph $G(P)$ is weakly connected, then $\mathcal{G}(J) \in \mathcal{G}^{\mathrm{WII}}$.*

Proof. First, we show that if the pattern graph $G(P)$ is weakly connected, then there exist $w_i > 0,\ i \in \mathcal{N}$, such that $D_i = w_i D_1,\ i \in \mathcal{N}$, where the matrices $D_i \in \mathbb{D}^N,\ i \in \mathcal{N}$, are as defined in (4.13). Because $\mathcal{G}(J) \in \mathcal{G}^{\mathrm{SI\&TPE}}$, it follows from (4.14) that $D_i P = P_i = P_i^{\mathrm{T}} = P^{\mathrm{T}} D_i,\ i \in \mathcal{N}$. Thus, $D_i P D_i^{-1} = P^{\mathrm{T}},\ i \in \mathcal{N}$. As a consequence, it follows that

$$D_i P D_i^{-1} = D_j P D_j^{-1}, \quad i, j \in \mathcal{N}. \tag{4.24}$$

Because $G(P)$ is weakly connected, it follows from (4.24) and Lemma 4.1 that there exist $w_i > 0,\ i \in \mathcal{N}$, such that $D_i = w_i D_1,\ i \in \mathcal{N}$. Hence, it follows from (4.14) and (4.15) that $P_i = w_i P_1$ and $q_i = w_i q_1$ for $i \in \mathcal{N}$. Let $\Phi(x) = J_1(x)$. Then, it follows that $J_i(x) = w_i \Phi(x) + r_i - w_i r_1,\ i \in \mathcal{N}$. The result then follows from the definition of weighted identical interest game. $\square$

Note that if $\mathcal{G}(J) \in \mathcal{G}^{\mathrm{WII}}$, then $\mathcal{G}(J) \in \mathcal{G}^{\mathrm{SI\&TPE}}$. In Theorem 4.7, we show that if $\mathcal{G}(J) \in \mathcal{G}^{\mathrm{SI\&TPE}}$ and the pattern graph $G(P)$ is weakly connected, then $\mathcal{G}(J) \in \mathcal{G}^{\mathrm{WII}}$. The converse of Theorem 4.7 does not hold. Consider the game $\mathcal{G}_1(J)$ by three agents, where each agent $i \in \mathcal{N} \triangleq \{1, 2, 3\}$ controls a scalar state $x_1, x_2, x_3 \in \mathbb{R}$ with quadratic

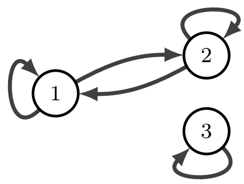


Fig. 4.1 The pattern graph $G(P)$ of the game $\mathcal{G}_1(J)$

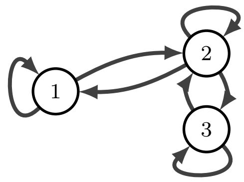


Fig. 4.2 The pattern graph $G(P)$ of the game $\mathcal{G}_2(J)$

payoff functions given by (4.8) with

$$P_1 \triangleq \begin{bmatrix} 6 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 14 \end{bmatrix}, \quad q_1 \triangleq \begin{bmatrix} 5 & 1 & 12 \end{bmatrix}^{\mathrm{T}},$$

$$P_2 \triangleq \begin{bmatrix} 12 & -2 & 0 \\ -2 & 4 & 0 \\ 0 & 0 & 28 \end{bmatrix}, \quad q_2 \triangleq \begin{bmatrix} 10 & 2 & 24 \end{bmatrix}^{\mathrm{T}},$$

$$P_3 \triangleq \begin{bmatrix} 3 & -1/2 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}, \quad q_3 \triangleq \begin{bmatrix} 5/2 & 1/2 & 6 \end{bmatrix}^{\mathrm{T}},$$

and $r_1 \triangleq 0,\ r_2 \triangleq 2$, and $r_3 \triangleq 1$. In this case, letting $\Phi(x) = J_1(x)$ and $w_1 = 1$, $w_2 = 2$, and $w_3 = 1/2$, we can see that $J_2(x) = w_2\Phi(x) + 2$ and $J_3(x) = w_3\Phi(x) + 1$. Hence, it follows from the definition of weighted identical interest game that $\mathcal{G}_1(J) \in \mathcal{G}^{\mathrm{WII}}$. However, the matrix P is given by

$$P = \begin{bmatrix} 6 & -1 & 0 \\ -2 & 4 & 0 \\ 0 & 0 & 7 \end{bmatrix},$$

and its pattern graph $G(P)$ is shown in Fig. 4.1, where we can see that the pattern graph $G(P)$ has two weakly connected components. Thus, the game $\mathcal{G}_1(J)$ is a weighted identical interest game where the pattern graph $G(P)$ is not weakly connected.

4.5 Numerical Example

To illustrate our main results, we consider the game $\mathcal{G}_2(J)$ by three agents, where each agent $i \in \mathcal{N} \triangleq \{1, 2, 3\}$ controls a scalar state $x_1, x_2, x_3 \in \mathbb{R}$ with quadratic payoff functions given by (4.8) with

$$P_1 \triangleq \begin{bmatrix} 6 & -1/2 & 0 \\ -1/2 & 2 & 4 \\ 0 & 4 & 14 \end{bmatrix}, \quad q_1 \triangleq \begin{bmatrix} 5 & 1 & 12 \end{bmatrix}^{\mathrm{T}},$$

$$P_2 \triangleq \begin{bmatrix} 12 & -1 & 0 \\ -1 & 4 & 8 \\ 0 & 8 & 28 \end{bmatrix}, \quad q_2 \triangleq \begin{bmatrix} 10 & 2 & 24 \end{bmatrix}^{\mathrm{T}},$$

$$P_3 \triangleq \begin{bmatrix} 3 & -1/4 & 0 \\ -1/4 & 1 & 2 \\ 0 & 2 & 7 \end{bmatrix}, \quad q_3 \triangleq \begin{bmatrix} 5/2 & 1/2 & 6 \end{bmatrix}^{\mathrm{T}},$$

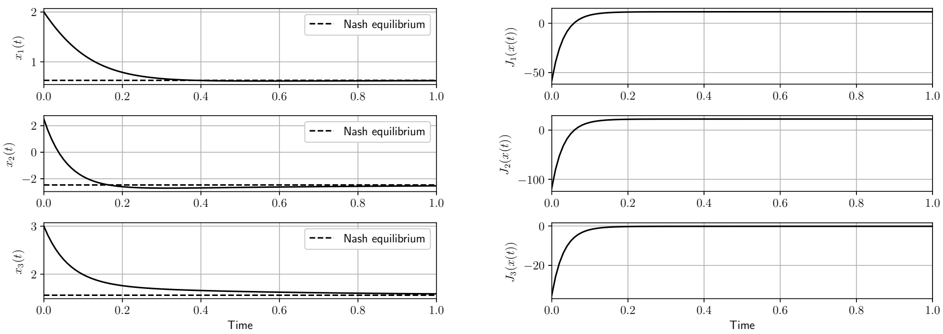
and $r_1 \triangleq 2, r_2 \triangleq 3$, and $r_3 \triangleq -5$. Since the matrices P_1, P_2, P_3 are positive definite, the game $\mathcal{G}_2(J)$ satisfies Assumption 4.1.

Now let $P \in \mathbb{R}^{3\times 3}$ be the matrix defined in (4.10) and $q \in \mathbb{R}^3$ be the vector defined in (4.11), that is,

$$P = \begin{bmatrix} 6 & -1/2 & 0 \\ -1 & 4 & 8 \\ 0 & 2 & 7 \end{bmatrix}, \quad q = \begin{bmatrix} 5 \\ 2 \\ 6 \end{bmatrix}.$$

With the matrices $D_1 = \mathrm{diag}[1, 1/2, 2]$, $D_2 = \mathrm{diag}[2, 1, 4]$, and $D_3 = \mathrm{diag}[1/2, 1/4, 1]$, it follows that (4.14) and (4.15) hold. As a consequence, it follows from Theorem 4.5 that $\mathcal{G}_2(J) \in \mathcal{G}^{\mathrm{SI\&TPE}}$. Furthermore, we can see from Fig. 4.2 that the pattern graph $G(P)$ is weakly connected. Letting $\Phi(x) = J_1(x), w_1 = 1, w_2 = 2$, and $w_3 = 1/2$, it follows that $J_2(x) = w_2\Phi(x) - 1$ and $J_3(x) = w_3\Phi(x) - 6$. Thus, as shown in Theorem 4.7, $\mathcal{G}_2(J) \in \mathcal{G}^{\mathrm{WII}}$.

By Theorem 4.6, the game $\mathcal{G}_2(J)$ has a unique Nash equilibrium given by $x^\star = P^{-1}q = [0.63, -2.47, 1.56]^{\mathrm{T}}$, which is asymptotically stable. Figure 4.3(a) shows the state trajectories of the pseudo-gradient dynamics (4.1) with $\lambda_1 = 2, \lambda_2 = 3$, and $\lambda_3 = 1$ versus time (Nash equilibrium $x^\star$ is shown in dashed lines). Furthermore, because the game $\mathcal{G}_2(J) \in \mathcal{G}^{\mathrm{SI\&TPE}}$, the payoffs $J_1(x), J_2(x), J_3(x)$ of each agent $i \in \mathcal{N}$ are increasing as shown in Fig. 4.3(b).



(a) States of agents $1, 2$, and 3 (b) Payoffs of agents $1, 2$, and 3

Fig. 4.3 The states and payoffs of the agents

4.6 Chapter Summary

We characterized the game by self-improving agents with totally positive externalities for the case with quadratic payoffs. We provided necessary and sufficient conditions for a game by agents with quadratic payoffs to be a game by self-improving agents with totally positive externalities. We also showed that a game by self-improving agents with totally positive externalities has a unique Nash equilibrium that is asymptotically stable. Finally, we showed that a game by self-improving agents and totally positive externalities is a weighted identical interest game.

Chapter 5

Positive Invariance of Set of Positive Externalities

5.1 Introduction

Continuing our descriptive approach in Chapter 4, in this chapter we answer the question whether there are noncooperative games where the payoff of each agent is nondecreasing. To answer this question, we introduce the notion of set of positive externalities. Informally, in the set of positive externalities, the agents influence each other positively. Thus, if the trajectories of all the agents are inside the set of positive externalities, then the payoffs of all the agents are nondecreasing. In this chapter, we characterize the set of positive externalities for game by two pseudo-gradient agents with quadratic payoffs. Furthermore, we show that only a certain subset of the set of positive externalities can be a positively invariant set of the pseudo-gradient dynamics. We also provide necessary and sufficient conditions such that this subset is a positively invariant set of the pseudo-gradient dynamics.

The contents of this chapter are as follows. In Section 5.2, we present the notation we use in this chapter and review some notions from geometry of polyhedral cone and matrices with invariance cone that we use later. In Section 5.3, we introduce the notion of game by pseudo-gradient agents with quadratic payoffs. In Section 5.4, we define the notion of set of positive externalities and characterize this set for games by two pseudo-gradient agents with quadratic payoffs. In Section 5.5, we show that the whole set of positive externalities cannot be a positively invariant set of pseudo-gradient dynamics. Furthermore, we provide necessary and sufficient conditions such that a certain subset of the set of totally externalities is a positively invariant set of the pseudo-gradient dynamics. In Section 5.6, we give an example to illustrate our results. Finally, we sumarize this chapter in Section 5.7.

5.2 Mathematical Preliminaries

5.2.1 Polyhedral cone

A *polyhedral cone* $\Omega \subseteq \mathbb{R}^n$ [82] is described by

$$\Omega \triangleq \{x \in \mathbb{R}^n : Gx \leq 0\}, \tag{5.1}$$

where $G \in \mathbb{R}^{m\times n}$. Note that a polyhedral cone is necessarily convex and closed. Furthermore, note that $\Omega \cap -\Omega = \{0\}$ and thus Ω is a *pointed cone*. If $\mathrm{Int}(\Omega) \neq \emptyset$, then Ω is a *solid cone*. A convex, closed, solid, pointed cone is called a *proper cone*. If there exists a nonsingular matrix $B \in \mathbb{R}^{n\times n}$ such that the polyhedral cone Ω can be described by

$$\Omega = \{y \in \mathbb{R}^n : y = Bx,\ x \geq 0,\ x \in \mathbb{R}^n\}, \tag{5.2}$$

then Ω is called a *simplicial cone*. In $\mathbb{R}^2$, we have the following lemma [83, Theorem 1].

Lemma 5.1. *Let $\Omega \subseteq \mathbb{R}^2$ be the polyhedral cone described in* (5.1). *Then Ω is a proper cone if and only if Ω is a simplicial cone.*

To check whether the interior of a polyhedral cone is not empty, the following lemma is useful.

Lemma 5.2. *Let $\Omega \subseteq \mathbb{R}^n$ be the polyhedral cone described in* (5.1). *Then, the interior of the polyhedral cone Ω is not empty if and only if*

$$y^{\mathrm{T}}G = 0 \text{ and } y \geq 0 \text{ implies } y = 0. \tag{5.3}$$

Proof. The result follows directly from a variant of Farkas Lemma [82, page 95]. □

5.2.2 Matrices with invariant Polyhedral Cone

Consider a continuous-time linear time-invariant dynamics

$$\dot{x}(t) \triangleq Ax(t), \quad x(0) = x_0, \quad t \geq 0, \tag{5.4}$$

where $x(t) \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$. A set $\Gamma \subseteq \mathbb{R}^n$ is a *positively invariant* set of the dynamics (5.4) if $x(0) \in \Gamma$ implies $x(t) \in \Gamma$ for $t \geq 0$, where $x(t)$ is the trajectory of the dynamics (5.4). We need the following lemma by Tarbouriech [84] (independently Castellan and Hennet [85]) that provides a necessary and sufficient condition such that the polyhedral cone Ω is a positively invariant set of the dynamics (5.4).

Lemma 5.3 ([84, 85])**.** *Consider the continuous-time linear time-invariant dynamics* (5.4) *and a polyhedral cone* Ω *described by* (5.1)*. Suppose that* Ω *is a proper cone. Then,* Ω *is a positively invariant set of the dynamics* (5.4) *if and only if there exists a Metzler matrix* $M \in \mathbb{R}^{m\times m}$ *such that*

$$MG = GA. \tag{5.5}$$

In the case where Ω is a simplicial cone, Tarbouriech and Burgat [86, Corollary 3] proved the following lemma.

Lemma 5.4 ([86, Corollary 3])**.** *Consider the continuous-time linear time-invariant dynamics* (5.4) *and the polyhedral cone* Ω (5.1)*. Suppose that* Ω *is a simplicial cone. If* Ω *is a positively invariant set of the dynamics* (5.4)*, then the matrix* A *is similar to a Metzler matrix* M*.*

A useful necessary condition such is given in the following lemma.

Lemma 5.5 ([87, Theorem 6])**.** *Consider the dynamics* (5.4) *and let* $\Omega \subseteq \mathbb{R}^n$ *be a proper polyhedral cone. If* Ω *is a positively invariant set of the dynamics* (5.4)*, then*

1. *there exists a real eigenvalue* $\nu(A)$ *such that* $\nu(A) > Re(\mu)$*, for any eigenvalue* $\mu \neq \nu(A)$ *of* A*, and*
2. *the corresponding eigenvector* v_ν *lies in* Ω*.*

The following result follows directly from Lemma 5.5.

Corollary 5.6. *Consider the dynamics* (5.4) *with* $n = 2$ *and let* $\Omega \subseteq \mathbb{R}^2$ *be a proper polyhedral cone. If* Ω *is a positively invariant set of the dynamics* (5.4)*, then* A *has real eigenvalues.*

Proof. Because Ω is a positively invariant set of the dynamics (5.4), it follows from Lemma 5.5 that A has 1 real eigenvalue. Because A is a real matrix in $\mathbb{R}^{2\times 2}$, it follows that the other eigenvalue of A must also be real. $\square$

5.3 Game by Pseudo-Gradient Agents with Quadratic Payoffs

Consider a game by 2 agents, controlling their individual state $x_i \in \mathbb{R},\ i \in \{1,2\}$. The agents' state profile is denoted by $x = [x_1, x_2]^{\mathrm{T}} \in \mathbb{R}^2$. Suppose that each agent $i \in \{1,2\}$, has a payoff function $J_i : \mathbb{R}^2 \to \mathbb{R}$, $(x_i, x_{-i}) \mapsto J_i(x_i, x_{-i})$ and wants to increase her own payoff. We assume that each agent knows her own payoff function, but not the other agents' payoff function. Furthermore, the payoff functions $J_i(x_i, x_{-i}),\ i \in \{1,2\}$, are quadratic functions described by

$$J_i(x) \triangleq -\frac{1}{2}x^{\mathrm{T}} P_i x + q_i^{\mathrm{T}} x + r_i, \quad i \in \{1,2\}, \tag{5.6}$$

where $P_i \in \mathbb{R}^{2\times 2},\ i \in \{1,2\}$, are symmetric positive-definite matrices, $q_i \in \mathbb{R}^2,\ i \in \{1,2\}$, and $r_i \in \mathbb{R},\ i \in \{1,2\}$. Each agent's state evolves under the *pseudo-gradient dynamics* [26] given by

$$\dot{x}_i(t) = \lambda_i \frac{\partial J_i(x(t))}{\partial x_i}, \quad x_i(0) \triangleq x_{i0}, \quad t \geq 0,$$

where $\lambda_i > 0,\ i \in \{1,2\}$. Note that each agent's dynamics depend on all agents' state profile. The dynamics of all the agents is described by

$$\dot{x}(t) = \Lambda(-Px(t) + q), \quad x(0) \triangleq x_0 \triangleq [x_{10}, x_{20}]^{\mathrm{T}}, \quad t \geq 0, \tag{5.7}$$

where

$$\Lambda \triangleq \mathrm{diag}[\lambda_1, \lambda_2] \in \mathbb{R}^{2\times 2}, \quad P \triangleq \begin{bmatrix} \mathrm{row}_1(P_1) \\ \mathrm{row}_2(P_2) \end{bmatrix} \in \mathbb{R}^{2\times 2}, \tag{5.8}$$

and $q \triangleq [q_1^1, q_2^2]^{\mathrm{T}} \in \mathbb{R}^2$. We denote this game by pseudo-gradient agents with quadratic payoffs as $\mathcal{G}(J)$ with $J \triangleq \{J_1, J_2\}$.

Now we define the Nash equilibrium of the game $\mathcal{G}(J)$ with (5.7). The state profile $x^\star \in \mathbb{R}^2$ is a *Nash equilibrium* of the game $\mathcal{G}(J)$ if

$$J_i(x_i^\star, x_{-i}^\star) \geq J_i(x_i, x_{-i}^\star), \quad x_i \in \mathbb{R}, \quad i \in \{1,2\}. \tag{5.9}$$

Because the payoff functions $J_i(x),\ i \in \{1,2\}$, are quadratic functions (5.6) and $P_i \succ 0,\ i \in \{1,2\}$, it follows that they are concave in $x_i \in \mathbb{R}$ for every fixed $x_{-i} \in \mathbb{R}$. Consequently, it follows from [46, Proposition 1.4.2] that the state profile $x^\star$ is a Nash equilibrium if and only if

$$\frac{\partial J_i(x^\star)}{\partial x_i} = (-\mathrm{row}_i(P_i)x^\star + q_i^i) = 0, \quad i \in \{1,2\}. \tag{5.10}$$

Note that it follows from (5.7) and (5.10) that the Nash equilibrium $x^\star$ is an equilibrium of the dynamics (5.7).

5.4 The Set of Positive Externalities

We consider the notion of *set of positive externalities* of the game $\mathcal{G}(J)$ described by

$$\begin{aligned} \Omega &\triangleq \left\{ x \in \mathbb{R}^2 : \frac{\partial J_i(x)}{\partial x_j}\frac{\partial J_j(x)}{\partial x_j} \geq 0,\ i \neq j \in \{1,2\} \right\} \\ &= \Omega_{12} \cap \Omega_{21}, \end{aligned} \tag{5.11}$$

where $\Omega_{ij} \triangleq \left\{ x \in \mathbb{R}^2 : \frac{\partial J_i(x)}{\partial x_j}\frac{\partial J_j(x)}{\partial x_j} \geq 0 \right\},\ i \neq j,\ i,j \in \{1,2\}$. Note that it is possible that $\Omega = \mathbb{R}^2$. In this case, the game $\mathcal{G}(J)$ is the game by two self-improving agents with totally positive externalities as defined and explored in Chapter 4. Thus, we assume that $\Omega \subsetneq \mathbb{R}^2$. Intuitively, in Ω, all agents influence each other positively. More formally, we have the following proposition:

Proposition 5.7. *Consider the game $\mathcal{G}(J)$ with (5.7). Let Ω be the set of positive externalities. If $x(t) \in \Omega$, then $\dot{J}_i(x(t)) \geq 0,\ i \in \{1,2\}$, where $x(t)$ is the trajectory of the dynamics (5.7).*

Proof. We have

$$\begin{aligned} \dot{J}_i(x(t)) &= \frac{\partial J_i(x(t))}{\partial x_i}\dot{x}_i(t) + \frac{\partial J_i(x(t))}{\partial x_j}\dot{x}_j(t) \\ &= \lambda_i\left(\frac{\partial J_i(x(t))}{\partial x_i}\right)^2 + \frac{\partial J_i(x(t))}{\partial x_j}\dot{x}_j(t) \geq 0, \end{aligned}$$

where $i \neq j,\ i,j \in \{1,2\}$, the second equality sign follows from the dynamics (5.7), and the last inequality follows from (5.11). □

To better understand Ω, we have the following proposition.

Proposition 5.8. *Consider the game $\mathcal{G}(J)$ with (5.7). Let Ω be the set of positive externalities,*

$$Z \triangleq \left[P_1^{\mathrm{T}}, P_2^{\mathrm{T}}\right] \in \mathbb{R}^{4\times 2}, \tag{5.12}$$

$$w \triangleq \left[q_1^{\mathrm{T}}, q_2^{\mathrm{T}}\right] \in \mathbb{R}^{4}, \tag{5.13}$$

$$D_{k,l} \triangleq diag[l,k,l,k] \in \mathbb{R}^{4\times 4}, \quad k,l \in \{-1,1\}, \tag{5.14}$$

and $W_{k,l} \subseteq \mathbb{R}^2$ be described by

$$W_{k,l} \triangleq \left\{x \in \mathbb{R}^2 : D_{k,l}Zx \leq D_{k,l}w\right\}, \quad k,l \in \{-1,1\}. \tag{5.15}$$

Then,

$$\Omega = \bigcup_{k,l\in\{-1,1\}} W_{k,l} = W_{-1,-1} \cup W_{-1,1} \cup W_{1,-1} \cup W_{1,1}. \tag{5.16}$$

Proof. Because $\frac{\partial J_i(x)}{\partial x_j} \in \mathbb{R},\ i,j \in \{1,2\}$, it follows that

$$\begin{aligned}\Omega_{ij} = &\left\{x \in \mathbb{R}^2 : \frac{\partial J_i(x)}{\partial x_j} \geq 0 \text{ and } \frac{\partial J_j(x)}{\partial x_j} \geq 0\right\} \\ &\cup \left\{x \in \mathbb{R}^2 : \frac{\partial J_i(x)}{\partial x_j} \leq 0 \text{ and } \frac{\partial J_j(x)}{\partial x_j} \leq 0\right\},\end{aligned} \tag{5.17}$$

where $i \neq j,\ i,j \in \{1,2\}$. Letting

$$\Omega_{ij}^k \triangleq \left\{x \in \mathbb{R}^2 : k \begin{bmatrix} \frac{\partial J_i(x)}{\partial x_j} \\ \frac{\partial J_j(x)}{\partial x_j} \end{bmatrix} \geq 0\right\}, \tag{5.18}$$

where $i \neq j,\ i,j \in \{1,2\}$, and $k \in \{-1,1\}$, it follows from (5.17) that $\Omega_{ij} = \Omega_{ij}^{-1} \cup \Omega_{ij}^{1},\ i \neq j,\ i,j \in \{1,2\}$. Hence, it follows from (5.11) that

$$\begin{aligned}\Omega &= \Omega_{12} \cap \Omega_{21} \\ &= (\Omega_{12}^{-1} \cup \Omega_{12}^{1}) \cap (\Omega_{21}^{-1} \cup \Omega_{21}^{1}) \\ &= \bigcup_{k,l\in\{-1,1\}} \Omega_{12}^{k} \cap \Omega_{21}^{l}.\end{aligned} \tag{5.19}$$

Let $W_{k,l} \triangleq \Omega_{12}^{k} \cap \Omega_{21}^{l}$. Then, it follows from (5.15) and (5.18) that

$$\begin{aligned} W_{k,l} &= \left\{ x \in \mathbb{R}^2 : k \left[\frac{\partial J_1(x)}{\partial x_2}, \frac{\partial J_2(x)}{\partial x_2} \right]^{\mathrm{T}} \geq 0 \right\} \\ &\cap \left\{ x \in \mathbb{R}^2 : l \left[\frac{\partial J_2(x)}{\partial x_1}, \frac{\partial J_1(x)}{\partial x_1} \right]^{\mathrm{T}} \geq 0 \right\} \\ &= \left\{ x \in \mathbb{R}^2 : \mathrm{diag}[k, k, l, l] \begin{bmatrix} \frac{\partial J_1(x)}{\partial x_2} \\ \frac{\partial J_2(x)}{\partial x_2} \\ \frac{\partial J_2(x)}{\partial x_1} \\ \frac{\partial J_1(x)}{\partial x_1} \end{bmatrix} \geq 0 \right\}. \end{aligned} \tag{5.20}$$

By rearranging (5.20) and noting that

$$\frac{\partial J_i(x)}{\partial x_j} = -\mathrm{row}_j(P_i)x + q_i^j, \quad i, j \in \{1, 2\},$$

the result follows. □

Note that each $W_{k,l}$ $k, l \in \{-1, 1\}$, is a solution to the linear inequalities (5.15) and hence a polyhedron [82]. Thus, each $W_{k,l}$ $k, l \in \{-1, 1\}$, is convex and connected. Although each $W_{k,l}$, $k, l \in \{-1, 1\}$, is connected, it is not necessarily the case that $\Omega = \bigcup_{k,l \in \{-1,1\}} W_{k,l}$ is connected. A sufficient condition such that Ω is connected is given in the following proposition.

Proposition 5.9. *Consider the game $\mathcal{G}(J)$ with* (5.7). *Let Ω be the set of positive externalities, $W_{k,l}$ $k, l \in \{-1, 1\}$, be defined in* (5.15) *and $\omega \subseteq \mathbb{R}^2$ be described by*

$$\omega \triangleq \{x \in \mathbb{R}^2 : Zx = w\}, \tag{5.21}$$

where Z is defined in (5.12) *and w is defined in* (5.13). *If $\omega \neq \emptyset$, then, Ω is connected.*

Proof. From the definition of $W_{k,l}$, $k, l \in \{-1, 1\}$, in (5.15), it follows that each $W_{k,l}$, $k, l \in \{-1, 1\}$, is convex and hence connected. Furthermore, it follows by calculation that $W_{-1,-1} \cap W_{-1,1} \cap W_{1,-1} \cap W_{1,1} = \omega$. Thus, because Ω is the union of $W_{k,l}$, $k, l \in \{-1, 1\}$, each of which is a connected set, the result followss from [88, Theorem 23.3]. □

Note that from Proposition 5.8, Ω is determined by the configuration γ of four lines in the plane described by the equations

$$\mathrm{row}_j(P_i)x + q_i^j = 0, \quad i, j \in \{1, 2\}. \tag{5.22}$$

When dealing with configurations of geometric objects such as points, lines, and surfaces, the notion of *general position* is frequently used. Intuitively, the notion of general position describes the configurations that are "most likely" to happen [89, page 3]. For configuration of lines, there are two conditions for N lines in the plane to be in general position: 1) no two lines are parallel, and 2) no three lines have a common intersection. We assume condition 1), that is, no two lines in the configuration γ of the lines described by (5.22) are parallel. We do not assume condition 2) because the connectedness of Ω depends on the intersection of those lines. Henceforth, we assume the following.

Assumption 5.1. *Any 2 rows of the matrix Z defined in* (5.12) *are linearly independent and the set ω defined in* (5.21) *is not empty.*

From Assumption 5.1 and Proposition 5.9, it follows that Ω is connected. It also follows that there exists a unique Nash equilibrium $x^\star$ of the game $\mathcal{G}(J)$. Assumption 5.1 also implies the following useful lemma.

Lemma 5.10. *Suppose Assumption 5.1 holds. Let*

$$B_{ij} \triangleq \begin{bmatrix} row_i(Z) \\ row_j(Z) \end{bmatrix} \in \mathbb{R}^{2\times 2}, \quad i \neq j, \quad i, j \in \{1, \dots, 4\}, \tag{5.23}$$

and

$$c_{ij} = \left[w^i, w^j\right]^{\mathrm{T}} \in \mathbb{R}^2, \quad i \neq j, \quad i, j \in \{1, \dots, 4\}, \tag{5.24}$$

wherer Z is defined in (5.12) *and w is defined in* (5.13)*. Then, the set of solutions to the linear equation*

$$B_{ij}x = c_{ij}, \quad i \neq j, \quad i, j \in \{1, \dots, 4\}, \tag{5.25}$$

is ω.

Proof. First, it follows from Assumption 5.1 that the matrix Z has full row rank. Thus, if $\omega \neq \emptyset$, it follows that $\omega = \{\tilde{x}\}$ for some $\tilde{x} \in \mathbb{R}^2$. It also follows from Assumption 5.1 that there exists a unique solution $\bar{x}$ to (5.25). By (5.21), $\tilde{x}$ necessarily satisfies (5.25). Thus, $\bar{x} = \tilde{x}$. $\square$

Lemma 5.10 gives us the following.

Corollary 5.11. *Consider the game* $\mathcal{G}(J)$ *with* (5.7) *satisfying Assumption 5.1. Let* $W_{k,l},\ k,l \in \{-1,1\}$, *be defined in* (5.15), ω *be defined in* (5.21), *and* $x^\star$ *be the Nash equilibrium of the game. Then, the followings hold:*

1. $\omega = \{x^\star\}$;
2. $W_p \cap W_q = \omega,\ p,q \in \{-1,1\}^2$, *and*
3. $\bigcap_{i\in\{1,\ldots,3\}} W_{p_i} = \omega,\ p_i \in \{-1,1\}^2,\ i \in \{1,\ldots,3\}$.

Proof. 1. It follows from (5.10) that the Nash equilibrium must satisfy (5.25) with $i = 1$ and $j = 4$. Thus, it follows from Lemma 5.10 that $\omega = \{x^\star\}$.

2. The case where $q = -p$ follows from direct calculation. Therefore, without loss of generality, suppose that $p = (1,1)$ and $q = (1,-1)$. Then, it follows by direct calculation that any element $W_p \cap W_q$ must satisfy the linear equality (5.25) for $i = 1$ and $j = 3$. Thus, it follows from Lemma 5.10 that $W_p \cap W_q = \omega$. The other cases follow from similar reasoning.

3. The result follows directly from 2).

□

Now, suppose Assumption 5.1 holds. Let $y \triangleq x - x^\star \in \mathbb{R}^2$, where $x^\star$ is the unique Nash equilibrium of the game $\mathcal{G}(J)$. Then, it follows from (5.7) that

$$\dot{y}(t) = -\Lambda P y(t), \quad y(0) \triangleq x(0) - x^\star, \quad t \geq 0, \tag{5.26}$$

where P and Λ are defined in (5.8). Let

$$Y_{k,l} \triangleq \left\{y \in \mathbb{R}^2 : D_{k,l} Z y \leq 0\right\}, \quad k,l \in \{-1,1\}, \tag{5.27}$$

where Z is defined in (5.12) and $D_{k,l}\ k,l \in \{-1,1\}$, are defined in (5.14). The next proposition shows that for any $k,l \in \{-1,1\}$, the set $W_{k,l}$ defined in (5.15) is a positively invariant set of the dynamics (5.7) if and only if the set $Y_{k,l}$ defined in (5.27) is a positively invariant set of the dynamics (5.26).

Proposition 5.12. *Consider the game* $\mathcal{G}(J)$ *with* (5.7) *satisfying Assumption 5.1. Let* $k,l \in \{-1,1\}$, *the set* $W_{k,l}$ *be defined in* (5.15), *and the set* $Y_{k,l}$ *be defined in* (5.27). *Then,* $W_{k,l}$ *is a positively invariant set of the dynamics* (5.7) *if and only if* $Y_{k,l}$ *is positively invariant set of the dynamics* (5.26).

Proof. Let $x(t)$ be the trajectory of the dynamics (5.7) and $y(t) = x(t) - x^\star$ be the trajectory of the dynamics (5.26). If $x(t) \in W_{k,l},\ t \geq 0$, then it follows from (5.15) that

$$\begin{aligned} D_{k,l}Zy(t) &= D_{k,l}Z(x(t) - x^\star) \\ &= D_{k,l}Zx(t) - D_{k,l}Zx^\star \\ &\leq D_{k,l}(w - Zx^\star) = 0, \quad t \geq 0. \end{aligned}$$

Thus, $y(t) \in Y_{k,l},\ t \geq 0$.

On the other hand, if $y(t) \in Y_{k,l},\ t \geq 0$, then it follows from (5.27) and $x(t) = y(t) + x^\star$ that

$$\begin{aligned} D_{k,l}Zx(t) &= D_{k,l}Z(y(t) + x^\star) \\ &\leq 0 + D_{k,l}Zx^\star \\ &= D_{k,l}w, \quad t \geq 0. \end{aligned}$$

Thus, $x(t) \in W_{k,l},\ t \geq 0$. $\square$

As a consequence, it follows from Proposition 5.12 that Ω is a positively invariant set of the dynamics (5.7) if and only if $\Omega' \subset \mathbb{R}^2$ described by

$$\Omega' \triangleq \bigcup_{k,l\in\{-1,1\}} Y_{k,l}, \tag{5.28}$$

where $Y_{k,l},\ k,l \in \{-1,1\}$, defined in (5.27), are positively invariant sets of the dynamics (5.26). Thus, it is sufficient to consider the dynamics (5.26) and the sets $Y_{k,l},\ k,l \in \{-1,1\}$.

Note that each $Y_{k,l},\ k,l \in \{-1,1\}$, is a polyhedral cone (5.1) and hence closed, convex, and pointed. Furthermore, also note that for any $k,l \in \{-1,1\}$, $y \in Y_{k,l}$, if and only if $-y \in Y_{-k,-l}$. Thus, $Y_{-k,-l} = -Y_{k,l},\ k,l \in \{-1,1\}$. Although it is not always the case that all $Y_{k,l}\ k,l \in \{-1,1\}$, are solid, we have the following proposition.

Proposition 5.13. *Consider the game $\mathcal{G}(J)$ with (5.26) satisfying Assumption 5.1. Let $Y_{k,l},\ k,l \in \{-1,1\}$, be defined in (5.27). Then, $\bigcup_{k,l\in\{-1,1\}} Int(Y_{k,l}) \neq \emptyset$.*

Proof. First, note that $y \in \mathrm{Int}(Y_{k,l})$ if and only if $-y \in \mathrm{Int}(Y_{-k,-l})$. Thus, it follows that $\bigcup_{k,l\in\{-1,1\}} \mathrm{Int}(Y_{k,l}) \neq \emptyset$ if and only if $\mathrm{Int}(Y_{1,1}) \cup \mathrm{Int}(Y_{-1,1}) \neq \emptyset$. It follows from Lemma 5.2 that $\mathrm{Int}(Y_{1,1}) \neq \emptyset$ (resp., $\mathrm{Int}(Y_{-1,1}) \neq \emptyset$) if and only if $y^{\mathrm{T}}Z = 0$ (resp., $y^{\mathrm{T}}D_{-1,1}Z = 0$)

and $y \geq 0$ implies $y = 0$. It follows from logical rule that $\mathrm{Int}(Y_{1,1}) \cup \mathrm{Int}(Y_{-1,1}) \neq \emptyset$ if and only if ($y^{\mathrm{T}}Z = 0$ and $y^{\mathrm{T}}D_{-1,1}Z = 0$ and $y \geq 0$) implies $y = 0$.

Now, consider the statement ($y^{\mathrm{T}}Z = 0$ and $y^{\mathrm{T}}D_{-1,1}Z = 0$). Expanding both equations, we have $y^{\mathrm{T}}Z = 0$ if and only if $\sum_{i=1}^{4} y^i \mathrm{row}_i(Z) = 0$ and $y^{\mathrm{T}}D_{-1,1}Z = 0$ if and only if $\sum_{i\in\{1,3\}} y^i \mathrm{row}_i(Z) = \sum_{j\in\{2,4\}} y^j \mathrm{row}_j(Z)$. Thus, if $y^{\mathrm{T}}Z = 0$ and $y^{\mathrm{T}}D_{-1,1}Z = 0$, then

$$\sum_{i=1}^{4} y^i \mathrm{row}_i(Z) = 2 \sum_{i\in\{1,3\}} y^i \mathrm{row}_i(Z) = 0.$$

It follows from Assumption 1 that any 2 rows of the matrix Z are linearly independent and thus $y^1 = y^3 = 0$. Repeating similar reasoning, we also have $y^2 = y^4 = 0$. Thus, for any $y \in \mathbb{R}^4$, if $y^{\mathrm{T}}Z = 0$ and $y^{\mathrm{T}}D_{-1,1}Z = 0$, then $y = 0$. Thus, the statement $y^{\mathrm{T}}Z = 0$ and $y^{\mathrm{T}}D_{-1,1}Z = 0$ and $y \geq 0$ implies $y = 0$ always hold. Hence, $\mathrm{Int}(Y_{1,1}) \cup \mathrm{Int}(Y_{-1,1})$ is not empty. □

Thus, the set Ω' always contains at least two proper polyhedral cones. From Lemma 5.1, it follows that Ω' always contains at least two simplicial cones. We assume that Ω' always contains four simplicial cones, that is, all $Y_{k,l},\ k,l \in \{-1,1\}$, are proper cones.

Assumption 5.2. *The sets $Y_{k,l},\ k,l \in \{-1,1\}$, defined in* (5.27) *are proper cones, that is, Int($Y_{k,l}$) $\neq \emptyset$ for all $k,l \in \{-1,1\}$.*

Note that Assumption 5.2 holds if and only if $\mathrm{Int}(W_{k,l}) \neq \emptyset$ for all $k,l \in \{-1,1\}$, where $W_{k,l},\ k,l \in \{-1,1\}$, are defined in (5.15).

5.5 Main Results

First, note that for any $k,l \in \{-1,1\}$, if $Y_{k,l}$, is a positively invariant set of the dynamics (5.26), then $y(0) \in Y_{k,l}$ implies $y(t) \in \Omega',\ t > 0$. If Assumption 5.1 holds, then the converse also holds.

Lemma 5.14. *Suppose that Assumptions 5.1 and 5.2 hold. Let the sets $Y_{k,l},\ k,l \in \{-1,1\}$, be defined in* (5.27)*, Ω' be defined in* (5.28)*, and $y(t)$ be the trajectory of the dynamics* (5.26)*. If $y(t) \in \Omega',\ t \geq 0$, then, there exists $k,l \in \{-1,1\}$, such that $Y_{k,l}$, is a positively invariant set of the dynamics* (5.26)*.*

Proof. Suppose that $y(0) \in \Omega'$, where $y(0) \neq 0$. Then, it follows from (5.28) that $y(0) \in Y_{k,l}$ for some $k,l \in \{-1,1\}$. There are two cases: 1) $y(t) \in Y_{k,l},\ t > 0$, or 2) there exists $T > 0$ such that $y(T) \in \mathrm{Int}(Y_{p,q})$ where $Y_{p,q} \neq Y_{k,l}$. In case 1), $Y_{k,l}$ is a positively invariant set of the dynamics (5.26) and the result follows.

We show that case 2) does not happen. Suppose, *reductio ad absurdum*, that there exists $T > 0$ such that $y(T) \in \mathrm{Int}(Y_{p,q})$. Let $S \triangleq \{x(t) \in \mathbb{R}^2 : 0 \leq t \leq T\} \subset \mathbb{R}^2$. Because the interval $[0, T]$ is connected and $y(t)$ is continuous, it follows from [88, Theorem 23.5] that S is connected. Because S is connected, it follows that there exists $0 < t_0 < T$ such that $y(t_0) \in Y_{k,l} \cap Y_{p,q} = \{0\}$. Because $y(t_0) = 0$ and 0 is the equilibrium of the dynamics (5.26), it follows that $y(t) = 0,\ t \geq t_0$. In particular $y(T) = 0$. Because $D_{p,q}Zy(T) = 0 \not< 0$, it follows from (5.27) that $y(T) \notin \mathrm{Int}(Y_{p,q})$. Thus, $y(t) \in Y_{k,l},\ t \geq 0$, and hence $Y_{k,l}$ is a positively invariant set of the dynamics (5.26). □

From Lemma 5.14, we have the following theorem.

Theorem 5.15. *Consider the game $\mathcal{G}(J)$ with* (5.7) *satisfying Assumptions 5.1 and 5.2. Let Ω be the subset of positive externalities, $x(t)$ be the trajectory of the dynamics* (5.7)*, and $W_{k,l},\ k, l \in \{-1, 1\}$, be defined in* (5.15)*. Suppose that $x(0) \in W_{k,l}$ for some $k, l \in \{-1, 1\}$. Then, $x(t) \in \Omega,\ t \geq 0$ if and only if $W_{k,l}$ is a positively invariant set of the dynamics* (5.7)*.*

Proof. If $W_{k,l}$ is a positively invariant set of the dynamics (5.7), then the result follows. Suppose $x(t) \in \Omega,\ t \geq 0$. Then, it follows from Proposition 5.12 that $y(t) \in \Omega',\ t \geq 0$, where $y(t)$ is the trajectory of the dynamics (5.26) with $y(0) \in Y_{k,l}$. Consequently, it follows from Lemma 5.14 that $Y_{k,l}$ is a positively invariant set of the dynamics (5.26). Because $Y_{k,l}$ is a positively invariant set of the dynamics (5.26), it also follows from Proposition 5.12 that $W_{k,l}$ is a positively invariant set of the dynamics (5.7). □

Theorem 5.15 does not imply that Ω is a positively invariant set of the dynamics (5.7). Instead, Theorem 5.15 only provides necessary and sufficient condition for the trajectory $x(t)$ of the dynamics (5.7) with $x(0) \in W_{k,l}$ for some $k, l \in \{-1, 1\}$, such that $x(t) \in \Omega,\ t \geq 0$. The set of positive externalities Ω is a positively invariant set of the dynamics (5.7) if for any $k, l \in \{-1, 1\}$, $x(0) \in W_{k,l}$, implies $x(t) \in \Omega,\ t \geq 0$. Thus, it follows from Theorem 5.15 that Ω is a positively invariant set of the dynamics (5.7) if and only if $W_{k,l},\ k, l \in \{-1, 1\}$, are positively invariant sets of the dynamics (5.7). However, as shown in the following theorem, it is impossible that all $W_{k,l},\ k, l \in \{-1, 1\}$, are positively invariant sets of the dynamics (5.7). We begin with the following lemmas.

Lemma 5.16. *Suppose that Assumptions 5.1 and 5.2 hold. Let $Y_{k,l},\ k, l \in \{-1, 1\}$, be defined in* (5.27)*. Let $y_{1,1} \in Y_{1,1} \backslash \{0\}$ and $y_{-1,1} \in Y_{-1,1} \backslash \{0\}$. Then, $y_{1,1}$ and $y_{-1,1}$ are linearly independent.*

Proof. Suppose, *reductio ad absurdum*, that $y_{-1,1} = \lambda y_{1,1}$ where $\lambda \neq 0$. If $\lambda > 0$, then $y_{-1,1} \in Y_{1,1}$. Thus $y_{-1,1} \in Y_{-1,1} \cap Y_{1,1}$. Because $Y_{-1,1} \cap Y_{1,1} = \{0\}$ (Corollary 5.11), it follows that $y_{-1,1} = 0$ contrary to Assumption 5.1. Thus, necessarily $\lambda < 0$. Because $Y_{-1,-1} = -Y_{1,1}$, it follows that $y_{-1,1} \in Y_{-1,-1}$ and thus $y_{-1,1} \in Y_{-1,1} \cap Y_{-1,-1}$. Hence, it follows from Corollary 5.11 that $y_{-1,1} = 0$ contrary to Assumption 5.1. Thus, $y_{1,1}$ and $y_{-1,1}$ are linearly independent. □

Lemma 5.17. *Suppose that Assumptions 5.1 and 5.2 hold. Let $k, l \in \{-1, 1\}$ and $Y_{k,l}$ be defined in* (5.27)*. Then, $Y_{k,l}$ is a positively invariant set of the dynamics* (5.26) *if and only if $Y_{-k,-l}$ is also a positively invariant set of the dynamics* (5.26)*.*

Proof. Suppose, without loss of generality, that $Y_{1,1}$ is a positively invariant set of the dynamics (5.26). Thus, because $Y_{1,1}$ is a polyhedral cone, it follows from Lemma 5.3 that there exists a Metzler matrix M such that $MZ = Z(-\Lambda P)$, where Z is defined in (5.12), and Λ and P are defined in (5.8). Note that the same Metzler matrix M satisfies $MD_{-1,-1}Z = -MZ = Z\Lambda P = D_{-1,-1}Z(-\Lambda P)$. Consequently, it follows one more time from Lemma 5.3 that $Y_{-1,-1}$ is a positively invariant set of the dynamics (5.26). The case $Y_{-1,1}$ and $Y_{1,-1}$ follows from similar reasoning. □

Now, we state the theorem that Ω is not a positively invariant set of the dynamics (5.7).

Theorem 5.18. *Consider the game $\mathcal{G}(J)$ with* (5.7) *satisfying Assumptions 5.1 and 5.2. Let Ω be the set of positive externalities and $W_{k,l} \in \{-1, 1\}$, be defined in* (5.15)*. Then, at most only one of the following statements holds:*

1. *both $W_{1,1}$ and $W_{-1,-1}$ are positively invariant sets of the dynamics* (5.7)*, or*
2. *both $W_{-1,1}$ and $W_{1,-1}$ are positively invariant sets of the dynamics* (5.7)*.*

Proof. Suppose, *reductio ad absurdum*, that both statements hold, that is, each $W_{k,l}$, $k, l \in \{-1, 1\}$, is a positively invariant set of the dynamics (5.7). Thus, it follows from Proposition 5.12 that each $Y_{k,l}$, $k, l \in \{-1, 1\}$, is a positively invariant set of the dynamics (5.27). Because $Y_{k,l}$ is a positively invariant set of the dynamics (5.26) if and only if $Y_{-k,-l}$ is a positively invariant set of the dynamics (5.26) (Lemma 5.17), it is sufficient to consider $Y_{1,1}$ and $Y_{-1,1}$. Because $Y_{1,1}$ and $Y_{-1,1}$ are positively invariant sets of the dynamics (5.26), it follows from Lemma 5.5 there exists eigenvectors $v_{1,1} \in Y_{1,1}$ and $v_{-1,1} \in Y_{-1,1}$ corresponding to the real eigenvalue $\nu(-\Lambda P)$, where Λ and P are defined in (5.8). Necessarily then, it follows from Lemma 5.16 that $v_{1,1}$ and $v_{-1,1}$ are linearly independent. Because $-\Lambda P \in \mathbb{R}^{2\times 2}$, having an eigenvalue with two linearly independent eigenvectors implies that $-\Lambda P$ is a diagonal matrix. Because Λ is a diagonal matrix, it follows that P is a diagonal matrix. However, it follows from (5.6) and (5.8) that P is a diagonal matrix if and only if $P_i,\ i \in \{1, 2\}$, are diagonal matrices. Necessarily, then, $\mathrm{row}_j(P_1) = \mu_j \mathrm{row}_j(P_2),\ \mu_j > 0,\ j \in \{1, 2\}$, contrary to Assumption 5.1. Thus, we arrive at a contradiction and at most only one of the statements holds. □

Thus, it follows from Theorems 5.15 and 5.18 that $x(t) \in \Omega,\ t \geq 0$, if and only if $x(0) \in W_{k,l}$ for some $k, l \in \{-1, 1\}$, and $W_{k,l}$ is a positively invariant set of the dynamics (5.7). Given $k, l \in \{-1, 1\}$, the next theorem provides necessary and sufficient condition such that the set $W_{k,l}$ defined in (5.15) is a positively invariant set of the dynamics (5.7).

Theorem 5.19. *Consider the game $\mathcal{G}(J)$ with (5.7) satisfying Assumptions 5.1 and 5.2. Let $k, l \in \{-1, 1\}$, and $W_{k,l}$ be defined in (5.15). Then, $W_{k,l}$ and $W_{-k,-l}$ are positively invariant sets of the dynamics (5.7) if and only if there exists a Metzler matrix $M_{k,l}$, satisfying*

$$MD_{k,l}Z = D_{k,l}(-\Lambda P), \tag{5.29}$$

where $D_{k,l}$ is defined in (5.14), Z is defined in (5.12), and both Λ and P are defined in (5.8).

Proof. It follows from Proposition 5.12 that $W_{k,l}$ is a positively invariant set of the dynamics (5.7) if and only if the set $Y_{k,l}$ defined in (5.27) is a positively invariant set of the dynamics (5.26). Furthermore, because $Y_{k,l}$ is a positively invariant set of the dynamics (5.26) if and only if $Y_{-k,-l}$ is a positively invariant set of the dynamics (5.26) (Lemma 5.17), it is sufficient to consider $Y_{k,l}$. Because $Y_{k,l}$ is a polyhedral cone (5.27), it follows from Lemma 5.3 that $Y_{k,l}$ is a positively invariant set of the dynamics (5.26) if and only if (5.29) holds. □

Although Theorem 5.19 provides a necessary and sufficient condition such that given $k, l \in \{-1, 1\}$, the set $W_{k,l}$ is a positively invariant set of the dynamics (5.26), in general, given the matrices $D_{k,l},\ Z,\ \Lambda$, and P, it is hard to check whether there exists a Metzler matrix $M_{k,l}$ satisfying (5.29). However, because we assume that each $Y_{k,l},\ k, l \in \{-1, 1\}$, is a simplicial cone (Assumption 5.2), we have the following proposition.

Proposition 5.20. *Consider the game $\mathcal{G}(J)$ with (5.7) satisfying Assumptions 5.1 and 5.2. Let $k, l \in \{-1, 1\}$, $W_{k,l}$, be defined in (5.15) and $Y_{k,l} \in \{-1, 1\}$, be defined in*

(5.27). *If $W_{k,l}$ is a positively invariant set of the dynamics* (5.7)*, then the following statements hold:*

1. *there exists an eigenvector v_ν of the matrix $-\Lambda P$ corresponding to the eigenvalue $\nu(-\Lambda P)$ such that $v_\nu \in Y_{k,l}$ and $\nu(-\Lambda P) > Re(\mu)$, for any eigenvalue $\mu \neq \nu(-\Lambda P)$ of $-\Lambda P$, where Λ and P are defined in* (5.8)*, and*

2. *the matrix $-\Lambda P$ is similar to a Metzler matrix M.*

Proof. First, it follows from Proposition 5.12 that $Y_{k,l}$ is a positively invariant set of the dynamics (5.26). Thus, because $Y_{k,l}$ is a polyhedral cone (5.27), 1) follows directly from Lemma 5.5. Furthermore, it follows from Assumption 5.2 that $Y_{k,l}$, is a simplicial cone. Thus, 2) follows directly from Lemma 5.4. □

Remark 5.21. *Note that as mentioned in Corollary 5.6, Proposition 5.20 implies that the matrix $-\Lambda P$ has real eigenvalues. However, it is not necessarily the case that the matrix $-\Lambda P$ has both negative eigenvalues.*

5.6 Numerical Example

To illustrate our main results, consider the following example. Consider the game $\mathcal{G}(J)$ by two pseudo-gradient agents with quadratic payoffs, where each agent $i \in \{1,2\}$, controls a scalar state $x_i \in \mathbb{R}$, $i \in \{1,2\}$, with quadratic payoff functions given by (5.6), where

$$P_1 \triangleq \begin{bmatrix} 4 & -1 \\ -1 & 5 \end{bmatrix}, \quad q_1 \triangleq \begin{bmatrix} 3 \\ 4 \end{bmatrix},$$
$$P_2 \triangleq \begin{bmatrix} 9 & -2 \\ -2 & 3 \end{bmatrix}, \quad q_2 \triangleq \begin{bmatrix} 7 \\ 1 \end{bmatrix},$$

$r_i \triangleq 0$, $i \in \{1,2\}$, and $\lambda_1 = \lambda_2 = 1$. The matrix Z and the vector w defined in (5.12) and (5.13) are given by

$$Z = \begin{bmatrix} 4 & -1 \\ -1 & 5 \\ 9 & -2 \\ -2 & 3 \end{bmatrix}, \quad w = \begin{bmatrix} 3 \\ 4 \\ 7 \\ 1 \end{bmatrix}.$$

By direct calculation, $Z[1,1]^{\mathrm{T}} = w$, and thus it follows that the set ω as defined in (5.21) is not empty. Thus, it follows from Corollary (5.11) that $x^\star = [1,1]^{\mathrm{T}}$ is the Nash equilibrium of the game $\mathcal{G}(J)$. Furthermore, by observation, any two rows of the matrix Z are linearly independent. Thus, Assumption 5.1 holds. Now, by direct calculation, it follows that $Z[2,2]^{\mathrm{T}} > w$, $Z[1/2,1/2]^{\mathrm{T}} < w$, $D_{1,-1}Z[1,-4]^{\mathrm{T}} < D_{1,-1}w$, and $D_{-1,1}Z[1,4]^{\mathrm{T}} < D_{-1,1}w$, where $D_{k,l},\ k,l \in \{-1,1\}$, are defined in (5.14). Thus, $\mathrm{Int}(W_{k,l}) \neq \emptyset$ for all $k,l \in \{-1,1\}$, and hence Assumption 5.2 holds (see Fig. 5.1).

Now, the matrices Λ and P defined in (5.8) are given by

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix}.$$

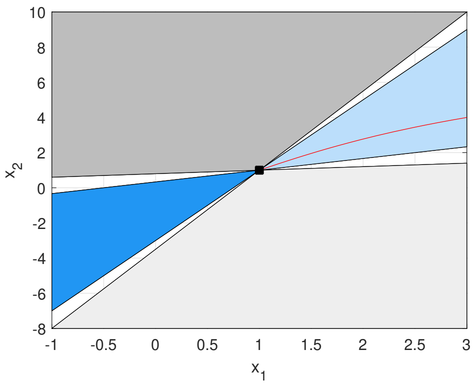
The eigenvalues of $-\Lambda P$ are -2 and -5, thus $\nu(\Lambda P) = -2$. The corresponding eigenvector $v_\nu = [0.4472, 0.8944]^{\mathrm{T}}$ and by direct calculation, $Zv_\nu > 0$ and thus $v_\nu \notin Y_{-1,1}$. Thus, it follows from Proposition 5.20 that $W_{-1,1}$ and $W_{1,-1}$ are not positively invariant sets of the dynamics (5.7). Furthermore, by direct calculation, the Metzler matrix $M_{1,1}$ given by

$$M_{1,1} \triangleq \begin{bmatrix} -4 & 0 & 0 & 1 \\ 2.1281 & -3.2297 & 0.4925 & 1.0872 \\ 0.228 & 0.7205 & -4.1422 & 1.0453 \\ 2 & 0 & 0 & -3 \end{bmatrix}$$

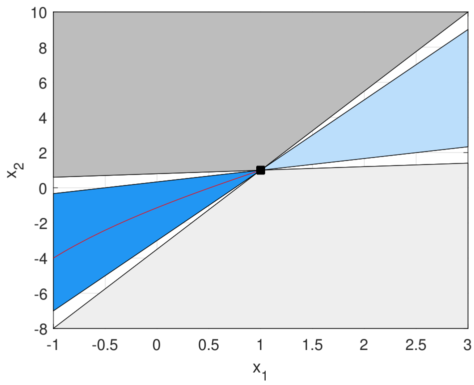
satisfies (5.29) for $k = l = 1$ and thus it follows from Theorem 5.19 that $W_{1,1}$ and $W_{-1,-1}$ are positively invariant sets of the dynamics (5.7).

Fig. 5.1 shows trajectories of the dynamics (5.7) for different initial conditions $x(0)$. In Fig. 5.1(a) (resp., 5.1(b)), the dynamics (5.7) starts from initial conditions $x(0) = [3,4]^{\mathrm{T}} \in W_{-1,-1}$ (resp., $x(0) = [-1,-4]^{\mathrm{T}} \in W_{1,1}$). Because $W_{-1,-1}$ (resp., $W_{1,1}$) is a positively invariant set of the dynamics (5.7), it follows that $x(t) \in W_{-1,-1},\ t \geq 0$ (resp., $x(t) \in W_{1,1},\ t \geq 0$). On the other hand, in In Fig. 5.1(c) (resp., 5.1(d)), the dynamics (5.7) starts from initial conditions $x(0) = [1,-8]^{\mathrm{T}} \in W_{-1,-1}$ (resp., $x(0) = [-1,6]^{\mathrm{T}} \in W_{1,1}$). Because $W_{-1,1}$ (resp., $W_{1,-1}$) is not a positively invariant set of the dynamics (5.7), we can see that there exists $T_{-1,1} > 0$ (resp., $T_{1,-1} > 0$) such that $x(T_{-1,1}) \notin W_{-1,1}$ (resp., $x(T_{1,-1}) \notin W_{1,-1}$))

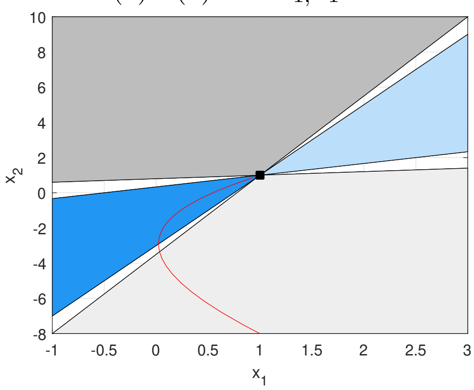
Because $W_{-1,-1}$ (resp., $W_{1,1}$) is a positively invariant set of the dynamics (5.7), it follows from Proposition that 5.7 if $x(0) \in W_{-1,-1}$ (resp., $x(0) \in W_{1,1}$), the payoff of each agent is nondecreasing. Fig. 5.2 shows the payoff of each agent when the the



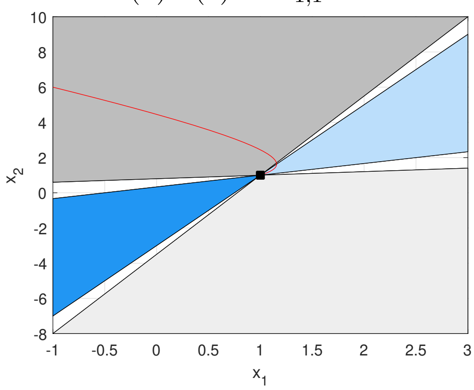
(a) $x(0) \in W_{-1,-1}$



(b) $x(0) \in W_{1,1}$

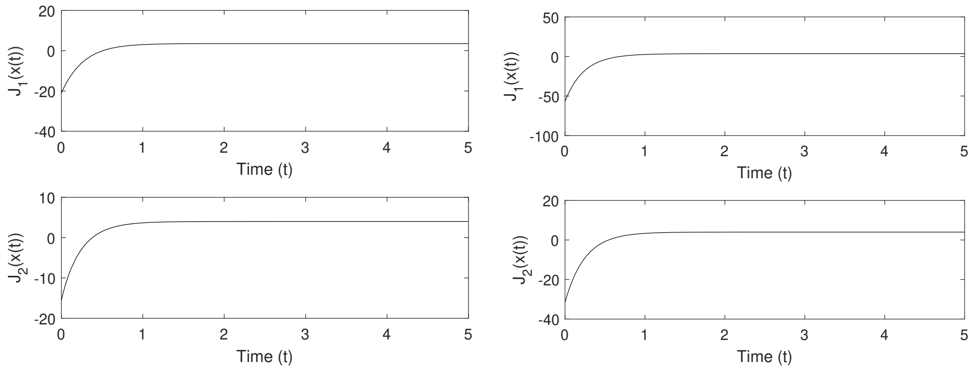


(c) $x(0) \in W_{-1,1}$



(d) $x(0) \in W_{1,-1}$

Fig. 5.1 Trajectory of the agents are shown by the red curve. The Nash equilibrium is at $(1, 1)$ shown by the black dot in the center. The colored areas are: 1) $W_{-1,-1}$ in light blue, 2) $W_{1,1}$ in blue, 3) $W_{1,-1}$ in light grey, and 4) $W_{-1,1}$ in grey.



(a) The payoff of each agent when the initial condition is $x(0) \in W_{-1,-1}$

(b) The payoff of each agent when the initial condition is $x(0) \in W_{1,1}$

Fig. 5.2 The payoff of the agents in the positively invariant sets $W_{-1,-1}$ and $W_{1,1}$

dynamics (5.7) starts from the initial condition $x(0) \in W_{-1,-1}$ (resp, $x(0) \in W_{1,1}$). We can see that in both cases (Fig. 5.2(a) and Fig. 5.2(b)), the payoff of each agent is nondecreasing.

5.7 Chapter Summary

We explored the notion of set of positive externalities. We characterized this set for games by two pseudo-gradient agents with quadratic payoffs. We showed that only a certain subset of the set of positive externalities can be a positively invariant set of the pseudo-gradient dynamics. We also provided necessary and sufficient conditions such that this subset is a positively invariant set of the pseudo-gradient dynamics. We illustrated our results with an example.

Chapter 6

Concluding Remarks and Future Research Recommendations

6.1 Conclusion

The focus of this thesis was the characterization of the behavior of dynamic agents in noncooperative games. In particular, we considered the Nash equilibrium seeking problem for multiple-order agents and we derived conditions for self-improving agents in noncooperative interaction such that their payoffs are nondecreasing.

We first developed Nash equilibrium seeking dynamics for second-order dynamic agents. Specifically, inspired by convex optimization algorithms, we designed the vanishing damping input and the Nesterov's gradient input that ensure convergence of the agents' trajectories to the Nash equilibrium. We note that both inputs have time-varying coefficients instead of constant coefficients. In deriving the inputs, we were able to relax the assumption on the pseudo-gradient. Specifically, instead of strong monotonicity assumption on the pseudo-gradient, we assumed cocoercivity to show convergence to the Nash equilibrium. Furthermore, in the case of vanishing damping input, we constructed a time-varying Lyapunov function to show that the Nash equilibrium is globally asymptotically stable.

In Chapter 3, we developed Nash equilibrium seeking dynamics for single-input single-output linear time-invariant dynamic agents. We used a passivation method to design static input such that the Nash equilibrium is asymptotically stable. Although in this case we have to assume that the pseudo-gradient is strongly monotone, our approach could handle agents possessing nonminimum phase linear time-invariant dynamics.

In the second part of this thesis, we derived conditions for self-improving agents with quadratic payoffs such that their payoffs are nondecreasing. First, in Chapter 4, we considered the case where each agent benefits each other everywhere in the state space, that is, the case of the game with totally positive externalities. We showed that when an agent evolves according to affine dynamics, the agent is self-improving if and only if she evolves according to the pseudo-gradient dynamics. Then, we provided necessary and sufficient conditions such that the game by self-improving agents with quadratic payoffs is a game with totally positive externalities. Furthermore, we proved that in this game, there exists a unique Nash equilibrium. We also provided a sufficient condition such that the unique Nash equilibrium is asymptotically stable. Lastly, considering the pattern graph of the game, we showed that if the pattern graph is connected, then the game by self-improving agents with totally positive externalities is a weighted game with identical interest.

In Chapter 5, we investigated the case where two agents possessing the pseudo-gradient dynamics benefit each other in a certain set. We call this set as the set of totally positive externalities. We showed that the set of positive externalities is a union of convex polyhedra. Because the union of convex polyhedra is not necessarily connected, we provided sufficient conditions such that the set of positive externalities is connected. Then, we proved that when the set of positive externalities is the union of proper convex cones intersecting only at the unique Nash equilibrium, it is not positively invariant. However, a certain subset of the set of positive externalities might be positively invariant. We provided necessary and sufficient conditions such that this certain subset is positively invariant.

6.2 Future Research Recommendations

Our results in Chapter 2 are applicable only in the case of second-order dynamic agents. It is beneficial to consider the case of general multiple-order nonlinear dynamic agents. Furthermore, we needed to assume the existence of a *unique* Nash equilibrium for this thesis. It is plausible that our results can be extended to the case of a set of multiple Nash equilibria. In Chapter 2, the agents are supposed to constitute cyberphysical systems. It is natural to ask how to extend our result to handle various kinds of disturbances. Recently, Romano and Pavel in [36] considered the case of a game by multi-integrator dynamic agents with disturbances and designed a Nash equilibrium seeking input by lthe linear disturbance rejection method. Guo and Persis in [90] considered the case of a game by LTI dynamic agents with disturbances and uncertain parameters. The authors then used the linear output regulation method to stabilize the Nash equilibrium. Because in Chapter 2 we assumed that the agents evolve according to nonlinear dynamics with uniform relative degree 2, we might be able to use nonlinear robust control techniques such as Lyapunov redesign and robust backstepping to handle disturbances and uncertainties [91].

In Chapter 3, we assumed that there is a central operator that communicates with all the agents. One interesting question is whether we could design a static input that does not require the central operator. Another research direction to pursue would be to suppose that the information from and to the central operator is corrupted by noise. Besides the consideration of central operator, we could also extend our approach to the case of agent with nonlinear dynamics that can be brought to the normal form. Recall that in the normal form, the nonlinear dynamics can be written as the cascade of kth-order integrator with the internal dynamics. Thus, by assuming that the zero dynamics possess some dissipativity or conic property, it should be possible to design a Nash equilibrium seeking input for a class of nonlinear dynamic agents. However, because the Nash equilibrium is not known, we need the notion of *equilibrium independent dissipativity* [92–95] instead of the usual notion of dissipativity. Lastly, in our Nash equilibrium seeking input, there is a single parameter $r \in \mathbb{R}$ that must be known by all the agents. As a consequence, it is difficult to extend our result to the case where each agent has only local information. To eliminate this parameter from our input, further study is needed.

Another line of research that we think will become more important in the near future is research on the game by dynamic agents with vector payoffs. As we are now entering the age of climate emergency, it is becoming more mundane for firms and corporations to include the objective of being sustainable in addition to the objective of maximizing profit. One way to model such firms and organizations with multiple objectives is to model them as agents with multidimensional or vector payoffs. Although there are results regarding existence and characterization of equilibria for such games [96–99], there is still very little results regarding the dynamics of such games.

In Chapter 4, we assume that each agent's state is scalar and unconstrained. We could investigate the case when each agent's state is not scalar. We could also investigate the case where each agent's state is scalar but the state space is constrained. Another direction of research would be to study games where there are relationships between the agents. Specifically, an agent might be in alliance with a subset of all the agents and in antagonistic relationship with another subset of all the agents. In this case, the agent will have positive (resp., negative) externalities from her allies (resp., enemies) everywhere in the whole state space. Hence, it might not be the case that there exists a unique Nash equilibrium that is asymptotically stable. It would be interesting to characterize sufficient conditions for stability and instability of the Nash equilibrium in such games.

In Chapter 5, we considered only the case where the set of positive externalities is the union of shifted proper convex cones that intersect only at the unique Nash equilibrium. Thus, further study is needed for more general case, that is, the case where the set of positive externalities is the union of convex polyhedra which might not be connected. We also only consider the case of two agents. Extending the characterization of set of positive externalities for N agents is another research direction worth pursuing. Furthermore, although we showed in Chapter 5 that the eigenvalues of the matrix in the agents' dynamics are real, we do not provide conditions for the eigenvalues to be strictly negative. An investigation in the line of [100] to provide full characterization of the matrix in the agents' dynamics might be fruitful.

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