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Invited Paper

The 1D Kardar–Parisi–Zhang equation: Height distribution and universality

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The Kardar–Parisi–Zhang (KPZ) equation, which was introduced in 1986 as a model equation to describe the dynamics of an interface motion, has been attracting renewed interest in recent years. In particular, the height distribution of its 1D version was determined exactly for a few special initial conditions. Its relevance in experiments was demonstrated and our understanding of the mathematical structures behind its tractability has deepened considerably. There are also new developments in the applicability of the KPZ universality in wider contexts. This paper is a short introductory review on the basics of the equation and on a few recent topics.

Subject Index A10, A50, A56

1. Introduction

The Kardar–Parisi–Zhang (KPZ) equation is a well known prototypical equation that describes a growing interface [1–3]. Since its introduction, the equation has attracted much attention from various fields of science. One reason for this popularity is that surface growth is a phenomenon that appears in a variety of contexts, from paper wetting and crystal growth to the expansion of the universe [4]. Another reason is that the equation can be thought of as one of the simplest nonequilibrium statistical mechanical models that has nonlinearity, noise effect, and many (infinite) degrees of freedom simultaneously. It also shows characteristic critical behaviors that are shared by many systems and serves as an interesting example of universality for nonequilibrium systems [5,6].

Its 1D version (describing the 1D interface in two dimensions) is already of much physical relevance and shows highly nontrivial behaviors. In this article we mostly focus on this 1D case. From the beginning, the 1D KPZ equation was associated with exact tractability. The exponent for the growth of fluctuations, which is known to be 1/3 and is regarded as a characteristic feature for the KPZ universality class, has already been identified in the original KPZ paper [1]. But it was only several years ago that the first exact solution for the KPZ equation was discovered [7,8]. One reason for this slow development may be that, from the mathematical point of view, there had been a serious difficulty with the equation from the outset. As it is usually written, the KPZ equation is not well defined; there is an intrinsic divergence in the equation, so it was difficult to make sense of the KPZ equation itself. The problem was resolved by Bertini and Giacomin in 1997, who found a reasonable way of defining the "solution" of the KPZ equation [9]. They use the Cole–Hopf transformation and hence

this solution is called the Cole–Hopf solution. But at this point, the techniques to handle the KPZ equation exactly had not been sufficiently developed.

On the other hand, if one is only interested in the universal aspects of the properties of the KPZ equation, one could study a lattice model, which is in the same KPZ universality class but does not have the difficulty of divergence. In fact, the 1/3 exponent had been established for a few lattice models and there had been other results as well [10,11]. But a breakthrough was brought into the field from an unexpected direction. In 1999, Johansson showed that the height *distribution* of a lattice growth model called the single-step model in the scaling limit is described by the Gaussian unitary ensemble (GUE) Tracy–Widom distribution from random matrix theory [12,13]. The result was obtained as a generalization of a previous result by Baik et al. on the statistics of the longest increasing subsequence of random permutations [14]. This was surprising, because it shows a connection between nonequilibrium current fluctuations and random matrix theory, which seem completely unrelated. Moreover, this work opened the way to study further universal properties of the KPZ class.

In the context of random permutations one can naturally consider some symmetries on them. For growth models this corresponds to considering different initial and boundary conditions [15,16]. One of the most important discoveries in the developments was that the height distribution depends on the geometry of the system. For instance, for surface growth with a flat initial condition, the limiting distribution is given by the Gaussian orthogonal ensemble (GOE) Tracy–Widom distribution.

These may sound like fairly mathematical developments. But one should notice that the results are on the statistics of the height of a growing surface, which should have direct relevance in real experiments. Of course, studying such statistics in experiments is a highly nontrivial task but, in 2010, Takeuchi and Sano employed the liquid crystal turbulence as a material and measured the height distribution of a growing region. Their data showed that the height distribution is in perfect agreement with the Tracy–Widom distributions [17–19].

Around the same time, all necessary ingredients were ready for treating the KPZ equation itself. In 2010, the first exact solution of the KPZ equation was given in Refs. [7,8] for the (technically simplest) narrow wedge initial condition. The method was a combination of the notion of the Cole–Hopf solution and the Tracy–Widom formula for the asymmetric simple exclusion process (ASEP) for the step initial conditions [20]. The height distribution of the KPZ equation for this special initial condition was written in the form of an integral of Fredholm determinants. The formula is fairly involved but simple enough that one can evaluate the value numerically with high precision [21].

Simultaneously there had been other attempts to try to obtain (basically) the same quantity from a different perspective. In Refs. [22,23], the authors used the replica approach and the Bethe ansatz solvability of the δ-Bose gas Hamiltonian. Soon after Refs. [7,8], they succeeded in obtaining the same formula as Refs. [7,8]. The advantage of the replica method is that it is suited for various generalizations. The exact solution for the stationary case was obtained in Ref. [24]. There are other works on the multipoint statistics [25], the flat initial condition [26,27], and so on. From this time on, the understanding of the solvability of these models has deepened considerably. O'Connell showed that a finite-temperature version of a semi-discrete directed polymer in random media is related to the quantum Toda lattice [28]. Borodin and Corwin introduced the Macdonald process as an underlying structure for the solvability of the models [29]. There have been many subsequent developments as well.

This paper aims to give a short account of recent developments in this area for nonexperts. We not only try to explain the recent results but also give basic facts related to the topic.

2. Surface growth

Surface growth is a phenomenon familiar to us in everyday life. If you put the edge of a sheet of paper in ink, you would observe that the interface between the dry and wet regions grows gradually. There are many examples of surface growth, such as growth of bacterial colonies, forest fires, crystal growth, etc. On the other hand, controlling surface growth is an important aspect for material science.

Surface growth is a dynamic process and is clearly an example of a nonequilibrium phenomenon. Compared to systems in thermal equilibrium, our understanding of nonequilibrium systems is still unsatisfactory. Although there has been much progress, a fundamental principle that governs nonequilibrium systems is still lacking. We have not found what would replace the maximum entropy principle for equilibrium systems. But nonequilibrium systems show interesting behaviors such as dissipative structures, which are not seen in equilibrium systems. Studying each phenomenon and each system still constitutes an important core of the current research into nonequilibrium systems. From this perspective, surface growth is one such example.

There are various types of surface growth. Here we are interested in cases like the paper wetting above, in which the interface grows in a fairly simple manner macroscopically (e.g. in the form of a line or circle) but the fluctuation grows in time. This type of phenomenon has been termed "kinetic roughening" and here our main interest is to understand how the fluctuation grows and what kinds of properties these fluctuations possess. In many other types of surface growth, we would observe a more interesting pattern formation at the interface due to nonlinearity. For these cases, understanding the mechanism of such structures would be a primary target and studying fluctuations would be much harder.

In principle one could try to describe a surface growth by using the Hamiltonian of the system. This may sound a most natural approach and in fact is employed in many studies but it has an intrinsic theoretical difficulty that the dynamics has a chaotic nature and it is extremely hard to study the long time behaviors of the systems. Here we take a more phenomenological approach. We describe surface growth using stochastic dynamics. An advantage of employing this approach is that one still keeps the microscopic degrees of freedom so that one can study the statistical properties of the system while avoiding the intrinsic difficulty in the first-principles approach.

Now let us describe what kind of phenomena we are interested in by using a simple example: ballistic deposition. This is a discrete time surface growth model on an integer lattice with the following stochastic rule. Let us denote by $h(x,t)$ the height at position x and at time t. Here x and t are both discrete. Initially we have a flat substrate: $h(x,0)=0, \forall x$. Then at each time t, a particle (a square of unit area) falls down from above at a random position x (site). The particle tries to go down to the position at which the neighboring position is already occupied. In other words, when the heights at neighboring positions are not higher than that at x(i.e., when $h(x\pm 1,t)\leq h(x,t)$), the particle goes all the way down to the surface at x($h(x,t+1)=h(x,t)+1$), but if the height at a neighboring position is higher than that at x (i.e., when $h(x+1,t)>h(x,t)$, say), the particle stops at the same height ($h(x,t+1)=h(x+1,t)$). This defines the stochastic dynamics of the surface and it is fairly easy to see how the surface grows under these rules by using computer simulations. On average, the surface grows linearly in time with a fixed velocity. One also observes that fluctuations around the average grow in time. We are interested in these fluctuations. If there were no influence from the neighboring sites in the above deposition rules, the height at each site would grow independently of other parts. Then, by the central limit theorem, the fluctuation of the surface would grow like $O\left(t^{1/2}\right)$ and the scaled height would obey the Gaussian distribution. The question is whether this behavior persists when we have the influence from the neighbors. In simulations, one observes that

the fluctuations grow like $O\left(t^{1/3}\right)$ rather than $O\left(t^{1/2}\right)$. One would guess that the surfaces at various points have strong correlations and as a result the fluctuation becomes no longer Gaussian. There is something nontrivial going on, which we want to understand. The same exponent is seen in many simulation models. From the experience in equilibrium statistical physics, we expect that there is some universality.

3. The KPZ equation as a surface growth model

Here we present the KPZ equation and explain how the equation appears as a model equation to describe surface growth [1,2].

3.1. *The equation*

Let $h(x,t)$ represent the height of the surface at position x and time $t \geq 0$. The 1D version of the KPZ equation reads

$$\frac{\partial h(x,t)}{\partial t} = \frac{\lambda}{2}\left(\frac{\partial h(x,t)}{\partial x}\right)^2 + \nu\frac{\partial^2 h(x,t)}{\partial x^2} + \eta(x,t), \tag{3.1}$$

where $\eta(x,t)$ is taken to be the Gaussian white noise with mean zero and covariance:

$$\left\langle \eta(x,t)\,\eta\left(x',t'\right)\right\rangle = D\delta\left(x-x'\right)\delta\left(t-t'\right). \tag{3.2}$$

Here $\langle\cdots\rangle$ indicates the average with respect to the randomness η. The parameters λ, ν, D represent the strengths of each term on the right-hand side of the equation. But there is a simple scaling (see, e.g., Ref. [30]) that one can set without loss of generality to $\nu = 1/2$, $\lambda = D = 1$. Hence, the KPZ equation that we study in the following is

$$\frac{\partial h(x,t)}{\partial t} = \frac{1}{2}\left(\frac{\partial h(x,t)}{\partial x}\right)^2 + \frac{1}{2}\frac{\partial^2 h(x,t)}{\partial x^2} + \eta(x,t), \tag{3.3}$$

with the noise covariance

$$\langle \eta(x,t)\,\eta\left(x',t'\right)\rangle = \delta\left(x-x'\right)\delta\left(t-t'\right). \tag{3.4}$$

3.2. *"Derivation"*

Here we provide a simple heuristic argument to "derive" the KPZ equation above. Suppose we want to describe a surface growth by some partial differential equation. Let $h(x,t)$ represent the height of the surface at position x and at time t. Their values are now continuous. We expect that there would be a sort of smoothening mechanism. Then the simplest equation for the surface growth may be taken to be the diffusion equation:

$$\frac{\partial h(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2 h(x,t)}{\partial x^2}. \tag{3.5}$$

But this is a deterministic equation and the surface has no fluctuations. The simplest way to include fluctuations is to add a noise term. Then we may employ the equation

$$\frac{\partial h(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2 h(x,t)}{\partial x^2} + \eta(x,t), \tag{3.6}$$

with η taken to be the Gaussian noise with mean zero and covariance (3.4). This is known as the Edwards–Wilkinson (EW) equation [31]. Since this is a linear equation, it can be solved exactly by using Fourier analysis. But since the noise is Gaussian and the equation is linear, the fluctuations seen

for this equation are still Gaussian. The EW equation is not enough to describe the surface growth of interest.

A great observation of KPZ was that, while a surface grows in a normal (to the surface) direction, we are measuring the height on the height axis, which is perpendicular to the space x axis, and hence there should appear a nonlinearity coming from this simple geometric effect. When the surface slope $\partial_x h$ is small, the effect takes the form $\sqrt{1+(\partial_x h)^2} \simeq 1 + \frac{1}{2}(\partial_x h)^2$ and the second term is exactly in the form of nonlinear term in the KPZ equation. One can eliminate the constant term by a simple change of coordinate h and then the equation we would get from these considerations is the KPZ equation.

Here it should be noted that, though the KPZ equation was "derived" for the case of a small slope, it does not necessarily mean that the slope should always be small. Once the equation is written down, the height function h can be any function, as long as it is single-valued.

4. Well-definedness of the KPZ equation

4.1. *Cole–Hopf transformation*

The nontrivial features of the KPZ equation appear from the nonlinear term in the equation. But through a simple nonlinear transformation,

$$Z(x,t) = \exp(h(x,t)), \tag{4.1}$$

known as the Cole–Hopf transformation, the KPZ equation can be (formally) rewritten as

$$\frac{\partial}{\partial t} Z(x,t) = \frac{1}{2}\frac{\partial^2 Z(x,t)}{\partial x^2} + \eta(x,t)\, Z(x,t). \tag{4.2}$$

One might think that the problem has now become trivial since Eq. (4.2) is now linear in Z. But this is not true (for instance, one cannot solve this by a simple Fourier analysis), because the noise η has now become multiplicative.

Still, the transformation has a few advantages. For instance, since the equation has the form of an imaginary time Schrödinger equation, the quantity Z in (4.2) can be interpreted as a (random) partition function for a directed polymer in a random environment η. Equation (4.2) is often called the stochastic heat equation (SHE).

From the point of view of the polymer problem, a natural initial condition is when the polymer starts from a single point, say, from the origin: $Z(x,0) = \delta(x)$. The corresponding initial condition for the height looks singular, since using the inverse of (4.1) one sees $h(x,0) = \log Z(x,0) = \log \delta(x)$. But if one writes $Z(x,0) = \lim_{\delta\to 0} c_\delta e^{-|x|/\delta}$, with c_δ some constant depending on δ, one can interpret the initial condition to be the sharp wedge initial condition, $h(x,0) = -\lim_{\delta\to 0}|x|/\delta$.

4.2. *"The KPZ equation is not well defined"*

When we say "the KPZ equation", it is usually referring to Eq. (3.3). But the KPZ equation as written in (3.3) is in fact not well defined. Though it may look just innocent, the equation in (3.3) contains intrinsic divergence. Hence all the results of Eq. (3.3) are meaningless in the sense that they are not completely on solid mathematical ground. Of course, they should contain some useful information on the equation in a formal sense. The question is how to make sense of them.

To consider the issues, it is more convenient to switch from the Langevin-type equation, which has a noise term as in (3.3), to a stochastic differential equation. The white noise $\eta(t)$, which depends only on time t, is formally associated with the time derivative of the Brownian motion: $dB(t)/dt = \eta(t)$.

As a generalization, the space–time white noise $\eta(x,t)$ is formally written as $\eta(x,t) = \partial B(x,t)/\partial t$, where $B(x,t)$ is the cylindrical Brownian motion with $dB(x,t)dB(x',t) = \delta(x-x')\,dt$. Then the equation for Z may be rewritten as

$$dZ(x,t) = \frac{1}{2}\frac{\partial^2 Z(x,t)}{\partial x^2}dt + Z(x,t) \times dB(x,t). \tag{4.3}$$

Here the question is the interpretation of the product $\times$ in (4.3). As is well known, there are two ways to do this: Itô's way and Stratonovich's [32]. A big difference appears when one considers a chain rule for a function of a stochastic process. For Itô, there is the famous Itô's formula, which says that there appears a correction term. On the other hand, for Stratonovich, there is no such term and one can use a usual chain rule.

When deriving (4.2) from the KPZ equation, we wrote "formally". By this, we meant that we did the calculations as if the variables such as $h(x,t)$ and $Z(x,t)$ are ordinary functions, i.e., not stochastic variables, so we did not include the Itô correction terms. Hence the interpretation of the product $Z(x,t) \times dB(x,t)$ in (4.3) should be Stratonovich: $Z(x,t) \circ dB(x,t)$. Switching to Itô by $Z(x,t) \circ dB(x,t) = Z(x,t)dB(x,t) + \frac{1}{2}dZ(x,t)dB(x,t)$ and using (4.3) for the second term, we encounter $\delta(0)$, which is not well defined. This is one way of seeing that the KPZ equation as written in the form of (3.3) contains a divergence and and hence is not well defined.

But in some sense we may think that the situation is not too bad. We could identify where and how the divergence appears in (3.3). It is given by a single $\delta(0)$. On the other hand, for lattice systems we do not observe such divergence. One may think that there is a renormalization working, which makes the systems finite.

For the KPZ equation, one notices that the SHE with the Itô interpretation from the beginning

$$dZ(x,t) = \frac{1}{2}\frac{\partial^2 Z(x,t)}{\partial x^2}dt + Z(x,t)\,dB(x,t) \tag{4.4}$$

is well defined. Then we can reverse the arguments: we can start from (4.4) and then *define* the "Cole–Hopf" solution of the KPZ equation by $h = \log Z$ [9]. The well defined version of the KPZ equation may be written as [33]

$$\partial_t h(x,t) = \frac{1}{2}\left(\frac{\partial h(x,t)}{\partial x}\right)^2 + \frac{1}{2}\frac{\partial^2 h(x,t)}{\partial x^2} - \infty + \eta(x,t). \tag{4.5}$$

As we have seen, the divergence comes from the delta covariance of the noise. By mollifying the noise, one can study the divergence in (4.5) more concretely [34]. The ∞ in (4.5) is replaced by a term that diverges as the mollification vanishes. More recently, Hairer found a way to define the KPZ equation without, but equivalent to, Cole–Hopf (using ideas from rough path and renormalization) [33]. This was further generalized to the theory of "regularity structures". There are a few other approaches to make sense of this type of stochastic PDE [35,36].

5. Exact solution for the KPZ equation

Though the KPZ equation looks like a fairly simple equation, it contains nonlinearity and noise effects. It is not at all easy to study its properties in detail. At the beginning of Chap. 6 of Ref. [2], in which the authors treat the KPZ equation, it is written "the KPZ equation cannot be solved in closed form due to its nonlinear character". In 2010, an exact solution was found for the height distribution at a single point, for the initial condition $Z(x,0) = \delta(x)$ (narrow wedge for KPZ). In its simplest

form, the result can be written as

$$\left\langle e^{-e^{h(x,t)+\frac{x^2}{2t}+\frac{t}{24}-\gamma_t s}} \right\rangle = \det\bigl(1-K_{s,t}\bigr)_{L^2(\mathbb{R}_+)} \tag{5.1}$$

where $\gamma_t = (t/2)^{1/3}$ and s is a parameter. The "det" on the RHS is the Fredholm determinant. For an integral operator K that acts on the space X and has a kernel $K(x, y)$, the Fredholm determiant is defined as

$$\det(1-K)_{L^2(X)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_X dx_1 \cdots \int_X dx_n \det\bigl(K\bigl(x_i, x_j\bigr)\bigr)_{i,j=1}^n \tag{5.2}$$

where the det on the RHS is the usual determinant for an $n \times n$ matrix. For (5.1), the kernel $K_{s,t}$ is given by

$$K_{s,t}(x, y) = \int_{-\infty}^{\infty} d\lambda \frac{\mathrm{Ai}(x+\lambda)\,\mathrm{Ai}(y+\lambda)}{e^{\gamma_t(s-\lambda)}+1} \tag{5.3}$$

where Ai is the Airy function.

The above formula (5.1) is equivalent to the explicit formula for height distribution,

$$\begin{aligned}\mathbb{P}\left[\frac{h(x,t)+\frac{x^2}{2t}+\frac{t}{24}}{(t/2)^{1/3}} \le s\right] &= 1 - \int_{-\infty}^{\infty} \exp\left[-e^{\gamma_t(s-u)}\right] \\ &\times \bigl(\det(1-P_u(B_t-P_{\mathrm{Ai}})\,P_u)_{L^2(\mathbb{R})} - \det(1-P_u B_t P_u)\bigr)_{L^2(\mathbb{R})}\, du,\end{aligned} \tag{5.4}$$

where $P_{\mathrm{Ai}}(x, y) = \mathrm{Ai}(x)\mathrm{Ai}(y)$, P_u is the projection onto $[u, \infty)$, and the kernel B_t is

$$B_t(x, y) = \int_{-\infty}^{\infty} d\lambda \frac{\mathrm{Ai}(x+\lambda)\,\mathrm{Ai}(y+\lambda)}{e^{\gamma_t\lambda}-1}. \tag{5.5}$$

This is the formula for any finite t. Once one finds the formula, it is not difficult to take the $t \to \infty$ limit. The result is

$$\lim_{t\to\infty} \mathbb{P}\left[\frac{h(x,t)+\frac{x^2}{2t}+\frac{t}{24}}{(t/2)^{1/3}} \le s\right] = F_2(s) \tag{5.6}$$

where $F_2(s)$ is the GUE Tracy–Widom (TW) distribution, which describes the largest eigenvalue distribution of the large GUE from random matrix theory.

The GUE is an ensemble of Hermitian matrices with the measure given by $P(H)dH \propto e^{-\mathrm{Tr}H^2}dH$, where H is an $N \times N$ Hermitian matrix and dH is the Lebesgue measure for all independent components (i.e., $H_{ii}, 1 \le i \le j, \Re H_{ij}, \Im H_{ij}, 1 \le i < j \le N$). Let $x_i, 1 \le i \le N$ denote the eigenvalues of the GUE matrix H. The joint eigenvalue density can be found explicitly to be

$$\frac{1}{Z}\Delta(x)^2 \prod_i e^{-x_i^2} \tag{5.7}$$

where $\Delta(x) = \prod_{1\le j<\le k\le N}(x_k - x_j)$ is the Vandelmonde determinant, and Z is a normalization constant. If we denote the largest eigenvalue by $x_{\max}$, its distribution in the limit of large matrix size

is the GUE Tracy–Widom distribution,

$$\lim_{N\to\infty} \mathbb{P}\left[\frac{x_{\max}-\sqrt{2N}}{2^{-1/2}N^{-1/6}} \le s\right] = F_2(s) = \det(1-P_sK_2P_s)_{L^2(\mathbb{R})}\,, \tag{5.8}$$

where P_s is the projection onto $[s,\infty)$ and K_2 is the Airy kernel:

$$K_2(x,y) = \int_0^\infty d\lambda \mathrm{Ai}(x+\lambda)\,\mathrm{Ai}(y+\lambda).$$

There is also a similar ensemble of real symmetric matrices called the GOE. The eigenvalue distribution for the GOE in the large-matrix-dimension limit is the GOE TW distribution and is denoted by F_1.

Equation (5.6) says that the distribution of the height described by the KPZ equation with the narrow wedge initial condition is, in the large-time limit, given by the GUE Tracy–Widom distribution. It is surprising that the height of a growth process and the largest eigenvalue of the random matrix ensemble share the same distribution in their scaling limits.

There are two obvious questions. One is how much one can generalize the analysis and the other is to understand the mechanism by which such a formula can be found for the KPZ equation.

6. Replica approach

The replica method is commonly used in studies of systems with randomness [37]. Suppose we have a Hamiltonian H that depends on randomness η. Then the partition function, $Z=\sum e^{-H}$, also depends on the randomness η. To know the thermodynamic quantity, our primary task is to compute the average free energy $\langle \log Z\rangle$. This is in general a hard problem. In some cases it turns out to be easier to consider the quantity $\langle Z^n\rangle$. Before the average with respect to randomness is taken, this can be interpreted as a partition function of n-replicated systems with the same randomness. This is the origin of the name "replica" and the quantity $\langle Z^n\rangle$ is often called the replica partition function. After the average over the randomness is performed, $\langle Z^n\rangle$ can be interpreted as a partition function for a nonrandom system with interaction among n-fold original systems. Once one finds some formula for $\langle Z^n\rangle$, one may find the average free energy by resorting to the identity

$$\langle \log Z\rangle = \lim_{n\to 0}\frac{\langle Z^n\rangle - 1}{n}. \tag{6.1}$$

The computation of the averaged free energy $\langle \log Z\rangle$ using this procedure is called the replica trick. The above formula can be "proved" by Taylor expanding $\langle Z^n\rangle = \langle e^{n\log Z}\rangle$ in n. But the problem is that in many cases $\langle Z^n\rangle$ can be computed only for integers $n=0,1,2,\ldots$. Then, to apply (6.1), one has to assume the analytic continuation of the formula at hand with respect to n, which in fact in many cases is not guaranteed [38]. There are still many successful applications and the replica method is one of the standard techniques for studying random systems.

For the case of the KPZ equation, the random system is the directed polymer in randomness η and its partition function Z is the solution to the SHE (4.2). Note that the logarithm of Z is just the height function h for the KPZ equation. A big difference from the above explanations is that, in the context of the KPZ equation, we do not use the replica trick because we are interested not only in the average but also in the full distribution of $\log Z$. This is much harder than computing only the average, but one may still hope that, if we have information on all the moments $\langle Z^n\rangle$, one can recover the full distribution of $\log Z$. This is related to the moment problem.

Since the SHE is a one-particle Schrödinger equation, there is a Feynman–Kac expression for the partition function:

$$Z(x,t) = \mathbb{E}_x\Big(e^{\int_0^t \eta(b(s),t-s)ds} Z(b(t),0)\Big), \tag{6.2}$$

where $b(s)$ is a Brownian motion with the initial position $b(0) = x$. Next we consider the replica partition function $\langle Z^N \rangle$ for a positive integer N. Because η is a Gaussian variable, one can take the average over the noise η to see that the replica partition function can be written as (for the narrow wedge case)

$$\langle Z^N(x,t) \rangle = \langle x | e^{-H_N t} | 0 \rangle \tag{6.3}$$

where H_N is the Hamiltonian of the (attractive) δ-Bose gas,

$$H_N = -\frac{1}{2}\sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} - \frac{1}{2}\sum_{j\neq k}^N \delta(x_j - x_k), \tag{6.4}$$

and $|x\rangle$ denotes the state with all N particles at the same position x [39]. More generally, the equal time N-point function of Z,

$$G(x_1,\dots,x_N;t) = \left\langle \prod_{i=1}^N Z(x_i,t) \right\rangle, \tag{6.5}$$

satisfies

$$\frac{\partial}{\partial t} G(x_1,\dots,x_N;t) = -H_N G(x_1,\dots,x_N;t). \tag{6.6}$$

The advantage of using the replica approach to the KPZ equation is that the δ-Bose gas is known to be exactly solvable. Through the Bethe ansatz, the eigenvalues and eigenfunctions for the Hamiltonian (6.4) can be constructed explicitly. Let us denote them by E_z and $|\Psi_z\rangle$, respectively, where z denotes the label of the eigenstate. We expand the quantity of interest on the LHS of (5.1) as

$$\left\langle e^{-e^{h(x,t)+\frac{x^2}{2t}+\frac{t}{24}-\gamma_t s}} \right\rangle = \sum_{N=0}^\infty \frac{\left(-e^{-\gamma_t s}\right)^N}{N!} \left\langle Z^N(x,t) \right\rangle e^{N\frac{\gamma_t^3}{12}} \tag{6.7}$$

By using the eigenstate expansion, we have

$$\left\langle Z^N \right\rangle = \sum_z \langle x|\Psi_z\rangle \langle \Psi_z|0\rangle e^{-E_z t}. \tag{6.8}$$

But here appears a difficulty of this replica analysis of the KPZ equation. Because the ground-state energy of the N particle system is $-\frac{1}{24}N(N^2-1)$, one sees $\langle Z^N \rangle \sim e^{N^3}$ when N is large and hence the generating function in (6.7) diverges. But if we proceed by using an appropriate analytic continuation, after some calculations one arrives at the Fredholm determinant formula (5.1).

The replica approach to the KPZ equation has in fact been very powerful and useful. For instance, in Ref. [26], the flat initial condition was studied. For this case, the limiting distribution is given by the GOE TW F_1. One can study some aspects of multipoint distribution as well [40,41], but this is not yet completely understood.

7. Stationary two-point correlation

In statistical mechanics, the two-point correlation is one of the most important quantities. In particular, for a system in equilibrium, the correlation function for two different space–time points

in equilibrium can be measured by scattering experiments. But computing this quantity is in many cases very difficult, even for solvable models.

For nonequilibrium systems, the corresponding quantity should be the correlation function for two different space–time points in its stationary situation. For the KPZ equation, this quantity has also been computed explicitly. The calculation was first done by a generalization of the replica analysis. More recently, a formula equivalent to the one in Ref. [42] was obtained rigorously by using the connection to the Macdonald process [43].

For the KPZ equation, the Brownian motion (BM) is stationary. If we set

$$h(x,0) = B(x)\,, \tag{7.1}$$

where $B(x), x \in \mathbb{R}$ is the two-sided BM, the correlation function in this stationary situation is written in the form

$$\langle \partial_x h(x,t)\, \partial_x h(0,0)\rangle = \tfrac{1}{2}(2t)^{-2/3}\, g_t''\Big(x/(2t)^{2/3}\Big). \tag{7.2}$$

The function g_t is written in a similar manner as for the height distribution for the narrow wedge case, but we do not give the formula here because it is fairly complicated. However, it is explicit and simple enough that one can draw the figure of the function.

8. Discrete models

8.1. Asymmetric simple exclusion process (ASEP)

The asymmetric simple exclusion process (ASEP) is a stochastic lattice gas model on a 1D lattice in which each particle performs an asymmetric random walk under exclusion interaction. The particle hopping rate to the right is p and that to the left is q. When a particle hops only in one direction ($p = 0$ or $q = 0$), the ASEP is called the totally asymmetric simple exclusion process (TASEP). By simple mapping, this can be mapped to a surface growth model known as the single-step model. In this mapping, the integrated current for ASEP corresponds to the surface height for the surface growth model. The single-step model is known to belong to the KPZ universality class. We also say that ASEP is in the KPZ class.

Moreover, there is a limiting procedure from ASEP to the KPZ equation. Let us consider a weakly asymmetric limit. We introduce a small parameter ϵ and set $j = x/\epsilon, t = \tau/\epsilon^2$ and at the same time we set $q - p = \sqrt{\epsilon}$. Then, as ϵ goes to zero, the ASEP tends to the KPZ equation [9]. In fact the explicit formula for the KPZ equation for the narrow wedge initial condition (5.4) was first derived by using this connection.

8.2. q-TASEP and q-boson TAZRP

Another discrete model that has played an important role in the recent developments on the KPZ equation is the q-TASEP. This is a generalization of TASEP. In the standard TASEP, a particle hops to the neighboring site (if the target site is empty), always at rate 1. In q-TASEP, a particle still hops to the neighboring site but at a rate depending on the distance to the next particle ahead. More concretely, if we denote the position of the jth particle by x_j, the jth particle hops at the rate $1 - q^{x_{j-1}-x_j-1}$.

One can instead focus on the dynamics of the "gap" $y_i = x_{i-1} - x_i - 1$ for q-TASEP. Then the dynamics of y_i is a version of the totally asymmetric zero-range process in which a particle hops to the right site at the rate $1 - q^{y_i}$. This is sometimes called the q-boson totally asymmetric zero-range process (q-TAZRP). The generator of the process can be written in terms of q-boson operators [44]. To have a surface growth picture, one can simply set the height to be $h(N,t) = x_N(t)$.

8.3. Rigorous replica analysis

For ASEP and q-TAZRP, an n-point function such as $\langle \prod_i q^{h(x_i,t)} \rangle$ satisfies the n particle dynamics of the same process (self-duality). This is a discrete generalization of the fact that the correlation function of $Z(x,t) = e^{h(x,t)}$, where $h(x,t)$ is the solution to the KPZ equation, satisfies the Schrödinger equation for the δ-Bose gas; see (6.6). One can apply the replica approach to ASEP and q-TAZRP to get a Fredholm determinant expression for a generating function for $h(x,t)$ [45].

In this case, the difficulty of the divergent series for the generating function is no longer present. In this sense, this analysis can be thought of as a rigorous version of the replica analysis for the KPZ equation. Put differently, the replica analysis for the KPZ equation (which is not rigorous) can be thought of as a shadow of the rigorous replica for ASEP or q-TAZRP. If one can apply the replica analysis to some problem for the KPZ equation, even though it is not completely rigorous, one could hope and expect that it would be possible to generalize the arguments to a discrete model so that the results for the KPZ equation can be proved mathematically.

In the analysis, one needs to use the completeness of the eigenfunctions written in the form of the Bethe wave function. The issue is often not easy, but for ASEP, q-TASEP, and some generalized processes, it has been established [46].

The existence of duality is often associated with the symmetry of the process. For ASEP, the duality is related to $U_q(sl_2)$ symmetry. One can generalize to other models (multiparticles, multispecies, etc) [47,48].

9. Stochastic integrability

Once the KPZ equation was solved, another natural question was to understand the mechanism and the mathematical structures that allow the exact computations. In the replica analysis, we encountered the δ-Bose gas, which is known to be exactly solvable. We also mentioned that the KPZ equation can be thought of as a limiting case of the ASEP, which is known to be related to a well known exactly solvable model, the XXZ spin chain. There are more connections between models in the KPZ class and the integrable systems. In the theory of integrable systems, there are mainly two classes: classical integrable systems and quantum integrable systems. The difference refers to the underlying dynamics of the systems in question. They have a lot in common but each of them contains its own concepts and techniques. The KPZ equation and related models are described by stochastic dynamics, neither classical nor quantum. Hence, in some sense, they are "stochastic integrable systems" [49]. If one looks more closely at the underlying mechanism of their solvability, they turn out to be closely related to the quantum integrability. But the studies of the KPZ equation and related models are not just simple applications of the methods from quantum integrable systems. There are various new aspects and techniques in the studies. They are in some way related to the stochasticity of the models, but this point should be clarified in future studies.

9.1. Semi-discrete polymer and Toda lattice

A finite-temperature directed polymer on a semi-discrete setting (discrete space and continuous time) was introduced by O'Connell and Yor [50]. Consider N semi-infinite lines $\{(j,s), s \geq 0\}$, $j = 1, 2, \ldots, N$ and a path π, which consists of segments $(j, t_{j-1}) \to (j, t_j)$, $j = 1, 2, \ldots N-1$, $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = t$. One can regard π as a polymer that starts from $(0,0)$, switches from the jth line to the $(j+1)$th line at time t_j for $j = 1, \ldots, N-1$, and ends at (N,t). Suppose that each line is associated with a Brownian motion, B_j, which is independent, and the polymer feels

the potential energy from them on each segment, so that, the energy of the polymer is given by

$$E[\pi] = \sum_{i=1}^{N} (B_i(t_i) - B_i(t_{i-1})). \tag{9.1}$$

The partition function is given by

$$Z_t^N(\beta) = \int_{0<t_1<\cdots<t_{N-1}<t} e^{\beta E[\pi]} dt_1 \cdots dt_{N-1}. \tag{9.2}$$

In an appropriate continuous space limit, this becomes the directed polymer for the KPZ equation (4.2).

In the zero-temperature ($\beta \to \infty$) limit, the free energy becomes the ground-state energy:

$$\lim_{\beta\to\infty} \beta \log Z_t^N(\beta) = \min_{\pi} E[\pi]. \tag{9.3}$$

For this case, the connection to the largest eigenvalue of GUE was known before Refs. [51,52].

O'Connell discovered that, for arbitrary β, the time evolution of the partition function is related to the Hamiltonian of the quantum Toda lattice [28]. The Hamiltonian of the Toda lattice is given by

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{N-1} e^{x_i - x_{i-1}}. \tag{9.4}$$

For the classical case, the Toda lattice is well known to have soliton solutions and have applications to several real physical systems [53]. The quantum version has also been studied. The eigenvalues and eigenfunctions have been identified. There does not seem to be a known experimental system described by the quantum Toda lattice, but it has found a possible realization in a directed polymer system.

9.2. Macdonald process

In Ref. [29], Borodin and Corwin showed that the measure written in the form

$$\frac{1}{Z} P_\lambda(a)\, Q_\lambda(b)$$

(as a measure for λ), where P, Q are Macdonald polynomials and a, b represent the set of parameters, is behind the solvability of the KPZ equation. The Macdonald polynomials are well known multivariable orthogonal polynomials, which play an important role [54]. The Macdonald polynomials have two parameters (t, q). For the special case of $q = t$, the Macdonald polynomials become the Schur function and the above Macdonald measure in this case is known as the Schur measure. The Schur function can be written as a single determinant. From this, it follows that the correlation function can be written as determinants. On the other hand, for the Macdonald polynomials, such representation as a single determinant is not known. Then it is not at all obvious whether some quantities can be written as a determinant. But Ref. [29] showed that expectation values of certain "observables" can be written as Fredholm determinants by using the properties of Macdonald operators and doing some contour integral calculations. The reason for the appearance of the Fredholm determinant is not yet well understood.

The connection to the models discussed above is as follows. One can construct a stochastic dynamics on the partition λ. For the special case of $t = 0$, the dynamics of λ_1 is related to the q-TASEP. On the other hand, in a certain limit, the dynamics of λ becomes a dynamics equivalent to the quantum Toda lattice.

9.3. Generalized ZRP and q-TASEP

There is a generalization of the ZRP and q-TASEP in another direction. In the one-step update rule, the weight for the q-Racah polynomial appears [55].

10. Universality

From the beginning, the KPZ equation has been associated with its universal aspects. In a sense, it was introduced as the simplest model that can explain the 1/3 exponent for the growth of surface fluctuations that had been seen in many simulation models.

As we have seen, for the KPZ equation and related discrete models, the height distributions have been determined exactly for a few initial and boundary conditions. In the large-t limit, they are given by the Tracy–Widom distributions of GUE and GOE type. These may sound like very mathematical results, but, if we believe in the universality, they should appear in all systems in the KPZ class. In this article, we have so far been mostly interested in solvable cases; most systems in the KPZ class are of course not solvable but they should show the same Tracy–Widom distributions.

10.1. Experiments

To test the KPZ universality beyond the exponents, a crucial question is whether universal quantities such as the Tracy–Widom distributions are observed in real experiments. There have been several attempts in this direction, but the clearest demonstration was given by experiments by Takeuchi and Sano using liquid crystal turbulence [17]. For more details, see Ref. [18].

10.2. Hamiltonian dynamics

As we have discussed, the KPZ universality also appears in transport systems like ASEP. For ASEP, the density is the only conserved quantity. If we recall a lesson from equilibrium critical phenomena that the universality class depends strongly on the number of components in the systems, we might think that the KPZ universality appears only for systems with a single component.

However, in 2012, van Beijeren presented an interesting observation that the scaled KPZ two-point function would appear in rather generic 1D fluid systems [56]. In Ref. [57], Spohn generalized the ideas and formulated the conjecture for more generic multicomponent systems.

In particular, the conjecture would apply to (deterministic) 1D Hamiltonian dynamics with three conserved quantities, such as the Fermi–Pasta–Ulam chain with the potential given by $V(x) = \frac{x^2}{2} + \alpha\frac{x^3}{3} + \beta\frac{x^4}{4}$. In these systems, there are two sound modes with velocities $\pm c$ and one heat mode with velocity 0. The two-point correlation function of the heat mode is described by a $\frac{5}{3}$-Levy distribution, whereas the sound modes are expected to be described by the KPZ correlation.

Of course, it would be extremely hard to prove the conjecture, because one has to treat many-body Hamiltonian dynamics, which are not integrable. But, since the conjecture itself is very concrete, we can try to verify it by tensing its consequences with numerical simulations. There have been many related works in the last few years; see, for instance, Ref. [58].

11. Summary

The KPZ equation was first introduced as a model equation that captures universal aspects of surface growth. For many years, the main focus was on the scaling exponents. More recently, for its 1D version, universal quantities such as the height distributions and the two-point correlation function have been determined exactly. The universal height distribution for the wedge initial condition turned out

to be given by the GUE Tracy–Widom distribution from random matrix theory. This was confirmed in experiments are well. The universal two-point function is expected to explain some aspects of rather generic 1D multicomponent systems including Hamiltonian dynamics. The connection to the random matrix theory has evolved to the notion of "stochastic integrability" and continues to provide deeper understanding of the beautiful mathematical structures behind the KPZ equation.

At the end of this article, let us briefly discuss the KPZ equation in higher dimensions. The first step is to find the values of the scaling exponents. There have been theoretical studies to try to find these values, but they have not been so conclusive. On the other hand, from numerical simulations, the exponent seems fairly consistent. One may use insights from the new developments for the 1D case to give a better understanding of higher-dimensional cases. A nice example is provided in Ref. [59], where new extensive Monte Carlo simulations in 2D on the distributions have been performed. It seems that the distributions show some universal behaviors. They are different from the Tracy–Widom distributions but they show a few similar features as their 1D counterparts. For instance, the distributions show geometry dependence. It would be nice to have a better understanding of universal behaviors for higher-dimensional cases.

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