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Difference equations arising from cluster algebras
（クラスター代数から生じる差分方程式）

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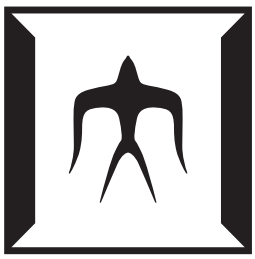
A thesis presented for the degree of
Doctor of Science



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Introduction

Cluster algebras were introduced by Fomin and Zelevinsky in the seminal paper [FZ02a]. A *cluster algebra* is a commutative ring equipped with a combinatorial structure called a *cluster pattern*. A cluster pattern is a graph whose vertices are *clusters*, which are tuples of *cluster variables*, and edges are *exchange relations*. Such combinatorial structures have been found in many areas of mathematics, and thus the theory of cluster algebra has many applications.

One of the main applications of the theory of cluster algebras is the study of discrete dynamical systems. In their fourth paper on cluster algebras [FZ07], Fomin and Zelevinsky introduced *bipartite belts*, which are discrete dynamical systems associated with bipartite symmetrizable generalized Cartan matrices. They proved that the bipartite belt associated with a generalized Cartan matrix A is periodic if and only if A is of finite type, that is, there exists a vector $v > 0$ such that $Av > 0$. Thus, periodic bipartite belts are classified by the Cartan-Killing classification. This result generalizes and refines the periodicity of Zamolodchikov's Y-systems, which was conjectured by Zamolodchikov [Zam91] in the study of thermodynamic Bethe ansatz, and proved by Fomin and Zelevinsky in [FZ03] prior to their fourth paper [FZ07]. They also proved that there is a bijection between the set of terms appear in a bipartite belt associated with a finite type Cartan matrix A and the set of almost positive roots in the root system associated with A. A key fact in the proof of these results is that terms in a bipartite belt are realized as cluster variables in some cluster algebra, and recurrence relations of this bipartite belt are realized as exchange relations in the same cluster algebra.

Bipartite belts are very special cases of discrete dynamical systems called *Y-systems* and *T-systems* in cluster algebras, in the sense of Nakanishi's paper [Nak11b]. These discrete dynamical systems have nice properties inherited from general properties of cluster algebras such as the Laurent phenomenon [FZ02a], the Laurent positivity [LS15, GHKK18], the synchronicity phenomenon [Nak19], and the quantization [BZ05, FG09]. It has been discovered that many interesting discrete dynamical systems can be realized as Y-systems or T-systems in cluster algebras, for example:

- periodic discrete dynamical systems that are generalization of Zamolodchikov's Y-systems [FZ07, GP19a, IIK+13a, IIK+13b, Kel13, NS16],
- non-periodic but integrable discrete dynamical systems such as Q-systems [DFK09, DFK10, Ked08], pentagram maps [GSTV16, Gli11], the q-Painlevé equations [BGM18, HI14, Oku15], and discrete dynamical systems associated with mutation-periodic quivers [FH14, FM11] and bipartite recurrent quivers [GP19b, GP20].

Because of these nice properties and interesting examples, it is natural to ask what discrete

dynamical systems arise from cluster algebras in general. In this paper, we give an answer to this question.

■Main result Let r be a positive integer, and we denote by $[1,r]$ the set $\{1,\dots,r\}$. Given a triple of matrices (N_0,N_+,N_-) in $\mathrm{Mat}_{r\times r}(\mathbb{Z}[z])$ whose entries are written as

$$N_\varepsilon=\Bigl(\sum_{p\in\mathbb{Z}_{\geq 0}} n^{\varepsilon}_{ab;p}z^p\Bigr)_{a,b\in[1,r]},$$

we consider the following relation for each $(a,u)\in[1,r]\times\mathbb{Z}$ among indeterminates in $\{T_a(u)\mid(a,u)\in[1,r]\times\mathbb{Z}\}$:

$$\prod_{b=1}^{r}\prod_{p\geq 0}T_b(u+p)^{n^0_{ba;p}}=\prod_{b=1}^{r}\prod_{p\geq 0}T_b(u+p)^{n^+_{ba;p}}+\prod_{b=1}^{r}\prod_{p\geq 0}T_b(u+p)^{n^-_{ba;p}}. \tag{0.0.1}$$

We impose the following conditions on (N_0,N_+,N_-):

(N1) $n^0_{ab;p}=\delta_{ab}\delta_{p0}+\delta_{a\sigma(b)}\delta_{pp_a}$ for some $\sigma\in\mathfrak{S}_r$ and $p_a\in\mathbb{Z}_{>0}$,
(N2) $n^+_{ab;p}\geq 0$ and $n^-_{ab;p}\geq 0$ for any a,b,p,
(N3) $n^+_{ab;p}=0$ and $n^-_{ab;p}=0$ unless $0<p<p_a$,
(N4) $n^+_{ab;p}n^-_{ab;p}=0$ for any a,b,p,

where $\mathfrak{S}_r$ is the symmetric group on $[1,r]$ and δ is the Kronecker delta. The condition (N1) says that the left-hand side in (0.0.1) is equal to $T_a(u)T_{\sigma(a)}(u+p_{\sigma(a)})$. The condition (N2) says that the right-hand side in (0.0.1) is a sum of two monomials. The condition (N3) together with (N1) implies that any $T_a(u)$ can be written as a rational function in the initial variables $(T_a(p))_{(a,p)\in R_{\mathrm{in}}}$, where

$$R_{\mathrm{in}}=\{(a,p)\in[1,r]\times\mathbb{Z}\mid 0\leq p<p_a\}.$$

The condition (N4) says that the two monomials in the right-hand side in (0.0.1) do not have common divisors.

Definition 0.0.1. We say that a triple of matrices $\alpha=(A_+,A_-,D)$ is a *T-datum* of size r if $A_\pm$ can be written as $A_\pm=N_0-N_\pm$ by a triple of matrices (N_0,N_+,N_-) in $\mathrm{Mat}_{r\times r}(\mathbb{Z}[z])$ satisfying (N1)–(N4), and D is a positive integer diagonal matrix satisfying the following conditions:

- $N_0D=DN_0$,
- $D^{-1}N_\pm D\in\mathrm{Mat}_{r\times r}(\mathbb{Z}[z])$,
- $A_+DA_-^\dagger=A_-DA_+^\dagger$,

where $A_\pm^\dagger:=(A_\pm|_{z=z^{-1}})^{\mathsf{T}}$.

Definition 0.0.2. Let α be a T-datum. Let $\mathcal{T}(\alpha)$ be the commutative ring generated by the indeterminates $(T_a(u)^{\pm 1})_{(a,u)\in[1,r]\times\mathbb{Z}}$ subject to the relations (0.0.1) and $T_a(u)T_a(u)^{-1}=1$ for any $(a,u)\in[1,r]\times\mathbb{Z}$. We define $\mathcal{T}^\circ(\alpha)$ to be the subring of $\mathcal{T}(\alpha)$ generated by $(T_a(u))_{(a,u)\in[1,r]\times\mathbb{Z}}$. We say that $\mathcal{T}^\circ(\alpha)$ is the *T-algebra* associated with α. We also say that the family of relations (0.0.1) is the *T-system* associated with α.

Let I be a finite index set. For a pair (B, x) of an $I \times I$ skew-symmetrizable integer matrix B and an I-tuple $x = (x_i)_{i \in I}$ of algebraically independent commuting variables, the cluster algebra associated with the initial seed (B, x) is defined [FZ02a, FZ07], which is denoted by $\mathcal{A}(B, x)$. In Section 2.2.4, we prove the following:

Theorem 0.0.3. *Let α be a T-datum of size r. Let $x = (x_{a,p})_{(a,p) \in R_{\mathrm{in}}}$ be an R_{in}-tuple of algebraically independent commuting variables. Then there exists a unique $R_{\mathrm{in}} \times R_{\mathrm{in}}$ skew-symmetrizable integer matrix B such that*

(1) there exists a unique injective ring homomorphism $\iota : \mathcal{T}^{\circ}(\alpha) \hookrightarrow \mathcal{A}(B, x)$ such that $\iota(T_a(p)) = x_{a,p}$ for any $(a, p) \in R_{\mathrm{in}}$,
(2) $\iota(T_a(u))$ is a cluster variable in $\mathcal{A}(B, x)$ for any $(a, u) \in [1, r] \times \mathbb{Z}$,
(3) the image of the relation (0.0.1) by ι is an exchange relation in $\mathcal{A}(B, x)$ for any $(a, u) \in [1, r] \times \mathbb{Z}$.

Conversely, we also prove that T-systems in cluster algebras (in the sense in [Nak11b]) yield T-data (Section 2.1 and 2.2.2). Therefore, our definition of T-data completely characterize when a system of difference equations of the form (0.0.1) is realized as a family of exchange relations in a cluster algebra. In the following, we give remarks and applications of Theorem 0.0.3.

■Sequences of mutations that preserve exchange matrices The matrix B in Theorem 0.0.3 is called the initial exchange matrix in the cluster algebra $\mathcal{A}(B, x)$. In the proof of Theorem 0.0.3, we give the explicit formula (2.2.9) expressing B using a matrix coefficients in a T-datum. We also construct a sequence of mutations, which are fundamental operations in the theory of cluster algebras, that preserves the exchange matrix B up to relabeling of indices. Such a sequence of mutations is called a *mutation loop*.

Mutation loops themselves are of interest from a geometric viewpoint: they are representatives of elements in cluster modular groups [FG09], which are cluster algebraic counterparts of mapping class groups of surfaces. We show that essentially all mutation loops are obtained by the formula (2.2.9) (Theorem 2.2.18). The formula (2.2.9) gives a effective way to find mutation loops since finding T-data is usually easier than finding mutation loops. We give many examples of T-data in Section 2.3, which recover or generalize mutation loops in the literature. In Section 2.3.1, we classify T-data of size 1 (Theorem 2.3.1), which turns out to recover the classification of period 1 quivers by Fordy and Marsh [FM11]. In Section 2.3.2, we define T-data associated with pairs of commuting Cartan matrices. They are generalization of bipartite belts by Fomin and Zelevinsky [FZ07]. In particular, our definition also works for non-bipartite cases such as the "tadpole type". In Section 2.3.3, we define T-data associated with level restricted T-systems for quantum affinizations [KNS09]. These T-systems are restricted version of T-systems for quantum affinizations discovered by Hernandez [Her07], where "T-systems for quantum affinizations" mean algebraic relations among q-characters of Kirillov-Reshetikhin modules over quantum affinizations. Although mutation loops corresponding to these T-data are already constructed in [IIK+13a, IIK+13b, KNS09, Nak11c], our method gives a simple systematic way to produce these mutation loops.

■T-systems with coefficients and Y-systems Theorem 0.0.3 can be extended to T-systems *with coefficients*. In fact, we show Theorem 0.0.3 in this generality (Theorem

2.2.19). Coefficients of T-systems are governed by Y-systems, which are generalization of Zamolodchikov's Y-systems [Zam91]. In terms of T-data, the coefficients of the T-system associated with a T-datum α is described by the *Langlands dual T-datum* $\alpha^\vee = (A_+^\vee, A_-^\vee, D^\vee)$. If we write the entries of the matrices in $\alpha^\vee$ as

$$N_\varepsilon^\vee = \left(\sum_{p \in \mathbb{Z}_{\geq 0}} \check{n}_{ab;p}^{\varepsilon} z^p \right)_{a,b \in [1,r]},$$

the coefficients of the T-system associated with α is governed by the following system of relations:

$$\prod_{b=1}^{r} \prod_{p \geq 0} Y_b(u-p)^{\check{n}_{ab;p}^0} = \frac{\prod_{b=1}^{r} \prod_{p \geq 0} \big(1 \oplus Y_b(u-p)\big)^{\check{n}_{ab;p}^-}}{\prod_{b=1}^{r} \prod_{p \geq 0} \big(1 \oplus Y_b(u-p)^{-1}\big)^{\check{n}_{ab;p}^+}},$$

where $\oplus$ is the "auxiliary addition" in the underlying semifield to which the coefficients belong. We call this family of relations the Y-system associated with α.

■Periodic T-systems and Y-systems We say that a T-datum is of *finite type* if the set $\{T_a(u) \in \mathcal{T}^\circ(\alpha) \mid (a,u) \in [1,r] \times \mathbb{Z}\}$ is a finite set. This is equivalent to saying that the T-system associated with α is periodic. By the synchronicity phenomenon of cluster algebras [Nak19], this is also equivalent to the periodicity of the *Y-system* associated with α in universal semifields.

Many examples of finite type T-data have been found in the literature, which are associated with the following data:

- finite type Cartan matrices [Zam91, FZ03, FZ07],
- tensor products of pairs of finite type Cartan matrices [RVT93, Kel13],
- untwisted quantum affine algebras [KN92, IIK+13a, IIK+13b],
- the sine-Gordon Y-systems and the reduced sine-Gordon Y-systems [Tat95, NS16], which are associated with continued fractions,
- admissible ADE bigraphs [GP19a].

In many cases in this list, the periodicities of Y-systems in universal semifields were conjectured in the 1990s in physics [Zam91, RVT93, KN92, Tat95], and proved in the 21st century by using the theory of cluster algebras [FZ03, FZ07, Kel13, IIK+13a, IIK+13b, NS16, GP19a].

Since there are many interesting examples of finite type T-data as in this list, the classification of finite type T-data is a interesting problem. Except for special cases [FZ07, GP19a], however, the classification of finite type T-data is still not well understood. In this paper, we prove that any finite type T-datum satisfies the following simultaneous positivity:

Theorem 0.0.4 (Theorem 3.1.5)**.** *Let $\alpha = (A_+, A_-, D)$ be a T-datum. If α is of finite type, then there exists a vector $v > 0$ such that $\mathring{A}_+^{\mathsf{T}} v > 0$ and $\mathring{A}_-^{\mathsf{T}} v > 0$, where $\mathring{A}_\pm = A_\pm|_{z=1}$.*

Theorem 0.0.4 gives a effective method to determine that a given T-datum is not of finite type (see Example 3.1.6). This theorem is also used in the next topic: relationship between cluster algebras and Nahm's conjecture.

■**Nahm's conjecture** In [Nah07], Nahm gave a connection between rational conformal field theories and torsion elements in Bloch groups. In particular, Nahm's conjecture states that the modularity of certain hypergeometric q-series is related to torsion elements in Bloch groups (see [Zag07, Chapter II, Section 3]). We give a version of Nahm's conjecture from a viewpoint of cluster algebras.

Let $\alpha = (A_+, A_-, D)$ be a Cartan-like T-datum (see Definition 3.2.1) of finite type. By Theorem 0.0.4, we can show that the system of equations

$$f_a = \prod_{b=1}^{r} (1 - f_b)^{\check{\kappa}_{ab}} \qquad (a \in [1, r]) \tag{0.0.2}$$

has a unique real solution such that $0 < f_a < 1$ for any $a \in [1, r]$, where we define $K^\vee = (\check{\kappa}_{ab}) \in \mathrm{Mat}_{r\times r}(\mathbb{Q})$ by $K^\vee = (\mathring{A}_+^\vee)^{-1}\mathring{A}_-^\vee$. For this solution, the value

$$c_\alpha := \frac{6}{\pi^2} \sum_{a=1}^{r} d_a L(f_a) \tag{0.0.3}$$

turns out to be a rational number, where $L(x)$ is the Rogers dilogarithm function. This fact follows from dilogarithm identities in cluster algebras that are proved by Nakanishi [Nak11b]. Moreover, for any solution $(f_1, \ldots, f_r) \in \overline{\mathbb{Q}}^r$ of (0.0.2), we can define a torsion element in the Bloch group $\mathcal{B}(F)$, where F is a number field containing the solution.

Motivated by Nahm's conjecture, we introduce a family of hypergeometric q-series $(\mathcal{Z}_{\alpha,\sigma}(q))_{\sigma\in S_\alpha}$ for any Cartan-like T-datum α of finite type, where S_α is a finite abelian group associated with α. We call these q-series the *partition q-series* of α. In fact, these are generalization of partition q-series of mutation loops introduced by Kato and Terashima [KT15]. We conjecture that partition q-series are modular functions:

Conjecture 0.0.5 (Conjecture 3.3.3)**.** *Let α be a Cartan-like T-datum of finite type. Then $q^{-c_\alpha/24}Z_{\alpha,\sigma}(q)$ is a modular function for any $\sigma \in S_\alpha$, where c_α is the rational number defined by* (0.0.3)*.*

We prove this conjecture for $r = 1$ using Rogers-Ramanujan type identities (Theorem 3.3.5). We also give the following examples (Example 3.3.6-3.3.9) supporting the conjecture for $r \geq 2$: Zagier's lists of 2×2 and 3×3 matrices concerning the Nahm's conjecture [Zag07], a q-series in the Andrew-Gordon identity [And74], fermionic formulas for quantum affine algebras [HKO+02], and q-series appear in the theory of nilpotent double affine Hecke algebras [CF13].

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- Yuma Mizuno, *Difference equations arising from cluster algebras*, Journal of Algebraic Combinatorics, 2020, https://doi.org/10.1007/s10801-020-00978-9

Chapter 1

Cluster algebras

We review cluster algebras following [FZ07].

1.1 Matrix mutations and quiver mutations

Let I be a finite index set. An $I \times I$ integer matrix $B = (B_{ij})_{i,j\in I}$ is called *skew-symmetrizable* if there exist a tuple of positive integers $d = (d_i)_{i\in I}$ such that $B_{ij}d_j = -B_{ji}d_i$. Such a tuple is called a (right) *symmetrizer* of B. For any $I \times I$ matrix $B = (B_{ij})_{i,j\in I}$ and bijection $\nu : I \to I'$ between finite index sets, we define an $I' \times I'$ matrix $\nu(B) = (B'_{i'j'})_{i',j'\in I'}$ by $B'_{\nu(i)\nu(j)} = B_{ij}$.

Definition 1.1.1. Let $B = (B_{ij})_{i,j\in I}$ be a skew-symmetrizable integer matrix, and let $k \in I$. The *matrix mutation* $\mu_k : B \mapsto B'$ is a transformation that transforms B into a new skew-symmetrizable integer matrix $B' = (B'_{ij})_{i,j\in I}$ defined as follows:

$$B'_{ij} = \begin{cases} -B_{ij} & \text{if } i = k \text{ or } j = k, \\ B_{ij} + [B_{ik}]_+[B_{kj}]_+ - [-B_{ik}]_+[-B_{kj}]_+ & \text{otherwise,} \end{cases} \tag{1.1.1}$$

where $[x]_+ := \max(0, x)$.

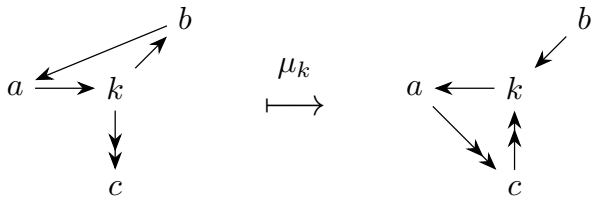
If d is a symmetrizer of B, then it is also a symmetrizer of $\mu_k(B)$. In particular, if B is a skew-symmetric integer matrix, then $\mu_k(B)$ is also a skew-symmetric integer matrix. In this case, it is convenient to describe matrix mutations in terms of quivers. A *quiver* is a finite oriented graph without loops and 2-cycles. For any skew-symmetric integer matrix B, we define a quiver $Q(B)$ as follows: the vertex set is I, and there are $[B_{ij}]_+$ arrows from i to j. Conversely, we can recover a skew-symmetric integer matrix $B(Q)$ from a quiver Q by $B(Q)_{ij} = Q_{ij} - Q_{ji}$, where Q_{ij} is the number of arrows from i to j. This gives a bijection between the set of $I \times I$ skew-symmetric integer matrices and the set of quivers whose vertex set is I.

Definition 1.1.2. Let Q be a quiver, and let k be a vertex of Q. The *quiver mutation* μ_k is a transformation that transforms Q into a quiver $\mu_k(Q)$ defined by the following three steps:

(1) For each length two path $i \to k \to j$, add a new arrow $i \to j$.
(2) Reverse all arrows incident to the vertex k.

(3) Remove all 2-cycles.

Matrix mutations and quiver mutations are compatible. The transformation



is an example of a quiver mutation.

1.2 Seed mutations

A set $\mathbb{P}$ is called a *semifield* if it is an abelian multiplicative group endowed with an binary operation $\oplus$ which is commutative, associative, and distributive with respect to the multiplication. We denote by $\mathbb{ZP}$ the group ring of $\mathbb{P}$ over $\mathbb{Z}$. This ring is an integral domain since the abelian multiplicative group of $\mathbb{P}$ is torsion-free. Throughout this paper, a $\mathbb{ZP}$-algebra means a commutative associative $\mathbb{ZP}$-algebra with an identity element, and we assume that a $\mathbb{ZP}$-algebra homomorphism sends the identity element to the identity element. We denote by $\mathbb{QP}$ the field of fractions of $\mathbb{ZP}$. We fix a field $\mathcal{F}$ that is isomorphic to the field of rational functions over $\mathbb{QP}$ in $|I|$ variables.

Example 1.2.1. Let J be a finite index set.

(1) Let $\mathrm{Trop}(u_j)_{j\in J}$ be the abelian multiplicative group generated by the indeterminates $(u_j)_{j\in J}$. We define a binary operation $\oplus$ on $\mathrm{Trop}(u_j)_{j\in J}$ by

$$\prod_{j\in J} u_j^{a_j} \oplus \prod_{j\in J} u_j^{b_j} = \prod_{j\in J} u_j^{\min(a_j,b_j)}.$$

This binary operation makes $\mathrm{Trop}(u_j)_{j\in J}$ a semifield, which is called a *tropical semifield*. If J is the empty set, $\mathrm{Trop}(u_j)_{j\in J} = \{1\}$ is called the *trivial semifield*.

(2) Let $\mathbb{Q}_{\mathrm{sf}}(u_j)_{j\in J}$ be the subset of $\mathbb{Q}(u_j)_{j\in J}$ consisting of all rational functions that can be written as subtraction-free expressions in $(u_i)_{i\in J}$. The set $\mathbb{Q}_{\mathrm{sf}}(u_j)_{j\in J}$ is a semifield with respect to the usual multiplication and addition, which is called a *universal semifield*.

Definition 1.2.2. An *(I-labeled) Y-seed* in $\mathbb{P}$ is a pair (B, y), where

- $B = (B_{ij})_{i,j\in I}$ is an $I \times I$ skew-symmetrizable integer matrix,
- $y = (y_i)_{i\in I}$ is an I-tuple in elements of $\mathbb{P}$.

Definition 1.2.3. An *(I-labeled) seed* in $\mathcal{F}$ is a pair (B, y, x), where

- (B, y) is an I-labeled Y-seed in $\mathbb{P}$,
- $x = (x_i)_{i\in I}$ is an I-tuple of elements in $\mathcal{F}$ forming a free generating set, that is, $(x_i)_{i\in I}$ is algebraically independent over $\mathbb{QP}$, and $\mathcal{F} = \mathbb{QP}(x_i)_{i\in I}$.

For a seed (B, y, x), we refer to B as the *exchange matrix*, y as the *coefficient tuple*, x as the *cluster*, y_i's as the *coefficients*, and x_i's as the *cluster variables*.

Definition 1.2.4. Let (B, y, x) be an I-labeled seed in $\mathcal{F}$, and let $k \in I$. The *seed mutation* $\mu_k : (B, y, x) \mapsto (B', y', x')$ is a transformation that transforms (B, x, y) into a new seed (B', y', x') defined as follows:

- $B' = (B'_{ij})_{i,j \in I}$ is given by (1.1.1),
- $y' = (y'_i)_{i \in I}$ is given by

$$y'_i = \begin{cases} y_k^{-1} & \text{if } i = k, \\ y_i (1 \oplus y_k)^{-B_{ki}} & \text{if } i \neq k \text{ and } B_{ki} \leq 0, \\ y_i (1 \oplus y_k^{-1})^{-B_{ki}} & \text{if } i \neq k \text{ and } B_{ki} \geq 0, \end{cases} \tag{1.2.1}$$

- $x' = (x'_i)_{i \in I}$ is given by $x'_i = x_i$ if $i \neq k$, and

$$x'_k = x_k^{-1} \left(\frac{y_k}{1 \oplus y_k} \prod_{j \in I} x_j^{[B_{jk}]_+} + \frac{1}{1 \oplus y_k} \prod_{j \in I} x_j^{[-B_{jk}]_+} \right). \tag{1.2.2}$$

We also say that the transformation $\mu_k : (B, y) \mapsto (B', y')$ is the *Y-seed mutation*.

The relation (1.2.2) is called the *exchange relation*. Seed mutations are involutions, that is, $\mu_k(\mu_k(B, y, x)) = (B, y, x)$.

1.3 Cluster algebras

The *I-regular* tree $\mathbb{T}_I$ is the tree such that all vertices have degree $|I|$ and the edges that are incident to each vertex are labeled by the elements in I. A *cluster pattern* is an assignment of an I-labeled seed to every vertex in $\mathbb{T}_I$, such that the two seeds assigned to the endpoints of any edge labeled by $k \in I$ are obtained from each other by the seed mutation μ_k.

Definition 1.3.1. The *cluster algebra* $\mathcal{A}$ associated with a given cluster pattern is the $\mathbb{ZP}$-subalgebra of $\mathcal{F}$ generated by all cluster variables in the pattern. We denote $\mathcal{A} = \mathcal{A}(B, y, x)$, where (B, y, x) is any seed in the underlying cluster pattern. We often denote $\mathcal{A}(B, y, x)$ by $\mathcal{A}(B, x)$ when $\mathbb{P}$ is the trivial semifield.

The *Laurent Phenomenon* is the one of the most important properties of cluster algebras.

Theorem 1.3.2 ([FZ02a, Theorem 3.1])**.** *Let x be a cluster in a cluster pattern. Then any cluster variables in the same cluster pattern is expressed as a Laurent polynomial in x with coefficients in $\mathbb{ZP}$.*

Chapter 2

Y-systems and T-systems

2.1 Y/T-systems in cluster algebras

In this section, we review T-systems and Y-systems in cluster algebras following [Nak11b]. Simply put, T-systems and Y-systems are algebraic relations that x_k's and y_k's, respectively, at mutation indices satisfy.

2.1.1 Mutation loops

Let $B = (B_{ij})_{i,j\in I}$ be a skew-symmetrizable integer matrix. Let r be a positive integer. For any sequence of indices $\mathbf{i} = (i_1, \dots, i_r) \in I^r$, we denote the composition of mutations $\mu_{i_r} \circ \cdots \circ \mu_{i_1}$ by $\mu_{\mathbf{i}}$. If $B_{i_a i_b} = 0$ for any $a, b \in [1, r]$, it is easy to see that $\mu_{\mathbf{i}}(B) = \mu_{\rho(\mathbf{i})}(B)$ for any permutation $\rho \in \mathfrak{S}_r$, where $\mathfrak{S}_r$ is the group of bijections on $[1, r]$ and $\rho(\mathbf{i}) := (i_{\rho^{-1}(1)}, \dots, i_{\rho^{-1}(r)})$. We say that a transformation $B \mapsto \mu_{\mathbf{i}}(B)$ is a *simultaneous mutation* if $B_{i_a i_b} = 0$ for any $a, b \in [1, r]$, and $a \neq b$ implies $i_a \neq i_b$ for any $a, b \in [1, r]$.

Let $\mathbf{i} = (i_1, \dots, i_r) \in I^r$ be a sequence of indices. Consider a partition of $\mathbf{i}$:

$$\begin{aligned} \mathbf{i} &= \mathbf{i}(0) \mid \mathbf{i}(1) \mid \cdots \mid \mathbf{i}(t-1), \\ \mathbf{i}(u) &= (i(u)_1, \dots, i(u)_{r_u}), \quad \sum_{u=0}^{t-1} r_u = r, \end{aligned} \tag{2.1.1}$$

where we allow $\mathbf{i}(u)$ to be the empty sequence. Formally, a partition of $\mathbf{i}$ is an order-preserving map $[1, r] \to \{0, \dots, t-1\}$ where t is a positive integer. A partition (2.1.1) is called a *partition into simultaneous mutations* if all $B(u) \mapsto B(u+1)$ in the following mutation sequence are simultaneous mutations:

$$B =: B(0) \overset{\mu_{\mathbf{i}(0)}}{\longmapsto} B(1) \overset{\mu_{\mathbf{i}(1)}}{\longmapsto} \cdots \overset{\mu_{\mathbf{i}(t-1)}}{\longmapsto} B(t). \tag{2.1.2}$$

Definition 2.1.1. We say that a quadruple $\gamma = (B, d, \mathbf{i}, \nu)$ is a *mutation loop* if

- $B = (B_{ij})_{i,j\in I}$ is a skew-symmetrizable integer matrix,
- $d = (d_i)_{i\in I}$ is a right symmetrizer of B,
- $\mathbf{i} = (i_1, \dots, i_r)$ is a sequence of elements in I equipped with a partition into simultaneous mutations $\mathbf{i} = \mathbf{i}(0) \mid \mathbf{i}(1) \mid \cdots \mid \mathbf{i}(t-1)$,
- $\nu : I \to I$ is a bijection such that $\mu_{\mathbf{i}}(B) = \nu(B)$ and $d = \nu(d)$.

The integer r is called the *length* of γ. Many examples of mutation loops are given in [Nak11b, Section 3].

The partition (2.1.1) decomposes $[1, r]$ into t parts. We define the subgroup $\mathfrak{S}_{r_0,\dots,r_{t-1}} \subseteq \mathfrak{S}_r$, which is isomorphic to $\mathfrak{S}_{r_0} \times \cdots \times \mathfrak{S}_{r_{t-1}}$, consisting of permutations that fix the each part as a set.

Definition 2.1.2. We say that two mutation loops $\gamma = (B, d, \mathbf{i}, \nu)$ and $\gamma' = (B', d', \mathbf{i}', \nu')$, where $\mathbf{i} = \mathbf{i}(0) \mid \cdots \mid \mathbf{i}(t-1)$ and $\mathbf{i}' = \mathbf{i}'(0) \mid \cdots \mid \mathbf{i}'(t'-1)$, are *equivalent* if there exists a bijection $f : I \to I'$ between the index sets of B and B', and a permutation $\rho \in \mathfrak{S}_{r_0,\dots,r_{t-1}}$ such that

- $B' = f(B)$,
- $d' = f(d)$,
- $t = t'$ and $\mathbf{i}'(u) = f(\rho(\mathbf{i}(u)))$ for each $u = 0, \dots, t-1$,
- $\nu' = f \circ \nu \circ f^{-1}$.

For any mutation loop $\gamma = (B, d, \mathbf{i}, \nu)$, we have the following infinite length mutation sequence that extends (2.1.2):

$$\begin{array}{ccccccccc}
 & & & & \cdots & \xrightarrow{\mu_{\mathbf{i}(-2)}} & B(-1) & \xrightarrow{\mu_{\mathbf{i}(-1)}} & \\
B(0) & \xrightarrow{\mu_{\mathbf{i}(0)}} & B(1) & \xrightarrow{\mu_{\mathbf{i}(1)}} & \cdots & \xrightarrow{\mu_{\mathbf{i}(t-2)}} & B(t-1) & \xrightarrow{\mu_{\mathbf{i}(t-1)}} & \\
B(t) & \xrightarrow{\mu_{\mathbf{i}(t)}} & B(t+1) & \xrightarrow{\mu_{\mathbf{i}(t+1)}} & \cdots & \xrightarrow{\mu_{\mathbf{i}(2t-2)}} & B(2t-1) & \xrightarrow{\mu_{\mathbf{i}(2t-1)}} & \\
B(2t) & \xrightarrow{\mu_{\mathbf{i}(2t)}} & \cdots & & & & & &
\end{array} \tag{2.1.3}$$

where $B(nt+k) = \nu^n(B(k))$ and $\mathbf{i}(nt+k) = \nu^n(\mathbf{i}(t+k))$ for any $n \in \mathbb{Z}$ and $0 \le k \le t-1$. Let P_γ be the set defined by

$$P_\gamma = \{(i, u) \in I \times \mathbb{Z} \mid i \in \mathbf{i}(u)\}.$$

Elements in P_γ are called *mutation points* of γ. We also define an integer $\lambda(i, u)$ for any $(i, u) \in I \times \mathbb{Z}$ by

$$\lambda(i, u) = \min\{v \in \mathbb{Z}_{\ge 0} \mid (i, u+v) \in P_\gamma\}$$

if there exists $v \in \mathbb{Z}_{\ge 0}$ such that $(i, u+v) \in P_\gamma$. Otherwise, we set $\lambda(i, u) = \infty$. The number $\lambda(i, u)$ is called the *latency* of (i, u). For any $(i, u) \in I \times \mathbb{Z}$ with $\lambda(i, u) < \infty$, we define an element $s(i, u) \in P_\gamma$ by

$$s(i, u) = \begin{cases} (i, u + \lambda(i, u)) & \text{if } (i, u) \notin P_\gamma, \\ (i, u + 1 + \lambda(i, u+1)) & \text{if } (i, u) \in P_\gamma. \end{cases}$$

The element $s(i, u)$ is called the *next mutation point* of (i, u).

A mutation loop is called *complete* if all latencies are finite, that is, $\lambda(i, u) < \infty$ for any $(i, u) \in I \times \mathbb{Z}$, or equivalently, for any $(i, 0) \in I \times \{0\}$. In the rest of this paper, we usually assume that mutation loops are complete.

In order to describe the T-system and the Y-system so that the relationship between them is apparent, we need another parameterization of the mutation points. For any

elements $(i,u),(j,v) \in P_\gamma$, we write $(i,u) \sim (j,v)$ if there exists $g \in \mathbb{Z}$ such that $j = \nu^g(i)$ and $v = u + gt$. Let $\pi : P_\gamma \to [1,r]$ be the surjective map defined by $\pi(i,u) = a$ where a is the unique element in $[1,r]$ such that $(i,u) \sim (i_a, v)$ and $0 \leq v \leq t-1$. We define a set R_γ by

$$R_\gamma = \{(\pi(i,u), u) \mid (i,u) \in P_\gamma\}. \tag{2.1.4}$$

Lemma 2.1.3. *The map $P_\gamma \to R_\gamma$ defined by $(i,u) \mapsto (\pi(i,u),u)$ is a bijection.*

Proof. The surjectivity is apparent since π is surjective. We assume that $(i,u),(j,u) \in P_\gamma$ satisfy $\pi(i,u) = \pi(j,u)$. Then we obtain $(i,u) \sim (j,u)$, and this implies that $i = j$ by the definition of the equivalence relation. □

Let $\sigma \in \mathfrak{S}_r$ be the bijection defined by

$$\sigma(a) = \pi(s(i,u)), \tag{2.1.5}$$

where $(i,u) \in \pi^{-1}(a)$ is any mutation point that maps to a by π. The definition of σ does not depend on the choice of (i,u). For any $a \in [1,r]$, we denote by λ_a the positive integer $1 + \lambda(i,u+1)$ where $(i,u) \in \pi^{-1}(a)$. In other words, λ_a is the positive integer satisfying $s(i,u) = (i,u+\lambda_a)$. The definition of λ_a also does not depend on the choice of (i,u).

2.1.2 Y-systems in cluster algebras

Let us describe a Y-system associated with a mutation loop γ. For any Y-seed (B,y), we have the following infinite length sequence of Y-seeds:

$$\begin{array}{ccccc} & & \cdots & (B(-1),y(-1)) & \overset{\mu_{\mathbf{i}(-1)}}{\longmapsto} \\ (B(0),y(0)) & \overset{\mu_{\mathbf{i}(0)}}{\longmapsto} & \cdots & (B(t-1),y(t-1)) & \overset{\mu_{\mathbf{i}(t-1)}}{\longmapsto} \\ (B(t),y(t)) & \overset{\mu_{\mathbf{i}(t)}}{\longmapsto} & \cdots & & \end{array} \tag{2.1.6}$$

where $(B(0),y(0)) = (B,y)$ and we define negative ones using the involution property of mutations. We define an element $Y_a(u) \in \mathbb{P}$ for any $(a,u) \in R_\gamma$ by

$$Y_a(u) = y_i(u), \tag{2.1.7}$$

where $i \in I$ is a unique index such that $(i,u) \in P_\gamma$ and $a = \pi(i,u)$.

Let $N_0^\vee = (\sum_{p\in\mathbb{Z}} \check{n}^0_{ab;p} z^p)_{a,b\in[1,r]} \in \mathrm{Mat}_{r\times r}(\mathbb{Z}[z])$ be the $r \times r$ matrix whose entries are integer coefficients polynomials in the variable z defined by

$$\sum_{p\in\mathbb{Z}} \check{n}^0_{ab;p} z^p = \delta_{ab} + \delta_{a'b} z^{\lambda_{a'}}, \tag{2.1.8}$$

where $a' = \sigma^{-1}(a)$. We also define two matrices $N_+^\vee = (\sum_{p\in\mathbb{Z}} \check{n}^+_{ab;p} z^p)_{a,b\in[1,r]}$ and $N_-^\vee = (\sum_{p\in\mathbb{Z}} \check{n}^-_{ab;p} z^p)_{a,b\in[1,r]}$ in $\mathrm{Mat}_{r\times r}(\mathbb{Z}[z])$ by

$$\sum_{p\in\mathbb{Z}} \check{n}^\pm_{ab;p} z^p = \sum_{\substack{(j,v)\in P_\gamma \\ s(k,v)=(k,u),\pi(j,v)=b}} [\pm B_{jk}(v)]_+ z^{\lambda(k,v)}, \tag{2.1.9}$$

where $(k,u) \in \pi^{-1}(a)$. The definition of $N_\pm^\vee$ does not depend on the choices of (k,u).

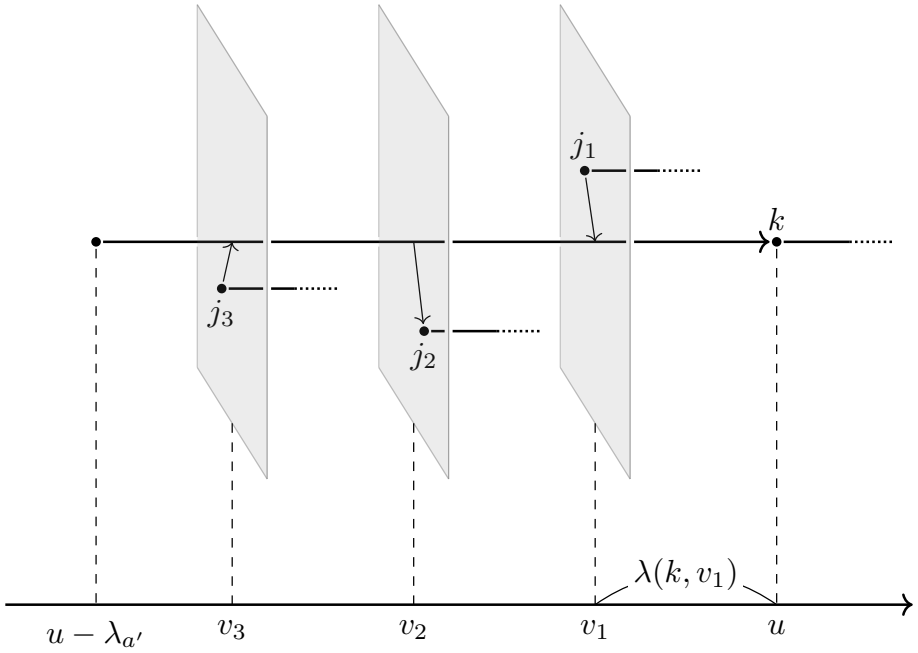


Fig. 2.1 A schematic description of a Y-system. A black point represents a mutation point. An arrow in a plane from (resp. to) a mutation point (j, v) to (resp. from) the right arrow that ends at (k, u) indicates that $[B_{jk}(v)]_+ \neq 0$ (resp. $[-B_{jk}(v)]_+ \neq 0$).

Proposition 2.1.4 ([Nak11b, Section 5.5]). *For any mutation loop γ, the family of elements $(Y_a(u))_{(a,u)\in R_\gamma}$ satisfy the following relation in $\mathbb{P}$ for any $(a, u) \in R_\gamma$:*

$$\prod_{b,p} Y_b(u-p)^{\check{n}^0_{ab;p}} = \frac{\prod_{b,p}\bigl(1 \oplus Y_b(u-p)\bigr)^{\check{n}^-_{ab;p}}}{\prod_{b,p}\bigl(1 \oplus Y_b(u-p)^{-1}\bigr)^{\check{n}^+_{ab;p}}},$$

where $\prod_{b,p} = \prod_{b=1}^{r}\prod_{p=0}^{\infty}$.

We call the family of relations in Proposition 2.1.4 the *Y-system* associated with γ, and the triple of matrices $(N^\vee_{\gamma,0}, N^\vee_{\gamma,+}, N^\vee_{\gamma,-})$ the *Y-system triple* of γ. From (2.1.8), the left-hand side in the Y-system can be rewritten as

$$\prod_{b,p} Y_b(u-p)^{\check{n}^0_{ab;p}} = Y_a(u)Y_{a'}(u-\lambda_{a'}).$$

If we define elements $P^\pm_a(u) \in \mathbb{P}$ by

$$P^+_a(u) = \frac{Y_a(u)}{1 \oplus Y_a(u)}, \quad P^-_a(u) = \frac{1}{1 \oplus Y_a(u)}, \tag{2.1.10}$$

the relation in Proposition 2.1.4 can be written in a simpler form as

$$\prod_{b,p} P^+_b(u-p)^{\check{n}^0_{ab;p}-\check{n}^+_{ab;p}} = \prod_{b,p} P^-_b(u-p)^{\check{n}^0_{ab;p}-\check{n}^-_{ab;p}}.$$

Figure 2.1 is a schematic description of a Y-system.

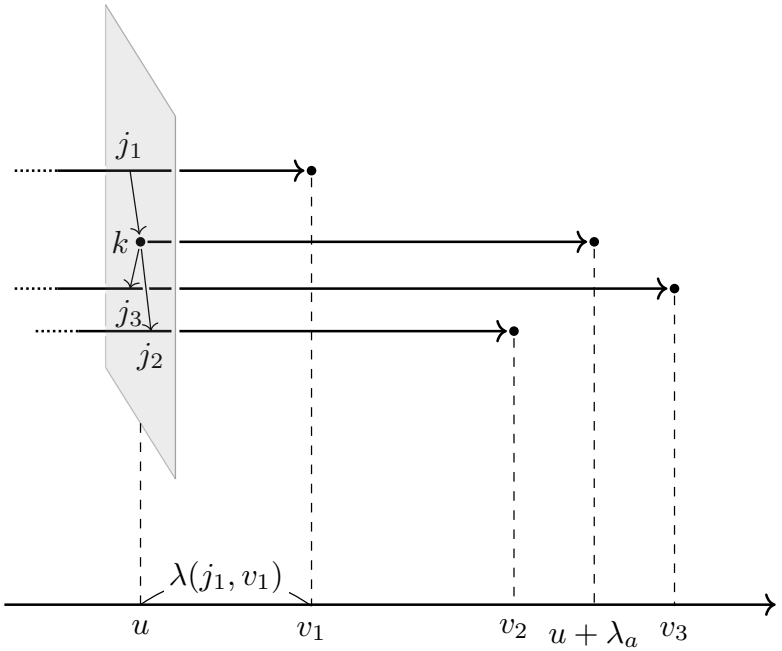


Fig. 2.2 A schematic description of a T-system. A black point represents a mutation point. An arrow in the plane from (resp. to) the mutation point (k, u) to (resp. from) a right arrow that ends at (j, v) indicates that $[-B_{jk}(u)]_+ \neq 0$ (resp. $[B_{jk}(u)]_+ \neq 0$).

2.1.3 T-systems in cluster algebras

Next we are going to describe T-systems. Let γ be a complete mutation loop. For any seed (B, y, x), we have the following infinite length sequence of seeds:

$$\begin{array}{lcll} & \cdots & (B(-1), y(-1), x(-1)) & \overset{\mu_{\mathbf{i}(-1)}}{\longmapsto} \\ (B(0), y(0), x(0)) \overset{\mu_{\mathbf{i}(0)}}{\longmapsto} & \cdots & (B(t-1), y(t-1), x(t-1)) & \overset{\mu_{\mathbf{i}(t-1)}}{\longmapsto} \\ (B(t), y(t), x(t)) \overset{\mu_{\mathbf{i}(t)}}{\longmapsto} & \cdots & & \end{array} \tag{2.1.11}$$

where $(B(0), y(0), x(0)) = (B, y, x)$. We define $Y_a(u) \in \mathbb{P}$ and $T_a(u) \in \mathcal{F}$ for any $(a, u) \in R_\gamma$ by

$$Y_a(u) = y_i(u), \quad T_a(u) = x_i(u), \tag{2.1.12}$$

where $i \in I$ is a unique index such that $(i, u) \in P_\gamma$ and $a = \pi(i, u)$.

We define a matrix $N_{\gamma,0} = (\sum_{p\in\mathbb{Z}} n^0_{ab;p} z^p)_{1\leq a,b\leq r} \in \mathrm{Mat}_{r\times r}(\mathbb{Z}[z])$ by

$$\sum_{p\in\mathbb{Z}} n^0_{ba;p} z^p = \delta_{ab} + \delta_{\sigma(a)b} z^{\lambda_a}. \tag{2.1.13}$$

We also define two matrices $N_{\gamma,+} = (\sum_{p\in\mathbb{Z}} n^+_{ab;p} z^p)_{a,b\in[1,r]}$ and $N_{\gamma,-} = (\sum_{p\in\mathbb{Z}} n^-_{ab;p} z^p)_{a,b\in[1,r]}$

in $\mathrm{Mat}_{r\times r}(\mathbb{Z}[z])$ by

$$\sum_{p\in\mathbb{Z}} n^{\pm}_{ba;p} z^p = \sum_{\substack{j\in I \\ \pi(s(j,u))=b}} [\mp B_{jk}(u)]_+ z^{\lambda(j,u)}, \tag{2.1.14}$$

where $(k,u)\in\pi^{-1}(a)$. The definition of $N_\pm$ does not depend on the choices of (k,u).

Proposition 2.1.5 ([Nak11b, Section 5.5]). *For any complete mutation loop, the family of elements $(Y_a(u))_{(a,u)\in R_\gamma}$ and $(T_a(u))_{(a,u)\in R_\gamma}$ satisfy the following relation in $\mathcal{F}$ for any $(a,u)\in R_\gamma$:*

$$\prod_{b,p} T_b(u+p)^{n^0_{ba;p}} = P^+_a(u)\prod_{b,p} T_b(u+p)^{n^-_{ba;p}} + P^-_a(u)\prod_{b,p} T_b(u+p)^{n^+_{ba;p}},$$

where $\prod_{b,p} = \prod_{b=1}^{r}\prod_{p=0}^{\infty}$ and $P^\pm_a(u)\in\mathbb{P}$ are defined by (2.1.10).

We call the family of relations in Proposition 2.1.5 the *T-system* associated with γ, and the triple of matrices $(N_{\gamma,0}, N_{\gamma,+}, N_{\gamma,-})$ the *T-system triple* of γ. From (2.1.13), the left-hand side in the T-system can be rewritten as

$$\prod_{b,p} T_b(u+p)^{n^0_{ba;p}} = T_a(u)T_{\sigma(a)}(u+\lambda_a).$$

Figure 2.2 is a schematic description of a T-system.

2.1.4 Relation between Y-systems and T-systems

Let $\gamma=(B,d,\mathbf{i},\nu)$ be a mutation loop.

Lemma 2.1.6. *The family of positive integers $(d_i(u))_{(i,u)\in I\times\mathbb{Z}}$ defined by $d_i(u)=d_i$ satisfies the following:*

(1) $B_{ij}(u)d_j(u) = -B_{ji}(u)d_i(u)$ for any $i,j\in I$ and $u\in\mathbb{Z}$,
(2) $d_i(u)=d_j(v)$ for any $(i,u),(j,v)\in P_\gamma$ such that $\pi(i,u)=\pi(j,v)$.

Proof. (1) holds since mutations preserve a symmetrizer. (2) follows from $d=\nu(d)$. □

From Lemma 2.1.6, the positive integers $d'_1,\dots,d'_r$ defined by $d'_a=d_i(u)$ where $(i,u)\in\pi^{-1}(a)$ do not depend on the choices of (i,u), and $d'_a=d_i(u)$ for any $(i,u)\in I\times\mathbb{Z}$ such that $\pi(s(i,u))=a$. We denote by D_γ the positive integer diagonal matrix $\mathrm{diag}(d'_1,\dots,d'_r)$.

Proposition 2.1.7 (cf. Proposition 5.13, [Nak11b]). *Let $(N^\vee_{\gamma,0}, N^\vee_{\gamma,+}, N^\vee_{\gamma,-})$ be the Y-system triple and $(N_{\gamma,0}, N_{\gamma,+}, N_{\gamma,-})$ be the T-system triple of a complete mutation loop γ. Then we have*

$$N^\vee_{\gamma,0} = N_{\gamma,0}$$

and

$$D_\gamma N^\vee_{\gamma,\varepsilon} = N_{\gamma,\varepsilon} D_\gamma$$

for any $\varepsilon\in\{0,+,-\}$.

Proof. The first identity follows from

$$\sum_p \check{n}^0_{ab;p} z^p = \delta_{ab} + \delta_{a'b} z^{\lambda_{a'}} = \delta_{ba} + \delta_{\hat{b}a} z^{\lambda_b} = \sum_p n^0_{ab;p} z^p.$$

The second identity for $\varepsilon = 0$ follows from Lemma 2.1.6. We now the second identity for $\varepsilon = \pm$. Let $(k,u) \in \pi^{-1}(b)$ and $(k',u') \in \pi^{-1}(a)$. Then we have

$$\begin{aligned}\sum_p \check{n}^{\pm}_{ab;p} z^p &= \sum_{\substack{(j,v)\in P_\gamma \\ s(k',v)=(k',u'),\pi(j,v)=b}} [\pm B_{jk'}(v)]_+ z^{\lambda(k,v)} \\ &= \sum_{\substack{j'\in I \\ \pi(s(j',u))=a}} [\pm B_{kj'}(u)]_+ z^{\lambda(j',u)}.\end{aligned}$$

On the other hand, Lemma 2.1.6 implies that

$$\begin{aligned}\sum_p n^{\pm}_{ab;p} z^p &= \sum_{\substack{j\in I \\ \pi(s(j,u))=a}} [\mp B_{jk}(u)]_+ z^{\lambda(j,u)} \\ &= \sum_{\substack{j\in I \\ \pi(s(j,u))=a}} [\pm d_j(u) d_k(u)^{-1} B_{kj}(u)]_+ z^{\lambda(j,u)} \\ &= d'_a (d'_b)^{-1} \sum_{\substack{j\in I \\ \pi(s(j,u))=a}} [\pm B_{kj}(u)]_+ z^{\lambda(j,u)}.\end{aligned}$$

□

For any matrix A, we denote the transpose of A by A^{T}. For any $A \in \mathrm{Mat}_{r\times r}(\mathbb{Z}[z^{\pm 1}])$, we define a matrix $A^\dagger \in \mathrm{Mat}_{r\times r}(\mathbb{Z}[z^{\pm 1}])$ by $A^\dagger = (A|_{z=z^{-1}})^{\mathsf{T}}$. Clearly, we have $(A^\dagger)^\dagger = A$ and $(AB)^\dagger = B^\dagger A^\dagger$ for any $A, B \in \mathrm{Mat}_{r\times r}(\mathbb{Z}[z^{\pm 1}])$.

Let $(A^\vee_{\gamma,+}, A^\vee_{\gamma,-})$ and $(A_{\gamma,+}, A_{\gamma,-})$ be the pairs of matrices defined by $A^\vee_{\gamma,\pm} = N^\vee_{\gamma,0} - N^\vee_{\gamma,\pm}$ and $A_{\gamma,\pm} = N_{\gamma,0} - N_{\gamma,\pm}$, respectively. We call them the *Y-system pair* and the *T-system pair* of γ. These pairs of matrices describe the following relation between the Y-system and the T-system:

Theorem 2.1.8. *Let γ be a complete mutation loop, and $(A^\vee_{\gamma,+}, A^\vee_{\gamma,-})$ and $(A_{\gamma,+}, A_{\gamma,-})$ be the Y-system pair and the T-system pair of γ, respectively. Then we have*

$$A^\vee_{\gamma,+} A^\dagger_{\gamma,-} = A^\vee_{\gamma,-} A^\dagger_{\gamma,+}.$$

Proof. The claim is equivalent to the following equality:

$$N^\vee_0 N^\dagger_- - N^\vee_0 N^\dagger_+ - N^\vee_- N^\dagger_0 + N^\vee_+ N^\dagger_0 = N^\vee_+ N^\dagger_- - N^\vee_- N^\dagger_+. \tag{2.1.15}$$

Let $a, b \in [1, r]$ and $p \in \mathbb{Z}$. Let us choose an element $(i,u) \in \pi^{-1}(a)$. Let $a' = \sigma^{-1}(a)$ and $b' = \sigma^{-1}(b)$. Let $p_a = \lambda_{a'}$ and $p_b = \lambda_{b'}$. Let $v = u - p$, $u' = u - p_a$, and $v' = v - p_b$. Then

the $(ab;p)$-th entry in the left-hand side in (2.1.15) is given by

$$\begin{aligned}
&n^-_{ba;-p} + n^-_{ba';p_a-p} - n^+_{ba;-p} - n^+_{ba';p_a-p} - \check{n}^-_{ab;p} - \check{n}^-_{ab';p+p_b} + \check{n}^+_{ab;p} + \check{n}^+_{ab';p+p_b} \\
&= \begin{cases}
+n^-_{ba;-p} & -n^+_{ba;-p} & -\check{n}^-_{ab';p+p_b} & +\check{n}^+_{ab';p+p_b} & \text{if } u \le v \text{ and } u' \le v', \\
+n^-_{ba;-p} & -n^+_{ba;-p} & +n^-_{ba';p_a-p} & -n^+_{ba';p_a-p} & \text{if } u \le v \text{ and } v' \le u', \\
-\check{n}^-_{ab;p} & +\check{n}^+_{ab;p} & -\check{n}^-_{ab';p+p_b} & +\check{n}^+_{ab';p+p_b} & \text{if } v \le u \text{ and } u' \le v', \\
-\check{n}^-_{ab;p} & +\check{n}^+_{ab;p} & +n^-_{ba';p_a-p} & -n^+_{ba';p_a-p} & \text{if } v \le u \text{ and } v' \le u',
\end{cases} \\
&= \sum_{\substack{j\in I \\ (j,v)\in P_\gamma,\pi(j,v)=b}} \begin{cases}
B_{ji}(u) + B_{ji}(v') & \text{if } u \le v \text{ and } u' \le v', \\
B_{ji}(u) + B_{ji}(u') & \text{if } u \le v \text{ and } v' \le u', \\
B_{ji}(v) + B_{ji}(v') & \text{if } v \le u \text{ and } u' \le v', \\
B_{ji}(v) + B_{ji}(u') & \text{if } v \le u \text{ and } v' \le u',
\end{cases} \\
&= \sum_{\substack{j\in I \\ (j,v)\in P_\gamma,\pi(j,v)=b}} (B_{ji}(\min(u,v)) + B_{ji}(\max(u',v'))).
\end{aligned}$$

On the other hand, the $(ab;p)$-th entry in the right-hand side in (2.1.15) is given by

$$\begin{aligned}
&\sum_{c=1}^{r} \sum_{w\in\mathbb{Z}} (\check{n}^+_{ac;u-w} n^-_{bc;v-w} - \check{n}^-_{ac;u-w} n^+_{bc;v-w}) \\
&= \sum_{\substack{j\in I,(k,w)\in P_\gamma \\ (j,v)\in P_\gamma,\pi(j,v)=b \\ \max(u',v')<w<\min(u,v)}} ([B_{ki}(w)]_+[B_{jk}(w)]_+ - [-B_{jk}(w)]_+[-B_{ki}(w)]_+).
\end{aligned}$$

These two entries coincide since

$$\begin{aligned}
&B_{ji}(\min(u,v)) + B_{ji}(\max(u',v')) \\
&= \sum_{\substack{(k,w)\in P_\gamma \\ \max(u',v')<w<\min(u,v)}} ([B_{ki}(w)]_+[B_{jk}(w)]_+ - [-B_{jk}(w)]_+[-B_{ki}(w)]_+)
\end{aligned}$$

by the rule of matrix mutations (1.1.1). □

2.2 Axiomatic approach to Y/T systems

In this section, we develop an axiomatic approach to Y-systems and T-systems based on the paper [Miz20a] written by the author of thesis.

2.2.1 T-data

Let r be a positive integer. As in the last section, we define an involution $\dagger : \mathrm{Mat}_{r\times r}(\mathbb{Q}(z)) \to \mathrm{Mat}_{r\times r}(\mathbb{Q}(z))$ by $A^\dagger = (A|_{z=z^{-1}})^{\mathsf{T}}$.

For a triple (N_0, N_+, N_-) of the matrices in $\mathrm{Mat}_{r\times r}(\mathbb{Z}[z])$ whose entries are given by

$$N_\varepsilon = \left(\sum_{p\in\mathbb{Z}_{\geq 0}} n^{\varepsilon}_{ab;p} z^p \right)_{a,b\in[1,r]}, \tag{2.2.1}$$

we consider the following conditions:

(N1) $n^0_{ab;p} = \delta_{ab}\delta_{p0} + \delta_{a\sigma(b)}\delta_{pp_a}$ for some $\sigma \in \mathfrak{S}_r$ and $p_a \in \mathbb{Z}_{>0}$,
(N2) $n^+_{ab;p} \geq 0$ and $n^-_{ab;p} \geq 0$ for any a, b, p,
(N3) $n^+_{ab;p} = 0$ and $n^-_{ab;p} = 0$ unless $0 < p < p_a$,
(N4) $n^+_{ab;p} n^-_{ab;p} = 0$ for any a, b, p.

Definition 2.2.1. We say that a triple of matrices $\alpha = (A_+, A_-, D)$ is a *T-datum* of size r if $A_\pm$ can be written as $A_\pm = N_0 - N_\pm$ by a triple of matrices (N_0, N_+, N_-) in $\mathrm{Mat}_{r\times r}(\mathbb{Z}[z])$ satisfying (N1)–(N4), and D is a positive integer diagonal matrix that satisfies the following conditions:

- $N_0 D = D N_0$,
- $D^{-1} N_\pm D \in \mathrm{Mat}_{r\times r}(\mathbb{Z}[z])$,
- $A_+ D A_-^\dagger = A_- D A_+^\dagger$.

It is clear that the triple (N_0, N_+, N_-) that satisfies the conditions (N1)–(N4) is uniquely recovered from $(A_+, A-)$ as follows: $N_0 = [A_+]_+ = [A_-]_+$, $N_+ = [-A_+]_+$, and $N_- = [-A_-]_+$, where we take $[\,]_+$ for each coefficient. Note that both matrices A_+ and A_- have non-zero determinants since their determinants are monic polynomials with constant terms 1, which follows from the conditions (N1) and (N3).

We say that the last equation

$$A_+ D A_-^\dagger = A_- D A_+^\dagger \tag{2.2.2}$$

in Definition 2.2.1 is the *symplectic relation* due to the following lemma, which can be easily verified:

Lemma 2.2.2. *Let $A_+, A_- \in \mathrm{Mat}_{r\times r}(\mathbb{Z}[z^{\pm 1}])$ be matrices with non-zero determinants, and D be a positive integer diagonal matrix. Then the following conditions are equivalent:*

(1) $A_+ D A_-^\dagger = A_- D A_+^\dagger$.
(2) $A_+ D A_-^\dagger$ *is a* $\dagger$*-invariant.*
(3) $A_- D A_+^\dagger$ *is a* $\dagger$*-invariant.*
(4) $(A_-)^{-1} A_+ D$ *is a* $\dagger$*-invariant.*
(5) $(A_+)^{-1} A_- D$ *is a* $\dagger$*-invariant.*
(6) $D^{-1}(A_-)^{-1} A_+$ *is a* $\dagger$*-invariant.*
(7) $D^{-1}(A_+)^{-1} A_-$ *is a* $\dagger$*-invariant.*
(8) The rows of the $r \times 2r$ *matrices* $[A_+\ A_-]$ *are pairwise orthogonal with respect to the symplectic pairing* $\langle\,,\,\rangle : (\mathbb{Z}[z^{\pm 1}])^{2r} \times (\mathbb{Z}[z^{\pm 1}])^{2r} \to \mathbb{Z}[z^{\pm 1}]$ *defined by*

$$\left\langle \begin{bmatrix} f(z) \\ g(z) \end{bmatrix}, \begin{bmatrix} f'(z) \\ g'(z) \end{bmatrix} \right\rangle = \begin{bmatrix} f(z)^{\mathsf{T}} & g(z)^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} O & D \\ -D & O \end{bmatrix} \begin{bmatrix} f'(z^{-1}) \\ g'(z^{-1}) \end{bmatrix},$$

where $f(z), g(z), f'(z), g'(z) \in (\mathbb{Z}[z^{\pm 1}])^r$.

Let $\alpha = (A_+, A_-, D)$ be a T-datum. Then the triple $\alpha^\vee = (A_+^\vee, A_-^\vee, D^\vee)$ defined by $N_\varepsilon^\vee = D^{-1} N_\varepsilon D$ and $D^\vee = \delta D^{-1}$ where δ is the product of the greatest common divisor and the least common multiple of all the entries in D, is also a T-datum. The T-datum $\alpha^\vee$ is called the *Langlands dual* of α. Clearly, we have $\alpha^{\vee\vee} = \alpha$. We write the entries of $N_\varepsilon^\vee$ as

$$N_\varepsilon^\vee = \left(\sum_{p \in \mathbb{Z}_{\geq 0}} \check{n}_{ab;p}^{\varepsilon} z^p \right)_{a,b \in [1,r]}. \tag{2.2.3}$$

Definition 2.2.3. Let $\alpha = (A_+, A_-, D)$ be a T-datum of size r. We say that a subset $R \subseteq [1,r] \times \mathbb{Z}$ is *consistent* for α if the following conditions are satisfied:

(R1) If $(a,u) \in R$ and $n_{ab;p}^0$, $n_{ab;p}^+$, or $n_{ab;p}^- \neq 0$, then $(b, u-p) \in R$.
(R2) If $(a,u) \in R$ and $n_{ba;p}^0$, $n_{ba;p}^+$, or $n_{ba;p}^- \neq 0$, then $(b, u+p) \in R$.
(R3) There exists a positive integer t such that $R = R^{(t)}$ and

$$[1,r] \times \mathbb{Z} = \bigsqcup_{k=0}^{t-1} R^{(k)},$$

where $R^{(k)} := \{(a, u+k) \mid (a,u) \in R\}$.

For example, $[1,r] \times \mathbb{Z}$ itself is always consistent since (R1) and (R2) are obvious, and (R3) is satisfied by setting $t = 1$. Note that the positive integer t in (R3) is uniquely determined from R. In the conditions (R1) and (R2), we can replace n with $\check{n}$. From (N1) together with (R1) and (R2), we have

$$\begin{aligned} (a,u) \in R \quad &\text{if and only if} \quad (\sigma(a), u + p_{\sigma(a)}) \in R \\ &\text{if and only if} \quad (\sigma^{-1}(a), u - p_a) \in R. \end{aligned}$$

Definition 2.2.4. Let (α, R) and (α', R') be pairs of T-data of size r and consistent subsets for them. They are called *equivalent* if there exists a permutation $\rho \in \mathfrak{S}_r$ such that $A'_\pm = \rho(A_\pm)$, $D' = \rho(D)$, and $R' = \rho(R)$, where $\rho(R) = \{(\rho(a), u) \mid (a,u) \in R\}$.

Definition 2.2.5. Let α be a T-datum of size r, and $R \subseteq [1,r] \times \mathbb{Z}$ be a consistent subset for α. We say that a family of elements $(Y_a(u))_{(a,u) \in R}$ is a *solution of the Y-system* associated with (α, R) in a semifield $\mathbb{P}$ if $Y_a(u) \in \mathbb{P}$ and the following relation holds in $\mathbb{P}$ for any $(a,u) \in R$:

$$\prod_{b,p} Y_b(u-p)^{\check{n}_{ab;p}^0} = \frac{\prod_{b,p} \bigl(1 \oplus Y_b(u-p)\bigr)^{\check{n}_{ab;p}^-}}{\prod_{b,p} \bigl(1 \oplus Y_b(u-p)^{-1}\bigr)^{\check{n}_{ab;p}^+}}, \tag{2.2.4}$$

where $\prod_{b,p} = \prod_{b=1}^r \prod_{p=0}^\infty$.

For any solution of the Y-system associated with (α, R) in $\mathbb{P}$, we define elements $P_a^{\pm}(u) \in \mathbb{P}$ $((a,u) \in R)$ by

$$P_a^+(u) = \frac{Y_a(u)}{1 \oplus Y_a(u)}, \quad P_a^-(u) = \frac{1}{1 \oplus Y_a(u)}.$$

Definition 2.2.6. Let α be a T-datum of size r, and $R \subseteq [1,r] \times \mathbb{Z}$ be a consistent subset for α. Let $Y = (Y_a(u))_{(a,u)\in R}$ be a solution of the Y-system associated with (α, R) in a semifield $\mathbb{P}$. Let $\mathcal{T}(\alpha, R, Y)$ be the $\mathbb{ZP}$-algebra generated by the indeterminates $(T_a(u)^{\pm 1})_{(a,u)\in R}$ subject to the relation

$$\prod_{b,p} T_b(u+p)^{n^0_{ba;p}} = P_a^+(u) \prod_{b,p} T_b(u+p)^{n^-_{ba;p}} + P_a^-(u) \prod_{b,p} T_b(u+p)^{n^+_{ba;p}} \tag{2.2.5}$$

for any $(a,u) \in R$, together with $T_a(u)T_a(u)^{-1} = 1$. We define $\mathcal{T}^\circ(\alpha, R, Y)$ to be the subalgebra of $\mathcal{T}(\alpha, R, Y)$ generated by $(T_a(u))_{(a,u)\in R}$. We say that $\mathcal{T}^\circ(\alpha, R, Y)$ is the *T-algebra* associated with (α, R, Y). We often denote $\mathcal{T}^\circ(\alpha, R, Y)$ by $\mathcal{T}^\circ(\alpha)$ when $R = [1,r] \times \mathbb{Z}$ and $\mathbb{P}$ is the trivial semifield.

The family of relations (2.2.4) is called the *Y-system* associated with (α, R), and the family of relations (2.2.5) is called the *T-system* associated with (α, R, Y).

Example 2.2.7 (Somos-4 recurrence)**.** The triple of 1×1 matrices $\alpha = (A_+, A_-, D)$ defined by

$$A_+ = \left[1 - 2z^2 + z^4\right], \quad A_- = \left[1 - z - z^3 + z^4\right], \quad D = \left[1\right]$$

is a T-datum, and the whole set $R = \{1\} \times \mathbb{Z}$ is consistent for α. The family $Y = (Y(u))_{(1,u)\in R}$ defined by $Y(u) = c_1 c_2^{-1}$ for any $u \in \mathbb{Z}$ is a solution of the Y-system associated with α in $\mathrm{Trop}(c_1, c_2)$, where we denote $Y_1(u)$ by $Y(u)$. The family of relations

$$T(u)T(u+4) = c_1 T(u+1)T(u+3) + c_2 T(u+2)^2$$

for $u \in \mathbb{Z}$ is the T-system associated with (α, R, Y), where we denote $T_1(u)$ by $T(u)$. This is called the *Somos-4 recurrence* [FZ02b].

Example 2.2.8 (Bipartite belt)**.** Let $A = (2\delta_{ab} - n_{ab})_{a,b\in[1,r]}$ be a symmetrizable generalized Cartan matrix, and D be a right symmetrizer of A. Suppose that A is bipartite, that is, there exists a function $\epsilon : [1,r] \to \{1,-1\}$ such that $n_{ab} > 0$ implies $\epsilon(a) \neq \epsilon(b)$ for any $a, b \in [1,r]$. Let $N = 2I_r - A$. Then the triple of $r \times r$ matrices $\alpha = (A_+, A_-, D)$ defined by

$$A_+ = (1+z^2)I_r, \quad A_- = (1+z^2)I_r - zN$$

is a T-datum since

$$A_+ D A_-^\dagger - A_- D A_+^\dagger = (z + z^{-1})(-DN^{\mathsf{T}} + ND) = 0,$$

and the set $R \subseteq [1,r] \times \mathbb{Z}$ defined by

$$R = \{(a,u) \in [1,r] \times \mathbb{Z} \mid \epsilon(a) = (-1)^{u-1}\}$$

u	0	1	2	3	4	5	6
$Y_1(u)$	y_1		$\dfrac{1 \oplus y_2 \oplus y_1 y_2}{y_1}$		$\dfrac{1}{y_2}$		$(1 \oplus y_1) y_2$
$Y_2(u)$		$(1 \oplus y_1) y_2$		$\dfrac{1 \oplus y_2}{y_1 y_2}$		y_1	
$T_1(u)$	x_1		$\dfrac{y_1 x_2 + 1}{(1 \oplus y_1) x_1}$		$\dfrac{x_1 + y_2}{(1 \oplus y_2) x_2}$		x_2
$T_2(u)$		x_2		$\dfrac{y_1 y_2 x_2 + x_1 + y_2}{(1 \oplus y_2 \oplus y_1 y_2) x_1 x_2}$		x_1	

Table 2.1 The bipartite belt associated with the Cartan matrix of type A_2.

is consistent for α. The family of relations

$$Y_a(u) Y_a(u-2) = \prod_{b=1}^{r} (1 \oplus Y_b(u-1))^{n_{ba}}$$

for $(a, u) \in R$ is the Y-system associated with α, and

$$T_a(u) T_a(u+2) = \frac{Y_a(u) \prod_{b=1}^{r} T_b(u+1)^{n_{ba}} + 1}{1 \oplus Y_a(u)}$$

for $(a, u) \in R$ is the T-system associated with (α, R, Y). The discrete dynamical system given by these relations are called the *bipartite belt* associated with A [FZ07, Section 8].

If A is the Cartan matrix of type A_2 for instance, the triple of matrices in α is given by

$$A_+ = \begin{bmatrix} 1+z^2 & 0 \\ 0 & 1+z^2 \end{bmatrix}, \quad A_- = \begin{bmatrix} 1+z^2 & -z \\ -z & 1+z^2 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Table 2.1 is the bipartite belt associated with the Cartan matrix of type A_2 with $\epsilon(1) = -1$ and $\epsilon(2) = 1$, where y_1 and y_2 are arbitrary elements in the underlying semifield $\mathbb{P}$, and we write $T_1(0)$ and $T_2(1)$ as x_1 and x_2, respectively.

2.2.2 T-data from mutation loops

Let us see that we can obtain T-data from mutation loops. Let γ be a complete mutation loop of length r. Let $(N_{\gamma,0}, N_{\gamma,+}, N_{\gamma,-})$ be the T-system triple and $(A_{\gamma,+}, A_{\gamma,-})$ be the T-system pair of γ, which are defined in Section 2.1. Let D_γ be the positive integer diagonal matrix in Proposition 2.1.7.

Lemma 2.2.9. *The triple $(N_{\gamma,0}, N_{\gamma,+}, N_{\gamma,-})$ satisfies the conditions (N1)–(N4).*

Proof. The condition (N1) is satisfied if $p_a = \lambda_{\sigma^{-1}(a)}$ and σ is as in (2.1.5). The condition (N2) is obvious from the definition. The definition (2.1.14) implies (N3) since $0 < \lambda(j,u) < \lambda_{\sigma^{-1}(b)}$ if $\pi(s(j,u)) = b$ and $(j,u) \notin P_\gamma$. The definition (2.1.14) also implies (N4) since at least one of $[b]_+$ and $[-b]_+$ is zero for any integer b. □

Proposition 2.2.10. *The triple* $\alpha_\gamma := (A_{\gamma,+}, A_{\gamma,-}, D_\gamma)$ *is a T-datum.*

Proof. We have $D_\gamma N_{\gamma,0} = D_\gamma N_{\gamma,0}^\vee = N_{\gamma,0} D_\gamma$, and $D_\gamma^{-1} N_{\gamma,\pm} D_\gamma = N_{\gamma,\pm}^\vee \in \mathrm{Mat}_{r\times r}(\mathbb{Z}[z])$ by Proposition 2.1.7. We also have

$$\begin{aligned} A_{\gamma,+} D_\gamma A_{\gamma,-}^\dagger - A_{\gamma,-} D_\gamma A_{\gamma,+}^\dagger &= D_\gamma A_{\gamma,+}^\vee D_\gamma^{-1} D_\gamma A_{\gamma,-}^\dagger - D_\gamma A_{\gamma,-}^\vee D_\gamma^{-1} D_\gamma A_{\gamma,+}^\dagger \\ &= D_\gamma (A_{\gamma,+}^\vee A_{\gamma,-}^\dagger - A_{\gamma,-}^\vee A_{\gamma,+}^\dagger) \\ &= 0 \end{aligned}$$

by Proposition 2.1.7 and Theorem 2.1.8. □

Let $D_\gamma^\vee$ the diagonal matrix defined by $D_\gamma^\vee = \delta D_\gamma^{-1}$ where δ is the product of the greatest common divisor and the least common multiple of all the entries in D_γ.

Corollary 2.2.11. *The triple* $\alpha_\gamma^\vee := (A_{\gamma,+}^\vee, A_{\gamma,-}^\vee, D_{\gamma,-}^\vee)$ *is a T-datum, and it is the Langlands dual of* α_γ.

Proposition 2.2.12. *The subset* $R_\gamma \subseteq [1,r]\times\mathbb{Z}$ *defined in* (2.1.4) *is consistent for* α_γ.

Proof. The conditions (R1) and (R2) follow from Proposition 2.1.4 and 2.1.5, respectively. The condition (R3) is satisfied if we define t in (R3) as the length of the partition of $\mathbf{i}$. □

2.2.3 Mutation loops from T-data

In this section, we prove all T-data can be obtained from mutation loops up to equivalence.

Theorem 2.2.13. *Suppose that* $\alpha = (A_+, A_-, D)$ *is a T-datum of size* r, *and* $R \subseteq [1,r]\times\mathbb{Z}$ *is consistent for* α. *Then there exists a complete mutation loop* γ *of length* r *such that* $(\alpha_\gamma, R_\gamma)$ *and* (α, R) *are equivalent, where* $\alpha_\gamma = (A_{\gamma,+}, A_{\gamma,-}, D_\gamma)$.

The rest of Section 2.2.3 is devoted to the proof of Theorem 2.2.13. Let $p_1,\dots,p_r$ be positive integers and σ be the permutation of $[1,r]$ in (N1). Let $\psi : [1,r]\times\mathbb{Z} \to [1,r]\times\mathbb{Z}$ be the bijection defined by $\psi(a,u) = (a,u+1)$. We define a family of subsets $R^{(k)}$ $(k\in\mathbb{Z})$ by $R^{(k)} = \psi^k(R)$ as in Definition 2.2.3. We also define a subset $R^{(k)}(u) \subseteq R^{(k)}$ for any $u\in\mathbb{Z}$ by $R^{(k)}(u) = \{(a,u+p) \in R^{(k)} \mid 0 \le p < p_a\}$. We denote $R^{(0)}(u)$ by $R(u)$. The map ψ restricts to a bijection $\psi|_{R^{(k)}(u)} : R^{(k)}(u) \to R^{(k+1)}(u+1)$. We will write this restriction simply ψ when no confusion can arise. Let t be the integer in (R3) in Definition 2.2.3. Then we have $R^{(k)}(u) = R^{(k+t)}(u)$. In particular, the map ψ^t restricts to a bijection $\psi^t|_{R^{(k)}(u)} : R^{(k)}(u) \to R^{(k)}(u+t)$. We also define a family of bijections $\varphi_u : R(u) \to R(u+1)$ $(u\in\mathbb{Z})$ by

$$\varphi_u(a,u+p) = \begin{cases} (a,u+p) & \text{if } p\neq 0, \\ (\sigma(a),u+p_{\sigma(a)}) & \text{if } p=0. \end{cases} \tag{2.2.6}$$

It is easy to check that ψ^t and φ commute in the sense that $\psi^t\circ\varphi_u = \varphi_{u+t}\circ\psi^t$. We define $R_0(u)\subseteq R(u)$ by $R_0(u) = \{(a,u+p)\in R(u) \mid p=0\}$, which is endowed with the linear order coming from the standard linear order on $[1,r]$.

For any $u \in \mathbb{Z}$, we define an $R(u) \times R(u)$ matrix $\bar{B}(u)$ by

$$\begin{aligned}\bar{B}_{(a,u+p)(b,u+q)}(u) &= -n^{+}_{ab;p-q} + n^{-}_{ab;p-q} + \check{n}^{+}_{ba;q-p} - \check{n}^{-}_{ba;q-p} \\ &\quad + \sum_{c=1}^{r} \sum_{v=0}^{\min(p,q)} \bigl(n^{+}_{ac;p-v}\check{n}^{-}_{bc;q-v} - n^{-}_{ac;p-v}\check{n}^{+}_{bc;q-v}\bigr),\end{aligned} \tag{2.2.7}$$

where $n^{\pm}$ and $\check{n}^{\pm}$ are defined in (2.2.1) and (2.2.3), respectively. Note that $\bar{B}(u)$ and $\bar{B}(v)$ may be different matrices if $u \neq v$, even though they have the same expression (2.2.7). Rather, these matrices are related by mutations, as the following lemma shows:

Lemma 2.2.14. *$\bar{B}(u+1) = \varphi_u(\mu_{R_0(u)}(\bar{B}(u)))$ for any $u \in \mathbb{Z}$.*

Proof. Let $\bar{B}'(u) = \mu_{R_0(u)}(\bar{B}(u))$ and $(a, u+p), (b, u+q) \in R(u)$. Then we have

$$\bar{B}'_{(a,u+p)(b,u+q)}(u) = \begin{cases} -\bar{B}_{(a,u+p)(b,u+q)}(u) & \text{if } p \text{ or } q = 0, \\ \bar{B}_{(a,u+p)(b,u+q)}(u) - \displaystyle\sum_{c=1}^{r} \bigl(n^{+}_{ac;p}\check{n}^{-}_{bc;q} - n^{-}_{ac;p}\check{n}^{+}_{bc;q}\bigr) & \text{if } p, q > 0, \end{cases} \tag{2.2.8}$$

since

$$\begin{aligned}&\sum_{\substack{c\in[1,r]\\(c,u)\in R_0(u)}} \bigl([\bar{B}_{(a,u+p)(c,u)}(u)]_+[\bar{B}_{(c,u)(b,u+q)}(u)]_+ \\ &\qquad - [-\bar{B}_{(a,u+p)(c,u)}(u)]_+[-\bar{B}_{(c,u)(b,u+q)}(u)]_+\bigr) \\ &= \sum_{c=1}^{r} \bigl(n^{-}_{ac;p}\check{n}^{+}_{bc;q} - n^{+}_{ac;p}\check{n}^{-}_{bc;q}\bigr)\end{aligned}$$

by (N2), (N4), and (R1).

The proof is divided into the following cases: (i) $p, q > 0$, (ii) $p = 0$ and $q > 0$, (iii) $p > 0$ and $q = 0$, and (iv) $p = q = 0$. For the case (i), we have

$$\bar{B}_{(a,u+p)(b,u+q)}(u) - \bar{B}_{(a,u+p)(b,u+q)}(u+1) = \sum_{c=1}^{r} \bigl(n^{+}_{ac;p}\check{n}^{-}_{bc;q} - n^{-}_{ac;p}\check{n}^{+}_{bc;q}\bigr)$$

by (2.2.7), and this yields the desired equality since $\varphi_u(a, u+p) = (a, u+p)$ and $\varphi_u(b, u+q) = (b, u+q)$. For the case (ii), we have $\varphi_u(a, u) = (\hat{a}, u + p_{\hat{a}})$ where $\hat{a} = \sigma(a)$. Then we have

$$\bar{B}'_{(a,u)(b,u+q)}(u) = -\bar{B}_{(a,u)(b,u+q)}(u) = -\check{n}^{+}_{ba;q} + \check{n}^{-}_{ba;q}$$

and

$$\begin{aligned}\bar{B}_{(\hat{a},u+p_{\hat{a}})(b,u+q)}(u+1) &= -n^{+}_{\hat{a}b;p_{\hat{a}}-q} + n^{-}_{\hat{a}b;p_{\hat{a}}-q} + \check{n}^{+}_{b\hat{a};q-p_{\hat{a}}} - \check{n}^{-}_{b\hat{a};q-p_{\hat{a}}} \\ &\quad + \sum_{c=1}^{r} \sum_{v=1}^{\min(p_{\hat{a}},q)} \bigl(n^{+}_{\hat{a}c;p_{\hat{a}}-v}\check{n}^{-}_{bc;q-v} - n^{-}_{\hat{a}c;p_{\hat{a}}-v}\check{n}^{+}_{bc;q-v}\bigr).\end{aligned}$$

These coincide by the $(b\hat{a}; q - p_{\hat{a}})$-th entry in the symplectic relation, together with (N1) and (N3). For the case (iii), we have $\varphi_u(b,u) = (\hat{b}, u + p_{\hat{b}})$ where $\hat{b} = \sigma(b)$. Then we have

$$\bar{B}'_{(a,u+p)(b,u)}(u) = -\bar{B}_{(a,u+p)(b,u)}(u) = n^+_{ab;p} - n^-_{ab;p}$$

and

$$\begin{aligned}\bar{B}_{(a,u+p)(\hat{b},u+p_{\hat{b}})}(u+1) &= -n^+_{a\hat{b};p-p_{\hat{b}}} + n^-_{a\hat{b};p-p_{\hat{b}}} + \check{n}^+_{\hat{b}a;p_{\hat{b}}-p} - \check{n}^-_{\hat{b}a;p_{\hat{b}}-p} \\ &+ \sum_{c=1}^{r} \sum_{v=1}^{\min(p,p_{\hat{b}})} \left(n^+_{ac;p-v}\check{n}^-_{\hat{b}c;p_{\hat{b}}-v} - n^-_{ac;p-v}\check{n}^+_{\hat{b}c;p_{\hat{b}}-v}\right)\end{aligned}$$

These coincide by the $(a\hat{b}; p - p_{\hat{b}})$-th entry in the symplectic relation, together with (N1) and (N3). For the case (iv), we have $\varphi_u(a,u) = (\hat{a}, u + p_{\hat{a}})$ and $\varphi_u(b,u) = (\hat{b}, u + p_{\hat{b}})$. Then we have

$$\bar{B}'_{(a,u)(b,u)}(u) = -\bar{B}_{(a,u)(b,u)}(u) = 0.$$

On the other hand, we have

$$\begin{aligned}\bar{B}_{(\hat{a},u+p_{\hat{a}})(\hat{b},u+p_{\hat{b}})}(u+1) &= -n^+_{\hat{a}\hat{b};p_{\hat{a}}-p_{\hat{b}}} + n^-_{\hat{a}\hat{b};p_{\hat{a}}-p_{\hat{b}}} + \check{n}^+_{\hat{b}\hat{a};p_{\hat{b}}-p_{\hat{a}}} - \check{n}^-_{\hat{b}\hat{a};p_{\hat{b}}-p_{\hat{a}}} \\ &+ \sum_{c=1}^{r} \sum_{v=1}^{\min(p_{\hat{a}},p_{\hat{b}})} \left(n^+_{\hat{a}c;p_{\hat{a}}-v}\check{n}^-_{\hat{b}c;p_{\hat{b}}-v} - n^-_{\hat{a}c;p_{\hat{a}}-v}\check{n}^+_{\hat{b}c;p_{\hat{b}}-v}\right) = 0\end{aligned}$$

by the $(\hat{a}\hat{b}; p_{\hat{a}} - p_{\hat{b}})$-th entry in the symplectic relation, together with (N1) and (N3). □

Lemma 2.2.15. *$\bar{B}(u+t) = \psi^t(\bar{B}(u))$ for any $u \in \mathbb{Z}$.*

Proof. Since $R(u) = R(u+t)$, the lemma follows from the fact that $\bar{B}(u+t)$ and $\bar{B}(u)$ have the same expression (2.2.7). □

We define an index set I by $I = R(0)$, and define an $I \times I$ integer matrices B by $B = \bar{B}(0)$, that is, $B = (B_{(a,p)(b,q)})_{(a,p),(b,q)\in R(0)}$ and

$$\begin{aligned}B_{(a,p)(b,q)} &= -n^+_{ab;p-q} + n^-_{ab;p-q} + \check{n}^+_{ba;q-p} - \check{n}^-_{ba;q-p} \\ &+ \sum_{c=1}^{r} \sum_{v=0}^{\min(p,q)} \left(n^+_{ac;p-v}\check{n}^-_{bc;q-v} - n^-_{ac;p-v}\check{n}^+_{bc;q-v}\right).\end{aligned} \tag{2.2.9}$$

We define a tuple of positive integer $d = (d_{a,u})_{(a,u)\in R(u)}$ by $d_{a,u} = d_a$, where d_a is the a-th entry in D. We also define $\mathbf{i} = \mathbf{i}(0) \mid \cdots \mid \mathbf{i}(t-1)$ by $\mathbf{i}(u) = (\varphi_{u-1} \circ \cdots \circ \varphi_0)^{-1}(R_0(u))$, where each $\mathbf{i}(u)$ is endowed with the linear order coming from the linear order on $R_0(u)$. Finally, we define $\nu = (\varphi_{t-1} \circ \cdots \circ \varphi_0)^{-1} \circ \psi^t$.

Lemma 2.2.16. *$\gamma = (B, d, \mathbf{i}, \nu)$ is a complete mutation loop of length r.*

Proof. It is easy to check that B is a skew-symmetrizable matrix with the symmetrizer d. Let us denote by $\vec{\varphi}_u$ the composition $\varphi_{u-1} \circ \cdots \circ \varphi_0$. We now prove

$$(\mu_{\mathbf{i}(u-1)} \circ \cdots \circ \mu_{\mathbf{i}(0)})(B) = (\vec{\varphi}_u)^{-1}(\bar{B}(u)) \tag{2.2.10}$$

for any $u = 0, \ldots, t-1$ by induction on u. The equation (2.2.10) holds when $u = 0$ since $B = \bar{B}(0)$ by definition. Suppose that $u > 0$. By the induction hypothesis and Lemma 2.2.14, we have

$$\begin{aligned}
&(\mu_{\mathbf{i}(u-1)} \circ \mu_{\mathbf{i}(u-2)} \circ \cdots \circ \mu_{\mathbf{i}(0)})(B) \\
&= (\mu_{\mathbf{i}(u-1)} \circ (\vec{\varphi}_{u-1})^{-1})(\bar{B}(u-1)) \\
&= ((\vec{\varphi}_{u-1})^{-1} \circ \mu_{R_0(u)})(\bar{B}(u-1)) \\
&= ((\vec{\varphi}_{u-1})^{-1} \circ (\varphi_{u-1})^{-1})(\bar{B}(u)) \\
&= (\vec{\varphi}_u)^{-1}(\bar{B}(u)),
\end{aligned}$$

and (2.2.10) is proved. Applying (2.2.10) for $u = t-1$ yields

$$\mu_{\mathbf{i}}(B) = (\vec{\varphi}_t)^{-1}(\bar{B}(t)) = ((\vec{\varphi}_t)^{-1} \circ \psi^t)(B).$$

This shows that γ is a mutation loop. By (R3) in Definition 2.2.3, we have

$$\{(a, 0) \mid a \in [1, r]\} = \bigsqcup_{u=0}^{t-1} \psi^{-u}(R_0(u))$$

as a set. This implies that the length of γ is r. The completeness follows from the fact that the latency of $((a, p), 0) \in I \times \{0\}$ is p. □

Now we complete the proof of Theorem 2.2.13 by showing the following lemma:

Lemma 2.2.17. *$(\alpha_\gamma, R_\gamma)$ and (α, R) are equivalent.*

Proof. By replacing (α, R) with a suitable equivalent one, we can assume that $u < v$ implies that $a < b$ for any $(a, u) \in R_0(u)$ and $(b, v) \in R_0(v)$ such that $0 \leq u, v \leq t-1$. Then the construction of γ ensures that $(\alpha_\gamma, R_\gamma) = (\alpha, R)$. □

2.2.4 Consequences

For any complete mutation loop γ, we denote by $F(\gamma)$ the pair $(\alpha_\gamma, R_\gamma)$ defined in Section 2.2.2. For any pair (α, R) of a T-datum and a consistent subset R for α, we denote $G(\alpha, R)$ by the complete mutation loop defined in Section 2.2.3.

We define

$$\mathrm{Ml}_r = \{[\gamma] \mid \gamma \text{ is a complete mutation loop of length } r\},$$

where $[\gamma]$ is the equivalence class of γ (see Definition 2.1.2). We also define

$$\mathrm{Td}_r = \{\alpha \mid \alpha \text{ is a T-datum of size } r\}$$

and

$$\mathrm{Td}_r' = \{[(\alpha, R)] \mid \alpha \in \mathrm{Td}_r \text{ and } R \subseteq [1, r] \times \mathbb{Z} \text{ is consistent for } \alpha\},$$

where $[(\alpha, R)]$ is the equivalence class of (α, R) (see Definition 2.2.4). We define

$$\hat{F}_r : \mathrm{Ml}_r \to \mathrm{Td}_r', \quad \hat{G}_r : \mathrm{Td}_r' \to \mathrm{Ml}_r$$

by $\hat{F}_r([\gamma]) = [F(\gamma)]$ and $\hat{G}_r([(\alpha, R)]) = [G(\alpha, R)]$.

Theorem 2.2.18. *$\hat{F}_r \circ \hat{G}_r = \mathrm{id}$ and $\hat{G}_r \circ \hat{F}_r = \mathrm{id}$.*

Proof. We first show that $\hat{F}_r$ and $\hat{G}_r$ are well-defined. Let γ and γ' be equivalent complete mutation loops, and ρ be the permutation in Definition 2.1.2. Then we have $A_{\gamma',\pm} = \rho(A_{\gamma,\pm})$ and $R_{\gamma'} = \rho(R_\gamma)$. Thus $\hat{F}_r$ is well-defined.

Let (α, R) and (α', R') be equivalent pairs of T-data and consistent subsets for them. Let $\rho \in \mathfrak{S}_r$ be the permutation in Definition 2.2.4. Let $\gamma = (B, d, \mathbf{i}, \nu)$ and $\gamma' = (B', d', \mathbf{i}', \nu')$ be the mutation loops given by $\gamma = G(\alpha, R)$ and $\gamma' = G(\alpha', R')$, respectively. Then the bijection $f_{\rho,u} : R(u) \to R'(u)$ defined by $f_{\rho,u}(a, u) = (\rho(a), u)$ satisfies $B' = f_{\rho,0}(B)$, $d' = f_{\rho,0}(d)$ and $\nu' = f_{\rho,0} \circ \nu \circ f_{\rho,0}^{-1}$. Moreover, $R_0'(u)$ and $f_{\rho,u}(R_0(u))$ coincide as sets. Thus γ and γ' are equivalent, and $\hat{G}_r$ is well-defined.

By Lemma 2.2.17, we have $\hat{F}_r \circ \hat{G}_r = \mathrm{id}$. It remains to show that $\hat{G} \circ \hat{F} = \mathrm{id}$. Let γ be any complete mutation loop, and I be the index set of γ. Let $\gamma' = G(\alpha_\gamma, R_\gamma)$. Then the index set I' of γ' is given by $I' = \{(a, u) \in R_\gamma \mid 0 \le u < p_a\}$. Let $f : I \to I'$ be the map defined by

$$f(i) = \begin{cases} (\pi(i, 0), 0) & \text{if } (i, 0) \in P_\gamma, \\ (\pi(s(i, 0)), \lambda(i, 0)) & \text{if } (i, 0) \notin P_\gamma, \end{cases}$$

where π, s, and λ, P_γ are defined in Section 2.1. It is easy to check that f is a bijection, and gives the equivalence between γ and γ'. □

For any consistent subset $R \subseteq [1, r] \times \mathbb{Z}$ for a T-datum of size r, we define the set R_{in} (the subscript "in" stands for initial) by $R_{\mathrm{in}} = \{(a, p) \in R \mid 0 \le p < p_a\}$.

Theorem 2.2.19. *Let α be a T-datum of size r. Suppose that $R \subseteq [1, r] \times \mathbb{Z}$ is consistent for α. Let $\mathbb{P}$ be a semifield, and $\mathcal{F}$ be a filed that is isomorphic to the field of rational functions over $\mathbb{QP}$ in $|R_{\mathrm{in}}|$ variables. Let $x = (x_{a,p})_{(a,p)\in R_{\mathrm{in}}}$ be an R_{in}-tuple of elements in $\mathcal{F}$ forming a free generating set. Let $Y = (Y_a(u))_{(a,u)\in R}$ be a solution of the Y-system associated with (α, R) in $\mathbb{P}$. Then there exists a unique R_{in}-labeled Y-seed (B, y) in $\mathbb{P}$ such that*

(1) there exists a unique injective $\mathbb{ZP}$-algebra homomorphism $\iota : \mathcal{T}^\circ(\alpha, R, Y) \hookrightarrow \mathcal{A}(B, y, x)$ such that $\iota(T_a(p)) = x_{a,p}$ for any $(a, p) \in R_{\mathrm{in}}$,
(2) $\iota(T_a(u))$ is a cluster variable in $\mathcal{A}(B, y, x)$ for any $(a, u) \in R$,
(3) the image of the relation (2.2.5) by ι is an exchange relation in $\mathcal{A}(B, y, x)$ for any $(a, u) \in R$.

Proof. Let $\gamma = (B, d, \mathbf{i}, \nu)$ be the mutation loop given by $\gamma = G(\alpha, R)$. By Lemma 2.2.17, we can assume that $\alpha = \alpha_\gamma$ and $R = R_\gamma$. We define a family of elements $y = (y_{a,p})_{(a,p)\in R_{\mathrm{in}}}$ in $\mathbb{P}$ by

$$y_{a,p} = Y_a(p) \frac{\prod_{b=1}^{r} \prod_{q=0}^{p} \bigl(1 \oplus Y_b(p-q)^{-1}\bigr)^{\check{n}^{+}_{ab;q}}}{\prod_{b=1}^{r} \prod_{q=0}^{p} \bigl(1 \oplus Y_b(p-q)\bigr)^{\check{n}^{-}_{ab;q}}}. \tag{2.2.11}$$

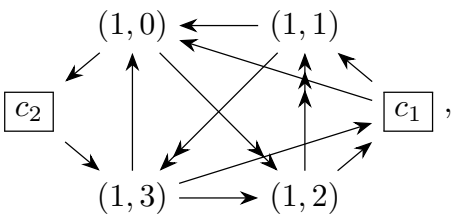
Then we can define a family of seeds $(B(u), y(u), x(u))_{u\in\mathbb{Z}}$ by (2.1.11), where $(B(0), y(0), x(0)) = (B, y, x)$. Note that $\mathbf{i}(u) = (\vec{\varphi}_u)^{-1}(R_0(u))$ for any $u \in \mathbb{Z}$ by the commutativity of ψ^t and φ, where we define $\vec{\varphi}_u : R(0) \to R(u)$ by

$$\vec{\varphi}_u = \begin{cases} \varphi_{u-1} \circ \cdots \circ \varphi_0 & \text{if } u \geq 0, \\ (\varphi_{-1} \circ \cdots \circ \varphi_u)^{-1} & \text{if } u < 0. \end{cases} \tag{2.2.12}$$

We define $Y' = (Y'_a(u))_{(a,u)\in R_\gamma}$ by (2.1.7), where $Y_a(u)$ in (2.1.7) is replaced with $Y'_a(u)$. Then we have $Y_a(p) = Y'_a(p)$ for any $(a, p) \in R_{\mathrm{in}}$ by (2.2.11). Moreover, Y and Y' are solutions of the same Y-system, we have $Y_a(u) = Y'_a(u)$ for any $(a, u) \in R$. We define $T'_a(u)$ for any $(a, u) \in R_\gamma$ by (2.1.12), where $T_a(u)$ in (2.1.12) is replaced with $T'_a(u)$. Let $\mathcal{A}_\gamma(B, y, x)$ be the $\mathbb{ZP}$-subalgebra of $\mathcal{A}(B, y, x)$ generated by $(x_{a,p}(u))_{(a,p)\in R_{\mathrm{in}}, u\in\mathbb{Z}}$. It is also generated by $(T'_a(u))_{(a,p)\in R_\gamma}$. We now show that $\mathcal{A}_\gamma(B, y, x)$ is isomorphic to $\mathcal{T}^\circ(\alpha, R, Y)$. Let $\bar{\iota} : \mathcal{T}^\circ(\alpha, R, Y) \to \mathcal{A}_\gamma(B, y, x)$ be the algebra homomorphism defined by $\bar{\iota}(T_a(u)) = T'_a(u)$. This is well-defined by Proposition 2.1.5. To construct the inverse of $\bar{\iota}$, we define an algebra homomorphism $\kappa : \mathbb{ZP}[x_{a,p}^{\pm 1}]_{(a,p)\in R_{\mathrm{in}}} \to \mathcal{T}(\alpha, R, Y)$ by $\kappa(x_{a,p}^{\pm 1}) = T_a(p)^{\pm 1}$. By the Laurent phenomenon of cluster algebras [FZ02a], $\mathcal{A}_\gamma(B, y, x)$ is a subalgebra of $\mathbb{ZP}[x_{a,p}^{\pm 1}]_{(a,p)\in R_{\mathrm{in}}}$. Then we have $\kappa(T'_a(u)) = T_a(u)$ since $(T'_a(u))_{(a,p)\in R_\gamma}$ and $(T_a(u))_{(a,p)\in R}$ satisfy the same T-system. Thus we obtain the algebra homomorphism $\bar{\kappa} : \mathcal{A}_\gamma(B, y, x) \to \mathcal{T}^\circ(\alpha, R, Y)$ as the restriction of κ, which is the inverse of $\bar{\iota}$. Therefore, the existence of a Y-seed satisfying (1)–(3) is proved.

The uniqueness follows from the following facts, which hold for any skew-symmetrizable cluster algebra: (i) given a cluster as a set, the exchange relations involving its elements are uniquely determined [CL20, Proposition 6.1], (ii) any two clusters that have $|I| - 1$ common cluster variables are related by an exchange relation [GSV08, Theorem 5]. □

Example 2.2.20 (Somos-4 recurrence)**.** Let α, R, and Y be as in Example 2.2.7. Then the Y-seed given by Theorem 2.2.19 is

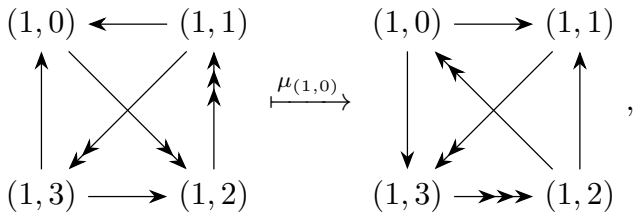


(2.2.13)

where we represent the Y-seed using a quiver with frozen vertices as in the cluster algebra literature (see e.g., [FWZ16]). For instance, $\boxed{c_2} \leftarrow i \leftarrow \boxed{c_1}$ means $y_i = c_1 c_2^{-1}$. The mutation loop $\gamma = (B, d, \mathbf{i}, \nu) = G(\alpha, R)$ is given by

- $B = B(Q)$,
- $d_i = 1$ for any $i \in I$,
- $\mathbf{i} = \mathbf{i}(0)$ with $\mathbf{i}(0) = ((1,0))$,
- $\nu = ((1,0)\,(1,1)\,(1,2)\,(1,3))$,

where Q is an underlying quiver in (2.2.13), and ν is the cyclic permutation corresponding to the right $\pi/2$ rotation of the quiver. In fact, the quiver mutation $\mu_{(1,0)}$ is given by



and we have $\mu_{(1,0)}(Q) = \nu(Q)$.

Example 2.2.21 (Bipartite belt)**.** Let α and R be as in Example 2.2.8. The mutation loop $\gamma = (B, d, \mathbf{i}, \nu) = G(\alpha, R)$ is given by

- $B = (B_{(a,p)(b,q)})_{(a,p),(b,q)\in I}$, where $B_{(a,p)(b,q)} = \epsilon(a)n_{ab}$,
- $d = (d_{a,p})_{(a,p)\in I}$, where $D = \mathrm{diag}(d_1, \ldots, d_r)$ and $d_{a,p} = d_a$,
- $\mathbf{i} = \mathbf{i}(0) \mid \mathbf{i}(1)$ with $\mathbf{i}(0) = \{(a,0) \mid \epsilon(a) = -1\}$ and $\mathbf{i}(1) = \{(a,1) \mid \epsilon(a) = 1\}$,
- $\nu = \mathrm{id}$,

where $I = \{(a,0) \mid \epsilon(a) = -1\} \sqcup \{(a,1) \mid \epsilon(a) = 1\}$. If A is the Cartan matrix of type A_2, for instance, the quiver $Q(B)$ is given by

$$Q(B) = \quad (1,0) \longleftarrow (2,1)\ .$$

If A is a Cartan matrix of finite type, $\mathcal{A}(B, y, x)$ is a finite type cluster algebra and the embedding $\iota : \mathcal{T}^\circ(\alpha, R, Y) \hookrightarrow \mathcal{A}(B, y, x)$ in Theorem 2.2.19 is an isomorphism [FZ07, Proposition 11.1].

2.2.5 Tropical T-system

By Theorem 2.2.19 and the Laurent phenomenon of cluster algebras [FZ02a], we obtain the following:

Corollary 2.2.22. *Let $\mathcal{T}^\circ(\alpha, R, Y)$ be a T-algebra. Then $T_a(u) \in \mathcal{T}^\circ(\alpha, R, Y)$ can be written as a Laurent polynomial in $(T_c(p))_{(c,p)\in R_{\mathrm{in}}}$ with coefficients in $\mathbb{ZP}$, for any $(a,u) \in R$.*

Let $R = [1,r] \times \mathbb{Z}$. By Corollary 2.2.22, any $T_a(u)$ can be uniquely written as

$$T_a(u) = \frac{N}{\prod_{(c,p)\in R_{\mathrm{in}}} T_c(p)^{d_{c,p}}}, \tag{2.2.14}$$

where N is a polynomial in $(T_c(p))_{(c,p)\in R_{\mathrm{in}}}$ with coefficients in $\mathbb{ZP}$ which is not divisible by any $T_c(p)$ $((c,p) \in R_{\mathrm{in}})$. We denote by $\mathfrak{t}_a^{(c)}(u)$ the integer $d_{c,0}$ in (2.2.14). The family

of integers $(\mathrm{t}_a^{(c)}(u))_{(a,u)\in[1,r]\times\mathbb{Z}}$ is uniquely determined by the initial conditions

$$\mathrm{t}_a^{(c)}(p) = \begin{cases} -1 & \text{if } (a,p) = (c,0), \\ 0 & \text{if } (a,p) \neq (c,0) \text{ and } 0 \leq p < p_a, \end{cases} \tag{2.2.15}$$

together with the following recurrence relation for each $(a,u) \in [1,r] \times \mathbb{Z}$:

$$\sum_{b,p} n_{ba;p}^{0} \mathrm{t}_b^{(c)}(u+p) = \max\Bigg(\sum_{b,p} n_{ba;p}^{-} \mathrm{t}_b^{(c)}(u+p), \sum_{b,p} n_{ba;p}^{+} \mathrm{t}_b^{(c)}(u+p)\Bigg). \tag{2.2.16}$$

In particular, the integer $\mathrm{t}_a^{(c)}(u)$ is independent of the choice of Y. The family of relations (2.2.16) is called the *tropical T-system* associated with α.

We also define a family of integers $(\hat{\mathfrak{y}}_a^{(c)}(u))_{(a,u)\in[1,r]\times\mathbb{Z}}$ by

$$\hat{\mathfrak{y}}_a^{(c)}(u) = \sum_{b,p} \big(n_{ba;p}^{-} \mathrm{t}_b^{(c)}(u+p) - n_{ba;p}^{+} \mathrm{t}_b^{(c)}(u+p)\big) \tag{2.2.17}$$

$$= \sum_{b,p} \big((n_{ba;p}^{0} - n_{ba;p}^{+}) \mathrm{t}_b^{(c)}(u+p) - (n_{ba;p}^{0} - n_{ba;p}^{-}) \mathrm{t}_b^{(c)}(u+p)\big). \tag{2.2.18}$$

By the relation (2.2.16), we have

$$[\pm\hat{\mathfrak{y}}_a^{(c)}(u)]_+ = \sum_{b,p} (n_{ba;p}^{0} - n_{ba;p}^{\pm}) \mathrm{t}_b^{(c)}(u+p). \tag{2.2.19}$$

The following lemma will be used in Section 3.

Lemma 2.2.23. *The following equalities hold for any T-datum:*

(1) For any $a \in [1,r]$, we have

$$\mathrm{t}_a^{(c)}(p_a) = \begin{cases} 1 & \text{if } a = \sigma(c), \\ 0 & \text{otherwise.} \end{cases}$$

(2) For any $a \in [1,r]$ and $0 \leq p \leq p_c$, we have

$$\mathrm{t}_a^{(c)}(-p) = \begin{cases} -1 & \text{if } (a,p) = (c,0), \\ 1 & \text{if } (a,p) = (\sigma^{-1}(c), p_c), \\ 0 & \text{otherwise.} \end{cases}$$

(3) For any $a \in [1,r]$ and $0 \leq p \leq p_c$, we have

$$[\pm\hat{\mathfrak{y}}_a^{(c)}(-p)]_+ = n_{ca;p}^{\pm}.$$

Proof. (1) is clear from (2.2.15) and (2.2.16). We prove (2) by induction on p. The case $p = 0$ follows from (2.2.15). Suppose that $p > 0$. Then we have

$$\begin{aligned}\mathfrak{t}_a^{(c)}(-p) &= -\mathfrak{t}_{\sigma(a)}^{(c)}(-p+p_{\sigma(a)}) + \max\left(\sum_{b,q} n_{ba;q}^{-}\mathfrak{t}_b^{(c)}(-p+q), \sum_{b,q} n_{ba;q}^{+}\mathfrak{t}_b^{(c)}(-p+q)\right)\\ &= \delta_{\sigma(a)c}\delta_{pp_c} + \max(-n_{ca;p}^{+}, -n_{ca;p}^{-})\\ &= \delta_{\sigma(a)c}\delta_{pp_c},\end{aligned}$$

and (2) is proved. We now prove (3). From (2.2.19), we have

$$\begin{aligned}[\pm\hat{\mathfrak{y}}_a^{(c)}(-p)]_+ &= \sum_{b,q}(n_{ba;q}^{0} - n_{ba;q}^{\pm})\mathfrak{t}_b^{(c)}(-p+q)\\ &= \mathfrak{t}_a^{(c)}(-p) + \mathfrak{t}_{\sigma(a)}^{(c)}(-p+p_{\sigma(a)}) - n_{ca;p}^{\pm}\mathfrak{t}_c^{(c)}(0)\\ &= \mathfrak{t}_a^{(c)}(-p) + \mathfrak{t}_{\sigma(a)}^{(c)}(-p+p_{\sigma(a)}) + n_{ca;p}^{\pm}.\end{aligned}$$

By (1) and (2) in this lemma, we have

$$\mathfrak{t}_a^{(c)}(-p) = \begin{cases} 1 & \text{if } (a,p) = (\sigma^{-1}(c), p_c),\\ -1 & \text{if } (a,p) = (c,0),\\ 0 & \text{otherwise,}\end{cases}$$

and

$$\mathfrak{t}_{\sigma(a)}^{(c)}(-p+p_{\sigma(a)}) = \begin{cases} 1 & \text{if } (a,p) = (c,0),\\ -1 & \text{if } (a,p) = (\sigma^{-1}(c), p_c),\\ 0 & \text{otherwise.}\end{cases}$$

Thus we have $\mathfrak{t}_a^{(c)}(-p) + \mathfrak{t}_{\sigma(a)}^{(c)}(-p+p_{\sigma(a)}) = 0$. This completes the proof of (3). □

2.2.6 Indecomposable T-data

If (A_+, A_-, D) and (A'_+, A'_-, D') are T-data, the direct sum

$$\left(\begin{bmatrix} A_+ & O\\ O & A'_+\end{bmatrix}, \begin{bmatrix} A_- & O\\ O & A'_-\end{bmatrix}, \begin{bmatrix} D & O\\ O & D'\end{bmatrix}\right)$$

is also a T-datum. A T-datum (A_+, A_-, D) is called *decomposable* if it can be written as a nontrivial direct sum after reordering the indices of matrices. A T-datum that is not decomposable is called *indecomposable*.

We say that a skew-symmetrizable matrix B is *connected* if it cannot be written as a nontrivial direct sum. We also say that a connected skew-symmetrizable matrix $B' = (B'_{ij})_{i,j\in I'}$ is a *connected component* of a skew-symmetrizable matrix $B = (B_{ij})_{i,j\in I}$ if $I' \subseteq I$, $B'_{ij} = B_{ij}$ for any $i, j \in I'$, and $B_{ij} = 0$ for any $i \in I'$ and $j \in I \setminus I'$.

Proposition 2.2.24. *Let α be an indecomposable T-datum. Let I' be the index set of a connected component of B, where B is the skew-symmetrizable matrix in the mutation loop $G(\alpha, [1,r]\times\mathbb{Z})$. Then the set*

$$R' := \bigcup_{u\in\mathbb{Z}} \vec{\varphi}_u(I')$$

is consistent for α, where $\vec{\varphi}_u$ is defined by (2.2.12). Moreover, the index set of B' is I', where B' is the skew-symmetrizable matrix in the mutation loop $G(\alpha, R')$.

Proof. Let $(B, d, \mathbf{i}, \nu) = G(\alpha, [1,r]\times\mathbb{Z})$. The bijection ν is given by $\nu = \varphi_0^{-1}\circ\psi$. Let I be the index set of B, that is, $I = \{(a,p) \mid a\in[1,r], 0\le p < p_a\}$. We recursively define subsets $I'_k \subseteq I$ for $k\in\mathbb{Z}_{>0}$ by $I'_0 = I'$ and $I'_k = \nu|_{I'_{k-1}}(I'_{k-1})$. Then $B|_{I'_k}$ is a connected component of B since mutations preserve connected components. We now prove (R1) and (R2) in Definition 2.2.3 for R'. The proof of (R1) and (R2) are almost the same, we only prove (R1). Suppose that $(a,u)\in R'$. If $\check{n}^0_{ab;p}\neq 0$, then $(b,u-p)\in R'$ by the definition of R'. Suppose that $\check{n}^+_{ab;p}$ or $\check{n}^-_{ab;p}\neq 0$. Then we have $(a,u),(b,u-p)\in\vec{\varphi}_{u-p}(I)$ and $\bar{B}_{(b,u-p)(a,u)}(u-p)\neq 0$, where $\bar{B}(u-p)$ is defined in (2.2.7). This shows that (a,p) and $(b,0)$ lie in the same connected component $B|_{I'_{p-u}}$. Thus we have $(b,u-p)\in R'$ since $\vec{\varphi}_{u-p}(\nu^{u-p}(b,0)) = (b,u-p)$ and $\nu^{u-p}(b,0)\in I'$.

We next prove (R3). Let t be the smallest positive integer such that $I'_t = I'$. We now show that

$$I = \bigsqcup_{k=0}^{t-1} I'_k. \tag{2.2.20}$$

The equality $I = \bigcup_{k=0}^{t-1} I'_k$ follows from the fact that α is indecomposable. Suppose that $I'_{k_1}\cap I'_{k_2}\neq\emptyset$ for some $0\le k_1 < k_1 < t$. Then we have $I'_{k_1} = I'_{k_2}$ since $B|_{I'_{k_1}}$ and $B|_{I'_{k_2}}$ are connected components of B. But this implies $I' = I'_{k_2-k_1}$, which contradicts the minimality of t. Thus (2.2.20) is proved. We now have

$$\begin{aligned}
\bigcup_{k=0}^{t-1}\psi^k(R') &= \bigcup_{k=0}^{t-1}\bigcup_{u\in\mathbb{Z}}(\psi^k\circ\vec{\varphi}_u)(I') = \bigcup_{k=0}^{t-1}\bigcup_{u\in\mathbb{Z}}(\vec{\varphi}_{u+k}\circ\nu^k)(I')\\
&= \bigcup_{k=0}^{t-1}\bigcup_{u\in\mathbb{Z}}\vec{\varphi}_{u+k}(I'_k) = \bigcup_{k=0}^{t-1}\bigcup_{u\in\mathbb{Z}}\vec{\varphi}_u(I'_k) = \bigcup_{u\in\mathbb{Z}}\vec{\varphi}_u\left(\bigcup_{k=0}^{t-1} I'_k\right)\\
&= \bigcup_{u\in\mathbb{Z}}\vec{\varphi}_u(I) = [1,r]\times\mathbb{Z}.
\end{aligned}$$

It remains to prove the disjointness. Suppose that there exists a element $(a,u)\in\psi^{k_1}(R')\cap\psi^{k_2}(R')$ for some $0\le k_1 < k_2 < t$. Then we have $((\vec{\varphi}_{u-k_i})^{-1}\circ\psi^{-k_i})(a,u)\in I'$ for $i=1,2$. Thus we have $(\nu^{k_i}\circ(\vec{\varphi}_{u-k_i})^{-1}\circ\psi^{-k_i})(a,u)\in I'_{k_i}$ for $i=1,2$. On the other hand, we have $\nu^{k_i}\circ(\vec{\varphi}_{u-k_i})^{-1}\circ\psi^{-k_i} = (\vec{\varphi}_u)^{-1}$. This implies that $(\vec{\varphi}_u)^{-1}(a,u)\in I'_{k_1}\cap I'_{k_2}$, which contradicts (2.2.20). □

Corollary 2.2.25. *Let $\alpha = (A_+, A_-, D)$ be an indecomposable T-datum of size r. Suppose that there exists $a \in [1, r]$ such that both the a-th columns in N_+ and N_- are zero vectors. Then both N_+ and N_- are zero matrices.*

Proof. Let $(B, d, \mathbf{i}, \nu) = G(\alpha, [1, r] \times \mathbb{Z})$. By the assumption and (2.2.9), the set $I' = \{(a, 0)\}$ is the index set of a connected component of B. Then the set R' defined in Proposition 2.2.24 is consistent for α. Let $(B', d', \mathbf{i}', \nu') = G(\alpha, R')$. By Proposition 2.2.24, B' is an $I' \times I'$ matrix. This implies that $B' = 0$ since any skew-symmetrizable matrix of size 1 should be the zero matrix. From (2.1.14) and Lemma 2.2.17, we have $N_\pm = 0$. □

2.3 Examples of T-data

2.3.1 Period 1 quivers

Theorem 2.3.1. *Let $p > 0$ be a positive integer, and $a(z) = 1 + n_1 z + \cdots + n_{p-1} z^{p-1} + z^p \in \mathbb{Z}[z]$ be a monic palindromic polynomial of degree p, that is, $n_q = n_{p-q}$ for any $0 < q < p$. Let d be any positive integer. Then the triple $\alpha = (A_+, A_-, [d])$ given by $A_\pm = N_0 - N_\pm$ where*

$$N_0 = \left[1 + z^p\right], \quad N_+ = \left[\sum_{q=1}^{p-1} [n_q]_+ z^q\right], \quad N_- = \left[\sum_{q=1}^{p-1} [-n_q]_+ z^q\right],$$

is a T-datum of size 1. Furthermore, any T-datum of size 1 is of this form.

Proof. The conditions (N1)–(N4) follow immediately from the definition. Since $a(z)$ is a monic palindromic polynomial of degree p, both $z^{-p/2} A_+$ and $z^{-p/2} A_-$ are †-invariant. This implies that $A_+ A_-^\dagger = A_- A_+^\dagger$. Thus α is a T-datum.

Conversely, let $\alpha = (A_+, A_-, [d]) = (N_0 - N_+, N_0 - N_-, [d])$ be any T-datum of size 1. We now identify 1×1 matrices with their entries. By the condition (N1), the matrix N_0 can be written as $N_0 = 1 + z^p$ for some positive integer $p > 0$. Then the matrices $N_\pm$ can be written as $N_\varepsilon = \sum_{q=1}^{p-1} n_q^\varepsilon z^q$ since the degrees of $N_\pm$ are greater that 0 and less than p by the condition (N3). We also have $n_q^\varepsilon \in \mathbb{Z}_{\geq 0}$ by the condition (N2). By the condition (N4), these numbers can be written as $n_q^\pm = [\pm n_q]_+$, where we define $n_q := n_q^+ - n_q^-$. We now show by induction on q that $n_q^\pm = n_{p-q}^\pm$ for any $0 \leq q \leq p$, where we set $n_0^\pm = n_p^\pm = 0$. The case $q = 0$ is obvious from the definition. Suppose that $q > 0$. Let m_q be the coefficient of z^{p-q} in the polynomial $N_0 N_+^\dagger + N_+ N_-^\dagger + N_- N_0^\dagger$, that is,

$$m_q = n_q^+ + n_{p-q}^- + \sum_{k=0}^{q} n_{p-q+k}^+ n_k^-. \tag{2.3.1}$$

On the other hand, m_q is also the coefficient of z^{p-q} in the polynomial $N_0 N_-^\dagger + N_- N_+^\dagger +$

$N_+ N_0^\dagger$ by the symplectic relation. Thus we obtain

$$m_q = n_q^- + n_{p-q}^+ + \sum_{k=0}^{q} n_{p-k}^- n_{q-k}^+. \tag{2.3.2}$$

The sum parts in (2.3.1) and (2.3.2) coincide by the induction hypothesis, so we obtain $n_q^+ + n_{p-q}^- = n_q^- + n_{p-q}^+$. Then we conclude from (N4) that $n_q^\pm = n_{p-q}^\pm$. □

Let α be a T-datum of size 1 given in Theorem 2.3.1. Let $\gamma = (B, d, \mathbf{i}, \nu)$ be the complete mutation loop given by $\gamma = G(\alpha, \{1\} \times \mathbb{Z})$. Then the index set I of B is given by $I = \{(1, i) \in \{1\} \times \mathbb{Z} \mid 0 \le i < p\}$. We identify I with the set $\{0, 1, \dots, p-1\}$. Then $\mathbf{i} = \mathbf{i}(0)$ with $\mathbf{i}(0) = (0)$, and ν is the cyclic permutation given by $\nu(i) = i + 1 \pmod t$. The exchange matrix $B = (B_{ij})_{i,j \in I}$ can be computed from the formula (2.2.9) as follows:

$$B_{ij} = -n_{i-j} + n_{j-i} + \sum_{k=0}^{\min(i,j)} \left(n_{i-k}^+ n_{j-k}^- - n_{i-k}^- n_{j-k}^+ \right), \tag{2.3.3}$$

where $n_i^\pm := [\pm n_i]_+$ and $n_i := 0$ unless $0 < i < p$.

Remark 2.3.2. The formula (2.3.3) is precisely the general solution of *period* 1 *quivers* given by Fordy and Marsh [FM11, Theorem 6.1]. We can regard Theorem 2.3.1 as another proof of the classification for period 1 quivers, which was also given in [FM11, Theorem 6.1].

In Example 2.2.7 and 2.2.20 (the Somos-4 recurrence), we give an example of a T-datum of size 1 and a period 1 quiver.

2.3.2 Commuting Cartan matrices

In this section, we give T-data associated with pairs of Cartan matrices, which are generalization of T-data associated with bipartite belts in Example 2.2.8 and 2.2.21.

Definition 2.3.3. A matrix $C = (c_{ab})_{a,b \in [1,r]} \in \mathrm{Mat}_{r \times r}(\mathbb{Z})$ is called a *symmetrizable weak generalized Cartan matrix* if

(1) $c_{aa} \le 2$ for any a,
(2) $c_{ab} \ge 0$ for any a, b,
(3) there exists a positive integer diagonal matrix D such that CD is a symmetric matrix.

The diagonal matrix D is called a (right) *symmetrizer* of C. Note that a symmetrizable generalized Cartan matrix is a symmetrizable weak generalized Cartan matrix satisfying $c_{aa} = 2$ for any $a \in [1, r]$.

Proposition 2.3.4. *Let A and A' be symmetrizable weak generalized Cartan matrices that have a common symmetrizer D. Let $N = (n_{ab})_{a,b \in [1,r]} := 2I_r - A$ and $N' = (n'_{ab})_{a,b \in [1,r]} := 2I_r - A'$. Then the triple $\alpha = (A_+, A_-, D)$ defined by*

$$N_0 = (1 + z^2) I_r, \quad N_+ = zN, \quad N_- = zN'$$

is a T-datum if and only if $AA' = A'A$ *and* $n_{ab}n'_{ab} = 0$ *for any* $a, b \in [1, r]$.

Proof. The conditions (N1)–(N3) are obvious by the definition. We also have $N_0 D = DN_0$ and $D^{-1}N_{\pm}D \in \mathrm{Mat}_{r\times r}(\mathbb{Z}[z])$ by the definition. The condition (N4) is equivalent to $n_{ab}n'_{ab} = 0$ for any $a, b \in [1, r]$. Thus it is sufficient to show that the symplectic relation is equivalent to $AA' = A'A$. Clearly, $AA' = A'A$ if and only if $NN' = N'N$. We now have

$$\begin{aligned}
&A_+DA_-^\dagger - A_-DA_+^\dagger \\
&= \big((1+z^2)I_r - zN\big)D\big((1+z^{-2})I_r - z^{-1}N'^{\mathsf{T}}\big) \\
&\quad - \big((1+z^2)I_r - zN'\big)D\big((1+z^{-2})I_r - z^{-1}N^{\mathsf{T}}\big) \\
&= (z+z^{-1})(-ND - DN'^{\mathsf{T}} + DN^{\mathsf{T}} + N'^{\mathsf{T}}D) + NDN'^{\mathsf{T}} - N'DN^{\mathsf{T}} \\
&= NDN'^{\mathsf{T}} - N'DN^{\mathsf{T}} \\
&= (NN' - N'N)D,
\end{aligned}$$

which completes the proof. □

Proposition 2.3.5. *Suppose that* α *given in Proposition 2.3.4 is a T-datum. Suppose further that there exists a function* $\epsilon : [1, r] \to \{1, -1\}$ *such that* n_{ab} *or* $n'_{ab} > 0$ *implies* $\epsilon(a) \neq \epsilon(b)$ *for any* $a, b \in [1, r]$. *Then the set*

$$R_\epsilon := \{(a, u) \in [1, r] \times \mathbb{Z} \mid \epsilon(a) = (-1)^{u-1}\}$$

is consistent for α.

Proof. The conditions (R1) and (R2) follow from the assumption on the function ϵ. The condition (R3) is satisfied by setting $t = 2$. □

Let $\gamma = (B, d, \mathbf{i}, \nu) = G(\alpha, R_\epsilon)$ be the mutation loop obtained from the data given in Proposition 2.3.5. The index set of B is given by

$$I = \{(a, 0) \mid \epsilon(a) = -1\} \sqcup \{(a, 1) \mid \epsilon(a) = 1\},$$

and we identify it with $[1, r]$ by taking the first components. Then $B = (B_{ab})_{a,b\in I}$ is given by

$$B_{ab} = \begin{cases} -\epsilon(a)n_{ab} & \text{if } n'_{ab} = 0, \\ +\epsilon(a)n'_{ab} & \text{if } n_{ab} = 0. \end{cases} \tag{2.3.4}$$

The symmetrizer d is given by $d = (d_a)_{a\in I}$, where d_a is the a-th entry in the common symmetrizer D. The sequence $\mathbf{i}$ is given by $\mathbf{i} = \mathbf{i}(0) \mid \mathbf{i}(1)$ with $\mathbf{i}(0) = \{a \in I \mid \epsilon(a) = -1\}$ and $\mathbf{i}(1) = \{a \in I \mid \epsilon(a) = 1\}$. The permutation ν is the trivial permutation.

Remark 2.3.6. If B is skew-symmetric, the corresponding quiver $Q(B)$ is called a *bipartite recurrent quiver* [GP19a]. Galashin and Pylyavskyy developed the classification theory of bipartite recurrent quivers [GP19a, GP19b, GP20]. In particular, they gave a complete classification of bipartite recurrent quivers with which the associated T-system is periodic.

Example 2.3.7 (Tensor product construction)**.** Let $\bar{A}$ and $\bar{A}'$ be symmetrizable weak generalized Cartan matrices of size $\bar{r}$ and $\bar{r}'$, respectively. Suppose that one of them is non-weak. Let D and D' be right symmetrizers of $\bar{A}$ and $\bar{A}'$, respectively. Let $A = \bar{A} \otimes I_{\bar{r}'}$, $A' = I_{\bar{r}} \otimes \bar{A}'$, and $D = \bar{D} \otimes \bar{D}'$. The matrices A and A' are symmetrizable weak generalized Cartan matrices that have the common symmetrizer D. Then the triple $\alpha = (A_+, A_-, D)$ given in Proposition 2.3.4 is a T-datum of size $\bar{r}\bar{r}'$ since $(\bar{A} \otimes I_{\bar{r}'})(I_{\bar{r}} \otimes \bar{A}') = (I_{\bar{r}} \otimes \bar{A}')(\bar{A} \otimes I_{\bar{r}'})$ and $n_{ab}\delta_{a'b'}\delta_{ab}n'_{a'b'} = 0$ for any $a, b \in [1, \bar{r}]$ and $a', b' \in [1, \bar{r}']$.

Suppose further that both $\bar{A}$ and $\bar{A}'$ are bipartite by functions $\bar{\epsilon}$ and $\bar{\epsilon}$, respectively. Then the function $\epsilon : [1, \bar{r}] \times [1, \bar{r}'] \to \{1, -1\}$ defined by $\epsilon(a, a') = \bar{\epsilon}(a)\bar{\epsilon}'(a')$ satisfies the assumption in Proposition 2.3.5. Thus we get the consistent subset R_ϵ for α. For example, let

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad A' = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

be Cartan matrices of types A_3 and A_2, respectively. Define $\bar{\epsilon}$ and $\bar{\epsilon}'$ by $\bar{\epsilon}(1) = \bar{\epsilon}(3) = \bar{\epsilon}'(2) = 1$ and $\bar{\epsilon}(2) = \bar{\epsilon}'(1) = -1$. Then the bipartite recurrent quiver $Q(B)$ is given by

$$Q(B) = \begin{array}{ccccc} (12,1) & \longleftarrow & (22,0) & \longrightarrow & (32,1) \\ \downarrow & & \uparrow & & \downarrow \\ (11,0) & \longrightarrow & (21,1) & \longleftarrow & (31,0) \end{array},$$

where we denote (a, a') by aa'.

Example 2.3.8 (Tadpole type)**.** Although Proposition 2.3.5 is for bipartite Cartan matrices, non-bipartite Cartan matrices are sometimes interesting. Let $A = 2I_r$ and $A' = (2\delta_{ab} - n'_{ab})_{a,b\in[1,r]}$ where

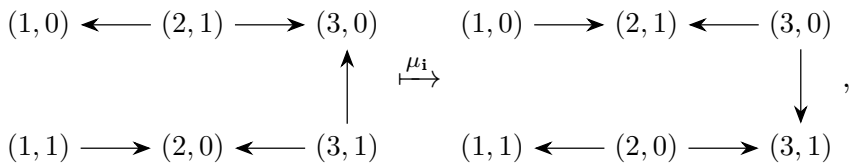
$$n'_{ab} = \begin{cases} 1 & \text{if } |a - b| = 1, \\ 1 & \text{if } a = b = r, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3.5}$$

The matrix A' is called the Cartan matrix of the *tadpole type* T_r. The tadpole type is non-bipartite since (2.3.5) has a non-zero entry in the diagonal. Let $D = I_r$. Then α in Proposition 2.3.4 is a T-datum. For example, $\alpha = (A_+, A_-, I_r)$ for $r = 3$ is given by

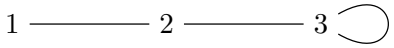
$$A_+ = \begin{bmatrix} 1+z^2 & 0 & 0 \\ 0 & 1+z^2 & 0 \\ 0 & 0 & 1+z^2 \end{bmatrix}, \quad A_- = \begin{bmatrix} 1+z^2 & -z & 0 \\ -z & 1+z^2 & -z \\ 0 & -z & 1-z+z^2 \end{bmatrix}.$$

Let $\gamma = (B, d, \mathbf{i}, \nu)$ be the mutation loop given by $\gamma = G(\alpha, [1, r] \times \mathbb{Z})$. Then the quiver

mutation $Q(B) \overset{\mu_{\mathbf{i}}}{\longmapsto} \mu_{\mathbf{i}}(Q(B)) = \nu(Q(B))$ is given as follows (we set $r = 3$ for simplicity) :



where $\mathbf{i} = \mathbf{i}(0) = \{(a, 0) \mid a \in [1, r]\}$, and $\nu(a, p) = (a, 1-p)$ is the permutation that swaps vertices at the same position in the top and bottom rows. Intuitively, this mutation loop is explained as follows. If we identify the vertices lying in the same ν-orbits and forget the orientation of the quiver, we obtain the following graph:



This is the Dynkin diagram of type T_r. Therefore, the mutation loop involves the "folding method" that constructs T_r diagram from A_{2r} diagram. In general, one advantage of the strategy of constructing mutation loops from T-data is that it can "automatically" perform such a folding method.

2.3.3 T-systems associated with quantum affinizations

In this section, we assign T-data to generalized Cartan matrices that satisfy a certain condition, including all finite and affine types. The T-data in this section are different from that in Section 2.3.2 even though both use Cartan matrices.

Fix a positive integer n. Let $C = (c_{ab})_{1 \le a,b \le n}$ be a symmetrizable generalized Cartan matrix. We assume that C is indecomposable. Let $\mathrm{diag}(c_1, \dots, c_n)$ be a *left* symmetrizer of C. We define integers t_a $(1 \le a \le n)$ by

$$t_a = c_a^{-1} \operatorname{lcm}(c_1, \dots, c_n).$$

We also define integers t_{ab} $(1 \le a, b \le n)$ by

$$t_{ab} = c_a^{-1} \operatorname{lcm}(c_a, c_b).$$

These integers do not depend on the choice of a symmetrizer. Let $[k]_z \in \mathbb{Z}[z^{\pm 1}]$ be the z-integer defined by

$$\begin{aligned} [k]_z &= \frac{z^k - z^{-k}}{z - z^{-1}} \\ &= z^{k-1} + z^{k-3} + \cdots + z^{-(k-3)} + z^{-(k-1)}. \end{aligned}$$

We denote $[k]_{z^{c_a}}$ by $[k]_{z_a}$.

Let ℓ be an integer with $\ell \ge 2$. Let H be the index set defined by

$$H = \{(a, m) \mid 1 \le a \le n, 1 \le m \le t_a \ell - 1\}.$$

We often denote an element $(a,m) \in H$ by am. For any $(a,m),(b,k) \in H$, we define polynomials $\tilde{n}^0_{am,bk}, \tilde{n}^+_{am,bk}, \tilde{n}^-_{am,bk} \in \mathbb{Z}[z^{\pm 1}]$ by

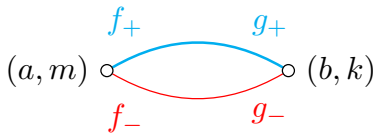
$$\begin{aligned}
\tilde{n}^0_{am,bk} &= [2]_{z_a}\delta_{ab},\\
\tilde{n}^+_{am,bk} &= \delta_{ab}(\delta_{m,k+1}+\delta_{m,k-1}),\\
\tilde{n}^-_{am,bk} &= \begin{cases} t_{ab}^{-1}|c_{ab}|[t_{ba}-|p-k|]_{z_b} & \text{if } a \sim b,\ p \in \mathbb{Z}, \text{ and } |p-k| < t_{ba},\\ 0 & \text{otherwise,}\end{cases}
\end{aligned}$$

where we write $a \sim b$ if $c_{ab} < 0$, and $p = mt_{ab}^{-1}t_{ba}$. We define two $H \times H$-matrices $\widetilde{A}_+$ and $\widetilde{A}_-$ by

$$\widetilde{A}_\pm = \left(\tilde{n}^0_{am,bk} - \tilde{n}^\pm_{am,bk}\right)_{am,bk \in H}.$$

To illustrate the pair of matrices $(\widetilde{A}_+, \widetilde{A}_-)$, it is useful to consider the graph $\Gamma(\widetilde{A}_+, \widetilde{A}_-)$ defined as follows:

- the set of vertices of $\Gamma(\widetilde{A}_+, \widetilde{A}_-)$ is H,
- for any pair of vertices $(a,m),(b,k) \in H$, we draw a blue edge equipped with the pair of polynomials $(f_+, g_+) := (\tilde{n}^+_{am,bk}, \tilde{n}^+_{bk,am})$, and a red edge equipped with the pair of polynomials $(f_-, g_-) := (\tilde{n}^-_{am,bk}, \tilde{n}^-_{bk,am})$:

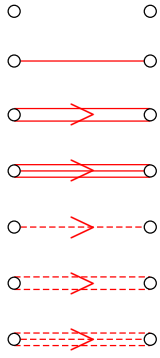


.

For a red edge

,

we use the following abbreviations:



if $(f_-, g_-) = (0,0)$,
if $(f_-, g_-) = (1,1)$,
if $(f_-, g_-) = (1, [2]_{z_a})$,
if $(f_-, g_-) = (1, [3]_{z_a})$,
if $(f_-, g_-) = (0,1)$,
if $(f_-, g_-) = (0, [2]_{z_a})$,
if $(f_-, g_-) = (0, [3]_{z_a})$.

We may use the same abbreviations for blue edges, but these are not needed here. When we use these abbreviations, we keep in mind the symmetrizer.

Example 2.3.9. Consider a Cartan matrix of type F_4:

$$C = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix},$$

and choose a symmetrizer as $\mathrm{diag}(2,2,1,1)$. When $\ell = 2$, the index set H is given by

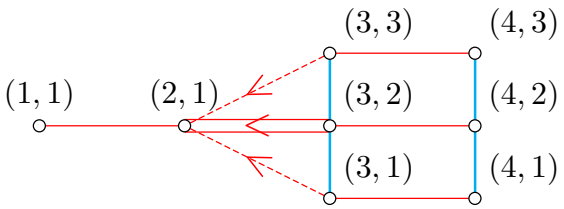
$$H = \{(1,1),(2,1),(3,1),(3,2),(3,3),(4,1),(4,2),(4,3)\},$$

and the matrices $\widetilde{A}_+$ and $\widetilde{A}_-$ are given by

$$\widetilde{A}_+ = \begin{bmatrix} [2]_{z^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & [2]_{z^2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & [2]_z & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & [2]_z & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & [2]_z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & [2]_z & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & [2]_z & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & [2]_z \end{bmatrix},$$

$$\widetilde{A}_- = \begin{bmatrix} [2]_{z^2} & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & [2]_{z^2} & -1 & -[2]_z & -1 & 0 & 0 & 0 \\ 0 & 0 & [2]_z & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & [2]_z & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & [2]_z & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & [2]_z & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & [2]_z & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & [2]_z \end{bmatrix}.$$

The diagram $\Gamma(\widetilde{A}_+, \widetilde{A}_-)$ is given by



.

More complicated examples are given in Figure 2.3. The left diagram is of

$$C = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{bmatrix}$$

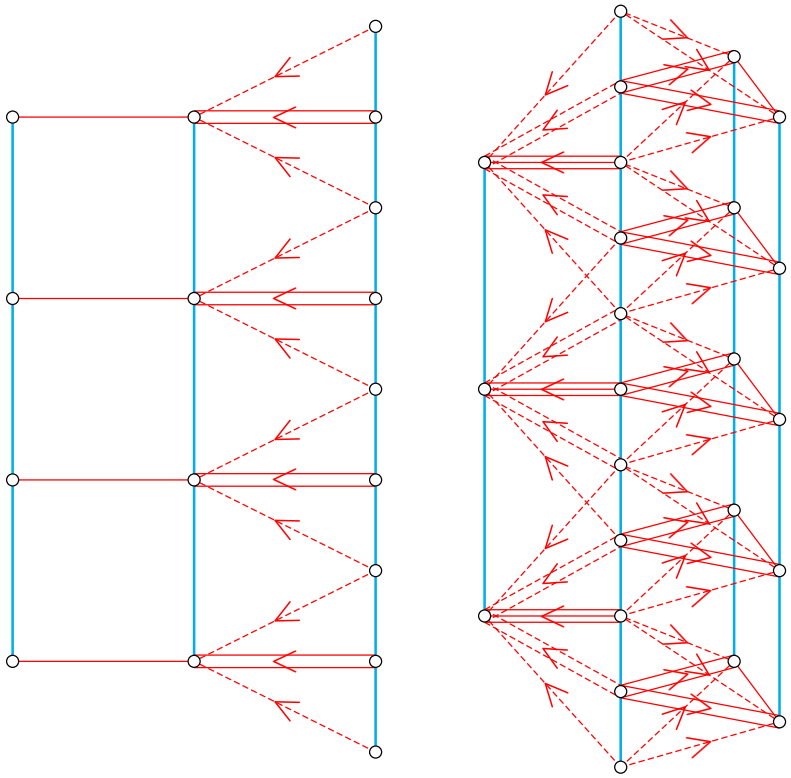


Fig. 2.3 Examples of the diagram $\Gamma(\widetilde{A}_+, \widetilde{A}_-)$.

(the Cartan matrix of type B_3) and $\ell = 5$, and the right diagram is of

$$C = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -3 & 2 & -2 & -2 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

and $\ell = 2$.

Proposition 2.3.10. *The red part of the diagram* $\Gamma(\widetilde{A}_+, \widetilde{A}_-)$ *contains the Dynkin diagram of the transpose of* C*. More precisely, we have*

$$(\widetilde{A}_-|_{z=1})_{am,bk} = c_{ba}$$

if $t_{ba}m = t_{ab}k$*.*

Proof. Suppose that $(a, m), (b, k) \in H$ satisfy $t_{ba}m = t_{ab}k$. Note that such pairs exist for any a, b since $m := t_{ab} \le t_a \le t_a\ell - 1$ and $k := t_{ba} \le t_b \le t_b\ell - 1$ satisfy the condition. If $a = b$, we have

$$(\widetilde{A}_-|_{z=1})_{am,bk} = (\tilde{n}^0_{am,am})|_{z=1} = 2,$$

and this is equal to c_{aa}. If $a \neq b$, we have

$$(\widetilde{A}_-|_{z=1})_{am,bk} = -(\tilde{n}^-_{am,bk})|_{z=1} = t_{ab}^{-1} c_{ab} t_{ba} = c_a c_{ab} c_b^{-1} = c_{ba}.$$

□

Lemma 2.3.11. *The matrix $\widetilde{A}_+ \widetilde{A}_-^{\mathsf{T}}$ is symmetric.*

Proof. The matrix $\widetilde{A}_+ \widetilde{A}_-^{\mathsf{T}}$ is symmetric if and only if

$$\begin{aligned} \sum_{(c,l)\in H} & (\tilde{n}^0_{am,cl}\tilde{n}^+_{bk,cl} + \tilde{n}^+_{am,cl}\tilde{n}^-_{bk,cl} + \tilde{n}^-_{am,cl}\tilde{n}^0_{bk,cl} \\ & - \tilde{n}^0_{am,cl}\tilde{n}^-_{bk,cl} - \tilde{n}^-_{am,cl}\tilde{n}^+_{bk,cl} - \tilde{n}^+_{am,cl}\tilde{n}^0_{bk,cl}) = 0 \end{aligned} \tag{2.3.6}$$

for any $(a,m),(b,k) \in H$. Let X be the left-hand side in (2.3.6). Then $X = 0$ is trivial except for the following cases:

(i) $a \sim b$ and $t_{ba}m = t_{ab}k$,
(ii) $a \sim b$, $p = mt_{ab}^{-1}t_{ba} \in \mathbb{Z}$, and $0 < |p-k| < t_{ba}$,
(ii') $a \sim b$, $p' = kt_{ba}^{-1}t_{ab} \in \mathbb{Z}$, and $0 < |p'-m| < t_{ab}$,
(iii) $a \sim b$, $p = mt_{ab}^{-1}t_{ba} \in \mathbb{Z}$, and $|p-k| = t_{ba}$,
(iii') $a \sim b$, $p' = kt_{ba}^{-1}t_{ab} \in \mathbb{Z}$, and $|p'-m| = t_{ab}$.

Moreover, the cases (ii') and (iii') reduce to the cases (ii) and (iii), respectively, since the left-hand side in (2.3.6) is skew-symmetric under $am \leftrightarrow bk$. For the case (i), we have

$$\begin{aligned} X &= 2t_{ba}^{-1}|c_{ba}|[t_{ab}-1]_{z_a} + t_{ab}^{-1}|c_{ab}|[t_{ba}]_{z_b}\cdot[2]_{z_b} \\ &\quad - [2]_{z_a}\cdot t_{ba}^{-1}|c_{ba}|[t_{ab}]_{z_a} - 2t_{ab}^{-1}|c_{ab}|[t_{ba}-1]_{z_b} \\ &= t_{ab}^{-1}|c_{ab}|([t_{ba}+1]_{z_b} - [t_{ba}-1]_{z_b}) - t_{ba}^{-1}|c_{ba}|([t_{ab}+1]_{z_a} - [t_{ab}-1]_{z_a}) \\ &= t_{ab}^{-1}|c_{ab}|(z^{t_{ba}c_b} + z^{-t_{ba}c_b}) - t_{ba}^{-1}|c_{ba}|(z^{t_{ab}c_a} + z^{-t_{ab}c_a}) \\ &= 0. \end{aligned}$$

Here, we use $[n]_z \cdot [2]_z = [n+1]_z + [n-1]_z$ to derive the second equality. For the case (ii), we have

$$\begin{aligned} X &= t_{ab}^{-1}|c_{ab}|[t_{ba}-|p-k|]_{z_b}\cdot[2]_{z_b} \\ &\quad - t_{ab}^{-1}|c_{ab}|([t_{ba}-|p-k|-1]_{z_b} + [t_{ba}-|p-k|+1]_{z_b}) \\ &= 0. \end{aligned}$$

For the case (iii), we have

$$X = t_{ab}^{-1}|c_{ab}|[1]_{z^b} - t_{ba}^{-1}|c_{ba}|[1]_{z^a} = 0.$$

□

Theorem 2.3.12. *Let $N_0, N_+, N_- \in \mathrm{Mat}_{H\times H}(\mathbb{Z}[z])$ be the matrices defined by*

$$N_\varepsilon = \left(z^{c_a}\tilde{n}^{\varepsilon}_{am,bk}\right)_{am,bk\in H} \qquad (\varepsilon \in \{0,+,-\}).$$

Then the triple (N_0, N_+, N_-) and the pair $(A_+, A_-) = (N_0 - N_+, N_0 - N_-)$ satisfy

(1) the conditions (N1), (N2), and (N4),
(2) the symplectic relation $A_+A_-^\dagger = A_-A_+^\dagger$,
(3) and the condition (N3) if and only if

$$c_{ab} \mid c_{ba} \text{ or } c_{ba} \mid c_{ab} \text{ for any } 1 \le a, b \le n. \tag{2.3.7}$$

Consequently, for any Cartan matrix C satisfies the condition (2.3.7) and any integer ℓ greater than or equal to 2, the triple (A_+, A_-, I_H) is a T-datum of size $|H|$.

Proof. The conditions (N1), (N2), and (N4) are obvious from the definition. The symplectic relation follows from Lemma 2.3.11 and the fact that $\tilde{n}^{\varepsilon}_{am,bk}$ are invariant under $z \mapsto z^{-1}$. The condition (N3) is equivalent to $(t_{ba}-1)c_b < c_a$ for any a, b such that $a \sim b$, and this is equivalent to $\mathrm{lcm}(c_a, c_b) < c_a + c_b$ for any a, b such that $a \sim b$. This happens if and only if $c_a \mid c_b$ or $c_b \mid c_a$ for any a, b such that $a \sim b$, and this is equivalent to the condition (2.3.7). □

Remark 2.3.13. If the Cartan matrix in Theorem 2.3.12 and its symmetrizer satisfy the condition

$$c_{ab} < -1 \;\Rightarrow\; c_a = -c_{ba} = 1,$$

which implies (2.3.7), the mutation loop corresponding to the T-datum (A_+, A_-, I_H) is explicitly constructed in [Nak11c]. The T-system associated with this T-datum is a certain truncation a T-system associated with Kirillov-Reshetikhin modules of the quantum affinization of a quantum Kac-Moody algebra [Her07, KNS09] (a truncation and a quantum Kac-Moody algebra are associated with ℓ and C, respectively).

Chapter 3

Periodic Y/T-systems

3.1 Finite type T-data

Definition 3.1.1. We say that a T-datum α is of *finite type* if the set $\{T_a(u) \in \mathcal{T}^\circ(\alpha, R, Y) \mid (a,u) \in R\}$ is a finite set.

Definition 3.1.1 does not depend on R since the set being considered is a finite set for some R if and only if it is a finite set for $[1,r] \times \mathbb{Z}$ by (R3). We will see that this is also independent of the choice of Y.

Definition 3.1.2. Let α be a T-datum and R be a consistent subset for α.

(1) We define $Y_{\mathrm{prin}}(\alpha, R)$ to be the solution of the Y-system associated with (α, R) in $\mathrm{Trop}(u_{a,p})_{(a,p)\in R_{\mathrm{in}}}$ such that $u_{a,p} = y_{a,p}$ for any $(a,p) \in R_{\mathrm{in}}$, where $y_{a,p}$ is defined by (2.2.11). By Theorem 2.2.19, the T-algebra $\mathcal{T}^\circ(\alpha, R, Y)$ is embedded into the cluster algebra with principal coefficients (see [FZ07] for the definition of cluster algebras with principal coefficients).

(2) We define $Y_{\mathrm{univ}}(\alpha, R)$ to be the solution of the Y-system associated with (α, R) in $\mathbb{Q}_{\mathrm{sf}}(u_{a,p})_{(a,p)\in R_{\mathrm{in}}}$ such that $u_{a,p} = y_{a,p}$ for any $(a,p) \in R_{\mathrm{in}}$, where $y_{a,p}$ is defined by (2.2.11).

Definition 3.1.3. Let α be a T-datum and R be a consistent subset for α. Let Ω be a integer with $t \mid \Omega$, where t is the integer in (R3) in Definition 2.2.3.

(1) We say that a solution $(Y_a(u))_{(a,u)\in R}$ of the Y-system associated with (α, R) is *periodic with period* Ω if $Y_a(u) = Y_a(u+\Omega)$ for any $(a,u) \in R$.

(2) We say that the T-system associated with (α, R, Y) is *periodic with period* Ω if Y is periodic with period Ω and $T_a(u) = T_a(u+\Omega)$ in $\mathcal{T}^\circ(\alpha, R, Y)$ for any $(a,u) \in R$.

Definition 3.1.3 also does not depend on R. By the synchronicity phenomenon in cluster algebras [Nak19], we have the following assertion:

Theorem 3.1.4. *Let α be a T-datum and R be a consistent subset for α. Let Ω be a integer with $t \mid \Omega$, where t is the integer in (R3) in Definition 2.2.3. Then the following conditions are equivalent:*

(1) The T-system associated with (α, R, Y) is periodic with period Ω for some Y.
(2) The T-system associated with (α, R, Y) is periodic with period Ω for any Y.

(3) $Y_{\mathrm{prin}}(\alpha, R)$ *is periodic with period* Ω.
(4) $Y_{\mathrm{univ}}(\alpha, R)$ *is periodic with period* Ω.

Proof. This follows from Theorem 2.2.19 together with the synchronicity phenomenon in cluster algebras [Nak19, Theorem 5.2 and 5.5]. □

It is easy to see that α is of finite type (for some Y) if and only if the condition (1) in Theorem 3.1.4 holds for some $\Omega > 0$. Therefore, Theorem 3.1.4 implies that Definition 3.1.1 does not depend on Y.

3.1.1 Simultaneous positivity of finite type T-data

For any matrix $A \in \mathrm{Mat}_{r\times r}(\mathbb{Z}[z^{\pm 1}])$, we define $\mathring{A} \in \mathrm{Mat}_{r\times r}(\mathbb{Z})$ by $\mathring{A} = A|_{z=1}$. For any vector $u, v \in \mathbb{R}^r$, we write $u > v$ and $u \geq v$ if all components of the vector $u - v$ are positive and non-negative, respectively. The following is the main theorem in this section, which gives a effective method to determine that a given T-datum is not of finite type.

Theorem 3.1.5. *Let* $\alpha = (A_+, A_-, D)$ *be a T-datum. If* α *is of finite type, then there exists a vector* $v > 0$ *such that* $\mathring{A}_+^{\mathsf{T}} v > 0$ *and* $\mathring{A}_-^{\mathsf{T}} v > 0$.

Proof. Without loss of generality we can assume that α is indecomposable. It is sufficient to find a vector $v \geq 0$ such that $\mathring{A}_+^{\mathsf{T}} v > 0$ and $\mathring{A}_-^{\mathsf{T}} v > 0$ since such a vector plus a sufficiently small positive vector is a desired vector. Let $c \in [1, r]$. Let $\mathfrak{t}^{(c)} = (\mathfrak{t}_a^{(c)}(u))_{(a,u)\in[1,r]\times\mathbb{Z}}$ be the family of integers defined in Section 2.2.5, that is, $\mathfrak{t}_a^{(c)}(u)$ is the minus of the lowest power of $T_c(0)$ in $T_a(u)$, where $T_a(u)$ is written as a Laurent polynomial in $(T_c(p))_{(c,p)\in R_{\mathrm{in}}}$. We also define the family of integers $\tilde{\mathfrak{t}}^{(c)} = (\tilde{\mathfrak{t}}_a^{(c)}(u))_{(a,u)\in[1,r]\times\mathbb{Z}}$, where $\tilde{\mathfrak{t}}_a^{(c)}(u)$ is the highest power of $T_c(0)$ in $T_a(u)$. By the definitions, we have $\mathfrak{t}_a^{(c)}(u) + \tilde{\mathfrak{t}}_a^{(c)}(u) \geq 0$ for any $(a, u) \in [1, r] \times \mathbb{Z}$. By Proposition 2.6 in [RS18], the family of integers $\tilde{\mathfrak{t}}^{(c)}$ is uniquely determined by the initial conditions

$$\tilde{\mathfrak{t}}_a^{(c)}(p) = \begin{cases} 1 & \text{if } (a,p) = (c,0), \\ 0 & \text{if } (a,p) \neq (c,0) \text{ and } 0 \leq p < p_a, \end{cases} \tag{3.1.1}$$

together with the following recurrence relation for each $(a, u) \in [1, r] \times \mathbb{Z}$:

$$\sum_{b,p} n^0_{ba;p} \tilde{\mathfrak{t}}_b^{(c)}(u+p) = \max\left(\sum_{b,p} n^-_{ba;p} \tilde{\mathfrak{t}}_b^{(c)}(u+p), \sum_{b,p} n^+_{ba;p} \tilde{\mathfrak{t}}_b^{(c)}(u+p) \right). \tag{3.1.2}$$

The family of integers $\mathfrak{t}^{(c)}$ and $\tilde{\mathfrak{t}}^{(c)}$ satisfy the same recurrence relation, but have the different initial conditions.

Let $v_a^{(c)}$ and $\tilde{v}_a^{(c)}$ be the integers defined by

$$v_a^{(c)} = \sum_{u=0}^{\Omega-1} \mathfrak{t}_a^{(c)}(u), \quad \tilde{v}_a^{(c)} = \sum_{u=0}^{\Omega-1} \tilde{\mathfrak{t}}_a^{(c)}(u),$$

where Ω is a period of the T-system. By the periodicity of the T-system, we have

$$v_a^{(c)} = \sum_{u=0}^{\Omega-1} \mathfrak{t}_a^{(c)}(u+p), \quad \tilde{v}_a^{(c)} = \sum_{u=0}^{\Omega-1} \tilde{\mathfrak{t}}_a^{(c)}(u+p)$$

for any $p \in \mathbb{Z}$. By summing up (2.2.16) with respect to the period, we have

$$\sum_b \mathring{n}_{ba}^0 v_b^{(c)} = \sum_{u=0}^{\Omega-1} \max\Bigl(\sum_{b,p} n_{ba;p}^- \mathfrak{t}_b^{(c)}(u+p), \sum_{ba;p} n_{ba;p}^+ \mathfrak{t}_b^{(c)}(u+p)\Bigr) \tag{3.1.3}$$

$$\geq \max\Bigl(\sum_b \mathring{n}_{ba}^- v_b^{(c)}, \sum_b \mathring{n}_{ba}^+ v_b^{(c)}\Bigr), \tag{3.1.4}$$

where $\mathring{n}_{ba}^{\varepsilon} = \sum_p n_{ba,p}$. This implies that $\mathring{A}_+^{\mathsf{T}} v^{(c)} \geq 0$ and $\mathring{A}_-^{\mathsf{T}} v^{(c)} \geq 0$. Similarly, we have $\mathring{A}_+^{\mathsf{T}} \tilde{v}^{(c)} \geq 0$ and $\mathring{A}_-^{\mathsf{T}} \tilde{v}^{(c)} \geq 0$ by summing up (3.1.2) with respect to the period. Let v and $\tilde{v}$ be the vectors defined by

$$v = \sum_{c=1}^{r} \begin{bmatrix} v_1^{(c)} \\ \vdots \\ v_r^{(c)} \end{bmatrix}, \quad \tilde{v} = \sum_{c=1}^{r} \begin{bmatrix} \tilde{v}_1^{(c)} \\ \vdots \\ \tilde{v}_r^{(c)} \end{bmatrix}.$$

We then define a vector v' by $v' = v + \tilde{v}$. We have $v' \geq 0$ since $\mathfrak{t}_a^{(c)}(u) + \tilde{\mathfrak{t}}_a^{(c)}(u) \geq 0$. We also have $\mathring{A}_+^{\mathsf{T}} v' \geq 0$ and $\mathring{A}_-^{\mathsf{T}} v' \geq 0$. Therefore, if we prove that $\mathring{A}_+^{\mathsf{T}} v > 0$ and $\mathring{A}_-^{\mathsf{T}} v > 0$, the assertion of the theorem follows.

From (2.2.17), (2.2.19), and (3.1.4), the a-th component of $\mathring{A}_{\pm}^{\mathsf{T}} v$ is positive if and only if there exists $(c,u) \in [1,r] \times \mathbb{Z}$ such that $[\pm \mathfrak{y}_a^{(c)}(u)]_+ \neq 0$. Therefore the a-th component of $\mathring{A}_{\pm}^{\mathsf{T}} v$ is positive if the a-th column of $N_{\pm}$ is non-zero by (3) in Lemma 2.2.23. It remains to prove that the a-th component of $\mathring{A}_{\pm}^{\mathsf{T}} v$ is also positive when the a-th column of $N_{\pm}$ is zero. If both the a-th columns of N_+ and N_- are zero, the assertion of the theorem follows from Corollary 2.2.25. Thus we can assume that either the a-th column of N_+ or N_- is non-zero. Without loss of generality we assume that the a-th column of N_+ is non-zero and the a-th column of N_- is zero. Let $n_{ca;p}^+ z^p$ be a term in the a-th column of N_+ with the minimal degree among the terms in this column. Now we have

$$\begin{aligned}
[-\hat{\mathfrak{y}}_a^{(\sigma(c))}(-p_{\sigma(c)} - p)]_+ &= \sum_{b,q} (n_{ba;q}^0 - n_{ba;q}^-) \mathfrak{t}_b^{(\sigma(c))}(-p_{\sigma(c)} - p + q) \\
&= \sum_{b,q} n_{ba;q}^0 \mathfrak{t}_b^{(\sigma(c))}(-p_{\sigma(c)} - p + q) \\
&= \max\Bigl(\sum_{b,q} n_{ba;q}^+ \mathfrak{t}_b^{(\sigma(c))}(-p_{\sigma(c)} - p + q), 0\Bigr) \\
&= \max\bigl(n_{ca;p}^+ \mathfrak{t}_c^{(\sigma(c))}(-p_{\sigma(c)}), 0\bigr) \\
&= \max(n_{ca;p}^+, 0) \\
&= n_{ca;p}^+,
\end{aligned}$$

and this implies that the a-th component of $\mathring{A}_-^{\mathsf{T}} v$ is positive. □

Example 3.1.6.

(1) A T-datum of size 1 (Theorem 2.3.1) is of finite type if and only if (A_+, A_-) is one of the following three pairs of matrices for some $p > 0$:

$$\begin{aligned} A_+ &= \left[1 + z^{2p}\right], & A_- &= \left[1 + z^{2p}\right], \\ A_+ &= \left[1 - z^p + z^{2p}\right], & A_- &= \left[1 + z^{2p}\right], \\ A_+ &= \left[1 + z^{2p}\right], & A_- &= \left[1 - z^p + z^{2p}\right]. \end{aligned}$$

The if part is proved by direct calculations, and the only if part follows from Theorem 3.1.5.

(2) A T-datum associated with a bipartite recurrent quiver, which is a special case of a T-datum in Proposition 2.3.4, is of finite type if and only if both A and A' are direct sums of ADE Cartan matrices [GP19a]. In fact, Theorem 3.1.5 generalizes Proposition 7.1 in [GP19a] to arbitrary T-data.

(3) A T-datum in Example 2.3.7 is of finite type if and only if both $\bar{A}$ and $\bar{A}'$ are of finite type Cartan matrices, except that one of them can be of tadpole type. The if part is proved in [Kel13], and the only if part follows from Theorem 3.1.5.

(4) A T-datum in Theorem 2.3.12 is of finite type if and only if C is of finite type Cartan matrix. The if part is proved in [IIK+13a, IIK+13b], and the only if part follows from Proposition 2.3.10 and Theorem 3.1.5.

3.2 Special values of the dilogarithm function

Definition 3.2.1. Let $\alpha = (A_+, A_-, D)$ be a T-datum. Let $P = \mathrm{diag}(z^{-p_a/2})_{a \in [1,r]}$, where p_a is the integer in (N1). We say that α is *Cartan-like* if both the matrices PA_+ and PA_- are invariant under $z \mapsto z^{-1}$.

This terminology comes from the fact that T-data in Section 2.3.2 satisfy this property. All examples in Section 2.3 are also Cartan-like. Note that $\mathring{A}_\pm$ are not Cartan matrices in general since they may not be sign-symmetric (see examples in Table 3.1 and 3.2). The matrix N_0 in the Cartan-like T-datum should be a diagonal matrix. This property is useful due to the following fact on real square matrices whose off-diagonal entries are non-positive. As a result, we assign a positive definite symmetric matrix to any Cartan-like T-datum of finite type (Proposition 3.2.3).

Lemma 3.2.2 ([FP62, Theorem 4.3])**.** *Let A be a real square matrix whose off-diagonal entries are all non-positive. Then the following conditions are equivalent:*

(1) there exists $v > 0$ such that $Av > 0$,
(2) all real eigenvalues of A are positive.

Proposition 3.2.3. *Let $\alpha = (A_+, A_-, D)$ is a Cartan-like T-datum of finite type. Then the following assertions hold:*

(1) $\mathring{A}_+$ and $\mathring{A}_-$ are invertible.

(2) Let $K = (\kappa_{ab})_{a,b\in[1,r]}$ be the matrix defined by $K = \mathring{A}_+^{-1}\mathring{A}_-$. Then KD is a positive definite symmetric matrix.

(3) Let $K^\vee = (\check{\kappa}_{ab})_{a,b\in[1,r]}$ be the matrix defined by $K^\vee = D^{-1}KD$. Then $K^\vee D^\vee$ is a positive definite symmetric matrix.

Proof. By Theorem 3.1.5 and Lemma 3.2.2, all the real eigenvalues of $\mathring{A}_\pm$ are positive. This implies (1). Since $K^\vee = (\mathring{A}_+^\vee)^{-1}\mathring{A}_-^\vee$, the assertion (3) follows from the assertion (2) for $\alpha^\vee$. We now prove (2). We first see that K is symmetric due to the symplectic relation. Suppose that there exists an eigenvector v of K with a non-positive eigenvalue. Let us denote by $-\lambda$ this eigenvalue. Then we have $(\mathring{A}_- + \lambda\mathring{A}_+)v = 0$. Thus 0 is an eigenvalue of $\mathring{A}_- + \lambda\mathring{A}_+$. Since $\lambda \geq 0$, all off-diagonal entries in $\mathring{A}_- + \lambda\mathring{A}_+$ are non-positive. Moreover, this matrix satisfies the condition (1) in Lemma 3.2.2 by Theorem 3.1.5. Thus its real eigenvalues are positive by Lemma 3.2.2, a contradiction. □

The function

$$\mathrm{Li}_2(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^2} \quad (|z| < 1)$$

is called the *dilogarithm function*. The *Rogers dilogarithm function* is a function on the interval $(0, 1)$ defined as follows:

$$L(x) = \mathrm{Li}_2(x) + \frac{1}{2}\log(x)\log(1-x).$$

We can define $L(0) = 0$ and $L(1) = \pi^2/6$ by continuity.

For any T-datum $\alpha = (A_+, A_-, D)$, we denote by d_a and $d_a^\vee$ the a-th entries in D and $D^\vee$, respectively.

Theorem 3.2.4. *Let $\alpha = (A_+, A_-, D)$ be a Cartan-like T-datum of finite type. Let $K^\vee = (\check{\kappa}_{ab})_{a,b\in[1,r]}$ be the matrix defined in Proposition 3.2.3.*

(1) The system of equations

$$f_a = \prod_{b=1}^{r}(1-f_b)^{\check{\kappa}_{ab}} \qquad (a \in [1,r]) \tag{3.2.1}$$

has a unique real solution such that $0 < f_a < 1$ for any $a \in [1, r]$.

(2) Let $(f_a)_{a\in[1,r]}$ be the unique solution in (1). Define the real number c_α by

$$c_\alpha := \frac{6}{\pi^2}\sum_{a=1}^{r} d_a L(f_a).$$

Then we have $c_\alpha \in \mathbb{Q}$.

Proof. We define a function $F_\alpha(x) : [0,\infty)^r \to \mathbb{R}$ by

$$F_\alpha(x) = \frac{1}{2}x^{\mathsf{T}}K^\vee D^\vee x + \sum_{a=1}^{r}(d_a^\vee)^{-1}\mathrm{Li}_2(\exp(-d_a^\vee x_a)).$$

A_+	A_-	K	c_α
$\begin{bmatrix} 1+z^2 & -z \\ -z & 1+z^2 \end{bmatrix}$	$\begin{bmatrix} 1+z^2 & 0 \\ 0 & 1+z^2 \end{bmatrix}$	$\begin{bmatrix} 4/3 & 2/3 \\ 2/3 & 4/3 \end{bmatrix}$	$4/5$
$\begin{bmatrix} 1+z^2 & -z \\ -z & 1+z^2 \end{bmatrix}$	$\begin{bmatrix} 1-z+z^2 & 0 \\ 0 & 1-z+z^2 \end{bmatrix}$	$\begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}$	1
$\begin{bmatrix} 1+z^2 & -z \\ -z-z^5 & 1+z^6 \end{bmatrix}$	$\begin{bmatrix} 1+z^2 & 0 \\ -z^3 & 1+z^6 \end{bmatrix}$	$\begin{bmatrix} 3/2 & 1 \\ 1 & 2 \end{bmatrix}$	$5/7$
$\begin{bmatrix} 1+z^2 & -z \\ -z-z^2 & 1+z^3 \end{bmatrix}$	$\begin{bmatrix} 1-z+z^2 & 0 \\ 0 & 1+z^3 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$	$3/4$
$\begin{bmatrix} 1+z^2 & -z \\ -z-z^5-z^9 & 1+z^{10} \end{bmatrix}$	$\begin{bmatrix} 1+z^2 & 0 \\ -z^3-z^7 & 1+z^{10} \end{bmatrix}$	$\begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}$	$4/7$

Table 3.1 Examples of Cartan-like T-data of finite type of size 2, where $D = I_2$ in these examples.

By setting $f_a = 1-\exp(-d_a^\vee x_a)$, we see that the statement (1) is equivalent to saying that the function $F_\alpha(x)$ has a unique critical point in $(0,\infty)^r$. This follows from the fact that $K^\vee D^\vee$ is a positive definite symmetric matrix, as in the proof of Lemma 2.1 in [VZ11].

We now prove (2). From the result in [Nak11b, Section 6], the value

$$\frac{6}{\pi^2} \sum_{\substack{(a,u)\in[1,r]\times\mathbb{Z} \\ 0\le u<\Omega}} d_a L\left(\frac{Y_a(u)}{1\oplus Y_a(u)}\right) \tag{3.2.2}$$

is an integer for any solution $Y = (Y_a(u))_{(a,u)\in[1,r]\times\mathbb{Z}}$ of the Y-system associated with $(\alpha, [1,r]\times\mathbb{Z})$ in the semifield $\mathbb{R}_{>0}$, where $\Omega > 0$ is a period of $Y_{\mathrm{univ}}(\alpha, [1,r]\times\mathbb{Z})$. Moreover, this value is independent of the choice of Y. In fact, Nakanishi [Nak11b] proved that these facts follow from the sign coherence property of cluster algebras, which was proved by Gross, Hacking, Keel, and Kontsevich [GHKK18] for skew-symmetrizable cluster algebras.

It is easy to see that the system of equations (3.2.1) is equivalent to

$$\prod_{b=1}^{r} f_b^{\sum_{p\in\mathbb{Z}}(\check{n}^0_{ab;p}-\check{n}^+_{ab;p})} = \prod_{b=1}^{r}(1-f_b)^{\sum_{p\in\mathbb{Z}}(\check{n}^0_{ab;p}-\check{n}^-_{ab;p})} \qquad (a\in[1,r]). \tag{3.2.3}$$

Thus the family $Y = (Y_a(u))_{(a,u)\in[1,r]\times\mathbb{Z}}$ defined by $Y_a(u) = f_a/(1-f_a)$ is a solution of the Y-system associated with $(\alpha, [1,r]\times\mathbb{Z})$ in $\mathbb{R}_{>0}$. Since this is a constant solution with respect to u, the integer (3.2.2) is equal to Ωc_α. Thus c_α is a rational number. □

Example 3.2.5. We give some examples of Cartan-like T-data of finite type of size 2 and size 3 in Table 3.1 and 3.2, respectively, where the matrix D in these examples are the identity matrices. We also show the positive definite symmetric matrix K and the rational number c_α associated with these T-data. The rational number c_α can be computed by using Theorem 6.8 in [Nak11b].

A_+	A_-	K	c_α
$\begin{bmatrix} 1+z^2 & -z & 0 \\ -z & 1+z^2 & -z \\ 0 & -z & 1+z^2 \end{bmatrix}$	$\begin{bmatrix} 1+z^2 & 0 & 0 \\ 0 & 1+z^2 & 0 \\ 0 & 0 & 1+z^2 \end{bmatrix}$	$\begin{bmatrix} 3/2 & 1 & 1/2 \\ 1 & 2 & 1 \\ 1/2 & 1 & 3/2 \end{bmatrix}$	1
$\begin{bmatrix} 1+z^2 & -z & 0 \\ -z & 1+z^2 & -z \\ 0 & -z & 1+z^2 \end{bmatrix}$	$\begin{bmatrix} 1-z+z^2 & 0 & 0 \\ 0 & 1-z+z^2 & 0 \\ 0 & 0 & 1-z+z^2 \end{bmatrix}$	$\begin{bmatrix} 3/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 3/4 \end{bmatrix}$	9/7
$\begin{bmatrix} 1+z^2 & -z & 0 \\ -z & 1+z^2 & -z \\ 0 & -z-z^2 & 1+z^3 \end{bmatrix}$	$\begin{bmatrix} 1-z+z^2 & 0 & 0 \\ 0 & 1-z+z^2 & 0 \\ 0 & 0 & 1+z^3 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$	9/10
$\begin{bmatrix} 1+z^2 & 0 & -z \\ -z^3 & 1+z^6 & 0 \\ -z-z^7 & -z^2-z^6 & 1+z^8 \end{bmatrix}$	$\begin{bmatrix} 1+z^2 & -z & 0 \\ -z-z^5 & 1+z^6 & 0 \\ 0 & 0 & 1+z^8 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 1 & 4 \end{bmatrix}$	1
$\begin{bmatrix} 1+z^2 & -z & 0 \\ -z & 1+z^2 & -z \\ 0 & -z & 1-z+z^2 \end{bmatrix}$	$\begin{bmatrix} 1+z^2 & 0 & 0 \\ 0 & 1+z^2 & 0 \\ 0 & 0 & 1+z^2 \end{bmatrix}$	$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 6 \end{bmatrix}$	2/3
$\begin{bmatrix} 1+z^2 & -z & 0 \\ -z-z^5 & 1+z^6 & -z^3 \\ 0 & -z^3 & 1+z^6 \end{bmatrix}$	$\begin{bmatrix} 1+z^2 & 0 & 0 \\ -z^3 & 1+z^6 & 0 \\ 0 & 0 & 1+z^6 \end{bmatrix}$	$\begin{bmatrix} 2 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 2 \end{bmatrix}$	4/5
$\begin{bmatrix} 1-z+z^2 & -z & 0 \\ -z & 1+z^2 & 0 \\ 0 & 0 & 1+z^5 \end{bmatrix}$	$\begin{bmatrix} 1+z^2 & 0 & 0 \\ 0 & 1+z^2 & -z \\ -z^2-z^3 & -z-z^4 & 1+z^5 \end{bmatrix}$	$\begin{bmatrix} 4 & 2 & -1 \\ 2 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix}$	3/2

Table 3.2 Examples of Cartan-like T-data of finite type of size 3, where $D = I_3$ in these examples.

3.3 Partition q-series

Let $\alpha = (A_+, A_-, D)$ be a Cartan-like T-datum of finite type of size r. We define two sets H_α and H'_α by

$$H_\alpha = \{(m, l) \in \mathbb{Z}^r \times \mathbb{Q}^r \mid \mathring{A}_- m = \mathring{A}_+ l\},$$
$$H'_\alpha = \{((\mathring{A}_+^\vee)^\mathsf{T} n, (\mathring{A}_-^\vee)^\mathsf{T} n) \mid n \in \mathbb{Z}^r\}.$$

These are free abelian groups of rank r, and the symplectic relation implies that H'_α is a subgroup of H_α. Let S_α be the quotient group of H_α by H'_α: $S_\alpha = H_\alpha / H'_\alpha$. This is a finite abelian group that is isomorphic to $\mathbb{Z}^r/(\text{the rows space of } \mathring{A}_+^\vee)$. In particular, the order of S_α is $\det \mathring{A}_+$. For any $\sigma \in S_\alpha$, we denote by $\sigma_{\geq 0}$ the set $\{(m, l) \in \sigma \mid m \geq 0\}$.

Definition 3.3.1. Let $\alpha = (A_+, A_-, D)$ be a Cartan-like T-datum of finite type. Let $\sigma \in S_\alpha$. We define the *partition q-series* of α at σ by

$$\mathcal{Z}_{\alpha,\sigma}(q) := \sum_{(m,l) \in \sigma_{\geq 0}} \frac{q^{\frac{1}{2}\langle m, l\rangle}}{\prod_{a=1}^r (q^{d_a^\vee})_{m_a}},$$

where $\langle m, l\rangle := m^\mathsf{T} D^\vee l$ and $(q)_n = \prod_{i=1}^n (1 - q^i)$ is the q-Pochhammer symbol. We also

define the *total partition q-series* of α by

$$\mathcal{Z}_{\alpha,\mathrm{tot}}(q) := \sum_{\sigma\in S_\alpha} \mathcal{Z}_{\alpha,\sigma}(q) = \sum_{m\in(\mathbb{Z}_{\geq 0})^r} \frac{q^{\frac{1}{2}m^{\mathsf{T}}K^\vee D^\vee m}}{\prod_{a=1}^r (q^{d_a^\vee})_{m_a}}.$$

Proposition 3.3.2.

(1) The partition q-series $\mathcal{Z}_{\alpha,\sigma}(q)$ with $q = e^{2\pi i\tau}$ converges to a holomorphic function on the upper half plane $\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}\,\tau > 0\}$, where we set $q^\kappa = e^{2\pi i\tau\kappa}$ for any $\kappa \in \mathbb{Q}$.

(2) We have

$$\lim_{\varepsilon\searrow 0} \varepsilon \log \mathcal{Z}_{\alpha,\mathrm{tot}}(e^{-\varepsilon}) = \frac{\pi^2}{6\delta} c_\alpha,$$

where $\delta = \mathrm{lcm}(d_1,\dots,d_r)\,\gcd(d_1,\dots,d_r)$ and c_α is the rational number in Theorem 3.2.4.

Proof. (1) follows from the fact that $K^\vee D^\vee$ is a positive definite symmetric matrix (Proposition 3.2.3). (2) follows from the asymptotic analysis in [VZ11]. □

Let $\Gamma \subseteq \mathrm{SL}(2,\mathbb{Z})$ be a congruence subgroup. We say that a holomorphic function $f(\tau)$ on the upper half plane is a *modular function* with respect to Γ if $f(\tau) = f(\frac{a\tau+b}{c\tau+d})$ for any $\tau \in \mathbb{H}$ and $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma$, and $f(\tau)$ is meromorphic at each cusp of Γ.

Conjecture 3.3.3. *Let $\alpha = (A_+, A_-, D)$ be a Cartan-like T-datum of finite type. Then there exists a congruence subgroup $\Gamma \subseteq \mathrm{SL}(2,\mathbb{Z})$ such that $q^{-c_\alpha/24}\mathcal{Z}_{\alpha,\sigma}(q)$ with $q = e^{2\pi i\tau}$ is a modular function with respect to Γ for any $\sigma \in S_\alpha$, where c_α is the rational number in Theorem 3.2.4.*

Remark 3.3.4. For any solution $(f_a)_{a\in[1,r]} \in \overline{\mathbb{Q}}^r$ of (3.2.1), we can define the element

$$\sum_{a=1}^r d_a[f_a] \in \mathcal{B}(F), \tag{3.3.1}$$

where F is a number field containing the solution, and $\mathcal{B}(F)$ is the Bloch group of F. By the result in [Nak11b, Section 6], we see that the element (3.3.1) is a torsion (see [Lee13]). Conjecture 3.3.3 can be regarded as a version of Nahm's Conjecture [Nah07, Zag07], which relates torsions in Bloch groups and the modularity of q-hypergeometric series.

Theorem 3.3.5. *Conjecture 3.3.3 holds for $r = 1$.*

Proof. From (1) in Example 3.1.6, it is sufficient to prove the following three cases:

$$\begin{aligned}
\alpha_1 &= \big(1+z^{2p}, 1+z^{2p}, d\big),\\
\alpha_2 &= \big(1-z^{p}+z^{2p}, 1+z^{2p}, d\big),\\
\alpha_3 &= \big(1+z^{2p}, 1-z^{p}+z^{2p}, d\big).
\end{aligned}$$

For these three cases, we have $S_{\alpha_1} \cong \mathbb{Z}/2\mathbb{Z}$, $S_{\alpha_2} \cong 0$, and $S_{\alpha_3} \cong \mathbb{Z}/2\mathbb{Z}$. We also have $L_{\alpha_1} = d/2$, $L_{\alpha_2} = 2d/5$, and $L_{\alpha_3} = 3d/5$.

We first consider α_2 because its proof is the simplest and follows from a well-known discussion (e.g., see [Zag07, Chapter II, Section 3]). In this case, the partition q-series is given by

$$\mathcal{Z}_{\alpha_2,0}(q) = \sum_{n\in\mathbb{Z}_{\geq 0}} \frac{q^{dn^2}}{(q^d)_n}.$$

Using the Rogers-Ramanujan identity

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \prod_{\substack{n>0 \\ n\equiv\pm 1 \pmod 5}} \frac{1}{1-q^n},$$

together with the Jacobi triple product identity, we have

$$q^{-d/60}\mathcal{Z}_{\gamma}(q) = \frac{1}{2\eta(q^d)} \sum_{n\in\mathbb{Z}} a(n) q^{dn^2/40}, \tag{3.3.2}$$

where $\eta(q) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ is the Dedekind eta, and

$$a(n) = \begin{cases} 1 & \text{if } n \equiv \pm 1 \pmod{20}, \\ -1 & \text{if } n \equiv \pm 9 \pmod{20}, \\ 0 & \text{otherwise.} \end{cases}$$

Since the right-hand side in (3.3.2) is the ratio of modular forms of weight $1/2$, it is a modular function. Thus we obtain the assertion for α_2.

We now prove the assertion for α_1 and α_3. The partition q-series in these cases are given by

$$\mathcal{Z}_{\alpha_3,0}(q) = \sum_{n\in\mathbb{Z}_{\geq 0}} \frac{q^{dn^2}}{(q^d)_{2n}}, \quad \mathcal{Z}_{\alpha_3,1}(q) = \sum_{n\in\mathbb{Z}_{\geq 0}} \frac{q^{d(n^2+n+\frac{1}{4})}}{(q^d)_{2n+1}},$$
$$\mathcal{Z}_{\alpha_1,0}(q) = \sum_{n\in\mathbb{Z}_{\geq 0}} \frac{q^{2dn^2}}{(q^d)_{2n}}, \quad \mathcal{Z}_{\alpha_1,1}(q) = \sum_{n\in\mathbb{Z}_{\geq 0}} \frac{q^{d(2n^2+2n+\frac{1}{2})}}{(q^d)_{2n+1}}.$$

To prove the assertion for α_1 and α_3, we use the following Rogers – Ramanujan type

identities (see [MLSZ08, S. 98, S. 94, S. 83, and S. 86] and references therein):

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_{2n}} = \prod_{\substack{n>0 \\ n\equiv\pm1,\pm3,\pm4,\pm5,\pm7,\pm9 \pmod{20}}} \frac{1}{1-q^n}, \tag{3.3.3}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_{2n+1}} = \prod_{\substack{n>0 \\ n\equiv\pm1,\pm2,\pm5,\pm6,\pm8,\pm9 \pmod{20}}} \frac{1}{1-q^n}, \tag{3.3.4}$$

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q)_{2n}} = \prod_{\substack{n>0 \\ n\equiv\pm2,\pm3,\pm4,\pm5 \pmod{16}}} \frac{1}{1-q^n}, \tag{3.3.5}$$

$$\sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q)_{2n+1}} = \prod_{\substack{n>0 \\ n\equiv\pm1,\pm4,\pm6,\pm7 \pmod{16}}} \frac{1}{1-q^n}. \tag{3.3.6}$$

Using (3.3.3) and (3.3.4) together with the quintuple product identity, we have

$$q^{-d/40}\mathcal{Z}_{\alpha_3,\sigma}(q) = \frac{1}{2\eta(q^d)} \sum_{n\in\mathbb{Z}} a_{3,\sigma}(n) q^{dn^2/60},$$

where

$$a_{3,0}(n) = \begin{cases} 1 & \text{if } n \equiv \pm1 \pmod{30}, \\ -1 & \text{if } n \equiv \pm11 \pmod{30}, \\ 0 & \text{otherwise}, \end{cases}$$

$$a_{3,1}(n) = \begin{cases} 1 & \text{if } n \equiv \pm4 \pmod{30}, \\ -1 & \text{if } n \equiv \pm14 \pmod{30}, \\ 0 & \text{otherwise}. \end{cases}$$

Thus we obtain the assertion for α_3. Similarly, using (3.3.5) and (3.3.6) together with the quintuple product identity, we have

$$q^{-d/48}\mathcal{Z}_{\alpha_1,\sigma}(q) = \frac{1}{2\eta(q^d)} \sum_{n\in\mathbb{Z}} a_{1,\sigma}(n) q^{dn^2/48},$$

where

$$a_{1,0}(n) = \begin{cases} 1 & \text{if } n \equiv \pm1 \pmod{24}, \\ -1 & \text{if } n \equiv \pm7 \pmod{24}, \\ 0 & \text{otherwise}, \end{cases}$$

$$a_{1,1}(n) = \begin{cases} 1 & \text{if } n \equiv \pm5 \pmod{24}, \\ -1 & \text{if } n \equiv \pm11 \pmod{24}, \\ 0 & \text{otherwise}. \end{cases}$$

Thus we obtain the assertion for α_1. □

We give some examples supporting Conjecture 3.3.3 for $r \geq 2$.

Example 3.3.6 (Zagier's lists)**.** Any 2×2 or 3×3 matrix K for the Cartan-like T-data in Table 3.1 and 3.2 appears in lists of Zagier [Zag07, Table 2 and 3] as an example where

$$\sum_{m \in (\mathbb{Z}_{\geq 0})^r} \frac{q^{\frac{1}{2}m^{\mathsf{T}}Km+B^{\mathsf{T}}m+C}}{\prod_{a=1}^{r}(q)_{m_a}}$$

appears to be a modular function for some $B \in \mathbb{Q}^r$ and $C \in \mathbb{Q}$. We can see that all sporadic examples with $B = 0$ in his lists are obtained from (A_+, A_-) or (A_-, A_+) in our Table 3.1 and 3.2.

Example 3.3.7 (Andrew-Gordon identity)**.** Let α be the Cartan-like T-datum associated with the tadpole type T_r (see Example 2.3.8). It is of finite type since its T-system can be obtained from the T-system associated with the bipartite belt of type A_{2r}, which is periodic, by an identification of variables. Since $\det \mathring{A}_+ = 1$, we have $S_\alpha = 0$. By using Theorem 6.1 in [Nak11b], we see that the rational number c_α is given by $c_\alpha = 1-3/(2r+3)$. The partition q-series of α is given by

$$\mathcal{Z}_{\alpha,0}(q) = \sum_{n \in (\mathbb{Z}_{\geq 0})^r} \frac{q^{N_1^2+\cdots+N_r^2}}{(q)_{n_1} \cdots (q)_{n_r}},$$

where $N_a = n_a + \cdots + n_r$. Using the Andrew-Gordon identity [And74]

$$\sum_{n \in (\mathbb{Z}_{\geq 0})^r} \frac{q^{N_1^2+\cdots+N_r^2}}{(q)_{n_1} \cdots (q)_{n_r}} = \prod_{\substack{n>0 \\ n \not\equiv 0, \pm(r+1) \pmod{2r+3}}} \frac{1}{1-q^n},$$

together with the Jacobi triple product identity, we have

$$q^{-c_\alpha/24}\mathcal{Z}_{\alpha,0}(q) = \frac{1}{2\eta(q)} \sum_{n \in \mathbb{Z}} a(n) q^{n^2/(8(2r+3))},$$

where

$$a(n) = \begin{cases} 1 & \text{if } n \equiv \pm 1 \pmod{4(2r+3)}, \\ -1 & \text{if } n \equiv \pm(4r+5) \pmod{4(2r+3)}, \\ 0 & \text{otherwise.} \end{cases}$$

This implies that $q^{-c_\alpha/24}\mathcal{Z}_{\alpha,0}(q)$ is a modular function.

Example 3.3.8 (Fermionic formulas)**.** For any quantum affine algebra $U_q(\hat{\mathfrak{g}})$ and positive integer with $\ell \geq 2$, the level ℓ restricted T-system and Y-system for $U_q(\hat{\mathfrak{g}})$ are defined (see [KNS11]). Reading the exponents in the T-system and Y-system in [KNS11, Section 2], we can obtain the Cartan-like T-datum $\alpha(U_q(\hat{\mathfrak{g}}), \ell)$, where we replace a normalization of the parameter u appropriately so that $u \in \mathbb{Z}$ and the T-datum satisfies (N1), and we also discard the parameter Ω in [KNS11, Section 2.4] for twisted $\hat{\mathfrak{g}}$. Explicitly, the

type of $\hat{\mathfrak{g}}$	T-datum $\alpha(U_q(\hat{\mathfrak{g}}), \ell)$
$X_n^{(1)}$	$\alpha(X_n, \ell)$
$A_{2n-2}^{(2)}$	$\alpha(A_{\ell-1} \otimes C_n)$
$A_{2n}^{(2)}$	$\alpha(A_{\ell-1} \otimes T_n)$
$D_{n+1}^{(2)}$	$\alpha(A_{\ell-1} \otimes B_n)$
$E_6^{(2)}$	$\alpha(A_{\ell-1} \otimes F_4)$
$D_4^{(3)}$	$\alpha(A_{\ell-1} \otimes G_2)$

Table 3.3 T-data associated with quantum affine algebras.

T-datum $\alpha(U_q(\hat{\mathfrak{g}}), \ell)$ is given in Table 3.3, where in the first line we denote by $\alpha(X_n, \ell)$ the T-datum in Theorem 2.3.12 associated with the Cartan matrix of type X_n and the integer ℓ, and in the remaining lines we denote by $\alpha(Y \otimes Z)$ the T-datum obtained by the tensor product construction in Example 2.3.7 from the Cartan matrices of types Y and Z. The T-datum $\alpha(U_q(\hat{\mathfrak{g}}), \ell)$ is of finite type for any $U_q(\hat{\mathfrak{g}})$ and ℓ by the periodicity results in [Kel13, IIK+10, IIK+13a, IIK+13b]. The partition q-series of $\alpha(U_q(\hat{\mathfrak{g}}), \ell)$ divided by a product of the Dedekind eta coincide with the q-series version of the fermionic formulas defined in [HKO+02, Section 5]. They conjectured that these q-series coincide with string functions of integrable highest modules of $\hat{\mathfrak{g}}$ [HKO+02, Conjecutre 5.3]. If this conjecture holds, Conjecture 3.3.3 for $\alpha(U_q(\hat{\mathfrak{g}}), \ell)$ follows from the results by Kac and Peterson [KP84].

Example 3.3.9 (q-series from Nil-DAHA)**.** Let X_n be the type of a finite type Cartan matrix, and p be an integer with $p \geq 2$. Consider the T-datum $\alpha(X_n \otimes A_{p-1})$, where the meaning of this notation is the same as that in Example 3.3.8. This is of finite type by [Kel13]. Then the partition q-series of $\alpha(X_n \otimes A_{p-1})$ are special cases of the q-series studied by Cherednik and Feigin in the theory of Fourier transform of nilpotent double affine Hecke algebras [CF13, Corollary 1.3]. In fact, they proved that their q-series are modular functions [CF13, Theorem 2.3].

3.4 Exponents

In Proposition 3.3.2, we see that the leading term of the asymptotics of the partition q-series is expressed by a special values of the dilogarithm function. In this section, we study exponents associated with T-datum, which is a sequence of integers that describe the sub-leading term of the asymptotics of the partition q-series.

Let $\alpha = (A_+, A_-, D)$ be a T-datum of finite type. We also assume that α is Cartan-like. Let $(f_a)_{a \in [1,r]}$ be the positive real solution of (3.2.1). We define two $r \times r$ diagonal matrices $\Delta_+ = \mathrm{diag}(f_a)$ and $\Delta_- = \mathrm{diag}(1 - f_a)$, and also define a polynomial

$$\bar{\tau}_\alpha(z) := \det(A_+\Delta_- + A_-\Delta_+) \in \mathbb{R}[z]. \tag{3.4.1}$$

The following proposition give a relationship between the polynomial $\bar{\tau}_\alpha(z)$ and the asymptotics of the partition q-series.

Proposition 3.4.1. *We have*

$$\lim_{\varepsilon \searrow 0} \mathcal{Z}_{\alpha,\mathrm{tot}}(e^{-\varepsilon}) e^{-c_\alpha \pi^2/6\delta\varepsilon} = \sqrt{\frac{\det \mathring{A}_+}{\bar{\tau}_\alpha(1)}},$$

where c_α and δ are as in Proposition 3.3.2.

Proof. As with (2) of Proposition 3.3.2, the assertion follows from the asymptotic analysis in [VZ11]. □

Remark 3.4.2. In [VZ11] (as well as [Ter94, Zag07]), they not only gave the limit of q-series, but also gave a formula on the asymptotic expansion in the powers of ε. Although it might be interesting to investigate higher order terms of the asymptotic expansion, it is not dealt with in this paper.

We give another description of the polynomial $\bar{\tau}_\alpha(z)$ in terms of a mutation loop. Let $R \subseteq [1, r] \times \mathbb{Z}$ be a consistent subset for α. Then we have a mutation loop associated with (α, R) by Section 2.2.2, which we denote by $\gamma = (B, d, \mathbf{i}, \nu)$. Considering the semifield $\mathbb{R}^I_{>0}$, the mutation loop γ gives a real analytic map $\mu : \mathbb{R}^I_{>0} \to \mathbb{R}^I_{>0}$ by the formula $y_i(0) \mapsto y_{\nu(i)}(t)$, where $y(0)$ and $y(t)$ as in (2.1.6). The map μ has a unique fixed point $\eta \in \mathbb{R}^I_{>0}$ by (1) in Theorem 3.2.4 and the formula (2.2.11). Then the differential $d\mu_\eta$ is an endomorphism on the tangent space of $\mathbb{R}^I_{>0}$ at the fixed point η.

Proposition 3.4.3. *The polynomial* $\det \bar{\tau}(z)$ *coincides with the characteristic polynomial of the inverse of $d\mu_\eta$ in the variable z^t:*

$$\det \bar{\tau}_\alpha(z) = \det(z^t - d\mu_\eta),$$

where t is the integer in (R3) in Definition 2.2.3.

Proof. First note that $\bar{\tau}_\alpha(z^{-1}) \doteq \bar{\tau}_\alpha(z)$ since we assume that α is Cartan-like, where $\doteq$ means the equality up to multiplying $\pm z^k$ for some $k \in \mathbb{Z}$.

Let $\eta(u)$ be the I-tuple real positive number defined as $y(u)$ in (2.1.6) associated with the initial condition $(B(0), y(0)) = (B, \eta)$. Let $L(u) = (L_{ij}(u))_{i,j \in I}$ be the matrix given by

$$L_{ij}(u) := \begin{cases} 1 & \text{if } i = j \notin \mathbf{i}(u), \\ -1 & \text{if } i = j \in \mathbf{i}(u), \\ [B_{ji}(u)]_+(1 - f_a) + [-B_{ji}(u)]_+ f_a & \text{if } j = (a, u) \in \mathbf{i}(u) \text{ and } i \neq j, \\ 0 & \text{otherwise.} \end{cases} \tag{3.4.2}$$

Then $L(u)$ is the matrix of the differential of the analytic map $\mu(u) : \mathbb{R}^I_{>0} \to \mathbb{R}^I_{>0}$ given by $y_i(u) \mapsto y_i(u+1)$ at $\eta(u)$ with respect to the bases $\left(\eta_i(u) \frac{\partial}{\partial y_i(u)}\big|_{\eta(u)}\right)_{i \in I}$ and $\left(\eta_i(u+1) \frac{\partial}{\partial y_i(u+1)}\big|_{\eta(u+1)}\right)_{i \in I}$ (see Proposition 4.1 [Miz20b]). By the chain rule for differentials, the matrix

$$L := P_\nu L(t-1) \cdots L(0)$$

is the matrix of $d\mu_\eta$ with respect to the basis $\left(\eta_i \frac{\partial}{\partial y_i}\big|_\eta\right)_{i\in I}$, where P_ν is the permutation matrix of ν. On the other hand, let $\widetilde{L}$ be the square matrix of size $I \times t$ defined by

$$\widetilde{L} \coloneqq \begin{bmatrix} O & \cdots & O & P_\nu L(t-1) \\ L(0) & \cdots & O & O \\ \vdots & \ddots & O & O \\ O & \cdots & L(t-2) & O \end{bmatrix}$$

Clearly, we have

$$\det(z^t - L) = \det(z - \widetilde{L}).$$

Let $H(u) = (H_{ij})_{i,j\in I}$ be the matrix given by

$$H_{ij} \coloneqq \begin{cases} 1 & \text{if } i = j \notin \mathbf{i}(u), \\ 0 & \text{otherwise,} \end{cases}$$

and let $\widetilde{H}$ be the square matrix of size $I \times t$ defined by

$$\widetilde{H} \coloneqq \begin{bmatrix} O & \cdots & O & P_\nu H(t-1) \\ H(0) & \cdots & O & O \\ \vdots & \ddots & O & O \\ O & \cdots & H(t-2) & O \end{bmatrix}$$

Explicitly, the entries in $\widetilde{H} = (\widetilde{H}_{(i,u),(j,v)})$ are given by

$$\widetilde{H}_{(i,u),(j,v)} = \begin{cases} 1 & \text{if } (j,v) \notin \mathbf{i}(v),\ (i,u+1) = (j,v), \text{ and } 0 \le v < t, \\ 1 & \text{if } (j,v) \notin \mathbf{i}(v),\ (\nu(i),0) = (j,v), \text{ and } v = t, \\ 0 & \text{otherwise,} \end{cases}$$

By the completeness of γ, the sum $\sum_{u=1}^{\infty} z^u \widetilde{H}^{u-1}$ is a finite sum, giving an inverse matrix of $z^{-1} - \widetilde{H}$. In particular, we have $\det(z^{-1} - \widetilde{H}) \doteq 1$. We now see that $(z^{-1} - \widetilde{H})^{-1}(z^{-1} - \widetilde{L})$ is the identity matrix except for the columns in $\mathbf{i}$ (we regard $\mathbf{i}$ as a subset of $I \times t$ by the obvious way), and $\mathbf{i} \times \mathbf{i}$ submatrix of this matrix coincides with $A_+^\vee \Delta_- + A_-^\vee \Delta_+$:

$$(z^{-1} - \widetilde{H})^{-1}(z^{-1} - \widetilde{L}) \sim \begin{bmatrix} A_+^\vee \Delta_- + A_-^\vee \Delta_+ & O \\ * & I \end{bmatrix}.$$

The first statement follows from the relation

$$(z^{-1} - \widetilde{H})^{-1}(z^{-1} - \widetilde{L}) = I + (z^{-1} - \widetilde{H})^{-1}(\widetilde{H} - \widetilde{L})$$

and the fact that $\widetilde{H} - \widetilde{L}$ is the zero matrix except for the columns in $\mathbf{i}$. Noting that $(z^{-1} - \widetilde{H})^{-1} = \sum_{u=1}^{\infty} z^u \widetilde{H}^{u-1}$, the second statement follows by comparing the formula

(2.1.8) and (2.1.9) with the definition of the matrices $\widetilde{H}$ and $\widetilde{L}$, respectively (see also Figure 2.1).

Therefore, we have

$$\det(z^t - d\mu_\eta) = \det(z^t - L) = \det(z - \tilde{L}) \doteq \bar{\tau}_\alpha(z^{-1}) \doteq \bar{\tau}_\alpha(z).$$

Comparing the highest degrees, we obtain the assertion. □

Definition 3.4.4. Let $\alpha = (A_+, A_-, D)$ be a Cartan-like T-datum of finite type. Suppose that α is indecomposable, and let t be the number of connected components of the skew-symmetrizable matrix B in Proposition 2.2.24. We define a polynomial $\tau_\alpha(z) \in \mathbb{R}[z]$ by $\tau_\alpha(z) \coloneqq \bar{\tau}_\alpha(z^{1/t})$.

We have

$$\tau_\alpha(z) = \det(z - d\mu_\eta) \tag{3.4.3}$$

by Proposition 3.4.3.

Theorem 3.4.5. *Any root of the polynomial $\tau_\alpha(z)$ is a root of unity. More precisely, we have $\lambda^{\Omega/t} = 1$ for any root λ of $\tau_\alpha(z)$, where Ω is a period in Definition 3.1.3.*

Proof. Let γ be a mutation loop associated with the pair $(\alpha, [1, r] \times \mathbb{Z})$. By the periodicity of the Y-system associated with α, we have $\mu^{\Omega/t} = \mathrm{id}$. Thus the assertion follows from Proposition 3.4.3 and the chain rule for differentials. □

For a polynomial $F(z)$ whose roots are all N-th roots of unity, the roots of $F(x)$ can be written as

$$e^{2\pi i m_1/N}, e^{2\pi i m_2/N}, \ldots, e^{2\pi i m_n/N},$$

where $0 \le m_1 \le m_2 \le \cdots \le m_n < N$ is a sequence of integers. We call this sequence of integers the exponents of $F(x)$, and denote by $\mathcal{E}(F(x))$.

Definition 3.4.6. The sequence $\mathcal{E}(\tau_\alpha(z))$ is called the *exponents* associated with α.

Example 3.4.7. Let α be a T-datum given by

$$A_+ = \begin{bmatrix} 1+z^2 & -z & 0 \\ -z & 1+z^2 & -z \\ 0 & -z & 1+z^2 \end{bmatrix}, \quad A_- = \begin{bmatrix} 1+z^2 & -z & 0 \\ -z & 1+z^2 & -z \\ 0 & -z & 1+z^2 \end{bmatrix},$$

and $D = I_3$. Then the matrix $K^\vee$ in Proposition 3.2.3 is given by

$$K^\vee = \begin{bmatrix} \frac{3}{2} & 1 & \frac{1}{2} \\ 1 & 2 & 1 \\ \frac{1}{2} & 1 & \frac{3}{2} \end{bmatrix},$$

and the solution of the equation (3.2.1) is given by $(f_1, f_2, f_3) = (1/3, 1/4, 1/3)$. We have

$$\begin{aligned} \bar{\tau}_\alpha(z) &= \det \begin{bmatrix} 1+z^2 & -\frac{3}{4}z & 0 \\ -\frac{2}{3}z & 1+z^2 & -\frac{2}{3}z \\ 0 & -\frac{3}{4}z & 1+z^2 \end{bmatrix} \\ &= z^6 + 2z^4 + 2z^2 + 1, \end{aligned}$$

and

$$\tau_\alpha(z) = z^3 + 2z^2 + 2z + 1 = (z - e^{\frac{2\pi i \cdot 2}{6}})(z - e^{\frac{2\pi i \cdot 3}{6}})(z - e^{\frac{2\pi i \cdot 4}{6}}).$$

Thus the exponents associated with α are $2, 3, 4$.

We see that $\tau_\alpha(z)$ is invariant under interchanging A_+ and A_-:

Proposition 3.4.8. *Let $\alpha = (A_+, A_-, D)$ be a Cartan-like T-datum of finite type. Then $\alpha^o := (A_-, A_+, D)$ is also a Cartan-like T-datum of finite type. Moreover, we have $\tau_\alpha(z) = \tau_{\alpha^o}(z)$.*

Proof. The first statement is obvious from the definition of T-data (Definition 2.2.1). Let f_a be the solution of the equation (3.2.1) for α. Then $1 - f_a$ is a solution of the equation (3.2.1) for α^o, which can be seen from another expression (3.2.3) of this equation. The identity $\tau_\alpha(z) = \tau_{\alpha^o}(z)$ now follows from (3.4.1). □

3.4.1 Exponents for T-datum of type (X_n, ℓ)

Fist we prepare some notations of root systems. Let C be an indecomposable Cartan matrix of finite type X_n, and ℓ be a positive integer such that $\ell \geq 2$, as in Section 2.3.3. Let Δ be a root system of type X_n on a $\mathbb{R}$-vector space with an inner product normalized as $(\alpha \mid \alpha) = 2$ for long roots α, where X_n is a finite type Dynkin diagram in the Figure 3.1. Let $\alpha_1, \ldots, \alpha_n$ be simple roots, where the numberings are consistent with the numberings of nodes in Figure 3.1. Let Δ^{long} and Δ^{short} be the set of long roots and short roots, respectively. Let Δ_+ be the set of positive roots, and ρ be the half of the sum of the positive roots:

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha.$$

Let c_a $(1 \leq a \leq n)$ be the entries in the left symmetrizer of C as in Section 2.3.3. We assume that $\gcd(c_1, \ldots, c_n) = 1$ for simplicity. We define an integer c by

$$c := \mathrm{lcm}(c_1, \ldots, c_n) = \begin{cases} 1 & \text{if } X_n = A_n, D_n, E_6, E_7 \text{ or } E_8, \\ 2 & \text{if } X_n = B_n, C_n \text{ or } F_4, \\ 3 & \text{if } X_n = G_2. \end{cases} \tag{3.4.4}$$

Let $h^\vee$ be the *dual Coxeter number*. The list of dual Coxeter numbers is given by

X_n	A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
$h^\vee$	$n+1$	$2n-1$	$n+1$	$2n-2$	12	18	30	9	4

. (3.4.5)

Let α be the T-datum associated with the pair (X_n, ℓ) given by Theorem 2.3.12, which is Cartan-like by definition. Moreover, α is of finite type with period $\Omega = 2c(\ell + h^\vee)$, which was prove in [IIK+13a, IIK+13b]. Thus we can define the polynomial $\tau_\alpha(z)$ by Definition 3.4.4. Note that $t = 2$ in Definition 3.4.4 for this α. We give a conjectural formula on exponents associated with α in terms of root systems.

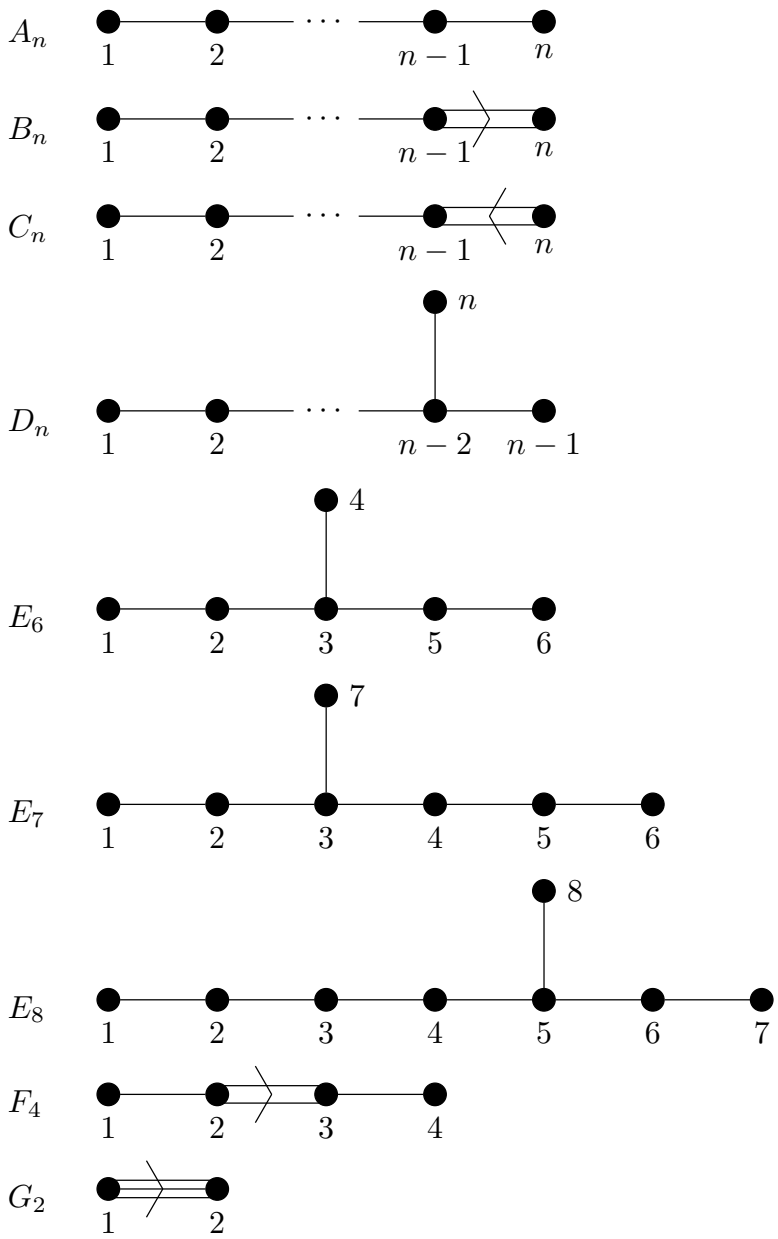


Fig. 3.1 The list of Dynkin diagrams of finite type.

We define two polynomials $N_{X_n,\ell}(z)$ and $D_{X_n,\ell}(z)$ by

$$N_{X_n,\ell}(z) = \prod_{a=1}^{n} \frac{z^{c(\ell+h^\vee)}-1}{z^{c_a}-1}, \tag{3.4.6}$$

$$D_{X_n,\ell}(z) = D^{\mathrm{long}}_{X_n,\ell}(z) D^{\mathrm{short}}_{X_n,\ell}(z), \tag{3.4.7}$$

where the polynomials $D^{\mathrm{long}}_{X_n,\ell}(z)$ and $D^{\mathrm{short}}_{X_n,\ell}(z)$ are defined by

$$D^{\mathrm{long}}_{X_n,\ell}(z) = \prod_{\alpha\in\Delta^{\mathrm{long}}} \left(z^c - e^{\frac{2\pi i(\rho|\alpha)}{\ell+h^\vee}}\right), \tag{3.4.8}$$

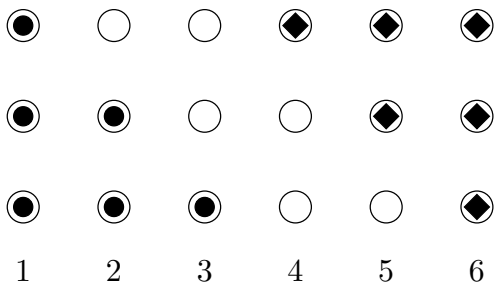


Fig. 3.2 The underlying white circles represent the exponents of $N_{A_3,3}(x)$, and the black marks represent the exponents of $D_{A_3,3}(x)$. The number of vertices in a same vertical line is a multiplicity. Furthermore, the black circles and diamonds represent the exponents that come from the positive roots and the negative roots, respectively. The unmarked white circles represent the exponents of $N_{A_3,3}(x)/D_{A_3,3}(x)$.

$$D^{\mathrm{short}}_{X_n,\ell}(z) = \prod_{\alpha\in\Delta^{\mathrm{short}}} \left(z - e^{\frac{2\pi i(\rho|\alpha)}{\ell+h^\vee}}\right). \tag{3.4.9}$$

When $X = A, D$ or E, the polynomials $N_{X_n,\ell}(z)$ and $D_{X_n,\ell}(z)$ can be written more simply:

$$N_{X_n,\ell}(z) = \left(\frac{z^{\ell+h^\vee}-1}{z-1}\right)^n, \tag{3.4.10}$$

$$D_{X_n,\ell}(z) = \prod_{\alpha\in\Delta} \left(z - e^{\frac{2\pi i(\rho|\alpha)}{\ell+h^\vee}}\right). \tag{3.4.11}$$

Conjecture 3.4.9. *Let C be an indecomposable Cartan matrix of finite type X_n, and ℓ be a positive integer such that $\ell \geq 2$. Let α be the T-datum associated with these data (X_n, ℓ) given by Theorem 2.3.12. Then the roots of $\tau_\alpha(z)$ is given by the formula*

$$\tau_\alpha(z) = \frac{N_{X_n,\ell}(z)}{D_{X_n,\ell}(z)}. \tag{3.4.12}$$

Example 3.4.10. Let $(X_n, \ell) = (A_3, 3)$. The dual Coxeter number is given by $h^\vee = 4$, so $\ell + h^\vee = 7$. Therefore, the exponents of $N_{A_3,3}(z)$ is given by

$$\mathcal{E}(N_{A_3,3}(z)) = (1,1,1,2,2,2,3,3,3,4,4,4,5,5,5,6,6,6).$$

On the other hand, by calculation on the root system of type A_3, we obtain

$$\mathcal{E}(D_{A_3,3}(z)) = (1,1,1,2,2,3,4,5,5,6,6,6).$$

As the result, the exponents of $N_{A_3,3}(z)/D_{A_3,3}(z)$ are given by

$$\mathcal{E}\left(\frac{N_{A_3,3}(z)}{D_{A_3,3}(z)}\right) = (2,3,3,4,4,5).$$

Figure 3.2 illustrates these exponents.

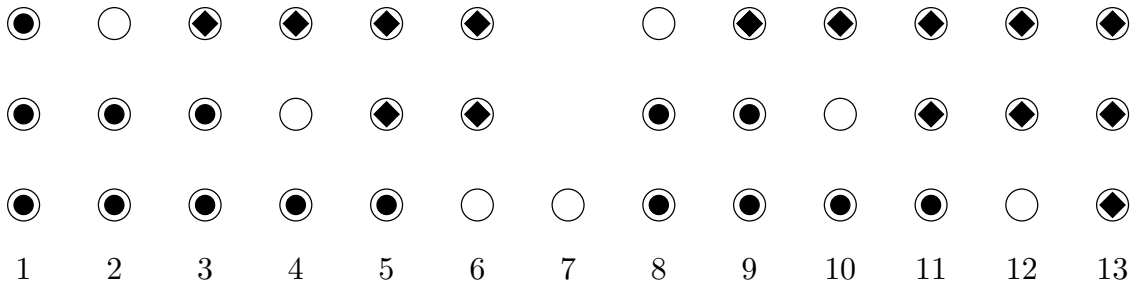


Fig. 3.3 The exponents of $N_{B_3,2}(z)$ and $D_{B_3,2}(z)$. The meanings of the symbols are the same as in Figure 3.2.

Example 3.4.11. Let $(X_n, \ell) = (B_3, 2)$. The dual Coxeter number is given by $h^\vee = 5$, so $c(\ell + h^\vee) = 14$. Therefore, the exponents of $N_{B_3,2}(z)$ is given by

$$\mathcal{E}(N_{B_3,2}(z)) = (1,1,1,2,2,2,3,3,3,4,4,4,5,5,5,6,6,6,7,8,8,8,9,9,9,\\ 10,10,10,11,11,11,12,12,12,13,13,13).$$

On the other hand, by calculation on the root system of type B_3, we obtain

$$\mathcal{E}(D^{\mathrm{long}}_{B_3,2}(z)) = (1,1,2,2,3,3,4,4,5,5,6,6,8,8,9,9,10,10,11,11,12,12,13,13),$$
$$\mathcal{E}(D^{\mathrm{short}}_{B_3,2}(z)) = (1,3,5,9,11,13).$$

As the result, the exponents of $N_{B_3,2}(z)/D_{B_3,2}(z)$ are given by

$$\mathcal{E}\left(\frac{N_{B_3,2}(z)}{D_{B_3,2}(z)}\right) = (2,4,6,7,8,10,12).$$

Figure 3.3 illustrates these exponents.

Theorem 3.4.12. *Conjecture 3.4.9 is true in the following cases:*

(1) (A_1, ℓ) *for all* $\ell \geq 2$,
(2) $(A_n, 2)$ *for all* $r \geq 1$.

We will prove Theorem 3.4.12 in Section 3.4.2.

3.4.2 Proofs of Theorem 3.4.12

(A_1, ℓ) case

Lemma 3.4.13. *The right-hand side of* (3.4.12) *is given by*

$$\frac{N_{A_1,\ell}(z)}{D_{A_1,\ell}(z)} = \prod_{a=2}^{\ell}(z - e^{\frac{2\pi i a}{\ell+2}}).$$

Proof. First note that the dual Coxeter number of A_1 is given by $h^\vee = 2$. Then, the lemma follows from

$$N_{A_1,\ell}(z) = \frac{z^{\ell+2} - 1}{x - 1},$$

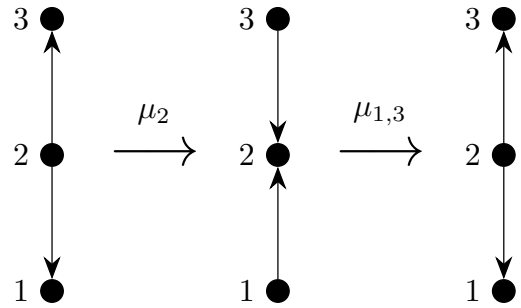


Fig. 3.4 The mutation loop for type $(A_1, 4)$.

and

$$\begin{aligned} D_{A_1,\ell}(z) &= (z - e^{\frac{2\pi i}{\ell+2}})(z - e^{\frac{-2\pi i}{\ell+2}}) \\ &= (z - e^{\frac{2\pi i}{\ell+2}})(z - e^{\frac{2\pi i(\ell+1)}{\ell+2}}). \end{aligned}$$

□

Let α be a T-datum of type (A_1, ℓ). What we have to show is that the exponents of $\tau_\alpha(z)$ are $2, 3, \ldots, \ell$. To prove this, we consider the mutation loop associated with (α, R), where

$$R = \{(m, p) \in [1, \ell - 1] \times \mathbb{Z} \mid m + p \equiv 0 \bmod 2\}$$

is a consistent subset for α. Then the index set of the mutation loop associated with (α, R) is given by $I = R_{\text{in}} = I(0) \sqcup I(0)$, where

$$\begin{aligned} I(0) &\coloneqq \{(m, 0) \mid 0 \le m \le \ell - 1, m \equiv 0 \bmod 2\}, \\ I(1) &\coloneqq \{(m, 1) \mid 0 \le m \le \ell - 1, m \equiv 1 \bmod 2\}. \end{aligned}$$

For example, Figure 3.4 shows the mutation loop for $(A_1, 4)$.

Let $\zeta = e^{\pi i/(\ell+2)}$. For $m = 1, 2, \ldots, \ell - 1$, we define non-zero real numbers

$$z_m = \frac{\sin\frac{\pi m}{\ell+2}\sin\frac{\pi(m+2)}{\ell+2}}{\sin^2\frac{\pi(m+1)}{\ell+2}} \tag{3.4.13}$$

$$= \frac{(\zeta^m - \zeta^{-m})(\zeta^{m+2} - \zeta^{-m-2})}{(\zeta^{m+1} - \zeta^{-m-1})^2}. \tag{3.4.14}$$

Then we see by direct calculations that

$$f_m = 1 - z_m = \frac{\sin^2\frac{\pi}{\ell+2}}{\sin^2\frac{(m+1)\pi}{\ell+2}}$$

is a solution of the equation (3.2.1).

Let $L(0) = (L_{mk}(0))_{m,k=1,\dots,\ell-1}$ and $L(1) = (L_{mk}(1))_{m,k=1,\dots,\ell-1}$ be matrices given by (3.4.2). Explicitly, these are given by

$$L_{mk}(p) = \begin{cases} \delta_{mk} & \text{if } k \in I(1-p), \\ -\delta_{mk} + z_k(\delta_{m,k-1} + \delta_{m,k+1}) & \text{if } k \in I(p). \end{cases} \tag{3.4.15}$$

for $p = 0, 1$.

For any non-zero complex number λ, we consider the following difference equation of the numbers $(\phi_m)_{0\le m\le \ell}$:

$$\phi_{m-1} + \phi_{m+1} = \phi_m z_m^{-1}(\lambda + \lambda^{-1}) \tag{3.4.16}$$

with the boundary conditions given by

$$\phi_0 = \phi_\ell = 0. \tag{3.4.17}$$

Lemma 3.4.14. *Let $(\phi_m)_{0\le m\le \ell}$ be a non-zero solution of the difference equation (3.4.16) satisfying the boundary conditions (3.4.17). Then the vector*

$$\psi = \begin{bmatrix} \psi_1 \\ \vdots \\ \psi_{\ell-1} \end{bmatrix}$$

defined by

$$\psi_m = \begin{cases} \lambda\phi_m & \text{if } m \in I(0), \\ \phi_m & \text{if } m \in I(1), \end{cases}$$

is an eigenvector of the transpose matrix of $L = L(1)L(0)$ with an eigenvalue λ^2, that is,

$$L^{\mathsf{T}}\psi = \lambda^2\psi.$$

Proof. Let $\psi' = L(1)^{\mathsf{T}}\psi$ and $\psi'' = L(0)^{\mathsf{T}}\psi' (= L^{\mathsf{T}}\psi)$. In the following equations, we assume that $\psi_0 = \psi_\ell = \psi'_0 = \psi'_\ell = 0$. Then we obtain

$$\psi'_m = \begin{cases} \psi_m & \text{if } m \in I(0), \\ -\psi_m + z_m(\psi_{m-1} + \psi_{m+1}) & \text{if } m \in I(1), \end{cases}$$

and

$$\psi''_m = \begin{cases} -\psi'_m + z_m(\psi'_{m-1} + \psi'_{m+1}) & \text{if } m \in I(0), \\ \psi'_m & \text{if } m \in I(1). \end{cases}$$

For any $m \in I(1)$, we compute

$$\begin{aligned}\psi''_m - \lambda^2\psi_m &= \psi'_m - \lambda^2\psi_m \\ &= -\psi_m + z_m(\psi_{m-1} + \psi_{m+1}) - \lambda^2\psi_m \\ &= -\phi_m + z_m\lambda(\phi_{m-1} + \phi_{m+1}) - \lambda^2\phi_m \\ &= z_m\lambda(-z_m^{-1}(\lambda + \lambda^{-1})\phi_m + \phi_{m-1} + \phi_{m+1}) \\ &= 0.\end{aligned}$$

In particular, we obtain $\psi'_m = \lambda^2\psi_m$ for $m \in I(1)$. Thus, for any $m \in I(0)$, we find that

$$\begin{aligned}\psi''_m - \lambda^2\psi_m &= -\psi'_m + z_m(\psi'_{m-1} + \psi'_{m+1}) - \lambda^2\psi_m \\ &= -\psi_m + z_m\lambda^2(\psi_{m-1} + \psi_{m+1}) - \lambda^2\psi_m \\ &= -\lambda\phi_m + z_m\lambda^2(\phi_{m-1} + \phi_{m+1}) - \lambda^3\phi_m \\ &= z_m\lambda^2(-z_m^{-1}(\lambda + \lambda^{-1})\phi_m + \phi_{m-1} + \phi_{m+1}) \\ &= 0,\end{aligned}$$

and this complete the proof. □

Now we focus on solving the difference equation (3.4.16) satisfying the boundary conditions (3.4.17). We will show that $(\phi_m^{(a)})_{m=0,\dots,\ell}$ for

$$\phi_m^{(a)} = \det\begin{bmatrix} 2\cos\frac{\pi a}{\ell+2} & 2\cos\frac{\pi a(m+1)}{\ell+2} \\ \sin\frac{\pi(a-1)}{\ell+2}/\sin\frac{\pi}{\ell+2} & \sin\frac{\pi(a-1)(m+1)}{\ell+2}/\sin\frac{\pi(m+1)}{\ell+2}\end{bmatrix} \tag{3.4.18}$$

is a non-zero solution of (3.4.16) and (3.4.17) for $\lambda = \zeta^a$ if $a = 2, \dots, \ell$ (Theorem 3.4.18).

To prove this, we introduce the following Laurent polynomials:

$$\alpha^{(a)}(z) = z^a + z^{-a}, \tag{3.4.19}$$

$$\beta^{(a)}(z) = \frac{z^{a-1} - z^{-a+1}}{z - z^{-1}}. \tag{3.4.20}$$

We write $\alpha^{(a)}(z^{m+1})$ and $\beta^{(a)}(z^{m+1})$ as $\alpha_m^{(a)}(z)$ and $\beta_m^{(a)}(z)$. We also define a Laurent polynomial $P_m^{(a)}(z)$ by

$$P_m^{(a)}(z) = \det\begin{pmatrix} \alpha_0^{(a)}(z) & \alpha_m^{(a)}(z) \\ \beta_0^{(a)}(z) & \beta_m^{(a)}(z)\end{pmatrix}. \tag{3.4.21}$$

Note that $\phi_m^{(a)}$ in (3.4.18) can be written as $P_m^{(a)}(\zeta)$.

Let us examine difference equations for $\alpha_m^{(a)}(z), \beta_m^{(a)}(z)$ and $P_m^{(a)}(z)$.

Lemma 3.4.15. *The Laurent polynomials $\alpha_m^{(a)}(z)$ satisfy the difference equation*

$$\alpha_{m-1}^{(a)}(z) + \alpha_{m+1}^{(a)}(z) = \alpha_0^{(a)}(z)\alpha_m^{(a)}(z). \tag{3.4.22}$$

Proof. We compute

$$\begin{aligned}
&\alpha^{(a)}_{m-1} + \alpha^{(a)}_{m+1}(z) \\
&= (z^{am} + z^{-am}) + (z^{a(m+2)} + z^{-a(m+2)}) \\
&= (z^{a} + z^{-a})(z^{a(m+1)} + z^{-a(m+1)}) \\
&= \alpha^{(a)}_{0}(z)\alpha^{(a)}_{m}(z),
\end{aligned}$$

and this proves the lemma. □

Lemma 3.4.16. *The Laurent polynomials $\beta^{(a)}_m(z)$ satisfy the difference equation*

$$\begin{aligned}
&\big(\beta^{(a)}_{m-1}(z) + \beta^{(a)}_{m+1}(z)\big)(z^m - z^{-m})(z^{m+2} - z^{-m-2}) \\
&= \alpha^{(a)}_0(z)\beta^{(a)}_m(z)(z^{m+1} - z^{-m-1})^2 - \alpha^{(a)}_m(z)\beta^{(a)}_0(z)(z - z^{-1})^2.
\end{aligned} \tag{3.4.23}$$

Proof. We compute

$$\begin{aligned}
&\big(\beta^{(a)}_{m-1}(z) + \beta^{(a)}_{m+1}(z)\big)(z^m - z^{-m})(z^{m+2} - z^{-m-2}) \\
&\quad - \alpha^{(a)}_0(z)\beta^{(a)}_m(z)(z^{m+1} - z^{-m-1})^2 \\
&= (z^{(a-1)m} - z^{-(a-1)m})(z^{m+2} - z^{-m-2}) \\
&\quad + (z^{(a-1)(m+2)} - z^{-(a-1)(m+2)})(z^m - z^{-m}) \\
&\quad - \alpha^{(a)}_0(z)(z^{(a-1)m} - z^{-(a-1)m})(z^{m+1} - z^{-m-1}) \\
&= (z^{am+2} + z^{-am-2} - z^{am-2m-2} - z^{-am+2m+2}) \\
&\quad + (z^{a(m+2)-2} + z^{-a(m+2)+2} - z^{a(m+2)-2m-2} - z^{-a(m+2)+2m+2}) \\
&\quad - (z^{a(m+2)} + z^{am} - z^{-am+2m+2} - z^{-a(m+2)+2m+2} \\
&\quad - z^{a(m+2)-2m-2} - z^{am-2m-2} + z^{-am} + z^{-a(m+2)}) \\
&= z^{am+2} + z^{-am-2} + z^{a(m+2)-2} + z^{-a(m+2)+2} \\
&\quad - z^{am} - z^{-am} - z^{a(m+2)} - z^{-a(m+2)} \\
&= (z - z^{-1})(z^{am+1} - z^{-am-1} + z^{a(m+2)-1} - z^{-a(m+2)+1}) \\
&= -(z - z^{-1})(z^{a(m+1)} + z^{-a(m+1)})(z^{a-1} - z^{-a+1}) \\
&= -\alpha^{(a)}_m(z)\beta^{(a)}_0(z)(z - z^{-1})^2,
\end{aligned}$$

and this proves the lemma. □

Lemma 3.4.17. *The Laurent polynomials $P^{(a)}_m(z)$ satisfy the difference equation*

$$\begin{aligned}
&\big(P^{(a)}_{m-1}(z) + P^{(a)}_{m+1}(z)\big)(z^m - z^{-m})(z^{m+2} - z^{-m-2}) \\
&= P^{(a)}_m(z)\alpha^{(a)}_0(z)(z^{m+1} - z^{-m-1})^2.
\end{aligned} \tag{3.4.24}$$

Proof. By using (3.4.22), (3.4.23) and the relation

$$(z^{m+1}-z^{-m+1})^2-(z-z^{-1})^2=(z^m-z^{-m})(z^{m+2}-z^{-m-2}),$$

we obtain

$$\begin{aligned}
&P_m^{(a)}(z)\alpha_0^{(a)}(z)(z^{m+1}-z^{-m-1})^2\\
&=\big(\alpha_0^{(a)}(z)\beta_m^{(a)}(z)-\alpha_m^{(a)}(z)\beta_0^{(a)}(z)\big)\alpha_0^{(a)}(z)(z^{m+1}-z^{-m-1})^2\\
&=\alpha_0^{(a)}(z)\Big(\big(\beta_{m-1}^{(a)}(z)+\beta_{m+1}^{(a)}(z)\big)(z^m-z^{-m})(z^{m+2}-z^{-m-2})\\
&\quad+\alpha_m^{(a)}(z)\beta_0^{(a)}(z)(z-z^{-1})^2\Big)\\
&\quad-\alpha_0^{(a)}(z)\alpha_m^{(a)}(z)\beta_0^{(a)}(z)(z^{m+1}-z^{-m-1})^2\\
&=\alpha_0^{(a)}(z)\big(\beta_{m-1}^{(a)}(z)+\beta_{m+1}^{(a)}(z)\big)(z^m-z^{-m})(z^{m+2}-z^{-m-2})\\
&\quad-\alpha_0^{(a)}(z)\alpha_m^{(a)}(z)\beta_0^{(a)}(z)\left((z^{m+1}-z^{-m+1})^2-(z-z^{-1})^2\right)\\
&=\Big(\alpha_0^{(a)}(z)\big(\beta_{m-1}^{(a)}(z)+\beta_{m+1}^{(a)}(z)\big)-\alpha_0^{(a)}(z)\alpha_m^{(a)}(z)\beta_0^{(a)}(z)\Big)\\
&\quad\cdot(z^m-z^{-m})(z^{m+2}-z^{-m-2})\\
&=\Big(\alpha_0^{(a)}(z)\big(\beta_{m-1}^{(a)}(z)+\beta_{m+1}^{(a)}(z)\big)-\big(\alpha_{m-1}^{(a)}(z)+\alpha_{m+1}^{(a)}(z)\big)\beta_0^{(a)}(z)\Big)\\
&\quad\cdot(z^m-z^{-m})(z^{m+2}-z^{-m-2})\\
&=\big(P_{m-1}^{(a)}(z)+P_{m+1}^{(a)}(z)\big)(z^m-z^{-m})(z^{m+2}-z^{-m-2}),
\end{aligned}$$

completing the proof. □

Using Lemma 3.4.17, we can construct non-zero solutions of the difference equation (3.4.16) satisfying the boundary conditions (3.4.17).

Theorem 3.4.18. *For any $a=2,3,\dots,\ell$ and $m=0,1,\dots,\ell$, let $\phi_m^{(a)}=P_m^{(a)}(\zeta)$. Then the following properties hold:*

(1) The numbers $\{\phi_m^{(a)}\mid 0\le m\le\ell\}$ satisfy the difference equation (3.4.16) for $\lambda=\zeta^a$.
(2) The boundary conditions (3.4.17) hold, i.e., $\phi_0^{(a)}=\phi_\ell^{(a)}=0$.
(3) There exists m such that $\phi_m^{(a)}\neq 0$.

Proof. The property (1) follows from Lemma 3.4.17 and the definition of z_m (3.4.13). Now we prove the property (2). The definition of $P_m^{(a)}(z)$ immediately implies that $P_0^{(a)}(z)=0$, hence $\phi_0^{(a)}=0$. To prove $\phi_\ell^{(a)}=0$, we define a Laurent polynomial $\widetilde{P}^{(a)}(x,y)$ of two variables by

$$\widetilde{P}^{(a)}(x,y)=\det\begin{pmatrix}\alpha^{(a)}(x) & \alpha^{(a)}(y)\\ \beta^{(a)}(x) & \beta^{(a)}(y)\end{pmatrix}.$$

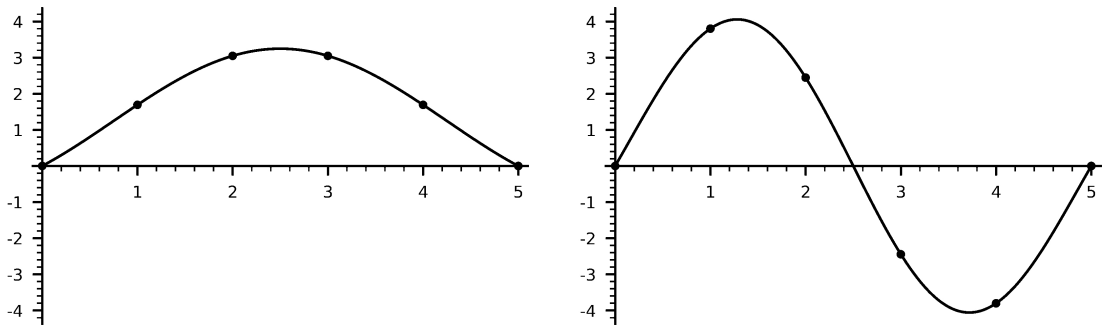


Fig. 3.5 The values of $(\phi_m^{(2)})_{m=0\ldots,\ell}$ and $(\phi_m^{(3)})_{m=0\ldots,\ell}$ for $\ell = 5$.

It is easy to see that the polynomial $\widetilde{P}^{(a)}(x,y)$ satisfies

$$\begin{aligned}
&\widetilde{P}^{(a)}(x,x^{m+1}) = P_m^{(a)}(x),\\
&\widetilde{P}^{(a)}(x,y^{-1}) = \widetilde{P}^{(a)}(x,y),\\
&\widetilde{P}^{(a)}(x,-y) = (-1)^a \widetilde{P}^{(a)}(x,y).
\end{aligned}$$

Thus we have

$$\begin{aligned}
\phi_\ell^{(a)} &= P_\ell^{(a)}(\zeta) = \widetilde{P}^{(a)}(\zeta,\zeta^{\ell+1}) = \widetilde{P}^{(a)}(\zeta,-\zeta^{-1})\\
&= (-1)^a \widetilde{P}^{(a)}(\zeta,\zeta^{-1}) = (-1)^a \widetilde{P}^{(a)}(\zeta,\zeta) = 0.
\end{aligned}$$

Now we prove the property (3). We will show that $\phi_1^{(a)} > 0$. The number $\phi_1^{(a)}$ can be written as

$$\begin{aligned}
\phi_1^{(a)} &= \zeta^a + \zeta^{-a} \cdot \frac{\zeta^{2a-2} - \zeta^{-2a+2}}{\zeta^2 - \zeta^{-2}} - \zeta^{2a} + \zeta^{-2a} \cdot \frac{\zeta^{a-1} - \zeta^{-a+1}}{\zeta - \zeta^{-1}}\\
&= \frac{\zeta^{a-1} - \zeta^{-a+1}}{\zeta^2 - \zeta^{-2}} \big((\zeta^a + \zeta^{-a})(\zeta^{a-1} + \zeta^{-a+1}) - (\zeta^{2a} + \zeta^{-2a})(\zeta + \zeta^{-1}) \big)\\
&= \frac{\zeta^{a-1} - \zeta^{-a+1}}{\zeta^2 - \zeta^{-2}} \big((\zeta + \zeta^{-1}) - (\zeta^{2a+1} + \zeta^{-2a-1}) \big)\\
&= \sin\frac{(a-1)\pi}{\ell+2} \left(\sin\frac{2\pi}{\ell+2} \right)^{-1} \left(2\cos\frac{\pi}{\ell+2} - 2\cos\frac{(2a+1)\pi}{\ell+2} \right).
\end{aligned}$$

This shows that $\phi_1^{(a)} > 0$ for $a = 2, 3, \ldots, \ell$. □

We plot the values of $(\phi_m^{(a)})_{m=0\ldots,\ell}$ for $\ell = 5$ and $a = 2, 3, 4, 5$ in Figure 3.5 and 3.6. The underlying graphs are plots of the function

$$\det \begin{bmatrix} 2\cos\frac{\pi a}{\ell+2} & 2\cos\frac{\pi a(u+1)}{\ell+2} \\ \sin\frac{\pi(a-1)}{\ell+2}/\sin\frac{\pi}{\ell+2} & \sin\frac{\pi(a-1)(u+1)}{\ell+2}/\sin\frac{\pi(u+1)}{\ell+2} \end{bmatrix}$$

in an interval $0 \le u \le \ell$, and the points on these graphs represent the values of $(\phi_m^{(a)})_{m=0\ldots,\ell}$.

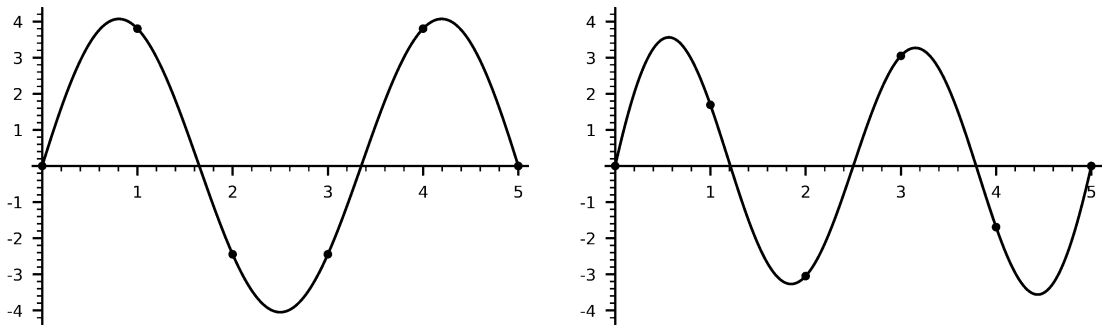


Fig. 3.6 The values of $(\phi_m^{(4)})_{m=0\ldots,\ell}$ and $(\phi_m^{(5)})_{m=0\ldots,\ell}$ for $\ell = 5$.

Corollary 3.4.19. *The exponents of $\tau_\alpha(z)$ for type (A_1, ℓ) are $2, 3, \ldots, \ell$, that is,*

$$\tau_\alpha(z) = \prod_{a=2}^{\ell} (z - e^{\frac{2\pi i a}{\ell+2}}).$$

In particular, Conjecture 3.4.9 is true in this case.

Proof. From Theorem 3.4.18, we find that $e^{\frac{2\pi i a}{\ell+2}}$ for $a = 2, 3, \ldots, \ell$ are eigenvalues of L, and thus of $d\mu_\eta$. These are all the eigenvalues and their multiplicities are one, since the size of the matrix L is $\ell - 1$. The assertion now follows by Proposition 3.4.3. □

$(A_n, 2)$ case

The following Lemma shows that the right-hand side of the conjectural formula (3.4.12) for $(A_n, 2)$ is the same as that for (A_1, ℓ) if we change the parameter as $n \leftrightarrow \ell - 1$. Such a phenomenon is known as the *level-rank duality*.

Lemma 3.4.20. *The right-hand side in (3.4.12) is given by*

$$\frac{N_{A_n,2}(z)}{D_{A_n,2}(z)} = \prod_{a=2}^{n+1} (z - e^{\frac{2\pi i a}{n+3}}).$$

Proof. Because $h^\vee = n + 1$, we obtain

$$N_{A_n,2}(z) = \left(\frac{z^{n+3} - 1}{z - 1}\right)^n,$$

and

$$D_{A_n,2}(z) = \prod_{\alpha \in \Delta} (z - e^{\frac{2\pi i (\rho \mid \alpha)}{n+3}}).$$

We can easily see that, say using a concrete realization of the root system of type A_n, the following holds:

$$\#\{\alpha \in \Delta_+ \mid (\rho \mid \alpha) = a\} = \begin{cases} n + 1 - a & \text{if } 1 \le a \le n + 1, \\ 0 & \text{if } a = n + 2. \end{cases}$$

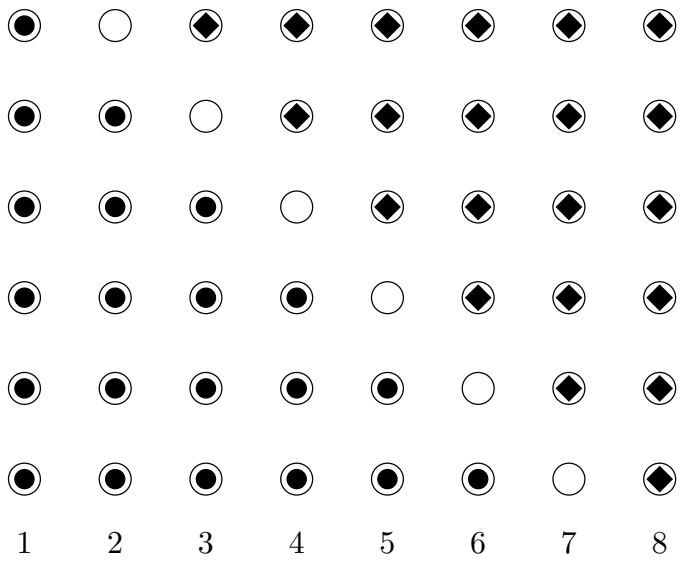


Fig. 3.7 The exponents of $N_{A_n,2}$ and $D_{A_n,2}$ for $n = 6$. The meanings of the symbols are the same as in Figure 3.2.

Thus for any $1 \leq a \leq n+2$, we obtain

$$\begin{aligned}
&\#\{\alpha \in \Delta_+ \mid (\rho \mid \alpha) = a \mod n+3\} \\
&= \#\{\alpha \in \Delta_+ \mid (\rho \mid \alpha) = a\} + \#\{\alpha \in \Delta_+ \mid (\rho \mid -\alpha) = a-n-3\} \\
&= \begin{cases} n & \text{if } a = 1, \\ n-1 & \text{if } 2 \leq a \leq n+1, \\ n & \text{if } a = n+2, \end{cases}
\end{aligned}$$

and this implies that

$$D_{A_n,2}(z) \prod_{a=2}^{n+1} (x - e^{\frac{2\pi i a}{n+3}}) = N_{A_n,2}(z),$$

completing the proof. Figure 3.7 illustrates these calculations. □

We now complete the proof of Theorem 3.4.12.

Corollary 3.4.21. *Conjecture 3.4.9 is true for* $(A_n, 2)$.

Proof. The assertion follows from Proposition 3.4.8 and Corollary 3.4.19. □

3.5 Relationships with affine Lie algebras

3.5.1 Partition q-series and fermionic formulas

First we study Example 3.3.8 in a little bit more detail, and give some examples. In this section, we will for simplicity assume that $\hat{\mathfrak{g}}$ is of untwisted type, that is, of type $X_n^{(1)}$.

Fix a pair (X_n, ℓ) as in Section 2.3.3 and 3.4.1, and let α be the T-datum associated with this pair. Recall the notations α_a and t_a in these sections.

Let Q be the free abelian group defined by

$$Q = \bigoplus_{a=1}^{n} \mathbb{Z}\alpha_a. \tag{3.5.1}$$

The free abelian group Q is called a root lattice. Let M be the free abelian subgroup of Q defined by

$$M = \bigoplus_{a=1}^{n} \mathbb{Z}t_a\alpha_a. \tag{3.5.2}$$

Let H be the index set given by

$$H = \{(a, m) \mid 1 \leq a \leq n, 1 \leq m \leq t_a\ell - 1\}.$$

For any element $u \in \mathbb{Z}^H$, let $u_m^{(a)}$ denote the (a, m)-entry of u.

Recall the abelian groups H_α, H'_α, and S_α in Section 3.3.

Lemma 3.5.1. *The group homomorphism*

$$\bar{F} : H_\alpha \longrightarrow Q$$

defined by

$$\bar{F}(u, v) = \sum_{(a,m)\in H} m u_m^{(a)} \alpha_a$$

is surjective, and the kernel of $\iota \circ \bar{F}$ is equal to H'_α, where $\iota : Q \to Q/\ell M$ is the projection. Thus we obtain the group isomorphism

$$F : S_\alpha \longrightarrow Q/\ell M. \tag{3.5.3}$$

Proof. Since $\mathring{A}_+$ is non-singular, the abelian group Z_γ is given by

$$H'_\alpha = \{(u, K^{-1}u) \mid u \in \mathbb{Z}^H\}.$$

Thus the map $\bar{F}$ is surjective since

$$\bar{F}(u, v) = \sum_{a=1}^{n} u_1^{(a)} \alpha_a$$

if $u_m^{(a)} = 0$ for all $m > 1$.

For any element $w \in \mathbb{Z}^H$, we compute

$$\begin{aligned}
&\bar{F}((\mathring{A}_+^\vee)^{\mathsf{T}} w, (\mathring{A}_-^\vee)^{\mathsf{T}} w) \\
&= \sum_{(a,m)\in H} m \left(\sum_{(b,k)\in H} \delta_{ab} \bar{C}^a_{mk} w_k^{(b)} \right) \alpha_a \\
&= \sum_{a=1}^{n} \left(\sum_{m=1}^{t_a\ell-1} 2m w_m^{(a)} - \sum_{m=1}^{t_a\ell-2} (m+1) w_m^{(a)} - \sum_{m=2}^{t_a\ell-1} (m-1) w_m^{(a)} \right) \alpha_a \\
&= \sum_{a=1}^{n} t_a \ell w^{(a)}_{t_a\ell-1} \alpha_a,
\end{aligned}$$

and this shows that $B_\gamma = \ker(\iota \circ \bar{F})$ as desired. □

Note that $K^\vee = K$ since $\alpha^\vee = \alpha$. Moreover, it is known that the positive symmetric matrix K is expressed as

$$K_{ab}^{mk} = \left(\min(t_b m, t_a k) - \frac{mk}{\ell} \right) (\alpha_a \mid \alpha_b)$$

where K_{ab}^{mk} is the $((a,m),(b,k))$-th entries of K (see [KNS11, (14.36)]).

From Lemma 3.5.1, we obtain the following result.

Theorem 3.5.2. *The partition q-series of α at $\sigma \in S_\alpha$ is given by*

$$\mathcal{Z}_{\alpha,\sigma}(q) = \sum_{u \in (\mathbb{Z}_{\geq 0})^H, (\lozenge)} \frac{q^{\frac{1}{2} u^{\mathsf{T}} K u}}{\prod_{(a,m)\in H} (q)_{u_m^{(a)}}},$$

where the sum runs over $u \in (\mathbb{Z}_{\geq 0})^H$ under the condition

$$\sum_{(a,m)\in H} m u_m^{(a)} \alpha_a \equiv \lambda \mod \ell M, \tag{$\lozenge$}$$

where λ is a representative of $F(\sigma)$.

Corollary 3.5.3. *The total partition q-series of α is given by*

$$\mathcal{Z}_\alpha(q) = \sum_{u \in (\mathbb{Z}_{\geq 0})^H} \frac{q^{\frac{1}{2} u^{\mathsf{T}} K u}}{\prod_{(a,m)\in H} (q)_{u_m^{(a)}}}.$$

The expression of the q-series $\mathcal{Z}_{\alpha,\sigma}(q)$ in Theorem 3.5.2 is called a fermionic formula [HKO+02]. See also Example 3.3.8.

Example 3.5.4. Let $(X_n, \ell) = (A_3, 3)$. The index set H is given by

$$H = \{(1,1), (1,2), (2,1), (2,2), (3,1), (3,2)\}.$$

The T-datum is given by

$$A_+ = \begin{bmatrix} 1+z^2 & -z & 0 & 0 & 0 & 0 \\ -z & 1+z^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1+z^2 & -z & 0 & 0 \\ 0 & 0 & -z & 1+z^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+z^2 & -z \\ 0 & 0 & 0 & 0 & -z & 1+z^2 \end{bmatrix},$$

$$A_- = \begin{bmatrix} 1+z^2 & 0 & -z & 0 & 0 & 0 \\ 0 & 1+z^2 & 0 & -z & 0 & 0 \\ -z & 0 & 1+z^2 & 0 & -z & 0 \\ 0 & -z & 0 & 1+z^2 & 0 & -z \\ 0 & 0 & -z & 0 & 1+z^2 & 0 \\ 0 & 0 & 0 & -z & 0 & 1+z^2 \end{bmatrix}.$$

The partition q-series are parametrized by elements of the set

$$S_\alpha = \{(a,0,b,0,c,0) \mid 0 \le a,b,c, \le 2\},$$

where $(u,v)+H'_\alpha \in H_\alpha$ is denoted by $(u_1^{(1)}, u_1^{(2)}, u_1^{(3)}, u_2^{(3)}, u_3^{(3)})$. Let us write $(a,0,b,0,c,0)$ more simply as abc. Then we have

$$\begin{aligned} S_\alpha = \{&000, 100, 200, 010, 110, 210, 020, 120, 220, \\ &001, 101, 201, 011, 111, 211, 021, 121, 221, \\ &002, 102, 202, 012, 112, 212, 022, 122, 222\}. \end{aligned}$$

We exhibit several low order terms of the partition q-series:

$$\mathcal{Z}_\gamma^\sigma(q) = 1 + 6q^2 + 20q^3 + 54q^4 + 144q^5 + 360q^6 + 804q^7 + O(q^8)$$

if $\sigma = 000$,

$$q^{\frac{2}{3}} + 3q^{\frac{5}{3}} + 13q^{\frac{8}{3}} + 38q^{\frac{11}{3}} + 108q^{\frac{14}{3}} + 264q^{\frac{17}{3}} + 622q^{\frac{20}{3}} + 1364q^{\frac{23}{3}} + O(q^{\frac{26}{3}})$$

if $\sigma = 100, 200, 010, 110, 020, 220, 001, 011, 111, 002, 022, 222$,

$$q + 6q^2 + 18q^3 + 56q^4 + 144q^5 + 357q^6 + 808q^7 + 1767q^8 + O(q^9)$$

if $\sigma = 210, 120, 211, 021, 221, 012, 112, 122$, and

$$2q^{\frac{4}{3}} + 8q^{\frac{7}{3}} + 28q^{\frac{10}{3}} + 76q^{\frac{13}{3}} + 199q^{\frac{16}{3}} + 468q^{\frac{19}{3}} + 1060q^{\frac{22}{3}} + 2256q^{\frac{25}{3}} + O(q^{\frac{28}{3}})$$

if $\sigma = 101, 201, 121, 102, 202, 212$.

Example 3.5.5. Let $(X_n, \ell) = (B_3, 2)$. The index set H is given by

$$H = \{(1,1), (2,1), (3,1), (3,2), (3,3)\}.$$

The T-datum is given by

$$A_+ = \begin{bmatrix} 1+z^4 & 0 & 0 & 0 & 0 \\ 0 & 1+z^4 & 0 & 0 & 0 \\ 0 & 0 & 1+z^2 & -z & 0 \\ 0 & 0 & -z & 1+z^2 & -z \\ 0 & 0 & 0 & -z & 1+z^2 \end{bmatrix},$$

$$A_- = \begin{bmatrix} 1+z^4 & -z^2 & 0 & 0 & 0 \\ -z^2 & 1+z^4 & -z^2 & -z-z^3 & -z^2 \\ 0 & 0 & 1+z^2 & 0 & 0 \\ 0 & -z & 0 & 1+z^2 & 0 \\ 0 & 0 & 0 & 0 & 1+z^2 \end{bmatrix}.$$

The partition q-series are parametrized by elements of the set

$$S_\alpha = \{(a,b,c,0,0) \mid 0 \le a,b \le 1, 0 \le c \le 3\},$$

where $(u,v) + Z_\gamma \in H_\gamma$ is denoted by $(u_1^{(1)}, u_1^{(2)}, u_1^{(3)}, u_2^{(3)}, u_3^{(3)})$. Let us write $(a,b,c,0,0)$ more simply as abc. Then we have

$$\begin{aligned} S_\alpha = \{&000, 100, 010, 110, 001, 101, 011, 111, \\ &002, 102, 012, 112, 003, 103, 013, 113\} \end{aligned}$$

We exhibit several low order terms of the partition q-series:

$$\mathcal{Z}_\gamma^\sigma(q) = 1 + 9q^2 + 21q^3 + 66q^4 + 144q^5 + 349q^6 + 723q^7 + O(q^8)$$

if $\sigma = 000$,

$$q^{\frac{1}{2}} + 4q^{\frac{3}{2}} + 13q^{\frac{5}{2}} + 38q^{\frac{7}{2}} + 97q^{\frac{9}{2}} + 228q^{\frac{11}{2}} + 504q^{\frac{13}{2}} + 1057q^{\frac{15}{2}} + O(q^{\frac{17}{2}})$$

if $\sigma = 100, 010, 110, 102, 012, 112$,

$$q^{\frac{3}{4}} + 5q^{\frac{7}{4}} + 17q^{\frac{11}{4}} + 48q^{\frac{15}{4}} + 120q^{\frac{19}{4}} + 279q^{\frac{23}{4}} + 608q^{\frac{27}{4}} + 1261q^{\frac{31}{4}} + O(q^{\frac{35}{4}})$$

if $\sigma = 001, 011, 111, 003, 013, 113$,

$$3q^{\frac{5}{4}} + 9q^{\frac{9}{4}} + 30q^{\frac{13}{4}} + 75q^{\frac{17}{4}} + 187q^{\frac{21}{4}} + 411q^{\frac{25}{4}} + 885q^{\frac{29}{4}} + 1783q^{\frac{33}{4}} + O(q^{\frac{37}{4}})$$

if $\sigma = 101, 103$, and

$$3q + 6q^2 + 25q^3 + 57q^4 + 156q^5 + 334q^6 + 744q^7 + 1491q^8 + O(q^9)$$

if $\sigma = 002$.

3.5.2 Affine Lie algebras

In this section, we review basic concepts of affine Lie algebras and their integrable highest weight modules. See [Kac90] for more detail. Let X_n (= A_n, B_n, C_n, D_n, $E_{6,7,8}$, F_4 or G_2) be a finite type Dynkin diagram, and $\mathfrak{g}$ be the finite dimensional simple Lie algebra of type X_n over $\mathbb{C}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and Δ be the set of roots. We also use the notations on root systems that we used in previous sections. We extend the inner product $(\cdot \mid \cdot)$ to the nondegenerate symmetric bilinear form on $\mathfrak{h}^*$. We define the following two free abelian groups:

$$Q = \bigoplus_{a=1}^{n} \mathbb{Z}\alpha_a, \quad M = \bigoplus_{a=1}^{n} \mathbb{Z}t_a\alpha_a.$$

We also define the following sets:

$$P = \left\{ \Lambda \in \mathfrak{h}^* \;\middle|\; \frac{2(\Lambda \mid \alpha_a)}{(\alpha_a \mid \alpha_a)} \in \mathbb{Z} \text{ for all } a = 1, \dots, r \right\},$$
$$P_+ = \left\{ \Lambda \in P \;\middle|\; \frac{2(\Lambda \mid \alpha_a)}{(\alpha_a \mid \alpha_a)} \geq 0 \text{ for all } a = 1, \dots, r \right\}.$$

Let

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$$

be the affine Lie algebra associated with $\mathfrak{g}$. The Lie bracket on $\hat{\mathfrak{g}}$ is defined as

$$[X \otimes t^m, Y \otimes t^n] = [X, Y] \otimes t^{m+n} + m\delta_{m+n,0} \cdot (X \mid Y)K,$$
$$[K, \hat{\mathfrak{g}}] = \{0\},$$
$$[d, X \otimes t^n] = nX \otimes t^n.$$

Fix a non-negative integer ℓ. Let us define the following set:

$$P_+^\ell = \{\Lambda \in P_+ \mid (\Lambda \mid \theta) \leq \ell\},$$

where θ is the highest root in Δ. For any $\Lambda \in P_+^\ell$, there is the unique level ℓ integrable highest weight $\hat{\mathfrak{g}}$-module such that the classical part of its highest weight is Λ, which is denoted by $L(\Lambda)$.

We define the following rational numbers associated with $L(\Lambda)$:

$$c(\ell) = \frac{\ell \dim \mathfrak{g}}{\ell + h^\vee},$$
$$h_\Lambda = \frac{(\Lambda \mid \Lambda + 2\rho)}{2(\ell + h^\vee)}.$$

Let $q = e^{2\pi i\tau}$. For any diagonalizable linear map $\alpha : V \to V$ with eigenvalues $\lambda_1, \lambda_2, \ldots$ with finite multiplicities $m_1, m_2, \ldots$, we define the trace $\mathrm{tr}_V q^\alpha$ by $\mathrm{tr}_V q^\alpha = \sum_i m_i q^{\lambda_i}$. We define the following two functions as in [KW88]:

$$\chi_\Lambda(\tau) = q^{-\frac{c(\ell)}{24}} \mathrm{tr}_{L(\Lambda)} q^{h_\Lambda - d}, \tag{3.5.4}$$

$$b^\Lambda_\lambda(\tau) = q^{-\frac{c(\ell)-r}{24}} \mathrm{tr}_{U(\Lambda,\lambda)} q^{h_\Lambda - \frac{(\lambda|\lambda)}{2\ell} - d}, \tag{3.5.5}$$

where $\lambda \in \mathfrak{h}^*$ and

$$U(\Lambda, \lambda) = \{v \in L(\Lambda) \mid (h \otimes t^n)v = \delta_{n,0}\lambda(h)v \text{ for all } h \in \mathfrak{h}, n \geq 0\}.$$

These series converge to holomorphic functions on the upper half-plane $\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}\,\tau > 0\}$. The function $\chi_\Lambda(\tau)$ is called the (specialized) *normalized character*[*1] of $L(\Lambda)$, and $b^\Lambda_\lambda(\tau)$ is called the *branching function* for the pair $(\mathfrak{g}, \mathfrak{h})$. We will simply call $b^\Lambda_\lambda(\tau)$ as the branching function. Note that dividing this branching function by the r-th power of the Dedekind eta function yields the *string function* in [KP84].

The asymptotics of the normalized character and the branching function were studied in detailed in [KP84, KW88]. Let $\tau \searrow 0$ denote the limit in the positive imaginary axis.

Theorem 3.5.6 ([KP84, KW88]). *Suppose that $\lambda \in \Lambda + Q$. Then*

$$\lim_{\tau \searrow 0} \chi_\Lambda(\tau) e^{-\frac{\pi i c(\ell)}{12\tau}} = a(\Lambda), \tag{3.5.6}$$

$$\lim_{\tau \searrow 0} b^\Lambda_\lambda(\tau) e^{-\frac{\pi i (c(\ell)-r)}{12\tau}} = |P/Q|^{\frac{1}{2}} |Q/\ell M|^{-\frac{1}{2}} a(\Lambda), \tag{3.5.7}$$

where $a(\Lambda)$ is the real number defined by

$$a(\Lambda) = |P/(\ell + h^\vee)M|^{-\frac{1}{2}} \prod_{\alpha \in \Delta_+} 2 \sin \frac{\pi(\Lambda + \rho \mid \alpha)}{\ell + h^\vee}. \tag{3.5.8}$$

The real number $a(\Lambda)$ is called the *asymptotic dimension* of $L(\Lambda)$ for the following reason. The module $L(\Lambda)$ is infinite dimensional unless Λ is trivial, and $\chi_\Lambda(0) = \dim L(\Lambda) = \infty$. However, (3.5.6) says that by multiplying by the appropriate exponential term and taking the limit, we can get the finite number $a(\Lambda)$. Therefore, we can think that the real number $a(\Lambda)$ is the "dimension" of $L(\Lambda)$.

3.5.3 Exponents and asymptotic dimension

The branching functions with $\Lambda = 0$ are expected to coincide with the partition q-series that we studied in Section 3.5.1 via the conjectural formula of the branching functions in [KNS93]. This was first observed in [KT15] for the total partition q-series with $X = ADE$. Let $b^\Lambda_\lambda(q)$ denote the formal q-series defined by the right-hand side of (3.5.5).

[*1] More precisely, $\chi_\Lambda(\tau)$ is the normalized character in [KW88] at $z = t = 0$.

Conjecture 3.5.7 ([KNS93]). *Suppose that $\ell \geq 2$. Let α be the T-datum associated with the pair (X_n, ℓ) as in Section 3.5.1. Suppose that $\sigma \in S_\alpha$, and $\lambda \in Q$ is a representative of $F(\sigma)$, where F is defined as (3.5.3). Then*

$$q^{-\frac{c(\ell)-r}{24}} \mathcal{Z}_{\alpha,\sigma}(q) = b^0_\lambda(q). \tag{3.5.9}$$

By summing (3.5.9) for all elements in $Q/\ell M$, we obtain the following conjectural identity on the total partition q-series:

$$q^{-\frac{c(\ell)-r}{24}} \mathcal{Z}_\alpha(q) = \sum_{\lambda \in Q/\ell M} b^0_\lambda(q). \tag{3.5.10}$$

By comparing the asymptotics in Proposition 3.3.2 and (3.5.7), and noting that $\det \mathring{A}_+ = |Q/\ell M|$, we find that (3.5.10) yields the following identities:

$$\frac{6}{\pi^2} \sum_{a=1}^{n} \sum_{m=1}^{t_a \ell - 1} L(f_m^{(a)}) = c(\ell) - n, \tag{3.5.11}$$

and

$$\frac{1}{\sqrt{\tau_\alpha(1)}} = |P/Q|^{\frac{1}{2}} a(0), \tag{3.5.12}$$

where $f_m^{(a)}$ is a positive real solution of the equation

$$f_m^{(a)} = \prod_{b=1}^{n} \prod_{k=1}^{t_a \ell - 1} (1 - f_k^{(b)})^{K_{ab}^{mk}} \quad (1 \leq a \leq n, 1 \leq m \leq t_a \ell - 1).$$

The identity (3.5.11) is called the dilogarithm identity in conformal field theories, and proved in [Nak11a, IIK+13a, IIK+13b] by using cluster algebras. On the other hand, the identity (3.5.12) is exactly Conjecture 3.4.9 at $z = 1$ because

$$\begin{aligned} \frac{N_{X_n,\ell}(1)}{D_{X_n,\ell}(1)} &= \prod_{a=1}^{r} t_a (\ell + h^\vee) \left(\prod_{\alpha \in \Delta_+} 2 \sin \frac{\pi(\rho \mid \alpha)}{\ell + h^\vee} \right)^{-2} \\ &= |Q/(\ell + h^\vee)M| \left(\prod_{\alpha \in \Delta_+} 2 \sin \frac{\pi(\rho \mid \alpha)}{\ell + h^\vee} \right)^{-2} \\ &= |P/Q|^{-1} a(0)^{-2}. \end{aligned}$$

Consequently, the conjectural identity (3.5.12) gives us a consistency between our conjecture on exponents (Conjecture 3.4.9) and the known conjecture on q-series (Conjecture 3.5.7), and also gives an interesting connection between the theory of cluster algebras and the representation theory of affine Lie algebras.

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