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Study on the large time behavior of solutions of compressible viscoelastic system

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Abstract

This thesis is concerned with a mathematical analysis of nonlinear system describing a motion of compressible viscoelastic fluid. We investigate the large time behavior of solutions around a motionless state or parallel flows. We first estabilish the L^p decay estimates of solutions for $1 < p \leq \infty$, provided that the initial data is sufficiently close to the motionless state in the whole space. In addition, we clarify the diffusion wave phenomena caused by the sound waves and the elastic shear waves. We next show that if the initial perturbation is sufficiently small, the parallel flow and the time-periodic parallel flow are asymptotically stable, provided that the Reynolds and the Mach numbers are small and the propagation speed of the shear wave is large.

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Supplement

1. Results of Chapter 5 were obtained in a joint research with Ayaka Haruki, and written in [7].

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Chapter 1

Introduction

In this thesis, we consider the compressible viscoelastic system

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho v) = 0, \\ \rho(\partial_t v + v \cdot \nabla v) - \nu \Delta v - (\nu + \nu')\nabla \mathrm{div} v + \nabla p(\rho) = \beta^2 \mathrm{div}(\rho F^\top F) + \rho g, \\ \partial_t F + v \cdot \nabla F = (\nabla v)F. \end{cases} \tag{1.1}$$

Here the superscript ${}^\top\cdot$ stands for the transposition; $\rho = \rho(x,t)$, $v = {}^\top(v^1(x,t), v^2(x,t), v^3(x,t))$, and $F = (F^{jk}(x,t))_{1\le j,k\le 3}$ are the unknown density, velocity field, and deformation tensor, respectively, at the time $t \ge 0$ and $x \in \Omega$; $p = p(\rho)$ is the given pressure ; ν and ν' are the viscosity coefficients satisfying

$$\nu > 0,\ 2\nu + 3\nu' \ge 0;$$

g is an given external force; $\beta > 0$ is the strength of the elasticity. We assume that $p'(1) > 0$, and we denote $\gamma \equiv \sqrt{p'(1)}$. If we set $\beta = 0$, the system (1.1) settles into the compressible Navier-Stokes equation:

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho v) = 0, \\ \rho(\partial_t v + v \cdot \nabla v) - \nu \Delta v - (\nu + \nu')\nabla \mathrm{div} v + \nabla p(\rho) = \rho g. \end{cases} \tag{1.2}$$

The system (1.1) is considered in a domain $D \subset \mathbb{R}^3$ with two cases; a whole space $D = \mathbb{R}^3$ or an infinite layer $D = \Omega$:

$$\Omega = \{x = (x', x_3);\ x' = (x_1, x_2) \in \mathbb{R}^2,\ 0 < x_3 < 1\}.$$

under the initial condition

$$(\rho, v, F)|_{t=0} = (\rho_0, v_0, F_0). \tag{1.3}$$

We also impose the following conditions

$$\mathrm{div}(\rho_0 {}^\top F_0) = 0, \tag{1.4}$$

$$\rho_0 \mathrm{det} F_0 = 1, \tag{1.5}$$

$$\sum_{m=1}^{3} (F_0^{ml} \partial_{x_m} F_0^{jk} - F_0^{mk} \partial_{x_m} F_0^{jl}) = 0, \ \ j,k,l = 1,2,3. \tag{1.6}$$

According to [11, Appendix A] and [28, Proposition 1], the conditions (1.5) and (1.6) are invariant for $t \geq 0$:

$$\rho \mathrm{det} F = 1, \tag{1.7}$$

$$\sum_{m=1}^{3} (F^{ml} \partial_{x_m} F^{jk} - F^{mk} \partial_{x_m} F^{jl}) = 0, \ \ j,k,l = 1,2,3. \tag{1.8}$$

Here the constraint (1.7) stands for the compressibility of fluid and the constraint (1.8) originates in a certain symmetric property of the first order derivatives of F in the Lagrangian coordinates. Furthermore, it is proved in [13, Appendix A] and [14, Appendix A] that the constraints (1.7) and (1.8) imply that the condition (1.6) is invariant for $t \geq 0$:

$$\mathrm{div}(\rho {}^\top F) = 0. \tag{1.9}$$

In the case of the infinite layer $D = \Omega$, we impose the non-slip boundary condition for v;

$$v|_{x_3=0,1} = 0. \tag{1.10}$$

As for g, we consider the following two cases;

$$g = g^1(x_3, t) e_1, \ \ e_1 = {}^\top(1,0,0), \ \ g^1(0,t) = g^1(1,t) = 0. \tag{1.11}$$

$$g = g^2(x_3, t) e_1, \ \ g^2(0,t) = g^2(1,t) = 0. \tag{1.12}$$

Here g^1 is a given smooth function of (x_3, t) converging to $g^1_\infty = g^1_\infty(x_3) \neq 0$ as t goes to infinity; g^2 is a T-periodic function of time t, where $T > 0$ is a constant. Under a suitable condition on g with (1.11), we see that there exists a parallel flow $\bar{u}_p = (\bar{\rho}, \bar{v}, \bar{F})$ of the problem (1.1)–(1.6) and (1.10) with the following properties:

$$\bar{\rho} = 1, \ \ \bar{v} = \bar{v}^1(x_3, t) e_1, \ \ \bar{F} = (\nabla(x - \bar{\psi}^1 e_1))^{-1}.$$

Here $\bar{\psi}^1(x_3,t) = \int_0^t \bar{v}^1(x_3,s)\mathrm{d}s$.

If g is assumed to have the form (1.12), the system (1.1) with (1.7)–(1.10) has a T-periodic solution $\bar{u}_T = (\bar{\rho}_T, \bar{v}_T, \bar{F}_T)$ satisfying the following properties:

$$\bar{\rho}_T = 1,\ \bar{v}_T = \bar{v}_T^1(x_3,t)e_1,\ \bar{F}_T = \bar{F}_T(x_3,t) = \left(\nabla(x - \bar{\psi}_T{}^1(x_3,t)e_1)\right)^{-1}.$$

Here $\bar{\psi}_T^1$ is a T-periodic function with respect to t satisfying

$$\partial_t \bar{\psi}_T^1 = \bar{v}_T^1.$$

In the case of the whole space $D = \mathbb{R}^3$, we assume that $g \equiv 0$ and $(\rho, v, F) \to (1, 0, I)$ as $|x| \to \infty$. Here I is the 3×3 identity matrix.

The purpose of this thesis is to study the stability of the motionless state $(1, 0, I)$ in the case of the whole space $D = \mathbb{R}^3$, and the stability of the parallel flows $\bar{u}_p$ and $\bar{u}_T$ in the case of the infinite layer $D = \Omega$.

The system (1.1) is derived by a motion of viscous compressible fluid under the effect of elastic body whose corresponding energy functional is given by $W(F) = \frac{\beta^2}{2}|F|^2$, called the Hookean linear elasticity. Moreover, we can classify the system (1.1) in a quasilinear parabolic-hyperbolic system since the system (1.1) is a composite system of the compressible Navier-Stokes equations and a first order hyperbolic system for F. We refer to [6, 22, 32] for more physical details.

We first recall the works about the case $\beta = 0$. The large time behavior of the solutions around $(\rho, v) = (1, 0)$ has been studied so far. Matsumura and Nishida [24] showed the global existence of the solutions of the problem (1.2)–(1.3) provided that the initial perturbation is sufficiently small in $H^3 \cap L^1$, and derived the decay estimate:

$$\|\nabla^k(\phi(t), m(t))\|_{L^2} \leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}},\ k = 0, 1,$$

where $(\phi, m) = (\rho - 1, \rho v)$. Hoff and Zumbrun [8] established the following L^p $(1 \leq p \leq \infty)$ decay estimates in $\mathbb{R}^n$, $n \geq 2$:

$$\|(\phi(t), m(t))\|_{L^p} \leq \begin{cases} C(1+t)^{-\frac{n}{2}\left(1-\frac{1}{p}\right)-\frac{n-1}{4}\left(1-\frac{2}{p}\right)} L(t), & 1 \leq p < 2, \\ C(1+t)^{-\frac{n}{2}\left(1-\frac{1}{p}\right)}, & 2 \leq p \leq \infty, \end{cases}$$

where $L(t) = \log(1+t)$ when $n = 2$, and $L(t) = 1$ when $n \geq 3$. Furthermore, the authors of [8] derived the following asymptotic property:

$$\left\|\left((\phi(t), m(t)) - \left(0, \mathcal{F}^{-1}\left(e^{-\nu|\xi|^2 t}\hat{\mathcal{P}}(\xi)\hat{m}_0\right)\right)\right)\right\|_{L^p} \leq C(1+t)^{-\frac{n}{2}\left(1-\frac{1}{p}\right)-\frac{n-1}{4}\left(1-\frac{2}{p}\right)}$$

for $2 \le p \le \infty$. Here $\hat{\mathcal{P}}(\xi) = I - \frac{\xi^\top \xi}{|\xi|^2}$, $\xi \in \mathbb{R}^n$. According to [19], the solution of the linearized system is expressed as the sum of two terms, one is the incompressible part given by $\mathcal{F}^{-1}\left(e^{-\nu|\xi|^2 t}\hat{\mathcal{P}}(\xi)\hat{m}_0\right)$ which solves the heat equation, and the other is the convolution of the heat kernel and the fundamental solution of the wave equation, called the diffusion wave. The authors of [8] found that the hyperbolic aspect of the sound waves plays a role of the spreading effect of the wave equation, and the decay rate of the solution becomes slower than the heat kernel when $1 \le p < 2$. On the other hand, if $2 < p \le \infty$, the compressible part of the solution $(\phi(t), m(t)) - \left(0, \mathcal{F}^{-1}\left(e^{-\nu|\xi|^2 t}\hat{\mathcal{P}}(\xi)\hat{m}_0\right)\right)$ tends to 0 faster than the heat kernel. See also [20] for the linearized problem. As for the mathematical study of the stability of parallel flows, Kagei [17] studied the stability of stationary parallel flow, and Brezina and Kagei [2] and Brezina [1] investigated the stability of time-periodic parallel flow.

We next review the related works in the case $\beta > 0$. The local in time existence of the strong solutions of the system (1.1) with (1.3) in the whole space was shown by Hu and Wang [10]. The global in time existence of the strong solutions of the system (1.1) with (1.3) was proved by Hu and Wang [11], Qian and Zhang [28], and Hu and Wu [12], provided that the initial perturbation $(\rho_0 - 1, v_0, F_0 - I)$ is sufficiently small. Hu and Wu [12] also showed that if the initial perturbation $(\rho_0 - 1, v_0, F_0 - I)$ belongs to $L^1(\mathbb{R}^3) \cap H^2(\mathbb{R}^3)$, the L^p decay estimates hold for the case $2 \le p \le 6$:

$$\|u(t)\|_{L^p} \le C(1+t)^{-\frac{3}{2}\left(1-\frac{1}{p}\right)}. \tag{1.13}$$

Here $u(t) = (\phi, w, G) = (\rho - 1, v, F - I)$. Li, Wei and Yao [21, 34] extended the above result to the case $2 \le p \le \infty$, and obtained the decay estimates in L^2 of higher order derivatives:

$$\|\nabla^k(\rho(t) - 1, v(t), F(t) - I)\|_{L^2} \le C(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \; k = 0, 1, \ldots, N-1, \tag{1.14}$$

provided that $u_0 = (\rho_0 - 1, v_0, F_0 - I)$ belongs to H^N, $N \ge 3$, and is small in $L^1 \cap H^3$. This follows from the diffusive aspect of the system (1.1). We also refer to [9, 23, 35] in recent progresses.

The main difficulty arises in the fact that the constraints (1.7)–(1.9) are nonlinear. To bypass the difficulty, Hu and Wu [12] found that the behavior of G is controlled by its skew-symmetric part $G - {}^\top G$ due to the constraint

(1.8). This property leads to the global in time existence theorem. The authors of [12] next used the Helmholtz decomposition of w and the skew-symmetric part of G to derive the decay estimates (1.13) with $2 \leq p \leq 6$ and (1.14) with $N = 2$. However, the decay rates in (1.13) reflect only the parabolic aspect of the system (1.1); it would be desirable to establish decay estimates which reflect the hyperbolic aspect of the system (1.1), which might give the optimal decay rates.

As for the mathematical study of the stability of parallel flows of viscoelastic fluids, the incompressible case was studied by [4]. It was shown in [4] that the parallel flow is exponentially stable. To the best knowledge of the author, there are no results for the stabilty analysis of parallel flows of compressible viscoelastic fluids. Comparing to the case around the motionless state $(1, 0, I)$, it is expected that the dynamics of solutions around parallel flows seems to be more complicated since the additional hyperbolic aspect arises from the advection term.

In Chapter 3 we consider the initial problem in the whole space $D = \mathbb{R}^3$. In view of the results in [8], it is expected that the system (1.1) has the diffusion wave phenomena affected by the sound wave and the elastic shear wave. In fact, let us consider the linearized system around $(1, 0, I)$:

$$\partial_t u + Lu = 0. \tag{1.15}$$

Here L is the linearized operator given by

$$L = \begin{pmatrix} 0 & \mathrm{div} & 0 \\ \gamma^2 \nabla & -\nu\Delta - \tilde{\nu}\nabla \mathrm{div} & -\beta^2 \mathrm{div} \\ 0 & -\nabla & 0 \end{pmatrix}.$$

We then see that the solenoidal part of the velocity $w_s = \mathcal{F}^{-1}(\hat{\mathcal{P}}(\xi)\hat{w})$ satisfies the following linear symmetric parabolic-hyperbolic system:

$$\begin{cases} \partial_t w_s - \nu\Delta w_s - \beta \mathrm{div} \tilde{G}_s = 0, \\ \partial_t \tilde{G}_s - \beta \nabla w_s = 0, \end{cases}$$

where $\tilde{G}_s = \beta \mathcal{F}^{-1}(\hat{\mathcal{P}}(\xi)\hat{G})$, while the complimentary part $w_c = w - w_s$ solves the following strongly damped wave equation:

$$\partial_t^2 w_c - (\beta^2 + \gamma^2)\Delta w_c - (\nu + \tilde{\nu})\partial_t \Delta w_c = 0.$$

In view of [30], the solution of the linearized system (1.15) behaves different to the case $\beta = 0$ ([8, 20]) by the additional hyperbolic aspect arising from the elastic shear wave. As a result, the principal part of the linearized system (1.15) can be regarded as a system of the strongly damped wave equation.

We shall show that if the initial perturbation $u_0 = (\rho_0 - 1, v_0, F_0 - I)$ is sufficiently small in $L^1 \cap H^3$, then the global strong solution satisfies the following L^p decay estimate

$$\|(\rho(t) - 1, v(t), F(t) - I)\|_{L^p} \leq C(1+t)^{-\frac{3}{2}\left(1-\frac{1}{p}\right)-\frac{1}{2}\left(1-\frac{2}{p}\right)},\ 1 < p \leq \infty,\ t \geq 0.$$

This result improves the L^p estimates (1.13) obtained in [12, 21] for $p > 2$.

We give an outline of the proof of the main result of Chapter 3. As we mentioned before, since the constraints (1.7)–(1.9) are nonlinear, straightforward application of the semigroup theory does not work well. To overcome this obstacle, we adopt a material coordinate transform which makes the constraint (1.9) a linear one. We first introduce a displacement vector $\tilde{\psi} = x - X$ as in [29]:

$$\tilde{\psi}(x,t) = x - X(x,t).$$

Here $x = x(X,t)$ is the material coordinate defined under the flow map

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = v(x(X,t),t), \\ x(X,0) = X, \end{cases}$$

and $X = X(x,t)$ denotes the inverse of x. Then we see that F has the form $F - I = \nabla\tilde{\psi} + h(\nabla\tilde{\psi})$. Here $h(\nabla\tilde{\psi})$ is a function satisfying $h(\nabla\tilde{\psi}) = O(|\nabla\tilde{\psi}|^2)$ for $|\nabla\tilde{\psi}| \ll 1$. We next make use of the nonlinear transform $\psi = \tilde{\psi} - (-\Delta)^{-1}\mathrm{div}^\top(\phi\nabla\tilde{\psi} + (1+\phi)h(\nabla\tilde{\psi}))$. It turns out that the constraint (1.9) becomes the linear condition $\phi + \mathrm{tr}(\nabla\psi) = \phi + \mathrm{div}\psi = 0$. Furthermore, the decay estimate of the L^p $(1 < p \leq \infty)$ norm of $u = (\phi, w, G)$ is obtained from $U = (\phi, w, \nabla\psi)$. Consequently, the L^p decay estimate can be obtained by employing the following integral equation

$$U(t) = e^{-tL}U(0) + \int_0^t e^{-(t-s)L}N(U)\mathrm{d}s,$$

where $N(U) = (N_1(U), N_2(U), N_3(U))$ is a nonlinearity satisfying $N_1 + \mathrm{tr}N_3 = 0$. We decompose U into the low-frequency part U_1 and the high-frequency part U_∞. We then apply the linearized analysis to U_1-part, and a variant of the Matsumura-Nishida energy method [25] to U_∞-part to establish the result in the case $2 \leq p \leq \infty$. On the other hand, for $1 < p < 2$, we derive the L^p estimate of $U(t)$ by employing the results in [20, 30]. Since the above mentioned nonlinear transform from $\tilde{\psi}$ to ψ includes the nonlocal operator $(-\Delta)^{-1}$, the case $p = 1$ is excluded here. See Remark 3.6 below.

In Chapter 4 we consider the initial boundary problem in $D = \{x = (x', x_3); x' = (x_1, x_2) \in \Pi_{j=1}^2 \mathbb{T}_{\frac{2\pi}{\alpha_j}},\ 0 < x_3 < 1\}$. Here $\mathbb{T}_{\frac{2\pi}{\alpha_j}} = \mathbb{R}/\left(\frac{2\pi}{\alpha_j}\right)\mathbb{Z}$,

$\alpha_j > 0$, $j = 1, 2$. We first prove that if g has the form with a suitable condition, the stationary parallel flow u_p exists and satisfies the following properties:

$$\|\bar{v}(t)\|_{H^5}^2 \le Ce^{-c_0\kappa t}\left(\|\bar{v}_0\|_{H^5}^2 + O\left(\frac{1}{\nu^2}\right) + O\left(\frac{1}{\kappa\nu^2}\right)\right),$$
$$\|\partial_t\bar{v}(t)\|_{H^3}^2 \le Ce^{-c_0\kappa t}\left(\frac{\beta^4}{\nu^2}\|\bar{v}_0\|_{H^5}^2 + O(1) + O\left(\frac{1}{\kappa}\right)\right),$$
$$\|\bar{F}(t) - \bar{F}_\infty\|_{H^4}^2 \le Ce^{-c_0\kappa t}\left(\frac{1}{\nu^2}\|\bar{v}_0\|_{H^5}^2 + O\left(\frac{1}{\beta^4}\right) + O\left(\frac{1}{\kappa\beta^4}\right)\right),$$

where $\kappa = \min\left\{\nu, \frac{\beta^2}{\nu}\right\}$, $\bar{\psi}_\infty^1 = \beta^{-2}(-\partial_{x_3}^2)^{-1}g_\infty^1$, and $\bar{F}_\infty = (\nabla(x - \bar{\psi}_\infty^1 e_1))^{-1}$. Here $(-\partial_{x_3}^2)^{-1}$ is the inverse of $-\partial_{x_3}^2$ with domain $D(-\partial_{x_3}^2) = H^2(0,1) \cap H_0^1(0,1)$.

We then show that if $\nu \gg 1$, $\gamma \gg 1$, $\beta \gg 1$ and $\|\bar{v}_0\|_{H^5(0,1)}^2 \ll 1$, then the system (1.1) with (1.3)-(1.6) and (1.10) has a unique global solution (ρ, v, F) such that $(\rho, v, F) \in C([0,\infty), H^2(D))$ and $\|(\rho(t), v(t), F(t)) - (1, \bar{v}(t), \bar{F}(t))\|_{H^2} \to 0$ exponentially as $t \to \infty$, provided that $(\rho_0 - 1, v_0 - \bar{v}_0, F_0 - \bar{F}_0) \in H^2(D)$ is sufficiently small. As a result, we have $\|(\rho(t), v(t), F(t)) - (1, 0, \bar{F}_\infty)\|_{H^2} \to 0$ exponentially as $t \to \infty$. We thus see that if $g_\infty = g_\infty^1 e_1 \neq 0$, then the viscoelastic compressible flow converges to the motionless state with nontrivial deformation $\bar{F}_\infty$ due to the elastic force. This is quite in contrast to the case of the usual viscous compressible fluid where nontrivial flows are in general observed when external forces are nontrivial. In fact, in the case of the usual viscous compressible fluid, under the action of $g_\infty = g_\infty^1 e_1 \neq 0$, a parallel flow with non-zero velocity field is stable for sufficiently small perturbations when $\nu \gg 1$ and $\gamma \gg 1$; see [17].

The proof of the main result of Chapter 4 is based on a variant of the Matsumura-Nishida energy method [26] which gives an appropriate a priori estimate of exponential decay type. To establish the a priori estimate, we use the displacement vector ψ. It then follows that F is written in terms of ψ as

$$F = \bar{F} - \bar{F}\nabla\left(\psi - \bar{\psi}^1 e_1\right)\bar{F} + h\left(\nabla(\psi - \bar{\psi}^1 e_1)\right),$$

where h satisfies $h(\nabla(\psi - \bar{\psi}^1(t)e_1)) = O(|\nabla(\psi - \bar{\psi}^1(t)e_1)|^2)$ for $|\nabla(\psi - \bar{\psi}^1(t)e_1)| \ll 1$. By using ψ, the problem for the perturbation is reduced to the one for $u(t) = (\phi(t), w(t), \zeta(t)) = \big(\rho(t) - 1, v(t) - \bar{v}(t), \psi(t) - \bar{\psi}^1(t)e_1\big)$

which takes the following form:

$$\begin{cases} \partial_t\phi + \mathrm{div} w = f^1, \\ \partial_t w - \nu\Delta w - \tilde{\nu}\nabla\mathrm{div} w + \gamma^2\nabla\phi - \beta^2(\Delta\zeta + K_\infty\zeta) = f^2, \\ \partial_t\zeta - w + w^3\partial_{x_3}\bar{\psi}_\infty = f^3, \\ w|_{x_3=0,1} = 0, \quad \zeta|_{x_3=0,1} = 0, \quad (\phi, w, \zeta)|_{t=0} = (\phi_0, w_0, \zeta_0). \end{cases} \tag{1.16}$$

Here $\tilde{\nu} = \nu + \nu'$ and $\bar{\psi}_\infty = \bar{\psi}^1_\infty e_1$; $K_\infty\zeta$ is a linear term of ζ satisfying $\|K_\infty\zeta\|_{L^2} \leq \frac{C}{\beta^2}\|\nabla\zeta\|_{H^1}$; f^j, $j = 1, 2, 3$ are written in a sum of nonlinear terms and linear terms with coefficients including $\bar{v}$ and $\bar{\psi}^1 - \bar{\psi}^1_\infty$ which decay exponentially in t. Applying a variant of the Matsumura-Nishida energy method given in [29] to (1.17) and estimating the interaction between the parallel flow and the perturbation, we establish the estimate:

$$\|u(t)\|^2_{H^2\times H^2\times H^3} + \int_0^t e^{-c_1(t-s)}\|u(s)\|^2_{H^2\times H^3\times H^3}\,\mathrm{d}s \leq Ce^{-c_1 t}\|u_0\|^2_{H^2\times H^2\times H^3},$$

provided that $\nu \gg 1$, $\gamma \gg 1$, $\beta \gg 1$, and the initial perturbation is sufficiently small.

We finally mention one remark. When $g(x_3, t) \equiv g^1_\infty(x_3)e_1$, $g^1_\infty \neq 0$, the stationary parallel flow in this thesis is a solution $\bar{u}_\infty = (1, 0, \bar{F}_\infty(x_3))$, which represents the motionless state with nontrivial deformation given by $\bar{F}_\infty$. When $\beta = 0$, we formally obtain the usual compressible Navier-Stokes equations (1.2). In this case, the system (1.2) has a parallel flow $\bar{u}_s = (1, \bar{v}_s(x_3))$ with $\bar{v}_s(x_3) \neq 0$; and it was shown by Kagei [17] that the parallel flow $\bar{u}_s$ is stable if $\nu \gg 1$ and $\gamma \gg 1$. On the other hand, the main result of this paper shows that the stationary parallel flow $\bar{u}$ in the viscoelastic compressible fluid is stable if $\nu \gg 1$, $\gamma \gg 1$ and $\beta \gg 1$. Namely, the motionless state $\bar{u}_\infty$ with nontrivial deformation is stable if $\beta \gg 1$, while the parallel flow with non-zero velocity field is stable if $\beta = 0$. This leads to an interesting question what happens when β decreases; it should occur some transition to nontrivial flows at some value of β. We will investigate this issue in the future work.

In Chapter 5 we extend the analysis of [16] to the case of time-periodic parallel flows. In contrast to the stationary case in [16], the velocity field is no more motionless even when $\beta \gg 1$. We shall show that the time-periodic parallel flow is exponentially stable under sufficiently small perturbations, if ν, γ and β are assumed to be sufficiently large compared to g^1. We briefly explain the main result of this paper in a more precise way.

Under a suitable condition on g, there exists a time-periodic parallel flow

$u_T = (\bar\rho_T, \bar v_T, \bar F_T)$ of (1.1) satisfying the following properties:

$$\sup_{t\in[0,T]} \|\bar F_T(t) - I\|_{H^5(0,1)} = O\left(\frac{1}{\beta^2}\right), \quad \sup_{t\in[0,T]} \|\bar v_T^1(t)\|_{H^4(0,1)} = O\left(\frac{1}{\nu}\right).$$

We define the periodic cell by D:

$$D = \{x = (x', x_3); x' = (x_1, x_2) \in \Pi_{j=1}^2 \mathbb{T}_{\frac{2\pi}{\alpha_j}},\ 0 < x_3 < 1\}.$$

Here $\mathbb{T}_{\frac{2\pi}{\alpha_j}} = \mathbb{R}/\left(\frac{2\pi}{\alpha_j}\right)\mathbb{Z}$, $\alpha_j > 0$, $j = 1, 2$.

The main result of Chapter 5 of this thesis states that if $\nu \gg 1$, $\gamma \gg 1$ and $\beta \gg 1$, then the system (1.1) with (1.3)–(1.6) and (1.10) has a unique global solution (ρ, v, F) such that $(\rho, v, F) \in C([0,\infty), H^2(D))$ and $\|(\rho(t), v(t), F(t)) - (1, \bar v_T(t), \bar F_T(t))\|_{H^2} \to 0$ exponentially as $t \to \infty$ under the small initial perturbation $(\rho_0 - 1, v_0 - \bar v_T(0), F_0 - \bar F_T(0)) \in H^2(D)$.

The proof of the main result of Chapter 5 is given by a similar argument of Qian [29] which is based on a variant of the Matsumura-Nishida energy method [26]. To establish the a priori estimate, we consider the following problem for the perturbation $u(t) = (\phi(t), w(t), G(t)) = (\rho(t) - 1, v(t) - \bar v_T(t), F(t) - \bar F_T(t))$:

$$\begin{cases} \partial_t\phi + \bar v_T^1\partial_{x_1}\phi + \mathrm{div} w = f_1, \\ \partial_t w + \bar v_T^1\partial_{x_1}w - \nu\Delta w - \tilde\nu\nabla\mathrm{div} w + \gamma^2\nabla\phi - \beta^2\mathrm{div} G \\ +(w^3\partial_{x_3}\bar v_T^1)e_1 + \nu(\phi\partial_{x_3}^2\bar v_T^1)e_1 - \beta^2\mathrm{div}(G^\top\bar E_T) - \beta^2(G^{33}\partial_{x_3}^2\bar\psi_T^1)e_1 = f_2, \\ \partial_t G + \bar v_T^1\partial_{x_1}G - \nabla w - (\nabla w)\bar E_T + w^3\partial_{x_3}^2\bar E_T - (\nabla\bar v_T)G = f_3, \\ \nabla\phi = -\mathrm{div}^\top G + {}^\top\bar E_T\mathrm{div}^\top G + f_4, \\ w|_{x_3=0,1} = 0, \quad (\phi, w, G)|_{t=0} = (\phi_0, w_0, G_0). \end{cases} \tag{1.17}$$

Here $\bar\psi_T = \bar\psi_T^1 e_1$ and $\bar E_T = \bar F_T - I$; f_j, $j = 1,2,3,4$ are nonlinear terms. We then show the following L^2 energy estimate of u:

$$\|u(t)\|_{H^2\times H^2\times H^3}^2 + \int_0^t e^{-c_1(t-s)}\|u(s)\|_{H^2\times H^3\times H^3}^2\,\mathrm{d}s \le Ce^{-c_1 t}\|u_0\|_{H^2\times H^2\times H^3}^2,$$

provided that $\nu \gg 1$, $\gamma \gg 1$, $\beta \gg 1$, and the initial perturbation is sufficiently small. In Chapter 4, it is assumed that the external force g converges to some stationary force g_∞ exponentially as $t \to \infty$, from which it is expected that the time dependence of g can be regarded as a simple perturbation of the stationary force. Indeed, by taking into account the exponential convergence of g, the non-stationary parallel flow $\bar{u}$ is decomposed into the stationary solution with zero velocity field and the exponentially decaying part; and the latter part is treated as a simple perturbation of the former part because of its exponential decay in time; in particular, the velocity field of the parallel flow $\bar{v}$ converges to zero, together with its time derivative $\partial_t \bar{v}$. In contrast to the situation of Chapter 4, in this thesis, the parallel flow $\bar{u}_T$ under consideration is time-periodic, and hence, the time fluctuation of the parallel flow $\bar{u}_T$ remains for all t. This requires a more detail analysis of the interaction between the time-periodic parallel flow and the perturbation.

This thesis is organized as follows. In Chapter 2, we introduce some notations which will be used throughout this thesis. In Chapter 3, we consider the asymptotic behavior of solutions around the motionless state $(1, 0, I)$ in $\mathbb{R}^3$ based on [15]. In Chapter 4, we discuss the stabilty of the parallel flow $\bar{u}_p$ based on [16]. In Chapter 5, we treat the stability of the time-periodic parallel flow $\bar{u}_T$ based on [7].

Chapter 2

Preliminaries

In this chapter, we prepare notations and function spaces which will be used throughout the thesis.

To consider functions $\frac{2\pi}{\alpha_j}$-periodic in $x_j,\ j = 1, 2$ in Chapter 4 and Chapter 5, we set

$$\mathbb{T}_{\frac{2\pi}{\alpha_j}} = \mathbb{R}/\left(\frac{2\pi}{\alpha_j}\right)\mathbb{Z},\ \alpha_j > 0, j = 1, 2.$$

For $1 \le p \le \infty$, we define $L^p(D)$ as the usual Lebesgue space on a domain D, and its norm is denoted by $\|\cdot\|_{L^p(D)}$. Similarly, $W^{m,p}(D)(1 \le p \le \infty, m \in \{0\} \cup \mathbb{N})$ denotes the m-th order L^p Sobolev space on D, and its norm is denoted by $\|\cdot\|_{W^{m,p}(D)}$. We set $H^m(D) = W^{m,2}(D)$ for an integer $m \ge 0$. For simplicity, we set $L^p = L^p(D) \times (L^p(D))^3 \times (L^p(D))^9$ (resp. $H^m = H^m(D) \times (H^m(D))^3 \times (H^m(D))^9$) when $D = \Pi_{j=1}^2 \mathbb{T}_{\frac{2\pi}{\alpha_j}} \times (0, 1)$ or $D = \mathbb{R}^3$. $H_0^1(D)$ denotes the completion of $C_0^\infty(D)$ in $H^1(D)$, where

$$C_0^\infty(D) := \{f \in C^\infty |\ \mathrm{supp}(f) : \text{compact in } D\}.$$

.

The inner product of $L^2(D)$ is denoted by

$$(f, g) := \int_D f(x)\overline{g(x)}dx,\ f, g \in L^2(D).$$

Here the symbol $\bar{\cdot}$ stands for its complex conjugate. The partial derivatives of a function u in x_j $(j = 1, 2, 3)$ and t are denoted by $\partial_{x_j} u$ and $\partial_t u$, respectively. Δ denotes the usual Laplacian with respect to x. For a multiindex $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in (\{0\} \cup \mathbb{N})^3$ and $\xi = {}^\top(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, we define ∂_x^α and ξ^α as $\partial_x^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}$ and $\xi^\alpha = \xi_1^{\alpha_1}\xi_2^{\alpha_2}\xi_3^{\alpha_3}$, respectively. For a function u and a nonnegative integer k, $\nabla^k u$ stands for $\nabla^k u = \{\partial_x^\alpha u \mid |\alpha| = k\}$.

For a scalar valued function $\rho = \rho(x)$, we define $\nabla\rho$ by its gradient with respect to x. For a vector valued function $w = w(x) = {}^\top(w^1(x), w^2(x), w^3(x))$, we define $\mathrm{div} w$ and $(\nabla w)^{jk} = (\partial_{x_k} w^j)$ as its divergence and Jacobian matrix with respect to x, respectively. When $D = \Pi_{j=1}^2 \mathbb{T}_{\frac{2\pi}{\alpha_j}} \times (0,1)$, we write m-th order tangential derivatives of u as $\partial^m u = \{\partial_x^\alpha u \mid |\alpha| = m, \alpha_3 = 0\}$ and we abbreviate ∂^1 to ∂. For a vector field $w = {}^\top(w^1, w^2, w^3)$ and a function u, we define $\nabla' \cdot w'$ and $\Delta' u$ as

$$\nabla' \cdot w' = \partial_{x_1} w^1 + \partial_{x_2} w^2$$

and

$$\Delta' u = \partial_{x_1}^2 u + \partial_{x_2}^2 u,$$

respectively. For a 3×3-matrix valued function $F = F(x) = (F^{jk}(x))$, we define its divergence $\mathrm{div} F$ and trace $\mathrm{tr} F$ as $(\mathrm{div} F)^j = \sum_{k=1}^3 \partial_{x_k} F^{jk}$ and $\mathrm{tr} F = \sum_{k=1}^3 F^{kk}$, respectively. For matrix-valued functions $F = (F^{jk})_{1 \le j,k \le 3}$ and $G = (G^{jk})_{1 \le j,k \le 3}$, we denote $F\nabla G$ by

$$(F\nabla G)^j = (\mathrm{div}(G^\top F) - G(\mathrm{div}{}^\top F))^j = \sum_{k,l=1}^3 F^{lk} \partial_{x_l} G^{jk}.$$

For functions $f = f(x)$ and $g = g(x)$, we denote the convolution of f and g by $f * g$:

$$(f * g)(x) = \int_{\mathbb{R}^3} f(x-y) g(y) \mathrm{d}y.$$

We denote the Fourier transform of a function $f = f(x)$ by $\hat{f}$ or $\mathcal{F} f$:

$$\hat{f}(\xi) = (\mathcal{F} f)(\xi) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} f(x) e^{-i\xi \cdot x} \mathrm{d}x \ (\xi \in \mathbb{R}^3),$$

where i is the imaginary unit. The Fourier inverse transform is denoted by $\mathcal{F}^{-1}$:

$$(\mathcal{F}^{-1} f)(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} f(\xi) e^{i\xi \cdot x} \mathrm{d}\xi \ (x \in \mathbb{R}^3).$$

We recall the Sobolev inequalities.

Lemma 2.1. *The following inequalities hold:*

$$\text{(i) } \|u\|_{L^\infty(0,1)} \le C\|u\|_{H^1(0,1)} \text{ for } u \in H^1(0,1).$$
$$\text{(ii) } \|u\|_{L^p} \le C\|u\|_{H^1} \text{ for } 2 \le p \le 6,\ u \in H^1.$$
$$\text{(iii) } \|u\|_{L^p} \le C\|u\|_{H^2} \text{ for } 2 \le p \le \infty,\ u \in H^2.$$

To estimate higher derivatives of solutions in Chapter 4 and Chapter 5, we prepare the following lemmata.

Lemma 2.2. *Let $Q_1 \in H^k(\Omega)$ and $Q_2 \in H^{k+1}(\Omega)$ with $\int_\Omega Q_1 \mathrm{d}x = 0$. If (u,p) satisfies the Stokes system*

$$\begin{cases} \mathrm{div} u = Q_1 \text{ in } \Omega, \\ -\Delta u + \nabla p = Q_2 \text{ in } \Omega, \\ u = 0 \text{ on } \{x_3 = 0, 1\}, \end{cases}$$

then there exists a positive constant C independent of (u,p) such that

$$\|\nabla^{k+2}u\|_{L^2} + \|\nabla^{k+1}p\|_{L^2} \le C(\|Q_1\|_{H^{k+1}} + \|Q_2\|_{H^k} + \|\nabla u\|_{L^2}). \tag{2.1}$$

Lemma 2.3. *Let $f \in H^1(D)$ satisfy the Poincaré type inequality $\|f\|_{L^2(D)} \le C\|\nabla f\|_{L^2(D)}$, and $g \in L^2(0,1)$. There holds the following inequality*

$$\|gf\|_{L^2(D)} \le C\|g\|_{L^2(0,1)}\|\nabla f\|_{L^2(D)}.$$

This can be proved in a similar manner to [4, Lemma 7.6]. So we omit the proof.

Lemma 2.4. [17, Lemma 8.3.] *Let m be a nonnegative integer and $1 \le k < s$. Suppose that $F(x,t,y)$ is a smooth function on $\Omega \times (0,\infty) \times I$, where I is a compact interval in $\mathbb{R}$. For $|\alpha| + 2j = k$ there hold*

$$\begin{aligned} &\|\partial_x^\alpha \partial_t^j [F(x,t,f_1)] f_2\|_{L^2} \\ &\quad \le \begin{cases} C_0(t, f_1(t))[[f_2]]_{k-1} + C_1(t, f_1(t))\{1 + |||Df_1|||_{m-1}^{|\alpha|+j-1}\}|||Df_1|||_{m-1}[[f_2]]_k, \\ C_0(t, f_1(t))[[f_2]]_{k-1} + C_1(t, f_1(t))\{1 + |||Df_1|||_{m-1}^{|\alpha|+j-1}\}|||Df_1|||_m[[f_2]]_{k-1}, \end{cases} \end{aligned}$$

where

$$[[f_2(t)]]_k := (\sum_{k=0}^{[\frac{k}{2}]} \|\partial_t^j f_2(t)\|^2_{H^{k-2j}})^{\frac{1}{2}},$$

$$|||Df_1|||_m := \begin{cases} \|\nabla f_1(t)\|_{L^2}, & m = 0, \\ ([[\nabla f_1(t)]]_m^2 + [[\partial_t f_1(t)]]_{m-1}^2)^{\frac{1}{2}}, & m \ge 1, \end{cases}$$

$$C_0(t, f_1(t)) := \sum_{(\beta,l)\le(\alpha,j),(\beta,l)\ne(0,0)} \sup_{x\in\Omega} |\partial_x^\beta \partial_t^l F(x,t,f_1(x,t))|,$$
$$C_1(t, f_1(t)) := \sum_{(\beta,l)\le(\alpha,j),1\le p\le|\alpha|+j} \sup_{x\in\Omega} |\partial_x^\beta \partial_t^l \partial_y^p F(x,t,f_1(x,t))|.$$

We use the following elementary inequality in Chapter 3. (See e.g., [33, Lemma 3.1] for the proof.)

Lemma 2.5. *If* $\max\{a,b\} > 1$, *then the following estimate holds:*

$$\int_0^t (1+t-s)^{-a}(1+s)^{-b}\mathrm{d}s \le C(1+t)^{-\min\{a,b\}},\ t\ge 0.$$

Chapter 3

Diffusion wave phenomena and L^p decay estimates of solutions of compressible viscoelastic system

In this chapter we consider the compressible viscoelastic system

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho v) = 0, \\ \rho(\partial_t v + v \cdot \nabla v) - \nu \Delta v - (\nu + \nu')\nabla \mathrm{div} v + \nabla P(\rho) = \beta^2 \mathrm{div}(\rho F^\top F), \\ \partial_t F + v \cdot \nabla F = (\nabla v) F \end{cases} \tag{3.1}$$

in the whole space $\mathbb{R}^3$.

We impose the following initial conditions

$$(\rho, v, F)|_{t=0} = (\rho_0, v_0, F_0) \tag{3.2}$$

$$\begin{cases} \mathrm{div}(\rho_0{}^\top F_0) = 0, \ \ \rho_0 \mathrm{det} F_0 = 1, \\ \displaystyle\sum_{m=1}^{3} (F_0^{ml} \partial_{x_m} F_0^{jk} - F_0^{mk} \partial_{x_m} F_0^{jl}) = 0, \ \ j, k, l = 1, 2, 3. \end{cases} \tag{3.3}$$

As we mentioned in the beginning of the introduction, the conditions (3.3) are invariant for $t \geq 0$:

$$\rho \mathrm{det} F = 1, \tag{3.4}$$

$$\sum_{m=1}^{3} (F^{ml} \partial_{x_m} F^{jk} - F^{mk} \partial_{x_m} F^{jl}) = 0, \ \ j, k, l = 1, 2, 3. \tag{3.5}$$

$$\mathrm{div}(\rho^\top F) = 0. \tag{3.6}$$

The aim of this chapter is to study the large time behavior of solutions of the problem (3.1)–(3.3) around a motionless state $(\rho, v, F) = (1, 0, I)$. We will show that if the initial perturbation $u_0 = (\rho_0 - 1, v_0, F_0 - I)$ is sufficiently small in $L^1 \cap H^3$, then the global strong solution satisfies the following L^p decay estimate

$$\|(\rho(t) - 1, v(t), F(t) - I)\|_{L^p} \le C(1+t)^{-\frac{3}{2}\left(1-\frac{1}{p}\right)-\frac{1}{2}\left(1-\frac{2}{p}\right)},\ 1 < p \le \infty,\ t \ge 0.$$

This result improves the decay estimate of the L^p norm of the perturbation u obtained in [12, 21] for $p > 2$.

This chapter is organized as follows. In Section 3.1 we state the main result of this paper on the L^p decay estimates. In Section 3.2 we reformulate the problem to prove the main result. In Section 3.3, we give a solution formula of the linearized problem and establish the L^p decay estimates in the case $p \ge 2$. In Section 3.4, we prove the L^p decay estimate in the remaining case $1 < p < 2$. In the Appendix 3.A, we derive the solution formula of the linearized problem.

3.1 Main result of Chapter 3

In this section, we state the main result of this chapter.

We set $u(t) = (\phi(t), w(t), G(t)) = (\rho(t) - 1, v(t), F(t) - I)$. Then $u(t)$ satisfies the following initial value problem

$$\begin{cases} \partial_t \phi + \mathrm{div} w = g_1, \\ \partial_t w - \nu \Delta w - \tilde{\nu} \nabla \mathrm{div} w + \gamma^2 \nabla \phi - \beta^2 \mathrm{div} G = g_2, \\ \partial_t G - \nabla w = g_3, \\ \nabla \phi + \mathrm{div}^\top G = g_4, \\ u|_{t=0} = u_0 = (\phi_0, w_0, G_0). \end{cases} \tag{3.7}$$

Here $g_j, j = 1, 2, 3, 4$, denote the nonlinear terms;

$$\begin{aligned} g_1 &= -\mathrm{div}(\phi w), \\ g_2 &= -w \cdot \nabla w + \frac{\phi}{1+\phi}(-\nu \Delta w - \tilde{\nu} \nabla \mathrm{div} w + \gamma^2 \nabla \phi) - \frac{1}{1+\phi} \nabla Q(\phi) \\ &\quad - \frac{\beta^2 \phi}{1+\phi} \mathrm{div} G + \frac{\beta^2}{1+\phi} \mathrm{div}(\phi G + G^\top G + \phi G^\top G), \\ g_3 &= -w \cdot \nabla G + \nabla w G, \\ g_4 &= -\mathrm{div}(\phi^\top G), \end{aligned}$$

where

$$Q(\phi) = \phi^2 \int_0^1 P''(1+s\phi)\mathrm{d}s,\ \nabla Q = O(\phi)\nabla\phi$$

for $|\phi| \ll 1$.

We recall the L^2 decay estimates obtained in [21].

Proposition 3.1. ([21]) *Let $u_0 \in H^N,\ N \geq 3$. There is a positive number ϵ_0 such that if u_0 satisfies $\|u_0\|_{L^1} + \|u_0\|_{H^3} \leq \epsilon_0$, then there exists a unique solution $u(t) \in C([0,\infty); H^N)$ of the problem* (3.7)*, and $u(t) = (\phi(t), w(t), G(t))$ satisfies*

$$\|u(t)\|_{H^N}^2 + \int_0^t (\|\nabla\phi(s)\|_{H^{N-1}}^2 + \|\nabla w(s)\|_{H^N}^2 + \|\nabla G(s)\|_{H^{N-1}}^2)\mathrm{d}s \leq C\|u_0\|_{H^N}^2,$$

$$\|\nabla^k u(t)\|_{L^2} \leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}}(\|u_0\|_{L^1} + \|u_0\|_{H^N})$$

for $k = 0, 1, 2, \ldots, N-1$ and $t \geq 0$.

We next state the main result of this chapter which reflects an effect of diffusion waves caused by an elastic aspect of the equations in decay properties.

Theorem 3.2. (i) *Let $2 \leq p \leq \infty$. Assume that ϕ_0, G_0, and F_0^{-1} satisfy $\nabla\phi_0 - \mathrm{div}^\top(I+G_0)^{-1} = 0$ and $F_0^{-1} = \nabla X_0$ for some vector field X_0. There is a positive number ϵ such that if $u_0 = (\phi_0, w_0, G_0)$ satisfies $\|u_0\|_{H^3} \leq \epsilon$ and $u_0 \in L^1$, then there exists a unique solution $u(t) \in C([0,\infty); H^3)$ of the problem* (3.7)*, and $u(t) = (\phi(t), w(t), G(t))$ satisfies*

$$\|u(t)\|_{L^p} \leq C(p)(1+t)^{-\frac{3}{2}\left(1-\frac{1}{p}\right)-\frac{1}{2}\left(1-\frac{2}{p}\right)}(\|u_0\|_{L^1} + \|u_0\|_{H^3})$$

uniformly for $t \geq 0$. Here $C(p)$ is a positive constant depending only on p.

(ii) *Let $1 < p < 2$. Assume that ϕ_0, G_0, and F_0^{-1} satisfy $\nabla\phi_0 - \mathrm{div}^\top(I + G_0)^{-1} = 0$ and $F_0^{-1} = \nabla X_0$ for some vector field X_0. There is a positive number ϵ_p such that if $u_0 = (\phi_0, w_0, G_0)$ satisfies $\|u_0\|_{H^3} \leq \epsilon_p$ and $u_0 \in L^1$, then there exists a unique solution $u(t) \in C([0,\infty); H^3)$ of the problem* (3.7)*, and $u(t) = (\phi(t), w(t), G(t))$ satisfies*

$$\|u(t)\|_{L^p} \leq C(p)(1+t)^{-\frac{3}{2}\left(1-\frac{1}{p}\right)+\frac{1}{2}\left(\frac{2}{p}-1\right)}(\|u_0\|_{L^1} + \|u_0\|_{L^p} + \|u_0\|_{H^3})$$

uniformly for $t \geq 0$. Here $C(p)$ is a positive constant depending only on p.

Remark 3.3. Since $\frac{1}{2}\left(1-\frac{2}{p}\right) > 0$ for $2 < p \leq \infty$, Theorem 3.2 (i) implies that the L^p norm of the perturbation $u = (\phi, w, G)$ tends to 0 faster than the heat kernel as $t \to \infty$. We thus improve the result in [21]. Furthermore, due to the elastic force $\beta^2\mathrm{div}(\rho F^\top F)$, we discover that if $2 < p \leq \infty$, then the decay rate of the L^p norm is faster than the result in [8] for the compressible Navier-Stokes equations.

3.2 Formulation of the problem

In this section, we rewrite the problem (3.7) into a specific form to prove Theorem 3.2.

Let $x = x(X,t)$ be the material coordinate defined by the solution of the flow map:

$$\begin{cases} \dfrac{dx}{dt}(X,t) = v(x(X,t),t), \\ x(X,0) = X, \end{cases}$$

and we denote its inverse by $X = X(x,t)$. According to [6, 32], F is defined by $F = \frac{\partial x}{\partial X}$. It is shown in [29] that its inverse F^{-1} is written as $F^{-1}(x,t) = \nabla X(x,t)$ if F_0^{-1} has the form $F_0^{-1} = \nabla X_0$. We set $\tilde{\psi} = x - X$. Then $\tilde{\psi}$ is a solution of

$$\partial_t \tilde{\psi} - v = -v \cdot \nabla \tilde{\psi},$$

and satisfies

$$G = \nabla \tilde{\psi} + h(\nabla \tilde{\psi}), \tag{3.8}$$

where $h(\nabla \tilde{\psi}) = (I - \nabla \tilde{\psi})^{-1} - I - \nabla \tilde{\psi}$.

We note that (3.8) is equivalent to

$$\nabla \tilde{\psi} = I - (I + G)^{-1}. \tag{3.9}$$

The following estimates hold for G and $\nabla \tilde{\psi}$.

Lemma 3.4. *Assume that G and $\tilde{\psi}$ satisfy* (3.8)*. There is a positive number δ_0 such that if $\|G\|_{H^3} \le \min\{1, \delta_0\}$, the following inequalities hold:*

$$C^{-1}\|\nabla \tilde{\psi}\|_{L^p} \le \|G\|_{L^p} \le C\|\nabla \tilde{\psi}\|_{L^p}, \ 1 \le p \le \infty, \tag{3.10}$$

$$\|\nabla^2 \tilde{\psi}\|_{L^2} \le C\|\nabla G\|_{L^2}, \tag{3.11}$$

$$\|\nabla^3 \tilde{\psi}\|_{L^2} \le C(\|\nabla G\|_{H^1}^2 + \|\nabla^2 G\|_{L^2}), \tag{3.12}$$

$$\|\nabla^4 \tilde{\psi}\|_{L^2} \le C(\|\nabla G\|_{H^1}\|\nabla^2 G\|_{H^1} + \|\nabla^3 G\|_{L^2}). \tag{3.13}$$

Proof. If $|G| < 1$, (3.9) implies

$$\nabla \tilde{\psi} = G - \sum_{l=2}^{\infty} (-G)^l. \tag{3.14}$$

Let $c > 0$ be a positive constant such that $\|G\|_{L^\infty} \le c\|G\|_{H^2}$. If $\|G\|_{H^3} \le \frac{1}{27c}$, then we have $|G| \le 9\|G\|_{L^\infty} \le 9c\|G\|_{H^2} \le \frac{1}{3}$, and hence

$$\left|\sum_{l=2}^{\infty}(-G)^l\right| \le \sum_{l=2}^{\infty}|G|^{l-1}|G| \le \sum_{l=2}^{\infty}\left(\frac{1}{3}\right)^{l-1}|G| \le \frac{1}{2}|G|.$$

Therefore, we obtain

$$\left\|\sum_{l=2}^{\infty}(-G)^l\right\|_{L^p} \le \frac{1}{2}\|G\|_{L^p} \text{ for } 1 \le p \le \infty. \tag{3.15}$$

Combining (3.14) and (3.15) yields (3.10).

To prove (3.11), we make use of (3.9), (3.10) and the following formula

$$\partial_{x_j}(F^{-1}) = -F^{-1}\partial_{x_j}FF^{-1},\ j = 1, 2, 3.$$

It then follows that

$$\begin{aligned}\|\nabla\partial_{x_j}\tilde{\psi}\|_{L^2} &= \|\partial_{x_j}(I+G)^{-1}\|_{L^2}\\ &= \|(I+G)^{-1}\partial_{x_j}G(I+G)^{-1}\|_{L^2}\\ &\le C\|(I+G)^{-1}\|_{L^\infty}^2\|\partial_{x_j}G\|_{L^2}\\ &\le C\|\partial_{x_j}G\|_{L^2}.\end{aligned}$$

This gives (3.11).

We next consider (3.12). Since

$$\begin{aligned}\nabla\partial_{x_j}\partial_{x_k}\tilde{\psi} &= -\partial_{x_j}\partial_{x_k}(I+G)^{-1}\\ &= -(I+G)^{-1}\partial_{x_j}G(I+G)^{-1}\partial_{x_k}G(I+G)^{-1}\\ &\quad + (I+G)^{-1}\partial_{x_j}\partial_{x_k}G(I+G)^{-1}\\ &\quad\quad - (I+G)^{-1}\partial_{x_k}G(I+G)^{-1}\partial_{x_j}G(I+G)^{-1},\end{aligned}$$

we have the following estimate by using Lemma 2.1

$$\begin{aligned}&\|\nabla^3\tilde{\psi}\|_{L^2}\\ &\le C(\|(I+G)^{-1}\|_{L^\infty}^3\|\nabla G\|_{L^4}^2 + \|(I+G)^{-1}\|_{L^\infty}\|\nabla^2 G\|_{L^2})\\ &\le C(\|\nabla G\|_{H^1}^2 + \|\nabla^2 G\|_{L^2}).\end{aligned}$$

We thus obtain (3.12). By a similar computation, we have (3.13). This completes the proof. ■

Based on Lemma 3.4, we consider $\tilde{\psi}$ instead of G. In terms of $\tilde{U} = (\phi, w, \nabla\tilde{\psi})$, the problem (3.7) is transformed into

$$\begin{cases} \partial_t \phi + \mathrm{div} w = f_1, \\ \partial_t w - \nu \Delta w - \tilde{\nu} \nabla \mathrm{div} w + \gamma^2 \nabla \phi - \beta^2 \Delta \tilde{\psi} = f_2, \\ \partial_t \nabla \tilde{\psi} - \nabla w = f_3, \\ \nabla \phi + \nabla \mathrm{div} \tilde{\psi} = f_4, \\ \tilde{U}|_{t=0} = \tilde{U}_0 = (\phi_0, w_0, \nabla \tilde{\psi}_0). \end{cases} \tag{3.16}$$

Here $f_j, j = 1, 2, 3, 4$, denote the nonlinear terms;

$$\begin{aligned} f_1 &= g_1, \\ f_2 &= g_2 + \beta^2 \mathrm{div} h(\nabla \tilde{\psi}), \\ f_3 &= -\nabla (w \cdot \nabla \tilde{\psi}), \\ f_4 &= -\mathrm{div}^\top (\phi \nabla \tilde{\psi} + (1 + \phi) h(\nabla \tilde{\psi})). \end{aligned}$$

We next introduce ψ by $\psi = \tilde{\psi} - (-\Delta)^{-1} \mathrm{div}^\top (\phi \nabla \tilde{\psi} + (1 + \phi) h(\nabla \tilde{\psi}))$, where $(-\Delta)^{-1} = \mathcal{F}^{-1} |\xi|^{-2} \mathcal{F}$, and set $\Psi = \nabla \psi$. By this transformation, the nonlinear constraint $\nabla \phi + \nabla \mathrm{div} \tilde{\psi} = f_4$ is transformed into the linear constraint $\phi + \mathrm{tr} \nabla \psi = 0$; and the problem (3.16) is rewritten as

$$\begin{cases} \partial_t \phi + \mathrm{div} w = N_1, \\ \partial_t w - \nu \Delta w - \tilde{\nu} \nabla \mathrm{div} w + \gamma^2 \nabla \phi - \beta^2 \mathrm{div} \Psi = N_2, \\ \partial_t \Psi - \nabla w = N_3, \\ \phi + \mathrm{tr} \Psi = 0, \ \Psi = \nabla \psi, \\ U|_{t=0} = U_0 = (\phi_0, w_0, \Psi_0). \end{cases} \tag{3.17}$$

Here $N_j, j = 1, 2, 3$, denote the nonlinear terms;

$$\begin{aligned} N_1 &= f_1, \\ N_2 &= f_2 - \beta^2 \mathrm{div}^\top (\phi \nabla \tilde{\psi} + (1 + \phi) h(\nabla \tilde{\psi})), \\ N_3 &= -\nabla (w \cdot \nabla \tilde{\psi}) - \nabla (-\Delta)^{-1} \nabla \mathrm{div} (\phi w) - \nabla (-\Delta)^{-1} \nabla \mathrm{div} (w \cdot \nabla \tilde{\psi}). \end{aligned}$$

We note that N_1 and N_3 satisfy $N_1 + \mathrm{tr} N_3 = 0$. The relations between ψ and $\tilde{\psi}$ are given as follows.

Lemma 3.5. (i) *Let $\tilde{U}_0$ and U_0 be the ones as in* (3.16) *and* (3.17)*, respectively. If ϕ_0 and $\tilde{\psi}_0$ satisfy $\nabla \phi_0 + \nabla \mathrm{div} \tilde{\psi}_0 = 0$, then it holds $U_0 = \tilde{U}_0 = (\phi_0, w_0, \nabla \tilde{\psi}_0)$.*

(ii) *There is a positive number* δ_0 *such that the following assertion holds true. Let*

$$\phi \in C([0,\infty); H^3), \ \ \psi \in C([0,\infty); H^4).$$

If $\|\phi\|_{C([0,\infty);H^3)} + \|\psi\|_{C([0,\infty);H^4)} \le \delta_0$*, then there uniquely exists* $\tilde{\psi} \in C([0,\infty); H^4)$ *such that*

$$\begin{aligned}
&\|\tilde{\psi}\|_{C([0,\infty);H^4)} \le \sqrt{\delta_0}, \\
&\tilde{\psi} = \psi + (-\Delta)^{-1}\mathrm{div}^\top(\phi\nabla\tilde{\psi} + (1+\phi)h(\nabla\tilde{\psi})).
\end{aligned} \tag{3.18}$$

(iii) *Let* $1 < p < \infty$*. There is a positive number* δ_p *such that if* $\|\phi\|_{C([0,\infty);H^3)} + \|\nabla\tilde{\psi}\|_{C([0,\infty);H^3)} \le \min\{\delta_0, \delta_p\}$*, the following inequalities hold for* $t \ge 0$*:*

$$C_p^{-1}\|\nabla\tilde{\psi}(t)\|_{L^p} \le \|\nabla\psi(t)\|_{L^p} \le C_p\|\nabla\tilde{\psi}(t)\|_{L^p}. \tag{3.19}$$

(iv) *There is a positive number* δ_1 *such that if* $\|\phi\|_{C([0,\infty);H^3)} + \|\nabla\tilde{\psi}\|_{C([0,\infty);H^3)} \le \min\{\delta_0, \delta_1\}$*, the following inequalities hold for* $t \ge 0$*:*

$$\begin{aligned}
\|\nabla\tilde{\psi}(t)\|_{L^\infty} &\le C(\|\phi(t)\|_{L^\infty} + \|\nabla\psi(t)\|_{L^\infty}) \\
&\quad + C(\|\nabla\phi(t)\|_{H^1} + \|\nabla^2\tilde{\psi}(t)\|_{H^1})^2,
\end{aligned} \tag{3.20}$$

$$\begin{aligned}
\|\nabla^2\psi(t)\|_{L^2} &\le C\|\nabla^2\tilde{\psi}(t)\|_{L^2} \\
&\quad + C(\|\phi(t)\|_{H^2} + \|\nabla\tilde{\psi}(t)\|_{H^2})\|\nabla\tilde{\psi}(t)\|_{H^2},
\end{aligned} \tag{3.21}$$

$$\begin{aligned}
\|\nabla^3\psi(t)\|_{L^2} &\le C(1 + \|\phi(t)\|_{H^2} + \|\nabla\tilde{\psi}(t)\|_{H^2})\|\nabla^3\tilde{\psi}(t)\|_{L^2} \\
&\quad + C(\|\nabla\phi(t)\|_{H^1} + \|\nabla^2\tilde{\psi}(t)\|_{H^1})\|\nabla\tilde{\psi}(t)\|_{H^2},
\end{aligned} \tag{3.22}$$

$$\|\nabla^4\psi(t)\|_{L^2} \le C\|\nabla^4\tilde{\psi}(t)\|_{L^2} + C(\|\phi(t)\|_{H^3} + \|\nabla\tilde{\psi}(t)\|_{H^3})^2. \tag{3.23}$$

Proof. (i) The condition $\nabla\phi_0 + \nabla\mathrm{div}\tilde{\psi}_0 = 0$ leads to $\phi_0\nabla\tilde{\psi}_0 + (1+\phi_0)h(\nabla\tilde{\psi}_0) = 0$. Therefore, we have $U_0 = \tilde{U}_0 = (\phi_0, w_0, \nabla\tilde{\psi}_0)$.

(ii) We set $\Gamma(\tilde{\psi}) = \psi + (-\Delta)^{-1}\mathrm{div}^\top(\phi\nabla\tilde{\psi} + (1+\phi)h(\nabla\tilde{\psi}))$ and $\mathcal{B}_{\sqrt{\delta_0}} = \{f \in C([0,\infty), H^4) | \ \|f\|_{C([0,\infty),H^4)} \le \sqrt{\delta_0}\}$. We then see that if $C_1\sqrt{\delta_0} \le 1$, then Γ is a mapping of $\mathcal{B}_{\sqrt{\delta_0}}$ into $\mathcal{B}_{\sqrt{\delta_0}}$. Indeed, since

$$h(\nabla\tilde{\psi}) = ((I - \nabla\tilde{\psi})^{-1} - I)\nabla\tilde{\psi},$$

we see that if $|\nabla\tilde{\psi}| < 1$, then

$$h(\nabla\tilde{\psi}) = \sum_{m=2}^{\infty}(\nabla\tilde{\psi})^m,$$

and hence,

$$(I-\nabla\tilde{\psi})^{-1}-I=\sum_{m=1}^{\infty}(\nabla\tilde{\psi})^m.$$

Furthermore, since

$$\partial_{x_j}((I-\nabla\tilde{\psi})^{-1}) =(I-\nabla\tilde{\psi})^{-1}\nabla\partial_{x_j}\tilde{\psi}(I-\nabla\tilde{\psi})^{-1},$$

we have

$$\begin{aligned}\|(-\Delta)^{-1}\mathrm{div}^{\top}h(\nabla\tilde{\psi})\|_{L^2} &\le C(\|h(\nabla\tilde{\psi})\|_{L^1}+\|h(\nabla\tilde{\psi})\|_{L^2})\\ &\le C\sum_{m=2}^{\infty}(\|(\nabla\tilde{\psi})^m\|_{L^1}+\|(\nabla\tilde{\psi})^m\|_{L^2})\\ &\le C\|\nabla\tilde{\psi}\|_{H^2}^2,\end{aligned}$$

and similarly,

$$\|\nabla(-\Delta)^{-1}\mathrm{div}^{\top}h(\nabla\tilde{\psi})\|_{L^2}\le C\|\nabla\tilde{\psi}\|_{H^2}\|\nabla\tilde{\psi}\|_{L^2}.$$

As for the estimate of the second order derivative of $(-\Delta)^{-1}\mathrm{div}^{\top}h(\nabla\tilde{\psi})$, since

$$\begin{aligned}\partial_{x_j}\partial_{x_k}((I-\nabla\tilde{\psi})^{-1}) =&(I-\nabla\tilde{\psi})^{-1}\nabla\partial_{x_j}\tilde{\psi}(I-\nabla\tilde{\psi})^{-1}\nabla\partial_{x_k}\tilde{\psi}(I-\nabla\tilde{\psi})^{-1}\\ &+(I-\nabla\tilde{\psi})^{-1}\nabla\partial_{x_j}\partial_{x_k}\tilde{\psi}(I-\nabla\tilde{\psi})^{-1}\\ &+(I-\nabla\tilde{\psi})^{-1}\nabla\partial_{x_k}\tilde{\psi}(I-\nabla\tilde{\psi})^{-1}\nabla\partial_{x_j}\tilde{\psi}(I-\nabla\tilde{\psi})^{-1},\end{aligned}$$

we have

$$\begin{aligned}&\|\nabla^2(-\Delta)^{-1}\mathrm{div}^{\top}h(\nabla\tilde{\psi})\|_{L^2}\\ &\le C\|\nabla h(\nabla\tilde{\psi})\|_{L^2}\\ &\le C(\|(I-\nabla\tilde{\psi})^{-1}-I\|_{L^\infty}\|\nabla^2\tilde{\psi}\|_{L^2}+\|(I-\nabla\tilde{\psi})^{-1}\|_{L^\infty}^2\|\nabla\tilde{\psi}\|_{L^2}\|\nabla^2\tilde{\psi}\|_{L^2})\\ &\le C\|\nabla\tilde{\psi}\|_{H^2}\|\nabla^2\tilde{\psi}\|_{L^2}.\end{aligned}$$

Similarly, one can show that

$$\|\nabla^3(-\Delta)^{-1}\mathrm{div}^{\top}h(\nabla\tilde{\psi})\|_{L^2}\le C(\|\nabla\tilde{\psi}\|_{H^2}\|\nabla^3\tilde{\psi}\|_{L^2}+\|\nabla^2\tilde{\psi}\|_{H^1}^2),$$

and

$$\|\nabla^4(-\Delta)^{-1}\mathrm{div}^{\top}h(\nabla\tilde{\psi})\|_{L^2}\le C\|\nabla\tilde{\psi}\|_{H^3}^2.$$

It then follows that if $\tilde{\psi} \in \mathcal{B}_{\sqrt{\delta_0}}$, then

$$
\begin{aligned}
&\|\Gamma(\tilde{\psi})\|_{H^4} \\
&\le C(\|\psi\|_{H^4} + \|(-\Delta)^{-1}\mathrm{div}^\top(\phi\nabla\tilde{\psi} + (1+\phi)h(\nabla\tilde{\psi}))\|_{H^4}) \\
&\le C(\|\psi\|_{H^4} + \|\phi\nabla\tilde{\psi} + (1+\phi)h(\nabla\tilde{\psi})\|_{L^1}) \\
&\qquad + C\|\phi\nabla\tilde{\psi} + (1+\phi)h(\nabla\tilde{\psi})\|_{H^3} \\
&\le C(\|\psi\|_{H^4} + \|\phi\|_{H^3}\|\nabla\tilde{\psi}\|_{H^3} + (1+\|\phi\|_{H^3})\|\nabla\tilde{\psi}\|_{H^3}^2) \\
&\le C_1\delta_0 \\
&\le \sqrt{\delta_0}.
\end{aligned}
$$

Therefore, $\Gamma(\tilde{\psi})$ belongs to $\mathcal{B}_{\sqrt{\delta_0}}$.

We next claim that if $\tilde{\psi}_j \in \mathcal{B}_{\sqrt{\delta_0}}$ $(j = 1, 2)$, then

$$
\|\Gamma(\tilde{\psi}_1) - \Gamma(\tilde{\psi}_2)\|_{C([0,\infty);H^4)} \le C_2\sqrt{\delta_0}\|\tilde{\psi}_1 - \tilde{\psi}_2\|_{C([0,\infty);H^4)}. \tag{3.24}
$$

To show this, we first have

$$
\begin{aligned}
&\|\Gamma(\tilde{\psi}_1) - \Gamma(\tilde{\psi}_2)\|_{H^4} \\
&\le C(\|\phi\nabla(\tilde{\psi}_1 - \tilde{\psi}_2)\|_{H^3} + \|(1+\phi)(h(\nabla\tilde{\psi}_1) - h(\nabla\tilde{\psi}_2))\|_{H^3} \\
&\le C\delta_0\|\tilde{\psi}_1 - \tilde{\psi}_2\|_{H^4} + C(1+\delta_0)\|h(\nabla\tilde{\psi}_1) - h(\nabla\tilde{\psi}_2)\|_{H^3}.
\end{aligned}
$$

As for the second term on the right-hand side, since

$$
\begin{aligned}
&h(\nabla\tilde{\psi}_1) - h(\nabla\tilde{\psi}_2) \\
&= ((I - \nabla\tilde{\psi}_1)^{-1} - I)\nabla(\tilde{\psi}_1 - \tilde{\psi}_2) \\
&\quad + \nabla(\tilde{\psi}_1 - \tilde{\psi}_2)((I - \nabla\tilde{\psi}_2)^{-1} - I) \\
&\qquad + ((I - \nabla\tilde{\psi}_1)^{-1} - I)\nabla(\tilde{\psi}_1 - \tilde{\psi}_2)((I - \nabla\tilde{\psi}_2)^{-1} - I),
\end{aligned}
$$

we see that

$$
\|h(\nabla\tilde{\psi}_1) - h(\nabla\tilde{\psi}_2)\|_{H^3} \le C(\sqrt{\delta_0} + \delta_0)\|\tilde{\psi}_1 - \tilde{\psi}_2\|_{H^4}.
$$

Therefore, we arrive at (3.24). Taking δ_0 small such that $C_2\sqrt{\delta_0} < 1$, we conclude that Γ is a contraction map in $\mathcal{B}_{\sqrt{\delta_0}}$. By the contraction mapping principle, we observe that there exists a unique $\tilde{\psi} \in \mathcal{B}_{\sqrt{\delta_0}}$ such that $\tilde{\psi} = \Gamma(\tilde{\psi})$. This indicates the unique existence of $\tilde{\psi}$ satisfying (3.18).

(iii) We assume that $\|\phi\|_{C([0,\infty);H^3)} + \|\tilde{\psi}\|_{C([0,\infty);H^4)} \le \delta$ with some small number $0 < \delta < 1$ to be determined later. Since the Riesz operator $\mathcal{R}_j f = \mathcal{F}^{-1}\left[\frac{\xi_j}{|\xi|}\hat{f}\right]$ is bounded from L^p to L^p for $1 < p < \infty$, we have

$$\begin{aligned}
&\|\nabla(-\Delta)^{-1}\mathrm{div}^\top(\phi\nabla\tilde{\psi} + (1+\phi)h(\nabla\tilde{\psi}))\|_{L^p} \\
&= \left\|\mathcal{F}^{-1}\left[\mathcal{F}({}^\top(\phi\nabla\tilde{\psi} + (1+\phi)h(\nabla\tilde{\psi})))\frac{\xi^\top\xi}{|\xi|^2}\right]\right\|_{L^p} \\
&\le C_p\|\phi\nabla\tilde{\psi} + (1+\phi)h(\nabla\tilde{\psi})\|_{L^p} \\
&\le C_p(\|\phi\|_{H^2}\|\nabla\tilde{\psi}\|_{L^p} + \|h(\nabla\tilde{\psi})\|_{L^p} + \|\phi\|_{H^2}\|h(\nabla\tilde{\psi})\|_{L^p}).
\end{aligned}$$

We see from $h(\nabla\tilde{\psi}) = \sum_{m=2}^\infty(\nabla\tilde{\psi})^m$ that

$$\|h(\nabla\tilde{\psi})\|_{L^p} \le C\|\nabla\tilde{\psi}\|_{L^\infty}\|\nabla\tilde{\psi}\|_{L^p}.$$

This leads to the estimate

$$\|\nabla(-\Delta)^{-1}\mathrm{div}^\top(\phi\nabla\tilde{\psi} + (1+\phi)h(\nabla\tilde{\psi}))\|_{L^p} \le C_p(\delta+\delta^2)\|\nabla\tilde{\psi}\|_{L^p}.$$

By taking δ small such that $C_p(\delta+\delta^2) \le \frac{1}{2}$, we obtain (3.19).

(iv) We assume that $\|\phi\|_{C([0,\infty);H^3)} + \|\tilde{\psi}\|_{C([0,\infty);H^4)} \le \delta$ with some small number $0 < \delta < 1$ to be determined later. It follows from the Sobolev inequality and the Plancherel theorem that

$$\begin{aligned}
&\|\nabla(-\Delta)^{-1}\mathrm{div}^\top(\phi\nabla\tilde{\psi} + (1+\phi)h(\nabla\tilde{\psi}))\|_{L^\infty} \\
&= \left\|\mathcal{F}^{-1}\left[\mathcal{F}({}^\top(\phi\nabla\tilde{\psi} + (1+\phi)h(\nabla\tilde{\psi})))\frac{\xi^\top\xi}{|\xi|^2}\right]\right\|_{L^\infty} \\
&\le C\left\|\mathcal{F}^{-1}\left[\mathcal{F}({}^\top(\phi\nabla\tilde{\psi} + (1+\phi)h(\nabla\tilde{\psi})))\frac{\xi^\top\xi}{|\xi|^2}\right]\right\|_{H^2} \\
&\le C(\|\phi\nabla\tilde{\psi}\|_{H^2} + \|h(\nabla\tilde{\psi})\|_{H^2} + \|\phi h(\nabla\tilde{\psi})\|_{H^2}).
\end{aligned}$$

Since

$$\begin{aligned}
\|\phi\nabla\tilde{\psi}\|_{H^2} &\le C(\|\phi\|_{H^2} + \|\nabla\tilde{\psi}\|_{H^2})(\|\phi\|_{L^\infty} + \|\nabla\tilde{\psi}\|_{L^\infty}) \\
&\quad + C(\|\nabla\phi\|_{H^1} + \|\nabla^2\tilde{\psi}\|_{H^1})^2, \\
\|h(\nabla\tilde{\psi})\|_{H^2} &\le C\|\nabla\tilde{\psi}\|_{H^2}\|\nabla\tilde{\psi}\|_{L^\infty} + C\|\nabla^2\tilde{\psi}\|_{H^1}^2, \\
\|\phi h(\nabla\tilde{\psi})\|_{H^2} &\le C\|\phi\|_{H^2}(\|\nabla\tilde{\psi}\|_{H^2}\|\nabla\tilde{\psi}\|_{L^\infty} + \|\nabla^2\tilde{\psi}\|_{H^1}^2),
\end{aligned}$$

we have

$$\|\nabla\tilde{\psi}\|_{L^\infty} = \|\nabla\psi + \nabla(-\Delta)^{-1}\mathrm{div}^\top(\phi\nabla\tilde{\psi} + (1+\phi)h(\nabla\tilde{\psi}))\|_{L^\infty}$$

$$
\begin{aligned}
&\le C(\delta+\delta^2)\|\nabla\tilde{\psi}\|_{L^\infty} \\
&\quad + C(\|\phi\|_{L^\infty} + \|\nabla\psi\|_{L^\infty}) + C(\|\nabla\phi\|_{H^1} + \|\nabla^2\tilde{\psi}\|_{H^1})^2.
\end{aligned}
$$

By taking δ small such that $C(\delta+\delta^2) \le \frac{1}{2}$, we arrive at (3.20). We can derive (3.21)–(3.23) as in the proof of (ii). This completes the proof. ■

Remark 3.6. Due to the restriction $p > 1$ in Lemma 3.5 (iii), the decay estimate of L^1 norm of $u(t)$ is excluded in Theorem 3.2.

3.3 Proof of Theorem 3.2 (i)

In this section, we prove Theorem 3.2 (i). The global existence and the L^2 decay estimates of higher order derivatives are guaranteed by Proposition 3.1. Hence we focus on the derivation of the L^p decay estimates except the case $p = 2$. In view of Lemmata 3.4–3.5 and the interpolation inequality: $\|u(t)\|_{L^p} \le \|u(t)\|_{L^2}^{\frac{2}{p}} \|u(t)\|_{L^\infty}^{1-\frac{2}{p}}$ $(2 \le p \le \infty)$, it suffices to obtain the L^∞ decay estimate of $U = (\phi, w, \Psi)$.

The problem (3.17) is written in the form:

$$
\begin{cases}
\partial_t U + LU = N, \\
\phi + \mathrm{div}\psi = 0, \\
U|_{t=0} = U_0,
\end{cases}
\tag{3.25}
$$

where

$$
L = \begin{pmatrix} 0 & \mathrm{div} & 0 \\ \gamma^2\nabla & -\nu\Delta - \tilde{\nu}\nabla\mathrm{div} & -\beta^2\mathrm{div} \\ 0 & -\nabla & 0 \end{pmatrix}, \quad N = \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix}.
$$

Theorem 3.2 (i) is proved by combining Lemma 3.4, Lemma 3.5 and the following L^∞ decay estimate of $U(t)$.

Proposition 3.7. *There exists a positive number δ_0 such that if $\|u_0\|_{L^1} + \|u_0\|_{H^3} \le \delta_0$, then the following inequality*

$$
\|U(t)\|_{L^\infty} \le C(1+t)^{-2}(\|u_0\|_{L^1} + \|u_0\|_{H^3})
$$

holds for $t \ge 0$.

To prove Proposition 3.7, we first give the following L^2 decay estimates for $\nabla^k U(t)$.

Proposition 3.8. *There exists a positive number δ_0 such that if $\|u_0\|_{L^1} + \|u_0\|_{H^3} \leq \delta_0$, then the following inequality*

$$\|\nabla^k U(t)\|_{L^2} \leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}}(\|u_0\|_{L^1} + \|u_0\|_{H^3})$$

holds for $k = 0, 1, 2$ and $t \geq 0$.

Proposition 3.8 follows from Proposition 3.1, Lemma 3.4 and Lemma 3.5.

We next investigate the linearized problem

$$\begin{cases} \partial_t U + LU = 0, \\ \phi + \mathrm{div}\psi = 0, \\ U|_{t=0} = U_0. \end{cases} \tag{3.26}$$

We denote by e^{-tL} the semigroup generated by $-L$. The solution of (3.26) is written as $U(t) = e^{-tL}U_0$.

To investigate the large time behavior of $U(t) = e^{-tL}U_0$, we take the Fourier transform with respect to x. We then obtain

$$\begin{cases} \partial_t \hat{U} + \hat{L}_\xi \hat{U} = 0, \\ \hat{\phi} + i\xi \cdot \hat{\psi} = 0, \\ \hat{U}|_{t=0} = \hat{U}_0, \end{cases} \tag{3.27}$$

where

$$\hat{L}_\xi \hat{U} = \begin{pmatrix} i\xi \cdot \hat{w} \\ i\gamma^2 \hat{\phi}\xi + (\nu|\xi|^2 I + \tilde{\nu}\xi^\top \xi)\hat{w} - i\beta^2 \hat{\Psi}\xi \\ -i\hat{w}^\top \xi \end{pmatrix}.$$

We have the following expression of $e^{-t\hat{L}_\xi}\hat{U}_0$.

Lemma 3.9. *If $|\xi| \neq 0, \frac{\beta}{\nu}, \frac{\sqrt{\beta^2+\gamma^2}}{\nu+\tilde{\nu}}$, the solution of (3.27) is written as*

$$\begin{pmatrix} \hat{\phi}(\xi,t) \\ \hat{w}(\xi,t) \\ \hat{\Psi}(\xi,t) \end{pmatrix} = \begin{pmatrix} \hat{K}^{11}(\xi,t) & \hat{K}^{12}(\xi,t) & \hat{K}^{13}(\xi,t) \\ \hat{K}^{21}(\xi,t) & \hat{K}^{22}(\xi,t) & \hat{K}^{23}(\xi,t) \\ \hat{K}^{31}(\xi,t) & \hat{K}^{32}(\xi,t) & \hat{K}^{33}(\xi,t) \end{pmatrix} \begin{pmatrix} \hat{\phi}_0(\xi) \\ \hat{w}_0(\xi) \\ \hat{\Psi}_0(\xi) \end{pmatrix}. \tag{3.28}$$

Here

$$\begin{aligned} \hat{K}^{11}(\xi,t) =& \frac{\mu_3(\xi)e^{\mu_4(\xi)t} - \mu_4(\xi)e^{\mu_3(\xi)t}}{\mu_3(\xi) - \mu_4(\xi)}, \\ \hat{K}^{12}(\xi,t) =& -i\frac{e^{\mu_3(\xi)t} - e^{\mu_4(\xi)t}}{\mu_3(\xi) - \mu_4(\xi)}{}^\top\xi, \\ \hat{K}^{13}(\xi,t) =& 0, \end{aligned}$$

$$
\begin{aligned}
\hat{K}^{21}(\xi,t) =& -i\gamma^2 \frac{e^{\mu_3(\xi)t} - e^{\mu_4(\xi)t}}{\mu_3(\xi)-\mu_4(\xi)}\xi,\\
\hat{K}^{22}(\xi,t) =& \frac{\mu_1(\xi)e^{\mu_1(\xi)t} - \mu_2(\xi)e^{\mu_2(\xi)t}}{\mu_1(\xi)-\mu_2(\xi)}\left(I - \frac{\xi^\top\xi}{|\xi|^2}\right)\\
&+ \frac{\mu_3(\xi)e^{\mu_3(\xi)t} - \mu_4(\xi)e^{\mu_4(\xi)t}}{\mu_3(\xi)-\mu_4(\xi)}\frac{\xi^\top\xi}{|\xi|^2},\\
\hat{K}^{31}(\xi,t) =& 0,\\
\hat{K}^{33}(\xi,t) =& \frac{\mu_1(\xi)e^{\mu_2(\xi)t} - \mu_2(\xi)e^{\mu_1(\xi)t}}{\mu_1(\xi)-\mu_2(\xi)}\left(I - \frac{\xi^\top\xi}{|\xi|^2}\right)\\
&+ \frac{\mu_3(\xi)e^{\mu_4(\xi)t} - \mu_4(\xi)e^{\mu_3(\xi)t}}{\mu_3(\xi)-\mu_4(\xi)}\frac{\xi^\top\xi}{|\xi|^2};
\end{aligned}
$$

$\hat{K}^{23}(\xi,t)\hat{\Psi}_0(\xi)$ and $\hat{K}^{32}(\xi,t)\hat{w}_0(\xi)$ are defined by

$$
\begin{aligned}
\hat{K}^{23}(\xi,t)\hat{\Psi}_0(\xi) =& i\beta^2\frac{e^{\mu_1(\xi)t} - e^{\mu_2(\xi)t}}{\mu_1(\xi)-\mu_2(\xi)}\left(I - \frac{\xi^\top\xi}{|\xi|^2}\right)\hat{\Psi}_0(\xi)\xi\\
&+ i\beta^2\frac{e^{\mu_3(\xi)t} - e^{\mu_4(\xi)t}}{\mu_3(\xi)-\mu_4(\xi)}\frac{\xi^\top\xi}{|\xi|^2}\hat{\Psi}_0(\xi)\xi,
\end{aligned}
$$

$$
\begin{aligned}
\hat{K}^{32}(\xi,t)\hat{w}_0(\xi) =& i\frac{e^{\mu_1(\xi)t} - e^{\mu_2(\xi)t}}{\mu_1(\xi)-\mu_2(\xi)}\left(I - \frac{\xi^\top\xi}{|\xi|^2}\right)\hat{w}_0(\xi)^\top\xi\\
&+ i\frac{e^{\mu_3(\xi)t} - e^{\mu_4(\xi)t}}{\mu_3(\xi)-\mu_4(\xi)}\frac{\xi^\top\xi}{|\xi|^2}\hat{w}_0(\xi)^\top\xi,
\end{aligned}
$$

where $\mu_j(\xi)$, $j = 1,2,3,4$, are given by

$$
\begin{aligned}
\mu_1(\xi) &= \frac{-\nu|\xi|^2 + \sqrt{\nu^2|\xi|^4 - 4\beta^2|\xi|^2}}{2},\\
\mu_2(\xi) &= \frac{-\nu|\xi|^2 - \sqrt{\nu^2|\xi|^4 - 4\beta^2|\xi|^2}}{2},\\
\mu_3(\xi) &= \frac{-(\nu+\tilde{\nu})|\xi|^2 + \sqrt{(\nu+\tilde{\nu})^2|\xi|^4 - 4(\beta^2+\gamma^2)|\xi|^2}}{2},\\
\mu_4(\xi) &= \frac{-(\nu+\tilde{\nu})|\xi|^2 - \sqrt{(\nu+\tilde{\nu})^2|\xi|^4 - 4(\beta^2+\gamma^2)|\xi|^2}}{2}.
\end{aligned}
$$

The proof of Lemma 3.9 will be given in Appendix 3.A.
The solution $U(t) = e^{-tL}U_0$ is thus given by

$$
U(t) = e^{-tL}U_0 = \mathcal{F}^{-1}e^{-t\hat{L}_\xi}\hat{U}_0.
$$

To study the asymptotic behavior of $U(t)$, we will make use of the following properties of μ_j $(j = 1, 2, 3, 4)$:

$$\mu_j(\xi)^2 + \nu|\xi|^2\mu_j(\xi) + \beta^2|\xi|^2 = 0,\ j = 1, 2,$$
$$\mu_j(\xi) \sim -\frac{\nu}{2}|\xi|^2 + i(-1)^{j+1}\beta|\xi|,\ \text{for } |\xi| \ll 1,\ j = 1, 2,$$
$$\mu_1(\xi) \sim -\frac{\beta^2}{\nu},\ \mu_2(\xi) \sim -\nu|\xi|^2,\ \text{for } |\xi| \gg 1,$$
$$\mu_j(\xi)^2 + (\nu + \tilde{\nu})|\xi|^2\mu_j(\xi) + (\beta^2 + \gamma^2)|\xi|^2 = 0,\ j = 3, 4,$$
$$\mu_j(\xi) \sim -\frac{\nu + \tilde{\nu}}{2}|\xi|^2 + i(-1)^{j+1}\sqrt{\beta^2 + \gamma^2}|\xi|,\ \text{for } |\xi| \ll 1,\ j = 3, 4,$$
$$\mu_3(\xi) \sim -\frac{\beta^2 + \gamma^2}{\nu + \tilde{\nu}},\ \mu_4(\xi) \sim -(\nu + \tilde{\nu})|\xi|^2,\ \text{for } |\xi| \gg 1.$$

We decompose the solution $U(t)$ of the problem (3.25) into its low and high frequency parts. Let $\hat{\varphi}_1, \hat{\varphi}_\infty \in C^\infty(\mathbb{R}^3)$ be cut-off functions such that

$$\hat{\varphi}_1(\xi) = \begin{cases} 1 & |\xi| \le \frac{M_1}{2}, \\ 0 & |\xi| \ge \frac{M_1}{\sqrt{2}}, \end{cases}\ \hat{\varphi}_1(-\xi) = \hat{\varphi}_1(\xi),$$

$$\hat{\varphi}_\infty(\xi) = 1 - \hat{\varphi}_1(\xi),$$

where

$$M_1 = \min\left\{\frac{\beta}{\nu}, \frac{\sqrt{\beta^2 + \gamma^2}}{\nu + \tilde{\nu}}\right\}.$$

We define the operators P_1 and P_∞ on L^2 by

$$P_1 u = \mathcal{F}^{-1}(\hat{\varphi}_1\hat{u}),\ P_\infty u = \mathcal{F}^{-1}(\hat{\varphi}_\infty\hat{u}) \text{ for } u \in L^2.$$

Lemma 3.10. *P_j $(j = 1, \infty)$ have the following properties.*
(i) $P_1 + P_\infty = I$.
(ii) $\partial_x^\alpha P_1 = P_1\partial_x^\alpha$, $\|\partial_x^\alpha P_1 f\|_{L^2} \le C_\alpha\|f\|_{L^2}$ *for* $\alpha \in (\{0\} \cup \mathbb{N})^3$ *and* $f \in L^2$.
(iii) $\partial_x^\alpha P_\infty = P_\infty\partial_x^\alpha$, $\|\partial_x^\alpha P_\infty f\|_{L^2} \le C\|\nabla\partial_x^\alpha P_\infty f\|_{L^2}$ *for* $\alpha \in (\{0\} \cup \mathbb{N})^3$ *with* $|\alpha| = k \ge 0$ *and* $f \in H^{k+1}$.

Lemma 3.10 immediately follows from the definitions of $P_j,\ j = 1, \infty$, and the Plancherel theorem. We omit the proof.

The solution $U(t)$ of (3.25) is decomposed as

$$U(t) = U_1(t) + U_\infty(t),\ U_1(t) = P_1 U(t),\ U_\infty(t) = P_\infty U(t).$$

It follows that $U_1(t) = (\phi_1(t), w_1(t), \nabla\psi_1(t))$ and $U_\infty(t) = (\phi_\infty(t), w_\infty(t), \nabla\psi_\infty(t))$ satisfy the equations

$$\begin{cases} U_1(t) = e^{-tL}U_1(0) + \displaystyle\int_0^t e^{-(t-s)L}P_1N(s)\mathrm{d}s, \\ \phi_1 + \mathrm{div}\psi_1 = 0, \end{cases} \tag{3.29}$$

and

$$\begin{cases} \partial_t U_\infty + LU_\infty = P_\infty N, \\ \phi_\infty + \mathrm{div}\psi_\infty = 0. \end{cases} \tag{3.30}$$

We first derive the L^∞ estimate of the low frequency part $U_1(t)$.

Proposition 3.11. *There exists a positive number δ_0 such that if $\|u_0\|_{L^1} + \|u_0\|_{H^3} \le \delta_0$, then the following inequality*

$$\|U_1(t)\|_{L^\infty} \le C(1+t)^{-2}(\|u_0\|_{L^1} + \|u_0\|_{H^3})$$

holds for $t \ge 0$.

To prove Proposition 3.11, we introduce the following lemmata.

Lemma 3.12. *Let $f \in L^1$. Then, the following estimates hold for $j \in \{0\} \cup \mathbb{N}$, $\alpha \in (\{0\} \cup \mathbb{N})^3$ and $t \ge 0$*

$$\left\|\partial_t^j \partial_x^\alpha \mathcal{F}^{-1}\left[\frac{e^{\mu_1(\xi)t} - e^{\mu_2(\xi)t}}{\mu_1(\xi) - \mu_2(\xi)}\hat{\eta}(\xi)\hat{\varphi}_1(\xi)\right]\right\|_{L^\infty} \le C(1+t)^{-\frac{3}{2}-\frac{j+|\alpha|}{2}},$$

$$\left\|\partial_t^j \partial_x^\alpha \mathcal{F}^{-1}\left[\frac{\mu_1(\xi)e^{\mu_2(\xi)t} - \mu_2(\xi)e^{\mu_1(\xi)t}}{\mu_1(\xi) - \mu_2(\xi)}\hat{\eta}(\xi)\hat{\varphi}_1(\xi)\right]\right\|_{L^\infty} \le C(1+t)^{-2-\frac{j+|\alpha|}{2}},$$

$$\left\|\partial_t^j \partial_x^\alpha \mathcal{F}^{-1}\left[\frac{e^{\mu_3(\xi)t} - e^{\mu_4(\xi)t}}{\mu_3(\xi) - \mu_4(\xi)}\hat{\eta}(\xi)\hat{\varphi}_1(\xi)\right]\right\|_{L^\infty} \le C(1+t)^{-\frac{3}{2}-\frac{j+|\alpha|}{2}},$$

$$\left\|\partial_t^j \partial_x^\alpha \mathcal{F}^{-1}\left[\frac{\mu_3(\xi)e^{\mu_4(\xi)t} - \mu_4(\xi)e^{\mu_3(\xi)t}}{\mu_3(\xi) - \mu_4(\xi)}\hat{\eta}(\xi)\hat{\varphi}_1(\xi)\right]\right\|_{L^\infty} \le C(1+t)^{-2-\frac{j+|\alpha|}{2}},$$

where $\hat{\eta}(\xi) = \tilde{\eta}(\frac{\xi}{|\xi|})$ with $\tilde{\eta} \in C^\infty(S^2)$ and $S^2 = \{\omega \in \mathbb{R}^3 | |\omega| = 1\}$.

Lemma 3.12 directly follows from [20, Theorem3.1].

We give the estimate of $\|e^{-tL}U_1(0)\|_{L^\infty}$ as follows.

Lemma 3.13. *It holds the following estimate:*

$$\|e^{-tL}U_1(0)\|_{L^\infty} \le C(1+t)^{-2}\|u_0\|_{L^1}.$$

Lemma 3.13 is a direct consequence of Lemma 3.12.

For simplicity, we set $\|u_0\|_{\mathcal{X}} = \|u_0\|_{L^1} + \|u_0\|_{H^3}$. We have the estimate of $\int_0^t \|e^{-(t-s)L} P_1 N(s)\|_{L^\infty} \mathrm{d}s$.

Lemma 3.14. *There exists a positive number δ_0 such that if $\|u_0\|_{\mathcal{X}} \le \delta_0$, then the following inequality*

$$\int_0^t \|e^{-(t-s)L} P_1 N(s)\|_{L^\infty} \mathrm{d}s \le C(1+t)^{-2} \|u_0\|_{\mathcal{X}} \tag{3.31}$$

holds for $t \ge 0$.

Proof. We first consider $\mathcal{F}^{-1}\left[\hat{\varphi}_1(\xi)\hat{K}^{23}(\xi, t-s)\hat{N}_3(\xi, s)\right]$ and $\mathcal{F}^{-1}\left[\hat{\varphi}_1(\xi)\hat{K}^{33}(\xi, t-s)\hat{N}_3(\xi, s)\right]$. Since

$$\hat{N}_3(\xi, s) = -\left(I - \frac{\xi^\top \xi}{|\xi|^2}\right)\mathcal{F}(\nabla(w \cdot \nabla\tilde{\psi}))(\xi, s) + \frac{\xi^\top \xi}{|\xi|^2}\mathcal{F}(\nabla(\phi w))(\xi, s),$$

we have

$$\begin{aligned} &\hat{K}^{23}(\xi, t-s)\hat{N}_3(\xi, s) \\ &\quad = -i\beta^2 \frac{e^{\mu_1(\xi)(t-s)} - e^{\mu_2(\xi)(t-s)}}{\mu_1(\xi) - \mu_2(\xi)}\left(I - \frac{\xi^\top \xi}{|\xi|^2}\right)\mathcal{F}(\nabla(w \cdot \nabla\tilde{\psi}))(\xi, s)\xi \\ &\qquad + i\beta^2 \frac{e^{\mu_3(\xi)(t-s)} - e^{\mu_4(\xi)(t-s)}}{\mu_3(\xi) - \mu_4(\xi)}\frac{\xi^\top \xi}{|\xi|^2}\mathcal{F}(\nabla(\phi w))(\xi, s)\xi, \end{aligned} \tag{3.32}$$

$$\begin{aligned} &\hat{K}^{33}(\xi, t-s)\hat{N}_3(\xi, s) \\ &\quad = \frac{\mu_1(\xi)e^{\mu_2(\xi)(t-s)} - \mu_2(\xi)e^{\mu_1(\xi)(t-s)}}{\mu_1(\xi) - \mu_2(\xi)}\left(I - \frac{\xi^\top \xi}{|\xi|^2}\right)\mathcal{F}(\nabla(w \cdot \nabla\tilde{\psi}))(\xi, s) \\ &\qquad + \frac{\mu_3(\xi)e^{\mu_4(\xi)(t-s)} - \mu_4(\xi)e^{\mu_3(\xi)(t-s)}}{\mu_3(\xi) - \mu_4(\xi)}\frac{\xi^\top \xi}{|\xi|^2}\mathcal{F}(\nabla(\phi w))(\xi, s)), \end{aligned} \tag{3.33}$$

We see from Lemma 3.12, (3.32) and (3.33) that

$$\begin{aligned} &\left\|\mathcal{F}^{-1}\left[\hat{\varphi}_1(\xi)\hat{K}^{j3}(\xi, t-s)\hat{N}_3(\xi, s)\right]\right\|_{L^\infty} \\ &\quad \le C(1+t-s)^{-2}(1+s)^{-2}\|u_0\|_{\mathcal{X}}, \ j = 1, 2, 3. \end{aligned} \tag{3.34}$$

Since

$$\|N_1(s)\|_{L^1} \le C\|u(s)\|_{L^2}\|\nabla u(s)\|_{L^2} \le C(1+s)^{-2}\|u_0\|_{\mathcal{X}}, \tag{3.35}$$

$$\|N_2(s)\|_{L^1} \leq C\|u(s)\|_{L^2}\|\nabla u(s)\|_{H^1} \leq C(1+s)^{-2}\|u_0\|_{\mathcal{X}}, \tag{3.36}$$

we see from Lemma 3.12 that

$$\begin{aligned} &\left\| \mathcal{F}^{-1}\left[\hat{\varphi}_1(\xi)\hat{K}^{jk}(\xi,t-s)\hat{N}_k(\xi,s)\right]\right\|_{L^\infty} \\ &\quad \leq C(1+t-s)^{-2}(1+s)^{-2}\|u_0\|_{\mathcal{X}},\ j=1,2,3,\ k=1,2. \end{aligned} \tag{3.37}$$

It follows from (3.34) and (3.37) that

$$\|e^{-(t-s)L}P_1N(s)\|_{L^\infty} \leq C(1+t-s)^{-2}(1+s)^{-2}\|u_0\|_{\mathcal{X}}.$$

By employing Lemma 2.5 with $a=b=2$, we have (3.31). This completes the proof. ■

Proof of Proposition 3.11. Taking L^∞ norm of the first equation of (3.29), we have

$$\|U_1(t)\|_{L^\infty} \leq \|e^{-tL}U_1(0)\|_{L^\infty} + \int_0^t \|e^{-(t-s)L}P_1N(s)\|_{L^\infty}\mathrm{d}s. \tag{3.38}$$

Together with (3.38), Lemma 3.13 and Lemma 3.14, then yields

$$\|U_1(t)\|_{L^\infty} \leq C(1+t)^{-2}\|u_0\|_{\mathcal{X}}. \tag{3.39}$$

This completes the proof of Proposition 3.11. ■

We next consider the high frequency part $U_\infty(t)$.

Proposition 3.15. *There exists a positive number δ_0 such that if $\|u_0\|_{L^1} + \|u_0\|_{H^3} \leq \delta_0$, then the following inequality*

$$\|U_\infty(t)\|_{L^\infty} \leq C(1+t)^{-2}(\|u_0\|_{L^1} + \|u_0\|_{H^3})$$

holds for $t \geq 0$.

Proof. We set $\tilde{\psi}_\infty = P_\infty\tilde{\psi}$, $\tilde{\Psi}_\infty = \nabla\tilde{\psi}_\infty$ and $\tilde{U}_\infty = (\phi_\infty, w_\infty, \nabla\tilde{\psi}_\infty)$. Since

$$\begin{aligned} \|\Psi_\infty\|_{L^\infty} &\leq c\|\Psi_\infty\|_{H^2} \\ &\leq c\|\tilde{\Psi}_\infty\|_{H^2} + c\|P_\infty\nabla(-\Delta)^{-1}\mathrm{div}^\top(\phi\nabla\tilde{\psi} + (1+\phi)h(\nabla\tilde{\psi}))\|_{H^2} \\ &\leq c\|\nabla^2\tilde{\Psi}_\infty\|_{L^2} + C\|\nabla P_\infty\nabla(-\Delta)^{-1}\mathrm{div}^\top(\phi\nabla\tilde{\psi} + (1+\phi)h(\nabla\tilde{\psi}))\|_{H^1} \\ &\leq c\|\nabla^2\tilde{\Psi}_\infty\|_{L^2} + C(1+t)^{-2}\|u_0\|_{\mathcal{X}}, \end{aligned}$$

we have $\|U_\infty(t)\|_{L^\infty} \leq c\|\nabla^2\tilde{U}_\infty(t)\|_{L^2} + C(1+t)^{-2}\|u_0\|_{\mathcal{X}}$. We thus estimate $\tilde{\psi}_\infty$ in substitution for ψ_∞.

We next consider $\|\tilde{U}_\infty\|_{L^\infty}$ instead of $\|U_\infty\|_{L^\infty}$. By applying P_∞ to the problem (3.16), we obtain

$$\partial_t\phi_\infty + \mathrm{div} w_\infty = f_{1,\infty}, \tag{3.40}$$
$$\partial_t w_\infty - \nu\Delta w_\infty - \tilde{\nu}\nabla\mathrm{div} w_\infty + \gamma^2\nabla\phi_\infty - \beta^2\mathrm{div}\tilde{\Psi}_\infty = f_{2,\infty}, \tag{3.41}$$
$$\partial_t\tilde{\Psi}_\infty - \nabla w_\infty = f_{3,\infty}, \tag{3.42}$$
$$\phi_\infty + \mathrm{div}\tilde{\psi}_\infty = \tilde{f}_{4,\infty}, \tag{3.43}$$
$$\tilde{U}_\infty|_{t=0} = P_\infty(\phi_0, w_0, \tilde{\Psi}_0),$$

where $f_{j,\infty} = P_\infty f_j$, $j = 1, 2, 3$, and $\tilde{f}_{4,\infty} = -P_\infty(-\Delta)^{-1}\mathrm{div} f_4$. We define $E[\tilde{U}_\infty]$ and $D[\tilde{U}_\infty]$ by

$$E[\tilde{U}_\infty] = \|\nabla_x^2\tilde{U}_\infty\|_{L^2}^2 + c_1\sum_{|\alpha|=2}(\partial_x^\alpha w_\infty, \partial_x^\alpha\tilde{\psi}_\infty),$$

$$D[\tilde{U}_\infty] = \sum_{|\alpha|=2}[\nu\|\nabla\partial_x^\alpha w_\infty\|_{L^2}^2 + \tilde{\nu}\|\mathrm{div}\partial_x^\alpha w_\infty\|_{L^2}^2 + c_1\gamma^2\|\partial_x^\alpha\phi_\infty\|_{L^2}^2 + c_1\beta^2\|\partial_x^\alpha\tilde{\Psi}_\infty\|_{L^2}^2].$$

Here c_1 is a positive constant to be determined later.

We establish the following energy estimate of $E[\tilde{U}_\infty]$.

Proposition 3.16. *The following estimate holds:*

$$\frac{\mathrm{d}}{\mathrm{d}t}E[\tilde{U}_\infty] + D[\tilde{U}_\infty] \le C_1\mathcal{N}. \tag{3.44}$$

Here

$$\mathcal{N} = \sum_{|\alpha|=2}\Big(\gamma^2|(\partial_x^\alpha f_{1,\infty}, \partial_x^\alpha\phi_\infty)| + |(\partial_x^\alpha f_{2,\infty}, \partial_x^\alpha w_\infty)| + \beta^2|(\partial_x^\alpha f_{3,\infty}, \partial_x^\alpha\tilde{\Psi}_\infty)|$$
$$+ c_1|(\partial_x^\alpha f_{2,\infty}, \partial_x^\alpha\tilde{\psi}_\infty)| + c_1|(\partial_x^\alpha(-\Delta)^{-1}\mathrm{div} f_{3,\infty}, \partial_x^\alpha w_\infty)|$$
$$+ c_1\gamma^2|(\nabla\partial_x^\alpha\tilde{f}_{4,\infty}, \partial_x^\alpha\tilde{\psi}_\infty)| + c_1\gamma^2\|\partial_x^\alpha\tilde{f}_{4,\infty}\|_{L^2}^2\Big).$$

Proof. We take the inner product of ∂_x^α(3.40) with $\partial_x^\alpha\phi_\infty$ to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^\alpha\phi_\infty\|_{L^2}^2 - (\nabla\partial_x^\alpha\phi_\infty, \partial_x^\alpha w_\infty) = (\partial_x^\alpha f_{1,\infty}, \partial_x^\alpha\phi_\infty). \tag{3.45}$$

We take the inner product of ∂_x^α(3.41) with $\partial_x^\alpha w_\infty$ to obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^\alpha w_\infty\|_{L^2}^2 + \nu\|\nabla\partial_x^\alpha w_\infty\|_{L^2}^2 + \tilde{\nu}\|\mathrm{div}\partial_x^\alpha w_\infty\|_{L^2}^2\\ &+\gamma^2(\nabla\partial_x^\alpha\phi_\infty, \partial_x^\alpha w_\infty) - \beta^2(\mathrm{div}\partial_x^\alpha\tilde{\Psi}_\infty, \partial_x^\alpha w_\infty) = (\partial_x^\alpha f_{2,\infty}, \partial_x^\alpha w_\infty).\end{aligned} \tag{3.46}$$

We take the inner product of $\partial_x^\alpha(3.42)$ with $\partial_x^\alpha\tilde{\Psi}_\infty$ to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^\alpha\tilde{\Psi}_\infty\|_{L^2}^2 + (\mathrm{div}\partial_x^\alpha\tilde{\Psi}_\infty, \partial_x^\alpha w_\infty) = (\partial_x^\alpha f_{3,\infty}, \partial_x^\alpha\tilde{\Psi}_\infty). \tag{3.47}$$

It then follows from $\gamma^2\times(3.45)+(3.46)+\beta^2\times(3.47)$ that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\gamma^2\|\partial_x^\alpha\phi_\infty\|_{L^2}^2 + \|\partial_x^\alpha w_\infty\|_{L^2}^2 + \beta^2\|\partial_x^\alpha\tilde{\Psi}_\infty\|_{L^2}^2) + D^0[\partial_x^\alpha w_\infty]\\ &\quad\le \gamma^2|(\partial_x^\alpha f_{1,\infty}, \partial_x^\alpha\phi_\infty)| + |(\partial_x^\alpha f_{2,\infty}, \partial_x^\alpha w_\infty)| + \beta^2|(\partial_x^\alpha f_{3,\infty}, \partial_x^\alpha\tilde{\Psi}_\infty)|.\end{aligned} \tag{3.48}$$

Here $D^0[\partial_x^\alpha w_\infty]$ is given by

$$D^0[\partial_x^\alpha w_\infty] = \nu\|\nabla\partial_x^\alpha w_\infty\|_{L^2}^2 + \tilde{\nu}\|\mathrm{div}\partial_x^\alpha w_\infty\|_{L^2}^2.$$

We next derive the dissipative estimate of $\|\partial_x^\alpha\tilde{\Psi}_\infty\|_{L^2}$. By substituting (3.43) to (3.41), we have

$$\partial_t w_\infty - \nu\Delta w_\infty - \tilde{\nu}\nabla\mathrm{div}w_\infty - \beta^2\Delta\tilde{\psi}_\infty - \gamma^2\nabla\mathrm{div}\tilde{\psi}_\infty = f_{2,\infty} - \gamma^2\nabla\tilde{f}_{4,\infty}. \tag{3.49}$$

We take the inner product of $\partial_x^\alpha(3.49)$ with $\partial_x^\alpha\tilde{\psi}_\infty$ to obtain

$$\begin{aligned}&(\partial_t\partial_x^\alpha w_\infty, \partial_x^\alpha\tilde{\psi}_\infty) - \nu(\Delta\partial_x^\alpha w_\infty, \partial_x^\alpha\tilde{\psi}_\infty) - \tilde{\nu}(\nabla\mathrm{div}\partial_x^\alpha w_\infty, \partial_x^\alpha\tilde{\psi}_\infty)\\ &\quad -\beta^2(\Delta\partial_x^\alpha\tilde{\psi}_\infty, \partial_x^\alpha\tilde{\psi}_\infty) - \gamma^2(\nabla\mathrm{div}\partial_x^\alpha\tilde{\psi}_\infty, \partial_x^\alpha\tilde{\psi}_\infty)\\ &\qquad = (\partial_x^\alpha f_{2,\infty}, \partial_x^\alpha\tilde{\psi}_\infty) - \gamma^2(\nabla\partial_x^\alpha\tilde{f}_{4,\infty}, \partial_x^\alpha\tilde{\psi}_\infty).\end{aligned} \tag{3.50}$$

The first term on the left-hand side of (3.50) is written as

$$\begin{aligned}&(\partial_t\partial_x^\alpha w, \partial_x^\alpha\tilde{\psi}_\infty)\\ &= -\frac{\mathrm{d}}{\mathrm{d}t}(\partial_x^\alpha w, \partial_x^\alpha\tilde{\psi}_\infty) + (\partial_x^\alpha w_\infty, \partial_t\partial_x^\alpha\tilde{\psi}_\infty)\\ &= -\frac{\mathrm{d}}{\mathrm{d}t}(\partial_x^\alpha w_\infty, \partial_x^\alpha\tilde{\psi}_\infty) - \|\partial_x^\alpha w_\infty\|_{L^2}^2 - (\partial_x^\alpha(-\Delta)^{-1}\mathrm{div}f_{3,\infty}, \partial_x^\alpha w_\infty).\end{aligned}$$

By integration by parts, the fourth term and fifth term of (3.50) are written as $-(\Delta\partial_x^\alpha\tilde{\psi}_\infty, \partial_x^\alpha\tilde{\psi}_\infty) = \|\nabla\partial_x^\alpha\tilde{\psi}_\infty\|_{L^2}^2$ and $-(\nabla\mathrm{div}\partial_x^\alpha\tilde{\psi}_\infty, \partial_x^\alpha\tilde{\psi}_\infty) = \|\mathrm{div}\partial_x^\alpha\tilde{\psi}_\infty\|_{L^2}^2$, respectively. It then follows from (3.50) that

$$\begin{aligned}&-\frac{\mathrm{d}}{\mathrm{d}t}(\partial_x^\alpha w_\infty, \partial_x^\alpha\tilde{\psi}_\infty) + \beta^2\|\nabla\partial_x^\alpha\tilde{\psi}_\infty\|_{L^2}^2 + \gamma^2\|\mathrm{div}\partial_x^\alpha\tilde{\psi}_\infty\|_{L^2}^2\\ &= \nu(\Delta\partial_x^\alpha w_\infty, \partial_x^\alpha\tilde{\psi}_\infty) + \tilde{\nu}(\nabla\mathrm{div}\partial_x^\alpha w_\infty, \partial_x^\alpha\tilde{\psi}_\infty) + \|\partial_x^\alpha w_\infty\|_{L^2}^2\\ &+(\partial_x^\alpha f_{2,\infty}, \partial_x^\alpha\tilde{\psi}_\infty) + (\partial_x^\alpha(-\Delta)^{-1}\mathrm{div}f_{3,\infty}, \partial_x^\alpha w_\infty) - \gamma^2(\nabla\partial_x^\alpha\tilde{f}_{4,\infty}, \partial_x^\alpha\tilde{\psi}_\infty).\end{aligned} \tag{3.51}$$

By integration by parts and the inequality $\|\partial_x^\alpha w_\infty\|_{L^2} \leq \|\nabla \partial_x^\alpha w_\infty\|_{L^2}$ following from Lemma 3.10 (iii), we have

$$
\begin{aligned}
&\nu(\Delta \partial_x^\alpha w_\infty, \partial_x^\alpha \tilde{\psi}_\infty) + \tilde{\nu}(\nabla \mathrm{div} \partial_x^\alpha w_\infty, \partial_x^\alpha \tilde{\psi}_\infty) + \|\partial_x^\alpha w_\infty\|_{L^2}^2 \\
&= -\nu(\nabla \partial_x^\alpha w_\infty, \nabla \partial_x^\alpha \tilde{\psi}_\infty) + \tilde{\nu}(\mathrm{div} \partial_x^\alpha w_\infty, \mathrm{div} \partial_x^\alpha \tilde{\psi}_\infty) + \|\partial_x^\alpha w_\infty\|_{L^2}^2 \\
&\leq \left(\frac{\nu}{2\beta^2} + \frac{\tilde{\nu}}{2\gamma^2} + \frac{1}{\nu} \right) D^0[\partial_x^\alpha w_\infty] + \frac{\beta^2}{2} \|\nabla \partial_x^\alpha \tilde{\psi}_\infty\|_{L^2}^2 + \frac{\gamma^2}{2} \|\mathrm{div} \partial_x^\alpha \tilde{\psi}_\infty\|_{L^2}^2.
\end{aligned}
$$

It then follows from (3.51) that

$$
\begin{aligned}
&-2\frac{\mathrm{d}}{\mathrm{d}t}(\partial_x^\alpha w_\infty, \partial_x^\alpha \tilde{\psi}_\infty) + \beta^2 \|\nabla \partial_x^\alpha \tilde{\psi}_\infty\|_{L^2}^2 + \gamma^2 \|\mathrm{div} \partial_x^\alpha \tilde{\psi}_\infty\|_{L^2}^2 \\
&\quad \leq \left(\frac{\nu}{\beta^2} + \frac{\tilde{\nu}}{\gamma^2} + \frac{2}{\nu} \right) D^0[\partial_x^\alpha w_\infty] + |(\partial_x^\alpha f_{2,\infty}, \partial_x^\alpha \tilde{\psi}_\infty)| \\
&\qquad + |(\partial_x^\alpha (-\Delta)^{-1} \mathrm{div} f_{3,\infty}, \partial_x^\alpha w_\infty)| + \gamma^2 |(\nabla \partial_x^\alpha \tilde{f}_{4,\infty}, \partial_x^\alpha \tilde{\psi}_\infty)|.
\end{aligned}
\tag{3.52}
$$

Adding (3.48) to $\frac{c_1}{2} \times$(3.52) and using $\|\partial_x^\alpha \phi_\infty\|_{L^2}^2 \leq 2(\|\mathrm{div} \partial_x^\alpha \tilde{\psi}_\infty\|_{L^2}^2 + \|\partial_x^\alpha \tilde{f}_{4,\infty}\|_{L^2}^2)$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\gamma^2 \|\partial_x^\alpha \phi_\infty\|_{L^2}^2 + \|\partial_x^\alpha w_\infty\|_{L^2}^2 + \beta^2 \|\partial_x^\alpha \tilde{\Psi}_\infty\|_{L^2}^2 - 2c_1(\partial_x^\alpha w_\infty, \partial_x^\alpha \tilde{\psi}_\infty)) \\
&+ \left(1 - c_1 \left(\frac{\nu}{\beta^2} + \frac{\tilde{\nu}}{\gamma^2} + \frac{2}{\nu} \right) \right) D^0[\partial_x^\alpha w_\infty] + \frac{c_1}{2} \gamma^2 \|\partial_x^\alpha \phi_\infty\|_{L^2}^2 + c_1 \beta^2 \|\partial_x^\alpha \tilde{\Psi}_\infty\|_{L^2}^2 \\
&\quad \leq \gamma^2 |(\partial_x^\alpha f_{1,\infty}, \partial_x^\alpha \phi_\infty)| + |(\partial_x^\alpha f_{2,\infty}, \partial_x^\alpha w_\infty)| + \beta^2 |(\partial_x^\alpha f_{3,\infty}, \partial_x^\alpha \tilde{\Psi}_\infty)| \\
&\qquad + 2c_1 |(\partial_x^\alpha f_{2,\infty}, \partial_x^\alpha \tilde{\psi}_\infty)| + 2c_1 |(\partial_x^\alpha (-\Delta)^{-1} \mathrm{div} f_{3,\infty}, \partial_x^\alpha w_\infty)| \\
&\qquad\quad + 2c_1 \gamma^2 |(\nabla \partial_x^\alpha f_{4,\infty}, \partial_x^\alpha \tilde{\psi}_\infty)| + c_1 \gamma^2 \|\partial_x^\alpha \tilde{f}_{4,\infty}\|_{L^2}^2.
\end{aligned}
\tag{3.53}
$$

We take $c_1 > 0$ small so that $1 - c_1 \left(\frac{\nu}{\beta^2} + \frac{\tilde{\nu}}{\gamma^2} + \frac{2}{\nu} \right) \geq \frac{1}{2}$ and $d_1 \|\tilde{U}_\infty(t)\|_{H^2}^2 \leq E[\tilde{U}_\infty](t) \leq d_2 D[\tilde{U}_\infty](t)$ for some positive numbers $d_1,\ d_2 > 0$. By summing (3.53) for $|\alpha| = 2$, we obtain (3.44). This completes the proof. ■

We next estimate $\mathcal{N}(t)$.

Proposition 3.17. *The following estimate holds uniformly in $t \geq 0$*

$$
\mathcal{N}(t) \leq C_2 \delta_0 D[\tilde{U}_\infty](t) + C_2 (1+t)^{-4} \|u_0\|_{\mathcal{X}}^2. \tag{3.54}
$$

Proposition 3.17 can be shown by using Lemma 2.1, Proposition 3.8, and integration by parts. We omit the proof.

Proof of Proposition 3.15 (continued). By taking δ_0 so that $C_1C_2\delta_0 \le \frac{1}{2}$, it follows from (3.44) and (3.54) that

$$\frac{\mathrm{d}}{\mathrm{d}t}E[\tilde{U}_\infty] + \frac{1}{2d_2}E[\tilde{U}_\infty] \le C(1+t)^{-4}\|u_0\|_{\mathcal{X}}^2. \tag{3.55}$$

Therefore, we see from Lemma 2.5 with $a = b = 4$ and (3.55) that

$$\begin{aligned} E[\tilde{U}_\infty](t) &\le e^{-ct}E[\tilde{U}_\infty](0) + C\int_0^t e^{-c(t-s)}(1+s)^{-4}\mathrm{d}s\|u_0\|_{\mathcal{X}}^2 \\ &\le C(1+t)^{-4}\|\tilde{U}_0\|_{H^2}^2 + C\int_0^t (1+t-s)^{-4}(1+s)^{-4}\mathrm{d}s\|u_0\|_{\mathcal{X}}^2 \\ &\le C(1+t)^{-4}\|u_0\|_{\mathcal{X}}^2. \end{aligned}$$

Since $\|U_\infty(t)\|_{L^\infty}^2 \le CE[\tilde{U}_\infty](t) + C(1+t)^{-4}\|u_0\|_{\mathcal{X}}^2$, we finally arrive at

$$\|U_\infty(t)\|_{L^\infty} \le C(1+t)^{-2}\|u_0\|_{\mathcal{X}}.$$

This completes the proof of Proposition 3.15. ■

Proof of Proposition 3.7. Proposition 3.7 immediately follows from Proposition 3.11 and Proposition 3.15. This completes the proof. ■

3.4 Proof of Theorem 3.2 (ii)

In this section, we give a proof of Theorem 3.2 (ii). In view of Lemmata 3.4–3.5, it suffices to obtain the following L^p decay estimate of $U = (\phi, w, \Psi)$ to prove Theorem 3.2 (ii).

Proposition 3.18. *There exists a positive number δ_0 such that if $\|u_0\|_{L^1} + \|u_0\|_{H^3} \le \delta_0$, then the following inequality*

$$\|U(t)\|_{L^p} \le C(p)(1+t)^{-\frac{3}{2}\left(1-\frac{1}{p}\right)+\frac{1}{2}\left(\frac{2}{p}-1\right)}(\|u_0\|_{L^1} + \|u_0\|_{L^p} + \|u_0\|_{H^3})$$

holds for $1 < p < 2$ and $t \ge 0$.

Proposition 3.18 is a direct consequence of the L^p estimates of $U_1(t)$ and $U_\infty(t)$ which will be established below in Proposition 3.19 and Proposition 3.25, respectively.

We first consider the low frequency part $U_1(t)$.

Proposition 3.19. *There exists a positive number δ_0 such that if $\|u_0\|_{\mathcal{X}} \le \delta_0$, then it holds the following estimate:*

$$\|U_1(t)\|_{L^p} \le C(1+t)^{-\frac{3}{2}\left(1-\frac{1}{p}\right)+\frac{1}{2}\left(\frac{2}{p}-1\right)}\|u_0\|_{\mathcal{X}}.$$

Since $\|U_1(t)\|_{L^p} \le \|U_1(t)\|_{L^1}^{\frac{2}{p}-1} \|U_1(t)\|_{L^2}^{2-\frac{2}{p}}$, $1 < p < 2$, it is enough to show the case $p = 1$ only:

$$\|U_1(t)\|_{L^1} \le C(1+t)^{\frac{1}{2}} \|u_0\|_{\mathcal{X}}. \tag{3.56}$$

To show (3.56), we introduce the following lemma.

Lemma 3.20. *Let $f \in L^1$. Then, the following estimates hold for $j \ge 0$, $\alpha \in (\{0\} \cup \mathbb{N})^3$ and $t \ge 0$:*

$$\left\| \partial_t^j \partial_x^\alpha \mathcal{F}^{-1} \left[\frac{e^{\mu_1(\xi)t} - e^{\mu_2(\xi)t}}{\mu_1(\xi) - \mu_2(\xi)} \hat{\varphi}_1(\xi) \right] \right\|_{L^1} \le C(1+t)^{1-\frac{j+|\alpha|}{2}},$$
$$\left\| \partial_t^j \partial_x^\alpha \mathcal{F}^{-1} \left[\frac{\mu_1(\xi)e^{\mu_2(\xi)t} - \mu_2(\xi)e^{\mu_1(\xi)t}}{\mu_1(\xi) - \mu_2(\xi)} \hat{\varphi}_1(\xi) \right] \right\|_{L^1} \le C(1+t)^{\frac{1}{2}-\frac{j+|\alpha|}{2}},$$
$$\left\| \partial_t^j \partial_x^\alpha \mathcal{F}^{-1} \left[\frac{e^{\mu_3(\xi)t} - e^{\mu_4(\xi)t}}{\mu_3(\xi) - \mu_4(\xi)} \hat{\varphi}_1(\xi) \right] \right\|_{L^1} \le C(1+t)^{1-\frac{j+|\alpha|}{2}},$$
$$\left\| \partial_t^j \partial_x^\alpha \mathcal{F}^{-1} \left[\frac{\mu_3(\xi)e^{\mu_4(\xi)t} - \mu_4(\xi)e^{\mu_3(\xi)t}}{\mu_3(\xi) - \mu_4(\xi)} \hat{\varphi}_1(\xi) \right] \right\|_{L^1} \le C(1+t)^{\frac{1}{2}-\frac{j+|\alpha|}{2}},$$
$$\left\| \partial_t^j \partial_x^\alpha \mathcal{F}^{-1} \left[\left(\frac{\mu_1(\xi)e^{\mu_1(\xi)t} - \mu_2(\xi)e^{\mu_2(\xi)t}}{\mu_1(\xi) - \mu_2(\xi)} - e^{-\nu|\xi|^2 t} \right) \frac{\xi^\top \xi}{|\xi|} \hat{\varphi}_1(\xi) \right] \right\|_{L^1}$$
$$\le C(1+t)^{\frac{1}{2}-\frac{j+|\alpha|}{2}},$$
$$\left\| \partial_t^j \partial_x^\alpha \mathcal{F}^{-1} \left[\left(\frac{\mu_3(\xi)e^{\mu_3(\xi)t} - \mu_4(\xi)e^{\mu_4(\xi)t}}{\mu_3(\xi) - \mu_4(\xi)} - e^{-\nu|\xi|^2 t} \right) \frac{\xi^\top \xi}{|\xi|} \hat{\varphi}_1(\xi) \right] \right\|_{L^1}$$
$$\le C(1+t)^{\frac{1}{2}-\frac{j+|\alpha|}{2}}.$$

Lemma 3.20 is obtained in [20, pp.216] and [30, pp.216] directly.

We have the estimate of $\|e^{-tL}U_1(0)\|_{L^1}$.

Lemma 3.21. *The following estimate holds for $t \ge 0$:*

$$\|e^{-tL}U_1(0)\|_{L^1} \le C(1+t)^{\frac{1}{2}} \|u_0\|_{L^1}.$$

Proof. The L^1 estimates of $\mathcal{F}^{-1}[\hat{\varphi}_1(\xi)\hat{K}^{j1}(\xi,t)\hat{\phi}_0(\xi)]$ $(j = 1,2,3)$, and $\mathcal{F}^{-1}[\hat{\varphi}_1(\xi)\hat{K}^{12}(\xi,t)\hat{w}_0(\xi)]$ immediately follow from Lemma 3.20:

$$\|\mathcal{F}^{-1}[\hat{\varphi}_1(\xi)\hat{K}^{j1}(\xi,t)\hat{\phi}_0(\xi)]\|_{L^1} \le C(1+t)^{\frac{1}{2}} \|\phi_0\|_{L^1},\ j = 1,2,3, \tag{3.57}$$
$$\|\mathcal{F}^{-1}[\hat{\varphi}_1(\xi)\hat{K}^{12}(\xi,t)\hat{w}_0(\xi)]\|_{L^1} \le C(1+t)^{\frac{1}{2}} \|w_0\|_{L^1}. \tag{3.58}$$

Since

$$\hat{K}^{22}(\xi,t)\hat{w}_0 = \frac{\mu_1(\xi)e^{\mu_1(\xi)t} - \mu_2(\xi)e^{\mu_2(\xi)t}}{\mu_1(\xi) - \mu_2(\xi)} \hat{w}_0(\xi)$$

$$
\begin{aligned}
&- \left(\frac{\mu_1(\xi) e^{\mu_1(\xi)t} - \mu_2(\xi) e^{\mu_2(\xi)t}}{\mu_1(\xi) - \mu_2(\xi)} - e^{-\nu|\xi|^2 t} \right) \frac{\xi^\top \xi}{|\xi|^2} \hat{w}_0(\xi) \\
&\quad + \left(\frac{\mu_3(\xi) e^{\mu_3(\xi)t} - \mu_4(\xi) e^{\mu_4(\xi)t}}{\mu_3(\xi) - \mu_4(\xi)} - e^{-\nu|\xi|^2 t} \right) \frac{\xi^\top \xi}{|\xi|^2} \hat{w}_0(\xi),
\end{aligned}
$$

$$
\begin{aligned}
\hat{K}^{33}(\xi,t)\hat{\Psi}_0 = {} & \frac{\mu_1(\xi) e^{\mu_2(\xi)t} - \mu_2(\xi) e^{\mu_1(\xi)t}}{\mu_1(\xi) - \mu_2(\xi)} \hat{\Psi}_0(\xi) \\
& - \left(\frac{\mu_1(\xi) e^{\mu_1(\xi)t} - \mu_2(\xi) e^{\mu_2(\xi)t}}{\mu_1(\xi) - \mu_2(\xi)} - e^{-\nu|\xi|^2 t} \right) \frac{\xi^\top \xi}{|\xi|^2} \hat{\Psi}_0(\xi) \\
& - \nu \frac{e^{\mu_1(\xi)t} - e^{\mu_2(\xi)t}}{\mu_1(\xi) - \mu_2(\xi)} \xi^\top \xi \hat{\Psi}_0(\xi) \\
& + \left(\frac{\mu_3(\xi) e^{\mu_3(\xi)t} - \mu_4(\xi) e^{\mu_4(\xi)t}}{\mu_3(\xi) - \mu_4(\xi)} - e^{-\nu|\xi|^2 t} \right) \frac{\xi^\top \xi}{|\xi|^2} \hat{\Psi}_0(\xi) \\
& + (\nu + \tilde{\nu}) \frac{e^{\mu_3(\xi)t} - e^{\mu_4(\xi)t}}{\mu_3(\xi) - \mu_4(\xi)} \xi^\top \xi \hat{\Psi}_0(\xi),
\end{aligned}
$$

we see from Lemma 3.20 that

$$
\|\mathcal{F}^{-1}[\hat{\varphi}_1(\xi)\hat{K}^{22}(\xi,t)\hat{w}_0(\xi)]\|_{L^1} \le C(1+t)^{\frac{1}{2}} \|w_0\|_{L^1}, \tag{3.59}
$$

$$
\|\mathcal{F}^{-1}[\hat{\varphi}_1(\xi)\hat{K}^{33}(\xi,t)\hat{\Psi}_0(\xi)]\|_{L^1} \le C(1+t)^{\frac{1}{2}} \|\Psi_0\|_{L^1}. \tag{3.60}
$$

It remains to estimate $\mathcal{F}^{-1}[\hat{\varphi}_1(\xi)\hat{K}^{23}(\xi,t)\hat{\Psi}_0]$ and $\mathcal{F}^{-1}[\hat{\varphi}_1(\xi)\hat{K}^{32}(\xi,t)\hat{w}_0]$.

We write $(\mathcal{F}^{-1}[\hat{\varphi}_1(\xi)\hat{K}^{23}(\xi,t)\hat{\Psi}_0])^j,\ j = 1,2,3$, and $(\mathcal{F}^{-1}[\hat{\varphi}_1(\xi)\hat{K}^{32}(\xi,t)\hat{w}_0])^{jk}$, $j,k = 1,2,3$, as

$$
\begin{aligned}
&(\mathcal{F}^{-1}[\hat{\varphi}_1(\xi)\hat{K}^{23}(\xi,t)\hat{\Psi}_0])^j \\
&= \beta^2 \sum_{k=1}^{3} (\mathcal{K}_0^k * \hat{\Psi}_0^{jk})(\xi) - \beta^2 \sum_{k,l=1}^{3} (\mathcal{K}_1 * \mathcal{L}_{j,k,l}^{\frac{\nu}{4}} * \hat{\Psi}_0^{lk})(\xi) \\
&\quad + \beta^2 \sum_{k,l=1}^{3} (\mathcal{K}_2 * \mathcal{L}_{j,k,l}^{\frac{\nu+\tilde{\nu}}{4}} * \hat{\Psi}_0^{lk})(\xi), \\
&(\mathcal{F}^{-1}[\hat{\varphi}_1(\xi)\hat{K}^{32}(\xi,t)\hat{w}_0])^{jk} \\
&= (\mathcal{K}_0^k * \hat{w}_0^j)(\xi) - \sum_{l=1}^{3} (\mathcal{K}_1 * \mathcal{L}_{j,k,l}^{\frac{\nu}{4}} * \hat{w}_0^l)(\xi) + \sum_{l=1}^{3} (\mathcal{K}_2 * \mathcal{L}_{j,k,l}^{\frac{\nu+\tilde{\nu}}{4}} * \hat{w}_0^l)(\xi),
\end{aligned}
$$

where

$$
\mathcal{K}_0^k = \mathcal{F}^{-1}\left[i\xi_k \frac{e^{\mu_1(\xi)t} - e^{\mu_2(\xi)t}}{\mu_1(\xi) - \mu_2(\xi)} \hat{\varphi}_1(\xi) \right],\ k = 1,2,3,
$$

$$\mathcal{K}_1 = \mathcal{F}^{-1}\left[\frac{e^{\tilde{\mu}_1(\xi)t} - e^{\tilde{\mu}_2(\xi)t}}{\tilde{\mu}_1(\xi) - \tilde{\mu}_2(\xi)}\hat{\varphi}_1(\xi)\right],$$
$$\mathcal{K}_2 = \mathcal{F}^{-1}\left[\frac{e^{\tilde{\mu}_3(\xi)t} - e^{\tilde{\mu}_4(\xi)t}}{\tilde{\mu}_3(\xi) - \tilde{\mu}_4(\xi)}\hat{\varphi}_1(\xi)\right],$$
$$\mathcal{L}^a_{l,jk} = \mathcal{F}^{-1}\left[i\xi_l\frac{\xi_j\xi_k}{|\xi|^2}e^{-a|\xi|^2}\right], \ a > 0, \ j,k,l = 1,2,3.$$

Here $\tilde{\mu}_j(\xi), \ j = 1,2,3,4$, are denoted by

$$\mu_j(\xi) = -\frac{\nu}{4}|\xi|^2 + \tilde{\mu}_j(\xi), \ j = 1,2,$$
$$\mu_j(\xi) = -\frac{\nu+\tilde{\nu}}{4}|\xi|^2 + \tilde{\mu}_j(\xi), \ j = 3,4.$$

The estimates of $\mathcal{K}_1$ and $\mathcal{K}_2$ follow from Lemma 3.20. As for $\mathcal{K}^k_0$, we use the following L^1 estimate of $\mathcal{L}^a_{l,jk}$ shown by Fujigaki and Miyakawa [5, pp.525–526].

Lemma 3.22. *Let $a > 0$ and $j,k,l = 1,2,3$. Then, the following inequality holds for $t \geq 0$:*

$$\|\mathcal{L}^a_{l,jk}(\cdot,t)\|_{L^1} \leq C_a t^{-\frac{1}{2}}.$$

By using Lemma 3.20, Lemma 3.22 and the Young inequality, we obtain

$$\|\mathcal{F}^{-1}[\hat{\varphi}_1(\xi)\hat{K}^{23}(\xi,t)\hat{\Psi}_0(\xi)]\|_{L^1} \leq C((1+t)^{\frac{1}{2}} + t^{-\frac{1}{2}})\|\Psi_0\|_{L^1}, \tag{3.61}$$
$$\|\mathcal{F}^{-1}[\hat{\varphi}_1(\xi)\hat{K}^{32}(\xi,t)\hat{w}_0(\xi)]\|_{L^1} \leq C((1+t)^{\frac{1}{2}} + t^{-\frac{1}{2}})\|w_0\|_{L^1}. \tag{3.62}$$

We next show the following uniform bounds with respect to $0 \leq t \leq 1$:

$$\|\mathcal{F}^{-1}[\hat{\varphi}_1(\xi)\hat{K}^{23}(\xi,t)\hat{\Psi}_0(\xi)]\|_{L^1} \leq C\|\Psi_0\|_{L^1}, \ t \geq 0, \tag{3.63}$$
$$\|\mathcal{F}^{-1}[\hat{\varphi}_1(\xi)\hat{K}^{32}(\xi,t)\hat{w}_0(\xi)]\|_{L^1} \leq C\|w_0\|_{L^1}, \ t \geq 0. \tag{3.64}$$

To derive (3.63) and (3.64), we prepare the following lemma proved in [31].

Lemma 3.23. ([31]) *Let $\alpha = N + \sigma - 3$, where $N \geq 0$ is an integer and $0 < \sigma \leq 1$. Let f be a function such that*

$$f \in C^\infty(\mathbb{R}^3 - \{0\}),$$
$$\partial^\eta_\xi f \in L^1(\mathbb{R}^3), \ |\eta| \leq N,$$
$$|\partial^\eta_\xi f(\xi)| \leq C_\eta|\xi|^{\alpha-|\eta|}, \ \xi \neq 0.$$

Then, we have

$$|\mathcal{F}^{-1}[f(\xi)](x)| \leq C_\alpha\left(\max_{|\eta|\leq N+2} C_\eta\right)|x|^{-3-|\alpha|}, \ x \neq 0.$$

By Taylor's formula we have

$$\frac{e^{\mu_1(\xi)t}-e^{\mu_2(\xi)t}}{\mu_1(\xi)-\mu_2(\xi)}\frac{\xi_j\xi_k\xi_l}{|\xi|^2}\varphi_1(\xi)=\frac{t}{2}e^{-\frac{\nu}{2}|\xi|^2t}\int_0^1 e^{i\beta|\xi|f(|\xi|)st}\mathrm{d}s\frac{\xi_j\xi_k\xi_l}{|\xi|^2}\varphi_1(\xi)$$

for $|\xi|\le\frac{M_1}{\sqrt{2}}$, where $f(|\xi|)=\sqrt{1-\frac{\nu^2}{4\beta^2}|\xi|^2}$.

It then follows from the above formula that

$$\left|\partial_\xi^\eta\left(\frac{e^{\mu_1(\xi)t}-e^{\mu_2(\xi)t}}{\mu_1(\xi)-\mu_2(\xi)}\frac{\xi_j\xi_k\xi_l}{|\xi|^2}\varphi_1(\xi)\right)\right|\le C_\eta|\xi|^{1-|\eta|}\text{ for }|\xi|\le\frac{M_1}{\sqrt{2}}.$$

We next use Lemma 3.23 with $(\alpha,N,\sigma)=(1,3,1)$ and calculate in a similar argument as in [20, pp.228–229] to obtain

$$\left\|\mathcal{F}^{-1}\left[\frac{e^{\mu_1(\xi)t}-e^{\mu_2(\xi)t}}{\mu_1(\xi)-\mu_2(\xi)}\frac{\xi_j\xi_k\xi_l}{|\xi|^2}\varphi_1(\xi)\right]\right\|_{L^1}\le C,\ 0\le t\le 1.$$

Similarly, we can prove

$$\left\|\mathcal{F}^{-1}\left[\frac{e^{\mu_3(\xi)t}-e^{\mu_4(\xi)t}}{\mu_3(\xi)-\mu_4(\xi)}\frac{\xi_j\xi_k\xi_l}{|\xi|^2}\varphi_1(\xi)\right]\right\|_{L^1}\le C,\ 0\le t\le 1.$$

We thus arrive at (3.63) and (3.64).

By (3.61)–(3.64), we have

$$\|\mathcal{F}^{-1}[\hat{\varphi}_1(\xi)\hat{K}^{23}(\xi,t)\hat{\varPsi}_0(\xi)]\|_{L^1}\le C(1+t)^{\frac{1}{2}}\|\varPsi_0\|_{L^1},\ t\ge 0,\tag{3.65}$$

$$\|\mathcal{F}^{-1}[\hat{\varphi}_1(\xi)\hat{K}^{32}(\xi,t)\hat{w}_0(\xi)]\|_{L^1}\le C(1+t)^{\frac{1}{2}}\|w_0\|_{L^1},\ t\ge 0.\tag{3.66}$$

We see from (3.57)–(3.60), (3.65) and (3.66) that

$$\|e^{-tL}U_1(0)\|_{L^1}\le C(1+t)^{\frac{1}{2}}\|u_0\|_{L^1}.$$

This completes the proof of Lemma 3.21. ■

We next estimate $\int_0^t\|e^{-(t-s)L}P_1N(s)\|_{L^1}\mathrm{d}s$.

Lemma 3.24. *There exists a positive number δ_0 such that if $\|u_0\|_{\mathcal{X}}\le\delta_0$, then the following estimate holds:*

$$\int_0^t\|e^{-(t-s)L}P_1N(s)\|_{L^1}\mathrm{d}s\le C(1+t)^{\frac{1}{2}}\|u_0\|_{\mathcal{X}},\ t\ge 0.$$

Proof. We obtain the following estimate in a similar argument as in the proof of Lemma 3.14 by using (3.32), (3.33), (3.35), (3.36) and Lemma 3.20 :

$$\begin{aligned}&\left\|\mathcal{F}^{-1}\left[\hat{\varphi}_1(\xi)\hat{K}^{jk}(\xi,t-s)\hat{N}_k(\xi,s)\right]\right\|_{L^1}\\&\quad\leq C(1+t-s)^{-2}(1+s)^{\frac{1}{2}}\|u_0\|_{\mathcal{X}},\ j,k=1,2,3.\end{aligned}\tag{3.67}$$

By using Lemma 2.5 with $a=b=2$, we have

$$\begin{aligned}&\int_0^t(1+t-s)^{-2}(1+s)^{\frac{1}{2}}\mathrm{d}s\\&\qquad\leq(1+t)^{\frac{5}{2}}\int_0^t(1+t-s)^{-2}(1+s)^{-2}\mathrm{d}s\\&\qquad\leq C(1+t)^{\frac{1}{2}}.\end{aligned}\tag{3.68}$$

We then see from (3.67) and (3.68) that

$$\int_0^t\|e^{-(t-s)L}P_1N(s)\|_{L^1}\mathrm{d}s\leq C(1+t)^{\frac{1}{2}}\|u_0\|_{\mathcal{X}},\ t\geq0.$$

This completes the proof. ■

Proof of Proposition 3.19. Taking L^1 norm of the first equation of (3.29), we obtain

$$\|U_1(t)\|_{L^1}\leq\|e^{-tL}U_1(0)\|_{L^1}+\int_0^t\|e^{-(t-s)L}P_1N(s)\|_{L^1}\mathrm{d}s.$$

This completes the proof. ■

We next consider the high frequency part U_∞.

Proposition 3.25. *There exists a positive number δ_p such that if $\|u_0\|_{\mathcal{X}}\leq\delta_p$, then it holds the following estimate for $t\geq0$:*

$$\|U_\infty(t)\|_{L^p}\leq C(1+t)^{-\frac{3}{2}\left(1-\frac{1}{p}\right)+\frac{1}{2}\left(\frac{2}{p}-1\right)}(\|u_0\|_{L^p}+\|u_0\|_{\mathcal{X}}).$$

In order to prove Proposition 3.25, we prepare the following lemma.

Lemma 3.26. *Let $1<p<\infty$ and $f\in L^p$. Then, the following estimates hold for $t\geq0$:*

$$\left\|\partial_t^j\partial_x^\alpha\mathcal{F}^{-1}\left[\frac{e^{\mu_1(\xi)t}-e^{\mu_2(\xi)t}}{\mu_1(\xi)-\mu_2(\xi)}\hat{\varphi}_\infty(\xi)\hat{f}(\xi)\right]\right\|_{L^p}\leq Ce^{-ct}\|f\|_{L^p}\ j+|\alpha|=1,$$

$$\left\|\mathcal{F}^{-1}\left[\frac{\mu_1(\xi)e^{\mu_2(\xi)t}-\mu_2(\xi)e^{\mu_1(\xi)t}}{\mu_1(\xi)-\mu_2(\xi)}\hat{\varphi}_\infty(\xi)\hat{f}(\xi)\right]\right\|_{L^p}\le Ce^{-ct}\|f\|_{L^p},$$
$$\left\|\partial_t^j\partial_x^\alpha\mathcal{F}^{-1}\left[\frac{e^{\mu_3(\xi)t}-e^{\mu_4(\xi)t}}{\mu_3(\xi)-\mu_4(\xi)}\hat{\varphi}_\infty(\xi)\hat{f}(\xi)\right]\right\|_{L^p}\le Ce^{-ct}\|f\|_{L^p},\ j+|\alpha|=1,$$
$$\left\|\mathcal{F}^{-1}\left[\frac{\mu_3(\xi)e^{\mu_4(\xi)t}-\mu_4(\xi)e^{\mu_3(\xi)t}}{\mu_3(\xi)-\mu_4(\xi)}\hat{\varphi}_\infty(\xi)\hat{f}(\xi)\right]\right\|_{L^p}\le Ce^{-ct}\|f\|_{L^p}.$$

Lemma 3.26 directly follows from [30, Theorem 4.1]. We first consider $\|e^{-tL}U_\infty(0)\|_{L^p}$.

Lemma 3.27. *The following estimate holds for $t\ge 0$:*

$$\|e^{-tL}U_\infty(0)\|_{L^p}\le Ce^{-ct}\|u_0\|_{L^p}. \tag{3.69}$$

Proof. The estimate (3.69) can be shown by using Lemma 3.5, Lemma 3.26 and the L^p boundedness of the Riesz operator. This completes the proof. ■

We next estimate $\int_0^t\|e^{-(t-s)L}P_\infty N(s)\|_{L^p}\mathrm{d}s$.

Lemma 3.28. *There exists a positive number δ_p such that if $\|u_0\|_{\mathcal{X}}\le\delta_p$, then the following estimate holds:*

$$\int_0^t\|e^{-(t-s)L}P_\infty N(s)\|_{L^p}\mathrm{d}s\le C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})+\frac{1}{2}(\frac{2}{p}-1)}\|u_0\|_{\mathcal{X}},\ t\ge 0.$$

Proof. We obtain the following estimate in a similar argument as in the proof of Lemma 3.27:

$$\begin{aligned}\left\|\mathcal{F}^{-1}\left[\hat{\varphi}_\infty(\xi)\hat{K}^{jk}(\xi,t-s)\hat{N}_k(\xi,s)\right]\right\|_{L^p}&\le Ce^{-c(t-s)}\|N_k(s)\|_{L^p},\\ &j=1,2,3,\ k=1,2.\end{aligned} \tag{3.70}$$

In view of Lemma 2.1 and the L^p boundedness of the Riesz operator, we have

$$\|N_k(s)\|_{L^p}\le C\|U(s)\|_{H^2}^2\le C(1+t)^{-\frac{3}{2}}\|u_0\|_{\mathcal{X}},\ k=1,2,3. \tag{3.71}$$

By employing Lemma 2.5 with $a=b=\frac{3}{2}$, we have

$$\begin{aligned}\int_0^t e^{-c(t-s)}(1+s)^{-\frac{3}{2}}\mathrm{d}s&\le C(1+t)^{-\frac{3}{2}}\\ &\le C(1+t)^{-2+\frac{5}{2p}}\\ &=C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})+\frac{1}{2}(\frac{2}{p}-1)}.\end{aligned} \tag{3.72}$$

Together with (3.70)–(3.72) yields

$$\int_0^t \|e^{-(t-s)L}P_\infty N(s)\|_{L^p}\mathrm{d}s \leq C(1+s)^{-\frac{3}{2}\left(1-\frac{1}{p}\right)+\frac{1}{2}\left(\frac{2}{p}-1\right)}\|u_0\|_{\mathcal{X}}.$$

This completes the proof. ■

Proof of Proposition 3.25. By taking L^p norm of the first equation of (3.30), we have

$$\|U_\infty(t)\|_{L^p} \leq \|e^{-tL}U_\infty(0)\|_{L^p} + \int_0^t \|e^{-(t-s)L}P_\infty N(s)\|_{L^p}\mathrm{d}s. \tag{3.73}$$

Combining Lemma 3.27, Lemma 3.28 and (3.73), we arrive at

$$\|U_\infty(t)\|_{L^p} \leq C(1+t)^{-\frac{3}{2}\left(1-\frac{1}{p}\right)+\frac{1}{2}\left(\frac{2}{p}-1\right)}(\|u_0\|_{L^p} + \|u_0\|_{\mathcal{X}}),\ t \geq 0.$$

This completes the proof of Proposition 3.25. ■

Appendix 3.A Proof of Lemma 3.9

In this appendix, we derive the solution formula (3.28).

Proof of Lemma 3.9. We write (3.27) as

$$\partial_t\hat{\phi} + i\xi\cdot\hat{w} = 0, \tag{3.A.1}$$

$$\partial_t\hat{w} + \nu|\xi|^2\hat{w} + \tilde{\nu}\xi^\top\xi\hat{w} + i\gamma^2\xi\hat{\phi} - i\beta^2\hat{\Psi}\xi = 0, \tag{3.A.2}$$

$$\partial_t\hat{\Psi} - i\hat{w}^\top\xi = 0, \tag{3.A.3}$$

$$\hat{\phi} + i\xi\cdot\hat{\psi} = 0,\ \hat{\Psi} = i\hat{\psi}^\top\xi, \tag{3.A.4}$$

$$(\hat{\phi},\hat{w},\hat{\Psi})|_{t=0} = (\hat{\phi}_0,\hat{w}_0,\hat{\Psi}_0),\ \hat{\phi}_0 + i\xi\cdot\hat{\psi}_0 = 0. \tag{3.A.5}$$

Setting $w_t = \partial_t w$, we see from (3.A.2)–(3.A.4) that

$$\partial_t\begin{pmatrix}\hat{w}\\ \hat{w}_t\end{pmatrix} + \mathcal{A}(\xi)\begin{pmatrix}\hat{w}\\ \hat{w}_t\end{pmatrix} = 0,\ \left.\begin{pmatrix}\hat{w}\\ \hat{w}_t\end{pmatrix}\right|_{t=0} = \begin{pmatrix}\hat{w}_0\\ \hat{w}_{t,0}\end{pmatrix}. \tag{3.A.6}$$

Here

$$\mathcal{A}(\xi) = \begin{pmatrix}0 & -I\\ \beta^2|\xi|^2I + \gamma^2\xi^\top\xi & \nu|\xi|^2I + \tilde{\nu}\xi^\top\xi\end{pmatrix}$$

and

$$\hat{w}_{t,0} = -i\gamma^2\xi\hat{\phi}_0 - (\nu|\xi|^2 I + \tilde{\nu}\xi^\top\xi)\hat{w}_0 + i\beta^2\hat{\Psi}_0\xi. \tag{3.A.7}$$

To solve (3.28), we first investigate the characteristic equation of $-\mathcal{A}(\xi)$. Let T be a 3×3 orthogonal matrix and set

$$\mathcal{T} = \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}.$$

We see that

$$\mathcal{A}(T\xi) = \mathcal{T}\mathcal{A}(\xi)^\top\mathcal{T}.$$

We choose T so that $T\xi = re_1$, where $r = |\xi|$ and $e_1 = {}^\top(1,0,0)$. Using this T, we have

$$\begin{aligned}
\det(\mu I_6 + \mathcal{A}(\xi)) &= \det(\mathcal{T}(\mu I_6 + \mathcal{A}(\xi))^\top\mathcal{T}) \\
&= \det(\mu I_6 + \mathcal{A}(T\xi)) \\
&= \det\begin{pmatrix} \mu I & -I \\ \beta^2 r^2 I + \gamma^2 r^2 e_1{}^\top e_1 & (\mu + \nu r^2)I + \tilde{\nu} r^2 e_1{}^\top e_1 \end{pmatrix} \\
&= (\mu^2 + \nu r^2\mu + \beta^2 r^2)^2[\mu^2 + (\nu+\tilde{\nu})r^2\mu + (\beta^2+\gamma^2)r^2] \\
&= (\mu - \mu_1(\xi))^2(\mu - \mu_2(\xi))^2(\mu - \mu_3(\xi))(\mu - \mu_4(\xi)).
\end{aligned}$$

Therefore, the eigenvalues of $-\mathcal{A}(\xi)$ are given by $\mu_j(\xi)$, $j = 1,2,3,4$. We note that

$$\mu_1(\xi)\mu_2(\xi) = \beta^2|\xi|^2,\ \mu_1(\xi) + \mu_2(\xi) = -\nu|\xi|^2, \tag{3.A.8}$$

$$\mu_3(\xi)\mu_4(\xi) = (\beta^2+\gamma^2)|\xi|^2,\ \mu_3(\xi) + \mu_4(\xi) = -(\nu+\tilde{\nu})|\xi|^2. \tag{3.A.9}$$

By using (3.A.8) and (3.A.9), the eigenprojections for $\mu_j(\xi)$ of $-\mathcal{A}(\xi)$ are written by

$$\Pi_1(\xi) = \frac{1}{\mu_1(\xi) - \mu_2(\xi)}\begin{pmatrix} -\mu_2(\xi)\left(I - \dfrac{\xi^\top\xi}{|\xi|^2}\right) & I - \dfrac{\xi^\top\xi}{|\xi|^2} \\ -\mu_1(\xi)\mu_2(\xi)\left(I - \dfrac{\xi^\top\xi}{|\xi|^2}\right) & \mu_1(\xi)\left(I - \dfrac{\xi^\top\xi}{|\xi|^2}\right) \end{pmatrix},$$

$$\Pi_2(\xi) = \frac{1}{\mu_1(\xi) - \mu_2(\xi)}\begin{pmatrix} \mu_1(\xi)\left(I - \dfrac{\xi^\top\xi}{|\xi|^2}\right) & -\left(I - \dfrac{\xi^\top\xi}{|\xi|^2}\right) \\ \mu_1(\xi)\mu_2(\xi)\left(I - \dfrac{\xi^\top\xi}{|\xi|^2}\right) & -\mu_2(\xi)\left(I - \dfrac{\xi^\top\xi}{|\xi|^2}\right) \end{pmatrix},$$

$$\Pi_3(\xi) = \frac{1}{\mu_3(\xi)-\mu_4(\xi)} \begin{pmatrix} -\mu_4(\xi)\dfrac{\xi^\top \xi}{|\xi|^2} & \dfrac{\xi^\top \xi}{|\xi|^2} \\ -\mu_3(\xi)\mu_4(\xi)\dfrac{\xi^\top \xi}{|\xi|^2} & \mu_3(\xi)\dfrac{\xi^\top \xi}{|\xi|^2} \end{pmatrix},$$

$$\Pi_4(\xi) = \frac{1}{\mu_3(\xi)-\mu_4(\xi)} \begin{pmatrix} \mu_3(\xi)\dfrac{\xi^\top \xi}{|\xi|^2} & -\dfrac{\xi^\top \xi}{|\xi|^2} \\ \mu_3(\xi)\mu_4(\xi)\dfrac{\xi^\top \xi}{|\xi|^2} & -\mu_4(\xi)\dfrac{\xi^\top \xi}{|\xi|^2} \end{pmatrix}.$$

The solution semigroup $e^{-t\hat{\mathcal{A}}(\xi)}$ is then expressed as

$$\begin{aligned}
&e^{-t\hat{\mathcal{A}}(\xi)} \\
&= e^{\mu_1(\xi)t}\Pi_1(\xi) + e^{\mu_2(\xi)t}\Pi_2(\xi) + e^{\mu_3(\xi)t}\Pi_3(\xi) + e^{\mu_4(\xi)t}\Pi_4(\xi) \\
&= \begin{pmatrix} \dfrac{\mu_1(\xi)e^{\mu_2(\xi)t}-\mu_2(\xi)e^{\mu_1(\xi)t}}{\mu_1(\xi)-\mu_2(\xi)}\left(I-\dfrac{\xi^\top \xi}{|\xi|^2}\right) & \dfrac{e^{\mu_1(\xi)t}-e^{\mu_2(\xi)t}}{\mu_1(\xi)-\mu_2(\xi)}\left(I-\dfrac{\xi^\top \xi}{|\xi|^2}\right) \\ -\mu_1(\xi)\mu_2(\xi)\dfrac{e^{\mu_1(\xi)t}-e^{\mu_2(\xi)t}}{\mu_1(\xi)-\mu_2(\xi)}\left(I-\dfrac{\xi^\top \xi}{|\xi|^2}\right) & \dfrac{\mu_1(\xi)e^{\mu_1(\xi)t}-\mu_2(\xi)e^{\mu_2(\xi)t}}{\mu_1(\xi)-\mu_2(\xi)}\left(I-\dfrac{\xi^\top \xi}{|\xi|^2}\right) \end{pmatrix} \\
&\quad + \begin{pmatrix} \dfrac{\mu_3(\xi)e^{\mu_4(\xi)t}-\mu_4(\xi)e^{\mu_3(\xi)t}}{\mu_3(\xi)-\mu_4(\xi)}\dfrac{\xi^\top \xi}{|\xi|^2} & \dfrac{e^{\mu_3(\xi)t}-e^{\mu_4(\xi)t}}{\mu_3(\xi)-\mu_4(\xi)}\dfrac{\xi^\top \xi}{|\xi|^2} \\ -\mu_3(\xi)\mu_4(\xi)\dfrac{e^{\mu_3(\xi)t}-e^{\mu_4(\xi)t}}{\mu_3(\xi)-\mu_4(\xi)}\dfrac{\xi^\top \xi}{|\xi|^2} & \dfrac{\mu_3(\xi)e^{\mu_3(\xi)t}-\mu_4(\xi)e^{\mu_4(\xi)t}}{\mu_3(\xi)-\mu_4(\xi)}\dfrac{\xi^\top \xi}{|\xi|^2} \end{pmatrix}
\end{aligned}$$

It then follows that $\hat{w}(\xi,t)$ is written as

$$\begin{aligned}
\hat{w}(\xi,t) = &\frac{\mu_1(\xi)e^{\mu_2(\xi)t}-\mu_2(\xi)e^{\mu_1(\xi)t}}{\mu_1(\xi)-\mu_2(\xi)}\left(I-\frac{\xi^\top \xi}{|\xi|^2}\right)\hat{w}_0(\xi) \\
&+\frac{e^{\mu_1(\xi)t}-e^{\mu_2(\xi)t}}{\mu_1(\xi)-\mu_2(\xi)}\left(I-\frac{\xi^\top \xi}{|\xi|^2}\right)\hat{w}_{t,0}(\xi) \\
&+\frac{\mu_3(\xi)e^{\mu_4(\xi)t}-\mu_4(\xi)e^{\mu_3(\xi)t}}{\mu_3(\xi)-\mu_4(\xi)}\frac{\xi^\top \xi}{|\xi|^2}\hat{w}_0(\xi) \\
&+\frac{e^{\mu_3(\xi)t}-e^{\mu_4(\xi)t}}{\mu_3(\xi)-\mu_4(\xi)}\frac{\xi^\top \xi}{|\xi|^2}\hat{w}_{t,0}(\xi).
\end{aligned} \tag{3.A.10}$$

Substituting (3.A.7) into (3.A.10) leads to

$$\begin{aligned}
\hat{w}(\xi,t) = &-i\gamma^2\frac{e^{\mu_3(\xi)t}-e^{\mu_4(\xi)t}}{\mu_3(\xi)-\mu_4(\xi)}\xi\hat{\phi}_0(\xi) \\
&+\frac{\mu_1(\xi)e^{\mu_1(\xi)t}-\mu_2(\xi)e^{\mu_2(\xi)t}}{\mu_1(\xi)-\mu_2(\xi)}\left(I-\frac{\xi^\top\xi}{|\xi|^2}\right)\hat{w}_0(\xi) \\
&+\frac{\mu_3(\xi)e^{\mu_3(\xi)t}-\mu_4(\xi)e^{\mu_4(\xi)t}}{\mu_3(\xi)-\mu_4(\xi)}\frac{\xi^\top\xi}{|\xi|^2}\hat{w}_0(\xi) \\
&+i\beta^2\frac{e^{\mu_1(\xi)t}-e^{\mu_2(\xi)t}}{\mu_1(\xi)-\mu_2(\xi)}\left(I-\frac{\xi^\top\xi}{|\xi|^2}\right)\hat{\Psi}_0(\xi)\xi \\
&+i\beta^2\frac{e^{\mu_3(\xi)t}-e^{\mu_4(\xi)t}}{\mu_3(\xi)-\mu_4(\xi)}\frac{\xi^\top\xi}{|\xi|^2}\hat{\Psi}_0(\xi)\xi.
\end{aligned} \tag{3.A.11}$$

We see from (3.A.1), (3.A.3), (3.A.5) and (3.A.11) that

$$\begin{aligned}
\hat{\phi}(\xi,t) &= \hat{\phi}_0(\xi) - i\xi\cdot\int_0^t \hat{w}(\xi,s)\mathrm{d}s \\
&= \frac{\beta^2}{\beta^2+\gamma^2}\left(\hat{\phi}_0(\xi)+\frac{{}^\top\xi\hat{\Psi}_0(\xi)\xi}{|\xi|^2}\right) \\
&\quad+\frac{\mu_3(\xi)e^{\mu_4(\xi)t}-\mu_4(\xi)e^{\mu_3(\xi)t}}{\mu_3(\xi)-\mu_4(\xi)}\left(\frac{\gamma^2}{\beta^2+\gamma^2}\hat{\phi}_0(\xi)-\frac{\beta^2}{\beta^2+\gamma^2}\frac{{}^\top\xi\hat{\Psi}_0(\xi)\xi}{|\xi|^2}\right) \\
&\quad-i\frac{e^{\mu_3(\xi)t}-e^{\mu_4(\xi)t}}{\mu_3(\xi)-\mu_4(\xi)}\xi\cdot\hat{w}_0(\xi) \\
&= \frac{\mu_3(\xi)e^{\mu_4(\xi)t}-\mu_4(\xi)e^{\mu_3(\xi)t}}{\mu_3(\xi)-\mu_4(\xi)}\hat{\phi}_0(\xi) - i\frac{e^{\mu_3(\xi)t}-e^{\mu_4(\xi)t}}{\mu_3(\xi)-\mu_4(\xi)}\xi\cdot\hat{w}_0(\xi),
\end{aligned}$$

$$\begin{aligned}
\hat{\Psi}(\xi,t) &= \hat{\Psi}_0(\xi) + i\left(\int_0^t \hat{w}(\xi,s)\mathrm{d}s\right)^\top\xi \\
&= \frac{\gamma^2}{\beta^2+\gamma^2}\left(\hat{\phi}_0(\xi)\frac{\xi^\top\xi}{|\xi|^2}+\frac{\xi^\top\xi}{|\xi|^2}\hat{\Psi}_0(\xi)\right) \\
&\quad+i\frac{e^{\mu_1(\xi)t}-e^{\mu_2(\xi)t}}{\mu_1(\xi)-\mu_2(\xi)}\left(I-\frac{\xi^\top\xi}{|\xi|^2}\right)\hat{w}_0(\xi)^\top\xi \\
&\quad+i\frac{e^{\mu_3(\xi)t}-e^{\mu_4(\xi)t}}{\mu_3(\xi)-\mu_4(\xi)}\frac{\xi^\top\xi}{|\xi|^2}\hat{w}_0(\xi)^\top\xi \\
&\quad+\frac{\mu_1(\xi)e^{\mu_2(\xi)t}-\mu_2(\xi)e^{\mu_1(\xi)t}}{\mu_1(\xi)-\mu_2(\xi)}\left(I-\frac{\xi^\top\xi}{|\xi|^2}\right)\hat{\Psi}_0(\xi)
\end{aligned}$$

$$
\begin{aligned}
&\quad + \frac{\mu_3(\xi)e^{\mu_4(\xi)t} - \mu_4(\xi)e^{\mu_3(\xi)t}}{\mu_3(\xi) - \mu_4(\xi)} \left(-\frac{\gamma^2}{\beta^2+\gamma^2}\hat{\phi}_0(\xi)\frac{\xi^\top \xi}{|\xi|^2} + \frac{\beta^2}{\beta^2+\gamma^2}\frac{\xi^\top \xi}{|\xi|^2}\hat{\Psi}_0(\xi) \right) \\
&= i\frac{e^{\mu_1(\xi)t} - e^{\mu_2(\xi)t}}{\mu_1(\xi) - \mu_2(\xi)} \left(I - \frac{\xi^\top \xi}{|\xi|^2} \right) \hat{w}_0(\xi)^\top \xi \\
&\quad + i\frac{e^{\mu_3(\xi)t} - e^{\mu_4(\xi)t}}{\mu_3(\xi) - \mu_4(\xi)} \frac{\xi^\top \xi}{|\xi|^2} \hat{w}_0(\xi)^\top \xi \\
&\quad + \frac{\mu_1(\xi)e^{\mu_2(\xi)t} - \mu_2(\xi)e^{\mu_1(\xi)t}}{\mu_1(\xi) - \mu_2(\xi)} \left(I - \frac{\xi^\top \xi}{|\xi|^2} \right) \hat{\Psi}_0(\xi) \\
&\quad + \frac{\mu_3(\xi)e^{\mu_4(\xi)t} - \mu_4(\xi)e^{\mu_3(\xi)t}}{\mu_3(\xi) - \mu_4(\xi)} \frac{\xi^\top \xi}{|\xi|^2} \hat{\Psi}_0(\xi).
\end{aligned}
$$

This completes the proof. ■

Chapter 4

Global existence of solutions of the compressible viscoelastic fluid around a parallel flow

In this chapter, we consider the stability of parallel flows of the compressible viscoelastic system

$$\partial_t \rho + \mathrm{div}(\rho v) = 0, \tag{4.1}$$

$$\rho(\partial_t v + v \cdot \nabla v) - \mu \Delta v - (\mu + \mu')\nabla \mathrm{div} v + \nabla P(\rho) = \alpha \mathrm{div}(\rho F^{\top} F) + \rho g, \tag{4.2}$$

$$\partial_t F + v \cdot \nabla F = \nabla v F, \tag{4.3}$$

in an infinite layer $\Omega_l = \mathbb{R}^2 \times (0, l)$:

$$\Omega_l = \{x = (x', x_3); \quad x' = (x_1, x_2) \in \mathbb{R}^2, \ 0 < x_3 < l\}.$$

Here $\rho = \rho(x, t)$, $v = v(x, t)$ and $F = F(x, t)$ are the unknown density, the velocity field and the deformation tensor, respectively, at the time $t > 0$ and position $x \in \Omega_l$; $P = P(\rho)$ is the pressure; μ and μ' are the viscosity coefficients satisfying $\mu > 0$ and $\frac{2}{3}\mu + \mu' > 0$; $\alpha > 0$ is the constant called the speed of propagation of shear wave; g is an external force which has the form

$$g = {}^{\top}(g^1(x_3, t), 0, 0), \ g^1(0, t) = g^1(l, t) = 0, \tag{4.4}$$

where g^1 is a given smooth function of (x_3, t) converging to $g^1_\infty = g^1_\infty(x_3) \neq 0$ as t goes to infinity. Here and in what follows ${}^{\top}\cdot$ stands for the transposition.

We assume that P is a smooth function of ρ and satisfies

$$P'(\rho_*) > 0$$

for a given positive number ρ_*.

X denotes the material coordinate; and $x = x(X,t)$ is a solution of the flow map defined as

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = v(x(X,t),t), \\ x(X,0) = X. \end{cases}$$

When g has the form of (4.4) and is suitably smooth, the system (4.1)–(4.3) has a solution representing a parallel flow, more precisely, a solution of the form $\bar{u} = {}^\top(\bar{\rho}, \bar{v}, \bar{F})$ with $\bar{\rho} = \rho_*$ and $\bar{v} = \bar{v}^1(x_3,t)e_1$, where $e_1 = {}^\top(1,0,0) \in \mathbb{R}^3$.

In this chapter we show that the parallel flow $\bar{u}$ is exponentially stable under sufficiently small perturbations, if μ, $P'(\rho_*)$ and α are sufficiently large compared to g^1. We briefly present the main result of this chapter in a more precise way. We introduce the following non-dimensional variables:

$$\tilde{x} = \frac{1}{l}x,\ \tilde{t} = \frac{V}{l}t,\ \tilde{v} = \frac{1}{V}v,\ \tilde{\rho} = \frac{1}{\rho_*}\rho,\ \tilde{F} = F,$$

$$\tilde{g} = \frac{l}{V^2}g,\ \tilde{P} = \frac{1}{\rho_* V^2}P,\ V = \frac{\rho_* \|g^1\|_{L^\infty} l^2}{\mu}.$$

The system (4.1)–(4.3) is then rewritten into the following dimensionless one on the layer $\Omega_1 = \mathbb{R}^2 \times (0,1)$:

$$\partial_{\tilde{t}}\tilde{\rho} + \mathrm{div}_{\tilde{x}}(\tilde{\rho}\tilde{v}) = 0, \tag{4.5}$$

$$\tilde{\rho}(\partial_{\tilde{t}}\tilde{v} + \tilde{v}\cdot\nabla_{\tilde{x}}\tilde{v}) - \nu\Delta_{\tilde{x}}\tilde{v} - (\nu+\nu')\nabla_{\tilde{x}}\mathrm{div}_{\tilde{x}}\tilde{v} + \nabla_{\tilde{x}}\tilde{P}(\tilde{\rho}) = \beta^2\mathrm{div}_{\tilde{x}}(\tilde{\rho}\tilde{F}{}^\top\tilde{F}) + \tilde{\rho}\tilde{g}, \tag{4.6}$$

$$\partial_{\tilde{t}}\tilde{F} + \tilde{v}\cdot\nabla_{\tilde{x}}\tilde{F} = \nabla_{\tilde{x}}\tilde{v}\tilde{F}. \tag{4.7}$$

Here ν, ν', γ and β are the non-dimensional parameters defined as

$$\nu = \frac{\mu}{\rho_* l V},\ \nu' = \frac{\mu'}{\rho_* l V},\ \gamma = \frac{\sqrt{P'(\rho_*)}}{V},\ \beta = \frac{\sqrt{\alpha}}{V}.$$

We note that $Re = \frac{1}{\nu}$ and $Ma = \frac{1}{\gamma}$ are the Reynolds number and the Mach number. We also assume that

$$\frac{\nu'}{\nu} \le \nu_1$$

for some positive constant $\nu_1 > 0$. We consider the system (4.5)–(4.7) under the non-slip boundary condition

$$\tilde{v}|_{x_3=0,1} = 0 \tag{4.8}$$

and the initial condition

$$\tilde\rho|_{t=0} = \tilde\rho_0,\ \tilde v|_{t=0} = \tilde v_0,\ \tilde F|_{t=0} = \tilde F_0. \tag{4.9}$$

We also impose the periodic boundary condition in $\tilde x'$:

$$\tilde\rho, \tilde v, \tilde F : \frac{2\pi}{\alpha_j}\text{−periodic in } \tilde x_j,\ j = 1, 2.$$

In what follows we abbreviate $\tilde x$, $\tilde t$, $\tilde\rho$, $\tilde v$, $\tilde F$, and $\tilde g$ as x, t, ρ, v, F, and g, respectively.

Under a suitable condition on g, we see that there exists a parallel flow $(\bar\rho, \bar v, \bar F)$ of (4.5)–(4.7) with the following properties:

$$\begin{aligned}
&\bar\rho = 1,\ \bar v = \bar v^1(x_3, t)e_1,\ \bar F = (\nabla(x - \bar\psi^1 e_1))^{-1},\\
&\|\bar v(t)\|_{H^5}^2 \le Ce^{-c_0\kappa t}\left(\|\bar v_0\|_{H^5}^2 + O\left(\frac{1}{\nu^2}\right) + O\left(\frac{1}{\kappa\nu^2}\right)\right),\\
&\|\partial_t \bar v(t)\|_{H^3}^2 \le Ce^{-c_0\kappa t}\left(\frac{\beta^4}{\nu^2}\|\bar v_0\|_{H^5}^2 + O(1) + O\left(\frac{1}{\kappa}\right)\right),\\
&\|\bar F(t) - \bar F_\infty\|_{H^4}^2 \le Ce^{-c_0\kappa t}\left(\frac{1}{\nu^2}\|\bar v_0\|_{H^5}^2 + O\left(\frac{1}{\beta^4}\right) + O\left(\frac{1}{\kappa\beta^4}\right)\right),
\end{aligned}$$

where $\kappa = \min\left\{\nu, \frac{\beta^2}{\nu}\right\}$, $\bar\psi^1(x_3, t) = \int_0^t \bar v^1(x_3, s)\mathrm{d}s$, $\bar\psi^1_\infty = \beta^{-2}(-\partial_{x_3}^2)^{-1}g^1_\infty$, and $\bar F_\infty = (\nabla(x - \bar\psi^1_\infty e_1))^{-1}$. Here $(-\partial_{x_3}^2)^{-1}$ is the inverse of $-\partial_{x_3}^2$ with domain $D(-\partial_{x_3}^2) = H^2(0, 1) \cap H^1_0(0, 1)$.

We introduce the displacement vector ψ as in [29]:

$$\psi(x, t) = x - X(x, t),$$

It then follows that F is written in terms of ψ as

$$F = \bar F + \bar F\nabla\left(\psi - \bar\psi^1 e_1\right)\bar F + h\left(\nabla(\psi - \bar\psi^1 e_1)\right),$$

where h satisfies $h(\nabla(\psi - \bar\psi^1(t)e_1)) = O(|\nabla(\psi - \bar\psi^1(t)e_1)|^2)$. By using ψ, the problem for the perturbation is reduced to the one for $u(t) = (\phi(t), w(t), \zeta(t)) = \big(\rho(t) - 1, v(t) - \bar v(t), \psi(t) - \bar\psi^1(t)e_1\big)$ which takes the following form:

$$\begin{cases}
\partial_t\phi + \mathrm{div}w = f^1,\\
\partial_t w - \nu\Delta w - \tilde\nu\nabla\mathrm{div}w + \gamma^2\nabla\phi - \beta^2(\Delta\zeta + K_\infty\zeta) = f^2,\\
\partial_t\zeta - w + w^3\partial_{x_3}\bar\psi_\infty = f^3,\\
w|_{x_3=0,1} = 0,\quad \zeta|_{x_3=0,1} = 0,\quad (\phi, w, \zeta)|_{t=0} = (\phi_0, w_0, \zeta_0).
\end{cases} \tag{4.10}$$

Here $\tilde{\nu} = \nu + \nu'$ and $\bar{\psi}_\infty = \bar{\psi}^1_\infty e_1$; $K_\infty \zeta$ is a linear term of ζ satisfying $\|K_\infty \zeta\|_{L^2} \le \frac{C}{\beta^2} \|\nabla \zeta\|_{H^1}$; f^j, $j = 1, 2, 3$ are written in a sum of nonlinear terms and linear terms with coefficients including $\bar{v}, \bar{\psi}^1 - \bar{\psi}^1_\infty$ which decay exponentially in t.

This chapter is organized as follows. In Section 4.2 we show the existence of the parallel flow and then state the main result of this chapter on the stability of the parallel flow. In Section 4.3 we establish the a priori estimate which ensures the global existence of the perturbation and its exponential decay as $t \to \infty$. In the Appendix 4.A, we give a proof of the existence of the parallel flow and its estimates.

4.1 Main result of Chapter 4

In this section, we first show the existence of the parallel flow. We then state the main result of this chapter on the stability of the parallel flow.

We impose the following conditions for ρ_0, F_0:

$$\mathrm{div}(\rho_0 {}^\top F_0) = 0, \tag{4.11}$$

$$\rho_0 \mathrm{det} F_0 = 1. \tag{4.12}$$

It then follows from (4.5) that these quantities are conserved:

$$\mathrm{div}(\rho {}^\top F) = 0, \tag{4.13}$$

$$\rho \mathrm{det} F = 1. \tag{4.14}$$

We next show the existence of a parallel flow $(\bar{\rho}, \bar{v}, \bar{F})$ of (4.5)–(4.7), as in [4], satisfying

$$\begin{aligned}
&\bar{\rho} = \bar{\rho}(x_3, t),\ \ \bar{\rho}|_{t=0} = 1,\\
&\bar{v} = \bar{v}^1(x_3, t) e_1,\ \ \bar{v}^1|_{x_3=0,1} = 0,\ \ \bar{v}^1|_{t=0} = \bar{v}^1_0,\\
&\bar{F} = \bar{F}(x_3, t),\ \ \bar{F}|_{t=0} = I.
\end{aligned}$$

Let $x(X,t) = {}^\top(x^1(X,t), x^2(X,t), x^3(X,t))$ be the flow map given by

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t}(X,t) = \bar{v}(x^3(X,t), t), \\ x(X,0) = X. \end{cases}$$

This yields

$$x(X,t) = X + \left(\int_0^t \bar{v}^1(x_3, s) \mathrm{d}s \right) e_1.$$

We set

$$\bar{\psi}^1(x_3,t)=\int_0^t \bar{v}^1(x_3,s)\mathrm{d}s. \tag{4.15}$$

By using the flow map $x(X,t)$, deformation tensor $\bar{F}$ is written by

$$\bar{F}=\frac{\partial x}{\partial X}=\left(\frac{\partial x_j}{\partial X_k}\right)_{1\le j,k\le 3}.$$

It is easy to see that

$$\bar{F}=\nabla(x+\bar{\psi}^1 e_1)=\big(\nabla(x-\bar{\psi}^1 e_1)\big)^{-1}=\begin{pmatrix}1 & 0 & \partial_{x_3}\bar{\psi}^1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

We set $\bar{\rho}=1$. Inserting $(\bar{\rho},\bar{v},\bar{F})$ into (4.6), we see that and $\bar{\psi}^1$ satisfies

$$\begin{cases}\partial_t^2\bar{\psi}^1-\beta^2\partial_{x_3}^2\bar{\psi}^1-\nu\partial_t\partial_{x_3}^2\bar{\psi}^1=g^1,\\ \bar{\psi}^1(x_3,0)=0,\ \partial_t\bar{\psi}^1(x_3,0)=\bar{v}^1(x_3,0).\end{cases} \tag{4.16}$$

We also assume the compatibility conditions for g^1:

$$\begin{aligned}&g^1(0,t)=g^1(1,t)=0,\ t\ge 0,\\ &\partial_{x_3}^2 g^1(0,t)=\partial_{x_3}^2 g^1(1,t)=0,\ t\ge 0.\end{aligned} \tag{4.17}$$

The existence of the parallel flow is now stated as follows.

Proposition 4.1. *Let* $\kappa=\min\left\{\nu,\frac{\beta^2}{\nu}\right\}$, *Assume that* $g^1\in H^1_{loc}([0,\infty);H^3(0,1))$ *satisfies the compatibility conditions* (4.17), *and that* $g^1_\infty\in H^3(0,1)$. *If* g^1 *satisfies* $e^{c_0\kappa t}\partial_t g^1\in L^2((0,\infty);H^3(0,1))$ *for some positive constant* c_0, *then the following assertions hold.*

If $(\bar{\rho},\bar{v},\bar{F})|_{t=0}=(1,\bar{v}_0,I)$ *and* $\bar{v}_0\in H^5(0,1)$, *then there exist a parallel flow* $(\bar{\rho},\bar{v},\bar{F})$ *of* (4.5)–(4.7) *that has the following properties:*

(i) $\bar{\rho}=1$.

(ii) *There exists* $\bar{\psi}^1=\bar{\psi}^1(x_3,t)\in\mathbb{R}$ *such that*

$$\bar{\psi}^1|_{t=0}=0,\ \ \bar{F}=\nabla(x+\bar{\psi}^1e_1)=\big(\nabla(x-\bar{\psi}^1e_1)\big)^{-1},\ \partial_t\bar{\psi}^1=\bar{v}^1.$$

(iii) $\bar{v}\in C^1([0,\infty);H^3(0,1))$.

(iv) *There hold the following estimates uniformly for* $t\ge 0$:

$$\begin{aligned}&\|\bar{v}^1(t)\|^2_{H^5(0,1)}\\ &\le Ce^{-c_0\kappa t}\left(\|\bar{v}_0\|^2_{H^3(0,1)}+\frac{1}{\nu^2}\|g^1(0)\|^2_{H^1(0,1)}+\frac{1}{\kappa\nu^2}\|e^{c_0\kappa t}\partial_t g^1\|^2_{L^2(0,\infty;H^1(0,1))}\right),\end{aligned} \tag{4.18}$$

$$\begin{aligned}
&\|\partial_t\bar{v}^1(t)\|^2_{H^3(0,1)}\\
&\le Ce^{-c_0\kappa t}\left(\frac{\beta^4}{\nu^2}\|\bar{v}_0\|^2_{H^5(0,1)}+\|g^1(0)\|^2_{H^1(0,1)}+\frac{1}{\kappa}\|e^{c_0\kappa t}\partial_t g^1\|^2_{L^2(0,\infty;H^1(0,1))}\right),
\end{aligned}\tag{4.19}$$

$$\begin{aligned}
&\|\bar{\psi}^1(t)-\bar{\psi}^1_\infty\|^2_{H^5(0,1)}\\
&\le Ce^{-c_0\kappa t}\left(\frac{1}{\nu^2}\|\bar{v}_0\|^2_{H^5(0,1)}+\frac{1}{\beta^4}\|g^1(0)\|^2_{H^3(0,1)}+\frac{1}{\kappa\beta^4}\|e^{c_0\kappa t}\partial_t g^1\|^2_{L^2(0,\infty;H^3(0,1))}\right).
\end{aligned}\tag{4.20}$$

Here $\bar{\psi}^1_\infty$ satisfies

$$-\beta^2\partial^2_{x_3}\bar{\psi}^1_\infty=g^1_\infty,\ \bar{\psi}^1_\infty|_{x_3=0,1}=0,$$

$$\bar{F}(x_3,t)\to\bar{F}_\infty(x_3)=\nabla(x+\bar{\psi}^1_\infty e_1)=\left(\nabla(x-\bar{\psi}^1_\infty e_1)\right)^{-1},$$

and the following estimate

$$\|\bar{\psi}^1_\infty\|_{H^5(0,1)}\le\frac{C}{\beta^2}\|g^1_\infty\|_{H^3(0,1)}.\tag{4.21}$$

The proof of Proposition 4.1 will be given in the Appendix 4.A.

We next consider the stability of the parallel flow $\bar{u}={}^\top(1,\bar{v},\bar{F})$. We will show that under some assumptions on ν, γ, and β, the perturbation of $\bar{u}$ exists globally in time and decay exponentially as $t\to\infty$. To this end, we first state the local time existence of the solution of the problem (4.5)–(4.9). By a similar argument to that in [18, 29, 36], one can prove the following local existence of solutions.

Proposition 4.2. *If $(\rho_0,v_0,F_0)\in H^2(\Omega)$ satisfies $v_0\in H^1_0(\Omega)$, (4.11)–(4.12) and $\rho_0\ge\frac{1}{2}$, then there exists positive numbers T and C such that the following assertion holds. The problem (4.5)–(4.9) has a unique solution $(\rho,v,F)\in C([0,T];H^2(\Omega))$ satisfying $\partial_t\rho,\partial_t F\in C([0,T];L^2(\Omega))$, $v\in L^2([0,T];H^3(\Omega))$, $\partial_t v\in C([0,T];L^2(\Omega))\cap L^2([0,T];H^1(\Omega))$ and*

$$\|(\rho(t),v(t),F(t))\|_{H^2}\le C\|(\rho_0,v_0,F_0)\|_{H^2}$$

for $0\le t\le T$.

As for the global existence, we have the following result.

Theorem 4.3. *Under the assumptions of the Propositions 4.1 and 4.2, there are positive numbers ν_0, γ_0 and β_0 such that if $\nu\ge\nu_0$, $\frac{\gamma^2}{\nu+\nu'}\ge\gamma_0^2$ and $\frac{\beta^2}{\gamma^2}\ge\beta_0^2$, then the following assertion holds. There is a positive number ϵ_0 such that* *if* $(\rho_0, v_0, F_0) \in H^2(\Omega)$ *and* $\bar{v}_0 \in H^5(0,1)$ *satisfies* $\|(\rho_0 - 1, v_0 - \bar{v}_0, F_0 - \bar{F}_0)\|^2_{H^2(\Omega)} + \|\bar{v}_0\|^2_{H^5(0,1)} \leq \epsilon_0$, $\int_\Omega (\rho_0 - 1)\mathrm{d}x = 0$, *then there exists a unique global solution* $(\rho, v, F) \in C([0,\infty); H^2(\Omega))$ *of the problem* (4.5)–(4.9), *and the perturbation* $U(t) = (\rho(t) - 1, v(t) - \bar{v}(t), F(t) - \bar{F}(t))$ *satisfies*

$$\|U(t)\|^2_{H^2} + \int_0^t e^{-c_1(t-s)} \|U(s)\|^2_{H^2\times H^3\times H^2}\, \mathrm{d}s \leq Ce^{-c_1 t}\|U_0\|^2_{H^2}$$

for $t \geq 0$.

4.2 A priori estimate

Theorem 4.3 is proved by combining Proposition 4.2 and the following a priori estimate.

Proposition 4.4. *There exist positive numbers* ν_0, γ_0 *and* β_0 *such that if* $\nu \geq \nu_0$, $\frac{\gamma^2}{\nu+\nu'} \geq \gamma_0^2$ *and* $\frac{\beta^2}{\gamma^2} \geq \beta_0^2$, *then the following assertion holds. Let* T *be an arbitrarily given positive number. Then there exists a positive constant* δ *such that if* $\|\bar{v}_0\|^2_{H^5(0,1)} + \tilde{E}(t) \leq \delta$ *uniformly for* $t \in [0,T]$, *it holds the following estimate:*

$$\tilde{E}(t) + \int_0^t e^{-c_1(t-s)} \tilde{D}(s)\mathrm{d}s \leq C\left(e^{-c_1 t}\tilde{E}(0) + \int_0^t e^{-c_1(t-s)}\tilde{\mathcal{R}}(s)\mathrm{d}s\right)$$

uniformly for $t \in [0,T]$, *where* C *is a positive constant independent of* T. *Here* $\tilde{E}(t)$ *and* $\tilde{D}(t)$ *are some quantities equivalent to*

$$\|U(t)\|^2_{H^2} + \|\partial_t U(t)\|^2_{L^2}$$

and

$$\|U(t)\|^2_{H^2\times H^3\times H^2} + \|\partial_t U(t)\|^2_{L^2\times H^1\times L^2},$$

respectivity; $\tilde{\mathcal{R}}(t)$ *is a function satisfying*

$$\tilde{\mathcal{R}}(t) \leq C\left(\frac{1}{\nu} + \frac{1}{\beta^2} + \frac{\sqrt{\nu}}{\beta} + \frac{\gamma}{\beta}\right)\tilde{D}(t) + (\tilde{E}(t)^{\frac{1}{2}} + \tilde{E}(t))\tilde{D}(t)$$

uniformly for $t \in [0,T]$ *with a positive constant* C *independent of* T .

By a standard argument, one can show that Propositions 4.2 and 4.4 imply Theorem 4.3 if $\|U_0\|_{H^2} + \|\bar{v}_0\|^2_{H^5(0,1)}$ is small enough and $\nu \geq \nu_0$, $\frac{\gamma^2}{\nu+\nu'} \geq \gamma_0^2$, $\frac{\beta^2}{\gamma^2} \geq \beta_0^2$ for some positive constants ν_0, γ_0 and β_0.

To prove Proposition 4.4, we introduce the displacement vector and write the perturbation equation by using the displacement vector in place of F. We denote the displacement vector by ψ:

$$\psi(x,t) = x - X(x,t).$$

We see that ψ satisfies

$$\begin{aligned} &\partial_t \psi - v = -v \cdot \nabla \psi, \\ &\psi|_{\{x_3=0,1\}} = 0. \end{aligned}$$

See [29]. We also note that F has its inverse F^{-1} for $t \geq 0$ by (4.14), and F^{-1} is written as

$$G = F^{-1} = \frac{\partial X}{\partial x}.$$

We assume that G and X satisfy

$$\begin{cases} G(x,0) = \nabla X(x,0) \text{ for } x \in D, \\ X = x \text{ on } \{x_3 = 0,1\} \text{ for } t \geq 0. \end{cases} \tag{4.22}$$

Lemma 4.5. *If $G = F^{-1}$ satisfies the condition* (4.22)*, then*

$$G = \nabla X^{-1} \tag{4.23}$$

for $x \in D$ and $t \geq 0$.

Proof. A direct computation shows that G is a solution of the following transport equation:

$$\partial_t G + v \cdot \nabla G + G \nabla v = 0. \tag{4.24}$$

We also see that ∇X satisfies the same equation as G. By (4.22), G and ∇X have the same initial value, and therefore, the uniqueness of solutions of (4.24) implies that $G = \nabla X$. This completes the proof. ■

In terms of ψ, we see from (4.23) that F is written as

$$F = (I - \nabla\psi)^{-1} = I + \nabla\psi + h(\nabla\psi).$$

Here

$$\begin{aligned} &h(\nabla\psi) = (I - \nabla\psi)^{-1} - I - \nabla\psi, \\ &h(\nabla(\bar{\psi}^1 e_1)) = 0. \end{aligned}$$

It then follows that (ρ, v, ψ) satisfies

$$
\begin{aligned}
&\partial_t \rho + \mathrm{div}(\rho v) = 0, \\
&\partial_t v + v \cdot \nabla v - \frac{\nu}{\rho}\Delta v - \frac{\nu + \nu'}{\rho}\nabla \mathrm{div} v + \frac{\nabla p(\rho)}{\rho} = \beta^2(\Delta \psi + N(\nabla \psi)) + g, \\
&\partial_t \psi - v = -v \cdot \nabla \psi, \\
&{}^\top(I - \nabla \psi)\nabla \rho = \rho \nabla \mathrm{div} \psi - \mathrm{div}(\rho^\top h), \\
&v|_{x_3=0,1} = 0, \quad \psi|_{x_3=0,1} = 0, \quad (\rho, v, \psi)|_{t=0} = (\rho_0, v_0, \psi_0),
\end{aligned}
$$

where

$$
\begin{aligned}
N(\nabla \psi) = {}& \mathrm{div}(h(\nabla \psi)) + (\nabla \psi)\nabla(\nabla \psi) \\
& + (\nabla \psi)\nabla(h(\nabla \psi)) + (h(\nabla \psi))\nabla(\nabla \psi) + (h(\nabla \psi))\nabla(h(\nabla \psi)).
\end{aligned}
$$

We set $\rho = 1 + \phi,\ v = \bar{v} + w$ and $\psi = \bar{\psi}^1 e_1 + \zeta$. Since

$$
F - \bar{F} = \bar{F}\nabla \zeta \bar{F} + h^1(\nabla \zeta).
$$

with

$$
\begin{aligned}
h^1 &= h(\bar{F}\nabla \zeta)\bar{F}, \\
|h^1| &= O(|\nabla \zeta|^2) \text{ for } |\nabla \zeta| \ll 1,
\end{aligned}
$$

we see that $u = (\psi, w, \zeta)$ is a solution of the following initial boundary problem:

$$
\partial_t \phi + \mathrm{div} w = f^1, \tag{4.25}
$$

$$
\partial_t w - \nu \Delta w - \tilde{\nu} \nabla \mathrm{div} w + \gamma^2 \nabla \phi - \beta^2(\Delta \zeta + K_\infty \zeta) = f^2, \tag{4.26}
$$

$$
\partial_t \zeta - w + w^3 \partial_{x_3} \bar{\psi}_\infty = f^3, \tag{4.27}
$$

$$
\nabla \phi = -\nabla \mathrm{div} \zeta + M_\infty \zeta + f^4, \tag{4.28}
$$

$$
w|_{x_3=0,1} = 0, \quad \zeta|_{x_3=0,1} = 0, \quad (\phi, w, \zeta)|_{t=0} = (\phi_0, w_0, \zeta_0). \tag{4.29}
$$

Here $\tilde{\nu} = \nu + \nu'$, $\bar{\psi}_\infty = \bar{\psi}^1_\infty e_1$; $K_\infty \zeta$ and $M_\infty \zeta$ are given by

$$
\begin{aligned}
K_\infty \zeta = {}& \mathrm{div}(\bar{E}_\infty \nabla \zeta + \bar{E}_\infty \nabla \zeta + \bar{E}_\infty \nabla \zeta \bar{E}_\infty) \\
& + (\bar{F}_\infty \nabla \zeta \bar{F}_\infty)\nabla \bar{E}_\infty + \bar{E}_\infty \nabla(\bar{F}_\infty \nabla \zeta \bar{F}_\infty), \\
M_\infty \zeta = {}& -\mathrm{div}^\top(\bar{E}_\infty \nabla \zeta + \nabla \zeta \bar{E}_\infty + \bar{E}_\infty \nabla \zeta \bar{E}_\infty) + {}^\top \bar{E}_\infty \mathrm{div}^\top(\bar{F}_\infty \nabla \zeta \bar{F}_\infty),
\end{aligned}
$$

and f^j, $j = 1, 2, 3, 4$, denote the sum of nonlinear terms and linear terms with coefficients including $\bar{v}$, $\bar{\psi}_{exp} = \bar{\psi} - \bar{\psi}_\infty$;

$$\begin{aligned}
&f^1 = f^1_L + f^1_N;\\
&f^1_L = -\bar{v}^1 \partial_{x_1} \phi,\ f^1_N = -\mathrm{div}(\phi w),\\
&f^2 = {}^\top(f^{2,1}, f^{2,2}, f^{2,3}) = f^2_L + f^2_N;\\
&f^2_L = -\nu\phi\partial^2_{x_3}\bar{v} - \bar{v}^1\partial_{x_1} w - w^3\partial_{x_3}\bar{v} - \beta^2 K_{exp}\zeta,\\
&f^2_N = -w\cdot\nabla w + \frac{\nu\phi}{1+\phi}(-\Delta w + \phi\partial^2_{x_3}\bar{v}) - \frac{\tilde{\nu}\phi}{1+\phi}\nabla\mathrm{div} w\\
&\qquad -\frac{\gamma^2\phi}{1+\phi}\nabla\phi - \frac{\gamma^2}{1+\phi}\nabla Q(\phi) + \beta^2 h^2,\\
&f^3 = {}^\top(f^{3,1}, f^{3,2}, f^{3,3}) = f^3_L + f^3_N;\\
&f^3_L = -w^3\partial_{x_3}\bar{\psi}_{exp} - \bar{v}^1\partial_{x_1}\zeta,\ f^3_N = -w\cdot\nabla\zeta,\\
&f^4 = {}^\top(f^{4,1}, f^{4,2}, f^{4,3}) = f^4_L + f^4_N;\\
&f^4_L = M_{exp}\zeta,\ f^4_N = -{}^\top\bar{F}^{-1}\mathrm{div}^\top(\phi(\bar{F}\nabla\zeta\bar{F}) + (1+\phi)h^1),
\end{aligned}$$

where

$$\begin{aligned}
&\bar{E}_\infty = \bar{F}_\infty - I = \nabla(\bar{\psi}^1_\infty e_1),\ \bar{E}_{exp} = \bar{F} - \bar{F}_\infty = \nabla(\bar{\psi}^1_{exp} e_1),\\
&Q(\phi) = \phi^2\int_0^1 P''(1+s\phi)\mathrm{d}s,\ \nabla Q = O(\phi)\nabla\phi \text{ for } |\phi| \ll 1,\\
&h^2 = \bar{F}\nabla h^1 + (\bar{F}\nabla\zeta\bar{F})\nabla(\bar{F}\nabla\zeta\bar{F} + h^1) + h^1\nabla(\bar{F} + \bar{F}\nabla\zeta\bar{F} + h^1),\\
&K_{exp}\zeta = \mathrm{div}(\bar{E}_{exp}\nabla\zeta\bar{F}_\infty + \bar{F}_\infty\nabla\zeta\bar{E}_{exp} + \bar{E}_{exp}\nabla\zeta\bar{E}_{exp})\\
&\qquad +(\bar{F}\nabla\zeta\bar{F})\nabla\bar{E}_{exp} + (\bar{F}_\infty\nabla\zeta\bar{E}_{exp} + \bar{E}_{exp}\nabla\zeta\bar{F}_\infty + \bar{E}_{exp}\nabla\zeta\bar{E}_{exp})\nabla\bar{E}_\infty\\
&\qquad\quad +\bar{E}_{exp}\nabla(\bar{F}\nabla\zeta\bar{F}) + \bar{F}_\infty\nabla(\bar{F}_\infty\nabla\zeta\bar{E}_{exp} + \bar{E}_{exp}\nabla\zeta\bar{F}_\infty + \bar{E}_{exp}\nabla\zeta\bar{E}_{exp}),\\
&M_{exp}\zeta = -{}^\top\bar{F}_\infty^{-1}\mathrm{div}^\top(\bar{E}_{exp}\nabla\zeta\bar{F}_\infty + \bar{F}_\infty\nabla\zeta\bar{E}_{exp}\\
&\qquad +\bar{E}_{exp}\nabla\zeta\bar{E}_{exp}) + {}^\top\bar{E}_{exp}\mathrm{div}^\top(\bar{F}\nabla\zeta\bar{F}).
\end{aligned}$$

Since $\rho_0 = 1 + \phi_0$ and $F_0 = I + \nabla\zeta_0 + h(\nabla\zeta_0)$, we see from (4.11), (4.12), and the fact $\int_D(\rho_0 - 1)\mathrm{d}x = 0$ that the following relations hold:

$$ {}^\top(I + \nabla\zeta_0 + h(\nabla\zeta_0))\nabla\phi_0 = -(1+\phi_0)(\nabla\mathrm{div}\zeta_0 + \mathrm{div}({}^\top(h(\nabla\zeta_0)))), \tag{4.30}$$

$$ (1+\phi_0)\det(I + \nabla\zeta_0 + h(\nabla\zeta_0)) = 1, \tag{4.31}$$

$$ \int_D \phi_0 \mathrm{d}x = 0. \tag{4.32}$$

We then have the following a priori estimate for the perturbation $u = (\phi, w, \zeta)$.

Proposition 4.6. *Under the assumption of Proposition 4.4, the following assertion holds. There exists a positive constant δ with $\delta < 1$ such that if $\|\bar{v}_0\|^2_{H^5(0,1)} + E(t) \le \delta$ uniformly for $t \in [0,T]$, then it holds the following estimate:*

$$E(t) + \int_0^t e^{-c_1(t-s)} D(s)\mathrm{d}s \le C\left(e^{-c_1 t}E(0) + \int_0^t e^{-c_1(t-s)}\mathcal{R}(s)\mathrm{d}s\right) \quad (4.33)$$

uniformly for $t \in [0,T]$ with a positive constant C independent of T. Here $E(t)$ and $D(t)$ are equivalent to

$$\|u(t)\|^2_{H^2\times H^2\times H^3} + \|\partial_t u(t)\|^2_{L^2\times L^2\times H^1},$$

and

$$\|u(t)\|^2_{H^2\times H^3\times H^3} + \|\partial_t u(t)\|^2_{L^2\times H^1\times H^1}$$

respectivity; $\mathcal{R}(t)$ is a function satisfying

$$\mathcal{R}(t) \le C\left(\frac{1}{\nu} + \frac{1}{\beta^2} + \frac{\sqrt{\nu}}{\beta} + \frac{\gamma}{\beta}\right) D(t) + (E(t)^{\frac{1}{2}} + E(t))D(t)$$

uniformly for $t \in [0,T]$ with a positive constant C independent of T .

Proposition 4.4 immediately follows from Proposition 4.6. In the remaining of this chapter, we will give a proof of Proposition 4.6.

4.3 Proof of Proposition 4.6

In this section, we prove Proposition 4.6 by a variant of the Matsumura-Nishida energy method ([26]). The argument is based on the one by Qian [29], where a variant of Matsumura-Nishida energy method for viscoelastic compressible system was given.

Let T be an given positive number. Throughout this section, we assume that $u(t) = (\phi(t), w(t), \zeta(t))$ is a solution of (4.25)–(4.32) on $[0,T]$.

Proposition 4.7. *Let j and k be nonnegative integers satisfying $0 \le 2j+k \le 2$. Then it holds the estimate:*

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\gamma^2\|\partial_t^j\partial^k\phi\|^2_{L^2} + \|\partial_t^j\partial^k w\|^2_{L^2} + \beta^2\|\nabla\partial_t^j\partial^k\zeta\|^2_{L^2}) \\
&\quad +\nu\|\nabla\partial_t^j\partial^k w\|^2_{L^2} + \tilde{\nu}\|\mathrm{div}\partial_t^j\partial^k w\|^2_{L^2} \\
&\qquad \le \beta^2\left(|(K_\infty\partial_t^j\partial^k\zeta, \partial_t^j\partial^k w)| + |(\nabla(\partial_t^j\partial^k w^3\partial_{x_3}\bar{\psi}_\infty), \nabla\partial_t^j\partial^k\zeta)|\right) + N^1_{j,k},
\end{aligned} \quad (4.34)$$

where

$$N^1_{j,k}=\gamma^2|(\partial_t^j\partial^k f^1,\partial_t^j\partial^k\phi)|+|(\partial_t^j\partial^k f^2,\partial_t^j\partial^k w)|+\beta^2|(\partial_t^j\partial^k f^3,\Delta\partial_t^j\partial^k\zeta)|.$$

Proof. We consider the case $j=k=0$ only. The other cases can be treated similarly. We take the inner product of (4.25) with ϕ to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\phi\|^2_{L^2}+(\mathrm{div}w,\phi)=(f^1,\phi).$$

By integration by parts, we have $(\mathrm{div}w,\phi)=-(w,\nabla\phi)$, and therefore,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\phi\|^2_{L^2}-(w,\nabla\phi)=(f^1,\phi). \tag{4.35}$$

We take the inner product of (4.26) with w to obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2_{L^2}-\nu(\Delta w,w)-\tilde{\nu}(\nabla\mathrm{div}w,w)\\&\quad+\gamma^2(\nabla\phi,w)-\beta^2(\Delta\zeta,w)-\beta^2(K_\infty\zeta,w)=(f^2,w)\end{aligned}$$

By integration by parts, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2_{L^2}+\nu\|\nabla w\|^2_{L^2}+\tilde{\nu}\|\mathrm{div}w\|^2_{L^2}\\&\quad+\gamma^2(\nabla\phi,w)-\beta^2(\Delta\zeta,w)-\beta^2(K_\infty\zeta,w)=(f^2,w).\end{aligned} \tag{4.36}$$

We take the inner product of (4.27) with $-\Delta\zeta$ to obtain

$$-(\partial_t\zeta,\Delta\zeta)+(w,\Delta\zeta)-(w^3\partial_{x_3}\bar{\psi}_\infty,\Delta\zeta)=-(f^3,\Delta\zeta).$$

By integration by parts, we have

$$-(\partial_t\zeta,\Delta\zeta)=(\partial_t\nabla\zeta,\nabla\zeta)=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\zeta\|^2_{L^2},$$

and

$$-(w^3\partial_{x_3}\bar{\psi}_\infty,\Delta\zeta)=(\nabla(w^3\partial_{x_3}\bar{\psi}_\infty),\nabla\zeta).$$

We thus obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\zeta\|^2_{L^2}+(w,\Delta\zeta)+(\nabla(w^3\partial_{x_3}\bar{\psi}_\infty),\nabla\zeta)=-(f^3,\Delta\zeta). \tag{4.37}$$

It then follows from $\gamma^2\times(4.35)+(4.36)+\beta^2\times(4.37)$ that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\gamma^2\|\phi\|^2_{L^2}+\|w\|^2_{L^2}+\beta^2\|\nabla\zeta\|^2_{L^2})+\nu\|\nabla w\|^2_{L^2}+\tilde{\nu}\|\mathrm{div}w\|^2_{L^2}\\&=\beta^2\left((K_\infty\zeta,w)-(\nabla(w^3\partial_{x_3}\bar{\psi}_\infty),\nabla\zeta)\right)\\&\quad+\gamma^2(f^1,\phi)+(f^2,w)-\beta^2(f^3,\Delta\zeta).\end{aligned} \tag{4.38}$$

This yields (4.34) for $j = k = 0$. The case $1 \le 2j + k \le 2$ can be proved similarly by applying $\partial_t^j \partial^k$ to (4.25), (4.26) and (4.27). This completes the proof. ■

Proposition 4.8. *Let k be a nonnegative integer satisfying $0 \le k \le 2$. Then it holds the estimate:*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\partial^k w, \partial^k \zeta) + \frac{\beta^2}{2}\|\nabla \partial^k \zeta\|_{L^2}^2 + \frac{\gamma^2}{2}\|\mathrm{div}\partial^k \zeta\|_{L^2}^2 \\
&\quad \le \left(1 + \frac{\nu^2}{2\beta^2}\right)\|\nabla \partial^k w\|_{L^2}^2 + \frac{\tilde{\nu}^2}{2\gamma^2}\|\mathrm{div}\partial^k w\|_{L^2}^2 \\
&\qquad + \gamma^2|(M_\infty \partial^k \zeta, \partial^k \zeta)| + \beta^2|(K_\infty \partial^k \zeta, \partial^k \zeta)| + |(\partial^k w^3 \partial_{x_3}\bar{\psi}_\infty, \partial^k w)| + N_k^2,
\end{aligned} \tag{4.39}$$

where

$$N_k^2 = |(\partial^k f^2, \partial^k \zeta)| + |(\partial^k f^3, \partial^k w)| + \gamma^2|(\partial^k f^4, \partial^k \zeta)|.$$

Proof. We consider the case $k = 0$ only. We take the inner product of (4.26) with ζ to obtain

$$\begin{aligned}
&(\partial_t w, \zeta) - \nu(\Delta w, \zeta) - \tilde{\nu}(\nabla \mathrm{div} w, \zeta) \\
&+\gamma^2(\nabla\phi, \zeta) - \beta^2(\Delta\zeta, \zeta) - \beta^2(K_\infty \zeta, \zeta) = (f^2, \zeta).
\end{aligned} \tag{4.40}$$

The first term on the left-hand side of (4.40) is written as

$$(\partial_t w, \zeta) = \frac{\mathrm{d}}{\mathrm{d}t}(w, \zeta) - (w, \partial_t \zeta).$$

By integration by parts, the fifth term of (4.40) is written as $-(\Delta\zeta, \zeta) = \|\nabla\zeta\|_{L^2}^2$. It then follows from (4.40) that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(w, \zeta) - (w, \partial_t \zeta) + \beta^2\|\nabla\zeta\|_{L^2}^2 + \gamma^2(\nabla\phi, \zeta) \\
&= \nu(\Delta w, \zeta) + \tilde{\nu}(\nabla \mathrm{div} w, \zeta) + \beta^2(K_\infty \zeta, \zeta) + (f^2, \zeta).
\end{aligned} \tag{4.41}$$

We take the inner product of (4.27) with w to obtain

$$(w, \partial_t \zeta) = \|w\|_{L^2}^2 - (w^3 \partial_{x_3}\bar{\psi}_\infty, w) + (f^3, w). \tag{4.42}$$

By (4.41) + (4.42), we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(w, \zeta) + \beta^2\|\nabla\zeta\|_{L^2}^2 + \gamma^2(\nabla\phi, \zeta) \\
&= \nu(\Delta w, \zeta)\tilde{\nu}(\nabla \mathrm{div} w, \zeta) + \|w\|_{L^2}^2 \\
&+\beta^2(K_\infty \zeta, \zeta) - (w^3 \partial_{x_3}\bar{\psi}_\infty, w) + (f^2, \zeta) + (f^3, w).
\end{aligned} \tag{4.43}$$

We take the inner product of (4.28) with ζ to obtain

$$(\nabla\phi, \zeta) = (\nabla \mathrm{div}\zeta, \zeta) + (M_\infty \zeta, \zeta) + (f^4, \zeta).$$

By integration by parts, we have $(\nabla \mathrm{div}\zeta, \zeta) = -\|\mathrm{div}\zeta\|_{L^2}^2$. We thus obtain

$$(\nabla\phi, \zeta) + \|\mathrm{div}\zeta\|_{L^2}^2 = (M_\infty \zeta, \zeta) + (f^4, \zeta). \tag{4.44}$$

By (4.43) $- \gamma^2 \times$ (4.44), we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(w, \zeta) + \beta^2 \|\nabla\zeta\|_{L^2}^2 + \gamma^2 \|\mathrm{div}\zeta\|_{L^2}^2 \\
&\quad = \nu(\Delta w, \zeta) + \tilde{\nu}(\nabla \mathrm{div} w, \zeta) + \|w\|_{L^2}^2 \\
&\qquad + \beta^2 (K_\infty \zeta, \zeta) + \gamma^2 (M_\infty \zeta, \zeta) - (w^3 \partial_{x_3} \bar{\psi}_\infty, w) \\
&\qquad\quad + (f^2, \zeta) + (f^3, w) - \gamma^2 (f^4, \zeta).
\end{aligned} \tag{4.45}$$

By integration by parts, we have

$$\nu(\Delta w, \zeta) = -\nu(\nabla w, \nabla\zeta) \leq \frac{\beta^2}{2}\|\nabla\zeta\|_{L^2}^2 + \frac{\nu^2}{2\beta^2}\|\nabla w\|_{L^2}^2,$$

$$\tilde{\nu}(\nabla \mathrm{div} w, \zeta) = -\tilde{\nu}(\mathrm{div} w, \mathrm{div}\zeta) \leq \frac{\gamma^2}{2}\|\mathrm{div}\zeta\|_{L^2}^2 + \frac{\tilde{\nu}^2}{2\gamma^2}\|\mathrm{div} w\|_{L^2}^2.$$

It then follows from (4.45) that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(w, \zeta) + \frac{\beta^2}{2} \|\nabla\zeta\|_{L^2}^2 + \frac{\gamma^2}{2} \|\mathrm{div}\zeta\|_{L^2}^2 \\
&\quad \leq \left(1 + \frac{\nu^2}{2\beta^2}\right) \|\nabla w\|_{L^2}^2 + \frac{\tilde{\nu}^2}{2\gamma^2}\|\mathrm{div} w\|_{L^2}^2 \\
&\qquad + \beta^2 |(K_\infty \zeta, \zeta)| + \gamma^2 |(M_\infty \zeta, \zeta)| + |(w^3 \partial_{x_3} \bar{\psi}_\infty, w)| \\
&\qquad\quad + |(f^2, \zeta)| + |(f^3, w)| + \gamma^2 |(f^4, \zeta)|.
\end{aligned} \tag{4.46}$$

This proves (4.39) for $k = 0$. The case $0 < k \leq 2$ can be proved similarly by applying ∂^k to (4.26), (4.27) and (4.28). This completes the proof. ■

Proposition 4.9. *Let k be a nonnegative integers satisfying $0 \leq k \leq 1$. Then it holds the estimate:*

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\nu \|\nabla\partial^k w(t)\|_{L^2}^2 + \tilde{\nu}\|\mathrm{div}\partial^k w(t)\|_{L^2}^2 \\
&-2\gamma^2 (\partial^k \phi, \mathrm{div}\partial^k w) + 2\beta^2 (\nabla\partial^k\zeta, \nabla\partial^k w)) + \frac{1}{2}\|\partial_t \partial^k w\|_{L^2}^2 \\
&\quad \leq \beta^2 \|\nabla\partial^k w\|_{L^2}^2 + \gamma^2 \|\mathrm{div}\partial^k w\|_{L^2}^2 \\
&\qquad + \beta^2 \left(|(K_\infty \partial^k\zeta, \partial_t \partial^k w)| + |(\nabla(\partial^k w \cdot \nabla\bar{\psi}_\infty), \nabla\partial^k w)|\right) + N_k^3,
\end{aligned} \tag{4.47}$$

where

$$N_k^3 = \gamma^2|(\partial^k f^1, \mathrm{div}\partial^k w)| + \frac{1}{2}\|\partial^k f^2\|_{L^2}^2 + \beta^2|(\nabla\partial^k f^3, \nabla\partial^k w)|.$$

Proof. We consider the case $k = 0$. We take the inner product of (4.25) with $-\mathrm{div}w$ to obtain

$$-(\partial_t\phi, \mathrm{div}w) - \|\mathrm{div}w\|_{L^2}^2 = -(f^1, \mathrm{div}w).$$

Since

$$-(\partial_t\phi, \mathrm{div}w) = -\frac{\mathrm{d}}{\mathrm{d}t}(\phi, \mathrm{div}w) + (\phi, \mathrm{div}\partial_t w),$$

we obtain

$$-\frac{\mathrm{d}}{\mathrm{d}t}(\phi, \mathrm{div}w) + (\phi, \mathrm{div}\partial_t w) = \|\mathrm{div}w\|_{L^2}^2 - (f^1, \mathrm{div}w). \tag{4.48}$$

We take the inner product of (4.26) with $\partial_t w$ to obtain

$$\begin{aligned}&\|\partial_t w\|_{L^2}^2 - \nu(\Delta w, \partial_t w) - \tilde{\nu}(\nabla\mathrm{div}w, \partial_t w)\\ &\quad+\gamma^2(\nabla\phi, \partial_t w) - \beta^2(\Delta\zeta, \partial_t w) - \beta^2(K_\infty\zeta, \partial_t w) = (f^2, \partial_t w).\end{aligned}$$

By integration by parts, we have

$$-(\Delta w, \partial_t w) = (\partial_t\nabla w, \nabla w) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla w\|_{L^2}^2,$$

$$-(\nabla\mathrm{div}w, \partial_t w) = (\partial_t\mathrm{div}w, \mathrm{div}w) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathrm{div}w\|_{L^2}^2,$$

$$(\nabla\phi, \partial_t w) = -(\phi, \mathrm{div}\partial_t w),$$

$$-(\Delta\zeta, \partial_t w) = (\nabla\zeta, \partial_t\nabla w).$$

We thus obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\nu\|\nabla w\|_{L^2}^2 + \tilde{\nu}\|\mathrm{div}w\|_{L^2}^2\right) + \|\partial_t w\|_{L^2}^2\\ &\quad-\gamma^2(\phi, \mathrm{div}\partial_t w) + \beta^2(\nabla\zeta, \nabla\partial_t w)\\ &\quad= \beta^2(K_\infty\zeta, \partial_t w) + (f^2, \partial_t w).\end{aligned} \tag{4.49}$$

We take the inner product of (4.27) with $-\Delta w$ to obtain

$$-(\partial_t\zeta, \Delta w) + (w, \Delta w) - (w \cdot \nabla\bar{\psi}_\infty, \Delta w) = -(f^3, \Delta w).$$

By integration by parts, we have

$$
\begin{aligned}
-(\partial_t\zeta, \Delta w) &= (\partial_t \nabla\zeta, \nabla w) = \frac{\mathrm{d}}{\mathrm{d}t}(\nabla\zeta, \nabla w) - (\nabla\zeta, \partial_t\nabla w),\\
(w, \Delta w) &= -\|\nabla w\|_{L^2}^2,\\
-(w\cdot\nabla\bar{\psi}_\infty, \Delta w) &= (\nabla(w\cdot\nabla\bar{\psi}_\infty), \nabla w).
\end{aligned}
$$

We thus obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\nabla\zeta, \nabla w) - (\nabla\zeta, \partial_t\nabla w)\\
&\quad = \|\nabla w\|_{L^2}^2 - (\nabla(w\cdot\nabla\bar{\psi}_\infty), \nabla w) - (\nabla f^3, \nabla w).
\end{aligned}
\tag{4.50}
$$

By $\gamma^2 \times (4.48) + (4.49) + \beta^2 \times (4.50)$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\nu\|\nabla w\|_{L^2}^2 + \tilde{\nu}\|\mathrm{div}w\|_{L^2}^2 - 2\gamma^2(\phi, \mathrm{div}w) + 2\beta^2(\nabla\zeta, \nabla w)) + \|\partial_t w\|_{L^2}^2\\
&= \beta^2\|\nabla w\|_{L^2}^2 + \gamma^2\|\mathrm{div}w\|_{L^2}^2\\
&\quad -\beta^2\left((K_\infty\zeta, \partial_t w) + (\nabla(w\cdot\nabla\bar{\psi}_\infty), \nabla w)\right)\\
&\qquad -\gamma^2(f^1, \mathrm{div}w) + (f^2, \partial_t w) + \beta^2(\nabla f^3, \nabla\zeta).
\end{aligned}
\tag{4.51}
$$

This, together with the inequality $|(f^2, \partial_t w)| \le \frac{1}{2}\|\partial_t w\|_{L^2}^2 + \frac{1}{2}\|f^2\|_{L^2}^2$, proves (4.47) for $k = 0$. The case $k = 1$ can be proved similarly by applying ∂ to (4.25), (4.26) and (4.27). This completes the proof. ■

We next estimate x_3-derivatives of ϕ. We introduce the following quantities:

$$
\dot{\phi} = \partial_t\phi + (\bar{v} + w)\cdot\nabla\phi, \tag{4.52}
$$

$$
q = \nu w + \beta^2\zeta. \tag{4.53}
$$

Note that

$$
\dot{\phi} = -\mathrm{div}w - \phi\mathrm{div}w. \tag{4.54}
$$

Proposition 4.10. *Let k and l be nonnegative integers satisfying $l \ge 1, 0 \le k + l - 1 \le 1$. Then it holds the estimate:*

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial^k\partial_{x_3}^l\phi\|_{L^2}^2 + \frac{1}{2}\frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\|\partial^k\partial_{x_3}^l\phi\|_{L^2}^2 + b_0\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\|\partial^k\partial_{x_3}^l\dot{\phi}\|_{L^2}^2\\
&\le \frac{C}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}\left\{\|\partial_t\nabla^{l-1}\partial^k w\|_{L^2}^2 + \|\nabla^l\partial^{k+1}q\|_{L^2}^2\right.\\
&\qquad \left.+\beta^4(\|\nabla^{l-1}K_\infty\partial^k\zeta)\|_{L^2}^2 + \|\nabla^{l-1}M_\infty\partial^k\zeta\|_{L^2}^2)\right\} + CN_{k,l}^4.
\end{aligned}
\tag{4.55}
$$

Here $b_0 > 0$ *is a constant independent of* ν, $\tilde{\nu}$, γ^2 *and* β^2, *and*

$$N^4_{k,l} = \frac{1}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}(\|\partial^k\nabla^{l-1}f^2\|^2_{L^2} + \beta^4\|\partial^k\nabla^{l-1}f^4\|^2_{L^2})$$
$$+\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\|\partial^k\nabla^l(\phi\mathrm{div}w)\|^2_{L^2} + |(\partial^k\partial^l_{x_3}((\bar{v}+w)\cdot\nabla\phi), \partial^k\partial^l_{x_3}\phi)|.$$

Proof. We consider the case $k=0$ and $l=1$ only. We see from the 3rd equation of (4.26)

$$\partial_t w^3 - \nu\Delta w^3 - \tilde{\nu}\partial_{x_3}\mathrm{div}w + \gamma^2\partial_{x_3}\phi - \beta^2(\Delta\zeta^3 + (K_\infty\zeta)^3) = f^{2,3},$$

which is rewritten by using (4.53) as

$$\begin{aligned} &-\partial^2_{x_3}q^3 - \tilde{\nu}\partial^2_{x_3}w^3 + \gamma^2\partial_{x_3}\phi \\ &\quad = -\partial_t w^3 + \Delta' q^3 + \tilde{\nu}\partial_{x_3}\nabla'\cdot w' + \beta^2(K_\infty\zeta)^3 + f^{2,3}. \end{aligned} \tag{4.56}$$

By applying ∂_{x_3} to (4.54), we have

$$\partial^2_{x_3}w^3 + \partial_{x_3}\dot{\phi} = -\partial_{x_3}\nabla'\cdot w' - \partial_{x_3}(\phi\mathrm{div}w). \tag{4.57}$$

By (4.56) + $\tilde{\nu}$ × (4.57), we obtain

$$\begin{aligned} &-\partial^2_{x_3}q^3 + \tilde{\nu}\partial_{x_3}\dot{\phi} + \gamma^2\partial_{x_3}\phi \\ &\quad = -\partial_t w^3 + \Delta' q^3 + \beta^2(K_\infty\zeta)^3 + f^{2,3} - \tilde{\nu}\partial_{x_3}(\phi\mathrm{div}w). \end{aligned} \tag{4.58}$$

We see from the 3rd equation of (4.28) that

$$\partial^2_{x_3}\zeta^3 + \partial_{x_3}\phi = -\partial_{x_3}\nabla'\cdot\zeta' + (M_\infty\zeta)^3 + f^{4,3}. \tag{4.59}$$

By ν × (4.57) + β^2 × (4.59), we obtain

$$\begin{aligned} &\partial^2_{x_3}q^3 + \nu\partial_{x_3}\dot{\phi} + \beta^2\partial_{x_3}\phi \\ &\quad = -\partial_{x_3}\nabla'\cdot q' + \beta^2(M_\infty\zeta)^3 + \beta^2 f^{4,3} - \nu\partial_{x_3}(\phi\mathrm{div}w). \end{aligned} \tag{4.60}$$

By (4.58) + (4.60), we obtain

$$\begin{aligned} &(\nu+\tilde{\nu})\partial_{x_3}\dot{\phi} + (\beta^2+\gamma^2)\partial_{x_3}\phi \\ &\quad = -\partial_t w^3 + \Delta' q^3 - \partial_{x_3}\nabla'\cdot q' + \beta^2((K_\infty\zeta)^3 + (M_\infty\zeta)^3) \\ &\qquad + f^{2,3} + \beta^2 f^{4,3} - (\nu+\tilde{\nu})\partial_{x_3}(\phi\mathrm{div}w). \end{aligned} \tag{4.61}$$

This gives

$$
\begin{aligned}
&\partial_{x_3}\dot{\phi} + \frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\partial_{x_3}\phi \\
&\quad = \frac{1}{\nu+\tilde{\nu}}(-\partial_t w^3 + \Delta' q^3 - \partial_{x_3}\nabla'\cdot q') - \frac{\beta^2}{\nu+\tilde{\nu}}((K_\infty\zeta)^3 + (M_\infty\zeta)^3) \\
&\qquad + \frac{1}{\nu+\tilde{\nu}} f^{2,3} + \frac{\beta^2}{\nu+\tilde{\nu}} f^{4,3} - \partial_{x_3}(\phi \mathrm{div} w).
\end{aligned}
\tag{4.62}
$$

We take the inner product of (4.62) with $\partial_{x_3}\phi$ to obtain

$$
\begin{aligned}
&(\partial_{x_3}\dot{\phi}, \partial_{x_3}\phi) + \frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\|\partial_{x_3}\phi\|_{L^2}^2 \\
&= \frac{1}{\nu+\tilde{\nu}}(-\partial_t w^3 + \Delta' q^3 - \partial_{x_3}\nabla'\cdot q', \partial_{x_3}\phi) \\
&\quad + \frac{\beta^2}{\nu+\tilde{\nu}}((K_\infty\zeta)^3 + (M_\infty\zeta)^3, \partial_{x_3}\phi) \\
&\qquad + \frac{1}{\nu+\tilde{\nu}}(f^{2,3}, \partial_{x_3}\phi) + \frac{\beta^2}{\nu+\tilde{\nu}}(f^{4,3}, \partial_{x_3}\phi) - (\partial_{x_3}(\phi \mathrm{div} w), \partial_{x_3}\phi).
\end{aligned}
\tag{4.63}
$$

By the definition of $\dot{\phi}$, we have

$$
(\partial_{x_3}\dot{\phi}, \partial_{x_3}\phi) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_{x_3}\phi\|_{L^2}^2 + (\partial_{x_3}((\bar{v}+w)\cdot\nabla\phi), \partial_{x_3}\phi). \tag{4.64}
$$

The right-hand side of (4.63) is estimated as

$$
\begin{aligned}
&\frac{1}{\nu+\tilde{\nu}}(-\partial_t w^3 + \Delta' q^3 - \partial_{x_3}\nabla'\cdot q', \partial_{x_3}\phi) - \frac{\beta^2}{\nu+\tilde{\nu}}((K_\infty\zeta)^3 + (M_\infty\zeta)^3, \partial_{x_3}\phi) \\
&+ \frac{1}{\nu+\tilde{\nu}}(f^{2,3}, \partial_{x_3}\phi) + \frac{\beta^2}{\nu+\tilde{\nu}}(f^{4,3}, \partial_{x_3}\phi) - (\partial_{x_3}(\phi \mathrm{div} w), \partial_{x_3}\phi) \\
&\le \frac{1}{2}\frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\|\partial_{x_3}\phi\|_{L^2}^2 + \frac{4}{(\nu+\tilde{\nu})(\beta^2+\gamma^2)}(\|\partial_t w^3\|_{L^2}^2 + 2\|\nabla\partial q\|_{L^2}^2) \\
&\quad + \frac{4\beta^4}{(\nu+\tilde{\nu})(\beta^2+\gamma^2)}(\|K_\infty\zeta\|_{L^2}^2 + \|M_\infty\zeta\|_{L^2}^2) \\
&\qquad + \frac{4}{(\nu+\tilde{\nu})(\beta^2+\gamma^2)}(\|f^2\|_{L^2}^2 + \beta^4\|f^4\|_{L^2}^2) + \frac{4(\nu+\tilde{\nu})}{\beta^2+\gamma^2}\|\partial_{x_3}(\phi \mathrm{div} w)\|_{L^2}^2.
\end{aligned}
\tag{4.65}
$$

It follows from (4.63)–(4.65) that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_{x_3}\phi\|_{L^2}^2+\frac{1}{2}\frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\|\partial_{x_3}\phi\|_{L^2}^2\\
&\le \frac{C}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}\left\{\|\partial_t w\|_{L^2}^2+\|\nabla\partial q\|_{L^2}^2+\beta^4(\|K_\infty\zeta\|_{L^2}^2+\|M_\infty\zeta\|_{L^2}^2)\right\}\\
&\quad+\frac{C}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}(\|f^2\|_{L^2}^2+\beta^4\|f^4\|_{L^2}^2)\\
&\quad+C\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\|\partial_{x_3}(\phi\mathrm{div}w)\|_{L^2}^2+C|(\partial_{x_3}((\bar{v}+w)\cdot\nabla\phi),\partial_{x_3}\phi)|.
\end{aligned}\tag{4.66}$$

We deduce from (4.63) and (4.66) that

$$\begin{aligned}
&\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\|\partial_{x_3}\dot{\phi}\|_{L^2}^2\\
&\le C\left(\frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\|\partial_{x_3}\phi\|_{L^2}^2+\frac{1}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}(\|\partial_t w\|_{L^2}^2+\|\nabla\partial q\|_{L^2}^2)\right.\\
&\quad+\frac{\beta^4}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}(\|K_\infty\zeta\|_{L^2}^2+\|(M_\infty\zeta)\|_{L^2}^2)\\
&\quad\quad\left.+\frac{1}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}(\|f^2\|_{L^2}^2+\beta^4\|f^4\|_{L^2}^2)+\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\|\partial_{x_3}(\phi\mathrm{div}w)\|_{L^2}^2\right).
\end{aligned}\tag{4.67}$$

We thus obtain (4.55) for $k=0$ and $l=1$ by adding (4.66) to $b_0\times(4.67)$ with $b_0>0$ satisfying $b_0C\le\frac{1}{4}$. The case $l\ge 1,\ 0<k+l-1\le 1$ can be proved similarly by applying $\partial^k\partial_{x_3}^{l-1}$ to (4.62). This completes the proof. ■

We next estimate higher order derivatives of w and ζ and tangential derivatives of ϕ.

Proposition 4.11. *It holds the following estimate:*

$$\begin{aligned}
&\|\nabla^2\partial q\|_{L^2}+\gamma^2\|\nabla\partial\phi\|_{L^2}\\
&\quad\le C\left(\|\partial_t\partial w\|_{L^2}+(\nu+\tilde{\nu})\|\partial\dot{\phi}\|_{H^1}+\beta^2\|\partial\phi\|_{H^1}+\beta^2\|K_\infty\partial\zeta\|_{L^2}+\beta^2\|M_\infty\zeta\|_{H^1}\right.\\
&\quad\quad\left.+(\nu+\tilde{\nu})\|\partial(\phi\mathrm{div}w)\|_{H^1}+\|f^2\|_{H^1}+\beta^2\|f^4\|_{H^1}+\|\nabla\partial q\|_{L^2}\right).
\end{aligned}\tag{4.68}$$

Proof. By (4.26), (4.52) and (4.53), we have

$$\begin{gathered}
\mathrm{div}(\partial q)=Q_1 \text{ in } \Omega,\\
-\Delta\partial q+\nabla(\gamma^2\phi)=Q_2 \text{ in } \Omega,\\
\partial q=0 \text{ on } \{x_3=0,1\},
\end{gathered}$$

where

$$Q_1 = \nu\partial\dot{\phi} - \beta^2\partial\phi + \beta^2(M_\infty\zeta)' - \nu\partial(\phi\mathrm{div}w) + \beta^2(f^4)',$$
$$Q_2 = -\partial_t\partial w - \tilde{\nu}\nabla\partial\dot{\phi} - \tilde{\nu}\nabla\partial(\phi\mathrm{div}w) + \beta^2 K_\infty\partial\zeta + \partial f^2.$$

We apply Lemma 2.2 with $k = 0$ to obtain

$$\|\nabla^2\partial q\|_{L^2} + \gamma^2\|\nabla\partial\phi\|_{L^2} \le C(\|Q_1\|_{H^1} + \|Q_2\|_{L^2} + \|\nabla\partial q\|_{L^2}). \tag{4.69}$$

This completes the proof. ■

Proposition 4.12. *Let j and l be nonnegative integers satisfying $j = 1, 2,\ l = 0, 1$. Then it hold the estimates:*

$$\begin{aligned}\|\partial_{x_3}^{l+2} q^j\|_{L^2} \le C \Big(&\|\partial_t\partial_{x_3}^l w^j\|_{L^2} + \|\partial_{x_3}^l\partial^2 q\|_{L^2} + \tilde{\nu}\|\partial_{x_3}^l\partial_{x_j}\dot{\phi}\|_{L^2} + \gamma^2\|\partial_{x_3}^l\partial\phi\|_{L^2} \\ &+\beta^2\|\partial_{x_3}^l(K_\infty\zeta)\|_{L^2} + \|\partial_{x_3}^l f^2\|_{L^2} + \tilde{\nu}\|\partial_{x_3}^l\partial_{x_j}(\phi\mathrm{div}w)\|_{L^2}\Big),\end{aligned} \tag{4.70}$$

and

$$\begin{aligned}\|\partial_{x_3}^{l+2} q^3\|_{L^2} \le C \Big(&\nu\|\partial_{x_3}^{l+1}\dot{\phi}\|_{L^2} + \beta^2\|\partial_{x_3}^{l+1}\phi\|_{L^2} + \|\partial_{x_3}^{l+1}\partial q\|_{L^2} \\ &+\beta^2\|\partial_{x_3}^l(M_\infty\zeta)\|_{L^2} + \nu\|\partial_{x_3}^{l+1}(\phi\mathrm{div}w)\|_{L^2} + \beta^2\|\partial_{x_3}^l f^4\|_{L^2}).\end{aligned} \tag{4.71}$$

Proof. We see from the j-th equation of (4.26) for $j = 1, 2$ that

$$\partial_t w^j - \Delta q^j + \tilde{\nu}\partial_{x_j}(\dot{\phi} + \phi\mathrm{div}w) + \gamma^2\partial_{x_j}\phi - \beta^2(K_\infty\zeta)^j = f^{2,j},$$

which is rewritten as

$$\partial_{x_3}^2 q^j = -\partial_t w^j + \Delta' q^j - \tilde{\nu}\partial_{x_j}\dot{\phi} - \gamma^2\partial_{x_j}\phi + \beta^2(K_\infty\zeta)^j + f^{2,j} - \tilde{\nu}\partial_{x_j}(\phi\mathrm{div}w)$$

This gives the estimate (4.70). The estimate (4.71) follows from (4.60) directly. This completes the proof. ■

Proposition 4.13. *It hold the estimates:*

$$\|\partial^k\phi\|_{L^2} \le \|\nabla\partial^k\zeta\|_{L^2} + \|\partial^{k-1}M_\infty\zeta\|_{L^2} + \|\partial^{k-1}f^4\|_{L^2},\ k = 1, 2, \tag{4.72}$$

$$\begin{aligned}\nu\|\partial_{x_3}^2 w\|_{L^2} \le &\|\partial_t w\|_{L^2} + (\nu + \tilde{\nu})\|\nabla\partial w\|_{L^2} + \gamma^2\|\nabla\phi\|_{L^2} \\ &+\beta^2\|\nabla^2\zeta\|_{L^2} + \beta^2\|K_\infty\zeta\|_{L^2} + \|f^2\|_{L^2},\end{aligned} \tag{4.73}$$

$$\|\partial_t\phi\|_{L^2} \le \|\nabla w\|_{L^2} + \|f^1\|_{L^2}, \tag{4.74}$$

$$\|\partial_t \nabla\zeta\|_{L^2} \le \|\nabla w\|_{L^2} + \|\nabla(w\cdot\nabla\bar{\psi}_\infty)\|_{L^2} + \|\nabla f^3\|_{L^2}, \tag{4.75}$$

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^k\zeta\|_{L^2}^2 + \frac{\beta^2}{2\nu}\|\nabla^k\zeta\|_{L^2}^2 \\ &\quad\le \frac{1}{2\nu\beta^2}\|\nabla^k q\|_{L^2}^2 + |(\nabla^k(w\cdot\nabla\bar{\psi}_\infty), \nabla^k\zeta)| + |(\nabla^k f^3, \nabla^k\zeta)|,\ \ k=2,3.\end{aligned} \tag{4.76}$$

Proof. We see from (4.28) that

$$\partial^k\phi = \partial^k \mathrm{div}\zeta + \partial^{k-1}(M_\infty\zeta)' + \partial^{k-1}(f^4)',$$

which gives (4.72).

As for (4.73), we see from (4.26) that

$$-\nu\partial_{x_3}^2 w' = -\partial_t w' + \nu\Delta' w' + \tilde{\nu}\partial \mathrm{div} w - \gamma^2\partial\phi + \beta^2(\Delta\zeta' + (K_\infty\zeta)') + (f^2)',$$

and

$$-(\nu+\tilde{\nu})\partial_{x_3}^2 w^3 = -\partial_t w^3 + \nu\Delta' w^3 + \tilde{\nu}\nabla'\cdot w' - \gamma^2\partial_{x_3}\phi + \beta^2(\Delta\zeta^3 + (K_\infty\zeta)^3) + f^{2,3},$$

which gives (4.73).

The estimate (4.74) immediately follows from the equation (4.25).

As for (4.75), we ∇^k to (4.27) to obtain

$$\partial_t\nabla^k\zeta = \nabla^k w - \nabla^k(w\cdot\nabla\bar{\psi}_\infty) + \nabla^k f^3 \text{ for } k=1,2,3, \tag{4.77}$$

which gives (4.75).

We take the inner product of (4.77) with $\nabla^k\zeta$ and use $w = \frac{1}{\nu}q - \frac{\beta^2}{\nu}\zeta$ from (4.53) to obtain,

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^k\zeta\|_{L^2}^2 + \frac{\beta^2}{\nu}\|\nabla^k\zeta\|_{L^2}^2 \\ &\quad -\frac{1}{\nu}(\nabla^k q, \nabla^k\zeta) + (\nabla^k(w\cdot\nabla\bar{\psi}_\infty), \nabla^k\zeta) = (\nabla^k f^3, \nabla^k\zeta).\end{aligned}$$

This leads to (4.76). This completes the proof. ■

We are now in a position to prove Proposition 4.6.

Proof of Proposition 4.6. By using the Poincaré inequality and integration of parts, we have

$$|(K_\infty\partial^k\zeta, \partial^k w)| + |(\nabla(\partial^k w\cdot\nabla\bar{\psi}_\infty), \nabla\partial^k\zeta)| \le \frac{C}{\beta^2}\|\nabla\partial^k\zeta\|_{L^2}^2\|\nabla\partial^k w\|_{L^2}^2.$$

From (4.34), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}E_0^k + d_0 D_0^k \le \frac{C}{\nu\gamma^2}\|\nabla\partial^k\zeta\|_{L^2}^2 + \frac{1}{\gamma^2}N_{0,k}^1, \tag{4.78}$$

where

$$E_0^k = \|\partial^k\phi\|_{L^2}^2 + \frac{1}{\gamma^2}\|\partial^k w\|_{L^2}^2 + \frac{\beta^2}{\gamma^2}\|\nabla\partial^k\zeta\|_{L^2}^2,$$
$$D_0^k = \frac{\nu}{\gamma^2}\|\nabla\partial^k w\|_{L^2}^2 + \frac{\tilde{\nu}}{\gamma^2}\|\mathrm{div}\partial^k w\|_{L^2}^2.$$

By using the Poincaré inequality and integration by parts, we have

$$\begin{aligned}&\gamma^2|(M_\infty\partial^k\zeta, \partial^k\zeta)| + \beta^2|(K_\infty\partial^k\zeta, \partial^k\zeta)| + |(\partial^k w\cdot\nabla\bar{\psi}_\infty, \partial^k w)|\\ &\le C\left(1+\frac{\gamma^2}{\beta^2}\right)\|\nabla\partial^k\zeta\|_{L^2}^2 + \frac{C}{\beta^2}\|\nabla\partial^k w\|_{L^2}^2.\end{aligned}$$

From (4.39), we obtain

$$\begin{aligned}&\frac{1}{\gamma^2}\frac{\mathrm{d}}{\mathrm{d}t}(\partial^k w, \partial^k\zeta) + \frac{\beta^2}{2\gamma^2}\|\nabla\partial^k\zeta\|_{L^2}^2 + \frac{1}{2}\|\mathrm{div}\partial^k\zeta\|_{L^2}^2\\ &\le \left(\frac{1}{\gamma^2}+\frac{\nu^2}{2\beta^2\gamma^2}+\frac{C}{\beta^2}\right)\|\nabla\partial^k w\|_{L^2}^2 + \frac{\tilde{\nu}^2}{2\gamma^4}\|\mathrm{div}\partial^k w\|_{L^2}^2\\ &\quad + C\left(\frac{1}{\beta^2}+\frac{1}{\gamma^2}\right)\|\nabla\partial^k\zeta\|_{L^2}^2 + \frac{1}{\gamma^2}N_k^2.\end{aligned} \tag{4.79}$$

It follows from (4.78) + (4.79) that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(E_0^k + \frac{2}{\gamma^2}(\partial^k w, \partial^k\zeta)\right) + \left(D_0^k + \frac{\beta^2}{2\gamma^2}\|\nabla\partial^k\zeta\|_{L^2}^2 + \frac{1}{2}\|\mathrm{div}\partial^k\zeta\|_{L^2}^2\right)\\ &\le \left(\frac{1}{\gamma^2}+\frac{\nu^2}{2\beta^2\gamma^2}+\frac{C}{\beta^2}\right)\|\nabla\partial^k w\|_{L^2}^2 + \frac{\tilde{\nu}^2}{2\gamma^4}\|\mathrm{div}\partial^k w\|_{L^2}^2\\ &\quad + C\left(\frac{1}{\beta^2}+\frac{1}{\gamma^2}+\frac{1}{\nu\gamma^2}\right)\|\nabla\partial^k\zeta\|_{L^2}^2 + \frac{1}{\gamma^2}(N_{0,k}^1 + N_k^2).\end{aligned} \tag{4.80}$$

We take ν, $\tilde{\nu}$, γ^2 and β^2 so that $\frac{1}{\gamma^2}+\frac{\nu^2}{2\beta^2\gamma^2}+\frac{C}{\beta^2} \le \frac{\nu}{4\gamma^2}$, $\frac{\nu+\tilde{\nu}}{\gamma^2} \le \frac{1}{2}$ and $C\left(\frac{1}{\beta^2}+\frac{1}{\gamma^2}+\frac{1}{\nu\gamma^2}\right) \le \frac{\beta^2}{4\gamma^2}$. It then follows

$$\frac{\mathrm{d}}{\mathrm{d}t}E_1^k + d_1 D_1^k \le C R_1^k. \tag{4.81}$$

Here

$$E_1^k = E_0^k + \frac{1}{\gamma^2}(\partial^k w, \partial^k \zeta),$$

$$D_1^k = D_0^k + \frac{\beta^2}{\gamma^2}\|\nabla\partial^k\zeta\|_{L^2}^2 + \|\mathrm{div}\partial^k\zeta\|_{L^2}^2,$$

$$R_1^k = \frac{1}{\gamma^2}(N_{0,k}^1 + N_k^2).$$

We observe that E_1^k is equivalent to E_0^k provided that $\beta^2 > 1$. By using the Poincaré inequality and integration by parts, we have

$$\begin{aligned}
&|(K_\infty\partial^k\zeta, \partial_t\partial^k w)| + |(\nabla(\partial^k w\cdot\nabla\bar{\psi}_\infty), \nabla\partial^k w)| \\
&\le \frac{C}{\beta^2}(\|\nabla\partial^k\zeta\|_{H^1}\|\partial_t\partial^k w\|_{L^2} + \|\nabla\partial^k w\|_{L^2}^2) \\
&\le \frac{1}{4\beta^2}\|\partial_t\partial^k w\|_{L^2}^2 + \frac{C}{\beta^2}(\|\nabla\zeta\|_{L^2}^2 + \|\nabla^2\zeta\|_{H^1}^2) + \frac{C}{\beta^2}\|\nabla\partial^k w\|_{L^2}^2.
\end{aligned}$$

This, together with (4.47), yields

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\nu\|\nabla\partial^k w\|_{L^2}^2 + \tilde{\nu}\|\mathrm{div}\partial^k w\|_{L^2}^2 - 2\gamma^2(\partial^k\phi, \mathrm{div}\partial^k w) + 2\beta^2(\nabla\partial^k\zeta, \nabla\partial^k w)) \\
&\quad + \frac{1}{4}\|\partial_t\partial^k w\|_{L^2}^2 \le C(\beta^2+1)\|\nabla\partial^k w\|_{L^2}^2 + C\gamma^2\|\mathrm{div}\partial^k w\|_{L^2}^2 \\
&\qquad\qquad + C\|\nabla\zeta\|_{L^2}^2 + C\|\nabla^2\zeta\|_{H^1}^2 + CN_k^3.
\end{aligned} \tag{4.82}$$

Let b_1 be a positive number which will be determined later. We set

$$\begin{aligned}
E_2 = b_1\frac{\beta^2\gamma^2}{\nu}\sum_{k=0}^{2}E_1^k + \sum_{k=0}^{1}\big(&\nu\|\nabla\partial^k w\|_{L^2}^2 + \tilde{\nu}\|\mathrm{div}\partial^k w\|_{L^2}^2 \\
&-2\gamma^2(\partial^k\phi, \mathrm{div}\partial^k w) + 2\beta^2(\nabla\partial^k\zeta, \nabla\partial^k w)\big),
\end{aligned}$$

$$D_2 = b_1\frac{\beta^2\gamma^2}{\nu}\sum_{k=0}^{2}D_1^k + \sum_{k=0}^{1}\|\partial_t\partial^k w\|_{L^2}^2,$$

$$R_2 = b_1\frac{\beta^2\gamma^2}{\nu}\sum_{k=0}^{2}R_1^k + \sum_{k=0}^{1}N_k^3.$$

By $b_1\frac{\beta^2\gamma^2}{\nu}\times\sum_{k=0}^{2}(4.81) + \sum_{k=0}^{1}(4.82)$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}E_2 + D_2 \le \frac{C}{b_1}\left(1 + \frac{1}{\beta^2} + \frac{\gamma^2}{\beta^2}\right)D_2 + C\|\partial_t\nabla\zeta\|_{H^1}^2 + CR_2. \tag{4.83}$$

We take ν, γ^2 and β^2 so large that $\frac{\nu}{\beta^2} \le \frac{1}{2}, \frac{\gamma^2}{\beta^2} \le \frac{1}{2}$ and $\frac{C}{b_1\nu}\left(1+\frac{1}{\beta^2}+\frac{\gamma^2}{\beta^2}\right) \le 2$, and then take b_1 so large that E_2 is equivalent to

$$\frac{\beta^2\gamma^2}{\nu}\sum_{k=0}^{2} E_1^k + \gamma^2 \sum_{k=0}^{1} D_0^k.$$

It then follows

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_2 + D_2 \le C\|\nabla^2\zeta\|_{H^1}^2 + CR_2. \tag{4.84}$$

From $(4.68)_{k=0,l=1}$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_{x_3}\phi\|_{L^2}^2 + \frac{1}{2}\frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\|\partial_{x_3}\phi\|_{L^2}^2 + b_0\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\|\partial_{x_3}\dot{\phi}\|_{L^2}^2 \\
&\le \frac{C}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}\big\{\|\partial_t w\|_{L^2}^2 + \|\nabla\partial q\|_{L^2}^2 \\
&\qquad + \beta^4(\|K_\infty\zeta\|_{L^2}^2 + \|M_\infty\zeta\|_{L^2}^2)\big\} + CN_{0,1}^4.
\end{aligned} \tag{4.85}$$

Since

$$\begin{aligned}
\|\nabla\partial q\|_{L^2}^2 \le C(\nu^2\|\nabla\partial w\|_{L^2}^2 + \beta^4\|\nabla\partial\zeta\|_{L^2}^2), \\
\beta^4(\|K_\infty\zeta\|_{L^2}^2 + \|M_\infty\zeta\|_{L^2}^2) \le C\|\nabla\zeta\|_{H^1}^2,
\end{aligned}$$

we see from (4.85) that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_{x_3}\phi\|_{L^2}^2 + \frac{1}{2}\frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\|\partial_{x_3}\phi\|_{L^2}^2 + b_0\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\|\partial_{x_3}\dot{\phi}\|_{L^2}^2 \\
&\le \frac{C}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}\big(\|\partial_t w\|_{L^2}^2 + \nu^2\|\nabla\partial w\|_{L^2}^2 + \|\nabla\zeta\|_{L^2}^2 \\
&\qquad + \beta^4\|\nabla\partial\zeta\|_{L^2}^2 + \|\nabla^2\zeta\|_{L^2}^2\big) + CN_{0,1}^4 \\
&\le \frac{C}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}\left(\left(1+\nu+\frac{1}{\beta^2}+\frac{\nu}{\beta^4}\right)D_2 + \|\nabla^2\zeta\|_{L^2}^2\right) + CN_{0,1}^4.
\end{aligned} \tag{4.86}$$

By $(4.72)_{k=1}$, we obtain

$$\begin{aligned}
\|\partial\phi\|_{L^2}^2 &\le C(\|\nabla\partial\zeta\|_{L^2}^2 + \|M_\infty\zeta\|_{L^2}^2 + \|f^4\|_{L^2}^2) \\
&\le C\left(\frac{1}{\beta^4}\|\nabla\zeta\|_{L^2}^2 + \|\nabla\partial\zeta\|_{L^2}^2 + \frac{1}{\beta^4}\|\nabla^2\zeta\|_{L^2}^2 + \|f^4\|_{L^2}^2\right) \\
&\le C\left(\left(\frac{\nu}{\beta^4}+\frac{\nu}{\beta^8}\right)D_2 + \frac{1}{\beta^4}\|\nabla^2\zeta\|_{L^2}^2 + \|f^4\|_{L^2}^2\right).
\end{aligned} \tag{4.87}$$

Furthermore, by (4.54), we obtain

$$\|\partial\dot{\phi}\|_{L^2}^2 \le C(\|\nabla\partial w\|_{L^2}^2 + \|\partial(\phi \mathrm{div} w)\|_{L^2}^2) \le C\left(\frac{1}{\beta^2}D_2 + \|\partial(\phi \mathrm{div} w)\|_{L^2}^2\right). \tag{4.88}$$

We set

$$E_3 = \|\partial_{x_3}\phi\|_{L^2}^2,$$
$$D_3 = \frac{1}{2}\frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\|\nabla\phi\|_{L^2}^2 + b_0\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\|\nabla\dot{\phi}\|_{L^2}^2,$$
$$R_3 = N_{0,1}^4 + \frac{1}{2}\frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\|f^4\|_{L^2} + b_0\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\|\partial(\phi \mathrm{div} w)\|_{L^2}^2.$$

It then follows from (4.86) + $\frac{1}{2}\frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}$ × (4.87) + $b_0\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}$ × (4.88) that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}E_3 + D_3 \\
&\quad \le C\left(\frac{1}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}\left(1+\nu+\frac{1}{\beta^2}+\frac{\nu}{\beta^4}\right)\right. \\
&\qquad\quad \left. +\frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\left(\frac{\nu}{\beta^4}+\frac{\nu}{\beta^8}\right) + \frac{\nu+\tilde{\nu}}{\beta^2(\beta^2+\gamma^2)}\right) D_2 \\
&\qquad\qquad +C\left(\frac{1}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})} + \frac{\beta^2+\gamma^2}{\beta^4(\nu+\tilde{\nu})}\right)\|\nabla^2\zeta\|_{L^2}^2 + CR_3.
\end{aligned} \tag{4.89}$$

From $(4.68)_{k=1,l=1}$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial\partial_{x_3}\phi\|_{L^2}^2 + \frac{1}{2}\frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\|\partial\partial_{x_3}\phi\|_{L^2}^2 + b_0\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\|\partial\partial_{x_3}\dot{\phi}\|_{L^2}^2 \\
&\le \frac{C}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}\left\{\|\partial_t\partial w\|_{L^2}^2 + \|\nabla\partial^2 q\|_{L^2}^2\right. \\
&\qquad\qquad\qquad \left. +\beta^4(\|K_\infty\partial\zeta\|_{L^2}^2 + \|M_\infty\partial\zeta\|_{L^2}^2)\right\} + CN_{1,1}^4.
\end{aligned} \tag{4.90}$$

Since

$$\|\nabla\partial^2 q\|_{L^2}^2 \le C(\nu^2\|\nabla\partial^2 w\|_{L^2}^2 + \beta^4\|\nabla\partial^2\zeta\|_{L^2}^2),$$
$$\beta^4(\|K_\infty\partial\zeta\|_{L^2}^2 + \|M_\infty\partial\zeta\|_{L^2}^2) \le C\|\nabla\partial\zeta\|_{H^1}^2,$$

we see from (4.90) that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial\partial_{x_3}\phi\|_{L^2}^2+\frac{1}{2}\frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\|\partial\partial_{x_3}\phi\|_{L^2}^2+b_0\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\|\partial\partial_{x_3}\dot{\phi}\|_{L^2}^2\\
&\le\frac{C}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}\big(\|\partial_t\partial w\|_{L^2}^2+\nu^2\|\nabla\partial^2 w\|_{L^2}^2\\
&\qquad\qquad+\|\nabla\partial\zeta\|_{L^2}^2+\beta^4\|\nabla\partial^2\zeta\|_{L^2}^2+\|\nabla^3\zeta\|_{L^2}^2\big)+CN_{1,1}^4\\
&\le\frac{C}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}\left(\left(1+\nu+\frac{1}{\beta^2}+\frac{\nu}{\beta^4}\right)D_2+\|\nabla^3\zeta\|_{L^2}^2\right)+CN_{1,1}^4.
\end{aligned}\tag{4.91}$$

By $(4.72)_{k=2}$, we obtain

$$\begin{aligned}
\|\partial^2\phi\|_{L^2}^2&\le C(\|\nabla\partial^2\zeta\|_{L^2}^2+\|M_\infty\partial\zeta\|_{L^2}^2+\|\partial f^4\|_{L^2}^2)\\
&\le C\left(\frac{1}{\beta^4}\|\nabla\partial\zeta\|_{L^2}^2+\|\nabla\partial^2\zeta\|_{L^2}^2+\frac{1}{\beta^4}\|\nabla^3\zeta\|_{L^2}^2+\|\partial f^4\|_{L^2}^2\right)\\
&\le C\left(\left(\frac{\nu}{\beta^4}+\frac{\nu}{\beta^8}\right)D_2+\frac{1}{\beta^4}\|\nabla^3\zeta\|_{L^2}^2+\|\partial f^4\|_{L^2}^2\right).
\end{aligned}\tag{4.92}$$

By (4.54), we obtain

$$\|\partial^2\dot{\phi}\|_{L^2}^2\le C(\|\nabla\partial^2 w\|_{L^2}^2+\|\partial^2(\phi\mathrm{div}w)\|_{L^2}^2)\le C\left(\frac{1}{\beta^2}D_2+\|\partial^2(\phi\mathrm{div}w)\|_{L^2}^2\right).\tag{4.93}$$

We set

$$E_4=\|\partial\partial_{x_3}\phi\|_{L^2}^2,$$

$$D_4=\frac{1}{2}\frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\|\nabla\partial\phi\|_{L^2}^2+b_0\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\|\nabla\partial\dot{\phi}\|_{L^2}^2,$$

$$R_4=N_{1,1}^4+\frac{1}{2}\frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\|\partial f^4\|_{L^2}+b_0\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\|\partial^2(\phi\mathrm{div}w)\|_{L^2}^2.$$

It then follows from $(4.90)+(4.91)+\frac{1}{2}\frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\times(4.92)+b_0\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\times(4.93)$ that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E_4+D_4\le C&\left(\frac{1}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}\left(1+\nu+\frac{1}{\beta^2}+\frac{\nu}{\beta^4}\right)\right.\\
&\left.+\frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\left(\frac{\nu}{\beta^4}+\frac{\nu}{\beta^8}\right)+\frac{\nu+\tilde{\nu}}{\beta^2(\beta^2+\gamma^2)}\right)D_2\\
&+C\left(\frac{1}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}+\frac{\beta^2+\gamma^2}{\beta^4(\nu+\tilde{\nu})}\right)\|\nabla^3\zeta\|_{L^2}^2+CR_4.
\end{aligned}\tag{4.94}$$

Let b_2 be a positive number which will be determined later. We set

$$\begin{aligned}E_5 &= b_2 E_2 + \beta^2(E_3 + E_4),\\ D_5 &= b_2 D_2 + \beta^2(D_3 + D_4),\\ R_5 &= b_2 R_2 + \beta^2(R_3 + R_4).\end{aligned}$$

We see from $b_2 \times (4.92) + \beta^2 \times (4.93) + \beta^2 \times (4.95)$ that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_5 + D_5 \le C &\left(\frac{1}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}\left(\beta^2 + \beta^2\nu + 1 + \frac{\nu}{\beta^2}\right)\right.\\ &\left. + \frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\left(\frac{\nu}{\beta^2} + \frac{\nu}{\beta^6}\right) + \frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\right) D_2\\ &+ C\left(1 + \frac{\beta^2}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})} + \frac{\beta^2+\gamma^2}{\beta^2(\nu+\tilde{\nu})}\right)\|\nabla^2\zeta\|_{H^1}^2 + CR_5.\end{aligned} \tag{4.95}$$

Taking b_2 large enough, we see from (4.95) that

$$\frac{\mathrm{d}}{\mathrm{d}t}E_5 + d_5 D_5 \le C\left(1 + \frac{\beta^2}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})} + \frac{\beta^2+\gamma^2}{\beta^2(\nu+\tilde{\nu})}\right)\|\nabla^2\zeta\|_{H^1}^2 + CR_5. \tag{4.96}$$

From $(4.68)_{k=0,l=2}$, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_{x_3}^2\phi\|_{L^2}^2 + \frac{1}{2}\frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\|\partial_{x_3}^2\phi\|_{L^2}^2 + b_0\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\|\partial_{x_3}^2\dot{\phi}\|_{L^2}^2\\ &\le \frac{C}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}\left\{\|\partial_t\nabla w\|_{L^2}^2 + \|\nabla^2\partial q\|_{L^2}^2\right.\\ &\qquad\left. + \beta^4(\|\nabla(K_\infty\zeta)\|_{L^2}^2 + \|\nabla(M_\infty\zeta)\|_{L^2}^2)\right\} + CN_{0,2}^4.\end{aligned} \tag{4.97}$$

From (4.70), we have

$$
\begin{aligned}
&\frac{1}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}(\|\nabla^2\partial q\|_{L^2}^2+\gamma^4\|\nabla\partial\phi\|_{L^2}^2)\\
&\le \frac{C}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}\Big(\|\partial_t\partial w\|_{L^2}^2+(\nu+\tilde{\nu})^2\|\partial\dot{\phi}\|_{H^1}^2+\beta^4\|\partial\phi\|_{H^1}^2\\
&\qquad\qquad +\beta^4\|K_\infty\partial\zeta\|_{L^2}^2+\beta^4\|M_\infty\zeta\|_{H^1}^2+\|\nabla\partial q\|_{L^2}^2\\
&\qquad\qquad\quad +(\nu+\tilde{\nu})^2\|\partial(\phi\mathrm{div}w)\|_{H^1}^2+\|f^2\|_{H^1}^2+\beta^4\|f^4\|_{H^1}^2\Big)\\
&\le C\left(\frac{1}{\beta^2}+\frac{1}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}\left(1+\nu+\frac{\nu}{\beta^4}+\frac{\nu^2}{\beta^2}\right)\right)D_5\\
&\quad +\frac{C}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}\|\nabla^2\zeta\|_{H^1}^2+\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\|\partial(\phi\mathrm{div}w)\|_{H^1}^2\\
&\quad\; +\frac{1}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}(\|f^2\|_{H^1}^2+\beta^4\|f^4\|_{H^1}^2).
\end{aligned}
\tag{4.98}
$$

From (4.100) and (4.98), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_{x_3}^2\phi\|_{L^2}^2+\frac{1}{2}\frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\|\partial_{x_3}^2\phi\|_{L^2}^2+b_0\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\|\partial_{x_3}^2\dot{\phi}\|_{L^2}^2\\
&\quad\le \frac{C}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}(\|\partial_t\nabla w\|_{L^2}^2+\|\nabla^2\zeta\|_{H^1}^2)\\
&\qquad\quad +C\left(\frac{1}{\beta^2}+\frac{1}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}\left(1+\nu+\frac{\nu}{\beta^4}+\frac{\nu^2}{\beta^2}\right)\right)D_5+C\tilde{N}_{0,2}^4,
\end{aligned}
\tag{4.99}
$$

where

$$
\tilde{N}_{0,2}^4=N_{0,2}^4+\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\|\partial(\phi\mathrm{div}w)\|_{H^1}^2+\frac{1}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}(\|f^2\|_{H^1}^2+\beta^4\|f^4\|_{H^1}^2).
$$

Let b_3 be a positive number to be determined later. We set

$$
\begin{aligned}
E_6&=b_3E_5+\beta^2\|\partial_{x_3}^2\phi\|_{L^2}^2,\\
D_6&=b_3D_5+\frac{1}{2}\frac{\beta^2(\beta^2+\gamma^2)}{\nu+\tilde{\nu}}\|\partial_{x_3}^2\phi\|_{L^2}^2+b_0\frac{\beta^2(\nu+\tilde{\nu})}{\beta^2+\gamma^2}\|\partial_{x_3}^2\dot{\phi}\|_{L^2}^2,\\
R_6&=b_3R_5+\beta^2\tilde{N}_{0,2}^4.
\end{aligned}
$$

We see from $b_3 \times$ (4.96) + (4.99) that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}E_6 + d_5 D_6 \\ &\leq \frac{C}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}(\|\partial_t \nabla w\|_{L^2}^2 + \|\nabla^2 \zeta\|_{H^1}^2 \\ &\qquad\qquad +\beta^4\|\partial\phi\|_{L^2}^2 + \beta^4\|\partial^2\phi\|_{L^2}^2) + CR_6. \end{aligned} \tag{4.100}$$

Taking b_3 large enough, we deduce from (4.100) that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}E_6 + d_6 D_6 \\ &\leq \frac{C\beta^2}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}\|\partial_t \nabla w\|_{L^2}^2 \\ &\quad +C\left(1+\frac{\beta^2}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}+\frac{\beta^2+\gamma^2}{\beta^2(\nu+\tilde{\nu})}\right)\|\nabla^2\zeta\|_{H^1}^2 + CR_6. \end{aligned} \tag{4.101}$$

By (4.76), we have

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^2\zeta\|_{H^1}^2 + \frac{\beta^2}{2\nu}\|\nabla^2\zeta\|_{H^1}^2 \\ &\leq \frac{1}{2\nu\beta^2}\|\nabla^2 q\|_{H^1}^2 + \sum_{k=2}^{3}|(\nabla^k(w\cdot\nabla\bar{\psi}_\infty),\nabla^k\zeta)| + \sum_{k=2}^{3}|(\nabla^k f^3,\nabla^k\zeta)| \end{aligned} \tag{4.102}$$

From (4.68), (4.70) and (4.71), we have

$$\begin{aligned} &\|\nabla^2 q\|_{H^1}^2 \\ &\leq C\Big(\|\partial_t w\|_{H^1}^2 + (\nu+\tilde{\nu})^2\|\partial\dot{\phi}\|_{H^1}^2 + \beta^4\|\partial\phi\|_{H^1}^2 + \beta^4\|K_\infty\partial\zeta\|_{L^2}^2 + \beta^4\|M_\infty\zeta\|_{H^1}^2 \\ &\quad +\gamma^4\|\partial\phi\|_{L^2}^2 + \beta^4\|\partial_{x_3}\phi\|_{L^2}^2 + \beta^4\|\partial_{x_3}^2\phi\|_{L^2}^2 + \nu^2\|\partial_{x_3}\dot{\phi}\|_{L^2}^2 + \nu^2\|\partial_{x_3}^2\dot{\phi}\|_{L^2}^2 \\ &\qquad +(\nu+\tilde{\nu})^2\|\partial(\phi\mathrm{div}w)\|_{H^1}^2 + \|f^2\|_{H^1}^2 + \beta^4\|f^4\|_{H^1}^2 + \|\nabla\partial q\|_{L^2}^2\Big) \\ &\leq C\Big(\|\partial_t w\|_{H^1}^2 + (\nu+\tilde{\nu})^2\|\partial\dot{\phi}\|_{H^1}^2 + \beta^4\|\partial\phi\|_{H^1}^2 + \|\nabla\zeta\|_{H^2}^2 \\ &\quad +\gamma^4\|\partial\phi\|_{L^2}^2 + \beta^4\|\partial_{x_3}\phi\|_{L^2}^2 + \beta^4\|\partial_{x_3}^2\phi\|_{L^2}^2 + \nu^2\|\partial_{x_3}\dot{\phi}\|_{L^2}^2 + \nu^2\|\partial_{x_3}^2\dot{\phi}\|_{L^2}^2 \\ &\qquad +(\nu+\tilde{\nu})^2\|\partial(\phi\mathrm{div}w)\|_{H^1}^2 + \|f^2\|_{H^1}^2 + \beta^4\|f^4\|_{H^1}^2 + \|\nabla\partial q\|_{L^2}^2\Big) \\ &\leq C\|\partial_t\nabla w\|_{L^2}^2 + \|\nabla^2\zeta\|_{H^1}^2) \\ &\quad +C\left(\nu+1+\frac{\nu^2}{\beta^2}+\frac{\nu}{\beta^4}+\frac{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}{\beta^2}\right)D_6 \\ &\qquad +C((\nu+\tilde{\nu})^2\|\partial(\phi\mathrm{div}w)\|_{H^1}^2 + \|f^2\|_{H^1}^2 + \beta^4\|f^4\|_{H^1}^2\Big). \end{aligned} \tag{4.103}$$

From (4.102) and (4.103), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^2\zeta\|_{H^1}^2+\frac{\beta^2}{2\nu}\|\nabla^2\zeta\|_{H^1}^2\\
&\le \frac{C}{\nu\beta^2}(\|\partial_t\nabla w\|_{L^2}^2+\|\nabla^2\zeta\|_{H^1}^2)\\
&\quad +C\left(\frac{1}{\beta^2}+\frac{1}{\nu\beta^2}+\frac{\nu}{\beta^4}+\frac{1}{\beta^6}+\frac{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}{\nu\beta^4}\right)D_6\\
&\quad\quad +\sum_{k=2}^{3}|(\nabla^k(w\cdot\nabla\bar{\psi}_\infty),\nabla^k\zeta)|+N^5,
\end{aligned}
\tag{4.104}
$$

where

$$
N^5=\frac{(\nu+\tilde{\nu})^2}{\nu\beta^2}\|\partial(\phi\mathrm{div}w)\|_{H^1}^2+\frac{1}{\nu\beta^2}\|f^2\|_{H^1}^2+\frac{\beta^2}{\nu}\|f^4\|_{H^1}^2+\sum_{k=2}^{3}|(\nabla^k f^3,\nabla^k\zeta)|.
$$

By (4.53), we have

$$
\frac{\nu}{\beta^2}\|\nabla^2 w\|_{H^1}^2\le C\left(\frac{1}{\nu\beta^2}\|\nabla^2 q\|_{H^1}^2+\frac{\beta^2}{\nu}\|\nabla^2\zeta\|_{H^1}^2\right)
\tag{4.105}
$$

We obtain by (4.104) + $b_4\times$ (4.105) with some positive constant b_4

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^2\zeta\|_{H^1}^2+\frac{\beta^2}{2\nu}\|\nabla^2\zeta\|_{H^1}^2+b_4\frac{\nu}{\beta^2}\|\nabla^2 w\|_{H^1}^2\\
&\le \frac{C}{\nu\beta^2}(\|\partial_t\nabla w\|_{L^2}^2+\|\nabla^2\zeta\|_{H^1}^2)\\
&\quad +C\left(\frac{1}{\beta^2}+\frac{1}{\nu\beta^2}+\frac{\nu}{\beta^4}+\frac{1}{\beta^6}+\frac{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}{\nu\beta^4}\right)D_6\\
&\quad\quad +\sum_{k=2}^{3}|(\nabla^k(w\cdot\nabla\bar{\psi}_\infty),\nabla^k\zeta)|+N^5.
\end{aligned}
\tag{4.106}
$$

Since

$$
\sum_{k=2}^{3}|(\nabla^k(w\cdot\nabla\bar{\psi}_\infty),\nabla^k\zeta)|\le\frac{b_4\nu}{2\beta^2}\|\nabla^2 w\|_{H^1}^2+\frac{C}{\beta^2}\|\nabla w\|_{L^2}^2+C\left(\frac{1}{\beta^2}+\frac{1}{\nu\beta^2}\right)\|\nabla^2\zeta\|_{H^1}^2,
$$

we see from (4.106) that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^2\zeta\|_{H^1}^2+\frac{\beta^2}{2\nu}\|\nabla^2\zeta\|_{H^1}^2+b_4\frac{\nu}{\beta^2}\|\nabla^2 w\|_{H^1}^2\\
&\le \frac{C}{\nu\beta^2}\|\partial_t\nabla w\|_{L^2}^2+C\left(\frac{1}{\beta^2}+\frac{1}{\nu\beta^2}\right)\|\nabla^2\zeta\|_{H^1}^2\\
&\quad +C\left(\frac{1}{\beta^2}+\frac{1}{\nu\beta^2}+\frac{\nu}{\beta^4}+\frac{1}{\beta^6}+\frac{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}{\nu\beta^4}\right)D_6+N^5.
\end{aligned}
\tag{4.107}
$$

Let b_5 be a positive number to be determined later. We set

$$E_7 = b_5 E_6 + \beta^2 \|\nabla^2 \zeta\|_{L^2}^2,$$
$$D_7 = b_5 D_6 + \frac{\beta^4}{2\nu} \|\nabla^2 \zeta\|_{H^1}^2 + b_4 \nu \|\nabla^2 w\|_{H^1}^2,$$
$$R_7 = b_5 R_6 + \beta^2 N^5.$$

By $b_5 \times$ (4.101) $+ \beta^2 \times$ (4.107), we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} E_7 + d_7 D_7 \\ &\leq C \left(\frac{1}{\nu} + \frac{\beta^2}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})} \right) \|\partial_t \nabla w\|_{L^2}^2 \\ &\quad + C \left(1 + \frac{1}{\nu} + \frac{\nu}{\beta^2} + \frac{1}{\beta^4} + \frac{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}{\nu\beta^2} \right) D_6 + C R_7. \end{aligned} \tag{4.108}$$

By taking b_5 large enough, we obtain from (4.108)

$$\frac{\mathrm{d}}{\mathrm{d}t} E_7 + d_7 D_7 \leq C \left(\frac{1}{\nu\beta^2} + \frac{\beta^2}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})} \right) \|\partial_t \nabla w\|_{L^2}^2 + C R_7. \tag{4.109}$$

By $\gamma^2 \times$ (4.74) $+ \beta^2 \times$ (4.75), we have

$$\gamma^2 \|\partial_t \phi\|_{L^2}^2 + \beta^2 \|\partial_t \nabla \zeta\|_{L^2}^2 \leq C((\beta^2+\gamma^2)\|\nabla w\|_{L^2}^2 + \gamma^2 \|f^1\|_{L^2}^2 + \beta^2 \|\nabla f^3\|_{L^2}^2). \tag{4.110}$$

From $(4.34)_{j=1,k=0}$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\gamma^2 \|\partial_t \phi\|_{L^2}^2 + \|\partial_t w\|_{L^2}^2 + \beta^2 \|\nabla \partial_t \zeta\|_{L^2}^2) + \nu \|\nabla \partial_t w\|_{L^2}^2 + \tilde{\nu} \|\mathrm{div} \partial_t w\|_{L^2}^2 \\ &\quad \leq \beta^2 \left(|(K_\infty \partial_t \zeta, \partial_t w)| + |(\nabla(\partial_t w \cdot \nabla \bar{\psi}_\infty), \nabla \partial_t \zeta)| \right) + N_{1,0}^1, \end{aligned} \tag{4.111}$$

By using the Poincaré inequality and integration by parts, we have

$$|(K_\infty \partial_t \zeta, \partial_t w)| + |(\nabla(\partial_t w \cdot \nabla \bar{\psi}_\infty), \nabla \partial_t \zeta)| \leq \frac{C}{\beta^2} \|\nabla \partial_t \zeta\|_{L^2} \|\nabla \partial_t w\|_{L^2}.$$

This, together with (4.111), implies that

$$\frac{\mathrm{d}}{\mathrm{d}t} E_8 + d_8 D_8 \leq \frac{C}{\nu} \|\partial_t \nabla \zeta\|_{L^2}^2 + C N_8, \tag{4.112}$$

where

$$E_8 = \gamma^2\|\partial_t\phi\|_{L^2}^2 + \|\partial_t w\|_{L^2}^2 + \beta^2\|\nabla\partial_t\zeta\|_{L^2}^2,$$
$$D_8 = \nu\|\nabla\partial_t w\|_{L^2}^2 + \tilde{\nu}\|\mathrm{div}\partial_t w\|_{L^2}^2,$$
$$R_8 = N_{1,0}^1.$$

Let b_6 be a positive number to be determined later. We set

$$E_9 = b_6 E_7 + E_8,$$
$$D_9 = b_6 D_7 + E_8 + (\gamma^2\|\partial_t\phi\|_{L^2}^2 + \beta^2\|\partial_t\nabla\zeta\|_{L^2}^2),$$
$$R_9 = b_6 R_7 + R_8 + (\gamma^2\|f^1\|_{L^2}^2 + \beta^2\|\nabla f^3\|_{L^2}^2).$$

By $b_6 \times (4.109) + \frac{\beta^2}{\nu} \times (4.110) + (4.112)$, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}E_9 + d_9 D_9\\ &\le C\left(\frac{1}{\nu\beta^2} + \frac{\beta^2}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}\right)\|\partial_t\nabla w\|_{L^2}^2 + \frac{C}{\nu}\|\partial_t\nabla\zeta\|_{L^2}^2 + CR_9.\end{aligned} \tag{4.113}$$

We take b_6 suitably large to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}E_9 + d_9 D_9 \le CR_9. \tag{4.114}$$

Since $D_9(t) \ge c_1 E_9(t)$ for some constant $c_1 > 0$, it follows from (4.114) that

$$\frac{\mathrm{d}}{\mathrm{d}t}E_9 + c_1 E_9 + d_9 D_9 \le CR_9. \tag{4.115}$$

This gives

$$E_9(t) + \int_0^t e^{-c_1(t-s)}D_9(s)\mathrm{d}s \le C\left(e^{-c_1 t}E_9(0) + \int_0^t e^{-c_1(t-s)}R_9(s)\mathrm{d}s\right). \tag{4.116}$$

From (4.73), we have

$$\begin{aligned}&\frac{\nu^2}{(\nu+\tilde{\nu})^2}\|\partial_{x_n}^2 w\|_{L^2}^2\\ &\quad\le C\left(\frac{1}{(\nu+\tilde{\nu})^2}\|\partial_t w\|_{L^2}^2 + \|\nabla\partial w\|_{L^2}^2 + \frac{\gamma^4}{(\nu+\tilde{\nu})^2}\|\nabla\phi\|_{L^2}^2\right.\\ &\qquad\left.+\frac{\beta^4}{(\nu+\tilde{\nu})^2}\|\nabla^2\zeta\|_{L^2}^2 + \frac{\beta^4}{(\nu+\tilde{\nu})^2}\|K_\infty\zeta\|_{L^2}^2 + \frac{1}{(\nu+\tilde{\nu})^2}\|f^2\|_{L^2}^2\right)\\ &\quad\le C\left(\frac{1}{(\nu+\tilde{\nu})^2}\|\partial_t w\|_{L^2}^2 + \|\nabla\partial w\|_{L^2}^2 + \frac{\gamma^4}{(\nu+\tilde{\nu})^2}\|\nabla\phi\|_{L^2}^2\right.\\ &\qquad\left.+\frac{\beta^4}{(\nu+\tilde{\nu})^2}\|\nabla^2\zeta\|_{L^2}^2 + \frac{1}{(\nu+\tilde{\nu})^2}\|\nabla\zeta\|_{H^1}^2 + \frac{1}{(\nu+\tilde{\nu})^2}\|f^2\|_{L^2}^2\right).\end{aligned} \tag{4.117}$$

Let b_7 a positive number to be determined later. We set

$$
\begin{aligned}
E &= b_7 E_9 + \frac{\nu^2}{(\nu+\tilde{\nu})^2}\|\partial_{x_3}^2 w\|_{L^2}^2,\\
D &= b_7 D_9,\\
\mathcal{R} &= b_7 R_9.
\end{aligned}
$$

By $b_7 \times$ (4.116) + (4.117) and taking b_7 large, we obtain

$$
\begin{aligned}
&E(t) + \int_0^t e^{-c_1(t-s)} D(s)\mathrm{d}s \\
&\leq C\left(e^{-c_1 t}E(0) + \frac{1}{(\nu+\tilde{\nu})^2}\|f^2(t)\|_{L^2}^2 + \int_0^t e^{-c_1(t-s)}\mathcal{R}(s)\mathrm{d}s\right).
\end{aligned} \tag{4.118}
$$

By using Lemmata 2.1 and 2.4 below, we have

$$
\frac{1}{(\nu+\tilde{\nu})^2}\|f^2\|_{L^2}^2 \leq C\left(\frac{1}{\nu^2} + \frac{1}{\beta^4} + E(t)\right)E(t),
$$

provided that $\|v_0\|_{H^5(0,1)}^2 \leq C\frac{\nu^2}{\beta^4}$. Therefore, there exists a positive constant δ such that if $E(t) \leq \delta$, then (4.118) yields

$$
E(t) + \int_0^t e^{-c_1(t-s)} D(s)\mathrm{d}s \leq C\left(e^{-c_1 t}E(0) + \int_0^t e^{-c_1(t-s)}\mathcal{R}(s)\mathrm{d}s\right). \tag{4.119}
$$

It remains to estimate $\mathcal{R}(t)$.

We see from (4.18)–(4.21) that if there exist a positive constant δ independent of ν, ν', γ, and β such that

$$
\|\bar{v}_0\|_{H^5(0,1)} \leq \delta\frac{\nu}{\beta^2},
$$

then the following estimates hold uniformly for $t \geq 0$;

$$
\begin{aligned}
\|\bar{\psi}_{exp}(t)\|_{H^5(0,1)} \leq \frac{C}{\beta^2},\ \|\bar{F}\|_{H^4(0,1)} \leq \frac{C}{\beta^2} + 1,\\
\|\bar{v}(t)\|_{H^5(0,1)} \leq \frac{C}{\nu},\ \|\partial_t \bar{v}(t)\|_{H^3(0,1)} \leq C.
\end{aligned}
$$

Using these estimates, together with Lemmata 2.1 and 2.4, one can obtain the following estimate for $\mathcal{R}(t)$.

Proposition 4.14. *There exists a positive constant δ with $\delta < 1$ such that if $\|\bar{v}_0\|^2_{H^5(0,1)} + E(t) \le \delta$, then we have the following estimate:*

$$\mathcal{R}(t) \le C\left(\frac{1}{\nu} + \frac{1}{\beta^2} + \frac{\sqrt{\nu}}{\beta} + \frac{\gamma}{\beta}\right) D(t) + C(E(t)^{\frac{1}{2}} + E(t))D(t) \tag{4.120}$$

uniformly for $t \in [0,T]$ with some positive constant C independent of T .

Proposition 4.6 now follows from (4.119) and Proposition 4.14. This completes the proof. ■

Appendix 4.A Proof of Proposition 4.1.

In this appendix, we give a proof of Proposition 4.1.
Proof of Proposition 4.1. We set $\bar{\rho} = 1$. It suffices to prove the existence of a solution $(\bar{\psi}^1, \bar{v}^1)$ of (4.15)–(4.16) with the properties in Proposition 4.1. For simplicity we assume that $\frac{2\beta}{\pi\nu}$ is not integer.

We set

$$A = \begin{pmatrix} 0 & -1 \\ -\beta^2\partial_{x_3}^2 & -\nu\partial_{x_3}^2 \end{pmatrix}.$$

It follows that the problem (4.15)–(4.16) is written as :

$$\partial_t \begin{pmatrix} \bar{\psi}^1 \\ \bar{v}^1 \end{pmatrix} + A \begin{pmatrix} \bar{\psi}^1 \\ \bar{v}^1 \end{pmatrix} = \begin{pmatrix} 0 \\ g^1 \end{pmatrix}, \quad \begin{pmatrix} \bar{\psi}^1 \\ \bar{v}^1 \end{pmatrix}\Bigg|_{t=0} = \begin{pmatrix} 0 \\ \bar{v}^1_0 \end{pmatrix}. \tag{4.A.1}$$

To solve (4.A.1), we consider the Fourier-sine expansions;

$$\bar{\psi}^1 = \sum_{k=1}^{\infty} \hat{\psi}_k(t)\sin(k\pi x_3), \;\; g^1 = \sum_{k=1}^{\infty} \hat{g}^1_k(t)\sin(k\pi x_3),$$
$$\bar{v}^1 = \sum_{k=1}^{\infty} \hat{v}_k(t)\sin(k\pi x_3), \;\; \bar{v}^1_0 = \sum_{k=1}^{\infty} \hat{v}_{0k}\sin(k\pi x_3).$$

We see from (4.A.1) that $(\hat{v}_k, \hat{\psi}_k)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} \hat{\psi}_k \\ \hat{v}_k \end{pmatrix} + \hat{A} \begin{pmatrix} \hat{\psi}_k \\ \hat{v}_k \end{pmatrix} = \begin{pmatrix} 0 \\ \hat{g}^1_k \end{pmatrix}, \quad \begin{pmatrix} \hat{\psi}_k \\ \hat{v}_k \end{pmatrix}\Bigg|_{t=0} = \begin{pmatrix} 0 \\ \hat{v}_{0k} \end{pmatrix}. \tag{4.A.2}$$

where

$$\hat{A} = \begin{pmatrix} 0 & -1 \\ \beta^2 k^2\pi^2 & \nu k^2\pi^2 \end{pmatrix}.$$

The solution of (4.A.2) is given by

$$\begin{pmatrix} \hat{\psi}_k(t) \\ \hat{v}_k(t) \end{pmatrix} = e^{-t\hat{A}} \begin{pmatrix} 0 \\ \hat{v}_{0k} \end{pmatrix} + \int_0^t e^{-(t-s)\hat{A}} \begin{pmatrix} 0 \\ \hat{g}_k^1(s) \end{pmatrix} \mathrm{d}s. \tag{4.A.3}$$

Since

$$\det(-\hat{A} - \lambda I) = \det \begin{pmatrix} -\lambda & 1 \\ -\beta^2 k^2 \pi^2 & -\lambda - \nu k^2 \pi^2 \end{pmatrix} = \lambda^2 + \nu k^2 \pi^2 \lambda + \beta^2 k^2 \pi^2,$$

we see that the characteristic roots $\lambda_\pm$ of $-\hat{A}$ are given by

$$\lambda_\pm = \frac{-\nu k^2 \pi^2 \pm \sqrt{\nu^2 k^4 \pi^4 - 4\beta^2 k^2 \pi^2}}{2}.$$

It is not difficult to see that

$$\lambda_+ = -\frac{\beta^2}{\nu^2} + O\left(\frac{1}{k^2}\right) \quad \text{for } k \gg 1,$$

$$\lambda_- = -\nu k^2 \pi^2 + O(1) \quad \text{for } k \gg 1,$$

$$\lambda_+ \lambda_- = \beta^2 k^2 \pi^2, B_k = \lambda_+ - \lambda_- = k\pi \sqrt{\nu^2 k^2 \pi^2 - 4\beta^2}.$$

The eigenprojections for $\lambda_\pm$ of $-\hat{A}$ are written as

$$P_+ = \frac{-\hat{A} - \lambda_-}{\lambda_+ - \lambda_-} = \frac{1}{\lambda_+ - \lambda_-} \begin{pmatrix} -\lambda_- & 1 \\ -\lambda_+ \lambda_- & \lambda_+ \end{pmatrix}.$$

$$P_- = \frac{-\hat{A} - \lambda_+}{\lambda_- - \lambda_+} = \frac{1}{\lambda_+ - \lambda_-} \begin{pmatrix} \lambda_+ & -1 \\ \lambda_+ \lambda_- & -\lambda_- \end{pmatrix}.$$

The solution semigroup $e^{-t\hat{A}}$ is then expressed as

$$\begin{aligned} e^{-t\hat{A}} &= e^{t\lambda_+} P_+ + e^{t\lambda_-} P_- \\ &= \frac{1}{\lambda_+ - \lambda_-} \left(e^{t\lambda_+} \begin{pmatrix} -\lambda_- & 1 \\ -\lambda_+ \lambda_- & \lambda_+ \end{pmatrix} + e^{t\lambda_-} \begin{pmatrix} \lambda_+ & -1 \\ \lambda_+ \lambda_- & -\lambda_- \end{pmatrix} \right) \end{aligned}$$

It then follows from (4.A.3) that $\hat{\psi}_k(t)$ and $\hat{v}_k(t)$ are written as

$$\begin{aligned} \hat{\psi}_k(t) &= \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} \hat{v}_{0k} + \int_0^t \frac{e^{\lambda_+ (t-s)} - e^{\lambda_- (t-s)}}{\lambda_+ - \lambda_-} \hat{g}_k^1(s) \mathrm{d}s, \\ \hat{v}_k(t) &= \frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-} \hat{v}_{0k} + \int_0^t \frac{\lambda_+ e^{\lambda_+ (t-s)} - \lambda_- e^{\lambda_- (t-s)}}{\lambda_+ - \lambda_-} \hat{g}_k^1(s) \mathrm{d}s. \end{aligned}$$

By integration by parts, we see that $\hat{\psi}_k(t)$ and $\hat{v}_k(t)$ are rewritten in the following forms :

$$\begin{aligned}\hat{\psi}_k(t) = &\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}\hat{v}_{0k} + \frac{\lambda_- e^{\lambda_+ t} - \lambda_+ e^{\lambda_- t}}{\lambda_+\lambda_-(\lambda_+ - \lambda_-)}\hat{g}_k^1(0) \\ &+\frac{1}{\lambda_+\lambda_-}\hat{g}_k^1(t) + \int_0^t \frac{\lambda_- e^{\lambda_+(t-s)} - \lambda_+ e^{\lambda_-(t-s)}}{\lambda_+\lambda_-(\lambda_+ - \lambda_-)}(\hat{g}_k^1)'(s)\mathrm{d}s,\end{aligned} \tag{4.A.4}$$

$$\begin{aligned}\hat{v}_k(t) = &\frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-}\hat{v}_{0k} + \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}\hat{g}_k^1(0) \\ &+ \int_0^t \frac{e^{\lambda_+(t-s)} - e^{\lambda_-(t-s)}}{\lambda_+ - \lambda_-}(\hat{g}_k^1)'(s)\mathrm{d}s.\end{aligned} \tag{4.A.5}$$

We next introduce the Fourier-sine expansion of g_∞^1

$$g_\infty^1 = \sum_{k=1}^\infty \hat{g}_{k,\infty}^1 \sin(k\pi x_3).$$

We set

$$\hat{\psi}_k^\infty = \frac{1}{\lambda_+\lambda_-}\hat{g}_{k,\infty}^1, \ \bar{\psi}_\infty^1 = \sum_{k=1}^\infty \hat{\psi}_k^\infty \sin(k\pi x_3).$$

It then follows that

$$-\beta^2\partial_{x_3}^2\bar{\psi}_\infty^1 = g_\infty^1, \ \|\bar{\psi}_\infty^1\|_{H^5(0,1)} \le \frac{C}{\beta^2}\|g_\infty^1\|_{H^3(0,1)}.$$

Let us estimate $\hat{\psi}_k$ and $\hat{v}_k$ by using (4.A.4) and (4.A.5).

(i) If $k^2\pi^2 \le \frac{2\beta^2}{\nu^2}$, then

$$4\beta^2k^2\pi^2 - \nu^2k^4\pi^4 \ge \begin{cases} 2\beta^2k^2\pi^2 \\ \nu^2k^4\pi^4, \end{cases} \quad |\lambda_\pm| \le \beta k\pi + \nu k^2\pi^2, \ |e^{\lambda_\pm t}| \le e^{-\frac{\nu k^2\pi^2}{2}t} \le e^{-\frac{\nu}{2}t}.$$

It follows that

$$\left|\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}\right| \le \frac{C}{\beta k\pi + \nu k^2\pi^2}e^{-c\nu t},$$

$$\left|\frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-}\right| \le Ce^{-c\nu t}, \ \left|\frac{\lambda_- e^{\lambda_+ t} - \lambda_+ e^{\lambda_- t}}{\lambda_+ - \lambda_-}\right| \le Ce^{-c\nu t}.$$

Based on these estimates, we see from (4.A.4) and (4.A.5) that

$$
\begin{aligned}
|\hat{v}_k(t)|^2 \leq & C \left(\left| \frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-} \right|^2 |\hat{v}_{0k}|^2 + \left| \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} \right|^2 |\hat{g}_k^1(0)|^2 \right. \\
& \left. + \int_0^t \left| \frac{e^{\lambda_+(t-s)} - e^{\lambda_-(t-s)}}{\lambda_+ - \lambda_-} \right| |(\hat{g}_k^1)'(s)| \mathrm{d}s \right)^2 \\
\leq & C e^{-c\nu t} \left(|\hat{v}_{0k}|^2 + \frac{1}{\beta k^2 \pi^2 + \nu^2 k^4 \pi^4} |\hat{g}_k^1(0)|^2 \right) \\
& + \frac{C}{\beta^2 k^2 \pi^2 + \nu^2 k^4 \pi^4} \left(\int_0^t e^{-c\nu(t-s)} |(\hat{g}_k^1)'(s)| \mathrm{d}s \right)^2,
\end{aligned}
$$

and

$$
\begin{aligned}
& |\hat{\psi}_k(t) - \hat{\psi}_{\infty,k}|^2 \\
& \leq C \left(\left| \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} \right|^2 |\hat{v}_{0k}|^2 + \left| \frac{\lambda_- e^{\lambda_+ t} - \lambda_+ e^{\lambda_- t}}{\lambda_+ \lambda_- (\lambda_+ - \lambda_-)} \right|^2 |\hat{g}_k^1(0)|^2 \right) \\
& \quad + \left| \frac{1}{\lambda_+ \lambda_-} \right|^2 |\hat{g}_k^1(t) - \hat{g}_{k,\infty}^1|^2 + \left(\int_0^t \left| \frac{\lambda_- e^{\lambda_+(t-s)} - \lambda_+ e^{\lambda_-(t-s)}}{\lambda_+ \lambda_- (\lambda_+ - \lambda_-)} \right| |(\hat{g}_k^1)'(s)| \mathrm{d}s \right)^2 \\
& \leq C e^{-c\nu t} \left(\frac{1}{\beta^2 k^2 \pi^2 + \nu^2 k^4 \pi^4} |\hat{v}_{0k}|^2 + \frac{1}{\beta^4 k^4 \pi^4} |\hat{g}_k^1(0)|^2 \right) \\
& \quad + \frac{1}{\beta^4 k^4 \pi^4} |\hat{g}_k^1(t) - \hat{g}_{k,\infty}^1|^2 + \frac{1}{\beta^4 k^4 \pi^4} \left(\int_0^t e^{-c\nu(t-s)} |(\hat{g}_k^1)'(s)| \mathrm{d}s \right)^2.
\end{aligned}
$$

(ii) If $\frac{2\beta^2}{\nu^2} \leq k^2\pi^2 < \frac{4\beta^2}{\nu^2}$ or $\frac{4\beta^2}{\nu^2} < k^2\pi^2 \leq \frac{8\beta^2}{\nu^2}$, then

$$
|e^{\lambda_\pm t}| \leq e^{-c\frac{\beta^2}{\nu} t}, \ C_1 \frac{\nu}{\beta^2} \leq \frac{1}{\beta k \pi + \nu k^2 \pi^2} \leq C_2 \frac{\nu}{\beta^2}, \ |\lambda_\pm| \leq C \frac{\beta^2}{\nu}.
$$

It follows that

$$
\begin{aligned}
\left| \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} \right| &= \left| \frac{1}{\lambda_+ - \lambda_-} \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta} \left(e^{\lambda_- t + \theta(\lambda_+ - \lambda_-)t} \right) \mathrm{d}\theta \right| \\
&= t \left| \int_0^1 e^{\lambda_- t + \theta(\lambda_+ - \lambda_-)t} \mathrm{d}\theta \right| \\
&\leq t \int_0^1 e^{(1-\theta)\mathrm{Re}\lambda_- t} e^{\theta \mathrm{Re}\lambda_+ t} \mathrm{d}\theta \\
&\leq t \left(\int_0^{\frac{1}{2}} e^{\frac{1}{2}\mathrm{Re}\lambda_- t} \mathrm{d}\theta + \int_{\frac{1}{2}}^1 e^{\frac{1}{2}\mathrm{Re}\lambda_+ t} \mathrm{d}\theta \right) \\
&\leq t \left(e^{\frac{1}{2}\mathrm{Re}\lambda_- t} + e^{\frac{1}{2}\mathrm{Re}\lambda_+ t} \right) \\
&\leq C \frac{\nu}{\beta^2} e^{-c\frac{\beta^2}{\nu} t} \leq C \frac{1}{\beta k \pi + \nu k^2 \pi^2} e^{-c\frac{\beta^2}{\nu} t},
\end{aligned}
$$

and

$$\begin{aligned}
&\left|\frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-}\right| \\
&= \left|\frac{1}{\lambda_+ - \lambda_-}\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta}\left((\lambda_- + \theta(\lambda_+ - \lambda_-))e^{\lambda_- t + \theta(\lambda_+ - \lambda_-)t}\right)\mathrm{d}\theta\right| \\
&= \left|\int_0^1 e^{((1-\theta)\lambda_- + \theta\lambda_+)t}\mathrm{d}\theta + \int_0^1 ((1-\theta)\lambda_- + \theta\lambda_+)te^{((1-\theta)\lambda_- + \theta\lambda_+)t}\mathrm{d}\theta\right| \\
&\le 2e^{-c\frac{\beta^2}{\nu}t} + \frac{|\lambda_+| + |\lambda_-|}{2}te^{-c\frac{\beta^2}{\nu}t} \le Ce^{-c\frac{\beta^2}{\nu}t}.
\end{aligned}$$

By using these estimates, we have

$$\begin{aligned}
|\hat{v}_k(t)|^2 \le Ce^{-c\frac{\beta^2}{\nu}t}&\left(|\hat{v}_{0k}|^2 + \frac{1}{\beta^2 k^2\pi^2 + \nu^2 k^4\pi^4}|\hat{g}_k^1(0)|^2\right) \\
&+ \frac{1}{\beta^2 k^2\pi^2 + \nu^2 k^4\pi^4}\left(\int_0^t e^{-c\frac{\beta^2}{\nu}(t-s)}|(\hat{g}_k^1)'(s)|\mathrm{d}s\right)^2,
\end{aligned}$$

$$\begin{aligned}
&|\hat{\psi}_k(t) - \hat{\psi}_{\infty,k}| \\
&\le Ce^{-c\frac{\beta^2}{\nu}t}\left(\frac{1}{\beta^2 k^2\pi^2 + \nu^2 k^4\pi^4}|\hat{v}_{0k}|^2 + \frac{1}{\beta^4 k^4\pi^4}|\hat{g}_k^1(0)|^2\right) \\
&\quad + \frac{1}{\beta^4 k^4\pi^4}e^{c\frac{\beta^2}{\nu}t}|\hat{g}_k^1(t) - \hat{g}_{k,\infty}^1|^2 + \frac{C}{\beta^4 k^4\pi^4}\left(\int_0^t e^{-c\frac{\beta^2}{\nu}(t-s)}|(\hat{g}_k^1)'(s)|\mathrm{d}s\right)^2.
\end{aligned}$$

(iii) If $k^2\pi^2 \ge \frac{8\beta^2}{\nu^2}$, then

$$\nu^2 k^4\pi^4 - 4\beta^2 k^2\pi^2 \ge \begin{cases} 4\beta^2 k^2\pi^2 \\ \frac{1}{2}\nu^2 k^4\pi^4, \end{cases} \quad |\lambda_\pm| \le \beta k\pi + \nu k^2\pi^2,\ |e^{\lambda_+ t}| \le e^{-\frac{\beta^2}{\nu}t},\ |e^{\lambda_- t}| \le e^{-\frac{\beta^2}{\nu}t}.$$

By combining these estimates, we obtain

$$\left|\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}\right| \le C\frac{1}{\beta k\pi + \nu k^2\pi^2}e^{-c\frac{\beta^2}{\nu}t}$$

$$\left|\frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-}\right| \le Ce^{-c\frac{\beta^2}{\nu}t}, \quad \left|\frac{\lambda_- e^{\lambda_+ t} - \lambda_+ e^{\lambda_- t}}{\lambda_+ - \lambda_-}\right| \le Ce^{-c\frac{\beta^2}{\nu}t},$$

and hence,

$$|\hat{v}_k(t)|^2 \le Ce^{-c\frac{\beta}{\nu}t}\left(|\hat{v}_{0k}|^2 + \frac{1}{\beta^2 k^2\pi^2 + \nu^2 k^4\pi^4}|\hat{g}_k^1(0)|^2\right)$$

$$+\frac{C}{\beta k\pi+\nu k^2\pi^2}\left(\int_0^t e^{-c\frac{\beta^2}{\nu}(t-s)}|(\hat{g}_k^1)'(s)|\mathrm{d}s\right)^2,$$

and

$$\begin{aligned}|\hat{\psi}_k(t)-\hat{\psi}_{\infty,k}| \leq & Ce^{-c\frac{\beta^2}{\nu}t}\left(\frac{1}{\beta^2k^2\pi^2+\nu^2k^4\pi^4}|\hat{v}_{0k}|^2+\frac{1}{\beta^4k^4\pi^4}|\hat{g}_k^1(0)|^2\right) \\ & +\frac{1}{\beta^4k^4\pi^4}|\hat{g}_k^1(t)-\hat{g}_{k,\infty}^1|^2+\frac{1}{\beta^4k^4\pi^4}\left(\int_0^t e^{-c\frac{\beta^2}{\nu}(t-s)}|(\hat{g}_k^1)'(s)|\mathrm{d}s\right)^2.\end{aligned}$$

Since

$$\begin{aligned}\left(\int_0^t e^{-c\kappa(t-s)}|(\hat{g}_k^1)'(s)|\mathrm{d}s\right)^2 &\leq \left(\int_0^t e^{-c\kappa(t-s)}\mathrm{d}s\right)\left(\int_0^t e^{-c\kappa(t-s)}|(\hat{g}_k^1)'(s)|^2\mathrm{d}s\right) \\ &\leq \frac{1}{c\kappa}e^{-c\kappa t}\int_0^\infty e^{c\kappa s}|(\hat{g}_k^1)'(s)|^2\mathrm{d}s\end{aligned}$$

and

$$\begin{aligned}|\hat{g}_k^1(t)-\hat{g}_{k,\infty}^1|^2=\left|\int_t^\infty(\hat{g}_k^1)'(s)\mathrm{d}s\right|^2 &\leq \left(\int_t^\infty|(\hat{g}_k^1)'(s)|\mathrm{d}s\right)^2 \\ &=\left(\int_t^\infty e^{-\frac{1}{2}c\kappa s}e^{\frac{1}{2}c\kappa s}|(\hat{g}_k^1)'(s)|\mathrm{d}s\right)^2 \\ &\leq \frac{1}{c\kappa}e^{-c\kappa t}\int_0^\infty e^{c\kappa s}|(\hat{g}_k^1)'(s)|^2\mathrm{d}s,\end{aligned}$$

we obtain the following estimates

$$\begin{aligned}|\hat{v}_k(t)|^2 \leq Ce^{-c\kappa t}\Bigg(&|\hat{v}_{0k}|^2+\frac{1}{\beta^2k^2\pi^2+\nu^2k^4\pi^4}|\hat{g}_k^1(0)|^2 \\ &+\frac{1}{c\kappa}\frac{1}{\beta^2k^2\pi^2+\nu^2k^4\pi^4}\int_0^\infty e^{c\kappa s}|(\hat{g}_k^1)'(s)|^2\mathrm{d}s\Bigg), \\ |\hat{\psi}_k(t)-\hat{\psi}_{\infty,k}|^2 \leq Ce^{-c\kappa t}\Bigg(&\frac{1}{\beta^2k^2\pi^2+\nu^2k^4\pi^4}|\hat{v}_{0k}|^2+\frac{1}{\beta^4k^4\pi^4}|\hat{g}_k^1(0)|^2 \\ &+\frac{1}{c\kappa}\frac{1}{\beta^4k^4\pi^4}\int_0^\infty e^{c\kappa s}|(\hat{g}_k^1)'(s)|^2\mathrm{d}s\Bigg).\end{aligned}$$

As a result, $\bar{v}^1(t)$ and $\bar{\psi}^1(t)-\bar{\psi}_\infty^1$ satisfy

$\|\bar{v}^1(t)\|_{H^5(0,1)}^2$

$$
\le Ce^{-c\kappa t}\left(\|\bar{v}^1(0)\|^2_{H^5(0,1)} + \frac{1}{\nu^2}\|g^1(0)\|^2_{H^3(0,1)} + \frac{1}{\kappa\nu^2}\|e^{c_0\kappa t}\partial_t g^1\|^2_{L^2(0,\infty;H^3(0,1))} \right),
$$

$$
\|\bar{\psi}^1(t) - \bar{\psi}^1_\infty\|^2_{H^5(0,1)}
$$

$$
\le Ce^{-c\kappa t}\left(\frac{1}{\nu^2}\|\bar{v}(0)\|^2_{H^5(0,1)} + \frac{1}{\beta^4}\|g^1(0)\|^2_{H^3(0,1)} + \frac{1}{\kappa\beta^4}\|e^{c_0\kappa t}\partial_t g^1\|^2_{L^2(0,\infty;H^3(0,1))} \right).
$$

The estimate of $\|\partial_t \bar{v}(t)\|_{H^3(0,1)}$ can be proved by using the formula

$$
\hat{v}_k'(t) = \frac{\lambda_+^2 e^{\lambda_+ t} - \lambda_-^2 e^{\lambda_- t}}{\lambda_+ - \lambda_-}\hat{v}_{0k} + \frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-}\hat{g}_k(0) + \int_0^t \frac{\lambda_+ e^{\lambda_+(t-s)} - \lambda_- e^{\lambda_-(t-s)}}{\lambda_+ - \lambda_-}\hat{g}_k^1(s)\mathrm{d}s,
$$

with the aid of the estimate

$$
\left| \frac{\lambda_+^2 e^{\lambda_+ t} - \lambda_-^2 e^{\lambda_- t}}{\lambda_+ - \lambda_-} \right| \le C\frac{\beta^2}{\nu}k^2\pi^2 e^{-c\kappa t}.
$$

This completes the proof of Proposition 4.1. ■

Chapter 5

Stability of time-periodic parallel flow of compressible viscoelastic system in an infinite layer

In this chapter, we investigate the stability of time-periodic flow of the compressible viscoelastic system

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho v) = 0, \\ \rho(\partial_t v + v \cdot \nabla v) - \nu \Delta v - (\nu + \nu')\nabla \mathrm{div} v + \nabla p(\rho) = \beta^2 \mathrm{div}(\rho F^\top F) + \rho g, \\ \partial_t F + v \cdot \nabla F = (\nabla v) F \end{cases} \tag{5.1}$$

in an infinite layer Ω:

$$\Omega = \{x = (x', x_3);\ x' = (x_1, x_2) \in \mathbb{R}^2,\ 0 < x_3 < 1\}.$$

Here $\rho = \rho(x,t)$, $v = {}^\top(v^1(x,t), v^2(x,t), v^3(x,t))$, and $F = (F^{jk}(x,t))_{1 \leq j,k \leq 3}$ are the unknown density, the velocity field, and the deformation tensor, respectivity, at the time $t \geq 0$ and $x \in \Omega$; $p = p(\rho)$ is the pressure; ν and ν' are the viscosity coefficients satisfying

$$\nu > 0,\ 2\nu + 3\nu' \geq 0;$$

We also assume that

$$\frac{\nu'}{\nu} \leq \nu_1$$

for some positive constant $\nu_1 > 0$. $\beta > 0$ is the strength of the elasticity. The pressure $p(\rho)$ is assumed to be a smooth function of ρ satisfying $p'(1) > 0$, and we denote $\gamma \equiv \sqrt{p'(1)}$. g is a given external force which has the form

$$g = g^1(x_3, t)e_1, \ e_1 = {}^\top(1, 0, 0), \ g^1(0, t) = g^1(1, t) = 0, \tag{5.2}$$

with g^1 being T-periodic function of time t, where $T > 0$.

The system is considered under the boundary condition

$$v|_{x_3=0,1} = 0, \tag{5.3}$$

and the initial condition

$$(\rho, v, F)|_{t=0} = (\rho_0, v_0, F_0). \tag{5.4}$$

We also assume that (ρ_0, F_0) satisfies the following condition

$$\mathrm{div}(\rho_0{}^\top F_0) = 0, \ \rho_0 \mathrm{det} F_0 = 1.. \tag{5.5}$$

As mentioned in the beginning of the introduction, the conditions (**??**) are invariant for $t \geq 0$:

$$\mathrm{div}(\rho^\top F) = 0, \rho \mathrm{det} F = 1.$$

If g is assumed to have the form (5.2), problem (5.1)–(5.3) has a T-periodic solution $\overline{u} = (\bar{\rho}, \bar{v}, \bar{F})$ satisfying the following properties:

$$\bar{\rho} = 1, \ \bar{v} = \bar{v}^1(x_3, t)e_1, \ \bar{F} = \bar{F}(x_3, t) = \left(\nabla(x - \bar{\psi}^1(x_3, t)e_1)\right)^{-1}.$$

Here $\bar{\psi}^1$ is a function satisfying

$$\bar{\psi}^1(x_3, t + T) = \bar{\psi}^1(x_3, t), \ \partial_t \bar{\psi}^1 = \bar{v}^1.$$

The aim of this chapter is to study the stability of time-periodic parallel flow $\bar{u}$.

Under a suitable condition on g, there exists a time-periodic parallel flow $(\bar{\rho}, \bar{v}, \bar{F})$ of (5.1) satisfying the following properties:

$$\sup_{t \in [0,T]} \|\bar{F}(t) - I\|_{H^5(0,1)} = O\left(\frac{1}{\beta^2}\right), \ \sup_{t \in [0,T]} \|\bar{v}^1(t)\|_{H^4(0,1)} = O\left(\frac{1}{\nu}\right).$$

Here I is the 3×3 identity matrix. We define the periodic cell by D:

$$D = \{x = (x', x_3); x' = (x_1, x_2) \in \Pi_{j=1}^2 \mathbb{T}_{\frac{2\pi}{\alpha_j}}, \ 0 < x_3 < 1\}.$$

Here $\alpha_j > 0$, $j = 1, 2$.

The main result of this chapter states that if $\nu \gg 1$, $\gamma \gg 1$, $\beta \gg 1$, then (5.1)–(5.5) has a unique global solution (ρ, v, F) such that $(\rho, v, F) \in C([0,\infty), H^2(D))$ and $\|(\rho(t), v(t), F(t)) - (1, \bar{v}(t), \bar{F}(t))\|_{H^2} \to 0$ exponentially as $t \to \infty$ if the initial perturbation $(\rho_0 - 1, v_0 - \bar{v}_0, F_0 - \bar{F}_0)$ is sufficiently small in $H^2(D)$.

The proof of the main result of this chapter is given by a similar argument of Qian [29] which is based on the Matsumura-Nishida energy method [26]. To establish the a priori estimate, we consider the following problem for the perturbation $u(t) = (\phi(t), w(t), G(t)) = \big(\rho(t) - 1, v(t) - \bar{v}(t), F(t) - \bar{F}(t)\big)$:

$$\begin{cases} \partial_t \phi + \bar{v}^1 \partial_{x_1} \phi + \mathrm{div} w = f_1, \\ \partial_t w + \bar{v}^1 \partial_{x_1} w - \nu \Delta w - \tilde{\nu} \nabla \mathrm{div} w + \gamma^2 \nabla \phi - \beta^2 \mathrm{div} G \\ +(w^3 \partial_{x_3} \bar{v}^1) e_1 + \nu(\phi \partial_{x_3}^2 \bar{v}^1) e_1 - \beta^2 \mathrm{div}(G^\top \bar{E}) - \beta^2 (G^{33} \partial_{x_3}^2 \bar{\psi}^1) e_1 = f_2, \\ \partial_t G + \bar{v}^1 \partial_{x_1} G - \nabla w - (\nabla w) \bar{E} + w^3 \partial_{x_3}^2 \bar{E} - (\nabla \bar{v}) G = f_3, \\ \nabla \phi = -\mathrm{div}^\top G + {}^\top \bar{E} \mathrm{div}^\top G + f_4, \\ w|_{x_3 = 0,1} = 0, \quad (\phi, w, G)|_{t=0} = (\phi_0, w_0, G_0). \end{cases} \tag{5.6}$$

Here $\tilde{\nu} = \nu + \nu'$, $\bar{\psi} = \bar{\psi}^1 e_1$ and $\bar{E} = \bar{F} - I$; f^j, $j = 1,2,3,4$ are nonlinear terms. To derive the L^2 energy estimate of u, we make use of the displacement vector $\psi(x,t) = x - X(x,t)$. Here, $X(x,t)$ is the inverse of the material coordinate $x(X,t)$ which is constructed by the flow map:

$$\begin{cases} \dfrac{dx}{dt}(X,t) = v(x(X,t),t), \ t > 0 \\ x(X,0) = X \in \Omega. \end{cases}$$

Under the suitable condition for F and X, F is written by using ψ as follows

$$F = \bar{F} - \bar{F} \nabla \left(\psi - \bar{\psi}^1 e_1\right) \bar{F} + h\left(\nabla(\psi - \bar{\psi}^1 e_1)\right).$$

Here h satisfies $h(\nabla(\psi - \bar{\psi}^1(t) e_1)) = O(|\nabla(\psi - \bar{\psi}^1(t) e_1)|^2)$. Applying a variant of the Matsumura-Nishida energy method given in [16, 29] to (5.6) and estimating the interaction between the time-periodic parallel flow and the perturbation, we obtain the estimate:

$$\|u(t)\|_{H^2 \times H^2 \times H^3}^2 + \int_0^t e^{-c_1(t-s)} \|u(s)\|_{H^2 \times H^3 \times H^3}^2 \, \mathrm{d}s \le C e^{-c_1 t} \|u_0\|_{H^2 \times H^2 \times H^3}^2,$$

provided that $\nu \gg 1$, $\gamma \gg 1$, $\beta \gg 1$, and the initial perturbation is sufficiently small.

This chapter is organized as follows. In Section 5.1, we state the existence of the time-periodic parallel flow and then give the main result the main result of this chapter. In Sections 5.2 and 5.3, we give a proof of the stability of the time-periodic parallel flow based on the Matsumura-Nishida energy method [26]. In Appendix 5.A, we prove the estimates for the time-periodic parallel flows.

5.1 Main result of Chapter 5

In this section, we first introduce a time-periodic parallel flow, and then give the main result on this chapter on the stability of the time-periodic parallel flow.

We introduce the time-periodic parallel flow defined as $(\bar{\rho}, \bar{v}, \bar{F})$, where

$$\bar{\rho} = 1,\ \bar{v} = \bar{v}^1(x_3, t)e_1,\ \bar{F} = \bar{F}(x_3, t) = \left(\nabla(x - \bar{\psi}^1(x_3, t)e_1)\right)^{-1},$$

$$\bar{v}^1(x_3, t+T) = \bar{v}^1(x_3, t),\ \bar{\psi}^1(x_3, t+T) = \bar{\psi}^1(x_3, t),\ \partial_t\bar{\psi}^1 = \bar{v}^1.$$

Here $T > 0$ is some constant.

We also assume the compatibility conditions for g^1:

$$\partial_{x_3}^{2j} g^1(0, t) = \partial_{x_3}^{2j} g^1(1, t) = 0,\ t \geq 0,\ j = 0, 1, 2. \tag{5.7}$$

Then the following assertions hold true.

Proposition 5.1. *Let* $\kappa = \min\left\{\nu, \frac{\beta^2}{\nu}\right\}$. *Assume that* $g^1 \in H^1(0, T; H^4(0, 1))$ *satisfies the compatibility conditions* (5.7). *Then there exist a time-periodic flow* $(\bar{\rho}, \bar{v}, \bar{F})$ *of* (5.1) *and* $(\bar{\psi}^1,\ \bar{v}^1)$*satisfies the following estimates*:

$$\sup_{t\in[0,T]} \|\bar{\psi}^1(t)\|^2_{H^6(0,1)} \leq \frac{C}{\beta^4}\left(1 + \frac{1}{\kappa^2}\right)\|g^1\|^2_{H^1(0,T;H^4(0,1))},$$

$$\sup_{t\in[0,T]} \|\bar{v}^1(t)\|^2_{H^4(0,1)} \leq \frac{C}{\nu^2}\left(1 + \frac{1}{\kappa^2}\right)\|g^1\|^2_{H^1(0,T;H^4(0,1))},$$

$$\sup_{t\in[0,T]} \|\partial_t\bar{v}^1(t)\|^2_{H^2(0,1)} \leq C\left(1 + \frac{1}{\kappa^2}\right)\|g^1\|^2_{H^1(0,T;H^4(0,1))}.$$

The proof of Proposition 5.1 can be shown as [16]. The detail will be given Appendix 5.A.

We next consider the stability of the time-periodic parallel flow $(1, \bar{v}, \bar{F})$.

We set $U(t) = (\phi(t), w(t), G(t)) = (\rho(t)-1, v(t)-\bar{v}(t), F(t)-\bar{F}(t))$. Then $U(t)$ satisfies the following initial-boundary problem

$$
\begin{cases}
\partial_t \phi + \bar{v}^1 \partial_{x_1} \phi + \mathrm{div} w = f_1, & (5.8) \\
\quad \partial_t w - \nu \Delta w - \tilde{\nu} \nabla \mathrm{div} w + \gamma^2 \nabla \phi - \beta^2 \mathrm{div} G + \bar{v}^1 \partial_{x_1} w + (w^3 \partial_{x_3} \bar{v}^1) e_1 \\
\qquad + \nu(\phi \partial_{x_3}^2 \bar{v}^1) e_1 - \beta^2 \mathrm{div}(G^\top \bar{E}) - \beta^2 (G^{33} \partial_{x_3}^2 \bar{\psi}^1) e_1 = f_2, & (5.9) \\
\partial_t G + \bar{v}^1 \partial_{x_1} G - \nabla w - \nabla w \bar{E} + w^3 \partial_{x_3} \bar{E} - \nabla \bar{v} G = f_3, & (5.10) \\
\nabla \phi = -\mathrm{div}^\top G + {}^\top \bar{E} \mathrm{div}^\top G + f_4, & (5.11) \\
w|_{x_3=0,1} = 0, U|_{t=0} = U_0 = (\phi_0, w_0, G_0).
\end{cases}
$$

Here $\tilde{\nu} = \nu + \nu'$, $\bar{E} = \bar{F} - I = \nabla(\bar{\psi}^1 e_1)$ and $f_j, j = 1,2,3,4$ denote the nonlinear terms;

$$
\begin{aligned}
f_1 &= -\mathrm{div}(\phi w), \\
f_2 &= -w \cdot \nabla w + \frac{\nu \phi}{1+\phi}(-\Delta w + \partial_{x_3}^2 \bar{v}^1 e_1) - \frac{\tilde{\nu}\phi}{1+\phi} \nabla \mathrm{div} w - \frac{\gamma^2 \phi}{1+\phi} \nabla \phi \\
&\quad - \frac{\gamma^2}{1+\phi} \nabla Q(\phi) + \frac{\beta^2 \phi}{1+\phi} \mathrm{div}((G^\top \bar{E}) + (\partial_{x_3}^2 \bar{\psi}^1) G^{33} e_1) \\
&\quad + \frac{\beta^2}{1+\phi} \mathrm{div}(G^\top G + \phi(\bar{F}^\top G + G^\top \bar{F} + G^\top G)), \\
f_3 &= -w \cdot \nabla G + \nabla w G, \\
f_4 &= -\mathrm{div}(\phi^\top G),
\end{aligned}
$$

where

$$
Q(\phi) = \phi^2 \int_0^1 P''(1+s\phi) \mathrm{d}s, \ \nabla Q = O(\phi) \nabla \phi \text{ for } |\phi| \ll 1.
$$

Now we mention the main result of this chapter about the stability of the time-periodic parallel flow $(1, \bar{v}, \bar{F})$.

Theorem 5.2. *Under the assumption of Proposition 5.1, there are positive numbers ν_1, γ_1, and $\beta_1 > 0$ such that if $\nu \geq \nu_1$, $\frac{\gamma^2}{\nu+\tilde{\nu}} \geq \gamma_1^2$, $\frac{\beta^2}{\gamma^2} \geq \beta_1^2$, the following assertion holds. There is a positive number ϵ_1 such that if $U_0 = (\phi_0, w_0, G_0)$ satisfies $\|U_0\|_{H^2 \times H^2 \times H^2} \leq \epsilon_1, w_0 \in H_0^1(\Omega)$, $\int_\Omega \phi_0 dx = 0$, then there exists a unique solution $(\phi(t), w(t), G(t)) \in C([0,\infty); H^2(D))$ of the problem* (5.8)-(5.11), *and the perturbation $U(t) = (\phi(t), w(t), G(t))$ satisfies*

$$
\|U(t)\|_{H^2 \times H^2 \times H^2}^2 + \int_0^t e^{-c_1(t-s)} \|U(s)\|_{H^2 \times H^3 \times H^2}^2 \mathrm{d}s \leq C e^{-c_1 t} \|U_0\|_{H^2 \times H^2 \times H^2}^2.
$$

for $t \geq 0$.

Theorem 5.2 is shown by combining local in time existence theorem and a priori estimate.

Proposition 5.3. *Let* $t_0 \geq 0$ *and* $(\phi, w, G)|_{t=t_0} = (\phi_0, w_0, G_0)$*. If we assume* $|\phi_0| \leq \frac{1}{2}, \int_\Omega \phi_0 dx = 0,\ w_0 \in H_0^1(\Omega), \mathrm{div}((1+\phi_0)^\top(G_0 + \overline{F}(t_0))) = 0$ *and* $(1+\phi_0)\mathrm{div}(G_0 + \overline{F}(t_0)) = 1$*, then there are some numbers* $t_1,\ C$ *independent on* t_0 *such that the solution* $(\phi, w, G) \in C([t_0, t_1]; H^2 \times H^2 \times H^2)$ *of* (5.8)–(5.10) *satisfying* $(\phi, w, G)|_{t=t_0} = (\phi_0, w_0, G_0)$ *exists uniquely and the following assertions hold:*

$$\begin{aligned}
&\partial_t \phi, \partial_t G \in C([t_0, t_1]; L^2(\Omega)),\\
&w \in L^2([t_0, t_1]; H^3(\Omega)),\ \ \partial_t w \in C([t_0, t_1]; L^2(\Omega)) \cap L^2([t_0, t_1]; H^1(\Omega)),\\
&\|(\phi(t), w(t), G(t))\|_{H^2\times H^2\times H^2} \leq C\|(\phi_0, w_0, G_0)\|_{H^2\times H^2\times H^2}.
\end{aligned}$$

Proposition 5.3 is proved by using Proposition 5.1 in a similar manner to [18, 29, 36]. We omit it.

We next state the a priori estimate.

Proposition 5.4. *There are positive numbers* $\nu_1,\ \gamma_1,\ \beta_1$ *such that if* $\nu \geq \nu_1$, $\frac{\gamma^2}{\nu+\tilde{\nu}} \geq \gamma_1^2$, $\frac{\beta^2}{\gamma^2} \geq \beta_1^2$*, the following assertion holds:*

There exists a positive number δ_0 *such that if* $\|u_0\|_{H^2\times H^2\times H^2} \leq \delta_0$*, then the following inequality holds:*

$$E(t) + \int_0^t e^{-C_1(t-s)} D(s) ds \leq C e^{-C_1 t} E(0).$$

for $t \geq 0$*.Here* $E(t)$ *and* $D(t)$ *are equivalent to* $\|u(t)\|^2_{H^2\times H^2\times H^2}$ *and* $\|u(t)\|^2_{H^2\times H^3\times H^2}$*, respectively.*

5.2 Basic estimates

In this section, we establish the basic estimates to show the Proposition 5.4. We first prepare the notations.

$$E_0 := \sum_{j=0}^{2} E_0^j, D_0 := \sum_{j=0}^{2} D_0^j, N_0 := \sum_{j=0}^{2} N_0^j.$$

Here

$$\begin{aligned}
E_0^j &:= \gamma^2 \|\partial^j \phi\|^2_{L^2} + \|\partial^j w\|^2_{L^2} + \beta^2 \|\partial^j G\|^2_{L^2},\\
D_0^j &:= \nu \|\nabla \partial^j w\|^2_{L^2} + \tilde{\nu} \|\mathrm{div} \partial^j w\|^2_{L^2},\\
N_0^j &:= \gamma^2 |(\partial^j f_1, \partial^j \phi)| + |(\partial^j f_2, \partial^j w)| + \beta^2 |(\partial^j f_3, \partial^j G)|.
\end{aligned}$$

Proposition 5.5. *There exists $\nu_1 > 0$ such that if $\nu \geq \nu_1$, the following estimate holds:*

$$\frac{d}{dt}E_0 + D_0 \leq \frac{c}{\nu}\Big(\sum_{j=1}^{2}\|\partial^j\phi\|_{L^2}^2 + \|\partial_{x_3}\phi\|_{L^2}^2\Big) + c\Big(\frac{1}{\nu} + \frac{\beta^2}{\nu}\Big)\sum_{j=0}^{2}\|\partial^j G\|_{L^2}^2 + CN_0.$$

Proof. We take the inner product of (5.8) with ϕ to obtain

$$\frac{1}{2}\frac{d}{dt}\|\phi\|_{L^2}^2 + (\overline{v}^1\partial_{x_1}\phi, \phi) + (\mathrm{div}w, \phi) = (f_1, \phi).$$

By integration by parts, we have $(\overline{v}^1\partial_{x_1}\phi, \phi) = 0$, and therefore

$$\frac{1}{2}\frac{d}{dt}\|\phi\|_{L^2}^2 + (\mathrm{div}w, \phi) = (f_1, \phi). \tag{5.12}$$

We take the inner product with (5.9) of w to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\|w\|_{L^2}^2 + (\overline{v}^1\partial_{x_1}w, w) + (w^3\partial_{x_3}\overline{v}^1, w^1)\\ &+ (-\nu\Delta w - \tilde{\nu}\nabla\mathrm{div}w, w) + \nu(\phi\partial_{x_3}^2\overline{v}^1, w^1) + \gamma^2(\nabla\phi, w)\\ &- \beta^2(\mathrm{div}(G^\top\overline{F}), w) - \beta^2(G^{33}\partial_{x_3}^2\overline{\psi}, w^1) = (f_2, w).\end{aligned}$$

By integration by parts, we have

$$(\overline{v}^1\partial_{x_1}w, w) = 0,$$

$$(-\nu\Delta w - \tilde{\nu}\nabla\mathrm{div}w, w) = \nu\|\nabla w\|_{L^2}^2 + \tilde{\nu}\|\mathrm{div}w\|_{L^2}^2,$$

and

$$(\nabla\phi, w) = -(\mathrm{div}w, \phi).$$

We thus obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2 &+ (w^3\partial_{x_3}\overline{v}^1, w^1) + \nu\|\nabla w\|_{L^2}^2 + \tilde{\nu}\|\mathrm{div}w\|_{L^2}^2 + \nu(\phi\partial_{x_3}^2\overline{v}^1, w^1)\\ &- \gamma^2(\mathrm{div}w, \phi) - \beta^2(\mathrm{div}(G^\top\overline{F}), w) - \beta^2(G^{33}\partial_{x_3}^2\overline{\psi}, w^1) = (f_2, w).\end{aligned} \tag{5.13}$$

We take the inner product of (5.10) with G to obtain

$$\frac{1}{2}\frac{d}{dt}\|G\|_{L^2}^2 + (\overline{v}^1\partial_{x_1}G, G) + (w^3\partial_{x_3}^2\overline{\psi}^1, G^{13}) - (\nabla w\overline{F}, G) - (\nabla\overline{v}G, G) = (f_3, G).$$

By integration by parts, we have

$$(\overline{v}^1\partial_{x_1}G, G) = 0,\ (\nabla w\overline{F}, G) = -(\mathrm{div}(G^\top\overline{F}), w).$$

We thus obtain

$$\frac{1}{2}\frac{d}{dt}\|G\|_{L^2}^2 + (w^3\partial_{x_3}^2\overline{\psi}^1, G^{13}) + (\mathrm{div}(G^\top\overline{F}), w) - (\nabla\overline{v}G, G) = (f_3, G). \quad (5.14)$$

It follows from $\gamma^2 \times (5.12) + (5.13) + \beta^2 \times (5.14)$ that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\gamma^2\|\phi\|_{L^2}^2 + \|w\|_{L^2}^2 + \beta^2\|G\|_{L^2}^2) + \nu\|\nabla w\|_{L^2}^2 + \tilde{\nu}\|\mathrm{div}w\|_{L^2}^2 \\
&= -(w^3\partial_{x_3}\overline{v}^1, w^1) - \nu(\phi\partial_{x_3}^2\overline{v}^1, w^1) \\
&\quad +\beta^2(G^{33}\partial_{x_3}^2\overline{\psi}^1, w^1) - \beta^2(w^3\partial_{x_3}^2\overline{\psi}^1, G^{13}) + \beta^2(\nabla\overline{v}G, G) \\
&\qquad +\gamma^2(f_1, \phi) + (f_2, w) + \beta^2(f_3, G).
\end{aligned} \quad (5.15)$$

Since $\|\bar{\psi}^1\|_{W^{5,\infty}(0,1)} = O\left(\frac{1}{\beta^2}\right)$ and $\|\bar{v}^1\|_{W^{3,\infty}(0,1)} = O\left(\frac{1}{\nu}\right)$ by combining Proposition 5.1 and Lemma 2.1, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\gamma^2\|\phi\|_{L^2}^2 + \|w\|_{L^2}^2 + \beta^2\|G\|_{L^2}^2) + \nu\|\nabla w\|_{L^2}^2 + \tilde{\nu}\|\mathrm{div}w\|_{L^2}^2 \\
&\le |(w^3\partial_{x_3}\overline{v}^1, w^1)| + \nu|(\phi\partial_{x_3}^2\overline{v}^1, w^1)| \\
&\quad + \beta^2|(G^{33}\partial_{x_3}^2\overline{\psi}^1, w^1)| + \beta^2|(w^3\partial_{x_3}^2\overline{\psi}^1, G^{13})| + \beta^2|(\nabla\overline{v}G, G)| \\
&\qquad + \gamma^2|(f_1, \phi)| + |(f_2, w)| + \beta^2|(f_3, G)| \\
&\le \left(\frac{C}{\nu} + \frac{3\nu}{8}\right)\|\nabla w\|_{L^2}^2 + \frac{C}{\nu}\|\phi\|_{L^2}^2 + C\left(\frac{1}{\nu} + \frac{\beta^2}{\nu}\right)\|G\|_{L^2}^2 \\
&\quad + \gamma^2|(f_1, \phi)| + |(f_2, w)| + \beta^2|(f_3, G)|.
\end{aligned}$$

We take $\nu > 0$ so that $\frac{C}{\nu^2} \le \frac{1}{8}$. It then follows from (5.15) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\gamma^2\|\phi\|_{L^2}^2 + \|w\|_{L^2}^2 + \beta^2\|G\|_{L^2}^2) + \nu\|\nabla w\|_{L^2}^2 + \tilde{\nu}\|\mathrm{div}w\|_{L^2}^2 \\
&\le \frac{C}{\nu}\|\phi\|_{L^2}^2 + C\left(\frac{1}{\nu} + \frac{\beta^2}{\nu}\right)\|G\|_{L^2}^2 \\
&\quad + \gamma^2|(f_1, \phi)| + |(f_2, w)| + \beta^2|(f_3, G)|.
\end{aligned}$$

Hence we obtain

$$\frac{d}{dt}E_0^0 + D_0^0 \le \frac{C}{\nu}\|\nabla\phi\|_{L^2}^2 + C\left(\frac{1}{\nu} + \frac{\beta^2}{\nu}\right)\|G\|_{L^2}^2 + CN_0^0.$$

Similarly we can show that

$$\frac{d}{dt}E_0^j + D_0^j \le \frac{C}{\nu}\|\partial^j\phi\|_{L^2}^2 + C\left(\frac{1}{\nu}+\frac{\beta^2}{\nu}\right)\|\partial^j G\|_{L^2}^2 + CN_0^j,\ j=1,2.$$

This completes the proof. $\square$

We next estimate $\sum_{j=0}^{2}\|\partial^j G\|_{L^2}^2$. Let $x = x(X,t)$ be the material coordinate defined by the solution of the flow map:

$$\begin{cases} \frac{dx}{dt}(X,t) = v(x(X,t),t) \\ x(X,0) = X. \end{cases}$$

We set $\psi = x - X$. Then we see that ψ satisfies

$$\begin{cases} \partial_t\psi - v = -v\cdot\nabla\psi \\ \psi|_{x_3=0,1} = 0. \end{cases}$$

F is rewritten as $F = I + \nabla\psi + (\nabla\psi)^2(I-\nabla\psi)^{-1}$. We set $\zeta = \psi - \overline{\psi}$. We then see that ζ satisfies $\zeta|_{\{x_3=0,1\}} = 0$ and

$$\partial_t\zeta + \overline{v}^1\partial_{x_1}\zeta - w + w^3\partial_{x_3}^2\overline{\psi}^1 e_1 = f_5, \tag{5.16}$$

where $f_5 = -v\cdot\nabla\zeta$. By using ζ, G is rewritten as

$$G = F - \overline{F} = \nabla\zeta + \overline{E}\nabla\zeta + \nabla\zeta\overline{E} + \overline{E}\nabla\zeta\overline{E} + h_1, \tag{5.17}$$

where $h_1 = -(\overline{F}\nabla\zeta)^2(I-\overline{F}\nabla\zeta)^{-1}\overline{F}$.

Lemma 5.6. *There are positive numbers β_0 and δ_0 such that if $\beta^2 \ge \beta_0^2$, $\|\nabla\zeta\|_{H^2} \le \delta_0$, the following inequality holds:*

$$C^{-1}\sum_{j=0}^{2}\|\nabla\partial^j\zeta\|_{L^2}^2 \le \sum_{j=0}^{2}\|\partial^j G\|_{L^2}^2 \le C\sum_{j=0}^{2}\|\nabla\partial^j\zeta\|_{L^2}^2.$$

Proof. Let δ be the positive number. We assume that $\|\nabla\zeta\|_{H^2} \le \delta_0 < 1$ and $\beta \ge 1$. It follows from (5.17) that

$$\begin{aligned}\|G\|_{L^2} &\ge \|\nabla\zeta\|_{L^2} - C\Big(\frac{1}{\beta^2}+\frac{1}{\beta^4}\Big)\|\nabla\zeta\|_{L^2} - C\|\nabla\zeta\|_{L^\infty}\|\nabla\zeta\|_{L^2} \\ &\ge \|\nabla\zeta\|_{L^2} - C\Big(\frac{1}{\beta^2}+\frac{1}{\beta^4}+\|\nabla\zeta\|_{H^2}\Big)\|\nabla\zeta\|_{L^2}.\end{aligned}$$

$$\|G\|_{L^2} \leq \|\nabla\zeta\|_{L^2} + C\Big(\frac{1}{\beta^2} + \frac{1}{\beta^4} + \|\nabla\zeta\|_{H^2}\Big)\|\nabla\zeta\|_{L^2}.$$

We take β and δ_0 so that $1 - C\Big(\frac{1}{\beta^2} + \frac{1}{\beta^4} + \delta_0\Big) \geq \frac{1}{\sqrt{2}}$. We thus have $\|G\|_{L^2}^2 \geq \frac{1}{2}\|\nabla\zeta\|_{L^2}^2$. Similarly we obtain $\|G\|_{L^2}^2 \leq 2\|\nabla\zeta\|_{L^2}^2$. Similary, we estimate $\partial^j G$ to have

$$\sum_{j=0}^{2} \|\partial^j G\|_{L^2}^2 \leq C \sum_{j=0}^{2} \|\nabla\partial^j\zeta\|_{L^2}^2.$$

□

We set

$$E_1 := \sum_{j=0}^{2} E_1^j, D_1 := \sum_{j=0}^{2} D_1^j, N_1 := \sum_{j=0}^{2} N_1^j,$$

where

$$\begin{aligned}
&E_1^j := (\partial^j w, \partial^j\zeta),\ \ D_1^j := \beta^2\|\nabla\partial^j\zeta\|_{L^2}^2,\\
&N_1^j := (\partial^j f_2, \partial^j\zeta) - \gamma^2(\partial^j f_4, \partial^j\zeta) + (\partial^j f_5, \partial^j w) - \gamma^2(\partial^j h_1, \nabla\partial^j\zeta).
\end{aligned}$$

We note that $E_0 + E_1$ and D_1 are equivalent to E_0 and $\sum_{j=0}^2 \|\partial^j G\|_{L^2}^2$, respectivity under the assumption in Lemma 5.6.

Proposition 5.7. *There are positive numbers $\beta_1, \gamma_1, \eta_1, \nu_2, \delta_1 > 0$ such that if $\beta^2 \geq \beta_1^2, \gamma^2 \geq \gamma_1^2, \frac{\beta^2}{\gamma^2} \geq \eta^2, \frac{\beta^2}{\nu+\tilde{\nu}} \geq \nu_2, \|\nabla\zeta\|_{H^2} \leq \delta_1$, the following estimate holds:*

$$\begin{aligned}
\frac{d}{dt}E_1 + \frac{1}{4}D_1 \leq & \Big\{\frac{1}{2} + C\left(\frac{1}{\nu} + \frac{1}{\nu\beta^2} + \frac{1}{\nu^2\beta^2}\right)\Big\}D_0\\
&+ \frac{C}{\beta^2}\Big(\sum_{j=1}^{2}\|\partial^j\phi\|_{L^2}^2 + \|\partial_{x_3}\phi\|_{L^2}^2\Big) + N_1.
\end{aligned}$$

Proof. We take the inner product of (5.9) with ζ

$$\begin{aligned}
&(\partial_t w, \zeta) - \nu(\Delta w, \zeta) - \tilde{\nu}(\nabla \mathrm{div} w, \zeta) + \gamma^2(\nabla\phi, \zeta) - \beta^2(\mathrm{div} G, \zeta)\\
&\quad + (\overline{v}^1\partial_{x_1} w, \zeta) + (w^3\partial_{x_3}\overline{v}^1, \zeta^1) + \nu(\phi\partial_{x_3}^2\overline{v}^1, \zeta^1)\\
&\qquad - \beta^2(\mathrm{div}(G^\top\overline{E}), \zeta) - \beta^2(G^{33}\partial_{x_3}^2\overline{\psi}^1, \zeta^1) = (\tilde{f}_2, \zeta).
\end{aligned}$$

By integration by parts, we have

$$-(\mathrm{div} G, \zeta) = (G, \nabla\zeta)$$

$$
\begin{aligned}
&= \|\nabla\zeta\|_{L^2}^2 + (\overline{E}\nabla\zeta + \nabla\zeta\overline{E} + \overline{E}\nabla\zeta\overline{E}, \nabla\zeta) + (h_1, \nabla\zeta) \\
&\geq \|\nabla\zeta\|_{L^2}^2 - C\left(\frac{1}{\beta^2} + \frac{1}{\beta^4}\right)\|\nabla\zeta\|_{L^2}^2 - C\|\nabla\zeta\|_{H^2}\|\nabla\zeta\|_{L^2} \\
&\geq \|\nabla\zeta\|_{L^2}^2 - \frac{1}{2}\|\nabla\zeta\|_{L^2}^2 = \frac{1}{2}\|\nabla\zeta\|_{L^2}^2
\end{aligned}
$$

and

$$
\begin{aligned}
\nu(\Delta w,&\zeta) + \tilde{\nu}(\nabla \mathrm{div} w, \zeta) \\
&= -\nu(\nabla w, \nabla\zeta) - \tilde{\nu}(\mathrm{div} w, \mathrm{div}\zeta) \\
&\leq \frac{\nu}{2}\|\nabla w\|_{L^2}^2 + \frac{\nu}{2}\|\nabla\zeta\|_{L^2}^2 + \frac{\tilde{\nu}}{2}\|\mathrm{div} w\|_{L^2}^2 + \frac{\tilde{\nu}}{2}\|\mathrm{div}\zeta\|_{L^2}^2 \\
&= \frac{1}{2}D_0^0 + \frac{\nu}{2}\|\nabla\zeta\|_{L^2}^2 + \frac{\tilde{\nu}}{2}\|\mathrm{div}\zeta\|_{L^2}^2.
\end{aligned}
$$

We thus obtain

$$
\begin{aligned}
&(\partial_t w, \zeta) + \frac{\beta^2}{2}\|\nabla\zeta\|_{L^2}^2 + \gamma^2(\nabla\phi, \zeta) \\
&\leq \frac{1}{2}D_0 + \frac{\nu}{2}\|\nabla\zeta\|_{L^2}^2 + \frac{\tilde{\nu}}{2}\|\mathrm{div}\zeta\|_{L^2}^2 \\
&\quad -(\overline{v}^1\partial_{x_1}w, \zeta) - (w^3\partial_{x_3}\overline{v}^1, \zeta^1) - \nu(\phi\partial_{x_3}^2\overline{v}^1, \zeta^1) \\
&\qquad +\beta^2(\mathrm{div}(G^\top\overline{E}), \zeta) + \beta^2(G^{33}\partial_{x_3}^2\overline{\psi}^1, \zeta^1) + (\tilde{f}_2, \zeta).
\end{aligned} \tag{5.18}
$$

It follows from (5.11) that

$$
\nabla\phi = -({}^\top\overline{F})^{-1}\mathrm{div}^\top G + \tilde{f}_4,
$$

where $\tilde{f}_4 = ({}^\top\overline{F})^{-1}f_4$. Since $({}^\top\overline{F})^{-1} = I - {}^\top\overline{E}$, we have

$$
\nabla\phi = -\mathrm{div}^\top G + {}^\top\overline{E}\mathrm{div}^\top G + \tilde{f}_4. \tag{5.19}
$$

We take the inner product of (5.19) with $-\zeta$ to obtain

$$
-(\nabla\phi, \zeta) = (\mathrm{div}^\top G, \zeta) - ({}^\top\overline{E}\mathrm{div}^\top G, \zeta) - (\tilde{f}_4, \zeta). \tag{5.20}
$$

Since $G = \nabla\zeta + \overline{E}\nabla\zeta + \nabla\zeta\overline{E} + \overline{E}\nabla\zeta\overline{E} + h_1$, we have

$$
(\mathrm{div}^\top G, \zeta) = (\mathrm{div}^\top(\nabla\zeta), \zeta) + (\mathrm{div}^\top(\overline{E}\nabla\zeta + \nabla\zeta\overline{E} + \overline{E}\nabla\zeta\overline{E}), \zeta) + (\mathrm{div}^\top h_1, \zeta).
$$

By integration by parts, we have

$$
(\mathrm{div}^\top(\nabla\zeta), \zeta) = (\nabla\mathrm{div}\zeta, \zeta) = -\|\mathrm{div}\zeta\|_{L^2}^2.
$$

Hence we see from (5.20) that

$$
\begin{aligned}
&-(\nabla\phi,\zeta)+\|\mathrm{div}\zeta\|_{L^2}^2\\
&=(\mathrm{div}^\top(\overline{E}\nabla\zeta+\nabla\zeta\overline{E}+\overline{E}\nabla\zeta\overline{E}),\zeta)-({}^\top\overline{E}\mathrm{div}^\top G,\zeta)\\
&\quad+(\mathrm{div}^\top h_1,\zeta)-(\tilde{f}_4,\zeta).
\end{aligned}
\tag{5.21}
$$

We take the inner product of (5.16) with w to obtain

$$
(\partial_t\zeta,w)+(\overline{v}^1\partial_{x_1}\zeta,w)-\|w\|_{L^2}^2+(w^3\partial_{x_3}^2\overline{\psi}^1,w^1)=(f_5,w).
$$

Since $\|w\|_{L^2}^2\le\frac{1}{\nu}D_0^0$, we see from this identity that

$$
(\partial_t\zeta,w)\le\frac{1}{\nu}D_0^0-(\overline{v}^1\partial_{x_1}\zeta,w)-(w^3\partial_{x_3}^2\overline{\psi}^1,w^1)+(f_5,w).
\tag{5.22}
$$

It follows from (5.18) $+\gamma^2\times$ (5.21) $+$ (5.22) and $\frac{d}{dt}(w,\zeta)=(\partial_t\zeta,w)+(\zeta,\partial_t w)$ that

$$
\begin{aligned}
&\frac{d}{dt}(w,\zeta)+\frac{\beta^2}{2}\|\nabla\zeta\|_{L^2}^2+\gamma^2\|\mathrm{div}\zeta\|_{L^2}^2\\
&\le\left(\frac{1}{2}+\frac{C}{\nu}\right)D_0^0+\frac{\nu}{2}\|\nabla\zeta\|_{L^2}^2+\frac{\tilde{\nu}}{2}\|\mathrm{div}\zeta\|_{L^2}^2-(\overline{v}^1\partial_{x_1}w,\zeta)-(w^3\partial_{x_3}\overline{v}^1,\zeta^1)\\
&\quad-\nu(\phi\partial_{x_3}^2\overline{v}^1,\zeta^1)+\beta^2(\mathrm{div}(G^\top\overline{E}),\zeta)+\beta^2(G^{33}\partial_{x_3}^2\overline{\psi}^1,\zeta^1)\\
&\qquad+\gamma^2(\mathrm{div}^\top(\overline{E}\nabla\zeta+\nabla\zeta\overline{E}+\overline{E}\nabla\zeta\overline{E}),\zeta)\\
&\qquad\quad-\gamma^2({}^\top\overline{E}\mathrm{div}^\top G,\zeta)-(\overline{v}^1\partial_{x_1}\zeta,w)-(w^3\partial_{x_3}^2\overline{\psi}^1,w^1)\\
&\qquad\qquad+(\tilde{f}_2,\zeta)-\gamma^2(\tilde{f}_4,\zeta)+(f_5,w)+\gamma^2(\mathrm{div}^\top h_1,\zeta).
\end{aligned}
\tag{5.23}
$$

We set

$$
\begin{aligned}
R_1:=&-(\overline{v}^1\partial_{x_1}w,\zeta)-(w^3\partial_{x_3}\overline{v}^1,\zeta^1)-\nu(\phi\partial_{x_3}^2\overline{v}^1,\zeta)+\beta^2(\mathrm{div}(G^\top\overline{E}),\zeta)\\
&+\beta^2(G^{33}\partial_{x_3}^2\overline{\psi}^1,\zeta^1)+\gamma^2(\mathrm{div}^\top(\overline{E}\nabla\zeta+\nabla\zeta\overline{E}+\overline{E}\nabla\zeta\overline{E}),\zeta)\\
&-\gamma^2({}^\top\overline{E}\mathrm{div}^\top G,\zeta)-(\overline{v}^1\partial_{x_1}\zeta,w)-(w^3\partial_{x_3}^2\overline{\psi}^1,w^1),\\
N_1^0:=&(\tilde{f}_2,\zeta)-\gamma^2(\tilde{f}_4,\zeta)+(f_5,w)+\gamma^2(\mathrm{div}^\top h_1,\zeta).
\end{aligned}
$$

Then we see from (5.23) that

$$
\begin{aligned}
&\frac{d}{dt}(w,\zeta)+\frac{\beta^2}{2}\|\nabla\zeta\|_{L^2}^2+\gamma^2\|\mathrm{div}\zeta\|_{L^2}^2\\
&\le\left(\frac{1}{2}+\frac{C}{\nu}\right)D_0^0+\frac{\nu}{2}\|\nabla\zeta\|_{L^2}^2+\frac{\tilde{\nu}}{2}\|\mathrm{div}\zeta\|_{L^2}^2+R_1+N_1^0.
\end{aligned}
\tag{5.24}
$$

By using the Schwartz inequality, we obtain

$$R_1 \leq \left\{4\epsilon + C\left(\frac{1}{\beta^2} + \frac{\gamma^2}{\beta^4} + \frac{\gamma^2}{\beta^6}\right)\right\}\beta^2\|\nabla\zeta\|_{L^2}^2 + \frac{C}{\epsilon}\left(\frac{1}{\nu^2\beta^2} + \frac{1}{\nu\beta^2}\right)D_0^0 + \frac{C}{\epsilon\beta^2}\|\nabla\phi\|_{L^2}^2$$

for any $\epsilon > 0$. This, together with (5.24), yields

$$\begin{aligned}
&\frac{d}{dt}(w,\zeta) + \frac{\beta^2}{2}\|\nabla\zeta\|_{L^2}^2 + \gamma^2\|\mathrm{div}\zeta\|_{L^2}^2 \\
&\leq \left\{\frac{1}{2} + C\left(\frac{1}{\nu} + \frac{1}{\nu\beta^2} + \frac{1}{\nu^2\beta^2}\right)\right\}D_0^0 \\
&+ \{4\beta^2\epsilon + C\beta^2(\frac{1}{\beta^2} + \frac{\gamma^2}{\beta^4} + \frac{\gamma^2}{\beta^6}) + \frac{\nu + 9\tilde{\nu}}{2}\}\|\nabla\zeta\|_{L^2}^2 \\
&\quad + \frac{C}{\beta^2}\|\nabla\phi\|_{L^2}^2 + N_1^0.
\end{aligned}$$

We set

$$I := \Big\{4\beta^2\epsilon + C\beta^2\Big(\frac{1}{\beta^2} + \frac{\gamma^2}{\beta^4} + \frac{\gamma^2}{\beta^6}\Big) + \frac{\nu + 9\tilde{\nu}}{2}\Big\}\|\nabla\zeta\|_{L^2}^2$$

and choose $\epsilon = \frac{1}{32}$. Then we see that

$$I = \frac{\beta^2}{8}\|\nabla\zeta\|_{L^2}^2 + C\beta^2\Big(\frac{1}{\beta^2} + \frac{\gamma^2}{\beta^4} + \frac{\gamma^2}{\beta^6} + \frac{\nu + 9\tilde{\nu}}{2\beta^2}\Big)\|\nabla\zeta\|_{L^2}^2.$$

We take $\beta^2, \gamma^2, \nu, \tilde{\nu}$ so that $C(\frac{1}{\beta^2} + \frac{\gamma^2}{\beta^4} + \frac{\gamma^2}{\beta^6} + \frac{\nu+9\tilde{\nu}}{2\beta^2}) \leq \frac{1}{8}$, then we have $I \leq \frac{\beta^2}{4}\|\nabla\zeta\|_{L^2}^2$.
Thus we obtain

$$\frac{d}{dt}E_1^0 + \frac{1}{4}D_1^0 \leq \Big\{\frac{1}{2} + C\Big(\frac{1}{\nu} + \frac{1}{\nu\beta^2} + \frac{1}{\nu^2\beta^2}\Big)\Big\}D_0^0 + \frac{C}{\beta^2}\|\nabla\phi\|_{L^2}^2 + N_1^0.$$

Similarily, we can show the following estimate for $j = 1, 2$:

$$\frac{d}{dt}E_1^j + \frac{1}{4}D_1^j \leq \Big\{\frac{1}{2} + C\Big(\frac{1}{\nu} + \frac{1}{\nu\beta^2} + \frac{1}{\nu^2\beta^2}\Big)\Big\}D_0^j + \frac{C}{\beta^2}\|\partial^j\phi\|_{L^2}^2 + N_1^j.$$

This completes the proof. □

We set

$$E_2 := E_0 + E_1,$$

$$D_2 := D_0 + D_1 + \frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}} \sum_{j=1}^{2} \|\partial^j \phi\|_{L^2}^2,$$

$$N_2 := N_0 + N_1 + \sum_{j=1}^{2} (\|\partial^{j-1} f_4\|_{L^2}^2 + \|\mathrm{div}\partial^{j-1\top} h_1\|_{L^2}^2),$$

Proposition 5.8. *It holds the following estimate:*

$$\frac{d}{dt}E_2 + \frac{1}{32}D_2 \le C\Big(\frac{1}{\nu} + \frac{1}{\beta^2}\Big)\|\partial_{x_3}\phi\|_{L^2}^2 + CN_2. \tag{5.25}$$

Proof. We first show that

$$\sum_{j=1}^{2} \|\partial^j \phi\|_{L^2}^2 \le C\Big(\frac{1}{\beta^2} + \frac{1}{\beta^6}\Big)D_1 + C\sum_{j=1}^{2}\Big(\|\partial^{j-1} f_4\|_{L^2}^2 + \|\mathrm{div}\partial^{j-1\top} h_1\|_{L^2}^2\Big). \tag{5.26}$$

We see from (5.20) that

$$\begin{aligned} \partial_{x_j}\phi = &-\partial_{x_j}(\mathrm{div}\zeta) - (\mathrm{div}^\top(\overline{E}\nabla\zeta + \nabla\zeta\overline{E} + \overline{E}\nabla\zeta\overline{E}))^j \\ &-(\mathrm{div}^\top h_1)^j + (\tilde{f}_4)^j,\ j = 1, 2. \end{aligned} \tag{5.27}$$

Therefore we obtain

$$\begin{aligned} \|\partial\phi\|_{L^2} \le \|\nabla\partial\zeta\|_{L^2} &+ \textstyle\sum_{j=1}^{2} \|(\mathrm{div}^\top(\overline{E}\nabla\zeta + \nabla\zeta\overline{E} + \overline{E}\nabla\zeta\overline{E}))^j\|_{L^2} \\ &+\|\mathrm{div}^\top h_1\|_{L^2} + \|\tilde{f}_4\|_{L^2}. \end{aligned}$$

Since

$$\mathrm{div}^\top(\overline{E}\nabla\zeta) = \mathrm{div}\begin{pmatrix} \partial_{x_1}\zeta^3\partial_{x_3}\overline{\psi}^1 & 0 & 0 \\ \partial_{x_2}\zeta^3\partial_{x_3}\overline{\psi}^1 & 0 & 0 \\ \partial_{x_3}\zeta^3\partial_{x_3}\overline{\psi}^1 & 0 & 0 \end{pmatrix} = \partial_{x_3}\overline{\psi}^1 \begin{pmatrix} \partial_{x_1}^2\zeta^3 \\ \partial_{x_1}\partial_{x_2}\zeta^3 \\ \partial_{x_1}\partial_{x_3}\zeta^3 \end{pmatrix},$$

$$\begin{aligned} \mathrm{div}^\top(\nabla\zeta\overline{E}) &= \mathrm{div}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \partial_{x_3}\overline{\psi}^1\partial_{x_1}\zeta^1 & \partial_{x_3}\overline{\psi}^1\partial_{x_1}\zeta^2 & \partial_{x_3}\overline{\psi}^1\partial_{x_1}\zeta^3 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ \partial_{x_3}\overline{\psi}^1\partial_{x_1}^2\zeta^1 + \partial_{x_3}\overline{\psi}^1\partial_{x_1}\partial_{x_2}\zeta^2 + \partial_{x_3}(\partial_{x_3}\overline{\psi}^1\partial_{x_1}\zeta^3) \end{pmatrix}, \end{aligned}$$

$$\mathrm{div}^\top(\overline{E}\nabla\zeta\overline{E}) = \mathrm{div}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ (\partial_{x_3}\overline{\psi}^1)^2\partial_{x_1}\zeta^3 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ (\partial_{x_3}\overline{\psi}^1)^2\partial_{x_1}^2\zeta^3 \end{pmatrix},$$

we have $\displaystyle\sum_{j=1}^{2}\|(\mathrm{div}^\top(\overline{E}\nabla\zeta + \nabla\zeta\overline{E} + \overline{E}\nabla\zeta\overline{E}))^j\|_{L^2}^2 \le \frac{C}{\beta^4}\|\nabla\partial\zeta\|_{L^2}^2$. It then follows

$$\|\partial\phi\|_{L^2}^2 \le C\Big(1+\frac{1}{\beta^4}\Big)\|\nabla\partial\zeta\|_{L^2}^2 + \|\mathrm{div}^\top h_1\|_{L^2}^2 + \|\tilde{f}_4\|_{L^2}^2.$$

Similarily, the following inequality holds :

$$\|\partial^2\phi\|_{L^2}^2 \le C\Big\{\Big(1+\frac{1}{\beta^4}\Big)\|\nabla\partial^2\zeta\|_{L^2}^2 + \|\partial f_4\|_{L^2}^2 + \|\mathrm{div}\partial^\top h_1\|_{L^2}^2\Big\}.$$

Therefore we obtain

$$\begin{aligned}\sum_{j=1}^{2}\|\partial^j\phi\|_{L^2}^2 &\le C\Big(1+\frac{1}{\beta^4}\Big)\sum_{j=1}^{2}\|\nabla\partial^j\zeta\|_{L^2}^2 \\ &\quad + C\sum_{j=1}^{2}\Big(\|\partial^{j-1}f_4\|_{L^2}^2 + \|\mathrm{div}\partial^{j-1\top}h_1\|_{L^2}^2\Big) \\ &\le C\Big(\frac{1}{\beta^2}+\frac{1}{\beta^6}\Big)D_1 + C\sum_{j=1}^{2}\Big(\|\partial^{j-1}f_4\|_{L^2}^2 + \|\mathrm{div}\partial^{j-1\top}h_1\|_{L^2}^2\Big).\end{aligned}$$

By combining Proposition 5.5 and Proposition 5.6, we have

$$\begin{aligned}&\frac{d}{dt}E_0 + D_0 \\ &\le \frac{C}{\nu}\Big(\sum_{j=1}^{2}\|\partial^j\phi\|_{L^2}^2 + \|\partial_{x_3}\phi\|_{L^2}^2\Big) + C\left(\frac{1}{\nu}+\frac{\beta^2}{\nu}\right)\sum_{j=0}^{2}\|\nabla\partial^j\zeta\|_{L^2}^2 + CN_0 \\ &= \frac{C}{\nu}\Big(\sum_{j=1}^{2}\|\partial^j\phi\|_{L^2}^2 + \|\partial_{x_3}\phi\|_{L^2}^2\Big) + C\left(\frac{1}{\nu}+\frac{1}{\nu\beta^2}\right)D_1 + CN_0.\end{aligned} \tag{5.28}$$

By (5.28) and Proposition 5.7, we obtain

$$\frac{d}{dt}(E_0+E_1) + D_0 + \frac{1}{4}D_1$$

$$\le \Big\{\frac{1}{2}+C\Big(\frac{1}{\nu}+\frac{1}{\nu\beta^2}+\frac{1}{\nu^2\beta^2}\Big)\Big\}D_0+C\Big(\frac{1}{\nu}+\frac{1}{\nu\beta^2}\Big)D_1$$
$$+C\Big(\frac{1}{\nu}+\frac{1}{\beta^2}\Big)\Big(\sum_{j=1}^{2}\|\partial^j\phi\|_{L^2}^2+\|\partial_{x_3}\phi\|_{L^2}^2\Big)+C(N_0+N_1).$$

We take ν, β^2 so that $\frac{1}{2}+C(\frac{1}{\nu}+\frac{1}{\nu\beta^2}+\frac{1}{\nu^2\beta^2})\le\frac{15}{16}, C(\frac{1}{\nu}+\frac{1}{\nu\beta^2})\le\frac{3}{16}$, and then we have

$$\begin{aligned}&\frac{d}{dt}(E_0+E_1)+\frac{1}{16}(D_0+D_1)\\&\le C\Big(\frac{1}{\nu}+\frac{1}{\beta^2}\Big)\Big(\sum_{j=1}^{2}\|\partial^j\phi\|_{L^2}^2+\|\partial_{x_3}\phi\|_{L^2}^2\Big)+C(N_0+N_1).\end{aligned}\tag{5.29}$$

It follows from $\frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\times(5.26)+(5.29)$ that

$$\frac{d}{dt}(E_0+E_1)+\frac{1}{16}(D_0+D_1)+\frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\sum_{j=1}^{2}\|\partial^j\phi\|_{L^2}^2$$
$$\le C\Big(\frac{1}{\nu}+\frac{1}{\beta^2}\Big)\Big(\sum_{j=1}^{2}\|\partial^j\phi\|_{L^2}^2+\|\partial_{x_3}\phi\|_{L^2}^2\Big)$$
$$+C\Big\{\frac{1}{\nu+\tilde{\nu}}\Big(1+\frac{\gamma^2}{\beta^2}\Big)+\frac{1}{(\nu+\tilde{\nu})\beta^4}\Big(1+\frac{\gamma^2}{\beta^2}\Big)\Big\}D_1$$
$$+C\Big\{N_0+N_1+\sum_{j=1}^{2}(\|\partial^{j-1}f_4\|_{L^2}^2+\|\mathrm{div}\partial^{j-1\top}h_1\|_{L^2}^2\Big\}.$$

We take $\nu+\tilde{\nu}, \gamma^2, \beta^2$ so that

$$C\Big(\frac{1}{\nu}+\frac{1}{\beta^2}\Big)\le\frac{1}{2}\cdot\frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\cdot C\Big\{\frac{1}{\nu+\tilde{\nu}}\Big(1+\frac{\gamma^2}{\beta^2}\Big)+\frac{1}{(\nu+\tilde{\nu})\beta^4}\Big(1+\frac{\gamma^2}{\beta^2}\Big)\Big\}\le\frac{1}{32}$$

. It then follows

$$\frac{d}{dt}(E_0+E_1)+\frac{1}{16}D_0+\frac{1}{32}D_1+\frac{1}{2}\frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\sum_{j=1}^{2}\|\partial^j\phi\|_{L^2}^2$$
$$\le C\Big(\frac{1}{\nu}+\frac{1}{\beta^2}\Big)\|\partial_{x_3}\phi\|_{L^2}^2$$
$$+C\Big\{N_0+N_1+\sum_{j=1}^{2}(\|\partial^{j-1}f_4\|_{L^2}^2+\|\mathrm{div}\partial^{j-1\top}h_1\|_{L^2}^2)\Big\},$$

This completes the proof. □

We set

$$E_3 := \sum_{j=0}^{1} E_3^j, D_3 := \sum_{j=0}^{1} D_3^j, N_3 := \sum_{j=0}^{1} N_3^j,$$

where

$$\begin{aligned}
&E_3^j := \nu\|\nabla\partial^j w\|_{L^2}^2 + \tilde{\nu}\|\mathrm{div}\partial^j w\|_{L^2}^2 - 2\gamma^2(\partial^j\phi, \mathrm{div}\partial^j w) + 2\beta^2(\partial^j G, \nabla\partial^j w),\\
&D_3^j := \|\partial_t\partial^j w\|_{L^2}^2,\\
&N_3^j := -\gamma^2(\partial^1 f_1, \mathrm{div}\partial^j w) + (\partial^j f_2, \partial_t\partial^j w) + \beta^2(\partial^j G, \nabla\partial^j w).
\end{aligned}$$

Proposition 5.9. *It holds the following inequality:*

$$\begin{aligned}
\frac{d}{dt}E_3 + D_3 \le C\|\partial_{x_3}\phi\|_{L^2}^2 + C\Big(&\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2} + \frac{\gamma^4(\nu+\tilde{\nu})}{\nu^2\tilde{\nu}(\beta^2+\gamma^2)}\\
&+ \frac{\beta^2}{\nu} + \frac{\gamma^2}{\tilde{\nu}} + 1 + \frac{1}{\nu} + \frac{1}{\nu^3} + \frac{1}{\beta^2} + \frac{\beta^2}{\nu^3}\Big)D_2 + CN_3.
\end{aligned}$$

Proof. We take the inner product of (5.8) with $\mathrm{div}w$ to obtain

$$(\partial_t\phi, \mathrm{div}w) + (\overline{v}^1\partial_{x_1}\phi, \mathrm{div}w) + \|\mathrm{div}w\|_{L^2}^2 = (f_1, \mathrm{div}w).$$

Since $(\partial_t\phi, \mathrm{div}w) = \frac{d}{dt}(\phi, \mathrm{div}w) - (\phi, \mathrm{div}\partial_t w)$, we have

$$\frac{d}{dt}(\phi, \mathrm{div}w) - (\phi, \mathrm{div}\partial_t w) + (\overline{v}^1\partial_{x_1}\phi, \mathrm{div}w) + \|\mathrm{div}w\|_{L^2}^2 = (f_1, \mathrm{div}w). \quad (5.30)$$

We take the inner product of (5.9) with $\partial_t w$ to obtain

$$\begin{aligned}
\|\partial_t w\|_{L^2}^2 - \nu(\Delta w, \partial_t w) - \tilde{\nu}(\nabla\mathrm{div}w, \partial_t w) + \gamma^2(\nabla\phi, \partial_t w) - \beta^2(\mathrm{div}G, \partial_t w)\\
+ (w^3\partial_{x_3}\overline{v}^1, \partial_t w^1) + (\overline{v}^1\partial_{x_1}w, \partial_t w) + \nu(\phi\partial_{x_3}^2\overline{v}^1, \partial_t w^1)\\
- \beta^2(\mathrm{div}(G^\top\overline{E}), \partial_t w) - \beta^2(G^{33}\partial_{x_3}^2\overline{\psi}^1, \partial_t w^1) = (f_2, \partial_t w).
\end{aligned}$$

We set

$$\begin{aligned}
\tilde{R}_2 := -(\overline{v}^1\partial_{x_1}w, \partial_t w) - (w^3\partial_{x_3}\overline{v}^1, \partial_t w^1) - \nu(\phi\partial_{x_3}^2\overline{v}^1, \partial_t w^1)\\
+ \beta^2(\mathrm{div}(G^\top\overline{E}), \partial_t w) + \beta^2(G^{33}\partial_{x_3}^2\overline{\psi}^1, \partial_t w^1).
\end{aligned}$$

By integration by parts, we have

$$\begin{aligned}
-\nu(\Delta w, \partial_t w) - \tilde{\nu}(\nabla\mathrm{div}w, \partial_t w) &= \nu(\nabla w, \nabla\partial_t w) + \tilde{\nu}(\mathrm{div}w, \mathrm{div}\partial_t w)\\
&= \frac{1}{2}\frac{d}{dt}(\nu\|\nabla w\|_{L^2}^2 + \tilde{\nu}\|\mathrm{div}w\|_{L^2}^2),
\end{aligned}$$

$$(\nabla\phi, \partial_t w) = -(\phi, \mathrm{div}\partial_t w), \ (\mathrm{div}G, \partial_t w) = -(G, \nabla\partial_t w).$$

We then obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\nu\|\nabla w\|_{L^2}^2 + \tilde{\nu}\|\mathrm{div}w\|_{L^2}^2) + \|\partial_t w\|_{L^2}^2 \\ &\quad -\gamma^2(\phi, \mathrm{div}\partial_t w) + \beta^2(G, \nabla\partial_t w) = \tilde{R}_2 + (f_2, \partial_t w).\end{aligned} \tag{5.31}$$

We take the inner product of (5.10) with ∇w to obtain

$$\begin{aligned}&(\partial_t G, \nabla w) - \|\nabla w\|_{L^2}^2 + (\overline{v}^1\partial_{x_1} G, \nabla w) \\ &\quad +(w^3\partial_{x_3}^2\overline{\psi}, \partial_{x_3} w^1) - (\nabla w\overline{E}, \nabla w) - (\nabla\overline{v}G, \nabla w) = (f_3, \nabla w).\end{aligned}$$

Since $(\partial_t G, w) = \frac{d}{dt}(G, \nabla w) - (G, \nabla\partial_t w)$, we have

$$\begin{aligned}&(\partial_t G, \nabla w) - \|\nabla w\|_{L^2}^2 + (\overline{v}^1\partial_{x_1} G, \nabla w) \\ &\quad +(w^3\partial_{x_3}^2\overline{\psi}^1, \partial_{x_3} w^1) - (\nabla w\overline{E}, \nabla w) - (\nabla\overline{v}G, \nabla w) = (f_3, \nabla w).\end{aligned} \tag{5.32}$$

It follows from $-\gamma^2 \times (5.30) + (5.31) + \beta^2 \times (5.32)$ that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\nu\|\nabla w\|_{L^2}^2 + \tilde{\nu}\|\mathrm{div}w\|_{L^2}^2 - 2\gamma^2(\phi, \mathrm{div}w) + 2\beta^2(G, \nabla w)) + \|\partial_t w\|_{L^2}^2 \\ &\quad = \beta^2\|\nabla w\|_{L^2}^2 + \gamma^2\|\mathrm{div}w\|_{L^2}^2 + \tilde{R}_2 + \gamma^2(\overline{v}^1\partial_{x_1}\phi, \mathrm{div}w) \\ &\qquad +\beta^2(w^3\partial_{x_3}^2\overline{\psi}, \partial_{x_3} w^1) + \beta^2(\overline{v}^1\partial_{x_1} G, \nabla w) - \beta^2(\nabla w\overline{E}, \nabla w) \\ &\qquad\quad -\beta^2(\nabla\overline{v}G, \nabla w) - \gamma^2(f_1, \mathrm{div}w) + (f_2, \partial_t w) + \beta^2(f_3, \nabla w).\end{aligned} \tag{5.33}$$

We set

$$\begin{aligned}R_2 &:= \tilde{R}_2 + \gamma^2(\overline{v}^1\partial_{x_1}\phi, \mathrm{div}w) - \beta^2(w^3\partial_{x_3}^2\overline{\psi}^1, \partial_{x_3} w^1) - \beta^2(\overline{v}^1\partial_{x_1} G, \nabla w) \\ &\quad + \beta^2(\nabla\overline{v}G, \nabla w) + \beta^2(\nabla w\overline{E}, \nabla w).\end{aligned}$$

We then see that (5.33) is rewritten as

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\nu\|\nabla w\|_{L^2}^2 + \tilde{\nu}\|\mathrm{div}w\|_{L^2}^2 - 2\gamma^2(\phi, \mathrm{div}w) + 2\beta^2(G, \nabla w)) + \|\partial_t w\|_{L^2}^2 \\ &= \gamma^2\|\mathrm{div}w\|_{L^2}^2 + \beta^2\|\nabla w\|_{L^2}^2 + R_2 + N_3^0.\end{aligned}$$

Since $\gamma^2\|\mathrm{div}w\|_{L^2}^2 + \beta^2\|\nabla w\|_{L^2}^2 \le \left(\frac{\beta^2}{\nu} + \frac{\gamma^2}{\tilde{\nu}}\right)D_0^0$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\nu\|\nabla w\|_{L^2}^2 + \tilde{\nu}\|\mathrm{div}w\|_{L^2}^2 - 2\gamma^2(\phi, \mathrm{div}w) + 2\beta^2(G, \nabla w)) + \|\partial_t w\|_{L^2}^2 \\ &\le \left(\frac{\beta^2}{\nu} + \frac{\gamma^2}{\tilde{\nu}}\right)D_0^0 + R_2 + N_3^0.\end{aligned} \tag{5.34}$$

By using the Schwartz inequality, R_2 is estimated as

$$\begin{aligned}
R_2 \le & \frac{1}{2}\|\partial_t w\|_{L^2}^2 + \left(\frac{1}{2} + \frac{C}{\nu} + \frac{C}{\nu^3}\right) D_0^0 \\
& + C\left(1 + \frac{\gamma^4}{\nu^2\tilde{\nu}}\right)\|\partial\phi\|_{L^2}^2 + C\|\partial_{x_3}\phi\|_{L^2}^2 + C\left(1 + \frac{\beta^4}{\nu^3}\right)(\|G\|_{L^2}^2 + \|\partial G\|_{L^2}^2).
\end{aligned} \tag{5.35}$$

By combining (5.34) and (5.35), we obtain

$$\begin{aligned}
& \frac{1}{2}\frac{d}{dt}(\nu\|\nabla w\|_{L^2}^2 + \tilde{\nu}\|\mathrm{div} w\|_{L^2}^2 - 2\gamma^2(\phi, \mathrm{div} w) + 2\beta^2(G, \nabla w)) + \frac{1}{2}\|\partial_t w\|_{L^2}^2 \\
& \le C\Big(\frac{\beta^2}{\nu} + \frac{\gamma^2}{\tilde{\nu}} + 1 + \frac{1}{\nu} + \frac{1}{\nu^3}\Big) D_0^0 \\
& \quad + C\left(1 + \frac{\gamma^4}{\nu^2\tilde{\nu}}\right)\|\partial\phi\|_{L^2}^2 + C\|\partial_{x_3}\phi\|_{L^2}^2 \\
& \qquad + C\Big(1 + \frac{\beta^4}{\nu^3}\Big)(\|G\|_{L^2}^2 + \|\partial G\|_{L^2}^2) + N_3^0.
\end{aligned}$$

Similarily the following estimate holds

$$\begin{aligned}
& \frac{d}{dt}(\nu\|\nabla\partial w\|_{L^2}^2 + \tilde{\nu}\|\mathrm{div}\partial w\|_{L^2}^2 - 2\gamma^2(\partial\phi, \mathrm{div}\partial w) + 2\beta^2(\partial G, \nabla\partial w)) + \|\partial_t\partial w\|_{L^2}^2 \\
& \le C\Big(\|\partial\phi\|_{L^2}^2 + \frac{\gamma^4}{\nu^2\tilde{\nu}}\|\partial^2\phi\|_{L^2}^2\Big) + C\Big(\frac{\beta^2}{\nu} + \frac{\gamma^2}{\tilde{\nu}} + 1 + \frac{1}{\nu} + \frac{1}{\nu^3}\Big) D_0^1 \\
& \quad + C\Big(1 + \frac{\beta^4}{\nu^3}\Big)(\|\partial G\|_{L^2}^2 + \|\partial^2 G\|_{L^2}^2) + N_3^1.
\end{aligned}$$

We thus obtain

$$\begin{aligned}
& \frac{d}{dt}E_3 + D_3 \\
& \le C\Big(\|\partial_{x_3}\phi\|_{L^2}^2 + \|\partial\phi\|_{L^2}^2\Big) + \frac{C\gamma^4}{\nu^2\tilde{\nu}}\sum_{j=1}^{2}\|\partial^j\phi\|_{L^2}^2 \\
& \quad + C\Big(\frac{\beta^2}{\nu} + \frac{\gamma^2}{\tilde{\nu}} + 1 + \frac{1}{\nu} + \frac{1}{\nu^3}\Big) D_0 + C\Big(1 + \frac{\beta^4}{\nu^3}\Big)\cdot\frac{1}{\beta^2} D_1 + CN_3 \\
& \le C\|\partial_{x_3}\phi\|_{L^2}^2 + C\Big(\frac{\nu + \tilde{\nu}}{\beta^2 + \gamma^2} + \frac{\gamma^4(\nu + \tilde{\nu})}{\nu^2\tilde{\nu}(\beta^2 + \gamma^2)} \\
& \quad + \frac{\beta^2}{\nu} + \frac{\gamma^2}{\tilde{\nu}} + 1 + \frac{1}{\nu} + \frac{1}{\nu^3} + \frac{1}{\beta^2} + \frac{\beta^2}{\nu^3}\Big) D_2 + CN_3.
\end{aligned}$$

This completes the proof. □

We estimate $\partial_{x_3}\phi$. We define $\dot{\phi}$ by

$$\dot{\phi} := \partial_t\phi + (\overline{v} + w) \cdot \nabla\phi.$$

We see from (5.8) that

$$\dot{\phi} = -\mathrm{div} w + \tilde{f}_1, \tag{5.36}$$

where $\tilde{f}_1 = -\phi \mathrm{div} w$. We next set

$$E_4 := \sum_{j=0}^{1} E_4^j, D_4 := \sum_{j=0}^{1} D_4^j, N_4 := \sum_{j=0}^{1} N_4^j,$$

where

$$\begin{aligned}
\tilde{D}_3 &= \frac{1}{64} D_2 + \frac{1}{\beta^2 + \gamma^2} D_3, \\
E_4^j &:= \|\partial^j \partial_{x_3}\phi\|_{L^2}^2, \\
D_4^j &:= \frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}} \|\partial^j \partial_{x_3}\phi\|_{L^2}^2 + b_0 \frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2} (\|\partial^j \dot{\phi}\|_{L^2}^2 + \|\partial^j \partial_{x_3}\dot{\phi}\|_{L^2}^2), \\
N_4^j &:= |(\partial^j \tilde{f}_6, \partial^j \partial_{x_3}\phi)| + \frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2} \|\partial^j \tilde{f}_1\|_{L^2}^2 + \frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2} \|\partial^j \tilde{f}_1\|_{L^2}^2 \\
&\quad + \frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2} \|\partial^j f_6\|_{L^2}^2.
\end{aligned}$$

Here $b_0 > 0$ is a constant independent of ν, $\tilde{\nu}$, γ^2, β^2, and $\tilde{f}_6 := \frac{1}{\nu+\tilde{\nu}} f_6 - \partial_{x_3}((\overline{v} + w) \cdot \nabla\phi)$, $f_6 := (\tilde{f}_2)^3 + \beta^2 (f_4)^3 + (\nu + \tilde{\nu}) \partial_{x_3} \tilde{f}_1$.

Proposition 5.10. *It holds the following inequality:*

$$\frac{d}{dt} E_4 + D_4 \leq \frac{1}{2} \tilde{D}_3 + C N_4.$$

Proof. It follows from (5.36) that

$$\sum_{j=0}^{1} \|\partial^j \dot{\phi}\|_{L^2}^2 \leq \frac{C}{\tilde{\nu}} D_0 + C \|\tilde{f}_1\|_{H^2}^2. \tag{5.37}$$

and

$$\partial_{x_3}^2 w^3 = -\partial_{x_3}\dot{\phi} - \partial_{x_3}(\nabla' \cdot w') + \partial_{x_3} \tilde{f}_1. \tag{5.38}$$

We see from the 3rd equation of (5.9) that

$$\begin{aligned}
&-(\nu+\tilde{\nu}) \partial_{x_3}^2 w^3 + \gamma^2 \partial_{x_3}\phi - \beta^2 \partial_{x_3} G^{33} \\
&\quad = -\partial_t w^3 + \nu \Delta' w^3 + \tilde{\nu} \partial_{x_3} (\nabla' \cdot w') + \beta^2 (\partial_{x_1} G^{31} + \partial_{x_2} G^{32}) \\
&\quad -\overline{v}^1 \partial_{x_1} w^3 + \beta^2 \partial_{x_3} \overline{\psi}^1 \partial_{x_1} G^{33} + (f_2)^3.
\end{aligned} \tag{5.39}$$

We see from (5.11) that

$$\partial_{x_3}\phi + \partial_{x_3}G^{33} = -\partial_{x_1}G^{13} - \partial_{x_2}G^{23} - \partial_{x_3}\overline{\psi}^1\partial_{x_1}\phi + (f_4)^3. \tag{5.40}$$

By $(\nu+\tilde{\nu})\times(5.38)+(5.39)+\beta^2\times(5.40)$ to eliminate $\partial_{x_3}^2 w^3$ and $\partial_{x_3}G^{33}$, we obtain

$$\begin{aligned}
&(\gamma^2+\beta^2)\partial_{x_3}\phi + (\nu+\tilde{\nu})\partial_{x_3}\dot{\phi} \\
&= -\partial_t w^3 + \nu\Delta' w^3 + \tilde{\nu}\partial_{x_3}(\nabla'\cdot w') + \beta^2(\partial_{x_1}(G^{31}-G^{13}) + \partial_{x_2}(G^{32}-G^{23})) \\
&\quad -\beta^2\partial_{x_3}\overline{\psi}^1\partial_{x_1}\phi + \beta^2\partial_{x_3}\overline{\psi}^1\partial_{x_1}G^{33} - \overline{v}^1\partial_{x_1}w^3 + f_6.
\end{aligned} \tag{5.41}$$

We take the inner product of (5.41) with $\partial_{x_3}\phi$ to obtain

$$\begin{aligned}
&(\gamma^2+\beta^2)\|\partial_{x_3}\phi\|_{L^2}^2 + (\nu+\tilde{\nu})(\partial_{x_3}\dot{\phi}, \partial_{x_3}\phi) \\
&= -(\partial_t w^3, \partial_{x_3}\phi) + \nu(\Delta' w^3, \partial_{x_3}\phi) + \tilde{\nu}(\partial_{x_3}(\nabla'\cdot w'), \partial_{x_3}\phi) \\
&\quad -\beta^2(\partial_{x_3}\overline{\psi}^1\partial_{x_1}\phi, \partial_{x_3}\phi) + \beta^2(\partial_{x_3}\overline{\psi}^1\partial_{x_1}G^{33}, \partial_{x_3}\phi) - (\overline{v}^1\partial_{x_1}w^3, \partial_{x_3}\phi) \\
&\qquad +\beta^2(\partial_{x_1}(G^{31}-G^{13}) + \partial_{x_2}(G^{32}-G^{23}), \partial_{x_3}\phi) + (f_6, \partial_{x_3}\phi).
\end{aligned} \tag{5.42}$$

By the definition of $\dot{\phi}$, the left hand side of (5.42) is calculated as

$$\begin{aligned}
&(\gamma^2+\beta^2)\|\partial_{x_3}\phi\|_{L^2}^2 + (\nu+\tilde{\nu})(\partial_{x_3}\dot{\phi}, \partial_{x_3}\phi) \\
&= \frac{1}{2}(\nu+\tilde{\nu})\frac{d}{dt}\|\partial_{x_3}\phi\|_{L^2}^2 + (\gamma^2+\beta^2)\|\partial_{x_3}\phi\|_{L^2}^2 \\
&\quad + (\nu+\tilde{\nu})(\partial_{x_3}((\overline{v}+w)\cdot\nabla\phi), \partial_{x_3}\phi).
\end{aligned}$$

On the other hand, the right hand side of (5.42) is estimated as

$$\begin{aligned}
&-(\partial_t w^3, \partial_{x_3}\phi) + \nu(\Delta' w^3, \partial_{x_3}\phi) + \tilde{\nu}(\partial_{x_3}(\nabla'\cdot w'), \partial_{x_3}\phi) - \beta^2(\partial_{x_3}\overline{\psi}^1\partial_{x_1}\phi, \partial_{x_3}\phi) \\
&\quad +\beta^2(\partial_{x_3}\overline{\psi}^1\partial_{x_1}G^{33}, \partial_{x_3}\phi) - (\overline{v}\partial_{x_1}w^3, \partial_{x_3}\phi) \\
&\qquad +\beta^2(\partial_{x_1}(G^{31}-G^{13}) + \partial_{x_2}(G^{32}-G^{23}), \partial_{x_3}\phi) + (f_6, \partial_{x_3}\phi) \\
&\le \frac{\beta^2+\gamma^2}{2}\|\partial_{x_3}\phi\|_{L^2}^2 + \frac{4}{\beta^2+\gamma^2}\|\partial_t w^3\|_{L^2}^2 + \frac{C\nu^2}{\beta^2+\gamma^2}\|\nabla\partial w\|_{L^2}^2 \\
&\quad +\frac{C}{\beta^2+\gamma^2}\|\partial\phi\|_{L^2}^2 + \frac{C}{\beta^2+\gamma^2}(\beta^2+1)\|\partial G\|_{L^2}^2 \\
&\qquad +\frac{C}{\nu^2(\beta^2+\gamma^2)}\|\nabla w\|_{L^2}^2 + (f_6, \partial_{x_3}\phi).
\end{aligned}$$

Hence we obtain

$$\frac{1}{2}(\nu+\tilde{\nu})\frac{d}{dt}\|\partial_{x_3}\phi\|_{L^2}^2+\frac{\beta^2+\gamma}{2}\|\partial_{x_3}\phi\|_{L^2}^2+(\nu+\tilde{\nu})(\partial_{x_3}((\overline{v}+w)\cdot\nabla\phi),\partial_{x_3}\phi)$$
$$\leq\frac{4}{\beta^2+\gamma^2}\|\partial_t w^3\|_{L^2}^2+\frac{C\nu^2}{\beta^2+\gamma^2}\|\nabla\partial w\|_{L^2}^2+\frac{C}{\beta^2+\gamma^2}\|\partial\phi\|_{L^2}^2$$
$$+\frac{C}{\beta^2+\gamma^2}(\beta^2+1)\|\partial G\|_{L^2}^2+\frac{C}{\nu^2(\beta^2+\gamma^2)}\|\nabla w\|_{L^2}^2+(f_6,\partial_{x_3}\phi).$$

Dividing this inequality by $\nu+\tilde{\nu}$, we have

$$\frac{1}{2}\frac{d}{dt}\|\partial_{x_3}\phi\|_{L^2}^2+\frac{1}{2}\frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\|\partial_{x_3}\phi\|_{L^2}^2$$
$$\leq\frac{C}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}(\|\partial_t w^3\|_{L^2}^2+\nu^2\|\nabla\partial w\|_{L^2}^2+\|\partial\phi\|_{L^2}^2$$
$$+\beta^2\|\partial G\|_{L^2}^2+\frac{1}{\nu^2}\|\nabla w\|_{L^2}^2)+(\tilde{f}_6,\partial_{x_3}\phi),$$

Similarly, we obtain the following inequality

$$\frac{d}{dt}\|\partial\partial_{x_3}\phi\|_{L^2}^2+\frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\|\partial\partial_{x_3}\phi\|_{L^2}^2$$
$$\leq\frac{C}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}(\|\partial_t\partial w\|_{L^2}^2+\nu^2\|\nabla\partial^2 w\|_{L^2}^2+\frac{1}{\nu^2}\|\nabla\partial w\|_{L^2}^2+\|\partial^2\phi\|_{L^2}^2$$
$$+(\beta^2+1)\|\nabla\partial^2\zeta\|_{L^2}^2)+(\partial\tilde{f}_6,\partial\partial_{x_3}\phi).$$

By $\frac{1}{\sqrt{(\nu+\tilde{\nu})(\beta^2+\gamma^2)}}\times$ (5.41), we have

$$\sqrt{\frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}}\partial_{x_3}\dot{\phi}+\sqrt{\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}}\partial_{x_3}\phi$$
$$=\frac{1}{\sqrt{(\nu+\tilde{\nu})(\beta^2+\gamma^2)}}(-\partial_t w^3+\nu\Delta' w^3+\tilde{\nu}\partial_{x_3}(\nabla'\cdot w')$$
$$+\beta^2(\partial_{x_1}(G^{31}-G^{13})+\partial_{x_2}(G^{32}-G^{23}))$$
$$-\beta^2\partial_{x_3}\overline{\psi}^1\partial_{x_1}\phi+\beta^2\partial_{x_3}\overline{\psi}^1\partial_{x_1}G^{33}-\overline{v}^1\partial_{x_1}w^3+f_6)$$

We thus obtain

$$
\begin{aligned}
&\frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\|\partial_{x_3}\dot{\phi}\|_{L^2}^2 \\
&\leq C\left(\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\|\partial_{x_3}\phi\|_{L^2}^2+\frac{1}{(\nu+\tilde{\nu})(\beta^2+\gamma^2)}\|\partial_t w\|_{L^2}^2\right. \\
&\quad+\left(1+\frac{1}{\nu^2(\nu+\tilde{\nu})^2}\right)\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\|\nabla\partial w\|_{L^2}^2 \\
&\qquad+\frac{\beta^4+1}{(\nu+\tilde{\nu})(\beta^2+\gamma^2)}\|\partial G\|_{L^2}^2+\frac{1}{(\nu+\tilde{\nu})(\beta^2+\gamma^2)}(\|\partial\phi\|_{L^2}^2 \\
&\qquad\quad\left.+\|\nabla w\|_{L^2}^2+\frac{1}{(\nu+\tilde{\nu})(\beta^2+\gamma^2)}\|f_6\|_{L^2}^2\right).
\end{aligned}
\tag{5.43}
$$

It holds the following estimate by adding (5.43)

$$
\begin{aligned}
&\frac{d}{dt}E_4+D_4 \\
&\leq \frac{C}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}\sum_{j=0}^{1}(\|\partial_t\partial^j w\|_{L^2}^2+\nu^2\|\nabla\partial^{j+1}w\|_{L^2}^2 \\
&\quad+\frac{1}{\nu^2}\|\nabla\partial^j w\|_{L^2}^2+\|\partial^{j+1}\phi\|_{L^2}^2+(\beta^2+1)\|\nabla\partial^{j+1}\zeta\|_{L^2}^2)+N_4 \\
&\leq \frac{C}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}\Big((\beta^2+\gamma^2)+\nu+\frac{1}{\nu^3}+\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}+\Big(1+\frac{1}{\beta^2}\Big)\Big)\tilde{D}_3+CN_4 \\
&\leq \frac{1}{2}\tilde{D}_3+CN_4.
\end{aligned}
$$

This completes the proof. □

We estimate $\partial_{x_3}^2\phi$. We set $\tilde{D}_4:=\tilde{D}_3+D_4$.

Proposition 5.11. *It holds the following inequality:*

$$
\begin{aligned}
&\frac{d}{dt}\|\partial_{x_3}^2\phi\|_{L^2}^2+\frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\|\partial_{x_3}^2\phi\|_{L^2}^2+b_0\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\|\partial_{x_3}^2\dot{\phi}\|_{L^2}^2 \\
&\leq \frac{C}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}\|\partial_t\nabla w\|_{L^2}^2+C\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\|\nabla^2\partial_{x_3}w\|_{L^2}^2 \\
&\quad+C\frac{\beta^4+1}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}\|\partial\partial_{x_3}G\|_{L^2}^2+C\left(\frac{1}{\nu+\tilde{\nu}}+\frac{1}{\beta^2+\gamma^2}\right)\tilde{D}_4+|(\partial_{x_3}\tilde{f}_6,\partial_{x_3}^2\phi)|.
\end{aligned}
$$

Proof. By applying $\frac{1}{\nu+\tilde{\nu}}\partial_{x_3}$ to (5.41), we have

$$
\begin{aligned}
&\partial_{x_3}^2\dot{\phi}+\frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\partial_{x_3}^2\phi\\
&=\frac{1}{\nu+\tilde{\nu}}\big(-\partial_t\partial_{x_3}w^3+\nu\partial_{x_3}\Delta' w^3+\tilde{\nu}\partial_{x_3}^2(\nabla'\cdot w')\\
&\quad+\beta^2(\partial_{x_1}\partial_{x_3}(G^{31}-G^{13})+\partial_{x_2}\partial_{x_3}(G^{32}-G^{23}))\\
&\qquad-\beta^2\partial_{x_3}(\partial_{x_3}\overline{\psi}^1\partial_{x_1}\phi)+\beta^2\partial_{x_3}(\partial_{x_3}\overline{\psi}^1\partial_{x_1}G^{33})-\partial_{x_3}(\overline{v}^1\partial_{x_1}w^3)+\partial_{x_3}f_6\big),
\end{aligned}
\tag{5.44}
$$

We take the inner product of $\partial_{x_3}^2\phi$ with (5.44) to obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial_{x_3}^2\phi\|_{L^2}^2+\frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\|\partial_{x_3}^2\phi\|_{L^2}^2+(\partial_{x_3}^2((\overline{v}+w)\cdot\nabla\phi),\partial_{x_3}^2\phi)\\
&=\frac{1}{\nu+\tilde{\nu}}\big(-\partial_t\partial_{x_3}w^3+\nu\partial_{x_3}\Delta' w^3+\tilde{\nu}\partial_{x_3}^2(\nabla'\cdot w')\\
&\quad+\beta^2(\partial_{x_1}\partial_{x_3}(G^{31}-G^{13})+\partial_{x_2}\partial_{x_3}(G^{32}-G^{23}))\\
&\qquad-\beta^2\partial_{x_3}(\partial_{x_3}\overline{\psi}^1\partial_{x_1}\phi)+\beta^2\partial_{x_3}(\partial_{x_3}\overline{\psi}^1\partial_{x_1}G^{33})\\
&\qquad\quad-\partial_{x_3}(\overline{v}^1\partial_{x_1}w^3)+\partial_{x_3}f_6,\partial_{x_3}^2\phi).
\end{aligned}
\tag{5.45}
$$

We have the estimate for the right hand side of (5.45) in a similar manner to Proposition 5.11

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial_{x_3}^2\phi\|_{L^2}^2+\frac{1}{2}\frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\|\partial_{x_3}^2\phi\|_{L^2}^2+(\partial_{x_3}^2((\overline{v}+w)\cdot\nabla\phi),\partial_{x_3}^2\phi)\\
&\le\frac{C}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}\|\partial_t\nabla w\|_{L^2}^2+C\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\|\nabla^2\partial_{x_3}w\|_{L^2}^2\\
&\quad+C\frac{\beta^4+1}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}\|\partial\partial_{x_3}G\|_{L^2}^2+C\left(\frac{1}{\nu+\tilde{\nu}}+\frac{1}{\beta^2+\gamma^2}\right)\tilde{D}_4+|(\partial_{x_3}\tilde{f}_6,\partial_{x_3}^2\phi)|.
\end{aligned}
$$

This completes the proof. □

We next estimate higher order derivatives of w and G. We set

$$
\begin{aligned}
E_5&:=E_2+\frac{1}{\beta^2+\gamma^2}E_3+E_4+\|\partial_{x_3}^2\phi\|_{L^2}^2,\\
D_5&:=\tilde{D}_4+\frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\|\partial_{x_3}^2\phi\|_{L^2}^2,\\
N_5&:=N_2+\frac{1}{\beta^2+\gamma^2}N_3+N_4+|(\partial_{x_3}\tilde{f}_6,\partial_{x_3}^2\phi)|,\\
\tilde{D}_5&:=D_5+\frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\|\partial_{x_3}^2\phi\|_{L^2}^2+\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\|\nabla\dot{\phi}\|_{H^1}^2.
\end{aligned}
$$

Proposition 5.12. *There exists a positive number $\beta_4^2 > 0$ such that if $\beta^2 > \beta_4^2$, then it holds the following inequality:*

$$\begin{aligned}
&\frac{d}{dt}\|\partial_{x_3} G\|_{L^2}^2 + \frac{\beta^2}{\nu}\|\partial_{x_3} G\|_{L^2}^2 + b_1 \frac{\nu}{\beta^2}\|\nabla^2 w\|_{L^2}^2 \\
&\le \frac{C}{\nu\beta^2}\|\nabla q\|_{H^1}^2 + C\left(\frac{1}{\nu\beta^4} + \frac{1}{\beta^6} + \frac{1}{\nu\beta^8}\right)\tilde{D}_5 + \left|\left(\partial_{x_3} f_3 + \frac{\beta^2}{\nu}\partial_{x_3} h_1, \partial_{x_3} G\right)\right|.
\end{aligned}$$

Proof. We first introduce the following quantity

$$q = \nu w + \beta^2 \zeta.$$

Since $w = \frac{1}{\nu} q - \frac{\beta^2}{\nu}\zeta$, we have

$$\begin{aligned}
\nabla w &= \frac{1}{\nu}\nabla q - \frac{\beta^2}{\nu}\nabla\zeta \\
&= \frac{1}{\nu}\nabla q - \frac{\beta^2}{\nu} G + \frac{\beta^2}{\nu}(\overline{E}\nabla\zeta + \nabla\zeta\overline{E} + \overline{E}\nabla\zeta\overline{E}) + \frac{\beta^2}{\nu} h_1.
\end{aligned}$$

By inserting this to (5.10), we obtain

$$\begin{aligned}
\partial_t G + \frac{\beta^2}{\nu} G = \frac{1}{\nu}\nabla q - w^3 \partial_{x_3}\overline{E} - \overline{v}^1 \partial_{x_1} G + \nabla w \overline{E} + \nabla \overline{v} G \\
+ \frac{\beta^2}{\nu}(\overline{E}\nabla\zeta + \nabla\zeta\overline{E} + \overline{E}\nabla\zeta\overline{E}) + \frac{\beta^2}{\nu} h_1 + f_3.
\end{aligned} \tag{5.46}$$

By applying ∂_{x_3} to (5.46) and taking the inner product of $\partial_{x_3} G$, it holds the following identity:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial_{x_3} G\|_{L^2}^2 + \frac{\beta^2}{\nu}\|\partial_{x_3} G\|_{L^2}^2 \\
&= \frac{1}{\nu}(\nabla\partial_{x_3} q, \partial_{x_3} G) - (\partial_{x_3}(w^3 \partial_{x_3}^2 \overline{\psi}^1), \partial_{x_3} G^{13}) - (\partial_{x_3}(\overline{v}^1 \partial_{x_1} G), \partial_{x_3} G) \\
&\quad + (\partial_{x_3}(\nabla w \overline{E}), \partial_{x_3} G) + (\partial_{x_3}(\nabla \overline{v} G), \partial_{x_3} G) + \frac{\beta^2}{\nu}(\partial_{x_3}(\overline{E}\nabla\zeta), \partial_{x_3} G) \\
&\quad + \frac{\beta^2}{\nu}(\partial_{x_3}(\nabla\zeta\overline{E}), \partial_{x_3} G) + \frac{\beta^2}{\nu}(\partial_{x_3}(\overline{E}\nabla\zeta\overline{E}), \partial_{x_3} G) \\
&\quad + \left(\partial_{x_3} f_3 + \frac{\beta^2}{\nu}\partial_{x_3} h_1, \partial_{x_3} G\right).
\end{aligned} \tag{5.47}$$

The right-hand side of (5.47) is estimated by

$$\frac{\beta^2}{\nu}\left(\frac{1}{4} + \frac{C}{\beta^2}\right)\|\partial_{x_3} G\|_{L^2}^2 + \frac{C}{\nu\beta^2}\|\nabla q\|_{H^1}^2$$

$$+ C\Big(\frac{1}{\nu\beta^4}+\frac{1}{\beta^6}+\frac{1}{\nu\beta^8}\Big)\tilde{D}_5 + \Big|\Big(\partial_{x_3} f_3 + \frac{\beta^2}{\nu}\partial_{x_3} h_1, \partial_{x_3} G\Big)\Big|.$$

We thus obtain

$$\begin{aligned}
&\frac{d}{dt}\|\partial_{x_3} G\|_{L^2}^2 + \frac{\beta^2}{\nu}\|\partial_{x_3} G\|_{L^2}^2 \\
&\leq \frac{C}{\nu\beta^2}\|\nabla q\|_{H^1}^2 + C\Big(\frac{1}{\nu\beta^4}+\frac{1}{\beta^6}+\frac{1}{\nu\beta^8}\Big)\tilde{D}_5 + \Big|\Big(\partial_{x_3} f_3 + \frac{\beta^2}{\nu}\partial_{x_3} h_1, \partial_{x_3} G\Big)\Big|.
\end{aligned}$$

This completes the proof. □

Proposition 5.13. *It holds the following inequality:*

$$\begin{aligned}
\|\nabla q\|_{H^1}^2 \leq C\Big(&\beta^2+\gamma^2+\frac{(\beta^2+\gamma^2)\tilde{\nu}^2}{\nu+\tilde{\nu}}+\frac{\gamma^2(\nu+\tilde{\nu})}{\beta^2} \\
&+\nu+\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}+\frac{1}{\nu^3}+\frac{1}{\beta^2}\Big)\tilde{D}_5 + C\|\partial_{x_3} G\|_{L^2}^2 \\
&+ C(\tilde{\nu}^2\|\tilde{f}_1\|_{H^1}^2 + \|f_2\|_{L^2}^2 + \beta^2\|h_1\|_{H^1}^2).
\end{aligned}$$

Proof. By simple calculation, we have

$$\begin{aligned}
\|\nabla q\|_{H^1}^2 &= \|\nabla q\|_{L^2}^2 + \|\nabla\partial q\|_{L^2}^2 + \|\partial_{x_3}^2 q\|_{L^2}^2 \\
&\leq C(\nu+\beta^2)D_6 + \|\partial_{x_3}^2 q\|_{L^2}^2.
\end{aligned}$$

By using $\dot{\phi}$ and q, it follows from (5.9) that

$$\begin{aligned}
&\partial_t w - \Delta q + \tilde{\nu}\nabla\dot{\phi} + \gamma^2\nabla\phi + \overline{v}^1\partial_{x_1} w + (w^3\partial_{x_3}\overline{v}^1)\boldsymbol{e}_1 \\
&\quad -\nu(\phi\partial_{x_3}^2\overline{v}^1)\boldsymbol{e}_1 - \beta^2\mathrm{div}(G^\top\overline{E}) - \beta^2 G^{33}(\partial_{x_3}\overline{\psi}^1)\boldsymbol{e}_1 \\
&\quad -\beta^2\mathrm{div}(\overline{E}\nabla\zeta + \nabla\zeta\overline{E} + \overline{E}\nabla\zeta\overline{E}) \\
&= -\tilde{\nu}\nabla\tilde{f}_1 + f_2 + \beta^2\mathrm{div}h_1,
\end{aligned} \tag{5.48}$$

which gives

$$\begin{aligned}
&\|\partial_{x_3}^2 q\|_{L^2}^2 \\
&\leq C(\|\partial_t w\|_{L^2}^2 + \tilde{\nu}^2\|\nabla\dot{\phi}\|_{L^2}^2 + (\gamma^4+1)\|\nabla\phi\|_{L^2}^2 + \|\partial^2 q\|_{L^2}^2 + \frac{1}{\nu^2}\|\nabla w\|_{L^2}^2 \\
&\quad + \|G\|_{L^2}^2 + \|\partial G\|_{L^2}^2 + \|\nabla\zeta\|_{H^1}^2 + \tilde{\nu}^2\|\tilde{f}_1\|_{H^1}^2 + \|f_2\|_{L^2}^2 + \beta^2\|h_1\|_{H^1}^2) \\
&\leq C\Big(\beta^2+\gamma^2+\frac{(\beta^2+\gamma^2)\tilde{\nu}^2}{\nu+\tilde{\nu}}+\frac{\gamma^2(\nu+\tilde{\nu})}{\beta^2}+\nu+\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}+\frac{1}{\nu^3}+\frac{1}{\beta^2}\Big)\tilde{D}_5 \\
&\quad + C\|\partial_{x_3} G\|_{L^2}^2 + C(\tilde{\nu}^2\|\tilde{f}_1\|_{H^1}^2 + \|f_2\|_{L^2}^2 + \beta^2\|h_1\|_{H^1}^2).
\end{aligned}$$

This completes the proof. □

We introduce the following quantities to estimate $\nabla^2 G$

$$
\begin{aligned}
E_6 &:= E_5 + \frac{1}{\tilde{\nu}(\beta^2+\gamma^2)}\|\partial_{x_3}G\|_{L^2}^2, \\
D_6 &:= \tilde{D}_5 + \frac{\beta^2}{\nu\tilde{\nu}(\beta^2+\gamma^2)}\|\partial_{x_3}G\|_{L^2}^2, \\
N_6 &:= N_5 + \frac{1}{\tilde{\nu}(\beta^2+\gamma^2)}\Big(\frac{\tilde{\nu}^2}{\nu\beta^2}\|\tilde{f}_1\|_{H^1}^2 + \frac{1}{\nu\beta^2}\|f_2\|_{L^2}^2 + \frac{1}{\nu}\|h_1\|_{H^1}^2 \\
&\qquad + \left|\left(\nabla^2\left(f_3 + \frac{\beta^2}{\nu}h_1\right), \nabla^2 G\right)\right|\Big).
\end{aligned}
$$

Proposition 5.14. *It holds the following inequality:*

$$
\begin{aligned}
&\frac{d}{dt}\|\nabla^2 G\|_{L^2}^2 + \frac{\beta^2}{\nu}\|\nabla^2 G\|_{L^2}^2 \\
&\le \frac{C}{\nu\beta^2}\|\partial_t\nabla w\|_{L^2}^2 + \frac{C\nu}{\beta^6}\|\nabla^2 w\|_{H^1}^2 + C\Big(\frac{1}{\nu} + \frac{1}{\beta^2} + \frac{\tilde{\nu}}{\beta^2} + 1 + \frac{\tilde{\nu}}{\nu}\Big)D_6 \\
&\quad + C\Big(\left|\left(\nabla^2\left(f_3 + \frac{\beta^2}{\nu}h_1\right), \nabla^2 G\right)\right| + \frac{\nu+\tilde{\nu}}{\nu\beta^2}\|\tilde{f}_1\|_{H^2}^2 \\
&\quad + \frac{1}{\nu\beta^2}\|f_2\|_{H^1}^2 + \frac{\beta^2}{\nu}\|f_4\|_{H^1}^2 + \frac{\beta^2}{\nu}\|h_1\|_{H^2}^2\Big).
\end{aligned}
$$

Proof. By applying ∇^2 to (5.46) and taking the inner product of $\nabla^2 G$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla^2 G\|_{L^2}^2 + \frac{\beta^2}{\nu}\|\nabla^2 G\|_{L^2}^2 \\
&\le \frac{\beta^2}{2\nu}\|\nabla^2 G\|_{L^2}^2 + C\Big(\frac{1}{\nu\beta^2}\|\nabla^3 q\|_{L^2}^2 + \frac{1}{\nu\beta^2}\|G\|_{H^2}^2 + \frac{\nu}{\beta^6}\|\nabla w\|_{H^2}^2\Big) \\
&\quad + |(\nabla^2(f_3 + \frac{\beta^2}{\nu}h_1), \nabla^2 G)|.
\end{aligned} \tag{5.49}
$$

The estimates of each terms are given as follows

$$
\begin{aligned}
\frac{1}{\nu\beta^2}\|G\|_{H^2}^2 &= \frac{1}{\nu\beta^2}\|\nabla^2 G\|_{L^2}^2 + \frac{1}{\nu\beta^2}(\|\partial_{x_3}G\|_{L^2}^2 + \|\partial G\|_{L^2}^2) + \frac{1}{\nu\beta^2}\|G\|_{L^2}^2 \\
&\le \frac{1}{\nu\beta^2}\|\nabla^2 G\|_{L^2}^2 + C\Big(\frac{1}{\nu\beta^4} + \frac{\tilde{\nu}}{\beta}\cdot\frac{\beta^2+\gamma^2}{\beta^2}\Big)D_6 \\
&\le \frac{1}{\nu\beta^2}\|\nabla^2 G\|_{L^2}^2 + C\Big(\frac{1}{\nu\beta^4} + \frac{\tilde{\nu}}{\beta^2}\Big)D_6.
\end{aligned}
$$

Since

$$
\|\nabla w\|_{H^2}^2 = \|\nabla^3 w\|_{L^2}^2 + \|\nabla^2 w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2
$$

$$\leq \frac{C}{\nu} D_6 + \|\nabla^2 w\|_{L^2}^2 + \|\nabla^3 w\|_{L^2}^2.$$

we need the estimate for $\|\nabla^k w\|_{L^2}^2$, $k = 2, 3$ to complete the proof. We mention about the estimate for $\|\nabla^3 q\|_{L^2}^2$.

Lemma 5.15. *It holds the following inequality:*

$$\begin{aligned}
&\frac{1}{\nu\beta^2}\|\nabla^3 q\|_{L^2}^2 \\
&\leq \frac{C}{\nu\beta^2}(\|\partial_t \nabla w\|_{L^2}^2 + \|\nabla^2 G\|_{L^2}^2) \\
&+ C\Big(\frac{1}{\nu} + \frac{1}{\beta^2} + \Big(1 + \frac{\gamma^2}{\beta^2}\Big)\Big(1 + \frac{\tilde{\nu}}{\nu}\Big) \\
&+ \frac{\nu + \tilde{\nu}}{\beta^2 + \gamma^2}(\beta^4 + \gamma^2) \cdot \frac{1}{\nu\beta^2} + \frac{\tilde{\nu}(\beta^2 + \gamma^2)}{\beta^4}\Big) D_6 \\
&+ \frac{(\nu + \tilde{\nu})^2}{\nu\beta^2}\|\tilde{f}_1\|_{H^2}^2 + \frac{1}{\nu\beta^2}\|f_2\|_{H^1}^2 + \frac{\beta^2}{\nu}\|f_4\|_{H^1}^2 + \frac{\beta^2}{\nu}\|h_1\|_{H^2}^2.
\end{aligned}$$

Proof. It follows from the easy calculation that

$$\frac{1}{\nu\beta^2}\|\nabla^3 q\|_{L^2}^2 = \frac{1}{\nu\beta^2}(\|\nabla^2 \partial q\|_{L^2}^2 + \|\partial_{x_3}^3 q\|_{L^2}^2).$$

We see from (5.27), (5.36) and (5.48) that $(\partial_{x_j}\phi, \partial_{x_j} q)$ $(j = 1, 2)$ satisfies the following problem

$$\begin{cases}
\mathrm{div}\partial_{x_j} q = r_j \text{ in } \Omega, \\
-\Delta \partial_{x_j} q + \gamma^2 \nabla \partial_{x_j}\phi = s_j \text{ in } \Omega, \\
\partial_{x_j} q = 0 \text{ on } \{x_3 = 0, 1\},
\end{cases}$$

where

$$\begin{aligned}
r_j &= -\nu\partial_{x_j}\dot{\phi} - \beta^2\partial_{x_j}\phi - \beta^2\partial_{x_3}\overline{\psi}^1\partial_{x_j}\partial_{x_1}\zeta^3 - \nu\partial_{x_j}\tilde{f}_1 - \beta^2(\mathrm{div}^\top h_1)^j + \beta^2(f_4)^j, \\
s_j &= -\partial_t\partial_{x_j} w - \overline{v}^1\partial_{x_1}\partial_{x_j} w - (\partial_{x_j} w^3 \partial_{x_3}\overline{v}^1)\boldsymbol{e}_1 - \tilde{\nu}\nabla\partial_{x_j}\dot{\phi} \\
&\quad +\nu(\partial_{x_j}\phi\partial_{x_3}^2\overline{v}^1)\boldsymbol{e}_1 + \beta^2\mathrm{div}(\partial_{x_j} G^\top \overline{E}) + \beta^2(\partial_{x_j} G^{33}\partial_{x_3}\overline{\psi}^1)\boldsymbol{e}_1 \\
&\quad\quad +\beta^2\mathrm{div}(\overline{E}\nabla\partial_{x_j}\zeta + \nabla\partial_{x_j}\zeta\overline{E} + \overline{E}\nabla\partial_{x_j}\zeta\overline{E}) \\
&\quad\quad\quad -\tilde{\nu}\nabla\partial_{x_j}\tilde{f}_1 + \partial_{x_j} f_2 + \beta^2\mathrm{div}\partial_{x_j} h_1.
\end{aligned}$$

It follows from Lemma 2.2 that

$$\|\nabla^2 \partial q\|_{L^2}^2 \leq C\Big((\beta^2 + \gamma^2)(1 + \nu + \tilde{\nu}) + \frac{\nu + \tilde{\nu}}{\beta^2 + \gamma^2}(\beta^4 + 1)$$

$$
\begin{aligned}
&+\frac{1}{\nu^3}+\frac{1}{\beta^2}+\frac{\nu\tilde{\nu}(\beta^2+\gamma^2)}{\beta^2}\Big)D_6\\
&+C((\nu+\tilde{\nu})^2\|\tilde{f}_1\|_{H^2}^2+\|f_2\|_{H^1}^2+\beta^4\|f_4\|_{H^1}^2+\beta^2\|h_1\|_{H^2}^2).
\end{aligned}
$$

We see from (5.48) that

$$
\begin{aligned}
\partial_{x_3}^2 q = {}& \partial_t w - \Delta' q + \tilde{\nu}\nabla\dot{\phi} + \gamma^2\nabla\phi + \overline{v}^1\partial_{x_1}w + (w^3\partial_{x_3}\overline{v}^1)\boldsymbol{e}_1 - \nu(\phi\partial_{x_3}^2\overline{v}^1)\boldsymbol{e}_1\\
&- \beta^2\mathrm{div}(G^\top\overline{E}) - \beta^2(G^{33}\partial_{x_3}^2\overline{\psi}^1)\boldsymbol{e}_1 - \beta^2\mathrm{div}(\overline{E}\nabla\zeta + \nabla\zeta\overline{E} + \overline{E}\nabla\zeta\overline{E})\\
&+ \tilde{\nu}\nabla\tilde{f}_1 - f_2 - \beta^2\mathrm{div}h_1.
\end{aligned}
$$

By differentiating in x_3 and taking L^2-norm to this equation, we have

$$
\begin{aligned}
\|\partial_{x_3}^3 q\|_{L^2}^2 \le{}& C\Big\{\|\partial_t\nabla w\|_{L^2}^2 + \|\nabla\partial^2 q\|_{L^2}^2 + \tilde{\nu}^2\|\nabla\dot{\phi}\|_{H^1}^2 + (\gamma^4+1)\|\nabla\phi\|_{H^1}^2\\
&+ \frac{1}{\nu^2}(\|\nabla w\|_{L^2}^2 + \|\nabla\partial w\|_{L^2}^2) + \|G\|_{H^2}^2\Big\}\\
&+ C(\tilde{\nu}^2\|\tilde{f}_1\|_{H^2}^2 + \|f_2\|_{H^1}^2 + \beta^4\|h_1\|_{H^2}^2)\\
\le{}& C(\|\partial_t\nabla w\|_{L^2}^2 + \|\nabla^2 G\|_{L^2}^2)\\
&+ C\Big(\beta^2 + \nu + \tilde{\nu}^2\cdot\frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\\
&+ (\gamma^4+1)\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2} + \frac{1}{\nu^3} + \frac{1}{\beta^2}\Big)\tilde{D}_5\\
&+ C((\nu+\tilde{\nu})^2\|\tilde{f}_1\|_{H^2}^2 + \|f_2\|_{H^1}^2 + \beta^4\|h_1\|_{H^2}^2).
\end{aligned}
$$

Hence we obtain

$$
\begin{aligned}
&\frac{1}{\nu\beta^2}\|\nabla^3 q\|_{L^2}^2\\
&\le \frac{C}{\nu\beta^2}(\|\partial_t\nabla w\|_{L^2}^2 + \|\nabla^2 G\|_{L^2}^2)\\
&\quad + C\Big(\frac{1}{\nu} + \frac{1}{\beta^2} + \Big(1+\frac{\gamma^2}{\beta^2}\Big)\Big(1+\frac{\tilde{\nu}}{\nu}\Big) + \frac{(\nu+\tilde{\nu})(\beta^4+\gamma^4)}{\nu\beta^2(\beta^2+\gamma^2)}\Big)\tilde{D}_5\\
&\quad + \frac{(\nu+\tilde{\nu})^2}{\nu\beta^2}\|\tilde{f}_1\|_{H^2}^2 + \frac{1}{\nu\beta^2}\|f_2\|_{H^1}^2 + \frac{\beta^2}{\nu}\|f_4\|_{H^1}^2 + \frac{\beta^2}{\nu}\|h_1\|_{H^2}^2.
\end{aligned}
\tag{5.50}
$$

$\square$

Proof of Proposition 5.14 (continued)
We see from (5.49) and (5.50) that

$$
\frac{1}{2}\frac{d}{dt}\|\nabla^2 G\|_{L^2}^2 + \frac{\beta^2}{\nu}\|\nabla^2 G\|_{L^2}^2
$$

$$
\begin{aligned}
&\le \frac{\beta^2}{2\nu}\|\nabla^2 G\|_{L^2}^2 + C\Big(\frac{1}{\nu\beta^2}\|\nabla^3 q\|_{L^2}^2 + \frac{1}{\nu\beta^2}\|G\|_{H^2}^2 + \frac{\nu}{\beta^6}\|\nabla w\|_{H^2}^2\Big)\\
&\quad + \left|\left(\nabla^2\left(f_3 + \frac{\beta^2}{\nu}h_1\right), \nabla^2 G\right)\right|\\
&\le \frac{\beta^2}{4\nu}\|\nabla^2 G\|_{L^2}^2 + \frac{C}{\nu\beta^2}\|\nabla^2 G\|_{L^2}^2 + \frac{C}{\nu\beta^2}\|\partial_t \nabla w\|_{L^2}^2 + \frac{C\nu}{\beta^6}(\|\nabla^2 w\|_{L^2}^2 + \|\nabla^3 w\|_{L^2}^2)\\
&\quad + C\Big(\frac{1}{\nu\beta^4} + \frac{\tilde\nu}{\beta^2} + \frac{1}{\nu} + \frac{1}{\beta^2} + \Big(1 + \frac{\gamma^2}{\beta^2}\Big)\Big(1 + \frac{\tilde\nu}{\nu}\Big) + \frac{\tilde\nu(\beta^2+\gamma^2)}{\beta^4}\Big)D_6\\
&\quad\quad + C\Big(\left|\left(\nabla^2\left(f_3 + \frac{\beta^2}{\nu}h_1\right), \nabla^2 G\right)\right| + \frac{\nu+\tilde\nu}{\nu\beta^2}\|\tilde f_1\|_{H^2}^2\\
&\quad\quad\quad + \frac{1}{\nu\beta^2}\|f_2\|_{H^1}^2 + \frac{\beta^2}{\nu}\|f_4\|_{H^1}^2 + \frac{\beta^2}{\nu}\|h_1\|_{H^2}^2\Big).
\end{aligned}
$$

We take ν, β so that $\frac{C}{\nu\beta^2} \le 1 \le \frac{\beta^2}{4\nu}$. Then we obtain

$$
\begin{aligned}
&\frac{d}{dt}\|\nabla^2 G\|_{L^2}^2 + \frac{\beta^2}{\nu}\|\nabla^2 G\|_{L^2}^2\\
&\le \frac{C}{\nu\beta^2}\|\partial_t \nabla w\|_{L^2}^2 + \frac{C\nu}{\beta^6}\|\nabla^2 w\|_{H^1}^2 + C\Big(\frac{1}{\nu} + \frac{1}{\beta^2} + \frac{\tilde\nu}{\beta^2} + 1 + \frac{\tilde\nu}{\nu}\Big)D_6\\
&\quad + C\Big(\left|\left(\nabla^2\left(f_3 + \frac{\beta^2}{\nu}h_1\right), \nabla^2 G\right)\right| + \frac{\nu+\tilde\nu}{\nu\beta^2}\|\tilde f_1\|_{H^2}^2\\
&\quad\quad + \frac{1}{\nu\beta^2}\|f_2\|_{H^1}^2 + \frac{\beta^2}{\nu}\|f_4\|_{H^1}^2 + \frac{\beta^2}{\nu}\|h_1\|_{H^2}^2\Big).
\end{aligned}
$$

This completes the proof. $\square$

We set

$$
\begin{aligned}
E_7 &:= \frac{\nu(\beta^2+\gamma^2)^2}{\beta^2}E_6 + \|\nabla^2 G\|_{L^2}^2,\\
D_7 &:= \frac{\nu(\beta^2+\gamma^2)^2}{\beta^4}D_6 + \frac{\nu}{\beta^6}\|\nabla^2 w\|_{L^2}^2 + \frac{\beta^2}{C\nu}\|\nabla^2 G\|_{L^2}^2,\\
N_7 &:= \frac{\nu(\beta^2+\gamma^2)^2}{\beta^4}N_6 + |(\nabla^2(f_3 + \frac{\beta^2}{\nu}h_1), \nabla^2 G)| + \frac{(\nu+\tilde\nu)}{\nu\beta^2}\|\tilde f_1\|_{H^2}^2\\
&\quad + \frac{1}{\nu\beta^2}\|f_2\|_{H^1}^2 + \frac{\beta^2}{\nu}\|f_4\|_{H^1}^2 + \frac{\beta^2}{\nu}\|h_1\|_{H^2}^2.
\end{aligned}
$$

Proposition 5.16. *It holds the following inequality:*

$$
\frac{\nu}{\beta^2}\|\nabla^3 w\|_{L^2}^2 \le \frac{C}{\nu\beta^2}\|\partial_t \nabla w\|_{L^2}^2 + C\Big(\frac{1}{\beta^2} + \frac{1}{\nu} + \frac{\nu+\tilde\nu}{\beta^2+\gamma^2}\Big)D_7
$$

$$+ C\Big\{\frac{(\nu+\tilde\nu)^2}{\nu\beta^6}\|\tilde f_1\|_{H^2}^2 + \frac{1}{\nu\beta^6}\|f_2\|_{H^1}^2 + \frac{1}{\nu\beta^2}\|f_4\|_{H^1}^2 + \frac{1}{\nu\beta^2}\|h_1\|_{H^2}^2\Big\}.$$

Proof. It follows from $\frac{\sqrt{\nu}}{\beta}w = \frac{1}{\sqrt{\nu}\beta}q - \frac{\beta}{\sqrt{\nu}}\zeta$ that

$$\begin{aligned}
&\frac{\nu}{\beta^2}\|\nabla^3 w\|_{L^2}^2\\
&\le C\Big(\frac{1}{\nu\beta^2}\|\nabla^3 q\|_{L^2}^2 + \frac{\beta^2}{\nu}\|G\|_{H^2}^2\Big)\\
&\le C\Big(\frac{C}{\nu\beta^2}(\|\partial_t\nabla w\|_{L^2}^2 + \|\nabla^2 G\|_{L^2}^2) + \frac{C}{\beta^4}\Big(\frac{1}{\nu} + \frac{1}{\beta^2} + \Big(1+\frac{\gamma^2}{\beta^2}\Big)\Big(1+\frac{\tilde\nu}{\nu}\Big)\\
&\quad + \frac{\nu+\tilde\nu}{\beta^2+\gamma^2}(\beta^4+\gamma^2)\cdot\frac{1}{\nu\beta^2} + \frac{\tilde\nu(\beta^2+\gamma^2)}{\beta^4}\Big)D_5\\
&\quad\quad + \frac{(\nu+\tilde\nu)^2}{\nu\beta^6}\|\tilde f_1\|_{H^2}^2 + \frac{1}{\nu\beta^6}\|f_2\|_{H^1}^2 + \frac{1}{\nu\beta^2}\|f_4\|_{H^1}^2\\
&\quad\quad\quad + \frac{1}{\nu\beta^2}\|h_1\|_{H^2}^2 + \frac{1}{\nu\beta^2}\|G\|_{H^2}^2\Big\}\\
&\le \frac{C}{\nu\beta^2}\|\partial_t\nabla w\|_{L^2}^2 + \Big(\frac{1}{\beta^2} + \frac{1}{\nu} + \frac{\nu+\tilde\nu}{\beta^2+\gamma^2}\Big)D_7\\
&\quad + C\Big\{\frac{(\nu+\tilde\nu)^2}{\nu\beta^6}\|\tilde f_1\|_{H^2}^2 + \frac{1}{\nu\beta^6}\|f_2\|_{H^1}^2 + \frac{1}{\nu\beta^2}\|f_4\|_{H^1}^2 + \frac{1}{\nu\beta^2}\|h_1\|_{H^2}^2\Big\}.
\end{aligned}$$

This completes the proof. □

We set

$$\tilde{D}_7 := \frac{1}{8}D_7 + \frac{C\nu}{\beta^6}\|\nabla^3 w\|_{L^2}^2.$$

Proposition 5.17. *It holds the following inequality:*

$$\|\partial_t\phi\|_{L^2}^2 \le C\Big(\frac{1}{\nu\tilde\nu}\cdot\frac{\beta^4}{(\beta^2+\gamma^2)^2} + \frac{1}{\nu^3}\cdot\frac{\nu+\tilde\nu}{(\beta^2+\gamma^2)^2}\Big)\tilde{D}_7 + C\|f_1\|_{L^2}^2.$$

$$\|\partial_t G\|_{L^2}^2 \le C\Big(\frac{\beta^4}{\nu^2(\beta^2+\gamma^2)^2} + \frac{1}{\nu^2(\beta^2+\gamma^2)^2} + \frac{\beta^2}{\nu^3(\beta^2+\gamma^2)^2}\Big)\tilde{D}_7 + C\|f_3\|_{L^2}^2.$$

Proof. We see from (5.8) that

$$\begin{aligned}
\|\partial_t\phi\|_{L^2}^2 &\le C(\|\mathrm{div} w\|_{L^2}^2 + \|\overline{v}^1\partial_{x_1}\phi\|_{L^2}^2 + \|f_1\|_{L^2}^2)\\
&\le C\Big(\frac{1}{\tilde\nu} + \frac{1}{\nu^2}\cdot\frac{\nu+\tilde\nu}{\beta^2+\gamma^2}\Big)D_6 + C\|f_1\|_{L^2}^2\\
&\le C\left(\frac{1}{\nu\tilde\nu}\frac{\beta^4}{(\beta^2+\gamma^2)^2} + \frac{1}{\nu^3}\frac{\nu+\tilde\nu}{(\beta^2+\gamma^2)^3}\right)\tilde{D}_7 + C\|f_1\|_{L^2}^2.
\end{aligned}$$

Similarly it follows from (5.10) that

$$
\begin{aligned}
\|\partial_t G\|_{L^2}^2 &\le C(\|\overline{v}^1\partial_{x_1}G\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 + \|\nabla w\overline{E}\|_{L^2}^2 \\
&\quad + \|w^3\partial_{x_3}^2\overline{\psi}^1\|_{L^2}^2 + \|\nabla\overline{v}G\|_{L^2}^2 + \|f_3\|_{L^2}^2 \\
&\le C\Big(\frac{1}{\nu} + \frac{1}{\nu\beta^4} + \frac{1}{\nu^2\beta^2}\Big)D_6 + C\|f_3\|_{L^2}^2 \\
&\le C\Big(\frac{\beta^4}{\nu^2(\beta^2+\gamma^2)^2} + \frac{1}{\nu^2(\beta^2+\gamma^2)^2} + \frac{\beta^2}{\nu^3(\beta^2+\gamma^2)^2}\Big)\tilde{D}_7 + C\|f_3\|_{L^2}^2.
\end{aligned}
$$

This completes the proof. □

Proposition 5.18. *It holds the following inequality:*

$$
\begin{aligned}
&\frac{1}{\beta^2+\gamma^2}\Big\{\frac{1}{2}\frac{d}{dt}(\gamma^2\|\partial_t\phi\|_{L^2}^2 + \|\partial_t w\|_{L^2}^2 + \beta^2\|\partial_t G\|_{L^2}^2) \\
&\quad + \nu\|\nabla\partial_t w\|_{L^2}^2 + \tilde{\nu}\|\mathrm{div}\partial_t w\|_{L^2}^2\Big\} \\
&\le \Big(\frac{1}{4}\cdot\frac{\nu}{\beta^2+\gamma^2} + \frac{C}{\nu(\beta^2+\gamma^2)}\Big)\|\nabla\partial_t w\|_{L^2}^2 \\
&\quad + C\Big(\frac{1}{\nu} + \frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\Big)D_6 + \frac{C}{\nu}(\|\partial_t\phi\|_{L^2}^2 + \|\partial_t G\|_{L^2}^2) \\
&\qquad + \frac{\gamma^2}{\beta^2+\gamma^2}|(\partial_t f_1, \partial_t\phi)| + \frac{1}{\beta^2+\gamma^2}|(\partial_t f_2, \partial_t w)| + \frac{\beta^2}{\beta^2+\gamma^2}|(\partial_t f_3, \partial_t G)|.
\end{aligned}
$$

Proof. We obtain the following estimate by similar argument to the proof of Proposition 5.5

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\gamma^2\|\partial_t\phi\|_{L^2}^2 + \|\partial_t w\|_{L^2}^2 + \beta^2\|\partial_t G\|_{L^2}^2) + \nu\|\partial_t\nabla w\|_{L^2}^2 + \tilde{\nu}\|\mathrm{div}\partial_t w\|_{L^2}^2 \\
&\le \gamma^2|(\partial_t(\overline{v}^1\partial_{x_1}\phi), \partial_t\phi)| + |(\partial_t(\overline{v}^1\partial_{x_1}w), \partial_t w)| + |(\partial_t(w^3\partial_{x_3}\overline{v}^1), \partial_t w^1)| \\
&\quad + \nu|(\partial_t(\phi\partial_{x_3}^2\overline{v}^1), \partial_t w^1)| + \beta^2|(\partial_t(\mathrm{div}(G^\top\overline{E})), \partial_t w)| \\
&\qquad + \beta^2|(\partial_t(G^{33}\partial_{x_3}^2\overline{\psi}^1), \partial_t w^1)| + \beta^2|(\partial_t(\overline{v}^1\partial_{x_1}G), \partial_t G)| + \beta^2|(\partial_t(\nabla w\overline{E}), \partial_t G)| \\
&\qquad\quad + \beta^2|(\partial_t(w^3\partial_{x_3}^2\overline{\psi}^1), \partial_t G^{13})| + \beta^2|(\partial_t(\nabla\overline{v}G), \partial_t G)| \\
&\qquad\qquad + \gamma^2|(\partial_t f_1, \partial_t\phi)| + |(\partial_t f_2, \partial_t w)| + \beta^2|(\partial_t f_3, \partial_t G)|.
\end{aligned}
$$

The right-hand side is estimated by

$$
\begin{aligned}
&\gamma^2|(\partial_t(\overline{v}^1\partial_{x_1}\phi), \partial_t\phi)| + |(\partial_t(\overline{v}^1\partial_{x_1}w), \partial_t w)| + |(\partial_t(w^3\partial_{x_3}\overline{v}^1), \partial_t w^1)| \\
&\quad + \nu|(\partial_t(\phi\partial_{x_3}^2\overline{v}^1), \partial_t w^1)| + \beta^2|(\partial_t(\mathrm{div}(G^\top\overline{E})), \partial_t w)|
\end{aligned}
$$

$$+\beta^2|(\partial_t(G^{33}\partial_{x_3}^2\overline{\psi}^1),\partial_t w^1)|+\beta^2|(\partial_t(\overline{v}^1\partial_{x_1}G),\partial_t G)|+\beta^2|(\partial_t(\nabla w\overline{E}),\partial_t G)|$$
$$+\beta^2|(\partial_t(w^3\partial_{x_3}^2\overline{\psi}^1),\partial_t G^{13})|+\beta^2|(\partial_t(\nabla\overline{v}G),\partial_t G)|$$
$$+\gamma^2|(\partial_t f_1,\partial_t\phi)|+|(\partial_t f_2,\partial_t w)|+\beta^2|(\partial_t f_3,\partial_t G)|$$
$$\le\left(\frac{\nu}{4}+\frac{C}{\nu}\right)\|\nabla\partial_t w\|_{L^2}^2$$
$$+C\Big(\frac{\beta^2+\gamma^2}{\nu}+\nu+\tilde{\nu}\Big)D_6+C\frac{\beta^2+\gamma^2}{\nu}(\|\partial_t\phi\|_{L^2}^2+\|\partial_t G\|_{L^2}^2)$$
$$+\gamma^2|(\partial_t f_1,\partial_t\phi)|+|(\partial_t f_2,\partial_t w)|+\beta^2|(\partial_t f_3,\partial_t G)|.$$

We thus obtain

$$\frac{1}{\beta^2+\gamma^2}\Big\{\frac{1}{2}\frac{d}{dt}(\gamma^2\|\partial_t\phi\|_{L^2}^2+\|\partial_t w\|_{L^2}^2+\beta^2\|\partial_t G\|_{L^2}^2)$$
$$+\nu\|\nabla\partial_t w\|_{L^2}^2+\tilde{\nu}\|\mathrm{div}\partial_t w\|_{L^2}^2\Big\}$$
$$\le\Big(\frac{1}{4}\cdot\frac{\nu}{\beta^2+\gamma^2}+\frac{C}{\nu(\beta^2+\gamma^2)}\Big)\|\nabla\partial_t w\|_{L^2}^2$$
$$+C\Big(\frac{1}{\nu}+\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\Big)D_6+\frac{C}{\nu}(\|\partial_t\phi\|_{L^2}^2+\|\partial_t G\|_{L^2}^2)$$
$$+\frac{\gamma^2}{\beta^2+\gamma^2}|(\partial_t f_1,\partial_t\phi)|+\frac{1}{\beta^2+\gamma^2}|(\partial_t f_2,\partial_t w)|+\frac{\beta^2}{\beta^2+\gamma^2}|(\partial_t f_3,\partial_t G)|.$$

This completes the proof. □

5.3 Proof of Proposition 5.4

We see from (5.25) and Proposition 5.9 that

$$\frac{d}{dt}\Big(E_2+\frac{1}{\beta^2+\gamma^2}E_3\Big)+\frac{1}{32}D_2+\frac{1}{\beta^2+\gamma^2}D_3$$
$$\le C\Big(\frac{1}{\nu}+\frac{1}{\beta^2}+\frac{1}{\beta^2+\gamma^2}\Big)\|\partial_{x_3}\phi\|_{L^2}^2$$
$$+C\Big\{\frac{\nu+\tilde{\nu}}{(\beta^2+\gamma^2)^2}+\Big(\frac{\gamma^2}{\beta^2+\gamma^2}\Big)^2\left(\frac{1}{\nu\tilde{\nu}}+\frac{1}{\nu^2}\right)$$
$$+\Big(\frac{1}{\nu}+\frac{1}{\nu^3}\Big)\frac{\beta^2}{\beta^2+\gamma^2}+\frac{1}{\tilde{\nu}}\frac{\gamma^2}{\beta^2+\gamma^2}+\frac{1}{\beta^2+\gamma^2}\Big(1+\frac{1}{\nu}+\frac{1}{\nu^3}+\frac{1}{\beta^2}\Big)\Big\}D_2$$
$$+C\Big(N_2+\frac{1}{\beta^2+\gamma^2}N_3\Big). \tag{5.51}$$

If $\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2} \ll 1, \frac{1}{\nu}, \frac{1}{\beta^2} \ll 1$, then the right-hand side of (5.51) is estimated as

$$
\begin{aligned}
& C\Big(\frac{1}{\nu}+\frac{1}{\beta^2}+\frac{1}{\beta^2+\gamma^2}\Big)\|\partial_{x_3}\phi\|_{L^2}^2 \\
& \quad +C\Big\{\frac{\nu+\tilde{\nu}}{(\beta^2+\gamma^2)^2}+\Big(\frac{\gamma^2}{\beta^2+\gamma^2}\Big)^2\Big(\frac{1}{\nu\tilde{\nu}}+\frac{1}{\nu^2}\Big) \\
& \qquad +\Big(\frac{1}{\nu}+\frac{1}{\nu^3}\Big)\frac{\beta^2}{\beta^2+\gamma^2}+\frac{1}{\tilde{\nu}}\frac{\gamma^2}{\beta^2+\gamma^2}+\frac{1}{\beta^2+\gamma^2}\Big(1+\frac{1}{\nu}+\frac{1}{\nu^3}+\frac{1}{\beta^2}\Big)\Big\}D_2 \\
& \qquad\quad +C\Big(N_2+\frac{1}{\beta^2+\gamma^2}N_3\Big) \\
& \leq C\Big(\frac{1}{\nu}+\frac{1}{\beta^2}+\frac{1}{\beta^2+\gamma^2}\Big)\|\partial_{x_3}\phi\|_{L^2}^2+\frac{1}{64}D_2+C\Big(N_2+\frac{1}{\beta^2+\gamma^2}N_3\Big),
\end{aligned}
$$

and hence,

$$
\begin{aligned}
& \frac{d}{dt}\Big(E_2+\frac{1}{\beta^2+\gamma^2}E_3\Big)+\frac{1}{64}D_2+\frac{1}{\beta^2+\gamma^2}D_3 \\
& \leq C\Big(\frac{1}{\nu}+\frac{1}{\beta^2}+\frac{1}{\beta^2+\gamma^2}\Big)\|\partial_{x_3}\phi\|_{L^2}^2+C\Big(N_2+\frac{1}{\beta^2+\gamma^2}N_3\Big).
\end{aligned} \tag{5.52}
$$

It follows from (5.52) and Proposition 5.10 that

$$
\begin{aligned}
& \frac{d}{dt}\Big(E_2+\frac{1}{\beta^2+\gamma^2}E_3+E_4\Big)+\tilde{D}_3+D_4 \\
& \leq \frac{1}{2}\tilde{D}_3+C\Big(\frac{1}{\nu}+\frac{1}{\beta^2}+\frac{1}{\beta^2+\gamma^2}\Big)\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}D_4 \\
& \quad +C\Big(N_2+\frac{1}{\beta^2+\gamma^2}N_3+N_4\Big).
\end{aligned}
$$

Moreover by taking $\nu, \tilde{\nu}, \beta^2, \gamma^2$ so that $C(\frac{1}{\nu}+\frac{1}{\beta^2}+\frac{1}{\beta^2+\gamma^2})\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2} \leq \frac{1}{2}$, we have

$$
\frac{d}{dt}\Big(E_2+\frac{1}{\beta^2+\gamma^2}E_3+E_4\Big)+\frac{1}{2}(\tilde{D}_3+D_4) \leq C\Big(N_2+\frac{1}{\beta^2+\gamma^2}N_3+N_4\Big). \tag{5.53}
$$

By combining (5.53) and Proposition 5.11, we obtain

$$
\begin{aligned}
& \frac{d}{dt}E_5+\frac{1}{2}D_5 \\
& \leq \frac{C}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}\|\partial_t\nabla w\|_{L^2}^2+C\frac{\beta^2}{\nu}\|\partial\partial_{x_3}G\|_{L^2}^2 \\
& \quad +C\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\|\nabla^2\partial w\|_{L^2}^2+CN_5.
\end{aligned} \tag{5.54}
$$

It follows from Proposition 5.12 and Proposition 5.13 that

$$
\frac{d}{dt}\|\partial_{x_3}G\|_{L^2}^2+\frac{\beta^2}{\nu}\|\partial_{x_3}G\|_{L^2}^2
$$

$$
\begin{aligned}
&\le C\Big(\frac{1}{\nu}+\frac{\gamma^2}{\nu\beta^2}+\frac{\beta^2+\gamma^2}{\beta^2}\cdot\frac{\tilde\nu^2}{\nu(\nu+\tilde\nu)}+\frac{\nu+\tilde\nu}{\nu}\cdot\frac{1+\gamma^2}{\beta^2}\cdot\frac{1+\gamma^2}{\beta^2+\gamma^2}\\
&\quad+\frac{1}{\beta^2}+\frac{1}{\nu^4\beta^2}+\frac{1}{\nu^2\beta^2}+\frac{1}{\nu\beta^4}+\frac{1}{\nu\beta^6}+\frac{1}{\nu\beta^8}\Big)\tilde D_5+\frac{C}{\nu\beta^2}\|\partial_{x_3}G\|_{L^2}^2\\
&\quad+C\Big(\frac{\tilde\nu^2}{\nu\beta^2}\|\tilde f_1\|_{H^1}^2+\frac{1}{\nu\beta^2}\|f_2\|_{L^2}^2+\frac{1}{\nu}\|h_1\|_{H^1}^2\Big)+\Big|\Big(\partial_{x_3}f_3+\frac{\beta^2}{\nu}\partial_{x_3}h_1,\partial_{x_3}G\Big)\Big|\\
&\le C\Big(\frac{1}{\nu}+\frac{1}{\beta^2}+\frac{\nu+\tilde\nu}{\nu}+\frac{\tilde\nu^2}{\nu(\nu+\tilde\nu)}\Big)\tilde D_5+\frac{C}{\nu\beta^2}\|\partial_{x_3}G\|_{L^2}^2\\
&\quad+C\Big(\frac{\tilde\nu^2}{\nu\beta^2}\|\tilde f_1\|_{H^1}^2+\frac{1}{\nu\beta^2}\|f_2\|_{L^2}^2+\frac{1}{\nu}\|h_1\|_{H^1}^2\Big)+\Big|\Big(\partial_{x_3}f_3+\frac{\beta^2}{\nu}\partial_{x_3}h_1,\partial_{x_3}G\Big)\Big|.
\end{aligned}
\tag{5.55}
$$

By combining (5.54) and (5.55), we have

$$
\frac{d}{dt}E_6+\frac14 D_6\le\frac{C}{(\beta^2+\gamma^2)(\nu+\tilde\nu)}\|\partial_t\nabla w\|_{L^2}^2+CN_6. \tag{5.56}
$$

We next estimate $\|\nabla^2 w\|_{L^2}^2$. We see from Proposition 5.13 that if $C\left(\frac{5}{\nu}+\frac{\gamma^2}{\beta^2}\right)\le\frac18$, then

$$
\begin{aligned}
&\frac{1}{\beta^2(\beta^2+\gamma^2)^2}\|\nabla^2 w\|_{L^2}^2\\
&\le\frac{C}{\nu^2\beta^2(\beta^2+\gamma^2)^2}\|\nabla^2 q\|_{L^2}^2+\frac{C\beta^2}{\nu^2(\beta^2+\gamma^2)^2}\|G\|_{H^1}^2\\
&\le\frac18 D_6+\frac{C}{\nu^2}(\tilde\nu^2\|\tilde f_1\|_{H^1}^2+\|f_2\|_{L^2}^2+\beta^2\|h_1\|_{H^1}^2).
\end{aligned}
\tag{5.57}
$$

It follows from (5.56) and (5.57) that

$$
\begin{aligned}
&\frac{d}{dt}E_6+\frac18 D_6+\frac{1}{\beta^2(\beta^2+\gamma^2)^2}\|\nabla^2 w\|_{L^2}^2\\
&\le\frac{C}{(\beta^2+\gamma^2)(\nu+\tilde\nu)}\|\partial_t\nabla w\|_{L^2}^2+CN_6+\frac{C}{\nu^2}(\tilde\nu^2\|\tilde f_1\|_{H^1}^2+\|f_2\|_{L^2}^2+\beta^2\|h_1\|_{H^1}^2).
\end{aligned}
\tag{5.58}
$$

By coupling Proposition 5.14 and (5.58), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\Big(\frac{\nu(\beta^2+\gamma^2)^2}{\beta^4}E_6+\|\nabla^2 G\|_{L^2}^2\Big)+\frac{\nu(\beta^2+\gamma^2)^2}{8\beta^4}D_6+\frac{\nu}{\beta^6}\|\nabla^2 w\|_{L^2}^2+\frac{\beta^2}{\nu}\|\nabla^2 G\|_{L^2}^2\\
&\le C\Big(\frac{\nu(\beta^2+\gamma^2)}{\beta^4(\nu+\tilde\nu)}+\frac{1}{\nu\beta^2}\Big)\|\partial_t\nabla w\|_{L^2}^2+\frac{C\nu}{\beta^6}\|\nabla^3 w\|_{L^2}^2
\end{aligned}
$$

$$
\begin{aligned}
&+ C\Big(\frac{1}{\nu} + \frac{1}{\beta^2} + \frac{\tilde\nu}{\beta^2} + 1 + \frac{\tilde\nu}{\nu}\Big) D_6 \\
&+ C\Big(\frac{\nu(\beta^2+\gamma^2)^2}{\beta^4} N_6 + |(\nabla^2(f_3 + \frac{\beta^2}{\nu} h_1), \nabla^2 G)| + \frac{\nu+\tilde\nu}{\nu\beta^2}\|\tilde f_1\|_{H^2}^2 \\
&\quad + \frac{1}{\nu\beta^2}\|f_2\|_{H^1}^2 + \frac{\beta^2}{\nu}\|f_4\|_{H^1}^2 + \frac{\beta^2}{\nu}\|h_1\|_{H^2}^2\Big).
\end{aligned}
$$

Hence we have

$$
\frac{d}{dt}E_7 + \frac{1}{16}D_7 \le C\Big(\frac{1}{\beta^2} + \frac{1}{\nu\beta^2}\Big)\|\partial_t \nabla w\|_{L^2}^2 + \frac{C\nu}{\beta^6}\|\nabla^3 w\|_{L^2}^2 + CN_7. \tag{5.59}
$$

We see from Proposition 5.16 and (5.59) that

$$
\begin{aligned}
&\frac{d}{dt}E_7 + \frac{1}{16}D_7 + \frac{C\nu}{\beta^2}\|\nabla^3 w\|_{L^2}^2 \\
&\le C\Big(\frac{1}{\beta^2} + \frac{1}{\nu\beta^2} + \frac{1}{\nu\beta^6}\Big)\|\partial_t\nabla w\|_{L^2}^2 + C\Big(\frac{1}{\beta^2} + \frac{1}{\nu} + \frac{\nu+\tilde\nu}{\beta^2+\gamma^2}\Big)D_7 + CN_7.
\end{aligned}
$$

Hence we obtain

$$
\frac{d}{dt}E_7 + \frac{1}{2}\tilde D_7 \le C\Big(\frac{1}{\beta^2} + \frac{1}{\nu\beta^2} + \frac{1}{\nu\beta^6}\Big)\|\partial_t\nabla w\|_{L^2}^2 + CN_7. \tag{5.60}
$$

We next estimate for time derivatives of (ϕ, w, G). By combining Proposition 5.17 and Proposition 5.18, we have

$$
\begin{aligned}
&\frac{d}{dt}\Big(\frac{\gamma^2}{\beta^2+\gamma^2}\|\partial_t\phi\|_{L^2}^2 + \frac{1}{\beta^2+\gamma^2}\|\partial_t w\|_{L^2}^2 + \frac{\beta^2}{\beta^2+\gamma^2}\|\partial_t G\|_{L^2}^2\Big) \\
&\quad + \|\partial_t\phi\|_{L^2}^2 + \frac{\nu}{\beta^2+\gamma^2}\|\nabla\partial_t w\|_{L^2}^2 + \frac{\tilde\nu}{\beta^2+\gamma^2}\|\mathrm{div}\partial_t w\|_{L^2}^2 + \|\partial_t G\|_{L^2}^2 \\
&\le C\Big(\frac{1}{\nu} + \frac{\nu+\tilde\nu}{\beta^2+\gamma^2}\Big)D_6 + C\Big(\|f_1\|_{L^2}^2 + \frac{\gamma^2}{\beta^2+\gamma^2}|(\partial_t f_1, \partial_t\phi)| \\
&\quad + \frac{1}{\beta^2+\gamma^2}|(\partial_t f_2, \partial_t w)| + \frac{\beta^2}{\beta^2+\gamma^2}|(\partial_t f_3, \partial_t G)| + \|f_3\|_{L^2}^2\Big).
\end{aligned} \tag{5.61}
$$

We see from (5.60) + (5.61) that

$$
\begin{aligned}
&\frac{d}{dt}\Big(E_7 + \frac{\gamma^2}{\beta^2+\gamma^2}\|\partial_t\phi\|_{L^2}^2 + \frac{1}{\beta^2+\gamma^2}\|\partial_t w\|_{L^2}^2 + \frac{\beta^2}{\beta^2+\gamma^2}\|\partial_t G\|_{L^2}^2\Big) \\
&\quad + \tilde D_7 + \|\partial_t\phi\|_{L^2}^2 + \frac{\nu}{4\beta^2}\|\partial_t\nabla w\|_{L^2}^2 + \|\partial_t G\|_{L^2}^2 \\
&\le C\Big(\frac{1}{\nu} + \frac{\nu+\tilde\nu}{\beta^2+\gamma^2}\Big)D_6 + C\Big(N_7 + \|f_1\|_{L^2}^2 + \frac{\gamma^2}{\beta^2+\gamma^2}|(\partial_t f_1, \partial_t\phi)|
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\beta^2+\gamma^2}|(\partial_t f_2,\partial_t w)|+\frac{\beta^2}{\beta^2+\gamma^2}|(\partial_t f_3,\partial_t G)|+\|f_3\|_{L^2}^2\Big)\\
&\le C\Big(\frac{1}{\nu}+\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\Big)\frac{\beta^4}{\nu(\beta^2+\gamma^2)}\tilde{D}_7+C\Big(N_7+\|f_1\|_{L^2}^2+\frac{\gamma^2}{\beta^2+\gamma^2}|(\partial_t f_1,\partial_t\phi)|\\
&+\frac{1}{\beta^2+\gamma^2}|(\partial_t f_2,\partial_t w)|+\frac{\beta^2}{\beta^2+\gamma^2}|(\partial_t f_3,\partial_t G)|+\|f_3\|_{L^2}^2\Big).
\end{aligned}
$$

We set

$$
\begin{aligned}
E_8 &:= E_7+\frac{\gamma^2}{\beta^2+\gamma^2}\|\partial_t\phi\|_{L^2}^2+\frac{1}{\beta^2+\gamma^2}\|\partial_t w\|_{L^2}^2+\frac{\beta^2}{\beta^2+\gamma^2}\|\partial_t G\|_{L^2}^2\\
D_8 &:= \tilde{D}_7+\|\partial_t\phi\|_{L^2}^2+\frac{\nu}{\beta^2}\|\nabla\partial_t w\|_{L^2}^2+\|\partial_t G\|_{L^2}^2\\
N_8 &:= N_7+\|f_1\|_{L^2}^2+\frac{\gamma^2}{\beta^2+\gamma^2}|(\partial_t f_1,\partial_t\phi)|\\
&\quad+\frac{1}{\beta^2+\gamma^2}|(\partial_t f_2,\partial_t w)|+\frac{\beta^2}{\beta^2+\gamma^2}|(\partial_t f_3,\partial_t G)|+\|f_3\|_{L^2}^2.
\end{aligned}
$$

It then follows

$$\frac{d}{dt}E_8+D_8\le CN_8.$$

We note that there exists a positive number $C_1>0$ such that $E_8\le C_1D_8$. This leads to

$$\frac{d}{dt}E_8+C_1(E_8+D_8)\le CN_8,$$

and hence,

$$E_8(t)+C_1\int_0^t e^{-C_1(t-s)}D_8(s)ds\le Ce^{-C_1t}E_8(0)+C\int_0^t e^{-C_1(t-s)}N_8(s)ds.$$

We see from the third equation of (5.9) that

$$
\begin{aligned}
\|\partial_{x_3}^2 w\|_{L^2}^2 &\le C\Big(\frac{1}{\nu^2}\|\partial_t w\|_{L^2}^2+\|\nabla\partial w\|_{L^2}^2+\frac{\gamma^4+1}{\nu^2}\|\nabla\phi\|_{L^2}^2+\frac{\beta^4}{\nu^2}\|\nabla G\|_{L^2}^2\\
&\quad+\frac{1}{\nu^4}\|\nabla w\|_{L^2}^2+\frac{1}{\nu^2}(\|G\|_{L^2}^2+\|\partial G\|_{L^2}^2)+\frac{1}{\nu^2}\|f_2\|_{L^2}^2\Big)\\
&\le C\frac{\beta^4+\gamma^4}{\nu^2}E_8+\frac{C}{\nu^2}E_8^2\le C\frac{\beta^4+\gamma^4}{\nu^2}E_8.
\end{aligned}
$$

We finally introduce the following quantities

$$E(t):=2CE_8(t)+\frac{\nu^2}{\beta^4+\gamma^4}\|\partial_{x_3}^2 w(t)\|_{L^2}^2$$

$$
\begin{aligned}
D(t) &:= 2CD_8(t) \\
N(t) &:= 2CN_8(t).
\end{aligned}
$$

It then holds the following inequality

$$E(t) + \int_0^t e^{-C_1(t-s)} D(s)ds \le Ce^{-C_1 t} E(0) + C\int_0^t e^{-C_1(t-s)} N(s)ds. \quad (5.62)$$

It remains to estimate $N(t)$ to complete the proof. We shall show that $N(t)$ is estimated in the following way by direct applications of Lemmata 2.1, 2.3 and 2.4 .

Proposition 5.19. *There exists a positive constant $\delta_1 > 0$ such that if $E(t) \le \delta$, then it holds the following estimate*

$$N(t) \le C(E(t)^{\frac{1}{2}} + E(t))D(t) \qquad (t \ge 0).$$

Proposition 5.4 now follows by combining (5.62) and Proposition 5.19.

□

Appendix 5.A Proof of Proposition 5.1.

In this appendix, we will give a proof of Proposition 5.1.
Proof of Proposition 5.1. We set $(\rho, v, F) = (\bar{\rho}, \bar{v}, \bar{F}) = (1, \bar{v}^1(x_3, t)e_1, \big(\nabla(x - \bar{\psi}^1(x_3, t)e_1)\big)^{-1})$ in (5.1). Since

$$\bar{F} = \nabla(x + \bar{\psi}^1 e_1) = \begin{pmatrix} 1 & 0 & \partial_{x_3}\bar{\psi}^1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\rho(\partial_t v + v\cdot\nabla v) - \nu\Delta v - (\nu+\nu')\nabla \mathrm{div} v + \nabla p(\rho) = (\partial_t \bar{v}^1 - \nu\partial_{x_3}^2 \bar{v}^1)e_1,$$
$$\beta^2 \mathrm{div}(\rho F^\top F) + \rho g = (\beta^2 \partial_{x_3}^2 \bar{\psi}^1 + g^1)e_1,$$

we see from the 2nd equation of (5.1) that $\bar{\psi}^1$ should satisfy the following time-periodic problem:

$$\begin{cases} \partial_t^2 \bar{\psi}^1 - \beta^2 \partial_{x_3}^2 \bar{\psi}^1 - \nu\partial_t \partial_{x_3}^2 \bar{\psi}^1 = g^1, & (5.A.1) \\ \bar{\psi}^1(0,t) = \bar{\psi}^1(1,t) = 0, & (5.A.2) \\ \bar{\psi}^1(x_3, t+T) = \bar{\psi}^1(x_3, t). & (5.A.3) \end{cases}$$

For simplicity we assume that $\frac{2\beta}{\pi\nu}$ is not integer.

We set

$$A = \left(\begin{array}{cc} 0 & -1 \\ -\beta^2 \partial_{x_3}^2 & -\nu \partial_{x_3}^2 \end{array} \right).$$

It follows that the problem (5.A.1)–(5.A.3) is written as

$$\partial_t \begin{pmatrix} \bar{\psi}^1 \\ \bar{v}^1 \end{pmatrix} + A \begin{pmatrix} \bar{\psi}^1 \\ \bar{v}^1 \end{pmatrix} = \begin{pmatrix} 0 \\ g^1 \end{pmatrix}, \ \begin{pmatrix} \bar{\psi}^1(x_3, t+T) \\ \bar{v}^1(x_3, t+T) \end{pmatrix} = \begin{pmatrix} \bar{\psi}^1(x_3, t) \\ \bar{v}^1(x_3, t) \end{pmatrix}. \tag{5.A.4}$$

To solve (5.A.4), we consider the Fourier-sine expansions;

$$\bar{\psi}^1 = \sum_{k=1}^{\infty} \hat{\psi}_k(t) \sin(k\pi x_3), \ \bar{v}^1 = \sum_{k=1}^{\infty} \hat{v}_k(t) \sin(k\pi x_3),$$
$$g^1 = \sum_{k=1}^{\infty} \hat{g}_k^1(t) \sin(k\pi x_3).$$

We see from (5.A.4) that $(\hat{v}_k, \hat{\psi}_k)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} \hat{\psi}_k \\ \hat{v}_k \end{pmatrix} + \hat{A} \begin{pmatrix} \hat{\psi}_k \\ \hat{v}_k \end{pmatrix} = \begin{pmatrix} 0 \\ \hat{g}_k^1 \end{pmatrix}, \ \begin{pmatrix} \hat{\psi}_k(t+T) \\ \hat{v}_k(t+T) \end{pmatrix} = \begin{pmatrix} \hat{\psi}_k(t) \\ \hat{v}_k(t) \end{pmatrix}. \tag{5.A.5}$$

where

$$\hat{A} = \left(\begin{array}{cc} 0 & -1 \\ \beta^2 k^2 \pi^2 & \nu k^2 \pi^2 \end{array} \right).$$

The solution of (5.A.5) is given by

$$\begin{pmatrix} \hat{\psi}_k(t) \\ \hat{v}_k(t) \end{pmatrix} = \int_{-\infty}^{t} e^{-(t-s)\hat{A}} \begin{pmatrix} 0 \\ \hat{g}_k^1(s) \end{pmatrix} \mathrm{d}s. \tag{5.A.6}$$

From [16], the solution semigroup $e^{-t\hat{A}}$ is represented as

$$e^{-t\hat{A}} = \frac{1}{\lambda_+ - \lambda_-} \left(e^{t\lambda_+} \left(\begin{array}{cc} -\lambda_- & 1 \\ -\lambda_+\lambda_- & \lambda_+ \end{array} \right) + e^{t\lambda_-} \left(\begin{array}{cc} \lambda_+ & -1 \\ \lambda_+\lambda_- & -\lambda_- \end{array} \right) \right),$$

where

$$\lambda_\pm = \frac{-\nu k^2 \pi^2 \pm \sqrt{\nu^2 k^4 \pi^4 - 4\beta^2 k^2 \pi^2}}{2}.$$

We note that $\lambda_\pm$ are the characteristic roots of $-\hat{A}$ satisfying the following properties

$$\lambda_+ = -\frac{\beta^2}{\nu} + O\left(\frac{1}{k^2}\right), \ \lambda_- = -\nu k^2 \pi^2 + O(1) \quad \text{for } k \gg 1,$$

$$\lambda_+\lambda_- = \beta^2 k^2 \pi^2.$$

It then follows form (5.A.6) that $\hat{\psi}_k(t)$ and $\hat{v}_k(t)$ are written as

$$\hat{\psi}_k(t) = \int_{-\infty}^{t} \frac{e^{\lambda_+(t-s)} - e^{\lambda_-(t-s)}}{\lambda_+ - \lambda_-} \hat{g}_k^1(s)\mathrm{d}s,$$
$$\hat{v}_k(t) = \int_{-\infty}^{t} \frac{\lambda_+ e^{\lambda_+(t-s)} - \lambda_- e^{\lambda_-(t-s)}}{\lambda_+ - \lambda_-} \hat{g}_k^1(s)\mathrm{d}s.$$

By integration by parts, we see that $\hat{\psi}_k(t)$ and $\hat{v}_k(t)$ are rewritten in the following forms :

$$\hat{\psi}_k(t) = \frac{1}{\lambda_+\lambda_-}\hat{g}_k^1(t) + \int_{-\infty}^{t} \frac{\lambda_- e^{\lambda_+(t-s)} - \lambda_+ e^{\lambda_-(t-s)}}{\lambda_+\lambda_-(\lambda_+ - \lambda_-)} (\hat{g}_k^1)'(s)\mathrm{d}s, \tag{5.A.7}$$
$$\hat{v}_k(t) = \int_{-\infty}^{t} \frac{e^{\lambda_+(t-s)} - e^{\lambda_-(t-s)}}{\lambda_+ - \lambda_-} (\hat{g}_k^1)'(s)\mathrm{d}s. \tag{5.A.8}$$

Let us estimate $\hat{\psi}_k$ and $\hat{v}_k$ by using (5.A.7) and (5.A.8).

We first prepare the following inequalities shown in [16] to estimate (5.A.7) and (5.A.8):

$$\left| \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} \right| \le C \frac{1}{\beta k \pi + \nu k^2 \pi^2} e^{-c\kappa t}, \tag{5.A.9}$$
$$\left| \frac{\lambda_- e^{\lambda_+ t} - \lambda_+ e^{\lambda_- t}}{\lambda_+ - \lambda_-} \right| \le C e^{-c\kappa t}. \tag{5.A.10}$$

Based on (5.A.7) and (5.A.10), we have

$$\begin{aligned}
|\hat{\psi}_k(t)|^2 &\le \frac{1}{\beta^4 k^4 \pi^4} \left(|\hat{g}_k^1(t)| + \int_{-\infty}^{t} e^{c\kappa(t-s)} |(\hat{g}_k^1)'(s)|\mathrm{d}s \right)^2 \\
&\le \frac{C}{\beta^4 k^4 \pi^4} |\hat{g}_k^1(t)|^2 + \frac{C}{\beta^4 k^4 \pi^4} \left(\int_{-\infty}^{t} e^{c\kappa(t-s)} |(\hat{g}_k^1)'(s)|\mathrm{d}s \right)^2 \\
&\le \frac{C}{\beta^4 k^4 \pi^4} |\hat{g}_k^1(t)|^2 + \frac{C}{\beta^4 k^4 \pi^4} \left(\int_{-\infty}^{t} e^{c\kappa(t-s)}\mathrm{d}s \right) \left(\int_{-\infty}^{t} e^{c\kappa(t-s)} |(\hat{g}_k^1)'(s)|^2\mathrm{d}s \right) \\
&\le \frac{C}{\beta^4 k^4 \pi^4} |\hat{g}_k^1(t)|^2 + \frac{C}{\kappa\beta^4 k^4 \pi^4} \int_{-\infty}^{t} e^{c\kappa(t-s)} |(\hat{g}_k^1)'(s)|^2\mathrm{d}s.
\end{aligned}$$

Hence, we obtain the following estimate for $0 \le l \le 6$

$$\|\partial_{x_3}^l \bar{\psi}^1(t)\|_{L^2(0,1)}^2 \le \frac{C}{\beta^4} \|\partial_{x_3}^{l-2} g^1(t)\|_{L^2(0,1)}^2 + \frac{C}{\kappa\beta^4} \int_{-\infty}^{t} e^{c\kappa(t-s)} \|\partial_t \partial_{x_3}^{l-2} g^1(s)\|_{L^2(0,1)}^2 \mathrm{d}s$$

$$
\begin{aligned}
&\le \frac{C}{\beta^4}\|g^1(t)\|^2_{H^4(0,1)} + \frac{C}{\kappa^2\beta^4}\sup_{t\in[0,T]}\|\partial_t g^1(t)\|^2_{H^4(0,1)}\\
&\le \frac{C}{\beta^4}\left(1+\frac{1}{\kappa^2}\right)\|g^1\|^2_{H^1(0,T,H^4(0,1))}.
\end{aligned}
$$

In the case that $\frac{2\beta}{\pi\nu}$ is a positive integer, we can deduce above estimates by using the following forms with $k = \frac{2\beta}{\pi\nu}$:

$$
\begin{aligned}
\hat{\psi}_k(t) =&\frac{1}{\beta^2k^2\pi^2}\hat{g}^1_k(t) + \frac{1}{\beta^2k^2\pi^2}\int_{-\infty}^{t} e^{-\frac{2\beta^2}{\nu}(t-s)}(\hat{g}^1_k)'(s)\mathrm{d}s\\
&+\frac{2}{\nu k^2\pi^2}\int_{-\infty}^{t}(t-s)e^{-\frac{2\beta^2}{\nu}(t-s)}(\hat{g}^1_k)'(s)\mathrm{d}s,\\
\hat{v}_k(t) =& -\int_{-\infty}^{t}(t-s)e^{-\frac{2\beta^2}{\nu}(t-s)}(\hat{g}^1_k)'(s)\mathrm{d}s - \frac{4}{\nu k^2\pi^2}\int_{-\infty}^{t} e^{-\frac{2\beta^2}{\nu}(t-s)}(\hat{g}^1_k)'(s)\mathrm{d}s.
\end{aligned}
$$

As a result, $\bar{\psi}^1(t)$ satisfies

$$
\sup_{t\in[0,T]}\|\bar{\psi}^1(t)\|^2_{H^6(0,1)} \le \frac{C}{\beta^4}\left(1+\frac{1}{\kappa^2}\right)\|g^1\|^2_{H^1(0,T,H^4(0,1))}.
$$

Similarly, we obtain the estimate of $\sup_{t\in[0,T]}\|\bar{v}^1(t)\|^2_{H^4(0,1)}$ by using (5.A.8) and (5.A.9). The estimate of $\sup_{t\in[0,T]}\|\partial_t\bar{v}^1(t)\|^2_{H^2(0,1)}$ is immediately proved by using (5.63). This completes the proof of Proposition 5.1. ■

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