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Vectorial prescribed mean curvature problem

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Abstract

We consider the prescribed mean curvature problem, which is a problem to find a surface with the mean curvature equal to a given function. In the present thesis, we consider the existence of a surface whose mean curvature vector coincides with the normal component of a given vector field. We obtained three major results under the following conditions.

In Chapter 2, we consider the problem under the Dirichlet boundary condition in a Euclidean space, and we proved the following. If the prescribed function and boundary value are sufficiently small in a dimensionally sharp Sobolev norm, there exists a solution near a graphical minimal surface.

In Chapter 3, we consider the problem on the n-dimensional torus. We proved the existence of solutions under the conditions that the Sobolev norm of a given vector field is sufficiently small, $n+1$-th component is monotonous, and the integrated value is zero.

In Chapter 4, we consider the Allen–Cahn equation with the advection term instead. Under the conditions that the Sobolev norm of the vector field and energy are uniformly bounded, we proved that the diffuse interface converges to a surface whose mean curvature vector coincides with the normal component of the limit of a given vector field.

Contents

Chapter 1

Introduction

Mean curvature has been studied in a wide range of mathematical fields such as variational calculus, differential geometry, nonlinear partial differential equations, and geometric measure theory. In this paper, we consider the prescribed mean curvature problem, which is a problem to find a surface with the mean curvature equal to a given function. The prescribed mean curvature problem has been studied by numerous researchers in the case that a given function depends on the position of the space and surface. On the other hand, if a given function depends on the first derivative of a surface, the problem has been little studied. In the present thesis, we consider the existence of a surface whose mean curvature vector coincides with the normal component of a given vector field. We obtained three major results, each of which comprises the following chapters.

1.1 Dirichlet problem in $\mathbb{R}^n$

In Chapter 2, we consider the following prescribed mean curvature problem with the Dirichlet condition,

$$\begin{cases} \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = H(x, u(x), \nabla u(x)) \quad \text{in } \Omega, \\ u = \phi \quad \text{on } \partial\Omega, \end{cases} \tag{1.1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$. The function $H(x,t,z) : \Omega\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ is given and we seek a solution u satisfying (1.1.1). Since the left hand side of (1.1.1) is the mean curvature of the graph of u, (1.1.1) is a prescribed mean curvature equation whose prescription depends on the location of the graph as well as the slope of the tangent space.

Prescribed mean curvature problems in a wide variety of formulation have been studied by numerous researchers. In the most classical case of $H = H(x)$, (1.1.1) has a solution if H and ϕ have suitable regularity and the mean curvature of $\partial\Omega$ satisfies a certain geometric condition (see [11, 12, 16, 19, 21, 36], for example). Giusti [13] determined a necessary and sufficient condition that a prescribed mean curvature problem without boundary conditions has solutions. In the case of $H = H(x, t)$, Gerhardt [10] constructed $H^{1,1}$ solutions, and Miranda [27] constructed BV solutions. In those papers, assumptions of the boundedness $|H| < \infty$ and the monotonicity $\frac{\partial H}{\partial t} \geq 0$ play important roles. If $|H| < \Gamma$ where Γ is determined by Ω, there exist solutions of (1.1.1), and the uniqueness of solutions is guaranteed by the monotonicity, that is, $\frac{\partial H}{\partial t} \geq 0$. Under the assumptions of boundedness, monotonicity and the convexity of Ω, Bergner [6] solved the Dirichlet problem in the case of $H = H(x, u, \nu(\nabla u))$ using the Leray-Schauder fixed point theorem. Here, ν is the unit normal vector of u, that is, $\nu(z) = \frac{1}{\sqrt{1+|z|^2}}(-z, 1)$. For the same problem as [6], Marquardt [26] gave a condition on $\partial\Omega$ depending on H which guarantees the existence of solutions even for a non-convex domain Ω.

The motivation of the present chapter comes from a singular perturbation problem studied in Chapter 4, where one considers the following problem on a domain $\tilde{\Omega} \subset \mathbb{R}^{n+1}$,

$$-\varepsilon\Delta\phi_\varepsilon + \frac{W'(\phi_\varepsilon)}{\varepsilon} = \varepsilon\nabla\phi_\varepsilon \cdot f_\varepsilon. \tag{1.1.2}$$

Here, W is a double-well potential, for example $W(\phi) = (1-\phi^2)^2$ and $\{f_\varepsilon\}_{\varepsilon>0}$ are given vector fields uniformly bounded in the Sobolev norm of $W^{1,p}(\tilde{\Omega})$, $p > \frac{n+1}{2}$. In Chapter 4, we proved under a natural assumption

$$\int_{\tilde{\Omega}} \left(\frac{\varepsilon|\nabla\phi_\varepsilon|^2}{2} + \frac{W(\phi_\varepsilon)}{\varepsilon} \right) dx + \|f_\varepsilon\|_{W^{1,p}(\tilde{\Omega})} \leq C \tag{1.1.3}$$

that the interface $\{\phi_\varepsilon = 0\}$ converges locally in the Hausdorff distance to a surface whose mean curvature H is given by $f \cdot \nu$ as $\varepsilon \to 0$. Here, f is the weak $W^{1,p}$ limit of f_ε. If the surface is represented locally as a graph of a function u over a domain $\Omega \subset \mathbb{R}^n$, the corresponding relation between the mean curvature and the vector field is expressed as

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = \nu(\nabla u(x)) \cdot f(x, u(x)) \quad \text{in } \Omega, \tag{1.1.4}$$

where $f \in W^{1,p}(\Omega \times \mathbb{R}; \mathbb{R}^{n+1})$ with $p > \frac{n+1}{2}$. Note that f is not bounded in L^∞ in general, unlike the cases studied in [6, 26]. In Chapter 2, we establish the well-posedness of the perturbative problem including (1.1.4) which has a $W^{1,p}$ norm control on the right-hand side of the equation. The following theorem is the main result of Chapter 2.

Theorem 1.1.1. *Let Ω be a $C^{1,1}$ bounded domain in $\mathbb{R}^n$ and fix constants $\varepsilon > 0$, $\frac{n+1}{2} < p < n+1$ and $q = \frac{np}{n+1-p}$. Suppose $h \in W^{2,\infty}(\Omega)$ satisfies the minimal surface equation, that is,*

$$\operatorname{div}\left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\right) = 0. \tag{1.1.5}$$

Then there exists a constant $\delta_1 > 0$ which depends only on n, p, Ω, $\|h\|_{W^{2,\infty}(\Omega)}$, and ε with the following property. Suppose $G \in W^{1,p}(\Omega\times\mathbb{R})$ and $\phi \in W^{2,q}(\Omega)$ satisfy

$$\|G\|_{W^{1,p}(\Omega\times\mathbb{R})} + \|\phi\|_{W^{2,q}(\Omega)} \leq \delta_1, \tag{1.1.6}$$

and a measurable function $H(x,t,z) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is such that $H(x,\cdot,\cdot)$ is a continuous function for a.e. $x \in \Omega$, *and for all $(t,z) \in \mathbb{R} \times \mathbb{R}^n$,*

$$|H(x,t,z)| \leq |G(x,t)| \quad \text{for a.e. } x \in \Omega. \tag{1.1.7}$$

Then, there exists a function $u \in W^{2,q}(\Omega)$ such that $u - h - \phi \in W_0^{1,q}(\Omega)$ and

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = H(x,u(x),\nabla u(x)) \quad \text{in } \Omega, \tag{1.1.8}$$

$$\|u-h\|_{W^{2,q}(\Omega)} < \varepsilon. \tag{1.1.9}$$

The claim proves that there exists a solution of (1.1.1) in a neighbourhood of any minimal surface if H and ϕ are sufficiently small in these norms. In particular, if we take $H(x,t,z) = \nu(z) \cdot f(x,t)$ and $G(x,t) = |f(x,t)|$, where $\|f\|_{W^{1,p}(\Omega\times\mathbb{R})}$ is sufficiently small, above conditions on G and H in Theorem 1.1.1 are satisfied and we can guarantee the existence of a solution for (1.1.1) nearby the given minimal surface (see Corollary 2.3.1). The method of proof is as follows. We prove that the linearized problem of (1.1.1) has a unique solution in $W^{2,q}(\Omega)$ and the norm of this solution is controlled by G and ϕ. When (1.1.6) is satisfied, there exist a suitable function space $\mathcal{A}$ and a mapping $T : \mathcal{A} \to \mathcal{A}$, and a fixed point of T is a solution of (1.1.8) with $u - h - \phi \in W_0^{1,q}(\Omega)$. We show that T satisfies assumptions of the Schauder fixed point theorem, and Theorem 1.1.1 follows.

1.2 Problem on torus

In Chapter 3, we consider the following prescribed mean curvature problem on torus $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$:

$$-\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \nu(\nabla u)\cdot g(x,u(x)) \quad \text{on } \mathbb{T}^n. \tag{1.2.1}$$

The vector field $g(x,x^{n+1}) : \mathbb{T}^n \times \mathbb{R} \to \mathbb{R}^{n+1}$ is given, and we seek a solution u satisfying (1.2.1). The left-hand side of (1.2.1) represents the mean curvature of the graph of u, and the right-hand side is the normal component of the vector field g on the graph.

As we noted in the previous section, prescribed mean curvature problems have been studied by numerous researchers in the case of Dirichlet conditions of a bounded domain $\Omega \subset \mathbb{R}^n$. In the case of a compact Riemannian manifold, Aubin [3] solved the linear elliptic problem $-\partial_i[a_{ij}(x)\partial_j u] = H(x)$ if the integrated value of H is zero. The assumption of the integrated value plays an important role in the existence of solutions to elliptic equations on a compact Riemannian manifold. Denny [8] solved the quasilinear elliptic problem $-\mathrm{div}(a(u(x))\nabla u) = H(x)$ on the torus $\mathbb{T}^n$ with $n = 2, 3$. Prescribed mean curvature problems on the one-dimensional torus $-\left(\frac{u'}{\sqrt{1+(u')^2}}\right)' = H(x,u,u')$ have been investigated for a wide variety of conditions H (refer to [9, 22, 24, 31, 32, 46], for example), but it was not considered in the general dimension torus $\mathbb{T}^n$. Therefore, we considered the special case of (1.2.1). The motivation for the present study comes from a singular perturbation problem as well as Chapter 2. In Chapter 3, we prove the existence of solutions to (1.2.1) assuming that the Sobolev norm of g is sufficiently small, g^{n+1} for the $n+1$th component is monotonous, and the integrated value of g^{n+1} is zero. The following theorem is the main result.

Theorem 1.2.1. *Fix* $\frac{n+1}{2} < p < n+1$ *and* $q = \frac{np}{n+1-p}$. *Then, there exists a constant* $\varepsilon_1 = \varepsilon_1(n,p) > 0$ *with the following property. If* $\varepsilon < \varepsilon_1$, *and* $g = (g^1,\ldots,g^n,g^{n+1}) = (g',g^{n+1}) \in W^{1,p}(\mathbb{T}^n \times (-1,1);\mathbb{R}^{n+1})$ *satisfies* (1.2.2)–(1.2.4),

$$\|g\|_{W^{1,p}(\mathbb{T}^n\times(-1,1))} < \varepsilon^{\frac{2}{3}}, \tag{1.2.2}$$

$$\partial_{n+1}g^{n+1}(x,x^{n+1}) > \varepsilon + \varepsilon^{\frac{1}{2}}|\partial_{n+1}g'(x,x^{n+1})|, \tag{1.2.3}$$

$$\int_{\mathbb{T}^n} g^{n+1}(x,0) = 0, \tag{1.2.4}$$

then there exists a function $u \in W^{2,q}(\mathbb{T}^n)$ *such that*

$$-\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \nu(\nabla u)\cdot g(x,u(x)) \quad \text{on } \mathbb{T}^n. \tag{1.2.5}$$

Moreover, the following inequality holds:

$$\left\| u - \int_{\mathbb{T}^n} u(y)\ dy \right\|_{W^{2,q}(\mathbb{T}^n)} \le \varepsilon^{\frac{1}{2}}. \tag{1.2.6}$$

The assumptions (1.2.2) and (1.2.3) guarantee the existence and uniqueness of solutions to the linearized problem of (1.2.1) where a given function depends on ∇u. (1.2.4) is necessary for the existence of solutions to elliptic equations on the torus. To our knowledge, prescribed mean curvature problems on the torus in the general dimension have been insufficiently studied. However, we have proved the existence of the solution under natural assumptions.

The following is method of proof. We first find the conditions of H for the linearized problem of (1.2.1) $-\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla v|^2}}\right) = H$ to have a unique solution. If we add a suitable constant term for any v, the function $\nu(\nabla v)\cdot g(x,v(x))$ satisfies the conditions. By estimating the norm of this solution with g, the mapping $T(v) = u$ has a fixed point using a fixed-point theorem, and Theorem 1.2.1 follows.

1.3 Allen–Cahn equation

The object of study in Chapter 4 is the energy functional appearing in the van der Waals–Cahn–Hillard theory [7, 15],

$$E_\varepsilon(u) = \int_\Omega \frac{\varepsilon|\nabla u|^2}{2} + \frac{W(u)}{\varepsilon}, \tag{1.3.1}$$

where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ $(n \ge 2)$ is the normalized density distribution of two phases of a material, $|\nabla u|^2 = \sum_{k=1}^n (\partial u/\partial x_k)^2$ and $W : \mathbb{R} \to [0,\infty)$ is a double-well potential with two global minima at ± 1. In the thermodynamic context, W corresponds to the Helmholtz free energy density and the typical example is $W(u) = (1-u^2)^2$. When the positive parameter ε is small relative to the size of the domain Ω and $E_\varepsilon(u)$ is bounded, it is expected that u is close to $+1$ or -1 on most of Ω while a spatial change between ± 1 occurs within a hypersurface-like region of $O(\varepsilon)$ thickness which we may call the *diffused interface* of u. In this case, the quantity $E_\varepsilon(u)$ is expected to be proportional to the surface area of the diffused interface. Due to the importance of the surface area in calculus of variations, it is interesting to investigate the validity of such expectation and other salient properties of E_ε.

In this direction, there have been a number of works studying the asymptotic behavior of E_ε as $\varepsilon \to 0+$ under various assumptions. For the energy minimizers with appropriate side conditions, it is well-known that it Γ-converges to the area functional of the limit interface [20, 25, 28, 29, 38]. On the other hand, due in part to the non-convex nature of the functional, there may exist multiple and even infinite number of critical points of E_ε different from the energy minimizers. For general critical points, Hutchinson and Tonegawa [17] proved that the limit is an integral stationary varifold [1]. For general *stable* critical points, Tonegawa and Wickramasekera [44] proved that the limit is an embedded real-analytic minimal hypersurface except for a closed singular set of codimension seven. More recently, Guaraco [14] showed that a uniform Morse index bound is sufficient to conclude the same regularity for $n \geq 3$ and gave a new proof of Almgren-Pitts theorem [33] as the application. The new proof significantly simplifies the existence part of the proof even though one needs to use Wickramasekera's hard regularity theorem [45].

While the investigations on the critical points of E_ε have direct links to the minimal surface theory as above, more generally, it turned out that suitable controls of the first variation of E_ε guarantee the analogous good asymptotic behaviors. For example, under the assumption that

$$\liminf_{\varepsilon \to 0+} \left(E_\varepsilon(u_\varepsilon) + \|f_\varepsilon\|_{W^{1,p}(\Omega)} \right) < \infty$$

with $f_\varepsilon := -\varepsilon \Delta u_\varepsilon + W'(u_\varepsilon)/\varepsilon$ and $p > n/2$, Tonegawa [40, 43] proved that the limit interface is an integral varifold whose generalized mean curvature belongs to L^q ($q = p(n-1)/(n-p) > n-1$) with respect to the surface measure. Here $W^{1,p}(\Omega) := \{u \in L^p(\Omega) : \nabla u \in L^p(\Omega)\}$. The mean curvature of the limit interface is characterized by the weak $W^{1,p}$ limit of f_ε [35]. Another example concerns one of De Giorgi's conjectures. Under the assumption that (with f_ε as above)

$$\liminf_{\varepsilon \to 0+} \left(E_\varepsilon(u_\varepsilon) + \varepsilon^{-1} \|f_\varepsilon\|^2_{L^2(\Omega)} \right) < \infty$$

and $n = 2, 3$, Röger-Schätzle [34] (independently [30] for the case of $n = 2$) proved the similar result. In this case, the limit interface has an L^2 generalized mean curvature.

In Chapter 4, along the line of research described above, we investigate the asymptotic behavior of u_ε satisfying

$$-\varepsilon\Delta u_\varepsilon + \frac{W'(u_\varepsilon)}{\varepsilon} = \varepsilon v_\varepsilon \cdot \nabla u_\varepsilon, \tag{1.3.2}$$

where v_ε is considered here as a given vector field and we assume that

$$\liminf_{\varepsilon\to 0+} \left(E_\varepsilon(u_\varepsilon) + \|v_\varepsilon\|_{W^{1,p}(\Omega)}\right) < \infty$$

and $p > n/2$. The problem is related to (parabolic) Allen-Cahn-type equations studied in [23, 39], for example. It is also natural to investigate the effect of advection term as $\varepsilon \to 0+$. We prove the analogous result Theorem 1.3.1 to [40, 43], namely, the limit is an integral varifold with L^q (the same as above) generalized mean curvature which is characterized by the weak $W^{1,p}$ limit of v_ε. Using this result, we give some existence theorem for a *vectorial prescribed mean curvature problem*, as described in Theorem 1.3.2. Despite the simplicity of the problem, this is the first existence result in the setting of the min-max method, with minimal regularity assumptions on the prescribed vector field. As for the existence problem for *scalar* constant or prescribed mean curvature using a min-max approach along the lines of Almgren-Pitts [33], we mention papers by X. Xhou and J. Zhu [47, 48].

As for the proof, just as in the case of [17, 40, 43], the key point is to prove a certain monotonicity-type formula which is the essential tool in the setting of Geometric Measure Theory. We wish to treat $\varepsilon v_\varepsilon \cdot \nabla u_\varepsilon$ as a perturbative term, and to do so, we need to control a certain "trace" norm of v_ε on diffused interface. If an ε-independent upper density ratio estimate of diffused surface measure is available, then we can control $\varepsilon v_\varepsilon \cdot \nabla u_\varepsilon$ by the $W^{1,p}$-norm of v_ε. For this purpose, we establish the key estimate, Theorem 4.1.7, which gives a local uniform upper density ratio estimate. Once this part is done, the rest proceeds just like [43] with minor modifications.

Chapter 4 is organized as follows. Section 1 contains the main estimates which ultimately give a monotonicity-type formula, Theorem 4.1.8. In Section 2, we prove the main theorem by modifying the proof in [40, 43], and in Section 3, we give some concluding remarks.

1.3.1 Assumptions and main results

We use the notation that $U_r(a) := \{x \in \mathbb{R}^n : |x-a| < r\}$, $B_r(a) := \{x \in \mathbb{R}^n : |x-a| \le r\}$, $U_r := U_r(0)$ and $B_r := B_r(0)$. Throughout Chapter 4, we assume that:

(a) The function $W : \mathbb{R} \to [0, \infty)$ is C^3 and has two strict minima $W(\pm 1) = W'(\pm 1) = 0$.

(b) For some $\gamma \in (-1, 1)$, $W' > 0$ on $(-1, \gamma)$ and $W' < 0$ on $(\gamma, 1)$.

(c) For some $\alpha \in (0, 1)$ and $\kappa > 0$, $W''(x) \geq \kappa$ for all $|x| \geq \alpha$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We assume that we are given $W^{1,2}(\Omega)$ functions $\{u_i\}_{i=1}^{\infty}$, $W^{1,p}(\Omega; \mathbb{R}^n)$ vector fields $\{v_i\}_{i=1}^{\infty}$ and positive constants $\{\varepsilon_i\}_{i=1}^{\infty}$ satisfying

$$-\varepsilon_i \Delta u_i + \frac{W'(u_i)}{\varepsilon_i} = \varepsilon_i v_i \cdot \nabla u_i \tag{1.3.3}$$

weakly on Ω for each $i \in \mathbb{N}$. In addition, assume that

$$\lim_{i \to \infty} \varepsilon_i = 0, \quad \frac{n}{2} < p < n \tag{1.3.4}$$

and that there exist constants c_0, E_0 and λ_0 such that, for all $i \in \mathbb{N}$, we have:

$$\|u_i\|_{L^\infty(\Omega)} \leq c_0, \tag{1.3.5}$$

$$\int_\Omega \left(\frac{\varepsilon_i |\nabla u_i|^2}{2} + \frac{W(u_i)}{\varepsilon_i} \right) \leq E_0, \tag{1.3.6}$$

$$\|v_i\|_{L^{\frac{np}{n-p}}(\Omega)} + \|\nabla v_i\|_{L^p(\Omega)} \leq \lambda_0. \tag{1.3.7}$$

The condition (1.3.5) is not essential and can be often derived from the PDE or the proof of existence. Here we assume (1.3.5) for simplicity. Next, define

$$\Phi(s) := \int_{-1}^{s} \sqrt{W(t)/2}\, dt, \quad w_i(x) := \Phi(u_i(x)).$$

By the Cauchy–Schwarz inequality and (1.3.6), we obtain

$$\int_\Omega |\nabla w_i| \leq \frac{1}{2} \int_\Omega \left(\frac{\varepsilon_i |\nabla u_i|^2}{2} + \frac{W(u_i)}{\varepsilon_i} \right) \leq \frac{1}{2} E_0.$$

Hence, by the compactness theorem for BV functions [49, Corollary 5.3.4], there exist a converging subsequence (which we denote by the same notation) $\{w_i\}$ in the L^1 norm and the limit BV function w. Define

$$u(x) := \Phi^{-1}(w(x)).$$

where Φ^{-1} is the inverse function of Φ. It follows that u_i converges to u a.e. on Ω. By Fatou's Lemma and (1.3.6), we have

$$\int_\Omega W(u) = \int_\Omega \lim_{i\to\infty} W(u_i) \leq \liminf_{i\to\infty} \int_\Omega W(u_i) = 0.$$

This shows that $u = \pm 1$ a.e. on Ω and u is a BV function. For simplicity we write $\partial^*\{u = 1\}$ as the reduced boundary [49] of $\{u = 1\}$ and $\|\partial^*\{u = 1\}\|$ as the boundary measure.

1.3.2 Associated varifolds

We associate to each solution of (1.3.2) a varifold in a natural way in the following. We refer to [1, 37] for a comprehensive treatment of varifolds.

Let $\mathbf{G}(n, n-1)$ be the Grassmannian, i.e. the space of unoriented $(n-1)$-dimensional subspaces in $\mathbb{R}^n$. We also regard $S \in \mathbf{G}(n, n-1)$ as the $n \times n$ matrix representing the orthogonal projection of $\mathbb{R}^n$ onto S. For two given square-matrices S_1 and S_2, we write $S_1 \cdot S_2 := \operatorname{trace}(S_1^t \circ S_2)$, where the upper-script t indicates the transpose of the matrix and $\circ$ is the matrix multiplication. We say that V is an $(n-1)$-dimensional varifold in $\Omega \subset \mathbb{R}^n$ if V is a Radon measure on $\mathbf{G}_{n-1}(\Omega) := \Omega \times \mathbf{G}(n, n-1)$. Let $\mathbf{V}_{n-1}(\Omega)$ be the set of all $(n-1)$-dimensional varifolds in Ω. Convergence in the varifold sense means convergence in the usual sense of measures. For $V \in \mathbf{V}_{n-1}(\Omega)$, we let $\|V\|$ be the weight measure of V. For $V \in \mathbf{V}_{n-1}(\Omega)$, we define the first variation of V by

$$\delta V(g) := \int_{\mathbf{G}_{n-1}(\Omega)} \nabla g(x) \cdot S \; dV(x, S) \tag{1.3.8}$$

for any vector field $g \in C_c^1(\Omega; \mathbb{R}^n)$. We let $\|\delta V\|$ be the total variation of δV. If $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$, then the Radon-Nikodym derivative $\delta V/\|V\|$ exists as a vector-valued $\|V\|$ measurable function. In this case, we define the generalized mean curvature vector of V by $-\delta V/\|V\|$ and we use the notation H_V.

We associate to each function u_i a varifold V_i as follows. First, we define a Radon measure μ_i on Ω by

$$d\mu_i := \frac{1}{\sigma}\Big(\frac{\varepsilon_i |\nabla u_i|^2}{2} + \frac{W(u_i)}{\varepsilon_i}\Big) d\mathcal{L}^n, \tag{1.3.9}$$

where $\mathcal{L}^n$ is the n-dimensional Lebesgue measure and $\sigma := \int_{-1}^1 \sqrt{2W(s)}\, ds$. Define $V_i \in \mathbf{V}_{n-1}(\Omega)$ by

$$V_i(\phi) := \int_{\{|\nabla u_i| \neq 0\}} \phi\Big(x, I - \frac{\nabla u_i(x)}{|\nabla u_i(x)|} \otimes \frac{\nabla u_i(x)}{|\nabla u_i(x)|}\Big) d\mu_i(x) \tag{1.3.10}$$

for $\phi \in C_c(\mathbf{G}_{n-1}(\Omega))$, where I is the $n \times n$ identity matrix and $\otimes$ is the tensor product of the two vectors. Note that $I - \frac{\nabla u_i(x)}{|\nabla u_i(x)|} \otimes \frac{\nabla u_i(x)}{|\nabla u_i(x)|}$ represents the orthogonal projection to the $(n-1)$-dimensional subspace $\{a \in \mathbb{R}^n : a \cdot \nabla u_i(x) = 0\}$. By definition, we have

$$\|V_i\| = \mu_i \llcorner_{\{|\nabla u_i| \neq 0\}}$$

and by (1.3.8), we have

$$\delta V_i(g) = \int_{\{|\nabla u_i| \neq 0\}} \nabla g \cdot \Big(I - \frac{\nabla u_i}{|\nabla u_i|} \otimes \frac{\nabla u_i}{|\nabla u_i|} \Big) d\mu_i \tag{1.3.11}$$

for each $g \in C_c^1(\Omega, \mathbb{R}^n)$.

1.3.3 Main Theorems

With the above assumptions and notation, we show:

Theorem 1.3.1. *Suppose that* u_i, v_i, ε_i *satisfy* (1.3.3)-(1.3.7) *and let* V_i *be the varifold associated with* u_i *as in* (1.3.10). *On passing to a subsequence we can assume that*

$$v_i \to v \text{ weakly in } W^{1,p}, \quad u_i \to u \text{ a.e.}, \quad V_i \to V \text{ in the varifold sense.}$$

Then we have the following properties.

(1) For each $\phi \in C_c(\Omega)$,

$$\begin{aligned} \frac{1}{2}\|V\|(\phi) &= \lim_{i\to\infty} \frac{1}{\sigma} \int_\Omega \frac{\varepsilon_i}{2} |\nabla u_i|^2 \phi = \lim_{i\to\infty} \frac{1}{\sigma} \int_\Omega \frac{W(u_i)}{\varepsilon_i} \phi \\ &= \lim_{i\to\infty} \frac{1}{\sigma} \int_\Omega |\nabla w_i| \phi. \end{aligned}$$

(2) $\operatorname{spt}\|\partial^*\{u=1\}\| \subset \operatorname{spt}\|V\|$ *and* $\{u_i\}$ *converges locally uniformly to* ± 1 *on* $\Omega \setminus \operatorname{spt}\|V\|$.

(3) For each $0 < b < 1$, $\{|u_i| \le 1-b\}$ *locally converges to* $\operatorname{spt}\|V\|$ *in the Hausdorff distance sense in* Ω.

(4) V *is an integral varifold and the density* $\theta(x)$ *of* V *satisfies*

$$\theta(x) = \begin{cases} \text{odd} & \mathcal{H}^{n-1}\text{a.e. } x \in \partial^*\{u=1\}, \\ \text{even} & \mathcal{H}^{n-1}\text{a.e. } x \in \operatorname{spt}\|V\| \setminus \partial^*\{u=1\}, \end{cases}$$

(5) the generalized mean curvature vector H_V of V is given by

$$H_V(x) = (T_x \operatorname{spt} \|V\|)^{\perp}(v(x)),$$

for $\|V\|$ a.e. $x \in \Omega$.

(6) For $\tilde{\Omega} \subset\subset \Omega$, there exists a constant λ_1 depending only on c_0,λ_0,n,p,W, E_0 and $\operatorname{dist}(\tilde{\Omega}, \partial\Omega)$ *such that*

$$\int_{\tilde{\Omega}} |H_V(x)|^{\frac{p(n-1)}{n-p}} \, d\|V\|(x) \le \int_{\tilde{\Omega}} |v(x)|^{\frac{p(n-1)}{n-p}} \, d\|V\|(x) \le \lambda_1.$$

Note that $\frac{p(n-1)}{n-p} > n-1$ due to (1.3.4).

Since V is integral and the generalized mean curvature vector is in the stated class, V satisfies various good properties described in [37, Section 17]. In particular, $\operatorname{spt} \|V\|$ is a closed countably $(n-1)$-rectifiable set (see [37, 17.9(1)]), and writing $\Gamma := \operatorname{spt} \|V\|$, for any $\phi \in C_c(\mathbf{G}_{n-1}(\Omega))$,

$$\int_{\mathbf{G}_{n-1}(\Omega)} \phi(x, S) \, dV(x, S) = \int_{\Gamma} \phi(x, T_x \Gamma) \theta(x) \, d\mathcal{H}^{n-1}(x).$$

Here, $T_x \Gamma \in \mathbf{G}(n, n-1)$ is the approximate tangent space of Γ at x which exists $\mathcal{H}^{n-1}$ a.e. $x \in \Gamma$. With this notation, (5) implies that $H_V(x) = (T_x \Gamma)^{\perp}(v(x))$ for $\mathcal{H}^{n-1}$ a.e. $x \in \Gamma$, i.e., the generalized mean curvature vector of V coincides with the projection of v to the orthogonal subspace $(T_x \Gamma)^{\perp}$ for $\mathcal{H}^{n-1}$ a.e. $x \in \Gamma$. We emphasize the difference of characterization of the mean curvature vector from [43], where the similar equality holds only on the reduced boundary of $\{u = 1\}$, while the equality here holds on the whole support of $\|V\|$ including on the "hidden boundary" $\Gamma \setminus \partial^*\{u = 1\}$. If we additionally assume that $\theta = 1$ for $\mathcal{H}^{n-1}$ a.e. $x \in \Gamma$, then because of the integrability of H_V and the Allard regularity theorem [1], except for a closed $\mathcal{H}^{n-1}$-null set, Γ is locally a $C^{1,2-\frac{n}{p}}$ hypersurface. Without the assumption $\theta = 1$, we can still conclude that $\operatorname{spt} \|V\|$ is $C^{1,2-\frac{n}{p}}$ hypersurface on a dense open set of $\operatorname{spt} \|V\|$, even though we do not know if the complement is $\mathcal{H}^{n-1}$-null or not.

1.3.4 Vectorial prescribed mean curvature problem

As an application of Theorem 1.3.1 with suitable modifications, we prove the following:

Theorem 1.3.2. *Let (M, g) be a smooth compact n-dimensional Riemannian manifold and let $\rho \in W^{2,p}(M)$ be a given function, where $p > \frac{n}{2}$. Then, there exists a non-zero integral varifold V in M such that*

$$H_V(x) = (T_x \operatorname{spt} \|V\|)^{\perp}(\nabla \rho(x))$$

for $\|V\|$ a.e. $x \in M$.

Proof. We may assume $p < n$. Consider the following functional for $\varepsilon > 0$ and $u \in W^{1,2}(M)$:

$$F_\varepsilon(u) := \int_M \Big(\frac{\varepsilon|\nabla u|^2}{2} + \frac{W(u)}{\varepsilon}\Big) \exp(\rho)\, d\omega_g.$$

By the Sobolev embedding, $\rho \in C^{0,2-\frac{n}{p}}(M)$ and thus $0 < \exp(\min \rho) \leq \exp(\rho) \leq \exp(\max \rho) < \infty$. By considering the path space in $W^{1,2}(M)$ connecting $u \equiv 1$ and $u \equiv -1$, the standard min-max method gives a non-trivial critical point u_ε for each $\varepsilon > 0$, with uniform strictly positive lower and upper bounds of $F_\varepsilon(u_\varepsilon)$. The critical point satisfies (1.3.3) with $v = \nabla\rho$ and $|u_\varepsilon| \leq 1$, with an appropriate modification of the equation. Take a sequence $\varepsilon_i \to 0+$ and a corresponding min-max critical points u_i. Then the sequence $u_i, \nabla\rho, \varepsilon_i$ satisfy all the assumptions of Theorem 1.3.1 with a small error terms coming from the metric of M (see Guaraco's work [14] for an adaptation of the results to Riemannian manifolds in the case $v_\varepsilon = 0$). The limit varifold V thus has the desired property. $\square$

For more remarks on the main results, see Section 4.

Chapter 2

Dirichlet problem in $\mathbb{R}^n$

Throughout this chapter, Ω is a bounded domain in $\mathbb{R}^n$ with $C^{1,1}$ boundary $\partial\Omega$. We define functions $A_{ij} : \mathbb{R}^n \to \mathbb{R}$ $(i, j = 1, \ldots, n)$ as

$$A_{ij}(z) := \frac{1}{\sqrt{1+|z|^2}} \left(\delta_{ij} - \frac{z_i z_j}{1+|z|^2} \right)$$

and the operator

$$L[z](u) := A_{ij}(z) D_{ij} u(x) \quad \text{for any } u \in W^{2,1}(\Omega),$$

where we omit the summation over $i, j = 1, \ldots, n$. By the Cauchy–Schwarz inequality, for any $\xi \in \mathbb{R}^n$,

$$\begin{aligned} A_{ij}(z)\xi_i\xi_j &= \frac{1}{\sqrt{1+|z|^2}} \left(\delta_{ij} - \frac{z_i z_j}{1+|z|^2} \right) \xi_i \xi_j \\ &= \frac{1}{\sqrt{1+|z|^2}} \left[|\xi|^2 - \left(\frac{z_i}{\sqrt{1+|z|^2}} \xi_i \right)^2 \right] \\ &\geq \frac{1}{\sqrt{1+|z|^2}} \left[|\xi|^2 - \left(\frac{|z|^2}{1+|z|^2} \right) |\xi|^2 \right] \\ &= \frac{1}{(1+|z|^2)^{\frac{3}{2}}} |\xi|^2. \end{aligned} \tag{2.0.1}$$

Hence, as is well-known, the operator $L[z]$ is elliptic.

2.1 Existence of solutions to the linearized problem

Theorem 2.1.1. *Suppose $v \in C^{1,\alpha}(\bar{\Omega})$ with $0 < \alpha < 1$, $B = (B_1, \ldots, B_n) \in L^\infty(\Omega; \mathbb{R}^n)$ with $\|B_i\|_{L^\infty(\Omega)} \leq K$ for all $i \in \{1, \ldots, n\}$, $f \in L^q(\Omega)$ and $\phi \in$*

$W^{2,q}(\Omega)$ with $q > n$. Then there exists a unique function $u \in W^{2,q}(\Omega)$ such that

$$\begin{cases} L[\nabla v](u) + B \cdot \nabla u = f \quad \text{in } \Omega, \\ u - \phi \in W_0^{1,q}(\Omega). \end{cases} \tag{2.1.1}$$

Moreover, there exists a constant c_1 which depends only on n, q, Ω, K, and $\|v\|_{C^{1,\alpha}(\bar{\Omega})}$ such that

$$\|u\|_{W^{2,q}(\Omega)} \le c_1(\|f\|_{L^q(\Omega)} + \|\phi\|_{W^{2,q}(\Omega)}). \tag{2.1.2}$$

Proof. By (2.0.1), for any $\xi \in \mathbb{R}^n$,

$$A_{ij}(\nabla v)\xi_i\xi_j \ge \frac{1}{(1 + \|v\|^2_{C^{1,\alpha}(\bar{\Omega})})^{\frac{3}{2}}}|\xi|^2 =: \lambda|\xi|^2, \tag{2.1.3}$$

where the constant λ depends only on $\|v\|_{C^{1,\alpha}(\bar{\Omega})}$. Since each A_{ij} is a smooth function of ∇v, there exists a constant Λ which depends only on $\|v\|_{C^{1,\alpha}(\bar{\Omega})}$ such that

$$\|A_{ij}(\nabla v)\|_{C^{0,\alpha}(\bar{\Omega})} \le \Lambda \quad \text{for all } i, j \in \{1, \dots, n\}. \tag{2.1.4}$$

By (2.1.3) and (2.1.4), there exists a unique solution $u \in W^{2,q}(\Omega)$ satisfying (2.1.1) (cf. [12, Theorem 9.15]). Using [12, Theorem 9.13], we can know that there exists a constant c_2 which depends only on n, q, Ω, λ, K, and Λ such that

$$\|u\|_{W^{2,q}(\Omega)} \le c_2(\|u\|_{L^q(\Omega)} + \|f\|_{L^q(\Omega)} + \|\phi\|_{W^{2,q}(\Omega)}). \tag{2.1.5}$$

Using the Aleksandrov maximum principle [12, Theorem 9.1], we can know that there exists a constant c_3 which depends only on n, Ω, K, and λ such that

$$\begin{aligned} \|u\|_{L^\infty(\Omega)} &\le \sup_{x \in \partial\Omega} |u| + c_3\|f\|_{L^n(\Omega)} \\ &= \sup_{x \in \partial\Omega} |\phi| + c_3\|f\|_{L^n(\Omega)}. \end{aligned} \tag{2.1.6}$$

By the Hölder and Sobolev inequalities, $\phi \in C(\bar{\Omega})$ and

$$\begin{aligned} \|u\|_{L^q(\Omega)} &\le c\|u\|_{L^\infty(\Omega)} \\ &\le c(\sup_{x \in \partial\Omega} |\phi| + \|f\|_{L^n(\Omega)}) \\ &\le c(\|\phi\|_{C(\bar{\Omega})} + \|f\|_{L^n(\Omega)}) \\ &\le c_4(\|f\|_{L^q(\Omega)} + \|\phi\|_{W^{2,q}(\Omega)}), \end{aligned} \tag{2.1.7}$$

where c_4 depends only on n, q, and Ω. By (2.1.5) and (2.1.7), there exists a constant c_1 which depends only on n, q, Ω, λ, K, and Λ such that

$$\|u\|_{W^{2,q}(\Omega)} \leq c_1(\|f\|_{L^q(\Omega)} + \|\phi\|_{W^{2,q}(\Omega)}). \tag{2.1.8}$$

Thus this theorem follows. □

2.2 Estimate for solutions

To proceed, we need the following theorem (cf. [49, Theorem 5.12.4]).

Theorem 2.2.1. *Let μ be a positive Radon measure on $\mathbb{R}^{n+1}$ satisfying*

$$K(\mu) := \sup_{B_r(x)\subset\mathbb{R}^{n+1}} \frac{1}{r^n}\mu(B_r(x)) < \infty.$$

Then there exists a constant c_5 which depends only on n such that

$$\left|\int_{\mathbb{R}^{n+1}} \phi\, d\mu\right| \leq c_5 K(\mu) \int_{\mathbb{R}^{n+1}} |\nabla\phi|\, d\mathcal{L}^{n+1}$$

for all $\phi \in C_c^1(\mathbb{R}^{n+1})$.

Lemma 2.2.1. *Suppose $v \in W^{1,\infty}(\Omega)$ with $\|v\|_{W^{1,\infty}(\Omega)} \leq V$ and $G \in W^{1,p}(\Omega\times\mathbb{R})$ with $\frac{n+1}{2} < p < n+1$. Let $q = \frac{np}{n+1-p}$. Then there exists a constant c_6 which depends only on n, p, Ω, and V such that*

$$\|G(\cdot, v(\cdot))\|_{L^q(\Omega)} \leq c_6\|G\|_{W^{1,p}(\Omega\times\mathbb{R})}. \tag{2.2.1}$$

Proof. Define

$$\Gamma := \{(x, v(x)) \in \Omega\times\mathbb{R}\}.$$

A set $B_r^n(x)$ is the open ball with center x and radius r in $\mathbb{R}^n$. In the following, $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure in $\mathbb{R}^{n+1}$ and $\mathcal{H}^n\llcorner_\Gamma$ is a Radon measure defined by

$$\mathcal{H}^n\llcorner_\Gamma(A) := \mathcal{H}^n(A\cap\Gamma) \quad \text{for all } A \subset \mathbb{R}^{n+1}.$$

Then the support of $\mathcal{H}^n\llcorner_\Gamma$ satisfies in particular $\mathrm{spt}\mathcal{H}^n\llcorner_\Gamma \subset \Omega\times(-2V, 2V)$. For any $B_r^{n+1}((x_0, x_0')) \subset \mathbb{R}^{n+1}$ with $(x_0, x_0') \in \mathbb{R}^n\times\mathbb{R}$,

$$\frac{1}{r^n}\mathcal{H}^n\llcorner_\Gamma(B_r^{n+1}((x_0, x_0'))) \leq \frac{1}{r^n}\int_{B_r^n(x_0)\cap\Omega} \sqrt{1+|\nabla v|^2}\, d\mathcal{L}^n \leq (1+V)\omega_n, \tag{2.2.2}$$

where ω_n is the volume of n-dimensional unit open ball. Using the standard Extension Theorem, we can know that there exists a function $\tilde{G} \in W^{1,p}(\mathbb{R}^{n+1})$ such that $\tilde{G} = G$ in $\Omega \times (-2V, 2V)$ and

$$\|\tilde{G}\|_{W^{1,p}(\mathbb{R}^{n+1})} \leq c_7 \|G\|_{W^{1,p}(\Omega\times(-2V,2V))}, \tag{2.2.3}$$

where c_7 depends only on n, p, Ω, and V. By Theorem 2.2.1 and smoothly approximating $\tilde{G}$,

$$\begin{aligned}
\int_\Omega |G(x, v(x))|^q \, dx &\leq \int_\Omega |\tilde{G}(x, v(x))|^q \sqrt{1 + |\nabla v|^2} \, dx \\
&= \int_\Gamma |\tilde{G}(x, x_{n+1})|^q \, d\mathcal{H}^n \\
&\leq c(n, V) \int_{\mathbb{R}^{n+1}} |\nabla \tilde{G}| |\tilde{G}|^{q-1} \, d\mathcal{L}^{n+1} \\
&\leq c(n, p, V) \|\nabla \tilde{G}\|_{L^p(\mathbb{R}^{n+1})} \|\tilde{G}\|^{q-1}_{W^{1,p}(\mathbb{R}^{n+1})} \\
&\leq c(n, p, V) c_7 \|G\|^q_{W^{1,p}(\Omega\times(-2V,2V))} \\
&\leq c(n, p, V) c_7 \|G\|^q_{W^{1,p}(\Omega\times\mathbb{R})}.
\end{aligned} \tag{2.2.4}$$

This lemma follows. □

2.3 Schauder fixed point theorem

We write the Schauder fixed point theorem needed later ([12, Corollary 11.2]).

Theorem 2.3.1. *Let $\mathcal{G}$ be a closed convex set in Banach space $\mathcal{B}$ and let T be a continuous mapping of $\mathcal{G}$ into itself such that the image $T(\mathcal{G})$ is precompact. Then T has a fixed point.*

We first prove Theorem 1.1.1 in the case that $h = 0$.

Theorem 2.3.2. *Assume that $G \in W^{1,p}(\Omega \times \mathbb{R})$ with $\frac{n+1}{2} < p < n + 1$ and $\phi \in W^{2,q}(\Omega)$ with $q = \frac{np}{n+1-p}$. Then there exists a constant $\delta_2 > 0$ which depends only on n, p, and Ω such that, if*

$$\|G\|_{W^{1,p}(\Omega\times\mathbb{R})} + \|\phi\|_{W^{2,q}(\Omega)} \leq \delta_2, \tag{2.3.1}$$

then, for any measurable function $H(x, t, z) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ such that $H(x, \cdot, \cdot)$ is a continuous function for a.e. $x \in \Omega$ *and*

$$|H(x, t, z)| \leq |G(x, t)| \quad \text{for a.e. } x \in \Omega, \text{ any } (t, z) \in \mathbb{R} \times \mathbb{R}^n, \tag{2.3.2}$$

there exists a function $u \in W^{2,q}(\Omega)$ *such that* $u - \phi \in W_0^{1,q}(\Omega)$ *and*

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = H(x, u(x), \nabla u(x)) \quad \text{in } \Omega. \tag{2.3.3}$$

Proof. Define

$$\mathcal{A} := \{v \in C^{1,\frac{1}{2}-\frac{n}{2q}}(\bar{\Omega}); \|v\|_{C^{1,\frac{1}{2}-\frac{n}{2q}}(\bar{\Omega})} \leq 1\}. \tag{2.3.4}$$

The set $\mathcal{A}$ is a closed convex set in Banach space $C^{1,\frac{1}{2}-\frac{n}{2q}}(\bar{\Omega})$. By (2.3.2) and Lemma 2.2.1, $H(\cdot, v(\cdot), \nabla v(\cdot)) \in L^q(\Omega)$ for any $v \in \mathcal{A}$. Using Theorem 2.1.1, we can know that there exist a unique function $w \in W^{2,q}(\Omega)$ and a constant $c_8 > 0$ which depends only on n, p, Ω, and not on v such that

$$\begin{cases} L[\nabla v](w) = H(x, v, \nabla v) \quad \text{in } \Omega, \\ w - \phi \in W_0^{1,q}(\Omega), \\ \|w\|_{W^{2,q}(\Omega)} \leq c_8(\|G\|_{W^{1,p}(\Omega\times\mathbb{R})} + \|\phi\|_{W^{2,q}(\Omega)}). \end{cases} \tag{2.3.5}$$

By the Sobolev inequality and (2.3.5), we obtain

$$\begin{aligned} \|w\|_{C^{1,\frac{1}{2}-\frac{n}{2q}}(\bar{\Omega})} &\leq c_9\|w\|_{C^{1,1-\frac{n}{q}}(\bar{\Omega})} \\ &\leq c_{10}\|w\|_{W^{2,q}(\Omega)} \\ &\leq c_{11}(\|G\|_{W^{1,p}(\Omega\times\mathbb{R})} + \|\phi\|_{W^{2,q}(\Omega)}), \end{aligned} \tag{2.3.6}$$

where $c_9, c_{10}, c_{11} > 0$ depend only on n, p, and Ω. Suppose that

$$\|G\|_{W^{1,p}(\Omega\times\mathbb{R})} + \|\phi\|_{W^{2,q}(\Omega)} \leq c_{11}^{-1} =: \delta_2(n, p, \Omega). \tag{2.3.7}$$

Let us define an operator $T : \mathcal{A} \to \mathcal{A}$ by $T(v) = w$ which satisfies (2.3.5). We show that $T(\mathcal{A})$ is precompact and T is a continuous mapping. For any sequence $\{v_m\}_{m\in\mathbb{N}} \subset \mathcal{A}$, we have $\sup_{m\in\mathbb{N}} \|T(v_m)\|_{C^{1,1-\frac{n}{q}}(\bar{\Omega})} \leq c_9^{-1}$ by (2.3.6) and (2.3.7). There exists a subsequence $\{T(v_{m_k})\}_{k\in\mathbb{N}} \subset \{T(v_m)\}_{m\in\mathbb{N}}$ which converges to a function $w_\infty \in C^1(\bar{\Omega})$ in the sense of $C^1(\bar{\Omega})$ by the Ascoli-Arzelà theorem. We see that $w_\infty \in C^{1,1-\frac{n}{q}}(\bar{\Omega})$ because

$$\frac{|\nabla w_\infty(x) - \nabla w_\infty(y)|}{|x-y|^{1-\frac{n}{q}}} = \lim_{k\to\infty} \frac{|\nabla T(v_{m_k})(x) - \nabla T(v_{m_k})(y)|}{|x-y|^{1-\frac{n}{q}}} \leq c_9^{-1}.$$

Let $\tilde{w}_k := T(v_{m_k}) - w_\infty$, and $\tilde{w}_k$ converges to 0 in the sense of $C^1(\bar{\Omega})$. Then we have

$$\begin{aligned} \frac{|\nabla \tilde{w}_k(x) - \nabla \tilde{w}_k(y)|}{|x-y|^{\frac{1}{2}-\frac{n}{2q}}} &\leq \left(\frac{|\nabla \tilde{w}_k(x) - \nabla \tilde{w}_k(y)|}{|x-y|^{1-\frac{n}{q}}}\right)^{\frac{1}{2}} |\nabla \tilde{w}_k(x) - \nabla \tilde{w}_k(y)|^{\frac{1}{2}} \\ &\leq 2c_9^{-\frac{1}{2}}(2\|\nabla \tilde{w}_k\|_{L^\infty(\Omega)})^{\frac{1}{2}}. \end{aligned} \tag{2.3.8}$$

Hence, $\{T(v_{m_k})\}_{k\in\mathbb{N}}$ converges to a function w_∞ in the sense of $C^{1,\frac{1}{2}-\frac{n}{2q}}(\bar{\Omega})$, and the operator T is a compact mapping. In particular, the set $T(\mathcal{A})$ is precompact.

Suppose that $\{v_m\}_{m\in\mathbb{N}}$ converges to v in the sense of $C^{1,\frac{1}{2}-\frac{n}{2q}}(\bar{\Omega})$. By (2.3.6) and (2.3.7), $\sup_{m\in\mathbb{N}}\|T(v_m)\|_{W^{2,q}(\Omega)}$ is bounded . Hence, there exists a subsequence $\{T(v_{m_k})\}_{k\in\mathbb{N}} \subset \{T(v_m)\}_{m\in\mathbb{N}}$ which weakly converges to a function $w \in W^{2,q}(\Omega)$. We show $T(v) = w$, that is,

$$A_{ij}(\nabla v(x))D_{ij}w(x) = H(x, v, \nabla v).$$

For any $\psi \in C_0^\infty(\Omega)$, by the weak convergence and the Hölder inequality,

$$\begin{aligned}
&\left|\int_\Omega \psi\{A_{ij}(\nabla v)D_{ij}w - A_{ij}(\nabla v_{m_k})D_{ij}(T(v_{m_k}))\}\right| \\
\leq&\left|\int_\Omega \psi A_{ij}(\nabla v)(D_{ij}w - D_{ij}(T(v_{m_k})))\right| \\
&+\left|\int_\Omega \psi D_{ij}(T(v_{m_k}))(A_{ij}(\nabla v) - A_{ij}(\nabla v_{m_k}))\right| \\
\leq&\left|\int_\Omega \psi A_{ij}(\nabla v)(D_{ij}w - D_{ij}(T(v_{m_k})))\right| \\
&+\|T(v_{m_k})\|_{W^{2,q}(\Omega)}\|\psi(A_{ij}(\nabla v) - A_{ij}(\nabla v_{m_k}))\|_{L^{\frac{q}{q-1}}(\Omega)} \\
\to&0 \quad (k\to\infty).
\end{aligned} \tag{2.3.9}$$

By (2.3.2) and $\|v_{m_k}\|_{L^\infty(\Omega)}$, $\|v\|_{L^\infty(\Omega)} \leq 1$, we compute

$$\begin{aligned}
&|H(x, v_{m_k}(x), \nabla v_{m_k}(x))| \\
\leq&|G(x, v_{m_k}(x)) - G(x, v(x))| + |G(x, v(x))| \\
\leq&\int_{-1}^1 |D_tG(x,t)|\ dt + |G(x, v(x))|.
\end{aligned} \tag{2.3.10}$$

$\int_{-1}^1 |D_tG(\cdot,t)|\ dt + |G(\cdot, v(\cdot))|$ is an integrable function by Lemma 2.2.1, $\|v\|_{C^1(\bar{\Omega})} \leq 1$, and Fubini's theorem. Since H is a continuous function with respect to t and z, using the dominated convergence theorem, we have

$$\int_\Omega \psi\{H(x, v(x), \nabla v(x)) - H(x, v_{m_k}(x), \nabla v_{m_k}(x))\} \to 0 \quad (k\to\infty). \tag{2.3.11}$$

By (2.3.9) and (2.3.11),

$$\begin{aligned}&\int_\Omega \psi\{A_{ij}(\nabla v)D_{ij}w - H(x, v(x), \nabla v(x))\}\\ =&\lim_{k\to\infty}\int_\Omega \psi\{A_{ij}(\nabla v_{m_k})D_{ij}(T(v_{m_k})) - H(x, v_{m_k}(x), \nabla v_{m_k}(x))\}\\ =&0. \qquad (2.3.12)\end{aligned}$$

Using the fundamental lemma of the calculus of variations, we have

$$A_{ij}(x, \nabla v)D_{ij}w - H(x, v(x), \nabla v(x)) = 0 \quad \text{for a.e. } x \in \Omega,$$

and $T(v) = w$. Hence, $\{T(v_m)\}_{m\in\mathbb{N}}$ weakly converges to $T(v)$ in $W^{2,q}(\Omega)$. By the compactness of T and the uniqueness of limit, we can show $\{T(v_m)\}_{m\in\mathbb{N}}$ converges to $T(v)$ in $C^{1,\frac{1}{2}-\frac{n}{2q}}(\bar{\Omega})$, and T is a continuous mapping. Using Theorem 2.3.1, we obtain a function $u \in W^{2,q}(\Omega)$ satisfying $u - \phi \in W_0^{1,q}(\Omega)$ and (2.3.3). □

Proof of Theorem 1.1.1. We should show that there exists a function $\tilde{u} \in W^{2,q}(\Omega)$ such that

$$A_{ij}(\nabla\tilde{u} + \nabla h)D_{ij}(\tilde{u} + h) = H(x, \tilde{u} + h, \nabla\tilde{u} + \nabla h), \qquad (2.3.13)$$

$$\tilde{u} - \phi \in W_0^{1,q}(\Omega), \qquad (2.3.14)$$

$$\|\tilde{u}\|_{W^{2,q}(\Omega)} < \varepsilon. \qquad (2.3.15)$$

Using the minimal surface equation (1.1.5) for h, we convert (2.3.13) as

$$\begin{aligned}&A_{ij}(\nabla\tilde{u} + \nabla h)D_{ij}\tilde{u} + \frac{D_{ij}h}{(1 + |\nabla\tilde{u} + \nabla h|^2)^{\frac{3}{2}}}((|\nabla\tilde{u}|^2 + 2\nabla\tilde{u}\cdot\nabla h)\delta_{ij}\\ &- D_i\tilde{u}D_j\tilde{u} - D_i\tilde{u}D_jh - D_j\tilde{u}D_ih)\\ =&H(x, \tilde{u} + h, \nabla\tilde{u} + \nabla h). \qquad (2.3.16)\end{aligned}$$

Define

$$\mathcal{A} := \{v \in C^{1,\frac{1}{2}-\frac{n}{2q}}(\bar{\Omega}); \|v\|_{C^{1,\frac{1}{2}-\frac{n}{2q}}(\bar{\Omega})} \leq \varepsilon\}. \qquad (2.3.17)$$

The set $\mathcal{A}$ is a closed convex set in Banach space $C^{1,\frac{1}{2}-\frac{n}{2q}}(\bar{\Omega})$. We consider the following differential equation,

$$\begin{aligned}&A_{ij}(\nabla v + \nabla h)D_{ij}w + \frac{D_{ij}h}{(1 + |\nabla v + \nabla h|^2)^{\frac{3}{2}}}((\nabla v\cdot\nabla w + 2\nabla w\cdot\nabla h)\delta_{ij}\\ &- D_ivD_jw - D_iwD_jh - D_jwD_ih)\\ =&H(x, v + h, \nabla v + \nabla h). \qquad (2.3.18)\end{aligned}$$

Define

$$\begin{aligned}B(\nabla v)\cdot\nabla w :=&\frac{D_{ij}h}{(1+|\nabla v+\nabla h|^2)^{\frac{3}{2}}}((\nabla v\cdot\nabla w+2\nabla w\cdot\nabla h)\delta_{ij}\\&-D_i vD_j w-D_i wD_j h-D_j wD_i h).\end{aligned}$$

Here, there exists a constant $c_{12}>0$ which depends only on n, p, Ω, ε, and $\|h\|_{W^{2,\infty}(\Omega)}$ such that

$$\|B_i(\nabla v)\|_{L^\infty(\Omega)}\le c_{12}\quad\text{for all } i\in\{1,\dots,n\},\tag{2.3.19}$$

where $B(\nabla v)=(B_1(\nabla v),\dots,B_n(\nabla v))\in L^\infty(\Omega;\mathbb{R}^n)$.

Using Theorem 2.1.1, we obtain a unique function $w\in W^{2,q}(\Omega)$ satisfying $w-\phi\in W_0^{1,q}(\Omega)$ and (2.3.18). By (2.3.19), Theorem 2.1.1, Lemma 2.2.1, and the Sobolev inequality, there exists a constant $c_{13}>0$ which depends only on n, p, Ω, ε, and $\|h\|_{W^{2,\infty}(\Omega)}$ such that

$$\|w\|_{C^{1,\frac{1}{2}-\frac{n}{2q}}(\bar\Omega)}\le c_{13}(\|G\|_{W^{1,p}(\Omega\times\mathbb{R})}+\|\phi\|_{W^{2,q}(\Omega)}).\tag{2.3.20}$$

Suppose that we have

$$\|G\|_{W^{1,p}(\Omega\times\mathbb{R})}+\|\phi\|_{W^{2,q}(\Omega)}\le c_{13}^{-1}\varepsilon:=\delta_1.\tag{2.3.21}$$

Let a operator $T:\mathcal{A}\to\mathcal{A}$ be defined by $T(v)=w$ which satisfies $w-\phi\in W_0^{1,q}(\Omega)$ and (2.3.18). The compactness of T can be proved by the argument of Theorem 2.3.2. In particular, the set $T(\mathcal{A})$ is precompact.

Suppose that $\{v_m\}_{m\in\mathbb{N}}\subset\mathcal{A}$ converges to v in the sense of $C^{1,\frac{1}{2}-\frac{n}{2q}}(\bar\Omega)$. Then there exists a subsequence $\{T(v_{m_k})\}_{k\in\mathbb{N}}\subset\{T(v_m)\}_{m\in\mathbb{N}}$ which weakly converges to a function $w\in W^{2,q}(\Omega)$. For any $\psi\in C_0^\infty(\Omega)$,

$$\begin{aligned}&\int_\Omega\psi\{B(\nabla v)\cdot\nabla w-B(\nabla v_{m_k})\cdot\nabla T(v_{m_k})\}\\=&\int_\Omega\psi B(\nabla v)\cdot(\nabla w-\nabla(T(v_{m_k})))\\&+\int_\Omega\psi\nabla(T(v_{m_k}))\cdot(B(\nabla v)-B(\nabla v_{m_k}))\\\to&0\quad(k\to\infty),\end{aligned}\tag{2.3.22}$$

since B is a continuous function and $T(v_{m_k})$ converges weakly to w. By (2.3.22) and the argument of Theorem 2.3.2, we can show that T is a continuous mapping. Using Theorem 2.3.1, we obtain a function $\tilde u\in W^{2,q}(\Omega)$ satisfying (2.3.13) and (2.3.14). Moreover, $\tilde u$ satisfies (2.3.15) by (2.3.20) and (2.3.21). Define $u:=\tilde u+h$. Then u satisfies $u-h-\phi\in W_0^{1,q}(\Omega)$, (1.1.8), and (1.1.9), and the proof is complete. □

Corollary 2.3.1. *Suppose* $f = (f_1, \dots, f_{n+1}) \in W^{1,p}(\Omega \times \mathbb{R}; \mathbb{R}^{n+1})$ *with* $\frac{n+1}{2} < p < n+1$ *and* $\phi \in W^{2,q}(\Omega)$ *with* $q = \frac{np}{n+1-p}$. *Let* $\varepsilon > 0$ *be arbitrary. Suppose* $h \in W^{2,\infty}(\Omega)$ *satisfies the minimal surface equation, that is,*

$$\operatorname{div}\left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\right) = 0. \tag{2.3.23}$$

Let $\delta_1 > 0$ *be the constant as in Theorem 1.1.1. If*

$$\sum_{i=1}^{n+1} \|f_i\|_{W^{1,p}(\Omega\times\mathbb{R})} + \|\phi\|_{W^{2,q}(\Omega)} \le \delta_1, \tag{2.3.24}$$

then there exists a function $u \in W^{2,q}(\Omega)$ *such that* $u - h - \phi \in W_0^{1,q}(\Omega)$ *and*

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \nu(\nabla u(x)) \cdot f(x, u(x)) \quad \text{in } \Omega, \tag{2.3.25}$$

$$\|u - h\|_{W^{2,q}(\Omega)} < \varepsilon. \tag{2.3.26}$$

Proof. Define

$$H(x,t,z) := \nu(z) \cdot f(x,t).$$

By $f \in W^{1,p}(\Omega \times \mathbb{R}; \mathbb{R}^{n+1})$, for a.e. $x \in \Omega$, $f(x,\cdot)$ is an absolutely continuous function. Hence $H(x,\cdot,\cdot)$ is a continuous function for almost every $x \in \Omega$. We have

$$|H(x,t,z)| \le \sum_{i=1}^{n+1} |f_i(x,t)| \quad \text{for a.e. } x \in \Omega, \text{ any } (t,z) \in \mathbb{R} \times \mathbb{R}^n,$$

and $\sum_{i=1}^{n+1} |f_i(x,t)| \in W^{1,p}(\Omega \times \mathbb{R})$. By the Minkowski inequality,

$$\left\| \sum_{i=1}^{n+1} |f_i(x,t)| \right\|_{W^{1,p}(\Omega\times\mathbb{R})} \le \sum_{i=1}^{n+1} \|f_i\|_{W^{1,p}(\Omega\times\mathbb{R})}.$$

Define

$$G(x,t) := \sum_{i=1}^{n+1} |f_i(x,t)|.$$

Then H and G satisfy the assumption of Theorem 1.1.1, and this corollary follows. □

Remark 2.3.2. *The uniqueness of solutions follows immediately using [12, Theorem 10.2]. Under the assumptions of Theorem 1.1.1, if we additionally assume that* H *is non-decreasing in* t *for each* $(x,z) \in \Omega \times \mathbb{R}^n$ *and continuously differentiable with respect to the* z *variables in* $\Omega \times \mathbb{R} \times \mathbb{R}^n$, *then the solution is unique in* $W^{2,q}(\Omega)$.

Chapter 3

Problem on torus

A theorem that holds in Euclidean space also holds on a torus, as we consider a function on a torus to be a periodic function in Euclidean space.

Let $X(\mathbb{T}^n)$ be a function space on $\mathbb{T}^n$. We define a subspace $X_{ave}(\mathbb{T}^n) \subset X(\mathbb{T}^n)$ as

$$X_{ave} := \{w \in X; \int_{\mathbb{T}^n} w = 0\}.$$

3.1 Existence of solutions to the linearized problem

Theorem 3.1.1. *Suppose $v \in C^1(\mathbb{T}^n)$ and $H \in L^2_{ave}(\mathbb{T}^n)$. Then, there exists a unique function $u \in W^{1,2}_{ave}(\mathbb{T}^n)$ such that*

$$\int_{\mathbb{T}^n} \frac{\nabla u \cdot \nabla \phi}{\sqrt{1+|\nabla v|^2}} = \int_{\mathbb{T}^n} H\phi$$

for all $\phi \in W^{1,2}(\mathbb{T}^n)$.

Proof. We define a function $B : W^{1,2}_{ave}(\mathbb{T}^n) \times W^{1,2}_{ave}(\mathbb{T}^n) \to \mathbb{R}$ as

$$B[w_1, w_2, v] := \int_{\mathbb{T}^n} \frac{\nabla w_1 \cdot \nabla w_2}{\sqrt{1+|\nabla v|^2}}.$$

By the Hölder inequality, we obtain

$$\begin{aligned} |B[w_1, w_2, v]| &\le \int_{\mathbb{T}^n} |\nabla w_1||\nabla w_2| \\ &\le \|\nabla w_1\|_{L^2(\mathbb{T}^n)} \|\nabla w_2\|_{L^2(\mathbb{T}^n)} \\ &\le \|w_1\|_{W^{1,2}(\mathbb{T}^n)} \|w_2\|_{W^{1,2}(\mathbb{T}^n)}. \end{aligned} \tag{3.1.1}$$

Using the Poincaré inequality, we have

$$
\begin{aligned}
|B[w,w,v]| &\geq \frac{1}{\sqrt{1+\|v\|^2_{C^1(\mathbb{T}^n)}}}\|\nabla w\|^2_{L^2(\mathbb{T}^n)} \\
&\geq \frac{1}{\sqrt{1+\|v\|^2_{C^1(\mathbb{T}^n)}}}\|\nabla w\|^2_{W^{1,2}(\mathbb{T}^n)}. \qquad (3.1.2)
\end{aligned}
$$

By (3.1.1), (3.1.2), and the Lax–Milgram theorem, for any $H \in L^2_{ave}(\mathbb{T}^n)$, there exists a unique function $u \in W^{1,2}_{ave}(\mathbb{T}^n)$ such that

$$
\int_{\mathbb{T}^n} \frac{\nabla u \cdot \nabla \psi}{\sqrt{1+|\nabla v|^2}} = \int_{\mathbb{T}^n} H\psi \qquad (3.1.3)
$$

for all $\psi \in W^{1,2}_{ave}(\mathbb{T}^n)$. For any $\phi \in W^{1,2}(\mathbb{T}^n)$, we define $c_\phi := \int_{\mathbb{T}^n} \phi$ and $\tilde{\phi} := \phi - c_\phi \in W^{1,2}_{ave}(\mathbb{T}^n)$. By (3.1.3) and $H \in L^2_{ave}(\mathbb{T}^n)$, we obtain

$$
\begin{aligned}
\int_{\mathbb{T}^n} \frac{\nabla u \cdot \nabla \phi}{\sqrt{1+|\nabla v|^2}} &= \int_{\mathbb{T}^n} \frac{\nabla u \cdot \nabla \tilde{\phi}}{\sqrt{1+|\nabla v|^2}} \\
&= \int_{\mathbb{T}^n} H\tilde{\phi} \\
&= \int_{\mathbb{T}^n} H\phi. \qquad (3.1.4)
\end{aligned}
$$

Thus, Theorem 3.1.1 follows. □

We define a mollifier as follows.

$$
\eta(x) := \begin{cases} C\exp\left(\frac{1}{|x|^2-1}\right) & \text{for } |x| < 1 \\ 0 & \text{for } |x| \geq 1, \end{cases}
$$

where the constant $C > 0$ is selected such that $\int_{\mathbb{R}^{n+1}} \eta = 1$. We define $\eta_\lambda(x) := \frac{1}{\lambda^n}\eta(\frac{x}{\lambda})$. For any $f \in L^2(\mathbb{T}^n \times (-1,1))$ and $x^{n+1} \in (-1+\lambda, 1-\lambda)$,

$$
\begin{aligned}
f_\lambda(x, x^{n+1}) :&= \int_{\mathbb{T}^n \times (-1,1)} \eta_\lambda(x-y, x^{n+1}-y^{n+1}), f(y, y^{n+1})\ dy \\
&= \int_{B^{n+1}(0,\lambda)} \eta_\lambda(y, y^{n+1}) f(x-y, x^{n+1}-y^{n+1})\ dy,
\end{aligned}
$$

where $B^{n+1}(x,\lambda)$ is an open ball with center x and radius λ in $\mathbb{T}^n \times \mathbb{R}$. Moreover, for any $g \in W^{1,p}(\mathbb{T}^n \times (-1,1); \mathbb{R}^{n+1})$, we define $g_\lambda := (g^1_\lambda, \ldots, g^n_\lambda, g^{n+1}_\lambda) = (g'_\lambda, g^{n+1}_\lambda)$.

Lemma 3.1.2. *Fix* $\beta_1 > 0$ *and* $0 < \lambda < 1$. *Suppose* $v \in C^1(\mathbb{T}^n)$ *satisfies* $\|v\|_{C^1(\mathbb{T}^n)} < \beta_1$ *and* $g \in W^{1,p}(\mathbb{T}^n \times (-1,1); \mathbb{R}^{n+1})$ *satisfies* $\partial_{n+1} g^{n+1}(x, x^{n+1}) > \beta_1 |\partial_{n+1} g'(x, x^{n+1})|$. *For any positive constant* $c_{14} > 0$, *if* $v(\mathbb{T}^n) + c_{14} \subset (-1+\lambda, 1-\lambda)$,

$$\int_{\mathbb{T}^n} \nu(\nabla v) \cdot g_\lambda(x, v) < \int_{\mathbb{T}^n} \nu(\nabla v) \cdot g_\lambda(x, v + c_{14}). \tag{3.1.5}$$

Proof. From the assumptions, we compute

$$\begin{aligned}
&\int_{\mathbb{T}^n} \nu(\nabla v) \cdot (g_\lambda(x, v + c_{14}) - g_\lambda(x, v)) \\
=&\int_{\mathbb{T}^n} \frac{1}{\sqrt{1 + |\nabla v|^2}} \int_v^{v+c_{14}} -\nabla v \cdot \partial_{n+1} g'_\lambda(x,t) + \partial_{n+1} g_\lambda^{n+1}(x,t)\ dt \\
\geq&\int_{\mathbb{T}^n} \frac{1}{\sqrt{1 + |\nabla v|^2}} \int_v^{v+c_{14}} -\beta_1 |\partial_{n+1} g'_\lambda(x,t)| + \partial_{n+1} g_\lambda^{n+1}(x,t)\ dt \\
\geq&\int_{\mathbb{T}^n} \frac{1}{\sqrt{1 + |\nabla v|^2}} \int_v^{v+c_{14}} \int_{\mathbb{T}^n \times (-1,1)} \eta_\lambda(x - y, t - y_{n+1}) \times \\
&\{-\beta_1 |\partial_{n+1} g'(y, y_{n+1})| + \partial_{n+1} g^{n+1}(y, y_{n+1})\}\ dt \\
>&0.
\end{aligned} \tag{3.1.6}$$

Lemma 3.1.2 follows. □

Lemma 3.1.3. *Suppose* $g \in W^{1,p}(\mathbb{T}^n \times (-1,1))$ *and* $v \in C^1(\mathbb{T}^n)$ *with* $\|v\|_{C^1(\mathbb{T}^n)} \leq \frac{7}{16}$. *Let* $q = \frac{np}{n+1-p}$. *Then, there exists a constant* $c_{15} = c_{15}(n,p) > 0$ *such that, if* $\lambda < \frac{1}{8}$,

$$\|g_\lambda(\cdot, v(\cdot))\|_{L^q(\mathbb{T}^n)} \leq c_{15} \|g\|_{W^{1,p}(\mathbb{T}^n \times (-1,1))}. \tag{3.1.7}$$

Proof. By the same proof as in Lemma 2.2.1, we obtain

$$\|g_\lambda(\cdot, v(\cdot))\|_{L^q(\mathbb{T}^n)} \leq c_{16} \|g_\lambda\|_{W^{1,p}(\mathbb{T}^n \times (-\frac{7}{8}, \frac{7}{8}))}, \tag{3.1.8}$$

where $c_{16} = c_{16}(n,p) > 0$. Using the Hölder inequality, we obtain

$$\begin{aligned}
&\int_{\mathbb{T}^n \times (-\frac{7}{8},\frac{7}{8})} |g_\lambda|^p \, dx \\
&\leq \int_{\mathbb{T}^n \times (-\frac{7}{8},\frac{7}{8})} \left(\int_{B^{n+1}(x,\lambda)} \eta_\lambda^{1-\frac{1}{p}+\frac{1}{p}}(x-y, x^{n+1}-y^{n+1}) |g(y,y^{n+1})| \, dy \right)^p dx \\
&\leq \int_{\mathbb{T}^n \times (-\frac{7}{8},\frac{7}{8})} \left(\int_{B^{n+1}(x,\lambda)} \eta_\lambda(x-y, x^{n+1}-y^{n+1}) |g(y,y^{n+1})|^p \, dy \right) dx \\
&\leq \int_{\mathbb{T}^n \times (-1,1)} |g(y,y^{n+1})|^p \left(\int_{B^{n+1}(y,\lambda)} \eta_\lambda(x-y, x^{n+1}-y^{n+1}) \, dx \right) dy \\
&= \int_{\mathbb{T}^n \times (-1,1)} |g(y,y^{n+1})|^p \, dy. \qquad (3.1.9)
\end{aligned}$$

We can show that $\|\nabla g_\lambda\|_{L^p(\mathbb{T}^n \times (-\frac{7}{8},\frac{7}{8}))} \leq \|\nabla g\|_{L^p(\mathbb{T}^n \times (-1,1))}$ in the exact same manner, and Lemma 3.1.3 follows by (3.1.8) and (3.1.9). $\square$

Theorem 3.1.4. *Suppose* $v \in C^1(\mathbb{T}^n)$ *and* $g \in W^{1,p}(\mathbb{T}^n \times (-1,1); \mathbb{R}^{n+1})$. *Then, there exist constants* $\varepsilon_2 = \varepsilon_2(n,p) > 0$*, if* $\lambda < \frac{1}{8}$*,* $\varepsilon < \varepsilon_2$*,* $\|v\|_{C^1(\mathbb{T}^n)} \leq \varepsilon^{\frac{1}{2}}$*, and* g *satisfies* (1.2.2)–(1.2.4)*. Then, there exist a unique function* $u \in W^{1,2}_{ave}(\mathbb{T}^n)$ *and a unique constant* $-\frac{1}{4} < c_v < \frac{1}{4}$ *such that*

$$\int_{\mathbb{T}^n} \frac{\nabla u \cdot \nabla \phi}{\sqrt{1+|\nabla v|^2}} = \int_{\mathbb{T}^n} \nu(\nabla v) \cdot g_\lambda(x, v + c_v)\phi \qquad (3.1.10)$$

for all $\phi \in W^{1,2}(\mathbb{T}^n)$.

Proof. We define

$$F(t) := \int_{\mathbb{T}^n} \nu(\nabla v) \cdot g_\lambda(x, v+t). \qquad (3.1.11)$$

The function F is continuous. Suppose that $\varepsilon < \frac{1}{16^2}$. We will consider that the domain of F is $[-\frac{1}{4}, \frac{1}{4}]$. By the mean value theorem, there exists a constant $c_{17} = c_{17}(n,p) > 0$ such that

$$\begin{aligned}
F\left(\frac{1}{4}\right) &= \int_{\mathbb{T}^n} (\nu(\nabla v) - \nu(0) + \nu(0)) \cdot g_\lambda\left(x, v + \frac{1}{4}\right) \\
&\geq -c_{17}\|v\|_{C^1(\mathbb{T}^n)} \left\| g_\lambda\left(\cdot, v(\cdot) + \frac{1}{4}\right)\right\|_{L^q(\mathbb{T}^n)} + \int_{\mathbb{T}^n} g_\lambda^{n+1}\left(x, v + \frac{1}{4}\right).
\end{aligned} \qquad (3.1.12)$$

By Lemma 3.1.3 and $\|v+\frac{1}{4}\|_{C^1(\mathbb{T}^n)} \le \frac{5}{16}$, we obtain

$$\left\| g_\lambda\left(\cdot, v(\cdot)+\frac{1}{4}\right)\right\|_{L^q(\mathbb{T}^n)} \le c_{15}\|g\|_{W^{1,p}(\mathbb{T}^n\times(-1,1))}. \tag{3.1.13}$$

By (1.2.3) and (1.2.4), there exists a constant $c_{18} = c_{18}(n) > 0$ such that

$$\begin{aligned}
&\int_{\mathbb{T}^n} g_\lambda^{n+1}\left(x, v+\frac{1}{4}\right) \\
=&\int_{\mathbb{T}^n}\int_{B^{n+1}(0,\lambda)} \eta_\lambda(y, y^{n+1}) g^{n+1}\left(x-y, v+\frac{1}{4}-y^{n+1}\right)\, dydx \\
>&\int_{\mathbb{T}^n}\int_{B^{n+1}(0,\lambda)} \eta_\lambda(y, y^{n+1}) g^{n+1}\left(x-y, \frac{1}{16}\right)\, dydx \\
>&\int_{\mathbb{T}^n}\int_{B^{n+1}(0,\lambda)} \eta_\lambda(y, y^{n+1}) \left(g^{n+1}\left(x-y, 0\right)+\frac{\varepsilon}{16}\right)\, dydx \\
>&\frac{c_{18}}{16}\varepsilon.
\end{aligned} \tag{3.1.14}$$

By (1.2.2), (3.1.12)–(3.1.14), and $\|v\|_{C^1(\mathbb{T}^n)} < \varepsilon^{\frac{1}{2}}$, if $\varepsilon < \left(\frac{c_{18}}{16c_{15}c_{17}}\right)^6 =: \varepsilon_2(n,p)$,

$$\begin{aligned}
F\left(\frac{1}{4}\right) &> -c_{15}c_{17}\|v\|_{C^1(\mathbb{T}^n)}\|g_\lambda\|_{W^{1,p}(\mathbb{T}^n\times(-1,1))} + \frac{c_{18}}{16}\varepsilon \\
&> -c_{15}c_{17}\varepsilon^{\frac{7}{6}} + \frac{c_{18}}{16}\varepsilon \\
&> \varepsilon\left(-c_{15}c_{17}\varepsilon^{\frac{1}{6}} + \frac{c_{18}}{16}\right) \\
&> 0.
\end{aligned} \tag{3.1.15}$$

Similarly, we can show that $F\left(-\frac{1}{4}\right) < 0$. By Lemma 3.1.2 and the mean value theorem, there exists a unique constant $-\frac{1}{4} < c_v < \frac{1}{4}$ that satisfies $F(c_v) = 0$. Using Theorem 3.1.1, Theorem 3.1.4 follows. □

3.2 $W^{2,p}$ estimate

Let us define an operator $T : \mathcal{A}(s) \to W^{1,2}_{ave}(\mathbb{T}^n) \times \left[-\frac{1}{4}, \frac{1}{4}\right]$ by $T(v) = (T_1(v), T_2(v)) := (u, c_v)$ that satisfies (3.1.10), where $\mathcal{A}(s) := \{w \in W^{2,q}_{ave}(\mathbb{T}^n) : \|w\|_{W^{2,q}(\mathbb{T}^n)} \le s\}$.

Theorem 3.2.1. *There exist constants* $\varepsilon_3 = \varepsilon_3(n,p) > 0$ *and* $c_{19} = c_{19}(n,p) > 0$*, if* $\lambda < \frac{1}{8}$*,* $\varepsilon < \min\{\varepsilon_2, \varepsilon_3\}$*,* $v \in \mathcal{A}(\varepsilon^{\frac{1}{2}})$*, and* $g \in W^{1,p}(\mathbb{T}^n \times (-1,1); \mathbb{R}^{n+1})$ *satisfies* (1.2.2)–(1.2.4)*. Then,*

$$\|T_1(v)\|_{W^{2,q}(\mathbb{T}^n)} \leq c_{19}\|g\|_{W^{1,p}(\mathbb{T}^n\times(-1,1))}. \tag{3.2.1}$$

Proof. We first assume that $v \in C^\infty(\mathbb{T}^n) \cap \mathcal{A}(\varepsilon^{\frac{1}{2}})$. Using [12, Corollary 8.11], we obtain $T_1(v) \in C^\infty(\mathbb{T}^n)$; thus, we can rewrite (3.1.10) as

$$\frac{\Delta T_1(v)}{\sqrt{1+|\nabla v|^2}} + \nabla T_1(v) \cdot \nabla\left(\frac{1}{\sqrt{1+|\nabla v|^2}}\right) = -\nu(\nabla v)\cdot g_\lambda(x, v + T_2(v)). \tag{3.2.2}$$

Using [12, Theorem 9.11], we find that there exists a constant $c_{20} = c_{20}(n,p) > 0$ such that

$$\begin{aligned}\|T_1(v)\|_{W^{2,q}(\mathbb{T}^n)} \leq & c_{20}\Bigg(\|T_1(v)\|_{L^q(\mathbb{T}^n)} + \|\nu(\nabla v)\cdot g_\lambda(x, v+T_2(v))\|_{L^q(\mathbb{T}^n)} \\ & + \left\|\nabla T_1(v)\cdot\nabla\left(\frac{1}{\sqrt{1+|\nabla v|^2}}\right)\right\|_{L^q(\mathbb{T}^n)}\Bigg).\end{aligned} \tag{3.2.3}$$

Using Lemma 3.1.3, we obtain

$$\|\nu(\nabla v)\cdot g_\lambda(x, v+T_2(v))\|_{L^q(\mathbb{T}^n)} \leq c_{15}\|g\|_{W^{1,p}(\mathbb{T}^n\times(-1,1))}. \tag{3.2.4}$$

Using the Sobolev inequality, we find that there exists a constant $c_{21} = c_{21}(n,p) > 0$ such that

$$\begin{aligned}&\left\|\nabla T_1(v)\cdot\nabla\left(\frac{1}{\sqrt{1+|\nabla v|^2}}\right)\right\|_{L^q(\mathbb{T}^n)} \\ \leq & \|T_1(v)\|_{C^1(\mathbb{T}^n)}\left\|\nabla\left(\frac{1}{\sqrt{1+|\nabla v|^2}}\right)\right\|_{L^q(\mathbb{T}^n)} \\ \leq & c_{21}\|T_1(v)\|_{W^{2,q}(\mathbb{T}^n)}\|v\|_{W^{2,q}(\mathbb{T}^n)}.\end{aligned} \tag{3.2.5}$$

Next, we estimate the term $\|T_1(v)\|_{L^q(\mathbb{T}^n)}$. If $q \leq 2$, then, by (3.1.2) and

Lemma 3.1.3, we obtain

$$\begin{aligned}
\|T_1(v)\|_{L^q(\mathbb{T}^n)} &\leq c_{22}(n,p)\|T_1(v)\|_{L^2(\mathbb{T}^n)}\\
&\leq c_{23}(n,p)B[T_1(v),T_1(v),v]^{\frac{1}{2}}.\\
&= c_2 3\left(\int_{\mathbb{T}^n}\frac{\nabla T_1(v)\cdot\nabla T_1(v)}{\sqrt{1+|\nabla v|^2}}\right)^{\frac{1}{2}}\\
&= c_2 3\left(\int_{\mathbb{T}^n}\nu(\nabla v)\cdot g_\lambda(x,v+T_2(v))T_1(v)\right)^{\frac{1}{2}}\\
&\leq c_{24}(n,p)\|g\|^{\frac{1}{2}}_{W^{1,p}(\mathbb{T}^n)}\|T_1(v)\|^{\frac{1}{2}}_{L^\infty(\mathbb{T}^n)}\\
&\leq c_{25}(n,p)\|g\|_{W^{1,p}(\mathbb{T}^n)}+\frac{1}{4c_{20}}\|T_1(v)\|_{W^{2,q}(\mathbb{T}^n)}. \qquad (3.2.6)
\end{aligned}$$

If $q>2$, by (3.2.6) and the Riesz–Thorin theorem, we obtain

$$\begin{aligned}
\|T_1(v)\|_{L^q(\mathbb{T}^n)} &\leq \|T_1(v)\|^{\frac{1}{q}}_{L^2(\mathbb{T}^n)}\|T_1(v)\|^{1-\frac{1}{q}}_{L^2(\mathbb{T}^n)}\\
&\leq c_{26}(n,p)\|g\|^{\frac{1}{2q}}_{W^{1,p}(\mathbb{T}^n)}\|T_1(v)\|^{\frac{1}{2q}+1-\frac{1}{q}}_{L^\infty(\mathbb{T}^n)}\\
&\leq c_{27}(n,p)\|g\|_{W^{1,p}(\mathbb{T}^n)}+\frac{1}{4c_{20}}\|T_1(v)\|_{W^{2,q}(\mathbb{T}^n)}. \qquad (3.2.7)
\end{aligned}$$

By (3.2.3)–(3.2.7), there exists a constant $c_{28}=c_{28}(n,p)>0$ such that

$$\begin{aligned}
\|T_1(v)\|_{W^{2,q}(\mathbb{T}^n)} \leq & c_{28}(\|g\|_{W^{1,p}(\mathbb{T}^n\times(-1,1))}+\|T_1(v)\|_{W^{2,q}(\mathbb{T}^n)}\|v\|_{W^{2,q}(\mathbb{T}^n)})\\
&+\frac{1}{4}\|T_1(v)\|_{W^{2,q}(\mathbb{T}^n)}. \qquad (3.2.8)
\end{aligned}$$

If $\varepsilon<\frac{1}{16c_{28}^2}$, we obtain

$$\|T_1(v)\|_{W^{2,q}(\mathbb{T}^n)}\leq 2c_{28}\|g\|_{W^{1,p}(\mathbb{T}^n\times(-1,1))}. \qquad (3.2.9)$$

For the general case of $v\in W^{2,q}(\mathbb{T}^n)$, suppose that $\{v_m\}_{m\in\mathbb{N}}\in C^\infty(\mathbb{T}^n)$ converges to v in the sense of $C^1(\mathbb{T}^n)$. By (3.2.9), there exists a subsequence $\{v_{m_k}\}_{k\in\mathbb{N}}\subset\{v_m\}_{m\in\mathbb{N}}$ such that $T_1(v_{m_k})$ converges to a function $w_\infty\in W^{2,q}(\mathbb{T}^n)$ in the sense of $C^1(\mathbb{T}^n)$ and $T_2(v_{m_k})$ converges to a constant

$d_\infty \in \left[-\frac{1}{4}, \frac{1}{4}\right]$. For any $\phi \in W^{1,2}(\mathbb{T}^n)$, we obtain

$$\begin{aligned}
&\int_{\mathbb{T}^n} \nu(\nabla v) \cdot g_\lambda(x, v + d_\infty)\phi - \nu(\nabla v_{m_k}) \cdot g_\lambda(x, v_{m_k} + T_2(v_{m_k}))\phi \\
\leq &\int_{\mathbb{T}^n} |\phi||\nu(\nabla v) - \nu(\nabla v_{m_k})||g_\lambda(x, v_{m_k} + T_2(v_{m_k}))| \\
&+ \int_{\mathbb{T}^n} |\phi| \left| \int_{v_{m_k}+T_2(v_{m_k})}^{v+d_\infty} \partial_{n+1} g_\lambda(x, s) \right| \\
\to &0 \quad (k \to \infty)
\end{aligned} \tag{3.2.10}$$

and

$$\begin{aligned}
&\int_{\mathbb{T}^n} \frac{\nabla w_\infty \cdot \nabla \phi}{\sqrt{1 + |\nabla v|^2}} - \frac{\nabla T_1(v_{m_k}) \cdot \nabla \phi}{\sqrt{1 + |\nabla v_{m_k}|^2}} \\
\leq &\int_{\mathbb{T}^n} \frac{(\nabla w_\infty - \nabla T_1(v_{m_k})) \cdot \nabla \phi}{\sqrt{1 + |\nabla v|^2}} \\
&+ \int_{\mathbb{T}^n} (\nabla T_1(v_{m_k}) \cdot \nabla \phi) \left(\frac{1}{\sqrt{1 + |\nabla v|^2}} - \frac{1}{\sqrt{1 + |\nabla v_{m_k}|^2}} \right) \\
\to &0 \quad (k \to \infty).
\end{aligned} \tag{3.2.11}$$

By (3.2.10) and (3.2.11), we obtain

$$\begin{aligned}
&\int_{\mathbb{T}^n} \frac{\nabla w_\infty \cdot \nabla \phi}{\sqrt{1 + |\nabla v|^2}} - \nu(\nabla v) \cdot g_\lambda(x, v + d_\infty)\phi \\
= &\lim_{k \to \infty} \int_{\mathbb{T}^n} \frac{\nabla T_1(v_{m_k}) \cdot \nabla \phi}{\sqrt{1 + |\nabla v_{m_k}|^2}} - \nu(\nabla v_{m_k}) \cdot g_\lambda(x, v_{m_k} + T_2(v_{m_k}))\phi \\
= &0,
\end{aligned} \tag{3.2.12}$$

that is, $T(v) = (w_\infty, d_\infty)$. By (3.2.9) and (3.2.12), Theorem 3.2.1 follows. □

3.3 Fixed point theorems

Next, we write the fixed-point theorem, which is needed later ([2, Theorem 1]). An operator $T : X \to A$ is considered weakly sequentially continuous if, for every sequence $\{x_m\}_{m \in \mathbb{N}} \subset X$ and $x_\infty \in X$ such that x_m weakly converges to x_∞, $T(x_m)$ weakly converges to $T(x_\infty)$.

Theorem 3.3.1. *Let X be a metrizable, locally convex topological vector space and Ω be a weakly compact convex subset of X. Then, any weakly sequentially continuous map $T : \Omega \to \Omega$ has a fixed point.*

We first prove Theorem 1.2.1 in the case of g_λ.

Theorem 3.3.2. *There exists a constant* $\varepsilon_4 = \varepsilon_4(n,p) > 0$*, if* $\lambda < \frac{1}{8}$, $\varepsilon < \varepsilon_4$*, and* $g \in W^{1,p}(\mathbb{T}^n \times (-1,1); \mathbb{R}^{n+1})$ *satisfies* (1.2.2)*–*(1.2.4)*. Then, there exists a function* $u_\lambda \in W^{2,q}(\mathbb{T}^n)$ *such that*

$$-\mathrm{div}\left(\frac{\nabla u_\lambda}{\sqrt{1+|\nabla u_\lambda|^2}}\right) = \nu(\nabla u_\lambda)\cdot g_\lambda(x, u_\lambda(x)) \quad \text{on } \mathbb{T}^n. \tag{3.3.1}$$

Proof. The set $W^{2,q}(\mathbb{T}^n)$ is a metrizable, locally convex topological vector space, and the set $\mathcal{A}(\varepsilon^{\frac{1}{2}})$ is a weakly compact convex subset of $W^{2,q}(\mathbb{T}^n)$. By (1.2.2) and Theorem 3.2.1, if $\varepsilon < \min\{\varepsilon_2, \varepsilon_3, c_{19}^{-6}\} =: \varepsilon_4$, we have

$$\begin{aligned}\|T_1(v)\|_{W^{2,q}(\mathbb{T}^n)} &\leq c_{19}\|g\|_{W^{1,p}(\mathbb{T}^n\times(-1,1))}\\ &\leq c_{19}\varepsilon^{\frac{1}{6}}\varepsilon^{\frac{1}{2}}\\ &\leq \varepsilon^{\frac{1}{2}} \quad \text{for any } v \in \mathcal{A}(\varepsilon^{\frac{1}{2}}),\end{aligned} \tag{3.3.2}$$

that is, $T_1(\mathcal{A}(\varepsilon^{\frac{1}{2}})) \subset \mathcal{A}(\varepsilon^{\frac{1}{2}})$. Suppose that $\{v_m\}_{m\in\mathbb{N}}$ weakly converges to v_∞ in the sense of $W^{2,q}(\mathbb{T}^n)$. According to Theorem 3.2.1, there exists a subsequence $\{v_{m_k}\}_{k\in\mathbb{N}} \subset \{v_m\}_{m\in\mathbb{N}}$ such that $T_1(v_{m_k})$ weakly converges to a function $w_\infty \in W^{2,q}(\mathbb{T}^n)$ in the sense of $W^{2,q}(\mathbb{T}^n)$ and $T_2(v_{m_k})$ converges to a constant $d_\infty \in \left[-\frac{1}{4}, \frac{1}{4}\right]$. By the same argument (3.2.10)–(3.2.12), for any $\phi \in W^{2,q}(\mathbb{T}^n)$,

$$\int_{\mathbb{T}^n} \frac{\nabla w_\infty \cdot \nabla\phi}{\sqrt{1+|\nabla v_\infty|^2}} - \nu(\nabla v_\infty)\cdot g_\lambda(x, v_\infty + d_\infty)\phi = 0, \tag{3.3.3}$$

that is, we obtain $\lim_{k\to\infty} T_1(v_{m_k}) = T_1(v_\infty)$ by the uniqueness of solution of Theorem 3.1.4. Therefore, every convergent subsequence of $\{T_1(v_m)\}$ converges to $T_1(v_\infty)$, and T_1 is a weakly sequentially continuous map. Using Theorem 3.3.1, we obtain a function $v_\lambda \in W^{2,q}_{ave}(\mathbb{T}^n)$ satisfying

$$-\mathrm{div}\left(\frac{\nabla v_\lambda}{\sqrt{1+|\nabla v_\lambda|^2}}\right) = \nu(\nabla v_\lambda)\cdot g_\lambda(x, v_\lambda(x) + T_2(v_\lambda)) \quad \text{on } \mathbb{T}^n, \tag{3.3.4}$$

that is, $u_\lambda := v_\lambda + T_2(v_\lambda) \in W^{2,q}(\mathbb{T}^n)$ satisfying (3.3.1). $\square$

Proof of Theorem 1.2.1. Suppose $u_\lambda \in W^{2,q}(\mathbb{T}^n)$ satisfies (3.3.1). By Theorem 3.2.1, there exists a convergent subsequence $\{u_{\lambda_k}\}_{k\in\mathbb{N}} \subset \{u_\lambda\}_{0<\lambda<\frac{1}{8}}$ with a limit $u_\infty \in W^{2,q}(\mathbb{T}^n)$ in the sense of $C^1(\mathbb{T}^n)$ and $\lambda_k \to 0$. We show that u_∞ satisfies (1.2.5). For any $\phi \in W^{1,2}(\mathbb{T}^n)$, we obtain

$$\begin{aligned}
&\int_{\mathbb{T}^n} -\mathrm{div}\left(\frac{\nabla u_{\lambda_k}}{\sqrt{1+|\nabla u_{\lambda_k}|^2}} - \frac{\nabla u_\infty}{\sqrt{1+|\nabla u_\infty|^2}}\right)\phi \\
=&\int_{\mathbb{T}^n} \left(\frac{\nabla u_{\lambda_k}}{\sqrt{1+|\nabla u_{\lambda_k}|^2}} - \frac{\nabla u_\infty}{\sqrt{1+|\nabla u_\infty|^2}}\right)\cdot \nabla\phi \\
\to&0. \qquad (3.3.5)
\end{aligned}$$

Using Lemma 3.1.3, we have

$$\begin{aligned}
&\int_{\mathbb{T}^n} \nu(\nabla u_{\lambda_k})\cdot g_{\lambda_k}(x,u_{\lambda_k}) - \nu(\nabla u_\infty)\cdot g(x,u_\infty) \\
=&\int_{\mathbb{T}^n} (\nu(\nabla u_{\lambda_k}) - \nu(\nabla u_\infty))\cdot g_{\lambda_k}(x,u_{\lambda_k}) \\
&+\int_{\mathbb{T}^n} \nu(\nabla u_\infty)\cdot (g_{\lambda_k}(x,u_{\lambda_k}) - g(x,u_{\lambda_k})) \\
&+\int_{\mathbb{T}^n} \nu(\nabla u_\infty)\cdot (g(x,u_{\lambda_k}) - g(x,u_\infty)) \\
=&c_{15}\|\nu(\nabla u_{\lambda_k}) - \nu(\nabla u_\infty)\|_{C^0(\mathbb{T}^n)}\|g_{\lambda_k}\|_{W^{1,p}(\mathbb{T}^n\times(-1,1))} \\
&+c_{15}\|g_{\lambda_k} - g_\infty\|_{W^{1,p}(\mathbb{T}^n\times(-1,1))} \\
&+\int_{\mathbb{T}^n}\left|\int_{u_\infty}^{u_{\lambda_k}} \partial_{n+1}g(x,s)\right| \\
\to&0. \qquad (3.3.6)
\end{aligned}$$

By (3.3.5) and (3.3.6), we obtain

$$\begin{aligned}
&\int_{\mathbb{T}^n} -\mathrm{div}\left(\frac{\nabla u_\infty}{\sqrt{1+|\nabla u_\infty|^2}}\right)\phi - \nu(\nabla u_\infty)\cdot g(x,u_\infty)\phi \\
=&\lim_{k\to\infty}\int_{\mathbb{T}^n} -\mathrm{div}\left(\frac{\nabla u_{\lambda_k}}{\sqrt{1+|\nabla u_{\lambda_k}|^2}}\right)\phi - \nu(\nabla u_{\lambda_k})\cdot g_{\lambda_k}(x,u_{\lambda_k})\phi \\
\to&0.
\end{aligned}$$

Thus, u_∞ satisfies (1.2.5) using the fundamental lemma of the calculus of variations. By (3.3.2), we obtain

$$\left\| u_\infty - \int_{\mathbb{T}^n} u_\infty(y)\,dy \right\|_{W^{2,q}(\mathbb{T}^n)} \le \varepsilon^{\frac{1}{2}}, \qquad (3.3.7)$$

and Theorem 1.2.1 follows.

□

Chapter 4

Allen–Cahn equation

4.1 Estimate for the upper density ratio

In this section, we prove Theorem 4.1.7-4.1.9, which give ε-independent estimates of the upper and lower density ratios of the energy. Throughout this section, we drop the index i and set $\Omega = U_1 = \{|x| < 1\}$ since the result is local. Assume $u \in W^{1,2}(U_1)$ and $v \in W^{1,p}(U_1; \mathbb{R}^n)$ satisfy (1.3.3) with a positive ε and (1.3.5)-(1.3.7) are satisfied for a given set of c_0, E_0, λ_0. The exponent p satisfies (1.3.4). We first derive two preliminary properties for u, Lemma 4.1.1 and 4.1.2.

Lemma 4.1.1. *There exists* $c_{29} > 0$ *depending only on* c_0, λ_0, n, p *and* W *such that*

$$\sup_{x \in U_{1-\varepsilon}} \varepsilon |\nabla u(x)| \le c_{29} \tag{4.1.1}$$

and

$$\sup_{x, x' \in U_{1-\varepsilon}} \varepsilon^{3-\frac{n}{p}} \frac{|\nabla u(x) - \nabla u(x')|}{|x - x'|^{2-\frac{n}{p}}} \le c_{29} \tag{4.1.2}$$

for $0 < \varepsilon < 1/2$. *If* $\varepsilon \ge 1/2$, *then we have for any* $0 < s < 1$

$$\sup_{x \in U_s} |\nabla u(x)| \le c_{29} \tag{4.1.3}$$

where c_{29} *depends additionally on* s. *In both cases, we have* $u \in W^{3,p}_{loc}(U_1)$.

Proof. Consider the case $0 < \varepsilon < 1/2$. Define $\tilde{u}(x) := u(\varepsilon x)$ and $\tilde{v}(x) := \varepsilon v(\varepsilon x)$ for $x \in U_{\varepsilon^{-1}}$. After this change of variables, we obtain from (1.3.3) that

$$-\Delta \tilde{u} + W'(\tilde{u}) = \tilde{v} \cdot \nabla \tilde{u} \quad \text{weakly on } U_{\varepsilon^{-1}}. \tag{4.1.4}$$

Under the change of variables, we obtain from (1.3.7)

$$\|\tilde{v}\|_{L^{\frac{np}{n-p}}(U_{\varepsilon^{-1}})} + \|\nabla\tilde{v}\|_{L^p(U_{\varepsilon^{-1}})} \leq \lambda_0 \varepsilon^{2-\frac{n}{p}}. \tag{4.1.5}$$

For any $U_2(x) \subset U_{\varepsilon^{-1}}$, let $\phi \in C^1_c(U_2(x))$ be a function such that $0 \leq \phi \leq 1$, $\phi = 1$ on $B_1(x)$ and $|\nabla\phi| \leq 4$ on $U_2(x)$. Use (4.1.4) with the test function $\tilde{u}\phi^2$. Using also (1.3.5), we obtain

$$\begin{aligned} \int |\nabla\tilde{u}|^2\phi^2 &\leq c_0 \int (2\phi|\nabla\phi||\nabla\tilde{u}| + |W'|\phi^2 + |\tilde{v}||\nabla\tilde{u}|\phi^2) \\ &\leq \frac{1}{2}\int |\nabla\tilde{u}|^2\phi^2 + \int (4c_0^2|\nabla\phi|^2 + c_0|W'|\phi^2 + c_0^2|\tilde{v}|^2\phi^2). \end{aligned} \tag{4.1.6}$$

Since $\frac{np}{n-p} > 2$, (4.1.5) and (4.1.6) give

$$\sup_{B_2(x)\subset U_{\varepsilon^{-1}}} \int_{B_1(x)} |\nabla\tilde{u}|^2 \leq c(c_0, \lambda_0, n, p, W). \tag{4.1.7}$$

We next note that the function $\tilde{u}\phi$ weakly satisfies the following equation:

$$-\Delta(\tilde{u}\phi) = -\tilde{u}\Delta\phi - 2\nabla\phi\cdot\nabla\tilde{u} + (\tilde{v}\cdot\nabla\tilde{u} - W'(\tilde{u}))\phi. \tag{4.1.8}$$

Using the standard L^p theory [12, Theorem 9.11] to (4.1.8), we may start a bootstrapping argument as follows. Staring with $q = 2$, we have

$$\begin{aligned} &\nabla\tilde{u} \in L^q_{loc} \implies \tilde{v}\cdot\nabla\tilde{u} \in L_{loc}^{\frac{npq}{np+q(n-p)}} \implies \tilde{u} \in W_{loc}^{2,\frac{npq}{np+q(n-p)}} \\ &\implies \nabla\tilde{u} \in L_{loc}^{\frac{npq}{np-q(2p-n)}} \end{aligned}$$

with the corresponding estimates relating these norms. Note that the exponent of integrability of $\nabla\tilde{u}$ is raised from q to $q\cdot\frac{np}{np-q(2p-n)}$, with the factor strictly larger than one. Thus, in a finite number of bootstrapping, we obtain the $W^{2,s}_{loc}$ (with $s > n$) estimate for $\tilde{u}$, and by the Sobolev inequality, the L^∞_{loc} estimate for $\nabla\tilde{u}$. Again by the L^p theory, we obtain the $W_{loc}^{2,\frac{np}{n-p}}$ estimate of $\tilde{u}$. In particular, by the Sobolev inequality, we obtain (4.1.1) and (4.1.2). Since the right-hand side of (4.1.8) is in $W^{1,p}_{loc}$ (note that $\tilde{v}\cdot\nabla^2\tilde{u} \in L_{loc}^{\frac{np}{2(n-p)}}$ and $\frac{np}{2(n-p)} > p$ by (1.3.4)), we have $\tilde{u} \in W^{3,p}_{loc}$ and the weak third-derivatives of $\tilde{u}$ exist. The case of $\varepsilon \geq 1/2$ does not require the change of variables as above and the proof is omitted. □

Lemma 4.1.2. *Given* $0 < s < 1$, *there exist constants* $0 < \varepsilon_5, \eta < 1$ *depending only on* c_0, λ_0, W, n, p *and* s *such that*

$$\sup_{x\in B_s} |u(x)| \leq 1 + \varepsilon^\eta \tag{4.1.9}$$

for $\varepsilon \leq \varepsilon_5$.

Proof. Let $q = \frac{np}{n-p} - 1$ and $\phi \in C_c^\infty(B_{\frac{s+1}{2}})$ with $\phi \geq 0$. Multiplying (1.3.3) by $[(u-1)_+]^q\phi^2$, we have

$$\begin{aligned} -\varepsilon \int q[(u-1)_+]^{q-1}|\nabla u|^2\phi^2 + 2[(u-1)_+]^q\phi\nabla\phi\cdot\nabla u \\ = \int \frac{W'}{\varepsilon}[(u-1)_+]^q\phi^2 - \int \varepsilon\nabla u\cdot v[(u-1)_+]^q\phi^2. \end{aligned} \tag{4.1.10}$$

By $W'(u) \geq \kappa(u-1)$ for $u \geq 1$ and (4.1.1), we obtain

$$\begin{aligned} & \frac{\kappa}{\varepsilon}\int[(u-1)_+]^{q+1}\phi^2 + \int \varepsilon q[(u-1)_+]^{q-1}|\nabla u|^2\phi^2 \\ \leq\ & 2\varepsilon\int[(u-1)_+]^q\phi|\nabla\phi||\nabla u| + c_{29}\int|v|[(u-1)_+]^q\phi^2 \\ \leq\ & \frac{q\varepsilon}{2}\int[(u-1)_+]^{q-1}|\nabla u|^2\phi^2 + \frac{8\varepsilon}{q}\int[(u-1)_+]^{q+1}|\nabla\phi|^2 \\ & + \frac{\kappa}{2\varepsilon}\int[(u-1)_+]^{q+1}\phi^2 + \frac{\varepsilon^q c(q,c_{29})}{\kappa^q}\int|v|^{q+1}\phi^2, \end{aligned} \tag{4.1.11}$$

which shows

$$\frac{\kappa}{2\varepsilon}\int[(u-1)_+]^{q+1}\phi^2 \leq \frac{8\varepsilon}{q}\int[(u-1)_+]^{q+1}|\nabla\phi|^2 + \frac{\varepsilon^q c(q,c_{29})}{\kappa^q}\int|v|^{q+1}\phi^2. \tag{4.1.12}$$

By (1.3.5), (1.3.7) and iterating the computation above with suitable ϕ, we obtain

$$\int_{B_s}[(u-1)_+]^{q+1} \leq c_{30}(s,q,\lambda_0,n,p,W,c_0,c_{29})\varepsilon^{q+1}. \tag{4.1.13}$$

To derive a contradiction, assume that $u(x_0) - 1 \geq \varepsilon^\eta$ for some $x_0 \in B_s$. By (4.1.1), for $y \in B_{\frac{\varepsilon^{1+\eta}}{2c_{29}}}(x_0)$,

$$u(y) - 1 \geq u(x_0) - 1 - \sup|\nabla u|\frac{\varepsilon^{1+\eta}}{2c_{29}} \geq \frac{\varepsilon^\eta}{2}. \tag{4.1.14}$$

Then we have

$$c_{30}\varepsilon^{q+1} \geq \int_{B_{\frac{\varepsilon^{1+\eta}}{2c_{29}}}(x_0)}[(u-1)_+]^{q+1} \geq \left(\frac{\varepsilon^\eta}{2}\right)^{q+1}\omega_n\left(\frac{\varepsilon^{1+\eta}}{2c_{29}}\right)^n, \tag{4.1.15}$$

which show by $q = \frac{np}{n-p} - 1$

$$\varepsilon^{\eta\frac{np}{n-p} - \frac{np}{n-p} + n + n\eta} \leq c_{31}(s,q,\lambda_0,n,p,W,c_0,c_{29}). \tag{4.1.16}$$

This is a contradiction if η and ε are sufficiently small. $u \geq -1-\varepsilon^\eta$ is proved similarly. $\square$

The next Lemma 4.1.3 is the starting point of the ultimate establishment of the monotonicity formula.

Lemma 4.1.3. *For $B_r(x) \subset U_1$, we have*

$$\frac{d}{dr}\left\{\frac{1}{r^{n-1}}\int_{B_r(x)}\left(\frac{\varepsilon|\nabla u|^2}{2}+\frac{W(u)}{\varepsilon}\right)\right\} = \frac{1}{r^n}\int_{B_r(x)}\left(\frac{W(u)}{\varepsilon}-\frac{\varepsilon|\nabla u|^2}{2}\right)$$
$$+\frac{\varepsilon}{r^{n+1}}\int_{\partial B_r(x)}((y-x)\cdot\nabla u)^2+\frac{\varepsilon}{r^n}\int_{B_r(x)}(v\cdot\nabla u)((y-x)\cdot\nabla u). \tag{4.1.17}$$

Proof. Multiply both sides of (1.3.3) by $\nabla u\cdot g$, where $g=(g^1,\cdots,g^n)\in C^1_c(U_1;\mathbb{R}^n)$. By integration by parts, we obtain

$$\int\Big(\Big(\frac{\varepsilon|\nabla u|^2}{2}+\frac{W}{\varepsilon}\Big)\mathrm{div}g-\varepsilon\sum_{i,j}u_{y_i}u_{y_j}g^i_{y_j}+\varepsilon(v\cdot\nabla u)(\nabla u\cdot g)\Big)=0 \tag{4.1.18}$$

We assume that $x=0$ after a suitable translation and let $g^j(y)=y_j\rho(|y|)$. Writing $r=|y|$, (4.1.18) becomes

$$\int\Big(\Big(\frac{\varepsilon|\nabla u|^2}{2}+\frac{W}{\varepsilon}\Big)(r\rho'+n\rho)-\varepsilon\frac{\rho'}{r}(y\cdot\nabla u)^2$$
$$-\varepsilon|\nabla u|^2\rho+\varepsilon(\nabla u\cdot v)(\nabla u\cdot y)\rho\Big)=0.$$

We choose ρ which is a smooth approximation of χ_{B_r}, the characteristic function of B_r, and then we take a limit $\rho\to\chi_{B_r}$. Then we have

$$-(n-1)\int_{B_r}\left(\frac{\varepsilon|\nabla u|^2}{2}+\frac{W}{\varepsilon}\right)+r\int_{\partial B_r}\left(\frac{\varepsilon|\nabla u|^2}{2}+\frac{W}{\varepsilon}\right)$$
$$=\int_{B_r}\left(\frac{W}{\varepsilon}-\frac{\varepsilon|\nabla u|^2}{2}\right)+\frac{\varepsilon}{r}\int_{\partial B_r}(y\cdot\nabla u)^2+\varepsilon\int_{B_r}(\nabla u\cdot v)(\nabla u\cdot y).$$

By dividing the above equation by r^n, the lemma follows. □

We need the following lemma to control the negative contribution of the right-hand side of (4.1.17).

Lemma 4.1.4. *Given $0<s<1$, there exist constants $0<\beta_2<1$ and $0<\varepsilon_6<1$ which depend only on c_0, λ_0, W, n, p and s such that, if $\varepsilon\le\varepsilon_6$,*

$$\sup_{B_s}\left(\frac{\varepsilon}{2}|\nabla u|^2-\frac{W(u)}{\varepsilon}\right)\le\varepsilon^{-\beta_2}. \tag{4.1.19}$$

The proof of Lemma 4.1.4 is deferred to the end of this section. Next, for $B_r(x) \subset U_1$ and $0 < r < \operatorname{dist}(x, \partial U_1)$, define

$$E(r,x) := \frac{1}{r^{n-1}} \int_{B_r(x)} \left(\frac{\varepsilon |\nabla u|^2}{2} + \frac{W(u)}{\varepsilon} \right).$$

Using Lemma 4.1.3 and Lemma 4.1.4, we prove:

Lemma 4.1.5. *Given $0 < s < 1$, there exist constants $0 < \varepsilon_7, c_{32}, c_{33} < 1$ which depend only on c_0, λ_0, W, n, p and s such that, if $B_{\varepsilon^{\beta_2}}(x) \subset U_s$, $|u(x)| \le \alpha$ and $\varepsilon \le \varepsilon_7$, then*

$$E(r,x) \ge c_{32} \quad \textit{for all } \varepsilon \le r \le c_{33}\varepsilon^{\beta_2}. \tag{4.1.20}$$

Proof. By integrating (4.1.17) over $[\varepsilon, r]$, we have

$$\begin{aligned} E(r,x) - E(\varepsilon,x) \ge & -\int_\varepsilon^r \frac{d\tau}{\tau^n} \int_{B_\tau(x)} \left(\frac{\varepsilon}{2}|\nabla u|^2 - \frac{W(u)}{\varepsilon} \right)_+ \\ & + \int_\varepsilon^r \frac{d\tau}{\tau^n} \int_{B_\tau(x)} \varepsilon(\nabla u \cdot v)(\nabla u \cdot (y-x)). \end{aligned} \tag{4.1.21}$$

By (4.1.19) and $B_r(x) \subset U_s$, we have

$$\int_\varepsilon^r \frac{d\tau}{\tau^n} \int_{B_\tau(x)} \left(\frac{\varepsilon}{2}|\nabla u|^2 - \frac{W(u)}{\varepsilon} \right)_+ \le \omega_n r \varepsilon^{-\beta_1}. \tag{4.1.22}$$

By (4.1.1) and (1.3.7), we have

$$\begin{aligned} \left| \int_\varepsilon^r \frac{d\tau}{\tau^n} \int_{B_\tau(x)} \varepsilon(\nabla u \cdot v)(\nabla u \cdot (y-x)) \right| & \le \int_0^r \frac{d\tau}{\tau^{n-1}} \int_{B_\tau(x)} c_{29}^2 \varepsilon^{-1} |v| \\ & \le c(\lambda_0, n, p, c_{29}) r^{3-\frac{n}{p}} \varepsilon^{-1}. \end{aligned} \tag{4.1.23}$$

Since $|u(x)| \le \alpha$, using (4.1.1), we have $|u(y)| \le \frac{\alpha+1}{2}$ for all $y \in B_{\frac{(1-\alpha)\varepsilon}{2c_{29}}}(x)$. By choosing a larger c_{29} if necessary, we may assume $\frac{(1-\alpha)}{2c_{29}} \le 1$. Define

$$c_{32} := \frac{\omega_n}{2} \frac{(1-\alpha)^n}{(2c_{29})^n} \min_{|t| \le \frac{1+\alpha}{2}} W(t) > 0.$$

With this choice, we obtain

$$\begin{aligned} E(\varepsilon, x) & \ge \frac{1}{\varepsilon^{n-1}} \int_{B_{\frac{(1-\alpha)\varepsilon}{2c_{29}}}(x)} \frac{W(u)}{\varepsilon} \\ & \ge \omega_n \frac{(1-\alpha)^n}{(2c_{29})^n} \min_{|t| \le \frac{1+\alpha}{2}} W(t) = 2c_{32}. \end{aligned} \tag{4.1.24}$$

Since a larger β_2 satisfies (4.1.19) as well, we may assume $(3-\frac{n}{p})\beta_2 - 1 > 0$ by choosing $\beta_2 < 1$ sufficiently close to 1. Then we may show that the sum of (4.1.22) and (4.1.23) may be bounded from above by c_{32} for sufficiently small ε and c_{33} if $r \le c_{33}\varepsilon^{\beta_2}$. Then, (4.1.20) follows from (4.1.21)-(4.1.24). $\square$

Theorem 4.1.6. *Given $0 < s < 1$, there exist constants $0 < \varepsilon_8, \beta_3 < 1$ and $0 < c_{34}$ which depend only on c_0, λ_0, W, n, p and s such that, if $B_r(x) \subset U_s$, $c_{33}\varepsilon^{\beta_2} < r$ and $\varepsilon \le \varepsilon_8$, then*

$$\frac{1}{r^n}\int_{B_r(x)} \left(\frac{\varepsilon}{2}|\nabla u|^2 - \frac{W(u)}{\varepsilon}\right)_+ \le \frac{c_{34}}{r^{1-\beta_3}}(E(r,x)+1). \tag{4.1.25}$$

Proof. The proof is similar to [43, Proposition 3.4] with a minor modification. Define $\beta_3 := \frac{1-\beta_2}{2\beta_2}$ and $\beta_4 := \frac{1+\beta_2}{2}$. β_3 and β_4 are chosen so that

$$\beta_2\beta_3 = \beta_4 - \beta_2, \tag{4.1.26}$$

$$0 < \beta_3 < 1, \quad 0 < \beta_2 < \beta_4 < 1. \tag{4.1.27}$$

We estimate the integral of (4.1.25) by separating $B_r(x)$ into three disjoint sets. Define

$$\begin{aligned} \mathcal{A} &:= B_r(x)\backslash B_{r-\varepsilon^{\beta_4}}(x), \\ \mathcal{B} &:= \{y \in B_{r-\varepsilon^{\beta_4}}(x) \,:\, \mathrm{dist}(\{|u| \le \alpha\}, y) < \varepsilon^{\beta_4}\}, \\ \mathcal{C} &:= \{y \in B_{r-\varepsilon^{\beta_4}}(x) \,:\, \mathrm{dist}(\{|u| \le \alpha\}, y) \ge \varepsilon^{\beta_4}\}. \end{aligned}$$

Note that $r > c_{33}\varepsilon^{\beta_2} > \varepsilon^{\beta_4}$ for small ε.

The estimates of the integral over $\mathcal{A}$ and $\mathcal{B}$ are exactly the same as in [43]. Namely, for $\mathcal{A}$, we use $\mathcal{L}^n(\mathcal{A}) \le n\omega_n r^{n-1}\varepsilon^{\beta_4}$ and (4.1.19) as well as $r^{-1} \le (c_{33}\varepsilon^{\beta_2})^{-1}$. For $\mathcal{B}$, we use (4.1.20) and prove that $\mathcal{L}^n(\mathcal{B}) \le c(n)\varepsilon^{n\beta_4}N$, where N is an integer satisfying $\mathcal{B} \subset \cup_{i=1}^N B_{5\varepsilon^{\beta_4}}(x_i)$. Here we only consider the estimate on $\mathcal{C}$ and refer the reader to the proof of [43, Proposition 3.4]. Define

$$\phi(z) := \min\{1, \varepsilon^{-\beta_4}\mathrm{dist}(\{y \,:\, |y-x| \ge r\} \cup \{|u| \le \alpha\}, z)\}.$$

ϕ is a Lipschitz function and is 0 on $\{y \,:\, |y-x| > r\} \cup \{|u| < \alpha\}$, 1 on $\mathcal{C}$ and $|\nabla\phi| \le \varepsilon^{-\beta_4}$. Differentiate (1.3.3) with respect to x_j, multiply it by $u_{x_j}\phi^2$ and sum over j. Then we have

$$\int \sum_j \varepsilon u_{x_j}\Delta u_{x_j}\phi^2 = \int \frac{W''}{\varepsilon}|\nabla u|^2\phi^2 - \varepsilon\sum_j \int (v\cdot\nabla u)_{x_j}\phi^2 u_{x_j}. \tag{4.1.28}$$

By integration by parts, the Cauchy–Schwarz inequality and (4.1.1), we obtain

$$\begin{aligned}
&\int \varepsilon|\nabla^2 u|^2\phi^2 + \frac{W''}{\varepsilon}|\nabla u|^2\phi^2 \\
= &\int -\sum_{i,j} 2\varepsilon u_{x_j}u_{x_ix_j}\phi\phi_{x_i} - \varepsilon\nabla u\cdot v(\Delta u\phi^2 + 2\phi\nabla u\cdot\nabla\phi) \\
\leq &\ \frac{1}{2}\int \varepsilon|\nabla^2 u|^2\phi^2 + c_{35}\int(\varepsilon|\nabla u|^2|\nabla\phi|^2 + |v|^2\phi^2\varepsilon^{-1}), \qquad (4.1.29)
\end{aligned}$$

where c_{35} depends on c_{29}. Since $|u| \geq \alpha$ on the support of ϕ, we have $W'' \geq \kappa$. Thus

$$\int \frac{\varepsilon}{2}|\nabla^2 u|^2\phi^2 + \frac{\kappa}{\varepsilon}|\nabla u|^2\phi^2 \leq c_{35}\int(\varepsilon|\nabla u|^2|\nabla\phi|^2 + |v|^2\phi^2\varepsilon^{-1}). \qquad (4.1.30)$$

By $|\nabla\phi| \leq \varepsilon^{-\beta_4}$ and (1.3.7), we have

$$\begin{aligned}
\int \frac{\kappa}{\varepsilon}|\nabla u|^2\phi^2 &\leq c_{35}\left(\varepsilon^{-2\beta_4}\int_{B_r}\varepsilon|\nabla u|^2 + \varepsilon^{-1}\|v\|^2_{L^{\frac{np}{n-p}}}(\omega_n r^n)^{\frac{np-2(n-p)}{np}}\right) \\
&\leq c_{36}\left(\varepsilon^{-2\beta_4}\int_{B_r}\varepsilon|\nabla u|^2 + \varepsilon^{-1}r^{n-\frac{2(n-p)}{p}}\right), \qquad (4.1.31)
\end{aligned}$$

where $c_{36} = c_{35} + c(n,p)\lambda_0^2$. Since $\phi = 1$ on $\mathcal{C}$, multiplying (4.1.31) by $\frac{\varepsilon^2}{\kappa r^n}$,

$$\frac{1}{r^n}\int_{\mathcal{C}}\frac{\varepsilon}{2}|\nabla u|^2 \leq \frac{c_{36}}{\kappa}\left(\frac{\varepsilon^{2-2\beta_4}}{r}E(r,x) + \varepsilon r^{2-\frac{2n}{p}}\right). \qquad (4.1.32)$$

By the definitions of β_2, β_3, β_4 and $r \geq c_{33}\varepsilon^{\beta_1}$, we have

$$\frac{\varepsilon^{2-2\beta_4}}{r} \leq \frac{\varepsilon^{\beta_2\beta_3}}{r^{1-\beta_3}c_{33}^{\beta_3}}, \qquad (4.1.33)$$

and using $\varepsilon \leq r$,

$$\varepsilon r^{2-\frac{2n}{p}} \leq \frac{1}{r^{\frac{2n}{p}-3}}, \qquad (4.1.34)$$

where $\frac{2n}{p} - 3 < 1$. Hence, we obtain

$$\frac{1}{r^n}\int_{\mathcal{C}}\frac{\varepsilon}{2}|\nabla u|^2 \leq \frac{c_{36}}{\kappa}\left(\frac{\varepsilon^{\beta_2\beta_3}}{r^{1-\beta_3}c_{33}^{\beta_2}}E(r,x) + \frac{1}{r^{\frac{2n}{p}-3}}\right). \qquad (4.1.35)$$

By re-defining $\beta_3 = \min\{\beta_3, 4 - \frac{2n}{p}\}$ and the estimates of integrals over $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, we proved (4.1.25). □

Theorem 4.1.7. *There exists a constant* $0 < c_{37}$ *which depends only on* c_0*,* λ_0*,* W*,* E_0*,* n *and* p *such that, if* $0 < \varepsilon < 1/2$ *and* $U_{2r}(x) \subset U_{1-\varepsilon}$*, then*

$$\operatorname{dist}(x, \partial U_{1-\varepsilon})^{n-1}\, E(r,x) \le c_{37}. \tag{4.1.36}$$

Proof. Define

$$E_1 := \sup_{U_{2r}(x)\subset U_{1-\varepsilon}} \operatorname{dist}(x, \partial U_{1-\varepsilon})^{n-1} E(r,x).$$

By Lemma 4.1.1, we have $\sup_{x\in U_{1-\varepsilon}} \varepsilon|\nabla u(x)| \le c_{29}$. Thus for any $U_{2r}(x) \subset U_{1-\varepsilon}$, we have

$$E(r,x) \le \omega_n r\Big(\frac{c_{29}^2}{2\varepsilon} + \sup_{|t|\le c_0} \frac{W(t)}{\varepsilon}\Big) \le \frac{c(n, c_{29}, W)}{\varepsilon}$$

and we have $E_1 < \infty$ for each ε. In the following, we give an estimate of E_1 depending only on c_0, λ_0, n, p, W and E_0. Let $U_{2r_0}(x_0) \subset U_{1-\varepsilon}$ be fixed such that

$$\operatorname{dist}(x_0, \partial U_{1-\varepsilon})^{n-1} E(r_0, x_0) > \frac{3}{4} E_1. \tag{4.1.37}$$

For simplicity, define

$$l := \operatorname{dist}(x_0, \partial U_{1-\varepsilon}) = 1 - \varepsilon - |x_0|$$

and change variables by $\tilde{x} = (x - x_0)/l$, $\tilde{r} = r/l$, $\tilde{\varepsilon} = \varepsilon/l$, $\tilde{u}(\tilde{x}) = u(x)$ and $\tilde{v}(\tilde{x}) = lv(x)$. Note that $U_{l+\varepsilon}(x_0) \subset U_1$. In particular, we write

$$\tilde{r}_0 := r_0/l \le 1/2.$$

By (1.3.3), (1.3.6) and (1.3.7), we have

$$-\tilde{\varepsilon}\Delta\tilde{u} + \frac{W'(\tilde{u})}{\tilde{\varepsilon}} = \tilde{\varepsilon}\tilde{v}\cdot\nabla\tilde{u} \quad \text{for } \tilde{x} \in U_{1+\tilde{\varepsilon}}, \tag{4.1.38}$$

$$\int_{U_{1+\tilde{\varepsilon}}} \left(\frac{\tilde{\varepsilon}|\nabla\tilde{u}|^2}{2} + \frac{W(\tilde{u})}{\tilde{\varepsilon}}\right) \le l^{1-n} E_0, \tag{4.1.39}$$

$$\|\tilde{v}\|_{L^{\frac{np}{n-p}}(U_{1+\tilde{\varepsilon}})} + \|\nabla\tilde{v}\|_{L^p(U_{1+\tilde{\varepsilon}})} \le l^{2-\frac{n}{p}}\lambda_0. \tag{4.1.40}$$

Define for $B_{\tilde{r}}(\tilde{x}) \subset U_{1+\tilde{\varepsilon}}$

$$\tilde{E}(\tilde{r}, \tilde{x}) := \frac{1}{\tilde{r}^{n-1}} \int_{B_{\tilde{r}}(\tilde{x})} \left(\frac{\tilde{\varepsilon}|\nabla\tilde{u}|^2}{2} + \frac{W(\tilde{u})}{\tilde{\varepsilon}}\right). \tag{4.1.41}$$

Under the above change of variables, note that we have $E(r,x) = \tilde{E}(\tilde{r},\tilde{x})$. Next, for any $x \in B_{3l/4}(x_0)$, we have $\operatorname{dist}(x,\partial U_{1-\varepsilon}) \geq l/4$. Hence for any $x \in B_{3l/4}(x_0)$ and $r < l/8 \leq \operatorname{dist}(x,\partial U_{1-\varepsilon})/2$, by the definition of E_1, we have

$$\operatorname{dist}(x,\partial U_{1-\varepsilon})^{n-1}E(r,x) \leq E_1.$$

This shows (again using $\operatorname{dist}(x,\partial U_{1-\varepsilon}) \geq l/4$)

$$\sup_{\tilde{x}\in B_{\frac{3}{4}},\, 0<\tilde{r}<\frac{1}{8}} \tilde{E}(\tilde{r},\tilde{x}) \leq 4^{n-1}l^{1-n}E_1. \tag{4.1.42}$$

We next let $c_{29}, c_{32}, c_{33}, c_{34}, \varepsilon_5, \varepsilon_6, \varepsilon_7, \varepsilon_8, \beta_2, \beta_3$ be constants obtained in Lemma 4.1.1–4.1.5 and Theorem 4.1.6 corresponding to the same c_0, λ_0, n, p, W and $s = 3/4$. Then note that the estimates up to Theorem 4.1.6 hold for $\tilde{u}$ and $\tilde{v}$ for $U_{3/4}$ and with respect to the new variables ($\tilde{x}$, $\tilde{r}$, $\tilde{\varepsilon}$ etc.) if

$$\tilde{\varepsilon} \leq \hat{\varepsilon} := \min\{\varepsilon_5, \varepsilon_6, \varepsilon_7, \varepsilon_8, 1/2\} \tag{4.1.43}$$

due to (4.1.38) and (4.1.40). It is important to note that $\hat{\varepsilon}$ depends only on c_0, λ_0, n, p and W. Note that (4.1.40) yields an upper bound for $\|\tilde{v}\|_{L^{\frac{np}{n-p}}(U_{1+\tilde{\varepsilon}})} + \|\nabla\tilde{v}\|_{L^p(U_{1+\tilde{\varepsilon}})}$ independent of l, as $l < 1$ and $2 - \frac{n}{p} > 0$. A priori, we do not know if (4.1.43) holds or not and we prove the desired estimate for E_1 by exhausting all the possibilities.

First consider the case $\tilde{\varepsilon} \geq \hat{\varepsilon}$. We use (4.1.3) and (4.1.1), respectively, for $\tilde{\varepsilon} > 1/2$ and $1/2 \geq \tilde{\varepsilon} \geq \hat{\varepsilon}$. Suppose that $\tilde{\varepsilon} > 1/2$. By (4.1.37) and (4.1.3), we have

$$\frac{3}{4l^{n-1}}E_1 < \tilde{E}(\tilde{r}_0, 0) \leq \omega_n \tilde{r}_0(\tilde{\varepsilon}c_{29}^2 + 2\sup_{|x|\leq c_0} W(x))$$

and since $l\tilde{\varepsilon} = \varepsilon \leq 1$, $l < 1$ and $\tilde{r}_0 \leq 1/2$, we obtain

$$E_1 < \frac{4}{3}\omega_n(c_{29}^2 + 2\sup_{|x|\leq c_0} W(x)). \tag{4.1.44}$$

If $1/2 \geq \tilde{\varepsilon} \geq \hat{\varepsilon}$, again by (4.1.37), (4.1.1) and $\tilde{r}_0 \leq 1/2$, we have

$$\begin{aligned} \frac{3}{4l^{n-1}}E_1 &< \tilde{E}(\tilde{r}_0, 0) \leq \omega_n(c_{29}^2 + \sup_{|x|\leq c_0} W(x))\frac{\tilde{r}_0}{\tilde{\varepsilon}} \\ &\leq \omega_n(c_{29}^2 + \sup_{|x|\leq c_0} W(x))\frac{1}{2\hat{\varepsilon}} \end{aligned}$$

and we obtain

$$E_1 < \omega_n(c_{29}^2 + \sup_{|x|\leq c_0} W(x))\frac{2}{3\hat{\varepsilon}}. \tag{4.1.45}$$

Thus by (4.1.44) and (4.1.45), if $\tilde\varepsilon \geq \hat\varepsilon$, E_1 is bounded by a constant which depends only on c_0, λ_0, n, p and W.

For the rest of the proof, consider the case $\tilde\varepsilon < \hat\varepsilon$ and consider the following four cases (a)–(d) depending on how large $\tilde r_0 = r_0/l$ is relative to $\tilde\varepsilon$ and $\tilde r_1$, where $\tilde r_1$ will be determined shortly depending only on c_0, λ_0, n, p, W and E_0:

$$(a)\, \tilde r_1 < \tilde r_0 \leq \frac{1}{2},\ (b)\, c_{33}\tilde\varepsilon^{\beta_2} < \tilde r_0 \leq \tilde r_1,\ (c)\, \tilde\varepsilon < \tilde r_0 \leq c_{33}\tilde\varepsilon^{\beta_2},\ (d)\, 0 < \tilde r_0 \leq \tilde\varepsilon.$$

To control the term involving v in (4.1.17), define a Radon measure

$$\mu(A) := \int_{A \cap B_{\frac{3}{4}}} \left(\frac{\tilde\varepsilon |\nabla \tilde u|^2}{2} + \frac{W(\tilde u)}{\tilde\varepsilon} \right).$$

By Theorem 2.2.1 and (4.1.42) (note that (4.1.42) has the restriction $\tilde r < 1/8$ but this can be dropped easily by replacing 4^{n-1} by a larger constant depending only on n), we have

$$\Big| \int_{B_{\frac{3}{4}}} \phi \, d\mu \Big| \leq c(n) l^{1-n} E_1 \int_{\mathbb{R}^n} |\nabla \phi| \, d\mathcal{L}^n \tag{4.1.46}$$

for all $\phi \in C^1_c(\mathbb{R}^n)$. By (4.1.17) and (4.1.25), if $c_{33}\tilde\varepsilon^{\beta_1} < \tilde r \leq \frac{1}{2}$, we have

$$\begin{aligned} \frac{d}{d\tilde r} \tilde E(\tilde r, 0) \geq & - \frac{c_{34}}{\tilde r^{1-\beta_2}} (\tilde E(\tilde r, 0) + 1) - \frac{1}{\tilde r^{n-1}} \int_{B_{\tilde r}} \tilde\varepsilon |\tilde v| |\nabla \tilde u|^2 \\ & + \frac{1}{\tilde r^n} \int_{B_{\tilde r}} \left(\frac{W(\tilde u)}{\tilde\varepsilon} - \frac{\tilde\varepsilon}{2} |\nabla \tilde u|^2 \right)_+ . \end{aligned} \tag{4.1.47}$$

Let $\phi \in C^1_c(B_{\frac{3\tilde r}{2}})$ be such that $0 \leq \phi \leq 1$, $\phi(y) = 1$ in $B_{\tilde r}$ and $|\nabla \phi| \leq \frac{4}{\tilde r}$. We use (4.1.46) with (4.1.40) by smoothly approximating $|\tilde v|$ as

$$\begin{aligned} \int_{B_{\tilde r}} \tilde\varepsilon |\tilde v| |\nabla \tilde u|^2 & \leq \int_{B_{\frac{3\tilde r}{2}}} \tilde\varepsilon \phi |\tilde v| |\nabla \tilde u|^2 \leq c(n) l^{1-n} E_1 \int_{B_{\frac{3\tilde r}{2}}} |\nabla (\phi |\tilde v|)| \\ & \leq c(n) l^{1-n} E_1 \int_{B_{\frac{3\tilde r}{2}}} \frac{4}{\tilde r} |\tilde v| + |\nabla \tilde v| \leq c(n) l^{3-n-\frac{n}{p}} \lambda_0 \tilde r^{n-\frac{n}{p}} E_1. \end{aligned} \tag{4.1.48}$$

Hence, for $c_{33}\tilde\varepsilon^{\beta_1} < \tilde r \leq \frac{1}{2}$, (4.1.47) with (4.1.48) and (4.1.42) give

$$\begin{aligned} \frac{d}{d\tilde r} \tilde E(\tilde r, 0) \geq & -c_{34} \tilde r^{\beta_2 - 1} (c(n) l^{1-n} E_1 + 1) - c(n) l^{3-n-\frac{n}{p}} \lambda_0 \tilde r^{1-\frac{n}{p}} E_1 \\ & + \frac{1}{\tilde r^n} \int_{B_{\tilde r}} \left(\frac{W(\tilde u)}{\tilde\varepsilon} - \frac{\tilde\varepsilon}{2} |\nabla \tilde u|^2 \right)_+ . \end{aligned} \tag{4.1.49}$$

By integrating (4.1.49) over $\tilde{r} \in (\tilde{s}_1, \tilde{s}_2)$ with $c_{33}\tilde{\varepsilon}^{\beta_1} < \tilde{s}_1 < \tilde{s}_2 \le \frac{1}{2}$, we obtain

$$\begin{aligned} \tilde{E}(\tilde{s}_2, 0) - \tilde{E}(\tilde{s}_1, 0) \ge& -c_{38}(\tilde{s}_2^{\beta_2} + l^{2-\frac{n}{p}} \tilde{s}_2^{2-\frac{n}{p}}) l^{1-n} E_1 - c_{38}\tilde{s}_2^{\beta_2} \\ &+ \int_{\tilde{s}_1}^{\tilde{s}_2} \frac{d\tilde{r}}{\tilde{r}^n} \int_{B_{\tilde{r}}} \left(\frac{W(\tilde{u})}{\tilde{\varepsilon}} - \frac{\tilde{\varepsilon}}{2} |\nabla \tilde{u}|^2 \right)_+ , \end{aligned} \tag{4.1.50}$$

where c_{38} depends only on c_0, λ_0, n, p and W. At this point, we choose $\tilde{r}_1 < 1/2$ depending only on c_0, λ_0, n, p and W so that

$$c_{38}(\tilde{r}_1^{\beta_2} + \tilde{r}_1^{2-\frac{n}{p}}) < \frac{1}{4}.$$

This in particular implies from (4.1.50) that if $c_{33}\tilde{\varepsilon}^{\beta_1} < \tilde{s}_1 < \tilde{s}_2 \le \tilde{r}_1$, then

$$\tilde{E}(\tilde{s}_2, 0) - \tilde{E}(\tilde{s}_1, 0) \ge -c_{38} - \frac{1}{4} l^{1-n} E_1. \tag{4.1.51}$$

With this $\tilde{r}_1$ being fixed, we proceed to check that E_1 is bounded in terms of $c_0, \lambda_0, n, p, W, E_0$ in each case (a)–(d).
Case (a): By (4.1.37), (4.1.39) and $\tilde{r}_1 < \tilde{r}_0$, we have

$$\frac{3}{4} l^{1-n} E_1 \le \tilde{E}(\tilde{r}_0, 0) \le \tilde{r}_0^{1-n} l^{1-n} E_0 \le \tilde{r}_1^{1-n} l^{1-n} E_0.$$

Hence

$$E_1 \le \frac{4}{3} \tilde{r}_1^{1-n} E_0$$

and E_1 is bounded by a constant depending only on c_0, λ_0, n, p, W and E_0.
Case (b): Since $c_{33}\tilde{\varepsilon}^{\beta_1} < \tilde{r}_0 \le \tilde{r}_1$, we may use (4.1.51) with $\tilde{s}_2 = \tilde{r}_1$ and $\tilde{s}_1 = \tilde{r}_0$. Then we obtain

$$\tilde{E}(\tilde{r}_1, 0) - \tilde{E}(\tilde{r}_0, 0) \ge -c_{38} - \frac{1}{4} l^{1-n} E_1.$$

Then, by (4.1.37) and (4.1.39), we obtain

$$E_1 \le 4(\tilde{E}(\tilde{r}_1, 0) + c_{38}) l^{n-1} \le 4(\tilde{r}_1^{1-n} E_0 + c_{38}),$$

which depends only on c_0, λ_0, n, p, W and E_0.
Case (c): By the same estimate used in the proof of Lemma 4.1.5, we have

$$\tilde{E}(c_{33}\tilde{\varepsilon}^{\beta_1}, 0) - \tilde{E}(\tilde{r}_0, 0) \ge -c_{32}. \tag{4.1.52}$$

We use (4.1.51) with $\tilde{s}_1 = c_{33}\tilde{\varepsilon}^{\beta_1}$ and $\tilde{s}_2 = \tilde{r}_1$ to obtain

$$\tilde{E}(\tilde{r}_1, 0) - \tilde{E}(c_{33}\tilde{\varepsilon}^{\beta_1}, 0) \ge -c_{38} - \frac{1}{4} l^{1-n} E_1, \tag{4.1.53}$$

and (4.1.52) and (4.1.53) combined with (4.1.37) give

$$E_1 \leq 4l^{n-1}(\tilde{E}(\tilde{r}_1, 0) + c_{32} + c_{38}) \leq 4\tilde{r}_1^{1-n}E_0 + 4(c_{32} + c_{38}),$$

which depends only on c_0, λ_0, n, p, W and E_0.
Case (d): Since $\tilde{r}_0 \leq \tilde{\varepsilon}$, we use (4.1.1) to obtain

$$\tilde{E}(\tilde{r}_0, 0) \leq \omega_n(c_{29}^2 + \sup_{|x|\leq c_0} W(x))\frac{\tilde{r}_0}{\tilde{\varepsilon}} \leq \omega_n(c_{29}^2 + \sup_{|x|\leq c_0} W(x)). \tag{4.1.54}$$

Then (4.1.54) and (4.1.37) gives

$$E_1 < \frac{4}{3}\omega_n(c_{29}^2 + \sup_{|x|\leq c_0} W(x))l^{n-1} \leq \omega_n(c_{29}^2 + \sup_{|x|\leq c_0} W(x)).$$

This completes the estimate for E_1. □

Once we obtain the upper density estimate, we may obtain the following monotonicity formula.

Theorem 4.1.8. *Given $0 < s < 1$, there exist constants $0 < c_{39}$ and $0 < \varepsilon_9 < 1$ depending only on $c_0, \lambda_0, n, p, W, E_0$ and s, such that, if $c_{33}\varepsilon^{\beta_2} \leq s_1 < s_2$, $B_{s_2}(x) \subset U_s$ and $\varepsilon < \varepsilon_9$, then*

$$\begin{aligned} E(s_2, x) - E(s_1, x) \geq &- c_{39}(s_2^{2-\frac{n}{p}} + s_2^{\beta_3}) \\ &+ \int_{s_1}^{s_2} \frac{d\tau}{\tau^n} \int_{B_\tau(x)} \Big(\frac{W(u)}{\varepsilon} - \frac{\varepsilon}{2}|\nabla u|^2\Big)_+. \end{aligned} \tag{4.1.55}$$

Proof. Let $\varepsilon_9 = \min\{\varepsilon_5, \varepsilon_6, \varepsilon_7, \varepsilon_8, (1-s)/2\}$ corresponding to the given s and suppose that $\varepsilon < \varepsilon_9$. For any $x \in U_s$ and $0 < r < (1-s)/2$, by Theorem 4.1.7, $E(r, x) \leq c_{37}(1 - s - \varepsilon)^{1-n}$, where the right-hand side is bounded by a constant depending only on $c_0, \lambda_0, n, p, W, E_0$ and s. For $B_{s_2}(x) \subset U_s$, we have (4.1.25) and (4.1.17). Arguing as (4.1.46)-(4.1.50) without change of variables (so $l = 1$) and with μ restricted to B_s in place of $B_{3/4}$, we obtain (4.1.55). □

Theorem 4.1.9. *Given $0 < s < 1$, there exist constants $0 < c_{40}$ depending only on $c_0, \lambda_0, n, p, W, E_0$ and s such that, if $\varepsilon < \varepsilon_9$, $|u(x)| \leq \alpha$, $\varepsilon \leq r$ and $B_r(x) \subset U_s$, then*

$$E(r, x) \geq c_{40}. \tag{4.1.56}$$

Proof. By Lemma 4.1.5, we may assume $c_{33}\tilde{\varepsilon}^{\beta_2} \leq r$ and $E(c_{33}\varepsilon^{\beta_2}, x) \geq c_{32}$. In (4.1.55), let $s_1 = c_{33}\varepsilon^{\beta_2}$ and $s_2 = r$. Fix $r_1 > 0$ depending only on $c_0, \lambda_0, n, p, W, E_0$ and s so that $c_{39}(r_1^{2-\frac{n}{p}} + r_1^{\beta_3}) \leq c_{32}/2$. Then for $c_{33}\varepsilon^{\beta_2} \leq r \leq r_1$, (4.1.55) shows that $E(r, x) \geq c_{32}/2$. For $1 > r > r_1$, $E(r, x) \geq r_1^{n-1}E(r_1, x) \geq r_1^{n-1}c_{32}/2$. Thus, setting $c_{40} = r_1^{n-1}c_{32}/2$, we have (4.1.56). $\square$

4.2 Discrepancy estimate

For the rest of the present section, we finish the proof of Lemma 4.1.4. We use the following result proved in [43, Lemma 3.9].

Lemma 4.2.1. *Given $0 < \eta, \beta_5 < 1$, $\eta \leq \beta_5$, $0 < c_{41}$, there exist $\varepsilon_{10} > 0$, $c_{42} > 0$ depending only on η, β_5, c_{41}, n and W with the following properties: Suppose $f \in C^3(U_{\varepsilon^{-\beta_5}})$, $g \in C^1(U_{\varepsilon^{-\beta_5}})$ and $0 < \varepsilon \leq \varepsilon_{10}$ satisfy*

$$-\Delta f + W'(f) = g$$

on $U_{\varepsilon^{-\beta_5}}$ and

$$\sup_{U_{\varepsilon^{-\beta_5}}} |f| \leq 1 + \varepsilon^{\eta}, \quad \sup_{U_{\varepsilon^{-\beta_5}}} \left(\frac{1}{2}|\nabla f|^2 - W(f)\right) \leq c_{41}.$$

Then

$$\sup_{B_{\frac{\varepsilon^{-\beta_5}}{2}}} \left(\frac{1}{2}|\nabla f|^2 - W(f)\right) \leq c_{42}(\varepsilon^{-\beta_5}\|g\|_{W^{1,n}(B_{\varepsilon^{-\beta_5}})} + \varepsilon^{\eta}). \tag{4.2.1}$$

We remark that the assumptions on W are essentially used for the proof of Lemma 4.2.1.

Proof of Lemma 4.1.4. As in the proof of Lemma 4.1.1, define $\tilde{u}(x) := u(\varepsilon x)$, $\tilde{v}(x) := \varepsilon v(\varepsilon x)$, and subsequently drop $\tilde{\cdot}$ for simplicity. We have

$$-\Delta u + W'(u) = \nabla u \cdot v$$

on $U_{\varepsilon^{-1}}$. With respect to the new variables, we need to prove

$$\sup_{U_{s\varepsilon^{-1}}} \left(\frac{1}{2}|\nabla u|^2 - W(u)\right) \leq \varepsilon^{1-\beta_1} \tag{4.2.2}$$

for some $0 < \beta_1 < 1$ for all sufficiently small ε. Let ϕ_λ be the standard mollifier, namely, define

$$\phi(x) := \begin{cases} C \exp\left(\frac{1}{|x|^2-1}\right) & \text{for } |x| < 1 \\ 0 & \text{for } |x| \geq 1, \end{cases}$$

where the constant $C > 0$ is selected so that $\int_{\mathbb{R}^n} \phi = 1$, and define $\phi_\lambda(x) := \frac{1}{\lambda^n}\phi(\frac{x}{\lambda})$. For $0 < \beta_6 < 1$ to be chosen depending only on n and p later, define for $x \in U_{\varepsilon^{-1}-1}$

$$f(x) := (u * \phi_{\varepsilon^{\beta_6}})(x) = \int u(x-y)\phi_{\varepsilon^{\beta_6}}(y)\, dy. \tag{4.2.3}$$

By (4.1.1) and (4.1.2), we have

$$\sup_{U_{\varepsilon^{-1}-1}} |f - u| \leq 2c_{29}\varepsilon^{\beta_6}, \tag{4.2.4}$$

$$\sup_{U_{\varepsilon^{-1}-1}} |\nabla f - \nabla u| \leq 2c_{29}\varepsilon^{\beta_6(2-\frac{n}{p})}. \tag{4.2.5}$$

We next define g to be

$$g := (\nabla u \cdot v) * \phi_{\varepsilon^{\beta_6}} + (W'(f) - W'(u) * \phi_{\varepsilon^{\beta_6}}), \tag{4.2.6}$$

so that we have

$$-\Delta f + W'(f) = g. \tag{4.2.7}$$

To use Lemma 4.2.1, we next estimate the $W^{1,n}$ norm of g on $U_{\varepsilon^{-\beta_5}}(x)$ with $x \in U_{s\varepsilon^{-1}}$, where $0 < \beta_4 < 1$ will be chosen depending only on n and p. In the following, let us write $U_{\varepsilon^{-\beta_5}}(x)$ as $U_{\varepsilon^{-\beta_5}}$ and $U_{\frac{\varepsilon^{-\beta_5}}{2}}(x)$ as $U_{\frac{\varepsilon^{-\beta_5}}{2}}$ for simplicity. For sufficiently small ε depending on s and β_4 so that $U_{2\varepsilon^{-\beta_4}}(x) \subset U_{\varepsilon^{-1}-1}$ (and so that we may use (4.1.1)), the first term of (4.2.6) can be estimated as

$$\|(\nabla u \cdot v) * \phi_{\varepsilon^{\beta_6}}\|_{W^{1,n}(U_{\varepsilon^{-\beta_4}})} \leq c_{43}(1 + \varepsilon^{-\beta_6})\|v\|_{L^n(U_{2\varepsilon^{-\beta_5}})} \tag{4.2.8}$$

where c_{43} depends only on ϕ, n and c_{29}. By (4.1.5), we obtain

$$\begin{aligned} \|v\|_{L^n(U_{2\varepsilon^{-\beta_5}})} &\leq \|v\|_{L^{\frac{np}{n-p}}(U_{2\varepsilon^{-\beta_5}})}\{\omega_n(2\varepsilon^{-\beta_5})^n\}^{\frac{2p-n}{np}} \\ &\leq \lambda_0\varepsilon^{(2-\frac{n}{p})(1-\beta_5)}(2^n\omega_n)^{\frac{2p-n}{np}}. \end{aligned} \tag{4.2.9}$$

Thus (4.2.8) and (4.2.9) show

$$\|(\nabla u \cdot v) * \phi_{\varepsilon^{\beta_6}}\|_{W^{1,n}(U_{\varepsilon^{-\beta_5}})} \leq c_{44}\varepsilon^{(2-\frac{n}{p})(1-\beta_5)-\beta_6} \tag{4.2.10}$$

where c_{44} depends only on ϕ, n, p, λ_0 and c_{29}. We next consider the second term of (4.2.6). By (4.2.4), (4.2.5) and

$$W'(f) - W'(u) * \phi_{\varepsilon^{\beta_6}} = (W'(f) - W'(u)) + (W'(u) - W'(u) * \phi_{\varepsilon^{\beta_6}}),$$

we compute

$$\sup|W'(f) - W'(u)| \leq \sup|W''| \sup|u - f| \leq c_{45}\varepsilon^{\beta_6}, \tag{4.2.11}$$

$$\begin{aligned} \sup|\nabla(W'(f) - W'(u))| \leq &\sup|W''| \sup|\nabla f - \nabla u| \\ &+ \sup|\nabla u| \sup|W'''| \sup|u - f| \\ \leq &\, c_{45}\varepsilon^{\beta_6(2-\frac{n}{p})}, \end{aligned} \tag{4.2.12}$$

$$\sup|W'(u) - W'(u) * \phi_{\varepsilon^{\beta_6}}| \leq c_{45}\varepsilon^{\beta_6}, \tag{4.2.13}$$

$$\sup|\nabla(W'(u) - W'(u) * \phi_{\varepsilon^{\beta_6}})| \leq c_{45}\varepsilon^{\beta_6(2-\frac{n}{p})}. \tag{4.2.14}$$

Here c_{45} depends only on $\phi, n, \lambda_0, c_{29}$ and W. Hence, (4.2.11)-(4.2.14) show

$$\|W'(f) - W'(u) * \phi_{\varepsilon^{\beta_6}}\|_{W^{1,n}(U_{\varepsilon^{-\beta_5}})} \leq 4c_{45}\varepsilon^{\beta_6(2-\frac{n}{p})-\beta_5}. \tag{4.2.15}$$

By (4.2.6), (4.2.10) and (4.2.15), we have

$$\|g\|_{W^{1,n}(U_{\varepsilon^{-\beta_5}})} \leq c_{44}\varepsilon^{(2-\frac{n}{p})(1-\beta_5)-\beta_6} + 4c_{45}\varepsilon^{\beta_6(2-\frac{n}{p})-\beta_5}. \tag{4.2.16}$$

We use Lemma 4.2.1 to f and g. Due to Lemma 4.1.2, we have $\sup|f| \leq \sup|u| \leq 1 + \varepsilon^{\eta}$ on $U_{\varepsilon^{-\beta_5}}$ and we may choose smaller η if necessary. Because of (4.1.1), we have $c_{41} \geq \sup_{U_{\varepsilon^{-\beta_5}}}(\frac{1}{2}|\nabla f|^2 - W(f))$ for a constant depending only on c_{29} and W (here again we restrict ε small so that $U_{\varepsilon^{-\beta_4}}(x) \subset U_{\varepsilon^{-1}-1} \cap U_{\varepsilon^{-1}(1+s)/2}$). Then we have all the assumptions for Lemma 4.2.1 and obtain

$$\begin{aligned} &\sup_{U_{\frac{\varepsilon^{-\beta_5}}{2}}} \Big(\frac{1}{2}|\nabla f|^2 - W(f)\Big) \\ &\qquad \leq c_{42}(\varepsilon^{(2-\frac{n}{p})(1-\beta_5)-\beta_5-\beta_6} + \varepsilon^{\beta_6(2-\frac{n}{p})-2\beta_5} + \varepsilon^{\eta}). \end{aligned} \tag{4.2.17}$$

At this point, we fix sufficiently small $0 < \beta_5, \beta_6 < 1$ depending only on n and p such that

$$(2 - \frac{n}{p})(1 - \beta_5) - \beta_5 - \beta_6 > 0, \quad \beta_6(2 - \frac{n}{p}) - 2\beta_5 > 0.$$

This shows that we may choose a $0 < \beta_2 < 1$ such that

$$\sup_{U_{\frac{\varepsilon^{-\beta_5}}{2}}} \left(\frac{1}{2}|\nabla f|^2 - W(f) \right) \le \varepsilon^{1-\beta_2} \tag{4.2.18}$$

for all sufficiently small $\varepsilon > 0$. We may take the center of $U_{\frac{\varepsilon^{-\beta_5}}{2}}$ ($= U_{\frac{\varepsilon^{-\beta_5}}{2}}(x)$) to be any $x \in U_{s\varepsilon^{-1}}$ so that we have the estimate on $U_{s\varepsilon^{-1}}$. By (4.2.5), (4.2.18) and

$$\sup |W(f) - W(u)| \le \sup |W'| \sup |u - f| \le c_{46}\varepsilon^{\beta_6},$$

we may also replace f by u in (4.2.18) by choosing a larger $0 < \beta_2 < 1$ if necessary. This proves the desired estimate. $\square$

4.3 Rectifiability and integrality of the limit varifold

In this section, we recover the index i and assume that $\{u_i\}_{i=1}^{\infty}$, $\{v_i\}_{i=1}^{\infty}$ and $\{\varepsilon_i\}_{i=1}^{n}$ satisfy (1.3.3)-(1.3.7). Define μ_i and V_i as in (1.3.9) and (1.3.10). By the standard weak compactness theorem of Radon measures, there exists a subsequence (denoted by the same index) and a Radon measure μ and a varifold V such that

$$\mu_i \to \mu, \ \ V_i \to V.$$

Lemma 4.3.1. *For $x \in \operatorname{spt}\mu$, there exists a subsequence $x_i \in U_1$ such that $u_i(x_i) \in [-\alpha, \alpha]$ and $\lim_{i\to\infty} x_i = x$.*

Proof. We prove this by contradiction and assume that there exists some $r > 0$ such that $|u_i| \ge \alpha$ on $U_r(x)$ for all large i. Without loss of generality, assume $u_i \ge \alpha$ on $U_r(x)$. Then we repeat the same argument leading to (4.1.30) with ϕ there replaced by $C_c^1(U_r(x))$. The argument shows that $\lim_{i\to\infty} \int \frac{\varepsilon_i}{2}|\nabla u_i|^2\phi^2 = 0$. Next, multiplying $u_i - 1$ to the equation (1.3.3) and using $W'(u_i)(u_i - 1) \ge \frac{\kappa}{2}(u_i - 1)^2$, we obtain

$$\begin{aligned}\int \frac{W}{\varepsilon_i}\phi^2 \le c(W) \int \frac{(u_i-1)^2}{\varepsilon_i}\phi^2 \le \frac{2c(W)}{\kappa} \int \frac{W'(u_i)(u_i-1)}{\varepsilon_i}\phi^2 \\ = \frac{2c(W)}{\kappa} \int (\varepsilon_i \Delta u_i (u_i - 1) + \varepsilon_i (v_i \cdot \nabla u_i)(u_i - 1))\phi^2.\end{aligned} \tag{4.3.1}$$

By integration by parts and (1.3.7), the right-hand side of (4.3.1) converges to 0. This shows that $\mu(U_r(x)) = 0$ and contradicts $x \in \operatorname{spt}\mu$. $\square$

Theorem 4.3.2. *There exist constants* $0 < D_1 \le D_2 < \infty$ *which depend only on* $c_0, \lambda_0, n, p, W, E_0$ *and* s *such that, for* $x \in \operatorname{spt}\mu \cap U_s$ *and* $B_r(x) \subset U_s$, *we have*

$$D_1 r^{n-1} \le \mu(B_r(x)) \le D_2 r^{n-1}. \tag{4.3.2}$$

Proof. This follows immediately from Theorem 4.1.7, 4.1.9 and Lemma 4.3.1. $\square$

For the subsequent use, define

$$\xi_i := \frac{\varepsilon_i |\nabla u_i|^2}{2} - \frac{W(u_i)}{\varepsilon_i}.$$

Once we have the monotonicity formula (4.1.55), we may prove the following "equi-partition of energy" by the same proof as in [40, Proposition 4.3]:

Theorem 4.3.3. ξ_i, $\frac{\varepsilon_i}{2}|\nabla u_i|^2 - |\nabla w_i|$ *and* $\frac{W(u_i)}{\varepsilon_i} - |\nabla w_i|$ *all converge to zero in* $L^1_{loc}(U_1)$.

Proof of Theorem 1.3.1. Recall that $\|V_i\| = \mu_i \llcorner_{\{|\nabla u_i| \neq 0\}}$. Since

$$\sigma \mu_i \llcorner_{\{|\nabla u_i| = 0\}} \le |\xi_i| \, d\mathcal{L}^n \to 0$$

by Theorem 4.3.3, $\mu_i \llcorner_{\{|\nabla u_i| \neq 0\}}$ converges to μ. We also know that $\|V_i\|$ converges to $\|V\|$ by definition, thus we have $\|V\| = \mu$. This proves (1). The claims (2) and (3) follows from Theorem 4.1.7, 4.1.9 and Lemma 4.3.1 (see also [40, Proposition 4.2]). Next, by (1.3.11), (4.1.18) and Theorem 4.3.3,

$$\begin{aligned} \delta V_i(g) &= \int_{\{|\nabla u_i| \neq 0\}} \operatorname{div} g \, d\mu_i \\ &\quad - \frac{1}{\sigma} \int_{\{|\nabla u_i| \neq 0\}} \nabla g \cdot \Big(\frac{\nabla u_i}{|\nabla u_i|} \otimes \frac{\nabla u_i}{|\nabla u_i|} \Big) (\varepsilon_i |\nabla u_i|^2 - \xi_i) \\ &= -\frac{1}{\sigma} \int \varepsilon_i (v_i \cdot \nabla u_i)(\nabla u_i \cdot g) + o(1) \end{aligned} \tag{4.3.3}$$

for $g \in C^1_c(U_1; \mathbb{R}^n)$, where $\lim_{i \to \infty} o(1) = 0$. By Theorem 4.1.7 and $\operatorname{spt} g \subset U_s$ for some $0 < s < 1$, we have a uniform bound on $E(r, x)$ (corresponding to u_i) for $B_r(x) \subset U_s$. Hence, by Theorem 2.2.1, we have

$$\begin{aligned} \int \varepsilon_i ((v_i - v) \cdot \nabla u_i)(\nabla u_i \cdot g) &\le c \Big(\int |v_i - v|^p |g|^p \varepsilon_i |\nabla u_i|^2 \Big)^{\frac{1}{p}} \\ &\le c \Big(\int |\nabla (|v_i - v|^p |g|^p)| \Big)^{\frac{1}{p}} \\ &\le c \left(\|\nabla v_i - \nabla v\|_{L^p} \|v_i - v\|_{L^p}^{p-1} + \|v_i - v\|_{L^p}^p \right)^{\frac{1}{p}} \end{aligned} \tag{4.3.4}$$

where the integrations are over $\operatorname{spt} g$. The above converges to 0 since we may choose a further subsequence of v_i which converges to v strongly in L^p_{loc}. Thus in the right-hand side of (4.3.3), we may replace v_i by v. Let $\epsilon > 0$ be arbitrary and let $\tilde{v}$ be a smooth vector field such that $\|v - \tilde{v}\|_{W^{1,p}(U_s)} < \epsilon$. By the varifold convergence $V_i \to V$, we have

$$\begin{aligned}\frac{1}{\sigma}\int \varepsilon_i(\tilde{v}\cdot\nabla u_i)(\nabla u_i \cdot g) &= \int S^{\perp}(\tilde{v})\cdot g\, dV_i(x,S) + o(1)\\ &\to \int S^{\perp}(\tilde{v})\cdot g\, dV(x,S).\end{aligned} \tag{4.3.5}$$

We may arbitrarily approximate the quantities in (4.3.5) by v by the same argument in (4.3.4), hence by (4.3.3)-(4.3.5), we obtain

$$\delta V(g) = -\int S^{\perp}(v)\cdot g\, dV(x,S). \tag{4.3.6}$$

Hence, $\|\delta V\|$ is a Radon measure on U_1. By (4.3.2) and Allard's rectifiability theorem [1, 5.5.(1)], V is rectifiable. Since V is rectifiable, there exist an $\mathcal{H}^{n-1}$ measurable and countably $n-1$-rectifiable set Γ such that

$$\int S^{\perp}(v)\cdot g\, dV(x,S) = -\int (T_x\,\Gamma)^{\perp}(v(x))\cdot g(x)\, d\|V\|(x). \tag{4.3.7}$$

The set Γ is the measure-theoretic support of $\|V\|$ and $T_x\,\Gamma$ is the approximate tangent space of Γ which exists for $\mathcal{H}^{n-1}$ a.e. on Γ. Next from (4.3.6), $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$ and the generalized mean curvature H_V exists. By (4.3.6) and (4.3.7), we have $H_V(x) = (T_x\,\Gamma)^{\perp}(v(x))$ holds for $\|V\|$ a.e. for x. This proves (5), except that we do not yet take $\Gamma = \operatorname{spt}\|V\|$. The proof of (4) is the same as [43] for the following reason. We may set $f = \varepsilon\nabla u\cdot v$ in [43] and we have $\|\varepsilon\nabla u \cdot v\|_{L^{\frac{np}{n-p}}(U_s)} \le c_{29}\lambda_0$ due to Lemma 4.1.1. In the proof, as long as we have the monotonicity formula (4.1.55) and the estimate Lemma 4.1.4, all the argument goes through. The point is that we do not need to take a derivative of f for the proof of integrality and we only need the control of $L^{\frac{np}{n-p}}$ norm as well as the estimate (4.1.2). See the comment in the proof of [40, Proposition 4.8] where it is explained that (4.1.2) is necessary. We should also point out that the L^n control of ∇f is not needed in the proof. Finally, by arguing as in (4.3.4) and the Hölder inequality, we have

$$\int_{U_s}\phi^{\frac{p(n-1)}{n-p}}\, d\|V\| \le c\|\nabla\phi\|_{L^p}\|\phi\|^{n(p-1)/(n-p)}_{L^{np/(n-p)}}$$

for any function $\phi \in C^1_c(U_s;\mathbb{R}^+)$ and we have the same inequality for $v \in W^{1,p}(U_1)$ by the density argument. Thus we have (6). By the well-known property of varifold having the generalized mean curvature in L^q with $q > n-1$ (see [37, 17.9(1)]), spt $\|V\|$ coincide with the measure-theoretic support Γ, so that we have (5) with $\Gamma = \operatorname{spt}\|V\|$. This concludes the proof of Theorem 1.3.1. □

4.4 Concluding remarks

In [23, 39], we studied the singular perturbation problem for

$$\partial_t u_\varepsilon + v_\varepsilon \cdot \nabla u_\varepsilon = \Delta u_\varepsilon - \frac{W'(u_\varepsilon)}{\varepsilon^2} \tag{4.4.1}$$

and proved that the time-parametrized family of limit varifolds satisfies the motion law of "normal velocity = mean curvavture vector + $v^\perp$" in a weak formulation (see [18, 41] for the case of $v_\varepsilon = 0$). In these works, we assumed that the prescribed initial data satisfies a boundedness of the upper density ratio. Part of the difficulty was to show that the upper density ratio bound can be controlled locally in time and uniformly with respect to ε. For the equilibrium problem, it is certainly not natural to assume such an upper density ratio estimate. It is interesting to see if one can drop the upper density ratio assumption for the initial data in the proof of [23, 39].

The vectorial prescribed mean curvature problem as in Theorem 1.3.2 seems, as far as we know, little studied so far. Traditionally, the prescription is the scalar version, i.e. given a scalar function (or constant) f, one looks for a hypersurface satisfying $H \cdot \nu = f$, where ν is the normal unit vector. The vectorial version is physically natural from the view point of force balance, in that the problem seeks the equality between the surface tension force and an external force acting on the surface. It must be said that the prescribed vector field in Theorem 1.3.2 is the gradient of a potential ρ, and not a general vector field. This is rather restrictive for applications and it is interesting to know if there can be a remedy for generalizations. If there may not exist a variational framework such as the min-max method to find solutions of (1.3.2), it should be still useful to have this diffused interface approach since the functional is well-behaved functional-analytically. As a further question, it is also interesting to investigate the asymptotic behavior of stable critical points of F_ε in the proof of Theorem 1.3.2, since we have a very successful analogy in [42, 44]. In this direction, we mention that a construction of prescribed scalar mean curvature hypersurfaces along the lines suggested here has been announced recently [5] (see also [4]).

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