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Essays on Bargaining Solutions and their Non-Cooperative Game-Theoretic Foundations

Doctor of Science Dissertation

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Abstract

The purpose of this dissertation is to analyze various bargaining situations by using game theory. As with Nash's seminal papers, we provide non-cooperative foundations for axiomatic bargaining solutions. That is, we provide bargaining processes and analyze the relationship between negotiators' equilibrium payoffs and axiomatic bargaining solutions.

In Chapter 1, we introduce Nash's seminal papers, Nash (1950) and Nash (1953), which initiate the analysis of non-cooperative foundations for axiomatic bargaining solutions. Also, we provide a brief summary of the studies in this dissertation.

In Chapter 2, we provide a bargaining process which is a generalization of the alternating-offers process in Rubinstein (1982) and analyze the relationship between its equilibrium payoffs and the asymmetric Nash bargaining solution (ANBS). In our process, the proposer in each period is decided stochastically and the probability to be a proposer depends on the history of proposers. In the bilateral model, there is a unique subgame perfect equilibrium (SPE). In the n-player model, although SPE may not be unique, an SPE similar to the SPE in the bilateral model exists. We show that, under these equilibria, the negotiators' SPE payoffs coincide with the ANBS weighted by the ratio of opportunities to be a proposer.

In Chapter 3, we consider how the Nash bargaining solution (NBS) can be achieved and how disagreement can be avoided under the simultaneous-offers process analyzed in Chatterjee and Samuelson (1990). In the simultaneous-offers process, all agreements on the Pareto frontier of the feasible utility set can be achieved and disagreement may occur in equilibria. To avoid disagreement, an arbitrator is often introduced into bargaining. However, if the arbitrator is biased, the NBS is never achieved in equilibrium. Thus, we consider introducing a mediator to avoid disagreement without eliminating the achievability of the NBS in equilibrium. In our model with a mediator, we obtain the following results. First, disagreement is not supported as a stationary SPE (SSPE) outcome. Second, even if a mediator is biased, the NBS is always one of the SSPE agreements. Finally, if a mediator is fair, negotiators always reach an agreement with the NBS in SSPE. Thus, by introducing a mediator, negotiators can avoid disagreement without eliminating the achievability of the NBS. Also, the NBS is always achieved under a process with a fair mediator.

In Chapter 4, we consider situations where negotiators have claims on a profit but the profit is not sufficient to cover the totality of these claims. Such situations are called claims problems and axiomatic solutions are called rules. As a central rule, we consider the constrained equal awards (CEA) rule. The CEA rule corresponds to the NBS in a special class of bargaining problems. Previous papers consider processes where claimants achieve the division of a profit chosen by the CEA rule. However, these processes are not "procedurally fair" or "multilateral." That is, claimants are not treated equally or some claimant is not involved in the negotiation at some stage. If at least one of the two features is missing, some claimant feels unfair and does not want to participate in the negotiation. Thus, we propose a process which is procedurally fair and multilateral. We show that, as a unique SPE division, the division chosen by the CEA rule is achieved under our process.

Chapter 5 is the concluding chapter of this dissertation. We summarize the results of our studies. Also, we discuss several remaining issues and future work.

Contents

Chapter 1

Introduction

A *bargaining problem* is a situation where individuals or firms decide how to divide a profit which can be made by their cooperation. For example, price negotiation in commodity trading, wage negotiation in labor disputes, or negotiation in international conflicts can be considered as bargaining problems. The purpose of this dissertation is to analyze various bargaining situations by using game theory. There are two approaches for analyzing bargaining problems. One is the *axiomatic approach* and the other is the *strategic approach*. These approaches were both initiated by Nash's studies. We first introduce the outlines of these studies.

Nash (1950) is the first paper which analyzes bargaining problems by the axiomatic approach. He defines a two-person bargaining problem as a pair (S, d). A compact and convex set $S \subset \mathbb{R}^2$ denotes a set of utilities which negotiators can achieve by their cooperation, and a point $d \in S$ denotes the negotiators' utilities when disagreement occurs. Under this setting, Nash (1950) defines a *bargaining solution* as a function assigning, for each bargaining problem (S, d), a pair of the negotiators' utilities in S. A bargaining solution specifies some agreement for each bargaining problem. In the axiomatic approach, a reasonable agreement for each bargaining situation is analyzed by finding bargaining solutions satisfying a number of desirable properties which the solutions should satisfy.

Nash (1950) considers the following four axioms: *Invariance to equivalent utility representations*, *Symmetry*, *Pareto efficiency*, and *Independence of irrelevant alternatives*.[1] He proves that there is a unique bargaining solution satisfying the above four axioms. Nowadays, this solution is called the *Nash bargaining solution* (NBS). Since the NBS satisfies several desirable properties, we can consider the NBS to be a reasonable agreement for each bargaining problem.

The axiomatic approach is a powerful tool to find a reasonable agreement such

[1] For the definitions of these axioms, for example, see Osborne and Rubinstein (1990).

as the NBS. However, the underlying bargaining process is not modeled explicitly, and therefore, it is not known how the negotiators achieve such an agreement. In order to know how a reasonable agreement is achieved, Nash also considers another approach for analyzing bargaining problem. In the study of Nash (1953), he provides a non-cooperative game which represents a process of bargaining and analyzes what agreement is achieved through negotiators' strategic behavior. Such an approach is called the strategic approach.

Nash (1953) considers a process of two-person bargaining as the following non-cooperative game. In his bargaining process, both negotiators simultaneously demand their utility levels. If their demands are compatible, the negotiators reach an agreement. Otherwise, disagreement occurs. This game is called the *Nash demand game*. In this game, Nash (1953) derives a refinement of Nash equilibrium which is robust to the negotiators' uncertainty in the feasible utility set. He proves that the negotiators' equilibrium payoffs coincides with the NBS as the possibility of uncertainty converges to zero. This result implies that a reasonable agreement is achieved when the negotiators behave strategically in the process of the Nash demand game. Through Nash (1953)'s research, we can know how a reasonable agreement is achieved.

The study of Nash (1950) analyzes *what* a reasonable bargaining agreement is under the concept of a *cooperative game*. On the other hand, the study of Nash (1953) analyzes *how* a reasonable agreement is achieved under the concept of a *non-cooperative game*. Nash's contribution is to connect a cooperative game with a non-cooperative game. Following Nash's seminal papers, several bargaining solutions and processes have been provided, and many papers have tried to connect these processes with these solutions. This line of research is known today as the *Nash program*.[2]

This dissertation also follows this line of research. That is, we provide bargaining processes and analyze the relationship between negotiators' equilibrium payoffs and bargaining solutions. In the following, we provide a brief summary of the studies in this dissertation with this context in mind.

In Chapter 2, we provide a bargaining process which is a generalization of the alternating-offers process in Rubinstein (1982) and analyze the relationship between its equilibrium payoffs and the generalized NBS. This chapter is based on Hanato (2020).

Rubinstein (1982) is one of the most important research papers regarding the Nash program. The bargaining model of Rubinstein (1982) incorporates a process

[2]For a comprehensive survey on the Nash program, see Serrano (2005) and Serrano (2020).

of renegotiation in contrast to the Nash demand game. In his model, bargaining is conducted between two negotiators each of whom takes turns offering a proposal to the opponent negotiator. If the proposal is accepted, the negotiators reach an agreement. Otherwise, a proposer is changed and they renegotiate at the next period. In this model, by the study of Binmore et al. (1986), it is known that the unique subgame perfect equilibrium (SPE) payoffs coincide with the NBS when the negotiators are sufficiently patient.

However, in real situations, negotiation may not be conducted exactly in the process of alternating-offers. For example, in price negotiation, if some negotiator is more aggressive, such a negotiator may offer proposals more frequently. Therefore, we consider a bargaining process where each negotiator may not take turns offering a proposal. In our study, we provide a process where the proposer in each period is decided stochastically. We suppose that the probability to be a proposer depends on the history of proposers.

The negotiators divide a profit of size 1 as in the model of Rubinstein (1982). We derive an SPE and analyze how its SPE payoffs are related to our process in the bilateral model and the n-player model. In the bilateral model, there is a unique SPE. In the n-player model, although SPE may not be unique, an SPE similar to the SPE in the bilateral model exists. We show that, under these equilibria, if the ratio of the negotiators' opportunities to be a proposer converges as the bargaining proceeds to later periods, the negotiators divide the profit according to this convergent ratio when they are sufficiently patient.

In our model, the negotiators' SPE payoffs coincides with the asymmetric NBS (ANBS) weighted by the convergent values of opportunities to be a proposer. The ANBS is a generalization of the NBS and the weights of the ANBS reflect negotiators' bargaining power. Thus, the opportunities to be a proposer can be considered as the negotiators' bargaining power. By our results, we can know that the ANBS is achieved when bargaining is conducted under the process where opportunities of proposals are asymmetric between negotiators.

In Chapter 3, we consider how the NBS is achieved in the simultaneous-offers process analyzed in Chatterjee and Samuelson (1990). This chapter is based on Hanato (2019).

Chatterjee and Samuelson (1990) consider a simultaneous-offers process which is a generalization of the Nash demand game in the sense that it incorporates a process of renegotiation similar to the Rubinstein (1982). In their model, two negotiators simultaneously propose their demands at each period. If their demands are compatible, the negotiators reach an agreement. Otherwise, they renegotiate. For example, such a simultaneous-offers process appears in negotiation of international conflicts.

In these situations, nations decide demands in advance and simultaneously reveal them on the negotiation table.

Chatterjee and Samuelson (1990) show that all agreements on the Pareto frontier of the feasible utility set can be achieved as well as disagreement may occur in equilibrium. The problem with this result is that an unreasonable agreement or an inefficient outcome may arise in equilibrium. Therefore, it is worth considering how a reasonable agreement such as the NBS can be achieved, and how disagreement can be avoided under the simultaneous-offers process.

One way to avoid disagreement is to introduce an arbitrator into bargaining. The role of an arbitrator is to impose some agreement when negotiators cannot reach an agreement. However, some studies that incorporate an arbitrator imply that, if the arbitrator is biased, a reasonable agreement is never achieved in equilibrium.

Thus, in our study, we consider introducing a mediator to avoid disagreement without eliminating the achievability of a reasonable agreement in equilibrium. Whereas an arbitrator imposes an agreement, a mediator has no such power and can only give advice to the negotiators. In contrast to bargaining with an arbitrator, negotiators have the right to reject the mediator's advice.

We analyze a simultaneous-offers bargaining model with such a mediator. Under a stationary SPE (SSPE) which is a refinement of an SPE, we obtain the following results. First, disagreement is not supported as an SSPE outcome. Second, even if a mediator is biased, the NBS is always one of the SSPE agreements. Finally, if a mediator is fair, negotiators always reach an agreement with the NBS in SSPE. These results imply that, by introducing a mediator, negotiators can avoid disagreement without eliminating the achievability of a reasonable agreement. Also, we can see that the NBS is achieved under the simultaneous-offers process with a fair mediator.

Chapter 4 is based on Hagiwara and Hanato (2021). We consider bargaining situations where negotiators (claimants) have claims on a profit. For example, when a firm goes bankrupt and its liquidation value has to be allocated, creditors have claims on it. Also, when an estate is allocated, heirs have claims on it. Especially, we consider the situation where the liquidation value or the estate is not sufficient to cover the totality of the claims. Such a problem is known as a *claims problem*. Although claims problems have been mainly analyzed by the axiomatic approach, some recent papers have analyzed the problems by using the strategic approach and connected a bargaining process with a solution. We also follow these studies.

An axiomatic solution of claims problems is called a *division rule* or simply a *rule*. As a central rule for claims problems, we consider the *constrained equal awards rule* (CEA rule). It is known that the division of a profit chosen by the CEA rule corresponds to the NBS in a special class of bargaining problems.

Li and Ju (2016) and Tsay and Yeh (2019) consider processes where claimants achieve the division chosen by the CEA rule. However, these processes are not "procedurally fair" or "multilateral." If a game is not procedurally fair, claimants are not treated equally. For example, only one claimant has the power to select a division in Li and Ju (2016). In addition, if a game is not multilateral, some claimant is not involved in the negotiation at some stage. Such a situation occurs in Tsay and Yeh (2019). If at least one of the two features is missing, the negotiation might not be initiated since some claimant feels unfair and does not want to participate in the negotiation.

Thus, we propose a process which is procedurally fair and multilateral. Our process resembles the simultaneous-offers process in the sense that in each period, all negotiators simultaneously make proposals and they can try again in the next period if they do not reach an agreement. We show that, as a unique SPE division, the division chosen by the CEA rule is achieved under our procedurally fair and multilateral process.

Chapter 5 is the concluding chapter of this dissertation. We summarize the results of our studies. Also, we discuss several remaining issues and future work.

Chapter 2

Equilibrium Payoffs and Proposal Ratios in Bargaining Models[1]

2.1 Introduction

In this chapter, we consider a basic non-cooperative bargaining problem in which negotiators divide a pie of size 1. We analyze the model which is a generalization of the model of Rubinstein (1982) from the viewpoint of the process of how a proposer is decided in each period. We consider a process where the proposer in each period is decided stochastically and a probability to be a proposer depends on the history of proposers. In this model, we derive the subgame perfect equilibrium (SPE) and analyze the relationship between its SPE payoffs and a bargaining solution.

Recent research on non-cooperative bargaining models based on Rubinstein (1982) has used one of the following tools in deciding the proposer in each period: alternating offers, constant probabilities across periods, or a Markov process (where a probability to be a proposer depends on the previous proposer). Alternating offers and constant probabilities across periods can be considered as a special case of a Markov process. For example, alternating offers is used in Rubinstein (1982), Shaked and Sutton (1984), Chae and Yang (1990), and Kultti and Vartiainen (2010), constant probabilities process is used in Fershtman and Seidmann (1993), Okada (1996), Kalandrakis (2006), and Laruelle and Valenciano (2008), a Markov process is used in Kalandrakis (2004), Britz et al. (2010), and Herings and Predtetchinski (2010).

In these processes, negotiators' probabilities to be a proposer do not depend on

[1] This chapter is based on Hanato (2020).

periods. Therefore, they cannot express the bargaining where situations change over time. However, in reality, bargaining situations may constantly and gradually change over time. For example, in price negotiation of commodity trading, if some individual can haggle over the price more tenaciously than the opponent, the opportunity to be a proposer of such an individual will gradually increase over time. Then, the probability to be a proposer of this individual will increase over time. Also, for a business person, there may exist some time periods when she is busier than other periods. During the busy time periods, it may be difficult for a business person to prepare a plan of proposal. Then, the probability that such a business person becomes a proposer may be small. In this situation, the probabilities to be a proposer depend on the periods rather than the previous proposer.

Such situations cannot be appropriately modeled within the framework of the aforementioned models since the probabilities to be a proposer in these models have nothing to do with periods. In contrast to these models, by dispensing the Markov assumption, we analyze a more flexible model in the sense that a negotiator's probability to be a proposer may depend on the history of proposers. That is, a Markov process is a special case of our process. By considering such a model, we can analyze more complex situations depending not only on the previous proposers but also on periods.

Although we consider the process which can depend on the history of proposers to analyze the situations depending on periods, it does not mean that the process needs to use the information of all previous proposers. For example, consider a process in which the probabilities to be a proposer depend on the previous finite number of proposers and the periods. Such a process is a special case of our process, but this process cannot be represented by a Markov process since it depends on periods.

Other related literatures are as follows. Mao (2017) and Mao and Zhang (2017) analyze the bilateral bargaining models which are not represented by a Markov process to handle complex situations from the viewpoint of the designer, but their procedures are deterministic. The model analyzed in this chapter is a generalization of their model in the sense that we consider a stochastic process. Although Merlo and Wilson (1995) and Merlo and Wilson (1998) each analyzes an n-player model where the probability may depend on the history of proposers, the goals of these studies are to characterize the set of SPE payoffs as a fixed point of a suitable mapping. Therefore, these papers do not derive the SPE payoffs explicitly so that one do not know how negotiators actually divide the pie other than the fact that one such division exists.

In our study, we derive an explicit expression for the SPE payoffs and analyze how negotiators divide the pie in the bilateral model and the n-player model, respectively. In the bilateral model, we show that there is a unique SPE and show how its SPE payoffs are related to the probability to be a proposer. In the n-player model, although we cannot derive the uniqueness of SPE, we can still obtain an SPE which has the same form as the SPE derived in the bilateral bargaining model. That is, we can obtain an SPE which does not involve punishments for negotiators who deviate from the strategy profile (similar to a stationary SPE derived in the models with a Markov process). Also, we can find that this SPE is a Markov perfect equilibrium (MPE). Under this MPE, the relationship between negotiators' MPE payoffs and the probability to be a proposer has the same relationship as the bilateral model.

In this chapter, we also analyze the case where the discount factor is sufficiently large. This case is not analyzed in Mao (2017), Mao and Zhang (2017), Merlo and Wilson (1995), and Merlo and Wilson (1998). In the bilateral model, we show that if the ratio of opportunities to be a proposer during some periods converges to some value in the long run, then negotiators divide the pie according to the ratio of this convergent value (we use the word "the proposal ratio" instead of the word "the ratio of opportunities to be a proposer" in the following). In reality, even if individuals propose the divisions freely in the beginning, the negotiation often calm down and the proposal ratio often stays in some value in the long run. Our result shows that the pie is divided according to the ratio of this value.

As corollaries, we can derive the result for models with alternating offers, constant probabilities across periods, and a Markov process. The main consequence of this result is that the process used in our model has less regularity than a Markov process, we can derive the same result as in the model that uses a Markov process. That is, the result that negotiators divide the pie according to the ratio of the convergent value is "robust" to departures from an exact Markov process. However, in contrast to the result as shown in Britz et al. (2010)'s Markov process model where all negotiators propose the same division in all states in a stationary SPE, under the SPE which we derive, the negotiators propose the divisions depending on the current state.

We also find the relationship between the SPE payoffs and asymmetric Nash bargaining solution (ANBS). The relationship between SPE payoffs and the ANBS is recently analyzed in Laruelle and Valenciano (2008), Kultti and Vartiainen (2010), and Britz et al. (2010). These papers analyze such a relationship in different bargaining processes. The processes of Laruelle and Valenciano (2008), Kultti and Vartiainen (2010), and Britz et al. (2010) are constant probabilities across periods, alternating offers, and a Markov process, respectively.

In our research, we analyze the relationship between the SPE payoffs and the

ANBS in a more flexible process than the processes in the aforementioned papers. Our research is actually not a generalization of these papers since these papers analyze such a relationship with a more general utility space than ours. However, finding a relationship between the SPE payoffs and the ANBS in our complex process clarifies how a bargaining procedure affects the bargaining outcome as "bargaining power." As a result of this analysis, we show that the limit of the SPE payoffs coincides with the ANBS weighted by the convergent values of the proposal ratio.

In the n-player model, all results about the limit payoffs in the bilateral model are also derived under the MPE which has the same form as the SPE derived in the bilateral model.

This chapter is organized as follows. In section 2.2, we define the bilateral bargaining model. In section 2.3, we show that there exists a unique SPE in the game defined in section 2.2 and how its SPE payoffs are related to the process of proposer. In section 2.4, we consider the case where the discount factor is sufficiently large in the bilateral model. In this section, we show that if the proposal ratio during some periods converges to some value in the long run, then negotiators divide the pie according to the ratio of this convergent value. In section 2.5, we analyze the relationship between the SPE payoffs and the ANBS. In section 2.6, we analyze the n-player model. In section 2.7, we conclude our study. Although some results in this chapter are based on the author's master thesis, there are additional discussions. Especially, the discussions about the relationship between the SPE payoffs and the ANBS are new results. For ease of understanding these additional discussions, this chapter contains the results in the author's master thesis.

2.2 The bilateral model

We consider the game in which two negotiators 1 and 2 divide a pie of size 1. We define $N = \{1, 2\}$ as the set of negotiators and $\delta \in (0, 1)$ as the common discount factor. Also, let $S = \{\,(x_1, x_2) \mid x_1 + x_2 = 1, x_1, x_2 \geq 0\}$ as the set of divisions. The game proceeds as follows.

At period $t \in \{1, 2, \dots\}$, nature selects one negotiator as a proposer. When negotiator $i \in N$ is selected as a proposer, then she proposes some division $x \in S$. After it, the responder $j (\neq i)$ responds with Yes or No. If the responder negotiator j accepts the opponent's proposal, then the game ends and negotiator i receives $\delta^{t-1} x_i$ and negotiator j receives $\delta^{t-1} x_j$. Conversely, if negotiator j rejects the opponent's proposal, the game continues to the next period $t + 1$ and repeat the above process.

In this model, we assume that a probability to be a proposer depends on the history of proposers. By such a model, we can consider the process depending not

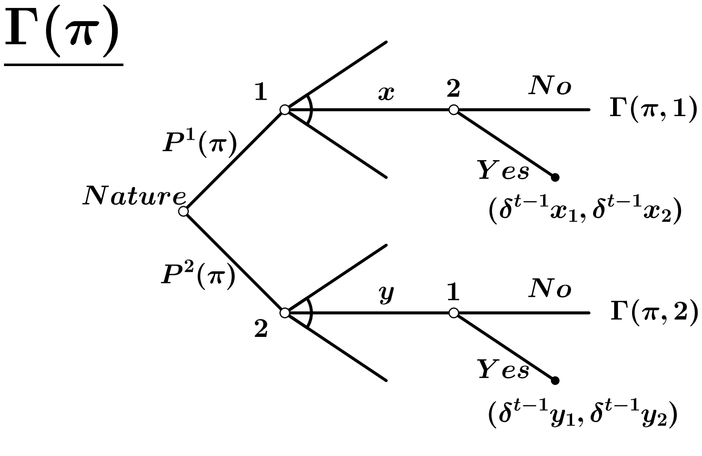

Figure 2.1: Subgame $\Gamma(\pi)$ corresponding to a history of proposers π

only on the previous proposers but also on periods.

Since $\bigcup_{t\in\mathbb{N}} N^{t-1}$ denotes the set of histories of proposers ($N^0 = \emptyset$), the probability that a proposer is chosen in the next period is represented by the function $P : \bigcup_{t\in\mathbb{N}} N^{t-1} \to \{(P^1, P^2) \mid P^1 + P^2 = 1, P^1, P^2 \geq 0\}$ where P^i denotes negotiator i's probability.

Histories are divided into three types H_t^a, H_t^b, and H_t^c where $H_t^a = (N \times S \times \{No\})^{t-1}$ denotes the set of histories at the beginning of period t ($H_1^a = \emptyset$), $H_t^b = H_t^a \times N$ denotes the set of histories after nature's selection, and $H_t^c = H_t^b \times S$ denotes the set of histories after the proposer's offer. Let $o(h_t^a) \in N^{t-1}$ be the history of proposers in $h_t^a \in H_t^a$. Then, negotiator i is selected as a proposer with probability $P^i(o(h_t^a))$ after $h_t^a \in H_t^a$.

Consider two histories $g_t^a, h_t^a \in H_t^a$ such that $o(g_t^a) = o(h_t^a) \in N^{t-1}$. Since $P(o(g_t^a)) = P(o(h_t^a))$, the subgames corresponding to g_t^a and h_t^a coincide. Therefore, subgames corresponding to H_t^a can be characterized by N^{t-1}. Now, we define $\Gamma(\pi)$ as the subgame corresponding to $\pi \in \bigcup_{t\in\mathbb{N}} N^{t-1}$. The original game is represented by $\Gamma(\emptyset)$. See Figure 2.1.

2.3 Uniqueness of SPE

We examine an SPE of this model. In this section, we prove the uniqueness of SPE. First, we prepare some notation. Let $\pi_r \in N^r$ be an order of proposers during r periods and let $\pi_r(k)$ be k-th proposer of the order π_r. Let $\pi_r^s = (\pi_r(1), \ldots, \pi_r(s))$ be the proposers of the order π_r from the first proposer to s-th proposer. Let $\pi_r\pi_s$ be an order of proposers in which π_s follows π_r. Then, we define $\Pi(\pi, \pi_r) =$

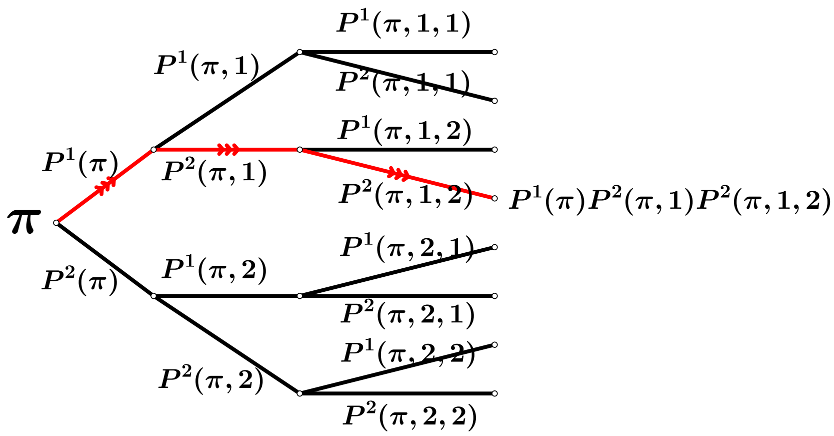

Figure 2.2: Extracted tree of three periods after a history of proposers π

$P^{\pi_r(1)}(\pi)\cdots P^{\pi_r(r)}(\pi\pi_r^{r-1})$.

To see the meaning of $\Pi(\pi,\pi_r)$, we extract edges of nature's action from the original game tree. Figure 2.2 represents the tree of three periods after the history of proposers π. For example, if $\pi_3 = (1,2,2)$, $\Pi(\pi,\pi_3) = P^1(\pi)P^2(\pi,1)P^2(\pi,1,2)$. Generally, $\Pi(\pi,\pi_r)$ is the probability that the order π_r occurs on condition that the history of proposers π occurred. Since the sum of the probabilities of all orders of length r (on condition that π occurred) equals 1, $\sum_{\pi_r\in N^r}\Pi(\pi,\pi_r) = 1$ for all $\pi \in \bigcup_{t\in\mathbb{N}} N^{t-1}$ and for all $r \in \mathbb{N}$.

Also, by the definition of $\Pi(\pi,\pi_r)$, for all $\pi \in \bigcup_{t\in\mathbb{N}} N^{t-1}$, $\pi_r \in N^r$, and $i \in N$,

$$P^i(\pi)\Pi((\pi,i),\pi_r) = \Pi(\pi,(i,\pi_r)). \tag{2.1}$$

Then, we define $p_t^i(\pi)$ as follows.

$$p_t^i(\pi) = \sum_{\pi_{t-1}\in N^{t-1}} \Pi(\pi,\pi_{t-1})P^i(\pi\pi_{t-1})$$

where $\Pi(\pi,\emptyset) = 1$. $\Pi(\pi,\pi_{t-1})P^i(\pi_{t-1})$ roughly represents the probability that the history of proposers $(\pi_{t-1},i) \in N^t$ occurs on condition that the history of proposers π occurred. Therefore, since $p_t^i(\pi)$ is the sum of these probabilities for all orders $\pi_{t-1} \in N^{t-1}$, $p_t^i(\pi)$ can be considered as negotiator i's probability to be a proposer at period t of the subgame $\Gamma(\pi)$. By its definition, $p_t^i(\pi) + p_t^j(\pi) = 1$.

Then, the following Lemma 2.3.1 holds. The meaning of Lemma 2.3.1 is that negotiator i's probability to be a proposer at period $t+1$ of the subgame $\Gamma(\pi)$ is equal to the sum of negotiator i's probabilities at period t of the subgame $\Gamma(\pi,i)$ and $\Gamma(\pi,j)$ where the probabilities are weighted by the probabilities that the subgame

$\Gamma(\pi)$ proceeds to the subgame $\Gamma(\pi, i)$ and $\Gamma(\pi, j)$, respectively.

Lemma 2.3.1. *For all* $\pi \in \bigcup_{t\in\mathbb{N}} N^{t-1}$ *and* $t \in \mathbb{N}$,

$$P^i(\pi) \cdot p_t^i(\pi, i) + P^j(\pi) \cdot p_t^i(\pi, j) = p_{t+1}^i(\pi)$$

where $j \neq i$.

Proof. By the definition of $p_t^i(\pi)$ and (2.1),

$$\begin{aligned}
&P^i(\pi) \cdot p_t^i(\pi, i) + P^j(\pi) \cdot p_t^i(\pi, j)\\
=&\sum_{\pi_{t-1}\in N^{t-1}} P^i(\pi)\Pi((\pi, i), \pi_{t-1})P^i(\pi, i, \pi_{t-1}) + \sum_{\pi_{t-1}\in N^{t-1}} P^j(\pi)\Pi((\pi, j), \pi_{t-1})P^i(\pi, j, \pi_{t-1})\\
=&\sum_{\pi_{t-1}\in N^{t-1}} \Pi(\pi, (i, \pi_{t-1}))P^i(\pi, i, \pi_{t-1}) + \sum_{\pi_{t-1}\in N^{t-1}} \Pi(\pi, (j, \pi_{t-1}))P^i(\pi, j, \pi_{t-1}).
\end{aligned}$$

The first term of the last equation can be considered as the sum of the orders of length t which starts from i, and the second term can be considered as the sum of the orders of length t which starts from j. Therefore, for all $i' \in N$,

$$\sum_{\pi_{t-1}\in N^{t-1}} \Pi(\pi, (i', \pi_{t-1}))P^i(\pi, i', \pi_{t-1}) = \sum_{\pi_t\in N^t, \pi_t(1)=i'} \Pi(\pi, \pi_t)P^i(\pi\pi_t).$$

Then,

$$\begin{aligned}
&P^i(\pi) \cdot p_t^i(\pi, i) + P^j(\pi) \cdot p_t^i(\pi, j)\\
=&\sum_{\pi_t\in N^t, \pi_t(1)=i} \Pi(\pi, \pi_t)P^i(\pi\pi_t) + \sum_{\pi_t\in N^t, \pi_t(1)=j} \Pi(\pi, \pi_t)P^i(\pi\pi_t)\\
=&\sum_{\pi_t\in N^t} \Pi(\pi, \pi_t)P^i(\pi\pi_t)\\
=&p_{t+1}^i(\pi).
\end{aligned}$$

□

By using the value $p_t^i(\pi)$, for all $\pi \in \bigcup_{t\in\mathbb{N}} N^{t-1}$ and $i \in N$, we also define

$$f_i(\pi) = \frac{\sum_{r=1}^{\infty} \delta^{r-1} p_r^i(\pi)}{\sum_{s=1}^{\infty} \delta^{s-1}}.$$

Since $p_t^i(\pi) + p_t^j(\pi) = 1$, $f_i(\pi) + f_j(\pi) = 1$ holds. Also, the following lemma about $f_i(\pi)$ holds.

Lemma 2.3.2. *For all* $\pi \in \bigcup_{t=1}^{\infty} N^{t-1}$ *and* $i \in N$,

$$f_i(\pi) = P^i(\pi)\left(1 - \delta f_j(\pi, i)\right) + P^j(\pi)\delta f_i(\pi, j)$$

where $j \neq i$.

Proof. By the definition of $f_i(\pi)$, the first term of the right hand side can be transformed as follows.

$$\begin{aligned}
P^i(\pi)\left(1 - \delta f_j(\pi, i)\right) =& \frac{P^i(\pi)}{\sum_{s=1}^{\infty} \delta^{s-1}} \left(\sum_{r=1}^{\infty} \delta^{r-1} - \delta \sum_{r=1}^{\infty} \delta^{r-1} p_r^j(\pi, i) \right) \\
=& \frac{P^i(\pi)}{\sum_{s=1}^{\infty} \delta^{s-1}} \left(1 + \sum_{r=1}^{\infty} \delta^r \left[1 - p_r^j(\pi, i)\right] \right) \\
=& \frac{1}{\sum_{s=1}^{\infty} \delta^{s-1}} \left(P^i(\pi) + \sum_{r=1}^{\infty} \delta^r P^i(\pi) p_r^i(\pi, i) \right).
\end{aligned}$$

Also, the second term can be transformed as follows.

$$\begin{aligned}
P^j(\pi)\delta f_i(\pi, j) =& \frac{P^j(\pi)}{\sum_{s=1}^{\infty} \delta^{s-1}} \delta \sum_{r=1}^{\infty} \delta^{r-1} p_r^i(\pi, j) \\
=& \frac{1}{\sum_{s=1}^{\infty} \delta^{s-1}} \sum_{r=1}^{\infty} \delta^r P^j(\pi) p_r^i(\pi, j).
\end{aligned}$$

Therefore, by summarizing the above two values,

$$\begin{aligned}
& P^i(\pi)\left(1 - \delta f_j(\pi, i)\right) + P^j(\pi)\delta f_i(\pi, j) \\
=& \frac{1}{\sum_{s=1}^{\infty} \delta^{s-1}} \left(P^i(\pi) + \sum_{r=1}^{\infty} \delta^r P^i(\pi) p_r^i(\pi, i) \right) + \frac{1}{\sum_{s=1}^{\infty} \delta^{s-1}} \sum_{r=1}^{\infty} \delta^r P^j(\pi) p_r^i(\pi, j) \\
=& \frac{1}{\sum_{s=1}^{\infty} \delta^{s-1}} \left(P^i(\pi) + \sum_{r=1}^{\infty} \delta^r p_{r+1}^i(\pi) \right) \\
=& \frac{\sum_{r=1}^{\infty} \delta^{r-1} p_r^i(\pi)}{\sum_{s=1}^{\infty} \delta^{s-1}} \\
=& f_i(\pi)
\end{aligned}$$

where we use Lemma 2.1 in the second equation. Thus, Lemma 2.3.2 holds.

□

We prove the existence of an SPE by using the above lemma.

Theorem 2.3.1. *Consider the following pair of strategies $\sigma = (\sigma_1, \sigma_2)$. Negotiator $i \in N$ proposes the division of the pie $(1 - \delta f_j(o(h_t^a), i), \delta f_j(o(h_t^a), i))$ to negotiator $j (\neq i)$ at the history $(h_t^a, i) \in H_t^b$. On the other hand, negotiator i accepts negotiator j's proposal x if $x_i \geq \delta f_i(o(h_t^a), j)$ and rejects if $x_i < \delta f_i(o(h_t^a), j)$ at the history $(h_t^a, j, x) \in H_t^c$. Then, $\sigma = (\sigma_1, \sigma_2)$ is an SPE of the game.*

Proof. Since the game is an infinite horizon discounted multi-stage game with observed actions, we can apply the one-shot deviation principle to prove Theorem 2.3.1. That is, σ is an SPE if there is no player who can become better off by deviating from σ for just one period (see Fudenberg and Tirole (1991)).

First, consider the history $(h_t^a, i) \in H_t^b$. If σ_i and σ_j are played after (h_t^a, i), negotiator i receives $\delta^{t-1} (1 - \delta f_j(o(h_t^a), i))$.

Suppose that negotiator i one-shot deviates from σ_i and proposes another division $x = (1 - x_j, x_j)$ which satisfies $x_j > \delta f_j(o(h_t^a), i)$. Then, negotiator j accepts it under σ_j and negotiator i receives $\delta^{t-1}(1 - x_j)$. However, $\delta^{t-1}(1 - x_j)$ is smaller than $\delta^{t-1}(1 - \delta f_j(o(h_t^a), i))$. Thus, negotiator i cannot improve her payoff by proposing the division $x = (x_i, x_j)$ satisfying $x_j > \delta f_j(o(h_t^a), i)$ after the history (h_t^a, i).

Next, suppose that negotiator i one-shot deviates from σ_i and proposes another division $x = (1 - x_j, x_j)$ which satisfies $x_j < \delta f_j(o(h_t^a), i)$. Then, negotiator j rejects the offer under σ_j and the game continues to the step after the history $(h_t^a, i, x, No) \in H_{t+1}^a$. After the history (h_t^a, i, x, No), negotiator i is selected as a proposer with probability $P^i(o(h_t^a), i)$ and receives $\delta^t (1 - \delta f_j(o(h_t^a), i, i))$ under σ_i and σ_j. On the other hand, negotiator i is selected as a responder with probability $P^j(o(h_t^a), i)$ and receives $\delta^{t+1} f_i(o(h_t^a), i, j)$ under σ_i and σ_j. Therefore, negotiator i receives $P^i(o(h_t^a), i)\delta^t (1 - \delta f_j(o(h_t^a), i, i)) + P^j(o(h_t^a), i)\delta^{t+1} f_i(o(h_t^a), i, j) = \delta^t f_i(o(h_t^a), i)$ (by Lemma 2.3.2) after the history (h_t^a, i, x, No) under σ_i and σ_j.

Now, since

$$
\begin{aligned}
&\delta^{t-1} (1 - \delta f_j(o(h_t^a), i)) - \delta^t f_i(o(h_t^a), i) \\
=&\delta^{t-1} [1 - \delta(f_j(o(h_t^a), i) + f_i(o(h_t^a), i))] \\
=&\delta^{t-1}(1 - \delta) \\
>&0,
\end{aligned}
$$

we can see

$$\delta^{t-1}(1 - \delta f_j(o(h_t^a), i)) > \delta^t f_i(o(h_t^a), i).$$

Therefore, negotiator i cannot improve her payoff by proposing the division $x = (x_i, x_j)$ satisfying $x_j < \delta f_j(o(h_t^a), i)$ after the history (h_t^a, i).

Subsequently, we consider the subgame after the history $(h_t^a, j, x) \in H_t^c$. If negotiator i accepts the offer, she receives $\delta^{t-1}x_i$. On the other hand, if she rejects the offer, the game continues to the step after the history $(h_t^a, j, x, No) \in H_{t+1}^a$. Then, negotiator i receives $\delta^t(1 - \delta f_j(o(h_t^a), j, i))$ with probability $P^i(o(h_t^a), j)$ and receives $\delta^{t+1} f_i(o(h_t^a), j, j)$ with probability $P^j(o(h_t^a), j)$ under σ_i and σ_j. Therefore, negotiator i receives $P^i(o(h_t^a), j)\delta^t(1 - \delta f_j(o(h_t^a), j, i))$
$+ P^j(o(h_t^a), j)\delta^{t+1} f_i(o(h_t^a), j, j) = \delta^t f_i(o(h_t^a), j)$ (by Lemma 2.3.2) after the history (h_t^a, j, x, No) under σ_i and σ_j.

Consider the case $x_i \geq \delta f_i(o(h_t^a), j)$. Suppose that negotiator i one-shot deviates from σ_i and rejects the offer x. Then, the game continues to the step after the history (h_t^a, j, x, No) and negotiator i receives $\delta^t f_i(o(h_t^a), j)$ under σ_i and σ_j. In this case, we can confirm that negotiator i cannot improve her payoff by deviating from σ_i since $\delta^{t-1}x_i \geq \delta^t f_i(o(h_t^a), j)$.

Consider the case $x_i < \delta f_i(o(h_t^a), j)$. Suppose that negotiator i one-shot deviates from σ_i and accepts the offer x. Then, negotiator i receives $\delta^{t-1}x_i$. However, she can receive larger payoff $\delta^t f_i(o(h_t^a), j)$ under σ_i and σ_j. Therefore, negotiator i cannot improve her payoff by deviating from σ_i.

Consequently, Theorem 2.3.1 holds since there is no profitable one-shot deviation.

□

We can find that this SPE is also an MPE, that is, an SPE in which negotiators' strategies depend only on the history of proposers and the current proposal (do not depend on the previous proposals).

When the SPE given in Theorem 2.3.1 is played, negotiators 1 and 2 receive $f_1(\emptyset)$ and $f_2(\emptyset)$ (by Lemma 2.3.2), respectively. We prove that the payoffs which are obtained in the SPE given in Theorem 2.3.1 are the unique SPE payoffs of the game.

Let $M_i(\pi_t)$ and $m_i(\pi_t)$ be the supremum and the infimum respectively of negotiator i's SPE payoffs of the game $\Gamma(\pi_t)$ in which all payoffs are multiplied by $1/\delta^t$. We have already confirmed that there is an SPE in the game. Therefore, for all $i \in N$ and for all $\pi \in \bigcup_{t=1}^{\infty} N^{t-1}$, $M_i(\pi)$ and $m_i(\pi)$ are well-defined. We derive two inequalities involving the supremum and the infimum of SPE payoffs.

Lemma 2.3.3. *For all $i \in N$, for all $t \in \{1, 2, \dots\}$, and for all $h_t^a \in H_t^a$, the following inequalities hold.*

$$M_i(\pi_{t-1}) \leq P^i(\pi_{t-1}) + \delta\left[P^j(\pi_{t-1})M_i(\pi_{t-1}, j) - P^i(\pi_{t-1})m_j(\pi_{t-1}, i)\right], \quad (2.2)$$

$$m_i(\pi_{t-1}) \geq P^i(\pi_{t-1}) + \delta\left[P^j(\pi_{t-1})m_i(\pi_{t-1}, j) - P^i(\pi_{t-1})M_j(\pi_{t-1}, i)\right] \quad (2.3)$$

where $\pi_{t-1} = o(h_t^a)$.

Proof. Let $G(h_t^a, i)$ be the subgame after the history $(h_t^a, i) \in H_t^b$ in which all payoffs are multiplied by $1/\delta^t$. Fix $i \in N$ and $h_t^a \in H_t^a$. First, we prove (2.2). Consider the game $G(h_t^a, i)$. When negotiator i proposes the division x which satisfies $x_j < \delta m_j(\pi_{t-1}, i)$ at the first period of $G(h_t^a, i)$, negotiator j rejects this proposal in all SPEs since negotiator j can receives a payoff of at least $\delta m_j(\pi_{t-1}, i)$ at the next period or later. Thus, if negotiator i's proposal is accepted at the first period in SPE, negotiator i's payoff is not larger than $1 - \delta m_j(\pi_{t-1}, i)$. Also, negotiator i can receive a payoff of at most $\delta M_i(\pi_{t-1}, i)$ at the next period or later. Since $M_i(\pi_{t-1}, i) + m_j(\pi_{t-1}, i) \leq 1$ by the definitions of $M_i(\pi_{t-1}, i)$ and $m_i(\pi_{t-1}, i)$, $\delta M_i(\pi_{t-1}, i) \leq \delta(1 - m_j(\pi_{t-1}, i)) < 1 - \delta m_j(\pi_{t-1}, i)$. Therefore, negotiator i can receive a payoff of at most $1 - \delta m_j(\pi_{t-1}, i)$ in the game $G(h_t^a, i)$.

Next, consider the game $G(h_t^a, j)$ $(j \neq i)$. Let $M_i^*(h_t^a, j)$ be the supremum of negotiator i's SPE payoffs in the game $G(h_t^a, j)$. Now, we show $M_i^*(h_t^a, j) \leq \delta M_i(\pi_{t-1}, j)$. Suppose $M_i^*(h_t^a, j) > \delta M_i(\pi_{t-1}, j)$. Then, there is an SPE $\sigma' = (\sigma_i', \sigma_j')$ in which negotiator j proposes the division $(x_i', 1 - x_i')$ satisfying $\delta M_i(\pi_{t-1}, j) < x_i' \leq M_i^*(h_t^a, j)$ and negotiator i accepts it at the first period of $G(h_t^a, j)$ since negotiator i cannot achieve a payoff larger than $\delta M_i(\pi_{t-1}, j)$ at the next period or later in all SPEs. Therefore, under σ', negotiator j obtains $1 - x_i'$. However, negotiator j can improve her payoff by proposing the division $(x_i^*, 1 - x_i^*)$ where x_i^* satisfies $x_i' > x_i^* > \delta M_i(\pi_{t-1}, j)$. This proposal is also accepted by negotiator i who follows the strategy σ_i' since negotiator i must accept all divisions satisfying $x_i > \delta M_i(\pi_{t-1}, j)$ in all SPEs. Then, negotiator j receives a payoff $1 - x_i^*$ $(> 1 - x_i')$. Therefore, for negotiator j, proposing the division $(x_i', 1 - x_i')$ is not a best response to σ_i'. This contradicts to the fact that σ' is an SPE of the game $G(h_t^a, j)$. Therefore, $M_i^*(h_t^a, j) \leq \delta M_i(\pi_{t-1}, j)$ holds, that is, negotiator i can receive a payoff of at most $\delta M_i(\pi_{t-1}, j)$ in the game $G(h_t^a, j)$.

Finally, consider the game $\Gamma(\pi_{t-1})$ in which all payoffs are multiplied by $1/\delta^{t-1}$. This game moves to the subgame $G(h_t^a, i)$ with probability $P^i(\pi_{t-1})$ and the game $G(h_t^a, j)$ with probability $P^j(\pi_{t-1})$. Therefore, from the above discussion, we can see

$$M_i(\pi_{t-1}) \leq P^i(\pi_{t-1})\,(1 - \delta m_j(\pi_{t-1}, i)) + P^j(\pi_{t-1})\delta M_i(\pi_{t-1}, j).$$

This inequality coincides with (2.2).

Next, we prove (2.3). First, consider the game $G(h_t^a, i)$. Let $m_i^*(h_t^a, i)$ be the infimum of negotiator i's SPE payoffs in the game $G(h_t^a, i)$. We show $m_i^*(h_t^a, i) \geq 1 - \delta M_j(\pi_{t-1}, i)$. Suppose $m_i^*(h_t^a, i) < 1 - \delta M_j(\pi_{t-1}, i)$. Then, there is an SPE

$\sigma'' = (\sigma''_i, \sigma''_j)$ in which negotiator i obtains some payoff x''_i satisfying $m^*_i(h^a_t, i) \leq x''_i < 1 - \delta M_j(\pi_{t-1}, i)$. However, negotiator i can improve her payoff by proposing the division $(x^{**}_i, 1 - x^{**}_i)$ at the first period of $G(h^a_t, i)$ where x^{**}_i satisfies $1 - x''_i > 1 - x^{**}_i > \delta M_j(\pi_{t-1}, i)$. This proposal is accepted by negotiator j who follows the strategy σ''_i since negotiator j must accept all divisions satisfying $x_j > \delta M_j(\pi_{t-1}, i)$ in all SPEs. Then, negotiator i receives a payoff x^{**}_i ($> x''_i$). Therefore, σ''_i is not a best response to σ''_j. This contradicts to the fact that σ'' is an SPE of the game $G(h^a_t, i)$. Therefore, $m^*_i(h^a_t, i) \geq 1 - \delta M_j(\pi_{t-1}, i)$ holds, that is, negotiator i can receive a payoff of at least $1 - \delta M_j(\pi_{t-1}, i)$ in the game $G(h^a_t, i)$.

Next, consider the game $G(h^a_t, j)$. For all SPEs, if negotiator j proposes the division x which satisfies $x_i < \delta m_i(\pi_{t-1}, j)$ at the first period of $G(h^a_t, j)$, negotiator i rejects this proposal since she can receive a payoff of at least $\delta m_i(\pi_{t-1}, j)$ at the next period or later. Therefore, for all SPEs, negotiator i can obtain a payoff of at least $\delta m_i(\pi_{t-1}, j)$ in the game $G(h^a_t, j)$.

Finally, consider the game $\Gamma(\pi_{t-1})$ in which all payoffs are multiplied by $1/\delta^{t-1}$. This game moves to the subgame $G(h^a_t, i)$ with probability $P^i(\pi_{t-1})$ and the subgame $G(h^a_t, j)$ with probability $P^j(\pi_{t-1})$. Therefore, from the above discussion, we can see

$$m_i(\pi_{t-1}) \geq P^i(\pi_{t-1}) \left(1 - \delta M_j(\pi_{t-1}, i)\right) + P^j(\pi_{t-1}) \delta m_i(\pi_{t-1}, j).$$

This inequality coincides with (2.3). □

Before proving the uniqueness of SPE payoffs, we provide an alternative form of $f_i(\pi)$. For all $r \in \{1, 2, \dots\}$, $\pi \in \bigcup_{t=1}^{\infty} N^{t-1}$, and $i \in N$, we define

$$q^i_r(\pi) = \sum_{\pi_{r-1} \in N^{r-1}} \Pi(\pi, \pi_{r-1}) P^i(\pi\pi_{r-1}) P^j(\pi\pi_{r-1}, i)$$

where $\Pi(\pi, \emptyset) = 1$ and $j \neq i$.

By using this value, we transform $f_i(\pi)$ into the form which is not a fractional expression.

Lemma 2.3.4. *For all $\pi \in \bigcup_{t=1}^{\infty} N^{t-1}$ and $i \in N$,*

$$f_i(\pi) = P^i(\pi) + \sum_{r=1}^{\infty} \delta^r \left(q^j_r(\pi) - q^i_r(\pi)\right).$$

Proof. Before transforming $f_i(\pi)$, we prove

$$p^i_{r+1}(\pi) - p^i_r(\pi) = q^j_r(\pi) - q^i_r(\pi). \tag{2.4}$$

First, we transform $p^i_{r+1}(\pi)$ as follows.

$$
\begin{aligned}
& p^i_{r+1}(\pi) \\
=& \sum_{\pi_{r-1}\in N^{r-1}} \Pi(\pi,\pi_{r-1})P^i(\pi\pi_{r-1})P^i(\pi\pi_{r-1},i) + \sum_{\pi_{r-1}\in N^{r-1}} \Pi(\pi,\pi_{r-1})P^j(\pi\pi_{r-1})P^i(\pi\pi_{r-1},j) \\
=& \sum_{\pi_{r-1}\in N^{r-1}} \Pi(\pi,\pi_{r-1})P^i(\pi\pi_{r-1})P^i(\pi\pi_{r-1},i) + q^j_r(\pi)
\end{aligned}
$$

where, in the first equation, we use the fact that the set of the orders of length r, N^r, can be divided into two sets of the orders in which the last proposers are i and j, respectively.

Therefore, (2.4) can be proved as follows.

$$
\begin{aligned}
& p^i_{r+1}(\pi) - p^i_r(\pi) \\
=& \left(\sum_{\pi_{r-1}\in N^{r-1}} \Pi(\pi,\pi_{r-1})P^i(\pi\pi_{r-1})P^i(\pi\pi_{r-1},i) + q^j_r(\pi) \right) - \sum_{\pi_{r-1}\in N^{r-1}} \Pi(\pi,\pi_{r-1})P^i(\pi\pi_{r-1}) \\
=& q^j_r(\pi) - \sum_{\pi_{r-1}\in N^{r-1}} \Pi(\pi,\pi_{r-1})P^i(\pi\pi_{r-1})P^j(\pi\pi_{r-1},i) \\
=& q^j_r(\pi) - q^i_r(\pi).
\end{aligned}
$$

Then, $f_i(\pi)$ can be transformed as follows.

$$
\begin{aligned}
f_i(\pi) =& (1-\delta)\sum_{r=1}^{\infty} \delta^{r-1} p^i_r(\pi) \\
=& \sum_{r=1}^{\infty} \delta^{r-1} p^i_r(\pi) - \sum_{r=1}^{\infty} \delta^{r} p^i_r(\pi) \\
=& P^i(\pi) + \sum_{r=1}^{\infty} \delta^{r} p^i_{r+1}(\pi) - \sum_{r=1}^{\infty} \delta^{r} p^i_r(\pi) \\
=& P^i(\pi) + \sum_{r=1}^{\infty} \delta^{r} (q^j_r(\pi) - q^i_r(\pi))
\end{aligned}
$$

where we use (2.4) in the last equation. □

By using Lemma 2.3.3 and 2.3.4, we prove the uniqueness of SPE payoffs.

Theorem 2.3.2. *For all $t \in \{0,1,\dots\}$ and $\pi \in N^t$, $(f_1(\pi), f_2(\pi))$ are the unique SPE payoffs of the game $\Gamma(\pi)$ in which all payoffs are multiplied by $1/\delta^t$. Especially, $(f_1(\emptyset), f_2(\emptyset))$ are the unique SPE payoffs of the original game.*

Proof. First, we prove that for all $i \in N$ and $m \in \{1, 2, \dots\}$,

$$M_i(\pi) \leq P^i(\pi) + \sum_{r=1}^{m} \delta^r \left(q_r^j(\pi) - q_r^i(\pi)\right) + \delta^{m+1}\alpha_m^i(\pi) \tag{2.5}$$

and

$$m_i(\pi) \geq P^i(\pi) + \sum_{r=1}^{m} \delta^r \left(q_r^j(\pi) - q_r^i(\pi)\right) + \delta^{m+1}\beta_m^i(\pi) \tag{2.6}$$

where

$$\alpha_m^i(\pi) = \sum_{\pi_m \in N^m} \Pi(\pi, \pi_m) \left[P^j(\pi\pi_m)M_i(\pi\pi_m, j) - P^i(\pi\pi_m)m_j(\pi\pi_m, i)\right]$$

and

$$\beta_m^i(\pi) = \sum_{\pi_m \in N^m} \Pi(\pi, \pi_m) \left[P^j(\pi\pi_m)m_i(\pi\pi_m, j) - P^i(\pi\pi_m)M_j(\pi\pi_m, i)\right].$$

We prove (2.5) and (2.6) by mathematical induction. The case of $m = 1$ is easily proved by using Lemma 2.3.3 few times. Now, suppose that (2.5) and (2.6) hold for $m = k$. Here, we only show the inequality (2.5). (2.6) is proved similarly.

To prove that (2.5) holds for $m = k + 1$, it is sufficient to prove

$$\alpha_k^i(\pi) \leq \left(q_{k+1}^j(\pi) - q_{k+1}^i(\pi)\right) + \delta\alpha_{k+1}^i(\pi). \tag{2.7}$$

By using Lemma 2.3.3, (2.7) is proved as follows.

$$\begin{aligned}
\alpha_k^i(\pi) &= \sum_{\pi_k \in N^k} \Pi(\pi, \pi_k) \left[P^j(\pi\pi_k)M_i(\pi\pi_k, j) - P^i(\pi\pi_k)m_j(\pi\pi_k, i)\right] \\
&\leq \left(q_{k+1}^j(\pi) - q_{k+1}^i(\pi)\right) \\
&\quad + \delta \sum_{\pi_k \in N^k} \Pi(\pi, \pi_k)P^j(\pi\pi_k) \left[P^j(\pi\pi_k, j)M_i(\pi\pi_k, j, j) - P^i(\pi\pi_k, j)m_j(\pi\pi_k, j, i)\right] \\
&\quad - \delta \sum_{\pi_k \in N^k} \Pi(\pi, \pi_k)P^i(\pi\pi_k) \left[P^i(\pi\pi_k, i)m_j(\pi\pi_k, i, i) - P^j(\pi\pi_k, i)M_i(\pi\pi_k, i, j)\right] \\
&= \left(q_{k+1}^j(\pi) - q_{k+1}^i(\pi)\right) \\
&\quad + \delta \sum_{\substack{\pi_{k+1} \in N^{k+1} \\ \pi_{k+1}(k+1) = j}} \Pi(\pi, \pi_{k+1}) \left[P^j(\pi\pi_{k+1})M_i(\pi\pi_{k+1}, j) - P^i(\pi\pi_{k+1})m_j(\pi\pi_{k+1}, i)\right]
\end{aligned}$$

$$
\begin{aligned}
&- \delta \sum_{\substack{\pi_{k+1} \in N^{k+1} \\ \pi_{k+1}(k+1)=i}} \Pi(\pi, \pi_{k+1}) \left[P^i(\pi\pi_{k+1}) m_j(\pi\pi_{k+1}, i) - P^j(\pi\pi_{k+1}) M_i(\pi\pi_{k+1}, j)\right] \\
&= \left(q^j_{k+1}(\pi) - q^i_{k+1}(\pi)\right) \\
&\quad + \delta \sum_{\pi_{k+1} \in N^{k+1}} \Pi(\pi, \pi_{k+1}) \left[P^j(\pi\pi_{k+1}) M_i(\pi\pi_{k+1}, j) - P^i(\pi\pi_{k+1}) m_j(\pi\pi_{k+1}, i)\right] \\
&= \left(q^j_{k+1}(\pi) - q^i_{k+1}(\pi)\right) + \delta \alpha^i_{k+1}(\pi)
\end{aligned}
$$

where we use Lemma 2.3.3 in the inequality.

Now, we can see that (2.5) (and (2.6)) holds for $m = k+1$. Therefore, for all $i \in N$ and $m \in \{1, 2, \dots\}$,

$$
M_i(\pi) \leq P^i(\pi) + \sum_{r=1}^{m} \delta^r \left(q^j_r(\pi) - q^i_r(\pi)\right) + \delta^{m+1} \alpha^i_m(\pi)
$$

and

$$
m_i(\pi) \geq P^i(\pi) + \sum_{r=1}^{m} \delta^r \left(q^j_r(\pi) - q^i_r(\pi)\right) + \delta^{m+1} \beta^i_m(\pi)
$$

hold.

Next, we consider taking the limit as $m \to \infty$ in (2.5). Focus on the third term of the right hand side in (2.5). Since $0 \leq P^i(\pi\pi_m), P^j(\pi\pi_m) \leq 1$ and $0 \leq M_i(\pi\pi_m, j), m_j(\pi\pi_m, i) \leq 1$ by their definitions,

$$
-1 \leq P^j(\pi\pi_m) M_i(\pi\pi_m, j) - P^i(\pi\pi_m) m_j(\pi\pi_m, i) \leq 1.
$$

Therefore, by the definition of $\alpha^i_m(\pi)$,

$$
-\delta^{m+1} = - \delta^{m+1} \sum_{\pi_m \in N^m} \Pi(\pi, \pi_m) \leq \delta^{m+1} \alpha^i_m(\pi) \leq \delta^{m+1} \sum_{\pi_m \in N^m} \Pi(\pi, \pi_m) = \delta^{m+1}
$$

where we use the fact $\sum_{\pi_m \in N^m} \Pi(\pi, \pi_m) = 1$.

Thus, by taking the limit of both sides of the inequality,

$$
\lim_{m \to \infty} \delta^{m+1} \alpha^i_m(\pi) = 0.
$$

As a result, by taking the limit in (2.5) and by Lemma 2.3.4,

$$
\begin{aligned}
M_i(\pi) \leq & P^i(\pi) + \sum_{r=1}^{\infty} \delta^r \left(q^j_r(\pi) - q^i_r(\pi)\right) + \lim_{m \to \infty} \delta^{m+1} \alpha^i_m(\pi) \\
= & f_i(\pi).
\end{aligned}
$$

Similarly,

$$m_i(\pi) \geq f_i(\pi)$$

holds. Thus, since $M_i(\pi) \geq m_i(\pi)$,

$$M_i(\pi) = m_i(\pi) = f_i(\pi)$$

holds. This equation means that for all $t \in \{0, 1, \dots\}$ and $\pi \in N^t$, $(f_1(\pi), f_2(\pi))$ are the unique SPE payoffs of the game $\Gamma(\pi)$ in which all payoffs are multiplied by $1/\delta^t$. □

By Theorem 2.3.2, we can also find that the SPE $\sigma = (\sigma_1, \sigma_2)$ given in Theorem 2.3.1 is the unique SPE of the game.

Theorem 2.3.3. *The SPE $\sigma = (\sigma_1, \sigma_2)$ given in Theorem 2.3.1 is the unique SPE of the game.*

Proof. Fix the history $(h_t^a, j, x) \in H_t^c$. When negotiator i rejects the proposal, by Theorem 2.3.2, she obtains $\delta^t f_i(o(h_t^a), j)$ at the next period or later in all SPEs. Therefore, in all SPEs, negotiator i accepts negotiator j's proposal x if $x_i > \delta f_i(o(h_t^a), j)$ and rejects if $x_i < \delta f_i(o(h_t^a), j)$.

Now, suppose that negotiator i rejects j's proposal x if $x_i = \delta f_i(o(h_t^a), j))$ in some SPE. Since $1 - \delta f_j(o(h_t^a), j) > \delta f_i(o(h_t^a), j)$, there exists some proposal x^* which satisfies $1 - \delta f_j(o(h_t^a), j) > x_i^* > \delta f_i(o(h_t^a), j)$. If negotiator j proposes the division x^*, negotiator i accepts it in the SPE. Then, negotiator j obtains $\delta^{t-1}(1 - x_i^*)(> \delta^t f_j(o(h_t^a), j))$. Therefore, after $(h_t^a, j) \in H_t^b$, it is better for negotiator j to propose the division x^* which is accepted by negotiator i than to propose some division which is rejected by negotiator i. Thus, in this SPE, negotiator j proposes some division which is accepted by negotiator i after $(h_t^a, j) \in H_t^b$. However, negotiator j's best response does not exist after (h_t^a, j) since negotiator i accepts the proposal x' if and only if $x_i' > \delta f_i(o(h_t^a), j)$. This is a contradiction. Hence, negotiator i accepts j's proposal x if $x_i = \delta f_i(o(h_t^a), j))$ in all SPEs.

By the above discussion, negotiator i accepts negotiator j's proposal x if $x_i \geq \delta f_i(o(h_t^a), j)$ and rejects if $x_i < \delta f_i(o(h_t^a), j)$ at the history $(h_t^a, j, x) \in H_t^c$ in all SPEs. Then, since $\delta^{t-1}(1 - \delta f_i(o(h_t^a), j) > \delta^t f_j(o(h_t^a), j)$, negotiator j's best response at the history $(h_t^a, j) \in H_t^b$ is only proposing the division of the pie $(1 - \delta f_i(o(h_t^a), j), \delta f_i(o(h_t^a), j))$ which is accepted by negotiator i. Therefore, the SPE given in Theorem 2.3.1 is the unique SPE of the game. □

Finally, we provide an interpretation of the unique SPE payoff $f_i(\emptyset)$. We can consider $p_t^i(\emptyset)$ as negotiator i's probability to be a proposer at period t in the original game. Thus, by the first form of $f_i(\emptyset)$, we can view each component game at period t involving negotiators dividing a pie of size δ^{t-1} according to the proposal ratio at period t, $p_t^1(\emptyset) : p_t^2(\emptyset)$. Then, $\delta^{t-1}p_t^i(\emptyset)$ is the value that negotiator i can obtain at period t. The numerator of $f_i(\emptyset)$ is the sum of these values and the denominator denotes the value of whole game. Therefore, the negotiator with more chances to be a proposer can obtain a higher payoff.

2.4 The limit of the SPE payoffs

Although it is generally difficult to examine the limit of the SPE payoffs in our model, if the process has some property during a time period of a certain length (which a Markov process satisfies), we can give a simple expression of the limit of the SPE payoffs. In this section, remember that $p_t^i(\emptyset)$ denotes negotiator i's probability to be a proposer at period t in the original game.

Theorem 2.4.1. *If there exists some $k \in \mathbb{N}$ such that $\lim_{m\to\infty} \sum_{t=(m-1)k+1}^{mk} p_t^i(\emptyset)$ converges to some value V_i ($\lim_{m\to\infty} \sum_{t=(m-1)k+1}^{mk} p_t^j(\emptyset)$ converges to V_j where $V_i + V_j = k$), then $\lim_{\delta\uparrow 1} f_i(\emptyset) = \frac{V_i}{k}$ and $\lim_{\delta\uparrow 1} f_j(\emptyset) = \frac{V_j}{k}$.*

Since $p_t^i(\emptyset)$ is negotiator i's probability to be a proposer at period t, $\sum_{t=(m-1)k+1}^{mk} p_t^i(\emptyset)$ is the sum of these probabilities during the periods $(m-1)k+1, \ldots, mk$. Theorem 2.4.1 means that if the proposal ratio during a time period of a certain length converges to some value, then negotiators divide the pie according to this ratio.

Proof. By assumption, for all $\epsilon > 0$, there exists some $N^*(\epsilon) \in \mathbb{N}$ such that

$$V_i - \epsilon \le \sum_{t=(m-1)k+1}^{mk} p_t^i(\emptyset) \le V_i + \epsilon$$

for $m \ge N^*(\epsilon)$. Therefore, for $m \ge N^*(\epsilon)$,

$$\delta^{mk-1}(V_i - \epsilon) \le \sum_{t=(m-1)k+1}^{mk} \delta^{t-1} p_t^i(\emptyset) \le \delta^{(m-1)k}(V_i + \epsilon). \tag{2.8}$$

We define

$$L(\delta) = A(\delta) + \frac{\sum_{m=N^*(\epsilon)}^{\infty} \delta^{mk-1}(V_i - \epsilon)}{\sum_{t=1}^{\infty} \delta^{t-1}}$$

and

$$R(\delta) = A(\delta) + \frac{\sum_{m=N^*(\epsilon)}^{\infty} \delta^{(m-1)k}(V_i + \epsilon)}{\sum_{t=1}^{\infty} \delta^{t-1}}$$

where

$$A(\delta) = \frac{\sum_{t=1}^{(N^*(\epsilon)-1)k} \delta^{t-1} p_t^i(\emptyset)}{\sum_{t=1}^{\infty} \delta^{t-1}}.$$

By the definitions of $L(\delta)$, $R(\delta)$, and (2.8),

$$L(\delta) \leq f_i(\emptyset) \leq R(\delta) \tag{2.9}$$

holds since

$$f_i(\emptyset) = A(\delta) + \frac{\sum_{m=N^*(\epsilon)}^{\infty} \sum_{t=(m-1)k+1}^{mk} \delta^{t-1} p_t^i(\emptyset)}{\sum_{t=1}^{\infty} \delta^{t-1}}.$$

Now, $L(\delta)$ can be transformed as follows.

$$\begin{aligned} L(\delta) =& A(\delta) + \frac{\sum_{m=N^*(\epsilon)}^{\infty} \delta^{mk-1}(V_i - \epsilon)}{\sum_{t=1}^{\infty} \delta^{t-1}} = A(\delta) + \frac{(V_i - \epsilon)\frac{\delta^{N^*(\epsilon)k-1}}{1-\delta^k}}{\frac{\sum_{t=1}^{k} \delta^{t-1}}{1-\delta^k}} \\ =& A(\delta) + \frac{(V_i - \epsilon)\delta^{N^*(\epsilon)k-1}}{\sum_{t=1}^{k} \delta^{t-1}}. \end{aligned}$$

Similarly, $R(\delta)$ can be transformed as follows.

$$\begin{aligned} R(\delta) =& A(\delta) + \frac{\sum_{m=N^*(\epsilon)}^{\infty} \delta^{(m-1)k}(V_i + \epsilon)}{\sum_{t=1}^{\infty} \delta^{t-1}} = A(\delta) + \frac{(V_i + \epsilon)\frac{\delta^{(N^*(\epsilon)-1)k}}{1-\delta^k}}{\frac{\sum_{t=1}^{k} \delta^{t-1}}{1-\delta^k}} \\ =& A(\delta) + \frac{(V_i + \epsilon)\delta^{(N^*(\epsilon)-1)k}}{\sum_{t=1}^{k} \delta^{t-1}}. \end{aligned}$$

Thus, $\lim_{\delta \uparrow 1} L(\delta) = \frac{V_i - \epsilon}{k}$ and $\lim_{\delta \uparrow 1} R(\delta) = \frac{V_i + \epsilon}{k}$ since $\lim_{\delta \uparrow 1} A(\delta) = 0$.

Hence, by (2.9),

$$
\begin{aligned}
&\frac{V_i - \epsilon}{k} \\
=&\lim_{\delta\uparrow 1} L(\delta) = \liminf_{\delta\uparrow 1} L(\delta) \\
\leq&\liminf_{\delta\uparrow 1} f_i(\emptyset) \leq \limsup_{\delta\uparrow 1} f_i(\emptyset) \\
\leq&\limsup_{\delta\uparrow 1} R(\delta) = \lim_{\delta\uparrow 1} R(\delta) \\
=&\frac{V_i + \epsilon}{k}.
\end{aligned}
\tag{2.10}
$$

Since (2.10) holds for all $\epsilon > 0$,

$$\lim_{\delta\uparrow 1} f_i(\emptyset) = \liminf_{\delta\uparrow 1} f_i(\emptyset) = \limsup_{\delta\uparrow 1} f_i(\emptyset) = \frac{V_i}{k}.$$

Thus,

$$\lim_{\delta\uparrow 1} f_j(\emptyset) = 1 - \lim_{\delta\uparrow 1} f_i(\emptyset) = \frac{k - V_i}{k} = \frac{V_j}{k}.$$

□

We have shown that if the proposal ratio during a time period of a certain length converges to some value, then negotiators divide the pie according to the ratio of this value. One interpretation of the condition "there exists some $k \in \mathbb{N}$ such that $\lim_{m\to\infty} \sum_{t=(m-1)k+1}^{mk} p_t^i(\emptyset)$ converges to some value V_i" is that, in reality, even if individuals propose the divisions freely in the beginning, the negotiation often calm down and the ratio of frequencies of proposal during a time period of a certain length often stays in some value in the long run.

Then, the number k in Theorem 2.4.1 can be considered as the length of this time period. That is, after the negotiation calm down, the ratio of individuals' opportunities to be a proposer are $V_i : V_j$ during these periods. Theorem 2.4.1 represents that individuals divide the pie with the ratio $V_i : V_j$ under this situation.

From Theorem 2.4.1, we obtain some corollaries.

Corollary 2.4.1. *If there exists some $k \in \mathbb{N}$ such that $\lim_{m\to\infty} \sum_{t=m+1}^{m+k} p_t^i(\emptyset)$ converges to some value V_i ($\lim_{m\to\infty} \sum_{t=m+1}^{m+k} p_t^j(\emptyset)$ converges to V_j where $V_i + V_j = k$), then $\lim_{\delta\uparrow 1} f_i(\emptyset) = \frac{V_i}{k}$ and $\lim_{\delta\uparrow 1} f_j(\emptyset) = \frac{V_j}{k}$.*

Proof. If the sequence $\{\sum_{t=m+1}^{m+k} p_t^i(\emptyset)\}_{m\in\mathbb{N}}$ converges to V_i, then the subsequence $\{\sum_{t=(m-1)k+1}^{mk} p_t^i(\emptyset)\}_{m\in\mathbb{N}}$ converges to V_i. Therefore, by Theorem 2.4.1, $\lim_{\delta\uparrow 1} f_i(\emptyset) = \frac{V_i}{k}$ and $\lim_{\delta\uparrow 1} f_j(\emptyset) = \frac{V_j}{k}$. □

Corollary 2.4.1 means that if the proposal ratio during k consecutive periods converges to $V_i : V_j$, then negotiators divide the pie according to this ratio. To help one understand, we give an example which is a generalization of the alternating offers process used in Rubinstein (1982) where $p^1_{2(m-1)+1}(\emptyset) = 1$, $p^1_{2m}(\emptyset) = 0$, $p^2_{2(m-1)+1}(\emptyset) = 0$, and $p^2_{2m}(\emptyset) = 1$ for all $m \in \mathbb{N}$.

Example 2.4.1. *Suppose that* $\lim_{m\to\infty} p^i_{2(m-1)+1}(\emptyset)$ *converges to* V^1_i *and* $\lim_{m\to\infty} p^i_{2m}(\emptyset)$ *converges to* V^2_i *(*$\lim_{m\to\infty} p^j_{2(m-1)+1}(\emptyset)$ *converges to* V^1_j *where* $V^1_i + V^1_j = 1$ *and* $\lim_{m\to\infty} p^j_{2m}(\emptyset)$ *converges to* V^2_j *where* $V^2_i + V^2_j = 1$*), then* $\lim_{\delta\uparrow 1} f_i(\emptyset) = \frac{V^1_i + V^2_i}{2}$ *and* $\lim_{\delta\uparrow 1} f_j(\emptyset) = \frac{V^1_j + V^2_j}{2}$.

In this example, the proposal ratio during 2 consecutive periods converges to $(V^1_i + V^2_i) : (V^1_j + V^2_j)$. Therefore, negotiators divide the pie according to this ratio. In Rubinstein's alternating offers model, this ratio is 1 : 1.

Proof. For all $\epsilon > 0$, there exists some $N^*(\epsilon) \in \mathbb{N}$ such that

$$V^1_i - \frac{\epsilon}{2} < p^i_{2(m-1)+1}(\emptyset) < V^1_i + \frac{\epsilon}{2}$$

for $m \geq N^*(\epsilon)$. Also, there exists some $N^{**}(\epsilon) \in \mathbb{N}$ such that

$$V^2_i - \frac{\epsilon}{2} < p^i_{2m}(\emptyset) < V^2_i + \frac{\epsilon}{2}$$

for $m \geq N^{**}(\epsilon)$.

Therefore,

$$V^1_i + V^2_i - \epsilon < \sum_{t=m+1}^{m+2} p^i_t(\emptyset) < V^1_i + V^2_i + \epsilon$$

for $m \geq \max\{2N^*(\epsilon), 2N^{**}(\epsilon)\}$.

Thus,

$$\lim_{m\to\infty} \sum_{t=m+1}^{m+2} p^i_t(\emptyset) = V^1_i + V^2_i.$$

Hence, by Corollary 2.4.1,

$$\lim_{\delta\uparrow 1} f_i(\emptyset) = \frac{V^1_i + V^2_i}{2}$$

and

$$\lim_{\delta\uparrow 1} f_j(\emptyset) = \frac{V^1_j + V^2_j}{2}.$$

□

The following Corollary 2.4.2 is also obtained by Theorem 2.4.1.

Corollary 2.4.2. *If* $\lim_{t\to\infty} p_t^i(\emptyset)$ *converges to some value* V_i *(*$\lim_{t\to\infty} p_t^j(\emptyset)$ *converges to* V_j *where* $V_i + V_j = 1$*), then* $\lim_{\delta\uparrow 1} f_i(\emptyset) = V_i$ *and* $\lim_{\delta\uparrow 1} f_j(\emptyset) = V_j$*.*

Proof. This is the case of $k = 1$. □

This corollary implies that the SPE payoff is equal to negotiator's probability to be a proposer in the limit.

A Markov process is used in Kalandrakis (2004), Britz et al. (2010), and Herings and Predtetchinski (2010) where a negotiator's probability to be a proposer in each period depends on the identity of the proposer in the last period. We prove that if negotiator's probability depends on the previous l periods, then this process satisfies the condition of Corollary 2.4.2.

Proposition 2.4.1. *Suppose that for all* $i \in N$ *and* $\pi \in \bigcup_{t=l+1}^{\infty} N^{t-1}$*,* $P^i(\pi) > 0$ *and* $P^i(\pi)$ *depends on the previous* l *periods (for* $\pi \in \bigcup_{t=1}^{l} N^{t-1}$*,* $P^i(\pi)$ *can take arbitrary values). Then,* $\lim_{t\to\infty} p_t^i(\emptyset)$ *exists.*

Proof. $\bigcup_{t=l+1}^{\infty} N^{t-1}$ can be divided into 2^l states which are characterized by the history of proposers during previous l periods $(i_1, \ldots, i_l) \in N^l$ (where i_l denotes the proposer in the last period). We define the set of these states as $\Theta = \{\theta_1, \ldots, \theta_{2^l}\}$. Then, for all $\pi, \pi' \in \theta_m$ ($m \in \{1, \ldots, 2^l\}$), $P^i(\pi) = P^i(\pi')$ by assumption. Therefore, for all $\pi \in \theta_m$, $P^i(\pi)$ can be expressed as a constant value $P^i(\theta_m) > 0$.

Now, define $Q_{t-1}(\theta) = \sum_{\pi_{t-1}\in N^{t-1}, \pi_{t-1}\in\theta} \Pi(\emptyset, \pi_{t-1})$ and $Q_{t-1}(\Theta) = (Q_{t-1}(\theta_1), \ldots, Q_{t-1}(\theta_{2^l}))$. Let θ' be the state corresponding to $(i_1, \ldots, i_l) \in N^l$. Also, let θ'' be the state corresponding to $(1, i_1, \ldots, i_{l-1})$ and θ''' be the state corresponding to $(2, i_1, \ldots, i_{l-1})$. Then,

$$Q_t(\theta') = P^{i_l}(\theta'')Q_{t-1}(\theta'') + P^{i_l}(\theta''')Q_{t-1}(\theta'''). \tag{2.11}$$

Therefore, we can express

$$Q_t(\Theta) = Q_{t-1}(\Theta)A$$

where A is the transition matrix satisfying (2.11). Under this setting, since

$$p_t^i(\emptyset) = \sum_{\pi_{t-1}\in N^{t-1}} \Pi(\emptyset, \pi_{t-1})P^i(\pi_{t-1})$$

$$
\begin{aligned}
&= \sum_{\theta \in \Theta} P^i(\theta) \sum_{\pi_{t-1} \in N^{t-1}, \pi_{t-1} \in \theta} \Pi(\emptyset, \pi_{t-1}) \\
&= \sum_{\theta \in \Theta} P^i(\theta) Q_{t-1}(\theta)
\end{aligned}
$$

for $t \geq l+1$, we can prove Proposition 2.4.1 by showing the $\lim_{t\to\infty} Q_t(\Theta)$ $\left(= \lim_{t\to\infty} Q_l(\Theta) A^{t-l}\right)$ exists. It is sufficient to show that A is ergodic. We show that all entries of A^l are positive, that is, show that we can arrive at any state from any state in l steps with positive probability.

Let θ_m be the state corresponding to $(m_1, \ldots, m_l) \in N^l$ and $\theta_{m'}$ be the state corresponding to $(m'_1, \ldots, m'_l) \in N^l$. We can arrive at the state θ_1 corresponding to $(m_2, \ldots, m_l, m'_1) \in N^l$ from θ_m in 1 step with probability $P^{m'_1}(\theta_m) > 0$ (since for all $\pi \in \bigcup_{t=l+1}^{\infty} N^{t-1}$, $P^i(\pi) > 0$). Also, we can arrive at the state θ_2 corresponding to $(m_3, \ldots, m_l, m'_1, m'_2) \in N^l$ from θ_m in 2 steps with probability $P^{m'_1}(\theta_m) P^{m'_2}(\theta_1) > 0$. Similarly, we can arrive at the state $\theta_{m'}$ from θ_m in l steps with probability $P^{m'_1}(\theta_m) P^{m'_2}(\theta_1) \cdots P^{m'_l}(\theta_{l-1}) > 0$. Since θ_m and $\theta_{m'}$ can be taken arbitrarily, we can arrive at any state from any state in l steps with positive probability. Therefore, A is ergodic. □

By Proposition 2.4.1, we can see that a Markov process satisfies the condition of Corollary 2.4.2 and negotiators divide the pie according to the proposal ratio in the limit. A special case of Proposition 2.4.1 is as follows.

Example 2.4.2. *Let $P^i(i) > 0$ and $P^i(j) > 0$ be constant values ($j \neq i$). Suppose that $P^i(\pi_t) = P^i(i)$ when $\pi_t(t) = i$ and $P^i(\pi_t) = P^i(j)$ when $\pi_t(t) = j$ ($P^i(\emptyset)$ and $P^j(\emptyset)$ can take arbitrary values). Then, $\lim_{\delta \uparrow 1} f_i(\emptyset) = \frac{P^i(j)}{P^i(j)+P^j(i)}$.*

Proof. First of all, we derive the recurrence relation of $p_t^i(\emptyset)$.

$$
\begin{aligned}
p_{t+1}^i(\emptyset) = &\sum_{\pi_t \in N^t, \pi_t(t)=i} \Pi(\emptyset, \pi_t) P^i(i) + \sum_{\pi_t \in N^t, \pi_t(t)=j} \Pi(\emptyset, \pi_t) P^i(j) \\
=&P^i(i) \sum_{\pi_{t-1} \in N^{t-1}} \Pi(\emptyset, \pi_{t-1}) P^i(\pi_{t-1}) + P^i(j) \sum_{\pi_{t-1} \in N^{t-1}} \Pi(\emptyset, \pi_{t-1}) P^j(\pi_{t-1}) \\
=&P^i(i) \cdot p_t^i(\emptyset) + P^i(j) \cdot p_t^j(\emptyset) \\
=&P^i(i) \cdot p_t^i(\emptyset) + P^i(j) \cdot (1 - p_t^i(\emptyset)) \\
=&\left(P^i(i) - P^i(j)\right) p_t^i(\emptyset) + P^i(j).
\end{aligned}
$$

Therefore, the following relationship between $p_{t+1}^i(\emptyset)$ and $p_t^i(\emptyset)$ holds.

$$
p_{t+1}^i(\emptyset) - \frac{P^i(j)}{1 + P^i(j) - P^i(i)} = \left(P^i(i) - P^i(j)\right) \left(p_t^i(\emptyset) - \frac{P^i(j)}{1 + P^i(j) - P^i(i)}\right)
$$

$$\Rightarrow \quad p^i_{t+1}(\emptyset) - \frac{P^i(j)}{P^i(j)+P^j(i)} = \left(P^i(i) - P^i(j)\right)\left(p^i_t(\emptyset) - \frac{P^i(j)}{P^i(j)+P^j(i)}\right).$$

Thus, the probability $p^i_{t+1}(\emptyset)$ can be calculated as follows.

$$p^i_{t+1}(\emptyset) = \frac{P^i(j)}{P^i(j)+P^j(i)} + \left(P^i(i) - P^i(j)\right)^t \left(P^i(\emptyset) - \frac{P^i(j)}{P^i(j)+P^j(i)}\right).$$

Since $-1 < P^i(i) - P^i(j) < 1$,

$$\lim_{t\to\infty} p^i_{t+1}(\emptyset) = \frac{P^i(j)}{P^i(j)+P^j(i)}.$$

By Corollary 2.4.2, we obtain

$$\lim_{\delta\uparrow 1} f_i(\emptyset) = \frac{P^i(j)}{P^i(j)+P^j(i)}.$$

□

A Markov process satisfies the condition of Corollary 2.4.2 and negotiators divide the pie according to the proposal ratio in the limit. The main consequence of Theorem 2.4.1 is that although the process used in Theorem 2.4.1 has less regularity than a Markov process, we can derive the same result as in the model that uses a Markov process. That is, the result that negotiators divide the pie according to the proposal ratio in the limit is "robust" to departures from an exact Markov process.

Actually, even if we assume an extra condition "there exists some $k \in \mathbb{N}$ such that $\lim_{m\to\infty} \sum_{t=(m-1)k+1}^{mk} p^i_t(\emptyset)$ converges to some value V_i" in Theorem 2.4.1 for the analysis of the limit payoffs, this condition is more general than assuming a Markov process in which probabilities depend on the previous finite number of proposers. In fact, if negotiators' probabilities to be a proposer depend on the periods and the probabilities converge to some values in the limit, it satisfies our condition but it cannot be represented by a Markov process with finite memory. Also, when we arbitrarily consider a deterministic process in which each negotiator proposes just once during 2 periods, all of such processes satisfy our condition, but in these processes, there exists a process which is not represented by a Markov process.[2]

[2]Let A be the order $1 \to 2$ where negotiator 1 proposes first and negotiator 2 proposes second. Also, let B be the order $2 \to 1$. Now, for example, consider the deterministic process $ABBAAABBBB\cdots$. In this process, the proposal ratio during 2 periods is $1:1$. Therefore, this process satisfies the condition in Theorem 2.4.1. However, for all $k \in \mathbb{N}$, this process is not represented by a Markov process in which probabilities depend on the previous k periods. Although this process is a very extreme example, even if we set A and B arbitrarily, such orders satisfy our condition. This fact implies that Theorem 2.4.1 can analyze not only simple situations which are represented by a Markov process but also more complex situations.

Therefore, our condition is more general than assuming a Markov process and we can still consider many complex situations.

If a Markov process is ergodic, the limiting distribution coincides with the unique stationary distribution. Therefore, in the models with a Markov process, the proposal ratio during one period in the limit coincides with the unique stationary distribution of the Markov process. Our condition in Theorem 2.4.1 generalizes this property. That is, by viewing k periods in the condition of Theorem 2.4.1 as a single unit, we can consider that the process which satisfies our condition has "a limiting distribution" and "a stationary distribution" of the unit. Then, the proposal ratio during the unit converges to "the stationary distribution" and this ratio coincides with the ratio of the limit SPE payoffs.

In a Markov process, a set of states B is called an absorbing set if we cannot reach a state outside of B with a positive probability from any state in B. In the models with a Markov process, the condition that there is no absorbing set is often assumed. In contrast to it, in our model, even if we consider a process with absorbing sets, we can calculate the limit of the SPE payoffs if this process satisfies the condition of Theorem 2.4.1.

As a simple example, consider the process where alternating offers which starts from negotiator 1 occurs with probability 1/2 and the deterministic process where negotiator 2 always becomes a proposer occurs with probability 1/2. That is, nature selects these two deterministic processes with equal probabilities at the first period. This process can be represented by a Markov process which depends on the previous 2 periods. In this process, there are mainly three states $(1,2)$, $(2,1)$, and $(2,2)$ where $(1,2)$, $(2,1)$, and $(2,2)$ denote the proposers in the previous 2 periods, respectively. After nature selects alternating offers, the transition probabilities from $(1,2)$ to $(2,1)$ and from $(2,1)$ to $(1,2)$ are 1, respectively. Conversely, after nature selects the deterministic process where negotiator 2 always becomes a proposer, the transition probability from $(2,2)$ to $(2,2)$ is 1. Then, this process has two absorbing sets $\{(1,2),(2,1)\}$ and $\{(2,2)\}$. However, we can calculate the limit of the SPE payoffs. Since $p^1_{2(m-1)+1}(\emptyset) = 1/2$ and $p^1_{2m}(\emptyset) = 0$ for all $m \in \mathbb{N}$, $\lim_{\delta\uparrow} f_1(\emptyset) = 1/4$ and $\lim_{\delta\uparrow} f_2(\emptyset) = 3/4$ by Example 2.4.1. Thus, even if the process has absorbing sets, we can calculate the limit of the SPE payoffs when the process satisfies the condition of Theorem 2.4.1.

Finally, we mention an interpretation of the proposals offered in the unique SPE $\sigma = (\sigma_1, \sigma_2)$ given in Theorem 2.3.1 when the discount factor is sufficiently large. The following proposition is derived from Theorem 2.3.1.

Proposition 2.4.2. *Suppose that, for all $\pi \in \bigcup_{t\in\mathbb{N}} N^{t-1}$, there exists some $k(\pi) \in$*

$\mathbb{N}$ *such that* $\lim_{m\to\infty}\sum_{t=(m-1)k(\pi)+1}^{mk(\pi)} p_t^i(\pi)$ *converges to some value* $V_i(\pi)$ *(thus,* $\lim_{m\to\infty}\sum_{t=(m-1)k(\pi)+1}^{mk(\pi)} p_t^j(\pi)$ *converges to* $V_j(\pi)$ *where* $V_i(\pi) + V_j(\pi) = k(\pi)$*). Then, when* $\delta \uparrow 1$*, negotiator* i *proposes the division* $\left(\frac{V_i(\pi,i)}{k(\pi,i)}, \frac{V_j(\pi,i)}{k(\pi,i)}\right)$ *in the unique SPE given in Theorem 2.3.1 if she is selected as a proposer after the history of proposer* π*.*[3]

Proof. By assumption, as with Theorem 2.4.1,

$$\lim_{\delta\uparrow 1} f_i(\pi,i) = \frac{V_i(\pi,i)}{k(\pi,i)} \quad \text{and} \quad \lim_{\delta\uparrow 1} f_j(\pi,i) = \frac{V_j(\pi,i)}{k(\pi,i)}.$$

Negotiator i proposes the division $(1-\delta f_j(\pi,i), \delta f_j(\pi,i))$ in the unique SPE if she is selected as a proposer after the history of proposer π. These values converge to $(\lim_{\delta\uparrow 1} f_i(\pi,i), \lim_{\delta\uparrow 1} f_j(\pi,i))$ when $\delta \uparrow 1$. Therefore, Proposition 2.4.2 holds. □

$V_i(\pi,i) : V_j(\pi,i)$ denotes the limit of the proposal ratio in the subgame $\Gamma(\pi,i)$. Thus, Proposition 2.4.2 implies that, when negotiator i is selected as a proposer after π, she proposes a division according to the limit of the proposal ratio after the current state (π,i).

In the model with a Markov process, if there is no absorbing set, all proposers propose the same division in all states in a stationary SPE when the discount factor is sufficiently large (see Britz et al. (2010)). In contrast, in our model, negotiators' proposals depend on the current state. Therefore, negotiators' proposals may change over time. Then, they propose a division according to the limit of the proposal ratio after the current state.

2.5 The relationship between the SPE payoffs and the ANBS in the bilateral model

In this section, we mention the relationship between the unique SPE payoffs and the ANBS. Finding the relationship between the unique SPE payoffs and the ANBS in our complex process clarifies how a bargaining procedure affects the bargaining outcome as "bargaining power." As a result, we show that the limit of the unique SPE payoffs coincides with the ANBS weighted by the convergent values of the proposal ratio.

[3] In Proposition 2.4.2, we have to assume the condition of convergence for all $\pi \in \bigcup_{t\in\mathbb{N}} N^{t-1}$. If this condition does not hold, after some history, the limit of the division which is proposed in the unique SPE may not exist. Even if we assume such a condition, we can still consider the process where negotiators' probabilities to be a proposer depends on the periods and the probabilities converge to some values in the limit. Therefore, this condition is more general than assuming a Markov process.

For a utility space S', the asymmetric Nash bargaining solution with weights w is defined as follows.

Definition 2.5.1. *The asymmetric Nash bargaining product with weights w, $\Phi_w : S' \to \mathbb{R}$, is defined by*

$$\Phi_w(x) = \Pi_{i \in N}(x_i)^{w_i}$$

where $w = (w_1, w_2)$ is a vector of nonnegative weights.

The asymmetric Nash bargaining solution (ANBS) with weights w, $x^A = (x_1^A, x_2^A) \in S'$, is the unique maximizer of the function Φ_w.

The weights w of the ANBS denotes "the bargaining power" of the negotiators. In our research, we consider the case $S' = \{(x_1, x_2) \mid x_1 + x_2 \leq 1, x_1, x_2 \geq 0\}$. For the limit of the unique SPE payoffs $(f_i(\emptyset), f_j(\emptyset))$ and the ANBS, the following relationship holds.

Theorem 2.5.1. *Suppose that there exists some $k \in \mathbb{N}$ such that $\lim_{m\to\infty} \sum_{t=(m-1)k+1}^{mk} p_t^i(\emptyset)$ converges to some value V_i ($\lim_{m\to\infty} \sum_{t=(m-1)k+1}^{mk} p_t^j(\emptyset)$ converges to V_j where $V_i + V_j = k$). Then, the ANBS with weights $V = (V_i, V_j)$ is $(\frac{V_i}{k}, \frac{V_j}{k})$. Therefore, the unique SPE payoffs $(f_i(\emptyset), f_j(\emptyset))$ converge to the ANBS with weights $V = (V_i, V_j)$ when $\delta \uparrow 1$.*

Proof. Notice that $\Phi_V(x) = \Pi_{i\in N}(x_i)^{V_i} = \Pi_{i:V_i>0}(x_i)^{V_i}$. Let $x' \in S'$ be the value that, for some $i \in N$ such that $V_i > 0$, $x_i' = 0$. Then, $\Phi_V(x') = 0$. Therefore, x' does not maximize the asymmetric Nash bargaining product. Thus, for $i \in N$ such that $V_i > 0$, $x_i^A > 0$. Conversely, let $x'' \in S'$ be the value that, for some $i \in N$ such that $V_i = 0$, $x_i'' > 0$. Then, $\Phi_V(x'')$ can be improved by reallocating the value $x_i'' > 0$ to the other negotiator j such that $V_j > 0$. Therefore, for $i \in N$ such that $V_i = 0$, $x_i^A = 0$.

Hence, it is sufficient to solve the following maximization problem to derive the ANBS.

$$\begin{aligned} \text{maximize} \quad & \sum_{i:V_i>0} V_i \ln x_i, \\ \text{subject to} \quad & \sum_{i:V_i>0} x_i = 1. \end{aligned} \tag{2.12}$$

Define

$$L(x,\lambda) = \sum_{i:V_i>0} V_i \ln x_i + \lambda \left(1 - \sum_{i:V_i>0} x_i\right)$$

where λ is a Lagrange multiplier.

The solution of (2.12) can be derived by solving

$$\frac{\partial L}{\partial x_i} = \frac{V_i}{x_i} - \lambda = 0, \quad \text{for } i \text{ such that } V_i > 0,$$
$$\frac{\partial L}{\partial \lambda} = 1 - \sum_{i:V_i>0} x_i = 0. \tag{2.13}$$

Then, we obtain

$$\frac{V_i}{x_i} = \frac{V_j}{x_j} \quad \text{for } i \text{ and } j \text{ such that } V_i, V_j > 0. \tag{2.14}$$

By combining (2.13), (2.14), and the fact that $\sum_{i\in N} V_i = k$, we find that the solution of (2.12) is $x_i = \frac{V_i}{k}$ for all i such that $V_i > 0$. Since $x_i^A = 0$ for i such that $V_i = 0$, the ANBS with weights V is $(\frac{V_1}{k}, \frac{V_2}{k})$. Therefore, Theorem 2.5.1 holds. □

Theorem 2.5.1 shows the relationship between the limit of the unique SPE payoffs and the ANBS. This proposition implies that when we consider the probability to be a proposer in the limit as "the bargaining power," the unique SPE payoffs converge to the ANBS when $\delta \uparrow 1$.

2.6 The n-player model

In this section, we consider the n-player model ($n > 2$). Although SPE (and SPE payoffs) may not be unique in the n-player model if δ is large (see Merlo and Wilson (1995)), we can still obtain an MPE which has the same form as the SPE derived in the bilateral model. That is, we can obtain an MPE in which there are no punishments for negotiators who deviate from the strategy profile. When we consider the process which depends only on the previous proposer and consider the proposer as the current state (that is, a Markov process), this MPE boils down to a stationary SPE (an SPE in which negotiators' strategies depend only on the current state and the current proposal). Under this MPE, we obtain the results which correspond to Theorem 2.4.1, Proposition 2.4.1, Proposition 2.4.2, and Theorem 2.5.1 in the n-player model.

2.6.1 The model

We consider the game in which n negotiators divide a pie of size 1. We redefine $N = \{1, 2, \ldots, n\}$ as the set of negotiators and $\delta \in (0, 1)$ as the common discount factor. Also, let $S = \{ (x_1, x_2, \ldots, x_n) \mid \sum_{i\in N} x_i = 1, x_i \geq 0\}$ as the set of divisions of the pie. As with the bilateral model, we assume that a probability to be a proposer depends on the history of proposers. Since $\bigcup_{t\in\mathbb{N}} N^{t-1}$ denotes the set of histories of proposers ($N^0 = \emptyset$), the probability that a proposer is chosen in the next period is represented by the function $P : \bigcup_{t\in\mathbb{N}} N^{t-1} \to \{(P^1, P^2, \ldots, P^n) \mid \sum_{i\in N} P^i = 1, P^i \geq 0\}$ where P^i denotes negotiator i's probability. The game proceeds as follows.

At period $t \in \{1, 2, \ldots\}$, nature selects one negotiator as a proposer. The negotiator who is selected as a proposer proposes some division $x \in S$. After it, all other negotiators respond with Yes or No sequentially (the order of responders does not affect our results). If all responders accept the proposal, then the game ends and negotiator $i \in N$ receives $\delta^{t-1}x_i$. Conversely, if some responder rejects the proposal, the game continues to the next period $t + 1$ and repeat the above process.

2.6.2 MPE

We use the same notation as the bilateral model. That is, $\pi_r \in N^r$ denotes an order of proposers during r periods and $\pi_r(k)$ denotes k-th proposer of the order π_r. $\pi_r^s = (\pi_r(1), \ldots, \pi_r(s))$ denotes the proposers of the order π_r from the first proposer to s-th proposer. $\pi_r\pi_s$ denotes an order of proposers in which π_s follows π_r.

We redefine $\Pi(\pi, \pi_r) = P^{\pi_r(1)}(\pi)P^{\pi_r(2)}(\pi\pi_r^1)\cdots P^{\pi_r(r)}(\pi\pi_r^{r-1})$ for all $\pi \in \bigcup_{t\in\mathbb{N}} N^{t-1}$ and $\pi_r \in N^r$. Then, $\sum_{\pi_r\in N^r} \Pi(\pi, \pi_r) = 1$.

Also, we redefine

$$p_t^i(\pi) = \sum_{\pi_{t-1}\in N^{t-1}} \Pi(\pi, \pi_{t-1})P^i(\pi\pi_{t-1})$$

and

$$f_i(\pi) = \frac{\sum_{r=1}^{\infty} \delta^{r-1} p_r^i(\pi)}{\sum_{s=1}^{\infty} \delta^{s-1}}$$

for all $i \in N$ and $\pi \in \bigcup_{t\in\mathbb{N}} N^{t-1}$ (As with the bilateral model, $p_t^i(\pi)$ denotes negotiator i's probability to be a proposer at period t of the subgame which starts after the history of proposers π). Then, $\sum_{i\in N} p_t^i(\pi) = 1$ and $\sum_{i\in N} f_i(\pi) = 1$.

As with Lemma 2.3.1, the following lemma holds.

Lemma 2.6.1. *For all $i \in N$, $\pi \in \bigcup_{t\in\mathbb{N}} N^{t-1}$, and $t \in \mathbb{N}$,*

$$\sum_{j\in N} P^j(\pi)p_t^i(\pi, j) = p_{t+1}^i(\pi).$$

Proof. The proof is the same as Lemma 2.3.1. □

Also, the following lemma holds.

Lemma 2.6.2. *For all* $\pi \in \bigcup_{t=1}^{\infty} N^{t-1}$ *and* $i \in N$,

$$f_i(\pi) = P^i(\pi)\left(1 - \sum_{j\neq i} \delta f_j(\pi, i)\right) + \sum_{j\neq i} P^j(\pi)\delta f_i(\pi, j).$$

Proof. By the definition of $f_i(\pi)$, the first term of the right hand side can be transformed as follows.

$$\begin{aligned}
P^i(\pi)\left(1 - \sum_{j\neq i} \delta f_j(\pi, i)\right) =& \frac{P^i(\pi)}{\sum_{s=1}^{\infty} \delta^{s-1}}\left(\sum_{r=1}^{\infty} \delta^{r-1} - \sum_{j\neq i} \delta \sum_{r=1}^{\infty} \delta^{r-1} p_r^j(\pi, i)\right) \\
=& \frac{P^i(\pi)}{\sum_{s=1}^{\infty} \delta^{s-1}}\left(1 + \sum_{r=1}^{\infty} \delta^r \left[1 - \sum_{j\neq i} p_r^j(\pi, i)\right]\right) \\
=& \frac{1}{\sum_{s=1}^{\infty} \delta^{s-1}}\left(P^i(\pi) + \sum_{r=1}^{\infty} \delta^r P^i(\pi) p_r^i(\pi, i)\right).
\end{aligned}$$

Also, the second term can be transformed as follows.

$$\begin{aligned}
\sum_{j\neq i} P^j(\pi)\delta f_i(\pi, j) =& \sum_{j\neq i} P^j(\pi)\delta \frac{\sum_{r=1}^{\infty} \delta^{r-1} p_r^i(\pi, j)}{\sum_{s=1}^{\infty} \delta^{s-1}} \\
=& \frac{1}{\sum_{s=1}^{\infty} \delta^{s-1}} \sum_{r=1}^{\infty} \delta^r \sum_{j\neq i} P^j(\pi) p_r^i(\pi, j).
\end{aligned}$$

Therefore, by summarizing the above two values,

$$\begin{aligned}
&P^i(\pi)\left(1 - \sum_{j\neq i} \delta f_j(\pi, i)\right) + \sum_{j\neq i} P^j(\pi)\delta f_i(\pi, j) \\
=& \frac{1}{\sum_{s=1}^{\infty} \delta^{s-1}}\left(P^i(\pi) + \sum_{r=1}^{\infty} \delta^r P^i(\pi) p_r^i(\pi, i)\right) + \frac{1}{\sum_{s=1}^{\infty} \delta^{s-1}} \sum_{r=1}^{\infty} \delta^r \sum_{j\neq i} P^j(\pi) p_r^i(\pi, j) \\
=& \frac{1}{\sum_{s=1}^{\infty} \delta^{s-1}}\left(P^i(\pi) + \sum_{r=1}^{\infty} \delta^r p_{r+1}^i(\pi)\right) \\
=& \frac{\sum_{r=1}^{\infty} \delta^{r-1} p_r^i(\pi)}{\sum_{s=1}^{\infty} \delta^{s-1}} \\
=& f_i(\pi)
\end{aligned}$$

where we use Lemma 2.6.1 in the second equation. Thus, Lemma 2.6.2 holds.

□

By using Lemma 2.6.2, we can see that there exists the following MPE in the n-player model.

Theorem 2.6.1. *Consider the following strategies $\sigma = (\sigma_1, \ldots, \sigma_n)$. After the history of proposers $\pi \in \bigcup_{t=1}^{\infty} N^{t-1}$, if negotiator $i \in N$ becomes a proposer, she proposes the division $x^* \in S$ where $x_j^* = \delta f_j(\pi, i)$ for $j \neq i$ and $x_i^* = 1 - \delta \sum_{j \neq i} f_j(\pi, i)$. Conversely, when negotiator i becomes a responder, she accepts negotiator j's proposal $x \in S$ if $x_i \geq \delta f_i(\pi, j)$ and rejects if $x_i < \delta f_i(\pi, j)$. Then, $\sigma = (\sigma_1, \ldots, \sigma_n)$ is an MPE of the n-player model.*

Proof. We apply the one-shot deviation principle. Consider the path after the history of proposers $\pi_{t-1} \in N^{t-1}$.

First, consider the case that negotiator i is selected as a proposer after π_{t-1}. If σ is played, negotiator i receives $\delta^{t-1}(1 - \delta \sum_{j \neq i} f_j(\pi_{t-1}, i))$. Suppose that negotiator i one-shot deviates from σ_i and proposes another division x which satisfies $x_j \geq \delta f_j(\pi_{t-1}, i)$ for all $j \neq i$ and $x_{j'} > \delta f_{j'}(\pi_{t-1}, i)$ for some $j' \neq i$. Then, all responders accept it under σ and negotiator i receives $\delta^{t-1}(1 - \sum_{j \neq i} x_j)$. However, $\delta^{t-1}(1 - \sum_{j \neq i} x_j)$ is smaller than $\delta^{t-1}(1 - \delta \sum_{j \neq i} f_j(\pi_{t-1}, i))$. Thus, negotiator i cannot improve her payoff by proposing the division x.

Next, suppose that negotiator i one-shot deviates from σ_i and proposes another division x' which satisfies $x'_{j^*} < \delta f_j(\pi_{t-1}, i)$ for some $j^* \neq i$. Then, negotiator j^* rejects the offer under σ_{j^*} and the game continues to the next period. Then, the history of proposers is (π_{t-1}, i). After this history, negotiator i is selected as a proposer with probability $P^i(\pi_{t-1}, i)$ and receives $\delta^t(1 - \delta \sum_{j \neq i} f_j(\pi_{t-1}, i, i))$ under σ. On the other hand, negotiator $j \neq i$ is selected as a proposer with probability $P^j(\pi_{t-1}, i)$ and negotiator i receives $\delta^{t+1} f_i(\pi_{t-1}, i, j)$ under σ. Therefore, negotiator i receives $P^i(\pi_{t-1}, i)\delta^t(1 - \delta \sum_{j \neq i} f_j(\pi_{t-1}, i, i)) + \sum_{j \neq i} P^j(\pi_{t-1}, i)\delta^{t+1} f_i(\pi_{t-1}, i, j) = \delta^t f_i(\pi_{t-1}, i)$ (by Lemma 2.6.2) under σ.

Now, since

$$\begin{aligned}
&\delta^{t-1}\left(1 - \delta \sum_{j \neq i} f_j(\pi_{t-1}, i)\right) - \delta^t f_i(\pi_{t-1}, i) \\
=&\delta^{t-1}\left(1 - \delta \sum_{j' \in N} f_{j'}(\pi_{t-1}, i)\right) \\
=&\delta^{t-1}(1 - \delta) \\
>&0,
\end{aligned}$$

we can see

$$\delta^{t-1}\left(1-\delta\sum_{j\neq i} f_j(\pi_{t-1},i)\right) > \delta^t f_i(\pi_{t-1},i).$$

Therefore, negotiator i cannot improve her payoff by proposing the division x'.

Subsequently, consider the case that negotiator $j^{**}\neq i$ is selected as a proposer after π_{t-1} and she proposes the division $x''\in S$. If negotiator i accepts the offer, she receives $\delta^{t-1}x''_i$. On the other hand, if she rejects the offer, the game continues to the next period. Then, negotiator i receives $\delta^t(1-\delta\sum_{j\neq i} f_j(\pi_{t-1},j^{**},i))$ with probability $P^i(\pi_{t-1},j^{**})$ and receives $\delta^{t+1}f_i(\pi_{t-1},j^{**},j)$ with probability $P^j(\pi_{t-1},j^{**})$ for $j\neq i$ under σ. Therefore, if negotiator i rejects negotiator j^{**}'s proposal x'', she receives $P^i(\pi_{t-1},j^{**})\delta^t(1-\delta\sum_{j\neq i} f_j(\pi_{t-1},j^{**},i))+$
$\sum_{j\neq i} P^j(\pi_{t-1},j^{**})\delta^{t+1}f_i(\pi_{t-1},j^{**},j)=\delta^t f_i(\pi_{t-1},j^{**})$ (by Lemma 2.6.2) under σ.

Consider the case $x''_i\geq \delta f_i(\pi_{t-1},j^{**})$. Suppose that negotiator i one-shot deviates from σ_i and rejects the offer x''. Then, the game continues to the next period and negotiator i receives $\delta^t f_i(\pi_{t-1},j^{**})$ under σ. In this case, we can confirm that negotiator i cannot improve her payoff by deviating from σ_i since $\delta^{t-1}x''_i\geq\delta^t f_i(\pi_{t-1},j^{**})$.

Consider the case $x''_i<\delta f_i(\pi_{t-1},j^{**})$. Suppose that negotiator i one-shot deviates from σ_i and accepts the offer x''. Then, negotiator i receives $\delta^{t-1}x''_i$. However, she can receive larger payoff $\delta^t f_i(\pi_{t-1},j^{**})$ under σ. Therefore, negotiator i cannot improve her payoff by deviating from σ_i.

Consequently, Theorem 2.6.1 holds since there is no profitable one-shot deviation.

□

Therefore, we can obtain an MPE which has the same form as the SPE derived in the bilateral model. That is, we can obtain an MPE in which there are no punishments for negotiators who deviate from the strategy profile. When we consider the process which depends only on the previous proposer and consider the proposer as the current state (that is, a Markov process), the MPE given in Theorem 2.6.1 boils down to a stationary SPE (negotiators' strategies depend only on the current state and the current proposal).

2.6.3 The limit of the MPE payoffs

If the MPE $\sigma=(\sigma_1,\ldots,\sigma_n)$ given in Theorem 2.6.1 is played, negotiator $i\in N$ receives the payoff $f_i(\emptyset)$. Under this MPE, we obtain the results which correspond to Theorem 2.4.1, Proposition 2.4.1, and Proposition 2.4.2 in the n-player model.

Theorem 2.6.2. *If there exists some* $k \in \mathbb{N}$ *such that* $\{(\sum_{t=(m-1)k+1}^{mk} p_t^1(\emptyset), \dots, \sum_{t=(m-1)k+1}^{mk} p_t^n(\emptyset))\}_{m\in\mathbb{N}}$ *converges to some values* $(V_1, \dots, V_n)$*, then* $\lim_{\delta\uparrow 1} f_i(\emptyset) = \frac{V_i}{k}$ *for all* $i \in N$*.*

Proof. The proof is the same as Theorem 2.4.1. □

Corollary 2.6.1. *If there exists some* $k \in \mathbb{N}$ *such that* $\{(\sum_{t=m+1}^{m+k} p_t^1(\emptyset), \dots, \sum_{t=m+1}^{m+k} p_t^n(\emptyset))\}_{m\in\mathbb{N}}$ *converges to some values* $(V_1, \dots, V_n)$*, then* $\lim_{\delta\uparrow 1} f_i(\emptyset) = \frac{V_i}{k}$ *for all* $i \in N$*.*

Corollary 2.6.2. *If* $\{(p_t^1(\emptyset), \dots, p_t^n(\emptyset))\}_{t\in\mathbb{N}}$ *converges to some values* $(V_1, \dots, V_n)$*, then* $\lim_{\delta\uparrow 1} f_i(\emptyset) = V_i$ *for all* $i \in N$*.*

Therefore, negotiators divide the pie according to the proposal ratio under the MPE given in Theorem 2.6.1.

If negotiators' probabilities to be a proposer depend on the previous l periods (a Markov process), this process satisfies the condition of Corollary 2.6.2.

Proposition 2.6.1. *Suppose that for all* $i \in N$ *and* $\pi \in \bigcup_{t=l+1}^{\infty} N^{t-1}$*,* $P^i(\pi) > 0$ *and* $P^i(\pi)$ *depends only on the previous* l *periods (for* $\pi \in \bigcup_{t=1}^{l} N^{t-1}$*,* $P^i(\pi)$ *can take arbitrary values). Then,* $\{(p_t^1(\emptyset), \dots, p_t^n(\emptyset))\}_{t\in\mathbb{N}}$ *converges.*

Proof. $\bigcup_{t=l+1}^{\infty} N^{t-1}$ can be divided into n^l states which are characterized by the history of proposers during previous l periods. The rest of the proof is the same as Proposition 2.4.1. □

We can also obtain the result corresponding to Proposition 2.4.2.

Proposition 2.6.2. *Suppose that, for all* $\pi \in \bigcup_{t\in\mathbb{N}} N^{t-1}$*, there exists some* $k(\pi) \in \mathbb{N}$ *such that* $\{\sum_{t=(m-1)k(\pi)+1}^{mk(\pi)} p_t^1(\pi), \dots, \sum_{t=(m-1)k(\pi)+1}^{mk(\pi)} p_t^n(\pi)\}$ *converges to some value* $V(\pi) = (V_1(\pi), \dots, V_n(\pi))$*. Then, when* $\delta \uparrow 1$*, negotiator* i *proposes the division* $\left(\frac{V_1(\pi,i)}{k(\pi,i)}, \dots, \frac{V_n(\pi,i)}{k(\pi,i)}\right)$ *in the MPE given in Theorem 2.6.1 if she is selected as a proposer after the history of proposer* π*.*

Proof. The proof is the same as Proposition 2.4.2. □

That is, when negotiator i is selected as a proposer after π, she proposes a division according to the limit of the proposal ratio after the current state (π, i). Therefore, negotiators' proposals depend on the current state and may change over time.

Hence, in the n-player model, under the MPE $\sigma = (\sigma_1, \dots, \sigma_n)$ given in Theorem 2.6.1, we obtain the same results as the bilateral model.

2.6.4 The relationship between the MPE payoffs and the ANBS

Finally, we mention the relationship between the MPE payoffs given in Theorem 2.6.1 and the ANBS as with the bilateral model. For a n-dimensional utility space S', the asymmetric Nash bargaining solution with weights w is defined as follows.

Definition 2.6.1. *The asymmetric Nash bargaining product with weights w, $\Phi_w : S' \to \mathbb{R}$, is defined by*

$$\Phi_w(x) = \Pi_{i \in N}(x_i)^{w_i}$$

where $w = (w_1, \ldots, w_n)$ is a vector of nonnegative weights.

The asymmetric Nash bargaining solution (ANBS) with weights w, $x^A = (x_1^A, \ldots, x_n^A) \in S'$, is the unique maximizer of the function Φ_w.

We consider the case $S' = \{(x_1, \ldots, x_n) \mid \sum_{i \in N} x_i \leq 1, x_i \geq 0\}$. Under the MPE given in Theorem 2.6.1, the following result corresponding to Theorem 2.5.1 holds in the n-player model.

Theorem 2.6.3. *Suppose that there exists some $k \in \mathbb{N}$ such that the sequence $\{\sum_{t=(m-1)k+1}^{mk} p_t^i(\emptyset), \ldots, \sum_{t=(m-1)k+1}^{mk} p_t^i(\emptyset)\}$ converges to some value $V = (V_1, \ldots, V_n)$. Then, the ANBS with weights $V = (V_1, \ldots, V_n)$ is $(\frac{V_1}{k}, \ldots, \frac{V_n}{k})$. Therefore, the MPE payoffs $(f_1(\emptyset), \ldots, f_n(\emptyset))$ which are obtained under the MPE given in Theorem 2.6.1 converge to the ANBS with weights $V = (V_1, \ldots, V_n)$ when $\delta \uparrow 1$.*

Proof. The proof is the same as Theorem 2.5.1. □

Therefore, under the MPE given in Theorem 2.6.1, the MPE payoffs converge to the ANBS weighted by the limit of the proposal ratio.

2.7 Conclusion

We analyzed the model which is a generalization of the model of Rubinstein (1982) from the viewpoint of the process of how a proposer is decided in each period. In our model, a negotiator's probability to be a proposer depends on the history of proposers and negotiators divide a pie of size 1. By considering such a model, we can analyze the situations depending not only on the previous proposers but also on periods.

In the bilateral bargaining model, we derived the unique SPE and analyzed how its SPE payoffs are related to the process. We saw each component game at period t involving negotiators dividing a pie of size δ^{t-1} according to the proposal ratio at period t in the unique SPE payoffs. Therefore, the negotiator with more chances to be a proposer can obtain a higher payoff.

In the case $\delta \uparrow 1$, we showed that if the proposal ratio converges to some value, then negotiators divide the pie according to this convergent value. The main consequence of Theorem 2.4.1 is that although the process used in Theorem 2.4.1 has less regularity than a Markov process, we can derive the same result as in the model that uses a Markov process. However, in contrast to the model with a Markov process where all negotiators propose the same division in all states in a stationary SPE, under the SPE which we derive, the negotiators propose the divisions depending on the current state. Also, we analyzed the relationship between the SPE payoffs and the ANBS. As a result, we showed that the limit of the SPE payoffs coincides with the ANBS weighted by the convergent values of the proposal ratio.

In the n-player model, we showed that there exists an MPE which has the same form as the unique SPE in the bilateral model in the sense that there are no punishments for negotiators who deviate from the strategy profile. Under this MPE, we showed that the same results as the bilateral model hold.

Chapter 3

Simultaneous-Offers Bargaining with a Mediator[1]

3.1 Introduction

In non-cooperative bargaining models, if negotiators cannot reach an agreement, the bargaining breaks down (disagreement). Especially, in the standard simultaneous-offers bargaining model, disagreement is supported as an equilibrium outcome.[2] However, in the sense that disagreement is unprofitable, such an outcome is undesirable. In reality, to avoid such disagreement, an arbitrator is often introduced into bargaining. The role of an arbitrator is to impose some agreement as a final bargaining outcome when negotiators cannot reach an agreement by themselves. For example, such an arbitrator is used to resolve conflicts in public-sector and to determine the salaries of major league baseball players. When an arbitrator is introduced into the bargaining, disagreement vanishes since the arbitrator forces negotiators to reach an agreement.[3]

Crawford (1979), Yildiz (2011), and Rong (2012) analyze the bargaining models with such an arbitrator. Crawford (1979) analyzes the simultaneous-offers bargaining model with an arbitrator, and Yildiz (2011) and Rong (2012) analyze the alternating-offers bargaining models with an arbitrator.[4] The games of these models proceed as follows. First, negotiators propose their demands simultaneously or alternately. If they can reach an agreement, then the bargaining ends. In contrast, if they cannot reach an agreement by themselves, the game proceeds to the arbitration

[1]This chapter is based on Hanato (2019).

[2]The standard simultaneous-offers bargaining is analyzed in Chatterjee and Samuelson (1990) as a generalization of the Nash demand game (Nash (1953)).

[3]More detailed roles of such an arbitrator are discussed in Crawford (1985).

[4]The model of Rong (2012) is a generalization of the model of Yildiz (2011).

process and the arbitrator decides a final bargaining outcome.

In these models, since the arbitrators have the authority to decide a final outcome, the equilibrium outcomes strongly depend on what agreement the arbitrator wishes to impose (especially, when the discount factor is sufficiently large). Therefore, a sufficiently "reasonable" agreement for negotiators which seems to be desirable as a bargaining outcome (e.g. Nash bargaining solution (NBS) (Nash (1950))) can be achieved in equilibrium if and only if the arbitrator is sufficiently fair. However, in real situations, it is observed that arbitrators are often biased and impose an agreement which seems to be unfair (for example, see Eylon et al. (2000) and Burger and Walters (2005)). Therefore, when an arbitrator is introduced, a reasonable agreement is eliminated from equilibrium if the arbitrator is biased. Actually, in the models of Crawford (1979) and Rong (2012), this is observed.

Given these facts, in our study, we consider introducing a mediator rather than an arbitrator to avoid disagreement without eliminating the achievability of a reasonable agreement in equilibrium. Whereas an arbitrator imposes an agreement, a mediator facilitates the reaching of an agreement by negotiators.[5] That is, a mediator can give advice but cannot impose an agreement. In contrast to the bargaining with an arbitrator, negotiators have the right to reject the mediator's advice. In this sense, a mediator has weaker authority than an arbitrator. Such a mediator is also often introduced into bargaining situations, but the role of a mediator in bargaining is not sufficiently analyzed. In this study, we focus on such a mediator.

In our bargaining model with a mediator, the game proceeds as follows. First, the negotiators simultaneously propose their demands. If these demands are compatible, the bargaining ends. If they are incompatible, the bargaining proceeds to the mediation process. In the mediation process, the mediator proposes a plan of an agreement. Then, what plan the mediator proposes depends on what kind of agreement the mediator considers to be ideal (appropriate) as an agreement of the concerned bargaining. After the mediator proposes a plan, the negotiators decide whether to accept the mediator's proposal or reject it. If both negotiators accept it, the bargaining ends with the mediator's plan. If some negotiator rejects it, the bargaining proceeds to the next step where the negotiators propose their demands again and the above process is repeated. Note that, in contrast to the aforementioned models with an arbitrator, which has finite periods, this model has infinite periods.

In our model, we assume that what kind of agreement the mediator considers to be ideal (appropriate) is known to both negotiators. We justify this assumption by

[5] This definition is by Muthoo (1999).

the following reasons. In reality, there are often cases that, as a reference point of the bargaining, a mediator reveals the agreement which she considers to be appropriate. The mediator's main purpose is talking with negotiators for reaching an agreement. Therefore, there seems to be no valid reasons to conceal her ideas. Then, in order to facilitate the discussion, the mediator may tell her idea of an appropriate agreement (however, the mediator's idea may be biased even if she believes herself to be fair, as arbitrators in the research of Eylon et al. (2000) are biased by various inevitable factors even if they believe themselves to be fair). On the other hand, even if a mediator does not reveal her ideal agreement, negotiators may be able to know what agreement the mediator intends to propose by searching the cases of the bargaining which the mediator handled in the past. That is, by the mediator's behavior in the past bargaining cases, negotiators can know approximate mediator's preference. In our model, by the above reasons, we (approximately) assume that what kind of agreement the mediator considers to be ideal (appropriate) is known to both negotiators.

By analyzing this model, we obtain the following desirable results where the NBS plays an important role as a reasonable agreement. First, we find that, although a mediator cannot impose an agreement, disagreement is not supported as an outcome of stationary subgame perfect equilibrium (SSPE). This result implies that a mediator can resolve conflicts as with an arbitrator. Second, although the set of SSPE agreements is biased towards the mediator's ideal agreement, the reasonable agreement in the sense of the NBS is always one of the SSPE agreements even if the mediator is biased. Therefore, the reasonable agreement is always achievable in SSPE (in contrast to models with an arbitrator). Additionally, we find that an agreement having such a property is only the NBS. Finally, we show that, conversely, if a mediator is fair in the sense that she wishes to achieve the NBS, the NBS is the unique SSPE agreement when the discount factor is sufficiently large. That is, the negotiators always reach an agreement with the NBS in SSPE. Thus, we find that the fair mediator facilitates the reaching of the reasonable agreement.

In addition to these desirable results, introducing a mediator instead of an arbitrator has another advantage. In reality, to call an arbitrator into bargaining often requires considerable effort since it may need legal processes. For example, the arbitration in labor dispute often needs it. Also, in reality, there are some bargaining situations where it is difficult to introduce an arbitrator due to negotiators' strong power. For example, in conflicts between nations, since nations have strong power, they may deviate from imposed decision forcibly after negotiation ends. Therefore, to impose an agreement surely, the introduced arbitrator needs to have sufficiently strong power, but it is difficult to find such an arbitrator. In contrast to these difficulties, since a mediator is merely an adviser, introducing it is easier than an arbitrator. That is, a mediator can resolve conflicts as with an arbitrator, but introducing it does not require much effort. This is another advantage of introducing a mediator.

In the remaining of this section, we introduce other related literatures. Although the role of a mediator in bargaining situations is not sufficiently analyzed, there are a few papers which analyze the bargaining with a mediator (e.g. Wilson (2001) and Jarque et al. (2003)). In most of these papers, a mediator is introduced as a system of the game. That is, a mediator does not make decision and does not have utility. However, since a mediator may have bias, it is natural to consider a mediator as a player of the game rather than a system. Therefore, in our model, we introduce a mediator as a player.

Camiña and Porteiro (2009) introduce a mediator as a player and analyze peace negotiations. In their alternating-offers bargaining model, the roles of the mediator are deciding which negotiator proposes first or deciding whether to submit an offer received from a negotiator to the other negotiator. Therefore, their mediator does not give advice about what agreement negotiators should reach. In contrast to their model, we consider the model where the mediator can propose a plan of an agreement to negotiators.

A mediator also appears in the literatures of mechanism design such as Myerson (1983), Myerson and Satterthwaite (1983), and Myerson (1986). Under these literatures, a mediator is introduced as a tool to exchange private information among players and a tool to suggest players' next actions for coordinating outcomes of the game.

Manzini and Mariotti (2001) and Manzini and Mariotti (2004) analyze alternating-offers bargaining models with an arbitrator. In these models, the arbitrator imposes an agreement if and only if both negotiators consent to proceed to the arbitration process. Our model and these models are quite different, but they are similar in the sense that the consent from both negotiators is necessary before the mediator's proposal or the arbitrator's decision is implemented.

Manzini and Ponsatí (2005), Manzini and Ponsati (2006), and Ponsatí (2004) analyze the bargaining with a stakeholder. A stakeholder is a third party who is interested in the resolution of the conflict and receives benefits when negotiators reach an agreement. For example, in conflicts in public-sector, the government can be considered as a stakeholder. In this situation, since the government wishes to improve social welfare, it makes effort to resolve the conflicts for its benefit. In our model, the mediator can also be considered as such a stakeholder. The most different point between the above existing literatures and our model is that, whereas the stakeholders in the above literatures are not interested in what agreement the negotiators reach, the mediator of our model is interested in it. At this point, our model can be applied to many bargaining situations such as the bargaining in public sector where the government is interested in how negotiators reach an agreement.

This chapter is organized as follows. In section 3.2, we define a simultaneous-offers bargaining model with a mediator. In section 3.3, we derive SSPEs of our model and analyze properties of the NBS as an SSPE agreement. In section 3.4, we compare our model with a model without a mediator and a model with an arbitrator. In this section, we analyze how the mediator affects the bargaining outcomes. In section 3.5, we conclude our study.

3.2 The model

We consider a bargaining model with three players, negotiators 1, 2, and the mediator. Let $S \subset \mathbb{R}^2_+$ be the feasible utility space for the negotiators. We assume that $d = (0,0)$ is an element of S, and there exists some $(x,y) \in S$ such that $(x,y) \gg d$.[6] Furthermore, we impose some assumptions on the set S as the same as existing literatures. That is, we assume that the set S is convex, compact, and strictly comprehensive.[7] Also, we define $u : S \to \mathbb{R}_+$ as the mediator's utility function. The mediator is interested in what agreement the negotiators reach. Here, we assume $u(x,y) \geq 0$ for all $(x,y) \in S$.

Additionally, we define $\overline{x} = \max\{x \mid (x,y) \in S\}$, $\overline{y} = \max\{y \mid (x,y) \in S\}$, and the function $f : [0,\overline{x}] \to [0,\overline{y}]$ as $f(x) = \max\{y \mid (x,y) \in S\}$. Since S is compact, these definitions are well-defined. By the assumptions of S, we can confirm that the function f is concave, strictly decreasing, and continuous. Then, $f(0) = \overline{y}$, $f(\overline{x}) = 0$, and f has the inverse function $f^{-1} : [0,\overline{y}] \to [0,\overline{x}]$ represented by $f^{-1}(y) = \max\{x \mid (x,y) \in S\}$ (f^{-1} is also concave, strictly decreasing, and continuous). Also, we define $p(x) = (x, f(x))$ for $x \in [0,\overline{x}]$. Then, the Pareto frontier of S can be represented as $\partial S = \{p(x) \mid x \in [0,\overline{x}]\}$. These are depicted in Figure 3.1.

Now, we describe the game. Let $\delta \in (0,1)$ be the common discount factor. The game starts from period 1 and proceeds as follows. At odd period t, the negotiators 1 and 2 simultaneously propose their demands $x \in [0,\overline{x}]$ and $y \in [0,\overline{y}]$, respectively. If $(x,y) \in S$, that is, if the negotiators' demands are compatible, the game ends and negotiators 1, 2, and the mediator receive $\delta^{t-1}x$, $\delta^{t-1}y$, and $\delta^{t-1}u(x,y)$, respectively.

[6] $(x,y) \gg (x',y')$ denotes $x > x'$ and $y > y'$.

[7] $(x,y) \geqslant (x',y')$ denotes $x \geq x'$ and $y \geq y'$. The set S is *comprehensive* if $(x'',y'') \geqslant (x',y') \geqslant (0,0)$ and $(x'',y'') \in S$ imply $(x',y') \in S$. The set S is *strictly comprehensive* if S is comprehensive and, for all $(x,y) \in S$ such that $(x',y') \geqslant (x,y)$ and $(x',y') \neq (x,y)$ for some $(x',y') \in S$, there exists some $(x'',y'') \in S$ such that $(x'',y'') \gg (x,y)$.

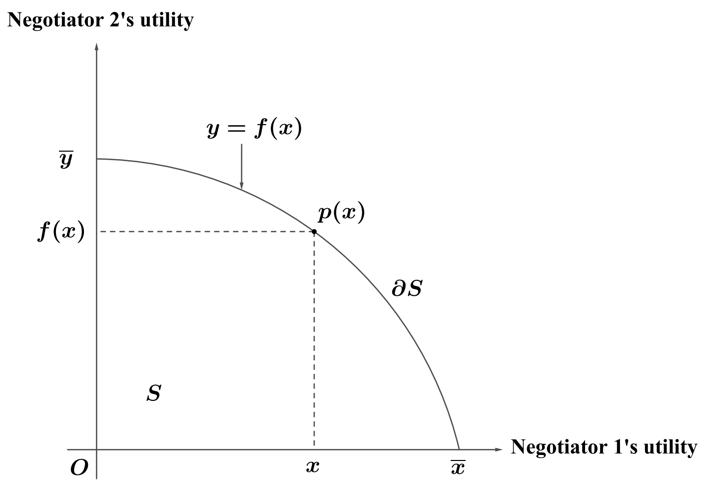

Figure 3.1: Feasible utility space S for the negotiators

If $(x, y) \notin S$, that is, if the negotiators' demands are incompatible, the game proceeds to the next period $t+1$. At even period $t+1$, the mediator proposes some $p(z) \in \partial S$ such that $z \in [f^{-1}(y), x]$ or chooses pass. Now, notice that, when $z \in [f^{-1}(y), x]$, $z \leq x$ and $f(x) \leq y$ hold (see Figure 3.2). That is, when the mediator gives advice, she recommends the negotiators to concede. In our model, we assume that the mediator can say nothing (pass) if she considers that the negotiators can reach an appropriate agreement by themselves.

If the mediator chooses pass, then the game proceeds to period $t + 2$. If the mediator proposes some $p(z)$, then the negotiators simultaneously decide whether to accept the mediator's proposal or reject it. If both negotiators accept it, the game ends and negotiators 1, 2, and the mediator receive $\delta^t z$, $\delta^t f(z)$, and $\delta^t u(p(z))$, respectively. If some negotiator rejects it, the game proceeds to the next period $t+2$. At period $t + 2$ or later, the process at period t is repeated at every odd period and the process at period $t + 1$ is repeated at every even period. This game continues until some agreement is reached. If the negotiation continues permanently in some strategy profile, then negotiator 1, 2, and the mediator receive payoffs of zero. The game tree is depicted in Figure 3.3.

In this study, we suppose that the mediator's utility function is single-peaked on ∂S. That is, we suppose that the mediator considers some agreement on ∂S as the ideal agreement of the bargaining, and suppose that the mediator's utility decreases as the distance from her ideal agreement increases.

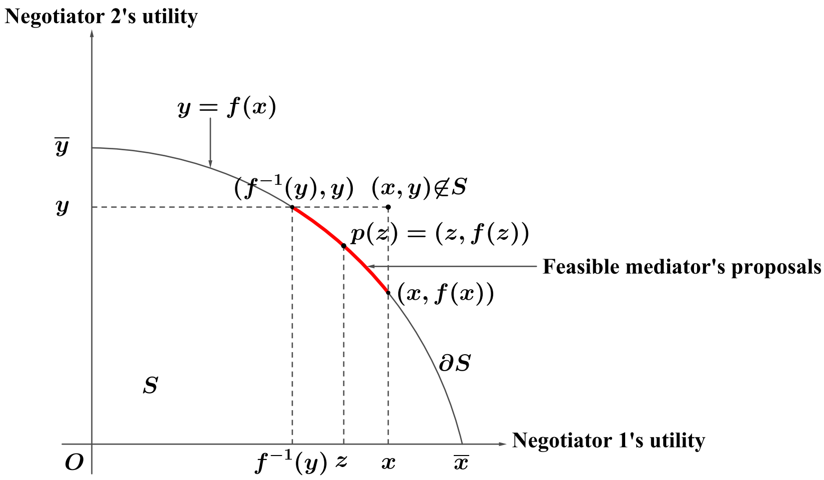

Figure 3.2: Feasible mediator's proposals

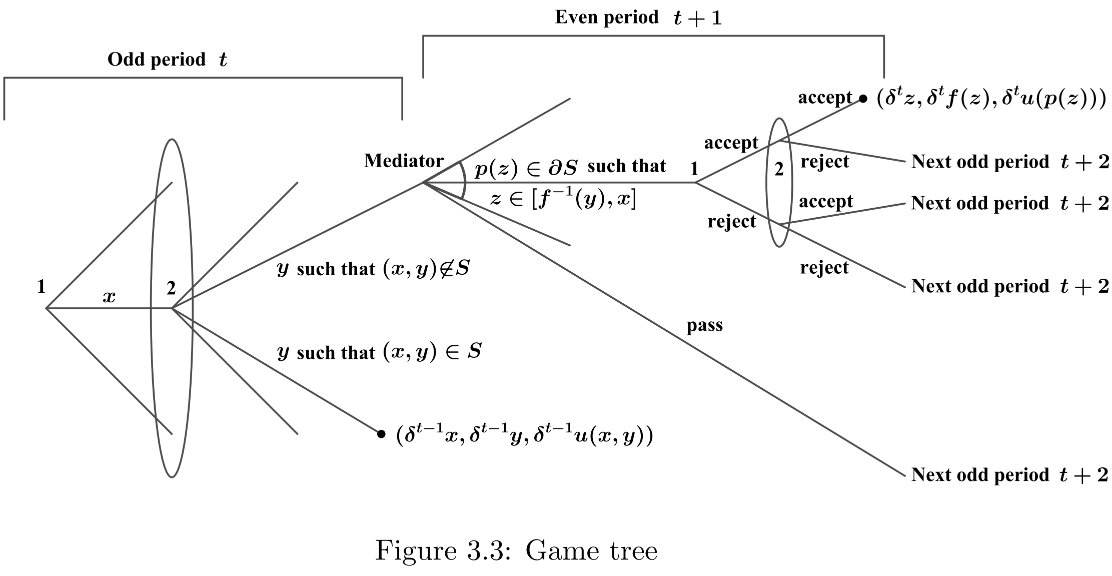

Figure 3.3: Game tree

Formally, we impose the following assumption on the mediator's utility function.

Assumption 3.2.1. *With respect to the mediator's utility function u, there exists some $\alpha \in [0, \overline{x}]$ such that,*

1. *for $x, x' \in [0, \overline{x}]$ such that $x < x' \leq \alpha$, $u(p(x)) < u(p(x'))$ and*

2. *for $x, x' \in [0, \overline{x}]$ such that $x > x' \geq \alpha$, $u(p(x)) < u(p(x'))$.*

In this assumption, the mediator's ideal agreement is $p(\alpha) \in \partial S$. The mediator favors negotiator 1 when α is close to $\overline{x}$ and favors negotiator 2 when α is close to zero. As mentioned in section 3.1, we assume that the mediator's preference is known to both negotiators.

As a solution concept of the above model, we use a stationary subgame perfect equilibrium (SSPE), that is, use a subgame perfect equilibrium (SPE) in which,

1. each negotiator's demand at every odd period is always the same value,

2. the mediator's proposal at every even period depends only on the negotiators' demands at the previous odd period, and

3. for each negotiator, whether she accepts the mediator's proposal or rejects it depends only on the negotiators' demands at the previous odd period and the mediator's proposal at the current period.

Then, we assume that, when some negotiator responses to the mediator's proposal, she accepts it if the mediator's proposal is not less profitable than rejecting it. Thus, if accepting the mediator's proposal and rejecting it are indifferent, the negotiator accepts the mediator's proposal.

3.3 SSPE outcomes

In this section, we derive SSPE outcomes of our model. Before proceeding to the analysis of SSPE, we prepare additional notation. When the curve $y = f(x)$ is scaled down by δ in the direction of y, we obtain $y = \delta f(x)$. In contrast, when $y = f(x)$ is scaled down by δ in the direction of x, we obtain $y = f(\frac{x}{\delta})$. Let $x^R(\delta)$ be the solution of $f(x) = \delta f(\delta x)$. Then, the unique intersection of the curves $y = \delta f(x)$ and $y = f(\frac{x}{\delta})$ is $(\delta x^R(\delta), f(x^R(\delta)))$. Now, notice that $p(x^R(\delta)) = (x^R(\delta), f(x^R(\delta)))$ and $p(\delta x^R(\delta)) = (\delta x^R(\delta), f(\delta x^R(\delta)))$ are the negotiators' offers proposed in the SPE of the Rubinstein's alternating-offers bargaining (Rubinstein (1982)) with the utility space S.

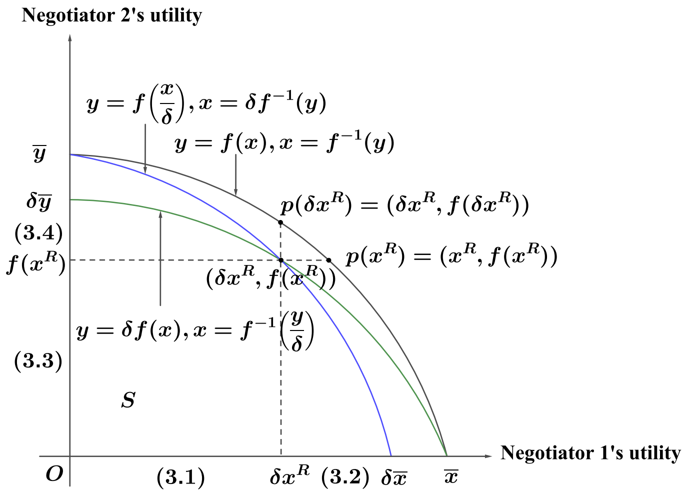

Figure 3.4: $y = \delta f(x)$ and $y = f(\frac{x}{\delta})$

Also, notice that

$$\delta f(x) < f(\frac{x}{\delta}) \text{ when } x \in [0, \delta x^R(\delta)), \tag{3.1}$$

$$\delta f(x) > f(\frac{x}{\delta}) \text{ when } x \in (\delta x^R(\delta), \delta\overline{x}], \tag{3.2}$$

$$\delta f^{-1}(y) < f^{-1}(\frac{y}{\delta}) \text{ when } y \in [0, f(x^R(\delta))), \text{ and} \tag{3.3}$$

$$\delta f^{-1}(y) > f^{-1}(\frac{y}{\delta}) \text{ when } y \in (f(x^R(\delta)), \delta\overline{y}]. \tag{3.4}$$

The equation $\delta f(x) = f(\frac{x}{\delta})$ holds if and only if $x = \delta x^R(\delta)$ and the equation $\delta f^{-1}(y) = f^{-1}(\frac{y}{\delta})$ holds if and only if $y = f(x^R(\delta))$. These are depicted in Figure 3.4. In this chapter, we may use x^R instead of $x^R(\delta)$ when we fix the value of δ.

Bargaining outcomes of our model can be divided into the following three cases.

1. The negotiators reach an agreement with some $(x, y) \in S$ at some odd period t by themselves. This outcome is denoted by $((x, y), t)$ where t is odd.

2. The mediator's proposal $p(z) \in \partial S$ is accepted at some even period t. This outcome is denoted by $(p(z), t)$ where t is even.

3. The negotiators never reach an agreement (disagreement).

In the following subsections, we sequentially analyze each case and derive SSPE outcomes. In this study, we use the one-shot deviation principle to derive SSPEs. That is, we use the fact that a stationary strategy profile σ is an SSPE if and only if there is no player who can become better off by deviating from σ for just one period

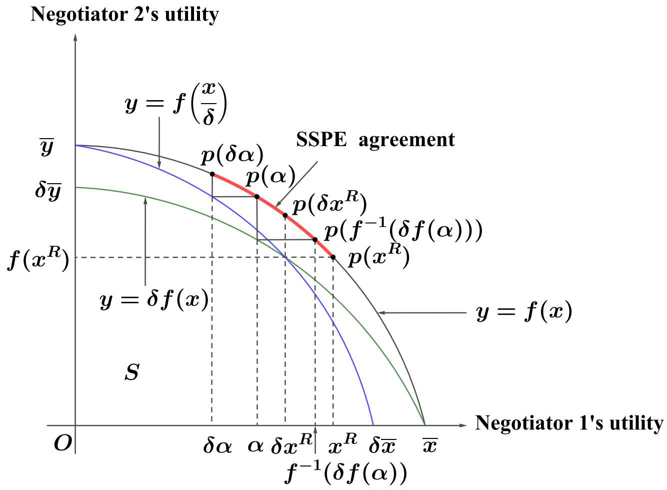

Figure 3.5: SSPE agreement at period 1 when $\alpha \in [0, \delta x^R)$

(for example, see Fudenberg and Tirole (1991)). Also, we analyze properties of an agreement with the NBS in our model.

3.3.1 Agreements at odd periods

In this subsection, we derive SSPE outcomes such that the negotiators reach an agreement at odd periods. By the definition of SSPE, in such SSPE outcomes, the negotiators reach an agreement at period 1. Regarding such SSPE outcomes, we obtain the following theorem.

Theorem 3.3.1. *The outcome $(p(x), 1)$ is supported as an SSPE outcome if and only if*

1. *$x \in [\delta\alpha, x^R(\delta)]$ when $\alpha \in [0, \delta x^R(\delta))$ (see Figure 3.5),*
2. *$x \in [\delta\alpha, f^{-1}(\delta f(\alpha))]$ when $\alpha \in [\delta x^R(\delta), x^R(\delta)]$ (see Figure 3.6), and*
3. *$x \in [\delta x^R(\delta), f^{-1}(\delta f(\alpha))]$ when $\alpha \in (x^R(\delta), \overline{x}]$ (see Figure 3.7).*

For all $(x, y) \in S \backslash \partial S$, the outcome $((x, y), 1)$ is not supported as an SSPE outcome. (In the SSPE, negotiators 1 and 2 demand x and $f(x)$, respectively, and the mediator follows the strategy described in Lemma 3.3.2.)

Since $\delta\alpha \leq \delta x^R(\delta)$ holds when $\alpha \in [0, x^R(\delta)]$ and $x^R(\delta) \leq f^{-1}(\delta f(\alpha))$ holds when $\alpha \in [\delta x^R(\delta), \overline{x}]$, an agreement on $\partial S^R = \{p(x) \mid x \in [\delta x^R(\delta), x^R(\delta)]\}$ is supported as an SSPE agreement for any $\alpha \in [0, \overline{x}]$. Now, notice that $p(\delta x^R(\delta))$ and $p(x^R(\delta))$

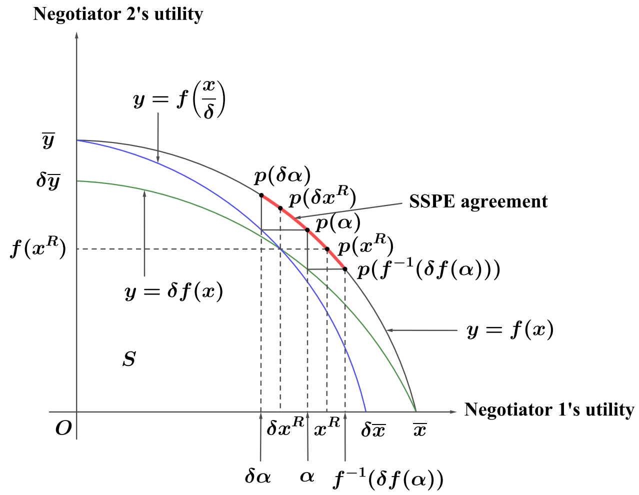

Figure 3.6: SSPE agreement at period 1 when $\alpha \in [\delta x^R, x^R]$

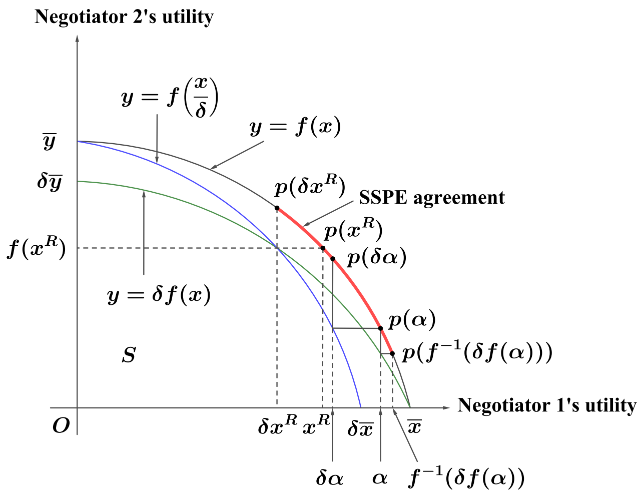

Figure 3.7: SSPE agreement at period 1 when $\alpha \in (\delta x^R, \overline{x}]$

are the negotiators' SPE offers in the Rubinstein's alternating-offers model. Since it is well-known that the NBS of the bargaining problem (S, d) lies on ∂S^R (for example, see Osborne and Rubinstein (1994)), we can find that, for all $\alpha \in [0, \overline{x}]$, the outcome that the negotiators reach an agreement with the NBS at period 1 is always supported as an SSPE outcome.

For the rest of this subsection, we prove Theorem 3.3.1. First of all, we prove that, for all $(x, y) \in S \backslash \partial S$, the outcome $((x, y), 1)$ is not supported as an SSPE outcome. That is, if the negotiators reach an agreement by themselves in some SSPE, they always reach an agreement on ∂S.

Lemma 3.3.1. *For all $(x, y) \in S \backslash \partial S$, the outcome $((x, y), 1)$ is not supported as an SSPE outcome.*

Proof. Suppose that there exists an SSPE such that the negotiators reach an agreement $(x, y) \in S \backslash \partial S$ at period 1. Then, by the assumption on S, negotiators 1 and 2 can improve their payoffs by deviating from the SSPE and proposing $f^{-1}(y)(> x)$ and $f(x)(> y)$, respectively. This is a contradiction. Thus, $((x, y), 1)$ is not supported as an SSPE outcome. □

Therefore, for deriving SSPE agreement at odd periods, it is sufficient to focus on an agreement on ∂S. Next, we describe the mediator's strategy in the SSPEs where negotiators 1 and 2 demand some x and $f(x)$ at odd periods, respectively.

Lemma 3.3.2. *Suppose that, in some SSPE σ, negotiators 1 and 2 demand x and $f(x)$ at odd periods, respectively, and reach an agreement with $p(x) \in \partial S$. Then, in the path after negotiators 1 and 2 demand x' and y', respectively (where $(x', y') \notin S$), the mediator chooses the following action at even periods under σ. Now, we define $A(x, x', y') = [f^{-1}(y'), x'] \cap [\delta x, f^{-1}(\delta f(x))]$ (see Figure 3.8).*

1. *When $\alpha \in A(x, x', y')$, the mediator proposes $p(\alpha) \in \partial S$ (it is accepted by both negotiators).*

2. *When $A(x, x', y') \neq \emptyset$, $\alpha < \min A(x, x', y')$, and $\min A(x, x', y') \leq x$, the mediator proposes $p(\min A(x, x', y')) \in \partial S$ (it is accepted by both negotiators).*

3. *When $A(x, x', y') \neq \emptyset$, $\alpha < \min A(x, x', y')$, and $\min A(x, x', y') > x$, the mediator proposes $p(\min A(x, x', y')) \in \partial S$ if $u(p(\min A(x, x', y'))) > \delta u(p(x))$ (it is accepted by both negotiators), and chooses pass (or offers some proposal rejected by some negotiator) if $u(p(\min A(x, x', y'))) < \delta u(p(x))$. If $u(p(\min A(x, x', y'))) = \delta u(p(x))$, the mediator proposes $p(\min A(x, x', y'))$ or chooses pass (or offers some proposal rejected by some negotiator).*

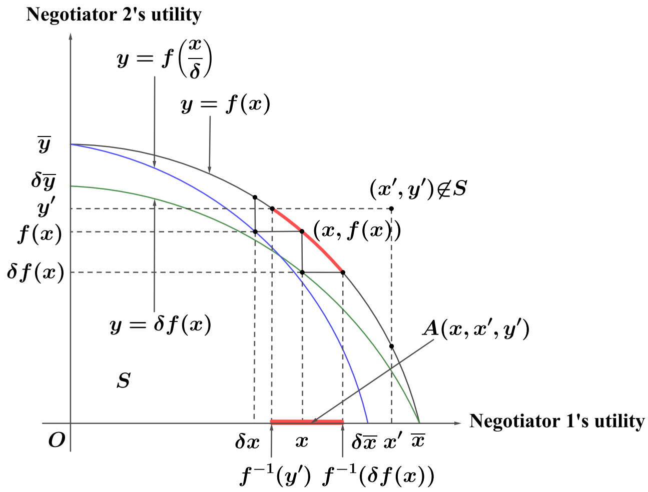

Figure 3.8: Example of $A(x, x', y')$

4. *When* $A(x, x', y') \neq \emptyset$, $\alpha > \max A(x, x', y')$, *and* $\max A(x, x', y') \geq x$, *the mediator proposes* $p(\max A(x, x', y')) \in \partial S$ *(it is accepted by both negotiators).*

5. *When* $A(x, x', y') \neq \emptyset$, $\alpha > \max A(x, x', y')$, *and* $\max A(x, x', y') < x$, *the mediator proposes* $p(\max A(x, x', y')) \in \partial S$ *if* $u(p(\max A(x, x', y'))) > \delta u(p(x))$ *(it is accepted by both negotiators), and chooses pass (or offers some proposal rejected by some negotiator) if* $u(p(\max A(x, x', y'))) < \delta u(p(x))$. *If* $u(p(\max A(x, x', y'))) = \delta u(p(x))$, *the mediator proposes* $p(\max A(x, x', y'))$ *or chooses pass (or offers some proposal rejected by some negotiator).*

6. *When* $A(x, x', y') = \emptyset$, *the mediator proposes some* $p(z) \in \partial S$ *satisfying* $z \in [f^{-1}(y'), x']$ *(it is rejected by some negotiator) or chooses pass.*

Proof. Without loss of generality, in the following proofs, we consider that the negotiators propose their demands at period t and the mediator proposes at period $t+1$. First of all, notice the following facts. By rejecting the mediator's proposal, negotiators 1 and 2 obtain payoffs $\delta^{t+1}x$ and $\delta^{t+1}f(x)$ at period $t+2$, respectively, under σ. Therefore, under σ, in the path after x' and y' are demanded by the negotiators, the mediator's proposal $p(z) \in \partial S$ is accepted by both negotiators if and only if $z \in A(x, x', y')$. Also, notice that, under σ, the mediator obtains $\delta^{t+1}u(p(x))$ at period $t+2$ by choosing pass (or by offering some proposal rejected by some negotiator) at period $t+1$. By the above facts, we prove each case.

1. When $\alpha \in A(x, x', y')$, the mediator's ideal agreement $p(\alpha)$ is accepted by both negotiators. Then, the mediator can obtain $\delta^t u(p(\alpha))$ $(\geq \delta^t u(p(x)) >$

$\delta^{t+1}u(p(x))$ by Assumption 3.2.1). Therefore, proposing $p(\alpha) \in \partial S$ is a best response to σ.

2. When $A(x, x', y') \neq \emptyset$, $\alpha < \min A(x, x', y')$, and $\min A(x, x', y') \leq x$, the most profitable proposal for the mediator in $p(A(x, x', y'))$ is $p(\min A(x, x', y')))$.[8] Then, she obtains $\delta^t u(p(\min A(x, x', y')))$ $(\geq \delta^t u(p(x)) > \delta^{t+1}u(p(x))$ by Assumption 3.2.1). Therefore, proposing $p(\min A(x, x', y'))) \in \partial S$ is a best response to σ.

3. When $A(x, x', y') \neq \emptyset$, $\alpha < \min A(x, x', y')$, and $\min A(x, x', y') > x$, the most profitable proposal for the mediator in $p(A(x, x', y'))$ is $p(\min A(x, x', y')))$. Then, she obtains $\delta^t u(p(\min A(x, x', y')))$. Therefore, under σ, the mediator proposes $p(\min A(x, x', y')) \in \partial S$ if $u(p(\min A(x, x', y'))) > \delta u(p(x))$, and chooses pass (or offers some proposal rejected by some negotiator) if $u(p(\min A(x, x', y'))) < \delta u(p(x))$. If $u(p(\min A(x, x', y'))) = \delta u(p(x))$, the mediator proposes $p(\min A(x, x', y'))$ or chooses pass (or offers some proposal rejected by some negotiator).

4. Since the proof of this case is analogous to the case 2, we omit it.

5. Since the proof of this case is analogous to the case 3, we omit it.

6. When $A(x, x', y') = \emptyset$, the mediator's proposal $p(z)$ such that $z \in [f^{-1}(y'), x']$ is rejected by some negotiator under σ. Then, by proposing some $p(z)$ or choosing pass, the mediator obtains $\delta^{t+1}u(p(x))$. Therefore, proposing some $p(z)$ satisfying $z \in [f^{-1}(y'), x']$ and choosing pass are best responses to σ.

□

By using Lemma 3.3.2, we derive all SSPE agreements at period 1. The analysis is divided into three cases, that is, when $\alpha \in [0, \delta x^R)$, when $\alpha \in [\delta x^R, x^R]$, and when $\alpha \in (x^R, \overline{x}]$. First, when $\alpha \in [0, \delta x^R)$, we obtain the following result.

Lemma 3.3.3. *When $\alpha \in [0, \delta x^R)$, the outcome $(p(x), 1)$ is supported as an SSPE outcome if and only if $x \in [\delta\alpha, x^R]$ (see Figure 3.5). In the SSPE, negotiators 1 and 2 demand x and $f(x)$, respectively, and the mediator follows the strategy described in Lemma 3.3.2.*

Proof. By Lemma 3.3.1, it is sufficient to consider the case where the negotiators reach an agreement on ∂S. We sequentially analyze five cases with respect to the value of x. Without loss of generality, we consider that the negotiators propose their demands at period t and the mediator proposes at period $t + 1$.

[8] For a function p and a set A, we define $p(A) = \{p(a) \mid a \in A\}$.

1. Suppose that, in some SSPE σ, negotiators 1 and 2 demand x and $f(x)$ such that $x \in [0, \delta\alpha)$, respectively. Consider the case that negotiator 1 deviates from σ and demands $f^{-1}(\delta f(x))$ (notice that $f^{-1}(\delta f(x)) > x$ by the facts that $f(x) > \delta f(x)$ and f^{-1} is strictly decreasing). Then, the game proceeds to the next period and the mediator proposes some $p(z)$ such that $z \in [x, f^{-1}(\delta f(x))]$ or chooses pass. Now, $A(x, f^{-1}(\delta f(x)), f(x)) = [x, f^{-1}(\delta f(x))] \cap [\delta x, f^{-1}(\delta f(x))] = [x, f^{-1}(\delta f(x))]$ $(\neq \emptyset)$.

   If $\alpha \leq f^{-1}(\delta f(x))$, since $x < \delta\alpha < \alpha$, the mediator proposes $p(\alpha)$ by the case 1 of Lemma 3.3.2, and negotiator 1 obtains $\delta^t \alpha$ $(> \delta^{t-1} x)$. If $\alpha > f^{-1}(\delta f(x))$, the mediator proposes $p(f^{-1}(\delta f(x)))$ by the case 4 of Lemma 3.3.2, and negotiator 1 obtains $\delta^t f^{-1}(\delta f(x))$. Now, since $x < \delta\alpha < \delta x^R$ and $\delta f(\delta x^R) = f(x^R)$, we find $\delta f(x) > \delta f(\delta x^R) = f(x^R)$. Thus, by the inequality (3.4), we obtain $\delta f^{-1}(\delta f(x)) > x$, that is, $\delta^t f^{-1}(\delta f(x)) > \delta^{t-1} x$.

   Therefore, negotiator 1 can improve her payoff by deviating from σ and demanding $f^{-1}(\delta f(x))$. This is a contradiction. Thus, the outcome $(p(x), 1)$ such that $x \in [0, \delta\alpha)$ is not supported as an SSPE outcome.

2. Suppose that, in some SSPE σ, negotiators 1 and 2 demand x and $f(x)$ such that $x \in [\delta\alpha, \alpha)$, respectively. Then, the mediator's proposals in σ are described in Lemma 3.3.2.

   Suppose that negotiator 1 deviates from σ and demands x^* such that $x^* < x$. Then, negotiator 1 obtains $\delta^{t-1} x^*$ $(< \delta^{t-1} x)$. Therefore, she cannot improve her payoff by demanding x^* $(< x)$. Also, suppose that negotiator 2 deviates from σ and demands y^* such that $y^* < f(x)$. Then, negotiator 2 obtains $\delta^{t-1} y^*$ $(< \delta^{t-1} f(x))$. Therefore, she also cannot improve her payoffs by demanding y^* $(< f(x))$.

   Next, suppose that negotiator 1 deviates from σ and demands x^{**} such that $x^{**} > x$. Then, the game proceeds to the next period and the mediator proposes some $p(z)$ such that $z \in [x, x^{**}]$ or chooses pass. Since $x < f^{-1}(\delta f(x))$, $A(x, x^{**}, f(x))$ can be transformed as $A(x, x^{**}, f(x)) = [x, x^{**}] \cap [\delta x, f^{-1}(\delta f(x))] = [x, \min\{x^{**}, f^{-1}(\delta f(x))\}]$ $(\neq \emptyset)$. If $\alpha \leq \min\{x^{**}, f^{-1}(\delta f(x))\}$, since $x < \alpha$, the mediator proposes $p(\alpha)$ by the case 1 of Lemma 3.3.2, and negotiator 1 obtains $\delta^t \alpha$ $(\leq \delta^{t-1} x)$. If $\alpha > \min\{x^{**}, f^{-1}(\delta f(x))\}$, the mediator proposes $p(\min\{x^{**}, f^{-1}(\delta f(x))\})$ by the case 4 of Lemma 3.3.2, and negotiator 1 obtains $\delta^t \min\{x^{**}, f^{-1}(\delta f(x))\}$ $(< \delta^t \alpha \leq \delta^{t-1} x)$. Therefore, negotiator 1 cannot improve her payoff by demanding x^{**} $(> x)$.

   Also, suppose that negotiator 2 deviates from σ and demands y^{**} such that $y^{**} > f(x)$. Then, the game proceeds to the next period and the mediator proposes some $p(z)$ such that $z \in [f^{-1}(y^{**}), x]$ or chooses pass. Now, since $x < f^{-1}(\delta f(x))$, $A(x, x, y^{**}) = [f^{-1}(y^{**}), x] \cap [\delta x, f^{-1}(\delta f(x))] = [\max\{f^{-1}(y^{**}), \delta x\}, x]$ $(\neq \emptyset)$. Therefore, since $x < \alpha$, the mediator proposes $p(x)$ by the case 4 of Lemma 3.3.2, and negotiator 2 obtains $\delta^t f(x)$ $(< \delta^{t-1} f(x))$. Thus, negotiator 2 cannot improve her payoff by demanding y^{**} $(> f(x))$.

Consequently, we can find that the SSPE σ is consistent. Therefore, the outcome $(p(x), 1)$ such that $x \in [\delta\alpha, \alpha)$ is supported as an SSPE outcome.

3. Suppose that, in some SSPE σ, negotiators 1 and 2 demand x and $f(x)$ such that $x \in [\alpha, f^{-1}(\delta f(\alpha))]$, respectively. Then, the mediator's proposals in σ are described in Lemma 3.3.2. By the same way as the case 2, we can find that negotiator 1 cannot improve her payoff by deviating from σ and demanding x^* $(< x)$. Also, negotiator 2 cannot improve her payoff by demanding y^* $(< f(x))$.

   Suppose that negotiator 1 deviates from σ and demands x^{**} such that $x^{**} > x$. Then, the game proceeds to the next period and the mediator proposes some $p(z)$ such that $z \in [x, x^{**}]$ or chooses pass. Since $x < f^{-1}(\delta f(x))$, $A(x, x^{**}, f(x))$ can be transformed as $A(x, x^{**}, f(x)) = [x, x^{**}] \cap [\delta x, f^{-1}(\delta f(x))] = [x, \min\{x^{**}, f^{-1}(\delta f(x))\}]$ $(\neq \emptyset)$. Then, since $x \geq \alpha$, the mediator proposes $p(x)$ by the case 2 (or 1) of Lemma 3.3.2, and negotiator 1 obtains $\delta^t x$ $(< \delta^{t-1} x)$. Thus, negotiator 1 cannot improve her payoff by demanding x^{**} $(> x)$.

   Also, suppose that negotiator 2 deviates from σ and demands y^{**} such that $y^{**} > f(x)$. Then, the game proceeds to the next period and the mediator proposes some $p(z)$ such that $z \in [f^{-1}(y^{**}), x]$ or chooses pass. Now, $A(x, x, y^{**}) = [f^{-1}(y^{**}), x] \cap [\delta x, f^{-1}(\delta f(x))] = [\max\{f^{-1}(y^{**}), \delta x\}, x]$ $(\neq \emptyset)$.

   If $\alpha \geq \max\{f^{-1}(y^{**}), \delta x\}$, since $\alpha \leq x$, the mediator proposes $p(\alpha)$ by the case 1 of Lemma 3.3.2, and negotiator 2 obtains $\delta^t f(\alpha)$. Since $x \leq f^{-1}(\delta f(\alpha))$, we obtain $\delta f(\alpha) \leq f(x)$, that is, $\delta^t f(\alpha) \leq \delta^{t-1} f(x)$. If $\alpha < \max\{f^{-1}(y^{**}), \delta x\}$, the mediator proposes $p(\max\{f^{-1}(y^{**}), \delta x\})$ by the case 2 of Lemma 3.3.2. Then, negotiator 2 obtains $\delta^t f(\max\{f^{-1}(y^{**}), \delta x\})$ $(< \delta^t f(\alpha))$. Since $x \leq f^{-1}(\delta f(\alpha))$, we obtain $\delta f(\alpha) \leq f(x)$. Therefore, $\delta^t f(\max\{f^{-1}(y^{**}), \delta x\}) < \delta^t f(\alpha) \leq \delta^{t-1} f(x)$ holds. By the above discussion, negotiator 2 cannot improve her payoff by demanding y^{**} $(> f(x))$.

   Consequently, we can find that the SSPE σ is consistent. Therefore, the outcome $(p(x), 1)$ such that $x \in [\alpha, f^{-1}(\delta f(\alpha))]$ is supported as an SSPE outcome.

4. Suppose that in some SSPE σ, negotiators 1 and 2 demand x and $f(x)$ such that $x \in (f^{-1}(\delta f(\alpha)), x^R]$, respectively. Then, the mediator's proposals in σ are described in Lemma 3.3.2. By the same way as the case 2, we can find that negotiator 1 cannot improve her payoff by deviating from σ and demanding x^* $(< x)$. Also, negotiator 2 cannot improve her payoff by demanding y^* $(< f(x))$.

   Suppose that negotiator 1 deviates from σ and demands x^{**} such that $x^{**} > x$. Then, the game proceeds to the next period and the mediator proposes some $p(z)$ such that $z \in [x, x^{**}]$ or chooses pass. Now, $A(x, x^{**}, f(x))$ can be transformed as $A(x, x^{**}, f(x)) = [x, x^{**}] \cap [\delta x, f^{-1}(\delta f(x))] = [x, \min\{x^{**}, f^{-1}(\delta f(x))\}]$ $(\neq \emptyset)$. Since $\alpha < f^{-1}(\delta f(\alpha))$, we obtain $x > f^{-1}(\delta f(\alpha)) > \alpha$. Therefore, the mediator proposes $p(x)$ by the case 2 of Lemma 3.3.2. Then, negotiator 1 obtains $\delta^t x$ $(< \delta^{t-1} x)$. Thus, negotiator 1 cannot improve her payoff by demanding x^{**} $(> x)$.

   Next, suppose that negotiator 2 deviates from σ and demands y^{**} such that $y^{**} > f(x)$. Then, the game proceeds to the next period and the mediator proposes some $p(z)$ such that $z \in [f^{-1}(y^{**}), x]$ or chooses pass. Now, $A(x, x, y^{**}) = [f^{-1}(y^{**}), x] \cap [\delta x, f^{-1}(\delta f(x))] = [\max\{f^{-1}(y^{**}), \delta x\}, x]$ $(\neq \emptyset)$.

   Since $\alpha < \delta x^R$, we obtain $\delta f(\alpha) > \delta f(\delta x^R) = f(x^R)$. By the inequality (3.4), $\delta f^{-1}(\delta f(\alpha)) > \alpha$ holds. Then, since $x > f^{-1}(\delta f(\alpha))$, we obtain $\delta x > \delta f^{-1}(\delta f(\alpha)) > \alpha$. Therefore, since $\alpha < \delta x \leq \max\{f^{-1}(y^{**}), \delta x\}$, the mediator proposes $p(\max\{f^{-1}(y^{**}), \delta x\})$ by the case 2 of Lemma 3.3.2, and negotiator 2 obtains $\delta^t f(\max\{f^{-1}(y^{**}), \delta x\})$. Since $\delta x \leq \delta x^R$, $\delta f(\delta x) \leq f(x)$ holds by the inequality (3.1). Then, we obtain $\delta f(\max\{f^{-1}(y^{**}), \delta x\}) \leq \delta f(\delta x) \leq f(x)$, that is, $\delta^t f(\max\{f^{-1}(y^{**}), \delta x\}) \leq \delta^{t-1} f(x)$. Therefore, negotiator 2 cannot improve her payoff by demanding y^{**} $(> f(x))$.

   Consequently, we can find that the SSPE σ is consistent. Therefore, the outcome $(p(x), 1)$ such that $x \in (f^{-1}(\delta f(\alpha)), x^R]$ is supported as an SSPE outcome.

5. Suppose that, in some SSPE σ, negotiators 1 and 2 demand x and $f(x)$ such that $x \in (x^R, \overline{x}]$, respectively. Consider the case that negotiator 2 deviates from σ and demands $f(\delta x)$. Then, the game proceeds to the next period and the mediator proposes some $p(z)$ such that $z \in [\delta x, x]$ or chooses pass. Now, $A(x, x, f(\delta x))$ can be transformed as $A(x, x, f(\delta x)) = [\delta x, x] \cap$

$[\delta x, f^{-1}(\delta f(x))] = [\delta x, x]$ ($\neq \emptyset$). Since $\alpha < \delta x^R < \delta x$, the mediator proposes $p(\delta x)$ by the case 2 of Lemma 3.3.2. Therefore, negotiator 2 obtains $\delta^t f(\delta x)$.

Since $\delta x > \delta x^R$, we obtain $\delta f(\delta x) > f(x)$ by the inequality (3.2). Then, $\delta^t f(\delta x) > \delta^{t-1} f(x)$. Therefore, negotiator 2 can improve her payoff by deviating from σ. This is a contradiction. Consequently, the outcome $(p(x), 1)$ such that $x \in (x^R, \overline{x}]$ is not supported as an SSPE outcome.

□

Roughly, Lemma 3.3.3 can be explained as follows. Consider the case where negotiators 1 and 2 demand x and $f(x)$, respectively, in some stationary strategy σ. When the negotiators deviate from σ, we can easily confirm that each negotiator cannot improve her payoff by proposing some demand smaller than the demand under σ. Therefore, it is sufficient to consider the case where the negotiators propose larger demands. Then, the game proceeds to the next period.

First, consider the case $x < \alpha$. In this case, even if negotiator 2 deviates from σ and proposes larger demand, the mediator proposes $p(x)$ at the next period. Therefore, negotiator 2 cannot improve her payoff by deviating from σ. Conversely, consider the case where negotiator 1 deviates from σ and proposes sufficiently large demand x^*. Then, the game proceeds to the next period and $\max A(x, x^*, f(x)) = f^{-1}(\delta f(x))$. Here, notice that the mediator's proposal which gives negotiator 1 a utility larger than $f^{-1}(\delta f(x))$ is rejected by negotiator 2. Therefore, the mediator proposes $p(\min\{f^{-1}(\delta f(x)), \alpha\})$ (notice that $\min A(x, x^*, f(x)) = x < \alpha$). When $x \in [0, \delta\alpha)$, reaching an agreement with the mediator's proposal $p(\min\{f^{-1}(\delta f(x)), \alpha\})$ at the next period is more profitable for negotiator 1 than reaching an agreement with $p(x)$ at the current period. Therefore, negotiator 1 deviates from σ, that is, σ is not an SSPE. When $x \in [\delta\alpha, \alpha)$, it is converse. Therefore, negotiator 1 does not deviate from σ, that is, σ is an SSPE.

The case $x \geq \alpha$ is similarly explained. In this case, negotiator 1 cannot improve her payoff by deviating from σ and proposing larger demands. When negotiator 2 deviates from σ and proposes sufficiently large demand y^*, the game proceeds to the next period and $\min A(x, x, y^*) = \delta x$. Then, the mediator proposes $p(\max\{\delta x, \alpha\})$ at the next period (notice that $\max A(x, x, y^*) = x \geq \alpha$). When $x \in [\alpha, x^R]$, negotiator 2 does not deviate from σ, that is, σ is an SSPE. When $x \in (x^R, \overline{x}]$, negotiator 2 deviates from σ, that is, σ is not an SSPE.

In the following, we analyze the cases of $\alpha \in [\delta x^R, x^R]$ and $\alpha \in (x^R, \overline{x}]$. Although the regions of SSPE agreements in these cases are different from the case of $\alpha \in [0, \delta x^R)$, the above discussion is similarly applied to these cases. Now, we analyze the case of $\alpha \in [\delta x^R, x^R]$.

Lemma 3.3.4. *When $\alpha \in [\delta x^R, x^R]$, the outcome $(p(x), 1)$ is supported as an SSPE outcome if and only if $x \in [\delta\alpha, f^{-1}(\delta f(\alpha))]$ (see Figure 3.6). In the SSPE, negotiators 1 and 2 demand x and $f(x)$, respectively, and the mediator follows the strategy described in Lemma 3.3.2.*

Proof. Without loss of generality, in the following proofs, we consider that the negotiators propose their demands at period t and the mediator proposes at period $t + 1$.

1. Suppose that, in some SSPE σ, negotiators 1 and 2 demand x and $f(x)$ such that $x \in [0, \delta\alpha)$, respectively. Consider the case that negotiator 1 deviates from σ and demands $f^{-1}(\delta f(x))$. By the same proof in the case 1 of Lemma 3.3.3, we can find that the outcome $(p(x), 1)$ such that $x \in [0, \delta\alpha)$ is not supported as an SSPE outcome.

2. Suppose that, in some SSPE σ, negotiators 1 and 2 demand x and $f(x)$ such that $x \in [\delta\alpha, \alpha)$, respectively. Then, the mediator's proposals in σ are described in Lemma 3.3.2. By the same proof in the case 2 of Lemma 3.3.3, we can find that the outcome $(p(x), 1)$ such that $x \in [\delta\alpha, \alpha)$ is supported as an SSPE outcome.

3. Suppose that, in some SSPE σ, negotiators 1 and 2 demand x and $f(x)$ such that $x \in [\alpha, f^{-1}(\delta f(\alpha))]$, respectively. Then, the mediator's proposals in σ are described in Lemma 3.3.2. By the same proof in the case 3 of Lemma 3.3.3, we can find that the outcome $(p(x), 1)$ such that $x \in [\alpha, f^{-1}(\delta f(\alpha))]$ is supported as an SSPE outcome.

4. Suppose that, in some SSPE σ, negotiators 1 and 2 demand x and $f(x)$ such that $x \in (f^{-1}(\delta f(\alpha)), \overline{x}]$, respectively. Consider the case that negotiator 2 deviates from σ and demands $f(\delta x)$. Then, the game proceeds to the next period and the mediator proposes some $p(z)$ such that $z \in [\delta x, x]$ or chooses pass. Now, $A(x, x, \delta x)$ can be transformed as $A(x, x, \delta x) = [\delta x, x] \cap [\delta x, f^{-1}(\delta f(x))] = [\delta x, x]$ $(\neq \emptyset)$.

   If $\alpha \geq \delta x$, since $\alpha < f^{-1}(\delta f(\alpha)) < x$, the mediator proposes $p(\alpha)$ by the case 1 of Lemma 3.3.2, and negotiator 2 obtains $\delta^t f(\alpha)$. Since $f^{-1}(\delta f(\alpha)) < x$, we obtain $\delta^t f(\alpha) > \delta^{t-1} f(x)$. If $\alpha < \delta x$, the mediator proposes $p(\delta x)$ by the case 2 of Lemma 3.3.2, and negotiator 2 obtains $\delta^t f(\delta x)$. Since $\delta x > \alpha \geq \delta x^R$, we obtain $\delta f(\delta x) > f(x)$ by the inequality (3.2). Thus, $\delta^t f(\delta x) > \delta^{t-1} f(x)$.

   Therefore, negotiator 2 can improve her payoff by deviating from σ and demanding $f(\delta x)$. This is a contradiction. Thus, the outcome $(p(x), 1)$ such that $x \in (f^{-1}(\delta f(\alpha)), \overline{x}]$ is not supported as an SSPE outcome.

□

Finally, we analyze the case of $\alpha \in (x^R, \overline{x}]$.

Lemma 3.3.5. *When $\alpha \in (x^R, \overline{x}]$, the outcome $(p(x), 1)$ is supported as an SSPE outcome if and only if $x \in [\delta x^R, f^{-1}(\delta f(\alpha))]$ (see Figure 3.7). In the SSPE, negotiators 1 and 2 demand x and $f(x)$, respectively, and the mediator follows the strategy described in Lemma 3.3.2.*

Proof. By exchanging the roles of negotiators 1 and 2 in Lemma 3.3.3, we can prove Lemma 3.3.5. □

Consequently, by summarizing Lemma 3.3.1, 3.3.3, 3.3.4, and 3.3.5, we obtain Theorem 3.3.1.

3.3.2 Agreements at even periods

Next, we derive SSPE outcomes such that the mediator's proposal is accepted at even periods. By the definition of SSPE, in such SSPE outcomes, the mediator's proposal is accepted at period 2. In this SSPE, an agreement is delayed. Regarding such SSPE outcomes, we obtain the following theorem.

Theorem 3.3.2. *The outcome $(p(x), 2)$ is supported as an SSPE outcome if and only if $x = \alpha$. (In the SSPE, negotiators 1 and 2 propose some $x' \in [f^{-1}(\delta f(\alpha)), \overline{x}]$ and $y' \in [f(\delta\alpha), \overline{y}]$, respectively, and the mediator follows the strategy described in Lemma 3.3.6. See Figure 3.9.)*

The SSPE of Theorem 3.3.2 can be interpreted as follows. In this SSPE, since the negotiators' demands are incompatible, they cannot reach an agreement by themselves. Thus, the mediator proposes a plan of an agreement to facilitate the reaching of an agreement. If some negotiator rejects the mediator's proposal, the negotiation breaks down and they cannot make a profit. Then, since accepting the mediator's proposal is better than disagreement for both negotiators, they decide to accept the mediator's proposal.

In the following, we prove Theorem 3.3.2. We first describe the mediator's strategy in the SSPE such that the mediator's proposal is accepted at even periods.

Lemma 3.3.6. *Suppose that the mediator's proposal $p(x)$ is accepted at even periods in some SSPE σ. Then, in the path after negotiators 1 and 2 demand x' and y', respectively (where $(x', y') \notin S$), the mediator chooses the following action at even periods under σ. Now, we define $B(x, x', y') = [f^{-1}(y'), x'] \cap [\delta^2 x, f^{-1}(\delta^2 f(x))]$.*

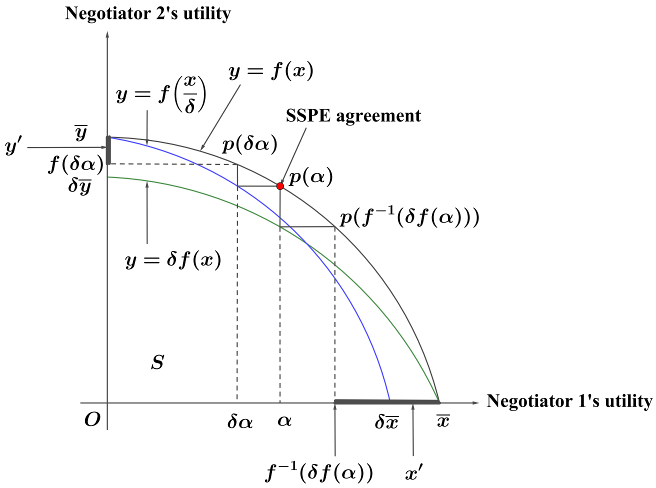

Figure 3.9: SSPE agreement at period 2

1. *When $\alpha \in B(x, x', y')$, the mediator proposes $p(\alpha) \in \partial S$ (it is accepted by both negotiators).*

2. *When $B(x, x', y') \neq \emptyset$, $\alpha < \min B(x, x', y')$, and $\min B(x, x', y') \leq x$, the mediator proposes $p(\min B(x, x', y')) \in \partial S$ (it is accepted by both negotiators).*

3. *When $B(x, x', y') \neq \emptyset$, $\alpha < \min B(x, x', y')$, and $\min B(x, x', y') > x$, the mediator proposes $p(\min B(x, x', y')) \in \partial S$ if $u(p(\min B(x, x', y'))) > \delta^2 u(p(x))$ (it is accepted by both negotiators), and chooses pass (or offers some proposal rejected by some negotiator) if $u(p(\min B(x, x', y'))) < \delta^2 u(p(x))$. If $u(p(\min B(x, x', y'))) = \delta^2 u(p(x))$, the mediator proposes $p(\min B(x, x', y'))$ or chooses pass (or offers some proposal rejected by some negotiator).*

4. *When $B(x, x', y') \neq \emptyset$, $\alpha > \max B(x, x', y')$, and $\max B(x, x', y') \geq x$, the mediator proposes $p(\max B(x, x', y')) \in \partial S$ (it is accepted by both negotiators).*

5. *When $B(x, x', y') \neq \emptyset$, $\alpha > \max B(x, x', y')$, and $\max B(x, x', y') < x$, the mediator proposes $p(\max B(x, x', y')) \in \partial S$ if $u(p(\max B(x, x', y'))) > \delta^2 u(p(x))$ (it is accepted by both negotiators), and chooses pass (or offers some proposal rejected by some negotiator) if $u(p(\max B(x, x', y'))) < \delta^2 u(p(x))$. If $u(p(\max B(x, x', y'))) = \delta^2 u(p(x))$, the mediator proposes $p(\max B(x, x', y'))$ or chooses pass (or offers some proposal rejected by some negotiator).*

6. *When $B(x, x', y') = \emptyset$, the mediator proposes some $p(z) \in \partial S$ satisfying $z \in [f^{-1}(y'), x']$ (it is rejected by some negotiator) or chooses pass.*

Proof. The proof is analogous to Lemma 3.3.2. □

By using Lemma 3.3.6, we prove Theorem 3.3.2.

Proof of Theorem 3.3.2

Proof. Suppose that σ is the SSPE such that negotiators 1 and 2 demand x' and y' ($(x', y') \notin S$), respectively, and the mediator's proposal $p(x)$ is accepted after x' and y' are demanded. Without loss of generality, in the following proofs, we consider that the negotiators propose their demands at period t and the mediator proposes at period $t + 1$.

1. Suppose $x = \alpha$. Since the mediator's proposal $p(\alpha)$ is accepted under σ, $\alpha \in B(\alpha, x', y')$ must hold by Lemma 3.3.6. Then, $f^{-1}(y') \leq \alpha \leq x'$, that is, $x' \geq \alpha$ and $y' \geq f(\alpha)$ must hold.

   Now, suppose $x' \in [\alpha, f^{-1}(\delta f(\alpha)))$. Consider the case that negotiator 2 deviates from σ and demands $f(x')$. Then, negotiator 2 obtains $\delta^{t-1} f(x')$ ($> \delta^t f(\alpha)$). Therefore, negotiator 2 can improve her payoff by deviating from σ and demanding $f(x')$. Also, suppose $y' \in [f(\alpha), f(\delta\alpha))$. Consider the case that negotiator 1 deviates from σ and demands $f^{-1}(y')$. Then, negotiator 1 obtains $\delta^{t-1} f^{-1}(y')$ ($> \delta^t \alpha$). Therefore, negotiator 1 can improve her payoff by deviating from σ and demanding $f^{-1}(y')$. Thus, x' and y' must satisfy $x' \in [f^{-1}(\delta f(\alpha)), \overline{x}]$ and $y' \in [f(\delta\alpha), \overline{y}]$, respectively.

   (a) We prove that negotiator 1 cannot improve her payoff by deviating from σ when $x' \in [f^{-1}(\delta f(\alpha)), \overline{x}]$ and $y' \in (f(\delta^2\alpha), \overline{y}]$.

   Consider the case that negotiator 1 deviates from σ and demands $x^* \in [0, f^{-1}(y')]$. Then, since $(x^*, y') \in S$, negotiator 1 obtains $\delta^{t-1} x^*$. Also, since $x^* \leq f^{-1}(y') < \delta^2\alpha < \delta\alpha$, we obtain $\delta^{t-1} x^* < \delta^t \alpha$. Therefore, negotiator 1 cannot improve her payoff by deviating from σ and demanding x^*.

   Consider the case that negotiator 1 deviates from σ and demands $x^{**} \in (f^{-1}(y'), \delta^2\alpha)$. Then, the game proceeds to the next period and the mediator proposes some $p(z)$ such that $z \in [f^{-1}(y'), x^{**}]$ or chooses pass. Now, since $x^{**} \in (f^{-1}(y'), \delta^2\alpha)$, $B(\alpha, x^{**}, y') = [f^{-1}(y'), x^{**}] \cap [\delta^2\alpha, f^{-1}(\delta^2 f(\alpha))] = \emptyset$. Therefore, the mediator's proposal is not accepted and the game proceeds to the next period. Then, negotiator 1 obtains $\delta^{t+2}\alpha$ at period $t + 3$ under σ. Since $\delta^{t+2}\alpha < \delta^t\alpha$, negotiator 1 cannot improve her payoff by deviating from σ and demanding x^{**}.

Consider the case that negotiator 1 deviates from σ and demands $x^{***} \in [\delta^2\alpha, \overline{x}]$. Then, the game proceeds to the next period and the mediator proposes some $p(z)$ such that $z \in [f^{-1}(y'), x^{***}]$ or chooses pass. Now, since $x^{***} \in [\delta^2\alpha, \overline{x}]$ and $y' \in (f(\delta^2\alpha), \overline{y}]$, $B(\alpha, x^{***}, y') = [f^{-1}(y'), x^{***}] \cap [\delta^2\alpha, f^{-1}(\delta^2 f(\alpha))] = [\delta^2\alpha, \min\{x^{***}, f^{-1}(\delta^2 f(\alpha))\}]$ $(\neq \emptyset)$. If $\alpha \leq \min\{x^{***}, f^{-1}(\delta^2 f(\alpha))\}$, the mediator proposes $p(\alpha)$ by the case 1 of Lemma 3.3.6, and negotiator 1 obtains $\delta^t\alpha$. If $\alpha > \min\{x^{***}, f^{-1}(\delta^2 f(\alpha))\}$, the mediator proposes $p(\min\{x^{***}, f^{-1}(\delta^2 f(\alpha))\})$ or chooses pass by the case 5 of Lemma 3.3.6. Then, negotiator 1 obtains $\delta^t \min\{x^{***}, f^{-1}(\delta^2 f(\alpha))\}$ at period $t+1$ or $\delta^{t+2}\alpha$ at period $t+3$. Now, $\delta^t \min\{x^{***}, f^{-1}(\delta^2 f(\alpha))\} < \delta^t\alpha$ and $\delta^{t+2}\alpha < \delta^t\alpha$ hold. Therefore, negotiator 1 cannot improve her payoff by deviating from σ and demanding x^{***}.

By the above discussion, we can find that negotiator 1 cannot improve her payoff by deviating from σ when $x' \in [f^{-1}(\delta f(\alpha)), \overline{x}]$ and $y' \in (f(\delta^2\alpha), \overline{y}]$.

(b) We prove that negotiator 1 cannot improve her payoff by deviating from σ when $x' \in [f^{-1}(\delta f(\alpha)), \overline{x}]$ and $y' \in [f(\delta\alpha), f(\delta^2\alpha)]$.

Consider the case that negotiator 1 deviates from σ and demands $x^* \in [0, f^{-1}(y')]$. Then, since $(x^*, y') \in S$, negotiator 1 obtains $\delta^{t-1}x^*$. Also, since $x^* \leq f^{-1}(y') \leq \delta\alpha$, we obtain $\delta^{t-1}x^* \leq \delta^t\alpha$. Therefore, negotiator 1 cannot improve her payoff by deviating from σ and demanding x^*.

Consider the case that negotiator 1 deviates from σ and demands $x^{**} \in (f^{-1}(y'), \overline{x}]$. Then, the game proceeds to the next period and the mediator proposes some $p(z)$ such that $z \in [f^{-1}(y'), x^{**}]$ or chooses pass. Now, since $y' \in [f(\delta\alpha), f(\delta^2\alpha)]$, we obtain $f(\delta^2\alpha) \geq y' \geq f(\delta\alpha) > \delta^2 f(\alpha)$. Therefore, $B(\alpha, x^{**}, y') = [f^{-1}(y'), x^{**}] \cap [\delta^2\alpha, f^{-1}(\delta^2 f(\alpha))] = [f^{-1}(y'), \min\{x^{**}, f^{-1}(\delta^2 f(\alpha))\}]$ $(\neq \emptyset)$. If $\alpha \leq \min\{x^{**}, f^{-1}(\delta^2 f(\alpha))\}$, since $f^{-1}(y') \leq \delta\alpha < \alpha$, the mediator proposes $p(\alpha)$ by the case 1 of Lemma 3.3.6, and negotiator 1 obtains $\delta^t\alpha$. If $\alpha > \min\{x^{**}, f^{-1}(\delta^2 f(\alpha))\}$, the mediator proposes $p(\min\{x^{**}, f^{-1}(\delta^2 f(\alpha))\})$ or chooses pass by the case 5 of Lemma 3.3.6. Then, negotiator 1 obtains $\delta^t \min\{x^{**}, f^{-1}(\delta^2 f(\alpha))\}$ at period $t+1$ or $\delta^{t+2}\alpha$ at period $t+3$. Now, $\delta^t \min\{x^{**}, f^{-1}(\delta^2 f(\alpha))\} < \delta^t\alpha$ and $\delta^{t+2}\alpha < \delta^t\alpha$ hold. Therefore, negotiator 1 cannot improve her payoff by deviating from σ and demanding x^{**}.

By the above discussion, we can find that negotiator 1 cannot improve her payoff by deviating from σ when $x' \in [f^{-1}(\delta f(\alpha)), \overline{x}]$ and $y' \in [f(\delta\alpha), f(\delta^2\alpha)]$.

Summarizing (a) and (b), we can find that negotiator 1 cannot improve her payoff by deviating from σ when $x' \in [f^{-1}(\delta f(\alpha)), \overline{x}]$ and $y' \in [f(\delta\alpha), \overline{y}]$. By exchanging the roles of negotiators 1 and 2 in the above proofs, we can also prove that negotiator 2 cannot improve her payoff by deviating from σ when $x' \in [f^{-1}(\delta f(\alpha)), \overline{x}]$ and $y' \in [f(\delta\alpha), \overline{y}]$. Therefore, the strategy profile where negotiators 1 and 2 propose x' and y' such that $x' \in [f^{-1}(\delta f(\alpha)), \overline{x}]$ and $y' \in [f(\delta\alpha), \overline{y}]$, respectively, and the mediator follows the strategy described in Lemma 3.3.6 is an SSPE. Under this SSPE, after x' and y' are demanded by the negotiators, the mediator proposes $p(\alpha)$ and it is accepted.

2. Suppose $x \in [0, \alpha)$. That is, in the SSPE σ, the mediator's proposal $p(x)$ such that $x < \alpha$ is accepted. Then, $x \in B(x, x', y')$ must hold. Now, we prove $x = \max B(x, x', y')$. Suppose $x < \max B(x, x', y')$. Then, if $\alpha > \max B(x, x', y')$, the mediator proposes $p(\max B(x, x', y'))$ under σ by the case 4 of Lemma 3.3.6. If $\alpha \leq \max B(x, x', y')$, since $\alpha > x \geq \min B(x, x', y')$, the mediator proposes $p(\alpha)$ under σ by the case 1 of Lemma 3.3.6. This contradicts to the fact that the mediator proposes $p(x)$ such that $x < \max B(x, x', y')$ and $x < \alpha$ under σ. Therefore, x must satisfy $x = \max B(x, x', y')$ $(< \alpha)$. Then, since $B(x, x', y') = [f^{-1}(y'), x'] \cap [\delta^2 x, f^{-1}(\delta^2 f(x))]$ and $\delta^2 x < x$ $(= \max B(x, x', y')) < f^{-1}(\delta^2 f(x))$, x' must satisfy $x' = \max B(x, x', y') = x$.

   Consider the case that negotiator 1 deviates from σ and demands $f^{-1}(\delta^2 f(x))$ $(> x = x')$. Then, the game proceeds to the next period and the mediator proposes some $p(z)$ such that $z \in [f^{-1}(y'), f^{-1}(\delta^2 f(x))]$ or chooses pass. Now, $B(x, f^{-1}(\delta^2 f(x)), y')$ can be transformed as $B(x, f^{-1}(\delta^2 f(x)), y') = [f^{-1}(y'), f^{-1}(\delta^2 f(x))] \cap [\delta^2 x, f^{-1}(\delta^2 f(x))] = [\max\{f^{-1}(y'), \delta^2 x\}, f^{-1}(\delta^2 f(x))]$ $(\neq \emptyset)$.

   If $\alpha \leq f^{-1}(\delta^2 f(x))$, since $f^{-1}(y') < x' = x < \alpha$ and $\delta^2 x < x < \alpha$ hold, the mediator proposes $p(\alpha)$ by the case 1 of Lemma 3.3.6, and negotiator 1 obtains $\delta^t \alpha$ $(> \delta^t x)$. If $\alpha > f^{-1}(\delta^2 f(x))$, the mediator proposes $p(f^{-1}(\delta^2 f(x)))$ by the case 4 of Lemma 3.3.6, and negotiator 1 obtains $\delta^t f^{-1}(\delta^2 f(x))$ $(> \delta^t x)$. Therefore, negotiator 1 can improve her payoff by deviating from σ and demanding $f^{-1}(\delta^2 f(x))$. This is a contradiction. Thus, the outcome $(p(x), 2)$ such that $x \in [0, \alpha)$ is not supported as an SSPE outcome.

3. Suppose $x \in (\alpha, \overline{x}]$. By the proof analogous to the case 2, we can find that the outcome $(p(x), 2)$ such that $x \in (\alpha, \overline{x}]$ is not supported as an SSPE outcome.

□

3.3.3 Disagreement

In this subsection, we analyze disagreement and obtain the following theorem.

Theorem 3.3.3. *Disagreement is not supported as an SSPE outcome.*

Proof. Suppose that disagreement occurs in some SSPE σ. Then, negotiators 1, 2, and the mediator obtain payoffs of zero. Since disagreement occurs, in the SSPE σ, negotiators 1 and 2 propose x' and y' such that $(x', y') \notin S$, respectively. After x' and y' are demanded, the mediator proposes some $p(z)$ such that $z \in [f^{-1}(y'), x']$ or chooses pass. Then, there exists some $z' \in [f^{-1}(y'), x']$ such that $z' > 0$ and $f(z') > 0$. Therefore, the proposal $p(z')$ is accepted by the negotiators. Now, $u(p(z')) > 0$ by $u(0, f(0)) \geq 0$, $u(\overline{x}, f(\overline{x})) \geq 0$, and Assumption 3.2.1. Thus, the mediator can obtain a payoff larger than zero by deviating from σ and proposing $p(z')$. This is a contradiction. Hence, disagreement is not supported as an SSPE outcome. □

Theorem 3.3.3 implies that the mediator can resolve conflict. As the same as Theorem 3.3.2, this result is caused by the fact that accepting the mediator's proposal is better than disagreement for both negotiators.

3.3.4 Agreement with the NBS

Now, we derived all SSPE outcomes. In this subsection, we analyze properties of an agreement with the NBS by using the aforementioned results. Let $p(x^N) = (x^N, f(x^N))$ be the NBS of the bargaining problem (S, d). In subsection 3.3.1, we saw that an agreement with the NBS at period 1 is always supported as an SSPE outcome. Now, we can derive a stronger result that an agreement with the NBS at period 1 is the "unique" outcome which is supported as an SSPE outcome for all $\delta \in (0, 1)$ and for all $\alpha \in [0, \overline{x}]$. To derive this result, first, we show the following proposition by Theorem 3.3.1.

Proposition 3.3.1. *An agreement with the NBS is the unique agreement which is supported as an SSPE agreement at period 1 for all $\delta \in (0, 1)$ and for all $\alpha \in [0, \overline{x}]$.*

Proof. First, notice that, since $p(\delta x^R(\delta))$ and $p(x^R(\delta))$ are the negotiators' SPE offers in the Rubinstein's alternating-offers model, $\lim_{\delta \uparrow 1} \delta x^R(\delta) = x^N$ and $\lim_{\delta \uparrow 1} x^R(\delta) = x^N$ hold (for example, see Binmore et al. (1986) and Osborne and Rubinstein (1994)).

Suppose $\alpha < x^N$. Since $\lim_{\delta \uparrow 1} \delta x^R(\delta) = x^N$, there exists some δ' such that $\alpha < \delta x^R(\delta)$ holds for all $\delta \in (\delta', 1)$. Then, by the case $\alpha \in [0, \delta x^R(\delta))$ of Theorem 3.3.1, for $\delta \in (\delta', 1)$, an agreement with $p(x)$ is an SSPE agreement if and only if $x \in [\delta\alpha, x^R(\delta)]$. Now, since $\lim_{\delta\uparrow 1} x^R(\delta) = x^N$, for $x > x^N$, there exists some $\delta^* \in (\delta', 1)$ such that $x > x^R(\delta^*)$. Therefore, an agreement with $p(x)$ such that $x > x^N$ is not supported as an SSPE agreement for some $\alpha \in [0, x^N)$ and $\delta^* \in (\delta', 1)$. This implies that, if $p(x)$ is supported as an SSPE agreement at period 1 for all $\delta \in (0,1)$ and $\alpha \in [0, \overline{x}]$, x must satisfy $x \leq x^N$. Conversely, suppose $\alpha > x^N$. By the proof analogous to the case of $\alpha < x^N$, we can prove that, if $p(x)$ is supported as an SSPE agreement at period 1 for all $\delta \in (0,1)$ and $\alpha \in [0, \overline{x}]$, x must satisfy $x \geq x^N$ (by the case $\alpha \in (x^R(\delta), \overline{x}]$ of Theorem 3.3.1). Therefore, if $p(x)$ is supported as an SSPE agreement at period 1 for all $\delta \in (0,1)$ and $\alpha \in [0, \overline{x}]$, x must satisfy $x = x^N$. Then, since $p(x^N)$ is an SSPE agreement at period 1 for all $\delta \in (0,1)$ and $\alpha \in [0, \overline{x}]$, we obtain Proposition 3.3.1. □

By Theorem 3.3.2, we can see that there is no agreement which is supported as an SSPE agreement at period 2 for all $\alpha \in [0, \overline{x}]$. Therefore, by combining Theorem 3.3.2, 3.3.3, and Proposition 3.3.1, we immediately obtain the following result.

Theorem 3.3.4. *The outcome $(p(x^N), 1)$ is the unique outcome which is supported as an SSPE outcome for all $\delta \in (0,1)$ and for all $\alpha \in [0, \overline{x}]$.*

Even if the mediator is biased, the NBS can always be achieved in SSPE. Now, the question is: Is there any other agreement which can always be achieved in SSPE? The result of Theorem 3.3.4 denies the existence of such an agreement. That is, an agreement other than the NBS may be eliminated from SSPE agreement. In contrast, the reasonable agreement in the sense of the NBS is the unique agreement which is always supported as an SSPE agreement.

Next, we consider the case where δ approaches to one. First, as a corollary of Theorem 3.3.1, 3.3.2, and 3.3.3, we obtain the following result. Notice that $\lim_{\delta\uparrow 1} \delta\alpha = \lim_{\delta\uparrow 1} f^{-1}(\delta f(\alpha)) = \alpha$ and $\lim_{\delta\uparrow 1} \delta x^R(\delta) = \lim_{\delta\uparrow 1} x^R(\delta) = x^N$.

Corollary 3.3.1. *The outcome $(p(x), 1)$ is supported as an SSPE outcome under $\delta \uparrow 1$ if and only if*

1. *$x \in [\alpha, x^N]$ when $\alpha \in [0, x^N)$,*
2. *$x = x^N$ when $\alpha = x^N$, and*
3. *$x \in [x^N, \alpha]$ when $\alpha \in (x^N, \overline{x}]$.*

The outcome $(p(x), 2)$ is supported as an SSPE outcome under $\delta \uparrow 1$ if and only if $x = \alpha$. Also, disagreement is not supported as an SSPE outcome.

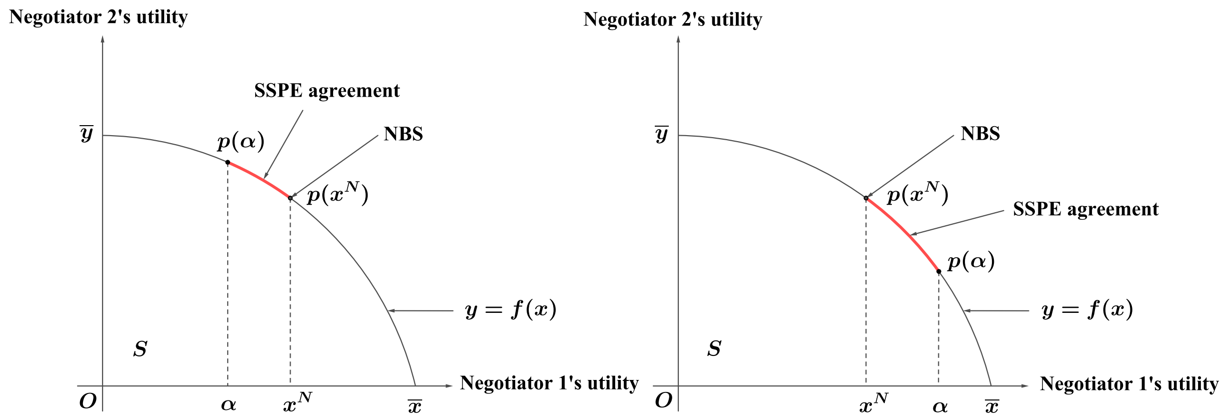

Figure 3.10: SSPE agreement when $\delta \uparrow 1$

This result shows that, when δ approaches to one, an agreement with $p(x)$ is an SSPE agreement if and only if $p(x)$ lies between the NBS $p(x^N)$ and the mediator's ideal agreement $p(\alpha)$ (see Figure 3.10). That is, as the mediator's ideal agreement approaches to the NBS, the set of SSPE agreements shrinks. Therefore, when the mediator is sufficiently fair, the agreement achieved in SSPE is sufficiently close to the NBS. Especially, when the mediator wishes to achieve the NBS, we obtain the following desirable result.

Theorem 3.3.5. *When $p(\alpha) = p(x^N)$, SSPE outcomes under $\delta \uparrow 1$ are $(p(x^N), 1)$ and $(p(x^N), 2)$. Therefore, when $\delta \uparrow 1$, the NBS is the unique agreement achieved in SSPEs.*

This result shows that, when the mediator wishes to achieve the NBS, the reasonable agreement (the NBS) is "surely" achieved in SSPEs under $\delta \uparrow 1$. That is, the fair mediator facilitates the reaching of the reasonable agreement.

3.4 Comparison with simultaneous-offers bargaining without a mediator and with an arbitrator

In this section, we compare the simultaneous-offers bargaining model with a mediator (defined in section 3.2) with bargaining models without a mediator and with an arbitrator. By comparison, we analyze how a mediator affects bargaining outcomes. In the following subsections, we use the same notation as section 3.2.

3.4.1 Comparison with a model without a mediator

First, we compare the model with a mediator with a model without a mediator. The model without a mediator is as follows. The game starts from period 1. At period t, negotiators 1 and 2 simultaneously propose their demands $x \in [0, \overline{x}]$ and $y \in [0, \overline{y}]$, respectively. If $(x, y) \in S$, then the game ends and negotiators 1 and 2 receive $\delta^{t-1}x$ and $\delta^{t-1}y$, respectively. If $(x, y) \notin S$, the game proceeds to the next period $t+1$ and repeat the above process. The game continues until an agreement is reached. We derive SSPE outcomes of this model, that is, derive outcomes induced by the SPE such that each negotiator's demand is always the same value. We obtain the following result.

Proposition 3.4.1. *In the model without a mediator, for all $x \in [0, \overline{x}]$, an agreement with $p(x)$ is supported as an SSPE outcome (see Figure 3.11) and disagreement is supported as an SSPE outcome.*

Proof. Without loss of generality, in the following proof, we consider that the negotiators propose their demands at period t.

Consider the stationary strategy profile $\sigma(x)$ where negotiators 1 and 2 demand x and $f(x)$, respectively. Then, negotiators 1 and 2 receive payoffs $\delta^{t-1}x$ and $\delta^{t-1}f(x)$, respectively. If negotiator 1 deviates from $\sigma(x)$ and demands x^* such that $x^* < x$, she obtains $\delta^{t-1}x^*$ ($< \delta^{t-1}x$). If negotiator 1 deviates from $\sigma(x)$ and demands x^{**} such that $x^{**} > x$, she obtains $\delta^t x$ ($< \delta^{t-1}x$) at the next period. Therefore, negotiator 1 cannot improve her payoff by deviating from $\sigma(x)$. Also, negotiator 2 cannot improve her payoff by deviating from $\sigma(x)$. Consequently, for all $x \in [0, \overline{x}]$, an agreement with $p(x)$ is supported as an SSPE outcome.

Next, consider the stationary strategy profile σ^d where negotiators 1 and 2 demand $\overline{x}$ and $f(0)$, respectively. Then, disagreement occurs and each negotiator receives a payoff of zero. Even if negotiator 1 deviates from σ^d and demands $x^* \in [0, \overline{x})$, she obtains a payoff of zero. Therefore, negotiator 1 cannot improve her payoff by deviating from σ^d. Also, negotiator 2 cannot improve her payoff by deviating from σ^d. Consequently, disagreement is supported as an SSPE outcome. □

Disagreement is supported as an SSPE outcome in the model without a mediator, but it does not appear as an SSPE outcome in the model with a mediator. These results imply that the mediator has the power to resolve conflict. Also, in the model without a mediator, since all agreements on the Pareto frontier ∂S can be achieved as an SSPE agreement, an unfair agreement may be achieved. In contrast to it, in the model with a mediator, when the mediator is sufficiently fair, the agreement achieved in SSPE is sufficiently close to the reasonable agreement (the NBS).

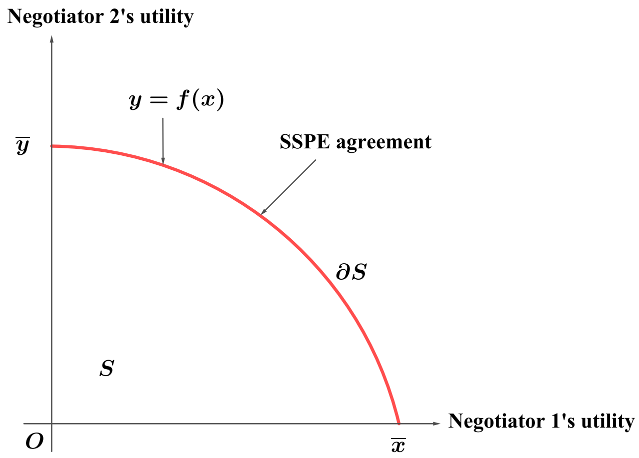

Figure 3.11: SSPE agreement in the model without a mediator

3.4.2 Comparison with a model with an arbitrator

Next, we compare the model with a mediator with a model with an arbitrator. The role of an arbitrator is to impose some agreement when negotiators cannot reach an agreement by themselves. The model with an arbitrator defined in this subsection has finite periods as with the models of Crawford (1979) and Rong (2012). This is in contrast to the model with a mediator in section 3.2, which has infinite periods. In this sense, the model with a mediator in section 3.2 and the model with an arbitrator in this subsection are not very comparable. Therefore, in order to clarify what factors cause differences in equilibrium outcomes of these models, we also define a two-period model with a mediator and compare equilibrium outcomes in the infinite-period model with a mediator, a two-period model with an arbitrator, and a two-period model with a mediator.

First, we define a two-period model with an arbitrator. The model is as follows. At period 1, negotiators 1 and 2 simultaneously propose their demands $x \in [0, \overline{x}]$ and $y \in [0, \overline{y}]$, respectively. If $(x, y) \in S$, then the game ends and negotiators 1 and 2 receive x and y, respectively. If $(x, y) \notin S$, the game proceeds to period 2. At period 2, the arbitrator imposes some $p(z)$ such that $z \in [f^{-1}(y), x]$ as an outcome of the bargaining. When the arbitrator imposes $p(z)$, negotiators 1, 2, and the arbitrator receive δz, $\delta f(z)$, and $\delta u(p(z))$, respectively, where $u : S \to \mathbb{R}_+$ is the arbitrator's utility function satisfying Assumption 3.2.1. In this model, since SSPE cannot be defined, we derive SPE. Then, we obtain the following result.

Proposition 3.4.2. *Consider the two-period model with an arbitrator. The outcome that the negotiators reach an agreement with $p(x) \in \partial S$ at period 1 is supported as* *an SPE outcome if and only if* $x \in [\delta\alpha, f^{-1}(\delta f(\alpha))]$. *Also, the outcome that the arbitrator imposes* $p(x) \in \partial S$ *at period 2 is supported as an SPE outcome if and only if* $x = \alpha$. *(In the SPE where the arbitrator imposes* $p(\alpha) \in \partial S$ *at period 2, negotiators 1 and 2 propose* $x' \in [f^{-1}(\delta f(\alpha)), \overline{x}]$ *and* $y' \in [f(\delta\alpha), \overline{y}]$*, respectively. See Figure 3.12.)*

Proof. First, notice that, in all SPEs, if negotiators 1 and 2 demand x' and y' such that $(x', y') \notin S$, respectively, the arbitrator imposes $p(x')$ if $\alpha > \max[f^{-1}(y'), x']$ $(= x')$, imposes $p(f^{-1}(y'))$ if $\alpha < \min[f^{-1}(y'), x']$ $(= f^{-1}(y'))$, and imposes $p(\alpha)$ if $\alpha \in [f^{-1}(y'), x']$.

Suppose that, in some SPE σ_1, the negotiators reach an agreement with $p(x)$ at period 1 (as with Lemma 3.3.1, it is sufficient to consider the case where the negotiators reach an agreement on ∂S). Then, negotiators 1 and 2 receive x and $f(x)$, respectively. If $x < \delta\alpha$, since negotiator 1 obtains $\delta\alpha$ by deviating from σ_1 and demanding α, she can improve her payoff. Similarly, if $x > f^{-1}(\delta f(\alpha))$, negotiator 2 can improve her payoff by demanding $f(\alpha)$. Therefore, x must satisfy $x \in [\delta\alpha, f^{-1}(\delta f(\alpha))]$.

When $x \in [\delta\alpha, \alpha]$, consider the case where negotiator 1 deviates from σ_1. Then, she obtains x^* by demanding $x^* \in [0, x)$ and obtains $\delta \min\{x^{**}, \alpha\}$ by demanding $x^{**} \in (x, \overline{x}]$. Since $x^* < x$ and $\delta \min\{x^{**}, \alpha\} \leq \delta\alpha \leq x$, negotiator 1 cannot improve her payoff by deviating from σ_1. Also, when $x \in [\delta\alpha, \alpha]$, consider the case where negotiator 2 deviates from σ_1. Then, she obtains y^* by demanding $y^* \in [0, f(x))$ and obtains $\delta f(x)$ by demanding $y^{**} \in (f(x), \overline{y}]$. Since $y^* < f(x)$ and $\delta f(x) < f(x)$, negotiator 2 cannot improve her payoff by deviating from σ_1. Therefore, the outcome that the negotiators reach an agreement with $p(x)$ such that $x \in [\delta\alpha, \alpha]$ at period 1 is supported as an SPE outcome. Similarly, the outcome that the negotiators reach an agreement with $p(x)$ such that $x \in (\alpha, f^{-1}(\delta f(\alpha)]$ at period 1 is supported as an SPE outcome.

Next, suppose that, in some SPE σ_2, the arbitrator imposes $p(x)$ at period 2 after negotiators 1 and 2 demand x' and y' $((x', y') \notin S)$, respectively. Then, x must satisfy $x = x'$ if $\alpha > \max[f^{-1}(y'), x']$ $(= x')$, x must satisfy $x = f^{-1}(y')$ if $\alpha < \min[f^{-1}(y'), x']$ $(= f^{-1}(y'))$, and x must satisfy $x = \alpha$ if $\alpha \in [f^{-1}(y'), x']$. If $\alpha > x'$, since negotiator 1 obtains $\delta\alpha$ $(> \delta x')$ by deviating from σ_2 and demanding α, she can improve her payoff. If $\alpha < f^{-1}(y')$, since negotiator 2 obtains $\delta f(\alpha)$ $(> \delta y')$ by deviating from σ_2 and demanding $f(\alpha)$, she can improve her payoff. Therefore, $\alpha \in [f^{-1}(y'), x']$, that is, $x' \geq \alpha$ and $y' \geq f(\alpha)$ must hold, and x must satisfy $x = \alpha$. Then, under σ_2, negotiators 1 and 2 receive $\delta\alpha$ and $\delta f(\alpha)$, respectively.

If $y' \in [f(\alpha), f(\delta\alpha))$, since negotiator 1 obtains $f^{-1}(y')$ $(> \delta\alpha)$ by deviating from

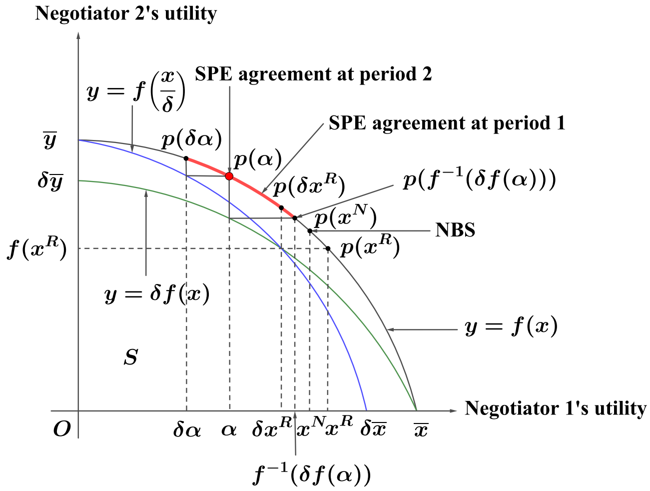

Figure 3.12: SPE agreement in the two-period models with an arbitrator and with a mediator

σ_2 and demanding $f^{-1}(y')$, she can improve her payoff. If $x' \in [\alpha, f^{-1}(\delta f(\alpha)))$, since negotiator 2 obtains $f(x')$ $(> \delta f(\alpha))$ by deviating from σ_2 and demanding $f(x')$, she can improve her payoff. Therefore, x' must satisfy $x' \in [f^{-1}(\delta f(\alpha)), \overline{x}]$ and y' must satisfy $y' \in [f(\delta\alpha), \overline{y}]$.

Finally, we prove that, when $x' \in [f^{-1}(\delta f(\alpha)), \overline{x}]$ and $y' \in [f(\delta\alpha), \overline{y}]$, each negotiator cannot improve her payoff by deviating from σ_2. Consider the case where negotiator 1 deviates from σ_2. Then, she obtains x^* by demanding $x^* \in [0, f^{-1}(y')]$ and obtains $\delta \min\{x^{**}, \alpha\}$ by demanding $x^{**} \in (f^{-1}(y'), \overline{x}]$. Since $x^* \leq f^{-1}(y') \leq \delta\alpha$ and $\delta \min\{x^{**}, \alpha\} \leq \delta\alpha$, negotiator 1 cannot improve her payoff by deviating from σ_2. Similarly, negotiator 2 cannot improve her payoff by deviating from σ_2. Therefore, it is consistent to the fact that σ_2 is an SPE. Thus, we obtain Proposition 3.4.2. □

Proposition 3.4.2 shows that the SPE outcomes in the model with an arbitrator strongly depend on what agreement the arbitrator wishes to impose. Especially, when $\delta \uparrow 1$, $p(\alpha)$ is the unique SPE agreement. That is, the arbitrator's ideal agreement is achieved in SPEs. This result is caused by the fact that the arbitrator has the authority to decide a final bargaining outcome. Therefore, if the arbitrator is biased, the NBS is eliminated from SPE agreement. Although the way of arbitration in the models of Crawford (1979) and Rong (2012) is different from the above model (they use the final-offer arbitration), a reasonable agreement is similarly eliminated from equilibrium (especially when the discount factor is sufficiently large).

Next, we define a two-period model with a mediator. This model is the same as the model with an arbitrator, except that an arbitrator is replaced with a mediator. The model is as follows. At period 1, negotiators 1 and 2 simultaneously propose their demands $x \in [0, \overline{x}]$ and $y \in [0, \overline{y}]$, respectively. If $(x, y) \in S$, then the game ends and negotiators 1 and 2 receive x and y, respectively. If $(x, y) \notin S$, the game proceeds to period 2. At period 2, the mediator proposes some $p(z) \in \partial S$ such that $z \in [f^{-1}(y), x]$ or chooses pass. If the mediator chooses pass, then the game ends with the outcome of disagreement and negotiators 1, 2, and the mediator receive payoffs of zero. If the mediator proposes some $p(z)$, then the negotiators simultaneously decide whether to accept the mediator's proposal or reject it. If both negotiators accept it, the game ends and negotiators 1, 2, and the mediator receive δz, $\delta f(z)$, and $\delta u(p(z))$, respectively, where $u : S \to \mathbb{R}_+$ is the mediator's utility function satisfying Assumption 3.2.1. If some negotiator rejects it, the game ends with the outcome of disagreement and negotiators 1, 2, and the mediator receive payoffs of zero. We assume that, when some negotiator responses to the mediator's proposal, she accepts it if the mediator's proposal is not less profitable than rejecting it. We derive SPE as with the two-period model with an arbitrator. Then, we obtain the following result.

Proposition 3.4.3. *Consider the two-period model with a mediator. The outcome that the negotiators reach an agreement with $p(x) \in \partial S$ at period 1 is supported as an SPE outcome if and only if $x \in [\delta\alpha, f^{-1}(\delta f(\alpha))]$. The outcome that the mediator's proposal $p(x) \in \partial S$ is accepted at period 2 is supported as an SPE outcome if and only if $x = \alpha$. Also, disagreement is not supported as an SPE outcome. (In the SPE where the mediator's proposal $p(\alpha) \in \partial S$ is accepted at period 2, negotiators 1 and 2 propose $x' \in [f^{-1}(\delta f(\alpha)), \overline{x}]$ and $y' \in [f(\delta\alpha), \overline{y}]$, respectively. See Figure 3.12.)*

Proof. First of all, we prove that, in all SPEs, any mediator's proposal is accepted at period 2 by both negotiators. Suppose that negotiators 1 and 2 demand x' and y' such that $(x', y') \notin S$, respectively, at period 1, and the mediator proposes $p(z')$ such that $z' \in [f^{-1}(y'), x']$ at period 2. Then, notice that, if some negotiator rejects the mediator's proposal, negotiators 1 and 2 receive payoffs of zero. Therefore, since $z' \geq 0$ and $f(z') \geq 0$ hold, both negotiators accept the mediator's proposal in SPE. Thus, in all SPEs, any mediator's proposal is accepted by both negotiators.

Next, we prove that, in all SPEs, the mediator never chooses pass. Suppose that negotiators 1 and 2 demand x' and y' such that $(x', y') \notin S$, respectively, at period 1. Then, there exists some $z' \in [f^{-1}(y'), x']$ such that $u(p(z')) > 0$ by $u(0, f(0)) \geq 0$, $u(\overline{x}, f(\overline{x})) \geq 0$, and Assumption 3.2.1. In all SPEs, if the mediator proposes $p(z')$, this proposal is accepted by both negotiators and the mediator receives a payoff larger than zero. Since the mediator receives a payoff of zero by choosing pass, proposing $p(z')$ is more profitable for the mediator than choosing pass. Therefore, in all SPEs, the mediator never chooses pass.

By the above discussions, we can see that disagreement is not supported as an SPE outcome. Also, since any mediator's proposal is accepted by both negotiators, we can see that, in all SPEs, if negotiators 1 and 2 demand x' and y' such that $(x', y') \notin S$, respectively, the mediator proposes $p(x')$ if $\alpha > \max[f^{-1}(y'), x']$ $(= x')$, proposes $p(f^{-1}(y'))$ if $\alpha < \min[f^{-1}(y'), x']$ $(= f^{-1}(y'))$, and proposes $p(\alpha)$ if $\alpha \in [f^{-1}(y'), x']$. This SPE strategy is the same as the SPE strategy of the arbitrator in the two-period model with an arbitrator. The rest of the proof is the same as Proposition 3.4.2. Thus, we obtain Proposition 3.4.3. □

By Proposition 3.4.3, we can confirm that the SPE outcomes of the two-period model with an arbitrator and the two-period model with a mediator coincide. The reason is explained as follows. In the two-period model with a mediator, since the negotiation breaks down if the negotiators reject the mediator's proposal, the negotiators need to accept the proposal to make profits. Therefore, any mediator's proposal at period 2 is accepted. This implies that the mediator in the two-period model with a mediator behaves as if she had the authority to decide a final bargaining outcome. That is, in this model, the mediator plays the same role as the arbitrator in the two-period model with an arbitrator. Thus, the SPE outcomes of the above two models coincide. Consequently, we can see that simply replacing an arbitrator with a mediator does not change the SPE outcomes.

Now, we compare the result in the infinite-period model with a mediator and the results in the above two-period models with an arbitrator and a mediator. In both two-period models, the SPE outcomes strongly depend on what the arbitrator's or the mediator's ideal agreement is. Therefore, if the arbitrator or the mediator is biased, the NBS is eliminated from SPE agreement. In contrast to it, in the infinite-period model with a mediator, even if the mediator is biased, the NBS can always be achieved in SSPE. To see why this difference occurs, consider the following situation. Suppose that, in all three models, the mediator and the arbitrator favor negotiator 2. Then, suppose $f^{-1}(\delta f(\alpha)) < x^N$. This is simply $\alpha < x^N$ when $\delta \uparrow 1$. Also, in all three models, consider the case where negotiators 1 and 2 demand x^N and $f(x^N)$, respectively. Now, notice that, when $f^{-1}(\delta f(\alpha)) < x^N$, the NBS is eliminated from SPE agreement in the two-period models with an arbitrator and a mediator (see Figure 3.12), but it is supported as an SSPE agreement in the infinite-period model with a mediator.

In the two-period models with an arbitrator and a mediator, if negotiator 2 deviates from demanding $f(x^N)$ and demands sufficiently large value, the game proceeds to the next period and the arbitrator's or the mediator's ideal agreement $p(\alpha)$ is achieved at period 2. Since the arbitrator and the mediator favor negotiator 2, reaching an agreement with $p(\alpha)$ at period 2 is more profitable for negotiator 2 than reaching an agreement with the NBS at period 1. Therefore, negotiator 2 has incentive to deviate and the NBS is eliminated from equilibrium agreement.

In contrast, in the infinite-period model with a mediator, even if negotiator 2 deviates from demanding $f(x^N)$ and demands sufficiently large value, the mediator does not propose her ideal agreement $p(\alpha)$ since this proposal is rejected by negotiator 1. Negotiator 1 rejects the mediator's proposal $p(\alpha)$ because she has the opportunity to continue the negotiation with negotiator 2 even if she rejects the mediator's proposal and can reach an agreement with the NBS at the next period, which is a more profitable agreement for negotiator 1 than an agreement with $p(\alpha)$. Such an opportunity to continue the negotiation after rejecting the mediator's proposal is a characteristic of the infinite-period model with a mediator. This implies that the mediator does not have the authority to decide a finial bargaining outcome. Then, since the mediator knows that the proposal $p(\alpha)$ is rejected by negotiator 1, she proposes some agreement close to the NBS. For negotiator 2, accepting this proposal at period 2 is less profitable than reaching an agreement with the NBS at period 1. Therefore, negotiator 2 does not deviate from demanding $f(x^N)$ and the negotiators reach an agreement with the NBS at period 1. The case where the mediator and the arbitrator favor negotiator 1 is similarly explained.

By the above discussions, we can see that, in the infinite-period model with a mediator, the negotiators' right to reject the mediator's proposal and the opportunity to continue the negotiation work as deterrents to an unfair proposal by a biased mediator. Therefore, the NBS can be achieved as an equilibrium agreement even if the mediator is biased. The different results of the model with a mediator in section 3.2 and the model with an arbitrator in this subsection are caused by these factors.

3.5 Conclusion

We considered introducing a mediator into bargaining instead of an arbitrator. An advantage of introducing a mediator is that it is easier than introducing an arbitrator since a mediator is merely an adviser. In this study, we analyzed the simultaneous-offers bargaining with a mediator and showed that the following desirable properties appear by introducing a mediator.

First, we found that disagreement is not supported as an SSPE outcome. This result implies that a mediator can resolve conflicts as with an arbitrator. Second, although the set of SSPE agreements is biased towards the mediator's ideal agreement, the reasonable agreement in the sense of the NBS is always one of the SSPE agreements even if the mediator is biased. An agreement having such a property is only the NBS. Therefore, in contrast to the bargaining with an arbitrator, the reasonable agreement is always achievable in equilibrium. Finally, if the mediator is fair in the sense that she wishes to achieve the NBS, the NBS is the unique SSPE agreement when the discount factor is sufficiently large. That is, the negotiators always reach an agreement with the NBS in SSPE. This implies that the fair mediator facilitates the reaching of the reasonable agreement.

Chapter 4

A Strategic Justification of the Constrained Equal Awards Rule through a Procedurally Fair Multilateral Bargaining Game[1]

4.1 Introduction

In this chapter, we consider situations where negotiators (claimants) have claims on a profit. For example, when a firm goes bankrupt and its liquidation value has to be allocated, creditors have claims on it. Also, when an estate is allocated, heirs have claims on it. Especially, we consider the situations where the endowment (the liquidation value or the estate) is not sufficient to cover the totality of the claims. Then, how is the endowment allocated? Such a problem is known as a claims problem.[2]

An axiomatic solution of claims problems is called a "rule."A rule is a single-valued mapping that associates, with each claims problem, an allocation of the endowment satisfying non-negativity, claims boundedness, and efficiency. We call such an allocation an "awards vector." As a central rule for claims problems, we consider the so-called constrained equal awards (CEA) rule. The CEA rule satisfies a number of desirable properties, and the rule has been characterized in multiple ways, reviewed in Thomson (2019).[3] Our purpose is to develop a strategic justification of the CEA rule.

[1]This chapter is based on Hagiwara and Hanato (2021).
[2]For a comprehensive survey on claims problems, see Thomson (2019).
[3]See, for example, Theorem 6.4 at pp 144 of Thomson (2019), which is due to Dagan (1996).

Nash (1953) initiates the study of strategic justifications of axiomatic solutions for bargaining problems through non-cooperative game. Specifically, he provides a strategic justification for the Nash bargaining solution (Nash (1950)) through the "Nash demand game." This line of research is known as the Nash program. The CEA rule corresponds to the Nash bargaining solution in a special class of bargaining problems (discussed in Remark 4.5.1). In this sense, we follow the original work of Nash (1953) as a part.

We provide the following game: At each period t, each claimant proposes a pair consisting of an awards vector and a permutation. If some awards vector is proposed by more than one claimant, then the awards vector which receives the highest number of votes is chosen as temporary awards vector.[4] The components of this temporary awards vector are subject to the composition of the reported permutations, and the game ends.[5] If no two claimants propose the same awards vector, the game proceeds to the period $t+1$ and we repeat the above process. The formal definition of the above game is proposed in section 4.4.

Our game is "procedurally fair" (claimants are treated equally) and "multilateral" (all claimants negotiate simultaneously). If a game is not procedurally fair, claimants are not treated equally. For example, only one claimant has the power to select a division in Li and Ju (2016). In addition, if a game is not multilateral, some claimant is not involved in the negotiation at some stage. Such a situation occurs in Tsay and Yeh (2019). If at least one of the two features is missing, the negotiation might not be initiated since some claimant feels unfair and does not want to participate in the negotiation. For detailed discussion, see section 4.2.

Our game resembles the simultaneous-offers bargaining game analyzed in Chatterjee and Samuelson (1990) in the sense that in each period, all agents simultaneously make proposals and if they do reach an agreement, they can try again in the next period.[6] In the games proposed by Li and Ju (2016) and Tsay and Yeh (2019), most claimants do not have the opportunity to renegotiate (see section 4.2), but in our game, all claimants do.

We show that for each claims problem, the awards vector chosen by the CEA rule achieved at period 1 is supported as an SPE outcome of the game associated

[4]When at least two awards vectors receive the highest number of votes, we use a tie-breaking. See section 4.4.

[5]The permutation idea is proposed in Thomson (2005) for the allocation of a social endowment of infinitely divisible resources and exploited by Doğan (2016), Hagiwara (2019), and others. Note that Chang and Hu (2008), Hayashi and Sakai (2009), Tsay and Yeh (2019), and Moreno-Ternero et al. (2020) apply the idea of letting each agent report a permutation as a strategy into their games to exchange the order of claimants not for exchanging an allocation.

[6]Note that, although Chatterjee and Samuelson (1990) consider the case of only two agents, our game is applicable to any number of agents.

with the problem (Proposition 4.5.1). In addition, for each claims problem, any SPE outcome of the game associated with the problem is the awards vector chosen by the CEA rule achieved at period 1 (Proposition 4.5.2). By Proposition 4.5.1 and 4.5.2, for each claims problem, the awards vector chosen by the CEA rule achieved at period 1 is the unique SPE outcome of the game associated with the problem (Theorem 4.5.1). If we change from the notion of subgame perfect equilibrium to the notion of Nash equilibrium, our result holds by the same way as in the proofs of Proposition 4.5.1 and 4.5.2 (Corollary 4.5.1).

Our results have two applications, one to bargaining problems and one to coalitional problems. For bargaining problems, the CEA rule corresponds to the Nash bargaining solution (Dagan and Volij (1993)). For coalitional problems, the CEA rule corresponds to the Dutta-Ray solution (Dutta and Ray (1989)). From these correspondences, our Theorem 4.5.1 and Corollary 4.5.1 imply that our game provides a strategic justification of the Nash bargaining solution and of the Dutta-Ray solution. For detailed discussions of these applications, see Remark 4.5.1 and 4.5.2.

This chapter is organized as follows. In section 4.2, we introduce related literature of our study. In section 4.3, we introduce the model of claims problems and define the CEA rule. In section 4.4, we provide our procedurally fair multilateral bargaining game. In section 4.5, we show that, for each claims problem, the awards vector chosen by the CEA rule achieved at period 1 is the unique SPE outcome of our game associated with the problem. This result holds even if we change from the notion of subgame perfect equilibrium to the notion of Nash equilibrium.

4.2 Related literature

Strategic justifications of the CEA rule for claims problems have been derived by Li and Ju (2016) and Tsay and Yeh (2019).[7,8]

Li and Ju (2016) propose the following n-stage game, where n is the number of claimants. Claimants are numbered in the reverse order of their claims. In Stage 1, the agent whose claim is the largest, claimant n, divides the endowment as proposal.

[7]For another strategic justification of the CEA rule, see Chun (1989). He considers a game where each agent reports a rule, while in each of the games of Li and Ju (2016), Tsay and Yeh (2019), and this study, an allocation is reported. Since each agent considers how to select allocations for all claims problems in Chun's game, it may be hard to use this game for real people compared to Li and Ju (2016), Tsay and Yeh (2019), and this study. Then, we compare our game with Li and Ju (2016) and Tsay and Yeh (2019).

[8]Strategic justifications of other rules have been also studied. For the family of f-just rules, see Dagan et al. (1997) and Chang and Hu (2008). For the constrained equal losses rule, see Li and Ju (2016) and Tsay and Yeh (2019). For the Talmud rule, see Li and Ju (2016), Tsay and Yeh (2019), and Moreno-Ternero et al. (2020). For the proportional rule, see Tsay and Yeh (2019).

In Stage $k \in \{2, \ldots, n\}$, claimant $k-1$ chooses a component of the proposal as their payoffs.[9] Claimant n is given the power to make the division, although agents with lower claims are only given priority to choose early on in the game but not the power to make the division. Therefore, claimants are not treated equally. In addition, at each stage of their game, only two claimants negotiate, so that the game is bilateral even in the case of more than two claimants. Furthermore, any claimant who negotiated with claimant n at a stage of their game cannot negotiate anymore.

Tsay and Yeh (2019) propose the following three-stage game for the CEA rule. In Stage 1, each claimant announces a pair consisting of an awards vector and a permutation. The composition of the reported permutations selects a claimant as coordinator. If all claimants, except for the coordinator, announce the same awards vector, then this awards vector is the proposal; otherwise, the awards vector announced by the coordinator is the proposal. In Stage 2, the coordinator either accepts or rejects the proposal. If he accepts it, the proposal is the outcome. If he rejects it, he selects one claimant to negotiate for the two claimants in Stage 3[10]; all the others receive their awards as specified in Stage 1. That is, in Stage 1, all claimants are given the opt-in of choosing the temporary awards vector, but in Stage 2, only the coordinator is given the power to reject a component of the proposal and to choose a claimant to negotiate the final awards for only the two claimants in Stage 3.[11] Therefore, only two claimants negotiate in Stage 3 of their game, so that the game is not multilateral in all stages. Moreover, in their game, only the coordinator and the claimant selected by the coordinator at Stage 2 have the opportunity to renegotiate.

By contrast, in our game, all claimants are treated equally; they negotiate multilaterally; and they all have opportunities to renegotiate.

4.3 The model

Let $N = \{1, \ldots, n\}$ be the set of agents with $n \geq 2$. Each agent $i \in N$ has a claim on a resource, $c_i \in \mathbb{R}_+$. Claimants are numbered so that $c_1 \leq \cdots \leq c_n$. Let

[9] This bilateral negotiation procedure is similar to the games of Chae and Yang (1988) and Sonn (1992).

[10] For the bilateral negotiation game for the CEA rule, see Tsay and Yeh (2019). They incorporate the axioms of rules, mainly *bilateral consistency* and *converse consistency*, into the corresponding games, as suggested by Krishna and Serrano (1996). The other researches of this line in claims problems are Dagan et al. (1997), Chang and Hu (2008), and Moreno-Ternero et al. (2020). Note that Tsay and Yeh (2019) and Moreno-Ternero et al. (2020) use non-cooperative bargaining procedures to solve bilateral negotiations, while Dagan et al. (1997) and Chang and Hu (2008) resolve bilateral negotiations by applying predetermined rules without claimants' strategic actions.

[11] Note that each claimant has an equal chance of becoming coordinators at the first stage of the game of Tsay and Yeh (2019).

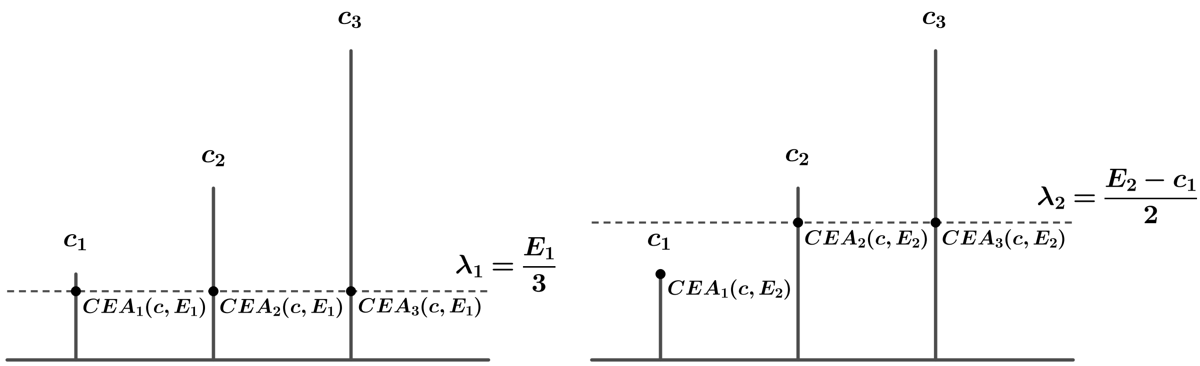

Figure 4.1: Examples of the constrained equal awards rule.

$c \equiv (c_1, \ldots, c_n)$ be a claims vector. There is an endowment E of the resource. The endowment is insufficient to honor the totality of the claims. Using $\mathbb{R}_+^N$ for the cross-product of n copies of $\mathbb{R}_+$ indexed by the members of N, a claims problem is a pair $(c, E) \in \mathbb{R}_+^N \times \mathbb{R}_{++}$ such that $E \leq \sum_{i \in N} c_i$. Let $\mathcal{C}^N$ denote the domain of all claims problems.

An awards vector for the claims problem $(c, E) \in \mathcal{C}^N$ is a vector $a \equiv (a_1, \ldots, a_n) \in \mathbb{R}_+^N$ (i.e., non-negativity) such that, for each $i \in N$, $a_i \leq c_i$ (i.e., claims boundedness) and $\sum_{i \in N} a_i = E$ (i.e., efficiency). Let $A(c, E) = \{a \in \mathbb{R}_+^n \mid \text{ for each } i \in N, a_i \leq c_i \text{ and } \sum_{i \in N} a_i = E\}$ be the set of awards vectors of the problem $(c, E) \in \mathcal{C}^N$. A division rule, or simply a rule, is a single-valued mapping which associates, with each problem $(c, E) \in \mathcal{C}^N$, an awards vector $a \in A(c, E)$.

The following is central to our study:[12]

Constrained equal awards rule, CEA**:** For each $(c, E) \in \mathcal{C}^N$ and each $i \in N$, $CEA_i(c, E) \equiv \min\{c_i, \lambda\}$, where $\lambda \in \mathbb{R}_+$ is chosen so as to satisfy efficiency.

In the two cases regarding the endowment when there are only three claimants, Figure 4.1 illustrates how allocates the endowment E_1 such that $\lambda_1 \equiv \frac{E_1}{3} < c_1$ and E_2 such that $c_1 < \lambda_2 \equiv \frac{E_2 - c_1}{2} < c_2$.

4.4 The game for a strategic justification of the CEA rule

We first provide an informal description of our game in words, and then we define the game mathematically.

[12]For other important rules for claims problems, see, for example, Ch.2 of Thomson (2019).

At each period t, each claimant proposes a pair consisting of an awards vector and a permutation. If some awards vector is proposed by more than one claimant, then the awards vector which receives the highest number of votes is chosen as temporary awards vector. If at least two awards vectors receive the highest number of votes, then the awards vector proposed by the claimant who has the lowest index among the claimants who announce one of these awards vectors is chosen as temporary awards vector. The components of this awards vector are subject to the composition of the reported permutations, and the game ends. Note that independently of the permutations reported by the other claimants and independently of the order of permutations for the composition, each claimant can assign any component of the temporary awards vector to himself by proposing an appropriate permutation. A claimant's payoff is the discounted present value of the minimum of his claim or his component of the resulting allocation adjusted so as to satisfy claims boundedness. If no two claimants propose the same awards vector, the game proceeds to the period $t+1$ and we repeat the above process. If the claimants cannot reach an agreement permanently, then negotiation breaks down and each claimant obtains a payoff of zero.

To define the game formally, let us introduce some notation. A permutation $\pi : N \to N$ is a one-to-one function from N to N. Let Π be the set of permutations. Let $\delta \in (0,1)$ be the claimants' common discount factor.

Let $(c, E) \in \mathcal{C}^N$ be given. The game $\Gamma(c, E)$ is as follows:

1. At period t, each claimant i proposes a pair $(a^i, \pi^i) \in A(c, E) \times \Pi$.

2. If for some $a \in A(c, E)$, $|\{i \in N \mid a^i = a\}| \geq 2$, we select $a^{\hat{i}}$, where $\hat{i} = \min\{i' \in N \mid a^{i'} \in \arg\max_{a \in A} |\{i \in N \mid a^i = a\}|\}$.

   The components of the awards vector $a^{\hat{i}}$ are subject to the composition $\pi^* \equiv \pi^n \circ \cdots \circ \pi^1$, and the game ends. Let $a^{\hat{i}}_{\pi^*} = (a^{\hat{i}}_{\pi^*(1)}, \ldots, a^{\hat{i}}_{\pi^*(n)})$. The payoff of claimant i is $\delta^{t-1} \min\{c_i, a^{\hat{i}}_{\pi^*(i)}\}$.

3. If for each $a \in A$, $|\{i \in N \mid a^i = a\}| \leq 1$, then the game proceeds to the next period $t+1$ and we repeat the above process.

   If the claimants cannot reach an agreement permanently, disagreement occurs and then each claimant obtains a payoff of zero.

Regarding our game, there are the following two remarks.

Remark 4.4.1. (Tie-breaking). We use a tie-breaking when at least two awards vectors receive the highest number of votes. Our results hold no matter what tie-breaking is used. One may say that this game does not treat claimants equally.

To resolve this problem for procedural fairness, let each claimant additionally report another permutation $\tilde{\pi}^i \in \Pi$ and according to the composition of these permutations $\pi^{**} \equiv \tilde{\pi}^n \circ \cdots \circ \tilde{\pi}^1$, a claimant who chooses one awards vector in those receiving the highest number of votes is selected as tie-breaker. Then, the awards vector reported by the tie-breaker is selected as temporary awards vector. In this modified game, independently of the permutations reported by the other claimants and independently of the order of the permutations for the composition, any claimant reporting one awards vector among those receiving the highest number of votes can be the tie-breaker by proposing an appropriate permutation. Therefore, this modified game is procedurally fair. Since the proofs of Proposition 4.5.1 and 4.5.2 are simpler, we propose those in this study.

Remark 4.4.2. (Inefficient allocations). In our game, some final allocation after exchange may not satisfy efficiency. Chang and Hu (2008) also use inefficient allocations in their game. In their game, each claimant reports a pair of consisting of an awards vector and a permutation. At the first stage, if some claimant reports a different awards vector from the awards vectors announced by the other claimants, then the coordinator, who is the first claimant selected by the composition of the reported permutations, gets a negative value and the other claimants get nothing. In their game, inefficient allocations have an important role to have Nash equilibria of their game, but in our game, these are just selected so as to satisfy claims boundedness.

4.5 Results

We first provide two propositions (Proposition 4.5.1 and 4.5.2) and then from these two results, we have our main result (Theorem 4.5.1).

In the following, we derive an SPE outcome of $\Gamma(c, E)$. The outcome that a_π is achieved at period t is denoted by $[a_\pi, t]$, where $a_\pi = (a_{\pi(1)}, \ldots, a_{\pi(n)})$.

Proposition 4.5.1. *For each* $(c, E) \in \mathcal{C}^N$, $[CEA(c, E), 1]$ *is supported as an SPE outcome of* $\Gamma(c, E)$.

Proof. Let $\sigma^* \equiv (\sigma_1^*, \ldots, \sigma_n^*)$ be the strategy profile in which each claimant $i \in N$ always proposes $(CEA(c, E), \pi_{id})$, where π_{id} is the identity permutation i.e., for each $j \in N$, $\pi_{id}(j) = j$. The outcome under σ^* is $[CEA(c, E), 1]$. Then, claimant i's payoff is $CEA_i(c, E)$ $(\leq c_i)$. We prove that σ^* is an SPE of $\Gamma(c, E)$.

We use the one-shot deviation principle: a strategy profile σ is an SPE if and only if no claimant gains by deviating from σ in a single action (for example, see Fudenberg and Tirole (1991)). Fix a pair consisting of $i \in N$ and a positive integer t arbitrarily. Suppose that claimant i deviates from σ_i^* in a single action and proposes some $(a^i, \pi^i) \neq (CEA(c, E), \pi_{id})$ at period t. First, we consider the case $E < \sum_{i \in N} c_i$. The proof is divided into two cases, $n \geq 3$ and $n = 2$.

Case $n \geq 3$. For each (a^i, π^i), $\arg\max_{a \in A} |\{i' \in N \mid a^{i'} = a\}| = \{CEA(c, E)\}$. If claimant i proposes (a^i, π^i), then since $\pi_{id} \circ \cdots \circ \pi^i \circ \cdots \pi_{id} = \pi^i$, he obtains $CEA_{\pi^i(i)}(c, E)$. Thus, to see that he cannot gain by deviating from σ_i^*, we prove that for each $\pi^i \in \Pi$, $\min\{c_i, CEA_{\pi^i(i)}(c, E)\} \leq CEA_i(c, E)$. It suffices to prove that for each $j \in N$, $\min\{c_i, CEA_j(c, E)\} \leq CEA_i(c, E)$.

By the definition of CEA, for each $j \in N$, $CEA_j(c, E) = \min\{c_j, \lambda\}$, where $\lambda \in \mathbb{R}_+$ is chosen so as to satisfy $\sum_{j \in N} CEA_j(c, E) = E$. If $CEA_i(c, E) = c_i$, then we immediately obtain that for each $j \in N$, $\min\{c_i, CEA_j(c, E)\} \leq CEA_i(c, E)(= c_i)$. If $CEA_i(c, E) < c_i$, then $\lambda = CEA_i(c, E)(< c_i)$. Since $\lambda < c_i$, we have that for each $j \in N$, $\min\{c_i, CEA_j(c, E)\} = \min\{c_i, \min\{c_j, \lambda\}\} = \min\{c_j, \lambda\} \leq \lambda = CEA_i(c, E)$. Therefore, claimant i cannot gain by deviating from σ_i^*.

Case $n = 2$. If claimant i proposes (a^i, π^i) such that $a^i = CEA(c, E)$ and $\pi^i \neq \pi_{id}$, then $\arg\max_{a \in A} |\{i' \in N \mid a^{i'} = a\}| = \{CEA(c, E)\}$. Therefore, since $\pi_{id} \circ \pi^i (= \pi^i \circ \pi_{id}) = \pi^i$ and $\pi^i \neq \pi_{id}$, claimant i obtains $CEA_j(c, E)$, where $j \neq i$. By a similar proof to the case $n \geq 3$, we obtain that $\min\{c_i, CEA_j(c, E)\} \leq CEA_i(c, E)$. Thus, claimant i cannot gain by deviating from σ_i^* if he proposes (a^i, π^i) such that $a^i = CEA(c, E)$ and $\pi^i \neq \pi_{id}$.

If claimant i proposes (a^i, π^i) such that $a^i \neq CEA(c, E)$, then for each $a \in A(c, E)$, $|\{i' \in N \mid a^{i'} = a\}| \leq 1$. Thus, the game proceeds to the next period $t + 1$. Since claimant i follows σ_i^* at period $t + 1$, he obtains $CEA_i(c, E)$ at period $t + 1$. If claimant i does not deviate from σ_i^*, he obtains $CEA_i(c, E)$ at period t. Since $CEA_i(c, E) \geq \delta CEA_i(c, E)$, he cannot gain by deviating from σ_i^*.

By the above discussion, if $\sum_{i \in N} c_i > E$, we have that σ^* is an SPE of $\Gamma(c, E)$. In the case $\sum_{i \in N} c_i = E$, since $A(c, E) = \{CEA(c, E)\}$, this case is proved by an analogous proof to that for $n \geq 3$. Therefore, $[CEA(c, E), 1]$ is supported as an SPE outcome of $\Gamma(c, E)$. $\square$

The following is the uniqueness part of our strategic justification of the CEA rule.

Proposition 4.5.2. *For each* $(c, E) \in \mathcal{C}^N$, *any SPE outcome of* $\Gamma(c, E)$ *is* $[CEA(c, E), 1]$.

Proof. First, we show that for any $b \neq CEA(c, E)$ and any period t, $[b, t]$ is not supported as an SPE outcome of $\Gamma(c, E)$. Suppose, by contradiction, that there exists an SPE σ of $\Gamma(c, E)$ whose outcome is $[b, t]$. We show that some claimant gains by deviating from σ. Let $\lambda \in \mathbb{R}^n_+$ be such that $\sum_{i \in N} \min\{c_i, \lambda\} = E$. Since $b \neq CEA(c, E)$, the proof is divided into the following three cases.

Case 1. For some $i^* \in N$, $b_{i^*} > c_{i^*}$.

Case 2. For each $i \in N$, $b_i \leq c_i$, and

- **2-1.** for some $i^{**} \in N$ such that $c_{i^{**}} \leq \lambda$, $b_{i^{**}} < c_{i^{**}}$, or
- **2-2.** for some $i^{***} \in N$ such that $c_{i^{***}} > \lambda$, $b_{i^{***}} \neq \lambda$.

The above three cases cover all possibilities of $b \neq CEA(c, E)$. By definition, for each claims problem, an allocation is selected by CEA if and only if the following three conditions are satisfied: (1) the selected allocation is efficient; (2) if someone's claim is less than or equal to λ, his assignment is equal to his claim; and (3) if someone's claim is more than λ, his assignment is equal to λ. Case 1 (resp. Case 2-1, Case 2-2) is that (1) (resp. (2), (3)) is not satisfied. Therefore, Cases 1, 2-1, and 2-2 are composed of all possibilities that $b \neq CEA(c, E)$.

We sequentially analyze each case.[13]

Case 1. Let $\hat{a}$ be the temporary awards vector before b is achieved by exchange at period t. For each $i \in N$, let π^i_σ be the permutation proposed by claimant i at period t under σ. That is, for each $i \in N$,

$$\hat{a}_{\pi^n_\sigma \circ \cdots \circ \pi^1_\sigma(i)} = b_i.$$

We show that for some $j \in N$, $b_j < \hat{a}_j (\leq c_j)$. Suppose, by contradiction, that for each $i \in N$, $\hat{a}_i \leq b_i$. If for some $k \in N$, $\hat{a}_k < b_k$, then $E = \sum_{i \in N} \hat{a}_i < \sum_{i \in N} b_i = E$, which is a contradiction. Thus, for each $i \in N$, $\hat{a}_i = b_i (= \hat{a}_{\pi^n_\sigma \circ \cdots \circ \pi^1_\sigma(i)})$. However, $c_{i^*} \geq \hat{a}_{i^*} = b_{i^*} > c_{i^*}$, which is also a contradiction. Consequently, for some $j \in N$, $\hat{a}_j > b_j (= \hat{a}_{\pi^n_\sigma \circ \cdots \circ \pi^1_\sigma(j)})$.

[13] When $\sum_{i \in N} c_i = E$, it suffices to consider Case 1 because of the following reason. When $\sum_{i \in N} c_i = E$, if for each $i \in N$, $b_i \leq c_i$, then $E = \sum_{i \in N} b_i \leq \sum_{i \in N} c_i = E$. This implies that for each $i \in N$, $b_i = c_i$, which contradicts the assumption that $b \neq CEA(c, E)$.

Claimant j can assign $\hat{a}_j$ to himself by proposing an appropriate π^j ($\neq \pi^j_\sigma$). Since $b_j < \hat{a}_j(\leq c_j)$, claimant j gains by deviating from σ_j, which contradicts the hypothesis that σ is an SPE of $\Gamma(c, E)$.

Case 2-1. We show that for some $j \in N$, $b_j > \lambda$. Suppose, by contradiction, that for each $i \in N$, $b_i \leq \lambda$. Since for each $i \in N$, $b_i \leq c_i$, then for each $k \in N$ such that $c_k \leq \lambda$, $b_k \leq c_k = CEA_k(c, E)$. In addition, for each $\ell \in N$ such that $c_\ell > \lambda$, $b_\ell \leq \lambda = CEA_\ell(c, E)$. Therefore, for each $i \in N$, $b_i \leq CEA_i(c, E)$. Since $b_{i^{**}} < c_{i^{**}}(= CEA_{i^{**}}(c, E))$, we have $\sum_{i\in N} b_i < \sum_{i\in N} CEA_i(c, E)$, which contradicts the assumption that $\sum_{i\in N} b_i = \sum_{i\in N} CEA_i(c, E) = E$. Thus, for some $j \in N$, $b_j > \lambda$. Claimant i^{**} can obtain b_j by proposing an appropriate $\pi^{i^{**}}$ ($\neq \pi^{i^{**}}_\sigma$). Since $b_j > \lambda \geq c_{i^{**}} > b_{i^{**}}$, claimant i^{**} gains by deviating from $\sigma_{i^{**}}$, which contradicts the hypothesis that σ is an SPE of $\Gamma(c, E)$.

Case 2-2. When $b_{i^{***}} < \lambda$, we show that for some $j \in N$, $b_j > \lambda$. Suppose, by contradiction, that for each $i \in N$, $b_i \leq \lambda$. Then, by the same proof as in Case 2-1, we obtain that for each $i \in N$, $b_i \leq CEA_i(c, E)$. Since $b_{i^{***}} < \lambda(= CEA_{i^{***}}(c, E))$, we have $\sum_{i\in N} b_i < \sum_{i\in N} CEA_i(c, E)$, which contradicts the assumption that $\sum_{i\in N} b_i = \sum_{i\in N} CEA_i(c, E) = E$. Thus, for some $j \in N$, $b_j > \lambda$.

Claimant i^{***} can obtain b_j by proposing an appropriate $\pi^{i^{***}}$ ($\neq \pi^{i^{***}}_\sigma$). Since $b_j > \lambda \geq c_{i^{***}} > b_{i^{***}}$, claimant i^{***} gains by deviating from $\sigma_{i^{***}}$, which contradicts the hypothesis that σ is an SPE of $\Gamma(c, E)$.

When $b_{i^{***}} > \lambda$, we show that for some $j \in N$ such that $c_j > \lambda$, $b_j < \lambda$. Suppose, by contradiction, that for each $i \in N$ such that $c_i > \lambda$, $b_i \geq \lambda(= CEA_i(c, E))$. In the case that for some $k \in N$ such that $c_k \leq \lambda$, $b_k < c_k$, this is in Case 2-1. Thus, we consider the case where, for each $\ell \in N$ such that $c_\ell \leq \lambda$, $b_\ell = c_\ell(= CEA_\ell(c, E))$. Since for each $i \in N$ such that $c_i > \lambda$, $b_i \geq \lambda(= CEA_i(c, E))$ and for each $\ell \in N$ such that $c_\ell \leq \lambda$, $b_\ell = c_\ell(= CEA_\ell(c, E))$, then for each $m \in N$, $b_m \geq CEA_m(c, E)$. Since $b_{i^{***}} > \lambda(= CEA_{i^{***}}(c, E))$, we have $\sum_{i\in N} b_i > \sum_{i\in N} CEA_i(c, E)$, which contradicts the assumption that $\sum_{i\in N} b_i = \sum_{i\in N} CEA_i(c, E) = E$. Thus, for some $j \in N$ such that $c_j > \lambda$, $b_j < \lambda$.

Claimant j can obtain $b_{i^{***}}$ by proposing an appropriate π^j ($\neq \pi^j_\sigma$). Since $\min\{c_j, b_{i^{***}}\} > \lambda > b_j$, claimant j gains by deviating from σ_j, which contradicts the hypothesis that σ is an SPE of $\Gamma(c, E)$.

By the above discussion, for any $b \neq CEA(c, E)$ and any period t, $[b, t]$ is not supported as an SPE outcome of $\Gamma(c, E)$.

Next, we show that for any $\tilde{t} \neq 1$, $[CEA(c,E), \tilde{t}]$ is not supported as an SPE outcome of $\Gamma(c,E)$. If $\sum_{i \in N} c_i = E$, then $A(c,E) = \{c\} = \{CEA(c,E)\}$. Therefore, each claimant proposes a pair $(a, \pi) \in A(c,E) \times \Pi$ such that $a = CEA(c,E)$ at period 1. This implies that the game ends at period 1. Thus, in the case $\sum_{i \in N} c_i = E$, for any $\tilde{t} \neq 1$, $[CEA(c,E), \tilde{t}]$ is not supported as an SPE outcome of $\Gamma(c,E)$.

We consider the case $\sum_{i \in N} c_i > E$. Suppose, by contradiction, that there exists an SPE σ' of $\Gamma(c,E)$ whose outcome is $[CEA(c,E), \tilde{t}]$ such that $\tilde{t} \neq 1$. Then, for each $i \in N$, claimant i's payoff is $\delta^{\tilde{t}-1} CEA_i(c,E)$. Let j be the claimant whose claim is minimal among the claims larger than zero. Since $\sum_{i \in N} c_i > E$, such a claimant exists. We show that claimant j can obtain a payoff larger than $\delta^{\tilde{t}-1} CEA_j(c,E) (> 0)$ by deviating from σ'_j.

Let (a^k, π^k) be claimant k's proposal at period 1 under σ', where $k \neq j$. Since under σ', the game ends at period $\tilde{t} \geq 2$, no two claimant propose the same awards vector at period 1, so that $a^k \neq a^j$, where a^j is claimant j's proposal of an awards vector at period 1 under σ'. We show that there exists $\ell \in N$ such that $a^k_\ell \geq CEA_j(c,E)$. Suppose that for each $i \in N$, $a^k_i < CEA_j(c,E)$. By the definitions of CEA and claimant j, for each $i \in N$ such that $c_i > 0$, $CEA_j(c,E) \leq CEA_i(c,E)$. Then, for each $i \in N$ such that $c_i > 0$, $a^k_i < CEA_i(c,E)$. In addition, for each $i \in N$ such that $c_i = 0$, $a^k_i = CEA_i(c,E) = 0$. Therefore, $\sum_{i \in N} a^k_i < \sum_{i \in N} CEA_i(c,E)$, which contradicts the assumption that $\sum_{i \in N} a^k_i = \sum_{i \in N} CEA_i(c,E) = E$. Thus, there exists $\ell \in N$ such that $a^k_\ell \geq CEA_j(c,E)$.

This implies that claimant j can obtain a payoff of $\min\{c_j, a^k_\ell\}$ at period 1 by deviating from σ'_j and proposing a pair consisting of a^k and an appropriate $\pi^j \in \Pi$, because, when claimant j changes his proposal regarding an awards vector a^j into a^k at period 1, a^k becomes the awards vector which receives the highest number of votes. Since $\min\{c_j, a^k_\ell\} \geq CEA_j(c,E) > \delta^{\tilde{t}-1} CEA_j(c,E) > 0$, claimant j gains by deviating from σ'_j, which contradicts the hypothesis that σ' is an SPE of $\Gamma(c,E)$. Therefore, $[CEA(c,E), \tilde{t}]$ such that $t \neq 1$ is not supported as an SPE outcome of $\Gamma(c,E)$.

Finally, we show that disagreement is not supported as an SPE outcome. The proof is analogous to the case of $[CEA(c,E), \tilde{t}]$ such that $\tilde{t} \neq 1$.

If $\sum_{i \in N} c_i = E$, then the game ends at period 1 as we discussed in the case of $[CEA(c,E), \tilde{t}]$ for $\tilde{t} \neq 1$. Thus, if $\sum_{i \in N} c_i = E$, disagreement is not supported as an SPE outcome of $\Gamma(c,E)$.

We consider the case $\sum_{i \in N} c_i > E$. Suppose, by contradiction, that there exists an SPE σ'' of $\Gamma(c,E)$ whose outcome is disagreement. Then, for each $i \in N$, claimant i's payoff is zero. Let j be the claimant whose claim is larger than zero. Since $\sum_{i \in N} c_i > E$, such a claimant exists. We show that claimant j can obtain a payoff larger than zero by deviating from σ''_j.

Let (a^k, π^k) be claimant k's proposal at period 1 under σ'', where $k \neq j$. Since under σ'', disagreement occurs, then no two claimant propose the same awards vector at period 1, so that $a^k \neq a^j$, where a^j is claimant j's proposal regarding an awards vector at period 1 under σ''.

Since $E > 0$, there exists $\ell \in N$ such that $a^k_\ell > 0$. This implies that claimant j can obtain the payoff of $\min\{c_j, a^k_\ell\}$ at period 1 by deviating from σ''_j and proposing a pair consisting of a^k and an appropriate $\pi^j \in \Pi$. Since $\min\{c_j, a^k_\ell\} > 0$, claimant j gains by deviating from σ''_j, which contradicts the hypothesis that σ'' is an SPE of $\Gamma(c, E)$. Therefore, disagreement is not supported as an SPE outcome of $\Gamma(c, E)$.

By the above discussions, any SPE outcome of $\Gamma(c, E)$ is $[CEA(c, E), 1]$. □

From Proposition 4.5.1 and 4.5.2, we have the following main result.

Theorem 4.5.1. *For each $(c, E) \in \mathcal{C}^N$, $[CEA(c, E), 1]$ is the unique SPE outcome of $\Gamma(c, E)$.*

Therefore, the awards vector chosen by the CEA rule is achieved under our procedurally fair and multilateral process.

We considered the notion of subgame perfect equilibrium for the above results. Even if we change from this notion to the notion of Nash equilibrium, our result holds by the same way as in the proofs of Proposition 4.5.1 and 4.5.2.

Corollary 4.5.1. *For each $(c, E) \in \mathcal{C}^N$, $[CEA(c, E), 1]$ is the unique Nash equilibrium outcome of $\Gamma(c, E)$.*

By the above result, we can know that, even if the claimants are not rational enough to implement a subgame perfect equilibrium, they achieve the awards vector chosen by the CEA rule.

Finally, in the following remarks, we describe relations between the CEA rule and some solution concept of the cooperative game theory. For this theory to be applicable, we need first to define a formal way of associating, with each claims problem, a cooperative problem. Two main classes of such problems have been studied, bargaining problems (Remark 4.5.1) and coalitional problems (Remark 4.5.2), and accordingly we establish two kinds of relations.

Remark 4.5.1. A bargaining problem is a pair (B, d), where B is a subset of $\mathbb{R}^N$ and d is a point of B. The set B is the feasible set consisting of all utility vectors attainable by the group N and d is the disagreement point. A bargaining solution is a function defined on a class of bargaining problems that associates, with each bargaining problem in the class, a unique point in the feasible set of the problem. The

Nash bargaining solution (Nash (1950)) selects the point maximizing the product of utility gains from d among all points of B dominating d.

Given a claims problem $(c, E) \in \mathcal{C}^N$, its associated bargaining problem is the problem with feasible set $B(c, E) = \{a \in \mathbb{R}^N_+ \mid \text{ for each } i \in N, a_i \leq c_i \text{ and } \sum_{i \in N} a_i = E\}$ and disagreement point $d = 0$.

For each $(c, E) \in \mathcal{C}^N$, the outcome chosen by the CEA rule coincides with the outcome chosen by the Nash bargaining solution when applied to $(B(c, E), d)$ (Dagan and Vojil (1993)). Therefore, for the bargaining problem associated with a claims problem, our game provides a strategic justification of the Nash bargaining solution.

Remark 4.5.2. A (transferable utility) coalitional problem is a vector $v \equiv (v(S))_{S \subseteq N} \in \mathbb{R}^{2|N|-1}$, where for each coalition $\emptyset \neq S \subseteq N$, $v(S) \in \mathbb{R}$ is the worth of S. A solution is a mapping that associates, with each such problem v, a point in $\mathbb{R}^N$ whose coordinates add up to $v(N)$. The Dutta-Ray solution (Dutta and Ray (1989)) selects, for each convex coalitional problem, the payoff vector in the core that is Lorenz-maximal.[14]

Given a claims problem $(c, E) \in \mathcal{C}^N$, its associated coalitional problem (O'Neill (1982)) is the problem $v(c, E) \in \mathbb{R}^{2|N|-1}$ defined by setting for each $\emptyset \neq S \subseteq N$, $v(c, E)(S) \equiv \max\{E - \sum_{i \in N \setminus S} c_i, 0\}$.

For each $(c, E) \in \mathcal{C}^N$, the outcome chosen by the CEA rule coincides with the outcome chosen by the Dutta-Ray solution when applied to $v(c, E)$ (see, for example, Theorem 14.2 at pp 373 of Thomson, 2019). Therefore, for the coalitional problem associated with a claims problem, our game provides a strategic justification of the Dutta-Ray solution.

[14]For the definitions of Lorenz-domination and the Dutta-Ray solution, see Dutta and Ray (1989).

Chapter 5

Conclusion

In this chapter, we summarize the results of our studies. Also, we discuss several remaining issues and future work.

5.1 Summary of this dissertation

This dissertation followed the line of research called the Nash program. We provided non-cooperative foundations for bargaining solutions that were originally defined axiomatically.

In Chapter 2, we analyzed the model which is a generalization of the model of Rubinstein (1982) from the viewpoint of the process of how a proposer is decided in each period. The negotiator's probability to be a proposer depends on the history of proposers and the negotiators divide a pie of size 1. In the bilateral bargaining model, we found each component game at period t involving negotiators dividing a pie of size δ^{t-1} according to the proposal ratio at period t in the unique SPE payoffs. The limit of this SPE payoffs coincides with the ANBS weighted by the convergent values of opportunities to be a proposer. In the n-player model, we showed that there exists an MPE which has the same form as the unique SPE in the bilateral model. Under this MPE, we showed that the same results as the bilateral model hold. By these results, we can see that the ANBS is achieved when bargaining is conducted under the process where opportunities of proposals are asymmetric between negotiators. Since the weights of the ANBS reflect the negotiators' bargaining power, the opportunities to be a proposer can be considered as their bargaining power.

In Chapter 3, we introduced a mediator into the simultaneous-offers bargaining process analyzed in Chatterjee and Samuelson (1990). In our model, we found that disagreement is not supported as an SSPE outcome. Also, although the set of SSPE agreements is biased towards the mediator's ideal agreement, a reasonable agreement in the sense of the NBS is always one of the SSPE agreements even if the mediator is biased. These results imply that, by introducing a mediator, the negotiators can avoid disagreement without eliminating the achievability of a reasonable agreement. In addition, if the mediator is fair in the sense that she wishes to achieve the NBS, the NBS is the unique SSPE agreement when the discount factor is sufficiently large. By this result, we can see that the NBS is achieved under the simultaneous-offers process with a fair mediator.

In Chapter 4, we considered claims problems. That is, we considered bargaining situations where negotiators (claimants) have claims on a profit and the profit is not sufficient to cover the totality of the claims. As a central rule for claims problems, we considered the CEA rule. The CEA rule corresponds to the NBS in a special class of bargaining problems. To develop a strategic justification of the CEA rule, we proposed a bargaining process which is procedurally fair and multilateral. Our process resembles the simultaneous-offers process in the sense that in each period, all negotiators simultaneously make proposals and, if they do not reach an agreement, they renegotiate in the next period. We showed that, as a unique SPE outcome and a unique Nash equilibrium outcome, the awards vector chosen by the CEA rule is achieved under our procedurally fair and multilateral process.

5.2 Remaining issues and future work

In the following, we discuss several remaining issues and future work.

First, we consider the assumption with respect to the negotiators' utility space in Chapter 2. In this chapter, we assumed that the negotiators divide a pie of size 1 and each negotiator receives a component of a division as her payoff. Under this assumption, the limit of the SPE payoffs coincides with the ANBS weighted by the convergent values of opportunities to be a proposer. On the other hand, with a more general utility space than our study, Laruelle and Valenciano (2008), Kultti and Vartiainen (2010), and Britz et al. (2010) show that stationary SPE payoffs converge to the ANBS under the processes of constant probabilities across periods, alternating offers, and a Markov process, respectively.

In their models, the crucial property is that all proposers propose the same division in all states when the discount factor is sufficiently large. However, in our model, negotiators' proposals depend on the current state and may change over time (Proposition 2.4.2 and 2.6.2). Therefore, even if we use the same method of the aforementioned papers, with a general utility space, we cannot prove the same relationship between SPE payoffs and the ANBS under our flexible process. Consequently, under our process with a general utility space, we do not know how the SPE payoffs are related to some bargaining solution. We leave analyzing such relationship with a general utility space as future work.

Second, we consider the assumption of the probability to be a proposer in Chapter 2. In our model, each negotiator's probability to be a proposer is given exogenously. However, in reality, who becomes a proposer is decided endogenously. From this viewpoint, Yildirim (2007) and Yildirim (2010) consider n-player models where each negotiator's probability to be a proposer depends on her effort to be a proposer. Also, Kambe (2009) and Rachmilevitch (2019) consider two-player models where both negotiators nominate a negotiator to decide a proposer in each period. As with these models, it is worth considering how probabilities to be a proposer are decided endogenously in our process.

Third, we consider the assumption of complete information in the bargaining model with a mediator in Chapter 3. We assumed that the mediator's ideal agreement is common knowledge. However, one might think that more realistic case is that the mediator's ideal agreement is her private information. We briefly discuss what may happen under this situation.

We conjecture that, if the mediator's ideal agreement is her private information, she obtains larger power, compared with the situation of complete information. Therefore, in this situation, agreements close to the mediator's ideal agreement may tend to be achieved as an equilibrium agreement. However, since negotiators do not know the mediator's ideal agreement, in order to deduce the mediator's preference, an agreement may tend to be delayed more severely than the situation with complete information. If this conjecture is true, contrary to the main role of a mediator, the mediator's private information may become an obstacle of facilitating the reaching of an agreement. We leave details of this situation as future work.

We also assumed that the information of negotiators' utilities is common knowledge. It may be more realistic to consider the situation where the information of negotiators' utilities is their private information. Jarque et al. (2003) analyze the role of a mediator under such a situation. In their model, a mediator is not a player. Therefore, the mediator does not have any bias. On the other hand, the situation with a biased mediator has not been analyzed. When a mediator is biased, equilibrium agreements may be affected by it. We also leave the analysis of this case as future work.

Fourth, it is also worth considering the n-negotiator model with a mediator as a natural extension of the model in Chapter 3. For example, in reality, the United Nations plays a role as a mediator in conflicts between several nations. In the n-negotiator model with a mediator, the feasible utility space is extended to n-dimensional space. Thus, it is expected that deriving equilibria in such a model is more difficult than the model in Chapter 3. As far as I know, the n-negotiator model with a mediator has not been analyzed in the literature of bargaining problems. On the other hand, as the literature of the mechanism design, some papers consider n-player models with a mediator where a mediator is introduced as a tool to exchange private information among players (for example, Myerson (1983) and Myerson (1986)). The knowledge from these papers may be useful to consider the n-negotiator model with a mediator in the literature of bargaining problems.

Also, in the model of Chapter 3, we assume that the discount factors have the same value at odd periods (where negotiators propose demands) and at even periods (where the mediator gives advice). That is, the negotiation periods by only negotiators and the mediation periods have the same time spans. However, these time spans may be different in reality. Thus, it may be worth considering the situation where the negotiation periods by only negotiators and the mediation periods have different time spans (discount factors).

Fifth, it may be worth considering claims problems with a mediator. Claims problems contain the situations of bankruptcy and inheritance. In these situations, a third party is often introduced to resolve conflict. For example, Ashlagi et al. (2012) analyze such a situation by the strategic approach. In their model, the authority imposes the division of an estate when there is conflict between negotiators' claims. However, the claims problems with a mediator (who can only give advice) has not been analyzed.

Since each claims problem is associated with some bargaining problem, it seems to be sufficient to consider only bargaining models with a mediator such as the models of Chapter 3 and the fourth discussion of this subsection. However, under the situation where negotiators have some claims, axioms for a reasonable agreement depends on these claims. Then, a reasonable agreement under a claims problem may be different from a reasonable agreement under a bargaining problem. Therefore, to introduce a mediator into not only bargaining problems but also claims problems is worth considering.

Also, there is a class so-called *bargaining problems with claims* which enriches both bargaining problems and claims problems. In bargaining problems with claims, negotiators have a general utility space rather than divide an endowment, and the negotiators have claims on the outside of the feasible utility space. For example, Albizuri et al. (2020), Chun and Thomson (1992), Dietzenbacher and Peters (2020), and Lombardi and Yoshihara (2010) analyze bargaining problems with claims by the axiomatic approach. To introduce a mediator into such problems may be also worth considering.

We leave analyzing such situations as future work. To analyze the situations where a mediator is introduced into claims problems and bargaining problems with claims by the strategic approach, knowledge from our studies in Chapter 3 and 4 may be useful.

Finally, we consider the permutation idea in the strategic approaches of claims problems. In our model in Chapter 4, we incorporated the permutation idea in our bargaining process to exchange an allocation. Also, Tsay and Yeh (2019) and Moreno-Ternero et al. (2020) incorporate the permutation idea to decide a coordinator of bargaining. However, it seems to be difficult for claimants to announce a permutation of claimants, especially when there are many claimants. In this sense, it may be unrealistic to incorporate a stage of announcing a permutation in a bargaining process. Therefore, it is worth considering a process of claims problems which does not depend on the permutation idea.

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