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Combinatorics of the quantum alcove model

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Abstract

We study the quantum alcove model, which plays an important role in Schubert calculus.

For a dominant weight λ, the collection of admissible subsets corresponding to a λ-chain is isomorphic to the crystal of quantum Lakshmibai-Seshadri (QLS) paths of shape λ. In this thesis, we give a generalization (interpolated QLS paths) of QLS paths for an arbitrary weight λ, and describe the relation between the collection of admissible subsets corresponding to a λ-chain and the set of interpolated QLS paths.

There exist two or more λ-chains associated to a weight λ. However, if λ is dominant, then all the collections of admissible subsets corresponding to these λ-chains are isomorphic through quantum Yang-Baxter moves. In this thesis, we also prove the existence of certain maps analogous to quantum Yang-Baxter moves for an arbitrary weight λ.

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Chapter 1

Introduction

1.1 Background

1.1.1 Chevalley formula for ordinary flag manifolds

Let G be a connected, simply-connected simple algebraic group over $\mathbb{C}$, with Borel subgroup $B \subset G$, maximal torus $H \subset G$, and weight lattice P. The *flag manifold* is defined as the quotient $X = G/B$. It is well-known that G/B has the orbit decomposition $G/B = \bigsqcup_{w \in W}(BwB/B)$ under the action of the Weyl group W. The Zariski closure $X_w := \overline{BwB}/B$ of the W-orbit $BwB/B \subset G/B$ for each $w \in W$ is called the *Schubert variety* corresponding to $w \in W$.

Schubert calculus is a branch of mathematics lying in the intersection of algebraic geometry, topology, and algebraic combinatorics, which studies various cohomology groups and K-groups of the flag manifold G/B. One of the fundamental problem of Schubert calculus is to describe the expansion of the product of classes corresponding to Schubert varieties X_w, called *Schubert classes*, into a linear combination of Schubert classes in the cohomology group or K-group of the flag manifold G/B. However, the description of such an expansion is, in general, difficult to obtain. Therefore, a Chevalley formula, which is the expansion formula for the product of the class of a line bundle associated to a weight of G and a Schubert class, has been studied. Namely, if we denote by $[\mathcal{L}(\lambda)]$ the class of the line bundle associated to $\lambda \in P$, and by $[X_w]$ the Schubert class corresponding to $w \in W$, then a Chevalley formula gives the description of the coefficients $c^v_{\lambda,w}$ in the expansion

$$[\mathcal{L}(-\lambda)] \cdot [X_w] = \sum_{v \in W} c^v_{\lambda,w}[X_v].$$

A Chevalley formula was first studied by Chevalley ([4]) as an equation in the cohomology group of the flag manifold G/B. After that, equations analogous to the Chevalley formula above has been studied for various kinds of algebras, such as K-groups and H-equivariant K-groups. Lenart-Postnikov ([22]) obtained a Chevalley formula for the H-equivariant K-group $K_H(G/B)$ of the flag manifold G/B; namely, they described the *K_H-Chevalley coefficient* $c_{w,v}^{\lambda,\xi} \in \mathbb{Z}$ in the following expansion:

$$[\mathcal{L}(-\lambda)] \cdot [\mathcal{O}_w] = [\mathcal{O}_w \otimes_{\mathcal{O}_X} \mathcal{L}(-\lambda)] = \sum_{v \in W} \sum_{\xi \in P} c_{w,v}^{\lambda,\xi} e^{-\xi} [\mathcal{O}_v],$$

where $\mathcal{O}_w$, $w \in W$, denotes the structure sheaf of X_w. They introduced the notion of *alcove model* to calculate K_H-Chevalley coefficients in their paper [22].

1.1.2 The quantum alcove model and equivariant K-group of semi-infinite flag manifolds

In [17], Lenart-Lubovsky introduced the notion of *quantum alcove model* for dominant weights, which is a generalization of the alcove model. In the theory of the quantum alcove model, one takes a λ-chain, which corresponds to an alcove path related to $\lambda \in P$, and then considers the collection of "admissible subsets" corresponding to such a λ-chain. The quantum alcove model is, for example, used for the expression of symmetric Macdonald polynomials at $t = 0$ in [30] and [21] (cf. [33]). On the other hand, in [21], Lenart-Naito-Sagaki-Schilling-Shimozono introduced *quantum Lakshmibai-Seshadri* (QLS) *paths*, which are combinatorial objects related to representation theory; for example, they are closely related to Kirillov-Reshetikhin modules and level-zero Demazure modules over quantum affine algebras. They constructed the *forgetful map*, which is a bijection between the collection of admissible subsets and the set of QLS paths. By using the forgetful map, they gave a symmetric Macdonald polynomial a certain representation-theoretic interpretation.

In [19], Lenart-Naito-Sagaki introduced a generalization of the quantum alcove model for arbitrary weights, to describe a Chevalley formula for the $(H \times \mathbb{C}^*)$-equivariant K-group of *semi-infinite flag manifolds*; here, a semi-infinite flag manifold is a certain ind-scheme of infinite type, which can be regarded as "loop spaces" of the ordinary flag manifold G/B. It is known by [11] that the $(H \times \mathbb{C}^*)$-equivariant K-group of the semi-infinite flag manifold is closely related to the H-equivariant quantum K-theory of the ordinary flag manifold G/B; here, the quantum K-theory is introduced by Givental and Lee (e.g., [7], [16]). As in the case of the ordinary flag manifold G/B, there exists a line bundle associated to $\lambda \in P$ over the semi-infinite flag manifold. Also, for each element x of the affine Weyl group W_{af}, there exists the corresponding semi-infinite Schubert variety in the semi-infinite flag manifold. For $\lambda \in P$ and $x \in W_{\mathrm{af}}$, we denote by $\mathcal{O}(\lambda)$ the sheaf corresponding to the line bundle associated to λ, and by $\mathcal{O}_x$ the structure sheaf of the semi-infinite Schubert variety corresponding to x. The Chevalley formula for the $(H \times \mathbb{C}^*)$-equivariant K-group of the semi-infinite flag manifold gives an explicit description of the coefficients $c_{x,\lambda}^{y}$ in the expansion

$$[\mathcal{O}(\lambda)] \cdot [\mathcal{O}_x] = \sum_{y \in W_{\mathrm{af}}} c_{x,\lambda}^{y} [\mathcal{O}_y],$$

where $c_{x,\lambda}^{y} \in \mathbb{Z}[P]((q^{-1}))$. The Chevalley formula is described and proved by Kato-Naito-Sagaki ([15]) in the case that λ is dominant, and by Naito-Orr-Sagaki ([28]) in the case that λ is anti-dominant. Finally, Lenart-Naito-Sagaki ([19]) described the Chevalley formula for an arbitrary weight λ in terms of the quantum alcove model.

1.1.3 Quantum Yang-Baxter moves

For a fixed $\lambda \in P$, there exist two or more λ-chains. Depending on these λ-chains, we can construct different collections of admissible subsets. Lenart-Lubovsky ([18]) proved that if λ is dominant, then these collections corresponding to reduced λ-chains are essentially identical, by constructing *quantum Yang-Baxter moves*. For two reduced λ-chains Γ_1, Γ_2, it is known that Γ_2 is obtained from Γ_1 by repeated application of a certain deformation procedure, called a *Yang-Baxter transformation* (cf. [22]). A quantum Yang-Baxter move is a bijection between two collections $\mathcal{A}(\Gamma_1)$ and $\mathcal{A}(\Gamma_2)$ of admissible subsets associated to two reduced λ-chains Γ_1 and Γ_2 such that Γ_2 is obtained from Γ_1 by a (single) Yang-Baxter transformation. A quantum Yang-Baxter move preserves important statistics, denoted by $\mathrm{end}(\cdot)$, $\mathrm{down}(\cdot)$, $\mathrm{wt}(\cdot)$, and $\mathrm{height}(\cdot)$. Thus collections $\mathcal{A}(\Gamma_1)$ and $\mathcal{A}(\Gamma_2)$ are "isomorphic".

1.2 Main results of this thesis

1.2.1 Interpolated QLS paths

The first topic of this thesis is *interpolated quantum Lakshmibai-Seshadri* (QLS) *paths*. We introduce interpolated QLS paths defined for an arbitrary weight, which generalizes QLS paths. We construct the forgetful map from the collection of admissible subsets to the corresponding set of interpolated QLS paths. As an application, we study the coefficients $c_{x,\lambda}^{y}$ in the following expansion of the graded characters $\operatorname{gch} V_x^-(\mu+\lambda)$ of the level-zero Demazure module $V_x^-(\mu+\lambda)$ over a quantum affine algebra:

$$\operatorname{gch} V_x^-(\mu+\lambda) = \sum_{y \in W_{\mathrm{af}}} c_{x,\lambda}^y \operatorname{gch} V_y(\mu),$$

where $c_{x,\lambda}^y \in \mathbb{Z}[P](\!(q^{-1})\!)$. Note that this expansion, together with an explicit description of the coefficients $c_{x,\lambda}^y$, provides a "representation-theoretic" analog of the Chevalley formula for the $(H \times \mathbb{C}^*)$-equivariant K-group of semi-infinite flag manifolds, given by Lenart-Naito-Sagaki ([19]); see [15] and [28]. Hence we call such an explicit expansion the *identity of Chevalley type* for graded characters. While the Chevalley formula is described in terms of the quantum alcove model in [19], we also give a description of the identity of Chevalley type in terms of interpolated QLS paths.

We give one more application. We know that the H-equivariant K-group $K_H(G/B)$ of the ordinary flag manifold G/B is a $\mathbb{Z}[P]$-module, where $\mathbb{Z}[P] (\simeq K_H(\mathrm{pt}))$ is the group algebra of the weight lattice P. As for the $\mathbb{Z}[P]$-module structure of $K_H(G/B)$, we study an *inverse Chevalley formula* for $K_H(G/B)$, which gives a description of the coefficients $c_w^{v,\xi}(\lambda)$ in the expansion

$$[\mathbb{C}(-\lambda) \otimes_{\mathbb{C}} \mathcal{O}_w] = \sum_{v \in W} \sum_{\xi \in P} c_w^{v,\xi}(\lambda) [\mathcal{O}_v \otimes_{\mathcal{O}_X} \mathcal{L}(\xi)],$$

where $\mathbb{C}(-\lambda)$ denotes 1-dimensional B-module of weight $-\lambda$, and $c_w^{v,\xi}(\lambda) \in \mathbb{Z}$. We show that the inverse Chevalley formula is given in terms of *interpolated LS paths*, which are the "$q = 0$ counterpart" of interpolated QLS paths. Also, by using the inverse Chevalley formula, we derive the character identity of the following form:

$$e^\lambda \operatorname{ch} D_w(\nu) = \sum_{v \in W} \sum_{\xi \in P} d_{w,\nu}^{v,\xi}(\lambda) \operatorname{ch} D_v(\xi),$$

where $\operatorname{ch} D_w(\nu)$ denotes the character of the Demazure module over B whose lowest weight is $w\nu$, and $d_{w,\nu}^{v,\xi}(\lambda) \in \mathbb{Z}$. This equation is, in fact, the specialization at $q = t = 0$ of the expansion formula for the product $e^\lambda E_{w\nu}(q,t)$, considered by Yip ([34]); here, $E_{w\nu}(q,t)$ denotes the non-symmetric Macdonald polynomial. Yip studied this equation in order to describe the Littlewood-Richardson rule for non-symmetric Macdonald polynomials; namely, the expansion formula for products of non-symmetric Macdonald polynomials. Littlewood-Richardson rule for the specialization at $q = t = 0$ of non-symmetric Macdonald polynomials, which are identical to characters of Demazure modules, is described under specific conditions by Haglund-Luoto-Mason-van Willigenburg ([8]), and the author refined their conditions in [12].

We write this character identity, which will play an important role in the Littlewood-Richardson rule, in terms of interpolated LS paths.

1.2.2 Generalized quantum Yang-Baxter moves

The second topic of this thesis is quantum Yang-Baxter moves. Let $\lambda \in P$ be an arbitrary weight, and take two λ-chains Γ_1 and Γ_2 such that Γ_2 is obtained from Γ_1 by a Yang-Baxter transformation. For each $w \in W$, we take the collections $\mathcal{A}(w, \Gamma_1)$, $\mathcal{A}(w, \Gamma_2)$ of admissible subsets corresponding to Γ_1, Γ_2. We prove that there exist certain subsets $\mathcal{A}_0(w, \Gamma_1) \subset \mathcal{A}(w, \Gamma_1)$, $\mathcal{A}_0(w, \Gamma_2) \subset \mathcal{A}(w, \Gamma_2)$ such that

- there exists a "sign-preserving" bijection from $\mathcal{A}_0(w, \Gamma_1)$ to $\mathcal{A}_0(w, \Gamma_2)$ which preserves statistics $\mathrm{end}(\cdot)$, $\mathrm{down}(\cdot)$, $\mathrm{wt}(\cdot)$, and $\mathrm{height}(\cdot)$,
- there exists a "sign-reversing" involution on $\mathcal{A}(w, \Gamma_1) \setminus \mathcal{A}_0(w, \Gamma_1)$ (resp., $\mathcal{A}(w, \Gamma_2) \setminus \mathcal{A}_0(w, \Gamma_2)$) which preserves statistics $\mathrm{end}(\cdot)$, $\mathrm{down}(\cdot)$, $\mathrm{wt}(\cdot)$, and $\mathrm{height}(\cdot)$.

The collection of these maps is a generalization of quantum Yang-Baxter moves.

Now, let $\lambda \in P$ be an arbitrary weight. Take a reduced λ-chain Γ, and fix $x \in W_{\mathrm{af}}$, and write $x = wt_\xi$ with $w \in W$ and $\xi \in Q^\vee$, where $Q^\vee$ is the coroot lattice. We consider the following generating function

$$\mathbf{G}_\Gamma(x) := \sum_{A \in \mathcal{A}(w,\Gamma)} (-1)^{n(A)} q^{-\,\mathrm{height}(A) - \langle \lambda, \xi \rangle} e^{\mathrm{wt}(A)}\, \mathrm{end}(A) t_{\xi + \mathrm{down}(A)};$$

here, $A \mapsto (-1)^{n(A)}$ denotes the "sign function" on $\mathcal{A}(w, \Gamma)$. If λ is dominant and w is the identity element e of W, then $\mathbf{G}_\Gamma(e)$ is a refinement of the specialization at $t = 0$ of the symmetric Macdonald polynomial $P_\lambda(q, 0)$, since we know from [21] that

$$P_\lambda(q, 0) = \sum_{A \in \mathcal{A}(e,\Gamma)} q^{\mathrm{height}(A)} e^{\mathrm{wt}(A)}.$$

By using our generalized quantum Yang-Baxter moves, we prove that the generating function $\mathbf{G}_\Gamma(x)$ does not depend on the choice of a "weakly reduced" λ-chain Γ for an arbitrary $\lambda \in P$; here, the definition of weakly reduced λ-chains is given in Definition 4.3.5. As an application of this, we obtain the identity of Chevalley type for graded characters of level-zero Demazure modules over a quantum affine algebra. Although the description and proof of the identity of Chevalley type for an arbitrary weight $\lambda \in P$ is parallel to those of the Chevalley formula ([19]), we can simplify the proof of the identity of Chevalley type by using our result on generating functions above. In this thesis, we give such a simple proof of the identity of Chevalley type.

1.3 Organization of the thesis

This thesis is organized as follows.

First, we prepare some basics in Chapter 2. In Section 2.1, we fix basic notation. In Section 2.2, we review the definition and basic facts of quantum Bruhat graphs. Also, we prepare specific reflection orders. In Section 2.3, we review the quantum alcove model. A part of this chapter is based on [14, Section 2].

The topic of Chapter 3 is interpolated QLS paths. In Section 3.1, we give the definition of interpolated QLS paths. Then, we describe the definition and properties of the forgetful map in Section 3.2. Moreover, we consider the equivariant K-groups of the semi-infinite flag manifold and ordinary flag manifold in Section 3.3. Also, in Section 3.3, we study the expansion formula for the product $e^{\lambda} \operatorname{ch} D_w(\nu)$. An extended abstract of this chapter is included in [13].

In Chapter 4, we treat quantum Yang-Baxter moves. First, we state a result that asserts the existence of generalized quantum Yang-Baxter moves in Section 4.1. Then, we give a proof of this result in Section 4.2. In Section 4.3, we study generating functions above, and give a simple proof of the identity of Chevalley type for graded characters of level-zero Demazure modules. This chapter is based on [14].

Chapter 2

Preliminaries

2.1 Basic notation

We fix basic notation of root systems and affine root systems, used throughout this thesis. A part of this section is based on [14, Section 2.1].

2.1.1 Notation for root systems

Let $\mathfrak{g}$ be a complex simple Lie algebra with Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. In Chapter 4, we may assume that $\mathfrak{g}$ is complex simple Lie algebra or complex Lie algebra of type $A_1 \times A_1$. We denote by $\langle \cdot, \cdot \rangle$ the canonical pairing of $\mathfrak{h}$ and $\mathfrak{h}^* := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$.

Let Δ be the root system of $\mathfrak{g}$, with $\Delta^+ \subset \Delta$ the set of all positive roots, and $\{\alpha_i\}_{i \in I}$ the set of all simple roots. Set $\rho := (1/2) \sum_{\alpha \in \Delta^+} \alpha$. For $\alpha \in \Delta^+$, we define $\mathrm{sgn}(\alpha) \in \{1, -1\}$ and $|\alpha| \in \Delta^+$ by

$$\mathrm{sgn}(\alpha) := \begin{cases} 1 & \text{if } \alpha \in \Delta^+, \\ -1 & \text{if } \alpha \in -\Delta^+, \end{cases}$$

$$|\alpha| := \mathrm{sgn}(\alpha)\alpha.$$

We set $Q := \sum_{i \in I} \mathbb{Z}\alpha_i$ and $Q^\vee := \sum_{i \in I} \mathbb{Z}\alpha_i^\vee$, where $\alpha^\vee$ is the coroot of $\alpha \in \Delta$; also, we set $Q^{\vee,+} := \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i^\vee$. Let $W = \langle s_i \mid i \in I \rangle$ be the Weyl group of $\mathfrak{g}$, with length function $\ell : W \to \mathbb{Z}_{\geq 0}$ and the longest element $w_\circ \in W$; here, for $\alpha \in \Delta^+$, $s_\alpha \in W$ denotes the reflection corresponding to α, and $s_i = s_{\alpha_i}$ is the simple reflection for $i \in I$.

For each $i \in I$, let ϖ_i denote the fundamental weight corresponding to α_i. Let $P := \sum_{i \in I} \mathbb{Z}\varpi_i$ be the weight lattice of $\mathfrak{g}$, with $P^+ := \sum_{i \in I} \mathbb{Z}_{\geq 0}\varpi_i$ the set of dominant weights; also, we set $\mathfrak{h}^*_{\mathbb{R}} := P \otimes_{\mathbb{Z}} \mathbb{R}$.

2.1.2 Affine root systems

Let $\mathfrak{g}_{\mathrm{af}} := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$ be the untwisted affine Lie algebra associated to $\mathfrak{g}$. We take the set of (affine) real roots Δ_{af} of $\mathfrak{g}_{\mathrm{af}}$, and the set of positive real roots Δ_{af}^{+}. We denote by $\delta \in \Delta_{\mathrm{af}}$ the null-root. Set $I_{\mathrm{af}} := I \sqcup \{0\}$. Then, we can take the set of simple roots $\{\alpha_i\}_{i \in I_{\mathrm{af}}} \subset \Delta_{\mathrm{af}}^{+}$ such that α_i, $i \in I$, is regarded as a simple root of $\mathfrak{g}$. We denote by $W_{\mathrm{af}} = \langle s_i \mid i \in I_{\mathrm{af}} \rangle$ the (affine) Weyl group of $\mathfrak{g}_{\mathrm{af}}$, where s_i, $i \in I_{\mathrm{af}}$, denotes the simple reflection corresponding to α_i. By [9, Section 6.5], we have $W_{\mathrm{af}} = \{t_\xi \mid \xi \in Q^\vee\} \rtimes W \simeq Q^\vee \rtimes W$, where t_ξ, $\xi \in Q^\vee$, is the translation corresponding to ξ.

Also, we take the set of real coroots $\Delta_{\mathrm{af}}^{\vee}$, the set of positive real coroots $\Delta_{\mathrm{af}}^{\vee,+} \subset \Delta_{\mathrm{af}}^{\vee}$, and the set of negative real coroots $\Delta_{\mathrm{af}}^{\vee,-} := -\Delta_{\mathrm{af}}^{\vee,+}$. We denote by $\widetilde{\delta} \in \Delta_{\mathrm{af}}^{\vee}$ the null-root, which is the coroot corresponding to δ. For $\beta^\vee \in \Delta_{\mathrm{af}}^{\vee}$, there uniquely exists a pair $(\gamma^\vee, k)$ of $\gamma \in \Delta$ and $k \in \mathbb{Z}$ such that $\beta^\vee = \gamma^\vee + k\widetilde{\delta}$. We set $\overline{\beta^\vee} := \gamma^\vee$, and $\deg(\beta^\vee) := k$.

Let $\mathfrak{g}_{\mathrm{af}}^{\vee}$ be the (affine) Lie algebra whose root system is $\Delta_{\mathrm{af}}^{\vee}$, and $\mathfrak{h}_{\mathrm{af}}^{\vee}$ its Cartan subalgebra. Let $W_{\mathrm{af}}^{\vee}$ be the (affine) Weyl group of $\mathfrak{g}_{\mathrm{af}}^{\vee}$. By abuse of notation, we also denote by s_i, $i \in I_{\mathrm{af}}$, the simple reflection corresponding to $\alpha_i^\vee$. Again, by [9, Section 6.5], we see that $W_{\mathrm{af}}^{\vee} = \{t_\xi \mid \xi \in Q\} \rtimes W \simeq Q \rtimes W$, where t_ξ, $\xi \in Q$, is the translation corresponding to ξ. Now, take the extended affine Weyl group $W_{\mathrm{ex}}^{\vee} = (P/Q) \rtimes W_{\mathrm{af}}^{\vee} = P \rtimes W$, and denote by $t_\mu \in W_{\mathrm{ex}}^{\vee}$, $\mu \in P$, the translation.

2.2 Quantum Bruhat graph

We review the definition of the *quantum Bruhat graph*, introduced in [3]. After that, we introduce the specific reflection order on Δ^+, and consider directed paths in $\mathrm{QBG}(W)$ which has increasing labels. A part of this section is based on [14, Section 2.2].

2.2.1 Definition of the quantum Bruhat graph

Definition 2.2.1 ([3, Definition 6.1]). The *quantum Bruhat graph* $\mathrm{QBG}(W)$ is the Δ^+-labeled directed graph whose vertices are the elements of W and whose edges are of the following form: $x \xrightarrow{\alpha} y$, with $x, y \in W$ and $\alpha \in \Delta^+$, such that $y = xs_\alpha$ and either of the following (B) or (Q) holds:

(B) $\ell(y) = \ell(x) + 1$;

(Q) $\ell(y) = \ell(x) - 2\langle \rho, \alpha^\vee \rangle + 1$.

If (B) (resp., (Q)) holds, then the edge $x \xrightarrow{\alpha} y$ is called a *Bruhat edge* (resp., *quantum edge*).

Definition 2.2.2 (cf. [2, Section 2.1]). The *Bruhat graph* $\mathrm{BG}(W)$ is the Δ^+-labeled directed graph whose vertices are the elements of W and whose edges are of the form $x \xrightarrow{\alpha} y$, with $x, y \in W$ and $\alpha \in \Delta^+$, such that $y = xs_\alpha$ and $\ell(y) = \ell(x) + 1$. In other words, the Bruhat graph $\mathrm{BG}(W)$ is the full subgraph of $\mathrm{QBG}(W)$ composed of all Bruhat edges.

Let $\mathbf{p} : w_0 \xrightarrow{\beta_1} w_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_r} w_r$ be a directed path in $\mathrm{QBG}(W)$. We set

$$\begin{aligned} \ell(\mathbf{p}) &:= r, \\ \mathrm{end}(\mathbf{p}) &:= w_r, \\ \mathrm{wt}(\mathbf{p}) &:= \sum_{\substack{k \in \{1,\ldots,r\} \\ w_{k-1} \xrightarrow{\beta_k} w_k \text{ is a quantum edge}}} \beta_k^\vee. \end{aligned}$$

Definition 2.2.3 (cf. [5, (2.2)]). Let $w \in W$. A total order $\lhd$ on $w\Delta^+$ is a *reflection order* if for all $\alpha, \beta \in w\Delta^+$ such that $\alpha + \beta \in w\Delta^+$, either $\alpha \lhd \alpha + \beta \lhd \beta$ or $\beta \lhd \alpha + \beta \lhd \alpha$ holds.

Remark 2.2.4. Let $w \in W$. If $\lhd$ is a reflection order on $w\Delta^+$, then the total order $\lhd^*$ defined by $\alpha \lhd^* \beta \Leftrightarrow \beta \lhd \alpha$ is also a reflection order on $w\Delta^+$.

It is known by [31, p. 662, Theorem] that the set of reflection orders on Δ^+ is in one-to-one correspondence with the set of reduced expressions of the longest element $w_\circ$ of W. More precisely, for each reduced expression $w_\circ = s_{i_1} \cdots s_{i_r}$, the total order $\lhd$ on Δ^+ defined by $\alpha_{i_1} \lhd s_{i_1}(\alpha_{i_2}) \lhd \cdots \lhd s_{i_1} \cdots s_{i_{r-1}}(\alpha_{i_r})$ is a reflection order. The correspondence $(i_1, \ldots i_r) \mapsto \lhd$ gives a desired bijection. With this fact, we can easily verify the following lemma.

Lemma 2.2.5. *Take a reflection order $\lhd$ on Δ^+. Let $\alpha, \beta \in \Delta^+$ be such that $\alpha \lhd \beta$. If $\alpha^\vee + \beta^\vee \in \Delta^{\vee,+} = \{\gamma^\vee \mid \gamma \in \Delta^+\}$, then one has $\alpha \lhd (\alpha^\vee + \beta^\vee)^\vee \lhd \beta$.*

Lemma 2.2.5 can be easily generalized to the argument for reflection orders $\prec$ on $w\Delta^+$ for arbitrary $w \in W$.

Corollary 2.2.6. *Let $w \in W$. Take a reflection order $\prec$ on $w\Delta^+$. Let $\alpha, \beta \in w\Delta^+$ be such that $\alpha \prec \beta$. If $\alpha^\vee + \beta^\vee \in w\Delta^{\vee,+}$, then one has $\alpha \prec (\alpha^\vee + \beta^\vee)^\vee \prec \beta$.*

Proof. We define a total order $\lhd$ on Δ^+ by $\alpha \lhd \beta \Leftrightarrow w\alpha \prec w\beta$ for each $\alpha, \beta \in \Delta^+$. Then $\lhd$ is a reflection order on Δ^+.

Let $\alpha, \beta \in \Delta^+$ be such that $w\alpha \prec w\beta$, and assume that $(w\alpha)^\vee + (w\beta)^\vee \in w\Delta^{\vee,+}$. Then we have $\alpha \lhd \beta$, and $\alpha^\vee + \beta^\vee = w^{-1}((w\alpha)^\vee + (w\beta)^\vee) \in \Delta^{\vee,+}$. Hence Lemma 2.2.5 implies that $\alpha \lhd (\alpha^\vee + \beta^\vee)^\vee \lhd \beta$. Therefore, we conclude that $w\alpha \prec ((w\alpha)^\vee + (w\beta)^\vee)^\vee \prec w\beta$, as desired. This proves the corollary. $\square$

Let $\lhd$ be a reflection order on Δ^+. A directed path $\mathbf{p}$ in $\mathrm{QBG}(W)$ of the form:

$$\mathbf{p} : w_0 \xrightarrow{\beta_1} w_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_r} w_r,$$

with $\beta_1 \lhd \cdots \lhd \beta_r$, is called a *label-increasing* directed path with respect to $\lhd$.

Theorem 2.2.7 ([3, Theorem 6.4])**.** *Let $\lhd$ be a reflection order on Δ^+. For all $v, w \in W$, there exists a unique label-increasing directed path from v to w in* $\mathrm{QBG}(W)$ *with respect to $\lhd$. Moreover, the unique label-increasing directed path from v to w has the minimum length.*

The property of $\mathrm{QBG}(W)$ in Theorem 2.2.7 is called the *shellability*.

For all $v, w \in W$, there exists at least one shortest directed path $\mathbf{p}$ from v to w; we set

$$\ell(v \Rightarrow w) := \ell(\mathbf{p}), \quad \mathrm{wt}(v \Rightarrow w) := \mathrm{wt}(\mathbf{p}).$$

Note that by [32, Lemma 1 (2)] or [20, Proposition 8.1], $\mathrm{wt}(v \Rightarrow w)$ is well-defined. Also, note that $\ell(v \Rightarrow w) \equiv \ell(w) - \ell(v) \mod 2$ for $v, w \in W$.

We consider a "generalization" of label-increasing directed paths. Let $\Pi = (\gamma_1, \dots, \gamma_r)$ be a sequence of roots, i.e., $\gamma_1, \dots, \gamma_r \in \Delta$. Assume that $\gamma_1, \dots, \gamma_r$ are distinct. Then we say that a directed path $\mathbf{p}$ is Π-*compatible* if $\mathbf{p}$ is of the following form:

$$\mathbf{p} : w_0 \xrightarrow{|\gamma_{j_1}|} w_1 \xrightarrow{|\gamma_{j_2}|} \cdots \xrightarrow{|\gamma_{j_p}|} w_p,$$

with $1 \le j_1 < \cdots < j_p \le r$. For $w \in W$, we denote by $\mathcal{P}(w, \Pi)$ the set of all Π-compatible directed paths in $\mathrm{QBG}(W)$ which start at w.

Remark 2.2.8. If $\{\gamma_1, \dots, \gamma_r\} \subset \Delta^+$, and if there exists a reflection order $\lhd$ on Δ^+ such that $\gamma_1 \lhd \cdots \lhd \gamma_r$, then a Π-compatible directed path in $\mathrm{QBG}(W)$ is a label-increasing directed path with respect to $\lhd$.

Let $\Pi = (\gamma_1, \dots, \gamma_r)$ be a sequence of root, with $\gamma_1, \dots, \gamma_r$ not necessarily distinct. For a directed path $\mathbf{p}$ of the form:

$$\mathbf{p} : w_0 \xrightarrow{|\gamma_{j_1}|} w_1 \xrightarrow{|\gamma_{j_2}|} \cdots \xrightarrow{|\gamma_{j_p}|} w_p,$$

with $1 \le j_1 < \cdots < j_p \le r$, we define $\mathrm{neg}(\mathbf{p})$ by

$$\mathrm{neg}(\mathbf{p}) := \#\{k \in \{1, \dots, p\} \mid \gamma_{j_k} \in -\Delta^+\}.$$

2.2.2 Specific reflection orders

Definition 2.2.9. For $\lambda \in P$, we set

$$\begin{aligned}\Delta^+(\lambda)_{>0} &:= \{\alpha \in \Delta^+ \mid \langle \lambda, \alpha^\vee \rangle > 0\},\\ \Delta^+(\lambda)_{=0} &:= \{\alpha \in \Delta^+ \mid \langle \lambda, \alpha^\vee \rangle = 0\},\\ \Delta^+(\lambda)_{<0} &:= \{\alpha \in \Delta^+ \mid \langle \lambda, \alpha^\vee \rangle < 0\}.\end{aligned}$$

Definition 2.2.10. For $\lambda \in P$, we define $\mathcal{RO}(\lambda, \Delta^+)$ to be the set of all reflection orders $\lhd$ on Δ^+ which satisfy $\alpha \lhd \beta \lhd \gamma$ for all $\alpha \in \Delta^+(\lambda)_{<0}$, $\beta \in \Delta^+(\lambda)_{=0}$, and $\gamma \in \Delta^+(\lambda)_{>0}$.

Remark 2.2.11. We can see that the set $\mathcal{RO}(\lambda, \Delta^+)$ is not empty for each $\lambda \in P$.

For $\lambda \in P$, there exists a unique $\lambda_+ \in W\lambda$ such that $\lambda_+ \in P^+$. Moreover, the set $\{w \in W \mid w\lambda_+ = \lambda\}$ has a unique maximal element (with respect to the Bruhat order), denoted by $w(\lambda)$.

Lemma 2.2.12. *For $\lambda \in P$, we have $w(\lambda)\Delta^+ = (\Delta^+(\lambda)_{>0}) \sqcup (-\Delta^+(\lambda)_{=0}) \sqcup (-\Delta^+(\lambda)_{<0})$.*

Proof. We set $I_{\lambda_+} := \{i \in I \mid \langle \lambda_+, \alpha_i^\vee \rangle = 0\}$, and $\Delta^+_{\lambda_+} := \Delta^+ \cap \sum_{i \in I_{\lambda_+}} \mathbb{Z}_{\geq 0}\alpha_i$.

If $\beta \in \Delta^+(\lambda)_{>0}$, then $\langle \lambda, \beta^\vee \rangle > 0$. Hence

$$\langle \lambda_+, w(\lambda)^{-1}\beta^\vee \rangle = \langle w(\lambda)\lambda_+, \beta^\vee \rangle = \langle \lambda, \beta^\vee \rangle > 0,$$

and since λ_+ is dominant, this implies $w(\lambda)^{-1}\beta \in \Delta^+ \setminus \Delta^+_{\lambda_+}$. Therefore, we have $\beta \in w(\lambda)\Delta^+ \setminus w(\lambda)\Delta^+_{\lambda_+}$.

If $\beta \in \Delta^+(\lambda)_{<0}$, then $\langle \lambda, \beta^\vee \rangle < 0$. Hence

$$\langle \lambda_+, w(\lambda)^{-1}\beta^\vee \rangle = \langle w(\lambda)\lambda_+, \beta^\vee \rangle = \langle \lambda, \beta^\vee \rangle < 0,$$

and this implies $w(\lambda)^{-1}\beta \in -(\Delta^+ \setminus \Delta^+_{\lambda_+})$. Therefore, we have $-\beta \in w(\lambda)\Delta^+ \setminus w(\lambda)\Delta^+_{\lambda_+}$.

The above arguments show that $(\Delta^+(\lambda)_{>0}) \sqcup (-\Delta^+(\lambda)_{<0}) \subset w(\lambda)\Delta^+ \setminus w(\lambda)\Delta^+_{\lambda_+}$. Next, we consider the relation between $\Delta^+(\lambda)_{=0}$ and $w(\lambda)\Delta^+_{\lambda_+}$.

We claim that $w(\lambda)\Delta^+_{\lambda_+} \subset -\Delta^+$. Let $i \in I_{\lambda_+}$. Since $\langle \lambda_+, \alpha_i^\vee \rangle = 0$, or equivalently, $s_i\lambda_+ = \lambda_+$, we have $w(\lambda)s_i\lambda_+ = w(\lambda)\lambda_+ = \lambda$. By the maximality of $w(\lambda)$, we obtain $w(\lambda)s_i < w(\lambda)$. This implies that $w(\lambda)\alpha_i \in -\Delta^+$. Hence we have $w(\lambda)\Delta^+_{\lambda_+} \subset -\Delta^+$, as desired.

Let $\beta \in \Delta^+_{\lambda_+}$. Since $\langle \lambda_+, w(\lambda)^{-1}\beta^\vee \rangle = \langle \lambda, \beta^\vee \rangle$, we deduce that $\beta \in w(\lambda)\Delta^+_{\lambda_+}$ if and only if $\beta \in -\Delta^+(\lambda)_{=0}$. Hence $w(\lambda)\Delta^+_{\lambda_+} = -\Delta^+(\lambda)_{=0}$.

Therefore, we have $(\Delta^+(\lambda)_{>0}) \sqcup (-\Delta^+(\lambda)_{=0}) \sqcup (-\Delta^+(\lambda)_{<0}) \subset w(\lambda)\Delta^+$. Since $|\Delta^+(\lambda)_{>0}|+|-\Delta^+(\lambda)_{=0}|+|-\Delta^+(\lambda)_{<0}| = |\Delta^+| = |w(\lambda)\Delta^+|(< \infty)$, we conclude that $(\Delta^+(\lambda)_{>0}) \sqcup (-\Delta^+(\lambda)_{=0}) \sqcup (-\Delta^+(\lambda)_{<0}) = w(\lambda)\Delta^+$, as desired. This proves the lemma. □

Based on this lemma, we define the set of reflection orders on $w(\lambda)\Delta^+$ which satisfy some additional conditions.

Definition 2.2.13. For $\lambda \in P$, we define $\mathcal{RO}(\lambda, w(\lambda)\Delta^+)$ to be the set of all reflection orders $\prec$ on $w(\lambda)\Delta^+$ which satisfy $\gamma \prec -\alpha \prec -\beta$ for all $\gamma \in \Delta^+(\lambda)_{>0}$, $\alpha \in \Delta^+(\lambda)_{<0}$ and $\beta \in \Delta^+(\lambda)_{=0}$.

Let $\lhd \in \mathcal{RO}(\lambda, \Delta^+)$. Write

$$\underbrace{\gamma_{-r} \lhd \cdots \lhd \gamma_{-q-2} \lhd \gamma_{-q-1}}_{\Delta^+(\lambda)_{<0}} \lhd \underbrace{\gamma_{-q} \lhd \cdots \lhd \gamma_{-1} \lhd \gamma_0}_{\Delta^+(\lambda)_{=0}} \lhd \underbrace{\gamma_1 \lhd \gamma_2 \lhd \cdots \lhd \gamma_p}_{\Delta^+(\lambda)_{>0}},$$

where $\Delta^+(\lambda)_{<0} = \{\gamma_{-r}, \dots, \gamma_{-q-2}, \gamma_{-q-1}\}$, $\Delta^+(\lambda)_{=0} = \{\gamma_{-q}, \dots, \gamma_{-1}, \gamma_0\}$, and $\Delta^+(\lambda)_{>0} = \{\gamma_1, \gamma_2, \dots, \gamma_p\}$. Then, we define the total order $\prec = \prec_\lhd$ on $w(\lambda)\Delta^+$ by

$$\begin{aligned} &\underbrace{\gamma_1 \prec \gamma_2 \prec \cdots \prec \gamma_p}_{\Delta^+(\lambda)_{>0}} \prec \underbrace{-\gamma_{-r} \prec \cdots \prec -\gamma_{-q-2} \prec -\gamma_{-q-1}}_{-\Delta^+(\lambda)_{<0}} \\ &\qquad\qquad \prec \underbrace{-\gamma_{-q} \prec \cdots \prec -\gamma_{-1} \prec -\gamma_0}_{-\Delta^+(\lambda)_{=0}}. \end{aligned} \tag{2.2.1}$$

Proposition 2.2.14. *The correspondence $\lhd \mapsto \prec_\lhd$ defines a bijection between $\mathcal{RO}(\lambda, \Delta^+)$ and $\mathcal{RO}(\lambda, w(\lambda)\Delta^+)$.*

Proof. We show that $\prec = \prec_\lhd$ is a reflection order on $w(\lambda)\Delta^+$. Let $\alpha, \beta \in w(\lambda)\Delta^+$ be such that $\alpha \prec \beta$ and $\alpha + \beta \in w(\lambda)\Delta^+$. Then α and β satisfy either of the following:

(1) $\alpha, \beta \in \Delta^+(\lambda)_{>0}$;

(2) $\alpha, \beta \in -\Delta^+(\lambda)_{=0}$;

(3) $\alpha, \beta \in -\Delta^+(\lambda)_{<0}$;

(4) $\alpha \in \Delta^+(\lambda)_{>0}$ and $\beta \in -\Delta^+(\lambda)_{<0}$;

(5) $\alpha \in \Delta^+(\lambda)_{>0}$ and $\beta \in -\Delta^+(\lambda)_{=0}$;

(6) $\alpha \in -\Delta^+(\lambda)_{<0}$ and $\beta \in -\Delta^+(\lambda)_{=0}$.

In the case (1)–(3) and (6), it is obvious that $\alpha \prec \alpha + \beta \prec \beta$ since $\lhd$ is a reflection order on Δ^+.

Assume the case (4) or (5). Suppose, for a contradiction, that $\alpha \prec \alpha+\beta \prec \beta$ fails. If $\alpha \prec \beta \prec \alpha + \beta$, then $\alpha + \beta \in (-\Delta^+(\lambda)_{<0}) \sqcup (-\Delta^+(\lambda)_{=0})$. By the definition of $\prec = \prec_\lhd$, we deduce that $-\beta \lhd -\alpha - \beta \lhd \alpha$. However, since $-\beta = (-\alpha - \beta) + \alpha$, this contradicts the assumption that $\lhd$ is a reflection order on Δ^+. Similarly, we can show that $\alpha + \beta \prec \alpha \prec \beta$ is also false. Therefore, we have $\alpha \prec \alpha + \beta \prec \beta$, as desired.

This completes the proof of $\prec \in \mathcal{RO}(\lambda, w(\lambda)\Delta^+)$.

Now, we claim that $\lhd \mapsto \prec_\lhd$ defines a bijection from $\mathcal{RO}(\lambda, \Delta^+)$ to $\mathcal{RO}(\lambda, w(\lambda)\Delta^+)$. Since the construction of the inverse map $\prec_\lhd \mapsto \lhd$ is obvious, and by the same argument as above, it is easy to verify that the resulting total order $\lhd$ is a reflection order on Δ^+. $\square$

2.3 The quantum alcove model

We review the quantum alcove model, introduced in [17] and [19]. A part of this section is based on [14, Section 2.3].

2.3.1 Alcove paths and admissible subsets

First, we recall from [22] the definition of alcove paths. For $\alpha \in \Delta$ and $k \in \mathbb{Z}$, we set $H_{\alpha,k} := \{\nu \in \mathfrak{h}^*_\mathbb{R} \mid \langle \nu, \alpha^\vee \rangle = k\}$; $H_{\alpha,k}$ is a hyperplane in $\mathfrak{h}^*_\mathbb{R}$. Also, for $\alpha \in \Delta$ and $k \in \mathbb{Z}$, we denote by $s_{\alpha,k}$ the reflection with respect to $H_{\alpha,k}$. Note that $s_{\alpha,k}(\nu) = \nu - (\langle \nu, \alpha^\vee \rangle - k)\alpha$ for $\nu \in \mathfrak{h}^*_\mathbb{R}$. Each connected component of the space

$$\mathfrak{h}^*_\mathbb{R} \setminus \bigcup_{\alpha \in \Delta^+, k \in \mathbb{Z}} H_{\alpha,k}$$

is called an *alcove*. If two alcoves A and B have a common wall, then we say that A and B are *adjacent*. For adjacent alcoves A and B, we write $A \xrightarrow{\beta} B$, $\beta \in \Delta$, if the common wall of A and B is contained in $H_{\beta,k}$ for some $k \in \mathbb{Z}$, and β points in the direction from A to B.

Definition 2.3.1 ([22, Definition 5.2]). A sequence $(A_0, \dots, A_r)$ of alcoves is called an *alcove path* if A_{i-1} and A_i are adjacent for all $i = 1, \dots, r$. If the length r of an alcove path $(A_0, \dots, A_r)$ is minimal among all alcove paths from A_0 to A_r, we say that $(A_0, \dots, A_r)$ is *reduced*.

The *fundamental alcove* $A_\circ$ is defined by

$$A_\circ := \{\nu \in \mathfrak{h}^*_\mathbb{R} \mid 0 < \langle \nu, \alpha^\vee \rangle < 1 \text{ for all } \alpha \in \Delta^+\}.$$

Also, for $\lambda \in P$, we define A_λ by

$$A_\lambda := A_\circ + \lambda = \{\nu + \lambda \mid \nu \in A_\circ\}.$$

Definition 2.3.2 ([22, Definition 5.4])**.** Let $\lambda \in P$. A sequence $(\beta_1, \dots, \beta_r)$ of roots $\beta_1, \dots, \beta_r \in \Delta$ is called a λ-*chain* if there exists an alcove path $(A_0, \dots, A_r)$, with $A_0 = A_\circ$ and $A_r = A_{-\lambda}$, such that

$$A_\circ = A_0 \xrightarrow{-\beta_1} A_1 \xrightarrow{-\beta_2} \cdots \xrightarrow{-\beta_r} A_r = A_{-\lambda}.$$

If such an alcove path $(A_0, \dots, A_r)$ is reduced, then we call $(\beta_1, \dots, \beta_r)$ a *reduced* λ-chain.

Now, following [19, Section 3.2], we review the quantum alcove model.

Definition 2.3.3 ([19, Definition 17])**.** Let $\lambda \in P$, and let $\Gamma = (\beta_1, \dots, \beta_r)$ be a λ-chain. Fix $w \in W$. A subset $A = \{j_1 < \cdots < j_p\} \subset \{1, \dots, r\}$ is said to be w-*admissible* if

$$\mathbf{p}(A) : w = w_0 \xrightarrow{|\beta_{j_1}|} w_1 \xrightarrow{|\beta_{j_2}|} \cdots \xrightarrow{|\beta_{j_p}|} w_p$$

is a directed path in $\mathrm{QBG}(W)$. Let $\mathcal{A}(w, \Gamma)$ denote the set of all w-admissible subsets of $\{1, \dots, r\}$.

Remark 2.3.4. The original definition of admissible subsets in [17] is only for $w = e \in W$. The notion of w-admissible subsets for an arbitrary $w \in W$ is introduced in [19].

Also, we prepare the "$q = 0$ part" of $\mathcal{A}(w, \Gamma)$.

Definition 2.3.5. Let $\lambda \in P$, and let $\Gamma = (\beta_1, \dots, \beta_r)$ be a λ-chain. Fix $w \in W$. We define the set $\mathcal{A}|_{q=0}(w, \Gamma)$ by

$$\mathcal{A}|_{q=0}(w, \Gamma) := \{A \in \mathcal{A}(w, \Gamma) \mid \mathbf{p}(A) \text{ is a directed path in } \mathrm{BG}(W)\}.$$

Let $\lambda \in P$, and let $\Gamma = (\beta_1, \dots, \beta_r)$ be a λ-chain. By the definition of λ-chains, there exists an alcove path $(A_\circ = A_0, \dots, A_r = A_{-\lambda})$ such that

$$A_\circ = A_0 \xrightarrow{-\beta_1} A_1 \xrightarrow{-\beta_2} \cdots \xrightarrow{-\beta_r} A_r = A_{-\lambda}.$$

For $k = 1, \dots, r$, we take $l_k \in \mathbb{Z}$ such that $H_{\beta_k, -l_k}$ contains the common wall of A_{k-1} and A_k, and set $\widetilde{l_k} := \langle \lambda, \beta_k^\vee \rangle - l_k$.

Fix $w \in W$. For $A = \{j_1 < \cdots < j_p\} \in \mathcal{A}(w, \Gamma)$, we set

$$\mathrm{end}(A) := w s_{|\beta_{j_1}|} \cdots s_{|\beta_{j_p}|}, \quad \mathrm{wt}(A) := -w s_{\beta_{j_1}, -l_{j_1}} \cdots s_{\beta_{j_p}, -l_{j_p}}(-\lambda);$$

we call $\mathrm{wt}(A)$ the *weight* of A. Also, we define a subset $A^- \subset A$ by

$$A^- := \left\{ j_k \in A \;\middle|\; ws_{|\beta_{j_1}|} \cdots s_{|\beta_{j_{k-1}}|} \xrightarrow{|\beta_{j_k}|} ws_{|\beta_{j_1}|} \cdots s_{|\beta_{j_k}|} \text{ is a quantum edge} \right\},$$

and set

$$\mathrm{down}(A) := \sum_{j \in A^-} |\beta_j|^\vee, \quad \mathrm{height}(A) := \sum_{j \in A^-} \mathrm{sgn}(\beta_j)\widetilde{l_j};$$

note that $\mathrm{end}(A) = \mathrm{end}(\mathbf{p}(A))$ and $\mathrm{down}(A) = \mathrm{wt}(\mathbf{p}(A))$. In addition, we define $n(A) \in \mathbb{Z}_{\geq 0}$ by $n(A) := \#\{j \in A \mid \beta_j \in -\Delta^+\}$; note that $n(A) = \mathrm{neg}(\mathbf{p}(A))$.

Remark 2.3.6. If $A \in \mathcal{A}|_{q=0}(w, \Gamma)$, then we have $A^- = \emptyset$. Hence $\mathrm{down}(A) = 0$ and $\mathrm{height}(A) = 0$.

2.3.2 Reduced chains of roots and reduced expressions

Let $\lambda \in P$ be an integral weight. We describe the relationship between reduced λ-chains and reduced expressions of $t_\lambda \in W_{\mathrm{ex}}^\vee$.

Take a reduced expression $t_{-\lambda} = \pi^\vee s_{i_1} s_{i_2} \cdots s_{i_r} \in W_{\mathrm{ex}}^\vee$, where $i_1, \ldots, i_r \in I_{\mathrm{af}}$ and $\pi^\vee$ is an automorphism of an (affine) Dynkin diagram of $\mathfrak{g}_{\mathrm{af}}^\vee$. For $k \in \{1, \ldots, r\}$, we set

$$(\beta_k^{\mathsf{L}})^\vee := \pi^\vee s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}^\vee),$$

and $\gamma_k^{\mathsf{L}} := \overline{\beta_k^{\mathsf{L}}}(= \overline{((\beta_k^{\mathsf{L}})^\vee)^\vee})$. Note that $\{(\beta_1^{\mathsf{L}})^\vee, \ldots, (\beta_r^{\mathsf{L}})^\vee\} = \Delta_{\mathrm{af}}^{\vee,+} \cap t_\lambda^{-1} \Delta_{\mathrm{af}}^{\vee,-}$. Then, [22, Lemma 5.3] implies the following.

Lemma 2.3.7. *The sequence $(\gamma_1^{\mathsf{L}}, \ldots, \gamma_r^{\mathsf{L}})$ is a reduced λ-chain.*

In addition, for the alcove path $(A_0, \ldots, A_r)$ with $A_0 = A_\circ$ and $A_r = A_{-\lambda}$ corresponding to the λ-chain $(\gamma_1^{\mathsf{L}}, \ldots, \gamma_r^{\mathsf{L}})$, i.e.,

$$A_\circ = A_0 \xrightarrow{-\gamma_1^{\mathsf{L}}} A_1 \xrightarrow{-\gamma_2^{\mathsf{L}}} \cdots \xrightarrow{-\gamma_r^{\mathsf{L}}} A_r = A_{-\lambda},$$

it is easy to verify that the hyperplane consisting the common wall of A_{k-1} and A_k for each $k = 1, \ldots, r$ is identical to $H_{\gamma_k^{\mathsf{L}}, -\deg((\beta_k^{\mathsf{L}})^\vee)}$.

Conversely, take a reduced λ-chain $(\beta_1, \ldots, \beta_r)$. Then there exists a reduced alcove path $(A_0, \ldots, A_r)$ with $A_0 = A_\circ$ and $A_r = A_{-\lambda}$ such that

$$A_\circ = A_0 \xrightarrow{-\beta_1} A_1 \xrightarrow{-\beta_2} \cdots \xrightarrow{-\beta_r} A_r = A_{-\lambda}.$$

By [22, Lemma 5.3], there exists $v \in W_{\mathrm{af}}^{\vee}$ and its reduced expression $v = s_{i_1} \cdots s_{i_r}$ such that $vA_{\circ} = A_{-\lambda}$, and that $A_k = s_{i_1} \cdots s_{i_k}(A_{\circ})$ for $k = 0, \dots, r$. Since $t_{-\lambda}A_{\circ} = A_{-\lambda}$, there exists an automorphism $\pi^{\vee}$ of the Dynkin diagram of $\mathfrak{g}_{\mathrm{af}}^{\vee}$ such that $t_{-\lambda} = \pi^{\vee}v$. Hence we obtain a reduced expression $t_{-\lambda} = \pi^{\vee}s_{i_1} \cdots s_{i_r}$ of $t_{-\lambda}$.

Therefore, the set of reduced λ-chains and the set of reduced expressions of $t_{\lambda} \in W_{\mathrm{ex}}^{\vee}$ are in one-to-one correspondence.

Chapter 3

Interpolated QLS paths

3.1 Definition of interpolated QLS paths

We introduce interpolated QLS paths, which is one of the analog of quantum Lakshmibai-Seshadri paths (QLS paths for short), defined in [21, Section 3]. We fix $\lambda \in P$ and $\lhd \in \mathcal{RO}(\lambda, \Delta^+)$.

3.1.1 Integrality conditions

We prepare some notation to introduce interpolated QLS paths.

Definition 3.1.1. Let $x, y \in W$, and $\sigma \in \mathbb{Q}$.

(1) If there exists a directed path $x = x_0 \xrightarrow{\gamma_1} x_1 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_r} x_r = y$ in $\mathrm{QBG}(W)$ such that $\gamma_1, \dots, \gamma_r \in \Delta^+(\lambda)_{>0}$, $\gamma_1 \rhd \cdots \rhd \gamma_r$, and $\sigma\langle \lambda, \gamma_k^\vee \rangle \in \mathbb{Z}$ for all $k = 1, \dots, r$, then we write $x \xRightarrow{(\lambda,+)}_\sigma y$.

(2) If there exists a directed path $x = x_0 \xrightarrow{\gamma_1} x_1 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_r} x_r = y$ in $\mathrm{BG}(W)$ such that $\gamma_1, \dots, \gamma_r \in \Delta^+(\lambda)_{>0}$, $\gamma_1 \rhd \cdots \rhd \gamma_r$, and $\sigma\langle \lambda, \gamma_k^\vee \rangle \in \mathbb{Z}$ for all $k = 1, \dots, r$, then we write $x \xRightarrow{(\lambda,+,q=0)}_\sigma y$.

(3) If there exists a directed path $x = x_0 \xrightarrow{\gamma_1} x_1 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_r} x_r = y$ in $\mathrm{QBG}(W)$ such that $\gamma_1, \dots, \gamma_r \in \Delta^+(\lambda)_{<0}$, $\gamma_1 \rhd \cdots \rhd \gamma_r$, and $\sigma\langle \lambda, \gamma_k^\vee \rangle \in \mathbb{Z}$ for all $k = 1, \dots, r$, then we write $x \xRightarrow{(\lambda,-)}_\sigma y$.

(4) If there exists a directed path $x = x_0 \xrightarrow{\gamma_1} x_1 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_r} x_r = y$ in $\mathrm{BG}(W)$ such that $\gamma_1, \dots, \gamma_r \in \Delta^+(\lambda)_{<0}$, $\gamma_1 \rhd \cdots \rhd \gamma_r$, and $\sigma\langle \lambda, \gamma_k^\vee \rangle \in \mathbb{Z}$ for all $k = 1, \dots, r$, then we write $x \xRightarrow{(\lambda,-,q=0)}_\sigma y$.

If $\sigma = 1$, then we omit σ: write $x \xRightarrow{(\lambda,\pm)} y$ or $x \xRightarrow{(\lambda,\pm,q=0)} y$.

3.1.2 Interpolated QLS paths

Definition 3.1.2. A tuple $(\underline{x}; \underline{y}; \underline{\sigma})$ of a sequence $\underline{x} : x_1, \dots, x_s$ of elements in W such that $x_i \neq x_{i+1}$ for all $1 \le i \le s-1$, a sequence $\underline{y} : y_1, \dots, y_{s-1}$ of elements in W such that $y_i \neq y_{i+1}$ for all $1 \le i \le s-2$, and a sequence $\underline{\sigma}$: $\sigma_0, \dots, \sigma_s$ of rational numbers is called an *interpolated quantum Lakshmibai-Seshadri* (QLS) *path* of shape λ if the following conditions hold:

(1) $0 = \sigma_0 < \sigma_1 < \cdots < \sigma_s = 1$;

(2) for all $1 \le i \le s-1$, it holds that $x_{i+1} \xRightarrow{(\lambda,-)}_{\sigma_i} y_i$;

(3) for all $1 \le i \le s-1$, it holds that $y_i \xRightarrow{(\lambda,+)}_{\sigma_i} x_i$.

We denote by $\mathrm{IQLS}(\lambda)$ the set of all interpolated QLS paths of shape λ.

For $\eta = (\underline{x}; \underline{y}; \underline{\sigma}) \in \mathrm{IQLS}(\lambda)$, a sequence $\underline{y}$ is possibly empty. In this case, we write $\eta = (\underline{x}; ; \underline{\sigma})$.

As (ordinary) QLS paths, we can define initial/final directions and weights of interpolated QLS paths.

Definition 3.1.3. Let $\eta = (x_1, \dots, x_s; y_1, \dots, y_{s-1}; \sigma_0, \dots, \sigma_s) \in \mathrm{IQLS}(\lambda)$ be an interpolated QLS path of shape λ.

(1) We set $\iota(\eta) := x_1$ and $\kappa(\eta) := x_s$. We call $\iota(\eta)$ the *initial direction* of η. Also, we call $\kappa(\eta)$ the *final direction* of η.

(2) We define $\mathrm{wt}(\eta)$ by

$$\mathrm{wt}(\eta) := \sum_{k=1}^{s} (\sigma_k - \sigma_{k-1}) x_k \lambda.$$

We call $\mathrm{wt}(\lambda)$ the *weight* of η.

Also, we define the negativity length of interpolated QLS paths, which is not defined for (ordinary) QLS paths.

Definition 3.1.4. Let $\eta = (x_1, \dots, x_s; y_1, \dots, y_{s-1}; \sigma_0, \dots, \sigma_s) \in \mathrm{IQLS}(\lambda)$ be an interpolated QLS path of shape λ. We define $\mathrm{neg}(\eta) \in \mathbb{Z}$ by

$$\mathrm{neg}(\eta) := \sum_{k=1}^{s-1} \ell(x_{k+1} \Rightarrow y_k).$$

We call $\mathrm{neg}(\eta)$ the *negativity length* of η.

Example 3.1.5. In this example, we assume that $\mathfrak{g}$ is of type A_2. Let $\lambda = -\varpi_1 + 2\varpi_2$. Let us calculate $\mathrm{IQLS}(\lambda)$.

Note that $\langle \lambda, \alpha_1^\vee \rangle = -1$, $\langle \lambda, \alpha_1^\vee + \alpha_2^\vee \rangle = 1$, and $\langle \lambda, \alpha_2^\vee \rangle = 2$. Hence $\alpha_1 \in \Delta^+(\lambda)_{<0}$ and $\alpha_1 + \alpha_2, \alpha_2 \in \Delta^+(\lambda)_{>0}$. We define a total order $\lhd$ on Δ^+ by $\alpha_1 \lhd \alpha_1 + \alpha_2 \lhd \alpha_2$. Then we have $\lhd \in \mathcal{RO}(\lambda, \Delta^+)$.

First, we see that $\mathrm{IQLS}(\lambda)$ has all straight paths $(x;;0,1)$, $x \in W$. Let us calculate other paths. Only the positive root $\alpha = \alpha_2 \in \Delta^+$ satisfies that there exists $\sigma \in \mathbb{Q} \cap (0,1)$ such that $\sigma\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}$. In this case, we have $\sigma = 1/2$. Hence paths $\eta \in \mathrm{IQLS}(\lambda)$ which are not straight paths are of the form $(xs_{\alpha_2}, x; x; 0, 1/2, 1)$ for some $x \in W$ such that $x \xrightarrow{\alpha_2} xs_{\alpha_2}$. Since α_2 is a simple root, there exists an edge $x \xrightarrow{\alpha_2} xs_{\alpha_2}$ for all $x \in W$. Therefore, we conclude that

$$\mathrm{IQLS}(\lambda) = \{(x;;0,1) \mid x \in W\} \sqcup \{(xs_{\alpha_2}, x; x; 0, 1/2, 1) \mid x \in W\}.$$

3.1.3 Interpolated LS paths

We also introduce interpolated Lakshmibai-Seshadri (LS) paths, which is a "$q = 0$ part" of the set of interpolated QLS paths. Interpolated LS paths are one of the analog of ordinary LS paths.

Definition 3.1.6. A tuple $(\underline{x}; \underline{y}; \underline{\sigma})$ of a sequence $\underline{x} : x_1, \dots, x_s$ of elements in W, a sequence $\underline{y} : y_1, \dots, y_{s-1}$ of elements in W, and a sequence $\underline{\sigma} : \sigma_0, \dots, \sigma_s$ of rational numbers is called an *interpolated Lakshmibai-Seshadri* (LS) *path* of shape λ if the following conditions hold:

(1) $0 = \sigma_0 < \sigma_1 < \cdots < \sigma_s = 1$;

(2) for all $1 \le i \le s-1$, it holds that $x_{i+1} \xrightarrow{(\lambda, -, q=0)}_{\sigma_i} y_i$;

(3) for all $1 \le i \le s-1$, it holds that $y_i \xrightarrow{(\lambda, +, q=0)}_{\sigma_i} x_i$.

We denote by $\mathrm{ILS}(\lambda)$ the set of all interpolated LS paths of shape λ.

By the definition of interpolated LS paths, it is obvious that $\mathrm{ILS}(\lambda) \subset \mathrm{IQLS}(\lambda)$.

Example 3.1.7. Again, we assume that $\mathfrak{g}$ is of type A_2, and let $\lambda = -\varpi_1 + 2\varpi_2$. Recall Example 3.1.5. It is obvious that $(x;;0,1) \in \mathrm{ILS}(\lambda)$ for all $x \in W$. For $x \in W$, we see that $(xs_{\alpha_2}, x; x; 0, 1/2, 1) \in \mathrm{ILS}(\lambda)$ if and only if $x \xrightarrow{\alpha_2} xs_{\alpha_2}$ is a Bruhat edge. Since $x \xrightarrow{\alpha_2} xs_{\alpha_2}$ is a Bruhat edge for $x = e, s_1, s_2s_1$, and that is a quantum edge for $x = s_2, s_1s_2, s_1s_2s_1$, we conclude that

$$\mathrm{ILS}(\lambda) = \{(x;;0,1) \mid x \in W\} \sqcup \{(xs_{\alpha_2}, x; x; 0, 1/2, 1) \mid x = e, s_1, s_2s_1\}.$$

3.2 Forgetful maps

We construct the map from $\mathcal{A}(w;\Gamma)$ to $\mathrm{IQLS}(\lambda)$ for $w \in W$, $\lambda \in P$, and a "suitable" λ-chain Γ. Note that our construction method of forgetful maps and proofs of properties of those is, in fact, a generalization of the argument in [27]. From now on, we fix $\lambda \in P$.

3.2.1 Construction of forgetful maps

First, we construct a rational number and a (finite) root from an affine root which belongs to $\Delta_{\mathrm{af}}^{\vee,+} \cap t_\lambda^{-1}\Delta_{\mathrm{af}}^{\vee,-}$.

Lemma 3.2.1. *For $\beta^\vee \in \Delta_{\mathrm{af}}^{\vee,+} \cap t_\lambda^{-1}\Delta_{\mathrm{af}}^{\vee,-}$, we have $\deg(\beta^\vee)/\langle\lambda,\overline{\beta^\vee}\rangle \in \mathbb{Q} \cap [0,1]$.*

Proof. Let $\chi : \Delta^\vee = \{\gamma^\vee \mid \gamma \in \Delta\} \to \{0,1\}$ be the characteristic function of $-\Delta^{\vee,+}$; i.e.,

$$\chi(\gamma^\vee) := \begin{cases} 0 & \text{if } \gamma^\vee \in \Delta^{\vee,+}, \\ 1 & \text{if } \gamma^\vee \in -\Delta^{\vee,+}. \end{cases}$$

By [25, eq. (1) in the proof of (2.4.1)], we have ($\langle\lambda,\overline{\beta^\vee}\rangle \neq 0$ and)

$$\Delta_{\mathrm{af}}^{\vee,+} \cap t_\lambda^{-1}\Delta_{\mathrm{af}}^{\vee,-} = \{\alpha^\vee + k\widetilde{\delta} \mid k \in \mathbb{Z},\ \chi(\alpha^\vee) \leq k < \chi(\alpha^\vee) + \langle\lambda,\alpha^\vee\rangle\}. \quad (3.2.1)$$

Since $\chi(\overline{\beta^\vee}) = 0$ or 1, this implies that $0 \leq \deg(\beta^\vee) \leq \langle\lambda,\overline{\beta^\vee}\rangle$, and hence $0 \leq \deg(\beta^\vee)/\langle\lambda,\overline{\beta^\vee}\rangle \leq 1$. Here, $\deg(\beta^\vee)/\langle\lambda,\overline{\beta^\vee}\rangle \in \mathbb{Q}$ is obvious since $\deg(\beta^\vee) \in \mathbb{Z}$ and $\langle\lambda,\overline{\beta^\vee}\rangle \in \mathbb{Z}$. □

Lemma 3.2.2. *Let $\beta^\vee \in \Delta_{\mathrm{af}}^{\vee,+} \cap t_\lambda^{-1}\Delta_{\mathrm{af}}^{\vee,-}$. Then we have $\overline{\beta} \in (\Delta^+(\lambda)_{>0}) \sqcup (-\Delta^+(\lambda)_{<0})$.*

Proof. By (3.2.1), we have $\langle\lambda,\overline{\beta^\vee}\rangle > 0$. If $\overline{\beta} \in \Delta^+$, then we have $\overline{\beta} \in \Delta^+(\lambda)_{>0}$. If $\overline{\beta} \in -\Delta^+$, then we have $-\overline{\beta} \in \Delta^+(\lambda)_{<0}$. This proves the lemma. □

By Lemmas 3.2.1 and 3.2.2, we obtain the map $\Phi : \Delta_{\mathrm{af}}^{\vee,+} \cap t_\lambda^{-1}\Delta_{\mathrm{af}}^{\vee,-} \to \mathbb{Q}_{\geq 0} \times ((\Delta^+(\lambda)_{>0}) \sqcup (-\Delta^+(\lambda)_{<0}))$ which is defined by

$$\Phi(\beta^\vee) := \left(\frac{\deg(\beta^\vee)}{\langle\lambda,\overline{\beta^\vee}\rangle}, \overline{\beta}\right).$$

It is obvious that Φ is injective.

Next, we prepare a "suitable" reduced expression $t_\lambda = s_{i_r} \cdots s_{i_2} s_{i_1} (\pi^\vee)^{-1}$ and a "suitable" λ-chain $\Gamma_\lhd(\lambda)$. Now, take a reflection order $\lhd \in \mathcal{RO}(\lambda,\Delta^+)$.

Let $\prec \in \mathcal{RO}(\lambda, w(\lambda)\Delta^+)$ be the reflection order on $w(\lambda)\Delta^+$ corresponding to $\lhd$ by the bijection $\mathcal{RO}(\lambda, \Delta^+) \to \mathcal{RO}(\lambda, w(\lambda)\Delta^+)$ which is discussed in Proposition 2.2.14. We define the total order $\prec^*$ on $w(\lambda)\Delta^+$ as follows: $\alpha \prec^* \beta$ if and only if $\beta \prec \alpha$. Then $\prec^*$ is also a reflection order of $w(\lambda)\Delta^+$. We consider $(\Delta^+(\lambda)_{>0}) \sqcup (-\Delta^+(\lambda)_{<0}) \subset w(\lambda)\Delta^+$ to be a totally ordered set under the total order $\prec^*$. By considering the ordinary order on $\mathbb{Q}_{\geq 0}$, we can define the lexicographic order on $\mathbb{Q}_{\geq 0} \times ((\Delta^+(\lambda)_{>0}) \sqcup (-\Delta^+(\lambda)_{<0}))$ and we see that $\mathbb{Q}_{\geq 0} \times ((\Delta^+(\lambda)_{>0}) \sqcup (-\Delta^+(\lambda)_{<0}))$ is a totally ordered set. Then the injection Φ induces a total order on $\Delta_{\mathrm{af}}^{\vee,+} \cap t_\lambda^{-1}\Delta_{\mathrm{af}}^{\vee,-}$, denoted by $<$. We see that $<$ is a reflection order on $\Delta_{\mathrm{af}}^{\vee,+} \cap t_\lambda^{-1}\Delta_{\mathrm{af}}^{\vee,-}$, defined as below.

Definition 3.2.3 (cf. [31, p. 662, Theorem])**.** Let $L \subset \Delta_{\mathrm{af}}^{\vee,+}$ be a totally ordered set, and $<$ a total order on L. An order $<$ is said to be a *reflection order* if the following (1) and (2) hold:

(1) For $\alpha^\vee, \beta^\vee \in L$, if $\alpha^\vee + \beta^\vee \in \Delta_{\mathrm{af}}^{\vee,+}$, then $\alpha^\vee + \beta^\vee \in L$ holds. Moreover, either of $\alpha^\vee < \alpha^\vee + \beta^\vee < \beta^\vee$ or $\beta^\vee < \alpha^\vee + \beta^\vee < \alpha^\vee$ holds;

(2) For $\alpha^\vee, \beta^\vee \in \Delta_{\mathrm{af}}^\vee$, if $\alpha^\vee + \beta^\vee \in L$, then either of the following (a) or (b) holds: (a) $\alpha^\vee \in L$ and $\alpha^\vee < \alpha^\vee + \beta^\vee$ or (b) $\beta^\vee \in L$ and $\beta^\vee < \alpha^\vee + \beta^\vee$.

Proposition 3.2.4. *The total order $<$ on $\Delta_{\mathrm{af}}^{\vee,+} \cap t_\lambda^{-1}\Delta_{\mathrm{af}}^{\vee,-}$ induced by the injection Φ is a reflection order.*

Proof. First, we show the condition (1) in Definition 3.2.3. Let $\alpha^\vee, \beta^\vee \in \Delta_{\mathrm{af}}^{\vee,+} \cap t_\lambda^{-1}\Delta_{\mathrm{af}}^{\vee,-}$, and assume that $\alpha^\vee + \beta^\vee \in \Delta_{\mathrm{af}}^{\vee,+}$. Then it is obvious that $\alpha^\vee + \beta^\vee \in \Delta_{\mathrm{af}}^{\vee,+} \cap t_\lambda^{-1}\Delta_{\mathrm{af}}^{\vee,-}$. Assume that $\alpha^\vee < \beta^\vee$. By the definition of $<$ on $\Delta_{\mathrm{af}}^{\vee,+} \cap t_\lambda^{-1}\Delta_{\mathrm{af}}^{\vee,-}$, we have either of the following:

(1) $\deg(\alpha^\vee)/\langle \lambda, \overline{\alpha^\vee} \rangle < \deg(\beta^\vee)/\langle \lambda, \overline{\alpha^\vee} \rangle$,

(2) $\deg(\alpha^\vee)/\langle \lambda, \overline{\alpha^\vee} \rangle = \deg(\beta^\vee)/\langle \lambda, \overline{\beta^\vee} \rangle$ and $\overline{\beta} \prec \overline{\alpha}$ (i.e., $\overline{\alpha} \prec^* \overline{\beta}$).

If (1) holds, then we have

$$\frac{\deg(\alpha^\vee)}{\langle \lambda, \overline{\alpha^\vee} \rangle} < \frac{\deg(\alpha^\vee) + \deg(\beta^\vee)}{\langle \lambda, \overline{\alpha^\vee} \rangle + \langle \lambda, \overline{\beta^\vee} \rangle} = \frac{\deg(\alpha^\vee + \beta^\vee)}{\langle \lambda, \overline{\alpha^\vee + \beta^\vee} \rangle} < \frac{\deg(\beta^\vee)}{\langle \lambda, \overline{\beta^\vee} \rangle},$$

this implies that $\alpha^\vee < \alpha^\vee + \beta^\vee < \beta^\vee$. If (2) holds, then we have

$$\frac{\deg(\alpha^\vee)}{\langle \lambda, \overline{\alpha^\vee} \rangle} = \frac{\deg(\alpha^\vee) + \deg(\beta^\vee)}{\langle \lambda, \overline{\alpha^\vee} \rangle + \langle \lambda, \overline{\beta^\vee} \rangle} = \frac{\deg(\alpha^\vee + \beta^\vee)}{\langle \lambda, \overline{\alpha^\vee + \beta^\vee} \rangle} = \frac{\deg(\beta^\vee)}{\langle \lambda, \overline{\beta^\vee} \rangle},$$

and $\overline{\beta} \prec (\overline{\alpha^\vee + \beta^\vee})^\vee \prec \overline{\alpha}$, since $\prec$ is a reflection order and by Corollary 2.2.6. Hence we conclude that $\alpha^\vee < \alpha^\vee + \beta^\vee < \beta^\vee$, as desired.

Next, we show the condition (2) in Definition 3.2.3. Assume that $\alpha^\vee + \beta^\vee \in \Delta_{\mathrm{af}}^{\vee,+} \cap t_\lambda^{-1}\Delta_{\mathrm{af}}^{\vee,-}$. If $\alpha^\vee, \beta^\vee \in \Delta_{\mathrm{af}}^{\vee,+} \cap t_\lambda^{-1}\Delta_{\mathrm{af}}^{\vee,-}$, then the assertion is obvious by the condition (1) in Definition 3.2.3. If $\alpha^\vee, \beta^\vee \in \Delta_{\mathrm{af}}^{\vee,+} \setminus t_\lambda^{-1}\Delta_{\mathrm{af}}^{\vee,-}$ and $\alpha^\vee + \beta^\vee \in \Delta_{\mathrm{af}}^\vee$, then it is obvious that $\alpha^\vee + \beta^\vee \in \Delta_{\mathrm{af}}^{\vee,+} \setminus t_\lambda^{-1}\Delta_{\mathrm{af}}^{\vee,-}$, a contradiction. Hence we can assume that $\alpha^\vee \in \Delta_{\mathrm{af}}^{\vee,+} \cap t_\lambda^{-1}\Delta_{\mathrm{af}}^{\vee,-}$ and $\beta^\vee \notin \Delta_{\mathrm{af}}^{\vee,+} \cap t_\lambda^{-1}\Delta_{\mathrm{af}}^{\vee,-}$. It is sufficient to show that $\alpha^\vee < \alpha^\vee + \beta^\vee$. First, we assume that $\overline{\beta} \in \Delta^+$.

Recall (3.2.1). Then we have $0 \leq \deg(\alpha^\vee)/\langle \lambda, \overline{\alpha^\vee} \rangle \leq 1$ by the assumption that $\alpha^\vee \in \Delta_{\mathrm{af}}^{\vee,+} \cap t_\lambda^{-1}\Delta_{\mathrm{af}}^{\vee,-}$. Also, the assumption $\beta^\vee \notin \Delta_{\mathrm{af}}^{\vee,+} \cap t_\lambda^{-1}\Delta_{\mathrm{af}}^{\vee,-}$ implies that one of the following holds: (a): $\langle \lambda, \overline{\beta^\vee} \rangle < 0 \leq \deg(\beta^\vee)$; (b-1): $0 \leq \langle \lambda, \overline{\beta^\vee} \rangle \leq \deg(\beta^\vee)$ and $\deg(\beta^\vee) \neq 0$; (b-2): $\langle \lambda, \overline{\beta^\vee} \rangle = \deg(\beta^\vee) = 0$.

Assume (a). Then we see that

$$\frac{\deg(\alpha^\vee + \beta^\vee)}{\langle \lambda, \overline{\alpha^\vee + \beta^\vee} \rangle} = \frac{\deg(\alpha^\vee) + \deg(\beta^\vee)}{\langle \lambda, \overline{\alpha^\vee} \rangle + \langle \lambda, \overline{\beta^\vee} \rangle} > \frac{\deg(\alpha^\vee) + \deg(\beta^\vee)}{\langle \lambda, \overline{\alpha^\vee} \rangle} \geq \frac{\deg(\alpha^\vee)}{\langle \lambda, \overline{\alpha^\vee} \rangle}.$$

Hence the definition of $<$ on $\Delta_{\mathrm{af}}^{\vee,+} \cap t_\lambda^{-1}\Delta_{\mathrm{af}}^{\vee,-}$ implies that $\alpha^\vee < \alpha^\vee + \beta^\vee$.

Next, assume (b-1). If $\langle \lambda, \overline{\beta^\vee} \rangle = 0$, then the assertion is obvious since

$$\frac{\deg(\alpha^\vee + \beta^\vee)}{\langle \lambda, \overline{\alpha^\vee + \beta^\vee} \rangle} = \frac{\deg(\alpha^\vee) + \deg(\beta^\vee)}{\langle \lambda, \overline{\alpha^\vee} \rangle} > \frac{\deg(\alpha^\vee)}{\langle \lambda, \overline{\alpha^\vee} \rangle}.$$

Assume that $\langle \lambda, \overline{\beta^\vee} \rangle > 0$. Since

$$\frac{\deg(\alpha^\vee)}{\langle \lambda, \overline{\alpha^\vee} \rangle} \leq 1 \leq \frac{\deg(\beta^\vee)}{\langle \lambda, \overline{\beta^\vee} \rangle},$$

we deduce that

$$\frac{\deg(\alpha^\vee) + \deg(\beta^\vee)}{\langle \lambda, \overline{\alpha^\vee} \rangle + \langle \lambda, \overline{\beta^\vee} \rangle} \geq \frac{\deg(\alpha^\vee)}{\langle \lambda, \overline{\alpha^\vee} \rangle};$$

here, the equality holds if and only if

$$\frac{\deg(\alpha^\vee)}{\langle \lambda, \overline{\alpha^\vee} \rangle} = 1 = \frac{\deg(\beta^\vee)}{\langle \lambda, \overline{\beta^\vee} \rangle}. \tag{3.2.2}$$

If (3.2.2) fails, then the assertion holds by the definition of $<$ on $\Delta_{\mathrm{af}}^{\vee,+} \cap t_\lambda^{-1}\Delta_{\mathrm{af}}^{\vee,-}$. Let us assume that (3.2.2) holds. We have

$$\deg(\alpha^\vee) = \langle \lambda, \overline{\alpha^\vee} \rangle, \tag{3.2.3}$$

$$\deg(\alpha^\vee + \beta^\vee) = \langle \lambda, \overline{\alpha^\vee + \beta^\vee} \rangle, \tag{3.2.4}$$

$$\deg(\beta^\vee) = \langle \lambda, \overline{\beta^\vee} \rangle > 0.$$

Suppose, for a contradiction, that $\overline{\alpha} \prec (\overline{\alpha^\vee + \beta^\vee})^\vee$. Then (3.2.3) and (3.2.1) imply that $\overline{\alpha} \in -\Delta^+(\lambda)_{<0}$ since $\alpha^\vee \in \Delta_{\mathrm{af}}^{\vee,+} \cap t_\lambda^{-1}\Delta_{\mathrm{af}}^{\vee,-}$. Similarly, we have $(\overline{\alpha^\vee + \beta^\vee})^\vee \in -\Delta^+(\lambda)_{<0}$ by (3.2.4) and (3.2.1). On the other hand, recall that $\overline{\beta} \in \Delta^+$. Hence we obtain $\overline{\beta} \in \Delta^+(\lambda)_{>0}$. By (2.2.1), we deduce that $\overline{\beta} \prec \overline{\alpha} \prec (\overline{\alpha^\vee + \beta^\vee})^\vee$. This contradicts that $\prec$ is a reflection order and Corollary 2.2.6. Hence we have $(\overline{\alpha^\vee + \beta^\vee})^\vee \prec \overline{\alpha}$. Therefore, we obtain $\overline{\alpha} \prec^* (\overline{\alpha^\vee + \beta^\vee})^\vee$ and $\alpha^\vee < \alpha^\vee + \beta^\vee$, as desired.

Now, assume (b-2). Then we have

$$\frac{\deg(\alpha^\vee + \beta^\vee)}{\langle \lambda, \overline{\alpha^\vee + \beta^\vee} \rangle} = \frac{\deg(\alpha^\vee)}{\langle \lambda, \overline{\alpha^\vee} \rangle}.$$

Hence we should prove that $(\overline{\alpha^\vee + \beta^\vee})^\vee \prec \overline{\alpha}$. Since $\langle \lambda, \overline{\beta^\vee} \rangle = 0$ and $\overline{\beta} \in \Delta^+$, we have $\overline{\beta} \in \Delta^+(\lambda)_{=0}$. Also, since $\alpha^\vee + \beta^\vee \in \Delta_{\mathrm{af}}^{\vee,+} \cap t_\lambda^{-1}\Delta_{\mathrm{af}}^{\vee,-}$, we deduce from Lemma 3.2.2 that $(\overline{\alpha^\vee + \beta^\vee})^\vee \in (\Delta^+(\lambda)_{>0}) \sqcup (-\Delta^+(\lambda)_{<0})$. By (2.2.1), we deduce that $(\overline{\alpha^\vee + \beta^\vee})^\vee \prec -\overline{\beta}$. Since $\prec$ is a reflection order, Corollary 2.2.6 implies that $(\overline{\alpha^\vee + \beta^\vee})^\vee \prec \overline{\alpha} \prec -\overline{\beta}$, as desired.

We consider the case $\overline{\beta} \in \Delta^-$. In this case, (3.2.1) implies that $\deg(\beta^\vee) > \langle \lambda, \overline{\beta^\vee} \rangle$. Note that $\deg(\beta^\vee) > 0$ since $\beta^\vee \in \Delta_{\mathrm{af}}^{\vee,+}$. Hence the assertion holds by the same argument as the case (a) and (b-1) with $\deg(\beta^\vee) \neq \langle \lambda, \overline{\beta^\vee} \rangle$ above. This proves the proposition. □

Hence, by [31, p. 662, Theorem], there exists a (unique) reduced expression $t_\lambda = s_{i_r} \cdots s_{i_2} s_{i_1} (\pi^\vee)^{-1}$ such that $\beta_1^{\mathsf{L}} < \cdots < \beta_r^{\mathsf{L}}$. We say that such reduced expression is *suitable* for $\lhd$. Also, we can take the λ-chain $\Gamma_\lhd(\lambda) = (\gamma_1^{\mathsf{L}}, \ldots, \gamma_r^{\mathsf{L}})$ which corresponds to such suitable reduced expression $t_\lambda = s_{i_r} \cdots s_{i_2} s_{i_1} (\pi^\vee)^{-1}$ by the relation between reduced expressions and reduced λ-chains, described in Section 2.3.2. We call $\Gamma_\lhd(\lambda)$ *suitable* for $\lhd$. From now on, we fix $\lhd \in \mathcal{RO}(\lambda, \Delta^+)$, and a suitable λ-chain $\Gamma_\lhd(\lambda)$ for $\lhd$, or equivalently, a suitable reduced expression $t_\lambda = s_{i_r} \cdots s_{i_2} s_{i_1} (\pi^\vee)^{-1}$ for $\lhd$.

Now we can define a map from $\mathcal{A}(w, \Gamma_\lhd(\lambda))$ to $\mathrm{IQLS}(\lambda)$ for each $w \in W$. Let $A = \{j_1, \ldots, j_s\} \in \mathcal{A}(w, \Gamma_\lhd(\lambda))$. Then by the definition of admissible subsets, we obtain the following directed path in $\mathrm{QBG}(W)$:

$$\mathbf{p}(A) : w = u_0 \xrightarrow{|\gamma_{j_1}^{\mathsf{L}}|} u_1 \xrightarrow{|\gamma_{j_2}^{\mathsf{L}}|} \cdots \xrightarrow{|\gamma_{j_s}^{\mathsf{L}}|} u_s;$$

here, $u_a = w s_{|\gamma_{j_1}^{\mathsf{L}}|} \cdots s_{|\gamma_{j_a}^{\mathsf{L}}|}$ for $a = 0, \ldots, s$. We set $d_i := \deg((\beta_i^{\mathsf{L}})^\vee)/\langle \mu, \overline{(\beta_i^{\mathsf{L}})^\vee} \rangle$ for $i = 1, \ldots, r$.

Since $\beta_1^{\mathsf{L}} < \ldots < \beta_r^{\mathsf{L}}$, the definition of the order $<$ implies that there exists

$m_1 < \cdots < m_t$ such that

$$\begin{aligned} 0 = d_{j_1} &= \cdots = d_{j_{m_1}} \\ &< d_{j_{m_1+1}} = \cdots = d_{j_{m_2}} \\ &< \cdots \\ &< d_{j_{m_t+1}} = \cdots = d_{j_s} = 1. \end{aligned}$$

If $d_{j_1} > 0$, then we set $m_1 := 0$. Also, if $d_{j_s} < 1$, then we set $m_t := s$.

For each $a = 1, \dots, t$, we consider the directed path

$$u_{m_a} \xrightarrow{|\gamma^{\mathsf{L}}_{j_{m_a+1}}|} u_{m_a+1} \xrightarrow{|\gamma^{\mathsf{L}}_{j_{m_a+2}}|} \cdots \xrightarrow{|\gamma^{\mathsf{L}}_{j_{m_{a+1}}}|} u_{m_{a+1}}.$$

By the definition of $<$ on $\Delta_{\mathrm{af}}^{\vee,+} \cap t_\lambda^{-1}\Delta_{\mathrm{af}}^{\vee,-}$ and the assumption $\beta_1^{\mathsf{L}} < \cdots < \beta_r^{\mathsf{L}}$, we have $\gamma^{\mathsf{L}}_{m_a+1} \succ \gamma^{\mathsf{L}}_{m_a+2} \succ \cdots \succ \gamma^{\mathsf{L}}_{m_{a+1}}$. Hence either of the following holds:

(1) $\gamma_p^{\mathsf{L}} \in \Delta^+(\lambda)_{>0}$ for all $m_a + 1 \leq p \leq m_{a+1}$; in this case, we set $n_a := m_a$;

(2) there exists $n_a \in \{m_a + 1, \dots, m_{a+1} - 1\}$ such that $\gamma_p^{\mathsf{L}} \in -\Delta^+(\lambda)_{<0}$ for all $m_a + 1 \leq p \leq n_a$ and $\gamma_p^{\mathsf{L}} \in \Delta^+(\lambda)_{>0}$ for all $n_a < p \leq m_{a+1}$;

(3) $\gamma_p^{\mathsf{L}} \in -\Delta^+(\lambda)_{<0}$ for all $m_a + 1 \leq p \leq m_{a+1}$; in this case, we set $n_a := m_{a+1}$.

Then, we wet

$$\begin{aligned} \Xi(A) := (&u_{m_t}, u_{m_{t-1}}, \dots, u_{m_1}; u_{n_{t-1}}, u_{n_{t-2}}, \dots, u_{n_1}; \\ &0, 1 - d_{j_{m_t}}, 1 - d_{j_{m_{t-1}}}, \dots, 1 - d_{j_{m_2}}, 1). \end{aligned}$$

The following lemma shows that Ξ defines a map $\mathcal{A}(w, \Gamma_\triangleleft(\lambda)) \to \mathrm{IQLS}(\lambda)$.

Lemma 3.2.5. *For $A \in \mathcal{A}(w, \Gamma_\triangleleft(\lambda))$, we have $\Xi(A) \in \mathrm{IQLS}(\lambda)$.*

Proof. Let $A = \{j_1, \dots, j_t\}$. It is sufficient to show that $(1 - d_{j_{m_a}})\langle \lambda, |\gamma_b^{\mathsf{L}}|^\vee \rangle \in \mathbb{Z}$ for all $2 \leq a \leq t$ and $m_{a-1} + 1 \leq b \leq m_a$. This follows from the following calculation:

$$\begin{aligned} (1 - d_{j_{m_a}})\langle \lambda, |\gamma_b^{\mathsf{L}}|^\vee \rangle &= (1 - d_b)\langle \lambda, |\gamma_b^{\mathsf{L}}|^\vee \rangle \\ &= \frac{\langle \lambda, \overline{(\beta_b^{\mathsf{L}})^\vee} \rangle - \deg((\beta_b^{\mathsf{L}})^\vee)}{\langle \lambda, \overline{(\beta_b^{\mathsf{L}})^\vee} \rangle} \cdot \langle \lambda, |\overline{(\beta_b^{\mathsf{L}})^\vee}| \rangle \\ &= \pm(\langle \lambda, \overline{(\beta_b^{\mathsf{L}})^\vee} \rangle - \deg((\beta_b^{\mathsf{L}})^\vee)) \in \mathbb{Z}. \end{aligned}$$

Hence we have $\Xi(A) \in \mathrm{IQLS}(\lambda)$. □

Finally, we define the forgetful map, which is an analog of the map Π^* in [21, Section 6.1], and the map Ξ in [27, Section 3.3].

Definition 3.2.6. The *forgetful map* $\widetilde{\Xi} : \mathcal{A}(w, \Gamma_\triangleleft(\lambda)) \to \mathrm{IQLS}(\lambda) \times W$ is defined by $\widetilde{\Xi}(A) := (\Xi(A), \mathrm{end}(A))$ for $A \in \mathcal{A}(w, \Gamma_\triangleleft(\lambda))$.

3.2.2 Injectivity of the forgetful map

Let us show the injectivity of the forgetful map, which is one of the main result of this thesis. First, we calculate the image of $\widetilde{\Xi}$.

Theorem 3.2.7. *One has the following.*

$$\mathrm{Im}(\widetilde{\Xi}) = \left\{ (\eta, u) \in \mathrm{IQLS}(\lambda) \times W \;\middle|\; \begin{array}{l} w \xRightarrow{(\lambda,+)} \kappa(\eta) \\ \iota(\eta) \xRightarrow{(\lambda,-)} u \end{array} \right\}. \tag{3.2.5}$$

Proof. We denote by $\mathcal{S}$ the right-hand side of (3.2.5). First, we show that $\mathrm{Im}(\widetilde{\Xi}) \subset \mathcal{S}$. Let $A = \{j_1, \ldots, j_s\} \in \mathcal{A}(w, \Gamma_{\lhd}(\lambda))$. It is sufficient to prove that $w \xRightarrow{(\lambda,+)} \kappa(\Xi(A))$ and $\iota(\Xi(A)) \xRightarrow{(\lambda,-)} \mathrm{end}(A)$.

Recall that $\kappa(\Xi(A)) = u_{m_1}$, and there exists a directed path

$$w \xrightarrow{|\gamma^{\mathsf{L}}_{j_1}|} u_1 \xrightarrow{|\gamma^{\mathsf{L}}_{j_2}|} \cdots \xrightarrow{|\gamma^{\mathsf{L}}_{j_{m_1}}|} u_{m_1}$$

in $\mathrm{QBG}(W)$. Also, $d_{j_1} = \cdots = d_{j_{m_1}} = 0$. Let $a = j_1, j_2, \ldots, j_{m_1}$. Then, we deduce that $\deg((\beta^{\mathsf{L}}_a)^\vee) = 0$ since

$$d_a = \frac{\deg((\beta^{\mathsf{L}}_a)^\vee)}{\langle \lambda, \overline{(\beta^{\mathsf{L}}_a)^\vee} \rangle} = 0.$$

By (3.2.1), we have $\gamma^{\mathsf{L}}_a = \overline{\beta^{\mathsf{L}}_a} \in \Delta^+$. This implies that $|\gamma^{\mathsf{L}}_a| = \gamma^{\mathsf{L}}_a \in \Delta^+(\lambda)_{>0}$. Since $\gamma^{\mathsf{L}}_{j_1} \succ \cdots \succ \gamma^{\mathsf{L}}_{j_{m_1}}$, it holds that $\gamma^{\mathsf{L}}_{j_1} \rhd \cdots \rhd \gamma^{\mathsf{L}}_{j_{m_1}}$. Therefore, we conclude that $w \xRightarrow{(\lambda,+)} \kappa(\Xi(A))$, as desired.

Next, we show that $\iota(\Xi(A)) \xRightarrow{(\lambda,-)} \mathrm{end}(A)$. Recall that $\iota(\Xi(A)) = u_{m_t}$, and there exists a directed path

$$u_{m_t} \xrightarrow{|\gamma^{\mathsf{L}}_{j_{m_t+1}}|} u_{m_t+1} \xrightarrow{|\gamma^{\mathsf{L}}_{j_{m_t+2}}|} \cdots \xrightarrow{|\gamma^{\mathsf{L}}_{j_s}|} u_s = \mathrm{end}(A)$$

in $\mathrm{QBG}(W)$. Also, $d_{j_{m_t+1}} = \cdots = d_{j_s} = 1$. Let $a = j_{m_t+1}, j_{m_t+2}, \ldots, j_s$. Then, we have $\deg((\beta^{\mathsf{L}}_a)^\vee) = \langle \lambda, \overline{(\beta^{\mathsf{L}}_a)^\vee} \rangle$ since

$$d_a = \frac{\deg((\beta^{\mathsf{L}}_a)^\vee)}{\langle \lambda, \overline{(\beta^{\mathsf{L}}_a)^\vee} \rangle} = 1.$$

Now, (3.2.1) implies that $\gamma^{\mathsf{L}}_a = \overline{\beta^{\mathsf{L}}_a} \in \Delta^-$, and this shows that $|\gamma^{\mathsf{L}}_a| = -\gamma^{\mathsf{L}}_a \in \Delta^+(\lambda)_{<0}$. Since $\gamma^{\mathsf{L}}_{j_{m_t+1}} \succ \cdots \succ \gamma^{\mathsf{L}}_{j_s}$, we have $|\gamma^{\mathsf{L}}_{j_{m_t+1}}| \rhd \cdots \rhd |\gamma^{\mathsf{L}}_{j_s}|$. Hence we deduce that $\iota(\Xi(A)) \xRightarrow{(\lambda,-)} \mathrm{end}(A)$, as desired. This completes the proof of $\mathrm{Im}(\widetilde{\Xi}) \subset \mathcal{S}$.

Next, we show that $\mathcal{S} \subset \mathrm{Im}(\Xi)$. Let $(\eta, u) \in \mathcal{S}$. Then we have $w \xRightarrow{(\lambda,+)} \kappa(\eta)$ and $\iota(\eta) \xRightarrow{(\lambda,-)} u$. We construct $A \in \mathcal{A}(w, \Gamma_{\triangleleft}(\lambda))$ such that $\Xi(A) = \eta$ and $\mathrm{end}(A) = u$.

Write $\eta = (x_1, \dots, x_s; y_1, \dots, y_{s-1}; \sigma_0, \dots, \sigma_s)$. Since $w \xRightarrow{(\lambda,+)} \kappa(\eta) = x_s$, there exists a directed path

$$u \xrightarrow{|\gamma^{\mathsf{L}}_{j_{s,1}}|} \cdots \xrightarrow{|\gamma^{\mathsf{L}}_{j_{s,m_s}}|} x_s$$

in $\mathrm{QBG}(W)$ such that $\gamma^{\mathsf{L}}_{j_{s,1}} \succ \cdots \succ \gamma^{\mathsf{L}}_{j_{s,m_s}}$ and $\gamma^{\mathsf{L}}_{j_{s,p}} \in \Delta^+(\lambda)_{>0}$ for all $p = 1, \dots, m_s$. Also, by the definition of interpolated QLS paths, for $i = 1, \dots, s-1$, we have the following directed paths

$$x_{i+1} \xrightarrow{|\gamma^{\mathsf{L}}_{j_{i,1}}|} \cdots \xrightarrow{|\gamma^{\mathsf{L}}_{j_{i,n_i}}|} y_i,$$

$$y_i \xrightarrow{|\gamma^{\mathsf{L}}_{j_{i,n_i+1}}|} \cdots \xrightarrow{|\gamma^{\mathsf{L}}_{j_{i,m_i}}|} x_i$$

in $\mathrm{QBG}(W)$ such that $\gamma^{\mathsf{L}}_{j_{i,1}} \succ \cdots \succ \gamma^{\mathsf{L}}_{j_{i,m_i}}$, and that $\gamma^{\mathsf{L}}_{j_{i,p}} \in -\Delta^+(\lambda)_{<0}$ for $p = 1, \dots, n_i$, and $\gamma^{\mathsf{L}}_{j_{i,p}} \in \Delta^+(\lambda)_{>0}$ for $p = n_i + 1, \dots, m_i$. Moreover, by $\iota(\eta) = x_1 \xRightarrow{(\lambda,-)} u$, there exists a directed path

$$x_1 \xrightarrow{|\gamma^{\mathsf{L}}_{j_{0,1}}|} \cdots \xrightarrow{|\gamma^{\mathsf{L}}_{j_{0,m_0}}|} u$$

in $\mathrm{QBG}(W)$ such that $\gamma^{\mathsf{L}}_{j_{0,1}} \succ \cdots \succ \gamma^{\mathsf{L}}_{j_{0,m_0}}$ and $\gamma^{\mathsf{L}}_{j_{0,p}} \in -\Delta^+(\lambda)_{<0}$ for $p = 1, \dots, m_0$.

For $i = 0, \dots, s$, we set $d_i := 1 - \sigma_i$. Also, for $i = 0, \dots, s$ and $p = 1, \dots, m_i$, we set $\beta^{\vee}_{j_{i,p}} := (\gamma^{\mathsf{L}}_{j_{i,p}})^{\vee} + d_i \langle \lambda, (\gamma^{\mathsf{L}}_{j_{i,p}})^{\vee} \rangle \widetilde{\delta}$. We show that $\beta^{\vee}_{j_{i,p}} \in \Delta^{\vee,+}_{\mathrm{af}} \cap t_{\lambda}^{-1} \Delta^{\vee,-}_{\mathrm{af}}$.

Assume that $i = s$. Then we have $d_s = 1 - \sigma_s = 0$. Since $\gamma^{\mathsf{L}}_{j_{s,p}} \in \Delta^+$, (3.2.1) implies that $\beta^{\vee}_{j_{i,p}} \in \Delta^{\vee,+}_{\mathrm{af}} \cap t_{\lambda}^{-1} \Delta^{\vee,-}_{\mathrm{af}}$.

Next, assume that $1 \le i \le s - 1$. In this case, we have $0 < \sigma_i < 1$. Since $0 < d_i \langle \lambda, (\gamma^{\mathsf{L}}_{j_{i,p}})^{\vee} \rangle < \langle \lambda, (\gamma^{\mathsf{L}}_{j_{i,p}})^{\vee} \rangle$ (note that $\langle \lambda, (\gamma^{\mathsf{L}}_{j_{i,p}})^{\vee} \rangle > 0$), and since $0 \le \chi((\gamma^{\mathsf{L}}_{j_{i,p}})^{\vee}) \le 1$, (3.2.1) implies that $\beta^{\vee}_{j_{i,p}} \in \Delta^{\vee,+}_{\mathrm{af}} \cap t_{\lambda}^{-1} \Delta^{\vee,-}_{\mathrm{af}}$.

Finally, assume that $i = 0$. Then we have $d_0 = 1 - \sigma_0 = 1$. Since $\gamma^{\mathsf{L}}_{j_{0,p}} \in \Delta^-$ (or equivalently $\chi((\gamma^{\mathsf{L}}_{j_{0,p}})^{\vee}) = 1$), (3.2.1) implies that $\beta^{\vee}_{j_{0,p}} \in \Delta^{\vee,+}_{\mathrm{af}} \cap t_{\lambda}^{-1} \Delta^{\vee,-}_{\mathrm{af}}$.

Therefore, we conclude that $\beta^\vee_{j_{i,p}} \in \Delta^{\vee,+}_{\mathrm{af}} \cap t_\lambda^{-1}\Delta^{\vee,-}_{\mathrm{af}}$ for all $i = 0, \dots, s$ and $p = 1, \dots, m_i$. Now, recall that $\gamma^{\mathsf{L}}_{j_{i,1}} \succ \cdots \succ \gamma^{\mathsf{L}}_{j_{i,m_i}}$ for all $i = 0, \dots, s$. Also, we have $0 = d_s < \cdots < d_1 < d_0 = 1$. Moreover, one has

$$\Phi(\beta^\vee_{j_{i,p}}) = \left(\frac{\deg(\beta^\vee_{j_{i,p}})}{\langle \lambda, \overline{\beta^\vee_{j_{i,p}}} \rangle}, \overline{\beta_{j_{i,p}}} \right) = (d_i, \gamma^{\mathsf{L}}_{j_{i,p}})$$

for $i = 0, \dots, s$ and $p = 1, \dots, m_i$. Therefore, by the definition of $<$ on $\Delta^{\vee,+}_{\mathrm{af}} \cap t_\lambda^{-1}\Delta^{\vee,-}_{\mathrm{af}}$, we have $\beta^\vee_{j_{s,1}} < \cdots < \beta^\vee_{j_{s,m_s}} < \beta^\vee_{j_{s-1,1}} < \cdots < \beta^\vee_{j_{s-1,m_{s-1}}} < \cdots < \beta^\vee_{j_{0,1}} < \cdots < \beta^\vee_{j_{0,m_0}}$. Hence there exists

$$A = \{j'_{s,1}, \dots, j'_{s,m_s}, j'_{s-1,1}, \dots, j'_{s-1,m_{s-1}}, \dots, j'_{0,1}, \dots, j'_{0,m_0}\}$$

such that $(\beta^{\mathsf{L}}_{j'_{i,p}})^\vee = \beta^\vee_{j_{i,p}}$ for all $i = 0, \dots, s$ and $p = 1, \dots, m_i$. It is obvious that $A \in \mathcal{A}(w, \Gamma_\lhd(\lambda))$, and that $\Xi(A) = \eta$ and $\mathrm{end}(A) = u$. This completes the proof of $(\eta, u) \in \mathrm{Im}(\widetilde{\Xi})$, and we conclude that $\mathrm{Im}(\widetilde{\Xi}) = \mathcal{S}$, as desired. □

Theorem 3.2.8. *The forgetful map $\widetilde{\Xi}$ is injective.*

Proof. This follows from the shellability of $\mathrm{QBG}(W)$ (see Theorem 2.2.7). □

3.2.3 Statistics for admissible subsets in terms of interpolated QLS paths

We rewrite statistics defined for admissible subsets in terms of interpolated QLS paths.

Proposition 3.2.9. *For $A \in \mathcal{A}(w, \Gamma_\lhd(\lambda))$, we have $n(A) = \mathrm{neg}(\Xi(A)) + \ell(\iota(\Xi(A)) \Rightarrow \mathrm{end}(A))$.*

Proof. Let $A = \{j_1, \dots, j_s\} \in \mathcal{A}(w, \Gamma_\lhd(\lambda))$. Then $n(A)$ counts the number of roots $\gamma^{\mathsf{L}}_{j_k}$, $k = 1, \dots, s$, such that $\gamma^{\mathsf{L}}_{j_k} \in \Delta^-$. Hence the desired identity holds since $u \xRightarrow{(\lambda,+)} \kappa(\Xi(A))$ and $\iota(\Xi(A)) \xRightarrow{(\lambda,-)} \mathrm{end}(A)$. □

Corollary 3.2.10. *For $A \in \mathcal{A}(w, \Gamma_\lhd(\lambda))$, we have $n(A) \equiv \mathrm{neg}(\Xi(A)) + \ell(\mathrm{end}(A)) - \ell(\iota(\Xi(A))) \mod 2$.*

Next, we rewrite $\mathrm{wt}(A)$ in terms of the weight of $\Xi(A)$.

Proposition 3.2.11. *For $A \in \mathcal{A}(w, \Gamma_\lhd(\lambda))$, we have $\mathrm{wt}(A) = \mathrm{wt}(\Xi(A))$.*

Proof. We show the proposition by the induction on $|A|$. If $|A| = 0$, i.e., $A = \emptyset$, then by the definition of $\mathrm{wt}(A)$, we have $\mathrm{wt}(A) = w\lambda$. On the other hand, we see that $\Xi(A) = (w; ; 0, 1)$. Hence $\mathrm{wt}(\Xi(A)) = w\lambda$. Therefore, it holds that $\mathrm{wt}(A) = \mathrm{wt}(\Xi(A))$.

Let $A = \{j_1, \dots, j_s\} \in \mathcal{A}(w, \Gamma_\lhd(\lambda)))$ and assume that $s \geq 1$. Set $B := \{j_1, \dots, j_{s-1}\}$. Our induction hypothesis is that $\mathrm{wt}(B) = \mathrm{wt}(\Xi(B))$.

Step 1: $\mathrm{wt}(A) = \mathrm{wt}(B) + (-\langle \lambda, \overline{(\beta^{\mathsf{L}}_{j_s})^\vee} \rangle + \deg((\beta^{\mathsf{L}}_{j_s})^\vee)) \,\mathrm{end}(B) \overline{\beta^{\mathsf{L}}_{j_s}}$**.**

By abuse of notation, we define $t_\nu : \mathfrak{h}^*_{\mathbb{R}} \to \mathfrak{h}^*_{\mathbb{R}}$ for $\nu \in \mathfrak{h}^*_{\mathbb{R}}$ by $t_\nu(\xi) := \xi + \nu$. We see that for $\nu \in \mathfrak{h}^*_{\mathbb{R}}$ and $\gamma \in \Delta^+$, $s_\gamma t_\nu = t_{s_\gamma(\nu)} s_\gamma$, and for $\nu_1, \nu_2 \in \mathfrak{h}^*_{\mathbb{R}}$, $t_{\nu_1} t_{\nu_2} = t_{\nu_1 + \nu_2}$. Also, for $\gamma \in \Delta$ and $k \in \mathbb{Z}$, we have $t_{k\gamma} s_{|\gamma|} = s_{\gamma,k}$.

By the definition of $\mathrm{wt}(A)$, we have

$$\begin{aligned}
\mathrm{wt}(A) &= -w s_{\gamma^{\mathsf{L}}_{j_1}, -\deg((\beta^{\mathsf{L}}_{j_1})^\vee)} \cdots s_{\gamma^{\mathsf{L}}_{j_{s-1}}, -\deg((\beta^{\mathsf{L}}_{j_{s-1}})^\vee)} s_{\gamma^{\mathsf{L}}_{j_s}, -\deg((\beta^{\mathsf{L}}_{j_s})^\vee)} (-\lambda) \\
&= -w s_{\gamma^{\mathsf{L}}_{j_1}, -\deg((\beta^{\mathsf{L}}_{j_1})^\vee)} \cdots s_{\gamma^{\mathsf{L}}_{j_{s-1}}, -\deg((\beta^{\mathsf{L}}_{j_{s-1}})^\vee)} (-\lambda \\
&\qquad\qquad - (\langle -\lambda, (\gamma^{\mathsf{L}}_{j_s})^\vee \rangle + \deg((\beta^{\mathsf{L}}_{j_s})^\vee)) \gamma^{\mathsf{L}}_{j_s}) \\
&= -w s_{\gamma^{\mathsf{L}}_{j_1}, -\deg((\beta^{\mathsf{L}}_{j_1})^\vee)} \cdots s_{\gamma^{\mathsf{L}}_{j_{s-1}}, -\deg((\beta^{\mathsf{L}}_{j_{s-1}})^\vee)} (-\lambda \\
&\qquad\qquad - (-\langle \lambda, \overline{(\beta^{\mathsf{L}}_{j_s})^\vee} \rangle + \deg((\beta^{\mathsf{L}}_{j_s})^\vee)) \overline{\beta^{\mathsf{L}}_{j_s}}).
\end{aligned}$$

We set $d := -\langle \lambda, \overline{(\beta^{\mathsf{L}}_{j_s})^\vee} \rangle + \deg((\beta^{\mathsf{L}}_{j_s})^\vee)$. By repeated application of identities $t_{k\gamma} s_{|\gamma|} = s_{\gamma,k}$ for $\gamma \in \Delta$ and $k \in \mathbb{Z}$, we see that there exists $\xi \in \mathfrak{h}^*_{\mathbb{R}}$ (in fact, $\xi \in Q$) such that

$$w s_{\gamma^{\mathsf{L}}_{j_1}, -\deg((\beta^{\mathsf{L}}_{j_1})^\vee)} \cdots s_{\gamma^{\mathsf{L}}_{j_{s-1}}, -\deg((\beta^{\mathsf{L}}_{j_{s-1}})^\vee)} = w s_{|\gamma^{\mathsf{L}}_{j_1}|} \cdots s_{|\gamma^{\mathsf{L}}_{j_{s-1}}|} t_\xi.$$

Hence

$$\begin{aligned}
\mathrm{wt}(A) &= -w s_{|\gamma^{\mathsf{L}}_{j_1}|} \cdots s_{|\gamma^{\mathsf{L}}_{j_{s-1}}|} t_\xi (-\lambda - d \overline{\beta^{\mathsf{L}}_{j_s}}) \\
&= -w s_{|\gamma^{\mathsf{L}}_{j_1}|} \cdots s_{|\gamma^{\mathsf{L}}_{j_{s-1}}|} (-\lambda - d \overline{\beta^{\mathsf{L}}_{j_s}} + \xi) \\
&= -w s_{|\gamma^{\mathsf{L}}_{j_1}|} \cdots s_{|\gamma^{\mathsf{L}}_{j_{s-1}}|} (-\lambda + \xi) + d w s_{|\gamma^{\mathsf{L}}_{j_1}|} \cdots s_{|\gamma^{\mathsf{L}}_{j_{s-1}}|} (\overline{\beta^{\mathsf{L}}_{j_s}}) \\
&= -w s_{|\gamma^{\mathsf{L}}_{j_1}|} \cdots s_{|\gamma^{\mathsf{L}}_{j_{s-1}}|} t_\xi (-\lambda) + d w s_{|\gamma^{\mathsf{L}}_{j_1}|} \cdots s_{|\gamma^{\mathsf{L}}_{j_{s-1}}|} (\overline{\beta^{\mathsf{L}}_{j_s}}) \\
&= \underbrace{-w s_{\gamma^{\mathsf{L}}_{j_1}, -\deg((\beta^{\mathsf{L}}_{j_1})^\vee)} \cdots s_{\gamma^{\mathsf{L}}_{j_{s-1}}, -\deg((\beta^{\mathsf{L}}_{j_{s-1}})^\vee)} (-\lambda)}_{=\mathrm{wt}(B)} \\
&\qquad\qquad + d \underbrace{w s_{|\gamma^{\mathsf{L}}_{j_1}|} \cdots s_{|\gamma^{\mathsf{L}}_{j_{s-1}}|}}_{=\mathrm{end}(B)} (\overline{\beta^{\mathsf{L}}_{j_s}}) \\
&= \mathrm{wt}(B) + (-\langle \lambda, \overline{(\beta^{\mathsf{L}}_{j_s})^\vee} \rangle + \deg((\beta^{\mathsf{L}}_{j_s})^\vee)) \,\mathrm{end}(B) (\overline{\beta^{\mathsf{L}}_{j_s}}),
\end{aligned}$$

as desired.

Step 2: $\mathrm{wt}(\Xi(A)) = \mathrm{wt}(\Xi(B)) + (-\langle \lambda, \overline{(\beta^{\mathsf{L}}_{j_s})^\vee} \rangle + \deg((\beta^{\mathsf{L}}_{j_s})^\vee)) \,\mathrm{end}(B) \overline{\beta^{\mathsf{L}}_{j_s}}$**.**

There are the following three cases:

(1) $d_{j_{s-1}} < d_{j_s} < 1$;

(2) $d_{j_{s-1}} = d_{j_s} < 1$;

(3) $d_{j_s} = 1$.

Assume the case (1). Then we have $m_{t-1} = s-1$ and $m_t = s$. Hence

$$\Xi(B) = (u_{m_{t-1}}, \dots, u_{m_1}; u_{n_{t-2}}, \dots, u_{n_1}; 0, 1 - d_{j_{m_{t-1}}}, \dots, 1 - d_{j_{m_2}}, 1),$$

while

$$\Xi(A) = (u_{m_t}, \dots, u_{m_1}; u_{n_{t-1}}, \dots, u_{n_1}; 0, 1 - d_{j_{m_t}}, \dots, 1 - d_{j_{m_2}}, 1).$$

Note that $u_{m_t} = u_{m_{t-1}} s_{|\gamma_{j_s}^{\mathsf{L}}|}$. Also, note that $\mathrm{end}(B) = u_{m_{t-1}}$.

Hence, by the definition of the weight, we calculate as the following:

$$\begin{aligned}
&\mathrm{wt}(\Xi(A))\\
&= \mathrm{wt}(\Xi(B)) - (1 - d_{j_{m_{t-1}}}) u_{m_{t-1}} \lambda\\
&\qquad\qquad + ((1 - d_{j_{m_{t-1}}}) - (1 - d_{j_{m_t}})) u_{m_{t-1}} \lambda + (1 - d_{j_{m_t}}) u_{m_t} \lambda\\
&= \mathrm{wt}(\Xi(B)) - (1 - d_{j_{m_t}}) u_{m_{t-1}} \lambda + (1 - d_{j_{m_t}}) u_{m_t} \lambda\\
&= \mathrm{wt}(\Xi(B)) - (1 - d_{j_{m_t}}) u_{m_{t-1}} \lambda + (1 - d_{j_{m_t}}) u_{m_{t-1}} s_{|\gamma_{j_s}^{\mathsf{L}}|} \lambda\\
&= \mathrm{wt}(\Xi(B)) - (1 - d_{j_{m_t}}) u_{m_{t-1}} \lambda + (1 - d_{j_{m_t}}) u_{m_{t-1}} (\lambda - \langle \lambda, |\gamma_{j_s}^{\mathsf{L}}|^\vee \rangle |\gamma_{j_s}^{\mathsf{L}}|)\\
&= \mathrm{wt}(\Xi(B)) - (1 - d_{j_{m_t}}) \langle \lambda, |\gamma_{j_s}^{\mathsf{L}}|^\vee \rangle u_{m_{t-1}} |\gamma_{j_s}^{\mathsf{L}}|\\
&= \mathrm{wt}(\Xi(B)) - (1 - d_{j_{m_t}}) \langle \lambda, (\gamma_{j_s}^{\mathsf{L}})^\vee \rangle u_{m_{t-1}} \gamma_{j_s}^{\mathsf{L}}\\
&= \mathrm{wt}(\Xi(B)) - \left(1 - \frac{\deg((\beta_{j_s}^{\mathsf{L}})^\vee)}{\langle \lambda, \overline{(\beta_{j_s}^{\mathsf{L}})^\vee} \rangle} \right) \langle \lambda, \overline{(\beta_{j_s}^{\mathsf{L}})^\vee} \rangle \,\mathrm{end}(B) \overline{\beta_{j_s}^{\mathsf{L}}}\\
&= \mathrm{wt}(\Xi(B)) + (-\langle \lambda, \overline{(\beta_{j_s}^{\mathsf{L}})^\vee} \rangle + \deg((\beta_{j_s}^{\mathsf{L}})^\vee)) \,\mathrm{end}(B) \overline{\beta_{j_s}^{\mathsf{L}}};
\end{aligned}$$

as desired.

Next, assume (2). Observe that

$$\Xi(B) = (u_{m_t - 1}, u_{m_{t-1}}, \dots, u_{m_1}; z, u_{n_{t-2}}, \dots, u_{n_1}; 0, 1 - d_{j_{m_t}}, \dots, 1 - d_{j_{m_2}}, 1),$$

where

$$z = \begin{cases} u_{n_{t-1}}, & \text{if } u_{n_{t-1}} \neq u_{m_t}, \\ u_{m_t - 1}, & \text{if } u_{n_{t-1}} = u_{m_t}, \end{cases}$$

while

$$\Xi(A) = (u_{m_t}, \dots, u_{m_1}; u_{n_{t-1}}, \dots, u_{n_1}; 0, 1 - d_{j_{m_t}}, \dots, 1 - d_{j_{m_2}}, 1).$$

Note that $m_t = s$, $m_{t-1} < s-1$, and that $u_{m_t} = u_{m_t-1} s_{|\gamma^{\mathsf{L}}_{j_s}|}$. Also, note that $\mathrm{end}(B) = u_{m_t-1}$.

Hence we calculate as the following:

$$\begin{aligned}
\mathrm{wt}(\Xi(A)) &= \mathrm{wt}(\Xi(B)) - (1-d_{j_{m_t}})u_{m_t-1}\lambda + (1-d_{j_{m_t}})u_{m_t}\lambda \\
&= \mathrm{wt}(\Xi(B)) - (1-d_{j_{m_t}})u_{m_t-1}\lambda + (1-d_{j_{m_t}})u_{m_t-1}s_{|\gamma^{\mathsf{L}}_{j_s}|}\lambda \\
&= \mathrm{wt}(\Xi(B)) - (1-d_{j_{m_t}})u_{m_t-1}\lambda \\
&\qquad\qquad + (1-d_{j_{m_t}})u_{m_t-1}(\lambda - \langle \lambda, |\gamma^{\mathsf{L}}_{j_s}|^{\vee}\rangle |\gamma^{\mathsf{L}}_{j_s}|) \\
&= \mathrm{wt}(\Xi(B)) - (1-d_{j_{m_t}})\langle \lambda, |\gamma^{\mathsf{L}}_{j_s}|^{\vee}\rangle u_{m_t-1}|\gamma^{\mathsf{L}}_{j_s}| \\
&= \mathrm{wt}(\Xi(B)) - (1-d_{j_{m_t}})\langle \lambda, (\gamma^{\mathsf{L}}_{j_s})^{\vee}\rangle u_{m_t-1}\gamma^{\mathsf{L}}_{j_s} \\
&= \mathrm{wt}(\Xi(B)) - \left(1 - \frac{\deg((\beta^{\mathsf{L}}_{j_s})^{\vee})}{\langle \lambda, \overline{(\beta^{\mathsf{L}}_{j_s})^{\vee}}\rangle}\right) \langle \lambda, \overline{(\beta^{\mathsf{L}}_{j_s})^{\vee}}\rangle\, \mathrm{end}(B)\overline{\beta^{\mathsf{L}}_{j_s}} \\
&= \mathrm{wt}(\Xi(B)) + (-\langle \lambda, \overline{(\beta^{\mathsf{L}}_{j_s})^{\vee}}\rangle + \deg((\beta^{\mathsf{L}}_{j_s})^{\vee}))\, \mathrm{end}(B)\overline{\beta^{\mathsf{L}}_{j_s}}.
\end{aligned}$$

This is the desired equation.

Finally, in the case (3), we see that $\Xi(A) = \Xi(B)$. Hence $\mathrm{wt}(\Xi(A)) = \mathrm{wt}(\Xi(B))$. Since $d_{j_s} = \deg((\beta^{\mathsf{L}}_{j_s})^{\vee})/\langle \lambda, \overline{(\beta^{\mathsf{L}}_{j_s})^{\vee}}\rangle = 1$, we have $-\langle \lambda, \overline{(\beta^{\mathsf{L}}_{j_s})^{\vee}}\rangle + \deg((\beta^{\mathsf{L}}_{j_s})^{\vee}) = 0$. This proves the desired identity.

Step 3: $\mathrm{wt}(A) = \mathrm{wt}(\Xi(A))$

By Steps 1 and 2, we obtain the following, as desired:

$$\begin{aligned}
\mathrm{wt}(A) &= \mathrm{wt}(B) + (-\langle \lambda, \overline{(\beta^{\mathsf{L}}_{j_s})^{\vee}}\rangle + \deg((\beta^{\mathsf{L}}_{j_s})^{\vee}))\, \mathrm{end}(B)\overline{\beta^{\mathsf{L}}_{j_s}} \\
&= \mathrm{wt}(\Xi(B)) + (-\langle \lambda, \overline{(\beta^{\mathsf{L}}_{j_s})^{\vee}}\rangle + \deg((\beta^{\mathsf{L}}_{j_s})^{\vee}))\, \mathrm{end}(B)\overline{\beta^{\mathsf{L}}_{j_s}} \\
&= \mathrm{wt}(\Xi(A));
\end{aligned}$$

here, we used our induction hypothesis in the 2nd equality. This completes the proof of the proposition. □

3.2.4 The "$q = 0$" counterpart of the forgetful map

If $A \in \mathcal{A}|_{q=0}(w, \Gamma_{\triangleleft}(\lambda))$, then $\mathbf{p}(A)$ is a directed path in $\mathrm{BG}(W)$. Hence, by the definition of $\mathrm{ILS}(\lambda)$, we deduce the following.

Corollary 3.2.12. *We have* $\Xi(\mathcal{A}|_{q=0}(w, \Gamma_{\triangleleft}(\lambda))) \subset \mathrm{ILS}(\lambda)$. *Moreover, we have*

$$\widetilde{\Xi}(\mathcal{A}|_{q=0}(w, \Gamma_{\triangleleft}(\lambda))) = \left\{ (\eta, u) \in \mathrm{ILS}(\lambda) \times W \;\middle|\; \begin{array}{l} w \xRightarrow{(\lambda,+,q=0)} \kappa(\eta) \\ \iota(\eta) \xRightarrow{(\lambda,-,q=0)} u \end{array} \right\}.$$

3.3 Equivariant K-theory

As an application of forgetful maps, we rewrite the Chevalley formula for the $(H \times \mathbb{C}^*)$-equivariant K-group of semi-infinite flag manifolds, described in [19]. Also, we write the inverse Chevalley formula for the H-equivariant K-group of (ordinary) flag manifolds. Moreover, by using the inverse Chevalley formula, we describe the Yip formula, the expansion formula for the product $e^{\lambda} \operatorname{ch} D_w(\nu)$, where $\lambda \in P$, and $\operatorname{ch} D_w(\nu)$ denotes the character of the Demazure submodule $D_w(\nu)$ of $\mathfrak{g}$ of lowest weight $w\nu$ for $w \in W$ and $\nu \in P^+$.

In this section, we fix $\lambda \in P$. We take a reflection order $\lhd \in \mathcal{RO}(\lambda, \Delta^+)$, and a suitable λ-chain $\Gamma_{\lhd}(\lambda)$ for $\lhd$.

3.3.1 Chevalley formula for semi-infinite flag manifolds

Let $U_{\mathsf{q}}(\mathfrak{g}_{\mathrm{af}})$ denote the quantum affine algebra associated to $\mathfrak{g}_{\mathrm{af}}$ with Chevalley generator $E_i, F_i \in U_{\mathsf{q}}(\mathfrak{g}_{\mathrm{af}})$, $i \in I_{\mathrm{af}} = I \sqcup \{0\}$, where q is an indeterminate. We denote by $U_{\mathsf{q}}^-(\mathfrak{g}_{\mathrm{af}}) := \langle F_i \rangle_{i \in I_{\mathrm{af}}} \subset U_{\mathsf{q}}(\mathfrak{g}_{\mathrm{af}})$ the subalgebra of $U_{\mathsf{q}}(\mathfrak{g}_{\mathrm{af}})$ generated by $\{F_i \mid i \in I_{\mathrm{af}}\}$.

For each $\lambda \in P^+$, we denote by $V(\lambda)$ the *level-zero extremal weight module* of extremal weight λ over $U_{\mathsf{q}}(\mathfrak{g}_{\mathrm{af}})$, which is equipped with a family $\{v_x\}_{x \in W_{\mathrm{af}}} \subset V(\lambda)$ of extremal weight vectors, where $v_x \in V(\lambda)$, $x \in W_{\mathrm{af}}$, is an extremal weight vector of weight $x\lambda$ (see [10, Proposition 8.2.2]). For $x \in W_{\mathrm{af}}$ and $\lambda \in P^+$, the *Demazure submodule* $V_x^-(\lambda)$ is defined by $V_x^-(\lambda) := U_{\mathsf{q}}^-(\mathfrak{g}_{\mathrm{af}}) v_x$. We denote by $\operatorname{gch} V_x^-(\lambda)$ the *graded character* of $V_x^-(\lambda)$ (see [15, §2.4]). If $x = wt_{\xi}$ with $w \in W$ and $\xi \in Q^{\vee}$, then we know that $\operatorname{gch} V_x^-(\lambda) \in \mathbb{Z}[P][\![q^{-1}]\!] q^{-\langle \lambda, \xi \rangle}$; in fact, we know that $\operatorname{gch} V_w^-(\lambda) \in \mathbb{Z}[\![q^{-1}]\!][P]$ for $w \in W$.

To describe the identity of Chevalley type for graded characters, we need some notation for partitions. Let $\lambda \in P^+$ and write it as $\lambda = \sum_{i \in I} m_i \varpi_i$. We define the set $\overline{\mathrm{Par}}(\lambda)$ by

$$\overline{\mathrm{Par}}(\lambda) := \left\{ \boldsymbol{\chi} = (\chi^{(i)})_{i \in I} \;\middle|\; \begin{array}{l} \chi^{(i)} \text{ is a partition whose length is} \\ \text{less than or equal to } \max\{m_i, 0\} \end{array} \right\}.$$

For $\boldsymbol{\chi} = (\chi^{(i)})_{i \in I} \in \overline{\mathrm{Par}}(\lambda)$, we write it as $\chi^{(i)} = (\chi_1^{(i)} \geq \chi_2^{(i)} \geq \cdots \geq \chi_{l_i}^{(i)} > 0)$, where $0 \leq l_i \leq \max\{m_i, 0\}$ and $\chi_1^{(i)}, \ldots, \chi_{l_i}^{(i)} \in \mathbb{Z}$, and set

$$|\boldsymbol{\chi}| := \sum_{i \in I} \sum_{k=1}^{l_i} \chi_k^{(i)}, \quad \iota(\boldsymbol{\chi}) := \sum_{i \in I} \chi_1^{(i)} \alpha_i^{\vee};$$

if $\chi^{(i)} = \emptyset$, then we understand that $l_i = 0$ and $\chi_1^{(i)} = 0$.

The following is the *identity of Chevalley type* for graded characters, which is a "representation-theoretic" analog of the Chevalley formula for the equivariant K-group of the semi-infinite flag manifold, described in [19]. We give a simpler proof of this identity in Section 4.3.3.

Theorem 3.3.1. *Let $\mu \in P^+$ and $x \in W_{\mathrm{af}}$. We write x as $x = wt_\xi$ with $w \in W$ and $\xi \in Q^\vee$. Take $\lambda \in P$ such that $\mu + \lambda \in P^+$, and let Γ be an arbitrary reduced λ-chain. Then we have*

$$\operatorname{gch} V_x^-(\mu+\lambda) = \sum_{A \in \mathcal{A}(w,\Gamma)} \sum_{\boldsymbol{\chi} \in \overline{\mathrm{Par}}(\lambda)} (-1)^{n(A)} q^{-\operatorname{height}(A) - \langle \lambda, \xi \rangle - |\boldsymbol{\chi}|} e^{\mathrm{wt}(A)} \operatorname{gch} V^-_{\mathrm{end}(A) t_{\xi + \mathrm{down}(A) + \iota(\boldsymbol{\chi})}}(\mu). \tag{3.3.1}$$

Remark 3.3.2. The right-hand side of (3.3.1) is identical to zero if $\mu+\lambda \notin P^+$; the proof is given in Section 4.3.4.

The purpose of this section is to rewrite this identity in terms of interpolated QLS paths. We prepare some additional statistics of interpolated QLS paths; the degree function Deg_w is one of the analog of that for semi-infinite LS paths ([29, Section 3.1]).

Definition 3.3.3. Let $\eta = (x_1, \dots, x_s; y_1, \dots, y_{s-1}; \sigma_0, \dots, \sigma_s) \in \mathrm{IQLS}(\lambda)$. Take $w, u \in W$ be such that $w \xRightarrow{(\lambda,+)} \kappa(\eta)$ and $\iota(\eta) \xRightarrow{(\lambda,-)} u$. We define $\xi(u, \eta, w)$ by

$$\xi(u,\eta,w) := \mathrm{wt}(w \Rightarrow \kappa(\eta)) + \sum_{k=1}^{s-1} (\mathrm{wt}(x_{k+1} \Rightarrow y_k) + \mathrm{wt}(y_k \Rightarrow x_k)) + \mathrm{wt}(\iota(\eta) \Rightarrow u).$$

Definition 3.3.4. Let $\eta = (x_1, \dots, x_s; y_1, \dots, y_{s-1}; \sigma_0, \dots, \sigma_s) \in \mathrm{IQLS}(\lambda)$ and $w \in W$ be such that $w \xRightarrow{(\lambda,+)} \kappa(\eta)$. We define $\mathrm{Deg}_w(\eta)$ by

$$\mathrm{Deg}_w(\eta) := -\langle \lambda, \mathrm{wt}(w \Rightarrow \kappa(\eta)) \rangle - \sum_{k=1}^{s-1} \sigma_k \langle \lambda, \mathrm{wt}(x_{k+1} \Rightarrow y_k) + \mathrm{wt}(y_k \Rightarrow x_k) \rangle.$$

We describe the relation between these statistics for interpolated QLS paths and those defined for admissible subsets. First, the following lemma is obvious by the definitions of $\mathrm{down}(A)$ and $\xi(u, \eta, w)$.

Lemma 3.3.5. *For $A \in \mathcal{A}(w, \Gamma_\triangleleft(\lambda))$, write $\widetilde{\Xi}(A) = (\eta, u)$. Then, it holds that* $\mathrm{down}(A) = \xi(u, \eta, w)$.

Next, we consider height(A) and $\mathrm{Deg}_w(\eta)$.

Lemma 3.3.6. *Let $w \in W$. For $A \in \mathcal{A}(w, \Gamma_{\lhd}(\lambda))$, write $\Xi(A) = \eta$. Then it holds that* $\mathrm{height}(A) = -\mathrm{Deg}_w(\eta)$.

Proof. We use the notation in Section 3.2. For example, we recall that

$$\begin{aligned}\Xi(A) = (&u_{m_t}, u_{m_{t-1}}, \dots, u_{m_1}; u_{n_{t-1}}, u_{n_{t-2}}, \dots, u_{n_1};\\ &0, 1 - d_{j_{m_t}}, 1 - d_{j_{m_{t-1}}}, \dots, 1 - d_{j_{m_2}}, 1).\end{aligned}$$

Then by the definition of $\mathrm{Deg}_w(\eta)$, we have

$$\begin{aligned}\mathrm{Deg}_w(\eta) = &-\langle \lambda, \mathrm{wt}(w \Rightarrow u_{m_1}) \rangle\\ &- \sum_{k=1}^{t-1} (1 - d_{j_{m_{k+1}}}) \langle \lambda, \mathrm{wt}(u_{m_k} \Rightarrow u_{n_k}) + \mathrm{wt}(u_{n_k} \Rightarrow u_{m_{k+1}}) \rangle.\end{aligned}$$

Let $k = 1, \dots, t-1$, and consider the following directed paths:

$$u_{m_k} \xrightarrow{|\gamma^{\mathsf{L}}_{j_{m_k+1}}|} u_{m_k+1} \xrightarrow{|\gamma^{\mathsf{L}}_{j_{m_k+2}}|} \cdots \xrightarrow{|\gamma^{\mathsf{L}}_{j_{n_k}}|} u_{n_k},$$

$$u_{n_k} \xrightarrow{|\gamma^{\mathsf{L}}_{j_{n_k+1}}|} u_{n_k+1} \xrightarrow{|\gamma^{\mathsf{L}}_{j_{n_k+2}}|} \cdots \xrightarrow{|\gamma^{\mathsf{L}}_{j_{m_{k+1}}}|} u_{m_{k+1}}.$$

Then

$$\mathrm{wt}(u_{m_k} \Rightarrow u_{n_k}) = \sum_{\substack{m_k+1 \le p \le n_k \\ u_{p-1} \to u_p \text{ is a quantum edge}}} |\gamma^{\mathsf{L}}_{j_p}|^{\vee}.$$

$$\mathrm{wt}(u_{n_k} \Rightarrow u_{m_{k+1}}) = \sum_{\substack{n_k+1 \le p \le m_{k+1} \\ u_{p-1} \to u_p \text{ is a quantum edge}}} |\gamma^{\mathsf{L}}_{j_p}|^{\vee}.$$

For $m_k + 1 \le p \le m_{k+1}$, we see that

$$\begin{aligned}(1 - d_{j_{m_{k+1}}}) \langle \lambda, |\gamma^{\mathsf{L}}_{j_p}|^{\vee} \rangle &= \mathrm{sgn}(\gamma^{\mathsf{L}}_{j_p}) (1 - d_{j_{m_{k+1}}}) \langle \lambda, (\gamma^{\mathsf{L}}_{j_p})^{\vee} \rangle\\ &= \mathrm{sgn}(\gamma^{\mathsf{L}}_{j_p}) \left(1 - \frac{\deg((\beta^{\mathsf{L}}_{j_p})^{\vee})}{\langle \lambda, \overline{(\beta^{\mathsf{L}}_{j_p})^{\vee}} \rangle} \right) \langle \lambda, \overline{(\beta^{\mathsf{L}}_{j_p})^{\vee}} \rangle\\ &= \mathrm{sgn}(\gamma^{\mathsf{L}}_{j_p}) (\langle \lambda, \overline{(\beta^{\mathsf{L}}_{j_p})^{\vee}} \rangle - \deg((\beta^{\mathsf{L}}_{j_p})^{\vee})).\end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{k=1}^{t-1}(1-d_{j_{m_{k+1}}})\langle \lambda, \mathrm{wt}(u_{m_k} \Rightarrow u_{n_k}) + \mathrm{wt}(u_{n_k} \Rightarrow u_{m_{k+1}})\rangle \\
&= \sum_{\substack{m_1+1\le p\le m_t \\ u_{p-1}\to u_p \text{ is a quantum edge}}} (1-d_{j_{m_{k+1}}})\langle \lambda, |\gamma^{\mathsf{L}}_{j_p}|^\vee\rangle \\
&= \sum_{\substack{m_1+1\le p\le m_t \\ u_{p-1}\to u_p \text{ is a quantum edge}}} \mathrm{sgn}(\gamma^{\mathsf{L}}_{j_p})(\langle \lambda, (\gamma^{\mathsf{L}}_{j_p})^\vee\rangle - \deg((\beta^{\mathsf{L}}_{j_p})^\vee)).
\end{aligned}$$

Also, consider the directed path

$$w \xrightarrow{|\gamma^{\mathsf{L}}_{j_1}|} u_1 \xrightarrow{|\gamma^{\mathsf{L}}_{j_2}|} \cdots \xrightarrow{|\gamma^{\mathsf{L}}_{j_{m_1}}|} u_{m_1}.$$

Note that $w \overset{(\lambda,+)}{\Longrightarrow} u_{m_1} = \kappa(\eta)$, and this implies that $\mathrm{sgn}(\gamma^{\mathsf{L}}_{j_p}) = 1$ and that $\langle \lambda, |\gamma^{\mathsf{L}}_{j_p}|^\vee\rangle = \langle \lambda, (\gamma^{\mathsf{L}}_{j_p})^\vee\rangle$ for $1 \le p \le m_1$. Moreover, we see that $\deg((\beta^{\mathsf{L}}_{j_p})^\vee) = 0$ for $1 \le p \le m_1$. Hence

$$\begin{aligned}
&\langle \lambda, \mathrm{wt}(w \Rightarrow \kappa(\eta))\rangle \\
&= \sum_{\substack{1\le p\le m_1 \\ u_{p-1}\to u_p \text{ is a quantum edge}}} \langle \lambda, |\gamma^{\mathsf{L}}_{j_p}|^\vee\rangle \\
&= \sum_{\substack{1\le p\le m_1 \\ u_{p-1}\to u_p \text{ is a quantum edge}}} \mathrm{sgn}(\gamma^{\mathsf{L}}_{j_p})(\langle \lambda, (\gamma^{\mathsf{L}}_{j_p})^\vee\rangle - \deg((\beta^{\mathsf{L}}_{j_p})^\vee)).
\end{aligned}$$

Finally, observe that $\langle \lambda, (\gamma^{\mathsf{L}}_{j_p})^\vee\rangle - \deg((\beta^{\mathsf{L}}_{j_p})^\vee) = 0$ for $m_t + 1 \le p \le s$. Therefore, we deduce that

$$\begin{aligned}
\mathrm{Deg}_w(\eta) &= -\sum_{\substack{1\le p\le m_t \\ u_{p-1}\to u_p \text{ is a quantum edge}}} \mathrm{sgn}(\gamma^{\mathsf{L}}_{j_p})(\langle \lambda, (\gamma^{\mathsf{L}}_{j_p})^\vee\rangle - \deg((\beta^{\mathsf{L}}_{j_p})^\vee)) \\
&= -\sum_{\substack{1\le p\le s \\ u_{p-1}\to u_p \text{ is a quantum edge}}} \mathrm{sgn}(\gamma^{\mathsf{L}}_{j_p})(\langle \lambda, (\gamma^{\mathsf{L}}_{j_p})^\vee\rangle - \deg((\beta^{\mathsf{L}}_{j_p})^\vee)) \\
&= -\,\mathrm{height}(A),
\end{aligned}$$

as desired. This proves the lemma. □

By Theorems 3.2.7 and 3.2.8, Corollary 3.2.10, Proposition 3.2.11, Lemmas 3.3.5 and 3.3.6, and Theorem 3.3.1, we obtain the following description of the identity of Chevalley type for graded characters in terms of interpolated QLS paths.

Theorem 3.3.7. *Let $\mu \in P^+$ and $x \in W_{\mathrm{af}}$. We write x as $x = wt_\xi$ with $w \in W$ and $\xi \in Q^\vee$. Take $\lambda \in P$ such that $\mu + \lambda \in P^+$, and let Γ be an arbitrary reduced λ-chain. Then we have*

$$\operatorname{gch} V_x^-(\mu+\lambda) = \sum_{\substack{\eta \in \mathrm{IQLS}(\lambda) \\ w \xRightarrow{(\lambda,+)} \kappa(\eta)}} \sum_{\substack{u \in W \\ \iota(\eta) \xRightarrow{(\lambda,-)} u}} \sum_{\boldsymbol{\chi} \in \overline{\mathrm{Par}}(\lambda)} (-1)^{\mathrm{neg}(\eta)+\ell(u)-\ell(\iota(\eta))} q^{\mathrm{Deg}_w(\eta)-\langle\lambda,\xi\rangle-|\boldsymbol{\chi}|} \times e^{\mathrm{wt}(\eta)} \operatorname{gch} V^-_{ut_{\xi+\xi(u,\eta,w)+\iota(\boldsymbol{\chi})}}(\mu).$$

3.3.2 Inverse Chevalley formula for flag manifolds

Let G be a connected, simply-connected simple algebraic group, whose Lie algebra is $\mathfrak{g}$. Take a Borel subgroup $B \subset G$ and a maximal torus $H \subset G$ such that $H \subset B$. Let $X = G/B$ be the flag manifold associated to G, and we denote by $\mathcal{O}_X$ the structure sheaf of X.

For $w \in W$, we denote by $\mathcal{O}_w$ the structure sheaf of the Schubert variety $X_w = \overline{BwB/B}$. Note that $X_{w_\circ} = X$. Also, for $\lambda \in P$, we denote by $\mathbb{C}(\lambda)$ the 1-dimensional B-module of weight λ. Then we define $\mathcal{L}(\lambda)$ by the sheaf of (holomorphic) sections of the line bundle $G \times_B \mathbb{C}(\lambda) \to G/B$. Now, we set $\mathcal{L}_w(\lambda) := \mathcal{O}_w \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda)$ for $w \in W$ and $\lambda \in P$.

Let $K_H(G/B)$ be the H-equivariant K-group of the flag manifold $X = G/B$. For H-equivariant coherent sheaf $\mathcal{M}$ on X, we denote by $[\mathcal{M}] \in K_H(G/B)$ the class of $\mathcal{M}$.

Now, we review the Chevalley formula, which is stated in [22]. For $\lambda \in P$ and $w \in W$, the Chevalley formula for $K_H(G/B)$ is expressed as follows:

$$[\mathcal{L}(-\lambda)] \cdot [\mathcal{O}_w] = [\mathcal{L}_w(-\lambda)] = \sum_{v \in W} \sum_{\xi \in P} c_{w,v}^{\lambda,\xi} e^{-\xi} [\mathcal{O}_v];$$

where $c_{w,v}^{\lambda,\xi} \in \mathbb{Z}$. In the Chevalley formula, integers $c_{w,v}^{\lambda,\xi}$ are called *K_H-Chevalley coefficients*.

In [22], Lenart-Postnikov gave an explicit description for K_H-Chevalley coefficients in terms of the alcove model.

Theorem 3.3.8 ([22, Theorem 6.1])**.** *Let $\lambda \in P$ and $w \in W$. For each $\xi \in P$ and $v \in W$, we have*

$$c_{w,v}^{\lambda,\xi} = \sum_J (-1)^{\mathrm{neg}(J)},$$

where the sum $\sum_J$ is over all $J = \{j_1, \ldots, j_s\} \subset \{1, \ldots, r\}$ which satisfy the following:

(1) $w \leftarrow ws_{|\gamma^{\mathsf{L}}_{j_1}|} \leftarrow \cdots \leftarrow ws_{|\gamma^{\mathsf{L}}_{j_1}|} \cdots s_{|\gamma^{\mathsf{L}}_{j_s}|} = v$ *is a directed path in* $\mathrm{BG}(W)$*;*

(2) *it holds that* $-\xi = ws_{\gamma^{\mathsf{L}}_{j_1}, -\deg((\beta^{\mathsf{L}}_{j_1})^\vee)} \cdots s_{\gamma^{\mathsf{L}}_{j_s}, -\deg((\beta^{\mathsf{L}}_{j_s})^\vee)}(-\lambda)$.

Also, we set $\mathrm{neg}(J) := \#\{j \in J \mid \gamma^{\mathsf{L}}_j \in \Delta^-\}$ *for each* $J \subset \{1, \dots, r\}$.

Remark 3.3.9. In [22], the above theorem is described by using an arbitrary (not necessarily reduced) λ-chain Γ instead of $(\gamma^{\mathsf{L}}_1, \dots, \gamma^{\mathsf{L}}_r)(= \Gamma_{\lhd}(\lambda))$.

Our aim is to describe the above Chevalley formula in terms of interpolated LS paths. To achieve this, we review the Mathieu's result.

For $v, w \in W$ and $\lambda \in P$, we set $S^v_w(\lambda) := \sum_{\xi \in P} c^{\lambda,\xi}_{w,v} e^{-\xi} \in \mathbb{Z}[P]$. Then it holds that

$$[\mathcal{L}_w(-\lambda)] = \sum_{v \in W} S^v_w(\lambda)[\mathcal{O}_v].$$

On the other hand, we define $m^{v,\xi}_w(\lambda) \in \mathbb{Z}$ for $v, w \in W$ and $\lambda, \xi \in P$ by

$$[\mathbb{C}(-\lambda) \otimes_{\mathbb{C}} \mathcal{O}_w] = \sum_{v \in W} \sum_{\xi \in P} m^{v,\xi}_w(\lambda)[\mathcal{L}_v(\xi)]. \tag{3.3.2}$$

Theorem 3.3.10 ([26, Section 5])**.** *For* $v, w \in W$ *and* $\lambda \in P$*, it holds that*

$$S^{v^{-1}}_{w^{-1}}(\lambda) = \sum_{\xi \in P} m^{v,\xi}_w(\lambda) e^{\xi}.$$

By this theorem, we obtain the following identity.

Corollary 3.3.11. *For* $v, w \in W$ *and* $\lambda, \xi \in P$*, we have* $m^{v,\xi}_w(\lambda) = c^{\lambda,-\xi}_{w^{-1},v^{-1}}$.

Now, we rewrite (3.3.2).

Lemma 3.3.12. *For* $\lambda \in P$ *and* $w \in W$*, we have*

$$[\mathbb{C}(-\lambda) \otimes_{\mathbb{C}} \mathcal{O}_w] = \sum_{A \in \mathcal{A}|_{q=0}(w_\circ w^{-1}, \Gamma_{\lhd}(\lambda))} (-1)^{n(A)} [\mathcal{L}_{(w_\circ \operatorname{end}(A))^{-1}}(-w_\circ \operatorname{wt}(A))].$$

Proof. By Corollary 3.3.11, we have

$$\begin{aligned} &[\mathbb{C}(-\lambda) \otimes_{\mathbb{C}} \mathcal{O}_w] \\ &= \sum_{v \in W} \sum_{\xi \in P} m^{v,\xi}_w(\lambda)[\mathcal{L}_v(\xi)] \\ &= \sum_{v \in W} \sum_{\xi \in P} c^{\lambda,-\xi}_{w^{-1},v^{-1}}[\mathcal{L}_v(\xi)] \end{aligned}$$

$$= \sum_{J=\{j_1,\dots,j_s\}} (-1)^{\mathrm{neg}(J)}$$

$$\times \left[\mathcal{L}_{\left(w^{-1}s_{|\gamma^{\mathsf{L}}_{j_1}|}\cdots s_{|\gamma^{\mathsf{L}}_{j_s}|}\right)^{-1}} \left(w^{-1} s_{\gamma^{\mathsf{L}}_{j_1}, -\deg((\beta^{\mathsf{L}}_{j_1})^\vee)} \cdots s_{\gamma^{\mathsf{L}}_{j_s}, -\deg((\beta^{\mathsf{L}}_{j_s})^\vee)} (-\lambda) \right) \right], \tag{3.3.3}$$

where the sum $\sum_J$ in the final equation is over all $J = \{j_1, \dots, j_s\} \subset \{1, \dots, r\}$ such that $w^{-1} \leftarrow w^{-1}s_{|\gamma^{\mathsf{L}}_{j_1}|} \leftarrow \cdots \leftarrow w^{-1}s_{|\gamma^{\mathsf{L}}_{j_1}|} \cdots s_{|\gamma^{\mathsf{L}}_{j_s}|}$ is a directed path in $\mathrm{BG}(W)$, or equivalently, $w_\circ w^{-1} \to w_\circ w^{-1}s_{|\gamma^{\mathsf{L}}_{j_1}|} \to \cdots \to w_\circ w^{-1}s_{|\gamma^{\mathsf{L}}_{j_1}|} \cdots s_{|\gamma^{\mathsf{L}}_{j_s}|}$ is a directed path in $\mathrm{BG}(W)$. This implies that

$$\begin{aligned}
&(3.3.3) \\
&= \sum_{A \in \mathcal{A}|_{q=0}(w_\circ w^{-1}, \Gamma_\triangleleft(\lambda))} (-1)^{n(A)} \\
&\quad \times \left[\mathcal{L}_{\left(w^{-1}s_{|\gamma^{\mathsf{L}}_{j_1}|}\cdots s_{|\gamma^{\mathsf{L}}_{j_s}|}\right)^{-1}} \left(w^{-1} s_{\gamma^{\mathsf{L}}_{j_1}, -\deg((\beta^{\mathsf{L}}_{j_1})^\vee)} \cdots s_{\gamma^{\mathsf{L}}_{j_s}, -\deg((\beta^{\mathsf{L}}_{j_s})^\vee)} (-\lambda) \right) \right] \\
&= \sum_{A \in \mathcal{A}|_{q=0}(w_\circ w^{-1}, \Gamma_\triangleleft(\lambda))} (-1)^{n(A)} \left[\mathcal{L}_{(w_\circ \,\mathrm{end}(A))^{-1}} (-w_\circ \,\mathrm{wt}(A)) \right],
\end{aligned}$$

as desired. this proves the lemma. □

By applying the restriction of the forgetful map $\widetilde{\widetilde{\Xi}}|_{\mathcal{A}|_{q=0}(w_\circ w^{-1}, \Gamma_\triangleleft(\lambda))} : \mathcal{A}|_{q=0}(w_\circ w^{-1}, \Gamma_\triangleleft(\lambda)) \to \mathrm{ILS}(\lambda) \times W$ (see Corollary 3.2.12), we obtain the following identity, which is the desired description of the inverse Chevalley formula in terms of interpolated LS paths.

Corollary 3.3.13. *For $\lambda \in P$ and $w \in W$, we have*

$$[\mathbb{C}(-\lambda) \otimes_{\mathbb{C}} \mathcal{O}_w] = \sum_{\substack{\eta \in \mathrm{ILS}(\lambda) \\ w_\circ w^{-1} \xRightarrow{(\lambda, +, q=0)} \kappa(\eta)}} \sum_{\substack{u \in W \\ \iota(\eta) \xRightarrow{(\lambda, -, q=0)} u}} (-1)^{\mathrm{neg}(\eta) + \ell(u) - \ell(\iota(\eta))} [\mathcal{L}_{(w_\circ u)^{-1}} (-w_\circ \,\mathrm{wt}(\eta))]. \tag{3.3.4}$$

3.3.3 Yip formula for characters of Demazure submodules

As an application of the inverse Chevalley formula, we describe the Yip formula explicitly. To obtain the Yip formula, we consider the cohomology groups of sheaves which appear in (3.3.4).

Definition 3.3.14 ([24, Section 4])**.** Let $\nu \in P^+$. The (ν-twisted) *Euler character* is the (unique) $\mathbb{Z}[P]$-linear map $\chi_\nu : K_H(G/B) \to \mathbb{Z}[P]$ which satisfies

$$\chi_\nu([\mathcal{M}]) = \sum_{i=0}^{\infty}(-1)^i \operatorname{ch} H^i(G/B, \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(-\nu))$$

for each H-equivariant coherent sheaf $\mathcal{M}$ on G/B.

Euler characters for sheaves which appear in the Chevalley formula are calculated as follows.

Lemma 3.3.15. *Let $\nu \in P^+$, $\lambda, \xi \in P$, and $v, w \in W$. Then we have the following.*

(1) *It holds that $\chi_\nu([\mathbb{C}(-\lambda) \otimes_{\mathbb{C}} \mathcal{O}_w]) = \overline{e^\lambda \operatorname{ch} D_w(\nu)}$.*

(2) *If $\xi + \nu \in P^+$, then we have $\chi_\nu([\mathcal{L}_v(-\xi)]) = \overline{\operatorname{ch} D_v(\xi + \nu)}$.*

Here, $\overline{\,\cdot\,}$ is a $\mathbb{Z}$-linear involution on $\mathbb{Z}[P]$ defined by $\overline{e^\zeta} = e^{-\zeta}$ for each $\zeta \in P$.

Proof. First, we recall from [26, Theorem 2.1] (see also [1, 4.3, Theorem]) that

$$\chi_0([\mathcal{L}_u(-\mu)]) = \sum_{i=0}^{\infty}(-1)^i \operatorname{ch} H^i(G/B, \mathcal{L}_u(-\mu)) = \overline{\operatorname{ch} D_u(\mu)}$$

for $u \in W$ and $\mu \in P^+$.

Then we calculate as follows:

$$\begin{aligned}
\chi_\nu([\mathbb{C}(-\lambda) \otimes_{\mathbb{C}} \mathcal{O}_w]) &= \sum_{i=0}^{\infty}(-1)^i \operatorname{ch} H^i(G/B, (\mathbb{C}(-\lambda) \otimes_{\mathbb{C}} \mathcal{O}_w) \otimes_{\mathcal{O}_X} \mathcal{L}(-\nu)) \\
&= \sum_{i=0}^{\infty}(-1)^i \operatorname{ch} H^i(G/B, \mathbb{C}(-\lambda) \otimes_{\mathbb{C}} (\mathcal{O}_w \otimes_{\mathcal{O}_X} \mathcal{L}(-\nu))) \\
&= \sum_{i=0}^{\infty}(-1)^i \operatorname{ch} H^i(G/B, \mathbb{C}(-\lambda) \otimes_{\mathbb{C}} \mathcal{L}_w(-\nu)) \\
&= e^{-\lambda}\left(\sum_{i=0}^{\infty}(-1)^i \operatorname{ch} H^i(G/B, \mathcal{L}_w(-\nu))\right) \\
&= \overline{e^\lambda \operatorname{ch} D_w(\nu)};
\end{aligned}$$

which proves (1). Also, we obtain

$$\begin{aligned}
\chi_\nu([\mathcal{L}_v(-\xi)]) &= \sum_{i=0}^{\infty}(-1)^i \operatorname{ch} H^i(G/B, (\mathcal{O}_v \otimes_{\mathcal{O}_X} \mathcal{L}(-\xi)) \otimes_{\mathcal{O}_X} \mathcal{L}(-\nu)) \\
&= \sum_{i=0}^{\infty}(-1)^i \operatorname{ch} H^i(G/B, \mathcal{O}_v \otimes_{\mathcal{O}_X} (\mathcal{L}(-\xi) \otimes_{\mathcal{O}_X} \mathcal{L}(-\nu)))
\end{aligned}$$

$$= \sum_{i=0}^{\infty} (-1)^i \operatorname{ch} H^i(G/B, \mathcal{O}_v \otimes_{\mathcal{O}_X} \mathcal{L}(-\xi - \nu))$$
$$= \overline{\operatorname{ch} D_v(\xi + \nu)},$$

which proves (2). □

Therefore, by taking the twisted Euler character for sheaves in (3.3.4) and applying the involution $\overline{\cdot}$, we obtain the following Yip formula, as desired.

Theorem 3.3.16. *Let $\lambda \in P$, $w \in W$, and $\nu \in P^+$. If $\nu + w_\circ \operatorname{wt}(\eta) \in P^+$ for all $\eta \in \mathrm{ILS}(\lambda)$ such that $w_\circ w^{-1} \xRightarrow{(\lambda,+,q=0)} \kappa(\eta)$, then the following identity holds.*

$$e^\lambda \operatorname{ch} D_w(\nu) = \sum_{\substack{\eta \in \mathrm{ILS}(\lambda) \\ w_\circ w^{-1} \xRightarrow{(\lambda,+,q=0)} \kappa(\eta)}} \sum_{\substack{u \in W \\ \iota(\eta) \xRightarrow{(\lambda,-,q=0)} u}} (-1)^{\operatorname{neg}(\eta) + \ell(u) - \ell(\iota(\eta))} \operatorname{ch} D_{(w_\circ u)^{-1}}(\nu + w_\circ \operatorname{wt}(\eta)).$$

Chapter 4

Quantum Yang-Baxter moves

4.1 Generalization of quantum Yang-Baxter moves

Quantum Yang-Baxter moves for a dominant weight are introduced in [18]. We review the Yang-Baxter transformation for λ-chains, and state main arguments about generalized quantum Yang-Baxter moves. This section is based on [14, Section 3]

4.1.1 Yang-Baxter transformation

Let $\lambda \in P$, and let $\Gamma = (\beta_1, \dots, \beta_r)$ be a λ-chain (of roots). The following procedure (YB) is called the *Yang-Baxter transformation*:

(YB) Take a segment $(\beta_{t+1}, \dots, \beta_{t+q})$ of Γ of the form

$$(\beta_{t+1}, \dots, \beta_{t+q}) = (\alpha, s_\alpha(\beta), s_\alpha s_\beta(\alpha), \dots, s_\beta(\alpha), \beta)$$

for some $\alpha, \beta \in \Delta$ with $\langle \alpha, \beta^\vee \rangle \leq 0$, or equivalently $\langle \beta, \alpha^\vee \rangle \leq 0$, and $\alpha \neq -\beta$, and set

$$\Gamma' := (\beta_1, \beta_2, \dots, \beta_t, \beta_{t+q}, \beta_{t+q-1}, \dots, \beta_{t+1}, \beta_{t+q+1}, \beta_{t+q+2}, \dots, \beta_r),$$

i.e., reverse the segment $(\beta_{t+1}, \dots, \beta_{t+q})$ of Γ.

Also, we define a procedure (D), called *deletion*, as follows:

(D) Take a segment $(\beta_{t+1}, \beta_{t+2})$ of Γ of the form $(\beta_{t+1}, \beta_{t+2}) = (\beta, -\beta)$ for some $\beta \in \Delta$, and set $\Gamma' = (\beta_1, \dots, \beta_t, \beta_{t+3}, \dots, \beta_q)$, i.e., delete the segment $(\beta_{t+1}, \beta_{t+2})$ of Γ.

It is known that every λ-chain can be transformed into an arbitrary reduced λ-chain by repeated application of the procedures (YB) and (D) (see [19, Remark 38], or [22, Lemma 9.3]).

4.1.2 Quantum Yang-Baxter moves

Let $\lambda \in P^+$ be a dominant weight, and let Γ_1, Γ_2 be λ-chains such that Γ_2 is obtained from Γ_1 by the Yang-Baxter transformation (YB). *Quantum Yang-Baxter moves*, introduced in [18, Section 3.1], give a bijection $\mathcal{A}(e, \Gamma_1) \to \mathcal{A}(e, \Gamma_2)$ which preserves weights and heights.

Our main result in this chapter is the existence of a generalization of quantum Yang-Baxter moves for an arbitrary (not necessarily dominant) weight $\lambda \in P$ and an arbitrary $w \in W$.

Let $\lambda \in P$ be an arbitrary weight, and let Γ_1 and Γ_2 be λ-chains such that Γ_2 is obtained from Γ_1 by the Yang-Baxter transformation (YB). If we write $\Gamma_1 = (\beta_1, \ldots, \beta_r)$ and $\Gamma_2 = (\beta'_1, \ldots, \beta'_r)$, then there exists $1 \le t \le r$ such that

- $(\beta_{t+1}, \ldots, \beta_{t+q}) = (\alpha, s_\alpha(\beta), s_\alpha s_\beta(\alpha), \ldots, s_\beta(\alpha), \beta)$ for some $q \ge 1$ and some $\alpha, \beta \in \Delta$ such that $\langle \alpha, \beta^\vee \rangle \le 0$ and $\alpha \ne -\beta$,
- $\Gamma_2 = (\beta'_1, \ldots, \beta'_r) =$ $(\beta_1, \beta_2, \ldots, \beta_t, \beta_{t+q}, \beta_{t+q-1}, \ldots, \beta_{t+1}, \beta_{t+q+1}, \beta_{t+q+2}, \ldots, \beta_r)$.

We take the alcove path $(A_\circ = A_0, \ldots, A_r = A_{-\lambda})$ corresponding to Γ_1, and take integers $l_k \in \mathbb{Z}$ for $k = 1, \ldots, r$ such that for each $k = 1, \ldots, r$, the hyperplane $H_{\beta_k, -l_k}$ contains the common wall of A_{k-1} and A_k. Also, we take the alcove path $(A_\circ = A'_0, \ldots, A'_r = A_{-\lambda})$ corresponding to Γ_2, and we take integers $l'_k \in \mathbb{Z}$ for $k = 1, \ldots, r$ such that for each $k = 1, \ldots, r$, the hyperplane $H_{\beta'_k, -l'_k}$ contains the common wall of A'_{k-1} and A'_k. Then it follows that $A'_k = A_k$ and $l'_k = l_k$ for $k = 1, \ldots, t, t+q+1, \ldots, r$, and that $l'_{t+p} = l_{t+q+1-p}$ for $p = 1, \ldots, q$.

Now, we divide Γ_1 into three parts $\Gamma_1^{(1)}$, $\Gamma_1^{(2)}$, and $\Gamma_1^{(3)}$ as follows:

$$\Gamma_1^{(1)} := (\beta_1, \ldots, \beta_t),\ \Gamma_1^{(2)} := (\beta_{t+1}, \ldots, \beta_{t+q}),\ \Gamma_1^{(3)} := (\beta_{t+q+1}, \ldots, \beta_r). \tag{4.1.1}$$

Also, we divide Γ_2 into three parts $\Gamma_2^{(1)}$, $\Gamma_2^{(2)}$, and $\Gamma_2^{(3)}$ as follows:

$$\Gamma_2^{(1)} := (\beta'_1, \ldots, \beta'_t),\ \Gamma_2^{(2)} := (\beta'_{t+1}, \ldots, \beta'_{t+q}),\ \Gamma_2^{(3)} := (\beta'_{t+q+1}, \ldots, \beta'_r). \tag{4.1.2}$$

Note that $\Gamma_1^{(1)} = \Gamma_2^{(1)}$ and $\Gamma_1^{(3)} = \Gamma_2^{(3)}$; in addition, $\beta_{t+1}, \ldots, \beta_{t+q}$ are distinct. Next, let $w \in W$. For a w-admissible subset $A \in \mathcal{A}(w, \Gamma_1)$, we define $A^{(1)}$, $A^{(2)}$, and $A^{(3)}$ by

$$\begin{aligned} &A^{(1)} := A \cap \{1, \ldots, t\},\ A^{(2)} := A \cap \{t+1, \ldots, t+q\}, \\ &A^{(3)} := A \cap \{t+q+1, \ldots, r\}. \end{aligned} \tag{4.1.3}$$

Also, for $B \in \mathcal{A}(w, \Gamma_2)$, we define $B^{(1)}$, $B^{(2)}$, and $B^{(3)}$ by

$$\begin{aligned} &B^{(1)} := B \cap \{1, \ldots, t\},\ B^{(2)} := B \cap \{t+1, \ldots, t+q\}, \\ &B^{(3)} := B \cap \{t+q+1, \ldots, r\}. \end{aligned} \tag{4.1.4}$$

Unlike the case that λ is dominant, there does not exist a bijection between $\mathcal{A}(w, \Gamma_1)$ and $\mathcal{A}(w, \Gamma_2)$ in general.

Example 4.1.1. Assume that $\mathfrak{g}$ is of type A_2. We set $\Gamma_1 := (\alpha_2, -\alpha_1, -\theta, -\alpha_1)$ and $\Gamma_2 := (-\theta, -\alpha_1, \alpha_2, -\alpha_1)$, where $\theta = \alpha_1 + \alpha_2$. Then we see that Γ_1 and Γ_2 are $(-2\varpi_1+\varpi_2)$-chains such that Γ_2 is obtained from Γ_1 by a Yang-Baxter transformation (YB). Let $w = s_2$. By direct calculation, we have

$$\begin{aligned} \mathcal{A}(w, \Gamma_1) = \{&\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,4\}, \{2,4\}, \{3,4\}, \\ &\{1,2,3\}, \{1,2,4\}, \{1,2,3,4\}\}, \\ \mathcal{A}(w, \Gamma_2) = \{&\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \\ &\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\}. \end{aligned}$$

Hence we have $|\mathcal{A}(w, \Gamma_1)| = 12$, while $|\mathcal{A}(w, \Gamma_2)| = 16$. This shows that there does not exist a bijection $\mathcal{A}(w, \Gamma_1) \to \mathcal{A}(w, \Gamma_2)$.

Thus, for a generalization of quantum Yang-Baxter moves, we need to consider certain subsets of $\mathcal{A}(w, \Gamma_1)$ and $\mathcal{A}(w, \Gamma_2)$. The following theorem is our main result; the proof is given in the next section.

Theorem 4.1.2. *There exist subsets $\mathcal{A}_0(w, \Gamma_1) \subset \mathcal{A}(w, \Gamma_1)$ and $\mathcal{A}_0(w, \Gamma_2) \subset \mathcal{A}(w, \Gamma_2)$ which satisfy the following.*

(1) *There exists a bijection $Y : \mathcal{A}_0(w, \Gamma_1) \to \mathcal{A}_0(w, \Gamma_2)$ such that for all $A \in \mathcal{A}_0(w, \Gamma_1)$, it holds that*

- $(Y(A))^{(1)} = A^{(1)}$, $\mathrm{end}((Y(A))^{(2)}) = \mathrm{end}(A^{(2)})$, $(Y(A))^{(3)} = A^{(3)}$,
- $\mathrm{down}(Y(A)) = \mathrm{down}(A)$, *and*
- $(-1)^{n(Y(A))} = (-1)^{n(A)}$.

(2) *For $k = 1, 2$, we set $\mathcal{A}_0^C(w, \Gamma_k) := \mathcal{A}(w, \Gamma_k) \setminus \mathcal{A}_0(w, \Gamma_k)$. There exists an involution I_k on $\mathcal{A}_0^C(w, \Gamma_k)$ such that for all $A \in \mathcal{A}_0^C(w, \Gamma_k)$, it holds that*

- $(I_k(A))^{(1)} = A^{(1)}$, $\mathrm{end}((I_k(A))^{(2)}) = \mathrm{end}(A^{(2)})$, $(I_k(A))^{(3)} = A^{(3)}$,
- $\mathrm{down}(I_k(A)) = \mathrm{down}(A)$, *and*
- $(-1)^{n(I_k(A))} = -(-1)^{n(A)}$.

Remark 4.1.3. In order to explain our maps Y, I_1, and I_2 in Theorem 4.1.2, we have a useful notion, called a *sijection*, introduced in [6]; for the definition of sijections, see [6, Section 2]. For sets S, T equipped with sign functions $S \to \{\pm 1\}$, $T \to \{\pm 1\}$, a sijection from S to T is the collection $(\iota_S, \iota_T, \varphi)$ of a sign-reversing involution ι_S on a subset S_0 of S, a sign-reversing involution ι_T on a subset T_0 of T, and a sign-preserving bijection φ from $S \setminus S_0$ to $T \setminus T_0$ (see [6, p.9]). In this terminology, our collection (I_1, I_2, Y) in Theorem 4.1.2 is a sijection from $\mathcal{A}(w, \Gamma_1)$ to $\mathcal{A}(w, \Gamma_2)$. This sijection can be thought of as a generalization of quantum Yang-Baxter moves.

As in the case that λ is dominant, we can prove that the maps Y, I_1, and I_2 preserve weights and heights.

Theorem 4.1.4. (1) *For all* $A \in \mathcal{A}_0(w, \Gamma_1)$*, it holds that* $\mathrm{wt}(Y(A)) = \mathrm{wt}(A)$ *and* $\mathrm{height}(Y(A)) = \mathrm{height}(A)$.

(2) *Let* $k = 1, 2$*. For all* $A \in \mathcal{A}_0^C(w, \Gamma_k)$*, it holds that* $\mathrm{wt}(I_k(A)) = \mathrm{wt}(A)$ *and* $\mathrm{height}(I_k(A)) = \mathrm{height}(A)$.

4.2 Proof of the existence of quantum Yang-Baxter moves

We prove Theorems 4.1.2 and 4.1.4 in this section. The proofs are based on a property analogous to the shellability of $\mathrm{QBG}(W)$ for the rank 2 root systems. This section is based on [14, Section 4 and Appendices A and B].

4.2.1 Quantum Bruhat operators

Let K be a field which contains the ring $\mathbb{C}[\![Q^{\vee,+}]\!] := \mathbb{C}[\![Q_i \mid i \in I]\!]$ of formal power series, where Q_i, $i \in I$, are variables, and set $Q^\xi := \prod_{i \in I} Q_i^{m_i}$ for $\xi = \sum_{i \in I} m_i \alpha_i^\vee \in Q^{\vee,+}$. For $\gamma \in \Delta^+$, following [3], we define the *quantum Bruhat operator* Q_γ on the group algebra $K[W]$ of W by

$$\mathsf{Q}_\gamma v := \begin{cases} v s_\gamma & \text{if } v \xrightarrow{\gamma} v s_\gamma \text{ is a Bruhat edge,} \\ Q^{\gamma^\vee} v s_\gamma & \text{if } v \xrightarrow{\gamma} v s_\gamma \text{ is a quantum edge,} \\ 0 & \text{otherwise.} \end{cases}$$

We set $\mathsf{Q}_{-\gamma} := -\mathsf{Q}_\gamma$ for $\gamma \in \Delta^+$, and then $\mathsf{R}_\gamma := 1 + \mathsf{Q}_\gamma$ for $\gamma \in \Delta$. The operators $\{\mathsf{R}_\gamma \mid \gamma \in \Delta\}$ satisfy the *Yang-Baxter equation*: for $\alpha, \beta \in \Delta$ such that $\langle \alpha, \beta^\vee \rangle \leq 0$ and $\alpha \neq -\beta$, it holds that

$$\mathsf{R}_\alpha \mathsf{R}_{s_\alpha(\beta)} \mathsf{R}_{s_\alpha s_\beta(\alpha)} \cdots \mathsf{R}_{s_\beta(\alpha)} \mathsf{R}_\beta = \mathsf{R}_\beta \mathsf{R}_{s_\beta(\alpha)} \cdots \mathsf{R}_{s_\alpha s_\beta(\alpha)} \mathsf{R}_{s_\alpha(\beta)} \mathsf{R}_\alpha; \qquad (4.2.1)$$

the proof of this equation is the same as that of [19, Proposition 36].

We give some properties of quantum Bruhat operators.

Lemma 4.2.1. *Let* $\Pi = (\beta_1, \dots, \beta_r)$ *be a sequence of roots (i.e.,* $\beta_1, \dots, \beta_r \in \Delta$*) such that* $\beta_1, \dots, \beta_r$ *are distinct.*

(1) *For* $v \in W$*, we have*

$$\mathsf{R}_{\beta_r}\mathsf{R}_{\beta_{r-1}} \cdots \mathsf{R}_{\beta_1} v = \sum_{\mathbf{p} \in \mathcal{P}(v,\Pi)} (-1)^{\mathrm{neg}(\mathbf{p})} Q^{\mathrm{wt}(\mathbf{p})} \, \mathrm{end}(\mathbf{p}).$$

(2) *For* $v \in W$*, we have*

$$\mathsf{R}_{|\beta_r|}\mathsf{R}_{|\beta_{r-1}|} \cdots \mathsf{R}_{|\beta_1|} v = \sum_{\mathbf{p} \in \mathcal{P}(v,\Pi)} Q^{\mathrm{wt}(\mathbf{p})} \, \mathrm{end}(\mathbf{p}).$$

Proof. For $J \subset \{1, \dots, r\}$, we set $\mathrm{neg}(J) := \{j \in J \mid \beta_j \in -\Delta^+\}$. We see that

$$\begin{aligned}
\mathsf{R}_{\beta_r}\mathsf{R}_{\beta_{r-1}} \cdots \mathsf{R}_{\beta_1} &= (1 + \mathsf{Q}_{\beta_r})(1 + \mathsf{Q}_{\beta_{r-1}}) \cdots (1 + \mathsf{Q}_{\beta_1}) \\
&= \sum_{\{j_1 < \dots < j_s\} \subset \{1,\dots,r\}} \mathsf{Q}_{\beta_{j_s}} \mathsf{Q}_{\beta_{j_{s-1}}} \cdots \mathsf{Q}_{\beta_{j_1}} \\
&= \sum_{\{j_1 < \dots < j_s\} \subset \{1,\dots,r\}} (\mathrm{sgn}(\beta_{j_s})\mathsf{Q}_{|\beta_{j_s}|})(\mathrm{sgn}(\beta_{j_{s-1}})\mathsf{Q}_{|\beta_{j_{s-1}}|}) \\
&\qquad\qquad \times \cdots (\mathrm{sgn}(\beta_{j_1})\mathsf{Q}_{|\beta_{j_1}|}) \\
&= \sum_{J = \{j_1 < \dots < j_s\} \subset \{1,\dots,r\}} (-1)^{\mathrm{neg}(J)} \mathsf{Q}_{|\beta_{j_s}|} \mathsf{Q}_{|\beta_{j_{s-1}}|} \cdots \mathsf{Q}_{|\beta_{j_1}|}. \quad (4.2.2)
\end{aligned}$$

Similarly, we see that

$$\mathsf{R}_{|\beta_r|}\mathsf{R}_{|\beta_{r-1}|} \cdots \mathsf{R}_{|\beta_1|} = \sum_{\{j_1 < \dots < j_s\} \subset \{1,\dots,r\}} \mathsf{Q}_{|\beta_{j_s}|} \mathsf{Q}_{|\beta_{j_{s-1}}|} \cdots \mathsf{Q}_{|\beta_{j_1}|}. \qquad (4.2.3)$$

For $J = \{j_1, \dots, j_s\} \subset \{1, \dots, r\}$, if we have the edge

$$v s_{|\beta_{j_1}|} \cdots s_{|\beta_{j_{a-1}}|} \xrightarrow{|\beta_{j_a}|} v s_{|\beta_{j_1}|} \cdots s_{|\beta_{j_a}|}$$

in $\mathrm{QBG}(W)$ for all $1 \le a \le s$, then we set $\delta(J) := 1$, and define a directed path $\mathbf{p}(J)$ in $\mathrm{QBG}(W)$ by

$$\mathbf{p}(J) : v \xrightarrow{|\beta_{j_1}|} v s_{|\beta_{j_1}|} \xrightarrow{|\beta_{j_2}|} \cdots \xrightarrow{|\beta_{j_s}|} v s_{|\beta_{j_1}|} \cdots s_{|\beta_{j_s}|};$$

otherwise, we set $\delta(J) := 0$. By the definition of quantum Bruhat operators, we have

$$\mathsf{Q}_{|\beta_{j_s}|}\mathsf{Q}_{|\beta_{j_{s-1}}|}\cdots\mathsf{Q}_{|\beta_{j_1}|}v = \begin{cases} Q^{\mathrm{wt}(\mathbf{p}(J))}\,\mathrm{end}(\mathbf{p}(J)) & \text{if } \delta(J)=1, \\ 0 & \text{if } \delta(J)=0. \end{cases}$$

If $\delta(J) = 1$, then we have $\mathrm{neg}(J) = \mathrm{neg}(\mathbf{p}(J))$. Therefore, by (4.2.2), we deduce that

$$\begin{aligned} \mathsf{R}_{\beta_r}\mathsf{R}_{\beta_{r-1}}\cdots\mathsf{R}_{\beta_1}v &= \sum_{J=\{j_1<\ldots<j_s\}\subset\{1,\ldots,r\}} (-1)^{\mathrm{neg}(J)}\mathsf{Q}_{|\beta_{j_s}|}\mathsf{Q}_{|\beta_{j_{s-1}}|}\cdots\mathsf{Q}_{|\beta_{j_1}|}v \\ &= \sum_{\substack{J\subset\{1,\ldots,r\}\\ \delta(J)=1}} (-1)^{\mathrm{neg}(\mathbf{p}(J))}Q^{\mathrm{wt}(\mathbf{p}(J))}\,\mathrm{end}(\mathbf{p}(J)) \\ &= \sum_{\mathbf{p}\in\mathcal{P}(v,\Pi)} (-1)^{\mathrm{neg}(\mathbf{p})}Q^{\mathrm{wt}(\mathbf{p})}\,\mathrm{end}(\mathbf{p}), \end{aligned}$$

as desired. This proves part (1) of the lemma.

Also, we see from (4.2.3) that

$$\begin{aligned} \mathsf{R}_{|\beta_r|}\mathsf{R}_{|\beta_{r-1}|}\cdots\mathsf{R}_{|\beta_1|}v &= \sum_{\{j_1<\ldots<j_s\}\subset\{1,\ldots,r\}} \mathsf{Q}_{|\beta_{j_s}|}\mathsf{Q}_{|\beta_{j_{s-1}}|}\cdots\mathsf{Q}_{|\beta_{j_1}|}v \\ &= \sum_{\substack{J\subset\{1,\ldots,r\}\\ \delta(J)=1}} Q^{\mathrm{wt}(\mathbf{p}(J))}\,\mathrm{end}(\mathbf{p}(J)) \\ &= \sum_{\mathbf{p}\in\mathcal{P}(v,\Pi)} Q^{\mathrm{wt}(\mathbf{p})}\,\mathrm{end}(\mathbf{p}). \end{aligned}$$

This proves part (2) of the lemma. □

Remark 4.2.2. If we set $\mathcal{P}(v,\Pi;w,\xi) := \{\mathbf{p}\in\mathcal{P}(v,\Pi) \mid \mathrm{end}(\mathbf{p}) = w,\ \mathrm{wt}(\mathbf{p}) = \xi\}$ for $v, w \in W$ and $\xi \in Q^{\vee,+}$, then by Lemma 4.2.1 (1), we deduce that

$$\mathsf{R}_{\beta_r}\cdots\mathsf{R}_{\beta_1}v = \sum_{w\in W}\sum_{\xi\in Q^{\vee,+}}\left(\sum_{\mathbf{p}\in\mathcal{P}(v,\Pi;w,\xi)}(-1)^{\mathrm{neg}(\mathbf{p})}\right)Q^{\xi}w. \tag{4.2.4}$$

Also, if we set $c^v_{\xi,w} := |\mathcal{P}(v,\Pi;w,\xi)|$ for $v, w \in W$ and $\xi \in Q^{\vee,+}$, then we deduce from Lemma 4.2.1 (2) that

$$\mathsf{R}_{|\beta_r|}\mathsf{R}_{|\beta_{r-1}|}\cdots\mathsf{R}_{|\beta_1|}v = \sum_{w\in W}\sum_{\xi\in Q^{\vee,+}} c^v_{\xi,w}Q^{\xi}w. \tag{4.2.5}$$

4.2.2 Key propositions to a generalization of quantum Yang-Baxter moves

We prove a certain property of $\mathrm{QBG}(W)$, which plays an important role in the proof of Theorem 4.1.2. Let $\alpha, \beta \in \Delta$ be such that $\langle \alpha, \beta^\vee \rangle \leq 0$ and $\alpha \neq -\beta$. We define sequences of roots Π, Π' by

$$\Pi = (\gamma_1, \ldots, \gamma_q) := (\alpha, s_\alpha(\beta), s_\alpha s_\beta(\alpha), \ldots, s_\beta(\alpha), \beta),$$
$$\Pi' = (\gamma'_1, \ldots, \gamma'_q) := (\beta, s_\beta(\alpha), \ldots, s_\alpha s_\beta(\alpha), s_\alpha(\beta), \alpha) = (\gamma_q, \ldots, \gamma_2, \gamma_1);$$

note that $\gamma_1, \ldots, \gamma_q$ are distinct. Also, let $\Delta_{\alpha,\beta}$ be the root subsystem of Δ generated by α and β. Then $\Delta_{\alpha,\beta}$ is a root system of rank 2. More precisely, we see that $\Delta_{\alpha,\beta}$ is isomorphic to the root system of type $A_1 \times A_1$, A_2, C_2, or G_2.

Assume temporarily that Δ is not of type G_2. Then we can prove the following property, which can be thought of as a generalization of the shellability of $\mathrm{QBG}(W)$ (Theorem 2.2.7) for the rank 2 root systems.

Proposition 4.2.3. *Let $v \in W$, and let $\mathbf{p}$ be a Π-compatible directed path in $\mathrm{QBG}(W)$ which starts at v, i.e., $\mathbf{p} \in \mathcal{P}(v, \Pi)$. Then only one of the following occurs.*

(1) *There exists a unique $\mathbf{p}' \in \mathcal{P}(v, \Pi) \setminus \{\mathbf{p}\}$ such that $\mathrm{end}(\mathbf{p}') = \mathrm{end}(\mathbf{p})$ and $\mathrm{wt}(\mathbf{p}') = \mathrm{wt}(\mathbf{p})$. This $\mathbf{p}'$ satisfies $(-1)^{\mathrm{neg}(\mathbf{p}')} = -(-1)^{\mathrm{neg}(\mathbf{p})}$. Moreover, there does not exist a path $\mathbf{q} \in \mathcal{P}(v, \Pi')$ such that $\mathrm{end}(\mathbf{q}) = \mathrm{end}(\mathbf{p})$ and $\mathrm{wt}(\mathbf{q}) = \mathrm{wt}(\mathbf{p})$.*

(2) *There exists a unique $\mathbf{p}' \in \mathcal{P}(v, \Pi')$ such that $\mathrm{end}(\mathbf{p}') = \mathrm{end}(\mathbf{p})$ and $\mathrm{wt}(\mathbf{p}') = \mathrm{wt}(\mathbf{p})$. This $\mathbf{p}'$ satisfies $(-1)^{\mathrm{neg}(\mathbf{p}')} = (-1)^{\mathrm{neg}(\mathbf{p})}$. Moreover, there does not exist a path $\mathbf{q} \in \mathcal{P}(v, \Pi) \setminus \{\mathbf{p}\}$ such that $\mathrm{end}(\mathbf{q}) = \mathrm{end}(\mathbf{p})$ and $\mathrm{wt}(\mathbf{q}) = \mathrm{wt}(\mathbf{p})$.*

The proof of this proposition can be reduced to the case that Δ is a root system of rank 2; in Section 4.2.8, we explain how to construct the explicit correspondence $\mathbf{p} \mapsto \mathbf{p}'$ through an example.

Now we assume that Δ is a root system of type $A_1 \times A_1$, A_2, or C_2. Then we see that there exists some $k = 1, \ldots, q$ such that γ_k and γ_{k+1} are the simple roots of Δ (for convenience of notation, we set $\gamma_{q+1} := \gamma_1$). We set

$$(\beta_1, \ldots, \beta_q) := (|\gamma_k|, |\gamma_{k-1}|, \ldots, |\gamma_1|, |\gamma_q|, \ldots, |\gamma_{k+1}|). \tag{4.2.6}$$

Then we have

$$(\beta_1, \ldots, \beta_q) = (\beta_1, s_{\beta_1}(\beta_q), s_{\beta_1} s_{\beta_q}(\beta_1), \ldots, s_{\beta_q}(\beta_1), \beta_q).$$

Also, if we set

$$\Pi^{\pm} := (\mp\beta_k, \mp\beta_{k-1}, \dots, \mp\beta_1, \pm\beta_q, \pm\beta_{q-1}, \dots, \pm\beta_{k+1}),$$

then $\Pi = \Pi^+$ or $\Pi = \Pi^-$. Note that the total order $\prec$ on $\{\beta_1, \dots, \beta_q\} = \Delta^+$ defined by

$$\beta_1 \prec \beta_2 \prec \cdots \prec \beta_q \tag{4.2.7}$$

is a reflection order; the total order $\prec'$ defined by

$$\beta_q \prec' \beta_{q-1} \prec' \cdots \prec' \beta_1 \tag{4.2.8}$$

is also a reflection order. We consider the following operators for $k = 0, 1, \dots, q$:

$$\begin{aligned} \mathsf{T}_k^{\pm} &:= \mathsf{R}_{\pm\beta_{k+1}} \cdots \mathsf{R}_{\pm\beta_q} \mathsf{R}_{\mp\beta_1} \cdots \mathsf{R}_{\mp\beta_k}, \\ \mathsf{S}_k &:= \mathsf{R}_{\beta_{k+1}} \cdots \mathsf{R}_{\beta_q} \mathsf{R}_{\beta_1} \cdots \mathsf{R}_{\beta_k}, \\ \mathsf{S}'_k &:= \mathsf{R}_{\beta_k} \cdots \mathsf{R}_{\beta_1} \mathsf{R}_{\beta_q} \cdots \mathsf{R}_{\beta_{k+1}}. \end{aligned}$$

In the following proposition, the matrices of operators on $K[W]$ are the representation matrices with respect to the basis W of $K[W]$. Note that for a W-linear operator $\mathsf{T} : K[W] \to K[W]$, the matrix of T is defined by $(c_{v,w})_{v,w \in W}$ if $\mathsf{T}w = \sum_{v \in W} c_{v,w} v$, $c_{v,w} \in K$.

Proposition 4.2.4. (1) *All the entries of the matrix of* S_k, $k = 0, 1, \dots, q$, *are of the form* $\sum_{j=1}^{r} m_j Q^{\xi_j}$, *where all* $\xi_j \in Q^{\vee,+}$ *are distinct, and* $m_j \in \{1, 2\}$.

(2) *Let* $v, w \in W$. *Assume that the* (v, w)*-entry of the matrix of* S_k *is of the form* $\sum_{j=1}^{r} m_j Q^{\xi_j}$ *as in (1). Also, assume that the* (v, w)*-entry of the matrix of* $\mathsf{T}_k^{\pm}$ *is of the form* $\sum_{\xi \in Q^{\vee,+}} n_{\xi}^{\pm} Q^{\xi}$. *For* $j = 1, \dots, r$, *if* $m_j = 2$, *then* $n_{\xi_j}^{\pm} = 0$, *and if* $m_j = 1$, *then* $n_{\xi_j}^{\pm} \in \{1, -1\}$. *Moreover, for* $\xi \in Q^{\vee,+} \setminus \{\xi_1, \dots, \xi_r\}$, *we have* $n_{\xi}^{\pm} = 0$.

(3) *Let* $v, w \in W$. *Assume that the* (v, w)*-entry of the matrix of* S_k *is of the form* $\sum_{j=1}^{r} m_j Q^{\xi_j}$ *as in (1). Also, assume that the* (v, w)*-entry of the matrix of* S'_k *is of the form* $\sum_{\xi \in Q^{\vee,+}} n_{\xi} Q^{\xi}$. *For* $j = 1, \dots, r$, *if* $m_j = 2$, *then* $n_{\xi_j} = 0$, *and if* $m_j = 1$, *then* $n_{\xi_j} = 1$.

The proof of Proposition 4.2.4 is based on direct calculations, which we give later.

Proof of Proposition 4.2.3. First, we show the proposition for the root system Δ of type $A_1 \times A_1$, A_2, or C_2. As in (4.2.6), we take a sequence

$$(\beta_1, \dots, \beta_q) := (|\gamma_k|, |\gamma_{k-1}|, \dots, |\gamma_1|, |\gamma_q|, \dots, |\gamma_{k+1}|)$$

of roots. Recall from (4.2.5) that

$$\mathsf{S}_k v = \sum_{w \in W} \sum_{\xi \in Q^{\vee,+}} c^v_{w,\xi} Q^\xi w,$$

where $c^v_{w,\xi} = |\mathcal{P}(v, \Pi; w, \xi)|$. By Proposition 4.2.4 (1), we see that $c^v_{w,\xi} \in \{0, 1, 2\}$. Also, again from (4.2.5), we see that

$$\mathsf{S}'_k v = \sum_{w \in W} \sum_{\xi \in Q^{\vee,+}} (c^v_{w,\xi})' Q^\xi w,$$

where $(c^v_{w,\xi})' = |\mathcal{P}(v, \Pi'; w, \xi)|$.

We write

$$\mathsf{T}^{\pm}_k v = \sum_{w \in W} \sum_{\xi \in Q^{\vee,+}} d^{v,\pm}_{w,\xi} Q^\xi w,$$

where $d^{v,\pm}_{w,\xi} \in \mathbb{Z}$. Then, by (4.2.4), if $\Pi = \Pi^+$, then

$$d^{v,+}_{w,\xi} = \sum_{\mathbf{q} \in \mathcal{P}(v,\Pi;w,\xi)} (-1)^{\mathrm{neg}(\mathbf{q})}, \quad d^{v,-}_{w,\xi} = \sum_{\mathbf{q} \in \mathcal{P}(v,\Pi;w,\xi)} (-1)^{\ell(\mathbf{q}) - \mathrm{neg}(\mathbf{q})}; \qquad (4.2.9)$$

if $\Pi = \Pi^-$, then

$$d^{v,+}_{w,\xi} = \sum_{\mathbf{q} \in \mathcal{P}(v,\Pi;w,\xi)} (-1)^{\ell(\mathbf{q}) - \mathrm{neg}(\mathbf{q})}, \quad d^{v,-}_{w,\xi} = \sum_{\mathbf{q} \in \mathcal{P}(v,\Pi;w,\xi)} (-1)^{\mathrm{neg}(\mathbf{q})}. \qquad (4.2.10)$$

We set $w := \mathrm{end}(\mathbf{p})$ and $\xi := \mathrm{wt}(\mathbf{p})$. Since $\mathbf{p} \in \mathcal{P}(v, \Pi; w, \xi)$, we have $c^v_{w,\xi} \neq 0$. First, assume that $c^v_{w,\xi} = 2$. Then there exists a unique $\mathbf{p}' \in \mathcal{P}(v, \Pi; w, \xi) \setminus \{\mathbf{p}\}$, i.e., there exists a unique $\mathbf{p}' \in \mathcal{P}(v, \Pi) \setminus \{\mathbf{p}\}$ such that $\mathrm{end}(\mathbf{p}') = \mathrm{end}(\mathbf{p}) = w$ and $\mathrm{wt}(\mathbf{p}') = \mathrm{wt}(\mathbf{p}) = \xi$. By Proposition 4.2.4 (2), we have $d^{v,\pm}_{w,\xi} = 0$. Hence, by (4.2.9) and (4.2.10), we obtain

$$\sum_{\mathbf{q} \in \mathcal{P}(v,\Pi;w,\xi)} (-1)^{\mathrm{neg}(\mathbf{q})} = (-1)^{\mathrm{neg}(\mathbf{p})} + (-1)^{\mathrm{neg}(\mathbf{p}')} = 0.$$

This shows that $(-1)^{\mathrm{neg}(\mathbf{p}')} = -(-1)^{\mathrm{neg}(\mathbf{p})}$. Here, by Proposition 4.2.4 (3), we deduce that $(c^v_{w,\xi})' = 0$. Hence there does not exist a $\mathbf{q} \in \mathcal{P}(v, \Pi')$ such that $\mathrm{end}(\mathbf{q}) = \mathrm{end}(\mathbf{p}) = w$ and $\mathrm{wt}(\mathbf{q}) = \mathrm{wt}(\mathbf{p}) = \xi$. This shows the proposition in the case $c^v_{w,\xi} = 2$.

Next, we assume that $c^v_{w,\xi} = 1$. In this case, there does not exist a $\mathbf{q} \in \mathcal{P}(v, \Pi) \setminus \{\mathbf{p}\}$ such that $\mathrm{end}(\mathbf{q}) = \mathrm{end}(\mathbf{p}) = w$ and $\mathrm{wt}(\mathbf{q}) = \mathrm{wt}(\mathbf{p}) = \xi$, since $\mathcal{P}(v, \Pi; w, \xi) = \{\mathbf{p}\}$. We set

$$(\mathsf{T}^{\pm}_k)' := \mathsf{R}_{\mp\beta_k} \cdots \mathsf{R}_{\mp\beta_1} \mathsf{R}_{\pm\beta_q} \cdots \mathsf{R}_{\pm\beta_{k+1}}.$$

Then, by the Yang-Baxter equation (4.2.1), we have $(\mathsf{T}^{\pm}_k)' = \mathsf{T}^{\pm}_k$. Hence, if we write

$$(\mathsf{T}^{\pm}_k)' v = \sum_{w \in W} \sum_{\xi \in Q^{\vee,+}} (d^{v,\pm}_{w,\xi})' Q^{\xi} w,$$

with $(d^{v,\pm}_{w,\xi})' \in \mathbb{Z}$, then we see that $(d^{v,\pm}_{w,\xi})' = d^{v,\pm}_{w,\xi}$. By Proposition 4.2.4 (2), we deduce that $(d^{v,\pm}_{w,\xi})' = d^{v,\pm}_{w,\xi} \in \{1, -1\}$. Again, by Proposition 4.2.4 (2) (by replacing $(\beta_1, \dots, \beta_q)$ with $(\beta_q, \dots, \beta_1)$), we deduce that $(c^v_{w,\xi})' = 1$. Hence there exists a unique $\mathbf{p}' \in \mathcal{P}(v, \Pi'; w, \xi)$, i.e., there exists a unique $\mathbf{p}' \in \mathcal{P}(v, \Pi')$ such that $\mathrm{end}(\mathbf{p}') = \mathrm{end}(\mathbf{p}) = w$ and $\mathrm{wt}(\mathbf{p}') = \mathrm{wt}(\mathbf{p}) = \xi$. If $\Pi = \Pi^+$, then

$$(-1)^{\mathrm{neg}(\mathbf{p})} = d^{v,+}_{w,\xi} = (d^{v,+}_{w,\xi})' = (-1)^{\mathrm{neg}(\mathbf{p}')};$$

if $\Pi = \Pi^-$, then

$$(-1)^{\mathrm{neg}(\mathbf{p})} = d^{v,-}_{w,\xi} = (d^{v,-}_{w,\xi})' = (-1)^{\mathrm{neg}(\mathbf{p}')}.$$

This shows that $(-1)^{\mathrm{neg}(\mathbf{p}')} = (-1)^{\mathrm{neg}(\mathbf{p})}$, as desired. This completes the proof of the proposition for the root system Δ of type $A_1 \times A_1$, A_2, or C_2.

Now, assume that the root system Δ is of an arbitrary type (except G_2), not necessarily of rank 2. Let $\overline{W}$ be the Weyl group of $\Delta_{\alpha,\beta}$. Note that $\overline{W}$ is a (dihedral) subgroup of W; the quantum Bruhat graph (denoted by $\mathrm{QBG}(\overline{W})$) of $\overline{W}$ is no longer a subgraph of $\mathrm{QBG}(W)$. By [18, Proposition 5.1 and Remarks 5.2 (2)], for each $u \in W$, there exist uniquely $\lfloor u \rfloor \in u\overline{W}$ and $\overline{u} \in \overline{W}$ such that

- $u = \overline{u}\lfloor u \rfloor$, and
- for a positive root γ of $\Delta_{\alpha,\beta}$, we have $\ell(\overline{u}) < \ell(\overline{u}s_{\gamma})$ if and only if $\ell(\lfloor u \rfloor\overline{u}) < \ell(\lfloor u \rfloor\overline{u}s_{\gamma})$.

We set $w := \mathrm{end}(\mathbf{p})$ and $\xi := \mathrm{wt}(\mathbf{p})$. Suppose, for a contradiction, that there exist two or more directed paths $\mathbf{q} \in \mathcal{P}(v, \Pi; w, \xi) \setminus \{\mathbf{p}\}$. Then we see that $|\mathcal{P}(v, \Pi, w, \xi)| \geq 3$. By [18, Theorem 5.3], there exists an injection $\mathcal{P}(v, \Pi; w, \xi) \hookrightarrow \mathcal{P}(\overline{v}, \Pi; \overline{w}, \gamma)$, where $\mathcal{P}(\overline{v}, \Pi; \overline{w}, \gamma)$ is the set of all Π-compatible directed paths in $\mathrm{QBG}(\overline{W})$ which starts at $\overline{v}$, ends at $\overline{w}$, and has weight ξ, where Π is considered to be a sequence of roots in the root system $\Delta_{\alpha,\beta}$. Hence we have $\#\mathcal{P}(\overline{v}, \Pi; \overline{w}, \xi) \geq 3$. This contradicts the proposition for the rank 2 root systems, shown above. Hence we conclude that there exists at most one directed path $\mathbf{q} \in \mathcal{P}(v, \Pi; w, \xi)$. If such a $\mathbf{q}$ exists, then, by the proposition for the rank 2 root systems and [18, Theorem 5.3], we have $(-1)^{\mathrm{neg}(\mathbf{q})} = -(-1)^{\mathrm{neg}(\mathbf{p})}$. Also, a similar argument shows that there exists at most one directed path $\mathbf{r} \in \mathcal{P}(v, \Pi'; w, \xi)$. If such an $\mathbf{r}$ exists, then we have $(-1)^{\mathrm{neg}(\mathbf{r})} = (-1)^{\mathrm{neg}(\mathbf{p})}$.

We show that at least one of the directed paths $\mathbf{q}$ and $\mathbf{r}$ exists. We write

$$\mathsf{R}_{\gamma_q} \cdots \mathsf{R}_{\gamma_1} v = \sum_{w \in W} \sum_{\xi \in Q^{\vee,+}} d^v_{w,\xi} Q^\xi w,$$
$$\mathsf{R}_{\gamma_1} \cdots \mathsf{R}_{\gamma_q} v = \sum_{w \in W} \sum_{\xi \in Q^{\vee,+}} (d^v_{w,\xi})' Q^\xi w.$$

If there does not exist a directed path $\mathbf{q} \in \mathcal{P}(v, \Pi; w, \xi) \setminus \{\mathbf{p}\}$, then by (4.2.4), we have $d^v_{w,\xi} = \pm 1$. By the Yang-Baxter equation (4.2.1), we deduce that $(d^v_{w,\xi})' = d^v_{w,\xi} = \pm 1$. By (4.2.4), we see that $\mathcal{P}(v, \Pi'; w, \xi) \neq \emptyset$. Therefore, we conclude that there exists a directed path $\mathbf{r} \in \mathcal{P}(v, \Pi'; w, \xi)$ in this case, as desired.

Finally, suppose, for a contradiction, that both $\mathbf{q}$ and $\mathbf{r}$ exist at the same time. Then, by [18, Theorem 5.3], we have $|\mathcal{P}(\overline{v}, \Pi; \overline{w}, \xi)| \geq 2$ and $|\mathcal{P}(\overline{v}, \Pi'; \overline{w}, \xi)| \geq 1$. This contradicts the proposition for the rank 2 root systems, shown above.

This completes the proof of Proposition 4.2.3. □

Thus it remains to prove Proposition 4.2.4. We assume temporarily that Δ is of type $A_1 \times A_1$, A_2, or C_2. If Δ is of type A_2 (resp., $A_1 \times A_1$, C_2), then we have $q = 3$ (resp., $q = 2, 4$). By the shellability of $\mathrm{QBG}(W)$, there exists a unique label-increasing directed path (with respect to $\prec$ or $\prec'$, defined in (4.2.7) and (4.2.8)) from v to w in $\mathrm{QBG}(W)$ for all $v, w \in W$. Hence we have

$$\mathsf{T}^+_0 v = \mathsf{T}^-_q v = \mathsf{S}_0 v = \mathsf{S}'_0 v = \sum_{w \in W} Q^{\mathrm{wt}(v \Rightarrow w)} w,$$
$$\mathsf{T}^-_0 v = \mathsf{T}^+_q v = \sum_{w \in W} (-1)^{\ell(v \Rightarrow w)} Q^{\mathrm{wt}(v \Rightarrow w)} w$$

for all $v \in W$. Therefore, the proposition is obvious in the case $k = 0, q$. Hence it suffices to prove the proposition in the case $k = 1, q-1$ for all types, and in the case $k = 2$ for type C_2.

4.2.3 Proof of Proposition 4.2.4: $k = 1, q-1$

We prove Proposition 4.2.4 in the case $k = 1, q-1$; recall that β_1 and β_q are the simple roots of Δ. By (4.2.4), for all $v \in W$, we have

$$\begin{aligned}\mathsf{T}_1^+ v &= \mathsf{R}_{\beta_2} \cdots \mathsf{R}_{\beta_q}(\mathsf{R}_{-\beta_1} v) \\ &= \mathsf{R}_{\beta_2} \cdots \mathsf{R}_{\beta_q}((1-\mathsf{Q}_{\beta_1})v) \\ &= \mathsf{R}_{\beta_2} \cdots \mathsf{R}_{\beta_q}(v - Q^{\mathrm{wt}(v\to vs_{\beta_1})} vs_{\beta_1}) \\ &= \sum_{\mathbf{q}\in\mathcal{P}(v,(\beta_q,\dots,\beta_2))} Q^{\mathrm{wt}(\mathbf{q})}\,\mathrm{end}(\mathbf{q}) \\ &\quad - \sum_{\mathbf{q}\in\mathcal{P}(vs_{\beta_1},(\beta_q,\dots,\beta_2))} Q^{\mathrm{wt}(v\to vs_{\beta_1})+\mathrm{wt}(\mathbf{q})}\,\mathrm{end}(\mathbf{q}).\end{aligned}$$

Recall that the total order $\prec'$ on $\Delta^+ = \{\beta_1, \dots, \beta_q\}$, defined by (4.2.8), is a reflection order. Hence, by the shellability of QBG(W), for all $w \in W$, there exists at most one directed path $\mathbf{q} \in \mathcal{P}(v, (\beta_q, \dots, \beta_2))$ such that $\mathrm{end}(\mathbf{q}) = w$. For such $\mathbf{q}$, we have $\mathrm{wt}(\mathbf{q}) = \mathrm{wt}(v \Rightarrow w)$ since $\mathbf{q}$ is a shortest directed path from v to w. The same argument shows that for all $w \in W$, there exists at most one directed path $\mathbf{q} \in \mathcal{P}(vs_{\beta_1}, (\beta_q, \dots, \beta_2))$ such that $\mathrm{end}(\mathbf{q}) = w$ and $\mathrm{wt}(\mathbf{q}) = \mathrm{wt}(vs_{\beta_1} \Rightarrow w)$. Hence, if we set

$$\delta_{v,w} := \begin{cases} 1 & \text{if there exists } \mathbf{q} \in \mathcal{P}(v,(\beta_q,\dots,\beta_2)) \text{ such that } \mathrm{end}(\mathbf{q}) = w, \\ 0 & \text{otherwise} \end{cases}$$

for $v, w \in W$, then we have

$$\begin{aligned}\mathsf{T}_1^+ v &= \sum_{w\in W} \delta_{v,w} Q^{\mathrm{wt}(v\Rightarrow w)} w - \sum_{w\in W} \delta_{vs_{\beta_1},w} Q^{\mathrm{wt}(v\to vs_{\beta_1})+\mathrm{wt}(vs_{\beta_1}\Rightarrow w)} w \\ &= \sum_{w\in W} (\delta_{v,w} Q^{\mathrm{wt}(v\Rightarrow w)} - \delta_{vs_{\beta_1},w} Q^{\mathrm{wt}(v\to vs_{\beta_1})+\mathrm{wt}(vs_{\beta_1}\Rightarrow w)}) w. \end{aligned} \tag{4.2.11}$$

Also, by the same argument, we see that

$$\mathsf{T}_1^- v = \sum_{w\in W} (-1)^{\ell(v\Rightarrow w)} (\delta_{v,w} Q^{\mathrm{wt}(v\Rightarrow w)} - \delta_{vs_{\beta_1},w} Q^{\mathrm{wt}(v\to vs_{\beta_1})+\mathrm{wt}(vs_{\beta_1}\Rightarrow w)}) w; \tag{4.2.12}$$

note that for a directed path $\mathbf{q}$ from vs_{β_1} to w, it follows that $(-1)^{\ell(\mathbf{q})} = (-1)^{\ell(vs_{\beta_1}\Rightarrow w)}$, and hence

$$(-1)^{\ell(vs_{\beta_1}\Rightarrow w)} = (-1)^{\ell(vs_{\beta_1}\Rightarrow v)+\ell(v\Rightarrow w)} = (-1)^{1+\ell(v\Rightarrow w)} = -(-1)^{\ell(v\Rightarrow w)}.$$

Let us consider S_1. By the same argument as for T_1^+, we deduce that

$$\begin{aligned}\mathsf{S}_1 v &= \sum_{\mathbf{q}\in\mathcal{P}(v,(\beta_q,\ldots,\beta_2))} Q^{\mathrm{wt}(\mathbf{q})}\,\mathrm{end}(\mathbf{q}) + \sum_{\mathbf{q}\in\mathcal{P}(vs_{\beta_1},(\beta_q,\ldots,\beta_2))} Q^{\mathrm{wt}(v\to vs_{\beta_1})+\mathrm{wt}(\mathbf{q})}\,\mathrm{end}(\mathbf{q})\\ &= \sum_{w\in W}(\delta_{v,w}Q^{\mathrm{wt}(v\Rightarrow w)} + \delta_{vs_{\beta_1},w}Q^{\mathrm{wt}(v\to vs_{\beta_1})+\mathrm{wt}(vs_{\beta_1}\Rightarrow w)})w. \qquad (4.2.13)\end{aligned}$$

Hence equations (4.2.11), (4.2.12), and (4.2.13) imply Proposition 4.2.4 (1) and (2) in the case $k = 1$, as desired.

Next, we consider the case $k = q-1$; recall that β_q is a simple root of Δ. By (4.2.4), we have

$$\begin{aligned}\mathsf{T}_{q-1}^+ v &= \mathsf{R}_{\beta_q}(\mathsf{R}_{-\beta_1}\cdots\mathsf{R}_{-\beta_{q-1}}v)\\ &= \mathsf{R}_{\beta_q}\left(\sum_{\mathbf{q}\in\mathcal{P}(v,(-\beta_{q-1},\ldots,-\beta_1))}(-1)^{\ell(\mathbf{q})}Q^{\mathrm{wt}(\mathbf{q})}\,\mathrm{end}(\mathbf{q})\right)\\ &= \sum_{\mathbf{q}\in\mathcal{P}(v,(-\beta_{q-1},\ldots,-\beta_1))}(-1)^{\ell(\mathbf{q})}\\ &\qquad\times(Q^{\mathrm{wt}(\mathbf{q})}\,\mathrm{end}(\mathbf{q}) + Q^{\mathrm{wt}(\mathbf{q})+\mathrm{wt}(\mathrm{end}(\mathbf{q})\to\mathrm{end}(\mathbf{q})s_{\beta_q})}\,\mathrm{end}(\mathbf{q})s_{\beta_q}).\end{aligned}$$

Hence, if we set

$$\delta'_{v,w} := \begin{cases} 1 & \begin{cases}\text{if there exists } \mathbf{q}\in\mathcal{P}(v,(-\beta_{q-1},\ldots,-\beta_1))\\ \text{such that } \mathrm{end}(\mathbf{q}) = w,\end{cases}\\ 0 & \text{otherwise}\end{cases}$$

for $v, w \in W$, then we have

$$\mathsf{T}_{q-1}^+ v = \sum_{w\in W}(-1)^{\ell(v\Rightarrow w)}(\delta'_{v,w}Q^{\mathrm{wt}(v\Rightarrow w)} - \delta'_{v,ws_{\beta_q}}Q^{\mathrm{wt}(v\Rightarrow ws_{\beta_q})+\mathrm{wt}(ws_{\beta_q}\to w)})w. \qquad (4.2.14)$$

Similarly, we have

$$\mathsf{T}_{q-1}^- v = \sum_{w\in W}(\delta'_{v,w}Q^{\mathrm{wt}(v\Rightarrow w)} - \delta'_{v,ws_{\beta_q}}Q^{\mathrm{wt}(v\Rightarrow ws_{\beta_q})+\mathrm{wt}(ws_{\beta_q}\to w)})w. \qquad (4.2.15)$$

Also, we see that

$$\mathsf{S}_{q-1} v = \sum_{w\in W}(\delta'_{v,w}Q^{\mathrm{wt}(v\Rightarrow w)} + \delta'_{v,ws_{\beta_q}}Q^{\mathrm{wt}(v\Rightarrow ws_{\beta_q})+\mathrm{wt}(ws_{\beta_q}\to w)})w. \qquad (4.2.16)$$

Hence equations (4.2.14), (4.2.15), and (4.2.16) imply Proposition 4.2.4 (1) and (2) in the case $k = q-1$.

It remains to prove Proposition 4.2.4 (3) in the case $k = 1, q-1$. It suffices to prove it in the case $k = 1$; indeed, if we replace $(\beta_1, \ldots, \beta_q)$ with $(\beta_q, \ldots, \beta_1)$ and consider the case $k = 1$, then we obtain the proposition in the case $k = q-1$. Recall equation (4.2.13). By the same argument, we see that

$$\mathsf{S}_1' v = \sum_{w \in W} (\varepsilon_{v,w} Q^{\mathrm{wt}(v \Rightarrow w)} + \varepsilon_{v, ws_{\beta_1}} Q^{\mathrm{wt}(v \Rightarrow ws_{\beta_1}) + \mathrm{wt}(ws_{\beta_1} \to w)}) w,$$

where

$$\varepsilon_{v,w} := \begin{cases} 1 & \text{if there exists } \mathbf{q} \in \mathcal{P}(v, (\beta_2, \ldots, \beta_q)) \text{ such that } \mathrm{end}(\mathbf{q}) = w, \\ 0 & \text{otherwise} \end{cases}$$

for $v, w \in W$. Assume that $c^v_{w,\xi} = 2$ for some $v, w \in W$ and $\xi \in Q^{\vee,+}$. It suffices to show that $(c^v_{w,\xi})' = 0$. In this case, we deduce from (4.2.13) that

$$\delta_{v,w} = \delta_{vs_{\beta_1},w} = 1, \tag{4.2.17}$$

$$\mathrm{wt}(v \Rightarrow w) = \mathrm{wt}(v \to vs_{\beta_1}) + \mathrm{wt}(vs_{\beta_1} \Rightarrow w) = \xi. \tag{4.2.18}$$

By (4.2.18), we see that the concatenation of the edge $v \to vs_{\beta_1}$ with any shortest directed path from vs_{β_1} to w in $\mathrm{QBG}(W)$ is a shortest directed path from v to w (cf. [3, Lemma 6.7], [32, Lemma 1 (2)], and [20, Proposition 8.1]). Now, take the (unique) label-increasing directed path $\mathbf{r}_0$ from vs_{β_1} to w in $\mathrm{QBG}(W)$ with respect to $\prec$ defined by (4.2.7), and let $\mathbf{r}$ be the concatenation of the edge $v \to vs_{\beta_1}$ with the path $\mathbf{r}_0$. Note that $\mathbf{r}_0$ is shortest, and hence $\mathbf{r}$ is also shortest. We claim that $\mathbf{r}_0 \in \mathcal{P}(v, (\beta_2, \ldots, \beta_q))$; otherwise, the concatenation

$$\mathbf{r} : v \xrightarrow{\beta_1} \underbrace{vs_{\beta_1} \xrightarrow{\beta_1} \cdots \to w}_{\mathbf{r}_0}$$

cannot be shortest. Hence $\mathbf{r}$ is the label-increasing directed path from v to w in $\mathrm{QBG}(W)$ such that $\mathbf{r} \notin \mathcal{P}(v, (\beta_2, \ldots, \beta_q))$. By the uniqueness of a label-increasing directed path, we conclude that $\varepsilon_{v,w} = 0$. Since $\delta_{v,w} = 1$ by (4.2.17), there exists $\mathbf{r}_1 \in \mathcal{P}(v, (\beta_q, \ldots, \beta_2))$ such that $\mathrm{end}(\mathbf{r}_1) = w$. Then the concatenation of the path $\mathbf{r}_1$ with the edge $w \to ws_{\beta_1}$ is label-increasing with respect to $\prec'$, defined by (4.2.8), and hence this concatenation is shortest. Also, since $\delta_{vs_{\beta_1},w} = 1$ by (4.2.17), there exists $\mathbf{r}_2 \in \mathcal{P}(vs_{\beta_1}, (\beta_q, \ldots, \beta_2))$ such that $\mathrm{end}(\mathbf{r}_2) = w$. Similarly, the concatenation of the path $\mathbf{r}_2$ with the edge $w \to ws_{\beta_1}$ is label-increasing with respect to $\prec'$, and hence this concatenation is shortest. Since the concatenation of the edge $v \to vs_{\beta_1}$ with any shortest directed path from vs_{β_1} to w is shortest, we obtain:

$$\begin{aligned}\ell(v \Rightarrow ws_{\beta_1}) &= \underbrace{\ell(v \Rightarrow w)}_{=\ell(\mathbf{r}_1)} + \ell(w \rightarrow ws_{\beta_1}) \\ &= \ell(v \rightarrow vs_{\beta_1}) + \underbrace{\ell(vs_{\beta_1} \Rightarrow w)}_{=\ell(\mathbf{r}_2)} + \ell(w \rightarrow ws_{\beta_1}) \\ &= \ell(v \rightarrow vs_{\beta_1}) + \ell(vs_{\beta_1} \Rightarrow ws_{\beta_1}).\end{aligned}$$

Hence the concatenation of the edge $v \rightarrow vs_{\beta_1}$ with any shortest directed path from vs_{β_1} to ws_{β_1} is shortest. Take the (unique) label-increasing directed path $\mathbf{r}_3$ from vs_{β_1} to ws_{β_1} in QBG(W) with respect to $\prec$. Then we deduce that $\mathbf{r}_3 \in \mathcal{P}(vs_{\beta_1}, (\beta_2, \dots, \beta_q))$; otherwise, the concatenation

$$v \xrightarrow{\beta_1} \underbrace{vs_{\beta_1} \xrightarrow{\beta_1} \cdots \rightarrow ws_{\beta_1}}_{\mathbf{r}_3}$$

cannot be shortest. Hence we conclude that $\varepsilon_{v,ws_{\beta_1}} = 0$. This completes the proof that $(c^v_{w,\xi})' = 0$.

It remains to show that if $c^v_{w,\xi} = 1$, then $(c^v_{w,\xi})' = 1$. Assume that $c^v_{w,\xi} = 1$. By the above argument (i.e., Proposition 4.2.4 (2) in the case $k = 1$), we have $d^{v,+}_{w,\xi} = \pm 1$. By the Yang-Baxter equation (4.2.1), we see that $(d^{v,+}_{w,\xi})' = d^{v,+}_{w,\xi} = \pm 1$. Hence we deduce again from the above argument (i.e., Proposition 4.2.4 (2) in the case $k = 1$, with $(\beta_1, \dots, \beta_q)$ replaced by $(\beta_q, \dots, \beta_1)$) that $(c^v_{w,\xi})' = 1$.

This completes the proof of Proposition 4.2.4 in the case $k = 1, q - 1$.

4.2.4 Proof of Proposition 4.2.4: case of type C_2

We consider the root system Δ of type C_2. We know that $q = 4$, and $(\beta_1, \beta_2, \beta_3, \beta_4) = (\alpha_1, 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2, \alpha_2)$ or $(\beta_1, \beta_2, \beta_3, \beta_4) = (\alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, \alpha_1)$.

Since only the case $k = 2$ is remaining, it suffices to calculate the matrices (with respect to the basis $W = \{e, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1, s_2s_1s_2, w_\circ\}$ of $K[W]$) of the following four operators:

(1) $\mathsf{R}_{\alpha_1+\alpha_2}\mathsf{R}_{\alpha_2}\mathsf{R}_{-\alpha_1}\mathsf{R}_{-2\alpha_1-\alpha_2} = \mathsf{R}_{-2\alpha_1-\alpha_2}\mathsf{R}_{-\alpha_1}\mathsf{R}_{\alpha_2}\mathsf{R}_{\alpha_1+\alpha_2}$;

(2) $\mathsf{R}_{2\alpha_1+\alpha_2}\mathsf{R}_{\alpha_1}\mathsf{R}_{-\alpha_2}\mathsf{R}_{-\alpha_1-\alpha_2} = \mathsf{R}_{-\alpha_1-\alpha_2}\mathsf{R}_{-\alpha_2}\mathsf{R}_{\alpha_1}\mathsf{R}_{2\alpha_1+\alpha_2}$;

(3) $\mathsf{R}_{\alpha_1+\alpha_2}\mathsf{R}_{\alpha_2}\mathsf{R}_{\alpha_1}\mathsf{R}_{2\alpha_1+\alpha_2}$; and

(4) $\mathsf{R}_{2\alpha_1+\alpha_2}\mathsf{R}_{\alpha_1}\mathsf{R}_{\alpha_2}\mathsf{R}_{\alpha_1+\alpha_2}$,

where the equalities in (1) and (2) follow from the Yang-Baxter equation (4.2.1). The following are the matrices (with respect to the basis W) of operators Q_γ, $\gamma \in \Delta^+ = \{\alpha_1, 2\alpha_1+\alpha_2, \alpha_1+\alpha_2, \alpha_2\}$ (cf. [18, Fig. 2 (B)]).

[1] Q_{α_1}:

$$\begin{pmatrix} 0 & Q_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & Q_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & Q_1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

[2] $\mathsf{Q}_{2\alpha_1+\alpha_2}$:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & Q_1Q_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_1Q_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

[3] $\mathsf{Q}_{\alpha_1+\alpha_2}$:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

[4] Q_{α_2}:

$$\begin{pmatrix} 0 & 0 & Q_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & Q_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

By explicit calculations (by using, e.g., SageMath), we obtain the following matrices.

(1) $\mathsf{R}_{\alpha_1+\alpha_2}\mathsf{R}_{\alpha_2}\mathsf{R}_{-\alpha_1}\mathsf{R}_{-2\alpha_1-\alpha_2}$:

$$\begin{pmatrix} 1 & Q_1Q_2-Q_1 & Q_2 & 0 & -Q_1Q_2 & -Q_1Q_2 & 0 & -Q_1Q_2^2 \\ -1 & 1 & 0 & Q_2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & -Q_1 & -Q_1Q_2 & 0 & -Q_1Q_2 \\ 0 & 1 & 1 & 1 & -Q_1 & -Q_1 & 0 & -Q_1Q_2 \\ 0 & -1 & -1 & -Q_2 & 1 & 0 & Q_2 & 0 \\ 0 & -1 & -1 & -1 & 1 & 1 & 0 & Q_2 \\ 0 & -1 & -1 & -1 & 1 & 0 & 1 & Q_1Q_2-Q_1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \end{pmatrix}$$

(2) $\mathsf{R}_{2\alpha_1+\alpha_2}\mathsf{R}_{\alpha_1}\mathsf{R}_{-\alpha_2}\mathsf{R}_{-\alpha_1-\alpha_2}$:

$$\begin{pmatrix} 1 & -Q_1Q_2+Q_1 & -Q_2 & 0 & -Q_1Q_2 & Q_1Q_2 & 0 & -Q_1Q_2^2 \\ 1 & 1 & 0 & -Q_2 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & Q_1 & -Q_1Q_2 & 0 & Q_1Q_2 \\ 0 & -1 & -1 & 1 & -Q_1 & Q_1 & 0 & -Q_1Q_2 \\ 0 & 1 & 1 & -Q_2 & 1 & 0 & -Q_2 & 0 \\ 0 & -1 & -1 & 1 & -1 & 1 & 0 & -Q_2 \\ 0 & -1 & -1 & 1 & -1 & 0 & 1 & -Q_1Q_2+Q_1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 \end{pmatrix}$$

(3) $\mathsf{R}_{\alpha_1+\alpha_2}\mathsf{R}_{\alpha_2}\mathsf{R}_{\alpha_1}\mathsf{R}_{2\alpha_1+\alpha_2}$:

$$\begin{pmatrix} 1 & Q_1Q_2+Q_1 & Q_2 & 0 & Q_1Q_2 & Q_1Q_2 & 0 & Q_1Q_2^2 \\ 1 & 1 & 0 & Q_2 & 0 & 2Q_1Q_2 & 0 & 0 \\ 1 & 2Q_1 & 1 & 0 & Q_1 & Q_1Q_2 & 0 & Q_1Q_2 \\ 2 & 2Q_1+1 & 1 & 1 & Q_1 & 2Q_1Q_2+Q_1 & 0 & Q_1Q_2 \\ 0 & 1 & 1 & Q_2 & 1 & 0 & Q_2 & 2Q_1Q_2 \\ 0 & 1 & 1 & 2Q_2+1 & 1 & 1 & 2Q_2 & 2Q_1Q_2+Q_2 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 & Q_1Q_2+Q_1 \\ 0 & 0 & 0 & 2 & 0 & 1 & 1 & 1 \end{pmatrix}$$

(4) $\mathsf{R}_{2\alpha_1+\alpha_2}\mathsf{R}_{\alpha_1}\mathsf{R}_{\alpha_2}\mathsf{R}_{\alpha_1+\alpha_2}$:

$$\begin{pmatrix} 1 & Q_1Q_2+Q_1 & 2Q_1Q_2+Q_2 & 2Q_1Q_2 & Q_1Q_2 & Q_1Q_2 & 0 & Q_1Q_2^2 \\ 1 & 1 & 2Q_2 & Q_2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 2Q_1Q_2+Q_1 & Q_1Q_2 & 2Q_1Q_2 & Q_1Q_2 \\ 0 & 1 & 1 & 1 & Q_1 & Q_1 & 0 & Q_1Q_2 \\ 2 & 1 & 2Q_2+1 & Q_2 & 1 & 0 & Q_2 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & Q_2 \\ 0 & 1 & 1 & 1 & 2Q_1+1 & 2Q_1 & 1 & Q_1Q_2+Q_1 \\ 0 & 0 & 0 & 0 & 2 & 1 & 1 & 1 \end{pmatrix}$$

This proves the proposition.

4.2.5 Case of type G_2

We consider the root system Δ of type G_2. We have $q = 6$, and

$$(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6) = (\alpha_1, 3\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2, \alpha_1 + \alpha_2, \alpha_2)$$

or

$$(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6) = (\alpha_2, \alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, \alpha_1).$$

In this case, a key proposition to the proof of Theorem 4.1.2 is slightly different from that in the other types. First, we state the following proposition for quantum Bruhat operators in type G_2.

Proposition 4.2.5. (1) *All the entries of the matrix of* S_k, $k = 0, 1, \ldots, 6$, *are of the form* $\sum_{j=1}^{r} m_j Q^{\xi_j}$, *where all* $\xi_j \in Q^{\vee,+}$ *are distinct, and* $m_j \in \{1, 2, 3\}$.

(2) *Let* $v, w \in W$. *Assume that the* (v, w)*-entry of the matrix of* S_k *is of the form* $\sum_{j=1}^{r} m_j Q^{\xi_j}$ *as in (1). Also, assume that the* (v, w)*-entry of the matrix of* $\mathsf{T}_k^{\pm}$ *is of the form* $\sum_{\xi \in Q^{\vee,+}} n_\xi^{\pm} Q^{\xi}$. *For* $j = 1, \ldots, r$, *if* $m_j = 2$, *then* $n_{\xi_j}^{\pm} = 0$, *and if* $m_j = 1$ *or if* $m_j = 3$, *then* $n_{\xi_j}^{\pm} \in \{1, -1\}$. *Moreover, for* $\xi \in Q^{\vee,+} \setminus \{\xi_1, \ldots, \xi_r\}$, *we have* $n_\xi^{\pm} = 0$.

(3) *Let* $v, w \in W$. *Assume that the* (v, w)*-entry of the matrix of* S_k *is of the form* $\sum_{j=1}^{r} m_j Q^{\xi_j}$ *as in (1). Also, assume that the* (v, w)*-entry of the matrix of* S'_k *is of the form* $\sum_{\xi \in Q^{\vee,+}} n_\xi Q^{\xi}$. *For* $j = 1, \ldots, r$, *if* $m_j = 2$, *then* $n_{\xi_j} = 0$ *or* $n_{\xi_j} = 2$, *and if* $m_j = 1$, *then* $n_{\xi_j} = 1$ *or* $n_{\xi_j} = 3$. *Moreover, if* $m_j = 3$, *then* $n_{\xi_j} = 1$.

The proof in the case $k = 0, 1, 5, 6$ is the same as that in types $A_1 \times A_1$, A_2, and C_2, given above. Since the case $k = 2, 3, 4$ is remaining, we need to calculate the matrices (with respect to the basis

$$W = \{e, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1, s_2s_1s_2, s_1s_2s_1s_2, s_2s_1s_2s_1, s_1s_2s_1s_2s_1, s_2s_1s_2s_1s_2, w_\circ\}$$

of $K[W]$) of the following 12 operators:

(1) $\mathsf{R}_{2\alpha_1+\alpha_2}\mathsf{R}_{3\alpha_1+2\alpha_2}\mathsf{R}_{\alpha_1+\alpha_2}\mathsf{R}_{\alpha_2}\mathsf{R}_{-\alpha_1}\mathsf{R}_{-3\alpha_1-\alpha_2}$;

(2) $\mathsf{R}_{3\alpha_1+2\alpha_2}\mathsf{R}_{\alpha_1+\alpha_2}\mathsf{R}_{\alpha_2}\mathsf{R}_{-\alpha_1}\mathsf{R}_{-3\alpha_1-\alpha_2}\mathsf{R}_{-2\alpha_1-\alpha_2}$;

(3) $\mathsf{R}_{\alpha_1+\alpha_2}\mathsf{R}_{\alpha_2}\mathsf{R}_{-\alpha_1}\mathsf{R}_{-3\alpha_1-\alpha_2}\mathsf{R}_{-2\alpha_1-\alpha_2}\mathsf{R}_{-3\alpha_1-2\alpha_2}$;

(4) $\mathsf{R}_{3\alpha_1+2\alpha_2}\mathsf{R}_{2\alpha_1+\alpha_2}\mathsf{R}_{3\alpha_1+\alpha_2}\mathsf{R}_{\alpha_1}\mathsf{R}_{-\alpha_2}\mathsf{R}_{-\alpha_1-\alpha_2}$;

(5) $\mathsf{R}_{2\alpha_1+\alpha_2}\mathsf{R}_{3\alpha_1+\alpha_2}\mathsf{R}_{\alpha_1}\mathsf{R}_{-\alpha_2}\mathsf{R}_{-\alpha_1-\alpha_2}\mathsf{R}_{-3\alpha_1-2\alpha_2}$;

(6) $\mathsf{R}_{3\alpha_1+\alpha_2}\mathsf{R}_{\alpha_1}\mathsf{R}_{-\alpha_2}\mathsf{R}_{-\alpha_1-\alpha_2}\mathsf{R}_{-3\alpha_1-2\alpha_2}\mathsf{R}_{-2\alpha_1-\alpha_2}$;

(7) $\mathsf{R}_{2\alpha_1+\alpha_2}\mathsf{R}_{3\alpha_1+2\alpha_2}\mathsf{R}_{\alpha_1+\alpha_2}\mathsf{R}_{\alpha_2}\mathsf{R}_{\alpha_1}\mathsf{R}_{3\alpha_1+\alpha_2}$;

(8) $\mathsf{R}_{3\alpha_1+2\alpha_2}\mathsf{R}_{\alpha_1+\alpha_2}\mathsf{R}_{\alpha_2}\mathsf{R}_{\alpha_1}\mathsf{R}_{3\alpha_1+\alpha_2}\mathsf{R}_{2\alpha_1+\alpha_2}$;

(9) $\mathsf{R}_{\alpha_1+\alpha_2}\mathsf{R}_{\alpha_2}\mathsf{R}_{\alpha_1}\mathsf{R}_{3\alpha_1+\alpha_2}\mathsf{R}_{2\alpha_1+\alpha_2}\mathsf{R}_{3\alpha_1+2\alpha_2}$;

(10) $\mathsf{R}_{3\alpha_1+2\alpha_2}\mathsf{R}_{2\alpha_1+\alpha_2}\mathsf{R}_{3\alpha_1+\alpha_2}\mathsf{R}_{\alpha_1}\mathsf{R}_{\alpha_2}\mathsf{R}_{\alpha_1+\alpha_2}$;

(11) $\mathsf{R}_{2\alpha_1+\alpha_2}\mathsf{R}_{3\alpha_1+\alpha_2}\mathsf{R}_{\alpha_1}\mathsf{R}_{\alpha_2}\mathsf{R}_{\alpha_1+\alpha_2}\mathsf{R}_{3\alpha_1+2\alpha_2}$;

(12) $\mathsf{R}_{3\alpha_1+\alpha_2}\mathsf{R}_{\alpha_1}\mathsf{R}_{\alpha_2}\mathsf{R}_{\alpha_1+\alpha_2}\mathsf{R}_{3\alpha_1+2\alpha_2}\mathsf{R}_{2\alpha_1+\alpha_2}$.

Proposition 4.2.5 can be verified by direct calculations. Here we give the list of the matrices of operators (1)–(12). First, the following are the matrices of operators Q_γ, $\gamma \in \Delta^+ = \{\alpha_1, 3\alpha_1+\alpha_2, 2\alpha_1+\alpha_2, 3\alpha_1+2\alpha_2, \alpha_1+\alpha_2, \alpha_2\}$ (cf. [18, Fig. 2 (C)]):

[1] Q_{α_1}:

$$\begin{pmatrix}
0 & Q_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & Q_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & Q_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}$$

[2] $\mathsf{Q}_{3\alpha_1+\alpha_2}$:

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & Q_1Q_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_1Q_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_1Q_2 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_1Q_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

[3] $\mathsf{Q}_{2\alpha_1+\alpha_2}$:

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

[4] $\mathsf{Q}_{3\alpha_1+2\alpha_2}$:

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_1Q_2^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_1Q_2^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

[5] $\mathsf{Q}_{\alpha_1+\alpha_2}$:

$$\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}$$

[6] Q_{α_2}:

$$\begin{pmatrix}
0&0&Q_2&0&0&0&0&0&0&0&0&0\\
0&0&0&Q_2&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&Q_2&0&0&0&0&0\\
0&0&0&0&0&0&0&Q_2&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&Q_2&0\\
0&0&0&0&0&0&0&0&0&0&0&Q_2\\
0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0
\end{pmatrix}$$

Hence the matrices of operators (1)–(12) can be calculated (by using, e.g., SageMath) as follows:

(1) $\mathsf{R}_{2\alpha_1+\alpha_2}\mathsf{R}_{3\alpha_1+2\alpha_2}\mathsf{R}_{\alpha_1+\alpha_2}\mathsf{R}_{\alpha_2}\mathsf{R}_{-\alpha_1}\mathsf{R}_{-3\alpha_1-\alpha_2}$:

$$\begin{pmatrix}
1&Q_1Q_2-Q_1&Q_2&0&-Q_1Q_2&-Q_1Q_2&-Q_1Q_2^2&-Q_1Q_2^2&0&0&Q_1Q_2^2&Q_1^2Q_2^3-Q_1^2Q_2^2\\
-1&1&0&Q_2&0&0&0&0&0&0&-Q_1Q_2^2&Q_1Q_2^2\\
1&0&1&0&-Q_1&-Q_1Q_2&0&0&-Q_1Q_2&0&0&0\\
0&1&1&1&-Q_1&-Q_1&0&0&-Q_1Q_2&-Q_1Q_2&0&0\\
0&-1&-1&0&1&0&Q_2&0&0&0&0&-Q_1Q_2^2\\
0&-1&-1&-1&1&1&Q_2&Q_2&0&0&0&-Q_1Q_2\\
0&0&0&1&-Q_1+1&-Q_1&1&0&-Q_1&-Q_1Q_2&0&-Q_1Q_2\\
0&0&0&0&-Q_1+1&-Q_1+1&1&1&-Q_1&-Q_1&0&-Q_1Q_2\\
0&0&0&-1&0&1&-1&0&1&0&Q_2&0\\
0&0&0&0&0&0&-1&-1&1&1&0&Q_2\\
0&0&0&0&0&0&-1&-1&1&0&1&Q_1Q_2-Q_1\\
0&0&0&0&0&0&0&0&0&1&-1&1
\end{pmatrix}$$

(2) $\mathsf{R}_{3\alpha_1+2\alpha_2}\mathsf{R}_{\alpha_1+\alpha_2}\mathsf{R}_{\alpha_2}\mathsf{R}_{-\alpha_1}\mathsf{R}_{-3\alpha_1-\alpha_2}\mathsf{R}_{-2\alpha_1-\alpha_2}$:

$$\begin{pmatrix}
1 & Q_1Q_2-Q_1 & Q_2 & 0 & 0 & -Q_1Q_2 & 0 & -Q_1Q_2^2 & 0 & 0 & Q_1Q_2^2 & Q_1^2Q_2^3-Q_1^2Q_2^2 \\
-1 & 1 & 0 & Q_2 & 0 & 0 & 0 & 0 & 0 & 0 & -Q_1Q_2^2 & Q_1Q_2^2 \\
1 & 0 & 1 & 0 & Q_1Q_2-Q_1 & -Q_1Q_2 & 0 & 0 & -Q_1Q_2 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & -Q_1 & 0 & 0 & -Q_1Q_2 & -Q_1Q_2 & 0 & 0 \\
0 & -1 & -1 & 0 & 1 & 0 & Q_2 & 0 & 0 & 0 & 0 & -Q_1Q_2^2 \\
0 & 0 & 0 & -1 & -1 & 1 & -Q_2 & Q_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & -Q_1 & 1 & 0 & -Q_1 & -Q_1Q_2 & 0 & -Q_1Q_2 \\
0 & 0 & 0 & -1 & -1 & 1 & -1 & 1 & 0 & Q_1Q_2-Q_1 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & 1 & -1 & 0 & 1 & 0 & Q_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & Q_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & Q_1Q_2-Q_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1
\end{pmatrix}$$

(3) $\mathsf{R}_{\alpha_1+\alpha_2}\mathsf{R}_{\alpha_2}\mathsf{R}_{-\alpha_1}\mathsf{R}_{-3\alpha_1-\alpha_2}\mathsf{R}_{-2\alpha_1-\alpha_2}\mathsf{R}_{-3\alpha_1-2\alpha_2}$:

$$\begin{pmatrix}
1 & Q_1Q_2-Q_1 & Q_2 & 0 & 0 & Q_1Q_2^2-Q_1Q_2 & 0 & 0 & -Q_1Q_2^2 & 0 & -Q_1Q_2^2 & -Q_1^2Q_2^3+Q_1^2Q_2^2 \\
-1 & 1 & 0 & Q_2 & 0 & 0 & 0 & 0 & 0 & -Q_1Q_2^2 & Q_1Q_2^2 & -Q_1Q_2^2 \\
1 & 0 & 1 & 0 & Q_1Q_2-Q_1 & 0 & 0 & 0 & -Q_1Q_2 & 0 & -Q_1Q_2^2 & 0 \\
0 & 1 & 1 & 1 & 0 & Q_1Q_2-Q_1 & 0 & 0 & -Q_1Q_2 & -Q_1Q_2 & 0 & -Q_1Q_2^2 \\
0 & -1 & -1 & -Q_2 & 1 & 0 & Q_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & Q_2-1 & -1 & 1 & -Q_2 & Q_2 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & -1 & 1 & -Q_1Q_2+Q_1 & 1 & 0 & Q_1Q_2-Q_1 & 0 & 0 & Q_1Q_2^2-Q_1Q_2 \\
0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 & 0 & Q_1Q_2-Q_1 & 0 & 0 \\
0 & 0 & 0 & -Q_2+1 & 0 & -1 & Q_2-1 & -Q_2 & 1 & 0 & Q_2 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 1 & 1 & 0 & Q_2 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 1 & Q_1Q_2-Q_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1
\end{pmatrix}$$

(4) $\mathsf{R}_{3\alpha_1+2\alpha_2}\mathsf{R}_{2\alpha_1+\alpha_2}\mathsf{R}_{3\alpha_1+\alpha_2}\mathsf{R}_{\alpha_1}\mathsf{R}_{-\alpha_2}\mathsf{R}_{-\alpha_1-\alpha_2}$:

$$\begin{pmatrix}
1 & -Q_1Q_2+Q_1 & -Q_2 & 0 & 0 & -Q_1Q_2^2+Q_1Q_2 & 0 & 0 & -Q_1Q_2^2 & 0 & Q_1Q_2^2 & -Q_1^2Q_2^3+Q_1^2Q_2^2 \\
1 & 1 & 0 & -Q_2 & 0 & 0 & 0 & 0 & 0 & -Q_1Q_2^2 & Q_1Q_2^2 & Q_1Q_2^2 \\
-1 & 0 & 1 & 0 & -Q_1Q_2+Q_1 & 0 & 0 & 0 & Q_1Q_2 & 0 & -Q_1Q_2^2 & 0 \\
0 & -1 & -1 & 1 & 0 & -Q_1Q_2+Q_1 & 0 & 0 & -Q_1Q_2 & Q_1Q_2 & 0 & -Q_1Q_2^2 \\
0 & 1 & 1 & -Q_2 & 1 & 0 & -Q_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -Q_2+1 & 1 & 1 & -Q_2 & -Q_2 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & 1 & -1 & -Q_1Q_2+Q_1 & 1 & 0 & -Q_1Q_2+Q_1 & 0 & 0 & -Q_1Q_2^2+Q_1Q_2 \\
0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & 0 & -Q_1Q_2+Q_1 & 0 & 0 \\
0 & 0 & 0 & -Q_2+1 & 0 & 1 & -Q_2+1 & -Q_2 & 1 & 0 & -Q_2 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 & -Q_2 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 1 & -Q_1Q_2+Q_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1
\end{pmatrix}$$

(5) $\mathsf{R}_{2\alpha_1+\alpha_2}\mathsf{R}_{3\alpha_1+\alpha_2}\mathsf{R}_{\alpha_1}\mathsf{R}_{-\alpha_2}\mathsf{R}_{-\alpha_1-\alpha_2}\mathsf{R}_{-3\alpha_1-2\alpha_2}$:

$$\begin{pmatrix}
1 & -Q_1Q_2+Q_1 & -Q_2 & 0 & 0 & Q_1Q_2 & 0 & -Q_1Q_2^2 & 0 & 0 & -Q_1Q_2^2 & Q_1^2Q_2^3-Q_1^2Q_2^2 \\
1 & 1 & 0 & -Q_2 & 0 & 0 & 0 & 0 & 0 & 0 & -Q_1Q_2^2 & -Q_1Q_2^2 \\
-1 & 0 & 1 & 0 & -Q_1Q_2+Q_1 & -Q_1Q_2 & 0 & 0 & Q_1Q_2 & 0 & 0 & 0 \\
0 & -1 & -1 & 1 & 0 & Q_1 & 0 & 0 & -Q_1Q_2 & Q_1Q_2 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & -Q_2 & 0 & 0 & 0 & 0 & -Q_1Q_2^2 \\
0 & 0 & 0 & 1 & 1 & 1 & -Q_2 & -Q_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & -Q_1 & 1 & 0 & Q_1 & -Q_1Q_2 & 0 & Q_1Q_2 \\
0 & 0 & 0 & -1 & -1 & -1 & 1 & 1 & 0 & -Q_1Q_2+Q_1 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & -1 & 1 & 0 & 1 & 0 & -Q_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & -Q_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & -Q_1Q_2+Q_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1
\end{pmatrix}$$

(6) $\mathsf{R}_{3\alpha_1+\alpha_2}\mathsf{R}_{\alpha_1}\mathsf{R}_{-\alpha_2}\mathsf{R}_{-\alpha_1-\alpha_2}\mathsf{R}_{-3\alpha_1-2\alpha_2}\mathsf{R}_{-2\alpha_1-\alpha_2}$:

$$\begin{pmatrix}
1 & -Q_1Q_2+Q_1 & -Q_2 & 0 & -Q_1Q_2 & Q_1Q_2 & Q_1Q_2^2 & -Q_1Q_2^2 & 0 & 0 & -Q_1Q_2^2 & Q_1^2Q_2^3-Q_1^2Q_2^2 \\
1 & 1 & 0 & -Q_2 & 0 & 0 & 0 & 0 & 0 & 0 & -Q_1Q_2^2 & -Q_1Q_2^2 \\
-1 & 0 & 1 & 0 & Q_1 & -Q_1Q_2 & 0 & 0 & Q_1Q_2 & 0 & 0 & 0 \\
0 & -1 & -1 & 1 & -Q_1 & Q_1 & 0 & 0 & -Q_1Q_2 & Q_1Q_2 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & -Q_2 & 0 & 0 & 0 & 0 & -Q_1Q_2^2 \\
0 & -1 & -1 & 1 & -1 & 1 & Q_2 & -Q_2 & 0 & 0 & 0 & Q_1Q_2^2 \\
0 & 0 & 0 & -1 & Q_1-1 & -Q_1 & 1 & 0 & Q_1 & -Q_1Q_2 & 0 & Q_1Q_2 \\
0 & 0 & 0 & 0 & -Q_1+1 & Q_1-1 & -1 & 1 & -Q_1 & Q_1 & 0 & -Q_1Q_2 \\
0 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 1 & 0 & -Q_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 & 0 & -Q_2 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 1 & -Q_1Q_2+Q_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1
\end{pmatrix}$$

(7) $\mathsf{R}_{2\alpha_1+\alpha_2}\mathsf{R}_{3\alpha_1+2\alpha_2}\mathsf{R}_{\alpha_1+\alpha_2}\mathsf{R}_{\alpha_2}\mathsf{R}_{\alpha_1}\mathsf{R}_{3\alpha_1+\alpha_2}$:

$$\begin{pmatrix}
1 & Q_1Q_2+Q_1 & Q_2 & 0 & Q_1Q_2 & Q_1Q_2 & Q_1Q_2^2 & Q_1Q_2^2 & 2Q_1Q_2^2 & 0 & Q_1Q_2^2 & Q_1^2Q_2^3+Q_1^2Q_2^2 \\
1 & 1 & 0 & Q_2 & 0 & 2Q_1Q_2 & 0 & 2Q_1Q_2^2 & 0 & 2Q_1Q_2^2 & Q_1Q_2^2 & Q_1Q_2^2 \\
1 & 2Q_1 & 1 & 0 & Q_1 & Q_1Q_2 & 0 & 0 & Q_1Q_2 & 0 & 0 & 0 \\
2 & 2Q_1+1 & 1 & 1 & Q_1 & 2Q_1Q_2+Q_1 & 0 & 0 & Q_1Q_2 & Q_1Q_2 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & Q_2 & 0 & 2Q_1Q_2 & 0 & 0 & Q_1Q_2^2 \\
0 & 1 & 1 & 1 & 1 & 1 & Q_2 & Q_2 & 2Q_1Q_2 & 2Q_1Q_2 & 0 & Q_1Q_2^2 \\
2 & 2Q_1+2 & 2 & 1 & Q_1+1 & 2Q_1Q_2+Q_1 & 1 & 0 & 2Q_1Q_2+Q_1 & Q_1Q_2 & 0 & Q_1Q_2 \\
2 & 2Q_1+2 & 2 & 2 & Q_1+1 & 2Q_1Q_2+Q_1+1 & 1 & 1 & 2Q_1Q_2+Q_1 & 2Q_1Q_2+Q_1 & 0 & Q_1Q_2 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 2Q_2 & 1 & 2Q_1Q_2 & Q_2 & 2Q_1Q_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 2Q_2+1 & 1 & 1 & 2Q_2 & 2Q_1Q_2+Q_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & Q_1Q_2+Q_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 1 & 1
\end{pmatrix}$$

(8) $\mathsf{R}_{3\alpha_1+2\alpha_2}\mathsf{R}_{\alpha_1+\alpha_2}\mathsf{R}_{\alpha_2}\mathsf{R}_{\alpha_1}\mathsf{R}_{3\alpha_1+\alpha_2}\mathsf{R}_{2\alpha_1+\alpha_2}$:

$$\begin{pmatrix}
1 & Q_1Q_2+Q_1 & Q_2 & 0 & 2Q_1Q_2 & Q_1Q_2 & 2Q_1Q_2^2 & Q_1Q_2^2 & 2Q_1Q_2^2 & 0 & Q_1Q_2^2 & Q_1^2Q_2^3+Q_1^2Q_2^2 \\
1 & 1 & 0 & Q_2 & 2Q_1Q_2 & 2Q_1Q_2 & 2Q_1Q_2^2 & 2Q_1Q_2^2 & 0 & 2Q_1Q_2^2 & Q_1Q_2^2 & Q_1Q_2^2 \\
1 & 2Q_1 & 1 & 0 & Q_1Q_2+Q_1 & Q_1Q_2 & 0 & 0 & Q_1Q_2 & 0 & 0 & 0 \\
2 & 2Q_1+1 & 1 & 1 & 2Q_1Q_2+2Q_1 & 2Q_1Q_2+Q_1 & 0 & 0 & Q_1Q_2 & Q_1Q_2 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & Q_2 & 0 & 2Q_1Q_2 & 0 & 0 & Q_1Q_2^2 \\
0 & 0 & 0 & 1 & 1 & 1 & Q_2 & Q_2 & 0 & 2Q_1Q_2 & 0 & 0 \\
2 & 2Q_1+2 & 2 & 1 & 2Q_1Q_2+2Q_1+1 & 2Q_1Q_2+Q_1 & 1 & 0 & 2Q_1Q_2+Q_1 & Q_1Q_2 & 0 & Q_1Q_2 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & Q_1Q_2+Q_1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 2Q_2+1 & 2Q_2 & 1 & 2Q_1Q_2 & Q_2 & 2Q_1Q_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 2Q_2+2 & 2Q_2+1 & 1 & 1 & 2Q_2 & 2Q_1Q_2+Q_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 1 & 0 & 1 & Q_1Q_2+Q_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 1 & 1 & 1
\end{pmatrix}$$

(9) $\mathsf{R}_{\alpha_1+\alpha_2}\mathsf{R}_{\alpha_2}\mathsf{R}_{\alpha_1}\mathsf{R}_{3\alpha_1+\alpha_2}\mathsf{R}_{2\alpha_1+\alpha_2}\mathsf{R}_{3\alpha_1+2\alpha_2}$:

$$\begin{pmatrix}
1 & Q_1Q_2+Q_1 & Q_2 & 0 & 2Q_1Q_2 & Q_1Q_2^2+Q_1Q_2 & 0 & 0 & Q_1Q_2^2 & 0 & Q_1Q_2^2 & Q_1^2Q_2^3+Q_1^2Q_2^2 \\
1 & 1 & 0 & Q_2 & 2Q_1Q_2 & 2Q_1Q_2 & 0 & 0 & 0 & Q_1Q_2^2 & Q_1Q_2^2 & Q_1Q_2^2 \\
1 & 2Q_1 & 1 & 0 & Q_1Q_2+Q_1 & 2Q_1Q_2 & 0 & 0 & Q_1Q_2 & 0 & Q_1Q_2^2 & 2Q_1^2Q_2^2 \\
2 & 2Q_1+1 & 1 & 1 & 2Q_1Q_2+2Q_1 & 3Q_1Q_2+Q_1 & 0 & 0 & Q_1Q_2 & Q_1Q_2 & 2Q_1Q_2^2 & 2Q_1^2Q_2^2+Q_1Q_2^2 \\
0 & 1 & 1 & Q_2 & 1 & 2Q_1Q_2 & Q_2 & 0 & 2Q_1Q_2 & 0 & 0 & 2Q_1Q_2^2 \\
0 & 0 & 0 & Q_2+1 & 1 & 1 & Q_2 & Q_2 & 0 & 2Q_1Q_2 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & Q_1Q_2+Q_1 & 1 & 0 & Q_1Q_2+Q_1 & 0 & 0 & Q_1Q_2^2+Q_1Q_2 \\
0 & 0 & 0 & 2 & 1 & 1 & 1 & 1 & 0 & Q_1Q_2+Q_1 & 0 & 0 \\
0 & 0 & 0 & Q_2+1 & 0 & 1 & Q_2+1 & Q_2 & 1 & 0 & Q_2 & 2Q_1Q_2 \\
0 & 0 & 0 & 2Q_2+2 & 0 & 1 & 2Q_2+2 & 2Q_2+1 & 1 & 1 & 2Q_2 & 2Q_1Q_2+Q_2 \\
0 & 0 & 0 & 2 & 0 & 1 & 2 & 1 & 1 & 0 & 1 & Q_1Q_2+Q_1 \\
0 & 0 & 0 & 2 & 0 & 0 & 2 & 2 & 0 & 1 & 1 & 1
\end{pmatrix}$$

(10) $\mathsf{R}_{3\alpha_1+2\alpha_2}\mathsf{R}_{2\alpha_1+\alpha_2}\mathsf{R}_{3\alpha_1+\alpha_2}\mathsf{R}_{\alpha_1}\mathsf{R}_{\alpha_2}\mathsf{R}_{\alpha_1+\alpha_2}$:

$$\begin{pmatrix}
1 & Q_1Q_2+Q_1 & 2Q_1Q_2+Q_2 & 2Q_1Q_2 & 0 & Q_1Q_2^2+Q_1Q_2 & 0 & 2Q_1Q_2^2 & 2Q_1^2Q_2^2+Q_1Q_2^2 & 2Q_1^2Q_2^2 & Q_1Q_2^2 & Q_1^2Q_2^3+Q_1^2Q_2^2 \\
1 & 1 & 2Q_2 & Q_2 & 0 & 0 & 0 & 0 & 2Q_1Q_2^2 & Q_1Q_2^2 & Q_1Q_2^2 & Q_1Q_2^2 \\
1 & 0 & 1 & 0 & Q_1Q_2+Q_1 & 0 & 2Q_1Q_2 & 0 & Q_1Q_2 & 0 & Q_1Q_2^2 & 0 \\
0 & 1 & 1 & 1 & 0 & Q_1Q_2+Q_1 & 0 & 2Q_1Q_2 & Q_1Q_2 & Q_1Q_2 & 0 & Q_1Q_2^2 \\
2 & 1 & 2Q_2+1 & Q_2 & 1 & 0 & Q_2 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 2Q_2+2 & Q_2+1 & 1 & 1 & Q_2 & Q_2 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & Q_1Q_2+Q_1 & 1 & 2Q_1Q_2 & 3Q_1Q_2+Q_1 & 2Q_1Q_2 & 2Q_1Q_2 & Q_1Q_2^2+Q_1Q_2 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2Q_1Q_2+2Q_1 & Q_1Q_2+Q_1 & 2Q_1Q_2 & 2Q_1Q_2 \\
2 & 2 & 2Q_2+2 & Q_2+1 & 2 & 1 & Q_2+1 & Q_2 & 1 & 0 & Q_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & Q_2 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 2Q_1+1 & 2Q_1 & 1 & Q_1Q_2+Q_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 1 & 1
\end{pmatrix}$$

(11) $\mathsf{R}_{2\alpha_1+\alpha_2}\mathsf{R}_{3\alpha_1+\alpha_2}\mathsf{R}_{\alpha_1}\mathsf{R}_{\alpha_2}\mathsf{R}_{\alpha_1+\alpha_2}\mathsf{R}_{3\alpha_1+2\alpha_2}$:

$$\begin{pmatrix}
1 & Q_1Q_2+Q_1 & 2Q_1Q_2+Q_2 & 2Q_1Q_2 & 0 & Q_1Q_2 & 0 & Q_1Q_2^2 & 0 & 0 & Q_1Q_2^2 & Q_1^2Q_2^3+Q_1^2Q_2^2 \\
1 & 1 & 2Q_2 & Q_2 & 0 & 0 & 0 & 0 & 0 & 0 & Q_1Q_2^2 & Q_1Q_2^2 \\
1 & 0 & 1 & 2Q_1Q_2 & Q_1Q_2+Q_1 & Q_1Q_2 & 2Q_1Q_2 & 0 & Q_1Q_2 & 0 & 2Q_1Q_2^2 & 0 \\
0 & 1 & 1 & 1 & 0 & 2Q_1Q_2+Q_1 & 0 & 2Q_1Q_2 & Q_1Q_2 & Q_1Q_2 & 0 & 2Q_1Q_2^2 \\
2 & 1 & 2Q_2+1 & 2Q_2 & 1 & 0 & Q_2 & 0 & 0 & 0 & 2Q_1Q_2^2 & Q_1Q_2^2 \\
2 & 2 & 2Q_2+2 & 2Q_2+1 & 1 & 1 & Q_2 & Q_2 & 0 & 0 & 2Q_1Q_2^2 & 2Q_1Q_2^2 \\
0 & 0 & 0 & 1 & 1 & 2Q_1Q_2+Q_1 & 1 & 0 & 2Q_1Q_2+Q_1 & Q_1Q_2 & 2Q_1Q_2 & Q_1Q_2 \\
0 & 0 & 0 & 1 & 1 & 2Q_1Q_2+2Q_1+1 & 1 & 1 & 2Q_1Q_2+2Q_1 & Q_1Q_2+Q_1 & 2Q_1Q_2 & 2Q_1Q_2 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & Q_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 1 & 1 & 0 & Q_2 \\
0 & 0 & 0 & 0 & 0 & 2Q_1+2 & 0 & 1 & 2Q_1+1 & 2Q_1 & 1 & Q_1Q_2+Q_1 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 1 & 1 & 1
\end{pmatrix}$$

(12) $\mathsf{R}_{3\alpha_1+\alpha_2}\mathsf{R}_{\alpha_1}\mathsf{R}_{\alpha_2}\mathsf{R}_{\alpha_1+\alpha_2}\mathsf{R}_{3\alpha_1+2\alpha_2}\mathsf{R}_{2\alpha_1+\alpha_2}$:

$$\begin{pmatrix}
1 & Q_1Q_2+Q_1 & 2Q_1Q_2+Q_2 & 2Q_1Q_2 & Q_1Q_2 & Q_1Q_2 & Q_1Q_2^2 & Q_1Q_2^2 & 0 & 0 & Q_1Q_2^2 & Q_1^2Q_2^3+Q_1^2Q_2^2 \\
1 & 1 & 2Q_2 & Q_2 & 0 & 0 & 0 & 0 & 0 & 0 & Q_1Q_2^2 & Q_1Q_2^2 \\
1 & 0 & 1 & 2Q_1Q_2 & 2Q_1Q_2+Q_1 & Q_1Q_2 & 2Q_1Q_2 & 0 & Q_1Q_2 & 0 & 2Q_1Q_2^2 & 0 \\
0 & 1 & 1 & 1 & 2Q_1Q_2+Q_1 & 2Q_1Q_2+Q_1 & 2Q_1Q_2 & 2Q_1Q_2 & Q_1Q_2 & Q_1Q_2 & 0 & 2Q_1Q_2^2 \\
2 & 1 & 2Q_2+1 & 2Q_2 & 1 & 0 & Q_2 & 0 & 0 & 0 & 2Q_1Q_2^2 & Q_1Q_2^2 \\
0 & 1 & 1 & 1 & 1 & 1 & Q_2 & Q_2 & 0 & 0 & 0 & Q_1Q_2^2 \\
0 & 0 & 0 & 1 & 2Q_1Q_2+Q_1+1 & 2Q_1Q_2+Q_1 & 1 & 0 & 2Q_1Q_2+Q_1 & Q_1Q_2 & 2Q_1Q_2 & Q_1Q_2 \\
0 & 0 & 0 & 0 & Q_1+1 & Q_1+1 & 1 & 1 & Q_1 & Q_1 & 0 & Q_1Q_2 \\
0 & 0 & 0 & 1 & 2 & 1 & 1 & 0 & 1 & 0 & Q_2 & 0 \\
0 & 0 & 0 & 0 & 2 & 2 & 1 & 1 & 1 & 1 & 0 & Q_2 \\
0 & 0 & 0 & 0 & 2Q_1+2 & 2Q_1+2 & 1 & 1 & 2Q_1+1 & 2Q_1 & 1 & Q_1Q_2+Q_1 \\
0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 1 & 1 & 1
\end{pmatrix}$$

Remark 4.2.6. From direct calculations of matrices of the operators above, we see that only in the following two cases, $m_j = 3$ in Proposition 4.2.5 (1), or $n_{\xi_j} = 3$ in Proposition 4.2.5 (3):

(1) The $(4, 6)$-entry (i.e., the $(s_1s_2, s_1s_2s_1)$-entry) of the matrix of the operator (9);

(2) The $(7, 9)$-entry (i.e., the $(s_2s_1s_2, s_2s_1s_2s_1)$-entry) of the matrix of the operator (10).

Let $v \in W$, and let $\mathbf{p}$ be a Π-compatible directed path in $\mathrm{QBG}(W)$ which starts at v, i.e., $\mathbf{p} \in \mathcal{P}(v, \Pi)$. First, we consider the cases except the following:

(E1) $\Pi = (\pm(3\alpha_1 + 2\alpha_2), \pm(2\alpha_1 + \alpha_2), \pm(3\alpha_1 + \alpha_2), \pm\alpha_1, \mp\alpha_2, \mp(\alpha_1 + \alpha_2))$, $v = s_1s_2s_1$, $\mathrm{end}(\mathbf{p}) = s_1s_2$, and $\mathrm{wt}(\mathbf{p}) = \alpha_1^\vee + \alpha_2^\vee$;

(E2) $\Pi = (\pm(\alpha_1 + \alpha_2), \pm\alpha_2, \mp\alpha_1, \mp(3\alpha_1 + \alpha_2), \mp(2\alpha_1 + \alpha_2), \mp(3\alpha_1 + 2\alpha_2))$, $v = s_1s_2s_1$, $\mathrm{end}(\mathbf{p}) = s_1s_2$, and $\mathrm{wt}(\mathbf{p}) = \alpha_1^\vee + \alpha_2^\vee$;

(E3) $\Pi = (\pm(3\alpha_1 + 2\alpha_2), \pm(2\alpha_1 + \alpha_2), \pm(3\alpha_1 + \alpha_2), \pm\alpha_1, \mp\alpha_2, \mp(\alpha_1 + \alpha_2))$, $v = s_2s_1s_2s_1$, $\mathrm{end}(\mathbf{p}) = s_2s_1s_2$, and $\mathrm{wt}(\mathbf{p}) = \alpha_1^\vee + \alpha_2^\vee$;

(E4) $\Pi = (\pm(\alpha_1 + \alpha_2), \pm\alpha_2, \mp\alpha_1, \mp(3\alpha_1 + \alpha_2), \mp(2\alpha_1 + \alpha_2), \mp(3\alpha_1 + 2\alpha_2))$, $v = s_2s_1s_2s_1$, $\mathrm{end}(\mathbf{p}) = s_2s_1s_2$, and $\mathrm{wt}(\mathbf{p}) = \alpha_1^\vee + \alpha_2^\vee$.

Then, we have the following:

Proposition 4.2.7. *Only one of the following occurs.*

(1) *There exists a unique* $\mathbf{p}' \in \mathcal{P}(v, \Pi)\setminus\{\mathbf{p}\}$ *such that* $\mathrm{end}(\mathbf{p}') = \mathrm{end}(\mathbf{p})$ *and* $\mathrm{wt}(\mathbf{p}') = \mathrm{wt}(\mathbf{p})$. *This* $\mathbf{p}'$ *satisfies* $(-1)^{\mathrm{neg}(\mathbf{p}')} = -(-1)^{\mathrm{neg}(\mathbf{p})}$. *Moreover, there does not exist a path* $\mathbf{q} \in \mathcal{P}(v, \Pi')$ *such that* $\mathrm{end}(\mathbf{q}) = \mathrm{end}(\mathbf{p})$ *and* $\mathrm{wt}(\mathbf{q}) = \mathrm{wt}(\mathbf{p})$.

(2) *There exists a unique* $\mathbf{p}' \in \mathcal{P}(v, \Pi')$ *such that* $\mathrm{end}(\mathbf{p}') = \mathrm{end}(\mathbf{p})$ *and* $\mathrm{wt}(\mathbf{p}') = \mathrm{wt}(\mathbf{p})$. *This* $\mathbf{p}'$ *satisfies* $(-1)^{\mathrm{neg}(\mathbf{p}')} = (-1)^{\mathrm{neg}(\mathbf{p})}$. *Moreover, there does not exist a path* $\mathbf{q} \in \mathcal{P}(v, \Pi) \setminus \{\mathbf{p}\}$ *such that* $\mathrm{end}(\mathbf{q}) = \mathrm{end}(\mathbf{p})$ *and* $\mathrm{wt}(\mathbf{q}) = \mathrm{wt}(\mathbf{p})$.

(3) *There exists a unique* $\mathbf{p}' \in \mathcal{P}(v, \Pi)\setminus\{\mathbf{p}\}$ *such that* $\mathrm{end}(\mathbf{p}') = \mathrm{end}(\mathbf{p})$ *and* $\mathrm{wt}(\mathbf{p}') = \mathrm{wt}(\mathbf{p})$. *This* $\mathbf{p}'$ *satisfies* $(-1)^{\mathrm{neg}(\mathbf{p}')} = -(-1)^{\mathrm{neg}(\mathbf{p})}$. *Moreover, there exist exactly two paths* $\mathbf{q}_1, \mathbf{q}_2 \in \mathcal{P}(v, \Pi')$ *such that* $\mathrm{end}(\mathbf{q}_1) = \mathrm{end}(\mathbf{q}_2) = \mathrm{end}(\mathbf{p})$ *and* $\mathrm{wt}(\mathbf{q}_1) = \mathrm{wt}(\mathbf{q}_2) = \mathrm{wt}(\mathbf{p})$. *These* $\mathbf{q}_1$, $\mathbf{q}_2$ *satisfy* $(-1)^{\mathrm{neg}(\mathbf{q}_2)} = -(-1)^{\mathrm{neg}(\mathbf{q}_1)}$.

Next, we consider the exceptional cases (E1)–(E4) above. Then, we have the following:

Proposition 4.2.8. (1) *In case (E1) or (E4), there exist exactly two paths* $\mathbf{p}_1, \mathbf{p}_2 \in \mathcal{P}(v, \Pi) \setminus \{\mathbf{p}\}$ *and a unique path* $\mathbf{q} \in \mathcal{P}(v, \Pi')$ *such that* $\mathrm{end}(\mathbf{p}_1) = \mathrm{end}(\mathbf{p}_2) = \mathrm{end}(\mathbf{q}) = \mathrm{end}(\mathbf{p})$ *and* $\mathrm{wt}(\mathbf{p}_1) = \mathrm{wt}(\mathbf{p}_2) = \mathrm{wt}(\mathbf{q}) = \mathrm{wt}(\mathbf{p})$. *Moreover, there exist two paths* $\mathbf{r}_1, \mathbf{r}_2 \in \{\mathbf{p}, \mathbf{p}_1, \mathbf{p}_2\}$ *such that for* $\mathbf{r}_3 \in \{\mathbf{p}, \mathbf{p}_1, \mathbf{p}_2\} \setminus \{\mathbf{r}_1, \mathbf{r}_2\}$, $(-1)^{\mathrm{neg}(\mathbf{r}_1)} = (-1)^{\mathrm{neg}(\mathbf{r}_2)} = -(-1)^{\mathrm{neg}(\mathbf{r}_3)}$ *and* $(-1)^{\mathrm{neg}(\mathbf{r}_1)} = (-1)^{\mathrm{neg}(\mathbf{q})}$.

(2) *In case (E2) or (E3), there exist exactly three paths* $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3 \in \mathcal{P}(v, \Pi')$ *such that* $\mathrm{end}(\mathbf{q}_1) = \mathrm{end}(\mathbf{q}_2) = \mathrm{end}(\mathbf{q}_3) = \mathrm{end}(\mathbf{p})$ *and* $\mathrm{wt}(\mathbf{q}_1) =$

$\mathrm{wt}(\mathbf{q}_2) = \mathrm{wt}(\mathbf{q}_3) = \mathrm{wt}(\mathbf{p})$. *In this case, there does not exist a path* $\mathbf{r} \in \mathcal{P}(v, \Pi) \setminus \{\mathbf{p}\}$ *such that* $\mathrm{end}(\mathbf{r}) = \mathrm{end}(\mathbf{p})$ *and* $\mathrm{wt}(\mathbf{r}) = \mathrm{wt}(\mathbf{p})$. *Moreover, there exist two paths* $\mathbf{r}_1, \mathbf{r}_2 \in \{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ *such that for* $\mathbf{r}_3 \in \{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\} \setminus \{\mathbf{r}_1, \mathbf{r}_2\}$, $(-1)^{\mathrm{neg}(\mathbf{r}_1)} = (-1)^{\mathrm{neg}(\mathbf{r}_2)} = -(-1)^{\mathrm{neg}(\mathbf{r}_3)}$ *and* $(-1)^{\mathrm{neg}(\mathbf{r}_1)} = (-1)^{\mathrm{neg}(\mathbf{p})}$.

By using Proposition 4.2.5 and Remark 4.2.6, we can prove Propositions 4.2.7 and 4.2.8 by the same argument as in types $A_1 \times A_1$, A_2, and C_2.

Remark 4.2.9. We have explicit descriptions of $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{q}$ in Proposition 4.2.8 (1), and those of $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$ in Proposition 4.2.8 (2).

(1) Case (E1). In this case, we have

$$\Pi = (\pm(3\alpha_1 + 2\alpha_2), \pm(2\alpha_1 + \alpha_2), \pm(3\alpha_1 + \alpha_2), \pm\alpha_1, \mp\alpha_2, \mp(\alpha_1 + \alpha_2))$$

and

$$\Pi' = (\mp(\alpha_1 + \alpha_2), \mp\alpha_2, \pm\alpha_1, \pm(3\alpha_1 + \alpha_2), \pm(2\alpha_1 + \alpha_2), \pm(3\alpha_1 + 2\alpha_2)).$$

If we set

$$\begin{aligned}
\mathbf{p}_1^{(\mathrm{E1})} &: s_1s_2s_1 \xrightarrow{3\alpha_1+2\alpha_2} s_2s_1s_2s_1 \xrightarrow{3\alpha_1+\alpha_2} s_2 \xrightarrow{\alpha_1+\alpha_2} s_1s_2, \\
\mathbf{p}_2^{(\mathrm{E1})} &: s_1s_2s_1 \xrightarrow{3\alpha_1+\alpha_2} e \xrightarrow{\alpha_1} s_1 \xrightarrow{\alpha_2} s_1s_2, \\
\mathbf{p}_3^{(\mathrm{E1})} &: s_1s_2s_1 \xrightarrow{3\alpha_1+\alpha_2} e \xrightarrow{\alpha_2} s_2 \xrightarrow{\alpha_1+\alpha_2} s_1s_2, \\
\mathbf{q}^{(\mathrm{E1})} &: s_1s_2s_1 \xrightarrow{\alpha_2} s_1s_2s_1s_2 \xrightarrow{\alpha_1} s_1s_2s_1s_2s_1 \xrightarrow{3\alpha_1+\alpha_2} s_1s_2,
\end{aligned}$$

then in Proposition 4.2.8 (1), we have $\{\mathbf{p}, \mathbf{p}_1, \mathbf{p}_2\} = \{\mathbf{p}_1^{(\mathrm{E1})}, \mathbf{p}_2^{(\mathrm{E1})}, \mathbf{p}_3^{(\mathrm{E1})}\}$ and $\mathbf{q} = \mathbf{q}^{(\mathrm{E1})}$.

(2) Case (E2). In this case, we have

$$\Pi = (\pm(\alpha_1 + \alpha_2), \pm\alpha_2, \mp\alpha_1, \mp(3\alpha_1 + \alpha_2), \mp(2\alpha_1 + \alpha_2), \mp(3\alpha_1 + 2\alpha_2))$$

and

$$\Pi' = (\mp(3\alpha_1 + 2\alpha_2), \mp(2\alpha_1 + \alpha_2), \mp(3\alpha_1 + \alpha_2), \mp\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2)).$$

If we set

$$\begin{aligned}
\mathbf{p}^{(\mathrm{E2})} &: s_1s_2s_1 \xrightarrow{\alpha_2} s_1s_2s_1s_2 \xrightarrow{\alpha_1} s_1s_2s_1s_2s_1 \xrightarrow{3\alpha_1+\alpha_2} s_1s_2, \\
\mathbf{q}_1^{(\mathrm{E2})} &: s_1s_2s_1 \xrightarrow{3\alpha_1+2\alpha_2} s_2s_1s_2s_1 \xrightarrow{3\alpha_1+\alpha_2} s_2 \xrightarrow{\alpha_1+\alpha_2} s_1s_2, \\
\mathbf{q}_2^{(\mathrm{E2})} &: s_1s_2s_1 \xrightarrow{3\alpha_1+\alpha_2} e \xrightarrow{\alpha_1} s_1 \xrightarrow{\alpha_2} s_1s_2, \\
\mathbf{q}_3^{(\mathrm{E2})} &: s_1s_2s_1 \xrightarrow{3\alpha_1+\alpha_2} e \xrightarrow{\alpha_2} s_2 \xrightarrow{\alpha_1+\alpha_2} s_1s_2,
\end{aligned}$$

then in Proposition 4.2.8 (2), we have $\mathbf{p} = \mathbf{p}^{(E2)}$ and $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\} = \{\mathbf{q}_1^{(E2)}, \mathbf{q}_2^{(E2)}, \mathbf{q}_3^{(E2)}\}$.

(3) Case (E3). In this case, we have

$$\Pi = (\pm(3\alpha_1 + 2\alpha_2), \pm(2\alpha_1 + \alpha_2), \pm(3\alpha_1 + \alpha_2), \pm\alpha_1, \mp\alpha_2, \mp(\alpha_1 + \alpha_2))$$

and

$$\Pi' = (\mp(\alpha_1 + \alpha_2), \mp\alpha_2, \pm\alpha_1, \pm(3\alpha_1 + \alpha_2), \pm(2\alpha_1 + \alpha_2), \pm(3\alpha_1 + 2\alpha_2)).$$

If we set

$$\begin{aligned}
\mathbf{p}^{(E3)} &: s_2s_1s_2s_1 \xrightarrow{3\alpha_1+\alpha_2} s_2 \xrightarrow{\alpha_1} s_2s_1 \xrightarrow{\alpha_2} s_2s_1s_2, \\
\mathbf{q}_1^{(E3)} &: s_2s_1s_2s_1 \xrightarrow{\alpha_1+\alpha_2} s_1s_2s_1s_2s_1 \xrightarrow{3\alpha_1+\alpha_2} s_1s_2 \xrightarrow{3\alpha_1+2\alpha_2} s_2s_1s_2 \\
\mathbf{q}_2^{(E3)} &: s_2s_1s_2s_1 \xrightarrow{\alpha_2} s_2s_1s_2s_1s_2 \xrightarrow{\alpha_1} w_\circ \xrightarrow{3\alpha_1+2\alpha_2} s_2s_1s_2 \\
\mathbf{q}_3^{(E3)} &: s_2s_1s_2s_1 \xrightarrow{\alpha_1+\alpha_2} s_1s_2s_1s_2s_1 \xrightarrow{\alpha_2} w_\circ \xrightarrow{3\alpha_1+\alpha_2} s_2s_1s_2,
\end{aligned}$$

then in Proposition 4.2.8 (2), we have $\mathbf{p} = \mathbf{p}^{(E3)}$, and $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\} = \{\mathbf{q}_1^{(E3)}, \mathbf{q}_2^{(E3)}, \mathbf{q}_3^{(E3)}\}$.

(4) Case (E4). In this case, we have

$$\Pi = (\pm(\alpha_1 + \alpha_2), \pm\alpha_2, \mp\alpha_1, \mp(3\alpha_1 + \alpha_2), \mp(2\alpha_1 + \alpha_2), \mp(3\alpha_1 + 2\alpha_2))$$

and

$$\Pi' = (\mp(3\alpha_1 + 2\alpha_2), \mp(2\alpha_1 + \alpha_2), \mp(3\alpha_1 + \alpha_2), \mp\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2)).$$

If we set

$$\begin{aligned}
\mathbf{p}_1^{(E4)} &: s_2s_1s_2s_1 \xrightarrow{\alpha_1+\alpha_2} s_1s_2s_1s_2s_1 \xrightarrow{3\alpha_1+\alpha_2} s_1s_2 \xrightarrow{3\alpha_1+2\alpha_2} s_2s_1s_2 \\
\mathbf{p}_2^{(E4)} &: s_2s_1s_2s_1 \xrightarrow{\alpha_2} s_2s_1s_2s_1s_2 \xrightarrow{\alpha_1} w_\circ \xrightarrow{3\alpha_1+2\alpha_2} s_2s_1s_2 \\
\mathbf{p}_3^{(E4)} &: s_2s_1s_2s_1 \xrightarrow{\alpha_1+\alpha_2} s_1s_2s_1s_2s_1 \xrightarrow{\alpha_2} w_\circ \xrightarrow{3\alpha_1+\alpha_2} s_2s_1s_2, \\
\mathbf{q}^{(E4)} &: s_2s_1s_2s_1 \xrightarrow{3\alpha_1+\alpha_2} s_2 \xrightarrow{\alpha_1} s_2s_1 \xrightarrow{\alpha_2} s_2s_1s_2,
\end{aligned}$$

then in Proposition 4.2.8 (1), we have $\{\mathbf{p}, \mathbf{p}_1, \mathbf{p}_2\} = \{\mathbf{p}_1^{(E4)}, \mathbf{p}_2^{(E4)}, \mathbf{p}_3^{(E4)}\}$ and $\mathbf{q} = \mathbf{q}^{(E4)}$.

4.2.6 Proof of Theorem 4.1.2

Based on Propositions 4.2.3, 4.2.7, and 4.2.8, we can prove the existence of a generalization of quantum Yang-Baxter moves. In the same way as in (4.1.3), we divide $\mathbf{p}(A)$ for $A \in \mathcal{A}(w, \Gamma_1)$ into three parts $\mathbf{p}(A)^{(1)}$, $\mathbf{p}(A)^{(2)}$, $\mathbf{p}(A)^{(3)}$. If we write $A = \{a_1, \ldots, a_l\}$, then $\mathbf{p}(A)$ is of the form:

$$\mathbf{p}(A) : w = w_0 \xrightarrow{|\beta_{a_1}|} \cdots \xrightarrow{|\beta_{a_l}|} w_l,$$

with $a_1 < \cdots < a_l$; we set $a_0 := 0$. Let $0 \leq i_1 \leq l$ be maximal such that $a_{i_1} \leq t$, and $0 \leq i_2 \leq l$ maximal such that $a_{i_2} \leq t + q$. Then, we set

$$\begin{aligned}
\mathbf{p}(A)^{(1)} &: w = w_0 \xrightarrow{|\beta_{a_1}|} \cdots \xrightarrow{|\beta_{a_{i_1}}|} w_{a_{i_1}}, \\
\mathbf{p}(A)^{(2)} &: w_{a_{i_1}} \xrightarrow{|\beta_{a_{i_1+1}}|} \cdots \xrightarrow{|\beta_{a_{i_2}}|} w_{a_{i_2}}, \\
\mathbf{p}(A)^{(3)} &: w_{a_{i_2}} \xrightarrow{|\beta_{a_{i_2+1}}|} \cdots \xrightarrow{|\beta_{a_l}|} w_{a_l}.
\end{aligned}$$

Note that the concatenation of $\mathbf{p}(A)^{(1)}$, $\mathbf{p}(A)^{(2)}$, and $\mathbf{p}(A)^{(3)}$ coincides with $\mathbf{p}(A)$.

Also, in the same way as in (4.1.4), we divide $\mathbf{p}(B)$ for each $B \in \mathcal{A}(w, \Gamma_2)$ into three parts $\mathbf{p}(B)^{(1)}$, $\mathbf{p}(B)^{(2)}$, $\mathbf{p}(B)^{(3)}$. If we write $B = \{b_1, \ldots, b_m\}$, then $\mathbf{p}(B)$ is of the form:

$$\mathbf{p}(B) : w = w_0 \xrightarrow{|\beta'_{b_1}|} \cdots \xrightarrow{|\beta'_{b_m}|} w_m,$$

with $b_1 < \cdots < b_m$; we set $b_0 := 0$. Let $0 \leq i_1 \leq m$ be maximal such that $b_{i_1} \leq t$, and $0 \leq i_2 \leq m$ maximal such that $b_{i_2} \leq t + q$. Then, we set

$$\begin{aligned}
\mathbf{p}(B)^{(1)} &: w = w_0 \xrightarrow{|\beta'_{b_1}|} \cdots \xrightarrow{|\beta'_{b_{i_1}}|} w_{b_{i_1}}, \\
\mathbf{p}(B)^{(2)} &: w_{b_{i_1}} \xrightarrow{|\beta'_{b_{i_1+1}}|} \cdots \xrightarrow{|\beta'_{b_{i_2}}|} w_{b_{i_2}}, \\
\mathbf{p}(B)^{(3)} &: w_{b_{i_2}} \xrightarrow{|\beta'_{b_{i_2+1}}|} \cdots \xrightarrow{|\beta'_{b_m}|} w_{b_m}.
\end{aligned}$$

Note that the concatenation of $\mathbf{p}(B)^{(1)}$, $\mathbf{p}(B)^{(2)}$, and $\mathbf{p}(B)^{(3)}$ coincides with $\mathbf{p}(B)$.

Proof of Theorem 4.1.2. We divide the proof of the theorem into two parts:

(1) Δ is not of type G_2;

(2) Δ is of type G_2.

Part 1: Δ is not of type G_2. We assume that Δ is not of type G_2. Let $A \in \mathcal{A}(w, \Gamma_1)$. Then, by Proposition 4.2.3 with $\Pi = \Gamma_1^{(2)}$ and $\Pi' = \Gamma_2^{(2)}$, we see that only one of the following occurs:

(1) there exists $\mathbf{r}_0 \in \mathcal{P}(\mathrm{end}(\mathbf{p}(A)^{(1)}), \Gamma_1^{(2)}) \setminus \{\mathbf{p}(A)^{(2)}\}$ such that $\mathrm{end}(\mathbf{r}_0) = \mathrm{end}(\mathbf{p}(A)^{(2)})$ and $\mathrm{wt}(\mathbf{r}_0) = \mathrm{wt}(\mathbf{p}(A)^{(2)})$;

(2) there exists $\mathbf{r}_0 \in \mathcal{P}(\mathrm{end}(\mathbf{p}(A)^{(1)}), \Gamma_2^{(2)})$ such that $\mathrm{end}(\mathbf{r}_0) = \mathrm{end}(\mathbf{p}(A)^{(2)})$ and $\mathrm{wt}(\mathbf{r}_0) = \mathrm{wt}(\mathbf{p}(A)^{(2)})$.

For convenience of explanation, we set

$$\varphi(A) := \begin{cases} 1 & \text{if (1) of the above holds,} \\ 2 & \text{if (2) of the above holds.} \end{cases}$$

We define a set $\mathcal{A}_0(w, \Gamma_1) \subset \mathcal{A}(w, \Gamma_1)$ by

$$\mathcal{A}_0(w, \Gamma_1) := \{A \in \mathcal{A}(w, \Gamma_1) \mid \varphi(A) = 2\}.$$

Then we have

$$\mathcal{A}_0^C(w, \Gamma_1) = \mathcal{A}(w, \Gamma_1) \setminus \mathcal{A}_0(w, \Gamma_1) = \{A \in \mathcal{A}(w, \Gamma_1) \mid \varphi(A) = 1\}.$$

Let us define a map $Y : \mathcal{A}_0(w, \Gamma_1) \to \mathcal{A}(w, \Gamma_2)$. Let $A \in \mathcal{A}_0(w, \Gamma_1)$. Then, by applying Proposition 4.2.3 with $\Pi = \Gamma_1^{(2)}$ and $\Pi' = \Gamma_2^{(2)}$, there exists a unique $\mathbf{r}_0 \in \mathcal{P}(\mathrm{end}(\mathbf{p}(A)^{(1)}), \Gamma_2^{(2)})$ such that $\mathrm{end}(\mathbf{r}_0) = \mathrm{end}(\mathbf{p}(A)^{(2)})$ and $\mathrm{wt}(\mathbf{r}_0) = \mathrm{wt}(\mathbf{p}(A)^{(2)})$. We write $\mathbf{r}_0$ as:

$$\mathbf{r}_0 : \mathrm{end}(\mathbf{p}(A)^{(1)}) = x_0 \xrightarrow{|\beta'_{j_1}|} \cdots \xrightarrow{|\beta'_{j_p}|} x_p.$$

Since $\mathbf{r}_0$ is $\Gamma_2^{(2)}$-compatible, we have $t + 1 \le j_1 < \cdots < j_p \le t + q$. Set $B^{(2)} := \{j_1, \ldots, j_p\}$, and define $Y(A)$ by $Y(A) := A^{(1)} \sqcup B^{(2)} \sqcup A^{(3)}$; note that $Y(A) \in \mathcal{A}(w, \Gamma_2)$. We define a set $\mathcal{A}_0(w, \Gamma_2)$ by

$$\mathcal{A}_0(w, \Gamma_2) := \{Y(A) \mid A \in \mathcal{A}_0(w, \Gamma_1)\}.$$

We claim that Y defines a bijection $Y : \mathcal{A}_0(w, \Gamma_1) \to \mathcal{A}_0(w, \Gamma_2)$. To verify this claim, it suffices to show that Y is injective.

Let $A_1, A_2 \in \mathcal{A}_0(w, \Gamma_1)$, and assume that $Y(A_1) = Y(A_2)$. We show that $A_1 = A_2$. First, we see that

$$A_1^{(1)} = (Y(A_1))^{(1)} = (Y(A_2))^{(1)} = A_2^{(1)},$$

and

$$A_1^{(3)} = (Y(A_1))^{(3)} = (Y(A_2))^{(3)} = A_2^{(3)}.$$

Hence it remains to show that $A_1^{(2)} = A_2^{(2)}$. By the definition of the map Y, we have $\mathrm{end}(\mathbf{p}(Y(A_1))^{(2)}) = \mathrm{end}(\mathbf{p}(A_1)^{(2)})$ and $\mathrm{wt}(\mathbf{p}(Y(A_1))^{(2)}) = \mathrm{wt}(\mathbf{p}(A_1)^{(2)})$. Also, we have $\mathrm{end}(\mathbf{p}(Y(A_2))^{(2)}) = \mathrm{end}(\mathbf{p}(A_2)^{(2)})$ and $\mathrm{wt}(\mathbf{p}(Y(A_2))^{(2)}) = \mathrm{wt}(\mathbf{p}(A_2)^{(2)})$. Since $\mathbf{p}(Y(A_1))^{(2)} = \mathbf{p}(Y(A_2))^{(2)}$, the uniqueness in Proposition 4.2.3 (2) (with $\Pi = \Gamma_2^{(2)}$ and $\Pi' = \Gamma_1^{(2)}$) implies that $\mathbf{p}(A_1)^{(2)} = \mathbf{p}(A_2)^{(2)}$, from which we obtain $A_1^{(2)} = A_2^{(2)}$, as desired. This shows the injectivity of Y.

To prove that Y satisfies the condition of Theorem 4.1.2 (1), it remains to show that $\mathrm{end}(Y(A)) = \mathrm{end}(A)$, $\mathrm{down}(Y(A)) = \mathrm{down}(A)$, and $(-1)^{n(Y(A))} = (-1)^{n(A)}$. The first equation is obvious, since

$$\begin{aligned}\mathrm{end}(Y(A)) &= \mathrm{end}(\mathbf{p}(Y(A))) = \mathrm{end}(\mathbf{p}(Y(A))^{(3)}) \\ &= \mathrm{end}(\mathbf{p}(A)^{(3)}) = \mathrm{end}(\mathbf{p}(A)) = \mathrm{end}(A).\end{aligned}$$

The second equation is shown as follows:

$$\begin{aligned}\mathrm{down}(Y(A)) &= \mathrm{wt}(\mathbf{p}(Y(A))) \\ &= \mathrm{wt}(\mathbf{p}(Y(A))^{(1)}) + \mathrm{wt}(\mathbf{p}(Y(A))^{(2)}) + \mathrm{wt}(\mathbf{p}(Y(A))^{(3)}) \\ &= \mathrm{wt}(\mathbf{p}(A)^{(1)}) + \mathrm{wt}(\mathbf{p}(Y(A))^{(2)}) + \mathrm{wt}(\mathbf{p}(A)^{(3)}) \\ &= \mathrm{wt}(\mathbf{p}(A)^{(1)}) + \mathrm{wt}(\mathbf{p}(A)^{(2)}) + \mathrm{wt}(\mathbf{p}(A)^{(3)}) \\ &= \mathrm{wt}(\mathbf{p}(A)) \\ &= \mathrm{down}(A).\end{aligned}$$

Since $(-1)^{\mathrm{neg}(\mathbf{p}(Y(A))^{(2)})} = (-1)^{\mathrm{neg}(\mathbf{p}(A)^{(2)})}$ by Proposition 4.2.3 (2), the remaining equation is shown as follows:

$$\begin{aligned}(-1)^{n(Y(A))} &= (-1)^{\mathrm{neg}(\mathbf{p}(Y(A)))} \\ &= (-1)^{\mathrm{neg}(\mathbf{p}(Y(A))^{(1)})}(-1)^{\mathrm{neg}(\mathbf{p}(Y(A))^{(2)})}(-1)^{\mathrm{neg}(\mathbf{p}(Y(A))^{(3)})} \\ &= (-1)^{\mathrm{neg}(\mathbf{p}(A)^{(1)})}(-1)^{\mathrm{neg}(\mathbf{p}(Y(A))^{(2)})}(-1)^{\mathrm{neg}(\mathbf{p}(A)^{(3)})} \\ &= (-1)^{\mathrm{neg}(\mathbf{p}(A)^{(1)})}(-1)^{\mathrm{neg}(\mathbf{p}(A)^{(2)})}(-1)^{\mathrm{neg}(\mathbf{p}(A)^{(3)})} \\ &= (-1)^{\mathrm{neg}(\mathbf{p}(A))} \\ &= (-1)^{n(A)}.\end{aligned}$$

Next, we construct an involution I_1 which satisfies the condition of Theorem 4.1.2 (2). Let $A \in \mathcal{A}_0^C(w, \Gamma_1)$. Then, by applying Proposition 4.2.3 with

$\Pi = \Gamma_1^{(2)}$, there exists a unique $\mathbf{r}_0 \in \mathcal{P}(\mathrm{end}(\mathbf{p}(A)^{(1)}), \Gamma_1^{(2)}) \setminus \{\mathbf{p}(A)^{(2)}\}$ such that $\mathrm{end}(\mathbf{r}_0) = \mathrm{end}(\mathbf{p}(A)^{(2)})$ and $\mathrm{wt}(\mathbf{r}_0) = \mathrm{wt}(\mathbf{p}(A)^{(2)})$. We write $\mathbf{r}_0$ as:

$$\mathbf{r}_0 : \mathrm{end}(\mathbf{p}(A)^{(1)}) = x_0 \xrightarrow{|\beta_{j_1}|} \cdots \xrightarrow{|\beta_{j_p}|} x_p.$$

Since $\mathbf{r}_0$ is $\Gamma_1^{(2)}$-compatible, we have $t+1 \le j_1 < \cdots < j_p \le t+q$. Set $B^{(2)} := \{j_1, \ldots, j_p\}$, and define $I_1(A)$ by $I_1(A) := A^{(1)} \sqcup B^{(2)} \sqcup A^{(3)}$; note that $I_1(A) \in \mathcal{A}(w, \Gamma_1)$. Since $\mathbf{p}(A)^{(2)} \in \mathcal{P}(\mathrm{end}(\mathbf{p}(I_1(A))^{(1)}), \Gamma_1^{(2)})$ satisfies the condition of Proposition 4.2.3 (1), with $\mathbf{p} = \mathbf{p}(I_1(A))^{(2)}$, we deduce that $I_1(A) \in \mathcal{A}_0^C(w, \Gamma_1)$, and that $I_1(I_1(A)) = I_1(A^{(1)} \sqcup B^{(2)} \sqcup A^{(3)}) = A^{(1)} \sqcup A^{(2)} \sqcup A^{(3)} = A$ by the definition of I_1. This shows that I_1 is an involution. Hence it remains to show that $\mathrm{end}(I_1(A)) = \mathrm{end}(A)$, $\mathrm{down}(I_1(A)) = \mathrm{down}(A)$, and $(-1)^{n(I_1(A))} = -(-1)^{n(A)}$, which can be shown by the same argument as that for Y. This completes the construction of I_1.

Finally, we show the existence of an involution I_2 on $\mathcal{A}_0^C(w, \Gamma_2)$. To do this, we examine the set $\mathcal{A}_0^C(w, \Gamma_2)$ in detail. Let $B \in \mathcal{A}(w, \Gamma_2)$. Then, in the same way as for $A \in \mathcal{A}(w, \Gamma_1)$, we see by Proposition 4.2.3, with $\Pi = \Gamma_2^{(2)}$ and $\Pi' = \Gamma_1^{(2)}$, that only one of the following occurs:

(1)′ there exists $\mathbf{r}_0 \in \mathcal{P}(\mathrm{end}(\mathbf{p}(B)^{(1)}), \Gamma_2^{(2)}) \setminus \{\mathbf{p}(B)^{(2)}\}$ such that $\mathrm{end}(\mathbf{r}_0) = \mathrm{end}(\mathbf{p}(B)^{(2)})$ and $\mathrm{wt}(\mathbf{r}_0) = \mathrm{wt}(\mathbf{p}(B)^{(2)})$;

(2)′ there exists $\mathbf{r}_0 \in \mathcal{P}(\mathrm{end}(\mathbf{p}(B)^{(1)}), \Gamma_1^{(2)})$ such that $\mathrm{end}(\mathbf{r}_0) = \mathrm{end}(\mathbf{p}(B)^{(2)})$ and $\mathrm{wt}(\mathbf{r}_0) = \mathrm{wt}(\mathbf{p}(B)^{(2)})$.

For convenience of explanation, we set

$$\varphi'(B) = \begin{cases} 1 & \text{if (1)}' \text{ of the above holds,} \\ 2 & \text{if (2)}' \text{ of the above holds.} \end{cases}$$

We claim that

$$\mathcal{A}_0(w, \Gamma_2) = \{B \in \mathcal{A}(w, \Gamma_2) \mid \varphi'(B) = 2\}. \tag{4.2.19}$$

If this equation is shown, then the following holds:

$$\mathcal{A}_0^C(w, \Gamma_2) := \mathcal{A}(w, \Gamma_2) \setminus \mathcal{A}_0(w, \Gamma_2) = \{B \in \mathcal{A}(w, \Gamma_2) \mid \varphi'(B) = 1\}.$$

First, we take $B \in \mathcal{A}_0(w, \Gamma_2)$. Then, there exists $A \in \mathcal{A}_0(w, \Gamma_1)$ such that $Y(A) = B$. Since $\mathbf{p}(A)^{(2)} \in \mathcal{P}(\mathrm{end}(\mathbf{p}(A)^{(1)}), \Gamma_1^{(2)})$ satisfies the condition of Proposition 4.2.3 (2), with $\Pi = \Gamma_2^{(2)}$, $\Pi' = \Gamma_1^{(2)}$, and $\mathbf{p} = \mathbf{p}(B)^{(2)}$, we have $\varphi'(B) = 2$. Next, we take $B \in \mathcal{A}(w, \Gamma_2)$ such that $\varphi'(B) = 2$. Then, there exists $\mathbf{r}_0 \in \mathcal{P}(\mathrm{end}(\mathbf{p}(B)^{(1)}), \Gamma_1^{(2)})$ such that $\mathrm{end}(\mathbf{r}_0) = \mathrm{end}(\mathbf{p}(B)^{(2)})$ and $\mathrm{wt}(\mathbf{r}_0) = \mathrm{wt}(\mathbf{p}(B)^{(2)})$. We write the $\mathbf{r}_0$ as:

$$\mathbf{r}_0 : \mathrm{end}(\mathbf{p}(B)^{(1)}) = x_0 \xrightarrow{|\beta_{j_1}|} \cdots \xrightarrow{|\beta_{j_p}|} x_p.$$

Then, we have $t+1 \leq j_1 < \cdots < j_p \leq t+q$. Set $A^{(2)} := \{j_1, \dots, j_p\}$, and then $A := B^{(1)} \sqcup A^{(2)} \sqcup B^{(3)}$. We see that $A \in \mathcal{A}_0(w, \Gamma_1)$ and $Y(A) = B$, and hence $B \in \mathcal{A}_0(w, \Gamma_2)$. Thus, equation (4.2.19) is shown. Hence the existence of the desired involution I_2 on $\mathcal{A}_0^C(w, \Gamma_2)$ can be shown by the same argument as that of the involution I_1 on $\mathcal{A}_0^C(w, \Gamma_1)$. This completes the proof of Theorem 4.1.2 for Δ not of type G_2.

Part 2: Δ is of type G_2. We prove the theorem for Δ of type G_2. As in Part 1, we define $\varphi(A)$ for $A \in \mathcal{A}(w, \Gamma_1)$. Let $A \in \mathcal{A}(w, \Gamma_1)$. We set $\Pi := \Gamma_1^{(2)}$, $\Pi' := \Gamma_2^{(2)}$, and $\mathbf{p} := \mathbf{p}(A)^{(2)}$, $v := \mathrm{end}(\mathbf{p}(A)^{(1)})$. We first consider the cases except cases (E1)–(E4). Then, by Proposition 4.2.7, only one of the following (1)–(3) occurs.

(1) There exists a unique $\mathbf{r}_0 \in \mathcal{P}(\mathrm{end}(\mathbf{p}(A)^{(1)}), \Gamma_1^{(2)}) \setminus \{\mathbf{p}(A)^{(2)}\}$ such that $\mathrm{end}(\mathbf{r}_0) = \mathrm{end}(\mathbf{p}(A)^{(2)})$ and $\mathrm{wt}(\mathbf{r}_0) = \mathrm{wt}(\mathbf{p}(A)^{(2)})$. This $\mathbf{r}_0$ satisfies $(-1)^{\mathrm{neg}(\mathbf{r}_0)} = -(-1)^{\mathrm{neg}(\mathbf{p}(A)^{(2)})}$.
Moreover, there does not exist a path $\mathbf{q} \in \mathcal{P}(\mathrm{end}(\mathbf{p}(A)^{(1)}), \Gamma_2^{(2)})$ such that $\mathrm{end}(\mathbf{q}) = \mathrm{end}(\mathbf{p}(A)^{(2)})$ and $\mathrm{wt}(\mathbf{q}) = \mathrm{wt}(\mathbf{p}(A)^{(2)})$.

(2) There exists a unique $\mathbf{r}_0 \in \mathcal{P}(\mathrm{end}(\mathbf{p}(A)^{(1)}), \Gamma_2^{(2)})$ such that $\mathrm{end}(\mathbf{r}_0) = \mathrm{end}(\mathbf{p}(A)^{(2)})$ and $\mathrm{wt}(\mathbf{r}_0) = \mathrm{wt}(\mathbf{p}(A)^{(2)})$. This $\mathbf{r}_0$ satisfies $(-1)^{\mathrm{neg}(\mathbf{r}_0)} = (-1)^{\mathrm{neg}(\mathbf{p}(A)^{(2)})}$.
Moreover, there does not exist a path $\mathbf{q} \in \mathcal{P}(\mathrm{end}(\mathbf{p}(A)^{(1)}), \Gamma_1^{(2)}) \setminus \{\mathbf{p}(A)^{(2)}\}$ such that $\mathrm{end}(\mathbf{q}) = \mathrm{end}(\mathbf{p}(A)^{(2)})$ and $\mathrm{wt}(\mathbf{q}) = \mathrm{wt}(\mathbf{p}(A)^{(2)})$.

(3) There exists a unique $\mathbf{r}_0 \in \mathcal{P}(\mathrm{end}(\mathbf{p}(A)^{(1)}), \Gamma_1^{(2)}) \setminus \{\mathbf{p}(A)^{(2)}\}$ such that $\mathrm{end}(\mathbf{r}_0) = \mathrm{end}(\mathbf{p}(A)^{(2)})$ and $\mathrm{wt}(\mathbf{r}_0) = \mathrm{wt}(\mathbf{p}(A)^{(2)})$. This $\mathbf{r}_0$ satisfies $(-1)^{\mathrm{neg}(\mathbf{r}_0)} = -(-1)^{\mathrm{neg}(\mathbf{p}(A)^{(2)})}$.
Moreover, there exist exactly two paths $\mathbf{q}_1, \mathbf{q}_2 \in \mathcal{P}(\mathrm{end}(\mathbf{p}(A)^{(1)}), \Gamma_2^{(2)})$ such that $\mathrm{end}(\mathbf{q}_1) = \mathrm{end}(\mathbf{q}_2) = \mathrm{end}(\mathbf{p}(A)^{(2)})$ and $\mathrm{wt}(\mathbf{q}_1) = \mathrm{wt}(\mathbf{q}_2) = \mathrm{wt}(\mathbf{p}(A)^{(2)})$. These $\mathbf{q}_1$, $\mathbf{q}_2$ satisfy $(-1)^{\mathrm{neg}(\mathbf{q}_2)} = -(-1)^{\mathrm{neg}(\mathbf{q}_1)}$.

We set

$$\varphi(A) := \begin{cases} 1 & \text{if (1) or (3) of the above holds,} \\ 2 & \text{if (2) of the above holds.} \end{cases}$$

Next, consider the exceptional cases (E1) and (E4). Recall Proposition 4.2.8 and Remark 4.2.9. For $A \in \mathcal{A}(w, \Gamma_1)$, we define $\varphi(A)$ by

$$\varphi(A) := \begin{cases} 3 & \text{if } \mathbf{p}(A)^{(2)} = \mathbf{p}_1^{(\mathrm{E1})}, \mathbf{p}_3^{(\mathrm{E1})}, \mathbf{p}_1^{(\mathrm{E4})}, \mathbf{p}_3^{(\mathrm{E4})}, \\ 4 & \text{if } \mathbf{p}(A)^{(2)} = \mathbf{p}_2^{(\mathrm{E1})}, \mathbf{p}_2^{(\mathrm{E4})}. \end{cases}$$

Finally, in the exceptional cases (E2) and case (E3), we set $\varphi(A) := 5$ for $A \in \mathcal{A}(w, \Gamma_1)$.

With $\varphi(A)$ as above, we define a subset $\mathcal{A}_0(w, \Gamma_1) \subset \mathcal{A}(w, \Gamma_1)$ by

$$\mathcal{A}_0(w, \Gamma_1) := \{A \in \mathcal{A}(w, \Gamma_1) \mid \varphi(A) = 2, 4, 5\}.$$

Also, we set

$$\mathcal{A}_0^C(w, \Gamma_1) := \mathcal{A}(w, \Gamma_1) \setminus \mathcal{A}_0(w, \Gamma_1) = \{A \in \mathcal{A}(w, \Gamma_1) \mid \varphi(A) = 1, 3\}.$$

Let us define a map $Y : \mathcal{A}_0(w, \Gamma_1) \to \mathcal{A}(w, \Gamma_2)$. Let $A \in \mathcal{A}_0(w, \Gamma_1)$. First, assume that $\varphi(A) = 2$. Take a unique $\mathbf{r}_0 \in \mathcal{P}(\mathrm{end}(\mathbf{p}(A)^{(1)}), \Gamma_2^{(2)})$ such that $\mathrm{end}(\mathbf{r}_0) = \mathrm{end}(\mathbf{p}(A)^{(2)})$ and $\mathrm{wt}(\mathbf{r}_0) = \mathrm{wt}(\mathbf{p}(A)^{(2)})$. Next, assume that $\varphi(A) = 4$. Then, $\mathbf{p}(A)^{(2)} = \mathbf{p}_2^{(\mathrm{E1})}$ or $\mathbf{p}(A)^{(2)} = \mathbf{p}_2^{(\mathrm{E4})}$. If $\mathbf{p}(A)^{(2)} = \mathbf{p}_2^{(\mathrm{E1})}$, then we set $\mathbf{r}_0 := \mathbf{q}^{(\mathrm{E1})}$; if $\mathbf{p}(A)^{(2)} = \mathbf{p}_2^{(\mathrm{E4})}$, then we set $\mathbf{r}_0 := \mathbf{q}^{(\mathrm{E4})}$. Also, assume that $\varphi(A) = 5$. Then, $\mathbf{p}(A)^{(2)} = \mathbf{p}^{(\mathrm{E2})}$ or $\mathbf{p}(A)^{(2)} = \mathbf{p}^{(\mathrm{E3})}$. If $\mathbf{p}(A)^{(2)} = \mathbf{p}^{(\mathrm{E2})}$, then we set $\mathbf{r}_0 := \mathbf{q}_2^{(\mathrm{E2})}$; if $\mathbf{p}(A)^{(2)} = \mathbf{p}^{(\mathrm{E3})}$, then we set $\mathbf{r}_0 := \mathbf{q}_2^{(\mathrm{E3})}$. We write the $\mathbf{r}_0$ as:

$$\mathbf{r}_0 : \mathrm{end}(\mathbf{p}(A)^{(1)}) = x_0 \xrightarrow{|\beta'_{j_1}|} x_1 \xrightarrow{|\beta'_{j_2}|} \cdots \xrightarrow{|\beta'_{j_p}|} x_p.$$

Set $B^{(2)} := \{j_1, \dots, j_p\}$, and define $Y(A) := A^{(1)} \sqcup B^{(2)} \sqcup A^{(3)}$. By the same argument as that in Part 1, if we set

$$\mathcal{A}_0(w, \Gamma_2) := \{Y(A) \mid A \in \mathcal{A}_0(w, \Gamma_1)\},$$

then Y defines a bijection between $\mathcal{A}_0(w, \Gamma_1)$ and $\mathcal{A}_0(w, \Gamma_2)$. Also, we can show that $(-1)^{n(Y(A))} = (-1)^{n(A)}$, $\mathrm{end}(Y(A)) = \mathrm{end}(A)$, and $\mathrm{down}(Y(A)) = \mathrm{down}(A)$.

Next, we construct an involution on $\mathcal{A}_0^C(w, \Gamma_1)$. Let $A \in \mathcal{A}_0^C(w, \Gamma_1)$. If $\varphi(A) = 1$, then there exists a unique $\mathbf{r}_0 \in \mathcal{P}(\mathrm{end}(\mathbf{p}(A)^{(1)}), \Gamma_1^{(2)}) \setminus \{\mathbf{p}(A)^{(2)}\}$ such that $\mathrm{end}(\mathbf{r}_0) = \mathrm{end}(\mathbf{p}(A)^{(2)})$ and $\mathrm{wt}(\mathbf{r}_0) = \mathrm{wt}(\mathbf{p}(A)^{(2)})$. If $\varphi(A) = 3$, then we set

$$\mathbf{r}_0 := \begin{cases} \mathbf{p}_3^{(\mathrm{E1})} & \text{if } \mathbf{p}(A)^{(2)} = \mathbf{p}_1^{(\mathrm{E1})}, \\ \mathbf{p}_3^{(\mathrm{E4})} & \text{if } \mathbf{p}(A)^{(2)} = \mathbf{p}_1^{(\mathrm{E4})}, \\ \mathbf{p}_1^{(\mathrm{E1})} & \text{if } \mathbf{p}(A)^{(2)} = \mathbf{p}_3^{(\mathrm{E1})}, \\ \mathbf{p}_1^{(\mathrm{E4})} & \text{if } \mathbf{p}(A)^{(2)} = \mathbf{p}_3^{(\mathrm{E4})}. \end{cases}$$

We write the $\mathbf{r}_0$ as:

$$\mathbf{r}_0 : \mathrm{end}(\mathbf{p}(A)^{(1)}) = x_0 \xrightarrow{|\beta_{j_1}|} x_1 \xrightarrow{|\beta'_{j_2}|} \cdots \xrightarrow{|\beta_{j_p}|} x_p.$$

Set $B^{(2)} := \{j_1, \ldots, j_p\}$, and define $I_1(A) := A^{(1)} \sqcup B^{(2)} \sqcup A^{(3)}$. We see that I_1 defines an involution on $\mathcal{A}_0^C(w, \Gamma_1)$ such that $(-1)^{n(I_1(A))} = -(-1)^{n(A)}$, $\mathrm{end}(I_1(A)) = \mathrm{end}(A)$, and $\mathrm{down}(I_1(A)) = \mathrm{down}(A)$.

Finally, we construct an involution I_2. Let $B \in \mathcal{A}(w, \Gamma_2)$. Set $\Pi := \Gamma_2^{(2)}$, $\Pi' := \Gamma_1^{(2)}$, and $\mathbf{p} := \mathbf{p}(B)^{(2)}$, $v := \mathrm{end}(\mathbf{p}(B)^{(1)})$. Let us consider the cases except cases (E1)–(E4). Then, by Proposition 4.2.7, only one of the following $(1)'$–$(3)'$ occurs.

$(1)'$ There exists a unique $\mathbf{r}_0 \in \mathcal{P}(\mathrm{end}(\mathbf{p}(B)^{(1)}), \Gamma_2^{(2)}) \setminus \{\mathbf{p}(B)^{(2)}\}$ such that $\mathrm{end}(\mathbf{r}_0) = \mathrm{end}(\mathbf{p}(B)^{(2)})$ and $\mathrm{wt}(\mathbf{r}_0) = \mathrm{wt}(\mathbf{p}(B)^{(2)})$. This $\mathbf{r}_0$ satisfies $(-1)^{\mathrm{neg}(\mathbf{r}_0)} = -(-1)^{\mathrm{neg}(\mathbf{p}(B)^{(2)})}$.
Moreover, there does not exist a path $\mathbf{q} \in \mathcal{P}(\mathrm{end}(\mathbf{p}(B)^{(1)}), \Gamma_1^{(2)})$ such that $\mathrm{end}(\mathbf{q}) = \mathrm{end}(\mathbf{p}(B)^{(2)})$ and $\mathrm{wt}(\mathbf{q}) = \mathrm{wt}(\mathbf{p}(B)^{(2)})$.

$(2)'$ There exists a unique $\mathbf{r}_0 \in \mathcal{P}(\mathrm{end}(\mathbf{p}(B)^{(1)}), \Gamma_1^{(2)})$ such that $\mathrm{end}(\mathbf{r}_0) = \mathrm{end}(\mathbf{p}(B)^{(2)})$ and $\mathrm{wt}(\mathbf{r}_0) = \mathrm{wt}(\mathbf{p}(B)^{(2)})$. This $\mathbf{r}_0$ satisfies $(-1)^{\mathrm{neg}(\mathbf{r}_0)} = (-1)^{\mathrm{neg}(\mathbf{p}(B)^{(2)})}$.
Moreover, there does not exist a path $\mathbf{q} \in \mathcal{P}(\mathrm{end}(\mathbf{p}(B)^{(1)}), \Gamma_2^{(2)}) \setminus \{\mathbf{p}(B)^{(2)}\}$ such that $\mathrm{end}(\mathbf{q}) = \mathrm{end}(\mathbf{p}(B)^{(2)})$ and $\mathrm{wt}(\mathbf{q}) = \mathrm{wt}(\mathbf{p}(B)^{(2)})$.

$(3)'$ There exists a unique $\mathbf{r}_0 \in \mathcal{P}(\mathrm{end}(\mathbf{p}(B)^{(1)}), \Gamma_2^{(2)}) \setminus \{\mathbf{p}(B)^{(2)}\}$ such that $\mathrm{end}(\mathbf{r}_0) = \mathrm{end}(\mathbf{p}(B)^{(2)})$ and $\mathrm{wt}(\mathbf{r}_0) = \mathrm{wt}(\mathbf{p}(B)^{(2)})$. This $\mathbf{r}_0$ satisfies $(-1)^{\mathrm{neg}(\mathbf{r}_0)} = -(-1)^{\mathrm{neg}(\mathbf{p}(B)^{(2)})}$.
Moreover, there exist exactly two paths $\mathbf{q}_1, \mathbf{q}_2 \in \mathcal{P}(\mathrm{end}(\mathbf{p}(B)^{(1)}), \Gamma_1^{(2)})$ such that $\mathrm{end}(\mathbf{q}_1) = \mathrm{end}(\mathbf{q}_2) = \mathrm{end}(\mathbf{p}(B)^{(2)})$ and $\mathrm{wt}(\mathbf{q}_1) = \mathrm{wt}(\mathbf{q}_2) = \mathrm{wt}(\mathbf{p}(B)^{(2)})$. These $\mathbf{q}_1$, $\mathbf{q}_2$ satisfy $(-1)^{\mathrm{neg}(\mathbf{q}_2)} = -(-1)^{\mathrm{neg}(\mathbf{q}_1)}$.

We define $\varphi'(B)$ by

$$\varphi'(B) := \begin{cases} 1 & \text{if } (1)' \text{ or } (3)' \text{ of the above holds,} \\ 2 & \text{if } (2)' \text{ of the above holds.} \end{cases}$$

For the exceptional cases (E1) and (E4), we set

$$\varphi'(B) := \begin{cases} 3 & \text{if } \mathbf{p}(B)^{(2)} = \mathbf{p}_1^{(\mathrm{E1})}, \mathbf{p}_3^{(\mathrm{E1})}, \mathbf{p}_1^{(\mathrm{E4})}, \mathbf{p}_3^{(\mathrm{E4})}, \\ 4 & \text{if } \mathbf{p}(B)^{(2)} = \mathbf{p}_2^{(\mathrm{E1})}, \mathbf{p}_2^{(\mathrm{E4})}; \end{cases}$$

for the exceptional cases (E2) and (E3), we set $\varphi'(B) := 5$.

Now, by the same argument as that in Part 1, we obtain

$$\mathcal{A}_0(w,\Gamma_2) = \{B \in \mathcal{A}(w,\Gamma_2) \mid \varphi'(B) = 2,4,5\},$$
$$\mathcal{A}_0^C(w,\Gamma_2) = \{B \in \mathcal{A}(w,\Gamma_2) \mid \varphi'(B) = 1,3\}.$$

Hence an involution I_2 on $\mathcal{A}_0^C(w,\Gamma_2)$ can be defined in the same way as I_1 on $\mathcal{A}_0(w,\Gamma_1)$. This completes the proof of the theorem. □

4.2.7 Proof of Theorem 4.1.4

We will prove that the maps Y, I_1 and I_2 preserve weights and heights.

We need additional notation. Let $\Psi = (\psi_1,\dots,\psi_p)$ be a sequence of roots $\psi_1,\dots,\psi_p \in \Delta$, $\mathbf{k} = (k_1,\dots,k_p)$ a sequence of integers $k_1,\dots,k_p \in \mathbb{Z}$, and $w \in W$. For a subset $J = \{j_1 < \cdots < j_a\} \subset \{1,\dots,p\}$ such that

$$w \xrightarrow{|\psi_{j_1}|} ws_{|\psi_{j_1}|} \xrightarrow{|\psi_{j_2}|} \cdots \xrightarrow{|\psi_{j_a}|} ws_{|\psi_{j_1}|}\cdots s_{|\psi_{j_a}|}$$

is a directed path in $\mathrm{QBG}(W)$ (note that if Ψ is a λ-chain for some $\lambda \in P$, then J is a w-admissible subset), we define $\mathrm{height}_{\mathbf{k},\Psi}(w,J)$ by

$$\mathrm{height}_{\mathbf{k},\Psi}(w,J) := \sum_{j \in J^-} \mathrm{sgn}(\psi_j) k_j,$$

where

$$J^- = \left\{ j_i \in J \;\middle|\; ws_{|\psi_{j_1}|}\cdots s_{|\psi_{j_{i-1}}|} \xrightarrow{|\psi_{j_i}|} ws_{|\psi_{j_1}|}\cdots s_{|\psi_{j_i}|} \text{ is a quantum edge} \right\}.$$

Also, we generalize the definition of down statistics:

$$\mathrm{down}_\Psi(w,J) := \sum_{j \in J^-} |\psi_j|^\vee.$$

Note that if $\Psi = \Gamma_1$, $\mathbf{k} = (\widetilde{l_1},\dots,\widetilde{l_r})$, $w \in W$, and $J = A \in \mathcal{A}(w,\Gamma_1)$, then we have

$$\mathrm{down}_\Psi(w,A) = \mathrm{down}(A), \quad \mathrm{height}_{\mathbf{k},\Psi}(w,A) = \mathrm{height}(A);$$

if $\Psi = \Gamma_2$, $\mathbf{k} = (\widetilde{l'_1},\dots,\widetilde{l'_r}) = (\widetilde{l_1},\dots,\widetilde{l_t},\widetilde{l_{t+q}},\dots,\widetilde{l_{t+1}},\widetilde{l_{t+q+1}},\dots,\widetilde{l_r})$, $w \in W$, and $J = B \in \mathcal{A}(w,\Gamma_2)$, then we have

$$\mathrm{down}_\Psi(w,B) = \mathrm{down}(B), \quad \mathrm{height}_{\mathbf{k},\Psi}(w,B) = \mathrm{height}(B).$$

In addition, for $A \in \mathcal{A}(w, \Gamma_1)$, it follows that

$$\begin{aligned}\mathrm{down}(A) &= \mathrm{down}_{\Gamma_1^{(1)}}(w, A^{(1)}) + \mathrm{down}_{\Gamma_1^{(2)}}(\mathrm{end}(\mathbf{p}(A)^{(1)}), A^{(2)}) \\ &\quad + \mathrm{down}_{\Gamma_1^{(3)}}(\mathrm{end}(\mathbf{p}(A)^{(2)}), A^{(3)}),\end{aligned}$$

and that

$$\begin{aligned}\mathrm{height}(A) &= \mathrm{height}_{(\widetilde{l_1},\dots,\widetilde{l_t}),\Gamma_1^{(1)}}(w, A^{(1)}) \\ &\quad + \mathrm{height}_{(\widetilde{l_{t+1}},\dots,\widetilde{l_{t+q}}),\Gamma_1^{(2)}}(\mathrm{end}(\mathbf{p}(A)^{(1)}), A^{(2)}) \\ &\quad + \mathrm{height}_{(\widetilde{l_{t+q+1}},\dots,\widetilde{l_r}),\Gamma_1^{(3)}}(\mathrm{end}(\mathbf{p}(A)^{(2)}), A^{(3)});\end{aligned} \tag{4.2.20}$$

for $B \in \mathcal{A}(w, \Gamma_2)$, it follows that

$$\begin{aligned}\mathrm{down}(B) &= \mathrm{down}_{\Gamma_2^{(1)}}(w, B^{(1)}) + \mathrm{down}_{\Gamma_2^{(2)}}(\mathrm{end}(\mathbf{p}(B)^{(1)}), B^{(2)}) \\ &\quad + \mathrm{down}_{\Gamma_2^{(3)}}(\mathrm{end}(\mathbf{p}(B)^{(2)}), B^{(3)}),\end{aligned}$$

and that

$$\begin{aligned}\mathrm{height}(B) &= \mathrm{height}_{(\widetilde{l'_1},\dots,\widetilde{l'_t}),\Gamma_2^{(1)}}(w, B^{(1)}) \\ &\quad + \mathrm{height}_{(\widetilde{l'_{t+1}},\dots,\widetilde{l'_{t+q}}),\Gamma_2^{(2)}}(\mathrm{end}(\mathbf{p}(B)^{(1)}), B^{(2)}) \\ &\quad + \mathrm{height}_{(\widetilde{l'_{t+q+1}},\dots,\widetilde{l'_r}),\Gamma_2^{(3)}}(\mathrm{end}(\mathbf{p}(B)^{(2)}), B^{(3)}). \\ &= \mathrm{height}_{(\widetilde{l_1},\dots,\widetilde{l_t}),\Gamma_2^{(1)}}(w, B^{(1)}) \\ &\quad + \mathrm{height}_{(\widetilde{l_{t+q}},\dots,\widetilde{l_{t+1}}),\Gamma_2^{(2)}}(\mathrm{end}(\mathbf{p}(B)^{(1)}), B^{(2)}) \\ &\quad + \mathrm{height}_{(\widetilde{l_{t+q+1}},\dots,\widetilde{l_r}),\Gamma_2^{(3)}}(\mathrm{end}(\mathbf{p}(B)^{(2)}), B^{(3)}).\end{aligned} \tag{4.2.21}$$

Next, we consider weights. For the above J, we set

$$\widehat{r}_{\mathbf{k},\Psi}(J) := s_{\psi_{j_1},-k_{j_1}} \cdots s_{\psi_{j_a},-k_{j_a}}.$$

Then, for $A \in \mathcal{A}(w, \Gamma_1)$, we have

$$\begin{aligned}\mathrm{wt}(A) &= -\widehat{r}_{(l_1,\dots,l_r),\Gamma_1}(A)(-\lambda). \\ &= -\widehat{r}_{(l_1,\dots,l_t),\Gamma_1^{(1)}}(A^{(1)})\widehat{r}_{(l_{t+1},\dots,l_{t+q}),\Gamma_1^{(2)}}(A^{(2)})\widehat{r}_{(l_{t+q+1},\dots,l_r),\Gamma_1^{(3)}}(A^{(3)})(-\lambda);\end{aligned} \tag{4.2.22}$$

for $B \in \mathcal{A}(w, \Gamma_2)$, we have

$$\begin{aligned}\mathrm{wt}(B) &= -\widehat{r}_{(l'_1,\dots,l'_r),\Gamma_2}(B)(-\lambda) \\ &= -\widehat{r}_{(l_1,\dots,l_t,l_{t+q},\dots,l_{t+1},l_{t+q+1},\dots,l_r),\Gamma_2}(B)(-\lambda) \\ &= -\widehat{r}_{(l_1,\dots,l_t),\Gamma_2^{(1)}}(B^{(1)})\widehat{r}_{(l_{t+q},\dots,l_{t+1}),\Gamma_2^{(2)}}(B^{(2)})\widehat{r}_{(l_{t+q+1},\dots,l_r),\Gamma_2^{(3)}}(B^{(3)})(-\lambda).\end{aligned} \tag{4.2.23}$$

Proof of Theorem 4.1.4. First, we consider heights. For $A \in \mathcal{A}(w, \Gamma_1)$, we see that

$$\begin{aligned}
&\mathrm{height}_{(\widetilde{l_{t+1}},\dots,\widetilde{l_{t+q}}),\Gamma_1^{(2)}}(\mathrm{end}(\mathbf{p}(A)^{(1)}), A^{(2)}) \\
&= \sum_{j \in (A^{(2)})^-} \mathrm{sgn}(\beta_j)(\langle \lambda, \beta_j^\vee \rangle - l_j) \\
&= \sum_{j \in (A^{(2)})^-} \mathrm{sgn}(\beta_j)\langle \lambda, \beta_j^\vee \rangle - \mathrm{height}_{(l_{t+1},\dots,l_{t+q}),\Gamma_1^{(2)}}(\mathrm{end}(\mathbf{p}(A)^{(2)}), A^{(2)}) \\
&= \left\langle \lambda, \sum_{j \in (A^{(2)})^-} \mathrm{sgn}(\beta_j)\beta_j^\vee \right\rangle - \mathrm{height}_{(l_{t+1},\dots,l_{t+q}),\Gamma_1^{(2)}}(\mathrm{end}(\mathbf{p}(A)^{(2)}), A^{(2)}) \\
&= \langle \lambda, \mathrm{down}_{\Gamma_1^{(2)}}(\mathrm{end}(\mathbf{p}(A)^{(1)}), A^{(2)}) \rangle \\
&\quad - \mathrm{height}_{(l_{t+1},\dots,l_{t+q}),\Gamma_1^{(2)}}(\mathrm{end}(\mathbf{p}(A)^{(2)}), A^{(2)}).
\end{aligned} \tag{4.2.24}$$

Let us assume that $A \in \mathcal{A}_0(w, \Gamma_1)$, and set $B := Y(A)$. Then we have

$$\begin{aligned}
&\mathrm{height}_{(\widetilde{l_{t+q}},\dots,\widetilde{l_{t+1}}),\Gamma_2^{(2)}}(\mathrm{end}(\mathbf{p}(B)^{(1)}), B^{(2)}) \\
&= \langle \lambda, \mathrm{down}_{\Gamma_2^{(2)}}(\mathrm{end}(\mathbf{p}(B)^{(1)}), B^{(2)}) \rangle \\
&\quad - \mathrm{height}_{(l_{t+q},\dots,l_{t+1}),\Gamma_2^{(2)}}(\mathrm{end}(\mathbf{p}(B)^{(2)}), B^{(2)}).
\end{aligned} \tag{4.2.25}$$

Here, since $\mathrm{down}(Y(A)) = \mathrm{down}(A)$, it follows that

$$\mathrm{down}_{\Gamma_2^{(2)}}(\mathrm{end}(\mathbf{p}(B)^{(1)}), B^{(2)}) = \mathrm{down}_{\Gamma_1^{(2)}}(\mathrm{end}(\mathbf{p}(A)^{(1)}), A^{(2)}).$$

Also, by [18, Lemma 3.5], we know that

$$\bigcap_{k=t+1}^{t+q} H_{\beta_k, -l_k} \neq \emptyset.$$

Therefore, by [19, Lemma 44], we have

$$\begin{aligned}
&\mathrm{height}_{(l_{t+q},\dots,l_{t+1}),\Gamma_2^{(2)}}(\mathrm{end}(\mathbf{p}(B)^{(2)}), B^{(2)}) \\
&= \mathrm{height}_{(l_{t+1},\dots,l_{t+q}),\Gamma_1^{(2)}}(\mathrm{end}(\mathbf{p}(A)^{(2)}), A^{(2)}),
\end{aligned}$$

and hence by (4.2.24) and (4.2.25), we obtain

$$\begin{aligned}
&\mathrm{height}_{(\widetilde{l_{t+q}},\dots,\widetilde{l_{t+1}}),\Gamma_2^{(2)}}(\mathrm{end}(\mathbf{p}(B)^{(1)}), B^{(2)}) \\
&= \mathrm{height}_{(\widetilde{l_{t+1}},\dots,\widetilde{l_{t+q}}),\Gamma_1^{(2)}}(\mathrm{end}(\mathbf{p}(A)^{(1)}), A^{(2)}).
\end{aligned}$$

Now, by (4.2.20) and (4.2.21), we deduce that $\mathrm{height}(B) = \mathrm{height}(A)$, as desired.

Assume that $A \in \mathcal{A}_0^C(w, \Gamma_1)$, and set $B := I_1(A)$. As in (4.2.24), we have

$$\begin{aligned}
&\mathrm{height}_{(\widetilde{l_{t+1}},\dots,\widetilde{l_{t+q}}),\Gamma_1^{(2)}}(\mathrm{end}(\mathbf{p}(B)^{(1)}), B^{(2)}) \\
&= \langle \lambda, \mathrm{down}_{\Gamma_1^{(2)}}(\mathrm{end}(\mathbf{p}(B)^{(1)}), B^{(2)})\rangle \\
&\quad - \mathrm{height}_{(l_{t+1},\dots,l_{t+q}),\Gamma_1^{(2)}}(\mathrm{end}(\mathbf{p}(B)^{(2)}), B^{(2)}).
\end{aligned}$$

Again, by the definition of I_1 and [19, Lemma 44], we have

$$\begin{aligned}
&\mathrm{height}_{(\widetilde{l_{t+1}},\dots,\widetilde{l_{t+q}}),\Gamma_1^{(2)}}(\mathrm{end}(\mathbf{p}(B)^{(1)}), B^{(2)}) \\
&= \langle \lambda, \mathrm{down}_{\Gamma_1^{(2)}}(\mathrm{end}(\mathbf{p}(B)^{(1)}), B^{(2)})\rangle \\
&\quad - \mathrm{height}_{(l_{t+1},\dots,l_{t+q}),\Gamma_1^{(2)}}(\mathrm{end}(\mathbf{p}(B)^{(2)}), B^{(2)}) \\
&= \langle \lambda, \mathrm{down}_{\Gamma_1^{(2)}}(\mathrm{end}(\mathbf{p}(A)^{(1)}), A^{(2)})\rangle \\
&\quad - \mathrm{height}_{(l_{t+1},\dots,l_{t+q}),\Gamma_1^{(2)}}(\mathrm{end}(\mathbf{p}(A)^{(2)}), A^{(2)}) \\
&= \mathrm{height}_{(\widetilde{l_{t+1}},\dots,\widetilde{l_{t+q}}),\Gamma_1^{(2)}}(\mathrm{end}(\mathbf{p}(A)^{(1)}), A^{(2)}),
\end{aligned}$$

and hence obtain $\mathrm{height}(B) = \mathrm{height}(A)$, as desired. Now, the assertion for I_2 is shown by the same argument as for I_1.

It remains to consider weights. Again, by [18, Lemma 3.5], we can take $\mu \in \mathfrak{h}_{\mathbb{R}}^*$ such that

$$\mu \in \bigcap_{k=t+1}^{t+q} H_{\beta_k, -l_k} \neq \emptyset.$$

Note that $\langle \mu, \beta_k^\vee \rangle = -l_k$ for $k = t+1, \dots, t+q$.

Let $A \in \mathcal{A}_0(w, \Gamma_1)$, and set $B := Y(A)$. Recall that $\mathrm{end}(\mathbf{p}(B)^{(2)}) = \mathrm{end}(\mathbf{p}(A)^{(2)})$. We set $v := \mathrm{end}(\mathbf{p}(A)^{(1)}) = \mathrm{end}(\mathbf{p}(B)^{(1)})$. Again, by abuse of notation, we define $t_\nu : \mathfrak{h}_{\mathbb{R}}^* \to \mathfrak{h}_{\mathbb{R}}^*$ for $\nu \in \mathfrak{h}_{\mathbb{R}}^*$ by $t_\nu(\xi) := \xi + \nu$, as in Section 3.2.3. If we write $A^{(2)} = \{j_1, \dots, j_a\}$, then we have

$$\begin{aligned}
t_\mu v^{-1} \mathrm{end}(\mathbf{p}(A^{(2)}))t_{-\mu} &= t_\mu v^{-1}(v s_{|\beta_{j_1}|} \cdots s_{|\beta_{j_a}|})t_{-\mu} \\
&= (t_\mu s_{|\beta_{j_1}|} t_{-\mu}) \cdots (t_\mu s_{|\beta_{j_a}|} t_{-\mu}) \\
&= (t_\mu t_{s_{|\beta_{j_1}|}(-\mu)} s_{|\beta_{j_1}|}) \cdots (t_\mu t_{s_{|\beta_{j_a}|}(-\mu)} s_{|\beta_{j_a}|}) \\
&= (t_{\langle \mu, |\beta_{j_1}|^\vee\rangle |\beta_{j_1}|} s_{|\beta_{j_1}|}) \cdots (t_{\langle \mu, |\beta_{j_a}|^\vee\rangle |\beta_{j_a}|} s_{|\beta_{j_a}|}) \\
&= (t_{\langle \mu, \beta_{j_1}^\vee\rangle \beta_{j_1}} s_{|\beta_{j_1}|}) \cdots (t_{\langle \mu, \beta_{j_a}^\vee\rangle \beta_{j_a}} s_{|\beta_{j_a}|}) \\
&= (t_{-l_{j_1}\beta_{j_1}} s_{|\beta_{j_1}|}) \cdots (t_{-l_{j_a}\beta_{j_a}} s_{|\beta_{j_a}|})
\end{aligned}$$

$$= s_{\beta_{j_1},-l_{j_1}} \cdots s_{\beta_{j_a},-l_{j_a}}$$
$$= \widehat{r}_{(l_{t+1},\dots,l_{t+q}),\Gamma_1^{(2)}}(A^{(2)}). \quad (4.2.26)$$

By the same calculation, we have

$$t_\mu v^{-1}\,\mathrm{end}(\mathbf{p}(B^{(2)}))t_{-\mu} = \widehat{r}_{(l_{t+q},\dots,l_{t+1}),\Gamma_2^{(2)}}(B^{(2)}).$$

Since $\mathrm{end}(\mathbf{p}(B)^{(2)}) = \mathrm{end}(\mathbf{p}(A)^{(2)})$, it follows that

$$\widehat{r}_{(l_{t+q},\dots,l_{t+1}),\Gamma_2^{(2)}}(B^{(2)}) = \widehat{r}_{(l_{t+1},\dots,l_{t+q}),\Gamma_1^{(2)}}(A^{(2)}).$$

Hence, by (4.2.22) and (4.2.23), we obtain $\mathrm{wt}(B) = \mathrm{wt}(A)$, as desired.

Next, assume that $A \in \mathcal{A}_0^C(w, \Gamma_1)$, and set $B := I_1(A)$. By the same calculation as for (4.2.26), we have

$$t_\mu v^{-1}\,\mathrm{end}(\mathbf{p}(B^{(2)}))t_{-\mu} = \widehat{r}_{(l_{t+1},\dots,l_{t+q}),\Gamma_1^{(2)}}(B^{(2)}).$$

Hence, from the equality $\mathrm{end}(\mathbf{p}(B)^{(2)}) = \mathrm{end}(\mathbf{p}(A)^{(2)})$, we deduce that

$$\widehat{r}_{(l_{t+1},\dots,l_{t+q}),\Gamma_1^{(2)}}(B^{(2)}) = \widehat{r}_{(l_{t+1},\dots,l_{t+q}),\Gamma_1^{(2)}}(A^{(2)}).$$

Therefore, we conclude by (4.2.22) and (4.2.23) that $\mathrm{wt}(B) = \mathrm{wt}(A)$, as desired.

The assertion for I_2 is shown by the same argument as for I_1. This completes the proof of the theorem. □

4.2.8 Example of quantum Yang-Baxter moves in type C_2

Based on Proposition 4.2.3, we explain how to construct quantum Yang-Baxter moves explicitly in a specific case. We assume that $\mathfrak{g}$ is of type C_2. Let Π, Π' be sequences of roots introduced in Section 4.2.2. We consider the case that $v = s_2$ and $\Pi = (-2\alpha_1 - \alpha_2, -\alpha_1, \alpha_2, \alpha_1 + \alpha_2)$. Note that $\Pi' = (\alpha_1 + \alpha_2, \alpha_2, -\alpha_1, -2\alpha_1 - \alpha_2)$. Let us construct an explicit matching between a certain subset of $\mathcal{P}(v, \Pi)$ and that of $\mathcal{P}(v, \Pi')$, and also sign-reversing involutions outside of those subsets.

Recall the matrices of $\mathsf{R}_{\alpha_1+\alpha_2}\mathsf{R}_{\alpha_2}\mathsf{R}_{\alpha_1}\mathsf{R}_{2\alpha_1+\alpha_2}$ and $\mathsf{R}_{2\alpha_1+\alpha_2}\mathsf{R}_{\alpha_1}\mathsf{R}_{\alpha_2}\mathsf{R}_{\alpha_1+\alpha_2}$ calculated in Section 4.2.4. In particular, the v-column of the matrix of $\mathsf{R}_{\alpha_1+\alpha_2}\mathsf{R}_{\alpha_2}\mathsf{R}_{\alpha_1}\mathsf{R}_{2\alpha_1+\alpha_2}$ (resp., $\mathsf{R}_{2\alpha_1+\alpha_2}\mathsf{R}_{\alpha_1}\mathsf{R}_{\alpha_2}\mathsf{R}_{\alpha_1+\alpha_2}$) is ${}^t(Q_2, 0, 1, 1, 1, 1, 1, 0)$ (resp., ${}^t(2Q_1Q_2 + Q_2, 2Q_2, 1, 1, 2Q_2 + 1, 1, 1, 0)$). For example, the (e, v)-entry of the matrix of the operator $\mathsf{R}_{2\alpha_1+\alpha_2}\mathsf{R}_{\alpha_1}\mathsf{R}_{\alpha_2}\mathsf{R}_{\alpha_1+\alpha_2}$ is $2Q_1Q_2 + Q_2$. Therefore, we deduce from equation (4.2.3) that there exist exactly three Π'-compatible directed paths $\mathbf{r}^{(1)}, \mathbf{r}^{(2)}, \mathbf{r}^{(3)}$ such that

- $\mathbf{r}^{(j)}$ starts at $v = s_2$ for $j = 1, 2, 3$,
- $\mathrm{end}(\mathbf{r}^{(j)}) = e$, $j = 1, 2, 3$,
- $\mathrm{down}(\mathbf{r}^{(j)}) = \alpha_1^\vee + \alpha_2^\vee$, $j = 1, 2$, and
- $\mathrm{down}(\mathbf{r}^{(3)}) = \alpha_2^\vee$;

remark that $Q^{\alpha_1^\vee+\alpha_2^\vee} = Q_1Q_2$ and $Q^{\alpha_2^\vee} = Q_2$. Similarly, we see that there exist six Π-compatible directed paths $\mathbf{p}_1, \dots, \mathbf{p}_6$ such that $\mathcal{P}(v, \Pi) = \{\mathbf{p}_1, \dots, \mathbf{p}_6\}$. Also, there exist twelve Π'-compatible directed paths $\mathbf{q}_1, \dots, \mathbf{q}_{12}$ such that $\mathcal{P}(v, \Pi') = \{\mathbf{q}_1, \dots, \mathbf{q}_{12}\}$. For $\mathbf{p} \in \mathcal{P}(v, \Pi)$ (resp., $\mathbf{q} \in \mathcal{P}(v, \Pi')$), the statistics $\mathrm{end}(\mathbf{p})$, $\mathrm{down}(\mathbf{p})$ (resp., $\mathrm{end}(\mathbf{q})$, $\mathrm{down}(\mathbf{q})$) are given in Tables 4.1, 4.2.

Table 4.1: Statistics of $\mathbf{p} \in \mathcal{P}(v, \Pi)$

$\mathbf{p} \in \mathcal{P}(v, \Pi)$	$\mathrm{end}(\mathbf{p})$	$\mathrm{down}(\mathbf{p})$
$\mathbf{p}_1$	e	$\alpha_2^\vee$
$\mathbf{p}_2$	s_2	0
$\mathbf{p}_3$	s_1s_2	0
$\mathbf{p}_4$	s_2s_1	0
$\mathbf{p}_5$	$s_1s_2s_1$	0
$\mathbf{p}_6$	$s_2s_1s_2$	0

Table 4.2: Statistics of $\mathbf{q} \in \mathcal{P}(v, \Pi')$

$\mathbf{q} \in \mathcal{P}(v, \Pi')$	$\mathrm{end}(\mathbf{q})$	$\mathrm{down}(\mathbf{q})$
$\mathbf{q}_1$	e	$\alpha_1^\vee + \alpha_2^\vee$
$\mathbf{q}_2$	e	$\alpha_1^\vee + \alpha_2^\vee$
$\mathbf{q}_3$	e	$\alpha_2^\vee$
$\mathbf{q}_4$	s_1	$\alpha_2^\vee$
$\mathbf{q}_5$	s_1	$\alpha_2^\vee$
$\mathbf{q}_6$	s_2	0
$\mathbf{q}_7$	s_1s_2	0
$\mathbf{q}_8$	s_2s_1	$\alpha_2^\vee$
$\mathbf{q}_9$	s_2s_1	$\alpha_2^\vee$
$\mathbf{q}_{10}$	s_2s_1	0
$\mathbf{q}_{11}$	$s_1s_2s_1$	0
$\mathbf{q}_{12}$	$s_2s_1s_2$	0

Note that $\mathbf{p} \in \mathcal{P}(v, \Pi)$ and $\mathbf{q} \in \mathcal{P}(v, \Pi')$ are explicitly written as follows:

$\mathbf{p}_1 : s_2 \xrightarrow{\alpha_2} e$;
$\mathbf{p}_2 : s_2$ (the trivial directed path);
$\mathbf{p}_3 : s_2 \xrightarrow{\alpha_1+\alpha_2} s_1s_2$;
$\mathbf{p}_4 : s_2 \xrightarrow{\alpha_1} s_2s_1$;
$\mathbf{p}_5 : s_2 \xrightarrow{\alpha_1} s_2s_1 \xrightarrow{\alpha_1+\alpha_2} s_1s_2s_1$;
$\mathbf{p}_6 : s_2 \xrightarrow{\alpha_1} s_2s_1 \xrightarrow{\alpha_2} s_2s_1s_2$;
$\mathbf{q}_1 : s_2 \xrightarrow{\alpha_1+\alpha_2} s_1s_2 \xrightarrow{\alpha_2} s_1 \xrightarrow{\alpha_1} e$;
$\mathbf{q}_2 : s_2 \xrightarrow{\alpha_1+\alpha_2} s_1s_2 \xrightarrow{\alpha_1} s_1s_2s_1 \xrightarrow{2\alpha_1+\alpha_2} e$;
$\mathbf{q}_3 : s_2 \xrightarrow{\alpha_2} e$;
$\mathbf{q}_4 : s_2 \xrightarrow{\alpha_1+\alpha_2} s_1s_2 \xrightarrow{\alpha_2} s_1$;
$\mathbf{q}_5 : s_2 \xrightarrow{\alpha_2} e \xrightarrow{\alpha_1} s_1$;
$\mathbf{q}_6 : s_2$ (the trivial directed path);
$\mathbf{q}_7 : s_2 \xrightarrow{\alpha_1+\alpha_2} s_1s_2$;
$\mathbf{q}_8 : s_2 \xrightarrow{\alpha_1+\alpha_2} s_1s_2 \xrightarrow{\alpha_2} s_1 \xrightarrow{2\alpha_1+\alpha_2} s_2s_1$;
$\mathbf{q}_9 : s_2 \xrightarrow{\alpha_2} e \xrightarrow{\alpha_1} s_1 \xrightarrow{2\alpha_1+\alpha_2} s_2s_1$;
$\mathbf{q}_{10} : s_2 \xrightarrow{\alpha_1} s_2s_1$;

$$\mathbf{q}_{11} : s_2 \xrightarrow{\alpha_1+\alpha_2} s_1s_2 \xrightarrow{\alpha_1} s_1s_2s_1; \qquad \mathbf{q}_{12} : s_2 \xrightarrow{\alpha_1+\alpha_2} s_1s_2 \xrightarrow{2\alpha_1+\alpha_2} s_2s_1s_2.$$

Thus, if we set

$$\begin{aligned}
\mathcal{P}_0(v,\Pi) &:= \mathcal{P}(v,\Pi),\\
\mathcal{P}_0(v,\Pi') &:= \{\mathbf{q}_3,\mathbf{q}_6,\mathbf{q}_7,\mathbf{q}_{10},\mathbf{q}_{11},\mathbf{q}_{12}\} \subset \mathcal{P}(v,\Pi'),\\
\mathcal{P}_0^C(v,\Pi') &:= \{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_4,\mathbf{q}_5,\mathbf{q}_8,\mathbf{q}_9\} = \mathcal{P}(v,\Pi') \setminus \mathcal{P}_0(v,\Pi'),
\end{aligned}$$

then we obtain the following bijection $Y^{v,\Pi} : \mathcal{P}_0(v,\Pi) \to \mathcal{P}_0(v,\Pi')$ and involution $I_2^{v,\Pi'}$ on $\mathcal{P}_0^C(v,\Pi')$, which preserve $\mathrm{end}(\cdot)$ and $\mathrm{down}(\cdot)$:

$$\begin{aligned}
Y^{v,\Pi} &: \mathbf{p}_1 \mapsto \mathbf{q}_3, \quad \mathbf{p}_2 \mapsto \mathbf{q}_6, \quad \mathbf{p}_3 \mapsto \mathbf{q}_7, \quad \mathbf{p}_4 \mapsto \mathbf{q}_{10}, \quad \mathbf{p}_5 \mapsto \mathbf{q}_{11}, \quad \mathbf{p}_6 \mapsto \mathbf{q}_{12};\\
I_2^{v,\Pi'} &: \mathbf{q}_1 \mapsto \mathbf{q}_2, \quad \mathbf{q}_2 \mapsto \mathbf{q}_1, \quad \mathbf{q}_4 \mapsto \mathbf{q}_5, \quad \mathbf{q}_5 \mapsto \mathbf{q}_4, \quad \mathbf{q}_8 \mapsto \mathbf{q}_9, \quad \mathbf{q}_9 \mapsto \mathbf{q}_8.
\end{aligned}$$

These maps give the correspondence $\mathbf{p} \mapsto \mathbf{p}'$ in Proposition 4.2.3.

Now, let us give an example of generalized quantum Yang-Baxter moves. Let $\lambda \in P$. Take λ-chains Γ_1, Γ_2 such that Γ_2 is obtained from Γ_1 by the Yang-Baxter transformation (YB). Let $w \in W$. As in equations (4.1.1) and (4.1.2), we take $\Gamma_1^{(k)}$, $\Gamma_2^{(k)}$, $k = 1,2,3$. Also, as in equations (4.1.3) and (4.1.4), we take $A^{(k)}$ (resp., $B^{(k)}$), $k = 1,2,3$, for $A \in \mathcal{A}(w,\Gamma_1)$ (resp., $B \in \mathcal{A}(w,\Gamma_2)$). In this example, we consider the case that $\Gamma_1^{(2)} = \Pi$ and $\Gamma_2^{(2)} = \Pi'$. By the consideration above, we can give an explicit description of quantum Yang-Baxter moves for $A \in \mathcal{A}(w,\Gamma_1)$ (resp., $B \in \mathcal{A}(w,\Gamma_2)$) such that $\mathrm{end}(A^{(1)}) = s_2$ (resp., $\mathrm{end}(B^{(1)}) = s_2$), as given in Tables 4.3, 4.4.

Table 4.3: List of $Y(A)$ for $A \in \mathcal{A}_0(w,\Gamma_1)$ such that $\mathrm{end}(A^{(1)}) = s_2$

$A^{(2)}$	$\mathbf{p}(A^{(2)})$	$\mathbf{p}(Y(A)^{(2)}) = Y^{v,\Pi}(\mathbf{p}(A^{(2)}))$	$Y(A)^{(2)}$
$\emptyset$	$\mathbf{p}_2$	$\mathbf{q}_6$	$\emptyset$
$\{t+2\}$	$\mathbf{p}_4$	$\mathbf{q}_{10}$	$\{t+3\}$
$\{t+3\}$	$\mathbf{p}_1$	$\mathbf{q}_3$	$\{t+2\}$
$\{t+4\}$	$\mathbf{p}_3$	$\mathbf{q}_7$	$\{t+1\}$
$\{t+2,t+3\}$	$\mathbf{p}_6$	$\mathbf{q}_{12}$	$\{t+1,t+4\}$
$\{t+2,t+4\}$	$\mathbf{p}_5$	$\mathbf{q}_{11}$	$\{t+1,t+3\}$

4.3 Generating functions and level-zero Demazure modules

We consider the generating function of certain statistics associated to an alcove path. We describe the relation between two such generating functions

Table 4.4: List of $I_2(B)$ for $B \in \mathcal{A}_0^C(w, \Gamma_2)$ such that $\mathrm{end}(B^{(1)}) = s_2$

$B^{(2)}$	$\mathbf{p}(B^{(2)})$	$\mathbf{p}(I_2(B)^{(2)}) = I_2^{v,\Pi'}(\mathbf{p}(B^{(2)}))$	$I_2(B)^{(2)}$
$\{t+1, t+2\}$	$\mathbf{q}_4$	$\mathbf{q}_5$	$\{t+2, t+3\}$
$\{t+2, t+3\}$	$\mathbf{q}_5$	$\mathbf{q}_4$	$\{t+1, t+2\}$
$\{t+1, t+2, t+3\}$	$\mathbf{q}_1$	$\mathbf{q}_2$	$\{t+1, t+3, t+4\}$
$\{t+1, t+2, t+4\}$	$\mathbf{q}_8$	$\mathbf{q}_9$	$\{t+2, t+3, t+4\}$
$\{t+1, t+3, t+4\}$	$\mathbf{q}_2$	$\mathbf{q}_1$	$\{t+1, t+2, t+3\}$
$\{t+2, t+3, t+4\}$	$\mathbf{q}_9$	$\mathbf{q}_8$	$\{t+1, t+2, t+4\}$

associated to two alcove paths which are related by the procedures (YB) and (D). As an application, we derive an identity of "Chevalley type" for the graded characters of Demazure submodules of (level-zero) extremal weight modules over a quantum affine algebra. This section is based on [14, Section 5 and Appendix C].

4.3.1 Generating functions

Take an indeterminate q, and consider the ring $R := \mathbb{Z}[q, q^{-1}]$ of Laurent polynomials in q. Recall that an element of the affine Weyl group W_{af} can be written $x = wt_\xi$, with w in the finite Weyl group W and ξ in the coroot lattice $Q^\vee$.

Definition 4.3.1. For each λ-chain Γ and $x = wt_\xi \in W_{\mathrm{af}}$, we define a *generating function* $\mathbf{G}_\Gamma(x) \in R[P][W_{\mathrm{af}}]$ associated to the set $\mathcal{A}(w, \Gamma)$ of w-admissible subsets by

$$\mathbf{G}_\Gamma(x) := \sum_{A \in \mathcal{A}(w,\Gamma)} (-1)^{n(A)} q^{-\,\mathrm{height}(A) - \langle \lambda, \xi \rangle} e^{\mathrm{wt}(A)} \,\mathrm{end}(A) t_{\xi + \mathrm{down}(A)}. \qquad (4.3.1)$$

We also think of $\mathbf{G}_\Gamma$ as a linear function on $R[P][W_{\mathrm{af}}]$.

Let $\lambda \in P$, and take λ-chains Γ_1, Γ_2. We consider the relation of the two generating functions $\mathbf{G}_{\Gamma_1}(x)$ and $\mathbf{G}_{\Gamma_2}(x)$ for $x = wt_\xi \in W_{\mathrm{af}}$.

First, we consider the case in which Γ_2 is obtained from Γ_1 by the procedure (YB). As a corollary of Theorems 4.1.2 and 4.1.4, we obtain the equality between the two generating functions.

Proposition 4.3.2. *Assume that Γ_2 is obtained from Γ_1 by (YB). Then we have $\mathbf{G}_{\Gamma_1}(x) = \mathbf{G}_{\Gamma_2}(x)$.*

Proof. As in Theorem 4.1.2, we take subsets $\mathcal{A}_0(w, \Gamma_1)$ and $\mathcal{A}_0^C(w, \Gamma_1)$ of $\mathcal{A}(w, \Gamma_1)$. Also, we take subsets $\mathcal{A}_0(w, \Gamma_2)$ and $\mathcal{A}_0^C(w, \Gamma_2)$ of $\mathcal{A}(w, \Gamma_2)$. Then we have the maps Y, I_1, I_2 as in Theorem 4.1.2. Note that by Theorem 4.1.4, Y, I_1, and I_2 preserve weights and heights.

Since I_1 is a sign-reversing involution which preserves weights, heights, and down statistics, we have

$$\sum_{A\in\mathcal{A}_0^C(w,\Gamma_1)} (-1)^{n(A)} q^{-\operatorname{height}(A)-\langle\lambda,\xi\rangle} e^{\operatorname{wt}(A)} \operatorname{end}(A) t_{\xi+\operatorname{down}(A)} = 0,$$

and hence

$$\mathbf{G}_{\Gamma_1}(x) = \sum_{A\in\mathcal{A}_0(w,\Gamma_1)} (-1)^{n(A)} q^{-\operatorname{height}(A)-\langle\lambda,\xi\rangle} e^{\operatorname{wt}(A)} \operatorname{end}(A) t_{\xi+\operatorname{down}(A)}.$$

We derive the similar result for $\mathbf{G}_{\Gamma_2}(x)$ via the sign-reversing involution I_2. Using the map Y given by a generalization of quantum Yang-Baxter moves, we deduce that

$$\begin{aligned}
&\mathbf{G}_{\Gamma_1}(x) \\
&= \sum_{A\in\mathcal{A}_0(w,\Gamma_1)} (-1)^{n(Y(A))} q^{-\operatorname{height}(Y(A))-\langle\lambda,\xi\rangle} e^{\operatorname{wt}(Y(A))} \operatorname{end}(Y(A)) t_{\xi+\operatorname{down}(Y(A))} \\
&= \mathbf{G}_{\Gamma_2}(x).
\end{aligned}$$

This concludes the proof. □

Next, we consider the case in which Γ_2 is obtained from Γ_1 by the procedure (D).

Proposition 4.3.3. *Assume that Γ_2 is obtained from Γ_1 by the procedure (D), which deletes the segment $(\pm\beta, \mp\beta)$ of Γ_1, where β is not a simple root. Then we have $\mathbf{G}_{\Gamma_1}(x) = \mathbf{G}_{\Gamma_2}(x)$.*

Proof. We write $\Gamma_1 = (\beta_1, \dots, \beta_r)$. By the assumption, there exists $u \in \{1, \dots, r-2\}$ such that

- $\beta_{u+2} = -\beta_{u+1}$,
- β_{u+1} and β_{u+2} are not simple roots, and
- $\Gamma_2 = (\beta_1, \dots, \beta_u, \beta_{u+3}, \dots, \beta_r)$.

Set $\beta := |\beta_{u+1}| = |\beta_{u+2}|$. Since β is not a simple root, there does not exist any path of the form $v \xrightarrow{|\beta_{u+1}|=\beta} v' \xrightarrow{|\beta_{u+2}|=\beta} v'' = v$. Hence, for $A \in \mathcal{A}(w, \Gamma_1)$, we have $A \cap \{u+1, u+2\} \neq \{u+1, u+2\}$. We define a subset $\mathcal{A}_\emptyset(w, \Gamma_1) \subset \mathcal{A}(w, \Gamma_1)$ by

$$\mathcal{A}_\emptyset(w, \Gamma_1) := \{A \in \mathcal{A}(w, \Gamma_1) \mid A \cap \{u+1, u+2\} = \emptyset\}.$$

Also, we define a subset $\mathcal{A}_\emptyset^C(w, \Gamma_1) \subset \mathcal{A}(w, \Gamma_1)$ by

$$\begin{aligned}\mathcal{A}_\emptyset^C(w, \Gamma_1) &:= \mathcal{A}(w, \Gamma_1) \setminus \mathcal{A}_\emptyset(w, \Gamma_1) \\ &= \{A \in \mathcal{A}(w, \Gamma_1) \mid A \cap \{u+1, u+2\} = \{u+1\}, \{u+2\}\}.\end{aligned}$$

We define a map $I_D : \mathcal{A}_\emptyset^C(w, \Gamma_1) \to \mathcal{A}_\emptyset^C(w, \Gamma_1)$ as follows. If $A \in \mathcal{A}_\emptyset^C(w, \Gamma_1)$ satisfies $A \cap \{u+1, u+2\} = \{u+1\}$, then we set

$$I_D(A) := (A \cap \{1, \ldots, u\}) \sqcup \{u+2\} \sqcup (A \cap \{u+3, \ldots, r\});$$

if $A \in \mathcal{A}_\emptyset^C(w, \Gamma_1)$ satisfies $A \cap \{u+1, u+2\} = \{u+2\}$, then we set

$$I_D(A) := (A \cap \{1, \ldots, u\}) \sqcup \{u+1\} \sqcup (A \cap \{u+3, \ldots, r\}).$$

We see that for all $A \in \mathcal{A}_\emptyset^C(w, \Gamma_1)$, we have $I_D(A) \in \mathcal{A}_\emptyset^C(w, \Gamma_1)$. Also, we see that $I_D(I_D(A)) = A$. Hence I_D defines an involution on $\mathcal{A}_\emptyset^C(w, \Gamma_1)$, and it is easy to see that it preserves the statistics $\mathrm{down}(\cdot)$, $\mathrm{end}(\cdot)$, $\mathrm{height}(\cdot)$, and $\mathrm{wt}(\cdot)$. Also, it is easy to verify that I_D negates the sign $(-1)^{n(\cdot)}$. Therefore, we obtain

$$\sum_{A \in \mathcal{A}_\emptyset^C(w, \Gamma_1)} (-1)^{n(A)} q^{-\mathrm{height}(A) - \langle \lambda, \xi \rangle} e^{\mathrm{wt}(A)} \mathrm{end}(A) t_{\xi + \mathrm{down}(A)} = 0.$$

Now, we define a bijection $D : \mathcal{A}_\emptyset(w, \Gamma_1) \to \mathcal{A}(w, \Gamma_2)$ by

$$D(A) := (A \cap \{1, \ldots, u\}) \sqcup \{j - 2 \mid j \in A \cap \{u+3, \ldots, r\}\}.$$

It is again easy to see that this bijection preserves all the statistics. Therefore, we deduce that

$$\begin{aligned}&\mathbf{G}_{\Gamma_1}(x) \\ &= \sum_{A \in \mathcal{A}(w, \Gamma_1)} (-1)^{n(A)} q^{-\mathrm{height}(A) - \langle \lambda, \xi \rangle} e^{\mathrm{wt}(A)} \mathrm{end}(A) t_{\xi + \mathrm{down}(A)} \\ &= \sum_{A \in \mathcal{A}_\emptyset(w, \Gamma_1)} (-1)^{n(A)} q^{-\mathrm{height}(A) - \langle \lambda, \xi \rangle} e^{\mathrm{wt}(A)} \mathrm{end}(A) t_{\xi + \mathrm{down}(A)} \\ &= \sum_{A \in \mathcal{A}_\emptyset(w, \Gamma_1)} (-1)^{n(D(A))} q^{-\mathrm{height}(D(A)) - \langle \lambda, \xi \rangle} e^{\mathrm{wt}(D(A))} \mathrm{end}(D(A)) t_{\xi + \mathrm{down}(D(A))} \\ &= \mathbf{G}_{\Gamma_2}(x),\end{aligned}$$

as desired. This concludes the proof. □

Remark 4.3.4. In the setup of of Proposition 4.3.3, assume that β is a simple root such that $\pm\beta$ appears in positions $u+1$ and $u+2$ in Γ_1. Then the equality $\mathbf{G}_{\Gamma_1}(x) = \mathbf{G}_{\Gamma_2}(x)$ does not hold. This is because there exists a directed path of the form $v \xrightarrow{\beta} v' \xrightarrow{\beta} v$ for all $v \in W$, in contrast to the case that β is not a simple root. In fact, in $\mathcal{A}(w, \Gamma_1)$ we can pair each A for which $A \cap \{u+1, u+2\} = \emptyset$ with $A' := A \sqcup \{u+1, u+2\}$. Let $h \in \mathbb{Z}$ be the contribution of (one of the) positions $u+1$, $u+2$ to height(A'); note that this is independent of A. By using the above pairing, as well as the map D and the cancellations given by the involution I_D in the proof of Proposition 4.3.3, we derive

$$\mathbf{G}_{\Gamma_1}(x) = \mathbf{G}_{\Gamma_2}(x)\,(1 - q^{-h} t_{\beta^\vee}).$$

We need the following weaker version of the notion of a reduced λ-chain.

Definition 4.3.5. A λ-chain is *weakly reduced* if it does not contain both a simple root and its negative.

Now let us consider arbitrary weakly reduced λ-chains Γ_1 and Γ_2. Then there exists a sequence $\Gamma_1 = \Psi_0, \Psi_1, \ldots, \Psi_p = \Gamma_1^*$ of λ-chains such that Γ_1^* is reduced, and each Ψ_k is obtained from Ψ_{k-1} by one of the procedure (YB) or (D). In a similar way, we relate Γ_2 to a reduced λ-chain Γ_2^*. Finally, we relate Γ_1^* to Γ_2^* by successively applying the procedure (YB). The weakly reduced property of Γ_1 and Γ_2 implies that, in the above process, the procedure (D) never deletes a segment $(\pm\beta, \mp\beta)$ where β is a simple root. By Propositions 4.3.2 and 4.3.3, we derive the following theorem.

Theorem 4.3.6. *For arbitrary weakly reduced λ-chains Γ_1 and Γ_2, we have $\mathbf{G}_{\Gamma_1}(x) = \mathbf{G}_{\Gamma_2}(x)$.*

4.3.2 Combinatorial realization of commutativity

In this section we realize combinatorially the symmetry of the general Chevalley formula in [19] coming from commutativity in equivariant K-theory. This realization involves the commutativity of the composition of two functions $\mathbf{G}_{\Gamma_1}$ and $\mathbf{G}_{\Gamma_2}$, and is based on the generalized quantum Yang-Baxter moves. The main result here will also play an important role in the proof of the character identity in Section 4.3.3.

We start by developing the notion of a weakly reduced chain of roots in Definition 4.3.5. Consider an arbitrary weight λ and an arbitrary decomposition of it $\lambda = \lambda_1 + \cdots + \lambda_p$. Let $\lambda_j = \sum_{i \in I} m_{ij} \varpi_i$.

Definition 4.3.7. The weight decomposition $\lambda = \lambda_1 + \cdots + \lambda_p$ is *cancellation free* if, for any $i \in I$, all the nonzero coefficients among $m_{i1}, \ldots, m_{ip}$ have the same sign.

Given the above weight decomposition, consider λ_j-chains of roots Γ_j, for $j = 1, \ldots, p$. Their concatenation, defined in the obvious way and denoted $\Gamma = \Gamma_1 * \cdots * \Gamma_p$, is clearly a λ-chain. Note that the alcove path corresponding to Γ is obtained by considering the shift of the alcove path for Γ_j by $\lambda_1 + \cdots + \lambda_{j-1}$, for $j = 1, \ldots, p$, and by concatenating them in this order.

Proposition 4.3.8. *The λ-chain Γ is a weakly reduced if and only if the weight decomposition $\lambda = \lambda_1 + \cdots + \lambda_p$ is cancellation free and each λ_j-chain Γ_j is weakly reduced.*

Proof. The result is easily derived from the following general fact about a (not necessarily reduced) λ-chain $\Gamma = (\beta_1, \ldots, \beta_r)$, for an arbitrary weight λ, where α is a positive root (see, e.g., [22, Lemma 5.3]):

$$\langle \lambda, \alpha^\vee \rangle = \#\{j \mid \beta_j = \alpha\} - \#\{j \mid \beta_j = -\alpha\}.$$

This fact is applied to a simple root $\alpha = \alpha_i$, by noting that $\langle \lambda, \alpha_i^\vee \rangle$ is the coefficient of ϖ_i in the expansion of λ. □

Let us now consider a cancellation free weight decomposition $\lambda = \mu + \nu$, and weakly reduced chains of roots Γ_1 and Γ_2 corresponding to μ and ν, respectively. Then, by Proposition 4.3.8, we have the weakly reduced λ-chain $\Gamma := \Gamma_1 * \Gamma_2$. Observe that there exists a natural bijection

$$\{(A, B) \mid A \in \mathcal{A}(w, \Gamma_1),\ B \in \mathcal{A}(\mathrm{end}(A), \Gamma_2)\} \to \mathcal{A}(w, \Gamma); \qquad (4.3.2)$$

let $A * B \in \mathcal{A}(w, \Gamma)$ denote the image of (A, B) under this bijection. The following lemma relates the statistics of interest under the bijection; its proof is based on completely similar arguments to those in the proof of [19, Theorem 46].

Lemma 4.3.9 ([19])**.** *For $A \in \mathcal{A}(w, \Gamma_1)$ and $B \in \mathcal{A}(\mathrm{end}(A), \Gamma_2)$, the following hold:*

(1) $n(A * B) = n(A) + n(B)$;

(2) $\mathrm{end}(A * B) = \mathrm{end}(B)$;

(3) $\mathrm{down}(A * B) = \mathrm{down}(A) + \mathrm{down}(B)$;

(4) $\mathrm{height}(A * B) = \mathrm{height}(A) + \mathrm{height}(B) + \langle \nu, \mathrm{down}(A) \rangle$;

(5) $\mathrm{wt}(A * B) = \mathrm{wt}(A) + \mathrm{wt}(B)$.

We are now ready to prove the main result of this section.

Theorem 4.3.10. *Given the above setup and any $x = wt_\xi \in W_{\mathrm{af}}$, we have*

$$\mathbf{G}_{\Gamma_1} \circ \mathbf{G}_{\Gamma_2}(x) = \mathbf{G}_{\Gamma_2} \circ \mathbf{G}_{\Gamma_1}(x) = \mathbf{G}_{\Gamma}(x).$$

These identities are realized combinatorially via the bijection (4.3.2) *and the generalized quantum Yang-Baxter moves.*

Proof. It suffices to prove the second equality. Indeed, this would imply that $\mathbf{G}_{\Gamma_1} \circ \mathbf{G}_{\Gamma_2}(x) = \mathbf{G}_{\Gamma'}(x)$, where $\Gamma' := \Gamma_2 * \Gamma_1$. The proof is then concluded by using Theorem 4.3.6 to show that $\mathbf{G}_{\Gamma}(x) = \mathbf{G}_{\Gamma'}(x)$. Recall that the mentioned theorem is proved by applying the generalized quantum Yang-Baxter moves.

By iterating the definition (4.3.1), we obtain

$$\begin{aligned}
&\mathbf{G}_{\Gamma_2} \circ \mathbf{G}_{\Gamma_1}(x) \\
&= \sum_{A \in \mathcal{A}(w,\Gamma_1)} (-1)^{n(A)} q^{-\operatorname{height}(A) - \langle \mu, \xi \rangle} e^{\mathrm{wt}(A)} \mathbf{G}_{\Gamma_2}\left(\mathrm{end}(A) t_{\xi + \mathrm{down}(A)}\right) \\
&= \sum_{A \in \mathcal{A}(w,\Gamma_1)} \sum_{B \in \mathcal{A}(\mathrm{end}(A),\Gamma_2)} (-1)^{n(A)+n(B)} q^{-\operatorname{height}(A) - \langle \mu, \xi \rangle - \operatorname{height}(B) - \langle \nu, \xi + \mathrm{down}(A) \rangle} \\
&\qquad\qquad \times e^{\mathrm{wt}(A)+\mathrm{wt}(B)} \,\mathrm{end}(B) t_{\xi + \mathrm{down}(A) + \mathrm{down}(B)} \\
&= \sum_{A \in \mathcal{A}(w,\Gamma_1)} \sum_{B \in \mathcal{A}(\mathrm{end}(A),\Gamma_2)} (-1)^{n(A*B)} q^{-\operatorname{height}(A*B) - \langle \lambda, \xi \rangle} \\
&\qquad\qquad \times e^{\mathrm{wt}(A*B)} \,\mathrm{end}(A*B) t_{\xi + \mathrm{down}(A*B)} \\
&= \mathbf{G}_{\Gamma}(x).
\end{aligned}$$

The last two equalities are based on the bijection (4.3.2) and Lemma 4.3.9. □

Theorem 4.3.10 immediately implies the following corollary involving a composition of more than two functions $\mathbf{G}_{\Gamma_i}$. Here we use the corresponding setup that was defined above. Namely, we consider the cancellation free weight decomposition $\lambda = \lambda_1 + \cdots + \lambda_p$, the weakly reduced λ_j-chains of roots Γ_j, for $j = 1, \ldots, p$, and their concatenation $\Gamma = \Gamma_1 * \cdots * \Gamma_p$.

Corollary 4.3.11. *In the above setup, the composite of generating functions $\mathbf{G}_{\Gamma_1} \circ \cdots \circ \mathbf{G}_{\Gamma_p}(x)$ is invariant under permuting the maps $\mathbf{G}_{\Gamma_\cdot}$, and coincides with $\mathbf{G}_{\Gamma}(x)$.*

We now generalize the function $\mathbf{G}_\Gamma$ on $R[P][W_{\mathrm{af}}]$ by defining the function $\widehat{\mathbf{G}}_\Gamma$, which expresses the general K-theory Chevalley formula for semi-infinite flag manifolds in [19].

Definition 4.3.12. For each λ-chain Γ and $x \in W_{\mathrm{af}}$, we define

$$\widehat{\mathbf{G}}_\Gamma(x) := \sum_{\boldsymbol{\chi} \in \overline{\mathrm{Par}}(\lambda)} q^{-|\boldsymbol{\chi}|} \mathbf{G}_\Gamma(x) t_{\iota(\boldsymbol{\chi})}. \tag{4.3.3}$$

Like above, we now consider a cancellation free weight decomposition $\lambda = \mu + \nu$, weakly reduced chains of roots Γ_1 and Γ_2 corresponding to μ and ν, respectively, and the weakly reduced λ-chain $\Gamma := \Gamma_1 * \Gamma_2$. Let $\mu = \sum_{i \in I} m_{i1} \varpi_i$ and $\nu = \sum_{i \in I} m_{i2} \varpi_i$, so $\lambda = \sum_{i \in I} m_i \varpi_i$ with $m_i = m_{i1} + m_{i2}$. We will show that there exists a natural bijection

$$\overline{\mathrm{Par}}(\mu) \times \overline{\mathrm{Par}}(\nu) \to \overline{\mathrm{Par}}(\lambda), \quad (\boldsymbol{\psi}, \boldsymbol{\omega}) \mapsto \boldsymbol{\chi} := \boldsymbol{\psi} * \boldsymbol{\omega}, \tag{4.3.4}$$

which is compatible with the corresponding statistics. The above map is constructed by defining the partition $\chi^{(i)}$ in terms of the partitions $\psi^{(i)}$ and $\omega^{(i)}$, for each $i \in I$; we will identify a partition with its Young diagram. We may assume that $m_{i1}, m_{i2} \geq 0$, and at least one is positive; indeed, otherwise $m_{i1}, m_{i2} \leq 0$, so $\psi^{(i)} = \omega^{(i)} = \emptyset$, and we let $\chi^{(i)} := \emptyset$. In the non-trivial case, we consider a rectangular partition with m_{i2} rows of size $\psi_1^{(i)}$; then $\chi^{(i)}$ is defined as the result of attaching $\omega^{(i)}$ at the right of the rectangle (top justified) and $\psi^{(i)}$ at the bottom of the rectangle (left justified). It is easy to verify that the result is indeed a partition of length at most m_i, as needed, as well as the fact that this map is invertible.

Lemma 4.3.13. *For $\boldsymbol{\psi} \in \overline{\mathrm{Par}}(\mu)$ and $\boldsymbol{\omega} \in \overline{\mathrm{Par}}(\nu)$, the following hold:*

(1) $\iota(\boldsymbol{\psi} * \boldsymbol{\omega}) = \iota(\boldsymbol{\psi}) + \iota(\boldsymbol{\omega})$;

(2) $|\boldsymbol{\psi} * \boldsymbol{\omega}| = |\boldsymbol{\psi}| + |\boldsymbol{\omega}| + \langle \nu, \iota(\boldsymbol{\psi}) \rangle$.

Proof. We use the above notation, in particular $\boldsymbol{\chi} := \boldsymbol{\psi} * \boldsymbol{\omega}$, as well as the fact that the weight decomposition $\lambda = \mu + \nu$ is cancellation free. The first relation is clear by construction. The second one follows from the fact that $\langle \nu, \iota(\boldsymbol{\psi}) \rangle = \sum_{i \in I} m_{i2} \psi_1^{(i)} = \sum_{i \in I} \max\{m_{i2}, 0\} \psi_1^{(i)}$; here we note that $m_{i2} \psi_1^{(i)}$ is the size of the rectangle used to construct $\chi^{(i)}$ in the non-trivial case. $\square$

Theorem 4.3.14. *Given the above setup and any $x = wt_\xi \in W_{\mathrm{af}}$, we have*

$$\widehat{\mathbf{G}}_{\Gamma_1} \circ \widehat{\mathbf{G}}_{\Gamma_2}(x) = \widehat{\mathbf{G}}_{\Gamma_2} \circ \widehat{\mathbf{G}}_{\Gamma_1}(x) = \widehat{\mathbf{G}}_\Gamma(x).$$

These identities are realized combinatorially via the bijections (4.3.2), (4.3.4), *and the generalized quantum Yang-Baxter moves.*

Proof. Like in the proof of Theorem 4.3.10, it suffices to prove the second equality. By iterating the definition (4.3.3) and by also using (4.3.1), we obtain

$$\begin{aligned}
&\widehat{\mathbf{G}}_{\Gamma_2} \circ \widehat{\mathbf{G}}_{\Gamma_1}(x) \\
&= \sum_{\boldsymbol{\psi} \in \overline{\mathrm{Par}}(\mu)} \sum_{A \in \mathcal{A}(w,\Gamma_1)} (-1)^{n(A)} q^{-\langle \mu, \xi \rangle - \mathrm{height}(A) - |\boldsymbol{\psi}|} \\
&\qquad\qquad \times e^{\mathrm{wt}(A)} \widehat{\mathbf{G}}_{\Gamma_2}(\mathrm{end}(A) t_{\xi + \mathrm{down}(A) + \iota(\boldsymbol{\psi})}) \\
&= \sum_{\boldsymbol{\psi} \in \overline{\mathrm{Par}}(\mu)} \sum_{\boldsymbol{\omega} \in \overline{\mathrm{Par}}(\nu)} \sum_{A \in \mathcal{A}(w,\Gamma_1)} (-1)^{n(A)} q^{-\langle \mu, \xi \rangle - \mathrm{height}(A) - |\boldsymbol{\psi}| - |\boldsymbol{\omega}|} \\
&\qquad\qquad \times e^{\mathrm{wt}(A)} \mathbf{G}_{\Gamma_2}(\mathrm{end}(A) t_{\xi + \mathrm{down}(A) + \iota(\boldsymbol{\psi})}) t_{\iota(\boldsymbol{\omega})} \\
&= \sum_{\boldsymbol{\psi} \in \overline{\mathrm{Par}}(\mu)} \sum_{\boldsymbol{\omega} \in \overline{\mathrm{Par}}(\nu)} q^{-|\boldsymbol{\psi}| - |\boldsymbol{\omega}| - \langle \nu, \iota(\boldsymbol{\psi}) \rangle} \sum_{A \in \mathcal{A}(w,\Gamma_1)} (-1)^{n(A)} q^{-\langle \mu, \xi \rangle - \mathrm{height}(A)} \\
&\qquad\qquad \times e^{\mathrm{wt}(A)} \mathbf{G}_{\Gamma_2}(\mathrm{end}(A) t_{\xi + \mathrm{down}(A)}) t_{\iota(\boldsymbol{\psi}) + \iota(\boldsymbol{\omega})} \\
&= \sum_{\boldsymbol{\psi} \in \overline{\mathrm{Par}}(\mu)} \sum_{\boldsymbol{\omega} \in \overline{\mathrm{Par}}(\nu)} q^{-|\boldsymbol{\psi}| - |\boldsymbol{\omega}| - \langle \nu, \iota(\boldsymbol{\psi}) \rangle} \mathbf{G}_{\Gamma_2} \circ \mathbf{G}_{\Gamma_1}(w t_\xi) t_{\iota(\boldsymbol{\psi}) + \iota(\boldsymbol{\omega})} \\
&= \sum_{\boldsymbol{\psi} \in \overline{\mathrm{Par}}(\mu)} \sum_{\boldsymbol{\omega} \in \overline{\mathrm{Par}}(\nu)} q^{-|\boldsymbol{\psi} * \boldsymbol{\omega}|} \mathbf{G}_{\Gamma}(w t_\xi) t_{\iota(\boldsymbol{\psi} * \boldsymbol{\omega})} \\
&= \widehat{\mathbf{G}}_{\Gamma}(x).
\end{aligned}$$

The last two equalities are based on the bijection (4.3.4), Lemma 4.3.13, and Theorem 4.3.10. □

Remark 4.3.15. (1) Theorem 4.3.14 exhibits a combinatorial realization of the symmetry of the general Chevalley formula [19, Theorem 33] coming from commutativity in equivariant K-theory.

(2) Corollary 4.3.11 can be extended to the setup of Theorem 4.3.14.

4.3.3 Identity of Chevalley type for graded characters

As an application of the results in Sections 4.3.1 and 4.3.2, we obtain the identity of Chevalley type for the graded characters of Demazure submodules of (level-zero) extremal weight modules over a quantum affine algebra (Theorem 3.3.1). Although Theorem 3.3.1 can be proved in a parallel way to [19, Theorem 33], we show that it follows immediately from the results in Sections 4.3.1 and 4.3.2.

Now we recall two special cases of Theorem 3.3.1, i.e., the cases that λ is dominant or anti-dominant. The following theorem gives the identity for dominant weights; this is a restatement of [28, Corollary C.1] in terms of the quantum alcove model, which is given by exactly the same argument as for [19, Theorem 29]. Here, for a dominant weight $\lambda \in P^+$, the *lex λ-chain* is a λ-chain constructed in [23, Proposition 4.2].

Theorem 4.3.16 (cf. [19, Theorem 29] and [28, Corollary C.1]; see also [15, Proposition D.1])**.** *Let $\mu, \lambda \in P^+$, and $x = wt_\xi \in W_{\mathrm{af}}$ with $w \in W$ and $\xi \in Q^\vee$. Let Γ be the lex λ-chain. Then we have*

$$\operatorname{gch} V_x^-(\mu+\lambda) = \sum_{A \in \mathcal{A}(w,\Gamma)} \sum_{\boldsymbol{\chi} \in \overline{\mathrm{Par}}(\lambda)} q^{-\operatorname{height}(A) - \langle \lambda, \xi \rangle - |\boldsymbol{\chi}|} \times e^{\mathrm{wt}(A)} \operatorname{gch} V^-_{\mathrm{end}(A) t_{\xi + \mathrm{down}(A) + \iota(\boldsymbol{\chi})}}(\mu).$$

Also, the following theorem gives the identity for anti-dominant weights; this is a restatement of [28, Corollary 3.15] in terms of the quantum alcove model, which is given by exactly the same argument as for [19, Theorem 32]. Here, following [19, Section 4.2], the *lex λ-chain* for an anti-dominant weight $\lambda \in -P^+$ is defined to be the reverse of the lex $(-\lambda)$-chain.

Theorem 4.3.17 (cf. [19, Theorem 32] and [28, Corollary 3.15]; see also [15, Proposition D.1])**.** *Let $\mu \in P^+$, and $x = wt_\xi \in W_{\mathrm{af}}$ with $w \in W$ and $\xi \in Q^\vee$. Take $\lambda \in -P^+$ such that $\mu + \lambda \in P^+$, and let Γ be the lex λ-chain. Then we have*

$$\operatorname{gch} V_x^-(\mu+\lambda) = \sum_{A \in \mathcal{A}(w,\Gamma)} (-1)^{|A|} q^{-\operatorname{height}(A) - \langle \lambda, \xi \rangle} e^{\mathrm{wt}(A)} \operatorname{gch} V^-_{\mathrm{end}(A) t_{\xi + \mathrm{down}(A)}}(\mu).$$

Proof of Theorem 3.3.1. Let μ, λ, x and Γ be as in the statement of Theorem 3.3.1. Write $\lambda = \lambda^+ + \lambda^-$, with $\lambda^+ \in P^+$ and $\lambda^- \in -P^+$ given by

$$\lambda^+ := \sum_{i \in I} \max\{\langle \lambda, \alpha_i^\vee \rangle, 0\} \varpi_i, \qquad \lambda^- := \sum_{i \in I} \min\{\langle \lambda, \alpha_i^\vee \rangle, 0\} \varpi_i.$$

Note that the weight decomposition $\lambda = \lambda^+ + \lambda^-$ is cancellation free. Take a lex λ^+-chain (resp., lex λ^--chain) Γ^+ (resp., Γ^-). Note that the two chains of roots are reduced, and Γ^+ consists of positive roots, while Γ^- consists of negative roots. Define a λ-chain Γ_0 as the concatenation $\Gamma^+ * \Gamma^-$, which is weakly reduced by Proposition 4.3.8.

By Theorems 4.3.14 and 4.3.6, for $x \in W_{\mathrm{af}}$ we have

$$\widehat{\mathbf{G}}_{\Gamma^-} \circ \widehat{\mathbf{G}}_{\Gamma^+}(x) = \widehat{\mathbf{G}}_{\Gamma_0}(x) = \widehat{\mathbf{G}}_{\Gamma}(x). \tag{4.3.5}$$

Now consider the correspondence $x \mapsto \operatorname{gch} V_x^-(\mu)$ for $x \in W_{\mathrm{af}}$, which defines an $R[P]$-module homomorphism $R[P][W_{\mathrm{af}}] \to \mathbb{Z}(\!(q^{-1})\!)[P]$. Under this homomorphism, $\widehat{\mathbf{G}}_{\Gamma}(x)$ is mapped to the right-hand side of (3.3.1). By (4.3.5), we obtain the same result by applying the homomorphism to $\widehat{\mathbf{G}}_{\Gamma^-} \circ \widehat{\mathbf{G}}_{\Gamma^+}(x)$. We observe that doing this parallels the process of expanding $\operatorname{gch} V_x^-(\mu+\lambda) = \operatorname{gch} V_x^-((\mu+\lambda^-)+\lambda^+)$ in terms of $\operatorname{gch} V_{\cdot}^-(\mu+\lambda^-)$ by Theorem 4.3.16, followed by expanding the result in terms of $\operatorname{gch} V_{\cdot}^-(\mu)$ by Theorem 4.3.17; here we use the fact that $\mu+\lambda \in P^+$ implies $\mu+\lambda^- \in P^+$. The mentioned observation proves that, by applying the above homomorphism to $\widehat{\mathbf{G}}_{\Gamma^-} \circ \widehat{\mathbf{G}}_{\Gamma^+}(x)$, we obtain $\operatorname{gch} V_x^-(\mu+\lambda)$. We conclude that the right-hand side of (3.3.1) coincides with $\operatorname{gch} V_x^-(\mu+\lambda)$. □

Remark 4.3.18. Theorem 3.3.1 can also be proved by using the λ-chain $\Gamma_0^* := \Gamma^- * \Gamma^+$ instead of Γ_0.

4.3.4 The right-hand side of the identity of Chevalley type for graded characters

We show that the right-hand side of (3.3.1) is identical to zero if $\mu+\lambda \notin P^+$.

Proposition 4.3.19. *Let $\mu \in P^+$, and $x = wt_\xi \in W_{\mathrm{af}}$ with $w \in W$ and $\xi \in Q^\vee$. Take $\lambda \in P$ such that $\mu+\lambda \notin P^+$, and let Γ be an arbitrary reduced λ-chain. Then we have*

$$\sum_{A \in \mathcal{A}(w,\Gamma)} \sum_{\boldsymbol{\chi} \in \overline{\mathrm{Par}}(\lambda)} (-1)^{n(A)} q^{-\operatorname{height}(A) - \langle \lambda, \xi \rangle - |\boldsymbol{\chi}|} \times e^{\mathrm{wt}(A)} \operatorname{gch} V^-_{\mathrm{end}(A) t_{\xi + \mathrm{down}(A) + \iota(\boldsymbol{\chi})}}(\mu) = 0.$$

In the proof of Proposition 4.3.19, we make use of the following equalities for graded characters.

Proposition 4.3.20 ([15, Proposition D.1])**.** *For each $x \in W_{\mathrm{af}}$, $\xi \in Q^\vee$, and $\lambda \in P^+$, one has*

$$\operatorname{gch} V^-_{xt_\xi}(\lambda) = q^{-\langle \lambda, \xi \rangle} \operatorname{gch} V_x^-(\lambda).$$

Proposition 4.3.21 (cf. [28, Appendix B])**.** *Let $\mu \in P^+$ and $x \in W$. Take $\lambda \in -P^+$ such that $\mu+\lambda \notin P^+$, and let Γ be the lex λ-chain. Then we have*

$$\sum_{A \in \mathcal{A}(x,\Gamma)} (-1)^{|A|} q^{-\operatorname{height}(A)} e^{\mathrm{wt}(A)} \operatorname{gch} V^-_{\mathrm{end}(A) t_{\mathrm{down}(A)}}(\mu) = 0.$$

Remark 4.3.22. In [28], Proposition 4.3.21 is stated and proved in terms of semi-infinite Lakshmibai-Seshadri paths.

Proof of Proposition 4.3.19. By considering $\lambda^{\pm}$, $\Gamma^{\pm}$, Γ_0, and by using Theorems 4.3.14 and 4.3.6 as in the proof of Theorem 3.3.1 (cf. (4.3.5)), we obtain:

$$\begin{aligned}
&\sum_{A\in\mathcal{A}(w,\Gamma)}\sum_{\boldsymbol{\chi}\in\overline{\mathrm{Par}}(\lambda)}(-1)^{n(A)}q^{-\operatorname{height}(A)-\langle\lambda,\xi\rangle-|\boldsymbol{\chi}|}e^{\mathrm{wt}(A)}\operatorname{gch}V^{-}_{\mathrm{end}(A)t_{\xi+\mathrm{down}(A)+\iota(\boldsymbol{\chi})}}(\mu)\\
&=\sum_{A\in\mathcal{A}(w,\Gamma_0)}\sum_{\boldsymbol{\chi}\in\overline{\mathrm{Par}}(\lambda)}(-1)^{n(A)}q^{-\operatorname{height}(A)-\langle\lambda,\xi\rangle-|\boldsymbol{\chi}|}\\
&\qquad\qquad\times e^{\mathrm{wt}(A)}\operatorname{gch}V^{-}_{\mathrm{end}(A)t_{\xi+\mathrm{down}(A)+\iota(\boldsymbol{\chi})}}(\mu)\\
&=\sum_{A\in\mathcal{A}(w,\Gamma^{+})}\sum_{B\in\mathcal{A}(\mathrm{end}(A),\Gamma^{-})}\sum_{\boldsymbol{\chi}\in\overline{\mathrm{Par}}(\lambda^{+})}(-1)^{|B|}\\
&\qquad\qquad\times q^{-\operatorname{height}(A)-\operatorname{height}(B)-\langle\lambda^{-},\mathrm{down}(A)+\iota(\boldsymbol{\chi})\rangle-\langle\lambda,\xi\rangle-|\boldsymbol{\chi}|}\\
&\qquad\qquad\qquad\qquad\times e^{\mathrm{wt}(A)+\mathrm{wt}(B)}\operatorname{gch}V^{-}_{\mathrm{end}(B)t_{\xi+\mathrm{down}(A)+\mathrm{down}(B)+\iota(\boldsymbol{\chi})}}(\mu)\\
&=\sum_{A\in\mathcal{A}(w,\Gamma^{+})}\sum_{B\in\mathcal{A}(\mathrm{end}(A),\Gamma^{-})}\sum_{\boldsymbol{\chi}\in\overline{\mathrm{Par}}(\lambda^{+})}(-1)^{|B|}\\
&\qquad\qquad\times q^{-\operatorname{height}(A)-\operatorname{height}(B)-\langle\lambda^{-},\mathrm{down}(A)+\iota(\boldsymbol{\chi})\rangle-\langle\lambda,\xi\rangle-|\boldsymbol{\chi}|}q^{-\langle\mu,\xi+\mathrm{down}(A)+\iota(\boldsymbol{\chi})\rangle}\\
&\qquad\qquad\qquad\qquad\times e^{\mathrm{wt}(A)+\mathrm{wt}(B)}\operatorname{gch}V^{-}_{\mathrm{end}(B)t_{\mathrm{down}(B)}}(\mu)\\
&=q^{-\langle\mu+\lambda,\xi\rangle}\sum_{A\in\mathcal{A}(w,\Gamma^{+})}\sum_{\boldsymbol{\chi}\in\overline{\mathrm{Par}}(\lambda^{+})}q^{-\operatorname{height}(A)-\langle\lambda^{-}+\mu,\mathrm{down}(A)+\iota(\boldsymbol{\chi})\rangle-|\boldsymbol{\chi}|}e^{\mathrm{wt}(A)}\\
&\qquad\qquad\times\sum_{B\in\mathcal{A}(\mathrm{end}(A),\Gamma^{-})}(-1)^{|B|}q^{-\operatorname{height}(B)}e^{\mathrm{wt}(B)}\operatorname{gch}V^{-}_{\mathrm{end}(B)t_{\mathrm{down}(B)}}(\mu);
\end{aligned}\tag{4.3.6}$$

here the third equality follows by Proposition 4.3.20. Since $\mu+\lambda\notin P^{+}$, it follows that $\mu+\lambda^{-}\notin P^{+}$. Therefore, we deduce by Proposition 4.3.21 that

$$\sum_{B\in\mathcal{A}(\mathrm{end}(A),\Gamma^{-})}(-1)^{|B|}q^{-\operatorname{height}(B)}e^{\mathrm{wt}(B)}\operatorname{gch}V^{-}_{\mathrm{end}(B)t_{\mathrm{down}(B)}}(\mu)=0$$

for each $A\in\mathcal{A}(w,\Gamma^{+})$, and hence that (4.3.6) is identical to zero, as needed. □

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