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Doctoral Thesis

# Cosmology<br>in<br>Palatini and Metric-affine Formalism

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# Chapter 1

# Introduction

I still remember seeing the full starry sky, during a small trip to the countryside as a child. Marveled, wondered, or feared, such are but small words.

However, the experience of being flushed by the universe is nothing peculiar. Since the dawn of time, mankind has been fascinated by the cosmos. In some ways, one could claim that cosmology is one of the oldest fields of study for humanity [5]. The questioning of the universe could have been first called 'mythology'. One of the oldest pieces of literature is *Epic of Gilgamesh* from 2750 B.C., a Mesopotamian mythology that supported the cultural belief of astrology, the fortune-telling of stars. Such trait was popular among preceding mythologies. When mankind started to venture off into the unknown, stars were also a prominent tool to know directions during the voyage of the seven seas. The history of mankind is thus the history of dissecting the universe.

The universe we know today does not revolve around the earth. However, it was not until the 16th century we have firmly obtained such a picture. Indeed, both heliocentrism and geocentrism were prominent among scholars. Although it was noticed that planets do not revolve in a perfect circle around the earth, by 'correcting' the orbits through smaller circles, geocentrism maintained its predictability. This changed when Nicolaus Copernicus published his book De revolutionibus orbium coelestium (On the Revolutions of the Celestial Spheres) on 1543 [6][1]. This book was scientifically supplemented by a series of observational data conducted by many, such as Galileo Galilei, Sir Isaac Newton. Thus the center of the universe was shifted from Earth to the Sun.

The center of the universe did not shift from the sun to the 'universe' until the late 18th century [7]. Cosmology was yet to be a scientific field, however, since it lacked data. One prominent research is the observation and analysis of stellar parallaxes conducted in the 1830s by Friedrich Bessel, Friedrich Georg Wilhelm von Struve and Thomas Henderson [8, 9, 10] see also [11]. Which led to William Herschel proposing a universe model from the observational data into what today is known as galaxies, Fig.1.1 of [12], see also [13]. Soon after John Michell modified Herschel's model, where the luminosities of the stars were fixed, into a different and more accurate galaxy model[2]. After a fair number of controversial discussions [16], in the year 1802 Herschel settled on Michell's analysis, and thus the first galaxy model was constructed and introduced. Hence, the center of the universe was shifted from the sun to our galaxy.

[1] Which tragically was also his death year.

[2] John Michell is also known to have invented the Cavendish experiment, found the square law, and first use statistics to analyze the universe. However, he is notably famous for being the first to consider 'black holes' [14, 15]

It was not until mid 20th century the center of the universe shifted from the fore-mentioned galaxy [17]. In 1915 Albert Einstein introduced General Relativity which in concept introduced the cosmological principle; the *belief* that 'there is no special place in the universe' [18]. Such theory is governed by the following equation,

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \,.$$

The construction of the theory was brought upon such a principle and along with the observational data shown how remarkably the theory fit to illustrate the universe. Moreover, the theory gained much publicity and was proven repeatedly and consistently that it was and still is the best gravitational theory mankind knows today [19, 20, 21, 22].

However, though Einstein's theory is still the best, there are problems that are yet to be answered [17]. One famous problem is the dark sector of the universe, namely dark matter, and dark energy. Even before the rise of General Relativity, observationalists have noticed that the inertial mass of the galaxy, thus the velocity of the galaxy arms, does not coincide with the mass calculated with observational light. For example, Lord Kelvin in his 1904 textbook, page 274, mentions that "*Many of our supposed thousand million stars, perhaps a great majority of them, may be dark bodies.*" [23]. This was later supplemented by Fritz Zwicky, in his fascinating 1933 paper *Die Rotverschiebung von extragalaktischen Nebeln*, using the virial theorem, quantitatively showed that there exists some unseen matter and thus calling it "dunkle materie" or "dark matter". This dark matter, which feebly if not at all interacts with anything besides the gravitational force, still could not be clarified by the Standard Model of Cosmology, which is constructed on the shoulders of two giants of modern physics; General Relativity and the Standard Model of particle physics. Another dark sector is called dark energy which was first observed in 1998 [24] and shows that the current universe is undergoing an accelerated expansion. Since gravity is currently only known to be attractive and never repulsive, it was commonly thought that the universe is retracting and if not at least decelerating. However recent observations indicate that the universe is actually expanding with acceleration. Thus this unknown type of energy, if matter at all, is called dark energy.

In contrast, the dawn of time also envelopes riddles of its own. One major problem is inflation. In 1979, Alexei Starobinski demonstrated that quantum correction to the Einstein equation computes that the early universe could have gone through an accelerated expansion [25, 26]. Meanwhile, big bang cosmology had certain problems called horizon and flatness problems raised in [27, 28] which needs fine-tuning for the initial conditions of the universe. Alan Guth, in the early 1980s, proposed a solution to these problems through

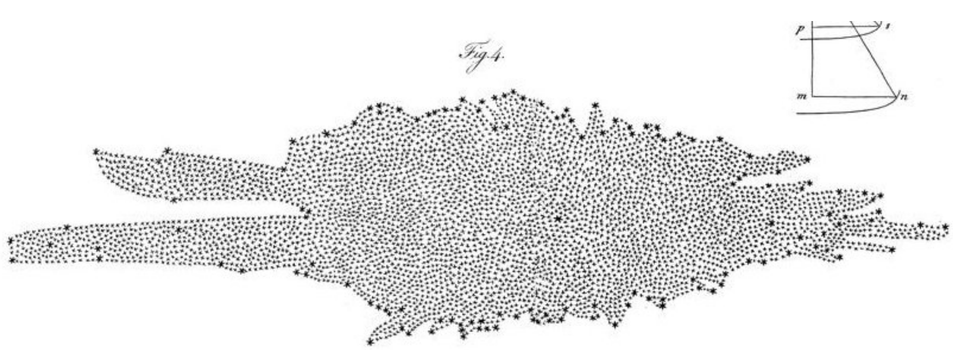


Figure 1.1: Herschel's Image of the 'universe' which what we know now as the Milky Way.

'inflationary' scenarios [29, 30], which may come from phase transitions of the early universe as shown by Katsuhiko Sato [31]. This scenario was later refined by Andrei Linde in [32] and analyzed through a model called "chaotic inflation" [33].

All these questions call for reexamination of the current framework of the standard model of cosmology.

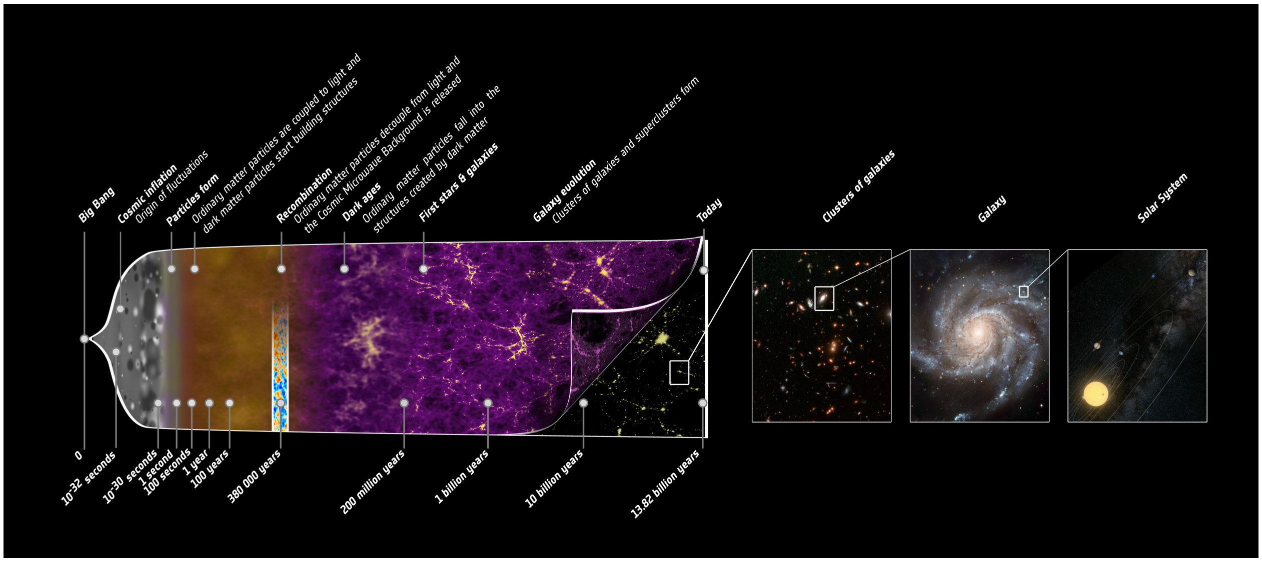


Figure 1.2: History of the Universe, taken from www.jpl.nasa.gov/

This thesis is constructed as follows,

**Part II:** The theoretical preliminaries for the thesis will be oriented. First, the framework of the standard model cosmology will be brought up. Then the troubles that arise from such a framework will be initiated and as concrete examples, inflationary mechanisms and the dark energy problem will be computed. To unravel such problems, modified gravity will be proposed with a focus on Palatini/metric-affine formalism and scalar-tensor theories.

**Part III:** The application of metric-affine gravity to inflation will be investigated. After presenting the ambiguity of covariantization in metric-affine gravity, such a loophole will be utilized to construct a viable yet simple inflationary scenario that is consistent with current observations. Thus this work deduces the possibility of probing the geometry of the universe through observations.
This section is based on the paper [34].

**Part IV:** Construction of viable scalar-tensor theories through metric-affine formalism will be established. Tackling the covariantization of Galileon theories and generalized Galileon theories, it will be shown that ghost-free properties are protected by a new symmetry called projective symmetry. This work is thus a pioneer for ghost-free metric-affine scalar-tensor theories.
This section is based on the paper [35]

**Part V:** Based on the previous part of the thesis, further investigation of scalar-tensor theories within metric-affine formalism will be presented. It will be both qualitatively and quantitatively shown that, although in general gauge projective symmetry does not imply ghost-freeness, in the unitary gauge such theory is totally expelled from Ostrogradsky instabilities.
This section is based on the paper [36]

**Part VI:** The cosmological perturbations for theories within Palatini formalism will be executed. After the orientation of three possible frames in such gravitational theories, it will be presented that each frame computes the same physical quantities. Furthermore, established on this result, Galileons are shown to be unstable in general. However, the ghost can be exorcised, and when done so the speed of gravitational waves is that of light. Thus eluding the current observational constraints.
This section is based on the paper [37].

**Part VII:** The summary of the thesis will be mentioned and concluded.

## Terminology

The following will be the terminology that will be used throughout the thesis.

| | |
|---|---|
| Einstein Gravity: | a theory in which the space-time metric follows the Einstein equations and non-interacting matter follows the metric's geodesics in the point particle limit |
| Metric-affine geometry: | differential metric geometry where the connection is not a priori Levi-Civita |
| Metric-affine Gravity: | gravitational theory in which the connection is not a priori Levi-Civita |
| Palatini Formalism: | a variational method in which the metric and connection are independent generally, the connection does not couple to matter |
| Metric-affine Formalism: | a variational method in which the metric and connection are independent, generally, the connection couple to matter |
| Einstein Cartan Gravity: | a theory in which the connection is taken to be metric, while torsion exists |
| Weyl geometry gravity: | a theory in which non-metricity is proportional to the metric. |
| Weyl-Cartan gravity: | a combination of Einstein Cartan Gravity and Weyl geometry gravity |
| Scalar-tensor theories: | theories that have both scalar(s) and a tensor, as a mediator of gravity |

# Notation

The following will be the notations that will be used throughout the thesis.

| | | |
|---|---|---|
| Metric: | $g_{\mu\nu}$ | a symmetric 2-rank tensor with the signature $(-,+,+,+)$ |
| Affine Connection: | $\Gamma^{\lambda}{}_{\mu\nu}$: | a asymmetric 3-rank matrix |
| Covariant Derivative: | $\overset{\Gamma}{\nabla}_{\mu}$ | derivative with respect to $\Gamma$ |
| | | acts on tensors as $\overset{\Gamma}{\nabla}_{\mu} B_{\nu}{}^{\lambda} = \partial_{\mu} B_{\nu}{}^{\lambda} - \Gamma^{\sigma}{}_{\mu\nu} B_{\sigma}{}^{\lambda} + \Gamma^{\lambda}{}_{\mu\sigma} B_{\nu}{}^{\sigma}$ |
| | | acts on scalar density of weight $w$ as $\overset{\Gamma}{\nabla}_{\mu}\rho = \partial_{\mu}\rho - w\rho\Gamma^{\nu}{}_{\mu\nu}$ |
| Lie Derivative: | $\pounds_{\xi}$ | derivative with respect to some vector $\xi^{\mu}$ |
| | | acts on tensors as $\pounds_{\xi} B_{\nu}{}^{\lambda} = \xi^{\mu}\partial_{\mu} B_{\nu}{}^{\lambda} + B_{\sigma}{}^{\lambda}\partial_{\nu}\xi^{\sigma} - B_{\nu}{}^{\sigma}\partial_{\sigma}\xi^{\lambda}$ |
| | | acts on scalar density of weight $w$ as $\pounds_{\xi}\rho = \xi^{\mu}\partial_{\mu}\rho + w\rho\partial_{\mu}\xi^{\mu}$ |
| Levi-Civita Connection: | $\left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\}$ | metric compatible and torsion-free affine connection defined solely from the metric |
| | | $\left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\} = \frac{1}{2} g^{\lambda\sigma}(\partial_{\mu} g_{\nu\sigma} + \partial_{\mu} g_{\mu\sigma} - \partial_{\sigma} g_{\mu\nu})$ |
| Torsion: | $\mathcal{T}^{\lambda}_{\mu\nu}$ | an anti-symmetric 3-rank tensor defined solely from the connection $\mathcal{T}^{\lambda}_{\mu\nu} = 2\Gamma^{\lambda}{}_{[\mu\nu]}$ |
| Non-metricity: | $\mathcal{Q}_{\lambda}{}^{\mu\nu}$ | a symmetric 3-rank tensor defined both from the metric and connection $\mathcal{Q}^{\mu\nu}_{\lambda} = \overset{\Gamma}{\nabla}_{\lambda} g^{\mu\nu}$ |
| Riemann Curvature: | $\overset{\Gamma}{R}{}^{\lambda}{}_{\sigma\mu\nu}$ | an anti-symmetric 4-rank tensor defined solely from the connection |
| | | $\overset{\Gamma}{R}{}^{\lambda}{}_{\sigma\mu\nu} = \partial_{\mu}\Gamma^{\lambda}{}_{\nu\sigma} - \partial_{\nu}\Gamma^{\lambda}{}_{\mu\sigma} + \Gamma^{\lambda}{}_{\mu\rho}\Gamma^{\rho}\ \nu\sigma - \Gamma^{\lambda}{}_{\nu\rho}\Gamma^{\rho}\ \mu\sigma$ |
| Ricci Tensor: | $\overset{\Gamma}{R}_{\mu\nu}$ | an asymmetric 2-rank tensor solely defined from Riemann curvature |
| | | $\overset{\Gamma}{R}_{\mu\nu} = \overset{\Gamma}{R}{}^{\sigma}{}_{\mu\sigma\nu}$ |
| Co-Ricci Tensor: | $\overset{\Gamma}{P}_{\mu\nu}$ | an asymmetric 2-rank tensor defined from the metric and Riemann curvature |
| | | $\overset{\Gamma}{P}_{\mu\nu} = g_{\rho\mu} g^{\lambda\sigma} \overset{\Gamma}{R}{}^{\rho}{}_{\lambda\nu\sigma}$ |
| Homothetic Tensor: | $\overset{\Gamma}{H}_{\mu\nu}$ | an anti-symmetric 2-rank tensor solely defined from Riemann curvature |
| | | $\overset{\Gamma}{H}_{\mu\nu} = \overset{\Gamma}{R}{}^{\sigma}{}_{\sigma\mu\nu}$ |
| Riemannian variables: | $\overset{g}{\square}$ | Variables of Riemann geometry, such as $\overset{g}{\nabla}_{\mu}$ and $\overset{g}{R}$, will be denoted with $g$ |
| 3 dimensional Curvature: | $\overset{\Gamma}{\mathcal{R}}{}^{\lambda}{}_{\sigma\mu\nu}$ | Riemann curvature tensor of the hyper-surface |
| | | For the precise definition, see §A.3.1 |

# Chapter 2

# Cosmology in Modified Gravity

## 2.1 The Standard Model of Cosmology

The Standard Model of cosmology is a model that consists of the Einstein equations, the Standard Model of physics, and their coupling. In this chapter, each of the topics will be briefly reviewed.

### 2.1.1 The Einstein equations and the Friedmann equations

Einstein, on November 25th, 1915, in his paper *Die Feldgleichungen der Graviation* equation (2), introduced the prized Einstein equations [18],[1]

$$\overset{g}{G}_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}\,, \tag{2.2}$$

which governs the dynamics of the space-time metric,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{2.3}$$

through the distribution of matter which is characterized by the energy-momentum tensor $T_{\mu\nu}$.

The tensor $G_{\mu\nu}$ is called the Einstein tensor and is expressed through a series of definitions which relates it to the metric as follows,

$$\overset{g}{G}_{\mu\nu} := \overset{g}{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \overset{g}{R}\,, \tag{2.4}$$

$$\overset{g}{R} := g^{\mu\nu} \overset{g}{R}_{\mu\nu}\,, \tag{2.5}$$

$$\overset{g}{R}_{\mu\nu} := \overset{g}{R}{}^{\lambda}{}_{\mu\lambda\nu}\,, \tag{2.6}$$

$$\overset{g}{R}{}^{\sigma}{}_{\lambda\mu\nu} := \partial_\mu \left\{ \begin{matrix} \sigma \\ \nu\lambda \end{matrix} \right\} - \partial_\nu \left\{ \begin{matrix} \sigma \\ \mu\lambda \end{matrix} \right\} + \left\{ \begin{matrix} \sigma \\ \mu\rho \end{matrix} \right\} \left\{ \begin{matrix} \rho \\ \nu\lambda \end{matrix} \right\} - \left\{ \begin{matrix} \sigma \\ \nu\rho \end{matrix} \right\} \left\{ \begin{matrix} \rho \\ \mu\lambda \end{matrix} \right\}, \tag{2.7}$$

$$\left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\} := \frac{1}{2} g^{\lambda\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\mu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})\,, \tag{2.8}$$

with $\overset{g}{R}$ the Ricci scalar, $\overset{g}{R}_{\mu\nu}$ the Ricci tensor, $\overset{g}{R}{}^{\sigma}{}_{\lambda\mu\nu}$ the Riemann tensor, and $\left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\}$ being the Levi-Civita connection. The Levi-Civita connection, is by definition symmetric in the lower indices and compatible with the metric, i.e. $\overset{g}{\nabla}_\mu g^{\sigma\lambda} = 0$.

[1] The celebrated Einstein equation in presence of matter $G_{\mu\nu} = T_{\mu\nu}$, does not appear in the paper of November 25th. The gravity-matter equations he derived was

$$\overset{g}{R}_{\mu\nu} = -\kappa (T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T) \tag{2.1}$$

as could be seen in equation (2a) in [18]. (The $G_{ij} = R_{ij} + S_{ij}$ notation Einstein uses is actually the modern day Ricci tensor $\overset{g}{R}_{ij}$, whereas the $S_{ij}$ is the term that disappears under the unimodular constraint $\sqrt{-g}$ =const. )

The formal investigation of Einstein equations and Einstein gravity will be conducted in §2.3.1.

Now consider the isotropic and homogeneous Friedmann-Lemaître-Robertson-Walker metric [38, 39, 40, 41],

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta)\right)\,, \tag{2.9}$$

where $a(t)$ is called the scale factor and it is taken as $c = 1$. Calculating the Ricci tensor, one obtains,

$$\overset{g}{R}_{00} = -3(\dot{H} + H^2)\,, \tag{2.10}$$

$$\overset{g}{R}_{ij} = \left(\dot{H} + 3H^2 + 2\frac{k}{a^2}\right)a^2\delta_{ij}\,, \tag{2.11}$$

$$\overset{g}{R} = 6\left(\dot{H} + 2H^2 + \frac{k}{a^2}\right)\,, \tag{2.12}$$

where the Hubble parameter was defined $H := \frac{1}{a}\frac{da}{dt}$. The term $k$ is called the spatial curvature term, since the 3-dimensional part of the metric,

$$ds_3^2 = \frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta)\,, \tag{2.13}$$

computes the 3 dimensional Ricci curvature,

$$\overset{g}{\mathcal{R}} = 6k\,. \tag{2.14}$$

As for matter, one may take the perfect fluid form of the energy-momentum tensor,

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}\,, \tag{2.15}$$

with $\rho$ being energy density, $p$ being pressure, and $u_\mu$ being the four-velocity time-like vector of the fluid.

Let this ansatz of the metric and matter be substituted into the Einstein equations with the cosmological constant,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}\,, \tag{2.16}$$

for future convenience. Then one obtains the celebrated Friedmann equations [38], which is the one Friedmann obtained in his 1922 paper *Über die Krümmung des Raumes* equation (3) and (4),

$$H^2 + \frac{k}{a^2} - \frac{1}{3}\Lambda = \frac{8\pi G}{3}\rho\,, \tag{2.17}$$

$$\dot{H} - \frac{k}{a^2} = -4\pi G(\rho + p)\,. \tag{2.18}$$

Taking the derivative of the first equation, and substituting the second equation one obtains,

$$\dot{\rho} + 3H(\rho + p) = 0\,, \tag{2.19}$$

which is none other than the energy-momentum conservation law,

$$0 = \overset{g}{\nabla}_\mu T^{\mu\nu}\,. \tag{2.20}$$

Now consider a scenario with multiple matter, $\rho = \sum_i \rho_i$, in such case, energy-momentum conservation holds for each type of matter,

$$\dot{\rho}_i + 3H(\rho_i + p_i) = 0\,. \tag{2.21}$$

Let each matter have a constant equation of state parameter of

$$w_i = \frac{p_i}{\rho_i}\,. \tag{2.22}$$

Then the equation (2.21), can be solved as,

$$\rho_i = \rho_{i,0}\left(\frac{a}{a_0}\right)^{-3(1+w_i)}, \tag{2.23}$$

with $a_0 = a(t = t_0)$,$\rho_{i,0} = \rho_i(t = t_0)$ for some moment $t = t_0$. For non-relativistic matter with $\rho_i \gg p_i$, $w_i \sim 0$ and for relativistic matter $\rho_i = 3p_i$, $w_i = \frac{1}{3}$. Former is usually called *matter* denoted as $\rho_m$ and the latter being *radiation* denoted by $\rho_r$. The spatial curvature term and the cosmological terms can also each be thought as matter by defining, $\rho_k = -\frac{3k}{8\pi G a^2}$ with $w_k = -\frac{1}{3}$ and $\rho_\Lambda = \frac{\Lambda}{8\pi G}$. When the universe is dominated by a single component, say $\rho_i$, then the scale factor and Hubble parameter can be solved as,

$$a(t) = \begin{cases} a_0\left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w_i)}} & w_i \neq -1 \\ a_0 e^{\sqrt{\frac{\Lambda}{3}}t} & w_i = -1 \end{cases} \tag{2.24}$$

$$H(t) = \begin{cases} \frac{2}{3(1+w_i)}\frac{t_0}{t} & w_i \neq -1 \\ \sqrt{\frac{\Lambda}{3}}t & w_i = -1 \end{cases} \tag{2.25}$$

Moreover, the Friedmann equation can be further rewritten as, defining $H_0 = H(t = t_0)$,

$$\frac{H^2}{H_0^2} = \sum_i \Omega_{i,0}\left(\frac{a}{a_0}\right)^{-3(1+w_i)}, \tag{2.26}$$

with the dimensionless energy density parameter defined as,

$$\Omega_{i,0} = \frac{8\pi G \rho_{i,0}}{3H_0^2}\,. \tag{2.27}$$

By taking $t = t_0$, one obtains that $\Omega_{i,0}$ represents the ratio of matter at that moment of time,

$$1 = \sum_i \Omega_{i,0} \tag{2.28}$$

Furthermore, defining redshift,

$$z = 1 - \frac{a_0}{a}\,, \tag{2.29}$$

one obtains the conventional dimensionless Friedmann equations,

$$\frac{H^2}{H_0^2} = \sum_i \Omega_{i,0}\left(1+z\right)^{3(1+w_i)}, \tag{2.30}$$

Thus by knowing the current ratio of matter, one can extrapolate back in time.

Using the current observational data from the Planck satellite [4], the future and past evolution of the cosmological parameters of the ΛCDM model can be computed as Fig.2.1

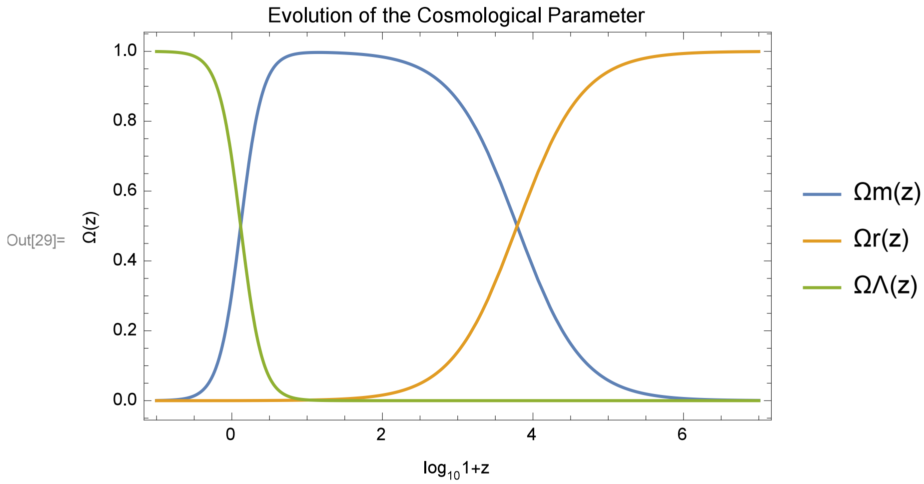


Figure 2.1: Evolution of the Cosmological parameters

### 2.1.2 Matter Coupling and Dynamics

It is said that Einstein got his idea of General Relativity through the equivalence principle. The equivalence principle is a principle of the coupling between the gravity sector and the matter sector and implies that there are some coordinates that 'turns-off' gravity. In this section, the dynamics of matter and its relation to the equivalence principle will be analyzed. First by computing autoparallels and geodesics directly from the equivalence principle, what type of matter follow such equations will be investigated.

#### 2.1.2.1 Autoparallels and Geodesics from the equivalence principle

Consider a certain coordinate $X^\mu$, a free-fall one, in which there are no forces acting on the particle,

$$0 = \frac{d^2X^\mu}{d\lambda^2}\,, \tag{2.31}$$

where $\lambda$ is an affine parameter that characterize the motion. This equation is not a tensor, and thus does not hold for every coordinate system. In order to construct a tensor equation of the dynamics, one can consider the following coordinate transformation $X^\mu \to x^\mu$. Then, from chain rule,

$$0 = \frac{d^2X^\mu}{d\lambda^2} = \frac{d^2x^\nu}{d\lambda^2}\frac{\partial X^\mu}{\partial x^\nu} + \frac{\partial^2 X^\mu}{\partial x^\rho \partial x^\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda}\,. \tag{2.32}$$

Since $X^\mu \to x^\mu$ is considered to be a one-to-one invertible transformation, there exists an inverse of $\frac{\partial X^\mu}{\partial x^\nu}$, that is $\frac{\partial x^\nu}{\partial X^\mu}$, thus

$$\frac{d^2x^\nu}{d\lambda^2} + \frac{\partial x^\nu}{\partial X^\mu}\frac{\partial^2 X^\mu}{\partial x^\nu \partial x^\sigma}\frac{dx^\nu}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0\,. \tag{2.33}$$

For this to be a *tensorial equation*, one needs to introduce a three-rank non-tensorial matrix $A^\lambda{}_{\mu\nu}$ that has the following transformation law,

$$A^\lambda{}_{(\mu\nu)}(x^\mu) = \frac{\partial x^\lambda}{\partial X^\alpha}\frac{\partial X^\beta}{\partial x^\mu}\frac{\partial X^\gamma}{\partial x^\nu}A^\alpha{}_{(\beta\gamma)}(X^\mu) + \frac{\partial^2 X^\alpha}{\partial x^\mu \partial x^\nu}\frac{\partial x^\lambda}{\partial X^\alpha}\,, \tag{2.34}$$

such that, the substitution into (2.33),

$$0 = \frac{d^2x^\mu}{d\lambda^2} + A^\mu{}_{(\nu\sigma)}(x^\alpha)\frac{dx^\nu}{d\lambda}\frac{dx^\sigma}{d\lambda}\,,$$

is a tensorial equation.

Since a *general* affine connection has the similar transformation law, which under coordinate transformation, is,

$$\Gamma^\lambda{}_{(\mu\nu)}(x^\mu) = \frac{\partial x^\lambda}{\partial X^\alpha}\frac{\partial X^\beta}{\partial x^\mu}\frac{\partial X^\gamma}{\partial x^\nu}\Gamma^\alpha{}_{(\beta\gamma)}(X^\mu) + \frac{\partial^2 X^\alpha}{\partial x^\mu \partial x^\nu}\frac{\partial x^\lambda}{\partial X^\alpha}\,, \tag{2.35}$$

the non-tensorial matrix $A^\lambda{}_{\mu\nu}$ can be *chosen* to be associated as an affine connection. Furthermore, since one can choose any general affine connection to be null in the original free falling coordinate frame $\Gamma^\alpha{}_{(\beta\gamma)}(X^\alpha) = 0$.[2] Substituting this back into (2.33), one obtains,

$$\begin{aligned} 0 &= \frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu{}_{(\nu\sigma)}(x^\alpha)\frac{dx^\nu}{d\lambda}\frac{dx^\sigma}{d\lambda} &(2.38)\\ &= u^\sigma \overset{\Gamma}{\nabla}_\sigma u^\nu\,, &(2.39)\end{aligned}$$

where $\overset{\Gamma}{\nabla}_\mu$ is the covariant derivative with respect to the affine connection $\Gamma^\lambda{}_{\mu\nu}(x^\alpha)$ and the four-velocity was defined as $u^\sigma = \frac{dx^\sigma}{d\lambda}$. This equation is called the auto-parallel equation. Such equation is invariant under affine transformation of the affine parameter, i.e. $\lambda \to \bar{\lambda} = a\lambda + b$ with $a, b$ being some constant.

Now *assuming* integrability of this equation, i.e. there exists some Lagrangian that computes this equation[3], one finally obtains,

$$\begin{aligned} 0 &= \frac{d^2x^\mu}{d\lambda^2} + \left\{ \begin{array}{c} \mu \\ \nu\sigma \end{array} \right\}(x^\alpha)\frac{dx^\nu}{d\lambda}\frac{dx^\sigma}{d\lambda} &(2.40)\\ &= u^\sigma \overset{g}{\nabla}_\sigma u^\nu\,, &(2.41)\end{aligned}$$

where

$$\left\{ \begin{array}{c} \mu \\ \nu\sigma \end{array} \right\} = \frac{1}{2}g^{\mu\rho}\left(\partial_\nu g_{\rho\sigma} + \partial_\sigma g_{\rho\nu} - \partial_\rho g_{\nu\sigma}\right)\,, \tag{2.42}$$

is the Levi-Civita connection of some (and not necessary space-time) metric $g_{\mu\nu}(x)$ and $\overset{g}{\nabla}$ being its covariant derivative. This equation is none other than the celebrated geodesic equation [43].

[2] It can be shown that the symmetric part of the connection always has a coordinate in which $\Gamma^\alpha{}_{(\beta\gamma)}(x'^\alpha) = 0$ [42].

Consider a coordinate where $\Gamma^\lambda{}_{(\mu\nu)}(x^\mu) \neq 0$, the transformation $x^\mu \to x'^\mu$, up to second-order is,

$$x'^\mu = x^\mu + \frac{1}{2}a^\mu{}_{\nu\sigma}x^\nu x^\sigma + \cdots\,, \tag{2.36}$$

with $a^\mu{}_{\nu\sigma}$ some constant independent of $x$ which is symmetric in its indices $(\nu\sigma)$. Substituting this into (2.35),

$$\Gamma^\lambda{}_{(\mu\nu)}(x'^\mu) = \Gamma^\lambda{}_{(\mu\nu)}(x^\mu) - a^\lambda{}_{\mu\nu}\,. \tag{2.37}$$

Thus by choosing $\Gamma^\lambda{}_{(\mu\nu)}(x^\mu) = a^\lambda{}_{\mu\nu}$, there is always a coordinate transformation $x^\mu \to x'^\mu$ such that $\Gamma^\lambda{}_{(\mu\nu)}(x'^\mu)$, at least locally.

Note that this indicates only the symmetric part of the connection (which includes torsion) can be made null, while the anti-symmetric part (which is a tensor) cannot. Nonetheless since it is only the symmetric part of the connection that couples in the autoparallel equations (2.39), it is sufficient enough [42].

[3] Strongly note that the equivalence principle itself *does not* imply the existence of the metric. Only when one assumes the existence of a Lagrangian, can the metric be introduced, which then has to be specified that such metric is the dynamical space-time metric nonetheless. This topic will be extensively discussed in §2.4.3.3 and §B.1.2.

To summarize, the existence of a freely falling frame in a certain coordinate indicates it to be, in any coordinates, to be a tensorial equation with a certain non-tensorial matrix ${A^\lambda}_{\mu\nu}$ that has the transforming property (2.34). Since a general affine connection ${\Gamma^\lambda}_{\mu\nu}$ has a similar transformation law, one can assume that ${A^\lambda}_{\mu\nu}$ is the affine connection ${\Gamma^\lambda}_{\mu\nu}$, which leads to the autoparallel equation (2.39). Further assuming the integrability of this equation, i.e. the existence of a Lagrangian, computes it to be the geodesic equation (2.41) which may or may not be the space-time metric. One must note that, although the equivalence principle hints at the existence of a metric, it does not precisely lead to one, and there are many (hidden) assumptions in between.

#### 2.1.2.2 Dynamics of 'Standard' Matter

Whether matter, from the Standard Model, follows the derived geodesics is a question that is asked next. This boils down to the minimal coupling ansatz between the (covariantized) Standard Model and (not necessary Einstein) gravity. If the action of the flat space-time Standard Model is pedagogically written as

$$S_{\mathrm{SM,flat}} = \int d^4x \mathcal{L}_{\mathrm{SM}}(\Psi_{\mathrm{SM}}, \partial\Psi_{\mathrm{SM}})\,, \tag{2.43}$$

with $\Psi$ being the Standard Model particles, it can be shown that covariantized action under the minimal coupling ansatz, written as

$$S_{\mathrm{SM,curved}} = \int d^4x \sqrt{-g}\mathcal{L}_{\mathrm{SM}}(\Psi_{\mathrm{SM}}, \overset{g}{\nabla}\Psi_{\mathrm{SM}})\,, \tag{2.44}$$

computes the classical point particle motion to be geodesics when not interacting with other matter.

Since theories of matter, in general, are field theories and not point particles, one must use some approximation. To do so, one can refer to geometric optics and use WKB approximation.

The Standard Model consists of scalars, vectors, and spinors. Therefore, instead of approximating the whole model with brute force, let each particle be simplified and then analyzed for this section.

First, consider calculating the dynamics of a scalar field. The (dimensionful) Lagrangian is,

$$\mathcal{L} = \frac{1}{2}\phi^*\left(\overset{g}{\Box} - \frac{m^2}{\hbar^2}\right)\phi\,, \tag{2.45}$$

which computes the equation of motion for the scalar field,

$$\left\{\overset{g}{\Box} + \left(\frac{m}{-i\hbar}\right)^2\right\}\phi = 0\,. \tag{2.46}$$

Now consider the wave expansion,

$$\phi = \sum_n e^{\frac{i}{\hbar}S}(-i\hbar)^n a_n\,, \tag{2.47}$$

The equation will then be expanded as,

$$0 \;=\; \sum_n e^{\frac{i}{\hbar}S}(-i\hbar)^n\left[\{(\partial_\mu S)^2 + m^2\}a_n + (\Box S + 2\partial^\mu S\partial_\mu)a_{n-1} + \Box a_{n-2}\right]\,. \tag{2.48}$$

Let the wave number vector be defined as

$$k_\mu := \partial_\mu S\,. \tag{2.49}$$

At zeroth order, the equation of motion computes,

$$[k^\mu k_\mu + m^2]a_0 = 0\,. \tag{2.50}$$

Thus by defining the velocity as,

$$\frac{k_\mu}{m} = u_\mu + \mathcal{O}(\hbar)\,, \tag{2.51}$$

which is a time-like unit vector normalized as $u^\mu u_\mu = -1$, one can shown it follows geodesics, since

$$\begin{aligned} 0 &= u^\mu \overset{g}{\nabla}_\nu u_\mu \\ &= u^\mu \overset{g}{\nabla}_\nu \left(\frac{1}{m}\partial_\mu S\right) + \mathcal{O}(\hbar) \\ &= u^\mu \overset{g}{\nabla}_\mu \left(\frac{1}{m}\partial_\nu S\right) + \mathcal{O}(\hbar) \\ &= u^\mu \overset{g}{\nabla}_\mu u_\nu + \mathcal{O}(\hbar)\,. \end{aligned}$$

Thus the scalar field, follows the geodesic equation up to zeroth order in $\hbar$.[4]

Similarly, for the vector field , consider the Maxwell Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\,, \tag{2.55}$$

which computes the equation of motion of the vector field, under the Lorentz gauge $\overset{g}{\nabla} \cdot A = 0$,

$$\overset{g}{\Box} A^\mu + \overset{g}{R}{}^{\mu\nu} A_\nu = 0\,. \tag{2.56}$$

Now consider the following wave expansion,

$$A^\mu = \sum_n e^{\frac{i}{\hbar}S}(-i\hbar)^n a_n^\mu\,, \tag{2.57}$$

with $a_n^\mu$ being a vector that includes the polarization modes of the original $A^\mu$.

The equation of motion, under this wave expansion computes,

$$0 = \sum_n e^{\frac{i}{\hbar}S}(-i\hbar)^n \left[\{(\partial_\mu S)^2\}a_n^\mu + (\Box S + 2\partial^\mu S\partial_\mu)a_{n-1}^\mu + (\overset{g}{R}{}^\mu{}_\nu + \delta^\mu_\nu)\Box a_{n-2}^\nu\right]\,. \tag{2.58}$$

[4]If the scalar is charged, and couples to some (external) electromagnetic field $A_\mu$, the action is

$$\mathcal{L} = -\frac{1}{2}\left((D_\mu\phi)^2 + \frac{m^2}{\hbar^2}\phi^2\right)\,, \tag{2.52}$$

with $D_\mu = \partial_\mu + i\frac{q}{\hbar}A_\mu$, then one can use the same WKB approximation to obtain,

$$u^\nu \overset{g}{\nabla}_\nu u_\mu = \frac{q}{m}u^\nu F_{\nu\mu} + \mathcal{O}(\hbar)\,, \tag{2.53}$$

at zeroth order, with the unit velocity being

$$u_\mu = \frac{1}{m}\left[\partial_\mu S - qA_\mu\right]\,. \tag{2.54}$$

Indeed this is the well-known motion of a charged particle in a presence of an electromagnetic field. Therefore, one has to keep note that, geodesics are obtained when the test particles that follow are not interacting with (external) fields other than (minimally coupled) gravity.

Thus at zeroth order,

$$(k^\nu k_\nu)a_0^\mu = 0 \tag{2.59}$$

with the wave number vector defined as $k_\mu = \partial_\mu S$. As expected, since the vector is massless, the resultant wave number vector is a null vector[5]. Again, using similar computation computes,

$$\begin{aligned} 0 &= k^\mu \overset{g}{\nabla}_\nu k_\mu && (2.60) \\ &= k^\mu \overset{g}{\nabla}_\mu k_\nu + \mathcal{O}(\hbar)\,. && (2.61) \end{aligned}$$

Thus a minimally coupled vector too follows the geodesics at zeroth order of $\hbar$.

As for a spinor field, it will be discussed later on in the context of torsion coupling in §2.4.3.4. However, (even in the presence of torsion), spinors also follow geodesics for zeroth order in $\hbar$. Nonetheless, all Standard Model particles follow geodesics when their covariantization is taken with the minimal coupling ansatz (2.44).

One can also take a macroscopic point of view, on the other hand. Consider a perfect fluid, which energy-momentum tensor takes the form,

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}\,, \tag{2.62}$$

with $\rho$ being the energy density and $p$ being pressure. Since energy-momentum tensor is conserved,

$$\begin{aligned} 0 &= \overset{g}{\nabla}_\mu T^{\mu\nu} \\ &= (\rho+p)u^\mu \overset{g}{\nabla}_\mu u^\nu + u^\nu \overset{g}{\nabla}_\mu\{(\rho+p)u^\mu\} + \overset{g}{\nabla}{}^\nu p\,, \end{aligned} \tag{2.63}$$

Taking a contraction with $u_\nu$, one obtains,

$$\overset{g}{\nabla}_\mu\{(\rho+p)u^\mu\} = u^\mu \overset{g}{\nabla}_\mu p\,. \tag{2.64}$$

Substituting this into (2.63), the final result is,

$$u^\mu \overset{g}{\nabla}_\mu u^\nu = -\frac{1}{\rho+p}\gamma^{\mu\nu}\overset{g}{\nabla}_\mu p \tag{2.65}$$

where the projection tensor $\gamma_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ was defined. This shows that when the pressure does not have a spatial gradient, perfect fluid follows geodesics. Moreover, pressureless dust follows geodesics.

To conclude, when the Standard Model of particle physics is minimally coupled to gravity, the non-interacting particles follow geodesics at zeroth order of $\hbar$ *regardless of the gravitational equations*. The gravitational equations govern the deformation of the metric that the geodesics follow and not the structure of the geodesic equations itself. On the other hand, even if one takes gravity as Einsteins and matter as the Standard Model particles, if the coupling between them is some non-trivial, the equivalence principle *may* be violated. Thus it is important that when one goes beyond the Standard Model of cosmology, the gravity-matter coupling must also be re-investigated.

[5]Similar calculations can be conducted for massive (Proca) vectors and although in such case the wave number vector becomes time-like, the calculations that precede do not differ.

## 2.2 The Unsolved Problems of the Universe

Although the current Standard Model of cosmology explained countless observations throughout the ages, some phenomena elude the predictability of the celebrated model. In this chapter, the giants of these incomprehensible problems will be reviewed. Firstly, the inflation mechanism, which questions the beginning of the universe, will be shown. After introducing the problems of the Standard Model of cosmology it solves, cosmological perturbation theory and its quantization will be used to deduce the observational parameters. Then dark energy, questioning the end of the universe, will be introduced. Walking through the observed late-time expansion and the theoretical severity of the cosmological constant, recent observational progress will then be examined. Finally, other cosmological problems, which are not directly related to this thesis, will be qualitatively inquired.

### 2.2.1 Inflation

#### 2.2.1.1 Inflation Preliminaries

From the Cosmic Microwave Background (CMB) observations [44] it could be said that the universe was highly homogeneous during the early epoch of time. However, currently, the observable universe is so large that the two arbitrary edges of the universe must be causally independent. To see this consider introducing a useful parameter called the conformal time, which is defined as

$$\begin{aligned} \tau &\equiv \int_0^t \frac{dt'}{a(t')} \\ &= \int_0^a d\ln a \left(\frac{1}{aH}\right) . \end{aligned} \tag{2.66}$$

The $\frac{1}{aH}$ that was factored in is called the comoving Hubble radius which shows the causally connected regions of our universe. Now using the Friedmann equations (2.17), the comoving Hubble radius can be shown to be,

$$\frac{1}{aH} = H_0^{-1} a^{\frac{1}{2}(1+3w)} . \tag{2.67}$$

Thus the conformal time has the following relation with the scale factor.

$$\tau \propto a^{\frac{1}{2}(1+3w)} \tag{2.68}$$

$$\propto \begin{cases} a & \text{RD} \\ a^{\frac{1}{2}} & \text{MD} \end{cases} , \tag{2.69}$$

where RD stands for *radiation dominant* with $w = \frac{1}{3}$ and MD stands for *matter dominant* with $w = 0$.

This implied that the comoving horizon is monotonically increasing for general matter. Thus the observed waves (of photons) that enter the horizon *now* was outside of the horizon during the CMB. Therefore, for the CMB to have such a high homogeneity, the universe during decoupling must have been even also highly homogeneous. This is called the horizon problem.

Let the causally independent regions be estimated through the temperature dependence of the Hubble parameter and scale factor. For the following $c$ and $\hbar$ is reintroduced for clarity. Recall that a distribution for relativistic momentum is

$$f(p) = \left(\exp\left[\frac{E(p) \pm \mu}{k_B T}\right]\right)^{-1} \begin{cases} + & \text{Fermi-Dirac} \\ - & \text{Bose-Einstein} \end{cases} , \tag{2.70}$$

with $E(p) = \sqrt{p^2c^2 + m^2c^4}$, $\mu$ being the chemical potential and $k_B$ being the Boltzmann factor. Then the energy density for $g$ degrees of freedom can be calculated as,

$$\rho = \frac{g}{(2\pi\hbar)^3}\int E(p)f(p)d^3p \tag{2.71}$$

$$= \frac{g(k_BT)^4}{2\pi^2c^3\hbar^3}\int_0^\infty \frac{\sqrt{x}\sqrt{x+\frac{2mc^2}{k_BT}}\left(x+\frac{mc^2}{k_BT}\right)^2}{\exp\left[x+\frac{mc^2-\mu}{k_BT}\pm 1\right]}dx\,. \tag{2.72}$$

Now consider the relativistic limit in which $k_BT \gg mc^2$ and $k_BT \gg |\mu|$. Then one obtains the relation of energy density with temperature as,

$$\rho = \frac{\pi^2 {k_B}^4}{30c^3\hbar^3}T^4 \times \begin{cases} \frac{7}{8} & \text{Fermi-Dirac} \\ 1 & \text{Bose-Einstein} \end{cases}\,. \tag{2.73}$$

If one considers the universe to be at thermal equilibrium, all of the particles have the same temperature. Thus by summing up the energy density for all particles,

$$\rho_\Gamma = \frac{\pi^2 k_B^4}{30c^3\hbar^3}g_*T^4\,, \tag{2.74}$$

where $g_*$ is called the effective degrees of freedom and defined as,

$$g_* = \sum_{b:bosons} g_b + \frac{7}{8}\sum_{f:fermions} g_f\,, \tag{2.75}$$

The Hubble parameter thus become a function of temperature as,

$$H = \sqrt{\frac{4\pi^3 G{k_B}^4 g_*}{45c^5\hbar^3}}T^2\,. \tag{2.76}$$

Since during the radiation dominative epoch $\rho \propto a^{-4}$, the scale factor is ,

$$\frac{a(t)}{a_0} = \frac{1}{g_*^{\frac{1}{4}}(t)T}\,. \tag{2.77}$$

If the big bang singularity was at GUT scale and thus $10^{15}$GeV and noting that the big bang nucleosynthesis is 0.1MeV, the ratio between the size of the universe, the comoving Hubble radius is,

$$\frac{1/aH|_{\mathrm{GUT}}}{1/aH|_{\mathrm{BBN}}} \sim 10^{19}\,, \tag{2.78}$$

with assuming that the degree of freedoms is approximately the same. Thus are the causally independent regions.

The other problem is the flatness problem. Recall that in §2.1.1, the energy density of spatial curvature was defined as,

$$\Omega_k = -\frac{k}{a^2H_0^2}\,,$$

and for the current observations it is constrained as,

$$\Omega_k = \begin{cases} -0.044^{+0.018}_{-0.015} & \text{Planck TT,TE,EE +low E} \\ 0.0007 \pm 0.0019 & \text{Planck TT,TE,EE +low E+lensing+BAO} \end{cases}\,, \tag{2.79}$$

each for 68% confidentiality of the combination analysis [4]. This indicates that the universe is (extremely) flat, which suggests that the Big Bang cosmology must be extremely unstable. To see this recall the Friedmann equations were

$$H^2 = \frac{8\pi G}{3}\sum_i \rho_i - \frac{k}{a^2}\,. \tag{2.80}$$

Defining $\Omega_i = \frac{8\pi G}{3}\frac{\rho_i}{H^2}$ ,

$$1 - \sum_i \Omega_i = -\frac{k}{(aH)^2}\,,$$

Taking the derivative of this equation with respect to $\ln a$,

$$\sum_i \frac{d\Omega_i}{d\ln a} = \frac{k}{(aH)^2}\sum_i (1+3w_i)\,\Omega_i\,,$$

and thus the (near) flat solution becomes unstable. Furthermore direct calculations [17] indicate that

$$|\sum_i \Omega_i(t_{\text{Big Bang Nucleosynthesis}}) - 1| \leq \mathcal{O}(10^{-16})\,, \tag{2.81}$$

$$|\sum_i \Omega(t_{GUT}) - 1| \leq \mathcal{O}(10^{-55})\,, \tag{2.82}$$

$$|\sum_i \Omega(t_{Planck}) - 1| \leq \mathcal{O}(10^{-61})\,. \tag{2.83}$$

Thus needs to be fine-tuned severely. This is called the flatness problem.

To solve these two Cauchy problems, the inflation theory was proposed[29]. This is simply done by considering that the comoving Hubble radius $\frac{1}{(aH)}$ is not increasing, which is what causes both of the problems, but actually was decreasing at the early epoch. For example, the horizon problem could be directly solved by allowing the comoving Hubble radius decrease into the observable universe scale and thus causally connecting the two arbitrary edges. The flatness problem could be solved through this mechanism where the comoving Hubble radius decreases. Since

$$|1 - \sum_i \Omega_i| = \frac{k}{(aH)^2}\,,$$

$\sum_i \Omega_i = 1$ becomes an attractor instead of an unstable point and thus solving the flatness problem. Without explicit dynamical calculations, it seems that the decreasing comoving Hubble radius solves the two problems of the Big Bang theory.

Now consider the necessary conditions of causing inflation. Since during inflation, it must be

$$\frac{d}{dt}\left(\frac{1}{(aH)}\right) < 0\,. \tag{2.84}$$

It can be calculated as

$$\frac{da}{dt} > 0\,. \tag{2.85}$$

Thus indicating the universe must have an accelerated expansion. During such a condition the Friedmann equation can be re-written as,

$$\frac{\ddot{a}}{a} = H^2\,(1-\varepsilon)\,. \tag{2.86}$$

With the *Hubble slow-roll parameter* $\varepsilon$ defined as,

$$\begin{aligned}\varepsilon &\equiv -\frac{\dot{H}}{H^2}\,,\\ &= -\frac{d\ln H}{dN}\,,\end{aligned}$$

with $dN \equiv Hdt = d\ln a$ is the parameter of necessary e-folds of inflation. Thus, the conditions of inflation is re-expressed as,

$$\varepsilon < 1\,. \tag{2.87}$$

Furthermore, from the Friedmann equations, during an accelerated expansion,

$$\rho + 3p \quad < \quad 0\,, \tag{2.88}$$

and thus pressure must be negative. Such is called the violation of the strong energy condition. All of these conditions are necessary conditions of inflationary dynamics.

Now for simplicity and the upcoming analysis consider an inflationary mechanism where a scalar causes the inflation. Such scalar is called the inflaton. Taking $8\pi G = 1$ for simplicity, consider the following action of,

$$\begin{aligned}S &= \int d^4x\sqrt{-g}\left[\frac{1}{2}\overset{g}{R} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]\\ &= S_{\rm EH} + S_\phi\,.\end{aligned}$$

Here $S_{EH}$ is the Einstein Hilbert action and $S_\phi$ is the canonical kinetic term of the considered scalar field. Since the energy-momentum of the scalar field is

$$\begin{aligned}T^{(\phi)}_{\mu\nu} &\equiv -\frac{2}{\sqrt{-g}}\frac{\delta S^\phi}{\delta g^{\mu\nu}}\\ &= \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}\partial^\sigma\phi\partial_\sigma\phi + V(\phi)\right)\,, \end{aligned}\tag{2.89}$$

and is the form of a perfect-fluid energy-momentum tensor, one can defined the energy density and pressure as,

$$\begin{aligned}\rho_\phi &= \frac{1}{2}\dot{\phi}^2 + V(\phi)\,,\\ p_\phi &= \frac{1}{2}\dot{\phi}^2 - V(\phi)\,,\end{aligned}$$

and its corresponding equation of state parameter as,

$$w_\phi \equiv \frac{p_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}\,. \tag{2.90}$$

The Friedmann equations for this scalar field can be re-arranged as

$$\frac{\ddot{a}}{a} = H^2\left[1 - \frac{3}{2}\left(1 + w_\phi\right)\right]\,,$$

Thus the *Hubble slow-roll parameter* has the relation,

$$\varepsilon = \frac{3}{2}(w_\phi + 1) = \frac{3}{2}\frac{\dot{\phi}^2}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}\,, \tag{2.91}$$

The left-hand side of this equation is purely geometrical while the right-hand is solely decided from the scalar field.

Since the condition of inflation was $\epsilon < 1$, for the scalar to induce a sufficient enough accelerated expansion,

$$1 \gg \frac{3}{2}\frac{\dot{\phi}^2}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}\,, \tag{2.92}$$

or simply,

$$\dot{\phi}^2 \ll V(\phi)\,. \tag{2.93}$$

Such analysis indicates that during inflation, the scalar has much more potential energy than kinetic energy. Furthermore, another slow-roll parameter could be defined to evaluate inflationary mechanisms. Since inflation must last for a sufficient time, it must be that

$$|\ddot{\phi}| \ll |3H\dot{\phi}|, |V_{,\phi}|\,. \tag{2.94}$$

Therefore a new parameter could be defined by using $\varepsilon$ as

$$\eta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}} \tag{2.95}$$

$$= \varepsilon - \frac{1}{2\varepsilon}\frac{d\varepsilon}{dN}\,. \tag{2.96}$$

Thus the two *Hubble slow-roll parameters* are defined and derived.

When one is deep within the inflationary regime, one could also introduce *potential slow-roll parameters* defined as,

$$\epsilon_V(\phi) \equiv \frac{1}{2}\left(\frac{V_{,\phi}}{V}\right)^2\,, \tag{2.97}$$

$$\eta_V(\phi) \equiv \frac{V_{,\phi\phi}}{V}\,, \tag{2.98}$$

These parameters are

$$\epsilon_V, |\eta_V| \ll 1\,, \tag{2.99}$$

during inflation and satisfy the approximation of

$$H^2 \approx \frac{1}{3}V(\phi) \approx \text{const.} \tag{2.100}$$

$$\dot{\phi} \approx -\frac{V_{,\phi}}{3H}\,. \tag{2.101}$$

During inflation the (Hubble) slow-roll parameters and the potential slow-roll parameters have the relation of

$$\varepsilon \approx \epsilon_V\,, \tag{2.102}$$

$$\eta \approx \eta_V - \epsilon_V\,, \tag{2.103}$$

and inflation ends when both sets of parameters satisfy

$$\varepsilon(\phi_{end}) \equiv 1, \quad \epsilon_V(\phi_{end}) \approx 1\,, \tag{2.104}$$

Thus could be used on different occasions; Hubble slow-roll parameters are used when the scalar dynamics are discussed, whereas potential slow-roll parameters are used when the potential is given. For a comprehensive review of many models of inflation, see for example [45, 46]

As an example consider $V(\phi) = \frac{1}{2}m^2\phi^2$ which is also called chaotic inflation and is one of the most simple inflationary models [45]. The potential slow-roll parameters could be derived straight forward as

$$\epsilon_V(\phi) = \frac{2}{\phi^2}\,, \tag{2.105}$$

$$\eta_V(\phi) = \frac{2}{\phi^2}\,. \tag{2.106}$$

Thus in order for inflation to last the condition is $\epsilon_V, |\eta_V| < 1$ and by reintroducing the Planck mass $M_{pl}$ by dimensional analysis the condition becomes

$$\phi > \sqrt{2}M_{pl} \equiv \phi_{end}\,. \tag{2.107}$$

Thus the e-folds would be

$$N(\phi) \approx \frac{\phi^2}{4{M_{pl}}^2} - \frac{1}{2}\,, \tag{2.108}$$

and to compare with the necessary e-folds of CMB, it could be approximated as$N(\phi) \approx \frac{\phi^2}{4{M_{pl}}^2}$. Therefore

$$\phi_{CMB} = 2\sqrt{N_{CMB}}M_{pl} \sim 14.1 - 15.5M_{pl}\,, \tag{2.109}$$

allowing the field value of the inflaton to be constrained. Here $N_{CMB}$ is taken as $50 \sim 60$. [6]

#### 2.2.1.2 Linear Perturbation Theory in a Single Field Scalar Theory

The following two sections are dedicated to deriving the observational variables of the CMB, namely spectral index and the tensor-to scalar ratio.
Instead of starting off with the usual perturbed metric, consider one using conformal time defined as,

$$\eta \sim \int_0^t \frac{dt}{a(t)}\,, \tag{2.111}$$

with the conformal Hubble parameter defined as $\mathcal{H} = \frac{1}{a}\frac{da}{d\eta} = aH$. The metric would then be

$$ds^2 = a(\eta)^2\left[-(1+2A)d\eta^2 - 2B_i d\eta dx^i + (\gamma_{ij} + 2N_{ij})dx^i dx^j\right]\,. \tag{2.112}$$

[6] The number of e-folds differ among models. However it is common to take about 50 to 60. Recall that $a \sim T^{-1}$ from (2.77). Thus, for example, the ratio between GUT scale $10^{15}$GeV and recombination 0.4eV, is

$$N \sim \ln\frac{a_{rec}}{a_{GUT}} \sim 55. \tag{2.110}$$

Again, consider a single scalar field with an energy-momentum tensor of

$$T^\mu_\nu = \partial^\mu\phi\partial_\nu\phi - \delta^\mu_\nu\left[\frac{1}{2}\partial^\sigma\phi\partial_\sigma\phi + V(\phi)\right] . \tag{2.113}$$

Perturbation of the scalar could be written as,

$$\phi(\eta, x^i) = \bar{\phi}(\eta) + \delta\phi(\eta, x^i) , \tag{2.114}$$

and thus the energy-momentum tensor could decomposed into,

$$T^\mu_\nu = \bar{T}^\mu_\nu + \delta T^\mu_\nu , \tag{2.115}$$

where

$$\bar{T}^\mu_\nu = \partial^\mu\bar{\phi}\partial_\nu\bar{\phi} - \delta^\mu_\nu\left[\frac{1}{2}\partial^\sigma\bar{\phi}\partial_\sigma\varepsilon\phi + V(\bar{\phi})\right] , \tag{2.116}$$

$$\delta T^\mu_\nu = \delta g^{\mu\lambda}\partial_\lambda\bar{\phi}\partial_\nu\bar{\phi} + 2\bar{g}^{\mu\lambda}\left(\partial_{(\lambda}\delta\phi\partial_{\nu)}\bar{\phi}\right) - \delta^\lambda_\nu\left[\frac{1}{2}\delta g^{\sigma\lambda}\partial_\lambda\bar{\phi}\partial_\sigma\bar{\phi} + \bar{g}^{\sigma\lambda}\partial_\lambda\bar{\phi}\partial_\sigma\delta\phi + V_{,\phi}\delta\phi\right] , \tag{2.117}$$

Now consider defining the Sasaki-Mukhanov variable,

$$u \equiv a\left(\delta\phi - \frac{\bar{\phi}'}{\mathcal{H}}\psi\right) , \tag{2.118}$$

which is an gauge invariant variable since from $\delta\phi \to \delta\phi - \bar{\phi}'T$ and $\psi \to \psi - \mathcal{H}T$,

$$u \to \tilde{u} = a\left\{\delta\phi - \bar{\phi}'T - \frac{\bar{\phi}'}{\mathcal{H}}(\psi - \mathcal{H}T)\right\} = u . \tag{2.119}$$

Constructing equations using gauge-invariant variables allows one to fix gauges without gauge-dependent modes causing instability.
Furthermore, the relation between the Sasaki-Mukhanov variable and the curvature perturbation could be written as

$$u = -\frac{a\bar{\phi}'}{\mathcal{H}}\zeta \tag{2.120}$$

From what follows, consider a spatially flat gauge $\psi = 0$ in which the Sasaki-Mukhanov variable could be written as,

$$u = a\delta\phi \tag{2.121}$$

Under such gauge, the linearized Einstein equations become,

$$\delta\phi'' + 2\mathcal{H}\delta\phi' - (\Delta - a^2V_{,\phi\phi})\delta\phi - \frac{1}{a^2}\left(\frac{a^2\bar{\phi}'^2}{\mathcal{H}}\right)'\delta\phi = 0 \tag{2.122}$$

Rewriting this using the Sasaki-Mukhanov variables result in

$$u'' - \left(\Delta - a^2V_{,\phi\phi} + \frac{a''}{a}\right)u - \frac{1}{a^2}\left(\frac{a^2\bar{\phi}'^2}{\mathcal{H}}\right)'u = 0 . \tag{2.123}$$

Such equation could be computed from an action of the form,

$$S = \frac{1}{2}\int d\eta d^3x\left[u'^2 - \partial^i u\partial_i u + \left\{-a^2V_{,\phi\phi} + \frac{a''}{a} + \frac{1}{a^2}\left(\frac{a^2\bar{\phi}'^2}{\mathcal{H}}\right)'\right\}\right] . \tag{2.124}$$

which is by construction, gauge invariant.

#### 2.2.1.3 Quantization of the Scalar Field and it's Quantum Fluctuations

As a next step, let the quantization of the action (2.124) be considered.

The conjugate momentum $\pi$ of the scalar fluctuation $u$ is computed as,

$$\pi(\eta, x^i) = \frac{\partial \mathcal{L}}{\partial u'} = u'(\eta, x^i)\,. \tag{2.125}$$

The canonical quantization of this fluctuation is,

$$\big[u(\eta, x^i), u(\eta, y^i)\big] = \big[\pi(\eta, x^i), \pi(\eta, y^i)\big] = 0\,, \tag{2.126}$$

$$\big[u(\eta, x^i), \pi(\eta, y^i)\big] = i\delta^3(x^i - y^i)\,. \tag{2.127}$$

With keeping the above in mind, one may see that the action (2.124), is none other than a harmonic oscillator with the effective mass of

$$2m_{eff} := \frac{a''}{a} + \frac{1}{a^2}\left(\frac{a^2\bar{\phi}'^2}{\mathcal{H}}\right)'\,. \tag{2.128}$$

Thus it is possible to expand the fluctuation with respect to plane waves, as

$$u(\eta, x^i) = \int \frac{d^3k}{(2\pi)^3}\left[u_k(\eta)a(k^i)e^{ik^ix_i} + u_k^*(\eta)a^\dagger(k^i)e^{-ik^ix_i}\right]\,, \tag{2.129}$$

where $k = |k^i|$. By finding out the commutation relation between the annihilation and creation operators $a(k^i)$ and $a^\dagger(k^i)$, one may define a well-defined vacuum.

The equation of motion that $u_k$ follows is,

$$u_k'' + \left(k^2 + a^2V_{,\phi\phi} - \frac{a''}{a}\right)u_k - \frac{1}{a^2}\left(\frac{a^2\bar{\phi}'^2}{\mathcal{H}}\right)'u_k = 0\,. \tag{2.130}$$

Note that since this equation is linear with respect to $u_k$, one must normalized the solution. It is common to take the Wronskian condition of

$$u_k\frac{\partial u_k^*}{\partial \eta} - \frac{\partial u_k}{\partial \eta}u_k^* = i\,. \tag{2.131}$$

The next step is to consider a scalar product defined as,

$$(\phi_1, \phi_2) = -i\int d^3x\left[\phi_1(\eta, x^i)\frac{\partial \phi_2^*(\eta, x^i)}{\partial \eta} - \frac{\partial \phi_1(\eta, x^i)}{\partial \eta}\phi_2^*(\eta, x^i)\right]\,, \tag{2.132}$$

for which, due to the Wronskian condition, the fluctuation is simply,

$$(u_k, u_k) = 1\,. \tag{2.133}$$

This shows that the operators could be expressed as,

$$a(k^i) = \left(u, u_ke^{ik^ix_i}\right)\,, \tag{2.134}$$

$$a^\dagger(k^i) = -\left(u, u_k^*e^{-ik^ix_i}\right)\,. \tag{2.135}$$

Thus the commutation relation for the operators could be calculated as,

$$\left[a(k_1^i), a^\dagger(k_2^i)\right] = (2\pi)^3\delta^3(k_1^i - k_2^i)\,, \tag{2.136}$$

$$\left[a(k_1^i), a(k_2^i)\right] = \left[a^\dagger(k_1^i), a^\dagger(k_2^i)\right] = 0\,. \tag{2.137}$$

The vacuum is then defined as,

$$a(k^i)\,|0\rangle = 0\,, \tag{2.138}$$

and indeed $a(k^i)$ becomes the annihilation operator.

Now, using the Fourier transformed fluctuation $\bar{u}(\eta k_a^i)$, define the power spectrum of the fluctuation as,

$$\left\langle 0\left|\bar{u}(\eta, k_1^i)\bar{u}(\eta, k_2^i)\right| 0\right\rangle = (2\pi)^3\delta^3(k^1 - k_2^i)P_u(\eta, k)\,, \tag{2.139}$$

with using the commutation relation, implies that,

$$P_u(\eta, k) = |u_k(\eta)|^2\,. \tag{2.140}$$

Noting the relation between Sasaki-Mukhanov variable and curvature fluctuation was (2.120), the power spectrum of the curvature perturbation could be written as,

$$P_\zeta(\eta, k) = \frac{\mathcal{H}^2}{a^2\bar{\phi}'^2}|u_k|^2\,. \tag{2.141}$$

Under the slow-roll approximation $|\bar{\phi}'' - \mathcal{H}\bar{\phi}'| \ll 3 - \mathcal{H}\bar{\phi}'$, the scalar field equation of motion and the Friedmann equations compute,[7]

$$\bar{\phi} \simeq -\frac{a^2 V_{,\phi}}{3\mathcal{H}}\,, \tag{2.144}$$

$$\mathcal{H}^2 = \frac{1}{3}a^2 V\,, \tag{2.145}$$

Inserting this into (2.130), the equation of motion of the perturbation under slow-roll conditions are computed as,

$$u_k'' + \left[k^2 - \frac{2 + 9\epsilon_V - 3\eta_V}{\eta^2}\right]u_k = 0\,. \tag{2.146}$$

Further defining a new variable,

$$\nu := \frac{3}{2} + 3\epsilon_V - \eta_V\,, \tag{2.147}$$

$$F_k := \frac{1}{\sqrt{-\eta}}u_k\,, \tag{2.148}$$

[7]These two equation could be used to give,

$$\mathcal{H}' = \mathcal{H}^2(1 - \epsilon_V)\,. \tag{2.142}$$

Since, under the slow-roll approximation $\epsilon_V$ could be considered constant, this could be solved as,

$$\mathcal{H} = -\frac{1}{(1 - \epsilon_V)\eta} \tag{2.143}$$

the perturbation equation becomes the form of

$$z^2\frac{d^2F_k}{dz^2} + z\frac{F_k}{dz} + (z^2 - \nu^2)F_k = 0\,, \tag{2.149}$$

with $z = -k\eta$. This is none other than the Bessel differential equation, and the solution is known to be,

$$J_\nu(z) = \left(\frac{z}{2}\right)^2 \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(\nu+k+1)} \left(\frac{z}{2}\right)^2 k\,, \tag{2.150}$$

$$N_\nu(z) = \frac{\cos(\nu\pi)J_\nu(z) - J_{-\nu}(z)}{\sin(\nu\pi)}\,, \tag{2.151}$$

with the Gamma function defined as $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$.

It is more convenient to use the following set of solutions of,

$$H_\nu^{(1)}(z) = J_\nu(z) + iN_\nu(z)\,, \tag{2.152}$$

$$H_\nu^{(2)}(z) = J_\nu(z) - iN_\nu(z)\,, \tag{2.153}$$

which is also known to be called the Hankel functions. Thus the general solution of the fluctuation $u_k(\eta)$ is solved as,

$$u_k(\eta) = \sqrt{-\eta}\left[\alpha_k H_\nu^{(1)}(-k\eta) + \beta_k H_\nu^{(2)}(-k\eta)\right]\,, \tag{2.154}$$

where the constants are constrained from the Wroskian conditon of

$$|\alpha_k|^2 - |\beta_k|^2 = \frac{4}{\pi}\,. \tag{2.155}$$

Finally, in order to find the value of these constant, consider a limit of the conformal time to the past as $(-k\eta \to -\infty)$. The perturbation equation (2.146) is then,

$$u_k'' + k^2u_k = 0\,. \tag{2.156}$$

The solution, that satisfies the Wronskian condition, is

$$u_k(\eta) = \frac{1}{\sqrt{2k}}e^{-ik\eta} \text{ where } (k\eta \to -\infty)\,. \tag{2.157}$$

The past limit of the Hankel functions are also given as,

$$H_\nu^{(1)}(z) = \sqrt{\frac{2}{\pi z}}e^{i\left(z - \frac{\pi}{4} - \frac{\pi\nu}{2}\right)}\,, \tag{2.158}$$

$$H_\nu^{(2)}(z) = \sqrt{\frac{2}{\pi z}}e^{-i\left(z - \frac{\pi}{4} - \frac{\pi\nu}{2}\right)}\,. \tag{2.159}$$

(2.160)

For the solution of the fluctuation (2.154) to hold at the past limit the constant are decided as,

$$\alpha_k = \sqrt{\frac{\pi}{4}}e^{\frac{i\pi}{2}\left(\nu+\frac{1}{2}\right)}\,, \tag{2.161}$$

$$\beta_k = 0\,. \tag{2.162}$$

Such vacuum that is defined through this procedure is called the Bunch-Davies vacuum .
Thus the final solution for the fluctuations are,

$$u_k(\eta) = \sqrt{-\frac{\pi\eta}{4}}e^{\frac{i\pi}{2}\left(\nu+\frac{1}{2}\right)}H_\nu^{(1)}(-k\eta)\,, \tag{2.163}$$

with $\nu = \frac{3}{2} + 3\epsilon_V - \eta_V$

#### 2.2.1.4 Observational variables and constraints from the Cosmic Microwave Background

Now consider the future limit with $z = -k\eta \to 0$. In such limit, the Hankel function of the first kind is approximated as,

$$H_\nu^{(1)}(z) \to \sqrt{2\pi}\frac{\Gamma(\nu)}{\Gamma\left(\frac{3}{2}\right)}e^{-i\frac{\pi}{2}}2^{\nu-\frac{3}{2}}z^{-\nu}\,. \tag{2.164}$$

Therefore,

$$u_k(\eta) \to \frac{1}{\sqrt{2k}}\frac{\Gamma(\nu)}{\Gamma\left(\frac{3}{2}\right)}e^{-i\frac{\pi}{2}\left(\nu+\frac{1}{2}\right)}2^{\nu-\frac{3}{2}}z^{-\nu+\frac{1}{2}}\,. \tag{2.165}$$

Under slow-roll approximation, the following holds,

$$\begin{aligned} 2^{\nu-\frac{3}{2}} &\simeq 1+(3\epsilon_V-\eta_V)\ln 2\,, \\ \Gamma(\nu) &\simeq \Gamma\left(\frac{3}{2}\right)\left[1+(3\epsilon_V-\eta_V)\psi\left(\frac{3}{2}\right)\right]\,, \end{aligned}$$

with the digamma function approximated as $\psi\left(\frac{3}{2}\right) \simeq 1.19$. Furthermore, since from (2.143) $-k\eta = (1-\epsilon_V)^{-1}\left(\frac{k}{\eta}\right)$, so

$$(-k\eta)^{\frac{1}{2}-\nu} \simeq (1-\epsilon_V)\left(\frac{k}{\mathcal{H}}\right)^{-1-3\epsilon_V+\eta_V}\,. \tag{2.166}$$

Thus the fluctuation, under the first-order of slow-roll parameters, are,

$$u_k(\eta) \simeq \frac{1}{\sqrt{2k}}e^{\frac{i\pi}{2}(1+3\epsilon_V-\eta_V)}\left(\frac{k}{\mathcal{H}}\right)^{-1-3\epsilon_V+\eta_V}\,. \tag{2.167}$$

Inserting this into (2.141)(2.144)(2.145), one finds the power spectrum of the scalar perturbations as,

$$P_\zeta(k) = \frac{H^2}{4k^3\epsilon_V}\left(\frac{k}{\mathcal{H}}\right)^{-6\epsilon_V+2\eta_V}\,. \tag{2.168}$$

It is sometimes convenient to use the dimensionless power spectrum defined as,

$$\mathcal{P}_\zeta(k) \equiv \frac{k^3}{2\pi^2}P_\zeta(k) = \frac{1}{2\epsilon_V}\left(\frac{H}{2\pi}\right)^2\left(\frac{k}{\mathcal{H}}\right)^{-6\epsilon_V+2\eta_V}\,, \tag{2.169}$$

By taking an appropriate wavelength, and by using $H \sim \frac{1}{3}V$, the amplitude for the scalar perturbation is determined fully by the potential as,

$$P_\zeta(k=\mathcal{H}) = \frac{V}{24\pi^2\epsilon_V}\,. \tag{2.170}$$

Defining the spectral index of the scalar perturbations as

$$\mathcal{P}_\zeta(k) \propto k^{n_s-1}\,, \tag{2.171}$$

the spectral index could be written purely with terms of the potential slow-roll parameters as,

$$n_s = 1-6\epsilon_V+2\eta_V\,, \tag{2.172}$$

which is an important observational variable of the CMB.
Following the same procedure for the tensor perturbations, one could obtain the power spectrum

$$P_T(\eta, k) = \frac{4H^2}{k^3}\left(\frac{k}{\mathcal{H}}\right)^{-2\epsilon_V} . \tag{2.173}$$

The spectral index for the tensor perturbation $n_T$ could also be defined as,

$$\mathcal{P}_T(k) \propto k^{n_T} , \tag{2.174}$$

with the dimensionless power spectrum defined as $\mathcal{P}_T(k) := \frac{k^3}{2\pi^2} P_T(k)$
Finally, define and compute the tensor-to-scalar ratioas

$$r := \frac{\mathcal{P}_T(k)}{\mathcal{P}_\zeta(k)} = 16\epsilon_V , \tag{2.175}$$

which is also an important observationally variable from the CMB. In Fig. 2.2 the most recent Planck 2018 results and the constraints on inflationary models are shown from [1].[8] For example, the chaotic inflation model considered earlier with the potential $V(\phi) = \frac{1}{2}m^2\phi^2$, can be seen to be excluded from observations.

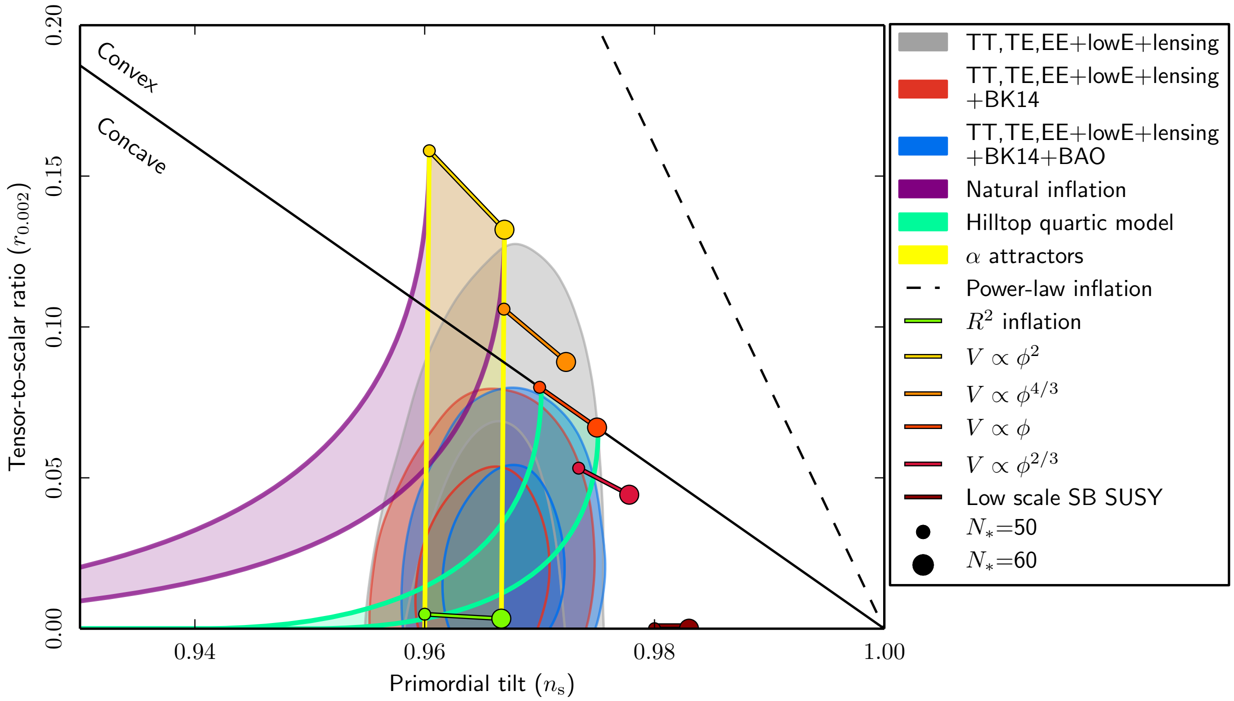


Figure 2.2: Planck Constraints on Inflation, taken from [1]

[8]One may wonder how the spectra of the early universe are precisely the one we observe today. This is ensured through conservation of curvature perturbation which is a direct consequence of energy-momentum conservation [47, 48].

### 2.2.2 Dark Energy

The first evidence for late-time accelerated cosmological expansion was found in 1998 by A. G.Riess et al. [24] through Type Ia supernova, and later was confirmed by numerous observational data. However, the origin and theoretical evidence of this expansion are still unknown. One could consider an exotic type of matter to obtain this accelerated expansion. Such substance is called dark energy and the approach to solve the fundamental background is called the dark energy problem. For this section, after showing observational evidence for such late-time accelerated expansion, the origin of this dark energy will be questioned. Then recent observations that are implied to be closely tied to dark energy will be introduced.

#### 2.2.2.1 Observational Implication of Late-time Accelerated Expansion

Here the explanation of observational evidence, especially the luminosity distance, would be conducted.
First recall that the dimensionless Friedmann equation could be written as (2.30),

$$\frac{H(z)^2}{H_0^2} = \Omega_{\gamma 0}(1+z)^4 + \Omega_{m0}(1+z)^3 + \Omega_{\Lambda 0} + \Omega_{k0}(1+z)^2 \,. \tag{2.176}$$

Where $\Omega_{\Lambda 0} = 1 - \Omega_{\gamma 0} - \Omega_{m0} - \Omega_{k0}$ By using the relation

$$dt \quad = \quad -(1+z)H dz \,. \tag{2.177}$$

The age of the universe could be calculated as

$$T \quad = \quad \int_0^\infty \frac{dz}{(1+z)H(z)} \,. \tag{2.178}$$

Since in the current epoch both radiation and spatial curvature could be neglected, $T$ could be approximated as

$$T \quad = \quad \frac{1}{3H_0\sqrt{\Omega_{\Lambda 0}}} \ln\left(\frac{1+\sqrt{\Omega_{\Lambda 0}}}{1-\sqrt{\Omega_{\Lambda 0}}}\right) \,. \tag{2.179}$$

When $\Omega_{\Lambda 0}$ is taken to be zero, thus in a universe without the cosmological constant, the current age of the universe becomes

$$T = \frac{2}{3H_0} \,. \tag{2.180}$$

Defining $H_0 = 100h\text{kms}^{-1}\text{Mpc}^{-1}$, one can calculate $H_0$ to be $\frac{1}{H_0} \sim 3.1 \times 10^{17} h^{-1}\text{s}$, the universe would be around the order of $10^{10}$ or ten billion years. This is inconsistent with the order of the galactic age which is around the same order, with some exceeding the supposed universe's age.
Now consider supernova observations. When the spectrum of a certain supernova *does not* include a hydrogen line this type is called Type I. Furthermore, when the spectrum includes an absorption line of silicon it is called Type Ia. Type Ia occurs when the mass of a white dwarf exceeds the Chandrasekhar mass, and the absolute luminosity that is emitted is almost always the same. This could be used as a beacon to measure the distance to the supernova, which is called luminosity distance.
When the universe expands absolute magnitude $M$ differs from the apparent magnitude $m$ due to redshift. This could be held as an expression such as

$$m - M \equiv \mu(z) = 5\log_{10}(d_L(z)/\text{Mpc}) + 25 \,, \tag{2.181}$$

where the luminosity distance $d_L$ is

$$d_L(z) \quad = \quad \frac{c}{H_0}(1+z)\int_0^z \left\{\Omega_{m0}(1+x)^3 + \Omega_{\Lambda 0}\right\}^{-2} dx \,. \tag{2.182}$$

The comparison between General Relativity without the cosmological constant and ΛCDM could be computed as the following figure 2.3. It could be seen that General Relativity without the cosmological constant cannot explain the observational data.

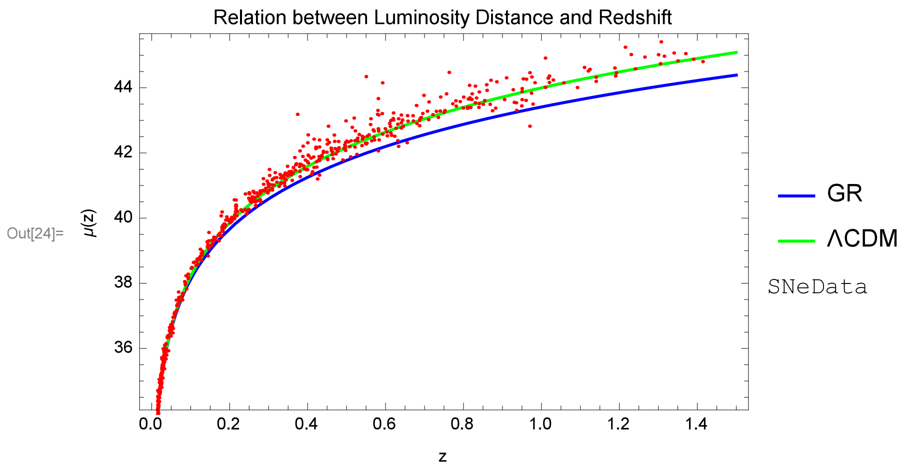


Figure 2.3: Comparison of Luminosity Distance of GR and ΛCDM [3]

#### 2.2.2.2 Cosmological Constant and the fine-tuning problem

A natural question, now that it is known that there is some kind of late-time accelerated expansion of the universe, is whether one can obtain such prediction from fundamental physics, such as the Standard Model of Particle Physics. [50, 51, 52, 53] The cosmological constant together with Cold Dark Matter seems to explain the universe quite well. By simply putting a constant in the Einstein equation

$$\overset{g}{G}_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}\,. \tag{2.183}$$

Recall that the Friedmann equations with perfect fluid were (2.17)-(2.18),

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}\,, \tag{2.184}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(1+w)\rho + \frac{\Lambda}{3}\,. \tag{2.185}$$

The cosmological constant, however, seems to have some problems. For the following, two major problems would be introduced. One is the fine-tuning problem and the other is the coincidence problem.

## Fine Tuning Problem

When one wonder about the origin of the cosmological constant, a vague guess would be vacuum energy. The energy of the vacuum could be calculated as

$$\rho_{vac} = 2\int_0^{k_{\text{cut off}}} \frac{d^3k}{(2\pi)^3}\sqrt{k^2+m^2} \tag{2.186}$$

[3]Data taken from https://supernova.lbl.gov/Union/ of the paper [49]

The contribution to the integral could be approximated by the largest momentum, which is the cutoff scale. Thus

$$\rho_{vac} \sim \frac{k^4_{\text{cut-off}}}{16\pi^2} \tag{2.187}$$

Since General Relativity is consider to be valid up to Planck scale the cut off scale could be taken as $k_{\text{cut-off}} \sim M_{pl} \sim 10^{19}$GeV. Therefore the energy of the vacuum could be approximated as

$$\rho_{vac} \simeq 10^{74}(\text{GeV})^4 \tag{2.188}$$

However when one estimates the energy of the cosmological constant, by considering the current epoch

$$\rho_\Lambda \sim \frac{\Lambda}{8\pi G} \sim \frac{3H_0^2}{8\pi G} \sim 8.1h \times 10^{-47}(\text{GeV})^4\,, \tag{2.189}$$

with $h$ defined through $H_0 = 100h\text{kms}^{-1}\text{Mpc}^{-1} \approx 2.1\times10^{-42}h\text{GeV}$. This indicates that when considering the cosmological constant to be the vacuum energy, the mismatch would be extraordinarily 120 worth in order[10]. To solve this problem one could consider a mechanism to subsequently vanish the vacuum energy in order to accomplish the cosmological constant. One such mechanism is the famous supersymmetry, where the bosonic vacuum energy is precisely the fermionic vacuum energy with the opposite sign and allows vacuum energy to disappear. Yet since the universe now has broken supersymmetry, the mechanism still implies very high fine-tuning to obtain the same order with the cosmological constant. Furthermore, since such a mechanism only addresses the cancellation of vacuum energy, the $10^{74}$, and not the cosmological constant $10^{-47}$, it is questionable whether the two mechanisms are the same or not.

## Coincidence Problem

The coincidence problem is simply could be put into the question, "How do two seemingly different things, the cosmological constant and ordinary matter, have the same order?"
As could be seen for the Planck data [4] the cosmological parameters, of TT,TE,EE+lowE 68% limits are

$$\Omega_{m,0} = 0.3166 \pm 0.0084 \tag{2.190}$$
$$\Omega_{\Lambda,0} = 0.6834 \pm 0.0084 \tag{2.191}$$

This indicates that the cosmological constant is the same order with matter, out of any ratio it could have taken. Recall in §2.1.1, the evolution of the energy density was investigated, and seemingly the ratio between matter and the cosmological constant could have taken any value. (See also fig. 2.1). The moment when the cosmological constant and matter was comparable was when it was the redshift,

$$z \sim 1 - \left(\frac{\Omega_{\Lambda,0}}{\Omega_{m,0}}\right)^{\frac{1}{3}} \sim 0.23\,, \tag{2.192}$$

which is relatively *recent* or merely about a billion years ago. Since the origin of the cosmological constant, and whether it couples to matter at all, is not well known, whether this coincidence problem is truly a problem is up for debate. One may say that although science needs no man, observation is done by mankind, thus the very existence of humanity bounds the possible observable. Such belief is called the anthropic principle and some use it to calculate the relevance of the cosmological constant [54]

[10]Of course one can lower the cut-off scale, but nonetheless, it has to be as low as $k_{cut-off} \sim 10^{-11}$GeV for the vacuum energy to be the cosmological constant in the absence of any mechanism.

#### 2.2.2.3 $H_0$ Tension

Whether the cosmological constant is truly a constant is an important question. This relates to whether dark energy is a new dynamical degree of freedom or not, which the celebrated ΛCDM model says otherwise. One example of such possible observation is the $H_0$ tension.

First results of SH0ES project which is based on Cepheids observed from SN Ia [55] and WMAP results [44] computed the Hubble parameter $H_0$ with some consistency. However, with the first release of the Planck satellite data, the tension between close and far observations of $H_0$ rose. With the most recent $H_0$ observation of Planck being from 2018 [4] and from SH0ES being from 2021 [2], the tension rose as high as $5.2\sigma$ for ΛCDM 2.4. Although it is still dangerous to conclude that ΛCDM is excluded, there seems to be some inconsistency between observation of the universe at low redshift ($z < 0.15$) and of high redshift ($z \geq 1100$). Future observation may find $H_0$ between these redshifts which may support the tension further.

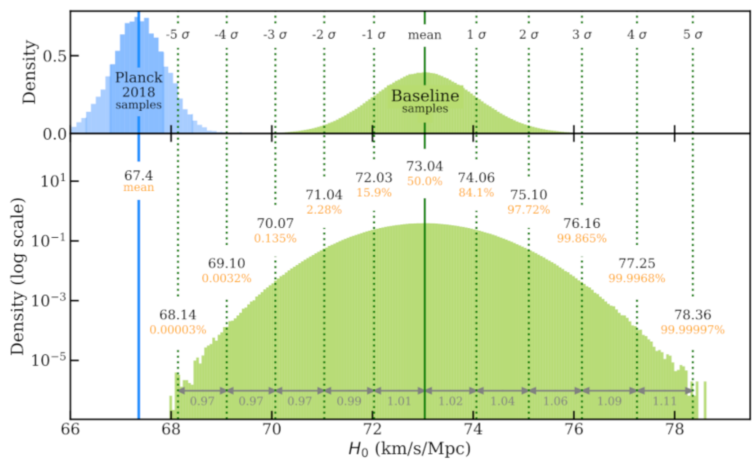


Figure 2.4: $H_0$ tension with Planck Data and SH0ES, taken from [2]

#### 2.2.2.4 Speed of Gravity and GW170817

One of the prominent characteristics of general relativity is that the speed that it propagates is light speed $c$.

Consider the following perturbation of the metric around the cosmological background,

$$g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2\left(\delta_{ij} + h_{ij} + \frac{1}{2}h_{ik}h^k_j\right)dx^i dx^j \,, \tag{2.193}$$

where $h_{ij}$ is a transverse-traceless tensor satisfying $\partial_i h^{ij} = h^i_i = 0$ while its indices are raised and lowered with the Kronecker delta $\delta_{ij}$. Substituting this to the Einstein-Hilbert action, one obtains,

$$S^{(2)}_{GW} = \frac{1}{8}\int dt d^3x \left[\dot{h}^2_{ij} - \frac{1}{a^2}\left(\partial h_{ij}\right)^2\right], \tag{2.194}$$

up to quadratic order. This indicates that the metric of Einstein gravity propagates in the speed of light.

There are other alternative theories of gravity that predict differently, however. Thus the question is whether one can differentiate among these theories by measuring the speed of gravitational waves. Indeed such was the idea proposed in [56, 57, 58, 59]. For example [56] estimates that simultaneous observations of gravitation waves and neutrinos or photons from gamma-ray burst and supernovae may constrain the speed of gravitational waves as,

$$|\delta_g| = \frac{|c - c_{GW}|}{c} = \begin{cases} 9.7\times 10^{-16} & \text{Supernova} \\ 4.6\times 10^{-16} & \text{Short gamma-ray burst} \end{cases} \,, \tag{2.195}$$

for future observations.

Indeed this was the case for the simultaneously ($\Delta t_{\text{SGRB-GW}} = 1.74$s) observed gravitational waves GW170817 and its gamma-ray burst counterpart GRB170817A of a neutron star and neutron star merger [60, 61]. Estimating that the short gamma-ray burst was emitted within $1.74 \pm 0.05$s after the gravitational waves, the speed of gravitational waves are constrained as,

$$-3\times 10^{-15} < \delta_g < 7\times 10^{-16}\,. \tag{2.196}$$

Thus if dark energy is of gravitational (spin-2) source, then a theory that explains must satisfy the constraint above. The constraints being strict led many to assume that $c_{GW}$ is indeed unity and thus 'eliminating' many alternative theories of gravity [62, 63, 64, 65]. General Relativity, of course, evades this constraint and again shows that it expresses the universe beautifully.

### 2.2.3 Other Unsolved Problems

#### 2.2.3.1 Reheating

Although inflation cures many of the big bang cosmology's pathology, since it implies an adiabatic expansion in the early universe, the temperature after inflation is very low. However, from big bang nucleosynthesis, it is known that at least the universe was 1keV. Therefore the mechanism to raise the temperature back up for the sufficient big bang temperature is called *reheating*. This mechanism, however, heavily relies on the theory and the matter couplings the inflaton has. Thus, just for simplicity, consider an inflation $\phi$ with mass $m$ coupled to a scalar field $\chi$ and spinor field $\psi$ as [17],

$$\Delta L_{int} = -g\phi\chi^2 - h\phi\bar{\psi}\psi\,. \tag{2.197}$$

Then the decay rates are calculable as,

$$\Gamma_{\phi\to\chi\chi} = \frac{g^2}{8\pi m}\,, \tag{2.198}$$

$$\Gamma_{\phi\to\psi\psi} = \frac{h^2 m}{8\pi}\,. \tag{2.199}$$

Since the inflation, at the end of its dynamics is expected to be oscillating around the bottom of the potential, the following form will be taken,

$$\phi(t) = \Phi(t)\cos(mt)\,. \tag{2.200}$$

The energy density of the inflation is,

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}m^2\phi^2 = \frac{1}{2}m^2\Phi^2\,. \tag{2.201}$$

The effective decay rate $\Gamma_{eff}$ can be defined through

$$\Gamma_{eff} = \frac{1}{a^3\rho_\phi}\frac{d(a^3\rho_\phi)}{dt}\,. \tag{2.202}$$

This gives a frictional contribution to the equation of motion of the inflaton as,

$$0 \sim \ddot{\phi} + (3H + \Gamma_{eff})\dot{\phi} + m^2\phi^2\,. \tag{2.203}$$

During radiation dominance, $H^2 = \frac{1}{4}\Gamma_{eff}^2$, using the relation between the Hubble parameter and temperature (2.76), the decay rate and temperature can be related as,

$$T_{RH} = \left(\frac{45c^5\hbar^3}{2\pi^2 {k_b}^4 g_*}\right)^{\frac{1}{4}} \sqrt{M_{pl}\Gamma_{eff}}\,. \tag{2.204}$$

Thus the temperature for reheating was derived.

#### 2.2.3.2 Baryogenesis

Currently, the universe is known to have extremely fewer anti-baryons compared to baryons. For example, the ratio of protons and anti-protons are observed by the PAMELA program as [66]

$$\frac{\bar{p}}{p} = 10^{-5} \sim 10^{-4} \tag{2.205}$$

and it seems certain that there is an asymmetry between baryons and anti-baryon numbers.

This problem first came out when Paul Dirac proposed the celebrated Dirac equation, which can be seen from the equation (9) and (12) of his 1928 paper [67]. The equation stated is as follows,

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0 \tag{2.206}$$

$$\bar{\psi}(x)(i\gamma^\mu\overleftarrow{\partial}_\mu - m) = 0 \tag{2.207}$$

This anti-matter was soon discovered by David Anderson in 1932 [68, 69]. Though it seemed that the relativistic quantum mechanics seemed to be relevant, it was wondered, quite early, why there is very few anti-matter.

Finally, after a third of a century, Andrei Sakharov proposed three conditions for baryon asymmetry to be achieved in 1967 [70]. This was because in 1964, a few years before, Christenson, Cronin, Fitch and Turlay discovered that a certain k meson decay provides concrete evidence for CP violation [71]. What they found was that the two eigenstates of a neutral k meson $K^0$, one with a short life span $K_S$ and the other with a long one $K_L$, does not decay with respect to CP symmetry. If CP symmetry is protected, the CP charge of the initial and final states of the decay are conserved. Thus there should be only the channel,

$$K_L^0 \rightarrow \pi^0\pi^0\pi^0, \pi^+\pi^-\pi^0, \tag{2.208}$$

$$K_S^0 \rightarrow \pi^+\pi-, \pi^0\pi^0. \tag{2.209}$$

However, what Cronin and Fitch's group found was that there exists the following channel,

$$K_L^0 \to \pi^+\pi^-, \tag{2.210}$$

which violates CP symmetry. Following this discovery, Sakharov proposed his criteria in which baryon number asymmetry could be attained. Through Sakharov's criteria, many baryogenesis mechanisms have been proposed. For example, the famous electro-weak baryogenesis explains the asymmetry through anomaly in the electro-weak sector and the phase transition that occurs.[72] Others are GUT baryogenesis which causes asymmetry in the GUT scale[73] and gravitational baryogenesis in which CP is violated through the gravitational interaction between curvature and baryon number currents[74]. Although many baryogenesis models are proposed, one interesting model is one that was proposed by Masataka Fukugita and Tsutomu Yanagida in 1986, where baryogenesis is obtained without introducing GUT scales[75]. This model looks into the fact that even if a sufficient baryogenesis is attained in the GUT era, the effect would be thinned out through inflation. Thus they consider a scenario where a Majorana particle $N$ decays into a lepton $l$ and a Higgs particle $h$ as follows

$$N_i \to l_i + \bar{h}, \bar{l}_i + h \tag{2.211}$$

(2.212)

and thus lepton asymmetry is accomplished.

### Sakharov's Criteria

First of all, Sakharov's criteria are necessary to understand baryogenesis[70]. The three conditions are as follows

**Sakharov's Three Conditions**

To achieve baryon number asymmetry macroscopically, the physics that governs baryon numbers must be also violated microscopically. Thus the necessary conditions are

1. Violation of Baryon Numbers
2. Violation of C and CP symmetries
3. Interactions Out of Thermal Equilibrium

This could be explained as the following

1. Violation of Baryon Numbers
   Firstly the asymmetry of baryon numbers is trivial. In order to achieve violation globally (in the cosmological sense), asymmetry must occur locally (microscopically).

2. Violation of C and CP symmetry
   CP symmetry, which is the combination of C symmetry of charge conjugation and P symmetry of spatial transformation, must be broken in order to achieve baryon number violation.
   First of all consider a non-baryonic X and Y and baryonic B, with the reaction

$$X \to Y + B\,. \tag{2.213}$$

   When C symmetry is conserved, the charge conjugate of the reaction has the same decay rate. Thus

$$\Gamma(X \to Y + B) - \Gamma(\bar{X} \to \bar{Y} + \bar{B}) = 0\,, \tag{2.214}$$

   and cannot achieve baryon number violation.
   Similarly if CP symmetry is conserved

$$\Gamma(X \to Y_L + B_L) = \Gamma(\bar{X} \to \bar{Y}_R + \bar{B}_R)\,, \tag{2.215}$$

$$\Gamma(X \to Y_R + B_R) = \Gamma(\bar{X} \to \bar{Y}_L + \bar{B}_L)\,. \tag{2.216}$$

Thus

$$\Gamma(X \to Y_L + B_L) + \Gamma(X \to Y_R + B_R)\,, \tag{2.217}$$
$$-\Gamma(\bar{X} \to \bar{Y}_L + \bar{B}_L) - \Gamma(\bar{X} \to \bar{Y}_R + \bar{B}_R) = 0\,, \tag{2.218}$$

and similarly cannot achieve baryon number violation.
Thus for global baryon number violation, both C symmetry and CP symmetry must be broken.

3. Interactions Out of Thermal Equilibrium
Baryon number violation cannot be achieved in thermal equilibrium. In thermal equilibrium, using the CPT theorem [11] and that the baryon number operator is odd.

$$\begin{aligned} < B > &= Z^{-1}\mathrm{tr}e^{-\beta\hat{H}}\hat{B} \\ &= Z^{-1}\mathrm{tr}(CPT)(CPT)^{-1}e^{-\beta\hat{H}}\hat{B} \\ &= Z^{-1}\mathrm{tr}e^{-\beta\hat{H}}(CPT)^{-1}\hat{B}(CPT) \\ &= Z^{-1}\mathrm{tr}e^{-\beta\hat{H}}(-\hat{B}) \\ &= -< B >\,, \end{aligned}$$

Thus

$$< B > \quad = \quad 0\,. \tag{2.219}$$

Where $Z \equiv \mathrm{tr}e^{-\beta\hat{H}}$
Thus interactions out of thermal equilibrium are necessary in order to violate baryon number conservation.

[11] Here notice that the satisfaction of the CPT theorem is needed. When considering a theory in which CPT theorem is broken, one could conduct leptogenesis within a thermal equilibrium [76]

## 2.3 Introduction to Modified Gravity

### 2.3.1 What is Einstein Gravity

The current Standard Model of cosmology is based on two pillars; the Standard Model of particle physics and General Relativity. In order to explain such puzzles, there are two major approaches. One is go beyond the Standard Model of particle physics and the other is to explore alternative theories of gravity. There are also other approaches where one may consider special initial conditions.

Therefore, at least for this thesis, the following will be called *Einstein gravity*, whereas any other theories of gravity will be called *Modified gravity*.

**Einstein Gravity**

When a metric $g_{\mu\nu}$ is governed by the following 4 dimensional second-order differential equation called the *Einstein equations* ,

$$\overset{g}{G}_{\mu\nu} = T_{\mu\nu}, \tag{2.220}$$

and classical non-interacting matter in the point particle limit, follows the geodesic equations of the fore-mentioned space-time metric,

$$\frac{d^2x^\sigma}{d\lambda^2} + \left\{ \begin{array}{c} \sigma \\ \mu\nu \end{array} \right\} \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0 \tag{2.221}$$

for some affine parameter $\lambda$, it will be called *Einstein gravity*.

Here the metric $g_{\mu\nu}$ is only a function of coordinates $g_{\mu\nu} = g_{\mu\nu}(x)$ and is a local, 4 dimensional, 2-rank and symmetric tensor.

Thus any *action* that both computes the Einstein and geodesic equations are not considered Modified gravity, whereas higher or lower dimensional Einstein equations will be considered as theories of Modified gravity in this thesis. Moreover, if the metric *does* follow Einstein equations, but matter *does not* follow geodesic equations in the point particle limit, it will also be called Modified gravity. Finally, all considerations are classical and not quantum.

#### 2.3.1.1 Uniqueness of the Einstein Equations: The Lovelock Theorem

In order to understand the distinctiveness of the Einstein equations, one needs to refer to the Lovelock theorem. In 1970-1972 David Lovelock proved a theorem[77, 78] where, when assumed locality, diffeomorphism, Lorentz invariance and the following conditions,

1. $A^{\mu\nu} = A^{\mu\nu}(g_{\mu\nu}, g_{\mu\nu,\rho}, g_{\mu\nu,\rho\sigma})$,
2. $\overset{g}{\nabla}_\mu A^{\mu\nu} = 0$,
3. $A^{\mu\nu} = A^{\nu\mu}$,

with $g_{\mu\nu,\rho} = \partial_\rho g_{\mu\nu}$ and $g_{\mu\nu,\rho\sigma} = \partial_\rho\partial_\sigma g_{\mu\nu}$, the only tensor satisfying these assumptions in D-dimensions is

$$A^\mu_\nu = \sum_{n=0}^{[D/2]} a_n \delta^{\mu\alpha_1\beta_1\cdots\alpha_n\beta_n}_{\nu\gamma_1\delta_1\cdots\gamma_n\delta_n} \prod_{r=1}^{n} \overset{g}{R}_{\alpha_r\beta_r}{}^{\gamma_r\delta_r}. \tag{2.222}$$

Here the Gauss symbol and the generalized Kronecker delta were defined as,

$$
\begin{aligned}
[D/2] &= \begin{cases} D/2 & D : \text{even} \\ (D-1)/2 & D : \text{odd} \end{cases} , \\
\delta^{\alpha_1 \cdots \alpha_n}_{\beta_1 \cdots \beta_n} &= \det \begin{vmatrix} \delta^{\alpha_1}_{\beta_1} & \cdots & \delta^{\alpha_1}_{\beta_n} \\ \vdots & \ddots & \vdots \\ \delta^{\alpha_n}_{\beta_1} & \cdots & \delta^{\alpha_n}_{\beta_n} \end{vmatrix} = n! \delta^{\alpha_1}_{[\beta_1} \cdots \delta^{\alpha_n}_{\beta_n]} .
\end{aligned}
$$

For the $D = 4$ case, the well known Einstein equations with the cosmological constant in the vacuum are obtained,

$$
A^{\mu\nu} = a_0 g^{\mu\nu} - 2a_1 \left( \overset{g}{R}{}^{\mu\nu} - \frac{1}{2} \overset{g}{R} g^{\mu\nu} \right) . \tag{2.223}
$$

Thus the following claim,

**Uniqueness of the Einstein Equations**

Einstein equations with a cosmological constant are unique under the following conditions,

1. Equation of motions is constructed purely from the metric $g_{\mu\nu}$[1]
2. 4 dimensional
3. Diffeomorphism invariant[2][3]
4. Lorentz invariant
5. Local[4]
6. Equation of motion is at most second-order in derivatives[5]
7. Exists an action[6]

Thus if any of these conditions are not satisfied, the uniqueness of the Einstein equations does not hold. Conversely, one can explore alternative theories beyond the Einstein equations by loosening one or more of these conditions.

1. non-metric or extra fields that mediate gravity

[1]Notice that this does not require that there could be more fields *in the action*, since if non-dynamical, one can eliminate these auxiliary fields. This later can be seen in the context of Palatini formalism where although there is an independent variable that is not the metric, the connection, it still computes the Einstein equations. See§2.4.2

[2]It can be shown that if the equation of motion are diffeomorphism invariant, i.e. tensorial, the action too is tensorial up to the surface term, see §B.2.3. Therefore ensuring the equation of motion being tensorial is enough for the action to be diffeomorphism invariant.

[3]Diffeomorphism, here, is the full $D = 4$ diffeomorphism invariance. Noting that spin is a D=3 concept, if one relaxes full diffeomorphism invariance, one can show that the Einstien equations are not the unique massless transverse and traceless spin-2 theory as shown in [79]

[4]The precise definition of locality differ among literature, while a naive one may be that a field is only dependent on the position as $\Psi = \Psi(x)$ while a more formal one would be ones using the Wightman formulation of field theories and following the classifications outlined by Jaffe in [80, 81], see also [82]

[5]This is due to the fact that higher derivatives in the equation of motion may cause instabilities. This will be reviewed in §2.5.1

[6]By assuming the existence of an action, $A_{\mu\nu} = A_{\nu\mu}$ is assured since it is the equation of motion of a symmetric metric. Furthermore, with diffeomorphism invariance of the action, one can also obtain $\overset{g}{\nabla}_\mu A^{\mu\nu} = 0$ for any action $\mathcal{L} = \mathcal{L}(g_{\mu\nu}$.

2. $D \neq 4$
3. Diffeomorphism violation
4. Lorentz violation
5. non-local
6. Inclusion of three or more derivatives
7. No action

Such is the landscape of Modified gravity.

#### 2.3.1.2 Proof of the Lovelock Theorem

For what follows, the proof of the Lovelock Theorem will be introduced for $D = 4$ following Lovelock's original paper in [77, 78]. For 4 dimensions, the symmetric condition is not necessary, thus the only two assumptions one has to take is,

$$
\begin{aligned}
A^{\mu\nu} &= A^{\mu\nu}(g_{\mu\nu}, g_{\mu\nu,\rho}, g_{\mu\nu,\rho\sigma})\,, && (2.224)\\
\overset{g}{\nabla}_\mu A^{\mu\nu} &= 0\,. && (2.225)
\end{aligned}
$$

First, introduce the following short-hand notation,

$$B^{;\mu\nu,\rho\sigma} = \frac{\partial B}{\partial g_{\mu\nu,\rho\sigma}}\,, \qquad (2.226)$$

where $B$ is some arbitrary rank tensor. The defined tensor has the following symmetric properties

$$
\begin{aligned}
&A^{\mu\nu;\alpha\beta,\gamma\delta} = A^{\mu\nu;\alpha\beta,\delta\gamma} = A^{\mu\nu;\beta\alpha,\gamma\delta}\,, && (2.227)\\
&A^{\mu\nu;\alpha\beta,\gamma\delta} + A^{\mu\nu;\alpha\gamma,\delta\beta} + A^{\mu\nu;\alpha\delta,\beta\gamma} = 0\,. && (2.228)
\end{aligned}
$$

Thus

$$A^{\mu\nu;\alpha\beta,\gamma\delta} = A^{\mu\nu;\gamma\delta,\alpha\beta}\,. \qquad (2.229)$$

Using $\overset{g}{\nabla}_\mu A^{\mu\nu} = 0$ (2.224),

$$A^{\mu\nu;\alpha\beta,\gamma\delta} + A^{\mu\gamma;\alpha\beta,\delta\nu} + A^{\mu\delta;\alpha\beta,\nu\gamma} = 0\,. \qquad (2.230)$$

The next step is to define a tenth order contravariant tensor as such

$$A^{\mu\nu;\alpha\beta,\gamma\delta;\epsilon\zeta,\eta\theta} \equiv \frac{\partial A^{\mu\nu;\alpha\beta,\gamma\delta}}{\partial g_{\epsilon\zeta,\eta\theta}}\,. \qquad (2.231)$$

From the definition of both sixth-order and tenth order tensors, one obtains the following symmetry,

$$A^{\mu\nu;\alpha\beta,\gamma\delta;\epsilon\zeta,\eta\theta} = A^{\mu\nu;\epsilon\zeta,\eta\theta;\alpha\beta,\gamma\delta}\,. \qquad (2.232)$$

Now when D=4, since any tensor with 5 or more indices have at least 2 indices that overlap, using cyclic symmetry and calculating each possibility of when the indices are equal with one another([78], Appendix A), then for any index the following holds true,

$$A^{\mu\nu;\alpha\beta,\gamma\delta;\epsilon\zeta,\eta\theta} = 0\,. \qquad (2.233)$$

This indicates that, there are no second-order derivatives for the following tensor,

$$A^{\mu\nu;\alpha\beta,\gamma\delta} = A^{\mu\nu;\alpha\beta,\gamma\delta}(g_{ab}, g_{ab,c}) \,, \tag{2.234}$$

Moreover since $A^{\mu\nu;\alpha\beta,\gamma\delta}$ is a tensor, and there are no sole tensors only consisting of $g_{ab,c}$,

$$A^{\mu\nu;\alpha\beta,\gamma\delta} = \alpha^{\mu\nu;\alpha\beta,\gamma\delta}(g_{ab}) \,. \tag{2.235}$$

Integrating the relation computes

$$\begin{aligned} A^{\mu\nu} &= \alpha^{\mu\nu;\alpha\beta,\gamma\delta} g_{\alpha\beta,\gamma\delta} + \beta^{\mu\nu}(g_{ab}) \\ &= \frac{2}{3}\alpha^{\mu\nu;\alpha\beta,\gamma\delta} \overset{g}{R}_{\alpha\beta\gamma\delta} + \beta g^{\mu\nu} \,. \end{aligned} \tag{2.236}$$

Here the last row was obtained by cyclic symmetry and noting that the only tensor that consists of the metric and has two free indices is the metric itself.
Finally, from cyclic symmetry, $\alpha^{\mu\nu;\alpha\beta,\gamma\delta}$ is either the combination of "only metric tensor" or "one metric tensor and one anti-symmetric tensor the Levi-Civita tensor $\epsilon^{\mu\nu\rho\sigma}$". Thus $\alpha^{\mu\nu;\alpha\beta,\gamma\delta}$ must have the form,

$$\alpha^{\mu\nu;\alpha\beta,\gamma\delta} = a^{\mu\nu;\alpha\beta,\gamma\delta} + b^{\mu\nu;\alpha\beta,\gamma\delta} \,, \tag{2.237}$$

with

$$\begin{aligned} a^{\mu\nu;\alpha\beta,\gamma\delta} = \; & a_1 g^{\mu\nu} g^{\alpha\beta} g^{\gamma\delta} + a_2 g^{\mu\nu} g^{\alpha\gamma} g^{\delta\beta} + a_3 g^{\mu\nu} g^{\alpha\delta} g^{\beta\gamma} \\ & + a_4 g^{\mu\alpha} g^{\nu\beta} g^{\gamma\delta} + a_5 g^{\mu\alpha} g^{\nu\gamma} g^{\beta\delta} + a_6 g^{\mu\alpha} g^{\nu\delta} g^{\gamma\beta} \\ & + a_7 g^{\mu\beta} g^{\nu\alpha} g^{\gamma\delta} + a_8 g^{\mu\beta} g^{\nu\gamma} g^{\alpha\delta} + a_9 g^{\mu\beta} g^{\nu\delta} g^{\gamma\alpha} \\ & + a_{10} g^{\mu\gamma} g^{\nu\alpha} g^{\beta\delta} + a_{11} g^{\mu\gamma} g^{\nu\beta} g^{\alpha\delta} + a_{12} g^{\mu\nu} g^{\alpha\delta} g^{\beta\gamma} \\ & + a_{13} g^{\mu\delta} g^{\nu\alpha} g^{\gamma\beta} + a_{14} g^{\mu\delta} g^{\nu\beta} g^{\alpha\gamma} + a_{15} g^{\mu\delta} g^{\nu\gamma} g^{\beta\alpha} \,. \end{aligned} \tag{2.238}$$

and

$$\begin{aligned} b^{\mu\nu;\alpha\beta,\gamma\delta} = \; & b_1 g^{\mu\nu} \epsilon^{\alpha\beta\gamma\delta} + b_2 g^{\mu\alpha} \epsilon^{\nu\gamma\delta\beta} + b_3 g^{\mu\beta} \epsilon^{\alpha\delta\nu\gamma} \\ & + b_4 g^{\mu\gamma} \epsilon^{\nu\beta\alpha\delta} + b_5 g^{\mu\delta} \epsilon^{\nu\gamma\beta\alpha} + b_6 g^{\nu\alpha} \epsilon^{\mu\delta\gamma\beta} \\ & + b_7 g^{\nu\beta} \epsilon^{\mu\alpha\gamma\delta} + b_8 g^{\nu\gamma} \epsilon^{\mu\beta\alpha\delta} + b_9 g^{\nu\delta} \epsilon^{\mu\beta\gamma\alpha} \\ & + b_{10} g^{\alpha\beta} \epsilon^{\nu\mu\gamma\delta} + b_{11} g^{\alpha\gamma} \epsilon^{\nu\beta\mu\delta} + b_{12} g^{\alpha\delta} \epsilon^{\mu\nu\beta\gamma} \\ & + b_{13} g^{\beta\gamma} \epsilon^{\nu\alpha\mu\delta} + b_{14} g^{\beta\delta} \epsilon^{\nu\mu\alpha\gamma} + b_{15} g^{\gamma\delta} \epsilon^{\nu\mu\beta\alpha} \,. \end{aligned} \tag{2.239}$$

By using cyclic symmetry the coefficients of $a^{\mu\nu;\alpha\beta,\gamma\delta}$ are computed as,

$$a_n = \begin{cases} \alpha & n = 5, 6, 8, 9, 10, 11, 13, 14 \\ -2\alpha & n = 2, 3, 4, 7, 12, 15 \\ 4\alpha & n = 1 \end{cases} \,.$$

$b^{\mu\nu;\alpha\beta,\gamma\delta}$ will disappear due to the characteristic of the symmetric properties of the Riemann tensor

$$b^{\mu\nu;\alpha\beta,\gamma\delta} R_{\gamma\alpha\beta\delta} = 0 \,. \tag{2.240}$$

Therefore, the Einstein equations and the cosmological constant

$$A^{\mu\nu} = -8\alpha G^{\mu\nu} + \beta g^{\mu\nu} \,, \tag{2.241}$$

is the sole equation that satisfies the conditions mentioned prior in (2.224) and (2.225).

Furthermore, in D-dimensions, the Lagrangian that computes the equations of motion that satisfied the conditions are [77],

$$L_n = \sqrt{-g}\delta^{\mu_1\cdots\mu_{2n}}_{\nu_1\cdots\nu_{2n}} \prod_{i=1}^{n} \overset{g}{R}_{\mu_{2i-1}\mu_{2i}}{}^{\nu_{2i-1}\nu_{2i}} , \tag{2.242}$$

which is called the Lovelock Lagrangian. For $D = 4$ one sees that there is another term, besides the cosmological constant and the Einstein-Hilbert action,

$$S = -\frac{1}{4}\int d^4x\sqrt{-g}\left(\overset{g}{R}^2 - 4\overset{g}{R}_{\mu\nu}\overset{g}{R}^{\mu\nu} + \overset{g}{R}^{\mu\nu\lambda\sigma}\overset{g}{R}_{\mu\nu\lambda\sigma}\right) . \tag{2.243}$$

This term is called the Gauss-Bonnet action, which in $D = 4$ is topological and does not effect the equation of motion. This can be shown by varying this action and show it is total derivative,

$$\begin{aligned}
\delta L_2 &= 4\sqrt{-g}\epsilon^{\alpha\beta}{}_{\gamma}{}^{\delta}\epsilon^{\mu\nu\rho\sigma}\overset{g}{R}_{\alpha\beta\mu\nu}\overset{g}{\nabla}_{\rho}\delta\left\{ \begin{array}{c}\gamma\\ \sigma\delta\end{array} \right\} && (2.244)\\
&= -4\sqrt{-g}\epsilon^{\alpha\beta}{}_{\gamma}{}^{\delta}\epsilon^{\mu\nu\rho\sigma}\overset{g}{\nabla}_{[\rho}\overset{g}{R}_{\mu\nu]\alpha\beta}\delta\left\{ \begin{array}{c}\gamma\\ \sigma\delta\end{array} \right\} + \text{s.t.} && (2.245)\\
&= 0 + \text{s.t.}\,, && (2.246)
\end{aligned}$$

due to Bianchi identity. Thus one can conclude, that the most general purely metric Lagrangian in four dimensions that computes an equation of motion that is at most 2nd-order is the Einstein-Hilbert action and the Cosmological constant.

$$S = \int d^4x\sqrt{-g}\left(\overset{g}{R} - 2\Lambda\right) \tag{2.247}$$

## 2.4 Palatini and Metric-affine Formalism

### 2.4.1 Metric-affine Geometry

Metric-affine geometry is a metric differential geometry with an affine connection that is generally is not Levi-Civita. It is a straightforward and minimal extension of Riemannian geometry and attracted much attention both in and out of gravitational theories. Many of the geometrical preliminaries can be found in the enlightening textbook by Schouten written in 1954 [83] and references therein. See also [84, 85, 86]

Recall that in §2.1.2.2, it was commented that the autoparallel equations can be derived from the equivalence principle, and then once one assumes the existence of a Lagrangian, these autoparallel equations (2.39) become that of geodesics (2.41). Which gives an important, but often overlooked conclusion; *the equivalence principle itself does not support the necessity of the metric but the connection.* Thus, one could question: what is the physics for a metric and a general affine connection?

#### 2.4.1.1 Curvature, Torsion, Non-metricity

Consider an affine connection $\Gamma$, which gives the covariant derivative that acts onto scalars $\phi$ and contravariant vectors $A^\mu$ as,

$$\overset{\Gamma}{\nabla}_\mu \phi = \partial_\mu \phi \tag{2.248}$$

$$\overset{\Gamma}{\nabla}_\mu A^\nu := \partial_\mu A^\nu + \Gamma^\nu{}_{\mu\alpha} A^\alpha \,. \tag{2.249}$$

Since the contraction between a contravariant vector $A^\mu$ and a covariant vector $B_\mu$ is a scalar, from chain rule, the rule for covariant derivatives acting on covariant vectors become,

$$\overset{\Gamma}{\nabla}_\mu B_\nu := \partial_\mu B_\nu - \Gamma^\alpha{}_{\mu\nu} B_\alpha \,. \tag{2.250}$$

Since now one obtained a covariant derivative, it is natural to ask what field strengths it may compute. As for acting on scalars, one obtains

$$[\overset{\Gamma}{\nabla}_\mu, \overset{\Gamma}{\nabla}_\nu]\phi = -\mathcal{T}^\lambda_{\mu\nu}\partial_\lambda \phi \,, \tag{2.251}$$

where $\mathcal{T}^\lambda_{\mu\nu}$ is called torsion and was defined as,

$$\mathcal{T}^\lambda_{\mu\nu} := \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu} \,. \tag{2.252}$$

If $\mathcal{T}^\lambda_{\mu\nu} = 0$ such connection is called a *symmetric connection.*. Gravitational theories that have torsion (and a metric) is known as Einstein-Cartan-Sciama-Kibble theories and was first proposed by Cartan in the paper *Sur une généralisation de la notion de courbure de Riemann et les espaces à torsion* in 1922 [87][18].

Field strengths of the covariant derivatives acting on a (contravariant) vector would be computed as,

$$[\overset{\Gamma}{\nabla}_\mu, \overset{\Gamma}{\nabla}_\nu]A^\lambda = \overset{\Gamma}{R}{}^\lambda{}_{\rho\mu\nu} A^\rho - \mathcal{T}^\sigma_{\mu\nu} \overset{\Gamma}{\nabla}_\sigma A^\lambda \,, \tag{2.253}$$

[18] Cartan, in the 1922 paper [87] however, only qualitatively mentions torsion and do not have concrete computation. In 1923, he releases a (lengthy) paper titled *Sur les variétés à connexion affine et la théorie de la relativité généralisée* in which he outlines the quantitative proposal for torsionful gravity [88]. For further details, see his (English) book [89] which also matches most of the computation of [88].

where $\overset{\Gamma}{R}{}^{\lambda}{}_{\rho\mu\nu}$ is called the Riemann curvature and is defined as,

$$\overset{\Gamma}{R}{}^{\alpha}{}_{\beta\mu\nu} \quad := \quad \partial_\mu \Gamma^{\alpha}{}_{\nu\beta} - \partial_\nu \Gamma^{\alpha}{}_{\mu\beta} + \Gamma^{\alpha}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\beta} - \Gamma^{\alpha}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\mu\beta} \,. \tag{2.254}$$

This Riemann curvature, which is purely defined from the affine connection, does not have the usual (anti-) symmetrical properties, i.e. $\overset{\Gamma}{R}_{\mu\nu\rho\sigma} \neq -\overset{\Gamma}{R}_{\nu\mu\rho\sigma}$ and $\overset{\Gamma}{R}_{\mu\nu\rho\sigma} \neq -\overset{\Gamma}{R}_{\rho\sigma\mu\nu}$. Furthermore, there are three possible contractions that lead to three Ricci tensors, the usual Ricci tensor $\overset{\Gamma}{R}_{\mu\nu} := \overset{\Gamma}{R}{}^{\lambda}{}_{\mu\lambda\nu}$, the co-Ricci tensor $\overset{\Gamma}{P}_{\mu\nu} := \overset{\Gamma}{R}_{\mu\lambda\nu}{}^{\lambda}$ and the homothetic tensor $\overset{\Gamma}{H}_{\mu\nu} = \overset{\Gamma}{R}{}^{\lambda}{}_{\lambda\mu\nu}$. The first two are asymmetric whereas the last one is anti-symmetric. The Ricci scalar is unique however, and defined as $\overset{\Gamma}{R} = g^{\mu\nu}\overset{\Gamma}{R}{}^{\lambda}_{\mu\lambda\nu}$.

Now consider introducing the metric $g_{\mu\nu}$, once done so, one may define the incompatibility of the connection to the metric as,

$$\mathcal{Q}_{\lambda}^{\ \mu\nu} = \overset{\Gamma}{\nabla}_\lambda g^{\mu\nu} = \partial_\lambda g^{\mu\nu} + \Gamma^{\mu}{}_{\lambda\sigma} g^{\nu\sigma} + \Gamma^{\nu}{}_{\lambda\sigma} g^{\mu\sigma} \,. \tag{2.255}$$

If $\mathcal{Q}_{\lambda}^{\ \mu\nu} = 0$ for a certain metric, such connection is called a *metric connection.*.

Geometry with a metric and a connection, expressed through curvature, torsion and non-metricity, is called metric-affine geometry, and gravitational theories expressed through metric-affine geometry is called metric-affine gravity.

Once torsion and non-metricity are defined, the connection can be decomposed into three parts, the Levi-Civita connection, the torsion tensor and the non-metricity tensor, as

$$\Gamma^{\lambda}{}_{\mu\nu} = \{^{\ \alpha}_{\beta\gamma}\} + \frac{1}{2}\left(\mathcal{T}^{\lambda}{}_{\mu\nu} + \mathcal{T}_{\nu}{}^{\lambda}{}_{\mu} - \mathcal{T}_{\mu\nu}{}^{\lambda} + \mathcal{Q}_{\nu}{}^{\lambda}{}_{\mu} + \mathcal{Q}_{\mu\nu}{}^{\lambda} - \mathcal{Q}^{\lambda}{}_{\mu\nu}\right) , \tag{2.256}$$

where the indices have been raised by the metric $g^{\mu\nu}$. Here recall that $\{^{\ \alpha}_{\beta\gamma}\}$ is the Levi-Civita connection that is defined by

$$\{^{\ \alpha}_{\mu\nu}\} := \frac{1}{2} g^{\alpha\beta}\left(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu}\right) \,. \tag{2.257}$$

From the expression (2.256), one immediately can see that for $\mathcal{T}^{\lambda}_{\ \mu\nu} = \mathcal{Q}_{\lambda}^{\ \mu\nu} = 0$, one uniquely obtains the Levi-Civita connection.

Using the expressions, (2.253) and the Jacobi identity, one may obtain the following metric-affine version of the Bianchi identities as,

$$\text{Zeroth:} \quad \overset{\Gamma}{R}{}^{(\alpha\beta)}{}_{\mu\nu} = \overset{\Gamma}{\nabla}_{[\mu}\mathcal{Q}_{\nu]}^{\ \alpha\beta} + \frac{1}{2}T^{\sigma}{}_{\mu\nu}\mathcal{Q}_{\sigma}^{\ \alpha\beta} \,, \tag{2.258}$$

$$\text{First:} \quad \overset{\Gamma}{R}{}^{\lambda}{}_{[\alpha\beta\gamma]} = \overset{\Gamma}{\nabla}_{[\alpha}\mathcal{T}^{\lambda}{}_{\beta\gamma]} - \mathcal{T}^{\sigma}{}_{[\alpha\beta}\mathcal{T}^{\lambda}{}_{\gamma]\sigma} \,, \tag{2.259}$$

$$\text{Second:} \quad \overset{\Gamma}{\nabla}_{[\alpha}\overset{\Gamma}{R}{}^{\mu}{}_{|\nu|\beta\gamma]} = \mathcal{T}^{\lambda}{}_{[\alpha\beta}\overset{\Gamma}{R}{}^{\mu}{}_{|\nu|\gamma]\lambda} \,. \tag{2.260}$$

These identities show that the covariant derivative acted on the Einstein tensor $\overset{\Gamma}{G}_{\mu\nu} := \overset{\Gamma}{R}_{\mu\nu} - \frac{1}{2}\overset{\Gamma}{R}g_{\mu\nu}$ does not conserve. This leads to a different type of energy-momentum conservation laws as shown in [84].

For the rest of this section, let the geometrical meanings of curvature, torsion, and non-metricity be explained and analyzed.

Firstly, a parallel transport of a vector $A^\mu$ from point $P$ to point $Q$ along a vector $B^\mu$ is

$$A^\mu_{P\to Q}(Q) = A^\mu(P) - \Gamma^\mu{}_{\nu\sigma}A^\sigma(P)B^\nu(P)\,, \tag{2.261}$$

up to first order in $B^\mu$. Similarly, a parallel transport of a vector $B^\mu$ from point $P$ to point $R$ along a vector $A^\mu$ is

$$B^\mu_{P\to R}(R) = B^\mu(P) - \Gamma^\mu{}_{\nu\sigma}B^\sigma(P)A^\nu(P)\,. \tag{2.262}$$

The parallelgram of the vectors $A^\mu_{P\to Q}(Q)$,$B^\mu_{P\to R}(R)$,$A^\mu(P)$, and $B^\mu(P)$ has a difference of

$$\{A^\mu_{P\to Q}(Q) + B^\mu(P)\} - \{A^\mu(P) + B^\mu_{P\to R}(R)\} = \mathcal{T}^\mu_{\nu\sigma}A^\nu(P)B^\sigma(P)\,. \tag{2.263}$$

Thus torsion quantifies the dislocation of the parallelgram of parallel transported vectors. Now consider the parallel transportation of a inner product between two vectors $A^\mu$ and $B^\mu$ along a vector $C^\mu$ from point $P$ to $Q$,

$$g_{\mu\nu,P\to Q}(Q)A^\mu_{P\to Q}(Q)B^\nu_{P\to Q}(Q) = g_{\mu\nu}(P)A^\mu(P)B^\nu(P) - \mathcal{Q}_{\lambda\mu\nu}(P)A^\mu(P)B^\nu(P)C^\lambda(P), \tag{2.264}$$

up to first order in $C^\mu$. Therefore, non-metricity quantifies the variation of an inner product, that is length and angle, along parallel transport. If the angle between the two vector $A^\mu$ and $B^\mu$ was defined as,

$$\theta = \cos^{-1}\left(\frac{g_{\mu\nu}A^\mu B^\nu}{\sqrt{g_{\mu\nu}A^\mu A^\nu}\sqrt{g_{\mu\nu}B^\mu B^\nu}}\right) \tag{2.265}$$

the change of the angle when the vectors were parallel transported along vector $C^\mu$ from point $P$ to point $Q$ is

$$\cos(\theta(Q)) - \cos(\theta(P)) = -\frac{1}{2}\cos(\theta)\mathcal{Q}_{\lambda\mu\nu}\left(\frac{A^\mu A^\nu}{g_{\alpha\beta}A^\alpha A^\beta} + \frac{B^\mu B^\nu}{g_{\alpha\beta}B^\alpha B^\beta} - 2\frac{A^\mu B^\nu}{g_{\alpha\beta}A^\alpha B^\beta}\right)C^\lambda\,, \tag{2.266}$$

where all terms of the right-hand side are evaluated at point $P$. Notice that for $A^\mu = B^\mu$ the right-hand side vanishes meaning that a single vector stays as a single vector for parallel transport.

Consider non-metricity to have a special form of

$$\mathcal{Q}_\lambda{}^{\mu\nu} = \frac{1}{4}\mathcal{W}_\lambda g^{\mu\nu}\,. \tag{2.267}$$

Such geometry is called Weyl geometry for torsion-less connection and Weyl-Cartan geometry for torsionful connection, and the vector $\mathcal{W}^\mu$ is called the Weyl vector. Furthermore, such connection is called a *semi-metric connection* . For this case, the right-hand side of (2.266) also disappears thus implying that for such geometry the angles preserve along parallel transport while the length doesn't.

Finally, in order to understand curvature, consider a parallel transport of a vector $C^\mu$ from point $P$ to $Q$ along $A^\mu$ and from point $Q$ to $S$ along $B^\mu$,

$$C^\mu_{P\to Q\to S}(S) = C^\mu(P) - \Gamma^\mu{}_{\nu\sigma}C^\sigma(B^\nu + A^\nu) - \partial_\lambda\Gamma^\mu{}_{\nu\sigma}A^\nu B^\lambda C^\sigma + \Gamma^\mu{}_{\nu\sigma}\Gamma^\sigma{}_{\alpha\beta}A^\nu B^\alpha C^\beta + \Gamma^\mu{}_{\nu\sigma}\Gamma^\nu{}_{\lambda\rho}A^\rho B^\lambda C^\sigma\,. \tag{2.268}$$

Similarly one may also compute for a parallel transport of a vector $C^\mu$ from point $P$ to $R$ along $B^\mu$ and from point $R$ to $S$ along $A^\mu$ and obtain $C^\mu_{P\to R\to S}(S)$. The difference of these are,

$$C^\mu_{P\to Q\to S}(S) - C^\mu_{P\to R\to S}(S) = -\overset{\Gamma}{R}{}^\mu{}_{\sigma\lambda\nu}A^\nu B^\lambda C^\sigma - \Gamma^\mu{}_{\nu\sigma}\mathcal{T}^\nu{}_{\alpha\beta}A^\alpha B^\beta C^\sigma\,, \tag{2.269}$$

up to second order in $A^\mu$ and $B^\mu$. Thus curvature shows the displacement of vectors along two parallel transports.

For further examples of metric-affine geometry and its structure on the manifold, refer to the book by Cartan [89]

#### 2.4.1.2 The Distortion Trick

Now notice that in (2.256) the torsion parts and non-metricity parts are tensorial while the general connection and the Levi-Civita connection is not. One may then introduce a tensor,

$$\kappa^{\alpha}{}_{\beta\gamma} := \Gamma^{\alpha}{}_{\beta\gamma} - \{^{\ \alpha}_{\beta\gamma}\} \ , \tag{2.270}$$

which is called the distortion tensor. The relation between distortion and torsion, non-metricity is thus,

$$\kappa^{\lambda}{}_{\mu\nu} = \frac{1}{2}\left(\mathcal{T}^{\lambda}{}_{\mu\nu} + \mathcal{T}_{\nu}{}^{\lambda}{}_{\mu} - \mathcal{T}_{\mu\nu}{}^{\lambda} + \mathcal{Q}_{\nu}{}^{\lambda}{}_{\mu} + \mathcal{Q}_{\mu\nu}{}^{\lambda} - \mathcal{Q}^{\lambda}{}_{\mu\nu}\right) . \tag{2.271}$$

Most literature uses non-metricity and torsion to express equations. However, throughout this work, both tensors will be implicit and the calculations will be done through the distortion. The translation between distortion to the two geometrical tensors could be done through,

$$\mathcal{T}^{\lambda}{}_{\mu\nu} = 2\kappa^{\lambda}{}_{[\mu\nu]} \tag{2.272}$$

$$\mathcal{Q}_{\alpha}{}^{\beta\gamma} = 2\kappa^{(\beta}{}_{\alpha}{}^{\gamma)} . \tag{2.273}$$

Now by using the distortion tensor $\kappa$, the Riemann curvature tensor could be re-expressed as

$$\overset{\Gamma}{R}{}^{\alpha}{}_{\beta\mu\nu} = \overset{g}{R}{}^{\alpha}{}_{\beta\mu\nu} + \overset{g}{\nabla}_{\mu}\kappa^{\alpha}{}_{\nu\beta} - \overset{g}{\nabla}_{\nu}\kappa^{\alpha}{}_{\mu\beta} + \kappa^{\alpha}{}_{\mu\lambda}\kappa^{\lambda}{}_{\nu\beta} - \kappa^{\alpha}{}_{\nu\lambda}\kappa^{\lambda}{}_{\mu\beta}, \ . \tag{2.274}$$

Recall that $\overset{g}{R}{}^{\alpha}{}_{\beta\mu\nu}$ is the Riemann tensor defined by the usual Levi-Civita connection $\{\ \}$, and $\overset{g}{\nabla}_{\mu}$, is the covariant derivative defined through the Levi-Civita connection. For further details of using the distortion trick see §A.2

### 2.4.2 Palatini Formalism

Metric-affine geometry, by itself cannot be a theory without being embed onto some Lagrangian. The most popular method that is usually considered is Palatini formalism, a misnomer, which was originally proposed by Einstein on June 9th 1925 in his paper *Einheitliche Feldtheorie von Gravitation und Elektrizitaet* as could be seen in equation (5) [90] , see also [91]. [19]

Palatini formalism is a variational method in which the Lagrangian consists of a metric $g_{\mu\nu}$ and $\Gamma^{\lambda}{}_{\mu\nu}$ and are varied independently. Furthermore, matter is usually taken such that it only couples with the metric. Thus an action in Palatini formalism would look like,

$$S(g,\Gamma,\Psi) = \int d^4x\sqrt{-g}\mathcal{L}(g,\Gamma) + S_m(g,\Psi), \tag{2.276}$$

with each of the variable's equation of motion being,

$$0 = \frac{\delta S}{\delta g^{\mu\nu}}, \tag{2.277}$$

$$0 = \frac{\delta S}{\delta \Gamma^{\lambda}{}_{\mu\nu}}, \tag{2.278}$$

$$0 = \frac{\delta S_m}{\delta \Psi}. \tag{2.279}$$

[19] As mentioned in [91], the well cited Palatini's paper *Deduzione invariantiva delle equazioni gravitazionali dal principio di Hamilton* [92] is about the *Palatini identity* that is,

$$\delta \overset{g}{R}_{\mu\nu} = \overset{g}{\nabla}_{\lambda}\delta\left\{\begin{array}{c}\lambda\\ \mu\nu\end{array}\right\} - \overset{g}{\nabla}_{\mu}\delta\left\{\begin{array}{c}\lambda\\ \nu\lambda\end{array}\right\}, \tag{2.275}$$

as seen in the first equation on page 208, and thus not related in any way to the "Palatini" formalism. Palatini's paper is phenomenal in the sense that it showed how to obtain the Einstein equations through variational methods from the *purely metric* Einstein-Hilbert action as seen in the third and fourth equation on page 209. Although a misnomer, this thesis will keep using the term *Palatini formalism* for the sake of conventionality.

In some literature, as is Einstein's original paper of [90], the arbitrary connection is taken to be symmetric, either a priori or a posteriori, however, in general, it does not have to be so. For what follows, unless specified, the connection is taken to be *asymmetric* and thus having all of its 64 components.

#### 2.4.2.1 Palatini Formalism with the Einstein-Hilbert action

As example, consider the Einstein-Hilbert action within Palatini formalism [90],

$$S=\int d^4x\sqrt{-g}\overset{\Gamma}{R}+S_m(g,\Psi). \tag{2.280}$$

The equation of motion is

$$0=\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \overset{\Gamma}{R}_{(\mu\nu)}-\frac{1}{2}g_{\mu\nu}\overset{\Gamma}{R}-T_{\mu\nu}, \tag{2.281}$$

$$0=\frac{\delta S}{\delta \Gamma^{\alpha}{}_{\beta\gamma}} = -\overset{\Gamma}{\nabla}_\alpha(\sqrt{-g}g^{\beta\gamma})+\delta^\beta_\alpha\overset{\Gamma}{\nabla}_\nu(\sqrt{-g}g^{\gamma\nu})+\sqrt{-g}\left\{\delta^\beta_\alpha g^{\gamma\mu}\mathcal{T}^\lambda_{\ \mu\lambda}+g^{\beta\gamma}\mathcal{T}^\lambda_{\ \lambda\alpha}-g^{\beta\mu}\mathcal{T}^\gamma_{\ \alpha\mu}-g^{\gamma\nu}\mathcal{T}^\beta_{\ \nu\alpha}\right\}, \tag{2.282}$$

where the energy-momentum tensor was defined as,

$$T_{\mu\nu} = -\frac{1}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}. \tag{2.283}$$

One notices that both of the equations of motion are first-order in derivatives, thus Palatini formalism is sometimes refered to as *first-order formalism of gravity*. However, since there are other first-order formalisms such as is electromagnetism[20], in this thesis the term will not be used.

Since both of the equations of motion are first-order, and thus are constraint equations, one can solve and eliminate either $g_{\mu\nu}$ or $\Gamma^\lambda{}_{\mu\nu}$. The former results in a pure connection theory and usually called Eddington formalism [93, 94][21] whereas the latter results in the usual metric formalism. For the Einstein-Hilbert case,

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[20] For example, consider the flat Maxwell action

$$S_{\text{2nd}} = \int d^4x\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) \tag{2.284}$$

$$= \int d^4x\left(\partial_\mu F^{\mu\nu}A_\nu+\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right)+\text{s.t.}, \tag{2.285}$$

with $F_{\mu\nu}=2\partial_{[\mu}A_{\nu]}$ and the equation of motion being $\partial_\mu F^{\mu\nu}=0$. Now, motivated by the action $S_{\text{2nd}}$ consider a new anti-symmetric field $\mathcal{F}_{\mu\nu}$, governed by the action,

$$S_{\text{1st}} = \int d^4x\left(\partial_\mu \mathcal{F}^{\mu\nu}A_\nu+\frac{1}{4}\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}\right). \tag{2.286}$$

Variation with respect to $\mathcal{F}_{\mu\nu}$ computes $\mathcal{F}_{\mu\nu}=F_{\mu\nu}$, whereas variation with respect to $A_\mu$ computes $\partial_\mu\mathcal{F}^{\mu\nu}$ which reduces to the Maxwell equations when substituted the solution of $\mathcal{F}_{\mu\nu}$.

[21] Einstein noticed early on the significance of Eddington's formulation as seen in his 1923 paper [95], which was written on his way back from Japan on SS Haruna Maru.

Here, very briefly, Eddington formalism will be introduced. Consider matter only being the vacuum energy, i.e. the cosmological constant with $\mathcal{L}=\overset{\Gamma}{R}-2\Lambda$. Then the Einstein equation computes the solution for the metric as,

$$g_{\mu\nu}=\frac{1}{\Lambda}\overset{\Gamma}{R}_{(\mu\nu)}. \tag{2.287}$$

Substituting this solution in the action, one obtains the resultant action is,

$$S=\int d^4x\frac{2\sqrt{\det\overset{\Gamma}{R}_{(\mu\nu)}}}{\Lambda}. \tag{2.288}$$

the connection can be solved as,

$$0 = \Gamma^{\lambda}{}_{\mu\nu} - \left\{ \begin{array}{c} \lambda \\ \mu\nu \end{array} \right\} - \frac{1}{4}\left(\Gamma^{\lambda}{}_{\lambda\mu} - \left\{ \begin{array}{c} \lambda \\ \lambda\mu \end{array} \right\}\right)\delta^{\lambda}_{\nu}\,, \tag{2.289}$$

which is traceless with respect to $(\lambda-\nu)$. Substituting this solution of the connection into the metric equation of motion, one obtains the Riemann geometric Einstein equations,

$$\overset{g}{R}_{(\mu\nu)} - \frac{1}{2}g_{\mu\nu}\overset{g}{R} = T_{\mu\nu} \tag{2.290}$$

Therefore, for the Einstein-Hilbert action, whether the formalism is metric or Palatini, one obtains the Einstein equations. Notice that in Palatini formalism, the connection is not uniquely Levi-Civita, but the equation of motion for the metric still follows the (metric) Einstein equations nonetheless,

#### 2.4.2.2 Projective Transformation

One may wonder why the 64 components of the connection are not uniquely determined in equation (2.289). Such trace-less property makes 4 components of the connection not fixed. This is due to the characteristic that the Ricci scalar and thus the Einstein-Hilbert action is invariant under projective transformation [83].

Projective transformation is the following transformation of the affine connection,

$$\Gamma^{\lambda}{}_{\mu\nu} \to \tilde{\Gamma}^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\mu\nu} + \xi_{\mu}\delta^{\lambda}_{\nu}. \tag{2.291}$$

Under such transformation, the geometrical variables are transformed as,

$$\overset{\tilde{\Gamma}}{R}{}^{\rho}{}_{\sigma\mu\nu} = \overset{\Gamma}{R}{}^{\rho}{}_{\sigma\mu\nu} + 2\partial_{[\mu}\xi_{\nu]}\delta^{\rho}_{\sigma}, \tag{2.292}$$

$$\tilde{\mathcal{T}}^{\lambda}{}_{\mu\nu} = \mathcal{T}^{\lambda}{}_{\mu\nu} + \xi_{[\mu}\delta^{\lambda}_{\nu]}, \tag{2.293}$$

$$\tilde{\mathcal{Q}}_{\lambda}{}^{\mu\nu} = \mathcal{Q}_{\lambda}{}^{\mu\nu} + 2\xi_{\lambda}g^{\mu\nu}. \tag{2.294}$$

Projective transformation does not change the symmetric part of the Ricci tensor and thus the Ricci scalar,

$$\overset{\Gamma}{R}_{(\mu\nu)} \to \overset{\tilde{\Gamma}}{R}_{(\mu\nu)} = \overset{\Gamma}{R}_{(\mu\nu)} \tag{2.295}$$

$$\overset{\Gamma}{R} \to \overset{\tilde{\Gamma}}{R} = \overset{\Gamma}{R} \tag{2.296}$$

Thus any action constructed with such terms, such as the Einstein-Hilbert action, will enjoy *projective invariance*. This is the reason the solution of the connection has 4 components, vectorial freedom, undetermined in (2.289).

Geometrically, projective transformation is a transformation that keeps the angles of vectors invariant while the length changes. Recall in (2.266), the relation between non-metricity and the angles of vectors were derived. Substituting (2.294) into (2.266), indeed projective transformation does not change the angles. Thus projective transformation can be thought of as a *conformal transformation* of the connection.

Furthermore, projective transformation is the most general transformation of the connection that keeps the autoparallel equations invariant under general transformations of the affine parameter.

Recall the autoparallel equations,

$$0 = \frac{d^2x^{\lambda}}{d\lambda^2} + \Gamma^{\lambda}{}_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda}\,, \tag{2.297}$$

Notice that for this action the cosmological constant is a necessity rather than just some parameter, unlike the Einstein-Hilbert action in the metric formalism nor the Palatini formalism.

which was invariant under affine transformation of the affine parameter $\lambda \to \bar{\lambda} = a\lambda + b$ for some constant $a, b$.

Now consider the following general transformation of the affine parameter,

$$\lambda \to \lambda = \lambda(\bar{\lambda})\,, \tag{2.298}$$

with $\lambda' := \frac{d\lambda}{d\bar{\lambda}} \neq 0$. The autoparallel equations then become,

$$0 \;=\; \frac{1}{\lambda'^2}\left\{\frac{d^2x^\lambda}{d\bar{\lambda}^2} + \Gamma^\lambda{}_{\mu\nu}\frac{dx^\mu}{d\bar{\lambda}}\frac{dx^\nu}{d\bar{\lambda}} - \frac{\lambda''}{\lambda'}\frac{dx^\lambda}{d\bar{\lambda}}\right\}. \tag{2.299}$$

The projective transformation of the affine connection $\Gamma^\lambda{}_{\mu\nu} = \bar{\Gamma}^\lambda{}_{\mu\nu} + \xi_\mu\delta^\lambda_\nu$, will compute,

$$0 \;=\; \frac{1}{\lambda'^2}\left\{\frac{d^2x^\lambda}{d\bar{\lambda}^2} + \bar{\Gamma}^\lambda_{\mu\nu}\frac{dx^\mu}{d\bar{\lambda}}\frac{dx^\nu}{d\bar{\lambda}} + \left(\xi_\mu\frac{dx^\mu}{d\bar{\lambda}} - \frac{\lambda''}{\lambda'}\right)\frac{dx^\lambda}{d\bar{\lambda}}\right\}. \tag{2.300}$$

Thus by taking $\xi_\mu$ such that

$$\lambda(\bar{\lambda}) = \int e^{\int \xi_\mu dx^\mu} d\bar{\lambda}\,, \tag{2.301}$$

one could obtain the same autoparallel equations.

### 2.4.3 Metric-affine Formalism

In Palatini formalism, the one assumption was that the connection does not couple to matter. One may loosen this assumption and introduce such coupling. Such formalism is called metric-affine formalism and the action will have the form,

$$S(g,\Gamma,\Psi) = \int d^4x\sqrt{-g}\mathcal{L}(g,\Gamma) + S_m(g,\Gamma,\Psi)\,. \tag{2.302}$$

Since the connection is coupled to the matter, a new energy-momentum tensor arises from the variation of the connection. Such tensor is called hyper-momentum tensor,

$$\Delta^\lambda{}_{\mu\nu} = \frac{\delta S_m}{\delta\Gamma^\lambda{}_{\mu\nu}} \tag{2.303}$$

#### 2.4.3.1 Classification of Metric-affine Gravity

This section is dedicated to classifying the Metric-affine gravity theories. The first part of this section defines the essential variables used in the rest of the chapter, the distortion tensor. The following part is then used to explain projective transformation and its invariance through using the distortion tensor, and then theories with or without the invariance will be shown. Moreover, the rest of the section is used to clarify the classification of different Metric-affine theories.

## Model I: Projective Invariant Models

Projective transformation and the invariance under it have been explained previously. Here the same will be re-explained with this time using the distortion tensor.

First, consider the Metric-affine Einstein-Hilbert action as

$$S_{\rm g} = \frac{M_{\rm Pl}^2}{2}\int {\rm d}^4x\sqrt{-g}\overset{\Gamma}{R}\,, \tag{2.304}$$

where $M_{\rm Pl}^2 := \frac{1}{8\pi G}$ is the reduced Planck mass and $\overset{\Gamma}{R} := g^{\mu\nu}\overset{\Gamma}{R}{}^{\lambda}{}_{\mu\lambda\nu}$ is the Ricci scalar.

The Ricci scalar is uniquely determined by contracting all the indices of the Riemann tensor. The uniqueness is only the property of the Ricci scalar since the Ricci tensor is not. The reason for this is that the Riemann tensor does not enjoy the (anti-) symmetries of the usual Riemann tensor has. Though irrelevant to this section, it is important to be noted of this difference.

Using the distortion tensor, the action (2.304) could be decomposed as,

$$S_{\rm g} = \frac{M_{\rm Pl}^2}{2}\int {\rm d}^4x\sqrt{-g}\Big[\overset{g}{R} + g^{\mu\nu}\left(\kappa^{\lambda}{}_{\mu\nu}\kappa^{\sigma}{}_{\sigma\lambda} - \kappa^{\lambda}{}_{\sigma\mu}\kappa^{\sigma}{}_{\nu\lambda}\right)\Big]\,. \tag{2.305}$$

Note that the 2nd and 3rd terms on the right-hand-side of Eq. (2.274) are surface terms when plugged into the action, and therefore are omitted. Looking at this action, it is only a usual Einstein gravity written in Riemannian geometry with a three-rank tensor field $\kappa$.

Now one can derive the two Euler-Lagrangian equations from the action. The variation, with respect to the metric, yields,

$$\frac{\delta S}{\delta g^{\mu\nu}} = \frac{M_{\rm Pl}^2}{2}\Big[\overset{g}{G}_{\mu\nu} + \kappa^{\lambda}{}_{\mu\nu}\kappa^{\sigma}{}_{\sigma\lambda} - \kappa^{\lambda}{}_{\sigma\mu}\kappa^{\sigma}{}_{\nu\lambda} - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\left(\kappa^{\lambda}{}_{\alpha\beta}\kappa^{\sigma}{}_{\sigma\lambda} - \kappa^{\lambda}{}_{\sigma\alpha}\kappa^{\sigma}{}_{\beta\lambda}\right)\Big]\,, \tag{2.306}$$

while the variation of the connection computes,

$$\frac{\delta S}{\delta \kappa^{\alpha}{}_{\beta\gamma}} = \frac{M_{\rm Pl}^2}{2}\left[g^{\beta\gamma}\kappa^{\sigma}{}_{\sigma\alpha} + \delta^{\beta}_{\alpha}\kappa^{\gamma\sigma}{}_{\sigma} - \kappa^{\beta\gamma}{}_{\alpha} - \kappa^{\gamma}{}_{\alpha}{}^{\beta}\right]\,. \tag{2.307}$$

One must first realize that there is no kinetic term of $\kappa$, as in $(\partial\kappa)^2$. This fact could also be noticed since the Euler-Lagrangian equations for the distortion do not include any derivatives of the tensor, i.e. it is a purely algebraic tensor equation. Thus, indicates that the distortion/connection is not dynamical, but is fixed by the constraint equations (2.307). This characteristic is crucial and later on, will be very important.

With the equations in hand, consider 'projective symmetry' for the Einstein-Hilbert action. The projective transformation in Metric-affine gravity is written as,

$$\Gamma^{\alpha}{}_{\beta\gamma} \to \tilde{\Gamma}^{\alpha}{}_{\beta\gamma} = \Gamma^{\alpha}{}_{\beta\gamma} + \delta^{\alpha}_{\gamma}U_{\beta}\,, \tag{2.308}$$

The Ricci scalar under this transformation is invariant as,

$$\overset{\Gamma}{R} \to \overset{\tilde{\Gamma}}{R} = \overset{\Gamma}{R}\,. \tag{2.309}$$

Since the Einstein-Hilbert term is constructed by this Ricci scalar, the action has "projective invariance". In general, however, Metric-affine theories do not withhold such invariance, as could be later seen in this paper. One may also consider and assume that the matter sector of the theory also has the projective invariance, which leads to making the full theory enjoy this projective symmetry. Such a theory will be called a projective invariant theory.

It is also important to note that, when the theory is projective invariant, the constraint equations for $\kappa$ do not fix all components of the connection. Such ambiguity could be thought of as a gauge of freedom, one that may fix later by hand since it does not affect the physics considered. This face could be seen explicitly. First, consider a vacuum or a case in which matter does not couple to the connection (this is normally called the Palatini theories). This then computes the constraint equations as

$$g^{\beta\gamma}\kappa^{\sigma}{}_{\sigma\alpha} + \delta^{\beta}_{\alpha}\kappa^{\gamma\sigma}{}_{\sigma} - \kappa^{\beta\gamma}{}_{\alpha} - \kappa^{\gamma}{}_{\alpha}{}^{\beta} = 0\,, \tag{2.310}$$

the algebraic characteristic of this equation allows one to precisely solve the distortion as,

$$\kappa^{\alpha}{}_{\beta\gamma} = \frac{1}{4}\delta^{\alpha}_{\gamma}\,\kappa_{\beta}\,, \tag{2.311}$$

with $\kappa_{\beta} := \kappa^{\lambda}{}_{\beta\lambda}$. Thus the distortion tensor $\kappa$ cannot be determined up to the trace $\kappa_{\beta}$ of the tensor. These remaining four degrees of freedom are not problematic and the reason for this is because the theory is projective invariant. More simply this trace vector directly corresponds to the projective transformation vector $U_{\beta}$. To see this clearer, one may define a trace-free tensor as

$$\bar{\kappa}^{\alpha}{}_{\beta\gamma} := \kappa^{\alpha}{}_{\beta\gamma} - \frac{1}{4}\delta^{\alpha}_{\gamma}\,\kappa_{\beta}\,. \tag{2.312}$$

Using the projective invariance of the theory, the action $S$ could be rewritten purely in the form of the trace-free tensor as,

$$S_{\rm g} \;=\; \frac{M_{\rm Pl}^2}{2}\int {\rm d}^4x\sqrt{-g}\Big[\overset{g}{R} + g^{\mu\nu}\left(\bar{\kappa}^{\lambda}{}_{\mu\nu}\bar{\kappa}^{\sigma}{}_{\sigma\lambda} - \bar{\kappa}^{\lambda}{}_{\sigma\mu}\bar{\kappa}^{\sigma}{}_{\nu\lambda}\right)\Big]\,. \tag{2.313}$$

The Einstein equations are then

$$G_{\mu\nu} = M_{\rm Pl}^{-2}\left[T_{\mu\nu} + \tau_{\mu\nu}\right]\,. \tag{2.314}$$

Here the energy-momentum tensor $T_{\mu\nu}$ is derived from the usual matter field by

$$T_{\mu\nu} = -2\frac{\delta S_{\rm m}}{\delta g^{\mu\nu}}\,, \tag{2.315}$$

whereas the hyper energy-momentum tensor $\tau_{\mu\nu}$ is

$$\tau_{\mu\nu} \;:=\; -\bar{\kappa}^{\lambda}{}_{\mu\nu}\bar{\kappa}^{\sigma}{}_{\sigma\lambda} + \bar{\kappa}^{\lambda}{}_{\sigma\mu}\bar{\kappa}^{\sigma}{}_{\nu\lambda} + \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\left(\bar{\kappa}^{\lambda}{}_{\alpha\beta}\bar{\kappa}^{\sigma}{}_{\sigma\lambda} - \bar{\kappa}^{\lambda}{}_{\sigma\alpha}\bar{\kappa}^{\sigma}{}_{\beta\lambda}\right)\,, \tag{2.316}$$

The hyper-momentum tensor could be thought as an extra term coming from the three-rank tensor field $\bar{\kappa}^{\alpha}{}_{\beta\gamma}$ on the usual Riemannian theory.

The variation of the connection is also written purely by the trace-free tensor $\bar{\kappa}^{\lambda}{}_{\mu\nu}$ as

$$\frac{\delta S}{\delta\kappa^{\alpha}{}_{\beta\gamma}} \;=\; \frac{M_{\rm Pl}^2}{2}\left[g^{\beta\gamma}\bar{\kappa}^{\sigma}{}_{\sigma\alpha} + \delta^{\beta}_{\alpha}\bar{\kappa}^{\gamma\sigma}{}_{\sigma} - \bar{\kappa}^{\beta\gamma}{}_{\alpha} - \bar{\kappa}^{\gamma}{}_{\alpha}{}^{\beta}\right]\,. \tag{2.317}$$

Thus the constraint equations for matter that is coupled with the connection with gravity induced by the Metric-affine Einstein-Hilbert term can be written as

$$g^{\beta\gamma}\bar{\kappa}^{\sigma}{}_{\sigma\alpha} + \delta^{\beta}_{\alpha}\bar{\kappa}^{\gamma\sigma}{}_{\sigma} - \bar{\kappa}^{\beta\gamma}{}_{\alpha} - \bar{\kappa}^{\gamma}{}_{\alpha}{}^{\beta} = -2M_{\rm Pl}^{-2}\frac{\delta S_{\rm m}}{\delta\bar{\kappa}^{\alpha}{}_{\beta\gamma}}\,, \tag{2.318}$$

and

$$\frac{\delta S_{\rm m}}{\delta \kappa_\beta} = 0\,. \tag{2.319}$$

Eq. (2.319) is trivially satisfied when the matter action is projective invariant. [22] Furthermore, when the matter field does not couple to the connection, the right-hand side of Eq. (2.318) becomes zero. Then, it could be solved as $\bar{\kappa}^\lambda{}_{\mu\nu} = 0$, and the usual Einstein gravity that is written with the Levi-Civita connection is restored. When one extends the theory and couple the matter with the connection/distortion tensor $\kappa^\lambda{}_{\mu\nu}$, an additional contribution to $\bar{\kappa}^\lambda{}_{\mu\nu}$ will need to be considered, solve the constraint equation (2.318). Here and now on, a projective invariant theory is classified as Model I.

## Model II and III: Non-projective Invariant Models

In general, matter does not have to be projective invariant. Thus it is important to discuss what happens when the theory is not projective invariant. Since the gravitational part is projective invariant while matter is not, the constraint equations of the connection induce constraint for matter. Thus one may either choose the following; matter is constrained by gravity, or constrain by hand. The former is irrelevant since additional constraints on matter may give rise to inconsistencies. The latter, although may not seem natural, is more popular and could be related to other types of theories. Here only the latter will be discussed. There are two ways to constrain the connection by hand, one is a priori or implicitly constrain the geometry, the other is to use Lagrangian multipliers. The torsion-less condition is frequently considered in theories under Palatini formulation, whereas metric compatible connections are assumed in Einstein Cartan theories. These two can be obtained by constraining the geometry of a fully Metric-affine theory.
As just stated, there are the following two common approaches. One is to assume there is no torsion ($\mathcal{T}^\lambda{}_{\mu\nu} = 0$), while the other is to consider a metric-compatible connection ($\mathcal{Q}_\alpha{}^{\beta\gamma} = 0$). These theories, in general, do not produce the same results. Hereby, classify these cases as the former into Model II(a) and the latter into Model II(b). Considering geometry is usually assumed *a priori* in the gravitational action. As it will be shown, for the Metric-affine Einstein-Hilbert action, both constraints compute the Einstein equations and induce Riemannian geometry if matter does not couple to the connection.

For Model II(a), the distortion has to satisfy, $\kappa^\lambda{}_{[\mu\nu]} = \mathcal{T}^\lambda{}_{\mu\nu}/2 = 0$. By using this and the solution (2.311) for the constraint equation, the connection could be simply solved as

$$\kappa^\alpha{}_{\beta\gamma} = 0\,. \tag{2.320}$$

In short, the affine connection is the Levi-Civita connection. Thus Riemannian geometry is obtained.

Moreover, for Model II(b), distortion must satisfy $\kappa^{(\beta}{}_\alpha{}^{\gamma)} = \mathcal{Q}_\alpha{}^{\beta\gamma}/2 = 0$. This too by using the constraint equations computes Eq. (2.320). Therefore, the solution of the affine connection is again the Levi-Civita connection.

The process is similar when one assumes the coupling between matter and the connection. The trace-free distortion tensor $\bar{\kappa}$ and the trace part of it $\kappa_\mu$ could both be solved by the set of constraint equations (2.318) and (2.319). In any case, the hyper energy-momentum tensor emerges in the matter sector of the Einstein equations. Furthermore, by the constraint of geometry, the trace part of the distortion tensor will be determined. Note that it will not appear in the Einstein equations.

It is known that, although constraining geometry a priori into the action, i.e. assuming torsion-free or the metric compatibility computes a consistent gravity theory, such a condition is too strict when the

[22] It is important to note that the Standard Model fundamental particles are projective invariant. However, composite particles and exotic matter are, in general, do not have projective symmetry.

projective invariance is broken. This could be seen, since the undetermined terms in the connection are only the projective vector, and thus four components. Therefore a minimized condition that one could impose is imposing a constraint on some vector $\mathcal{C}^\mu$, which is constructed by the distortion tensor, using Lagrange multipliers, as

$$S_{\rm g} = \frac{M_{\rm Pl}^2}{2}\int {\rm d}^4x\sqrt{-g}\left\{R(g) + g^{\mu\nu}\left(\kappa^{\lambda}{}_{\mu\nu}\kappa^{\sigma}{}_{\sigma\lambda} - \kappa^{\lambda}{}_{\sigma\mu}\kappa^{\sigma}{}_{\nu\lambda}\right) + \lambda^\mu\mathcal{C}_\mu(\kappa)\right\}\,. \tag{2.321}$$

By varying the Lagrange multiplier $\lambda^\mu$, one obtains four sets of equations $\mathcal{C}_\mu(\kappa) = 0$, which constrains the four undetermined vector degrees of freedom that are inside the connection.

Now by performing projective transformation (2.308), torsion and non-metricity computes

$$\mathcal{T}^{\lambda}_{\ \mu\nu} \quad\to\quad \tilde{\mathcal{T}}^{\lambda}_{\ \mu\nu} = \mathcal{T}^{\lambda}_{\ \mu\nu} + \Delta\mathcal{T}^{\lambda}_{\ \mu\nu} = \mathcal{T}^{\lambda}_{\ \mu\nu} + 2\delta^{\lambda}_{[\nu}U_{\mu]} \tag{2.322}$$

$$\mathcal{Q}^{\lambda}{}_{\mu\nu} \quad\to\quad \tilde{\mathcal{Q}}^{\lambda}{}_{\mu\nu} = \mathcal{Q}^{\lambda}{}_{\mu\nu} + \Delta\mathcal{Q}^{\lambda}{}_{\mu\nu} = \mathcal{Q}^{\lambda}{}_{\mu\nu} + 2g_{\mu\nu}U^{\lambda}\,. \tag{2.323}$$

This could be used to solve the gauge freedom as $U_\mu = \Delta\mathcal{T}^{\lambda}_{\ \mu\lambda}/3 = \Delta\mathcal{Q}^{\lambda}{}_{\mu\lambda}/2 = \Delta\mathcal{Q}_{\mu}{}^{\lambda}{}_{\lambda}/8$. From this result, breaking the projective invariance could be done by constraining either of the three geometrical vectors. In other words, if the vector that will be constrained by the Lagrange multiplier $\mathcal{C}_\mu(\kappa)$ is chosen by either of the following three vectors, one may obtain a well-defined theory.
(a) the torsion vector: $\mathcal{T}_\mu := \mathcal{T}^{\lambda}{}_{\mu\lambda}$,
(b) the non-metricity trace vector : $\mathcal{Q}_\mu := \mathcal{Q}^{\lambda}{}_{\mu\lambda}$ ,
(c) the Weyl vector : $\mathcal{W}_\mu := \frac{1}{4}\mathcal{Q}_{\mu}{}^{\lambda}{}_{\lambda}$.
From now on, the above will be classified as models III (a), III (b), and III (c), respectively.

As for the Metric-affine Einstein-Hilbert action, each of the models computes a constraint equation as $\mathcal{C}_\mu(\kappa) = 0$ which then computes the solution for the connection as $\kappa^{\beta}{}_{\alpha\beta} = 0$. Where the face of $\mathcal{T}_\mu = 2\kappa^{\lambda}{}_{[\mu\lambda]}$, $\mathcal{Q}^\mu = 2\kappa^{(\lambda}{}_{\lambda}{}^{\mu)}$, and $\mathcal{W}_\mu = \frac{1}{2}\kappa^{\lambda}{}_{\mu\lambda}$ were used. The affine connection is again the Levi-Civita connection and the conventional Einstein equations are restored, if matter does not couple with the connection.

When matter couples the connection, the hyper energy-momentum tensor will appear in the basic equations. Such a term will have a contribution from the Lagrange multiplier. The modified Einstein equations read,

$$\tag{2.324}$$

$$G_{\mu\nu} = M_{\rm Pl}^{-2}\left(T_{\mu\nu} + \tau_{\mu\nu} + \Delta\tau_{\mu\nu}\right)\,, \tag{2.325}$$

with

$$\Delta\tau_{\mu\nu} = -\lambda^\alpha\frac{\delta C_\alpha}{\delta g^{\mu\nu}}\,. \tag{2.326}$$

For Models III(a) and (c), $C_\alpha$ does not contain the metric $g_{\mu\nu}$, thus there is no contribution from the Lagrangian multiplier. However for Model III (b), since $\mathcal{Q}_\lambda = g_{\alpha\beta}g^{\lambda\sigma}\kappa^{\beta}{}_{\lambda\sigma} + \kappa^{\lambda}{}_{\mu\lambda}$ there appears an extra term of $\Delta\tau_{\mu\nu}$. In such case, it could be calculated as

$$\Delta\tau_{\mu\nu} = -\lambda_{(\mu}\kappa_{\nu)}{}^{\lambda}{}_{\lambda} + \lambda_\alpha\kappa^{\alpha}{}_{(\mu\nu)}\,. \tag{2.327}$$

Nonetheless, the constraint equation for $\kappa$ becomes

$$g^{\beta\gamma}\kappa^{\sigma}{}_{\sigma\alpha} + \delta^{\beta}_{\alpha}\kappa^{\gamma\sigma}{}_{\sigma} - \kappa^{\beta\gamma}{}_{\alpha} - \kappa^{\gamma}{}_{\alpha}{}^{\beta} + \lambda^\mu\frac{\delta C_\mu}{\delta\kappa^{\alpha}{}_{\beta\gamma}} = -2M_{\rm Pl}^{-2}\frac{\delta S_{\rm m}}{\delta\kappa^{\alpha}{}_{\beta\gamma}}\,, \tag{2.328}$$

whereas

$$\lambda^\mu \frac{\delta C_\mu}{\delta \kappa^\alpha{}_{\beta\gamma}} = \begin{cases} \delta_\alpha{}^\gamma \lambda^\beta - \delta_\alpha{}^\beta \lambda^\gamma & \text{(Model III(a))} \\ \delta_\alpha{}^\beta \lambda^\gamma + g^{\beta\gamma}\lambda_\alpha & \text{(Model III(b))} \\ \frac{1}{2}\delta_\alpha{}^\gamma \lambda^\beta & \text{(Model III(c))} \end{cases} . \tag{2.329}$$

Although not explicitly shown, these constraint equations (2.328) with $\mathcal{C}_\mu(\kappa) = 0$, could be solved and the solution of the distortion tensor $\kappa^\lambda{}_{\mu\nu}$ is calculable.

As a summary, the following table is the classification of Metric-affine gravity theories.(Table 2.1)

| Models | constraint | properties | Palatini formalism | Metric-affine formalism |
|---|---|---|---|---|
| I | $\frac{\delta S_G}{\delta \Gamma^\lambda{}_{\mu\nu}}\delta^\lambda_\nu = 0$ | projective invariant | Einstein equations $G_{\mu\nu}(g) = M_{\rm Pl}^{-2} T_{\mu\nu}$ | $+\tau_{\mu\nu}$ |
| II (a) | $\mathcal{T}^\lambda{}_{\mu\nu} = 0$ | torsion-free | | |
| II (b) | $\mathcal{Q}^\lambda{}_{\mu\nu} = 0$ | metric compatible | | |
| III (a) | $\mathcal{T}^\lambda = 0$ | $\mathcal{T}^\lambda{}_{\mu\nu} \neq 0$ and $\mathcal{Q}^\lambda{}_{\mu\nu} \neq 0$ in general | | |
| III (b) | $\mathcal{Q}^\lambda = 0$ | | | $+\tau_{\mu\nu} + \Delta\tau_{\mu\nu}$ |
| III (c) | $\mathcal{W}^\lambda = 0$ | | | $+\tau_{\mu\nu}$ |

Table 2.1: The classification of Metric-affine gravity theories.

#### 2.4.3.2 The Dirac Lagrangian in Metric-affine Gravity

Since matter couples to the connection in metric-affine gravity, a simple question is *how* it couples to gravity.

Consider a minimal scalar field $\phi$, a vector field $A^\mu$, and a Dirac field $\psi$. Such action with the matter above could be written as,

$$S = S_{\rm EH}(g, \Gamma) + S_{\rm m}(g, \Gamma, \phi, A, \psi)\,, \tag{2.330}$$

with $S_{\rm m}$ as a matter action which includes the connection. To make the calculation simplied consider the distortion tensor

$$\kappa^\lambda{}_{\mu\nu} := \Gamma^\lambda{}_{\mu\nu} - \{{}^{\,\lambda}_{\mu\nu}\}\,. \tag{2.331}$$

Then the action could be rewritten as,

$$S = S_{\rm EH}(g, \kappa) + S_{\rm m}(g, \kappa, \phi, A, \psi)\,. \tag{2.332}$$

Firstly, the 'conventional' scalar field action is $-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ which, when defined so, by definition does not couple to the connection[23]

For the vectors, the appropriate definition of the covariant field strength is

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu\,, \tag{2.333}$$

which is a tensor regardless of the definition through partial derivatives. Thus, the vector fields do not couple to the connection. This also holds for the Yang-Mills fields. One may propose a field strength like term of,

$${}^\Gamma F_{\mu\nu} := \overset{\Gamma}{\nabla}_\mu A_\nu - \overset{\Gamma}{\nabla}_\nu A_\mu,\,, \tag{2.334}$$

[23] One can couple the connection through the covariantization of the action $L = \frac{1}{2}\phi\Box\phi$. This will be revisited in §3.1.1 and §6.2.

However, this does not have $U(1)$ symmetry and will not be discussed further. For more discussion see for example [96, 97, 98].

As shown, the Standard Model bosons do not couple to the connection, the Dirac field, however, can couple to the connection. Therefore the Dirac field can be a source of inducing connection. In order to see this, one has to use the tetrad-spin connection formalism, since fermions cannot be expressed purely through the metric (and the affine connection). Consider a tetrad $e^a_\mu$ and a spin connection $\omega^{ab}{}_\mu$. Let it be assumed that the tetrad could map coordinate indices to spin indices at any point i.e.,

$$A^a_{\mathrm{PT}}(x+dx) = e^a_\mu(x+dx) A^\mu_{\mathrm{PT}}(x+dx)\,, \tag{2.335}$$

with

$$A^a_{\mathrm{PT}}(x+dx) = A^a(x) - \omega^a{}_{b\mu} A^b(x) dx^\mu\,, \tag{2.336}$$

$$e^a_\mu(x+dx) = e^a_\mu + \partial_\alpha e^a_\mu(x) dx^\alpha\,. \tag{2.337}$$

When done so $(g, \Gamma)$ and $(e, \omega)$ are uniquely related as

$$g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu\,, \tag{2.338}$$

$$\overset{\Gamma}{\mathcal{D}}_\mu e^a_\nu = \partial_\mu e^a_\nu - \Gamma^\alpha{}_{\mu\nu} e^a_\alpha + \omega^a{}_{b\mu} e^b_\nu = 0\,, \tag{2.339}$$

which the latter is none other than the tetrad postulate[24].
The second equation insist that the spin connection could be decomposed as,

$$\omega^{ab}{}_\mu = \Delta^{ab}{}_\mu + \kappa^a{}_\mu{}^b \tag{2.340}$$

where $\Delta^{ab}{}_\mu$ is the Ricci rotation coefficients and

$$\kappa^a{}_\mu{}^b = e^a_\alpha e^b_\beta \kappa^\alpha{}_\mu{}^\beta\,, \tag{2.341}$$

is distortion with respect to the spin indicies.
The covariant derivative of the Dirac field $\psi$ is

$$\overset{\Gamma}{\mathcal{D}}_\mu \psi = \left(\partial_\mu + \frac{1}{8}\omega^{ab}{}_\mu [\gamma_a, \gamma_b]\right)\psi\,, \tag{2.342}$$

with $\gamma_a$ being the gamma matrix which satisfies $\{\gamma_a, \gamma_b\} = -2\eta_{ab}$. One may note that due to the anti-symmetric $[\gamma_a, \gamma_b]$ (2.342) is actually projective invariant.

In metric-affine geometry, there are two possibilities for the Dirac field Lagrangian,

$$\mathcal{L}_D = \frac{i}{2}\left(\bar{\psi}\gamma^\mu \overset{\Gamma}{\mathcal{D}}_\mu \psi - (\overset{\Gamma}{\mathcal{D}}_\mu \bar{\psi})\gamma^\mu \psi\right) - m\bar{\psi}\psi\,, \tag{2.343}$$

or

$$\mathcal{L}'_D = i\bar{\psi}\gamma^\mu \overset{\Gamma}{\mathcal{D}}_\mu \psi - m\bar{\psi}\psi\,, \tag{2.344}$$

[24]Notice that the tetrad postulate (2.339) is not the metricity condition. Non-metricity is related to the spin connection via $Q^{\mu\nu}_\lambda e^a_\mu e^b_\nu = \overset{\Gamma}{D}_\mu \eta^{ab} = \omega^{(ab)}{}_\mu$. Notice that even though, a purely spin $\frac{1}{2}$ only couples to the anti-symmetric part of the spin connection from (2.342) it still couples to some components of non-metricity since $\kappa^{[\mu}{}_\lambda{}^{\nu]} = \frac{1}{2}\left(\mathcal{Q}^{[\mu\nu]}{}_\lambda - T_\lambda{}^{\mu\nu} - \mathcal{T}^{[\mu\nu]}{}_\lambda\right)$. Furthermore, one could expect that spin $\frac{3}{2}$ fields, such as the gravitinos or Rarita-Schwinger particles, full components of non-metricity couples to fermions.

with $\gamma^\mu = e^\mu_a \gamma^a$. In Riemannian geometry, these Lagrangian are equivalent up to surface terms. However, for metric-affine geometry, this is not the case. This could be see since, the two Lagrangians each computes a different type of interaction as,

$$\mathcal{L}_{\rm int} = -\frac{1}{4}\epsilon^{\alpha\beta\gamma\delta}\kappa_{\alpha\beta\gamma}j^5_\delta\,, \tag{2.345}$$

and

$$\mathcal{L}'_{\rm int} = -\frac{1}{4}\epsilon^{\alpha\beta\gamma\delta}\kappa_{\alpha\beta\gamma}j^5_\delta + \frac{i}{2}\kappa^{[\alpha}{}_\beta{}^{\beta]}j_\alpha\,. \tag{2.346}$$

Here the current and the axial current of the Dirac field were defined as,

$$j_\mu = \bar{\psi}\gamma_\mu\psi\,, \quad j^\mu_5 = \bar{\psi}\gamma_\mu\gamma^5\psi\,, \tag{2.347}$$

with $\gamma^5 = -\frac{i}{4!}\epsilon^{\alpha\beta\gamma\delta}\gamma_\alpha\gamma_\beta\gamma_\gamma\gamma_\delta$ and $\epsilon^{\alpha\beta\gamma\delta}$ is the Levi-Civita tensor. In both cases, the interaction between the connection and the Dirac field, distortion is induced i.e. $\kappa^\mu{}_{\alpha\beta} \neq 0$. In such a case, even in Einstein gravity the equivalence between the metric formalism and the metric-affine formalism does not hold anymore. This is quite well known in the context of the Einstein-Cartan-Sciama-Kibble theory and the theory where the fermion dependent connection is integrated out is called the Hehl-Datta Lagrangian [99, 96, 84].

#### 2.4.3.3 Dynamics in Metric-affine Formalism: Autoparallel Equations

Metric-affine formalism is a formalism in which matter couples not only the metric but also the connection. Thus, one would expect that the dynamics of matter will also be affected by the connection. A naive expectation would be that matter follows autoparallels,

$$u^\sigma \overset{\Gamma}{\nabla}_\sigma u^\nu = 0\,, \tag{2.348}$$

instead of the geodesics. However, as it was briefly mentioned in §2.1.2.1, autoparallel equations are not integrable, i.e. does not have a Lagrangian. Therefore, when one uses the Lagrangian picture as a basis for physics, *one cannot obtain autoparallels as dynamics of any matter.* One can show this using the Helmholtz conditions and basic tensorial properties as outlined in §B.1.2.

Thus, with inconsistency aside, if one *wants* to impose that matter follows autoparallels, instead of geodesics, one has to

1. Use a purely equation of motion picture without using Lagrangians
2. Redefine the variational principle itself to include connectionful effects within dynamics

The latter approach was proposed by Kleinert and Pelster in 1996 [100] for metric compatible connection.

#### 2.4.3.4 Dynamics in Metric-affine Formalism: WKB Approximation

Since general metric-affine matter does not seem to follow autoparallels, the question arises as to what kind of dynamics does it have. In §2.1.2.2 it was shown that better ways to find what dynamics matter follows are through using geometric optics, WKB approximation, and the Fermat principle. Standard Model scalars and vectors, generally do not couple to the connection[25], and thus will follow geodesics. However, as seen in §2.4.3.2, fermions in general couple to torsion. Thus one may use WKB approximation to see what dynamics these torsion coupled fermions follow [101, 102, 103].

[25] Naively, one would expect the covariantization $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ couple to the connection. However, this would violate the U(1) symmetry. Furthermore, $F_{\mu\nu}$ is already a tensor even when it is written with partial derivatives. Thus, unless exotic situations, one would expect that the connection does not couple to the vector fields.

First covariant derivatives of spinors for some connection are

$$\overset{\Gamma}{\mathcal{D}}_\mu \Psi = \partial_\mu \Psi + \frac{1}{4} e^b_\nu (\overset{\Gamma}{\mathcal{D}}_\mu e^\nu_a) \gamma_b \gamma^a \Psi \,, \tag{2.349}$$

$$\overset{\Gamma}{\mathcal{D}}_\mu \bar{\Psi} = \partial_\mu \bar{\Psi} - \frac{1}{4} \bar{\Psi} e^b_\nu (\overset{\Gamma}{\mathcal{D}}_\mu e^\nu_a) \gamma_b \gamma^a \,. \tag{2.350}$$

So one could transform to another connection, using the distortion trick as

$$\gamma^\mu \overset{\Gamma}{\mathcal{D}}_\mu \Psi = \gamma^\mu \overset{g}{\mathcal{D}}_\mu \Psi + \frac{1}{4} \kappa_{\nu\mu\lambda} \gamma^\mu \gamma^{[\nu} \gamma^{\lambda]} \Psi \,. \tag{2.351}$$

Then consider a minimally coupled Dirac Lagrangian

$$\mathcal{L}_{Dirac} = \frac{i}{2} (\bar{\Psi} \gamma^\mu \overset{\Gamma}{\mathcal{D}}_\mu \Psi - (\overset{\Gamma}{\mathcal{D}}_\mu \bar{\Psi}) \gamma^\mu \Psi) - m \bar{\Psi} \Psi \,, \tag{2.352}$$

which has the equation of motion,

$$0 = i\hbar \left( \gamma^\mu \overset{\Gamma}{\mathcal{D}}_\mu \Psi - \frac{1}{2} \gamma^\mu \kappa^\lambda{}_{\lambda\mu} \Psi \right) - m\Psi \tag{2.353}$$

$$= i\hbar \left( \gamma^\mu \overset{g}{\mathcal{D}}_\mu \Psi - \frac{1}{4} \kappa_{[\alpha\beta\gamma]} \gamma^{[\alpha} \gamma^\beta \gamma^{\gamma]} \Psi \right) - m\Psi \,, \tag{2.354}$$

since,

$$\gamma^\mu \gamma^{[\nu} \gamma^{\lambda]} = \gamma^{[\mu} \gamma^\nu \gamma^{\lambda]} + 2 g^{\mu[\nu} \gamma^{\lambda]} \,.$$

Obviously, of all the components, the fully anti-symmetric part of the connection solely couples to fermions. Thus for what follows, the connection

$$\Gamma^{*\lambda}{}_{\mu\nu} = \left\{ \begin{array}{c} \lambda \\ \mu\nu \end{array} \right\} + g^{\lambda\sigma} \mathcal{T}_{[\sigma\mu\nu]} \,, \tag{2.355}$$

and its covariant derivative $\nabla^*_\mu$ and $\mathcal{D}^*_\mu$ will be used.

Now consider expanding the spinor as

$$\Psi(x) = \sum_n e^{\frac{i}{\hbar} S(x)} (-i\hbar)^n a_n(x) \,, \tag{2.356}$$

with $a_n$ having the properties of a spinor.
Substituting this into the Dirac equations lead to

$$0 = \sum_n \left[ (i\hbar)^n \left\{ \gamma^\mu \partial_\mu S(x) + m \right\} a_n(x) + (i\hbar)^{n+1} \left\{ \gamma^\mu \overset{g}{\mathcal{D}}_\mu a_n(x) - \frac{1}{4} \mathcal{T}_{[\alpha\beta\gamma]} \gamma^\alpha \gamma^\beta \gamma^\gamma a_n(x) \right\} \right] \,, \tag{2.357}$$

Which computes for zeroth order of $\hbar$,

$$\left\{ \gamma^\mu \partial_\mu S(x) + m \right\} a_0(x) = 0 \,, \tag{2.358}$$

and first-order of $\hbar$

$$\left\{ \gamma^\mu \partial_\mu S(x) + m \right\} a_1(x) = -\gamma^\mu \overset{g}{\mathcal{D}}_\mu a_0(x) + \frac{1}{4} \mathcal{T}_{[\alpha\beta\gamma]} \gamma^\alpha \gamma^\beta \gamma^\gamma a_0(x) \,, \tag{2.359}$$

In order for the first equation to have a non-trivial $a_0$ it must be $\det(\gamma^\mu \overset{g}{\nabla}_\mu S(x) + m) = 0$ leads to using $\partial_\mu S(x) = k_\mu$ and $\gamma^{(\mu}\gamma^{\nu)} = -g^{\mu\nu}$

$$\begin{aligned} k^\mu k_\mu &= \gamma^{(\mu}\gamma^{\nu)}\partial_\mu S \partial_\nu S \\ &= -m^2\,, \end{aligned}$$

which is the Hamilton-Jacobi equation.
Now redefining the unit velocity $u_\mu = \frac{1}{m} k_\mu$ and $\overset{g}{\nabla}_\mu u_\nu = \frac{1}{m}\overset{g}{\nabla}_\mu \overset{g}{\nabla}_\nu S = \overset{g}{\nabla}_\nu u_\mu$

$$\begin{aligned} 0 &= \overset{g}{\nabla}_\mu (u^\nu u_\nu) \\ &= u^\nu \overset{g}{\nabla}_\mu u_\nu \\ &= u^\nu \overset{g}{\nabla}_\nu u_\mu\,, \end{aligned} \tag{2.360}$$

(2.361)

This indicates that up to the zeroth order of $\hbar$ the Dirac particle follows the metric geodesic and not of auto-parallels.

In order to calculate the first-order in $\hbar$, one needs to introduce Gordon decomposition [26] of the current $j_\mu = \bar{\Psi}\gamma_\mu \Psi$ as

$$j^\mu = j^\mu_M + j^\mu_c\,, \tag{2.362}$$

where

$$j^\mu_M = \frac{\hbar}{2m}\overset{g}{\mathcal{D}}_\nu (\bar{\Psi}\sigma^{\mu\nu}\Psi)\,, \tag{2.363}$$

$$j_{c\mu} = \frac{\hbar}{2mi}\left[(\mathcal{D}^*_\mu \bar{\Psi})\Psi - \bar{\Psi}\mathcal{D}^*_\mu \Psi\right]\,. \tag{2.364}$$

The current $j_M$ could be thought as the magnetization current and the current $j_c$ as the convection current. Physically, a particle orbits is characterized through electromagnetic measurements, thus the motion of a particle $v^\mu$ should be defined from the convection current as

$$v^\mu = \frac{j^\mu_c}{\sqrt{-j^\nu_c j_{c\nu}}}\,, \tag{2.365}$$

which is normalized to $v^\mu v_\mu = -1$

[26]The decomposition of the current is calculated from

$$\begin{aligned} \frac{2m}{\hbar} j^\mu_M &= i\overset{g}{\mathcal{D}}_\nu \bar{\Psi}(g^{\mu\nu} - \gamma^\nu\gamma^\mu)\Psi - i\bar{\Psi}(g^{\mu\nu} - \gamma^\mu\gamma^\nu)\overset{g}{\mathcal{D}}_\nu \Psi \\ &= ig^{\mu\nu}\left((\overset{g}{\mathcal{D}}_\nu\bar{\Psi})\Psi - \bar{\Psi}\overset{g}{\mathcal{D}}_\nu\Psi\right) \\ &\quad -\bar{\Psi}\left(-\frac{m}{\hbar} + \frac{1}{4}\kappa_{\alpha\beta\gamma}\gamma^{[\alpha}\gamma^\beta\gamma^{\gamma]}\right)\gamma^\mu\Psi + \bar{\Psi}\gamma^\mu\left(\frac{m}{\hbar} + \frac{1}{4}\kappa_{\alpha\beta\gamma}\gamma^{[\alpha}\gamma^\beta\gamma^{\gamma]}\right)\Psi \\ &= ig^{\mu\nu}\left((\overset{g}{\mathcal{D}}_\nu\bar{\Psi})\Psi - \bar{\Psi}\overset{g}{\nabla}_\nu\Psi\right) + \bar{\Psi}\kappa_{\alpha\beta\gamma}\left[\gamma^\mu, \gamma^{[\alpha}\gamma^\beta\gamma^{\gamma]}\right]\Psi + \frac{2m}{\hbar}j^\mu \\ &= -\frac{2m}{\hbar}j^\mu_c + \frac{2m}{\hbar}j^\mu \end{aligned}$$

Note *the particle must be massive*

The convection current could be expanded as, using $\partial_\mu S = m u_\mu$

$$j_{c\mu} = f^2 u_\mu - i\hbar u_\mu(\bar{a}_0 a_1 - \bar{a}_1 a_0) + \frac{\hbar}{2mi} f^2 \left[(\mathcal{D}^*_\mu \bar{b}_0) b_0 - \bar{b}_0 \mathcal{D}^*_\mu b_0\right] + \mathcal{O}(\hbar^2)\,. \tag{2.366}$$

The derivation of above is as follows. First let the spinor part be normalized as $a_0(x) = f(x) b_0(x)$ with $\bar{b}_0 b_0 = 1$
Using $\Psi(x) = \sum_n e^{\frac{i}{\hbar} S(x)} (-i\hbar)^n a_n(x)$,

$$\begin{aligned} j_{c\mu} &= \frac{\hbar}{2mi} \left[(\mathcal{D}^*_\mu \bar{\Psi})\Psi - \bar{\Psi} \mathcal{D}^*_\mu \Psi\right] \\ &= \frac{\hbar}{mi} \sum_n (-i\hbar)^{n+m} (-1)^m \left[\frac{2mi}{\hbar} \bar{a}_m a_n u_\mu + (\mathcal{D}^*_\mu \bar{a}_m) a_n - \bar{a}_m \mathcal{D}^*_\mu a_n\right] \\ &= \frac{\hbar}{2mi} \left[\frac{2mi}{\hbar} \bar{a}_0 a_0 u_\mu + (-i\hbar) \frac{2mi}{\hbar} (\bar{a}_0 a_1 - \bar{a}_1 a_0) u_\mu + (\mathcal{D}^*_\mu \bar{a}_0) a_0 - \bar{a}_0 \mathcal{D}^*_\mu a_0\right] + \mathcal{O}(\hbar^2) \\ &= f^2 u_\mu - i\hbar(\bar{a}_0 a_1 - \bar{a}_1 a_0) u_\mu + \frac{\hbar}{2mi} f^2 \left[(\mathcal{D}^*_\mu \bar{b}_0) b_0 - \bar{b}_0 \mathcal{D}^*_\mu b_0\right] + \mathcal{O}(\hbar^2)\,, \end{aligned}$$

Since $u^\mu \mathcal{D}^*_\mu b_0 = 0$

$$v_\mu = \frac{j_{c\mu}}{\sqrt{-j_{c\nu} j_c^\nu}} = +\frac{\hbar}{2mi} \left[(\mathcal{D}^*_\mu \bar{b}_0) b_0 - \bar{b}_0 \mathcal{D}^*_\mu b_0\right] + \mathcal{O}(\hbar^2)\,.$$

So the motion $v^\mu$ could be expanded up to second order in $\hbar$ as

$$v_\mu = u_\mu + \frac{\hbar}{2mi} \left[(\mathcal{D}^*_\mu \bar{b}_0) b_0 - \bar{b}_0 \mathcal{D}^*_\mu b_0\right] + \mathcal{O}(\hbar^2)\,. \tag{2.367}$$

Now by using

$$\mathcal{D}^*_{[\alpha} \mathcal{D}^*_{\beta]} \Psi = \frac{1}{8} R^*_{\mu\nu\alpha\beta} \gamma^{[\mu} \gamma^{\nu]} \Psi - \Gamma^{*\lambda}_{\ [\alpha\beta]} \mathcal{D}^*_\lambda \Psi\,, \tag{2.368}$$

$$\mathcal{D}^*_{[\alpha} \mathcal{D}^*_{\beta]} \bar{\Psi} = -\frac{1}{8} R^*_{\mu\nu\alpha\beta} \bar{\Psi} \gamma^{[\mu} \gamma^{\nu]} - \Gamma^{*\lambda}_{\ [\alpha\beta]} \mathcal{D}^*_\lambda \bar{\Psi}\,. \tag{2.369}$$

Then the particle motion for the Dirac particle becomes

$$m v^\lambda \overset{g}{\nabla}_\lambda v_\alpha = \frac{\hbar}{4} R^*_{\mu\nu\alpha\beta} \bar{b}_0 \sigma^{\mu\nu} b_0 u^\beta + \mathcal{O}(\hbar^2) \tag{2.370}$$

$$= \frac{\hbar}{4} R^*_{\mu\nu\alpha\beta} S_0^{\mu\nu} v^\beta + \mathcal{O}(\hbar^2)\,. \tag{2.371}$$

With zeroth order spin density tensor defined as $S_0^{\mu\nu} = \bar{b}_0 \sigma^{\mu\nu} b_0$ from,

$$S^{\mu\nu} \equiv \frac{\bar{\Psi} \sigma^{\mu\nu} \Psi}{\bar{\Psi}\Psi} \tag{2.372}$$

$$= \bar{b}_0 \sigma^{\mu\nu} b_0 + \mathcal{O}(\hbar)\,, \tag{2.373}$$

and $u^\mu = v^\mu + \mathcal{O}(\hbar)$
As a result, a charged Dirac particle follows geodesics at zeroth order of $\hbar$ and deviates from the first order where the torsion effects then take place. This is important when one considers metric-affine gravity and their subsets since this difference is expected to directly appear in observations[104].

## 2.5 Scalar Tensor Theories

### 2.5.1 The Quest for the Most General Scalar-Tensor Theory and the Ostrogradsky Ghost

As outlined in §2.2, the incomprehensible phenomena of the universe suggest a possibility of theories beyond Einstein gravity. Since the universe is ridiculously homogeneous, if any other field besides the metric could be key, it would be scalars. However, one cannot randomly conjure theories with scalars and tensors. This is due to what is known as *Ostrogradsky instability* [105, 106, 107].

Firstly, before going to field theories, recall a simple harmonic oscillator having some mass $m$ and frequency $\omega$ [107],

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2 \,. \tag{2.374}$$

The Hamiltonian, with the conjugate momenta $\pi_x = m\dot{x}$, is

$$H = \frac{\pi_x^2}{2m} + \frac{1}{2}m\omega^2 x^2 \,, \tag{2.375}$$

which is clearly bounded from below.

Now consider a theory with higher-derivatives [107],

$$L = -\frac{\epsilon m}{2\omega^2}\ddot{x}^2 + \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2 \,. \tag{2.376}$$

Such theory has 4 derivatives in its equation of motion and thus needs 4 initial data in order for the dynamics to be determined. The general solution for the dynamics is,

$$x(t) = C_+ \cos(k_+ t) + S_+ \sin(k_+ t) + C_- \cos(k_- t) + S_- \sin(k_- t) \,, \tag{2.377}$$

with

$$k_\pm = \omega\sqrt{\frac{1 \mp \sqrt{1-4\epsilon}}{2\epsilon}} \,. \tag{2.378}$$

In order to go to the Hamiltonian picture properly, one has to redefine the higher derivative into lower derivatives using a Lagrangian multiplier,

$$L' = -\frac{\epsilon m}{2\omega^2}\dot{y}^2 + \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2 + \lambda(y - \dot{x}) \,. \tag{2.379}$$

Then, the conjugate momenta being $\pi_x = m\dot{x} - \lambda$,$\pi_y = -\frac{\epsilon m}{\omega^2}\dot{y}$,$\pi_\lambda = 0$, the Hamiltonian becomes,

$$H = \frac{\pi_x^2}{2m} - \frac{\omega^2}{2\epsilon m}\pi_y^2 + \frac{\lambda^2}{2m} + \frac{1}{2}m\omega^2 x^2 - \lambda y \tag{2.380}$$

which is not bounded from below[27]. Furthermore, by substituting the solution, the Hamiltonian becomes,

$$H = \frac{1}{2}m\sqrt{1-4\epsilon}\,k_+^2(C_+^2 + S_+^2) - \frac{1}{2}m\sqrt{1-4\epsilon}\,k_-^2(C_-^2 + S_-^2) \,, \tag{2.381}$$

[27] There are two second-class constraints for this system $C_1 = \pi_\lambda \approx 0$ and $C_2 = my - \lambda$. Thus there are 4 degrees of freedom in phase space, which computes 2 dynamical degrees of freedom. Thus, even though the original Lagrangian apparently has one degree of freedom $x(t)$, there exists a hidden (ghostly) degree of freedom.

which indicates a mode $(-)$ that has negative energy. Such unstable ghost mode is called Ostrogradsky ghost. Ghosts, interacting with other matter, take energy from other matter and render the physical system unstable.

This result is similar for field theories: higher derivatives could possibly cause instability. Thus the question is: *what is the most general equation of motion with up to second-order of derivatives of $\phi$ and $g_{\mu\nu}$?*. This question will be revisited in §2.5.3

### 2.5.2 Galileons

Originally proposed by Nicolis, Rattazzi, and Trincherini in 2008 [108], Galileons were initially introduced to construct theories with ghost-less self-accelerating solutions, which also naturally arises in braneworld models such as the DGP Model [109].

$$\pi \to \pi + b_\mu x^\mu + c \,, \tag{2.382}$$

with $b_\mu$ and $c$ being some constants. The most general theory having such symmetry is [108],

$$\begin{aligned} \mathcal{L} &= c_1\phi + c_2\bar{X} - c_3\bar{X}\overset{\eta}{\Box}\phi + c_4\bar{X}\left[(\overset{\eta}{\Box}\phi)^2 - \partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi\right] \\ &\quad -\frac{1}{3}c_5\bar{X}\left[(\overset{\eta}{\Box}\phi)^3 - 3\overset{\eta}{\Box}\phi\partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi + 2\partial_\mu\partial_\nu\phi\partial^\nu\partial^\lambda\phi\partial_\lambda\partial^\mu\phi\right] \,, \end{aligned} \tag{2.383}$$

with the flat space kinetic scalar defined as $\bar{X} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. This theory cannot be simply covariantized by replacing $\partial_\mu \to \overset{g}{\nabla}_\mu$, however, since in curved space covariant derivatives do not commute. For example terms such as,

$$\overset{g}{\nabla}_\mu\left(\overset{g}{\nabla}^\mu\overset{g}{\Box}\phi - \overset{g}{\Box}\partial^\mu\phi\right) = -\overset{g}{\nabla}^\mu\left(\overset{g}{R}_{\mu\nu}\partial^\nu\phi\right) \,, \tag{2.384}$$

compute higher-derivatives, thus having Ostrogradky instability. To avoid this, one needs to add curvature terms in the original action to eliminate such higher-derivatives. Once done so, one obtains the covariant Galileon theory,

$$\begin{aligned} \mathcal{L} &= c_1\phi + c_2X - c_3X\overset{g}{\Box}\phi + \frac{1}{2}c_4X^2\overset{g}{R} + c_4X\left[(\overset{g}{\Box}\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}\right] \\ &\quad c_5X^2\overset{g}{G}^{\mu\nu}\phi_{\mu\nu} - \frac{1}{3}c_5X\left[(\overset{g}{\Box}\phi)^3 - 3\overset{g}{\Box}\phi\phi_{\mu\nu}\phi^{\mu\nu} + 2\phi_{\mu\nu}\phi^{\nu\lambda}\phi^\mu_\lambda\right] \,, \end{aligned} \tag{2.385}$$

with the curved space-time kinetic scalar defined as $X = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. Here the shorthand notation $\phi_{\mu\nu} = \overset{g}{\nabla}_\mu\overset{g}{\nabla}_\nu\phi$ was introduced.

### 2.5.3 Horndeski

The discovery and construction of the Galileons further led to the investigation of the generalization of scalar-tensor theories. During such quest, the Horndeski theory was re-discovered [110]. Horndeski theory is known to be the most general scalar-tensor theory that has at most second-order derivatives in its equation of motion. For what follows, the proof of Horndeski will be shown. For a comprehensive review of the subject, see [111].

Firstly, consider a Lagrangian constructed with respect to $g_{\mu\nu}$ and $\phi$ with arbitrary number of derivatives,

$$\mathcal{L} = \mathcal{L}(g_{\mu\nu}, \partial_\lambda g_{\mu\nu}, \cdots; \phi, \phi_\mu, \cdots). \tag{2.386}$$

The equations of motion for both $g_{\mu\nu}$ and $\phi$ are,

$$E^{\mu\nu} = \frac{1}{\sqrt{-g}}\frac{\delta\sqrt{-g}L}{\delta g_{\mu\nu}}, \tag{2.387}$$

$$E_\phi = \frac{\delta L}{\delta\phi}. \tag{2.388}$$

Now consider that the Lagrangian is invariant with respect to diffeomorphism, i.e. $\delta_\xi g_{\mu\nu} = 2\overset{g}{\nabla}_{(\mu}\xi_{\nu)}$ and $\delta_\xi\phi = \xi^\mu\phi_\mu$. Thus

$$\begin{aligned} 0 &= \delta_\xi\left(\int d^4x\sqrt{-g}\mathcal{L}\right) \\ &= \int d^4x\sqrt{-g}\left(2E^{\mu\nu}\overset{g}{\nabla}_\mu\xi_\nu + E_\phi\phi^\mu\xi_\mu\right) \\ &= -2\int d^4xs\sqrt{-g}\left(\overset{g}{\nabla}_\mu E^{\mu\nu} - \frac{1}{2}E_\phi\phi^\nu\right)\xi_\nu + \text{s.t.}, \end{aligned} \tag{2.389}$$

Since this must hold for any coordinate transformation, so for arbitrary $\xi^\mu$, the following relation is derived,

$$\overset{g}{\nabla}_\mu E^{\mu\nu} = \frac{1}{2}E_\phi\phi^\nu. \tag{2.390}$$

Then the question then boils down to finding the most general (symmetric) tensor of the form $A^{\mu\nu} = A^{\mu\nu}(g_{\mu\nu}g_{\mu\nu,\lambda}, g_{\mu\nu,\lambda\sigma}, \phi, \phi_\mu, \phi_{,\mu\nu})$ such that $\overset{g}{\nabla}_\mu A^{\mu\nu}$ is also up to second-derivatives of $\phi$ and $g_{\mu\nu}$. Here $\phi_{,\mu\nu} = \partial_\mu\partial_\nu\phi$, $g_{\mu\nu,\lambda} := \partial_\lambda g_{\mu\nu}$ and $g_{\mu\nu,\lambda\sigma} := \partial_\sigma\partial_\lambda g_{\mu\nu}$.

Again, introduce the following shorthand notation,

$$B^{;\mu\nu,\rho\sigma} = \frac{\partial B}{\partial g_{\mu\nu,\rho\sigma}}, \tag{2.391}$$

$$B^{|\mu\nu} = \frac{\partial B}{\partial\phi_{,\mu\nu}}, \tag{2.392}$$

where $B$ is some arbitrary rank tensor.

As a next step, consider explicitly calculating $\overset{g}{\nabla}_\mu A^{\mu\nu}$ as,

$$\overset{g}{\nabla}_\mu A^{\mu\nu} = A^{\mu\nu;\alpha\beta,\gamma\delta}g_{\alpha\beta,\gamma\delta\mu} + A^{\mu\nu|\alpha\beta}\phi_{,\alpha\beta\mu} + \cdots, \tag{2.393}$$

with $g_{\alpha\beta,\gamma\delta\mu} := \partial_\mu\partial_\gamma\partial_\delta g_{\alpha\beta}$ and$\phi_{,\alpha\beta\mu} = \partial_\alpha\partial_\beta\partial_\mu\phi$. For $\overset{g}{\nabla}_\mu A^{\mu\nu}$ to not have third-order derivatives,

$$0 = A^{\mu\nu;\alpha\beta,\gamma\delta}\frac{\partial g_{\alpha\beta,\gamma\delta\mu}}{\partial g_{\nu\epsilon,\eta\lambda\sigma}}, \tag{2.394}$$

$$0 = A^{\mu\nu|\alpha\beta}\frac{\partial\phi_{,\alpha\beta\mu}}{\partial\phi_{,\eta\lambda\sigma}}, \tag{2.395}$$

that reduces to

$$0 = A^{\mu(\nu;|\alpha\beta|,\gamma\delta)} \propto A^{\mu\nu;\alpha\beta,\gamma\delta} + A^{\mu\gamma;\alpha\beta,\delta\nu} + A^{\mu\delta;\alpha\beta,\nu\gamma} \tag{2.396}$$

$$0 = A^{\mu(\nu|\alpha\beta)} \propto A^{\mu\nu|\alpha\beta} + A^{\mu\alpha|\beta\nu} + A^{\mu\beta|\nu\alpha}, \tag{2.397}$$

which, with repeated permutation, further reduces to

$$A^{\mu\nu;\alpha\beta,\gamma\delta} = A^{\gamma\delta;\alpha\beta,\mu\nu} = A^{\alpha\beta;\mu\nu,\gamma\delta}, \tag{2.398}$$

$$A^{\mu\nu|\alpha\beta} = A^{\alpha\beta|\mu\nu}. \tag{2.399}$$

Following [110, 112], let the following "Property S" be defined.

Property S

A tensor $A^{\alpha_1\alpha_2\cdots\alpha_{2n-1}\alpha_{2n}}$ is said to have property S when it follows the conditions;

1. $A$ is symmetric for some arbitrary $(\alpha_{2l-1}\alpha_{2l})$ for $l = 1, 2, \cdots, n$.
2. $A$ is symmetric for interchanging of some pair of arbitrary $(\alpha_{2l-1}\alpha_{2l})$ and $(\alpha_{2m-1}\alpha_{2m})$ for $l, m = 1, 2, \cdots, n$.
3. $A$ vanishes for symmetrization of three out of four arbitrary indices of $(\alpha_{2l-1}\alpha_{2l})$ and $(\alpha_{2m-1}\alpha_{2m})$ for $l, m = 1, 2, \cdots, n$.

When a tensor $A^{\alpha_1\alpha_2\cdots\alpha_{2n-1}\alpha_{2n}}$ of $n > 4$ has property S, such tensor is null in 4 dimensions [110].[2]

From (2.398)-(2.399), one obtains that the following tenth-order tensors are of property S and thus null,

$$A^{\mu\nu;\alpha\beta,\gamma\delta;\epsilon\eta,\rho\sigma} = 0, \tag{2.400}$$

$$A^{\mu\nu;\alpha\beta,\gamma\delta|\epsilon\eta|\rho\sigma} = 0, \tag{2.401}$$

$$A^{\mu\nu|\alpha\beta|\gamma\delta|\epsilon\eta|\rho\sigma} = 0. \tag{2.402}$$

These equations show that $A^{\mu\nu}$ is at most linear with respect to $g_{\alpha\beta,\gamma\delta}$ thus $R^{\lambda}{}_{\sigma\mu\nu}$ and cubic in $\phi_{,\mu\nu}$ thus in $\phi_{\mu\nu}$. Now using the same techniques of Lovelock (see (2.237)), one can write all possible independent terms that are constructed from the metric.

Thus the most general symmetric tensor $A^{\mu\nu}$ that is up to second-derivatives of $g_{\mu\nu}$ and $\phi$ such that $\overset{g}{\nabla}_\mu A^{\mu\nu}$ also is so, while still respecting the symmetry of $A^{\mu\nu}$ is,

$$\begin{aligned} A^{\alpha\beta} &= \delta^{\alpha\gamma\delta\epsilon}_{\mu\nu\rho\sigma} g^{\mu\beta} \left\{ (K_1\phi^{\nu}_{\gamma} + K_3\phi_\gamma\phi^\nu) R_{\delta\epsilon}{}^{\rho\sigma} + (K_4\phi^{\nu}_{\gamma} + K_6\phi_\gamma\phi^\nu)\phi^{\rho}_{\delta}\phi^{\sigma}_{\epsilon} \right\} \\ &\quad + \delta^{\alpha\gamma\delta}_{\epsilon\mu\nu} g^{\epsilon\beta} \left\{ K_2 R_{\gamma\delta}{}^{\mu\nu} + (K_5\phi^{\mu}_{\gamma} + K_8\phi_\gamma\phi^\mu)\phi^{\nu}_{\delta} \right\} + K_7\delta^{\alpha\gamma}_{\delta\epsilon} g^{\delta\beta}\phi^{\epsilon}_{\gamma} + K_9 g^{\alpha\beta} + K_{10}\phi^\alpha\phi^\beta, \end{aligned} \tag{2.403}$$

[2]Although the definition looks tedious, it is straightforward to see that indeed such a tensor is null when one uses concrete indices. For example a 4 dimensional tensor that has this property can be a tenth-order tensor $A^{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6\mu_7\mu_8\mu_9\mu_{10}}$. This tensor $A$ can have at least have 2×3 indices that overlap due to the pigeonhole principle. For example $A^{0123012301}$ has 3 0s and 1s. One can then 'collect' these indices together using conditions 1. and 2. , such as

$$A^{0123012301} = A^{0101012323}.$$

Then,

$$\begin{aligned} 0 &= A^{(010)1012323} + A^{(011)0012323} \\ &\propto A^{0101012323} + A^{1001012323} + A^{0011012323} + A^{0110012323} + A^{1100012323} + A^{1100012323} \\ &= 3A^{0101012323} + 3A^{1100012323} \end{aligned}$$

where the first line used condition 3. and the third line condition 1. and 2.. The final term is zero since $A^{1100012323} = A^{11(000)12323} = 0$ from condition 3.

Similar calculations can be done for an arbitrary combination of indices of $A$. Thus indeed this tenth-order tensor is null if it has property S.

where $K_i$ are functions of $K_i = K_i(\phi, X)$. These 10 functions are not all independent in the light of integrability, however, since the following must be also satisfied,

$$\overset{g}{\nabla}_\mu A^{\mu\nu} = Q\phi^\mu. \tag{2.404}$$

By first calculating $\overset{g}{\nabla}_\mu A^{\mu\nu}$ and then imposing the terms that are not proportional to $\phi^\nu$ to disappear, the following 8 conditions for the functions $K_i$ are obtained,

$$\begin{aligned}
0 &= 2K_{1\phi} - 2K_3 + K_5 - 2XK_6, &(2.405)\\
0 &= -K_{2X} - K_{1\phi} + K_3 + 2XK_{3X}, &(2.406)\\
0 &= -2K_{3X} + K_6, &(2.407)\\
0 &= -K_{1X} + \frac{3}{2}K_4, &(2.408)\\
0 &= 2K_{2\phi} + \frac{1}{2}K_7 - XK_8, &(2.409)\\
0 &= -K_{5X} + 3K_6 - 3K_{4\phi} - XK_{6X}, &(2.410)\\
0 &= -\frac{1}{2}K_{7X} - K_{5\phi} + K_8 + XK_{8X}, &(2.411)\\
0 &= -K_{9X} + K_{10} - K_{7\phi} &(2.412)
\end{aligned}$$

where $K_{i\phi} = \frac{\partial K_i}{\partial \phi}$ and $K_{iX} = \frac{\partial K_i}{\partial X}$. Out of these 8 conditions only 6 are independent, thus eliminating 6 functions of $K_i$ as

$$\begin{aligned}
K_2 &= \frac{1}{2}F + W, &(2.413)\\
K_4 &= \frac{2}{3}K_{1X}, &(2.414)\\
K_5 &= 2K_3 - 2K_{1\phi} + 4XK_{3X}, &(2.415)\\
K_6 &= +2K_{3X}, &(2.416)\\
K_7 &= -2F_\phi - 4W_\phi + 2XK_8, &(2.417)\\
K_{10} &= -2F_{\phi\phi} - 4W_{\phi\phi} + 2XK_{8\phi} + K_{9X}. &(2.418)
\end{aligned}$$

Here $W$ is some function of $\phi$ and $F = F(\phi, X)$ is defined as,

$$F = -2\int^X \left(K_{1\phi}(\phi, X') - K_3(\phi, X') - 2X'K_{3X}(\phi, X')\right) dX' \tag{2.419}$$

Finally, using methods such as the Helmholtz conditions and the Vainberg-Tonti trick §B.2, the Lagrangian $\mathcal{L}_H$ that computes the equation of motion $\frac{1}{\sqrt{-g}}\frac{\delta\sqrt{-g}\mathcal{L}_H}{\delta g_{\mu\nu}} = A^{\mu\nu}$ and $\frac{\delta \mathcal{L}_H}{\delta\phi}$ can be derived. Thus the *original* Horndeski Lagrangian is of the form [110, 113],

$$\begin{aligned}
\mathcal{L}_H &= \delta^{\alpha\beta\gamma}_{\mu\nu\sigma}\left[\kappa_1\phi^\mu_\alpha R_{\beta\gamma}{}^{\nu\sigma} + \frac{2}{3}\kappa_{1X}\phi^\mu_\alpha\phi^\nu_\beta\phi^\sigma_\gamma + \kappa_3\phi_\alpha\phi^\mu R_{\beta\gamma}{}^{\nu\sigma} + 2\kappa_{3X}\phi_\alpha\phi^\mu\phi^\nu_\beta\phi^\sigma_\gamma\right]\\
&\quad + \delta^{\alpha\beta}_{\mu\nu}\left[(\mathcal{F} - 2W)R^{\mu\nu}_{\alpha\beta} + 2\mathcal{F}_X\phi^\mu_\alpha\phi^\nu_\beta + 2\kappa_8\phi_\alpha\phi^\mu\phi^\nu_\beta\right]\\
&\quad - 6(\mathcal{F}_\phi - 2W_\phi - X\kappa_8)\overset{g}{\Box}\phi + \kappa_9,
\end{aligned} \tag{2.420}$$

where

$$\kappa_1 = \int^X \frac{1}{X'} K_1(\phi, X') dX' \tag{2.421}$$

$$\kappa_3 = \int^X \frac{1}{X'} K_3(\phi, X') dX' \tag{2.422}$$

$$\kappa_8 = \int^X \frac{1}{X'} K_8(\phi, X') dX' \tag{2.423}$$

$$\kappa_9 = X^2 \int^X \frac{1}{X'^3} K_9(\phi, X') dX' \tag{2.424}$$

$$\mathcal{F} = -2 \int^X \left(\kappa_{1\phi}(\phi, X') - \kappa_3(\phi, X') - 2X'\kappa_{3X}(\phi, X')\right) dX' \tag{2.425}$$

This action, rewritten by the current convention, is the one that Horndeski derived in 1974 [110].

By defining the functions,

$$K = \kappa_9 + 4X \int^X dX'(\kappa_{8\phi} - 2\kappa_{3\phi\phi}), \tag{2.426}$$

$$G_3 = 6\mathcal{F}_\phi - 12W_\phi - 2X\kappa_8 - 8X\kappa_{3\phi} + 2\int^X dX'(\kappa_8 - 2\kappa_{3\phi}), \tag{2.427}$$

$$G_4 = 2\mathcal{F} - 4W - 4X\kappa_3, \tag{2.428}$$

$$G_5 = -4\kappa_1, \tag{2.429}$$

the modern form of the Horndeski Lagrangian is then achieved [113].

$$\mathcal{L}_{Horndeski} = \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5, \tag{2.430}$$

$$\tag{2.431}$$

with

$$\mathcal{L}_2 = K, \tag{2.432}$$

$$\mathcal{L}_3 = -G_3 \overset{g}{\Box}\phi, \tag{2.433}$$

$$\mathcal{L}_4 = G_4 \overset{g}{R} + G_{4X}\left[(\overset{g}{\Box}\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}\right], \tag{2.434}$$

$$\mathcal{L}_5 = G_5 G^{\mu\nu} \overset{g}{\nabla}_\mu \overset{g}{\nabla}_\nu \phi - \frac{1}{6} G_{5X}\left[(\overset{g}{\Box}\phi)^3 - 3\phi_{\mu\nu}\phi^{\mu\nu}\overset{g}{\Box}\phi + 2\phi_{\mu\nu}\phi^{\nu\lambda}\phi^{\mu}_{\lambda}\right]. \tag{2.435}$$

Here the functions $K$ and $G_i$ are all functions of $\phi$ and $X$ as $G_i = G_i(\phi, X)$. The first term $K$ is the so-called K-essence term [114, 115, 116]. One could immediately see that such theories are a straightforward generalization of the covariant Galileons shown in (2.385), and thus Horndeski is sometimes called the Generalized Galileons. For the details of the equation of motion see the original paper by Kobayashi, Yamaguchi, Yokoyama [113].

Thus the Horndeski theory, the most general 4-dimensional action that has most second-order derivatives with respect to $g_{\mu\nu}$ and $\phi$, was derived.

### 2.5.4 DHOST

Since the most general scalar-tensor theory that has at most second-order derivatives in its equation of motion was derived, one would expect that there are no more ghost-free theories to consider. However, if one uses *degeneracy*, one could construct more scalar-tensor theories.

Consider a normal scalar field action

$$L = \frac{1}{2}\dot{\phi}^2 \,, \tag{2.436}$$

and introduce an *invertible* transformation $\phi = \dot{\chi} + \psi$, which then computes a seemingly higher derivative Lagrangian,

$$L' = \frac{1}{2}\dot{\psi}^2 + \dot{\psi}\ddot{\chi} + \frac{1}{2}\ddot{\chi}^2 \,. \tag{2.437}$$

However, substituting the higher derivative into a lower one using a Lagrange multiplier,

$$L'' = \frac{1}{2}\dot{\psi}^2 + \dot{\psi}\dot{\eta} + \frac{1}{2}\dot{\eta}^2 + \lambda(\eta - \dot{\chi}) \,, \tag{2.438}$$

reveals that the determinant of the kinetic matrix is null,

$$\det \frac{\partial^2 L}{\partial \dot{\Psi}_i \partial \dot{\Psi}_j} = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0 \,, \tag{2.439}$$

with $\Psi_i = (\psi, \chi)$.

Therefore although the theory is seemingly higher-derivative and has extra degrees of freedom, it does not. This can of course quantitatively analyzed through Hamilton analysis.

A simple way to find a degenerate theory is to conduce an invertible transformation of variables from an non-degenerate theory. For example, one may consider the following disformal transformations [117],

$$\bar{g}_{\mu\nu} = C(X,\phi)g_{\mu\nu} + D(X,\phi)\partial_\mu\phi\partial_\nu\phi \,, \tag{2.440}$$

which is invertible if $\det\bar{g}_{\mu\nu} \sim C^3(C - 2DX) \neq 0$. Such invertible metric has the following inverse,

$$(\bar{g}^{-1})^{\mu\nu} = \frac{1}{C}\left(g^{\mu\nu} - \frac{D}{C - 2XD}\partial^\mu\phi\partial^\nu\phi\right) \,. \tag{2.441}$$

In order for a local map $X \to \bar{X} := -\frac{1}{2}(\bar{g}^{-1})^{\mu\nu}\partial_\mu\phi_\nu\phi$ to be one-to-one, one must also impose [118],

$$\frac{\partial \bar{X}}{\partial X} = \frac{C - XC_X + 2X^2 D_X}{(C - 2XD)^2} \neq 0 \,, \tag{2.442}$$

which also ensures that the variation is also well-defined for both frames,

$$\delta\bar{g}_{\mu\nu} = \left\{ C\delta^\alpha_\mu\delta^\beta_\nu - \frac{1}{2}\left(C_X g_{\mu\nu} + D_X\partial_\mu\phi\partial_\nu\phi\right)\partial^\alpha\phi\partial^\beta\phi \right\}\delta g_{\alpha\beta} + \cdots \tag{2.443}$$

One could also consider higher-order transformations [118], but as for this section, it will not be considered further.

After Horndeski theory was re-discovered, it was noticed that certain 'beyond-Horndeski' theories are also healthy [119], such as the terms,

$$\begin{aligned}\mathcal{L}_4^{BH} &= F_4(\phi,X)\epsilon^{\mu\nu\rho}{}_{\sigma}\epsilon^{\mu'\nu'\rho'\sigma'}\phi_\mu\phi_{\mu'}\phi_{\nu\nu'}\phi_{\rho\rho'}\,, &(2.444)\\ \mathcal{L}_5^{BH} &= F_5(\phi,X)\epsilon^{\mu\nu\rho\sigma}\epsilon^{\mu'\nu'\rho'\sigma'}\phi_\mu\phi_{\mu'}\phi_{\nu\nu'}\phi_{\rho\rho'}\phi_{\sigma\sigma'}\,. &(2.445)\end{aligned}$$

$\mathcal{L}_4^{BH}$ for example, can be obtained through disformal transformation of the Horndeski Lagrangian, defined as [119]

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + \Gamma(\phi,X)\partial_\mu\phi\partial_\nu\phi\,, \tag{2.446}$$

with

$$\Gamma(\phi,X) = \int^X \left.\frac{-2F_4}{G_4 - 2XG_{4X} + 4X^2F_4}\right|_{X=\tilde{X}} d\tilde{X} \tag{2.447}$$

These constructions of beyond-Horndeski theories and degeneracy conditions led to the Degenerate Higher-order Scalar-Tensor theories or DHOST theories. The DHOST theories were first constructed up to quadratic order of $\phi_{\mu\nu}$ in [120], which was followed by the Hamilton analysis in [121, 122] and later up to cubic order in [123]. Since then, many phenomena were researched, for a review see [3, 124, 125]. For example, in [126] the cosmological evolution was explored, where the sufficient conditions of a de-Sitter attraction and avoidance of ghost and gradient instabilities were shown. As for [127], the authors researched DHOST classes that have the same speed of gravitational waves with the speed of light to satisfy the GW170817 and GRB170817A simultaneous observation[60, 61]. For an overview of DHOST theories, see [3].
The DHOST action up to quadratic terms in $\phi_{\mu\nu}$, which is called qDHOST, is given as

$$\mathcal{L} = fR(g) + P + Q_1 g^{\mu\nu}\phi_{\mu\nu} + Q_2\phi^\mu\phi_{\mu\nu}\phi^\nu + C^{\mu\nu,\rho\sigma}\phi_{\mu\nu}\phi_{\rho\sigma}\,, \tag{2.448}$$

where

$$C^{\mu\nu,\rho\sigma} = \alpha_1 g^{\rho(\mu}g^{\nu)\sigma} + \alpha_2 g^{\mu\nu}g^{\rho\sigma} + \frac{1}{2}\alpha_3(\phi^\mu\phi^\nu g^{\rho\sigma} + \phi^\rho\phi^\sigma g^{\mu\nu}) + \frac{1}{2}\alpha_4(\phi^\rho\phi^{(\mu}g^{\nu)\sigma} + \phi^\sigma\phi^{(\mu}g^{\nu)\rho}) + \alpha_5\phi^\mu\phi^\nu\phi^\rho\phi^\sigma\,. \tag{2.449}$$

However, the functions $f$,$P$,$Q_i$,$\alpha_i$ are not independent, and have to be imposed the following degeneracy conditions such that the theory has at most 3 degrees of freedom,

$$D_0 = D_1 = D_2 = 0\,, \tag{2.450}$$

with

$$\begin{aligned}D_0 &:= -4(\alpha_1+\alpha_2)\times[Xf(2\alpha_1 + X\alpha_4 + 4f_X) - 2f^2 - 8X^2f_X^2]\,, &(2.451)\\ D_1 &:= 4[X^2\alpha_1(\alpha_1+3\alpha_2) - 2f^2 - 4Xf\alpha_2]\alpha_4 + 4X^2f(\alpha_1+\alpha_2)\alpha_5 + 8X\alpha_1^3\\ &\quad -4(f+4Xf_X-6X\alpha_2)\alpha_1^2 - 16(f+5Xf_X)\alpha_1\alpha_2 + 4X(3f-4Xf_X)\alpha_1\alpha_3 - X^2f\alpha_3^2\\ &\quad +32f_X(f+2Xf_X)\alpha_2 - 16ff_X\alpha_1 - 8f(f-Xf_X)\alpha_3 + 48ff_X^2\,, &(2.452)\\ D_2 &:= 4[2f^2 + 4Xf\alpha_2 - X^2\alpha_1(\alpha_1+3\alpha_2)]\alpha_5 + 4\alpha_1^3 + 4(2\alpha_2 - X\alpha_3 - 4f_X)\alpha_1^2\\ &\quad +3X^2\alpha_1\alpha_3^2 - 4Xf\alpha_3^2 + 8(f+Xf_X)\alpha_1\alpha_3 - 32f_X\alpha_1\alpha_2 + 16f_X^2\alpha_1 + 32f_X^2\alpha_2 - 16ff_X\alpha_3\,. &(2.453)\end{aligned}$$

The rest of the section is dedicated to deriving the above and the classification of DHOST.

First consider the ADM decomposition of space-time. The metric is given as,

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt)\,, \tag{2.454}$$

or in the matrix form,

$$(g_{\mu\nu}) = \begin{pmatrix} -N^2 + h_{ij}N^iN^j & h_{ij}N^j \\ h_{ij}N^i & h_{ij} \end{pmatrix}, \tag{2.455}$$

$$(g^{\mu\nu}) = \frac{1}{N^2}\begin{pmatrix} -1 & N^j \\ N^i & N^2h^{ij} - N^iN^j \end{pmatrix}, \tag{2.456}$$

with the lapse function being $N$ and the shift vector being $N^i$.

Then define a time-like unit vector $n^\mu$ which satisfies the normalization condition of $n^\mu n_\mu = -1$. The induced metric, which is also the projection operator is,

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu\,. \tag{2.457}$$

Using the above, one can decompose an arbitrary vector $Q_\mu$ into temporal and spatial terms as,

$$Q_\mu = \hat{Q}_\mu - Q_* n_\mu\,, \tag{2.458}$$

where

$$Q_* := Q_\mu n^\mu\,, \tag{2.459}$$

$$\hat{Q}_\mu := h^\nu_\mu Q_\nu\,, \tag{2.460}$$

are the temporal and spatial projection of the vector, respectively. The extrinsic curvature could also be defined and computed as,

$$K_{\mu\nu} = D_\mu n_\nu = \frac{1}{2N}\left(\dot{h}_{\mu\nu} - D_\mu N_\nu - D_\nu N_\mu\right)\,. \tag{2.461}$$

with the induced covariant derivative $D_\mu$ being calculated as $D_\mu N_\nu = h^\sigma_\mu h^\rho_\nu \overset{g}{\nabla}_\sigma N_\rho$. Here the 'time derivative' $\dot{h}_{\mu\nu} = \mathcal{L}_t h_{\mu\nu}$ is defined through the time direction vector of,

$$t^\mu = \delta^\mu_0 \partial_0 = Nn^\mu + N^\mu\,. \tag{2.462}$$

Now consider $Q_\mu = \partial_\mu\phi$. The terms that inhibit higher time derivatives are $\overset{g}{\nabla}_\mu Q_\nu$, with noting the indices are symmetric. The relevant terms for the kinetic terms are,

$$\left(\overset{g}{\nabla}_\mu Q_\nu\right)_{\text{kin}} = \lambda_{\mu\nu}\dot{Q}_* + \Lambda_{\mu\nu}{}^{\rho\sigma}K_{\rho\sigma}\,, \tag{2.463}$$

where it was defined,

$$\lambda_{\mu\nu} := \frac{1}{N}n_\mu n_\nu\,, \tag{2.464}$$

$$\Lambda_{\mu\nu}{}^{\rho\sigma} := -Q_* h^\rho_{(\mu}h^\sigma_{\nu)} + 2n_{(\mu}h^{(\rho}_{\nu)}\hat{Q}^{\sigma)}\,. \tag{2.465}$$

Substituting this to the action, the relevant terms become,

$$L_{kin} = \mathcal{A}\dot{Q}_*^2 + 2\dot{Q}_*\mathcal{B}^{\mu\nu}K_{\mu\nu} + \mathcal{C}^{\mu\nu,\rho\sigma}K_{\mu\nu}K_{\rho\sigma} \tag{2.466}$$

with,

$$\mathcal{A} := C^{\alpha\beta,\gamma\delta}\lambda_{\alpha\beta}\lambda_{\gamma\delta} \tag{2.467}$$

$$\mathcal{B}^{\mu\nu} := 2f_X\frac{Q_*}{N}h^{\mu\nu} + C^{\mu\nu,\rho\sigma}\Lambda_{\mu\nu}{}^{\rho\sigma}\lambda_{\rho\sigma} \tag{2.468}$$

$$\mathcal{C}^{\mu\nu,\rho\sigma} := f\left(h^{\mu(\rho}h^{\sigma)\nu} - h^{\mu\nu}h^{\rho\sigma}\right) + 2f_X\left(\hat{Q}^\mu\hat{Q}^\nu h^{\rho\sigma} + \hat{Q}^\rho\hat{Q}^\sigma h^{\mu\nu} + C^{\alpha\beta,\gamma\delta}\Lambda_{\mu\nu}{}^{\alpha\beta}\Lambda_{\rho\sigma}{}^{\gamma\delta}\right) \tag{2.469}$$

Therefore the kinetic matrix of qDHOST is of the form,

$$\begin{pmatrix} \mathcal{A} & \mathcal{B}^{\rho\sigma} \\ \mathcal{B}^{\mu\nu} & C^{\mu\nu,\rho\sigma} \end{pmatrix}, \tag{2.470}$$

which determinant can be calculated as,

$$D_0 + D_1Q_*^2 + D_2Q_*^4, \tag{2.471}$$

where the relation $\hat{Q}^\mu\hat{Q}_\mu = X + Q_*^2$ was used.

Thus for this to vanish, each coefficient must be zero, which is precisely the degeneracy conditions of (2.450)

Using the degeneracy conditions one may classify DHOST into three classes, which each having subclasses as follows.

1. Class I: $\alpha_1 = -\alpha_2$
   (a) Subclass Ia: $f \neq X\alpha_1$ with three arbitrary functions.
   (b) Subclass Ib: $f = X\alpha_1$ with three arbitrary functions.
2. Class II: $f \neq 0$ and $\alpha_2 \neq -\alpha_1$
   (a) Subclass IIa: $f \neq X\alpha_1$ with three arbitrary functions.
   (b) Subclass IIb: $f \neq X\alpha_1$ with three arbitrary functions.
3. ClassIII: $f = 0$
   (a) Subclass IIIa: $\alpha_1 + 3\alpha_2 \neq 0$ with three arbitrary functions.
   (b) Subclass IIIb: $\alpha_1 + 3\alpha_2 = 0$ with three arbitrary functions.
   (c) Subclass IIIc: $\alpha_1 = 0$ with four arbitrary functions.

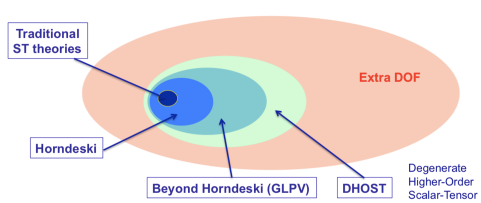



Figure 2.5: The Landscape of Scalar-tensor theories, taken from [3]

### 2.5.5 U-DHOST

First considered in [128], the U-degenerate HOST theories are theories that have a degenerate kinetic matrix *in the unitary gauge* and not when in an arbitrary gauge. When a theory is constructed to be gauge invariant, gauge fixing does not change the 'physics'. So if truly the unitary gauge $\phi = t$ *can* be taken in the theory, it is enough to investigate in such gauge. Thus to impose the degeneracy of the kinetic matrix in the gauge-invariant form may be too strict, as in what DHOST theories are, which led to U-DHOST.[29] Moreover, in the unitary gauge $\delta\phi = 0$ , this theory is a subset of *spatially covariant gravity*, investigated in [129, 130, 131, 132], where spatially covariant theories with at most 3 degrees of freedom were constructed. Note that, when unitary gauge cannot be taken, for example, backgrounds that are time and space dependent, an extra mode appears and could be ghostly. These modes are called "shadowy" modes [128] and will be explained in the next section.

The kinetic matrix was the form of (2.466) and by taking the unitary gauge, the action takes the form,

$$L_{kin} = \mathcal{A}_U \dot{Q}_*^2 + 2\dot{Q}_* \mathcal{B}_U^{\mu\nu} K_{\mu\nu} + \mathcal{C}_U^{\mu\nu,\rho\sigma} K_{\mu\nu} K_{\rho\sigma} \,, \tag{2.472}$$

with the coefficients being [128],

$$\begin{aligned} \mathcal{A}_U &= \alpha_1 + \alpha_2 + (\alpha_3 + \alpha_4) X_U + \alpha_5 X_U^2 \,, && (2.473) \\ \mathcal{B}_U^{\mu\nu} &= 4 f_X + 2\alpha_2 + \alpha_3 X_U \,, && (2.474) \\ \mathcal{C}_U^{\mu\nu,\rho\sigma} &= (f - \alpha_1 X_U) h^{i(k} h^{l)j} - (f - \alpha_2 X_U) \gamma^{ij} \gamma^{kl} \,, && (2.475) \end{aligned}$$

where $X_U = -Q_*^2 = -\frac{\dot{\phi}^2}{N^2}$. The degeneracy conditions for this kinetic matrix imposes four conditions to the six free functions, thus leaving only two. In [128], $f$ and $\alpha_5$ were chosen to the independent functions and the rest have to be imposed as,

$$\begin{aligned} \alpha_1 &= -\alpha_2 = \frac{f}{X} \,, && (2.476) \\ \alpha_3 &= -\frac{2}{X} \left( 2 f_X - \frac{f}{X} \right) , && (2.477) \\ \alpha_4 &= \frac{2}{X} \left( 2 f_X - \frac{f}{X} \right) - X \alpha_5 \,. && (2.478) \end{aligned}$$

Thus inserting this into the action computes,

$$\begin{aligned} & L_{tUd}[f, \alpha_5] \\ = \; & f R + \frac{f}{X} \left( L_1^{(2)} - L_2^{(2)} \right) + \frac{2}{X^2} (f - 2 X f_X) \left( L_3^{(2)} - L_4^{(2)} \right) - \alpha_5 \left( X L_4^{(2)} - L_5^{(2)} \right) . \end{aligned} \tag{2.479}$$

Now consider decomposing the Lagrangian as

$$L = L_{tUD}[f, 0] + \tilde{L}_\phi \,, \tag{2.480}$$

with $\tilde{L}_\phi = \sum_{I=1}^5 \alpha_I L_I^{(2)}$ $L_{tUD}[f, 0]$ is obviously degenerate under unitary gauge. So simply, when $\tilde{L}_\phi$ is degenerate under unitary gauge, the whole theory becomes U-DHOST. The kinetic matrix of $\tilde{L}_\phi$ could be written in the form of

$$\tilde{L}_{\phi,\mathrm{kin}} = \hat{\mathcal{K}}_U^{ij,kl} \left( K_{ij} + \sigma h_{ij} \dot{Q}_* \right) \left( K_{kl} + \sigma h_{kl} \dot{Q}_* \right) \,, \tag{2.481}$$

[29] If one wishes to obtain a fully gauge-invariant theory, one may use the Stueckelberg trick and recover gauge invariance.

with

$$\hat{\mathcal{K}}^{ij,kl} = -X_U(\alpha_1\gamma^{i(k}\gamma^{l)j} - \alpha_2\gamma^{ij}\gamma^{kl})\,. \tag{2.482}$$

The degeneracy of this matrix directly relates the the degeneracy of the whole theory. So thus for the theory to be U-degenerate, the kinetic matrix's degeneracy conditions compute,

$$4(\alpha + 3\alpha_2)(\alpha_1 + \alpha_2 + X(\alpha_3 + \alpha_4) + X^2\alpha_5) = 2(2\alpha_2 + X\alpha_3)^2\,. \tag{2.483}$$

#### 2.5.5.1 "Shadowy" modes in U-DHOST

As mentioned, U-DHOST is healthy only when the unitary gauge can be taken, but may be plagued in some different background. This extra mode is called "shadowy" modes are the generalizations of instantaneous modes in khronometric theories [133, 134].

Consider the following Lagrangian in coordinates $(t, x, y, z)$ [128],

$$L[\psi] = -\frac{1}{2}\left[(v\partial_t\psi + \partial_x\psi)^2 + (\partial_y\psi)^2 + (\partial_z\psi)^2\right]\,. \tag{2.484}$$

which has a time derivative acting on the field $\psi$. Now consider going to a different coordinate of $t \to t' - vx$, then the Lagrangian becomes,

$$L[\psi] = -\frac{1}{2}\left[(\partial_x\psi)^2 + (\partial_y\psi)^2 + (\partial_z\psi)^2\right]\,. \tag{2.485}$$

Thus, for this specific set of coordinates, the Lagrangian apparently do not seem to have a degree of freedom. Hence this is how the shadowy mode can be interpreted in U-DHOST.

For an explicit example, consider [128],

$$L = X + \mu\left\{2X\phi^\mu\phi_{\mu\nu}\phi^{\nu\sigma}\phi_\sigma + (\phi^\mu\phi_{\mu\nu}\phi^\nu)^2\right\}\,. \tag{2.486}$$

This does not satisfy the degeneracy conditions of DHOST (2.450), but does for the conditions of U-DHOST (2.478).

Now consider the following perturbations around the unitary gauge $\bar{\phi} = t$,

$$\phi = \bar{\phi} + \chi(t, x)\,, \tag{2.487}$$

then the quadratic Lagrangian is given as,

$$\mathcal{L}_0 = \frac{1}{2}\{\dot{\chi}^2 - (\partial_x\chi)^2\} + \mu(\partial_x\dot{\chi})^2\,, \tag{2.488}$$

which has the dispersion relation of

$$(1 + 2\mu k^2)\omega^2 - k^2 = 0\,, \tag{2.489}$$

and has two solutions that correspond to 1 degree of freedom.

However, if the background has the solution of $\bar{\phi} = t + \alpha x$, and the perturbations are of (2.487), the quadratic action computes,

$$\mathcal{L}_\alpha = \frac{1}{2}\{\dot{\chi}^2 - (\partial_x\chi)^2\} + \mu\left[\alpha^2\left\{\ddot{\chi} + (\partial_x\partial_x\chi)^2\right\} - 2\alpha(1+\alpha^2)\left\{\ddot{\chi}\partial_x\dot{\chi} + \partial_x\partial_x\chi\partial_x\dot{\chi}\right\} + (1 + 4\alpha^2 + \alpha^4)(\partial_x\dot{\chi})^2\right]\,. \tag{2.490}$$

The dispersion relation of this seemingly 1 degree of freedom is,

$$2\alpha^2\mu\omega^4 + 4\alpha\mu(1+\alpha^2)k\omega^3 + \{1 + 2\mu(1+4\alpha^2+\alpha^4)k^2\}\omega^2 - k^2 + 4\alpha\mu(1+\alpha^2)\omega k^3 + 2\alpha^2\mu k^4 = 0 \quad (2.491)$$

and there are 4 solutions. This corresponds to the theory having 2 degrees of freedom. Taking $\alpha \to 0$, thus the unitary gauge, hides this degree of freedom for 'observers' that are in that frame.

Whether this shadowy mode is 'dangerous' has to be debated, however. For example, in [135], the authors have found that propagate for three-dimensional space-like hypersurface since it has to satisfy a certain elliptic equation.

### 2.5.6 Inflation with Horndeski Gravity

Once a theory is fixed, it is natural to question what kind of cosmology it can achieve. In this section, following [113], the cosmological perturbation for inflationary scenarios within Horndeski gravity (2.430) will be introduced.

For such action, the quadratic actions could be derived by first assuming an ADM decomposition of the metric, such that,

$$g_{\mu\nu}dx^\mu dx^\nu = -N^2dt^2 + \gamma_{ij}(dx^i + N^idt)(dx^j + N^jdt)\,, \quad (2.492)$$

where $N$ is the lapse, $N_i$ is the shift and $\gamma_{ij}$ is the spatial metric.
Under the unitary gauge $\delta\phi = 0$ ,these variables could be further taken as

$$N = 1 + \alpha\,, \quad (2.493)$$
$$N_i = \partial_i\beta\,, \quad (2.494)$$
$$\gamma_{ij} = a^2(t)e^{2\zeta}\left(\delta_{ij} + h_{ij} + \frac{1}{2}h_{ik}h^k{}_j\right) \quad (2.495)$$

with $\alpha, \beta, \zeta$ being scalar perturbations. Here $h_{ij}$ is the tensor perturbation satisfying the traceless condition $h_{ii} = 0$ and the transverse condition $\partial_j h_{ij} = 0$.

First, the quadratic action for the tensor perturbations could be calculated as [113],

$$S_T^{(2)} = \frac{1}{8}\int dtd^3x\left[\mathcal{G}_T\dot{h}_{ij}^2 - \frac{\mathcal{F}_T}{a^2}(\nabla h_{ij})^2\right]\,, \quad (2.496)$$

with

$$\mathcal{F}_T \equiv 2\left[G_4 - X(\ddot{\phi}G_{5X} + G_{5\phi})\right]\,, \quad (2.497)$$
$$\mathcal{G}_T \equiv 2\left[G_4 - 2XG_{4X} - X(H\dot{\phi}G_{5X} - G_{5\phi})\right]\,. \quad (2.498)$$

Then the speed of the tensor mode could be expressed as,

$$c_T^2 = \frac{\mathcal{F}_T}{\mathcal{G}_T}\,, \quad (2.499)$$

In order to avoid ghost and gradient instabilities within the cosmological background,

$$\mathcal{F}_T > 0, \quad \mathcal{G}_T > 0\,. \quad (2.500)$$

If one takes constant roll approximation as,

$$\epsilon \equiv -\frac{\dot{H}}{H^2} \sim \text{const.}\,. \quad (2.501)$$

Then the following variables, that are defined, are constant as

$$f_T \equiv \frac{\dot{F}_T}{HF_T} \sim \text{const.}\,, \tag{2.502}$$

$$g_T \equiv \frac{\dot{G}_T}{HG_T} \sim \text{const.}\,. \tag{2.503}$$

By further defining,

$$\nu_T \equiv \frac{3-\epsilon+g_T}{2-2\epsilon-f_T+g_T}\,, \tag{2.504}$$

$$\gamma_T \equiv 2^{2\nu_T-3}\left|\frac{\Gamma(\nu_T)}{\Gamma\left(\frac{3}{2}\right)}\right|^2\left(1-\epsilon-\frac{1}{2}f_T+\frac{1}{2}g_T\right) \tag{2.505}$$

The spectral index and power spectrum of the tensor perturbations could be calculated as,

$$n_T = 3-2\nu_T\,, \tag{2.506}$$

$$\mathcal{P}_T = 8\gamma_T\frac{\mathcal{G}_T^{\frac{1}{2}}}{\mathcal{F}_T^{\frac{3}{2}}}\frac{H^2}{4\pi^2}\bigg|_{1=-ky_T} \tag{2.507}$$

Similarly, after eliminating $\alpha$ and $\beta$ through constraints, one may compute the quadratic action for the scalar perturbation as [113],

$$S_S^{(2)} = \int dtd^3xa^3\left[\mathcal{G}_s\dot{\zeta}^2-\frac{\mathcal{F}_S}{a^2}(\nabla\zeta)^2\right]\,, \tag{2.508}$$

where

$$\mathcal{F}_S = \frac{1}{a}\frac{d}{dt}\left(\frac{a}{\Sigma}\mathcal{G}_T^2\right)-\mathcal{F}_T\,, \tag{2.509}$$

$$\mathcal{G}_S = \frac{\Sigma}{\Theta^2}\mathcal{G}_T^2+3\mathcal{G}_T\,, \tag{2.510}$$

which is further defined by

$$\begin{aligned}\Sigma \equiv\ & XK_X+2X^2K_{XX}+12H\dot{\phi}XG_{3X}+6H\dot{\phi}X^2G_{3XX}-2XG_{3\phi}-2X^2G_{3\phi X}-6H^2G_4\\ & +6\left[H^2(7XG_{4X}+16X^2G_{4XX}+4X^3G_{4XXX})-H\dot{\phi}\left(G_{4\phi}+5XG_{4\phi X}+2X^2G_{4\phi XX}\right)\right]\\ & +30H^3\dot{\phi}XG_{5X}+26H^3\dot{\phi}X^2G_{5XX}+4H^3\dot{\phi}X^3G_{5XXX}\\ & -6H^2X\left(6G_{5\phi}+9G_{5\phi X}+2X^2G_{5\phi XX}\right)\,,\end{aligned} \tag{2.511}$$

$$\begin{aligned}\Theta \equiv\ & -\dot{\phi}XG_{3X}+2HG_4-8HXG_{4X}-8HX^2G_{4XX}+\dot{\phi}G_{4\phi}+2X\dot{\phi}G_{4\phi X}\\ & -H^2\dot{\phi}(5XG_{5X}+2X^2G_{5XX})+2HX(3G_{5\phi}+2XG_{5\phi X})\end{aligned} \tag{2.512}$$

To avoid ghosts and gradient instabilities, the following conditions must be satisfied,

$$\mathcal{F}_S>0,\quad \mathcal{G}_S>0\,. \tag{2.513}$$

Define the following variables, which are constant during the constant roll approximation as,

$$f_S \equiv \frac{\dot{F}_S}{HF_S} \sim \text{const.}\,, \tag{2.514}$$

$$g_S \equiv \frac{\dot{G}_S}{HG_S} \sim \text{const.}\,. \tag{2.515}$$

By defining,

$$\nu_S \equiv \frac{3-\epsilon+g_S}{2-2\epsilon-f_S+g_S}\,, \tag{2.516}$$

$$\gamma_T \equiv 2^{2\nu_T-3}\left|\frac{\Gamma(\nu_T)}{\Gamma\left(\frac{3}{2}\right)}\right|^2\left(1-\epsilon-\frac{1}{2}f_T+\frac{1}{2}g_T\right)\,, \tag{2.517}$$

the spectral index and power spectrum of the tensor perturbations could be calculated as,

$$n_S = 1+3-2\nu_S\,, \tag{2.518}$$

$$\mathcal{P}_S = 8\gamma_S\frac{\mathcal{G}_S^{\frac{1}{2}}}{\mathcal{F}_S^{\frac{3}{2}}}\frac{H^2}{4\pi^2}\bigg|_{1=-ky_S} \tag{2.519}$$

Under the limit of $\epsilon, f_T, g_T, f_S, G_S \ll 1$, the tensor-to-scalar ratio is given as,

$$r = 16\left(\frac{\mathcal{F}_S}{\mathcal{F}_T}\right)^{\frac{3}{2}}\left(\frac{\mathcal{G}_S}{\mathcal{G}_T}\right)^{-\frac{1}{2}} \tag{2.520}$$

Thus the observational variables for the CMB for Horndeski-type of inflation are now fully derived.

# Chapter 3

# Metric-affine Gravity and Inflation

| | |
|---|---|
| Based on | *"Metric-affine Gravity and Inflation"* |
| Authors: | Keigo Shimada, Katsuki Aoki, Kei-ichi Maeda |
| Journal: | Phys. Rev. D 99, 104020 (2019) |

In the previous chapter, scalar-tensor theories and their application to cosmology, and metric-affine gravity were reviewed. Furthermore, it was explained that both may arise for a fundamental theory. Thus it is quite natural to consider the combination of these frameworks and see if there exists anything new. Indeed, as it will be shown later, the metric-affine framework exhibits different observational results compared to the metric formalism. As an example, chaotic inflation in metric formalism is fully excluded, as was seen in §2.2.1.4, whereas in the metric-affine formalism it is still not.

Metric-affine formalism (or Palatini) and its application to inflation have received attention in recent years. A common approach is the coupling between the curvature scalar and the inflaton [136, 137, 138, 139]. There have been studies of how attractors behave in such theory [140] and multifield extensions were explored [141]. Applying the Palatini formalism to the Higgs inflation, where inflaton is the Higgs, are also considered and loop correction are investigated [142, 143, 144]. There are other approaches such as considering the Eddington formalism, where gravity is purely affine, and applying it to inflation[145, 146, 147] see also (2.288). However, one must note that, in all of these works, *connection is only induced by the curvature tensor*. There are other ways to consider coupling connection to the inflaton. One prominent way is through covariant derivatives, and this chapter is used to formulate precisely that.

## 3.1 The 'Minimal' Coupled Scalar-tensor Theory

### 3.1.1 "Canonical" Scalar Field

In metric-affine gravity, connection couples to matter.

Since $\overset{\Gamma}{\nabla}_\mu \phi = \partial_\mu \phi$ one would expect that scalars do not couple to the connection. This is usually the case. However, in this section, it will be shown that under certain prescriptions a "canonical" scalar that is "minimally" coupled can couple to the connection.

First, recall the action of a real scalar field in flat Minkowski space,

$$S_{\phi,\text{flat}} = \int d^4x \left( -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) \right) . \tag{3.1}$$

This action could be also rewritten, using integration by parts, as

$$S_{\phi,\text{flat}} = \int d^4x \left( \frac{1}{2}\phi\overset{\eta}{\Box}\phi - V(\phi) \right) , \tag{3.2}$$

where $\overset{\eta}{\Box} := \eta^{\mu\nu}\partial_\mu\partial_\nu$ is the flat-space d'Alembertian operator. These two actions are *equivalent up to the surface term.*

To curve space-time, it is necessary to covariantize the action. Covariantization in Riemannian geometry is a straightforward; one simply substitutes the volume density $d^4x \to \sqrt{-g}d^4x$ and $\partial_\mu \to \overset{g}{\nabla}_\mu$. The covariantization of (3.1) is equivalent to the covariantization of (3.2) up to the surface term, just as it is in flat space.

Now consider, covariantization in Metric-affine geometry, which actually gives different results. Starting from the action (3.1), the scalar field does not couple to the connection because $\overset{\Gamma}{\nabla}_\mu\phi = \partial_\mu\phi$. Thus covariantizing (3.1) in the Metric-affine formalism is no different than in Riemannian geometry. On the other hand, covariantization of (3.2) computes interesting results. First, there arises an ambiguity when trying to define the d'Alembertian operator $\overset{\Gamma}{\Box}$ in a Metric-affine curved space-time. Since, in general, the connection is not compatible with the metric, one can introduce two different second-order covariant derivatives; $\overset{\Gamma}{\nabla}_\mu\overset{\Gamma}{\nabla}^\mu$ and $\overset{\Gamma}{\nabla}^\mu\ \overset{\Gamma}{\nabla}_\mu$ with $\overset{\Gamma}{\nabla}^\mu := g^{\mu\nu}\overset{\Gamma}{\nabla}_\nu$. In what follows, it will be shown that the two actions (3.1) and (3.1) are equivalent in flat space-time, but differ in Metric-affine curved space.

To obtain the correct action in flat space, it is necessary that it is $\overset{\Gamma}{\Box} \to \overset{\eta}{\Box}$ in the limit of a flat space-time. Noting the above, the d'Alembertian operator in curved space-time will be defined as

$$\overset{\Gamma}{\Box} \;=\; \alpha\overset{\Gamma}{\nabla}^\mu\overset{\Gamma}{\nabla}_\mu + (1-\alpha)\overset{\Gamma}{\nabla}_\mu\overset{\Gamma}{\nabla}^\mu . \tag{3.3}$$

Here to show the difference between the two operators, a constant $\alpha$ was introduced.
This does not include all possible candidates of the metric-affine d'Alembertian, which will be further investigated in §6.2. However, for simplicity and clarity, in this part of the thesis, only the term(3.3) will be considered.

With the Metric-affine d'Alembertian operator defined, the proposed scalar field action in Metric-affine space is, therefore,

$$S_\phi = \int d^4x\sqrt{-g}\left( \frac{1}{2}\phi\overset{\Gamma}{\Box}\phi - V(\phi) \right) . \tag{3.4}$$

This is the covariantized (3.2).

Now using distortion, or the three geometrical vectors, $\overset{\Gamma}{\Box}\phi$ could be decomposed as,

$$\begin{aligned}\overset{\Gamma}{\Box}\phi &= \overset{g}{\Box}\phi + \left[(1-\alpha)g^{\alpha\beta}\kappa^{\gamma}{}_{\gamma\beta} - \alpha g^{\beta\gamma}\kappa^{\alpha}{}_{\beta\gamma}\right]\partial_\alpha\phi \\ &= \overset{g}{\Box}\phi - \left(\alpha\mathcal{Q}^\lambda - 2\mathcal{W}^\lambda + \mathcal{T}^\lambda\right)\partial_\lambda\phi\,, \end{aligned} \tag{3.5}$$

where the Riemann geometric d'Alembertian is $\overset{g}{\Box}\phi := \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi)$. Using the above, the variation of the considered action (3.4) is

$$\frac{\delta S_\phi}{\delta\kappa^{\alpha}{}_{\beta\gamma}} = -\frac{\alpha}{2}g^{\beta\gamma}\phi\partial_\alpha\phi + \frac{(1-\alpha)}{2}\delta^\beta_\alpha\phi\partial^\gamma\phi\,. \tag{3.6}$$

Now everything is set to solve the constraint equations for $\kappa^{\lambda}{}_{\mu\nu}$ in each classification of metric-affine gravity, as outlined in §2.4.3.1.

### 3.1.2 Computation of the Riemann Frame for each Model

#### Model I

The Einstein-Hilbert was projective invariant by construction. However, as stated, in general, metric-affine models are not so. Thus one may enforce a theory to become projective invariant, by imposing certain conditions.

The projective transformation (2.308) upon the metric-affine d'Alembertian operator 3.3 results in,

$$\overset{\Gamma}{\Box}\phi \to \overset{\Gamma}{\Box}\phi + (1-2\alpha)U^\lambda\partial_\lambda\phi\,. \tag{3.7}$$

Looking at this, for a certain value of $\alpha = 1/2$, the theory is projective invariant.

The constraint equations for such value of $\alpha$ is

$$\frac{M_{\rm Pl}^2}{2}\left(g^{\beta\gamma}\bar{\kappa}^{\sigma}{}_{\sigma\alpha} + \delta^\beta_\alpha\bar{\kappa}_\sigma{}^{\gamma\sigma} - \bar{\kappa}_\alpha{}^{\beta\gamma} - \bar{\kappa}^{\gamma}{}_{\alpha}{}^{\beta}\right) - \frac{1}{4}g^{\beta\gamma}\phi\partial_\alpha\phi + \frac{1}{4}\delta^\beta_\alpha\phi\partial^\gamma\phi = 0\,, \tag{3.8}$$

which could be solved as,

$$\bar{\kappa}^{\alpha}{}_{\beta\gamma} = \frac{\phi}{4M_{\rm Pl}^2}\left(\delta^\alpha_\beta\partial_\gamma\phi - g_{\beta\gamma}\partial^\alpha\phi\right)\,, \tag{3.9}$$

up to gauge freedom of the projective mode. This could be directly translated to giving torsion and non-metricity as

$$\mathcal{T}^{\lambda}{}_{\mu\nu} = \frac{\phi}{2M_{\rm Pl}^2}\delta^\lambda{}_{[\mu}\partial_{\nu]}\phi, \tag{3.10}$$

$$\mathcal{Q}_\lambda{}^{\mu\nu} = 0, \tag{3.11}$$

also up to gauge freedom.

These results indicate that for a projective invariant 'minimally' coupled scalar field, there are two certain special gauges; one that cancels out torsion and another that cancels out non-metricity. These two gauges do not coincide with one another, i.e. there is no gauge in which the connection becomes Levi-Civita. This is similar to Metric-affine $f(R)$ gravity. This theory is projective invariant and allows both a metric-compatible gauge and a torsion-free gauge [148, 149, 150]. Another thing to note is that under projective transformation, Weyl geometry emerges. This is because, under projective transformation, non-metricity is shifted as,

$$\mathcal{Q}_\alpha{}^{\beta\gamma} \overset{\Gamma\to\tilde{\Gamma}}{\to} \tilde{\mathcal{Q}}_\alpha{}^{\beta\gamma} = 2U_\alpha g^{\beta\gamma} = 8\mathcal{W}_\alpha g^{\beta\gamma} . \tag{3.12}$$

Now as repeatedly noted the equation for the connection is algebraic and it does not induce new degrees of freedom. In such a case, one may substitute the solution of the distortion tensor right into the actions (2.313) and (3.4), and thus obtaining a Riemann frame Lagrangian. The resultant action $S_{\mathrm{g}\phi} := S_{\mathrm{g}} + S_\phi$ is written purely in terms of the metric and the scalar as

$$S_{\mathrm{g}\phi} = \int d^4x\sqrt{-g}\left[\frac{M_{\mathrm{Pl}}^2}{2}\overset{g}{R} - \frac{1}{2}\left(1 - \frac{3\phi^2}{8M_{\mathrm{Pl}}^2}\right)(\partial\phi)^2 - V(\phi)\right] . \tag{3.13}$$

This effective theory is none other than a scalar-tensor theory that has a modified kinetic term embedded in Riemannian geometry, which in this thesis was called the Riemann frame.

## Models II (a) and (b)

For this section, reconsider the Einstein-Hilbert action and the same action of the scalar field (3.4). This time, however, without imposing projective invariance. For Model II(a), the torsion-free condition $\mathcal{T}^\lambda_{\ \mu\nu} = 0$ is assumed. Furthermore, the parameter $\alpha$ is not constrained due to the fact that projective symmetry is not assumed.

From the constraint equation of the connection $\kappa^\lambda{}_{\mu\nu}$,

$$\frac{M_{\mathrm{Pl}}^2}{2}\left[g^{\beta\gamma}\kappa^\sigma{}_{\sigma\alpha} + \delta_\alpha^{(\beta}\kappa^{\gamma)\sigma}{}_\sigma - \kappa^{(\beta\gamma)}{}_\alpha - \kappa^{(\gamma}{}_\alpha{}^{\beta)}\right] - \frac{\alpha}{2}g^{\beta\gamma}\phi\partial_\alpha\phi + \frac{(1-\alpha)}{2}\delta_\alpha^{(\beta}\phi\partial^{\gamma)}\phi = 0 . \tag{3.14}$$

The solution for the distortion could be calculated as,

$$\kappa^\alpha{}_{\beta\gamma} = \frac{1}{6M_{\mathrm{Pl}}^2}\left[3(\alpha-1)g_{\beta\gamma}\phi\partial^\alpha\phi + 2(\alpha+1)\delta^\alpha_{(\beta}\phi\partial_{\gamma)}\phi\right] , \tag{3.15}$$

which computed non-metricity as

$$\mathcal{Q}_\lambda{}^{\mu\nu} = \frac{1}{3M_{\mathrm{Pl}}^2}\left[(\alpha+1)g^{\mu\nu}\phi\partial_\lambda\phi + 2(2\alpha-1)\delta_\lambda^{(\mu}\phi\partial^{\nu)}\phi\right] . \tag{3.16}$$

This solution could again be substituted into the original action, and again the Riemann frame for this action is obtained as

$$S_{\mathrm{g}\phi} = \int d^4x\sqrt{-g}\Big[\frac{M_{\mathrm{Pl}}^2}{2}\overset{g}{R} - \frac{1}{2}f(\phi)(\partial\phi)^2 - V(\phi)\Big] , \tag{3.17}$$

with

$$f(\phi) := 1 + \frac{(11\alpha^2 - 8\alpha - 1)}{6M_{\mathrm{Pl}}^2}\phi^2 . \tag{3.18}$$

Again, this is nothing but a scalar-tensor theory with a modified kinetic term written in Riemannian geometry.

A similar calculation is conducted for Model II(b), where it is assumed that $\mathcal{Q}^{\lambda}{}_{\mu\nu} = 0$, i.e., the connection is metric a priori. The distortion is solved as,

$$\kappa^{\alpha}{}_{\beta\gamma} = \frac{\phi}{4M_{\rm Pl}^2}\left(\delta^{\alpha}_{\beta}\partial_{\gamma}\phi - g_{\beta\gamma}\partial^{\alpha}\phi\right) , \tag{3.19}$$

and thus the torsion becomes,

$$\mathcal{T}^{\lambda}{}_{\mu\nu} = \frac{\phi}{2M_{\rm Pl}^2}\delta^{\lambda}{}_{[\mu}\partial_{\nu]}\phi \, . \tag{3.20}$$

As a result, the distortion of Model II(b) looks as if it's the same with Model I (3.9). This is incorrect due to the fact that the latter allows gauge transformations, while the former does not.

The Riemann frame for this action is precisely that of (3.13).

## Models III (a), (b) and (c)

Instead of implicitly assuming geometry, one could also impose it thorough the usage of Lagrangian multipliers.

The variation of the Einstein-Hilbert action with adding both the Lagrange multiplier (2.321) and the scalar action (3.4), becomes

$$g^{\beta\gamma}\kappa^{\sigma}{}_{\sigma\alpha} + \delta^{\beta}_{\alpha}\kappa^{\gamma\sigma}{}_{\sigma} - \kappa^{\beta\gamma}{}_{\alpha} - \kappa^{\gamma}{}_{\alpha}{}^{\beta} + \lambda^{\mu}\frac{\delta C_{\mu}}{\delta\kappa^{\alpha}{}_{\beta\gamma}} + M_{\rm Pl}^{-2}\Big[-\alpha g^{\beta\gamma}\phi\partial_{\alpha}\phi + (1-\alpha)\delta^{\beta}_{\alpha}\phi\partial^{\gamma}\phi\Big] = 0 \, . \tag{3.21}$$

Contraction by $\delta^{\gamma}_{\alpha}$, computes the Lagrange multiplier as

$$\lambda_{\mu} = \begin{cases} \frac{(2\alpha-1)}{3M_{\rm Pl}^2}\phi\partial_{\mu}\phi & \text{(Model III(a))} \\ \frac{(2\alpha-1)}{2M_{\rm Pl}^2}\phi\partial_{\mu}\phi & \text{(Models III(b) and (c))} \end{cases} . \tag{3.22}$$

First, it could be seen that the results in Models III(a) and (b) are precisely Models II(a) and (b), respectively. The connections $\kappa^{\lambda}{}_{\mu\nu}$ are also the same, which also computes the same Riemann frame action (3.17) embedded as Riemannian geometry.

The difference arises for Model III(c). For this model neither metric compatibility nor the torsion-free is satisfied. The connection is solved as,

$$\kappa^{\alpha}{}_{\beta\gamma} = \frac{1}{16M_{\rm Pl}^2}\Big[2(2\alpha-3)g_{\beta\gamma}\phi\partial^{\alpha}\phi + (2\alpha+3)\delta^{\alpha}_{(\beta}\phi\partial_{\gamma)}\phi + (6\alpha+1)\delta^{\alpha}_{[\beta}\phi\partial_{\gamma]}\phi\Big] , \tag{3.23}$$

which then results the torsion and non-metricity as

$$\mathcal{T}^{\alpha}_{\beta\gamma} = \frac{6\alpha+1}{8M_{\rm Pl}^2}\delta^{\alpha}_{[\beta}\phi\partial_{\gamma]}\phi, \tag{3.24}$$

$$\mathcal{Q}_{\alpha}{}^{\beta\gamma} = \frac{2\alpha-1}{8M_{\rm Pl}^2}\left(-g^{\beta\gamma}\phi\partial_{\alpha}\phi + 4\delta^{(\beta}_{\alpha}\phi\partial^{\gamma)}\phi\right) . \tag{3.25}$$

The resulting Riemann frame action is given by (3.17) with a different functional form of,

$$f(\phi) = 1 + \frac{3(12\alpha^2 - 12\alpha - 1)}{32M_{\rm Pl}^2}\phi^2 \tag{3.26}$$

## 3.2 Application to Inflation

### 3.2.1 Inflationary Scenario in Metric-affine Gravity

Now the ingredients to consider inflation are prepared. The important result was in Metric-affine gravity theory with a "canonical" scalar could be re-casted to the Riemann frame action (3.17) as

$$f(\phi) \quad = \quad 1 + B(\alpha)\frac{\phi^2}{M_{\rm Pl}^2}\,, \tag{3.27}$$

where $B(\alpha)$ differs depending on which models that are given as,

$$B(\alpha) = \begin{cases} \frac{1}{6}(11\alpha^2 - 8\alpha - 1) & \text{Models II(a) and III(a)} \\ -\frac{3}{8} & \text{Models I, II(b), and III(b)} \\ \frac{3}{32}(12\alpha^2 - 12\alpha - 1) & \text{Model III(c)} \end{cases} \tag{3.28}$$

This function has the following properties; $B(\alpha) \geq -\frac{9}{22}$ for Models II(a) and III(a) and $B(\alpha) \geq -\frac{3}{8}$ for Model III(c). Furthermore, all of $B(\alpha)$ coincides at a $\alpha = 1/2$ with the value being $B\left(\frac{1}{2}\right) = -\frac{3}{8}$. The function $B(\alpha)$ is purely determined by the considered geometry. One value of $\alpha$ is directly one value of $B(\alpha)$. For the rest of the section $B(\alpha)$ could be re-taken as a parameter which has one-to-one correspondence between the geometry that is considered.

To make calculations easier, the scalar field of (3.17) could be canonically normalized through the redefinition of the scalar field as

$$\mathrm{d}\Phi = \sqrt{1 + B(\alpha)\frac{\phi^2}{M_{\rm Pl}^2}}\mathrm{d}\phi\,, \tag{3.29}$$

which has a analytic solution of,

$$\Phi = \begin{cases} \frac{1}{2}\left[\phi\sqrt{1+\frac{B\phi^2}{M_{\rm Pl}^2}} + \frac{M_{\rm Pl}}{B^{1/2}}\sinh^{-1}\left(\frac{B^{1/2}\phi}{M_{\rm Pl}}\right)\right] & (B>0) \\ \phi & (B=0) \\ \frac{1}{2}\left[\phi\sqrt{1-\frac{|B|\phi^2}{M_{\rm Pl}^2}} + \frac{M_{\rm Pl}}{|B|^{1/2}}\sin^{-1}\left(\frac{|B|^{1/2}\phi}{M_{\rm Pl}}\right)\right] & (B<0) \end{cases} . \tag{3.30}$$

Although, one started with a Metric-affine scalar-tensor theory which is recasted into a scalar with a modified kinetic term, this then turned out to be just a canononical scalar (in the conventional sense) described by $\Phi$with a modified potential, as

$$S_{\rm g\Phi} \quad = \quad \int \mathrm{d}^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}\overset{g}{R} - \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - U(\Phi))\right]\,, \tag{3.31}$$

with $U(\Phi) := V(\phi(\Phi))$.

When $B = 0$, the action is no different from the one constructed from the conventional Riemannian geometry. On the other hand, for $B < 0$, the canonical scalar is always $\Phi \leq \phi$. To be more exact, $\phi$ will be constrained as

$$0 \leq \phi \leq \frac{M_{\rm Pl}}{\sqrt{|B|}}\,, \tag{3.32}$$

to avoid ghost instabilities. Thus, the value of $\phi$ cannot go over the Planck mass, which then cannot introduce new phenomena. The cases of $\phi \ll M_{\rm pl}/\sqrt{|B|}$do not introduce new features for inflationary cosmology, and thus will not be discussed furthermore.

The most interesting case is when $B > 0$, which is admitted by Models II(a), III(a) and III(c). The new canonical scalar field $\Phi$ acts differently depending on whatever energy scale one is considering. Such that

$$\Phi \approx \begin{cases} \phi & (\phi \ll M_{\rm Pl}/\sqrt{B}) \\ \frac{\sqrt{B}}{2M_{\rm Pl}}\phi^2 & (\phi \gg M_{\rm Pl}/\sqrt{B}) \end{cases} , \tag{3.33}$$

To put it into words, when the scalar $\phi$ is small, there is only a little difference between Metric-affine gravity and its purely metric counterpart. This difference needs to be taken into consideration when $\phi$ becomes larger than $M_{\rm Pl}/\sqrt{B}$. In an inflationary scenario, a scalar field may exceed Planck mass.[1] When the scalar field does exceed, the effective potential for the new canonical field, which acts as $\Phi \propto \phi^2$, becomes significantly flat. The current CMB observation [4] indicates that a smaller tensor-to-scalar ratio is preferred, thus it is quite relevant to apply this Metric-affine scalar-tensor theory and assume the scalar as an inflaton.

To connect the model to an inflationary scenario, one must analyze the scalar perturbations, and compute the observational variables. The amplitude (2.170)was,

$$P_\zeta \sim \frac{U}{24\pi^2\epsilon_U} , \tag{3.34}$$

and the spectral index (2.171)and tensor-to-scalar ratio (2.175) was

$$n_s \sim 1 + 2\eta_U - 6\epsilon_U , \tag{3.35}$$

$$r \sim 16\epsilon_U \tag{3.36}$$

Here the potential slow-roll parameters are,

$$\epsilon_U(\Phi) = \frac{M_{\rm Pl}^2}{2}\left(\frac{U_{,\Phi}}{U}\right)^2 , \tag{3.37}$$

$$\eta_U(\Phi) = M_{\rm Pl}^2\frac{U_{,\Phi\Phi}}{U} . \tag{3.38}$$

### 3.2.2 Chaotic Inflation in Metric-affine Gravity

To compute the value of the $P_\zeta$,$n_s$ and $r$, one must specify the potential. Here assume the potential is polynomial, i.e. a chaotic inflation scenario[33] also briefly mentioned in §2.2.1. The observationary significant potential is when,

$$V = \frac{1}{2}m^2\phi^2 , \tag{3.39}$$

From this, the deformed potential is,

$$U \approx \begin{cases} \frac{1}{2}m^2\Phi^2 & (\Phi \ll M_{\rm Pl}/\sqrt{B}) \\ \frac{m^2 M_{\rm Pl}}{\sqrt{B}}\Phi & (\Phi \gg M_{\rm Pl}/\sqrt{B}) \end{cases} . \tag{3.40}$$

When $B$ is small, the conventional chaotic inflation is restored. However, when $B \sim \mathcal{O}(1)$ the effective potential is similar to that of a linear potential. This modifies the inflationary scenario, and thus the observational parameters.

Fig.3.1 shows the relation between the mass of the inflaton $m$ and the parameter $B(\alpha)$ using the observed amplitude of the density fluctuation[4]. One could see that when $B(\alpha)$ is sufficiently small, the value is that of the conventional chaotic inflationary model. The $n_s$-$r$ diagram is shown in Fig. 3.2. From Fig.3.2, again it

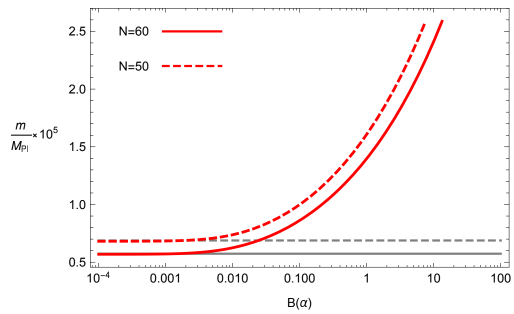



Figure 3.1: The relation between the inflaton mass $m$ and the parameter $B(\alpha)$ constrained from the observational amplitude of density fluctuations. The solid and dashed lines correspond to N=60 and 50, respectively.

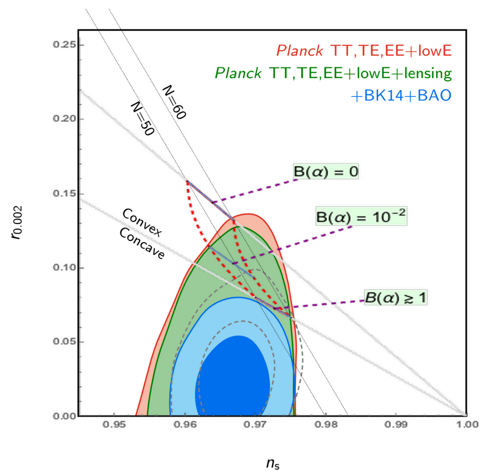



Figure 3.2: The $n_s$-$r$ diagram for different values of $B(\alpha)$. The observational constraint is taken from Planck 2018[4].

could be seen that for a small value of $B(\alpha)$ the variables are the same of the conventional chaotic potential. One the other hand, when $B(\alpha) \sim \mathcal{O}(1)$ the tensor-to-scalar ratio $r$ decrease. Thus, this model is not fully excluded from the current CMB observations.

The current observation indicates that $r < 0.10$. This could be recasted on the constraint on $B$ as $B(\alpha) \gtrsim 0.034$ for $N = 50$ e-folds. This can then be used to constrain the parameter $\alpha$, for the Models II(a) and III(a), as

$$\begin{aligned} &\alpha \gtrsim 0.86 \quad \text{or} \quad \alpha \lesssim -0.13 \quad \text{(Models II(a) and III(a))} \\ &\alpha \gtrsim 1.10 \quad \text{or} \quad \alpha \lesssim -0.10 \qquad \text{(Model III(c))}\,. \end{aligned} \tag{3.41}$$

One could say that if inflation was actually induced by chaotic inflation and the space-time geometry is actually metric-affine, $\overset{\Gamma}{\nabla}{}^{\mu}\overset{\Gamma}{\nabla}{}_{\mu}$ which is $(\alpha = 1)$ is observationally favored than $\overset{\Gamma}{\nabla}{}_{\mu}\overset{\Gamma}{\nabla}{}^{\mu}$ $(\alpha = 0)$.

[1] It is of course not trivial that the current physics could be applicable to the energy scale above Planck. One normally must introduce quantum effects into consideration and may need quantum gravity in such a scenario. However, for this work, the physics are only analyzed classical due to; first for simplicity, second for the non-triviality of quantum effects under curved Metric-affine space-time, and third for the fact that quantum gravity is not complete.

## 3.3 Emergence of G-Inflation

### 3.3.1 Scalar field with Galilean symmetry

Scalar-tensor theories had a great turn in the recent decade or two. The Horndeski scalar-tensor gravity theory or its extended version[110, 113, 151] is a theory in which, the equation of motion in such theories consists of up to the second-order derivatives.(See §2.5 for a review). These theories are constructed from a symmetry of the scalar called the Galileon symmetry. This symmetry is found in the decoupling limit of the DGP(Dvali-Gabadadze-Porrati) brane world model [109, 152] (For reviews see [153]).

The Galilean symmetry is defined as

$$\phi \to \phi + b^{\mu} x_{\mu} + c, \tag{3.42}$$

where $b^{\mu}$ and $c$ are some constants.

Assuming Galilean symmetry fixes the action of a scalar field as

$$\begin{aligned}
\mathcal{L}_{(1,0)} &= \phi \\
\mathcal{L}_{(2,0)} &= \partial_{\mu}\phi\partial^{\mu}\phi \\
\mathcal{L}_{(3,0)} &= \partial_{\mu}\phi\partial^{\mu}\phi\overset{\eta}{\Box}\phi - \partial_{\mu}\phi\partial_{\nu}\phi\partial^{\mu}\partial^{\nu}\phi & (3.43)\\
&= \frac{3}{2}\partial_{\mu}\phi\partial^{\mu}\phi\overset{\eta}{\Box}\phi + (\text{surface terms}) & (3.44)
\end{aligned}$$

which is written up to cubic terms.

Now consider the following covariantized action.

$$S_{g\phi} = \int d^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}\overset{\Gamma}{R} - X - \frac{X}{M^3}\overset{\Gamma}{\Box}\phi\right], \tag{3.45}$$

with $X := -\frac{1}{2}(\partial\phi)^2$ and $M$ is some parameter that has mass dimension. These are purely Galileon in the flat limit. Similar to the previous section, one could obtain the action in the Riemann frame as,

$$S_{g\phi} = \int d^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}\overset{g}{R} - X + \frac{4B(\alpha)}{M_{\rm Pl}^2 M^6}X^3 - \frac{X}{M^3}\overset{g}{\Box}\phi\right], \tag{3.46}$$

with again $B(\alpha)$ being the same function of $\alpha$ that was introduced earlier. 3.2.1.

### 3.3.2 de-Sitter phase for Metric-affine G-inflation

The action (3.46) looks quite similar to the G-inflation action which was introduced in [154]. Here, however, the non-linear term of $X$ naturally appears, as a result of integrating out the connection. One must notice that the third term in this action is of $X^3$, whereas it is $X^2$ for the example that was considered in [154]. Similar to their result, this action also has a de-Sitter solution, as it will be shown.

Now assume a flat FLRW space-time, which leads to the Friedmann equations and the eom of the scalar.

$$\begin{aligned}
0 &= -3M_{\rm Pl}^2 H^2 - \frac{1}{2}\dot{\phi}^2 + \frac{3}{M^3}H\dot{\phi}^3 + \frac{5B}{2}\frac{\dot{\phi}^6}{M^6 M_{\rm Pl}^2}, \\
0 &= M_{\rm Pl}^2(3H^2 + 2\dot{H}) - \frac{1}{2}\dot{\phi}^2 + \frac{B}{2}\frac{\dot{\phi}^6}{M_{\rm Pl}^2 M^6} - \frac{1}{M^3}\ddot{\phi}\dot{\phi}^2, \\
0 &= \left(-1 + 3B\frac{\dot{\phi}^4}{M_{\rm Pl}M^6}\right)(\ddot{\phi} + 3H\dot{\phi}) + 12B\ddot{\phi}\frac{\dot{\phi}^4}{M_{\rm Pl}^2 M^6} + \frac{3}{M^3}\left(\dot{H}\dot{\phi}^2 + 2H\ddot{\phi}\dot{\phi} + 3H^2\dot{\phi}^2\right),
\end{aligned}$$

here $H = \dfrac{\dot{a}}{a}$ is the Hubble parameter defined by the scale factor $a(t)$.

First, in order to check if inflation truly occurs or not, the existence of a de Sitter solution will be shown. For a de-Sitter space-time $H = H_{\mathrm{dS}}$=constant and $\dot{\phi} = \dot{\phi}_{\mathrm{dS}}$=constant, by plugging this into the equation one finds that there are two de-Sitter branches of

$$
\begin{aligned}
X &= X_{\mathrm{dS}_\pm} := \frac{M^3 M_{\mathrm{Pl}}}{\sqrt{3(1+4B) \pm \sqrt{3(3+16B)}}}, \\
H &= H_{\mathrm{dS}_\pm} := \frac{4M^3}{3(1 \pm \sqrt{3(3+16B)})\dot{\phi}_{\mathrm{dS}_\pm}},
\end{aligned}
$$

with $X_{\mathrm{dS}_\pm} = \dot{\phi}^2_{\mathrm{dS}_\pm}/2$. The two branches are labeld $+$ and $-$ respectively.

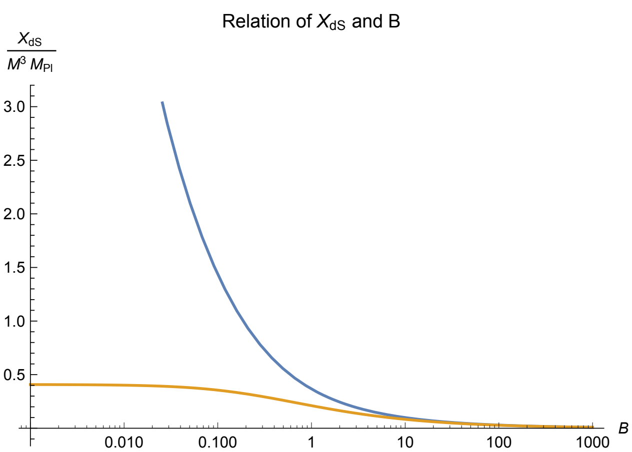
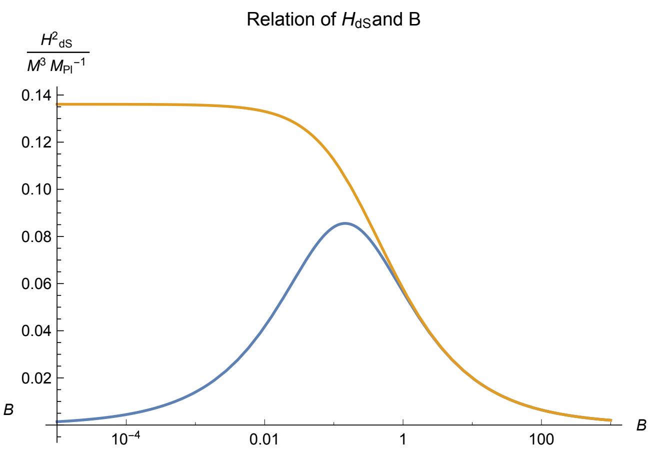



Figure 3.3: The relation of $X$ and $B$ for the two branches with the orange being the $+$ branch and blue the $-$ branch




The relation of $X$ and $B$ for the two branches with the orange being the $+$ branch and blue the $-$ branch

For the $+$ branch, $B > -3/16$ is required. Whereas for the $-$ branch, an additional constraint of $B > 0$ or $-3/16 < B < -1/6$ is required. To obtain an expanding scenario, $H$ must be positive. For the $+$ branch $\dot{\phi}_{\mathrm{dS}_+}$ is always positive while for the $-$ branch $\dot{\phi}_{\mathrm{dS}_-} > 0$ for $-3/16 < B < -1/6$ or $\dot{\phi}_{\mathrm{dS}_-} < 0$ for $B > 0$. In order to have an inflationary scenario, Models I, II (b), and III(b) cannot be considered since $B = -3/8$ for these three models.

To study the stability of the de Sitter phase, perturbation of the action up to the second order is necessary. In [154, 113] the computation of the quadratic action is given. The scalar perturbation $\mathcal{R}_\phi$ while imposing the unitary gauge ($\delta\phi = 0$) is obtained as (2.508). In the current case, it becomes as

$$
S^{(2)}_\pm = \frac{\dot{\phi}^2_{\mathrm{dS}_\pm}}{2(H_{\mathrm{dS}_\pm} - \dot{\phi}^3_{\mathrm{dS}_\pm}/2M^2_{\mathrm{Pl}}M^3)^2} \times \int d\eta d^3x a^2 \left[G_S(\mathcal{R}'_\phi)^2 - F_S(\partial\mathcal{R}_\phi)^2\right], \tag{3.47}
$$

where the prime used to show that the derivative with respect of conformal time $\eta$, and

$$\begin{aligned} F_S &= \frac{1}{3} - \frac{2(1+2B)X_{\mathrm{dS}_\pm}^2}{M_{\mathrm{Pl}}^2 M^6} \\ &= \frac{8B}{3\left[3+14B \pm (1+2B)\sqrt{3(3+16B)}\right]}, \\ G_S &= 1 + \frac{6(1+6B)X_{\mathrm{dS}_\pm}^2}{M_{\mathrm{Pl}}^2 M^6} \\ &= \frac{\left[3+16B \pm (1+2B)\sqrt{3(3+16B)}\right]}{4B}.. \end{aligned}$$

As it was shown in [154, 113], when either $F_S$ or $G_S$ is negative, the de Sitter solution becomes unstable. When the $-$ branch ($X_{\mathrm{dS}_-}$ and $H_{\mathrm{dS}_-}$) is chosen ($F_S < 0$) always occurs. Whereas for the $+$ branch ($X_{\mathrm{dS}_+}$ and $H_{\mathrm{dS}_+}$), $G_S$ is always positive for $B > -3/16$, while $F_S$ is negative when $-3/16 < B < 0$. For this theory, one de Sitter solution ($X_{\mathrm{dS}_+}$ and $H_{\mathrm{dS}_+}$) is stable only when $B > 0$, while the other solution is always unstable. This is also seen through computer calculation where the evolution of inflation goes to the $+$ branch attractor.

Furthermore the sound speed $c_s$ is given by

$$c_s^2 = \frac{F_s}{G_s}. \tag{3.48}$$

Now that the solution of de Sitter attractor phase is derived, it is possible to calculate the tensor-to-scalar ratio and the amplitude of the scalar perturbations, which formula is given in [154, 113] and (2.519)( 2.520). The results are,

$$\begin{aligned} \mathcal{P}_\zeta &\sim \frac{B^2 M^3}{27\pi^2 M_{Pl}^3}\sqrt{\frac{3+16B+\sqrt{3(3+16B)}}{[2+11B+B\sqrt{3(3+16B)}]^3}}, \\ r &\sim \frac{6(1+6B)}{B^2}\sqrt{\frac{6(1+6B)[1+\sqrt{3(3+16B)}]}{3+16B+\sqrt{3(3+16B)}}}. \end{aligned}$$

The observational upper limit of the tensor-to-scalar ratio is constrained as $r < 0.10$[4], the bound on $B(\alpha)$ then is constrained as $B(\alpha) \gtrsim 1.6 \times 10^4$ (see Fig. 3.5 , which is $\alpha \gtrsim 93.7$ or $\alpha \lesssim -92.9$ for Models II(a) and III(a), and $\alpha \gtrsim 119.6$ or $\alpha \lesssim -118.6$ for Model III(c). The mass parameter $M$ could also be bounded from the amplitude of the scalar perturbations and the constraint of $B(\alpha)$ and thus $M \lesssim 0.0060 M_{\mathrm{Pl}}$.

Since this action was first considered through Galilean symmetry, and thus inhibits shift symmetry in the action, the de Sitter phase does not end for this theory. The solution for this was given in [154], where one adds two certain functions into the theory. One of these functions is some coefficient for the terms in the action that breaks the scale invariance and flips the sign of the ghost. Computer calculations indicate that the flip function, whether they are polynomial or exponential, can stop the de-Sitter phase. Furthermore, one also needs to allow the spectral index to tilt so it satisfies observational limits. The modified action of the scalar field is written as,

$$
\begin{aligned}
S_\phi &= \int \mathrm{d}^4x\sqrt{-g}\left[-g_1(\phi)X + g_2(\phi)\frac{X}{M^3}\overset{\Gamma}{\Box}\phi\right] , \\
&= \int \mathrm{d}^4x\sqrt{-g}\left[-g_1(\phi)X + \frac{4g_2^2(\phi)B}{M_{\rm Pl}^2 M^6}X^3 + \frac{g_2(\phi)}{M^3}X\overset{g}{\Box}\phi\right] .
\end{aligned}
\tag{3.49}
$$

where $g_1(\phi)$ and $g_2(\phi)$ are appropriate functions of $\phi$, which break the Galilean symmetry. Here from the first to the second row, the connection was solved and integrated out. For example, when one chooses $g_1(\phi) = \tanh[\lambda(\phi - \phi_{\rm end})/M_{\rm Pl}]$ the inflationary phase could end. On the other hand if $g_2(\phi) = \exp[\epsilon_g\phi/M_{\rm Pl}]$ could tilt the spectral index [154]. It should be noted that it is possible to find appropriate functions to satisfy the observational data.

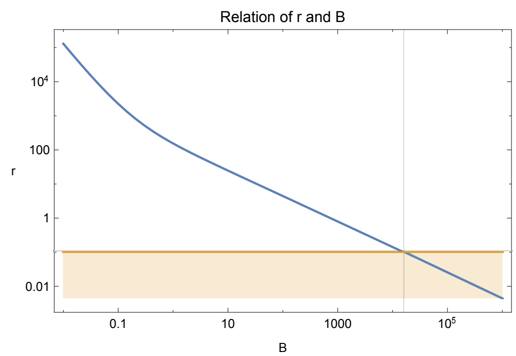



Figure 3.5: The tensor-to-scalar ratio $r$ in terms of $B(\alpha)$ in Metric-affine G-inflation.

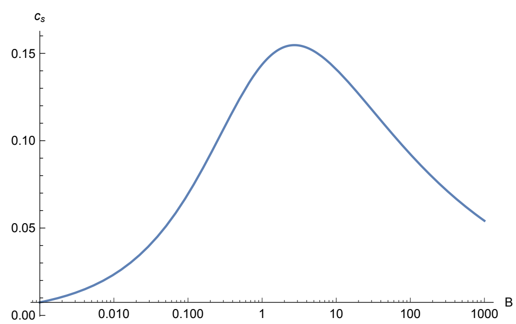



Figure 3.6: Relation of Speed of Scalar and B

## 3.4 An Extended d'Alembertian and Constraints from Observations

The d'Alembertian (3.5) could be extended further as,

$$\overset{\Gamma}{\Box}\phi := \overset{g}{\Box}\phi + \left(\alpha_{\mathcal{Q}}\mathcal{Q}^{\lambda} + \alpha_{\mathcal{W}}\mathcal{W}^{\lambda} + \alpha_{\mathcal{T}}\mathcal{T}^{\lambda}\right)\partial_{\lambda}\phi\,. \tag{3.50}$$

with $\mathcal{T}_{\mu} = T^{\lambda}{}_{\mu\lambda}\,, \mathcal{W}_{\mu} = \frac{1}{4}Q_{\mu\lambda}{}^{\lambda}\,, \mathcal{Q}_{\mu} = Q^{\lambda}{}_{\lambda\mu}$. When it is chosen as $\alpha_{\mathcal{Q}} = -\alpha, \alpha_{\mathcal{W}} = 2, \alpha_{\mathcal{T}} = -1$ the d'Alembertian reduces to the one in the previous section. (3.5). Now, assume that a canonical scalar field action is (3.4). The Riemann frame action, in which the connection is integrated out, becomes

$$S_{g\phi} = \int d^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}\overset{g}{R} - \frac{1}{2}\left(1 + \frac{B(\alpha_I)}{M_{\rm Pl}^2}\phi^2\right)(\partial\phi)^2 - V(\phi)\right]. \tag{3.51}$$

Using this reduced action it is possible to show the constraints on the coefficients $\alpha_I$ ($I = Q$,$W$, and $T$) using the observational data from the CMB just like the previous sector. Furthermore, for the further calculations, all of $\alpha_I$ could be also substituted into arbitrary functions of $\phi$ and $X$ as $\alpha_I(\phi, X)$. However, constraints on functions are far difficult than parameters. Thus, for simplicity and concreteness, the following constraints would be done assuming $\alpha_I$ is a constant. Another way to look at this action is that it could be considered as a scalar-tensor theory in which the scalar is minimally coupled to the connection. In [155, 156], similar action was brought up to study the classification of torsionless Metric-affine scalar-tensor theories by using a different type of transformation of the metric and the connection.

## Model I (Projective Invariant Model)

The projective transformation (2.308) computes

$$\begin{aligned}\overset{\Gamma}{\Box}\phi \to \overset{\bar{\Gamma}}{\Box}\phi &= \overset{g}{\Box}\phi + \alpha_{\mathcal{T}}(\mathcal{T}^{\mu} + 3U^{\mu})\partial_{\mu}\phi + \alpha_{\mathcal{W}}(\mathcal{W}^{\mu} + 2U^{\mu})\partial_{\mu}\phi + \alpha_{Q}(\mathcal{Q}^{\mu} + 2U^{\mu})\partial_{\mu}\phi \\ &= \overset{\Gamma}{\Box}\phi + [3\alpha_{T} + 2(\alpha_{W} + \alpha_{Q})]U^{\mu}\partial_{\mu}\phi\,. \end{aligned} \tag{3.52}$$

For this theory to withhold projective invariance, there must be a relation between the three parameters as, $3\alpha_T + 2(\alpha_W + \alpha_Q) = 0$. Keeping the above in mind, the solutions of the geometrical tensors are,

$$\bar{\kappa}^{\alpha}{}_{\beta\gamma} = \frac{\phi}{4M_{Pl}^2}\left[(3\alpha_{\mathcal{T}} + \alpha_{\mathcal{W}})g_{\beta\gamma}\partial^{\alpha}\phi + (\alpha_{\mathcal{T}} + \alpha_{\mathcal{W}})\delta^{\alpha}_{\beta}\partial_{\gamma}\phi\right], \tag{3.53}$$

$$\mathcal{T}^{\lambda}{}_{\mu\nu} = \frac{\alpha_{\mathcal{T}} + \alpha_{\mathcal{W}}}{4M_{Pl}^2}\phi\delta^{\lambda}_{[\mu}\partial_{\nu]}\phi\,, \tag{3.54}$$

$$\mathcal{Q}_{\lambda}{}^{\mu\nu} = \frac{2\alpha_{\mathcal{T}} + \alpha_{\mathcal{W}}}{M_{Pl}^2}\phi\delta^{(\mu}_{\lambda}\partial^{\nu)}\phi\,, \tag{3.55}$$

up to gauge freedom.
Then the effectively the Riemann frame action becomes (3.51) with

$$B(\alpha_I) = -\frac{1}{8}\left(27\alpha_{\mathcal{T}}^2 + 11\alpha_{\mathcal{W}}^2 + 34\alpha_{\mathcal{T}}\alpha_{\mathcal{W}} + 20\alpha_{\mathcal{W}}\alpha_{\mathcal{Q}} + 40\alpha_{\mathcal{T}}\alpha_{\mathcal{Q}}\right). \tag{3.56}$$

# Model II

## Model II(a) (Torsion-free Model)

Now consider a theory in which $\mathcal{T}^{\mu}{}_{\nu\rho} = 0$ (and thus $\alpha_{\mathcal{T}} = 0$) is satisfied. The solution of the distortion is,

$$\kappa^{\alpha}{}_{\beta\gamma} = -\frac{\phi}{12M_{Pl}^2}\left[3(\alpha_W + 2\alpha_Q)g_{\beta\gamma}\partial^{\alpha}\phi - 2(\alpha_W - 2\alpha_Q)\delta^{\alpha}_{(\beta}\partial_{\gamma)}\phi\right] , \tag{3.57}$$

$$\mathcal{Q}_{\lambda}{}^{\mu\nu} = \frac{\phi}{6M_{Pl}^2}\left[(\alpha_{\mathcal{W}} - 2\alpha_{\mathcal{Q}})g^{\mu\nu}\partial_{\lambda}\phi - 2(\alpha_{\mathcal{W}} + 4\alpha_{\mathcal{Q}})\delta_{\lambda}^{(\mu}\partial^{\nu)}\phi\right] . \tag{3.58}$$

Then the action becomes (3.51) with

$$B(\alpha_I) = -\frac{\alpha_W^2 - 16\alpha_W\alpha_Q - 44\alpha_Q^2}{24} . \tag{3.59}$$

## Model II(b) (Metric-Compatible Model)

Assuming Einstein-Cartan geometry, in which $\mathcal{Q} = 0$ ($\alpha_{\mathcal{W}} = \alpha_{\mathcal{Q}} = 0$), the computed distortion is

$$\kappa^{\alpha}{}_{\beta\gamma} = \frac{\alpha_{\mathcal{T}}}{4M_{Pl}^2}\phi\left(g_{\beta\gamma}\partial^{\alpha}\phi - \delta^{\alpha}_{\beta}\partial_{\gamma}\phi\right) , \tag{3.60}$$

$$\mathcal{T}^{\alpha}{}_{\beta\gamma} = -\frac{\alpha_{\mathcal{T}}}{2M_{Pl}^2}\phi\delta^{\alpha}_{[\beta}\partial_{\gamma]}\phi . \tag{3.61}$$

The resultant action becomes (3.51) with

$$B(\alpha_I) = -\frac{3}{8}\alpha_{\mathcal{T}}^2 . \tag{3.62}$$

# Model III (constraint with a Lagrange Multiplier)

In the Model III, the Lagrange Multiplier $\lambda_\mu$ is introduced to fix the gauge freedom. The following solutions can be computed from each model.

## Model III(a) ($\mathcal{T}_\mu = 0, \alpha_{\mathcal{T}} = 0$)

The solution is

$$\kappa^{\alpha}{}_{\beta\gamma} = -\frac{\phi}{12M_{Pl}^2}\left[3(\alpha_W + 2\alpha_Q)g_{\beta\gamma}\partial^{\alpha}\phi - 2(\alpha_W - 2\alpha_Q)\delta^{\alpha}_{(\beta}\partial_{\gamma)}\phi\right], \tag{3.63}$$

$$\mathcal{T}^{\alpha}{}_{\beta\gamma} = 0 , \tag{3.64}$$

$$\mathcal{Q}_{\lambda}{}^{\mu\nu} = \frac{\phi}{6M_{Pl}^2}\left[(\alpha_{\mathcal{W}} - 2\alpha_{\mathcal{Q}})g^{\mu\nu}\partial_{\lambda}\phi - 2(\alpha_{\mathcal{W}} + 4\alpha_{\mathcal{Q}})\delta_{\lambda}^{(\mu}\partial^{\nu)}\phi\right] , \tag{3.65}$$

with

$$\lambda^{\mu} = \lambda^{\mu}_{T} := -\frac{2}{3M_{Pl}^2}(\alpha_{\mathcal{W}} + \alpha_{\mathcal{Q}})\phi\partial^{\mu}\phi . \tag{3.66}$$

$$\tag{3.67}$$

The Riemann equivalent action becomes (3.51) with

$$B(\alpha_I) = -\frac{\alpha_W^2 - 16\alpha_W\alpha_Q - 44\alpha_Q^2}{24} . \tag{3.68}$$

### Model.III(b) ($\mathcal{Q}_\mu = 0, \alpha_\mathcal{Q} = 0$)

The solution is

$$\kappa^\alpha{}_{\beta\gamma} = \frac{3C_T + C_W}{4M_{Pl}^2}\phi g_{\beta\gamma}\partial^\alpha\phi + \frac{\alpha_\mathcal{T} + \alpha_\mathcal{W}}{4M_{Pl}^2}\phi\delta^\alpha_\beta\phi\partial_\gamma - \frac{5(\alpha_\mathcal{T} + 2\alpha_\mathcal{W})}{4M_{Pl}^2}\phi\delta^\alpha_\gamma\phi\partial_\beta\,, \quad (3.69)$$

$$\mathcal{T}^\alpha{}_{\beta\gamma} = \frac{11\alpha_\mathcal{T} + 6\alpha_\mathcal{W}}{8M_{Pl}^2}\phi\delta^\alpha_{[\beta}\partial_{\gamma]}\phi\,, \quad (3.70)$$

$$\mathcal{Q}_\alpha{}^{\beta\gamma} = -\frac{2\alpha_\mathcal{T} + \alpha_\mathcal{W}}{2M_{Pl}^2}\phi\left(5g^{\beta\gamma}\partial_\alpha\phi - 2\delta_\alpha^{(\beta}\partial^{\gamma)}\phi\right)\,, \quad (3.71)$$

with

$$\lambda^\mu = \lambda^\mu_W := -\frac{1}{2M_{Pl}^2}(3\alpha_\mathcal{T} + 2\alpha_\mathcal{W})\phi\partial^\mu\phi\,. \quad (3.72)$$

The Riemann geometry equivalent action becomes (3.51) with

$$B(\alpha_I) = \frac{3}{8}(11\alpha_\mathcal{T}^2 + 12\alpha_\mathcal{T}\alpha_\mathcal{W} + 3\alpha_\mathcal{W}^2)\,. \quad (3.73)$$

### Model III(c) ($\mathcal{W}_\mu = 0, \alpha_\mathcal{W} = 0$)

The solution is

$$\kappa^\alpha{}_{\beta\gamma} = \frac{3\alpha_\mathcal{T} - 2\alpha_\mathcal{Q}}{8M_{Pl}^2}\phi g_{\beta\gamma}\partial^\alpha\phi - \frac{\alpha_\mathcal{T} + 2\alpha_\mathcal{Q}}{8M_{Pl}^2}\phi\delta^\alpha_\beta\phi\partial_\gamma - \frac{\alpha_\mathcal{T} - 2\alpha_\mathcal{Q}}{16M_{Pl}^2}\phi\delta^\alpha_\gamma\phi\partial_\beta\,, \quad (3.74)$$

$$\mathcal{T}^\alpha{}_{\beta\gamma} = -\frac{\alpha_\mathcal{T} + 6\alpha_\mathcal{Q}}{8M_{Pl}^2}\phi\delta^\alpha_{[\beta}\partial_{\gamma]}\phi\,, \quad (3.75)$$

$$\mathcal{Q}_\alpha{}^{\beta\gamma} = -\frac{\alpha_\mathcal{T} - 2\alpha_\mathcal{Q}}{8M_{Pl}^2}\phi\left(g^{\beta\gamma}\partial_\alpha\phi - 4\delta_\alpha^{(\beta}\partial^{\gamma)}\phi\right)\,, \quad (3.76)$$

with

$$\lambda^\mu = \lambda^\mu_Q := -\frac{1}{M_{Pl}^2}(3\alpha_\mathcal{T} - 2\alpha_\mathcal{Q})\phi\partial^\mu\phi\,. \quad (3.77)$$

The Riemann frame action becomes (3.51) with

$$B(\alpha_I) = -\frac{3}{32}(\alpha_\mathcal{T}^2 + 12\alpha_\mathcal{T}\alpha_\mathcal{Q} - 12\alpha_\mathcal{Q}^2)\,. \quad (3.78)$$

### 3.4.1 Relation between the three models

One interesting fact is that there is some relation between the three models I, II, and III.

Model I is a model in which the thoery is constrained to be projective invariant. Thus it is possible to eliminate one of the three constants by using the gauge freedom.
For example, by choosing $U^\mu = -\mathcal{T}^\mu/3$, the connection term of $\mathcal{T}^\mu$ and the related constant $\alpha_\mathcal{T}$disappears. The two parameters that are left are$\alpha_\mathcal{Q}$ and $\alpha_\mathcal{W}$ and the extended d' Alembertian (3.50) is written with them. Another example is when the choice is made as $U^\mu = -\mathcal{W}^\mu/2$ and $U^\mu = -\mathcal{Q}^\mu/2$, for each case the two parameters that are left are $\alpha_\mathcal{T}, \alpha_\mathcal{Q}$ and $\alpha_\mathcal{T}, \alpha_\mathcal{W}$ are used to represent (3.50).

For each case becomes, the trace-free distortion tensor could be calculated as,

$$\bar{\kappa}^{\alpha}{}_{\beta\gamma} = \begin{cases} \frac{\phi}{12M_{\mathrm{Pl}}^2}\left[-3(\alpha_W + 2\alpha_Q)g_{\beta\gamma}\partial^\alpha\phi + (\alpha_W - 2\alpha_Q)\delta^\alpha_\beta\partial_\gamma\phi\right] \\ \frac{\phi}{8M_{\mathrm{Pl}}^2}\left[(3\alpha_T - 2\alpha_Q)g_{\beta\gamma}\partial^\alpha\phi - (\alpha_T + 2\alpha_Q)\delta^\alpha_\beta\partial_\gamma\phi\right] \\ \frac{\phi}{4M_{\mathrm{Pl}}^2}\left[(3\alpha_T + \alpha_W)g_{\beta\gamma}\partial^\alpha\phi + (\alpha_T + \alpha_W)\delta^\alpha_\beta\partial_\gamma\phi\right] \end{cases}, \tag{3.79}$$

with each gauge choice of $U^\mu$ as

$$U_\mu = \begin{cases} \frac{\alpha_{\mathcal{W}} - 2\alpha_{\mathcal{Q}}}{12M_{Pl}^2}\phi\partial_\mu\phi \\ -\frac{\alpha_{\mathcal{T}} - 2\alpha_{\mathcal{Q}}}{16M_{Pl}^2}\phi\partial_\mu\phi \\ -\frac{5(2\alpha_{\mathcal{T}} + \alpha_{\mathcal{W}})}{4M_{Pl}^2}\phi\partial_\mu\phi \end{cases}. \tag{3.80}$$

With the parameter $B(\alpha_I)$ being

$$B(\alpha_I) = \begin{cases} -\frac{1}{24}(\alpha_W^2 - 16\alpha_W\alpha_Q - 44\alpha_Q^2) \\ -\frac{3}{32}(\alpha_{\mathcal{T}}^2 + 12\alpha_{\mathcal{T}}\alpha_{\mathcal{Q}} - 12\alpha_{\mathcal{Q}}^2) \\ \frac{3}{8}(11\alpha_{\mathcal{T}}^2 + 12\alpha_{\mathcal{T}}\alpha_{\mathcal{W}} + 3\alpha_{\mathcal{W}}^2) \end{cases}, \tag{3.81}$$

which could be obtained from (3.56) by substituting one constant by $3\alpha_{\mathcal{T}} + 2(\alpha_{\mathcal{W}} + \alpha_{\mathcal{Q}}) = 0$; the projective invariance condition.

Comparing the above to those of Models II or III, Models III (a), (b), and (c) correspond to the three values that interchange with projective transformation in Model I (3.81), respectively. Furthermore, the result of Model II (a) is of Model I (3.81). Whereas Model II (b), cannot be mapped from Model I with any gauge choice except for a special case of the constants. When specifically, $\alpha_{\mathcal{W}} = -2\alpha_{\mathcal{T}}$ and $\alpha_{\mathcal{Q}} = \frac{1}{2}\alpha_{\mathcal{T}}$. These satisfy the projective invariance condition and it is possible to obtain the same results for Model II (b) and Model I.

### 3.4.2 Combination of Model I and III: Complete fixing of Projective Gauge

Recently, the necessary and sufficient conditions of estimating whether certain gauge fixings are complete or not were formulated [157]. When a certain theory inhibits gauge invariance, one may fix the gauge *a priori* or *a posteriori* the variation of the action. In both cases, if the gauge fixing is complete, one will obtain the full set of the same equations. However, if the gauge fixing is incomplete, one may find unwanted degrees of freedom when deriving the equations of motion. The authors of [157] have found a way to determine whether the gauge conditions are sufficient or not; if the gauge fixing is complete the solution(s) of the Lagrange multipliers are $\lambda_I^\mu = 0$.

Now consider Model I, where a relation is imposed as, $3\alpha_{\mathcal{T}}+2(\alpha_{\mathcal{W}}+\alpha_{\mathcal{Q}}) = 0$ to withhold projective invariance. Then, just as in Model III, one can impose constraints on either $\mathcal{T}$, $\mathcal{Q}$ or $\mathcal{W}$. By the equations of motion, the Lagrangian multipliers could be solved, as seen in the previous section. Now since the theory is projective invariant one could interchange the parameters between $\mathcal{T}$, $\mathcal{Q}$ or $\mathcal{W}$ as

$$\lambda_T^\mu = -\frac{2}{3M_{Pl}^2}(\alpha_{\mathcal{W}} + \alpha_{\mathcal{Q}})\phi\partial^\mu\phi = \frac{\alpha_{\mathcal{T}}}{M_{Pl}^2}\phi\partial^\mu\phi\,, \tag{3.82}$$

$$\lambda_Q^\mu = -\frac{1}{2M_{Pl}^2}(3\alpha_{\mathcal{T}} + 2\alpha_{\mathcal{W}})\phi\partial^\mu\phi = \frac{\alpha_{\mathcal{Q}}}{M_{Pl}^2}\phi\partial^\mu\phi\,, \tag{3.83}$$

$$\lambda_W^\mu = -\frac{1}{M_{Pl}^2}(3\alpha_{\mathcal{T}} - 2\alpha_{\mathcal{Q}})\phi\partial^\mu\phi = \frac{2\alpha_{\mathcal{W}}}{M_{Pl}^2}\phi\partial^\mu\phi\,, \tag{3.84}$$

Since each Lagrangian multiplier fixes each corresponding vector, the corresponding coefficient in the extended d'Alembertian becomes trivially zero, and thus, in a projective invariant theory, $\lambda_I^\mu = 0$ is satisfied. This shows that in the extended d'Alembertian case, the constraints with Lagrangian multipliers are not only just gauge fixings but actually complete gauge fixings. To summarize, when considering how to fix the projective gauge in Model I, constraining either of the three vectors with Lagrangian multipliers that are considered in Model III is a good choice since these lead to complete gauge fixings.

### 3.4.3 Observational constraints on the parameters $\alpha_I$

Finally, it is possible to compute the observational bounds for the coefficients of the extended d'Alembertian. First assume, just like in the previous section, an inflationary scenario with a chaotic potential $V(\phi) = \frac{1}{2}m^2\phi^2$. Recall that the tensor-to-scalar ratio $r$ bounds show that $B(\alpha_I) \gtrsim 0.034$. Since Model I is constructed from three parameters, that are constrained by the projective condition $3\alpha_T + 2(\alpha_W + \alpha_Q) = 0$, one must consider a three-dimensional parameter space constrained on a two-dimensional plane with bounds. The observational bounds of $B(\alpha_I)$ (3.56) can be projected onto two-parameter plane with the projective conditions taken into account. The allowed regions for each two parameters are shown in Figs. 3.7, 3.8 and 3.9.

Recall that Models III(a), III(b) and III(c) give the ditto function $B(\alpha_I)$ under specific gauges ($\mathcal{T}^\mu = 0$), ($\mathcal{Q}_\mu = 0$) and ($\mathcal{W}_\mu = 0$), respectively. Therefore, the bounds on the two constants are given also by Figs. 3.7, 3.8 and 3.9. Model II(a) is ditto with Model III(a), which imposes a bound on the two constants and are shown in Fig. 3.7. Model II(b) is excluded because $B(\alpha_I) < 0$.

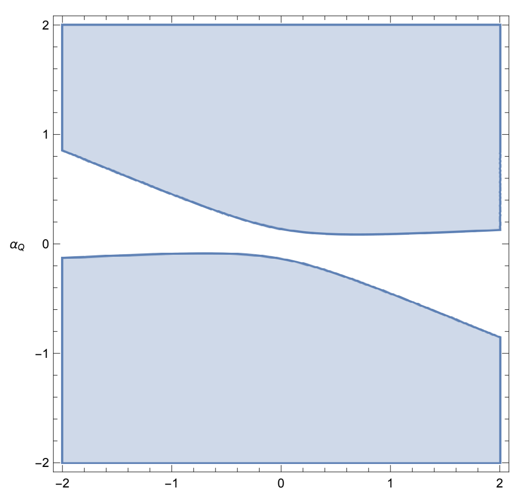



Figure 3.7: Constraints on $\alpha_W$ and $\alpha_Q$ in Model I with two parameters $(\alpha_W, \alpha_Q)$, and Model II(a) and Model III(a).The shaded region is consistent with the observational data for the tensor-scalar ratio $r$.

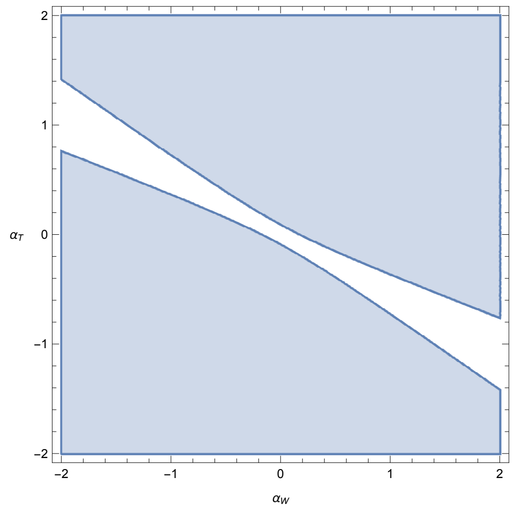



Figure 3.8: Constraints on $\alpha_T$ and $\alpha_W$ in Model I with two parameters $(\alpha_T, \alpha_W)$, and Model III(b).The shaded region is consistent with the observational data for the tensor-scalar ratio $r$.

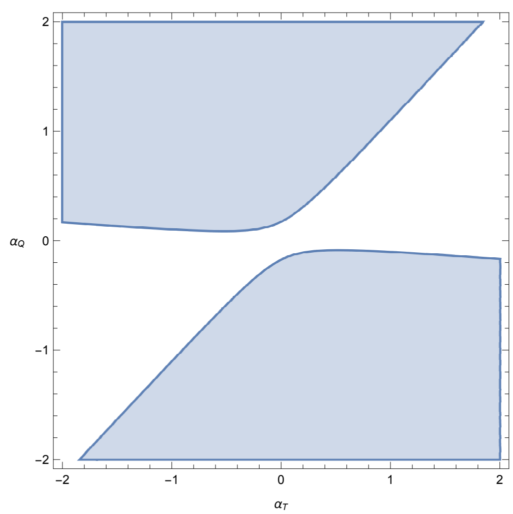



Figure 3.9: Constraints on $\alpha_Q$ and $\alpha_T$ in Model I with two parameters $(\alpha_Q, \alpha_T)$, and Model III(c).The shaded region is consistent with the observational data for the tensor-scalar ratio $r$.

## 3.5 Summary

In this part of the thesis, inflation was investigated within the context of metric-affine gravity. After showing that there are ambiguities for the covariantization of scalars in metric-affine gravity, the loop-hole was utilized to construct a 'minimal' coupled scalar theory that differs from theories that are covariantized in Riemann

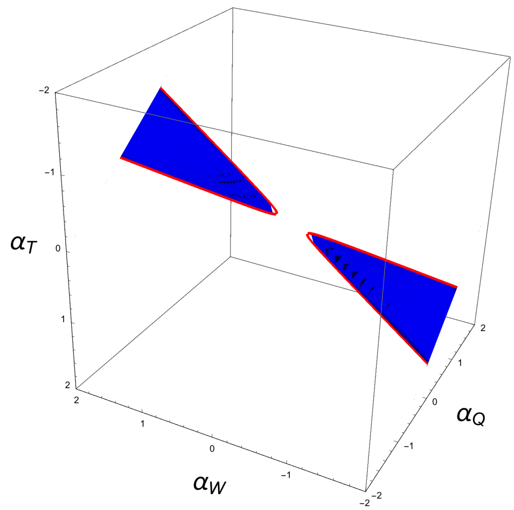



Figure 3.10: Systematic Figure of 3-dimensional Constraints on $\alpha_Q$, $\alpha_W$ and $\alpha_T$ in Model I

geometry. The novel theory has different models that each have a Riemann frame, a frame of which the connection is solved and substituted such that the resultant theory is written in Riemann geometry but has equivalent dynamics with the original theory. Then observational variables for each model were computed and compared. It was found that, in certain models, chaotic inflation is consistent with observations. This is in sharp contrast with the result in the metric formalism where chaotic inflation is eliminated from observations. Other relevant models were also considered and computed. This work hints at the possibility of probing the 'geometry' of the universe, whether Riemann or metric-affine, through observations.

# Chapter 4

# Galileon and generalized Galileon with projective invariance in metric-affine formalism

| | |
|---|---|
| Based on | *"Galileon and generalized Galileon with projective invariance in a metric-affine formalism"* |
| Authors: | Katsuki Aoki, Keigo Shimada |
| Journal: | Phys.Rev.D 98 (2018) 4, 044038 |

In the previous part of the thesis, candidates of covariantization of scalar fields and their derivatives were investigated in metric-affine formalism. A natural question after such application is the question of whether one can extend scalar-tensor theories further within metric-affine formalism. In this part and the preceding part of the thesis, such theories will be constructed and explored. One important feature that one has to take into account, as mentioned in §2.5, is the ghost-freeness of such theories. It will be shown later on that such ghost-free property is closely related to *projective symmetry* when constructed in metric-affine formalism. Combining scalar theories with metric-affine formalism is rather a new and young approach and this work will hopefully serve as an important beacon for investigating metric-affine scalar-tensor theories.

## 4.1 Galileon in metric-affine formalism

Galileons were phenomenal for the investigation and the surge of popularity for scalar-tensor theories. Recall Galileons are theories, that in flat space-time enjoy the Galilean invariance,

$$\phi \to \phi + b_\mu x^\mu + c\,, \tag{4.1}$$

with $b_\mu$ and $c$ being some constant parameters [108]. Then the flat space-time Galileon Lagrangian is uniquely given as,

$$\mathcal{L} = \sum_{n\geq 2}^{5} \frac{c_n}{\Lambda_3^{3(n-2)}} \mathcal{L}_n^{\rm gal}\,, \tag{4.2}$$

with each term being,

$$\begin{aligned}
\mathcal{L}_2^{\rm gal} &:= \epsilon^{\alpha\beta\gamma\delta}\epsilon^{\alpha'}{}_{\beta\gamma\delta}\partial_\alpha\phi\partial_{\alpha'}\phi\,, && (4.3)\\
\mathcal{L}_3^{\rm gal} &:= \epsilon^{\alpha\beta\gamma\delta}\epsilon^{\alpha'\beta'}{}_{\gamma\delta}\partial_\alpha\phi\partial_{\alpha'}\phi\partial_\beta\partial_{\beta'}\phi\,, && (4.4)\\
\mathcal{L}_4^{\rm gal} &:= \epsilon^{\alpha\beta\gamma\delta}\epsilon^{\alpha'\beta'\gamma'}{}_{\delta}\partial_\alpha\phi\partial_{\alpha'}\phi\partial_\beta\partial_{\beta'}\phi\partial_\gamma\partial_{\gamma'}\phi\,, && (4.5)\\
\mathcal{L}_5^{\rm gal} &:= \epsilon^{\alpha\beta\gamma\delta}\epsilon^{\alpha'\beta'\gamma'\delta'}\partial_\alpha\phi\partial_{\alpha'}\phi\partial_\beta\partial_{\beta'}\phi\partial_\gamma\partial_{\gamma'}\phi\partial_\delta\partial_{\delta'}\phi && (4.6)
\end{aligned}$$

$c_n$ are dimensionless constants while $\Lambda_3$ is the strong coupling scale. By expanding the Levi-Civita tensors, one obtain the forms

$$\begin{aligned}
\mathcal{L}_2^{\rm gal} &= -6(\partial\phi)^2\,, && (4.7)\\
\mathcal{L}_3^{\rm gal} &= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi\epsilon^{\alpha\beta\gamma\delta}\epsilon^{\alpha'}{}_{\beta\gamma\delta}\partial_{\alpha'}\partial_\alpha\phi\,,\\
&= -(\partial\phi)^2\overset{\eta}{\Box}\phi\,, && (4.8)\\
\mathcal{L}_4^{\rm gal} &= \partial_\mu\phi\partial^\mu\phi\epsilon^{\alpha\beta\gamma\delta}\epsilon^{\alpha'\beta'}{}_{\gamma\delta}\partial_{\alpha'}\partial_\alpha\phi\partial_{\beta'}\partial_\beta\phi\\
&= -2(\partial\phi)^2\left[(\overset{\eta}{\Box}\phi)^2 - (\partial_\alpha\partial_\beta\phi)^2\right]\,, && (4.9)\\
\mathcal{L}_5^{\rm gal} &= \frac{5}{2}\partial_\mu\phi\partial^\mu\phi\epsilon^{\alpha\beta\gamma\delta}\epsilon^{\alpha'\beta'\gamma'}{}_{\delta}\partial_{\alpha'}\partial_\alpha\phi\partial_{\beta'}\partial_\beta\phi\partial_{\gamma'}\partial_\gamma\phi\\
&= -\frac{5}{2}(\partial\phi)^2\left[(\overset{\eta}{\Box}\phi)^3 - 3\overset{\eta}{\Box}\phi(\partial_\alpha\partial_\beta\phi)^2 + 2(\partial_\alpha\partial_\beta\phi)^3\right]\,, && (4.10)
\end{aligned}$$

, with $\overset{\eta}{\Box} = \eta^{\mu\nu}\partial_\mu\partial_\nu\phi$ understood as the flat space-time d'Alembertian operator. Here some terms were integrated by parts to obtain the result.

The integration by parts of such terms can be schematically given as,

$$\begin{aligned}
\mathcal{L}_n^{\rm gal} &= \epsilon\epsilon(\partial\phi)^2(\partial\partial\phi)^{n-2}\\
&= (\partial\phi)^2\epsilon\epsilon(\partial\partial\phi)^{n-2} + \text{ total divergence}\,. && (4.11)
\end{aligned}$$

Some comments should be taken into account of these terms. In metric formalism these Galileon terms do not have unique covariantization, which leads to the covariant Galileon [158] and the covariantized Galileon [108], respectively. The covariant Galileon is the covariantization of $(\partial\phi)^2\epsilon\epsilon(\partial\partial\phi)^{n-2}$, whereas the covariantized Galileon is of $\epsilon\epsilon(\partial\phi)^2(\partial\partial\phi)^{n-2}$. Recall that the covariant Galileon becomes a subset of

Horndeski gravity, whereas the covariantized Galileon is of GLPV theory. Thus is the status of construction in metric formalism.

Now consider instead covariantizing the Galileon theory is metric affine formalism. As mentioned in §2.4.2.2, projective invariance is an important feature in metric-affine gravity which the Einstein-Hilbert action also inheres. Therefore, it is natural to think that projective invariance plays a key role in constructing metric-affine scalar-tensor theories [1]. Thus projective invariance will be assumed for the following discussion of constructing metric-affine Galileon theories [2].

To construct covariantizations of Galileon theories, one could introduce the *"minimal assumption"*, i.e. the theories are the form,

$$\mathcal{L}_n^{\mathrm{gal}\Gamma} = \mathcal{L}_n^{\mathrm{gal}\Gamma}(g, \phi, \overset{\Gamma}{\nabla}_\mu \phi, \overset{\Gamma}{\nabla}_\mu \overset{\Gamma}{\nabla}_\nu \phi)\,. \tag{4.12}$$

Recall that in §3.1.1, it was noticed that there are multiple ways to construct derivative of scalars in metric-affine formalism due to the presence of non-metricity, since

$$\begin{aligned}\overset{\Gamma}{\nabla}{}^\mu \overset{\Gamma}{\nabla}{}^\nu \phi &= g^{\mu\alpha} g^{\nu\beta} \overset{\Gamma}{\nabla}_\alpha \overset{\Gamma}{\nabla}_\beta \phi + \mathcal{Q}^{\mu\nu\gamma} \overset{\Gamma}{\nabla}_\gamma \phi \\ &\neq g^{\mu\alpha} g^{\nu\beta} \overset{\Gamma}{\nabla}_\alpha \overset{\Gamma}{\nabla}_\beta \phi\,. \end{aligned} \tag{4.13}$$

Therefore a Lagrangian which contains $\overset{\Gamma}{\nabla}{}^\mu \overset{\Gamma}{\nabla}{}^\nu \phi$ implicitly contains non-metricity tensor in its construction,

$$\begin{aligned}&\mathcal{L}(g, \phi, \overset{\Gamma}{\nabla}_\mu \phi, \overset{\Gamma}{\nabla}_\mu \overset{\Gamma}{\nabla}_\nu \phi, \overset{\Gamma}{\nabla}{}^\mu \overset{\Gamma}{\nabla}{}^\nu \phi) \\ &= \mathcal{L}(g, \phi, \overset{\Gamma}{\nabla}_\mu \phi, \overset{\Gamma}{\nabla}_\mu \overset{\Gamma}{\nabla}_\nu \phi, \mathcal{Q}_\alpha{}^{\beta\gamma})\,. \end{aligned} \tag{4.14}$$

When this "minimal assumption" (4.12) is assumed, one finds that the Lagrangian of the covariant Galileons are uniquely constructed up to quartic order in the presence of projective invariance. (In§6.2 the "minimal assumption" will be loosened and the most general Galileon terms will be constructed)

This is due to the fact that the covariantization of $\epsilon\epsilon(\partial\phi)^2(\partial\partial\phi)^{n-2}$ is projective invariant, whereas $(\partial\phi)^2\epsilon\epsilon(\partial\partial\phi)^{n-2}$ is not. Thus, in contrast with metric formalism, the metric-affine Galileon is just simply the covariantization of (4.3)-(4.6), by replacing $\overset{\Gamma}{\nabla}_\mu$ with $\partial_\mu$, and is unique. The explicit form of projective Galileons are,

$$\mathcal{L}_2^{\mathrm{gal}\Gamma} = \epsilon^{\alpha\beta\gamma\delta} \epsilon^{\alpha'}{}_{\beta\gamma\delta} \overset{\Gamma}{\nabla}_\alpha \phi \overset{\Gamma}{\nabla}_{\alpha'} \phi\,, \tag{4.15}$$

$$\mathcal{L}_3^{\mathrm{gal}\Gamma} = \epsilon^{\alpha\beta\gamma\delta} \epsilon^{\alpha'\beta'}{}_{\gamma\delta} \overset{\Gamma}{\nabla}_\alpha \phi \overset{\Gamma}{\nabla}_{\alpha'} \phi \overset{\Gamma}{\nabla}_\beta \overset{\Gamma}{\nabla}_{\beta'} \phi\,, \tag{4.16}$$

$$\mathcal{L}_4^{\mathrm{gal}\Gamma} = \epsilon^{\alpha\beta\gamma\delta} \epsilon^{\alpha'\beta'\gamma'}{}_{\delta} \overset{\Gamma}{\nabla}_\alpha \phi \overset{\Gamma}{\nabla}_{\alpha'} \phi \overset{\Gamma}{\nabla}_\beta \overset{\Gamma}{\nabla}_{\beta'} \phi \overset{\Gamma}{\nabla}_\gamma \overset{\Gamma}{\nabla}_{\gamma'} \phi\,, \tag{4.17}$$

$$\mathcal{L}_5^{\mathrm{gal}\Gamma} = \epsilon^{\alpha\beta\gamma\delta} \epsilon^{\alpha'\beta'\gamma'\delta'} \overset{\Gamma}{\nabla}_\alpha \phi \overset{\Gamma}{\nabla}_{\alpha'} \phi \overset{\Gamma}{\nabla}_\beta \overset{\Gamma}{\nabla}_{\beta'} \phi \overset{\Gamma}{\nabla}_\gamma \overset{\Gamma}{\nabla}_{\gamma'} \phi \overset{\Gamma}{\nabla}_\delta \overset{\Gamma}{\nabla}_{\delta'} \phi\,, \tag{4.18}$$

and

$$\mathcal{L}_4^{\mathrm{gal}\Gamma'} = \epsilon^{\alpha\beta\gamma\delta} \epsilon^{\alpha'\beta'\gamma'}{}_{\delta} \overset{\Gamma}{\nabla}_\alpha \phi \overset{\Gamma}{\nabla}_{\alpha'} \phi \overset{\Gamma}{\nabla}_\beta \overset{\Gamma}{\nabla}_{\beta'} \phi \overset{\Gamma}{\nabla}_{\gamma'} \overset{\Gamma}{\nabla}_{\gamma} \phi\,, \tag{4.19}$$

$$\mathcal{L}_5^{\mathrm{gal}\Gamma'} = \epsilon^{\alpha\beta\gamma\delta} \epsilon^{\alpha'\beta'\gamma'\delta'} \overset{\Gamma}{\nabla}_\alpha \phi \overset{\Gamma}{\nabla}_{\alpha'} \phi \overset{\Gamma}{\nabla}_\beta \overset{\Gamma}{\nabla}_{\beta'} \phi \overset{\Gamma}{\nabla}_\gamma \overset{\Gamma}{\nabla}_{\gamma'} \phi \overset{\Gamma}{\nabla}_{\delta'} \overset{\Gamma}{\nabla}_{\delta} \phi\,. \tag{4.20}$$

[1] As will be shown later, ghost-free theories seem to be projective invariant, although the opposite is not generally true. For example, in §6.3 a ghostly mode appears because projective invariance is violated.

[2] In past literature, projective invariance is often overlooked for metric-affine scalar-tensor theories. Indeed the first literature on metric-affine scalar-tensor theories uses an additional constraint to eliminate the projective mode present in the connection [159]

Notice that the final two terms are allowed due to the existence of torsion, which induces asymmetry of the indices of $\overset{\Gamma}{\nabla}_\mu \overset{\Gamma}{\nabla}_\nu \phi$, since,

$$2\overset{\Gamma}{\nabla}_{[\mu} \overset{\Gamma}{\nabla}_{\nu]}\phi = \mathcal{T}^{\alpha}{}_{\mu\nu}\partial_\alpha\phi \neq 0\,. \tag{4.21}$$

Therefore both (4.19) and (4.20) differs from (4.17) and (4.18). However, as it will shown in §4.4, (4.17) and (4.19) does not change the stucture of the theory and computes the same result. Therefore, as for this section the terms (4.19) and (4.20) will be ignored. Adding the new found Galileon terms to the Einstein-Hilbert action, the complete action of the covariant Galileon in the metric-affine formalism is thus,

$$\mathcal{L}(g,\Gamma,\phi) = \frac{M_{\rm pl}^2}{2} g^{\mu\nu} \overset{\Gamma}{R}_{\mu\nu} + \sum_{n\geq 2}^{5} \frac{c_n}{\Lambda_3^{3(n-2)}} \mathcal{L}_n^{\rm gal\Gamma}\,. \tag{4.22}$$

This action, at least up to the quartic order, is the unique covariant Galileon theory in metric-affine formalism.

In order to analyze this action, however, one can only take up to the $n = 4$ term. This is because (4.22) is third-order in affine connection and thus the equation of motion is non-linear which, although algebraic, not solvable. Therefore, here and after, the term $\mathcal{L}_5^{\rm gal\Gamma}$ will be omitted in order to compute concrete solutions of this theory.

Now that (4.22) is considered up to $n = 4$, the connection can be solved. Using the distortion trick, see §A.2, one obtains the following solution for the equation of motion of the connection,

$$\begin{aligned}\kappa^{\mu}{}_{\alpha\beta} &= \\ \frac{-1}{M_{\rm pl}^2\left(1+2c_4X^2/(\Lambda_2)^8\right)}\Bigg[ &\frac{c_3}{\Lambda_3^3}\left(X\delta^\mu_\beta\phi_\alpha - X\phi^\mu g_{\alpha\beta} + 2\phi^\mu\phi_\alpha\phi_\beta\right) \\ &+\frac{2c_4}{\Lambda_3^6}\left\{2X\phi^\mu{}_{(\alpha}\phi_{\beta)} - X\phi^\mu\phi_{\alpha\beta} + \phi_\alpha\phi_\beta(\phi^\mu\phi^\gamma{}_\gamma - 2\phi^{\mu\gamma}\phi_\gamma)\right\}\Bigg]\,,\end{aligned} \tag{4.23}$$

up to the projective gauge. Recall that $\phi_\mu = \partial_\mu\phi$, $\phi_{\mu\nu} = \overset{g}{\nabla}_\mu \overset{g}{\nabla}_\nu\phi$, and as for this part of the thesis $X = \phi^\mu\phi_\mu$.

Here the scale $\Lambda_2$ is defined as

$$(\Lambda_2)^4 = (\Lambda_3)^3\, M_{\rm pl}\,. \tag{4.24}$$

Notice that the solution of distortion implies that Levi-Civita is generally not the solution for the connection due to the scalar field, and it is neither torsionless nor metric compatible. Recall that for many metric-affine theories, f(R) theories, for example, have a torsionless or metric-compatible connection as a solution once the appropriate gauge is chosen. This is not the case for the projective Galileon theory.

Substituting the solution of the connection into (4.22), the resultant Riemann frame action is thus,

$$\mathcal{L} = \frac{M_{\rm pl}^2}{2}\overset{g}{R} + \frac{3(c_3^2-4c_2c_4)X^3/(\Lambda_2)^8}{1+2c_4X^2/(\Lambda_2)^8} + \frac{1}{1+2c_4X^2/(\Lambda_2)^8}\left(c_2\mathcal{L}_2^{\rm galg} + \frac{c_3}{(\Lambda_3)^3}\mathcal{L}_3^{\rm galg} + \frac{c_4}{(\Lambda_3)^6}\mathcal{L}_4^{\rm galg}\right)\,, \tag{4.25}$$

with

$$\mathcal{L}_2^{\rm galg} = \epsilon^{\alpha\beta\gamma\delta}\epsilon^{\alpha'\beta'}{}_{\gamma\delta}\phi_\alpha\phi_{\alpha'}\,, \tag{4.26}$$

$$\mathcal{L}_3^{\rm galg} = \epsilon^{\alpha\beta\gamma\delta}\epsilon^{\alpha'}{}_{\beta'\gamma\delta}\phi_\alpha\phi_{\alpha'}\phi_{\beta\beta'}\,, \tag{4.27}$$

$$\mathcal{L}_4^{\rm galg} = \epsilon^{\alpha\beta\gamma\delta}\epsilon^{\alpha'\beta'\gamma'}{}_{\delta}\phi_\alpha\phi_{\alpha'}\phi_{\beta\beta'}\phi_{\gamma\gamma'}\,. \tag{4.28}$$

This action is neither the covariant Galileon not the covariantized Galileon in the metric formalism. Furthermore, for scales,

$$|X| \ll (\Lambda_2)^4 \,, \tag{4.29}$$

the action (4.25) reduces to that of the metric formalism. The relevance of this new scale $\Lambda_2$ naturally appears in metric-affine formalism.

Therefore, as for the covariantization of Galileons, there are three theories: the covariant Galileon and the covariantized Galileon in metric formalism, and the metric-affine projective invariant Galileon (4.22) which is dynamically equivalent to (4.25). These theories, when one "turns off" gravity, which corresponds to taking the limit of $M_{\rm pl} \to \infty$, reduce back to the original flat Galileon primary introduced in (4.2).

## 4.2 Generalized Galileon in metric-affine formalism is DHOST

### 4.2.1 Equivalent Lagrangian to class $^2$ N-I/Ia of DHOST

In this section, based upon the analysis done previously, the generalizations of curved space Galileons that respect projective invariance will be explored in metric-affine formalism.

Recall that, the currently known most general scalar-tensor theories without Ostrogradsky instability are known as DHOST theories, as introduced in §2.5.4. In DHOST theory, in order to evade the Ostrogradsky ghost, one needs to fine-tune the functions within the theory to satisfy the degeneracy conditions (6.105) [120].

In this section, it will be shown that, out of the different classes of DHOST, the following Lagrangian is equivalent to class $^2$N-I/Ia of DHOST theory without the need for fine-tuning.

$$\mathcal{L}(g,\Gamma,\phi) = f_1(\phi,X) g^{\mu\nu} \overset{\Gamma}{R}_{\mu\nu} + f_2(\phi,X) \overset{\Gamma}{G}{}^{\mu\nu} \overset{\Gamma}{\nabla}_\mu \phi \overset{\Gamma}{\nabla}_\nu \phi + F_2(\phi,X) + F_3(\phi,X)\mathcal{L}_3^{\rm gal\Gamma} + F_4(\phi,X)\mathcal{L}_4^{\rm gal\Gamma} \,. \tag{4.30}$$

Here the introduced $f_1, f_2, F_2, F_3, F_4$ are some arbitrary functions of $\phi$ and $X := g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. The final three terms given in (4.30) are generalizations of the Galileon terms analyzed in the previous sections. Furthermore, the Einstein tensor is defined such that it will be projective invariant,

$$\overset{\Gamma}{G}{}^{\alpha\beta} := \frac{1}{4} \epsilon^{\gamma\alpha\mu\nu} \epsilon_\gamma{}^{\beta\mu'\nu'} \overset{\Gamma}{R}_{\mu\nu\mu'\nu'} \,. \tag{4.31}$$

Thus, this action (4.30) is the straightforward generalization of the Galileon field in the metric-affine formalism which includes non-minimal couplings to curvature.

The solution of distortion is computed as,

$$\begin{aligned} \kappa^\mu{}_{\alpha\beta} \;=\;& k^1_{1,0} g_{\alpha\beta}\phi^\mu + k^2_{1,0}\delta^\mu_\alpha\phi_\beta + k^3_{1,0}\delta^\mu_\beta\phi_\alpha + k^1_{3,0}\phi^\mu\phi_\alpha\phi_\beta + k^1_{1,1} g_{\alpha\beta}\phi^\mu\phi^\gamma_\gamma + k^2_{1,1} g_{\alpha\beta}\phi_\gamma\phi^{\mu\gamma} + k^3_{1,1}\delta^\mu_\alpha\phi_\beta\phi^\gamma_\gamma \\ &+ k^4_{1,1}\delta^\mu_\beta\phi_\alpha\phi^\gamma_\gamma + k^5_{1,1}\delta^\mu_\alpha\phi^\gamma\phi_{\beta\gamma} + k^6_{1,1}\delta^\mu_\beta\phi^\gamma\phi_{\alpha\gamma} + k^7_{1,1}\phi^\mu\phi_{\alpha\beta} + k^8_{1,1}\phi_\alpha\phi^\mu_\beta + k^9_{1,1}\phi_\beta\phi^\mu_\alpha \\ &+ k^1_{3,1} g_{\alpha\beta}\phi^\mu\phi^\gamma\phi^\delta\phi_{\gamma\delta} + k^2_{3,1}\delta^\mu_\alpha\phi_\beta\phi^\gamma\phi^\delta\phi_{\gamma\delta} + k^3_{3,1}\delta^\mu_\beta\phi_\alpha\phi^\gamma\phi^\delta\phi_{\gamma\delta} + k^4_{3,1}\phi^\mu\phi_\alpha\phi_\beta\phi^\gamma_\gamma \\ &+ k^5_{3,1}\phi^\mu\phi_\alpha\phi^\gamma\phi_{\beta\gamma} + k^6_{3,1}\phi^\mu\phi_\beta\phi^\gamma\phi_{\alpha\gamma} + k^7_{3,1}\phi_\alpha\phi_\beta\phi_\gamma\phi^{\mu\gamma} + k^1_{5,1}\phi^\mu\phi_\alpha\phi_\beta\phi^\gamma\phi^\delta\phi_{\gamma\delta} \,, \end{aligned} \tag{4.32}$$

with each of the coefficients being,

$$
\begin{aligned}
k^1_{1,0} &= -\frac{f_{1\phi} - F_3 X}{2f_1 - f_2 X + 2F_4 X^2}\,, \\
k^3_{1,0} &= -k^1_{1,0} = \frac{f_{1\phi} - F_3 X}{2f_1 - f_2 X + 2F_4 X^2}\,, \\
k^1_{3,0} &= -\frac{2f_1 F_3 - f_2 F_3 X + 2f_{1\phi} F_4 X}{f_1(2f_1 - f_2 X + 2F_4 X^2)}\,, \\
k^1_{1,1} &= 0\,, \\
k^2_{1,1} &= -\frac{f_{1X}}{f_1}\,, \\
k^4_{1,1} &= 0\,, \\
k^6_{1,1} &= -k^2_{1,1} = \frac{f_{1X}}{f_1}\,, \\
k^7_{1,1} &= -\frac{f_2 - 2F_4 X}{2f_1 - f_2 X + 2F_4 X^2}\,, \\
k^8_{1,1} &= -k^7_{1,1} = \frac{f_2 - 2F_4 X}{2f_1 - f_2 X + 2F_4 X^2}\,, \\
k^9_{1,1} &= -\frac{2F_4 X}{2f_1 - f_2 X + 2F_4 X^2}\,, \\
k^1_{3,1} &= -\frac{f_{1X}(f_2 - 2F_4 X)}{f_1(2f_1 - f_2 X + 2F_4 X^2)}\,, \\
k^3_{3,1} &= -k^1_{3,1} = \frac{f_{1X}(f_2 - 2F_4 X)}{f_1(2f_1 - f_2 X + 2F_4 X^2)}\,, \\
k^4_{3,1} &= -\frac{2F_4}{2f_1 - f_2 X + 2F_4 X^2}\,, \\
k^5_{3,1} &= 0\,, \\
k^6_{3,1} &= \frac{1}{2f_1^2}\left[f_{1X} f_2 - 2f_1 f_{2X} + \frac{f_1 f_2 (f_2 - 2F_4 X)}{2f_1 - f_2 X + 2F_4 X^2}\right], \\
k^7_{3,1} &= -\frac{1}{2f_1^2}\left[f_{1X} f_2 - 2f_1 f_{2X} + \frac{f_1 (f_2^2 - 8f_1 F_4 - 2f_2 F_4 X))}{2f_1 - f_2 X + 2F_4 X^2}\right], \\
k^1_{5,1} &= -\frac{4f_{1X} F_4}{f_1(2f_1 - f_2 X + 2F_4 X^2)}\,.
\end{aligned}
$$

Thus substituting the solution (4.32) into (4.30), one obtains the Riemann frame action of,

$$
\mathcal{L} = f\overset{g}{R} + P + Q_1 g^{\mu\nu}\phi_{\mu\nu} + Q_2 \phi^\mu \phi_{\mu\nu}\phi^\nu + C^{\mu\nu,\rho\sigma}\phi_{\mu\nu}\phi_{\rho\sigma}\,, \tag{4.33}
$$

where it was defined,

$$
C^{\mu\nu,\rho\sigma} = \alpha_1 g^{\rho(\mu} g^{\nu)\sigma} + \alpha_2 g^{\mu\nu} g^{\rho\sigma} + \frac{1}{2}\alpha_3(\phi^\mu\phi^\nu g^{\rho\sigma} + \phi^\rho\phi^\sigma g^{\mu\nu}) + \frac{1}{2}\alpha_4(\phi^\rho\phi^{(\mu} g^{\nu)\sigma} + \phi^\sigma\phi^{(\mu} g^{\nu)\rho}) + \alpha_5 \phi^\mu\phi^\nu\phi^\rho\phi^\sigma\,. \tag{4.34}
$$

. The coefficients of the Lagrangian (4.33) are given as,

$$f = f_1 - \frac{1}{2} f_2 X \,, \tag{4.35}$$

$$P = F_2 + \frac{3X(f_{1\phi} - F_3 X)^2}{2f_1 - f_2 X + 2F_4 X^2} \,, \tag{4.36}$$

$$Q_1 = -2f_\phi + \frac{4f_1(f_{1\phi} - F_3 X)}{2f_1 - f_2 X + 2F_4 X^2} \,, \tag{4.37}$$

$$Q_2 = \frac{2f_\phi}{X} - \frac{4(f_1 - 3f_{1X})(f_{1\phi} - F_3 X)}{X(2f_1 - f_2 X + 2F_4 X^2)} \,, \tag{4.38}$$

$$\alpha_1 = -\alpha_2 = -\frac{f_2}{2} - \frac{f_1(f_2 - 2F_4 X)}{2f_1 - f_2 X + 2F_4 X^2} \,, \tag{4.39}$$

$$\alpha_3 = 2f_{2X} + \frac{4f_1 F_4 + (4f_{1X} - f_2)(f_2 - 2F_4 X)}{2f_1 - f_2 X + 2F_4 X^2} \,, \tag{4.40}$$

$$\alpha_4 = -2f_{2X} + 2\frac{f_{1X}}{f_1}(3f_{1X} - f_2) + \frac{f_{1X}}{f_1^2} X(f_{1X} f_2 - 4f_1 f_{2X}) + \frac{f_2^2 - 4f_1 F_4 - 2f_2 F_4 X}{2f_1 - f_2 X + 2F_4 X^2} \,, \tag{4.41}$$

$$\alpha_5 = -\frac{f_{1X}}{f_1^2}(f_{1X} f_2 - 4f_1 f_{2X}) + \frac{2f_{1X}\{4f_1 F_4 + (3f_{1X} - f_2)(f_2 - 2F_4 X)\}}{f_1(2f_1 - f_2 X + 2F_4 X^2)} \,, \tag{4.42}$$

where $f_{i\phi} = \frac{\partial f_i}{\partial \phi}$ and $f_{iX} = \frac{\partial f_i}{\partial X}$. Although tedious, the derived coefficients (4.35)-(4.42) indeed satisfy the degeneracy conditions (6.105). Out of the classes of degeneracy conditions the resultant action is of class $^2$N-I/Ia quadratic DHOST or qDHOST. qDHOST depends on five arbitrary functions which is precisely the same with the number of arbitrary functions that (4.30) has.

### 4.2.2 Specific models

In this section, some specific models of the analyzed ghost-free action of (4.30) will be introduced.

For example, a non-minimally coupled scalar field of the form,

$$\mathcal{L} = \frac{M_{\rm pl}^2 - \xi\phi^2}{2} g^{\mu\nu} \overset{\Gamma}{R}_{\mu\nu} - \frac{1}{2}(\partial\phi)^2 - V(\phi) \,, \tag{4.43}$$

is often investigated in metric formalism. Furthermore, for the special case $\xi = 1/6$, the theory is known to be conformally invariant. As for such $\xi\phi^2 R$ couplings in metric-affine formalism, the Riemann frame of the Lagrangian (4.43) is,

$$\mathcal{L} = \frac{M_{\rm pl}^2 - \xi\phi^2}{2} \overset{g}{R} - \frac{M_{\rm pl}^2 - \xi(1+6\xi)\phi^2}{2(M_{\rm pl}^2 - \xi\phi^2)} (\partial\phi)^2 - V(\phi) \,, \tag{4.44}$$

and $\xi = 1/6$ is not a conformal coupling since the kinetic term is not non-canonical. Such action was first investigated in [142] where they assumed torsionless and used the Einstein frame to compute observational variables. By analyzing this Riemann frame action, and thus using metric formalism techniques, it can be shown that their analysis indeed matches the Riemann frame action.

One may also introduce a non-minimal coupling term to the Einstein tensor as,

$$\mathcal{L} = \frac{M_{\rm pl}^2}{2} g^{\mu\nu} \overset{\Gamma}{R}_{\mu\nu} - \frac{1}{2}\left(g^{\mu\nu} - \frac{\overset{\Gamma}{G}{}^{\mu\nu}}{M^2}\right) \partial_\mu\phi \partial_\nu\phi - V(\phi) \,. \tag{4.45}$$

Recall that in metric formalism, the scalar derivatives couples to the Einstein tensor $G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ is in class of $\mathcal{L}_5$ in Horndeski theory, see (2.430). Meanwhile the Riemann frame of (4.45) computes,

$$\mathcal{L} = \frac{M_{\rm pl}^2}{2}\overset{g}{R} - \frac{1}{2}\left(g^{\mu\nu} - \frac{\overset{g}{G}{}^{\mu\nu}}{M^2}\right)\partial_\mu\phi\partial_\nu\phi - V(\phi) - \frac{1}{4M^4M_{\rm pl}^2(2-X/M^2M_{\rm pl}^2)}\mathcal{L}_4^{{\rm gal}g}\,, \tag{4.46}$$

which is not Horndeski but GLPV.

Another interesting theory is kinetic coupling to curvature. The most general action $\mathcal{L}=\mathcal{L}(g,\Gamma,X)$ up to linear in curvature is,

$$\mathcal{L} = f(X)g^{\mu\nu}\overset{\Gamma}{R}_{\mu\nu} + P(X)\,, \tag{4.47}$$

which has the Riemann frame of

$$\mathcal{L} = f\overset{g}{R} + P + \frac{6f_X^2}{f}\phi^\alpha\phi^\beta\phi_{\alpha\gamma}\phi^\gamma_\beta\,. \tag{4.48}$$

The (almost) simultaneous observation of light and gravitational waves strongly constrains the speed of gravitational waves in the late-time universe [60, 61]. For DHOST to explain the late-time universe, for example dark energy §2.2.2, the functions have to be fine tuned such that $\alpha_1=\alpha_2=0$ and $f=f(\phi,X)$ as shown in [62, 63, 64] (see also [160, 127, 161]). However, the Lagrangian (4.48) does not require such fine-tuning even in the presence of non-minimal coupling $f(X)R$ since (4.48) naturally has the "counter-term" $\phi^\alpha\phi^\beta\phi_{\alpha\gamma}\phi^\gamma_\beta$ to eliminate the Ostrogradsky ghost and constrain the speed of gravitational waves to be unity. Later on, in §6.1 and in §6.3, it will be shown explicitly how such mechanism occur and quantitatively analyzed how the observable of Jordan and Riemann frame coincides.

## 4.3 Higher orders of connection

Both for Galileons (4.22) and Generalized Galileons (4.30) in metric-affine formalism, only connection up to quadratic order has been considered in the action. This is due to the equation of motion being non-linear for higher orders and thus rendering it unsolvable, although algebraic.

Guessing from the solution (4.32) ,when a metric-affine scalar-tensor Lagrangian consists cubic order in the connection or higher, the solution of the connection may be given as

$$\kappa^\mu{}_{\alpha\beta} = \sum_{i,j,k}^{\infty} k^k_{i,j}(\phi,X)[(\nabla\phi)^i(\nabla\nabla\phi)^j]^\mu{}_{\alpha\beta}\,, \tag{4.49}$$

with the label $k$ classifying the numbers of all possible contractions of $(\nabla\phi)^i(\nabla\nabla\phi)^j$ which has free indices $\mu,\alpha,\beta$ for $i$ and $j$. Up to $j=1$, for example,

$$\begin{aligned}\kappa^\mu{}_{\alpha\beta} \;=\;& k^1_{1,0}g_{\alpha\beta}\phi^\mu + k^2_{1,0}\delta^\mu_\alpha\phi_\beta + k^3_{1,0}\delta^\mu_\beta\phi_\alpha + k^1_{3,0}\phi^\mu\phi_\alpha\phi_\beta + k^1_{1,1}g_{\alpha\beta}\phi^\mu\phi^\gamma_\gamma + k^2_{1,1}g_{\alpha\beta}\phi_\gamma\phi^{\mu\gamma} + k^3_{1,1}\delta^\mu_\alpha\phi_\beta\phi^\gamma_\gamma\\ &+k^4_{1,1}\delta^\mu_\beta\phi_\alpha\phi^\gamma_\gamma + k^5_{1,1}\delta^\mu_\alpha\phi^\gamma\phi_{\beta\gamma} + k^6_{1,1}\delta^\mu_\beta\phi^\gamma\phi_{\alpha\gamma} + k^7_{1,1}\phi^\mu\phi_{\alpha\beta} + k^8_{1,1}\phi_\alpha\phi^\mu_\beta + k^9_{1,1}\phi_\beta\phi^\mu_\alpha\\ &+k^1_{3,1}g_{\alpha\beta}\phi^\mu\phi^\gamma\phi^\delta\phi_{\gamma\delta} + k^2_{3,1}\delta^\mu_\alpha\phi_\beta\phi^\gamma\phi^\delta\phi_{\gamma\delta} + k^3_{3,1}\delta^\mu_\beta\phi_\alpha\phi^\gamma\phi^\delta\phi_{\gamma\delta} + k^4_{3,1}\phi^\mu\phi_\alpha\phi_\beta\phi^\gamma_\gamma\\ &+k^5_{3,1}\phi^\mu\phi_\alpha\phi^\gamma\phi_{\beta\gamma} + k^6_{3,1}\phi^\mu\phi_\beta\phi^\gamma\phi_{\alpha\gamma} + k^7_{3,1}\phi_\alpha\phi_\beta\phi_\gamma\phi^{\mu\gamma} + k^1_{5,1}\phi^\mu\phi_\alpha\phi_\beta\phi^\gamma\phi^\delta\phi_{\gamma\delta} + \mathcal{O}(\phi^2_{\mu\nu})\,.\end{aligned} \tag{4.50}$$

If the theory is projective invariant, terms such as $k^2_{1,0},k^3_{1,1},k^5_{1,1},k^2_{3,1}$ can be removed through projective transformation.

Recall that the second derivative of the scalar field is given as

$$\overset{\Gamma}{\nabla}_\mu \overset{\Gamma}{\nabla}_\nu \phi = \phi_{\mu\nu} - \kappa^\alpha{}_{\mu\nu}\phi_\alpha\,, \tag{4.51}$$

thus $\overset{\Gamma}{\nabla}_\mu \overset{\Gamma}{\nabla}_\nu \phi$ will only include linear terms in $\phi_{\alpha\beta}$ if the higher-order terms are eliminated, i,e, if $\kappa$ admits a solution

$$k^k_{i,j} = 0 \quad \text{for} \quad j \geq 2\,. \tag{4.52}$$

However, if one allows quintic Galileon (4.22), so thus up to $n = 5$, it can be calculated that (4.22) does not have any solution (4.52) if $c_5 \neq 0$. This implies that (the Riemann frame of) the quintic Galileon generates more than cubic terms in $\phi_{\mu\nu}$ and thus possibly not within the cubic DHOST theories. Of course, some cancellations may occur in higher orders of $\phi_{\mu\nu}$. Thus it is an open question whether quintic Galileon in metric-affine formalism is still ghost-free.

## 4.4 Projective invariant scalar-tensor theories

Curvature and higher-derivative couplings terms that violate projective invariance, such as $\mathcal{T}^\lambda_{\mu\nu}\partial_\lambda\phi\overset{\Gamma}{\nabla}{}^\mu\overset{\Gamma}{\nabla}{}^\nu\phi$ or $\mathcal{Q}_\lambda{}^{\mu\nu}\partial^\lambda\phi\overset{\Gamma}{\nabla}_\mu\overset{\Gamma}{\nabla}_\nu\phi$, introduce Ostrogradky ghosts. Thus, although yet a conjecture, ghost-free theories seem to be projective invariant, see also [162]. However, projective invariance itself does not insure ghost-freeness. In this section, the most general projective scalar-tensor theory up to quadratic order in connection will be explored. Such action is given as

$$\begin{aligned}
\mathcal{L} = {} & f g^{\mu\nu}\overset{\Gamma}{R}_{\mu\nu} + g_1 g^{\mu\alpha}g^{\nu\beta}\overset{\Gamma}{R}_{\mu\nu}\partial_\alpha\phi\partial_\beta\phi \\
& + g_2 g^{\alpha\beta}g^{\mu\nu}\overset{\Gamma}{R}{}^\rho{}_{\alpha\mu\beta}\partial_\rho\phi\partial_\nu\phi + F_2 + F_3\mathcal{L}_3^{\mathrm{gal}\Gamma} + F_4\mathcal{L}_4^{\mathrm{gal}\Gamma} \\
& + C_1\epsilon^{\mu\nu\rho\sigma}\epsilon^{\mu'\nu'\rho'}{}_\sigma\partial_\mu\phi\partial_{\mu'}\phi\overset{\Gamma}{\nabla}_\nu\overset{\Gamma}{\nabla}_{\nu'}\phi\overset{\Gamma}{\nabla}_{[\rho}\overset{\Gamma}{\nabla}_{\rho']}\phi + C_2(\mathcal{L}_3^{\mathrm{gal}\Gamma})^2 \\
& + C_3(g^{\mu\beta}g^{\nu\delta}g^{\alpha\gamma} - g^{\mu\nu}g^{\alpha\gamma}g^{\beta\delta})\partial_\mu\phi\partial_\nu\phi\overset{\Gamma}{\nabla}_\alpha\overset{\Gamma}{\nabla}_\beta\phi\overset{\Gamma}{\nabla}_\gamma\overset{\Gamma}{\nabla}_\delta\phi\,,
\end{aligned} \tag{4.53}$$

with $f, g_1, g_2, F_2, F_2, F_4, C_1, C_2, C_3$ being some arbitrary functions of $\phi$ and $X := g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$.

When one computes the solution for distortion $\kappa$ which is given as the form (4.50) with (4.52), it can be shown that $k^k_{i,j}$ includes $f, g_1, g_2, F_2, F_3, F_,4, C_2, C_3$ but does not have any dependency to $C_1$. Thus, as stated earlier in §4.1 when excluding the term (4.19) from the Galileon action (4.22), the action

$$\mathcal{L}_4^{\mathrm{gal}\Gamma} = \epsilon^{\alpha\beta\gamma\delta}\epsilon^{\alpha'\beta'\gamma'}{}_\delta\overset{\Gamma}{\nabla}_\alpha\phi\overset{\Gamma}{\nabla}_{\alpha'}\phi\overset{\Gamma}{\nabla}_\beta\overset{\Gamma}{\nabla}_{\beta'}\phi\overset{\Gamma}{\nabla}_\gamma\overset{\Gamma}{\nabla}_{\gamma'}\phi\,, \tag{4.54}$$

and

$$\mathcal{L}_4^{\mathrm{gal}\Gamma'} = \epsilon^{\alpha\beta\gamma\delta}\epsilon^{\alpha'\beta'\gamma'}{}_\delta\overset{\Gamma}{\nabla}_\alpha\phi\overset{\Gamma}{\nabla}_{\alpha'}\phi\overset{\Gamma}{\nabla}_\beta\overset{\Gamma}{\nabla}_{\beta'}\phi\overset{\Gamma}{\nabla}_{\gamma'}\overset{\Gamma}{\nabla}_\gamma\phi\,, \tag{4.55}$$

a structurally the same and computes the same distortion although $\overset{\Gamma}{\nabla}_\gamma\overset{\Gamma}{\nabla}_{\gamma'}\phi$ is not symmetric.

Recalling the degeneracy conditions of DHOST theory (4.33) ,see (2.450), is

$$D_0 = 0\,,\ D_1 = 0\,,\ D_2 = 0\,, \tag{4.56}$$

with

$$D_0 := -4(\alpha_1+\alpha_2)[Xf(2\alpha_1+X\alpha_4+4f_X)-2f^2-8X^2f_X^2]\,, \tag{4.57}$$

$$\begin{aligned} D_1 :=& \,4[X^2\alpha_1(\alpha_1+3\alpha_2)-2f^2-4Xf\alpha_2]\alpha_4 \\ &+4X^2f(\alpha_1+\alpha_2)\alpha_5+8X\alpha_1^3 \\ &-4(f+4Xf_X-6X\alpha_2)\alpha_1^2-16(f+5Xf_X)\alpha_1\alpha_2 \\ &+4X(3f-4Xf_X)\alpha_1\alpha_3-X^2f\alpha_3^2 \\ &+32f_X(f+2Xf_X)\alpha_2-16ff_X\alpha_1 \\ &-8f(f-Xf_X)\alpha_3+48ff_X^2\,, \end{aligned} \tag{4.58}$$

$$\begin{aligned} D_2 :=& \,4[2f^2+4Xf\alpha_2-X^2\alpha_1(\alpha_1+3\alpha_2)]\alpha_5+4\alpha_1^3 \\ &+4(2\alpha_2-X\alpha_3-4f_X)\alpha_1^2+3X^2\alpha_1\alpha_3^2-4Xf\alpha_3^2 \\ &+8(f+Xf_X)\alpha_1\alpha_3-32f_X\alpha_1\alpha_2+16f_X^2\alpha_1 \\ &+32f_X^2\alpha_2-16ff_X\alpha_3\,. \end{aligned} \tag{4.59}$$

Furthermore,

$$\alpha_1+\alpha_2=0\,, \tag{4.60}$$

is called class I, whereas

$$Xf(2\alpha_1+X\alpha_4+4f_X)-2f^2-8X^2f_X^2=0\,,\ f\neq 0 \tag{4.61}$$

is called class II. Class I and class II may be further classified to class Ia/IIa for $f \neq X\alpha_1$ or class Ib/IIb for $f = X\alpha_1$. Furthermore, $f = 0$ is called class III.

For the current theory in hand, (4.53), $D_i$ are computed as,

$$D_0 = -\frac{8X(C_3-4C_2X)(f+g_2X)^2D^2}{E}\,, \tag{4.62}$$

$$D_1 = -\frac{8(C_3-8C_2X)(f+g_2X)^2D^2}{E}\,, \tag{4.63}$$

$$D_2 = \frac{32C_2(f+g_2X)^2D^2}{E}\,, \tag{4.64}$$

with

$$D := 2fg-4f_XgX+fg_XX\,, \tag{4.65}$$

$$\begin{aligned} E :=& \,[2f^2+2g_2^2X^2+fX(4g_2-C_3X)] \\ &\times[f^2+(F_4-C_3)g_1X^3+fX\{g_2+(F_4-C_3)X\}] \\ &\times[2f^2+g_1X^3(2F_4+C_3-12C_2X) \\ &\quad +fX(2g_2+X\{2F_4+C_3-12C_2X\}]\,, \end{aligned} \tag{4.66}$$

and $g := (f+g_1X)(f+g_2X)$.

Thus for the Lagrangian (4.53), it can be classified to the DHOST classes as,

$$\text{class Ia}: C_2=C_3=0\,, \tag{4.67}$$

$$\text{class IIa}: D=0\,,\ g\neq 0\,,\ f\neq 0\,, \tag{4.68}$$

$$\text{class Ib}\cap\text{IIb}: g=0 \tag{4.69}$$

$$\text{class III}: f=0\,, \tag{4.70}$$

which enforces the theory to be free from Ostrogradsky instability.

## 4.5 Summary

In this section, Galileons and Generalized Galileons were extended for projective invariant metric-affine formalism. It was found that, unlike metric formalism, the covariantization of Galileons is unique once one admits projective invariance. Furthermore, the solution of the connection was computed and its corresponding Riemann frame was derived. One sees that when the Galileon scales as (4.29), the theory is similar to the Galileons in metric formalism, whereas it deviates in higher scales. As for the Generalized Galileons, the projective invariance alone does not enforce ghost-freeness, however, for special cases of (4.30) it was shown that the theory boils down to Class $^2$N-I/Ia of DHOST theory in the Riemann frame. Furthermore, since the action (4.30) has the same numbers of arbitrary functions as Class $^2$N-I/Ia of DHOST theory, it is precisely equivalent. Certain specific models were also explored and compared to their metric formalism counterparts. Finally, quintic Galileons and the most general projective invariant scalar-tensor theory were analyzed and were investigated their relevance.

# Chapter 5

# Scalar-metric-affine theories: Can we get ghost-free theories from symmetry?


| | |
|---|---|
| Based on | *"Scalar-metric-affine theories: Can we get ghost-free theories from symmetry?"* |
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In the previous part of the thesis, scalar-tensor theories in metric-affine formalism were investigated. It was hinted that projective invariance plays a key role in ghost-free properties. On the other hand, projective symmetry is not enough to protect a theory from ghostly instabilities. In this part of the thesis, further investigation of projective invariance will be conducted. It will be shown that, although projective invariance does not protect the theory in an arbitrary gauge, it does so in the unitary gauge for a large class of theories. Thus a metric-affine scalar-tensor theory boils down to a U-DHOST theory that was introduced in §2.5.5. Both through qualitative and quantitative analysis, ghost-free properties of projective invariant theories will be explored.

## 5.1 Ghost-free scalar field from projective symmetry

In this section, it will be shown that the projective invariant Lagrangian of the form,

$$\mathcal{L}(g,\Gamma,\phi) = \frac{M_{\rm pl}^2}{2}\overset{\Gamma}{R} + \mathcal{L}_\phi(g,\phi,\overset{\Gamma}{\nabla}_\mu\phi,\overset{\Gamma}{\nabla}_\mu\overset{\Gamma}{\nabla}_\nu\phi)\,. \tag{5.1}$$

where,

$$\mathcal{L}_\phi = \mathcal{L}_\phi(g,\phi,\overset{\Gamma}{\nabla}_\mu\phi,\overset{\Gamma}{\nabla}_\mu\overset{\Gamma}{\nabla}_\nu\phi)\,, \tag{5.2}$$

is not plagued with Ostrogradsky ghosts in the unitary gauge.

Recall that, the covariant derivatives are defined as,

$$\overset{\Gamma}{\nabla}_\mu\phi = \partial_\mu\phi\,,\quad \overset{\Gamma}{\nabla}_\mu\overset{\Gamma}{\nabla}_\nu\phi = \partial_\mu\partial_\nu\phi - \Gamma^\alpha{}_{\mu\nu}\partial_\alpha\phi\,. \tag{5.3}$$

Naively, one would expect that $\mathcal{L}$ or more specifically the $\overset{\Gamma}{\nabla}_\mu\overset{\Gamma}{\nabla}_\nu\phi$ in $\mathcal{L}_\phi$ will cause problems. However, as it will be shown, this will not be the case due to projective symmetry.

Firstly, the second-order derivative of $\phi$ contains the connection and therefore the distortion $\kappa^\lambda{}_{\mu\nu}$ as

$$\overset{\Gamma}{\nabla}_\mu\overset{\Gamma}{\nabla}_\nu\phi = \overset{g}{\nabla}_\mu\overset{g}{\nabla}_\nu\phi - \kappa^\alpha{}_{\mu\nu}\partial_\alpha\phi\,. \tag{5.4}$$

Since there exists $\overset{\Gamma}{\nabla}\overset{\Gamma}{\nabla}\phi$, the constraint equation for $\kappa$ will also change with the end result being that the distortion could be solved into the form $\kappa = \kappa(g,\phi,\overset{g}{\nabla}\phi,\overset{g}{\nabla}{}^2\phi)$ [1] Since $\kappa^\lambda{}_{\mu\nu}$ is not dynamical, and thus an auxillary field, one may substituting it into the Lagrangian, and obtain a certain form of a scalar-tensor theory in the Riemann frame as,

$$\mathcal{L}(g,\phi) = \frac{M_{\rm pl}^2}{2}\overset{g}{R} + \mathcal{L}'_\phi(g,\phi,\overset{g}{\nabla}\phi,\overset{g}{\nabla}{}^2\phi)\,. \tag{5.5}$$

In general, (5.5) has second-order time derivatives that produce the Ostrogradsky ghost, and one must impose *degeneracy condtions* in order to eliminate such modes from the theory as DHOST theories were in §2.5.4. Such *"non-trivial"* conditions will not have to be assumed here, however. Instead one may assume projective symmetry, introduced in §2.4.2.2, which will protect the theory from ghosts and absorb the ghostly time derivatives into its gauge mode.

First of all, recall that the EH action is invariant under the projective transformation §2.4.2.2. Similarly, one may suppose that the scalar field Lagrangian $\mathcal{L}_\phi$ also inhibits projective symmetry. Since, within $\mathcal{L}_\phi$, the connection appears only in the covariant derivative of the scalar field, the projective symmetry of $\mathcal{L}_\phi$ has to be realized to be the invariance under,

$$\overset{\Gamma}{\nabla}_\mu\overset{\Gamma}{\nabla}_\nu\phi \to \overset{\Gamma}{\nabla}_\mu\overset{\Gamma}{\nabla}_\nu\phi - \xi_\mu\partial_\nu\phi\,. \tag{5.6}$$

As a next step, one should see the relation between this projective invariance and the ghost-free properties. The clearest way is to first apply the 3+1 decomposition. Consider a unit normal vector $n_\alpha$ to 3-dimensional spacelike hypersurfaces and a projection tensor onto these hypersurfaces, which are defined here as,

$$\gamma_{\mu\nu} := g_{\mu\nu} + n_\mu n_\nu\,. \tag{5.7}$$

[1] See §4.2.1 for an example of an explicit solution of the solution of the constraint in a scalar-metric-affine theory.

Then any tensor can be decomposed into *temporal* parts and the *spatial* parts. For example the first derivative of $\phi$ can be decomposes as,

$$A_* := n^\mu A_\mu \,, \quad \hat{A}_\mu := \gamma_\mu^\nu A_\nu \tag{5.8}$$

with $A_\mu := \partial_\mu \phi$. Similarly, the second-order derivative is decomposed as,

$$\begin{aligned}\overset{g}{\nabla}_\mu \overset{g}{\nabla}_\nu \phi = \overset{g}{\nabla}_\mu A_\nu \phi = D_\mu \hat{A}_\nu - A_* K_{\mu\nu} + 2n_{(\mu}(K_{\nu)\alpha}\hat{A}^\alpha - D_{\nu)}A_*) \\ + n_\mu n_\nu (\pounds_n A_* - \hat{A}_\alpha a^\alpha).\end{aligned} \tag{5.9}$$

Here the following were defined; $D_\mu$ is the covariant derivative associated with the spatial metric $\gamma_{\mu\nu}$, $\pounds_n$ is the Lie derivative with respect to $n^\mu$, $a_\mu := n^\alpha \nabla_\alpha n_\mu$ is the acceleration, $N$ is the lapse function, and $K_{\mu\nu} := \frac{1}{2}\pounds_n \gamma_{\mu\nu}$ is the extrinsic curvature.

$\pounds_n A_*$, in which $\ddot{\phi}$ and $\dot{N}$ are included, is the problematic term that causes Ostrogradsky ghost.

Now instead of fully imposing the degeneracy conditions, one should recall the discussion of U-DHOST, introduced in §2.5.5. The paper [128], which first brought the discussion, argued that the degeneracy conditions of DHOST, obtained by [120], are actually too strict for a theory to be free from Ostrogradsky mode(s). Since, in a general metric theory, one may fix onto an arbitrary gauge and discuss the properties there. Thus one may take the unitary gauge $\phi = \phi(t)$, and discuss the ghost exists in this gauge or not. Note that in the unitary gauge where $\delta\phi = 0$, the original scalar $\phi$ is actually not taken as a dynamical variable. However, the lapse $N$ is the one that holds the degrees of freedom. Even so, considering the term $\pounds_n A_*$ is enough since this is the only term that includes the derivative of $N$.

Let it be assumed that the unitary gauge can be always chosen in original projective invariant theory of (5.1). Since in the unitary gauge $\hat{A}_\mu = 0$, the projective transformation of $\overset{\Gamma}{\nabla}\overset{\Gamma}{\nabla}\phi$ is,

$$\overset{\Gamma}{\nabla}_\mu \overset{\Gamma}{\nabla}_\nu \phi \to \overset{\Gamma}{\nabla}_\mu \overset{\Gamma}{\nabla}_\nu \phi + A_* \xi_\mu n_\nu \,, \tag{5.10}$$

for some arbitrary vector $\xi_\mu(x)$. Now recall from (5.9) that,

$$\overset{\Gamma}{\nabla}_\mu \overset{\Gamma}{\nabla}_\nu \phi \supset n_\mu n_\nu \pounds_n A_* \,. \tag{5.11}$$

Thus by choosing $\xi_\mu = -\frac{1}{A_*}\pounds_n A_*$, one could always eliminate $\pounds_n A_*$ from the action.[2] Thus the theory is trivially U-degenerate, i.e. there are no dependence on $\pounds_n A_*$ in the unitary gauge, and thus have the form,

$$\mathcal{L}_\phi(g, \phi, \overset{\Gamma}{\nabla}_\mu \phi, \overset{\Gamma}{\nabla}_\mu \overset{\Gamma}{\nabla}_\nu \phi) = \mathcal{L}_\phi(t, N, \gamma_{\mu\nu}, K_{\mu\nu}, \kappa; D_\mu) \,. \tag{5.12}$$

Here recall that, since the theory is diffeomorphism invariant, it has no explicit dependence on the shift. Finally, since the action is algebraic in terms of the distortion tensor $\kappa$, one may solve $\kappa$ and integrating it out. Therefore, the Lagrangian will be given by the form,

$$\mathcal{L} = \mathcal{L}(t, N, \gamma_{\mu\nu}, K_{\mu\nu}; D_\mu) \,, \tag{5.13}$$

Now, consider sketching the procedure to find the degrees of freedoms are indeed 3. First, the Hamiltonian (5.13) is linear in shift $N^i$ this gives 6 first-class constraints with 3 being primary and 3 secondary. This, of course, comes from spatial diffeomorphism invariance, just like in general relativity. On the other hand, since the Hamiltonian of (5.13) does not include $\dot{N}$ it also gives 2 constraints, 1 being primary and another being secondary, but is not first-class. This can be interpreted as the consequence of breaking temporal diffeomorphism. Thus, at the most, this theory has $10 - 6 \times 1 - \frac{1}{2} \times 2 = 3$ degrees of freedom and thus indeed there are no extra Ostrogradsky ghostly degrees of freedom. Later on, a more quantitative Hamiltonian analysis will be conducted.

[2] Obviously since the theory is *gauge-invariant*, once explicitly computed $\pounds_n A_*$ does not appear into the Lagrangian at all.

## 5.2 Non-minimal coupling to curvature

In this section, the approach from the previous section will be extended and non-minimal coupling of curvature with scalars and their derivatives will be considered. A non-minimal coupled termed could potentially yield higher-orders in derivatives. For example consider,

$$f_1 \overset{\Gamma}{R} \supset 2 f_1 \overset{g}{\nabla}_\alpha \kappa^{[\alpha\beta]}{}_\beta \,, \tag{5.14}$$

where $f_1$ is some function. Integrating by parts compute the term,

$$-2(\overset{g}{\nabla}_\alpha f_1)\kappa^{[\alpha\beta]}{}_\beta \,.$$

Therefore, even when $f_1$ itself does not contain $\pounds_n A_*$, it may generate one through non-minimal couplings. Thus, in this section, whether non-minimal coupling theories are also ghost-free will be explored.[3].

### 5.2.1 Trivially U-degenerate couplings

Similar to the previous section, there are non-minimal coupling terms that are trivially U-degenerate. Such terms are,

$$f_2 \overset{\Gamma}{G}{}^{\mu\nu} \overset{\Gamma}{\nabla}_\mu \phi \overset{\Gamma}{\nabla}_\nu \phi \,, \;\; f_3 \overset{\Gamma}{G}{}^{\mu\alpha\nu\beta} \overset{\Gamma}{\nabla}_\mu \phi \overset{\Gamma}{\nabla}_\nu \phi \overset{\Gamma}{\nabla}_\alpha \overset{\Gamma}{\nabla}_\beta \phi, \tag{5.15}$$

which do not lead to Ostrogradsky instability. Here $f_2$ and $f_3$ are projective invariant functions of $\phi$, $\overset{\Gamma}{\nabla}_\mu \phi$, $\overset{\Gamma}{\nabla}_\mu \overset{\Gamma}{\nabla}_\nu \phi$, and the dual Riemann tensor and the Einstein tensor is defined as,

$$\overset{\Gamma}{G}{}^{\mu\nu\alpha\beta} := \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\gamma\delta}\overset{\Gamma}{R}_{\rho\sigma\gamma\delta} \,, \tag{5.16}$$

$$\overset{\Gamma}{G}{}^{\mu\nu} := \overset{\Gamma}{G}{}^{\mu\alpha\nu}{}_\alpha \,. \tag{5.17}$$

which are both projective invariant. Note that the Einstein tensor is not the one defined naively, $\overset{\Gamma}{G}{}^{\mu\nu} \neq \overset{\Gamma}{R}{}^{\mu\nu} - \frac{1}{2}\overset{\Gamma}{R} g^{\mu\nu}$, but

$$\overset{\Gamma}{G}_{\mu\nu} = \frac{1}{2}\left(\overset{\Gamma}{R}_{\mu\nu} + P_{\mu\nu} - g_{\mu\nu}\overset{\Gamma}{R}\right) \,. \tag{5.18}$$

where the co-Ricci tensor was defined as $P_{\mu\nu} = \overset{\Gamma}{R}_{\mu\alpha\nu}{}^\alpha$

Once taken the unitary gauge, $\overset{\Gamma}{\nabla}_\mu \phi \propto n_\mu$, the non-null components of the non-minimal couplings (5.15) are the terms,

$$\overset{\Gamma}{G}{}^{\mu\nu} n_\mu n_\nu \,, \;\; \overset{\Gamma}{G}{}^{\mu\alpha\nu\beta} n_\mu n_\nu \gamma_\alpha^{\alpha'} \gamma_\beta^{\beta'} \,. \tag{5.19}$$

[3]For simplicity, torsion or non-metricity couplings will be ignored. Since both torsion and non-metricity do not have derivatives with respect to distortion, it can be guessed that such non-minimal couplings will not be problematic.

Making use of the ADM decomposition of metric-affine curvature, outlined in §A.3.1, the ADM decomposed (5.15) becomes,

$$\overset{\Gamma}{G}^{\mu\nu} n_\mu n_\nu = \frac{1}{2}\left(\overset{\Gamma}{\mathcal{R}}^{\mu\nu}{}_{\mu\nu} + \overset{\Gamma}{\mathcal{K}}^{1\mu}{}_{\mu}\overset{\Gamma}{\mathcal{K}}^{2\mu}{}_{\mu} - \overset{\Gamma}{\mathcal{K}}^{1\mu\nu}\overset{\Gamma}{\mathcal{K}}^{2}_{\nu\mu}\right), \tag{5.20}$$

$$\begin{aligned}\overset{\Gamma}{G}^{\mu\alpha\nu\beta} n_\mu n_\nu \gamma^{\alpha'}_{\alpha}\gamma^{\beta'}_{\beta} = \frac{1}{2}\Bigg[&\overset{\Gamma}{\mathcal{K}}^{1\mu\alpha'}\overset{\Gamma}{\mathcal{K}}^{2\beta'}{}_{\mu} + \overset{\Gamma}{\mathcal{K}}^{1\beta'\mu}\overset{\Gamma}{\mathcal{K}}^{2}{}_{\mu}{}^{\alpha'} - \overset{\Gamma}{\mathcal{K}}^{1\mu}{}_{\mu}\overset{\Gamma}{\mathcal{K}}^{2\beta'\alpha'} - \overset{\Gamma}{\mathcal{K}}^{1\beta'\alpha'}\overset{\Gamma}{\mathcal{K}}^{2\mu}{}_{\mu} \\ &+ \left(\overset{\Gamma}{\mathcal{K}}^{1\mu}{}_{\mu}\overset{\Gamma}{\mathcal{K}}^{2\mu}{}_{\mu} - \overset{\Gamma}{\mathcal{K}}^{1\mu\nu}\overset{\Gamma}{\mathcal{K}}^{2}_{\nu\mu}\right)\gamma^{\alpha'\beta'} - \overset{\Gamma}{\mathcal{R}}_{\mu}{}^{\beta'\mu\alpha'} - \overset{\Gamma}{\mathcal{R}}^{\beta'\mu\alpha'}{}_{\mu} + \overset{\Gamma}{\mathcal{R}}^{\mu\nu}{}_{\mu\nu}\gamma^{\alpha'\beta'}\Bigg],\end{aligned} \tag{5.21}$$

which indeed show that they do not have time derivatives of the distorsion tensor in the unitary gauge.

Since, $3+1$ decomposition shows that such terms do not have $\pounds_n K_{\mu\nu}$ nor $\pounds_n \kappa^{\alpha}{}_{\mu\nu}$ Therefore, even if (5.15) are included in the earlier action $\mathcal{L} = \frac{M_{\rm pl}^2}{2}\overset{\Gamma}{R} + \mathcal{L}_\phi$, the Riemann frame of such action still only includes,

$$t, N, \gamma_{\mu\nu}, K_{\mu\nu}, \kappa^{\alpha}{}_{\mu\nu}, D_\mu\,. \tag{5.22}$$

To conclude, the Lagrangian,

$$\mathcal{L} = \frac{M_{\rm pl}^2}{2}\overset{\Gamma}{R} + f_2\overset{\Gamma}{G}^{\mu\nu}\overset{\Gamma}{\nabla}_\mu\phi\overset{\Gamma}{\nabla}_\nu\phi + f_3\overset{\Gamma}{G}^{\mu\alpha\nu\beta}\overset{\Gamma}{\nabla}_\mu\phi\overset{\Gamma}{\nabla}_\nu\phi\overset{\Gamma}{\nabla}_\alpha\overset{\Gamma}{\nabla}_\beta\phi + \mathcal{L}_\phi\,, \tag{5.23}$$

does not contain $\pounds_n A_*$ in the Riemann frame, which is obtained through integrating out $\kappa$, and is a trivially U-degenerate theory.

### 5.2.2 U-degenerate theories via conformal and disformal transformations

In this section, theories that explicitly contain $\pounds_n A_*$ but are free from Ostrogradsky instability due to degeneracy will be explored. Degeneracy implies that the kinetic matrix has zero eigenvalues. Therefore the kinetic matric could be block diagonalized into null and non-null kinetic matrices. In such *"frame"*, the theory is trivially degenerate. By taking this path the other way, one may *"generate"* a degenerate theory by transforming the variables of a trivially degenerate theory, such as (5.23). This method is the same ones that were used to construct was beyond Horndeski theories (the degenerate theories) from Horndeski theory (a trivially degenerate theory) by the use of disformal transformations [163, 164].

Using (5.23) as a sort of "seed theory", one can generate more general ghost-free theories through field redefinitions. Furthermore for theory (5.23), since in metric-affine formalism the metric and connection are taken as independent variables, the transformation of the metric does not introduce new derivatives of variables due to the lack of metric derivatives in (5.23). For example, consider the following conformal transformation,

$$g_{\mu\nu} \to \bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}\,, \tag{5.24}$$

where $\Omega$ is some function of $\phi$ and its $n$-th derivatives. The theory (5.23) will then become,

$$\sqrt{-\bar{g}}\mathcal{L}|_{g_{\mu\nu}\to\bar{g}_{\mu\nu}} = \sqrt{-g}\left[\frac{M_{\rm pl}^2}{2}\Omega^2\overset{\Gamma}{R} + \bar{f}_2\overset{\Gamma}{G}^{\mu\nu}\overset{\Gamma}{\nabla}_\mu\phi\overset{\Gamma}{\nabla}_\nu\phi + \Omega^{-2}\bar{f}_3\overset{\Gamma}{G}^{\mu\alpha\nu\beta}\overset{\Gamma}{\nabla}_\mu\phi\overset{\Gamma}{\nabla}_\nu\phi\overset{\Gamma}{\nabla}_\alpha\overset{\Gamma}{\nabla}_\beta\phi + \Omega^4\bar{\mathcal{L}}_\phi\right], \tag{5.25}$$

where the functions with denoted with bars are the functions of the seed Lagrangian (5.23), $\bar{f}_i = \bar{f}_2(\bar{g}, \phi, \overset{\Gamma}{\nabla}_\mu\phi, \overset{\Gamma}{\nabla}_\mu\overset{\Gamma}{\nabla}_\nu\phi) = \bar{f}_i(\Omega^2 g, \phi, \overset{\Gamma}{\nabla}_\mu\phi, \overset{\Gamma}{\nabla}_\mu\overset{\Gamma}{\nabla}_\nu\phi)$.

As long the transformation is invertible , the degrees of freedom will neither increase nor decrease in the Jordan frame and the conformal frame. However, one may (re-)introduce matter *in the conformal frame* which may lead to additional degrees of freedom [165, 35]. For example, a minimally coupled (metric-affine) matter of ,

$$\sqrt{-g}\left[\frac{M_{\rm pl}^2}{2}\Omega^2\overset{\Gamma}{R}+\cdots+\mathcal{L}_{\rm m}(g,\Gamma,\psi)\right] , \tag{5.26}$$

is equivalent to (5.23) with a non-minimal coupling to matter,

$$\sqrt{-\bar{g}}\left[\frac{M_{\rm pl}^2}{2}\overset{\Gamma}{R}+\cdots+\Omega^{-4}\mathcal{L}_{\rm m}(\Omega^{-2}\bar{g},\Gamma,\psi)\right] . \tag{5.27}$$

The simplest matter that one could consider is the cosmological constant,

$$\mathcal{L}_{\rm m}=-M_{\rm pl}^2\Lambda\,, \tag{5.28}$$

which yields the term

$$-\Omega^{-4}M_{\rm pl}^2\Lambda \tag{5.29}$$

in the Lagrangian (5.27). Therefore if $\Omega$ contains $\pounds_n A_*$ and its derivatives, the theory (5.27) with just a constant term (5.28) introduces extra degrees of freedom. Hence, the conformal factor $\Omega$ is commonly imposed to be a function up to the first derivative of $\phi$. i.e. $X$. However, as shown in the previous section, $\overset{\Gamma}{\nabla}\overset{\Gamma}{\nabla}\phi$ does not contain $\pounds_n A_*$ in the unitary gauge as long it's constructed to be projective invariant. Thus one may include $\overset{\Gamma}{\nabla}\overset{\Gamma}{\nabla}\phi$ in the conformal.

Therefore, here and on, it will be assumed that

$$\Omega=\Omega(g,\phi,\overset{\Gamma}{\nabla}_\mu\phi,\overset{\Gamma}{\nabla}_\mu\overset{\Gamma}{\nabla}_\nu\phi) \tag{5.30}$$

and is projective invariant, such that non-minimal couplings of matter do not introduce the Ostrogradsky modes.

Noting that the cosmological constant in (5.28) can be absorbed into the definition of $\mathcal{L}_\phi$, the conformal transformation of the action (5.23) becomes,

$$f_1\overset{\Gamma}{R}+f_2\overset{\Gamma}{G}{}^{\mu\nu}\overset{\Gamma}{\nabla}_\mu\phi\overset{\Gamma}{\nabla}_\nu\phi+f_3\overset{\Gamma}{G}{}^{\mu\alpha\nu\beta}\overset{\Gamma}{\nabla}_\mu\phi\overset{\Gamma}{\nabla}_\nu\phi\overset{\Gamma}{\nabla}_\alpha\overset{\Gamma}{\nabla}_\beta\phi+\mathcal{L}_\phi\,, \tag{5.31}$$

with the four arbitrary functions $f_1, f_2, f_3, \mathcal{L}_\phi$ as,

$$f_1=\frac{M_{\rm pl}^2}{2}\Omega^2\,,\ f_2=\bar{f}_2\,,\ f_3=\bar{f}_3\Omega^{-2}\,,\ \mathcal{L}_\phi=\Omega^4\bar{\mathcal{L}}_\phi\,. \tag{5.32}$$

Now, consider further generalization of conformal transformation [163],

$$g^{\mu\nu}\to\bar{g}^{\mu\nu}=\Omega^{-2}(g^{\mu\nu}+\Upsilon^{\mu\nu})\,, \tag{5.33}$$

where $\Upsilon^{\mu\nu}=\Upsilon^{\mu\nu}g^{\mu\nu},\phi,\overset{\Gamma}{\nabla}_\mu\phi,\overset{\Gamma}{\nabla}_\mu\overset{\Gamma}{\nabla}_\nu\phi$ is a projective invariant and symmetric tensor. Furthermore, $g^{\mu\nu}$ is assumed to be non-degenerate and thus invertible. Note that the transformation is defined through the inverse of the metric, rather than the metric itself just for convenience. Obviously the inverse of $g^{\mu\nu}$ exists when,

$$\det(\bar{g}_{\mu\nu}) = \frac{\Omega^8}{\det(\delta^\mu_\nu + \Upsilon^\mu_\nu)}\det(g_{\mu\nu})\,, \tag{5.34}$$

is non-zero.

For simplicity, consider disformal transformation [117]

$$\Upsilon^{\mu\nu} = \Upsilon \overset{\Gamma}{\nabla}{}^\mu \phi \overset{\Gamma}{\nabla}{}^\nu \phi\,, \tag{5.35}$$

where $\Upsilon = \Upsilon(g^{\mu\nu}, \phi, \overset{\Gamma}{\nabla}_\mu \phi, \overset{\Gamma}{\nabla}_\mu \overset{\Gamma}{\nabla}_\nu \phi)$ is some projective invariant scalar function. Since both the Einstein tensor and the dual Riemann tensor are constructed through the Levi-Civita tensor, the novel coupling only arises from the Ricci scalar. Thus in the disformal frame, the seed theory (5.23) becomes,

$$\mathcal{L}_{\rm UD} = f_1 \overset{\Gamma}{R} + f_2 \overset{\Gamma}{G}{}^{\mu\nu} \overset{\Gamma}{\nabla}_\mu \phi \overset{\Gamma}{\nabla}_\nu \phi + f_3 \overset{\Gamma}{G}{}^{\mu\alpha\nu\beta} \overset{\Gamma}{\nabla}_\mu \phi \overset{\Gamma}{\nabla}_\nu \phi \overset{\Gamma}{\nabla}_\alpha \overset{\Gamma}{\nabla}_\beta \phi + f_4 \overset{\Gamma}{R}_{\mu\nu} \overset{\Gamma}{\nabla}{}^\mu \phi \overset{\Gamma}{\nabla}{}^\nu \phi + \mathcal{L}_\phi\,, \tag{5.36}$$

with

$$\begin{aligned} f_1 &= \frac{M_{\rm pl}^2 \Omega^2}{2\sqrt{1+\Upsilon X}}\,,\ f_2 = \bar{f}_2 \sqrt{1+\Upsilon X}\,, \\ f_3 &= \Omega^{-2} \bar{f}_3 \sqrt{1+\Upsilon X}\,,\ f_4 = \frac{M_{\rm pl}^2 \Omega^2 \Upsilon}{2\sqrt{1+\Upsilon X}}\,, \\ \mathcal{L}_\phi &= \Omega^4 \bar{\mathcal{L}}_\phi \sqrt{1+\Upsilon X}\,, \end{aligned} \tag{5.37}$$

where $X = g^{\mu\nu} \overset{\Gamma}{\nabla}_\mu \phi \overset{\Gamma}{\nabla}_\nu \phi$. This new theory have five arbitrary functions $f_1, f_2, f_3, f_4, \mathcal{L}_\phi$, however cannot be taken freely. This is because the regularity/invertability of the disformal transformation constrains that $\Omega$ and $1 + \Upsilon X$ must not cross zero nor diverge, and thus $f_1, f_1 + f_4 X \neq 0$.

Recall that the Einstein tensor was defined as

$$\overset{\Gamma}{G}_{\mu\nu} = \frac{1}{2}\left( \overset{\Gamma}{R}_{\mu\nu} + \overset{\Gamma}{R}_{\mu\alpha\nu}{}^\alpha - g_{\mu\nu} \overset{\Gamma}{R} \right)\,.$$

Therefore, the Ricci tensor coupling $\overset{\Gamma}{R}_{\mu\nu} \overset{\Gamma}{\nabla}{}^\mu \phi \overset{\Gamma}{\nabla}{}^\nu \phi$ that appeared cannot be absorbed into the Einstein tensor through the redefinitions of functions.

## 5.3 Quadratic scalar-metric-affine theory

In this section, a concrete scalar-metric-affine Lagrangian and its Riemann frame counterpart will be considered. The most general projective invariant Lagrangian, constructed up to the quadratic order of the connection and scalar $\phi$ and its derivatives is,

$$\begin{aligned} \mathcal{L}_{\rm qPI}(g, \Gamma, \phi) = {} & f_1 \overset{\Gamma}{R} + f_2 \overset{\Gamma}{G}{}^{\mu\nu} \overset{\Gamma}{\nabla}_\mu \phi \overset{\Gamma}{\nabla}_\nu \phi + f_4 \overset{\Gamma}{R}_{\mu\nu} \overset{\Gamma}{\nabla}{}^\mu \phi \overset{\Gamma}{\nabla}{}^\nu \phi + F_2 + F_3 \mathcal{L}_3^{\rm gal\Gamma} + F_4 \mathcal{L}_4^{\rm gal\Gamma} \\ & + C_1 \epsilon^{\mu\nu\rho\sigma} \epsilon^{\mu'\nu'\rho'}{}_\sigma \overset{\Gamma}{\nabla}_\mu \phi \overset{\Gamma}{\nabla}_{\mu'} \phi \overset{\Gamma}{\nabla}_\nu \overset{\Gamma}{\nabla}_{\nu'} \phi \overset{\Gamma}{\nabla}_{[\rho} \overset{\Gamma}{\nabla}_{\rho']} \phi + C_2 (\mathcal{L}_3^{\rm gal\Gamma})^2 \\ & + C_3 (g^{\mu\beta} g^{\nu\delta} g^{\alpha\gamma} - g^{\mu\nu} g^{\alpha\gamma} g^{\beta\delta}) \partial_\mu \phi \partial_\nu \phi \overset{\Gamma}{\nabla}_\alpha \overset{\Gamma}{\nabla}_\beta \phi \overset{\Gamma}{\nabla}_\gamma \overset{\Gamma}{\nabla}_\delta \phi\,. \end{aligned} \tag{5.38}$$

Here, the Galileon terms are constructed as,

$$\mathcal{L}_3^{\mathrm{gal}\Gamma} = \epsilon^{\alpha\beta\gamma\delta}\epsilon^{\alpha'\beta'}{}_{\gamma\delta}\overset{\Gamma}{\nabla}_\alpha\phi\overset{\Gamma}{\nabla}_{\alpha'}\phi\overset{\Gamma}{\nabla}_\beta\overset{\Gamma}{\nabla}_{\beta'}\phi\,, \tag{5.39}$$

$$\mathcal{L}_4^{\mathrm{gal}\Gamma} = \epsilon^{\alpha\beta\gamma\delta}\epsilon^{\alpha'\beta'\gamma'}{}_{\delta}\overset{\Gamma}{\nabla}_\alpha\phi\overset{\Gamma}{\nabla}_{\alpha'}\phi\overset{\Gamma}{\nabla}_\beta\overset{\Gamma}{\nabla}_{\beta'}\phi\overset{\Gamma}{\nabla}_\gamma\overset{\Gamma}{\nabla}_{\gamma'}\phi\,, \tag{5.40}$$

and $f_1, f_2, f_4, F_2, F_3, F_4, C_1, C_2, C_3$ are some arbitrary functions of $\phi$ and $X := (\partial\phi)^2$.

The constructed most general projective invariant Lagrangian up to quadratic connection (5.38) is a concrete example of the ghost-free scalar-metric-affine theory that was analyzed earlier (5.36). Furthermore, $C_1$ does not appear in the final expression and can be set as $C_1 = 0$ without loss of generality as shown in §4.4.

Recall that in §4.4 it was shown that for $C_2 = C_3 = 0$, quadratic DHOST is derived (in the Riemann frame) and scalar self interactions are given as (first-order) Galileon terms $\mathcal{L}_3^{\mathrm{gal}\Gamma}$ and $\mathcal{L}_4^{\mathrm{gal}\Gamma}$. Thus as for (5.38), it is not a DHOST theory and thus may inhibit Ostrograsky ghost. In the rest of the section it will be shown that in unitary gauge, however, (5.38) is ghost-free.

(5.38), once integrated out $\kappa$ computes,

$$\begin{aligned}\mathcal{L}_{\mathrm{qU}}(g,\phi) = {} & f\overset{g}{R} + P + Q_1 g^{\mu\nu}\phi_{\mu\nu} + Q_2\phi^\mu\phi_{\mu\nu}\phi^\nu + \left(\kappa_1 + \frac{f}{X}\right)L_1^{(2)} + \left(\kappa_2 - \frac{f}{X}\right)L_2^{(2)} \\ & + \left(\frac{2f}{X^2} - \frac{4f_X}{X} + 2\sigma\kappa_1 + 2\left[3\sigma - \frac{1}{X}\right]\kappa_2\right)L_3^{(2)} + \left(\alpha + \frac{2f_X}{X} - \frac{2f}{X^2} - \frac{2\kappa_1}{X}\right)L_4^{(2)} \\ & + \left(-\frac{\alpha}{X} + \frac{2f_X}{X^2} + \kappa_1\left[\frac{1}{X^2} + 3\sigma^2 - \frac{2\sigma}{X}\right] + \kappa_2\left[3\sigma - \frac{1}{X}\right]^2\right)L_5^{(2)}\,,\end{aligned} \tag{5.41}$$

where

$$L_1^{(2)} = \phi_{\mu\nu}\phi^{\mu\nu}\,,\ L_2^{(2)} = (\phi^\mu{}_\mu)^2\,,\ L_3^{(2)} = \phi^\mu\phi^\nu\phi_{\mu\nu}\phi^\rho{}_\rho\,,\ L_4^{(2)} = \phi_{\mu\nu}\phi^\mu\phi^{\nu\rho}\phi_\rho\,,\ L_5^{(2)} = (\phi^\mu\phi^\nu\phi_{\mu\nu})^2\,. \tag{5.42}$$

The functions $f, P, Q_1, Q_2, \alpha, \kappa_1, \kappa_2, \sigma$ are,

$$f = f_1 - \frac{f_2 X}{2}\,, \tag{5.43}$$

$$P = F_2 + \frac{3X(g_\phi - 2F_3'X)^2}{8fg + 4X^2[C_3' + 2(F_4' - 6C_2'X)]}\,, \tag{5.44}$$

$$Q_1 = -2f_\phi + \frac{2g(g_\phi - 2F_3'X)}{2fg + X^2[C_3' + 2(F_4' - 6C_2'X)]}\,, \tag{5.45}$$

$$Q_2 = \frac{2f_\phi}{X} - \frac{(g_\phi - 2F_3'X)(2g - 3g_X X)}{X[2fg + X^2\{C_3' + 2(F_4' - 6C_2'X)\}]}\,, \tag{5.46}$$

$$\alpha = -\frac{fg_X(4g + g_X X) - 4f_X g(g + 2g_X X) + 2C_3'X(f^2 - 3ff_X X + 4f_X^2 X^2)}{2g^2X - C_3' f X^3}\,, \tag{5.47}$$

$$\kappa_1 = -\frac{g^2}{fgX - (C_3' - F_4')X^3}\,, \tag{5.48}$$

$$\kappa_2 = \frac{g^2(2fg + X^2(2F_4' - 4C_2'X - C_3'))}{X(fg - (C_3' - F_4')X^2)[2fg + X^2\{C_3' + 2(F_4' - 6C_2'X)\}]}\,, \tag{5.49}$$

$$\sigma = \frac{g_X}{2g} \tag{5.50}$$

with

$$g = f_1(f_1 + f_4X)\,,\ F_3' = F_3(f_1 + f_4X)\,,\ F_4' = F_4(f_1 + f_4X)^2\,,\ C_2' = C_2(f_1 + f_4X)^2\,,\ C_3' = C_3(f_1 + f_4X)^2\,. \tag{5.51}$$

Taking the unitary gauge, the Lagrangian (5.41) becomes,

$$\begin{aligned}\mathcal{L}_{\mathrm{qU}} &= A_*^2 \hat{K}^{\mu\nu,\alpha\beta} \left(K_{\mu\nu} - \sigma\gamma_{\mu\nu} A_* \pounds_n A_*\right) \left(K_{\alpha\beta} - \sigma\gamma_{\alpha\beta} A_* \pounds_n A_*\right) \\ &\quad + P - (Q_1 - A_*^2 Q_2)\pounds_n A_* - A_*(2f_\phi + Q_1) K^\mu{}_\mu + f^3 R + (2f_X + A_*^2\alpha) D_\mu A_* D^\mu A_* \end{aligned} \tag{5.52}$$

where

$$\hat{K}^{\mu\nu,\alpha\beta} = (\kappa_1 \gamma^{\mu(\alpha}\gamma^{\beta)\nu} + \kappa_2 \gamma^{\mu\nu}\gamma^{\alpha\beta})\,. \tag{5.53}$$

Recalling the degeneracy conditions of U-DHOST given as (2.483), the theory is indeed U-DHOST.

Finally, note that for cases $f_1 = 0$ and $f_1 + f_4 X = 0$ the transformation to Lagrangian (5.23) is not regular and thus the dynamical degrees of freedom may differ. Indeed for $g = f_1(f_1 + f_4 X) = 0$, $\kappa_1 = \kappa_2 = 0$ and thus $\hat{K}^{\mu\nu,\alpha\beta} = 0$ which from (5.52) renders the theory having no degrees of freedom.

Also notice that when $\sigma = 0$ and thus $g_X = 0$,

$$Q_1 - A_*^2 Q_2 = g_X \cdot \frac{3X(g_\phi - 2F_3' X)}{2fg + X^2[C_3' + 2(F_4' - 6C_2' X)]} = 0\,, \tag{5.54}$$

(5.52) does not have explicit dependence of $\pounds_n A_*$ and leads to a trivially U-degenerate theory.

## 5.4 Hamiltonian Analysis of the Theories

In this section, a detailed Hamiltonian analysis will be conducted in the Jordan frame and showed that indeed that the theory is at the least Ostrogradsky ghost-free in the unitary gauge. The notation and related topics of ADM decomposition in metric-affine formalism are given in §A.3.1.

First of all, the connection dependent second-order derivative $\overset{\Gamma}{\nabla}_\mu \overset{\Gamma}{\nabla}_\nu \phi$ of the scalar field reduces under the unitary gauge as,

$$\overset{\Gamma}{\nabla}_\mu \overset{\Gamma}{\nabla}_\nu \phi = n_\mu n_\nu (\pounds_n A_* + A_* \kappa_*) - n_\mu (D_\nu A_* + A_* \hat{\kappa}^2_\nu) - n_\nu (D_\mu A_* + A_* \hat{\kappa}^1_\mu) - A_* \overset{\Gamma}{\mathcal{K}}{}^1_{\nu\mu}\,. \tag{5.55}$$

The terms $\kappa_*$ and $\hat{\kappa}^1_\mu$ are projective modes and thus do not appear in projective invariant theories. Now by defining a new variable,

$$V_\mu := \hat{\kappa}^2_\mu - D_\mu N / N\,. \tag{5.56}$$

The second-order derivative can be expressed as,

$$\overset{\Gamma}{\nabla}_\mu \overset{\Gamma}{\nabla}_\nu \phi = -A_* n_\mu V_\nu - A_* \overset{\Gamma}{\mathcal{K}}{}^1_{\nu\mu} + \text{projective modes}\,. \tag{5.57}$$

Therefore, for any projective invariant theory, the second-order derivative will become arbitrary functions of $t, N, \gamma^{\mu\nu}, V_\mu, \overset{\Gamma}{\mathcal{K}}{}^1_{\mu\nu}$ in the unitary gauge.

Consider the following Lagrangian

$$\mathcal{L} = F^{\mu\nu\rho\sigma} \overset{\Gamma}{R}_{\mu\nu\rho\sigma} + \mathcal{L}_\phi\,, \tag{5.58}$$

where $F^{\mu\nu\rho\sigma}$ and $\mathcal{L}_\phi$ consists of $g^{\mu\nu}, \phi, \overset{\Gamma}{\nabla}_\mu \phi, \overset{\Gamma}{\nabla}_\mu \overset{\Gamma}{\nabla}_\nu \phi$.

Since the curvature tensor is anti-symmetric in the last two indices, without loss of generality, it can be taken as $F^{\mu\nu(\rho\sigma)} = 0$. Just for simplicity consider $F^{\mu\nu\rho\sigma}$ given as the form,

$$F^{\mu\nu\rho\sigma} = \hat{F}^{\mu\nu\rho\sigma} + F_1 \gamma^{\nu[\rho} n^{\sigma]} n^\mu - F_2 \gamma^{\mu[\rho} n^{\sigma]} n^\nu + \hat{F}^\alpha{}_\alpha{}^{\rho\sigma} n^\mu n^\nu\,. \tag{5.59}$$

Recall that the projective transformation of the Riemann tensor is,

$$\overset{\Gamma}{R}_{\mu\nu\rho\sigma} \to \overset{\Gamma}{R}_{\mu\nu\rho\sigma} + 2g_{\mu\nu}\partial_{[\mu}U_{\nu]}\,. \tag{5.60}$$

Therefore, by considering $g_{\mu\nu}F^{\mu\nu\rho\sigma}=0$ the projective invariance of the Lagrangian is guaranteed for $F^{\mu\nu\rho\sigma}$ and $\mathcal{L}_\phi$ are projective invariant. Thus the considered functions $F_1, F_2, \hat{F}^{\mu\nu\rho\sigma}$ and $\mathcal{L}_\phi$ becomes functions of $t, N, \gamma^{\mu\nu}, V_\mu, \overset{\Gamma}{\mathcal{K}}{}^1_{\mu\nu}$ in the unitary gauge.

(5.36), considered in the previous section. reduces under the unitary gauge as,

$$\mathcal{L}_{\mathrm{UD}} = \left[\hat{F}^{\mu\nu\rho\sigma}_{\mathrm{UD}} + f_1\gamma^{\nu[\rho}n^{\sigma]}n^\mu - (f_1 - A_*^2 f_4)\gamma^{\mu[\rho}n^{\sigma]}n^\nu\right]\overset{\Gamma}{R}_{\mu\nu\rho\sigma} + \mathcal{L}_\phi\,, \tag{5.61}$$

where

$$\hat{F}^{\mu\nu\rho\sigma}_{\mathrm{UD}} = \frac{1}{2}(2f_1 + A_*^2 f_2 - A_*^3 f_3 \overset{\Gamma}{\mathcal{K}}{}^{1\alpha}{}_\alpha)\gamma^{\mu[\rho}\gamma^{\sigma]\nu} + \frac{1}{2}A_*^3 f_3 \overset{\Gamma}{\mathcal{K}}{}^{1\mu[\rho}\gamma^{\sigma]\nu} - \frac{1}{2}A_*^3 f_3 \overset{\Gamma}{\mathcal{K}}{}^{1\nu[\rho}\gamma^{\sigma]\mu}\,. \tag{5.62}$$

Thus, (5.36) is a subclass of (5.58) by choosing appropriate functions in (5.59). With (5.58) being more general, the rest of the ghost-free analysis will be conducted from (5.58).

Following §A.3.1, the Lagrangian (5.58) under the unitary gauge can be ADM decomposed as,

$$\begin{aligned} N\mathcal{L} = N\Big[ & F_1\gamma^{\mu\nu}\pounds_n\overset{\Gamma}{\mathcal{K}}{}^1_{\mu\nu} + F_2\gamma^{\mu\nu}\pounds_n\overset{\Gamma}{\mathcal{K}}{}^2_{\mu\nu} - (F_1\overset{\Gamma}{\mathcal{K}}{}^{1\mu\nu} + F_2\overset{\Gamma}{\mathcal{K}}{}^{2\mu\nu})K_{\mu\nu} \\ & + \hat{F}^{\mu\nu\rho\sigma}(\mathcal{R}_{\mu\nu\rho\sigma} + 2D_\rho\kappa^{\mathrm{PI}}_{\mu\nu\sigma} + 2\kappa^{\mathrm{PI}\alpha}_{\mu}{}_\rho\kappa^{\mathrm{PI}}_{\alpha\nu\sigma} + 2\overset{\Gamma}{\mathcal{K}}{}^1_{\nu\sigma}\overset{\Gamma}{\mathcal{K}}{}^2_{\mu\rho}) - 2\hat{F}_\sigma{}^{\sigma\mu\nu}\overset{\Gamma}{\mathcal{K}}{}^1_{\rho\mu}\overset{\Gamma}{\mathcal{K}}{}^{2\rho}{}_\nu \\ & - F_1V_\mu\kappa^{\mathrm{PI}\mu\nu}{}_\nu - V_\mu D^\mu F_1 + D^2F_1 + \kappa^{\mathrm{PI}\mu\nu}{}_\mu D_\nu F_2 + F_2D_\mu\kappa^{\mathrm{PI}\nu\mu}{}_\nu \\ & - (F_1\overset{\Gamma}{\mathcal{K}}{}^1_{\mu\nu} - F_2\overset{\Gamma}{\mathcal{K}}{}^2_{\nu\mu})\kappa^{\mathrm{PI}\mu\nu} + (D^\nu F_2 - F_2\kappa^{\mathrm{PI}\mu\nu}{}_\mu)\hat{\kappa}^3_\nu + \mathcal{L}_\phi\Big] \end{aligned} \tag{5.63}$$

where the following projective invariant variables were defined,

$$\kappa^{\mathrm{PI}}_{\mu\nu\rho} := \hat{\kappa}_{\mu\nu\rho} + \gamma_{\mu\nu}\hat{\kappa}^1_\rho\,, \quad \kappa^{\mathrm{PI}}_{\mu\nu} := \hat{\kappa}^3_{\mu\nu} + \gamma_{\mu\nu}\kappa_*\,. \tag{5.64}$$

Recall that, since the theory is projective invariant $F_1, F_2, \hat{F}^{\mu\nu\rho\sigma}, \mathcal{L}_\phi$ are functions of only $t, N, \gamma^{\mu\nu}, V_\mu, \overset{\Gamma}{\mathcal{K}}{}^1_{\mu\nu}$. Therefore, the Lagrangian is linear with respect to $\kappa^{\mathrm{PI}}_{\alpha\beta}$ and $\hat{\kappa}^3_\mu$ which can be considered as the Lagrangian multipliers.

The time derivative of the given variables are to be given as $\pounds_t$, where the time vector is defined as,

$$t^\mu = Nn^\mu + N^\mu\,, \tag{5.65}$$

with $N$ and $N^\mu$ being the lapse and the shift vectors respectively.

Thus this theory has the following 116 canonical variables of

$$(N,\pi_N),\ (N^\mu,\pi_\mu),\ (\gamma_{\mu\nu},\pi^{\mu\nu}),\ (\overset{\Gamma}{\mathcal{K}}{}^1_{\mu\nu},\Pi^{\mu\nu}_1),\ (\overset{\Gamma}{\mathcal{K}}{}^2_{\mu\nu},\Pi^{\mu\nu}_2),\ (\kappa^{\mathrm{PI}}_{\mu\nu\rho},\Pi^{\mu\nu\rho}_{\mathrm{PI}}),\ (V_\mu,\Pi^\mu)\,,$$

There are 70 primary constraints,

$$\begin{aligned} \pi_N \approx 0\,, \quad \pi_\mu \approx 0\,, \quad \Pi^{\mu\nu\rho}_{\mathrm{PI}} \approx 0\,, \quad \Pi^\alpha \approx 0\,, \\ \Pi^{\mu\nu}_1 - \sqrt{\gamma}F_1\gamma^{\mu\nu} \approx 0\,, \quad \Pi^{\mu\nu}_2 - \sqrt{\gamma}F_2\gamma^{\mu\nu} \approx 0\,, \\ \pi^{\mu\nu} + \frac{\sqrt{\gamma}}{2}(F_1\overset{\Gamma}{\mathcal{K}}{}^{1(\mu\nu)} + F_2\overset{\Gamma}{\mathcal{K}}{}^{2(\mu\nu)}) \approx 0\,, \end{aligned} \tag{5.66}$$

and

$$F_1\overset{\Gamma}{\mathcal{K}}{}^1_{\mu\nu}-F_2\overset{\Gamma}{\mathcal{K}}{}^2_{\nu\mu}\approx 0\,,\;\; D^\nu F_2-F_2\kappa^{\mathrm{PI}\mu\nu}{}_\mu\approx 0\,. \tag{5.67}$$

Using the constraints, $\Pi_1^{\mu\nu}-\sqrt{\gamma}F_1\gamma^{\mu\nu}\approx 0$ and $\Pi_2^{\mu\nu}-\sqrt{\gamma}F_2\gamma^{\mu\nu}\approx 0$, the functions $F_1$ and $F_2$ can be replaced as $\tilde{\Pi}_1:=\gamma_{\mu\nu}\Pi_1^{\mu\nu}/\sqrt{\gamma}\approx 3F_1$ and $\tilde{\Pi}_2:=\gamma_{\mu\nu}\Pi_2^{\mu\nu}/\sqrt{\gamma}\approx 3F_2$ by redefining the Lagrangian multipliers. The total Hamiltonian which has to be considered is thus,

$$H_{\mathrm{tot}}=\int d^3x\left(\mathcal{H}_V+\lambda^\mu\pi_\mu+N^\mu\mathcal{H}_\mu+\lambda_I\Phi^I\right)\,, \tag{5.68}$$

where

$$\begin{aligned}\mathcal{H}_V=-N\sqrt{\gamma}\Big[&\hat{F}^{\mu\nu\rho\sigma}(\mathcal{R}_{\mu\nu\rho\sigma}+2D_\rho\kappa^{\mathrm{PI}}_{\mu\nu\sigma}+2\kappa_\mu^{\mathrm{PI}\alpha}{}_\rho\kappa^{\mathrm{PI}}_{\alpha\nu\sigma}+2\overset{\Gamma}{\mathcal{K}}{}^1_{\nu\sigma}\overset{\Gamma}{\mathcal{K}}{}^2_{\mu\rho})-2\hat{F}_\sigma{}^{\sigma\mu\nu}\overset{\Gamma}{\mathcal{K}}{}^1_{\rho\mu}\overset{\Gamma}{\mathcal{K}}{}^{2\rho}{}_\nu\\&-\frac{1}{3}\tilde{\Pi}_1V_\mu\kappa^{\mathrm{PI}\mu\nu}{}_\nu-\frac{1}{3}V_\mu D^\mu\tilde{\Pi}_1+\frac{1}{3}D^2\tilde{\Pi}_1+\frac{1}{3}\kappa^{\mathrm{PI}\mu\nu}{}_\mu D_\nu\tilde{\Pi}_2+\frac{1}{3}\tilde{\Pi}_2D_\mu\kappa^{\mathrm{PI}\nu\mu}{}_\nu+\mathcal{L}_\phi\Big]\,,\end{aligned} \tag{5.69}$$

and

$$\mathcal{H}_\mu(x)=\frac{\delta}{\delta N^\mu(x)}\int d^3yP^A(y)\pounds_NQ_A(y)\,. \tag{5.70}$$

$\pounds_N$ is the Lie derivative with respect to the $N^\mu$, whereas $Q_A$ and $P^A$ are the following canonical variables; $Q_A=\{N,\gamma_{\mu\nu},\overset{\Gamma}{\mathcal{K}}{}^1_{\mu\nu},\overset{\Gamma}{\mathcal{K}}{}^2_{\mu\nu},\kappa^{\mathrm{PI}}_{\mu\nu\rho},V_\mu\}$ and $P^A=\{\pi_N,\pi^{\mu\nu},\Pi_1^{\mu\nu},\Pi_2^{\mu\nu},\Pi_{\mathrm{PI}}^{\mu\nu\rho},\Pi^\mu\}$. Finally, $\Phi^I$ and $\lambda_I$ the 67 primary constraints and their associated Lagrangian multipliers,

$$\begin{aligned}\lambda_I\Phi^I=&\lambda_N\pi_N+\lambda^{\mathrm{PI}}_{\mu\nu\rho}\Pi_{\mathrm{PI}}^{\mu\nu\rho}+\lambda^V_\mu\Pi^\mu+\lambda^1_{\mu\nu}(\Pi_1^{\mu\nu}-\sqrt{\gamma}F_1\gamma^{\mu\nu})+\lambda^2_{\mu\nu}(\Pi_2^{\mu\nu}-\sqrt{\gamma}F_2\gamma^{\mu\nu})\\&+\lambda_{\mu\nu}\left[\pi^{\mu\nu}+\frac{\sqrt{\gamma}}{6}(\tilde{\Pi}_1\overset{\Gamma}{\mathcal{K}}{}^{1(\mu\nu)}+\tilde{\Pi}_2\overset{\Gamma}{\mathcal{K}}{}^{2(\mu\nu)})\right]+\frac{1}{3}\sqrt{\gamma}\lambda_\kappa^{\mu\nu}(\tilde{\Pi}_1\overset{\Gamma}{\mathcal{K}}{}^1_{\mu\nu}-\tilde{\Pi}_2\overset{\Gamma}{\mathcal{K}}{}^2_{\nu\mu})+\frac{1}{3}\sqrt{\gamma}\lambda_\kappa^\nu(D_\nu\tilde{\Pi}_2-\tilde{\Pi}_2\kappa^{\mathrm{PI}}_{\mu\nu}{}^\mu)\,,\end{aligned} \tag{5.71}$$

with

$$\lambda_\kappa^{\mu\nu}=N\kappa^{\mathrm{PI}\mu\nu}\,,\quad \lambda_\kappa^\mu=-N\hat{\kappa}^{3\mu}\,, \tag{5.72}$$

since there are just Lagrange multipliers.

Similarly to general relativity, the time preservation of $\pi_\mu\approx 0$ lwill lead to the momentum constraint $\mathcal{H}_\mu\approx 0$ which are first-class since spatial diffeomorphism invariance is assured. Likewise, the time preservation of the other constraints computes,

$$\frac{d}{dt}\Phi^I(t,x)\approx\int d^3y\lambda_J(y)\mathcal{M}^{IJ}(t,x,y)+\{\Phi^I(t,x),\int d^3y\mathcal{H}_V(t,y)\}+\frac{\partial}{\partial t}\Phi^I(t,x)\approx 0\,. \tag{5.73}$$

with the matric $\mathcal{M}^{IJ}$ defined as,

$$\mathcal{M}^{IJ}:=\{\Phi^I(t,x),\Phi^J(t,y)\}\,. \tag{5.74}$$

If this matrix $\mathcal{M}^{IJ}$ has eigenvalues that are zero, this leads to some of the Lagrange multipliers to be not determined which may then compute secondary constraints if $\frac{d}{dt}\Phi^I\approx 0$ are not satisfied.

To see how many secondary constraints are obtained, it is clearer to see in the following ansatz,

$$\hat{F}^{\mu\nu\rho\sigma}=\hat{F}\gamma^{\mu[\rho}\gamma^{\nu|\sigma]}\,, \tag{5.75}$$

with the functions $F_1, F_2, \hat{F}$ being constant. Then 31 components of the set of Lagrangian multipliers will be left undetermined,

$$\lambda_N, \quad \lambda^\mu_\kappa, \quad F_1\lambda^1_{[\mu\nu]} - F_2\lambda^2_{[\mu\nu]}, \quad \lambda^{\rm PI}_{\mu\nu\rho} - \frac{1}{3}\gamma_{\mu\rho}\lambda^{\rm PI\,\sigma}_{\sigma\nu}\,, \tag{5.76}$$

which will then lead to the following 31 secondary constraints,

$$\begin{aligned} \frac{d}{dt}\pi_N &\approx \mathcal{L}_\phi + N\frac{\partial\mathcal{L}_\phi}{\partial N} - F_1 V^\alpha \kappa^{\rm PI}_{\alpha\beta}{}^\beta \\ &\quad + \hat{F}\big(\overset{\Gamma}{\mathcal{K}}{}^{1\alpha}{}_\alpha \overset{\Gamma}{\mathcal{K}}{}^{2\beta}{}_\beta - \overset{\Gamma}{\mathcal{K}}{}^{1\alpha\beta}\overset{\Gamma}{\mathcal{K}}{}^2_{\beta\alpha} + \mathcal{R}(\gamma) - \kappa^{\rm PI\alpha\beta\gamma}\kappa^{\rm PI}_{\beta\gamma\alpha} + D_\alpha\kappa^{\rm PI\alpha\beta}{}_\beta\big) \approx 0\,, \end{aligned} \tag{5.77}$$

$$\frac{d}{dt}\Pi^\mu \approx N\left(\frac{\partial\mathcal{L}_\phi}{\partial V_\mu} - F_1\kappa^{\rm PI\mu\alpha}{}_\alpha\right) \approx 0\,, \tag{5.78}$$

$$\frac{1}{F_1}\frac{d}{dt}\Pi_1^{[\mu\nu]} - \frac{1}{F_2}\frac{d}{dt}\Pi_2^{[\mu\nu]} \approx -N\left(\frac{2\hat{F}}{F_2}\overset{\Gamma}{\mathcal{K}}{}^{1[\mu\nu]} - \frac{1}{F_1}\frac{\partial\mathcal{L}_\phi}{\partial\overset{\Gamma}{\mathcal{K}}{}^1_{[\mu\nu]}}\right) \approx 0\,, \tag{5.79}$$

$$\begin{aligned} \frac{d}{dt}\left(\Pi^{\mu\nu\rho} - \frac{1}{3}\gamma^{\mu\rho}\Pi^{\alpha\nu}{}_\alpha\right) &\approx -NF_1V^\mu\gamma^{\nu\rho} - N\hat{F}\kappa^{\rm PI\nu\rho\mu} - N\hat{F}\kappa^{\rm PI\rho\mu\nu} - \hat{F}\gamma^{\nu\rho}D^\mu N \\ &\quad + \frac{1}{3}\gamma^{\mu\rho}\left(NF_1V^\nu + N\hat{F}\kappa^{\rm PI\alpha\nu}_\alpha + N\hat{F}\kappa^{\rm PI\nu\alpha}{}_\alpha + \hat{F}D^\nu N\right) \approx 0\,. \end{aligned} \tag{5.80}$$

Let the secondary constraints be denoted as $\Psi^i \approx 0$.

For this theory, the number of degrees in phase space is thus reduced to

$$\#_{\rm d.o.f} \leq 116 - \underbrace{6\times 2}_{\pi_\mu\approx 0,\mathcal{H}_\mu\approx 0} - \underbrace{67}_{\Phi^I\approx 0} - \underbrace{31}_{\Psi^i\approx 0} = 6\,. \tag{5.81}$$

Therefore, at the most, there are three degrees of freedom and thus there are no Ostrogradsky ghost is the Lagrangian [4].

One could extend the discussion to generic functions of $F_1, F_2, \hat{F}^{\mu\nu\rho\sigma}$. Although the analysis becomes tedious, the structure does not change and it can be confirmed that even in general cases the degrees of freedom are at the most 3.

Again, similarly, the undetermined Lagrangian multipliers are,

$$\lambda_N, \quad \lambda^\mu_\kappa, \quad F_1\lambda^1_{[\mu\nu]} - F_2\lambda^2_{[\mu\nu]}, \quad \lambda^{\rm PI}_{\mu\nu\rho} - \frac{1}{3}\gamma_{\mu\rho}\lambda^{\rm PI\,\sigma}_{\sigma\nu}\,, \tag{5.82}$$

which will lead to secondary constraints,

$$\frac{d}{dt}\pi_N \approx \frac{\partial F_1}{\partial N}\lambda^{1\alpha}{}_\alpha + \frac{\partial F_2}{\partial N}\lambda^{2\alpha}{}_\alpha + \cdots \approx 0\,, \tag{5.83}$$

$$\frac{d}{dt}\Pi^\mu \approx \frac{\partial F_1}{\partial V_\mu}\lambda^{1\alpha}{}_\alpha + \frac{\partial F_2}{\partial V_\mu}\lambda^{2\alpha}{}_\alpha + \cdots \approx 0\,, \tag{5.84}$$

$$\frac{1}{F_1}\frac{d}{dt}\Pi_1^{[\mu\nu]} - \frac{1}{F_2}\frac{d}{dt}\Pi_2^{[\mu\nu]} \approx \frac{1}{F_1}\frac{\partial F_1}{\partial\overset{\Gamma}{\mathcal{K}}{}^1_{[\mu\nu]}}\lambda^{1\alpha}{}_\alpha + \frac{1}{F_1}\frac{\partial F_2}{\partial\overset{\Gamma}{\mathcal{K}}{}^1_{[\mu\nu]}}\lambda^{2\alpha}{}_\alpha + \cdots \approx 0\,, \tag{5.85}$$

$$\frac{d}{dt}\left(\Pi^{\mu\nu\rho} - \frac{1}{3}\gamma^{\mu\rho}\Pi^{\alpha\nu}{}_\alpha\right) \approx \cdots \approx 0 \tag{5.86}$$

[4] There are possibilities where the degrees of freedom may reduce further resulting in theories that have less than 3 degrees of freedom. For example a first-class constraint in $\Phi^I \approx 0, \Psi^i \approx 0$ or a tertiary constraint $\frac{d}{dt}\Psi \approx 0$ can reduce the degrees of freedom. Theories that have two degrees of freedom in such a manner are known in minimally modified gravity theories [166, 167] and cuscuton theories [168, 169, 170]. In this chapter, however, since the interest lies in the absence of Ostrogradsky instability, such discussion will not be conducted.

The equations (5.83)-(5.86) again give the 31 secondary constraints. Therefore, the Lagrangian (5.58) with (5.59) has at most 6 degrees of freedom in the phase space and does not have Ostrogradsky instability in the unitary gauge.

## 5.5 Summary

In this part of the thesis, the properties of projective invariant scalar-metric-affine theories were investigated in the unitary gauge. It was first shown that the projective invariant function of the form $\mathcal{L}_\phi = \mathcal{L}_\phi(\overset{\Gamma}{\nabla}, \overset{\Gamma}{\nabla}\phi, \overset{\Gamma}{\nabla}\overset{\Gamma}{\nabla}\phi)$ is trivially U-degenerate, i.e. does not have $\pounds_N A_*$ in its construction and thus Ostrograsky ghost-free. Then based on the action with trivially U-degenerate couplings (5.23), U-degenerate theories were constructed through an (invertible) transformation of the metric. Then, as an example, a concrete theory (5.38) was considered, which is also the most general scalar-metric-affine theory up to the quadratic order of the connection. It was shown that this theory is U-degenerate and thus a U-DHOST theory. Furthermore, it was shown through Hamiltonian analysis that indeed such theory is free from Ostrogradsky ghost in the unitary gauge.

# Chapter 6

# Cosmological Perturbations in Palatini Formalism


| | |
|---|---|
| Based on | *"Cosmological Perturbations in Palatini Formalism"* |
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In the previous parts of the thesis, many Palatini and metric-affine theories were analyzed in the Riemann frame, i.e. the frame where the solution of the connection was substituted. Riemann frame is, in general, useful since tools of the usual metric formalism can be utilized. However, it is non-trivial that the Jordan frame and the Riemann frame compute the same quantities even in perturbation theory. Therefore, this part of the thesis analyzes both frames and shows that indeed these frames compute the same physical variables within cosmological perturbation.

## 6.1 K-essence with its non-minimal coupling to Ricci scalar

In this section, a theory of K-essence with a non-minimal coupled Ricci scalar will be considered. The action is

$$S_4{}^{\text{Jordan}} := \int d^4x\sqrt{-g}\left[G_4(\phi,X)\overset{\Gamma}{R} + K(\phi,X)\right], \tag{6.1}$$

with $X := -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi$. The first term is a coupling of scalar and its kinetic term to the Ricci scalar where as the second term is the so-called K-essence term, recall (2.430).

Notice that, unlike in metric formalism, the following counter-term for the $L_4$ Horndeski action,

$$\mathcal{L}_4{}^{\text{metric}} = G_4(\phi,X)\overset{g}{R} + G_{4X}\left[(\phi_{\mu\nu})^2 - (\phi_\mu{}^\mu)^2\right], \tag{6.2}$$

is not necessary in order to keep the equation of motion second-order in derivatives as shown in §2.5.3. For the Palatini Lagrangian (6.1), however, this counter term is unnecessary to keep the equation of motion to be second-order. Furthermore, as will be mentioned later, such covariantization of this counter term can lead to dynamical connections and dynamical and introduce new degrees of freedom in Palatini formalism.

There are 3 ways to analyze the theory (6.1); Einstein frame, Jordan Frame, and Riemann frame. Einstein frame, which is the most conventional method, is the result of the conformal transformation of the metric which leads to an Einstein-Hilbert action, which usually makes the calculations drastically simpler. Jordan frame (6.1), on the other hand, is the most physically straightforward, although in general is tedious. Finally, recall that the Riemann frame is the frame in which the solution of the connection is substituted, which is possible since the connection is an auxiliary field and non-dynamical. Such frame is on-shell fully written in Riemannian geometry and is neither Einstein nor Jordan.

All of these frames are related to each other through an *invertible* redefinition of the variables and should compute same results, which are the observational variables, in the different frames coincide. Through the sections that follows it will be shown to be indeed the case.

### 6.1.1 Analysis in Einstein frame

Similar to the analysis done in it's metric formalism counterpart, consider the following conformal transformation of (6.1) under,

$$\tilde{g}_{\mu\nu} = G_4 g_{\mu\nu}. \tag{6.3}$$

Then the Einstein frame action is obtained as,

$$S_4{}^{\text{Einstein}} = \int d^4\tilde{x}\sqrt{-\tilde{g}}\left[\tilde{g}^{\mu\nu}\overset{\Gamma}{R}_{\mu\nu} + \frac{K(\phi, G_4\tilde{X})}{G_4{}^2(\phi, G_4\tilde{X})}\right]. \tag{6.4}$$

Here the Einstein frame variables are denoted with a tilde and in the coordinate $\tilde{x}^\mu$; $\tilde{X} := -\frac{1}{2}\tilde{g}^{\mu\nu}\tilde{\partial}_\mu\phi\tilde{\partial}_\nu\phi = G_4^{-1}X$.

This resultant frame is nothing but the Einstein-Hilbert action plus some k-essence term. Thus, when assumed no torsion, the connection can be uniquely solved as

$$\Gamma^\lambda{}_{\mu\nu} \quad = \quad \left\{{}^{\lambda}_{\mu\nu}\right\}_{\tilde{g}}. \tag{6.5}$$

Then one obtains the Einstein-frame action written purely with the conformal metric $\tilde{g}_{\mu\nu}$ as,

$$S_4{}^{\text{Einstein}}\big|_{\Gamma=\{\}_{\tilde{g}}} = \int d^4\tilde{x}\sqrt{-\tilde{g}}\left[\tilde{g}^{\mu\nu}\overset{\tilde{g}}{R}_{\mu\nu} + \tilde{K}(\phi,\tilde{X})\right]. \tag{6.6}$$

Here $\overset{\tilde{g}}{R}_{\mu\nu}$ is the Ricci tensor defined through the Levi-Civita connection of the $\tilde{g}_{\mu\nu}$. Finally, the Einstein K-essence function is defined through the Jordan variables as,

$$\tilde{K}(\phi, \tilde{X}) = \frac{K(\phi, X)}{{G_4}^2(\phi, X)} \,. \tag{6.7}$$

Recall that the 'Einstein' frame of the action (6.2) also is a minimally coupled k-essence action. However, the relation of the function $\tilde{K}(\phi, \tilde{X})$ between Einstein and Jordan variables differ.

Now consider proceeding into calculating the cosmological perturbation of such action. The cosmologigcal ansatz of the Einstein frame metric and coordinates are taken as,

$$d\tilde{s}^2 = -\tilde{N}^2 d\tilde{t}^2 + \tilde{\gamma}_{ij}\left(d\tilde{x}^i + \tilde{N}^i d\tilde{t}\right)\left(d\tilde{x}^j + \tilde{N}^j d\tilde{t}\right), \tag{6.8}$$

where $\tilde{N}$ and $\tilde{N}^i$ are the lapse and shift. The perturbation of the scalar field is,

$$\phi(\tilde{t}, \tilde{x}^i) = \phi_{\rm B}(\tilde{t}) + \delta\phi(\tilde{t}, \tilde{x}^i). \tag{6.9}$$

Firstly, consider calculating the tensor perturbations. The ansatz is as follows,

$$\tilde{N} = 1, \tag{6.10}$$
$$\tilde{N}_i = 0, \tag{6.11}$$
$$\tilde{\gamma}_{ij} = \tilde{a}(\tilde{t})^2\left(\delta_{ij} + \tilde{h}_{ij} + \frac{1}{2}\delta^{kl}\tilde{h}_{ik}\tilde{h}_{jl}\right), \tag{6.12}$$
$$\delta\phi = 0. \tag{6.13}$$

Substitution of this ansatz and computing up to second-order results to the quadratic action of the tensor perturbation of [113],

$$\delta^{(2)}{S_4}^{\rm Einstein,tensor} = \frac{1}{4}\int d\tilde{t}\, d^3\tilde{x}\, \tilde{a}^3\left[\tilde{h}'^2_{ij} - \frac{1}{\tilde{a}^2}(\tilde{\partial}_k\tilde{h}_{ij})^2\right], \tag{6.14}$$

where the prime denotes the derivative with $\tilde{t}$.

As is obvious from (6.14), the sound speed of the tensor perturbation is unity. Thus the gravitational waves propagate at the same speed as that of electromagnetic waves. Recall that in metric formalism, however, the speed of gravitational waves differs from that of light as was shown in (2.496) for $G_4 = G_4(\phi, X)$. In Palatini formalism even when $G_4$ has $X$ dependency.

Next, let scalar perturbations be considered. The ansatz of the metric is taken as,

$$\tilde{N} = 1 + \tilde{\alpha}, \tag{6.15}$$
$$\tilde{N}_i = \tilde{\partial}_i\tilde{\beta}, \tag{6.16}$$
$$\tilde{\gamma}_{ij} = \tilde{a}(\tilde{t})^2 e^{2\tilde{\psi}}\delta_{ij}, \tag{6.17}$$
$$\tilde{\zeta} = -\tilde{\psi} + \frac{\tilde{H}(\tilde{t})}{\phi'_{\rm B}(\tilde{t})}\,\delta\phi\,, \tag{6.18}$$

where the Hubble parameter of the Einstein frame is defined as$\tilde{H} = \frac{\tilde{a}'}{\tilde{a}}$.

After substituting the ansatz and solving the auxiliary fields $\tilde{\alpha}$ and $\tilde{\beta}$, the quadratic action for the scalar perturbations are [113],

$$\delta^{(2)}{S_4}^{\rm Einstein,scalar} = \int d\tilde{t}\, d^3\tilde{x}\, \tilde{a}^3\left[\tilde{\mathcal{G}}_S\zeta'^2 - \frac{\tilde{\mathcal{F}}_S}{\tilde{a}^2}(\tilde{\partial}_i\tilde{\zeta})^2\right], \tag{6.19}$$

with

$$\begin{aligned}
\tilde{\mathcal{F}}_S &= \frac{6\tilde{X}K_{\tilde{X}}}{-\tilde{K}+2\tilde{X}\tilde{K}_{\tilde{X}}} = 2\tilde{\epsilon} \\
&= \frac{6X(2KG_{4X}-K_XG_4)}{(K-2XK_X)G_4+3KG_{4X}X}, \qquad (6.20)\\
\tilde{\mathcal{G}}_S &= \frac{6(\tilde{X}\tilde{K}_{\tilde{X}}+2\tilde{X}^2\tilde{K}_{\tilde{X}\tilde{X}})}{-\tilde{K}+2\tilde{X}\tilde{K}_{\tilde{X}}} \\
&= \frac{6X}{(G_4-G_{4X}X)^2\{-K(G_4+3G_{4X})+2XK_XG_4\}} \\
&\quad\times\left[-6X^2KG_{4X}^3+X(8K+5K_XX)G_4G_{4X}^2+(K_X+2K_{XX}X)G_4^3\right.\\
&\quad\left.-2\{K(G_{4X}+2G_{4XX}X)+XK_X(3G_{4X}-XG_{4XX})+X^2K_{XX}G_{4X}\}G_4^2\right]. \qquad (6.21)
\end{aligned}$$

Here $\tilde{\epsilon} := -\frac{\tilde{H}'}{\tilde{H}^2}$ and all of the functions on the right-hand sides are evaluated at the background.

The ghost and gradient instabilities make the predictability of the theory ill-defined, thus taking the following conditions,

$$\begin{aligned}
\tilde{\mathcal{F}}_S = 2\tilde{\epsilon} &> 0\,,\\
\tilde{\mathcal{G}}_S &> 0\,,
\end{aligned}$$

are necessary to avoid such instabilities.

The sound speed of curvature perturbations can be then read off from (6.19) as,

$$\begin{aligned}
\tilde{c}_S^2 &= \frac{2\tilde{\epsilon}}{\tilde{\mathcal{G}}_S} \\
&= \frac{\tilde{K}_{\tilde{X}}}{\tilde{K}_{\tilde{X}}+2\tilde{X}\tilde{K}_{\tilde{X}\tilde{X}}} \\
&= (G_4-G_{4X}X)^2\,(-2KG_{4X}+K_XG_4) \\
&\quad\times\Big[-6X^2KG_{4X}^3+X(8K+5XK_X)G_4G_{4X}^2+(K_X+2XK_{XX})G_4^3 \\
&\quad -2[K(G_{4X}+2XG_{4XX})+XK_X(3G_{4X}-XG_{4XX})+X^2K_{XX}G_{4X})]G_4^2\Big]^{-1},
\end{aligned} \qquad (6.22)$$

and again all of the functions are evaluated with respect to the background.

When assumed $G_4 = G_4(\phi)$, the sound speed of scalar perturbation reduces to

$$\tilde{c}_S^2 = \frac{K_X}{K_X+2XK_{XX}}, \qquad (6.23)$$

which is the same as that of k-essence in the metric formalism as was in (2.508). One notices that the form of $G_4(\phi)$ does not affect sound speed of curvature perturbation.

Similar to [113], consider defining the variables,

$$\tilde{f}_S := \frac{\tilde{\epsilon}'}{\tilde{H}\tilde{\epsilon}}, \quad \tilde{g}_S := \frac{\tilde{\mathcal{G}}_S'}{\tilde{H}\tilde{\mathcal{G}}_S}. \qquad (6.24)$$

Using the constant-roll assumption of $\tilde{\epsilon} \sim$ const., $\tilde{f}_S \sim$ const., and $\tilde{g}_S \sim$ const., the power spectrum and the spectral index of the scalar perturbation are given as

$$\tilde{\mathcal{P}}_{\tilde{\zeta}} = \frac{\tilde{\gamma}_S}{4}\frac{1}{\tilde{\epsilon}\tilde{c}_S}\frac{\tilde{H}^2}{4\pi^2}\bigg|_{-\tilde{k}\tilde{y}_S=1}, \tag{6.25}$$

$$\tilde{n}_S - 1 = 3 - \tilde{\nu}_S, \tag{6.26}$$

with,

$$d\tilde{y}_S := \frac{\tilde{c}_S}{\tilde{a}}d\tilde{t}, \tag{6.27}$$

$$\tilde{\nu}_S := \frac{3-\tilde{\epsilon}+\tilde{g}_S}{2-2\tilde{\epsilon}-\tilde{f}_S+\tilde{g}_S}, \tag{6.28}$$

$$\tilde{\gamma}_S := 2^{2\tilde{\nu}_S-3}\left|\frac{\Gamma(\tilde{\nu}_S)}{\Gamma(\frac{3}{2})}\right|^2\left(1-\tilde{\epsilon}-\frac{\tilde{f}_S}{2}+\frac{\tilde{g}_S}{2}\right), \tag{6.29}$$

where $\Gamma(s)$ is the Gamma function.

In the limit of $\epsilon, f_S, g_S \ll 1$, the tensor-to-scalar ratio becomes,

$$\tilde{r} = 16\tilde{\epsilon}\tilde{c}_S. \tag{6.30}$$

Therefore, all observational variables in the Einstein frame are computed.

### 6.1.2 Analysis in Jordan frame

In the previous section, it could be noticed that the Einstein frame analysis was fairly simple. In contrast, as it will be shown, the Jordan frame analysis is tedious and lengthy. First, recall the original Lagrangian was,

$$S_4^{\text{Jordan}} = \int d^4x\sqrt{-g}\left[G_4(\phi,X)\overset{\Gamma}{R} + K(\phi,X)\right]. \tag{6.31}$$

From the equation of motion of the connection, which is algebraic, the connection can be uniquely solved under the torsionless condition as,

$$\Gamma^\lambda{}_{\mu\nu} = \left\{{}^{\lambda}_{\mu\nu}\right\}_g + \frac{1}{2}g^{\lambda\sigma}\left(2g_{\sigma(\mu}\partial_{\nu)}\ln G_4 - g_{\mu\nu}\partial_\sigma \ln G_4\right), \tag{6.32}$$

which coincides with that of the Einstein frame (6.5). When $\partial_\mu\phi = 0$ the connection reduces to the Levi-Civita connection, and $G_4$ effectively becomes the Planck mass. Thus at the end of the dynamics of the scalar, such as the end of inflation, the theory becomes simply Einstein gravity.

Under the cosmological ansatz, non-null components of the background connection $\bar{\Gamma}^\lambda{}_{\mu\nu}$ is

$$\bar{\Gamma}^0{}_{00} = \frac{1}{2}\frac{d}{dt}\ln G_4, \tag{6.33}$$

$$\bar{\Gamma}^0{}_{ij} = a^2\delta_{ij}\left(H + \frac{1}{2}\frac{d}{dt}\ln G_4\right), \tag{6.34}$$

$$\bar{\Gamma}^i{}_{j0} = \delta^i_j\left(H + \frac{1}{2}\frac{d}{dt}\ln G_4\right), \tag{6.35}$$

with $\frac{d}{dt}\ln G_4 = \frac{\partial}{\partial t}\ln G_4$.

Now upon this background connection, let the perturbations be calculated. Firstly, for the tensor perturbation, metric is perturbed as (6.10)-(6.13) with tilde-less quantities. The connection is perturbed around the background solution as,

$$\Gamma^0{}_{00} = \frac{1}{2}\frac{d}{dt}\ln G_4, \tag{6.36}$$

$$\Gamma^i{}_{00} = \Gamma^0{}_{i0} = 0, \tag{6.37}$$

$$\Gamma^0{}_{ij} = a^2\delta_{ij}\left(H + \frac{1}{2}\frac{d}{dt}\ln G_4\right) + D_{1,ij}, \tag{6.38}$$

$$\Gamma^i{}_{j0} = \delta^i_j\left(H + \frac{1}{2}\frac{d}{dt}\ln G_4\right) + D_{2,j}^{\ i}, \tag{6.39}$$

$$\Gamma^i{}_{jk} = \partial^i D_{3jk} + 2\partial_{(j}D_{4,k)}^{\ i}\,, \tag{6.40}$$

where the tensor perturbations of the (torsionless) connection are denoted by $D_{s,ij}$ .

The non-null components of the Ricci tensor can then be calculated as

$$\overset{\Gamma}{R}_{00} = -3\left(\dot{H} + H^2 + \frac{1}{2}H\frac{d}{dt}\ln G_4 + \frac{1}{2}\ln\ddot{G}_4\right) - D_{2,j}^{\ i}D_{2,i}^{\ j}, \tag{6.41}$$

$$\begin{aligned}
\overset{\Gamma}{R}_{ij} &= a^2\delta_{ij}\left[\dot{H} + 3H^2 + \frac{1}{2}\ln\ddot{G}_4 + \frac{5}{2}H\frac{d}{dt}\ln G_4 + \frac{1}{2}\left(\frac{d}{dt}\ln G_4\right)^2\right] \\
&\quad + \dot{D}_{1,ij} + \left(H + \frac{d}{dt}\ln G_4\right)D_{1,ij} - 2a^2\left(H + \frac{1}{2}\frac{d}{dt}\ln G_4\right)D_{2,ij} + \partial^k\partial_k D_{3,ij} \\
&\quad - D_{1,ik}D_{2,j}^{\ k} - D_{1,jk}D_{2,i}^{k} - \left(\partial^l D_{3ik} + 2\partial_{(i}D_{4,k)}^{\ l}\right)\left(\partial^k D_{3jl} + 2\partial_{(j}D_{4,l)}^{\ k}\right).
\end{aligned} \tag{6.42}$$

The quadratic action of the tensor perturbation then becomes,

$$\begin{aligned}
&\delta^{(2)}\left\{\int d^4x\sqrt{-g}\left(G_4\overset{\Gamma}{R} + K\right)\right\} \\
= &\int dt d^3x G_4 a^3\left[D_{2,ij}D_2^{\ ij} - \frac{1}{a^2}\left\{2D_{1,ij}D_2^{\ ij} + \delta^{ij}\left(\partial^l D_{3ik} + 2\partial_{(i}D_{4,k)}^{\ l}\right)\left(\partial^k D_{3jl} + 2\partial_{(j}D_{4,l)}^{\ k}\right)\right.\right. \\
&\left. + h^{ij}\left(\dot{D}_{1,ij} + \left(H + \frac{d}{dt}\ln G_4\right)D_{1,ij} - 2a^2\left(H + \frac{1}{2}\frac{d}{dt}\ln G_4\right)D_{2,ij} + \partial^k\partial_k D_{3,ij}\right)\right\} \\
&\left. + \frac{1}{2}h^{ij}h_{ij}\left\{\dot{H} + 3H^2 + \frac{1}{2}\ln\ddot{G}_4 + \frac{5}{2}H\frac{d}{dt}\ln G_4 + \frac{1}{2}\left(\frac{d}{dt}\ln G_4\right)^2\right\}\right].
\end{aligned} \tag{6.43}$$

Noticing that all of the tensor perturbations of the connection are auxillary fields, i.e. they are not dynamical, they can be solved by using their own equations of motion,

$$\begin{aligned}
\frac{\delta(\delta^{(2)}S)}{\delta D_{1,ij}} &= -aG_4\left\{2D_2^{\ ij} + h^{ij}(H + \frac{d}{dt}\ln G_4)\right\} + \frac{d}{dt}(aG_4h^{ij}), \\
\frac{1}{a^3G_4}\frac{\delta(\delta^{(2)}S)}{\delta D_{2,ij}} &= 2D_2^{\ ij} - \frac{2}{a^2}D_1^{\ ij} + 2h^{ij}\left(H + \frac{1}{2}\frac{d}{dt}\ln G_4\right), \\
\frac{1}{aG_4}\frac{\delta(\delta^{(2)}S)}{\delta D_{3,ij}} &= 2\partial^k\partial_k D_4^{\ ij} - \partial^k\partial_k h^{ij}, \\
\frac{1}{aG_4}\frac{\delta(\delta^{(2)}S)}{\delta D_{4,ij}} &= 2\partial^k\partial_k D_3^{\ ij} - 2\partial^k\partial_k D_4^{\ ij}\ .
\end{aligned}$$

Which can be solved as,

$$D_2{}^{ij} = \frac{1}{2}\dot{h}^{ij}, \tag{6.44}$$

$$D_1{}^{ij} = a^2\left[\frac{1}{2}\dot{h}^{ij} + h^{ij}\left(H + \frac{1}{2}\frac{d}{dt}\ln G_4\right)\right], \tag{6.45}$$

$$D_4{}^{ij} = \frac{1}{2}h^{ij}, \tag{6.46}$$

$$D_3{}^{ij} = \frac{1}{2}h^{ij}. \tag{6.47}$$

Recalling the analysis of the Einstein frame, this indeed matches the perturbation of the background connection (6.32) itself.

Which in regular tensor notation is,

$$\begin{aligned}\delta\Gamma^\lambda{}_{\mu\nu} &= \frac{1}{2}\bar{g}^{\lambda\sigma}\left(2\overset{g}{\nabla}_{(\mu}\delta g_{\nu)\sigma} - \overset{g}{\nabla}_\sigma\delta g_{\mu\nu}\right) + \frac{1}{2}\left(\bar{g}_{\mu\nu}\delta g^{\lambda\sigma} - \bar{g}^{\lambda\sigma}\delta g_{\mu\nu}\right)\partial_\sigma\ln G_4 \\ &\quad + \frac{1}{2G_4}\left(\delta^\lambda_{(\mu}\delta^\sigma_{\nu)} - \frac{1}{2}\bar{g}^{\lambda\sigma}\bar{g}_{\mu\nu}\right) \qquad (6.48) \\ &\quad \times\left(G_{4X}\phi_\alpha\phi_\beta\overset{g}{\nabla}_\sigma\delta g^{\alpha\beta} + 2G_{4X}\phi_{\sigma(\alpha}\phi_{\beta)}\delta g^{\alpha\beta} + \phi_\alpha\phi_\beta\delta g^{\alpha\beta}\overset{g}{\nabla}_\sigma G_{4X} - G_{4X}\phi_\alpha\phi_\beta\delta g^{\alpha\beta}\overset{g}{\nabla}_\sigma\ln G_4\right),\end{aligned} \tag{6.49}$$

where perturbed variables of the metric and connection were defined as, $g_{\mu\nu} \to \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$ and $\Gamma^\lambda{}_{\mu\nu} \to \bar{\Gamma}^\lambda{}_{\mu\nu} + \delta\Gamma^\lambda{}_{\mu\nu}$. This solution can also be directly obtained by perturbing the $S_4{}^{\text{Jordan}}$ action (6.1) and solving it with respect to $\delta\Gamma^\lambda{}_{\mu\nu}$.

Nonetheless, the tensor perturbations of the connection (6.36)-(6.40) is indeed consistent.

Finally substituting the solutions of $D_i{}^{ij}$, the final form of the quadratic action for the tensor-perturbations are derived as,

$$\begin{aligned}\delta^{(2)}S_4{}^{\text{Jordan,tensor}} &= \delta^{(2)}\left\{\int d^4x\sqrt{-g}\left(G_4\overset{\Gamma}{R} + K\right)\right\}, \\ &= \frac{1}{4}\int dt d^3x G_4 a^3\left[\dot{h}_{ij}^2 - \frac{1}{a^2}(\partial_k h_{ij})^2\right].\end{aligned} \tag{6.50}$$

Indeed it is confirmed that, the speed of gravitational waves is unity, and matched the analysis of the Einstein frame.

The Einstein frame variables and the Jordan frame variables under the cosmological ansatz can be mapped upon by $d\tilde{s}^2 = G_4(t)ds^2$, when written explicitly, is the form

$$\begin{aligned}d\tilde{t} &= \sqrt{G_4(t)}\,dt, \\ d\tilde{x} &= dx, \\ \tilde{a}(\tilde{t}) &= \sqrt{G_4(t)}\,a(t), \\ \tilde{h}_{ij} &= h_{ij}.\end{aligned}$$

Substituting this into the Einstein frame result of tensor perturbations (6.14), indeed one obtains the result of the Jordan frame (6.50).

The same calculation can be done for the scalar perturbation. The ansatz for the scalar perturbations of the Jordan-frame metric are the same as the curvature perturbations given in (6.15)–(6.18) which in the current scenario is tildeless. The scalar perturbation of the torsionless connection takes the following ansatz of

$$\begin{aligned}
\Gamma^0{}_{00} &= \frac{1}{2}\frac{d}{dt}\ln G_4 + c_1, && (6.51)\\
\Gamma^0{}_{i0} &= c_{2,i}, && (6.52)\\
\Gamma^0{}_{ij} &= a^2\delta_{ij}\left(H + \frac{1}{2}\frac{d}{dt}\ln G_4\right) + c_3\delta_{ij} + c_{4,ij}, && (6.53)\\
\Gamma^i{}_{00} &= c_5{}^{,i} && (6.54)\\
\Gamma^i{}_{j0} &= \delta^i_j\left(H + \frac{1}{2}\frac{d}{dt}\ln G_4\right) + c_6\delta^i_j + c_7{}^{,i}{}_{,j}, && (6.55)\\
\Gamma^i{}_{jk} &= \delta_{jk}c_8{}^{,i} + \delta^i_{(j}c_{9,k)} + c_{10}{}^{,i}{}_{,jk}. && (6.56)
\end{aligned}$$

Substituting both the metric and the connection ansatz, once the action is taken up to quadratic order in perturbation variables, the scalar perturbation of the connection $c_i$ can be solved through their own equation of motion as,

$$\begin{aligned}
c_1 &= \left(1 + \frac{G_{4X}\dot{\phi}^2}{2G_4}\right)\dot{\alpha} + \frac{\alpha G_{4X}}{2G_4}\left[2\dot{\phi}\ddot{\phi} + \dot{\phi}^2\frac{d}{dt}\ln\left(\frac{G_{4X}}{G_4}\right)\right], && (6.57)\\
c_2 &= \alpha + H\beta + \frac{1}{2}\beta\frac{d}{dt}\ln G_4 - \frac{G_{4X}\dot{\phi}^2}{2G_4}(\alpha + 2H\beta), && (6.58)\\
c_3 &= a^2\left[\dot{\zeta} + \left(2H + \frac{d}{dt}\ln G_4\right)(\zeta - \alpha) + \frac{G_{4X}\dot{\phi}^2\alpha}{2G_4}\left\{\frac{\dot{\alpha}}{\alpha} + 2\frac{\ddot{\phi}}{\dot{\phi}} + \frac{d}{dt}\ln\left(\frac{G_{4X}}{G_4}\right)\right\}\right], && (6.59)\\
c_4 &= -\beta, && (6.60)\\
c_5 &= \frac{1}{a^2}\left[\alpha + \dot{\beta} + \frac{1}{2}\beta\frac{d}{dt}\ln G_4 + \frac{G_{4X}\dot{\phi}^2}{2G_4}(\alpha + 2H\beta)\right], && (6.61)\\
c_6 &= \dot{\zeta} + \frac{G_{4X}\dot{\phi}^2\alpha}{2G_4}\left\{\frac{\dot{\alpha}}{\alpha} + 2\frac{\ddot{\phi}}{\dot{\phi}} + \frac{d}{dt}\ln\left(\frac{G_{4X}}{G_4}\right)\right\}, && (6.62)\\
c_7 &= 0, && (6.63)\\
c_8 &= -(\zeta + H\beta) + \frac{G_{4X}\dot{\phi}^2}{2G_4}(\alpha + 2H\beta), && (6.64)\\
c_9 &= 2\zeta - \frac{G_{4X}\dot{\phi}^2}{G_4}(\alpha + 2H\beta), && (6.65)\\
c_{10} &= 0. && (6.66)
\end{aligned}$$

which also matches the Lorentz invariant tensor notation derived in (6.49)

Substituting the solutions, and again solving $\alpha$ and $\beta$ results as the quadratic action for the scalar-perturbations,

$$\delta^{(2)}S_4{}^{\text{Jordan,scalar}} = \delta^{(2)}\left(\int d^4x\sqrt{-g}G_4\overset{\Gamma}{R}\right) = \int dt\, d^3x\, a^3\left[\mathcal{G}_S\dot{\zeta}^2 - \frac{\mathcal{F}_S}{a^2}(\partial_i\zeta)^2\right]. \qquad (6.67)$$

Here $\mathcal{F}_S = G_4(t)\tilde{\mathcal{F}}_S$ and $\mathcal{G}_S = G_4(t)\tilde{\mathcal{F}}_S$ with $\tilde{\mathcal{F}}_S$ and $\tilde{\mathcal{G}}_S$ being the same form of (6.20) and (6.21). Therefore it is indeed confirmed that the sound speed $c_S$ coinciding with the computed result of (6.22).

Recall that the conformal relation of Einstein and Jordan metrics were, $d\tilde{s}^2 = G_4(t, x^i)ds^2$, which computes the following relation for the perturbed variables [171]:

$$\tilde{N} = N + \frac{2\delta G_4}{2G_4(t)}, \tag{6.68}$$

$$\tilde{N}_i = G_4(t)N_i. \tag{6.69}$$

$$\tilde{\psi} = \psi + \frac{2\delta G_4}{2G_4(t)}, \tag{6.70}$$

$$\tilde{\zeta} = \zeta, \tag{6.71}$$

with $\delta G_4(t, x^i) \equiv G_4(t, x^i) - G_4(t)$. This result shows that indeed the obtained quadratic action is the same one obtained in the Einstein frame analysis under the redefinition,

$$\begin{aligned} d\tilde{t} &= \sqrt{G_4(t)}\, dt, \\ d\tilde{x} &= dx, \\ \tilde{a}(\tilde{t}) &= \sqrt{G_4(t)}\, a(t), \\ \tilde{\zeta} &= \zeta. \end{aligned}$$

It is known that conformal transformation does not change observables [171, 172]. Thus, the amplitudes and the spectral index of scalar and tensor perturbations are also invariant. Thus the tensor-to-scalar ratio is also unaffected,

$$r = 16\, \frac{\mathcal{F}_S}{2G_4}\, c_s = 16\tilde{\epsilon}\tilde{c}_s = \tilde{r}. \tag{6.72}$$

Therefore, all observational variables in the Jordan frame, although tedious, are computed.

### 6.1.3 Analysis in Riemann frame

In the previous sections, the perturbation theory of Einstein and Jordan frame was considered. The former is obviously only possible when such a frame exists, while the latter is tedious and lengthy. In Palatini formalism, it is common that the connection is non-dynamical and thus can be integrated out. Thus in many Palatini theories, a Riemann frame exists while Einstein's does not. Therefore it is useful to see how perturbations work in the Riemann frame

For the current case, consider substituting the solution of the affine connection (6.32) into the action(6.1) [1]. Thus the Riemann frame action is obtained as,

$$\begin{aligned} \mathcal{L}_4^{\text{Riemann}} &= \sqrt{-g}\left[G_4 \overset{g}{R} + \frac{3}{2}\frac{(\overset{g}{\nabla} G_4)^2}{G_4} + K\right] \\ &= \sqrt{-g}\left[G_4 \overset{g}{R} - \frac{3}{2G_4}\left(2G_{4\phi}^2 X + 2G_{4\phi}G_{4X}\phi^\alpha\phi_{\alpha\beta}\phi^\beta - G_{4X}^2\phi^\alpha\phi_{\alpha\beta}\phi^{\beta\gamma}\phi_\gamma\right) + K\right]. \end{aligned} \tag{6.73}$$

This frame computes the same equation of motions of the metric and scalar as (6.1) and thus could be considered dynamically equivalent.[2]

[1] Although it is less known, it is possible to integrate out the metric, since it too can be taken as a non-dynamical variable in the context of Palatini formalism. The formalism of purely affine connection theory is called Eddington formalism, and one may consider inflation in such scenarios as outlined in [146, 147, 173].

[2] Note that although the physics of the Einstein frame and the Jordan frame are equivalent, the physical variables are not the same. On the other hand, the Riemann frame and Jordan frame uses the same variables.

As commented in the previous parts of this thesis, this action is a qDHOST of class ${}^2N$-I/Ia [122, 123], as first noticed in [35]. Cosmological perturbations of such theories were conducted in [123, 174, 175], which will be the that will be followed here. In order to directly compare with previous works, consider defining the following functions,

$$\begin{aligned}
f &= G_4, \\
\bar{K} &= K - \frac{3G_{4\phi}^2 X}{G_4} + 6X\frac{\partial}{\partial\phi}\int \frac{G_{4\phi}G_{4X}}{G_4}dX, \\
Q &= -3\int \frac{G_{4\phi}G_{4X}}{G_4}dX, \\
A_4 &= \frac{3G_{4X}^2}{2G_4}\,.
\end{aligned}$$

Then the action (6.73) reduces to the DHOST action, see §2.5.4 considered in [123, 174, 175] as [3]

$$\mathcal{L} = f\overset{g}{R} + \bar{K} + Q\overset{g}{\Box}\phi + A_4\phi^\mu\phi_{\mu\nu}\phi^{\nu\rho}\phi_\rho. \tag{6.74}$$

with the degeneracy conditions (2.450) automatically satisfied.

The quadratic ADM action is then [123, 174, 175],

$$\begin{aligned}
\delta^{(2)}\mathcal{L}_4^{\text{Riemann}} &= a^3\frac{M^2}{2}\Bigg\{\delta\mathcal{K}_{ij}\delta\mathcal{K}^{ij} - \delta\mathcal{K}^2 + \overset{3}{R}\frac{\delta\sqrt{h}}{a^3} + \delta^{(2)}\overset{3}{R} + \alpha_K H^2\delta N^2 + 4H\alpha_B\delta\mathcal{K}\delta N \\
&\quad + (1+\alpha_H)\overset{3}{R}\delta N + 4\beta_1\delta\mathcal{K}\delta\dot{N} + \beta_2\delta\dot{N}^2 + \frac{\beta_3}{a^2}(\partial_i\delta N)^2\Bigg\}\,.
\end{aligned} \tag{6.75}$$

Here, $\overset{3}{R}$ is the spatial Ricci scalar, and $\mathcal{K}_{ij}$ is the extrinsic curvature. The functions were defined as follows,

$$\begin{aligned}
\frac{M^2}{2} &= f, \\
2HM^2\alpha_B &= -4X\dot{X}A_4 + 2(3\dot{X} - 4HX)f_X + 2\sqrt{-2X}XQ_{1X} + 4X\dot{X}\bar{K}_{XX}, \\
\frac{M^2}{2}H^2\alpha_K &= (24HX\dot{X} - 3X^2 + 12X\ddot{X})A_4 + \frac{1}{2}X(12HX\dot{X} + 7\dot{X}^2 + 4X\ddot{X})A_{4X}, \\
&\quad + 2X^2\dot{X}^2A_{4XX} + 6X(2H^2 + 3\dot{H})f_X + 12X^2(\dot{H} + 2H^2)f_{XX} + 4X^2Q_{1X} \\
&\quad - 6HX^2\sqrt{-2X}Q_{1XX}, \\
\alpha_H &= -2\frac{f_X X}{f}, \\
\alpha_T &= \alpha_L = 0, \\
\beta_1 &= \frac{f_X X}{f}, \\
\beta_2 &= -6\left(\frac{f_X X}{f}\right)^2, \\
\beta_3 &= -\frac{2f_X X}{f}\left(2 - \frac{3f_X X}{f}\right).
\end{aligned}$$

[3] One must note that the original papers of DHOST theories [123, 174, 175] have different notations with this thesis since, the authors took $X = (\partial\phi)^2$.

Now, substituting the ansatz for tensor perturbations of tildeless (6.10)-(6.13) , results in,

$$\delta^{(2)}\mathcal{L}_4^{\text{Riemann,tensor}} = \frac{1}{8}\int dtd^3xM^2a^3\left[\dot{h}_{ij}^2 - \frac{1}{a^2}(\partial_k h_{ij})^2\right] = \delta^{(2)}\mathcal{L}_4^{\text{Jordan,tensor}}, \tag{6.76}$$

which precisely matches that of the Jordan frame.

Similarly, substituting the scalar perturbations of the metric (6.15)-(6.18), the quadratic action becomes,

$$\begin{aligned}\delta^{(2)}\mathcal{L}_4^{\text{Riemann,scalar}} \quad = \quad & a^3\frac{M^2}{2}\Big\{-6(\dot{\zeta}-\beta_1\dot{\alpha})^2 + 12H\left[(1+\alpha_B)\dot{\zeta}-\beta_1\dot{\alpha}\right]\alpha \\ & H^2(\alpha_K - 6 - 12\alpha_B)\alpha^2 + 4\left[\dot{\zeta}-\beta_1\dot{\alpha} - H(1+\alpha_B)\alpha\right]\beta \\ & \frac{1}{a^2}\left[2(\partial\zeta)^2 + 4(1+\alpha_H\partial_i\zeta\partial_i\alpha + \beta_3(\partial_i\alpha)^2)\right]\Big\}. \end{aligned} \tag{6.77}$$

Variation of $\beta$ computes,

$$\alpha = \frac{\dot{\tilde{\zeta}}}{H(1+\alpha_B - \dot{\beta}_1)}, \tag{6.78}$$

where $\zeta$ was redefined to $\tilde{\zeta}$ as,

$$\tilde{\zeta} = \zeta - \beta_1\alpha, \tag{6.79}$$

Which computes the quadratic action for the curvature perturbation as

$$\delta^{(2)}S_4^{\text{Riemann,scalar}} = \int d^2xdta^3\frac{M^2}{2}\left[A_{\tilde{\zeta}}\dot{\tilde{\zeta}}^2 + B_{\tilde{\zeta}}\frac{\left(\partial_i\tilde{\zeta}\right)^2}{a^2}\right], \tag{6.80}$$

with

$$A_{\tilde{\zeta}} \quad = \quad \frac{1}{\left(1+\alpha_B - \frac{\dot{\beta}_1}{H}\right)^2}\left[\alpha_K + 6\alpha_B^2 - \frac{6}{a^3H^2M^2}\frac{d}{dt}(a^3HM^2\alpha_B\beta_1)\right], \tag{6.81}$$

$$B_{\tilde{\zeta}} \quad = \quad 2 - \frac{2}{aM^2}\frac{d}{dt}\left[\frac{aM^2(1+\alpha_H+\beta_1)}{H(1+\alpha_B)-\dot{\beta}_1)}\right]. \tag{6.82}$$

Substituting the forms of the functions with respect to the original Riemann frame action variables, it can be shown that the scalar perturbation matches with (6.20) and (6.21), i.e. $\frac{M^2}{2}A_{\tilde{\zeta}} = \mathcal{G}_S$ and $\frac{M^2}{2}B_{\tilde{\zeta}} = \mathcal{F}_S$. Again, just as in the tensor perturbation analysis, the quadratic action of the Riemann frame is the same as the one of the Jordan frame (6.67). Therefore, all observational variables in the Riemann frame are computed.

As a conclusion, in this chapter, the perturbation analysis around the cosmological background for Einstein, Jordan, and Riemann frames were conducted and as a result, it is confirmed that the observables are indeed the same for all frames, i,e, the power spectra, and the sound velocities for both the scalar and tensor perturbations. Thus one can also conclude that the Jordan frame perturbation (6.51)-(6.56) is consistent. Therefore, one may use this perturbation ansatz for theories that have neither an Einstein frame nor a Riemann frame.

## 6.2 Possible Galileon terms in Palatini formalism

In §3.1.1, the covariantization of the flat space-time action

$$\mathcal{L}_3{}^{\text{flat}} = G_3 \eta^{\mu\nu} \partial_\mu \partial_\nu \phi, \tag{6.83}$$

where $G_3 = G_3\left(\phi, -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right)$, and thus was the definition of the d'Alembertian operator in (3.3). It was mentioned that such covariantization in Palatini formalism is not unique, which is sharp contrast to the metric formalism in which the covariantization is unique as follows,

$$\mathcal{L}_3{}^{\text{metric}} = G_3 g^{\mu\nu} \overset{g}{\nabla}_\mu \overset{g}{\nabla}_\nu \phi. \tag{6.84}$$

The reason was that, in Palatini formalism, metric incompatibility of the connection results in

$$\overset{\Gamma}{\nabla}{}^\mu \overset{\Gamma}{\nabla}_\mu \phi \neq \overset{\Gamma}{\nabla}_\mu \overset{\Gamma}{\nabla}{}^\mu \phi \tag{6.85}$$

In §3.1.1, however, not all possible terms of the d'Alembertian operator was considered. Indeed one may consider, for example,

$$\mathcal{L}_3{}^{\text{Palatini}} = \begin{cases} G_3 g^{\mu\nu} \overset{\Gamma}{\nabla}_\mu \overset{\Gamma}{\nabla}_\nu \phi \\ G_3 \overset{\Gamma}{\nabla}_\mu (g^{\mu\nu} \overset{\Gamma}{\nabla}_\nu \phi) \\ G_3 \overset{\Gamma}{\nabla}_\mu \{\overset{\Gamma}{\nabla}_\nu (g^{\mu\nu}\phi)\} \\ G_3 g^{\mu\nu} g^{\alpha\beta} \overset{\Gamma}{\nabla}_\mu (g_{\alpha\beta} \overset{\Gamma}{\nabla}_\nu \phi) \\ \vdots \end{cases}, \tag{6.86}$$

which all reduces to (6.83)in flat space-time. Since one may introduce arbitrary numbers of metrics between the covariant derivative, one may wonder if the candidates of the d'Alembertian operator are finite in Palatini formalism. Luckily this is the case. To understand this there are three points to consider.

Firstly, recall that any (torsionless) connection can be decomposed into the Levi-Civita connection and the non-metricity respectively. Schematically, it can be written as

$$\overset{\Gamma}{\nabla} = \overset{g}{\nabla} + \text{ terms containing } \mathcal{Q}_\lambda^{\ \mu\nu} , \tag{6.87}$$

which is the result of,

$$\Gamma^\lambda_{\ \mu\nu} \quad = \quad \left\{{\lambda \atop \mu\nu}\right\}_g + \frac{1}{2}\left(\mathcal{Q}_{\mu\nu}{}^\lambda + \mathcal{Q}_{\nu\mu}{}^\lambda - \mathcal{Q}^\lambda{}_{\mu\nu}\right). \tag{6.88}$$

The second point is that the covariantized terms can be written in the form,

$$G_3 \cdot \overset{\Gamma}{\nabla} \cdot \overset{\Gamma}{\nabla} \cdot \phi \ , \tag{6.89}$$

with $\cdot$ representing some arbitrary number of metrics. Here the first derivative acts either on (some number of) a metric, the second covariant derivative, or $\phi$, on the other hand, the second derivative may operate either on a metric or $\phi$. The final third point is that the result must be a scalar, and thus all of the space-time indices are contracted.

Through the three points, one may guess that the covariantization of the flat action (6.83) in Palatini formalism can be constructed by the following terms, $\overset{g}{\nabla}\overset{g}{\nabla}\phi$, $\mathcal{Q} \times \partial\phi$, $\mathcal{Q} \times Q$, $\overset{g}{\nabla}\mathcal{Q}$, with each term possibly having more than one way to be contracted. As a result, the possible independent terms are,

1. $\overset{g}{\Box}\phi$,
2. $\mathcal{W}^\mu\partial_\mu\phi$, $\tilde{\mathcal{Q}}^\mu\partial_\mu\phi$,
3. $\phi\mathcal{Q}_{\alpha\beta\gamma}\mathcal{Q}^{\alpha\beta\gamma}$, $\phi\mathcal{Q}_{\alpha\beta\gamma}\mathcal{Q}^{\beta\gamma\alpha}$, $\phi\mathcal{W}^\mu\mathcal{W}_\mu$, $\phi\mathcal{W}_\mu\tilde{\mathcal{Q}}^\mu$, $\phi\tilde{\mathcal{Q}}_\mu\tilde{\mathcal{Q}}^\mu$,
4. $\phi\overset{g}{\nabla}_\mu\mathcal{W}^\mu$, $\phi\overset{g}{\nabla}_\mu\tilde{\mathcal{Q}}^\mu$.

Recall that the Weyl vector and the non-metricity vector was defined as,

$$\mathcal{W}_\mu := \frac{1}{4}\mathcal{Q}_{\mu\nu}{}^{\nu}, \tag{6.90}$$

$$\tilde{\mathcal{Q}}^\mu := \mathcal{Q}_\nu{}^{\nu\mu}. \tag{6.91}$$

Combining all of these term one obtained the most general Palatini $\mathcal{L}_3$ action, which consists of 10 functions as,

$$\begin{aligned}
\mathcal{L}_3{}^{\text{Jordan}} :=\; & G_{3,0}\overset{g}{\Box}\phi + G_{3,1}\mathcal{W}^\mu\partial_\mu\phi + G_{3,2}\tilde{\mathcal{Q}}^\mu\partial_\mu\phi \\
& + G_{3,3}\phi\mathcal{Q}_{\alpha\beta\gamma}\mathcal{Q}^{\alpha\beta\gamma} + G_{3,4}\phi\mathcal{Q}_{\alpha\beta\gamma}\mathcal{Q}^{\beta\gamma\alpha} + G_{3,5}\phi\mathcal{W}^\mu\mathcal{W}_\mu + G_{3,6}\phi\mathcal{W}_\mu\tilde{\mathcal{Q}}^\mu \\
& + G_{3,7}\phi\tilde{\mathcal{Q}}_\mu\tilde{\mathcal{Q}}^\mu + G_{3,8}\phi\overset{g}{\nabla}_\mu\mathcal{Q}^\mu + G_{3,9}\phi\overset{g}{\nabla}_\mu\tilde{\mathcal{Q}}^\mu \qquad (6.92) \\
=:\; & \sum_{i=0}^{9} G_{3,i}\,\overset{\Gamma}{\Box}_{(i)}\phi\,. \qquad (6.93)
\end{aligned}$$

Here all of the functions are dependent on $\phi$ and $X$, thus. $G_{3,i} = G_{3,i}(\phi, X)$.

Indeed under the flat space-time limit, $g_{\mu\nu} \to \eta_{\mu\nu}$ and $\Gamma^\lambda{}_{\mu\nu} \to 0$, the obtained L3 action in Palatini (6.93) reduces to the flat space-time action (6.83).

The terms that were obtained, in some cases, have been obtained throughout the literature. For example, first three terms in (6.92) were considered in §3.1.1 and [34, 156, 176, 177], on the other hand the first eight terms were considered in [178] although the functions $G_{3,i}$ were only functions of $\phi$.

Since one obtained the most general Palatini $\mathcal{L}_3$ action, it is natural to question what will the observables be and how they differ from that of metric formalism.

## 6.3 Tensor and scalar perturbations with the Galileon terms in Palatini formalism

In this section, the following action

$$\mathcal{L}_{3+4}^{\text{Jordan}} := \mathcal{L}_3{}^{\text{Jordan}} + \mathcal{L}_4{}^{\text{Jordan}} = G_4\overset{\Gamma}{R} + K + \sum_{i=0}^{9} G_{3,i}\,\overset{\Gamma}{\Box}_{(i)}\phi, \tag{6.94}$$

where $G_4$, $K$,$G_{3,i}$ are functions of $\phi$ and $X = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$, will be investigated in the context of cosmological perturbation.

Firstly, unlike Sec. 6.1, there is no Einstein frame for this action. This is because the conformal transformation of the metric cannot bring the action solely to the Einstein-Hilbert action since there are connection-full terms in $\mathcal{L}_3{}^{\text{Jordan}}$. The Riemann frame, however, does exist since the connection is non-dynamical and the solution can be computed. Thus, recalling the analysis of the previous section, cosmological perturbation in the Riemann frame will be conducted.

The solution of the connection for (6.94) is computed as,

$$\Gamma^{\lambda}{}_{\mu\nu} = \left\{{}^{\lambda}_{\mu\nu}\right\}_g + \frac{1}{D}\left[\left\{A^X\partial^\lambda X + A^\phi\partial^\lambda\phi\right\}g_{\mu\nu} + 2\left\{B^X\partial_{(\mu}X + B^\phi\partial_{(\mu}\phi\right\}\delta^\lambda_{\nu)}\right], \tag{6.95}$$

with,

$$\begin{aligned} A^\phi &= -6G_4(2G_{4\phi} + G_{3,1} + 2G_{3,2} - G_{3,8} - 2G_{3,9}) \\ &- \{-6(G_{3,8\phi} + 2G_{3,9\phi})G_4 + 2[32(G_{3,3} + G_{3,4}) + 5G_{3,5} + 17G_{3,6} + 56G_{3,7}]G_{4,\phi} \\ &\quad +(8G_{3,3} + 12G_{3,4} + 5G_{3,6} + 28G_{3,7})G_{3,1} - 2(16G_{3,3} + 8G_{3,4} + 5G_{3,5} + 7G_{3,6})G_{3,2} \\ &\quad -8G_{3,3}G_{3,8} - 12G_{3,4}G_{3,8} - 5G_{3,6}G_{3,8} - 28G_{3,7}G_{3,8} + 32G_{3,3}G_{3,9} + 16G_{3,4}G_{3,9} \\ &\quad +10G_{3,5}G_{3,9} + 14G_{3,6}G_{3,9}\}\,\phi \\ &+\{(8G_{3,3} + 12G_{3,4} + 5G_{3,6} + 28G_{3,7})G_{3,8\phi} - 2(16G_{3,3} + 8G_{3,4} + 5G_{3,5} + 7G_{3,6})G_{3,9\phi}\}\phi^2, \end{aligned} \tag{6.96}$$

$$\begin{aligned} A^X &= 6G_4\{-2G_{4X} + (G_{3,8X} + 2G_{3,9X})\phi\} \\ &-2\{32(G_{3,3} + G_{3,4}) + 5G_{3,5} + 17G_{3,6} + 56G_{3,7}\}G_{4X}\phi \\ &+\{(8G_{3,3} + 12G_{3,4} + 5G_{3,6} + 28G_{3,7})G_{3,8X} - 2(16G_{3,3} + 8G_{3,4} + 5G_{3,5} + 7G_{3,6})G_{3,9X}\}\phi^2, \end{aligned} \tag{6.97}$$

$$\begin{aligned} B^\phi &= 4G_4(6G_{4\phi} + G_{3,1} - 2G_{3,2} - G_{3,8} + 2G_{3,9}) \\ &+2\{-2(G_{3,8\phi} - 2G_{3,9\phi})G_4 + 2[16(G_{3,3} + G_{3,4}) + G_{3,5} + 7G_{3,6} + 40G_{3,7}]G_{4,\phi} \\ &+(8G_{3,3} + 4G_{3,4} + G_{3,6} + 20G_{3,7})G_{3,1} - 2(8G_{3,4} + G_{3,5} + 5G_{3,6})G_{3,2} - 8G_{3,3}G_{3,8} \\ &-4G_{3,4}G_{3,8} - G_{3,6}G_{3,8} - 20G_{3,7}G_{3,8} + 16G_{3,4}G_{3,9} + 2G_{3,5}G_{3,9} + 10G_{3,6}G_{3,9}\}\,\phi \\ &+2\{(8G_{3,3} + 4G_{3,4} + G_{3,6} + 20G_{3,7})G_{3,8\phi} + 2(8G_{3,4} + G_{3,5} + 5G_{3,6})G_{3,9\phi}\}\phi^2, \end{aligned} \tag{6.98}$$

$$\begin{aligned} B^X &= 4G_4\{6G_{4X} - (G_{3,8X} - 2G_{3,9X})\phi\} + 4\{16(G_{3,3} + G_{3,4}) + G_{3,5} + 7G_{3,6} + 40G_{3,7}\}G_{4X}\phi \\ &+\{2(8G_{3,3} + 4G_{3,4} + G_{3,6} + 20G_{3,7})G_{3,8X} - 4(8G_{3,4} + G_{3,5} + 5G_{3,6})G_{3,9X}\}\phi^2, \end{aligned} \tag{6.99}$$

$$\begin{aligned} D &= 24G_4^2 + 4(8G_{3,3} + 20G_{3,4} - G_{3,5} + 8G_{3,6} + 44G_{3,7})G_4\phi \\ &-2\left\{64G_{3,3}^2 - 32G_{3,4}^2 + 16G_{3,3}(2G_{3,4} + G_{3,5} + G_{3,6} + 10G_{3,7}) - 9(G_{3,6}^2 - 4G_{3,5}G_{3,7})\right. \\ &\qquad \left. +4G_{3,4}[G_{3,5} - 8(G_{3,6} + G_{3,7})]\right\}\phi^2\,. \end{aligned} \tag{6.100}$$

One can see that, when $G_{3,i} = 0$, this solution reduces to the one obtained in (6.32). Similarly, when the scalar has finished its dynamics, $\partial_\mu\phi = 0$ the connection becomes Levi-Civita.

Inserting the solution of the connection into (6.94), one obtains the Riemann frame, which is,

$$\mathcal{L}^{\text{Riemann}}_{3+4} = G_4\overset{g}{R} + K + G_{3,0}\overset{g}{\Box}\phi + E_{\phi\phi} + E_{\phi X}\phi^\alpha\phi_{\alpha\beta}\phi^\beta + E_{XX}\phi^\alpha\phi_{\alpha\beta}\phi^{\gamma\delta}\phi_\delta\ , \tag{6.101}$$

with the functions being,

$$\begin{aligned}
E_{\phi\phi} &= -\frac{X}{8D^2}\left[4A^{\phi 2}\{12G_4+(40G_{3,3}+28G_{3,4}+G_{3,5}+10G_{3,6}+100G_{3,7})\phi\}\right.\\
&\quad +B^{\phi 2}\{12G_4+(136G_{3,3}+124G_{3,4}+25G_{3,5}+70G_{3,6}+196G_{3,7})\phi\}\\
&\quad +4A^{\phi}B^{\phi}\{24G_4+(56G_{3,3}+68G_{3,4}+5G_{3,5}+32G_{3,6}+140G_{3,7})\phi\}\\
&\quad -8DA^X\{6G_{4\phi}-G_{3,1}-10G_{3,2}+G_{3,8}+10G_{3,9}+(G_{3,8\phi}+10G_{3,9\phi})\phi\}\\
&\quad \left.+4DB^X\{6G_{4\phi}+5G_{3,1}+14G_{3,2}-5G_{3,8}-14G_{3,9}-(5G_{3,8\phi}+14G_{3,9\phi})\phi\}\right],
\end{aligned} \tag{6.102}$$

$$\begin{aligned}
E_{\phi X} &= -\frac{1}{8D^2}\left[A^{\phi}A^X\{48G_4+4(40G_{3,3}+28G_{3,4}+G_{3,5}+10G_{3,6}+100G_{3,7})\phi\}\right.\\
&\quad +(B^{\phi}A^X+A^{\phi}B^X)\{48G_4+2(56G_{3,3}+68G_{3,4}+5G_{3,5}+32G_{3,6}+140G_{3,7})\phi\}\\
&\quad +B^{\phi}B^X\{12G_4+(136G_{3,3}+124G_{3,4}+25G_{3,5}+70G_{3,6}+196G_{3,7})\phi\}\\
&\quad -4DA^{\phi}(6G_{4X}+G_{3,8X}\phi+10G_{3,9X}\phi)\\
&\quad +2DB^{\phi}(6G_{4X}-5G_{3,8X}\phi-14G_{3,9X}\phi)\\
&\quad -4DA^X\{6G_{4\phi}-G_{3,1}-10G_{3,2}+G_{3,8}+10G_{3,9}+(G_{3,8\phi}+10G_{3,9\phi})\phi\}\\
&\quad \left.+2DB^X\{6G_{4\phi}+5G_{3,1}+14G_{3,2}-5G_{3,8}-14G_{3,9}-(5G_{3,8\phi}+14G_{3,9\phi})\phi\}\right],
\end{aligned} \tag{6.103}$$

$$\begin{aligned}
E_{XX} &= \frac{1}{16D^2}\left[4A^{X2}\{12G_4+(40G_{3,3}+28G_{3,4}+G_{3,5}+10G_{3,6}+100G_{3,7})\phi\}\right.\\
&\quad +B^{X2}\{12G_4+(136G_{3,3}+124G_{3,4}+25G_{3,5}+70G_{3,6}+196G_{3,7})\phi\}\\
&\quad +4A^XB^X\{24G_4+(56G_{3,3}+68G_{3,4}+5G_{3,5}+32G_{3,6}+140G_{3,7})\phi\}\\
&\quad -8DA^X\{6G_{4X}+(G_{3,8X}+10G_{3,9X})\phi\}\\
&\quad \left.+4DB^X\{6G_{4X}-(5G_{3,8X}+14G_{3,9X})\phi\}\right].
\end{aligned} \tag{6.104}$$

Again, indeed under $G_{3,i}=0$, this reduces to the action (6.73).

Notice that this action does not satisfy the degeneracy conditions given as (2.450). Therefore this theory, in general, has Ostrogradsky ghost, and thus unstable. However, thanks to the numerous number of free functions, one may tune the functions such that the degeneracy conditions hold. As for the theory (6.101), the degeneracy conditions boil down to,

$$E_{XX}=\frac{3G_{4X}^2}{2G_4}, \tag{6.105}$$

which will make the theory have at most 2 tensor and 1 scalar degrees of freedom. Again such theory is a qDHOST of class $^2$N-I/Ia [122, 123].

The cosmological perturbation of this theory is tedious but calculable. The tensor perturbation of (6.101) , under the degeneracy condition (6.105), becomes (6.50). Thus the sound velocity of tensor perturbation becomes unity as a result of the ghost-free condition.

As for the scalar perturbations, redefining the functions as,

$$f = G_4, \tag{6.106}$$

$$Q = G_{3,0}+\int E_{\phi X}dX, \tag{6.107}$$

$$A_4 = \frac{3G_{4X}^2}{2G_4}=E_{XX}, \tag{6.108}$$

the action (6.101) becomes the DHOST action given in (6.74). Computation of such action is lengthy, but nonetheless possible, and one may obtain the form of (6.81) and (6.82) similar to the calculations in §6.1

## 6.4 Summary

In this part of the thesis, cosmological perturbation of $\mathcal{L}_3$ (6.94) and $\mathcal{L}_4$ (6.1) were investigated in different frames. It was shown that there are three frames; Einstein, Jordan, and Riemann. In cosmological perturbation theory, each frame computes the same quadratic action and observables up to the redefinition of the variables. The speed of gravitational waves of $\mathcal{L}_4$ theory is unity which differs from its metric formalism counterpart. Furthermore, $\mathcal{L}_3$ too takes part in effecting the velocity of gravitational waves, unlike in metric formalism. Although $\mathcal{L}_3$ is, in general, plagued with Ostrogradsky ghost, once tuned to be eliminated, the speed of tensor perturbation reduces back to unity. Thus both of these theories can be candidates of dark energy as well as inflation.

# Chapter 7

# Conclusion

In this thesis, theoretical and observational aspects of cosmology in Palatini and metric-affine formalism were surveyed in the context of scalar-tensor theories.

In Chapter 2, the theoretical preliminaries of Standard Model of cosmology, its problems, and modified gravity as a solution were familiarized. First, the Friedmann equations from the Einstein equations were derived. Using these the evolution of the Hubble parameter and matter were computed. Then the dynamics of matter were questioned. It was shown that the equivalence principle computes the autoparallel equations and with assuming the integrability of the geodesics equations. Then whether such dynamics can be derived from matter was debated. Utilizing geometric optics, minimal coupling ansatz of standard matter follows the geodesic equations under the point particle limit.

The section preceding was discussions of the unsolved problems of the universe. Inflation, the accelerated expansion of the early universe, crack two major problems in big bang cosmology; the horizon and flatness problem. The slow-roll approximation was oriented in the context of canonical inflaton, and linear perturbation employing gauge invariant variables was achieved. Selecting the vacuum carefully, one was able to quantize the inflation and obtain the observational variables; spectral index and tensor-to-scalar ratio. Then dark energy and its observational evidence were delivered. It was implied that, if this dark energy comes from vacuum energy, one requires extreme fine-tuning. Then certain recent observational results were briefly noted. Finally, the section ends with mentioning other problems that are relevant for comprehending the universe, such as reheating and baryogenesis.

The following section was to clarify the notion of Einstein gravity, which is a combination of Einstein equations and geodesic equations. It was demonstrated that for a certain number of assumptions the Einstein equations are unique, which is called the Lovelock theorem. Then, based on the theorem, a minimal extension of Riemann geometry, metric-affine geometry was proposed. Such geometry has a non-Levi-Civita connection and has torsion and non-metricity. After geometrical and algebraic preliminaries, methods of embedding such geometry onto a theory were presented; Palatini and metric-affine formalism. As an example, variation of the Einstein-Hilbert action with respect to the Palatini formalism was executed showing that the results match that of the metric formalism up to gauge freedom. Such gauge freedom is called projective symmetry and had numerous properties. Based on this analysis, metric-affine gravity was classified into certain different models. Meanwhile, it was clarified that the notion of point particle dynamics in metric-affine gravity is not of auto-parallels but of geodesics. Such differences may arise from quantum properties.

The final section was dedicated to the explanation of scalar-tensor theories. It was first remarked that one cannot just construct arbitrary scalar-tensor theories due to an instability that appears from more than 2nd derivatives; the Ostrogradsky instability. Respecting this, Galileons were shown that, due to well-tuned symmetries, have second-order in derivatives. The generalization was extended further to Horndeski theories, which is the most general theory with a second-order equation of motion for the metric and a scalar. On the contrary, theories can evade such theorems through degeneracy conditions and assembling further scalar-tensor theories as seen in DHOST and U-DHOST theories.

In Chapter 3, inflationary scenarios in metric-affine gravity were explored. First, noticing that covariantization is non-trivial in metric-affine gravity, one can construct a "minimal" coupled scalar field theory that differs from the metric gravity counterpart. After the classification of these models, the connection for each model was solved, and once substituted in the action the Riemann frame was acquired. Such frame is written in Riemann geometrical variables and the dynamics are equivalent to that of the original frame. Then for each of the models, the possibility of a viable inflationary scenario was surveyed. A chaotic inflation model, which is excluded from observations for the metric gravity case, can be resurrected in some models of metric-affine gravity. Finally, other actions such as G-inflation and extended d'Alembertians were analyzed and their observational variables were calculated respectively.

In Chapter 4, Galileons and generalized Galileon that respects projective symmetry were constructed in metric-affine formalism. After a brief review of non-trivial covariantizations of Galileons in the metric formalism, a complete and unique covariantization in the metric-affine formalism was constructed. Solving the connection, one obtains a Riemann frame which is neither covariant Galileon nor covariantized Galileon, implying that there is another Galileon in curved space than the former two. Based on this result, the generalization of Galileons in metric-affine formalism was examined. It was shown that a certain simple projective invariant metric-affine scalar-tensor theory computes class $^2$N-I/Ia of DHOST. This shines a light on the geometrical description of DHOST and its degeneracy conditions. After analyzing some specific models, the most general projective invariant scalar-tensor theory up to quadratic order in connection was built.

In Chapter 5, further analysis of projective invariant metric-affine scalar-tensor theories was completed. Recalling that projective invariance does not necessarily protect the theories from ghost instability, if one takes the unitary gauge, however, it looks like there are no degrees of freedom besides the initial 3, which are classes of U-DHOST. After showing how the projective mode eats the ghost in the unitary gauge, models were classified through trivially U-degenerate theories and their transformations thereof. Finally, for a wide class of theories, Hamiltonian analysis was conducted showing that indeed there are no hidden ghosts in the unitary gauge.

In Chapter 6, cosmological perturbations in Palatini formalism and their consistency among different frames were investigated. First considering a simple theory with K-essence and a non-minimal coupled Ricci scalar, it was shown that there are three possible frames to calculate upon. One is the original frame called the Jordan frame. Another is obtained through the conformal transformation of the metric, which is called the Einstein frame with gravity being purely Einsteinian. The third one is the Riemann frame. It was shown that, although the methodology of cosmological perturbations differs, the final result of observational variables are the same and thus consistent. Based on these results, the most general L3 Galileons were constructed. In general, such a theory is plagued with a ghost. However, it is possible to exorcise the ghost which also enforces the speed of gravitational waves to be that of light, which eludes the current constraints.

To conclude, this thesis and the works therein is the first to shine light to the construction of ghost-free scalar tensor theories and their cosmology in the context of Palatini and metric-affine formalism. Such results unify the approaches of past literature and serves to show a deeper connection between extensions of geometry and ghost-free properties of alternative theories of gravity.

Investigations of scalar-tensor theories in Palatini and metric-affine formalism are far from complete, however. Since, although this thesis has shown how to construct certain ghost-free scalar-tensor theories, it is still far from 'the most general' theory. Whether such theory, for example the most general theory with up to second-order in derivatives in it's equations of motion, exists and how much it differs from it's metric formalism counterpart is a question worth pursuing. Furthermore, phenomenology can be expected to differ from the usual metric formalism models that can be probed through current or future observations. Nonetheless, it can be said that our quest has just begun.

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# Index

# Appendix

## Appendix A

# Useful Formulae

### A.1 Some Formulae in Metric-affine Gravity

For an arbitrary scalar $\phi$, an arbitrary vector $A^\mu$ and an arbitrary tensor $B^\mu{}_\nu$

$$[\overset{\Gamma}{\nabla}_\mu, \overset{\Gamma}{\nabla}_\nu]\phi = -\mathcal{T}^\lambda_{\mu\nu}\partial_\lambda\phi \tag{A.1}$$

$$[\overset{\Gamma}{\nabla}_\mu, \overset{\Gamma}{\nabla}_\nu]A^\lambda = \overset{\Gamma}{R}{}^\lambda{}_{\rho\mu\nu}A^\rho - \mathcal{T}^\sigma_{\mu\nu}\overset{\Gamma}{\nabla}_\sigma A^\lambda \tag{A.2}$$

$$[\overset{\Gamma}{\nabla}_\mu, \overset{\Gamma}{\nabla}_\nu]B^\lambda{}_\sigma = \overset{\Gamma}{R}{}^\lambda{}_{\rho\mu\nu}B^\rho{}_\sigma - \overset{\Gamma}{R}{}^\rho{}_{\sigma\mu\nu}B^\lambda{}_\rho - \mathcal{T}^\rho_{\mu\nu}\overset{\Gamma}{\nabla}_\rho B^\lambda{}_\sigma \tag{A.3}$$

Using this and the Jacobi identity one finds the following three Bianchi identities for the metric-affine Riemann curvature,

$$\text{Zeroth:} \quad \overset{\Gamma}{R}{}^{(\alpha\beta)}{}_{\mu\nu} = \overset{\Gamma}{\nabla}_{[\mu}\mathcal{Q}_{\nu]}{}^{\alpha\beta} + \frac{1}{2}T^\sigma{}_{\mu\nu}\mathcal{Q}_\sigma{}^{\alpha\beta} \tag{A.4}$$

$$\text{First:} \quad \overset{\Gamma}{R}{}^\lambda{}_{[\alpha\beta\gamma]} = \overset{\Gamma}{\nabla}_{[\alpha}\mathcal{T}^\lambda{}_{\beta\gamma]} - \mathcal{T}^\sigma{}_{[\alpha\beta}\mathcal{T}^\lambda{}_{\gamma]\sigma}, \tag{A.5}$$

$$\text{Second:} \quad \overset{\Gamma}{\nabla}_{[\alpha}\overset{\Gamma}{R}{}^\mu{}_{|\nu|\beta\gamma]} = \mathcal{T}^\lambda{}_{[\alpha\beta}\overset{\Gamma}{R}{}^\mu{}_{|\nu|\gamma]\lambda} \tag{A.6}$$

The Ricci tensor, co-Ricci tensor, and Homothetic tensor can be related as,

$$\overset{\Gamma}{R}_{[\mu\nu]} = \frac{1}{2}\overset{\Gamma}{H}_{\mu\nu} - \overset{\Gamma}{\nabla}_{[\mu}\mathcal{T}_{\nu]} - \frac{1}{2}\overset{\Gamma}{\nabla}_\lambda\mathcal{T}^\lambda_{\mu\nu} + \mathcal{T}^\sigma_{\rho[\mu}\mathcal{T}^\rho_{\nu]\sigma} - \frac{1}{2}\mathcal{T}^\lambda_{\mu\nu}\mathcal{T}_\lambda \tag{A.7}$$

$$\overset{\Gamma}{R}{}^\mu{}_\nu = \overset{\Gamma}{P}{}^\mu{}_\nu + \overset{\Gamma}{\nabla}_\lambda\mathcal{Q}_\nu{}^{\lambda\mu} - \overset{\Gamma}{\nabla}_\nu\mathcal{Q}^\mu - \mathcal{T}^\lambda_{\nu\sigma}\mathcal{Q}_\lambda{}^{\mu\sigma} \tag{A.8}$$

$$\overset{\Gamma}{H}_{\mu\nu} = 4\overset{\Gamma}{\nabla}_{[\mu}\mathcal{W}_{\nu]} + 2\mathcal{T}^\lambda_{\mu\nu}\mathcal{W}_\lambda \tag{A.9}$$

where it was defined,

$$\mathcal{T}_\mu := \mathcal{T}^\lambda{}_{\mu\lambda}\,, \tag{A.10}$$

$$\mathcal{Q}_\mu := \mathcal{Q}^\lambda{}_{\mu\lambda}\,, \tag{A.11}$$

$$\mathcal{W}_\mu := \frac{1}{4}\mathcal{Q}_\mu{}^\lambda{}_\lambda \tag{A.12}$$

For the variation $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$ $\Gamma^\lambda{}_{\mu\nu} \to \Gamma^\lambda{}_{\mu\nu} + \delta\Gamma^\lambda{}_{\mu\nu}$, the geometrical quantities vary as,

$$\begin{aligned}
\delta\mathcal{T}^\lambda_{\mu\nu} &= 2\delta\Gamma^\lambda{}_{[\mu\nu]} && \text{(A.13)}\\
&= 2\delta^\lambda_\alpha\delta^\beta_{[\mu}\delta^\gamma_{\nu]}\delta\Gamma^\alpha{}_{\beta\gamma}\,, && \text{(A.14)}\\
\delta\mathcal{Q}_\lambda^{\ \mu\nu} &= \overset{\Gamma}{\nabla}_\lambda\delta g^{\mu\nu} + 2g^{\rho(\mu}\delta\Gamma^{\nu)}{}_{\lambda\rho} && \text{(A.15)}\\
&= \overset{\Gamma}{\nabla}_\lambda\delta g^{\mu\nu} + 2g^{\gamma(\mu}\delta^{\nu)}_\alpha\delta^\beta_\lambda\delta\Gamma^\alpha{}_{\beta\gamma}\,, && \text{(A.16)}\\
\delta\overset{\Gamma}{R}{}^\lambda{}_{\sigma\mu\nu} &= 2\overset{\Gamma}{\nabla}_{[\mu}\delta\Gamma^\lambda_{\nu]\sigma} + \mathcal{T}^\rho_{\mu\nu}\delta\Gamma^\lambda{}_{\rho\sigma} && \text{(A.17)}\\
&= \left(-2\delta^\lambda_\alpha\delta^\gamma_\sigma\delta^\lambda_{[\mu}\overset{\Gamma}{\nabla}_{\nu]} + \delta^\lambda_\alpha\delta^\gamma_\sigma\mathcal{T}^\beta_{\mu\nu}\right)\delta\Gamma^\alpha{}_{\beta\gamma}\,. && \text{(A.18)}
\end{aligned}$$

The Lie derivative of an affine connection is[1]

$$\begin{aligned}
\pounds_\xi\Gamma^\lambda{}_{\mu\nu} &= \xi^\alpha\partial_\alpha\Gamma^\lambda{}_{\mu\nu} - \Gamma^\sigma{}_{\mu\nu}\partial_\sigma\xi^\lambda + \Gamma^\lambda{}_{\sigma\nu}\partial_\mu\xi^\sigma + \Gamma^\lambda{}_{\mu\sigma}\partial_\nu\xi^\sigma + \partial_\mu\partial_\nu\xi^\lambda && \text{(A.19)}\\
&= \overset{\Gamma}{\nabla}_\mu\overset{\Gamma}{\nabla}_\nu\xi^\lambda - \overset{\Gamma}{\nabla}_\mu\left(\mathcal{T}^\lambda_{\nu\rho}\xi^\rho\right) - \overset{\Gamma}{R}{}^\lambda{}_{\nu\mu\rho}\xi^\rho && \text{(A.20)}
\end{aligned}$$

## A.2 Distortion Trick

### A.2.1 Introducing Distorsion

The connection, unlike the metric, is not a tensor. However, the difference of two connections is a tensor. Therefore, consider taking the Levi-Civita connection $\left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\}$ of some metric $g_{\mu\nu}$ as a 'reference connection' and define the distortion *tensor* as,

$$\kappa^\lambda{}_{\mu\nu} \;:=\; \Gamma^\lambda{}_{\mu\nu} - \left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\}, \tag{A.21}$$

[1] This can be derived both from using, the formal method of coordinate transformation

$$\Gamma^\lambda{}_{(\mu\nu)}(x^\mu) = \frac{\partial x^\lambda}{\partial X^\alpha}\frac{\partial X^\beta}{\partial x^\mu}\frac{\partial X^\gamma}{\partial x^\nu}\Gamma^\alpha{}_{(\beta\gamma)}(X^\mu) + \frac{\partial^2 X^\alpha}{\partial x^\mu\partial x^\nu}\frac{\partial x^\lambda}{\partial X^\alpha}\,,$$

or noticing that distortion $\kappa = \Gamma - \{\}$ is a tensor and thus

$$\pounds_\xi\kappa^\lambda{}_{\mu\nu} = \xi^\alpha\partial_\alpha\kappa^\lambda{}_{\mu\nu} - \kappa^\sigma{}_{\mu\nu}\partial_\sigma\xi^\lambda + \kappa^\lambda{}_{\sigma\nu}\partial_\mu\xi^\sigma + \kappa^\lambda{}_{\mu\sigma}\partial_\nu\xi^\sigma\,,$$

and

$$\begin{aligned}
\pounds_\xi\left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\} &= \frac{1}{2}g^{\lambda\sigma}\left(\overset{g}{\nabla}_\mu\pounds_\xi g_{\nu\sigma} + \overset{g}{\nabla}_\nu\pounds_\xi g_{\mu\sigma} - \overset{g}{\nabla}_\sigma\pounds_\xi g_{\mu\nu}\right)\\
&= \overset{g}{\nabla}_{(\mu}\overset{g}{\nabla}_{\nu)}\xi^\lambda - \overset{g}{R}{}^\lambda{}_{(\mu\nu)\rho}\xi^\rho
\end{aligned}$$

for any affine connection $\Gamma^{\lambda}{}_{\mu\nu}$.

On the other-hand, recall that the definition of torsion, non-metricity and curvature tensors were

$$\begin{aligned} \mathcal{T}^{\lambda}_{\mu\nu} &:= \Gamma^{\lambda}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\mu\nu}\,, && \text{(A.22)} \\ \mathcal{Q}_{\lambda}^{\mu\nu} &:= \overset{\Gamma}{\nabla}_{\lambda} g^{\mu\nu}\,, && \text{(A.23)} \\ R^{\alpha}{}_{\beta\mu\nu} A^{\beta} &:= \left[\overset{\Gamma}{\nabla}_{\mu}, \overset{\Gamma}{\nabla}_{\nu}\right] A^{\alpha} + \mathcal{T}^{\beta}_{\mu\nu} \overset{\Gamma}{\nabla}_{\beta} A^{\alpha}\,, && \text{(A.24)} \end{aligned}$$

where $A^{\alpha}$ is some arbitrary vector.

The relation between these geometrical tensors $\mathcal{T}$,$\mathcal{Q}$ and $\overset{\Gamma}{R}$ and the distortion tensor $\kappa$ is

$$\begin{aligned} \kappa^{\lambda}{}_{\mu\nu} &= \frac{1}{2}\left(\mathcal{T}^{\lambda}{}_{\mu\nu} + \mathcal{T}_{\nu}{}^{\lambda}{}_{\mu} - \mathcal{T}_{\mu\nu}{}^{\lambda} + \mathcal{Q}_{\nu}{}^{\lambda}{}_{\mu} + \mathcal{Q}_{\mu\nu}{}^{\lambda} - \mathcal{Q}^{\lambda}{}_{\mu\nu}\right)\,, && \text{(A.25)} \\ \mathcal{T}^{\lambda}_{\mu\nu} &= \kappa^{\lambda}{}_{\mu\nu} - \kappa^{\lambda}{}_{\nu\mu}\,, && \text{(A.26)} \\ \mathcal{Q}_{\lambda}^{\mu\nu} &= \kappa^{\mu}{}_{\lambda}{}^{\nu} + \kappa^{\nu}{}_{\lambda}{}^{\mu}\,, && \text{(A.27)} \\ \overset{\Gamma}{R}{}^{\alpha}{}_{\beta\mu\nu} &= \overset{g}{R}{}^{\alpha}{}_{\beta\mu\nu} + \overset{g}{\nabla}_{\mu}\kappa^{\alpha}{}_{\nu\beta} - \overset{g}{\nabla}_{\nu}\kappa^{\alpha}{}_{\mu\beta} + \kappa^{\alpha}{}_{\mu\lambda}\kappa^{\lambda}{}_{\nu\beta} - \kappa^{\alpha}{}_{\nu\lambda}\kappa^{\lambda}{}_{\mu\beta}\,. && \text{(A.28)} \end{aligned}$$

Using the above, it is straightforward to write the equations in terms of $\mathcal{T}$ and $\mathcal{Q}$.

### A.2.2 Integrating by Parts in Metric-affine Formalism

In general, due to metric incompatibility, a general affine-connection does not satisfy the usual integration by parts formula of

$$\int d^4x\sqrt{-g}\left[A^{\alpha\beta\gamma\cdots}{}_{\alpha'\beta'\gamma'\cdots}\overset{\Gamma}{\nabla}_{\mu}B^{\mu\alpha'\beta'\gamma'\cdots}{}_{\alpha\beta\gamma\cdots}\right] \neq -\int d^4x\sqrt{-g}\left[B^{\mu\alpha'\beta'\gamma'\cdots}{}_{\alpha\beta\gamma\cdots}\overset{\Gamma}{\nabla}_{\mu}A^{\alpha\beta\gamma\cdots}{}_{\alpha'\beta'\gamma'\cdots}\right] \qquad \text{(A.29)}$$

for some arbitrary tensor $A$ and $B$. Using the distortion trick and the general knowledge of integration by parts in Riemann geometry of,

$$\int d^4x\sqrt{-g}\left[A^{\alpha\beta\gamma\cdots}{}_{\alpha'\beta'\gamma'\cdots}\overset{g}{\nabla}_{\mu}B^{\mu\alpha'\beta'\gamma'\cdots}{}_{\alpha\beta\gamma\cdots}\right] = -\int d^4x\sqrt{-g}\left[B^{\mu\alpha'\beta'\gamma'\cdots}{}_{\alpha\beta\gamma\cdots}\overset{g}{\nabla}_{\mu}A^{\alpha\beta\gamma\cdots}{}_{\alpha'\beta'\gamma'\cdots}\right]\,, \qquad \text{(A.30)}$$

it will be shown how the integration by parts in metric-affine geometry are conducted.

Straightforwardly decomposing the general affine connection into the Levi-Civita connection and distorsion, the following can be calculated.

$$
\begin{aligned}
&\int d^4x\sqrt{-g}\left[A^{\alpha\beta\gamma\cdots}_{\alpha'\beta'\gamma'\cdots}\overset{\Gamma}{\nabla}_\mu B^{\mu\alpha'\beta'\gamma'\cdots}_{\alpha\beta\gamma\cdots}\right] \\
= &\int d^4x\sqrt{-g}\left[A^{\alpha\beta\gamma\cdots}_{\alpha'\beta'\gamma'\cdots}\overset{g}{\nabla}_\mu B^{\mu\alpha'\beta'\gamma'\cdots}_{\alpha\beta\gamma\cdots} + \kappa^\mu{}_{\mu\sigma}A^{\alpha\beta\gamma\cdots}_{\alpha'\beta'\gamma'\cdots}B^{\sigma\alpha'\beta'\gamma'\cdots}_{\alpha\beta\gamma\cdots}\right. \\
&\qquad -\kappa^\sigma{}_{\mu\alpha}A^{\alpha\beta\gamma\cdots}_{\alpha'\beta'\gamma'\cdots}B^{\mu\alpha'\beta'\gamma'\cdots}_{\sigma\beta\gamma\cdots} - \kappa^\sigma{}_{\mu\beta}A^{\alpha\beta\gamma\cdots}_{\alpha'\beta'\gamma'\cdots}B^{\mu\alpha'\beta'\gamma'\cdots}_{\alpha\sigma\gamma\cdots} - \cdots \\
&\qquad \left. +\kappa^{\alpha'}{}_{\mu\sigma}A^{\alpha\beta\gamma\cdots}_{\alpha\beta'\gamma'\cdots}B^{\mu\sigma\beta'\gamma'\cdots}_{\alpha\beta\gamma\cdots} + \kappa^{\beta'}{}_{\mu\sigma}A^{\alpha\beta\gamma\cdots}_{\alpha\beta'\gamma'\cdots}B^{\mu\alpha'\sigma\gamma'\cdots}_{\alpha\beta\gamma\cdots} + \cdots\right] \\
= &\int d^4x\sqrt{-g}\left[-B^{\mu\alpha'\beta'\gamma'\cdots}_{\alpha\beta\gamma\cdots}\overset{g}{\nabla}_\mu A^{\alpha\beta\gamma\cdots}_{\alpha'\beta'\gamma'\cdots} + \kappa^\mu{}_{\mu\sigma}A^{\alpha\beta\gamma\cdots}_{\alpha'\beta'\gamma'\cdots}B^{\sigma\alpha'\beta'\gamma'\cdots}_{\alpha\beta\gamma\cdots}\right. \\
&\qquad -\kappa^\sigma{}_{\mu\alpha}A^{\alpha\beta\gamma\cdots}_{\alpha'\beta'\gamma'\cdots}B^{\mu\alpha'\beta'\gamma'\cdots}_{\sigma\beta\gamma\cdots} - \kappa^\sigma{}_{\mu\beta}A^{\alpha\beta\gamma\cdots}_{\alpha'\beta'\gamma'\cdots}B^{\mu\alpha'\beta'\gamma'\cdots}_{\alpha\sigma\gamma\cdots} - \cdots \\
&\qquad \left. +\kappa^{\alpha'}{}_{\mu\sigma}A^{\alpha\beta\gamma\cdots}_{\alpha\beta'\gamma'\cdots}B^{\mu\sigma\beta'\gamma'\cdots}_{\alpha\beta\gamma\cdots} + \kappa^{\beta'}{}_{\mu\sigma}A^{\alpha\beta\gamma\cdots}_{\alpha\beta'\gamma'\cdots}B^{\mu\alpha'\sigma\gamma'\cdots}_{\alpha\beta\gamma\cdots} + \cdots\right] \\
= &\int d^4x\sqrt{-g}\left[-B^{\mu\alpha'\beta'\gamma'\cdots}_{\alpha\beta\gamma\cdots}\overset{\Gamma}{\nabla}_\mu A^{\alpha\beta\gamma\cdots}_{\alpha'\beta'\gamma'\cdots} + \kappa^\mu{}_{\mu\sigma}A^{\alpha\beta\gamma\cdots}_{\alpha'\beta'\gamma'\cdots}B^{\sigma\alpha'\beta'\gamma'\cdots}_{\alpha\beta\gamma\cdots}\right. \\
&\qquad -\kappa^\sigma{}_{\mu\alpha}A^{\alpha\beta\gamma\cdots}_{\alpha'\beta'\gamma'\cdots}B^{\mu\alpha'\beta'\gamma'\cdots}_{\sigma\beta\gamma\cdots} - \kappa^\sigma{}_{\mu\beta}A^{\alpha\beta\gamma\cdots}_{\alpha'\beta'\gamma'\cdots}B^{\mu\alpha'\beta'\gamma'\cdots}_{\alpha\sigma\gamma\cdots} - \cdots \\
&\qquad +\kappa^{\alpha'}{}_{\mu\sigma}A^{\alpha\beta\gamma\cdots}_{\alpha\beta'\gamma'\cdots}B^{\mu\sigma\beta'\gamma'\cdots}_{\alpha\beta\gamma\cdots} + \kappa^{\beta'}{}_{\mu\sigma}A^{\alpha\beta\gamma\cdots}_{\alpha\beta'\gamma'\cdots}B^{\mu\alpha'\sigma\gamma'\cdots}_{\alpha\beta\gamma\cdots} + \cdots \\
&\qquad +\kappa^{\alpha}{}_{\mu\sigma}A^{\sigma\beta\gamma\cdots}_{\alpha'\beta'\gamma'\cdots}B^{\mu\alpha'\beta'\gamma'\cdots}_{\alpha\beta\gamma\cdots} + \kappa^{\beta}{}_{\mu\sigma}A^{\alpha\sigma\gamma\cdots}_{\alpha'\beta'\gamma'\cdots}B^{\mu\alpha'\beta'\gamma'\cdots}_{\alpha\beta\gamma\cdots} + \cdots \\
&\qquad \left. -\kappa^{\sigma}{}_{\mu\alpha'}A^{\alpha\beta\gamma\cdots}_{\sigma\beta'\gamma'\cdots}B^{\mu\alpha'\beta'\gamma'\cdots}_{\alpha\beta\gamma\cdots} - \kappa^{\sigma}{}_{\mu\beta'}A^{\alpha\beta\gamma\cdots}_{\alpha'\sigma\gamma'\cdots}B^{\mu\alpha'\beta'\gamma'\cdots}_{\alpha\beta\gamma\cdots} - \cdots\right] \\
= &\int d^4x\sqrt{-g}\left[-B^{\mu\alpha'\beta'\gamma'\cdots}_{\alpha\beta\gamma\cdots}\overset{\Gamma}{\nabla}_\mu A^{\alpha\beta\gamma\cdots}_{\alpha'\beta'\gamma'\cdots} + \kappa^\mu{}_{\mu\sigma}A^{\alpha\beta\gamma\cdots}_{\alpha'\beta'\gamma'\cdots}B^{\sigma\alpha'\beta'\gamma'\cdots}_{\alpha\beta\gamma\cdots}\right] \\
= &\int d^4x\sqrt{-g}\left[-B^{\mu\alpha'\beta'\gamma'\cdots}_{\alpha\beta\gamma\cdots}\left(\overset{\Gamma}{\nabla}_\mu - 2\mathcal{W}_\mu + \mathcal{T}_\mu\right)A^{\alpha\beta\gamma\cdots}_{\alpha'\beta'\gamma'\cdots}\right].
\end{aligned}
$$

Recall that the Weyl vector and the torsion vector can be written as,

$$
\begin{aligned}
W_\mu = &\quad \tfrac{1}{4}g_{\alpha\beta}\mathcal{Q}_\mu{}^{\alpha\beta} &= \frac{1}{2}\kappa^\sigma{}_{\mu\sigma}. \\
\mathcal{T}_\mu = &\quad \mathcal{T}^\lambda_{\mu\lambda} &= \kappa^\lambda{}_{\mu\lambda} - \kappa^\lambda{}_{\lambda\mu}
\end{aligned}
$$

### A.2.3 Computing Torsionless Gravity

Consider the torsionless case $\mathcal{T} = 0$ and consider the Einstein-Hilbert action $\mathcal{L} = \overset{\Gamma}{R} = g^{\beta\nu}\overset{\Gamma}{R}{}^\alpha{}_{\beta\alpha\nu}$ ,

$$
\overset{\Gamma}{R} = \overset{g}{R} + 4\mathcal{Q}_{\lambda\mu\nu}\mathcal{Q}^{\lambda\mu\nu} - 2\mathcal{Q}_{\lambda\mu\nu}\mathcal{Q}^{\mu\lambda\nu} + 2\mathcal{Q}_\mu\mathcal{W}^\mu - 4\mathcal{W}^\mu\mathcal{W}_\mu\,. \tag{A.31}
$$

The equation of motion of the connection under the torsionless constraint is

$$
\sqrt{-g}\frac{\delta S_g}{\delta\kappa^\alpha{}_{\beta'\gamma'}} \times \frac{1}{2}\left(\delta^\beta_{\beta'}\delta^\gamma_{\gamma'} + \delta^\gamma_{\beta'}\delta^\beta_{\gamma'}\right) \quad = \quad -\mathcal{Q}_\alpha{}^{\beta\gamma} + 2\mathcal{Q}_\alpha g^{\beta\gamma} - 2\mathcal{W}^{(\beta}\delta^{\gamma)}_\alpha + \mathcal{Q}^{(\beta}\delta^{\gamma)}_\alpha\,, \tag{A.32}
$$

where the symmetry of the last two indices of the connection was taken into account.

Now by varying the action (A.31) with respect of the non-metricity tensor results as,

$$\sqrt{-g}\frac{\delta S_g}{\delta \mathcal{Q}_\mu{}^{\nu\lambda}} = \frac{1}{2}\mathcal{Q}^\mu{}_{\nu\lambda} - \frac{1}{4}\mathcal{Q}_{(\lambda\nu)}{}^\mu + \frac{1}{2}g_{\nu\lambda}\mathcal{Q}^\mu + 2\delta^\mu_{(\nu}\mathcal{W}_{\lambda)} - 2g_{\nu\lambda}\mathcal{W}^\mu \,. \tag{A.33}$$

These two could be connected via,

$$\sqrt{-g}\frac{\delta S_g}{\delta \mathcal{Q}_\mu{}^{\nu\lambda}} \times \left(\delta^\beta_\mu g^{\gamma(\nu} + \delta^\gamma_\mu g^{\beta(\nu}\right)\delta^{\lambda)}_\alpha = \sqrt{-g}\frac{\delta S_g}{\delta \kappa^\alpha{}_{\beta'\gamma'}} \times \frac{1}{2}\left(\delta^\beta_{\beta'}\delta^\gamma_{\gamma'} + \delta^\gamma_{\beta'}\delta^\beta_{\gamma'}\right) \,. \tag{A.34}$$

Because,

$$\begin{aligned}
\frac{\delta \mathcal{Q}_\mu{}^{\nu\lambda}}{\delta \kappa^\alpha{}_{\beta'\gamma'}}\frac{1}{2}\left(\delta^\beta_{\beta'}\delta^\gamma_{\gamma'} + \delta^\gamma_{\beta'}\delta^\beta_{\gamma'}\right) &= \delta^{\beta'}_\mu \delta^{(\lambda}_\alpha g^{\nu)\gamma'}\left(\delta^\beta_{\beta'}\delta^\gamma_{\gamma'} + \delta^\gamma_{\beta'}\delta^\beta_{\gamma'}\right) \,, \\
&= \left(\delta^\beta_\mu g^{\gamma(\nu} + \delta^\gamma_\mu g^{\beta(\nu}\right)\delta^{\lambda)}_\alpha \,.
\end{aligned}$$

When matter is taken into consideration, this symmetry should also hold as,

$$\frac{\delta S_M}{\delta \mathcal{Q}_\mu{}^{\nu\lambda}} \times \left(\delta^\beta_\mu g^{\gamma(\nu} + \delta^\gamma_\mu g^{\beta(\nu}\right)\delta^{\lambda)}_\alpha = \frac{\delta S_M}{\delta \kappa^\alpha{}_{\beta'\gamma'}} \,.$$

Furthermore, the equation of motion for the metric is,

$$\begin{aligned}
&\sqrt{-g}\frac{\delta S_g}{\delta g^{\mu\nu}} \\
=\; &\overset{\Gamma}{R}_{(\mu\nu)} - \frac{1}{2}g_{\mu\nu}\overset{\Gamma}{R} \\
=\; &G_{\mu\nu}(g,\{\}) + 2g_{\mu\nu}\mathcal{Q}^\lambda\mathcal{Q}_\lambda - g_{\mu\nu}\mathcal{W}^\mu\mathcal{Q}_\mu + 2\mathcal{W}^\alpha\mathcal{Q}_{(\mu\nu)\alpha} - \mathcal{W}^\alpha\mathcal{Q}_{\alpha\mu\nu} \\
&+4g_{\mu\nu}\mathcal{Q}^{\alpha\beta\gamma}\mathcal{Q}_{\beta\alpha\gamma} - \frac{1}{8}g_{\mu\nu}\mathcal{Q}^{\alpha\beta\gamma}\mathcal{Q}_{\alpha\beta\gamma} + \frac{1}{2}\mathcal{Q}^{\alpha\beta}{}_\mu\mathcal{Q}_{\alpha\beta\nu} \\
&-2\mathcal{Q}^{\alpha\beta}{}_\mu\mathcal{Q}_{\beta\alpha\nu} - 4\mathcal{Q}_{\mu\alpha\beta}\mathcal{Q}_\nu{}^{\alpha\beta} \,.
\end{aligned} \tag{A.35}$$

### A.2.4 Computing Metric-compatible Gravity

For $\mathcal{Q} = 0$ and $\mathcal{L}_g = \overset{\Gamma}{R}$ the Einstein-Hilbert could be written in terms of,

$$\overset{\Gamma}{R} = \overset{g}{R} - \mathcal{T}^\mu\mathcal{T}_\mu + 4\mathcal{T}_{\lambda\mu\nu}\mathcal{T}^{\lambda\mu\nu} + \frac{1}{2}\mathcal{T}_{\lambda\mu\nu}\mathcal{T}^{\mu\lambda\nu} \,. \tag{A.36}$$

The equation of motion of the connection under the torsionless constraint is

$$\sqrt{-g}\frac{\delta S_g}{\delta \kappa^{\alpha'}{}_{\beta\gamma'}} \times \frac{1}{2}\left(\delta^{\alpha'}_\alpha\delta^\gamma_{\gamma'} - g_{\alpha\gamma'}g^{\alpha'\gamma}\right) = \mathcal{T}^{\beta}_{\alpha}{}^{\gamma} - g^{\beta\gamma}\mathcal{T}_\alpha + \delta^\beta_\alpha\mathcal{T}^\gamma \,. \tag{A.37}$$

Now by varying the action (A.36) with respect of the non-metricity tensor results as,

$$\frac{1}{\sqrt{-g}}\frac{\delta S_g}{\delta \mathcal{T}^\lambda{}_{\mu\nu}} = \mathcal{T}_\lambda{}^{\mu\nu} - \mathcal{T}^{[\mu\nu]}{}_\lambda + \delta^{[\mu}_\lambda\mathcal{T}^{\nu]} \,, \tag{A.38}$$

These two could be connected via,

$$\frac{1}{\sqrt{-g}}\frac{\delta S_g}{\delta \mathcal{T}^{\lambda}{}_{\mu\nu}} \times \delta^{\beta}_{[\mu}\left(\delta^{\gamma}_{\nu]}\delta^{\lambda}_{\alpha} - g_{\nu]\alpha}g^{\lambda\gamma}\right) = \sqrt{-g}\frac{\delta S_g}{\delta \kappa^{\alpha'}{}_{\beta\gamma'}} \times \frac{1}{2}\left(\delta^{\alpha'}_{\alpha}\delta^{\gamma}_{\gamma'} - g_{\alpha\gamma'}g^{\alpha'\gamma}\right) . \tag{A.39}$$

The equation for the metric is

$$\begin{aligned} \sqrt{-g}\frac{\delta S_g}{\delta g^{\mu\nu}} &= \overset{\Gamma}{R}_{(\mu\nu)} - \frac{1}{2}g_{\mu\nu}\overset{\Gamma}{R} \\ &= \overset{g}{G}_{\mu\nu} + \frac{1}{2}g_{\mu\nu}\mathcal{T}^{\alpha}\mathcal{T}_{\alpha} + 2\mathcal{T}_{\lambda}\mathcal{T}_{(\mu\nu)}{}^{\lambda} \\ &\quad -\frac{1}{8}g_{\mu\nu}\mathcal{T}_{\alpha\beta\gamma}\mathcal{T}^{\alpha\beta\gamma} - \frac{1}{4}g_{\mu\nu}\mathcal{T}_{\alpha\beta\gamma}\mathcal{T}^{\beta\alpha\gamma} - \frac{1}{4}\mathcal{T}_{\alpha\beta(\mu}\mathcal{T}_{\nu)}{}^{\alpha\beta} + \frac{1}{4}\mathcal{T}_{\mu\alpha\beta}\mathcal{T}_{\nu}{}^{\alpha\beta} . \end{aligned} \tag{A.40}$$

## A.3 ADM Decomposition of the Connection

Using distorsion, the decomposition of a 3-rank tensor with respect to ADM formalism will be shown.

For a time-like norm-vector $n^{\mu}$, where $g_{\mu\nu}n^{\mu}n^{\nu} = -1$, orthogonal to a surface $\Sigma$, the following projection $\gamma^{\mu}{}_{\nu}$ can be defined,

$$\gamma^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + n^{\mu}n_{\nu} . \tag{A.41}$$

The distorsion tensor can be then decomposed as,

$$\kappa_{*} := \kappa_{\alpha\beta\gamma}n^{\alpha}n^{\beta}n^{\gamma} , \tag{A.42}$$
$$\hat{\kappa}^{1}_{\mu} := \kappa_{\alpha\beta\gamma}n^{\alpha}n^{\beta}\gamma^{\gamma}_{\mu} , \tag{A.43}$$
$$\hat{\kappa}^{2}_{\mu} := \kappa_{\alpha\beta\gamma}n^{\alpha}\gamma^{\beta}_{\mu}n^{\gamma} , \tag{A.44}$$
$$\hat{\kappa}^{3}_{\mu} := \kappa_{\alpha\beta\gamma}\gamma^{\alpha}_{\mu}n^{\beta}n^{\gamma} , \tag{A.45}$$
$$\hat{\kappa}^{1}_{\mu\nu} := \kappa_{\alpha\beta\gamma}n^{\alpha}\gamma^{\beta}_{\mu}\gamma^{\gamma}_{\nu} , \tag{A.46}$$
$$\hat{\kappa}^{2}_{\mu\nu} := \kappa_{\alpha\beta\gamma}\gamma^{\alpha}_{\mu}n^{\beta}\gamma^{\gamma}_{\nu} , \tag{A.47}$$
$$\hat{\kappa}^{3}_{\mu\nu} := \kappa_{\alpha\beta\gamma}\gamma^{\alpha}_{\mu}\gamma^{\beta}_{\nu}n^{\gamma} , \tag{A.48}$$
$$\hat{\kappa}_{\mu\nu\rho} := \kappa_{\alpha\beta\gamma}\gamma^{\alpha}_{\mu}\gamma^{\beta}_{\nu}\gamma^{\gamma}_{\rho} , \tag{A.49}$$

Note that under projective transformation, $\kappa_{\mu\nu\rho} \to \kappa_{\mu\nu\rho} + g_{\mu\nu}U_{\rho}$, the components transform as,

$$\kappa_{*} \to \kappa_{*} - U_{*} ,\ \hat{\kappa}^{3}_{\mu\nu} \to \hat{\kappa}^{3}_{\mu\nu} + \gamma_{\mu\nu}U_{*} , \hat{\kappa}^{1}_{\mu} \to \hat{\kappa}^{1}_{\mu} - \hat{U}_{\mu} ,\ \hat{\kappa}_{\mu\nu\rho} \to \hat{\kappa}_{\mu\nu\rho} + \gamma_{\mu\nu}\hat{U}_{\rho} ,$$

whereas the other components do not change. Here it was defined,

$$U_{*} := U_{\alpha}n^{\alpha} ,\ \hat{U}_{\mu} := U_{\alpha}\gamma^{\alpha}_{\mu} . \tag{A.50}$$

To decompose the Riemann curvature, it is useful to define the variables,

$$\overset{\Gamma}{\mathcal{K}}{}^{1}_{\mu\nu} := (\overset{\Gamma}{\nabla}_{\beta}n_{\alpha})\gamma^{\alpha}_{\mu}\gamma^{\beta}_{\nu} = K_{\mu\nu} - \hat{\kappa}^{1}_{\mu\nu} , \tag{A.51}$$
$$\overset{\Gamma}{\mathcal{K}}{}^{2}_{\mu\nu} := (\overset{\Gamma}{\nabla}_{\beta}n^{\alpha})\gamma_{\alpha\mu}\gamma^{\beta}_{\nu} = K_{\mu\nu} + \hat{\kappa}^{2}_{\mu\nu} , \tag{A.52}$$

which could be considered as the extrinsic curvature of metric-affine geometry.

### A.3.1 ADM Decomposition of the Riemann Curvature

With the tools in hand, all of the independent components of the $3+1$ decomposed curvature are as follows,

$$\overset{\Gamma}{R}_{\alpha\beta\gamma\delta} n^\alpha n^\beta n^\gamma \gamma^\delta_\rho = \pounds_n \hat{\kappa}^1_\rho - D_\rho \kappa_* - a_\rho \kappa_* + a^\gamma (\overset{\Gamma}{\mathcal{K}}{}^1_{\gamma\rho} - \overset{\Gamma}{\mathcal{K}}{}^2_{\gamma\rho}) + \overset{\Gamma}{\mathcal{K}}{}^{1\gamma}{}_\rho \hat{\kappa}^3_\gamma + \overset{\Gamma}{\mathcal{K}}{}^{2\gamma}{}_\rho \hat{\kappa}^2_\gamma \,, \tag{A.53}$$

$$\overset{\Gamma}{R}_{\alpha\beta\gamma\delta} n^\alpha \gamma^\beta_\mu n^\gamma \gamma^\delta_\rho = -\pounds_n \overset{\Gamma}{\mathcal{K}}{}^1_{\mu\rho} - D_\rho \hat{\kappa}^2_\mu + \overset{\Gamma}{\mathcal{K}}{}^1_{\alpha\rho} K^\alpha{}_\mu - a^\alpha \hat{\kappa}_{\alpha\mu\rho} - a_\mu \hat{\kappa}^1_\rho - a_\rho \hat{\kappa}^2_\mu + \overset{\Gamma}{\mathcal{K}}{}^1_{\mu\rho} \kappa_* + \overset{\Gamma}{\mathcal{K}}{}^{1\alpha}{}_\rho \hat{\kappa}^3_{\alpha\mu} + \hat{\kappa}^1_\rho \hat{\kappa}^2_\mu + \hat{\kappa}^\alpha{}_{\mu\rho} \hat{\kappa}^2_\alpha \tag{A.54}$$

$$\overset{\Gamma}{R}_{\alpha\beta\gamma\delta} \gamma^\alpha_\mu n^\beta n^\gamma \gamma^\delta_\rho = \pounds_n \overset{\Gamma}{\mathcal{K}}{}^2_{\mu\rho} - D_\rho \hat{\kappa}^3_\mu - \overset{\Gamma}{\mathcal{K}}{}^2_{\alpha\rho} K^\alpha{}_\mu - a^\alpha \hat{\kappa}_{\mu\alpha\rho} - a_\mu \hat{\kappa}^1_\rho - a_\rho \hat{\kappa}^3_\mu + \overset{\Gamma}{\mathcal{K}}{}^2_{\mu\rho} \kappa_* + \overset{\Gamma}{\mathcal{K}}{}^{2\alpha}{}_\rho \hat{\kappa}^3_{\mu\alpha} - \hat{\kappa}^1_\rho \hat{\kappa}^3_\mu - \hat{\kappa}_\mu{}^\alpha{}_\rho \hat{\kappa}^3_\alpha \,, \tag{A.55}$$

$$\begin{aligned} \overset{\Gamma}{R}_{\alpha\beta\gamma\delta} \gamma^\alpha_\mu \gamma^\beta_\nu n^\gamma \gamma^\delta_\rho = {} & \pounds_n \hat{\kappa}_{\mu\nu\rho} - D_\rho \hat{\kappa}^3_{\mu\nu} - 2 D_{[\mu} K_{\nu]\rho} - a_\mu (K_{\nu\rho} - \overset{\Gamma}{\mathcal{K}}{}^1_{\nu\rho}) + a_\nu (K_{\mu\rho} - \overset{\Gamma}{\mathcal{K}}{}^2_{\mu\rho}) - a_\rho \hat{\kappa}^3_{\mu\nu} \\ & + (\hat{\kappa}^{3\,\alpha}_\mu - K_\mu{}^\alpha) \hat{\kappa}_{\alpha\nu\rho} - (\hat{\kappa}^{3\alpha}{}_\nu + K^\alpha{}_\nu) \hat{\kappa}_{\mu\alpha\rho} + \overset{\Gamma}{\mathcal{K}}{}^1_{\nu\rho} \hat{\kappa}^3_\mu + \overset{\Gamma}{\mathcal{K}}{}^2_{\mu\rho} \hat{\kappa}^2_\nu \,, \end{aligned} \tag{A.56}$$

$$\overset{\Gamma}{R}_{\alpha\beta\gamma\delta} n^\alpha n^\beta \gamma^\gamma_\rho \gamma^\delta_\sigma = 2 D_{[\rho} \hat{\kappa}^1_{\sigma]} - 2 \overset{\Gamma}{\mathcal{K}}{}^{1\gamma}{}_{[\rho} \overset{\Gamma}{\mathcal{K}}{}^2_{|\gamma|\sigma]} \,, \tag{A.57}$$

$$\overset{\Gamma}{R}_{\alpha\beta\gamma\delta} n^\alpha \gamma^\beta_\mu \gamma^\gamma_\rho \gamma^\delta_\sigma = -2 D_{[\rho} \overset{\Gamma}{\mathcal{K}}{}^1_{|\mu|\sigma]} - 2 \overset{\Gamma}{\mathcal{K}}{}^1{}_{\mu[\rho} \hat{\kappa}^1_{\sigma]} - 2 \overset{\Gamma}{\mathcal{K}}{}^{1\alpha}{}_{[\rho} \hat{\kappa}_{|\alpha\mu|\sigma]} \,, \tag{A.58}$$

$$\overset{\Gamma}{R}_{\alpha\beta\gamma\delta} \gamma^\alpha_\mu n^\beta \gamma^\gamma_\rho \gamma^\delta_\sigma = 2 D_{[\rho} \overset{\Gamma}{\mathcal{K}}{}^2_{|\mu|\sigma]} - 2 \overset{\Gamma}{\mathcal{K}}{}^2_{\mu[\rho} \hat{\kappa}^1_{\sigma]} - 2 \overset{\Gamma}{\mathcal{K}}{}^{2\alpha}{}_{[\rho} \hat{\kappa}_{|\mu\alpha|\sigma]} \,, \tag{A.59}$$

$$\overset{\Gamma}{R}_{\alpha\beta\gamma\delta} \gamma^\alpha_\mu \gamma^\beta_\nu \gamma^\gamma_\rho \gamma^\delta_\sigma = \overset{\Gamma}{\mathcal{R}}_{\mu\nu\rho\sigma} + 2 \overset{\Gamma}{\mathcal{K}}{}^1_{\nu[\sigma} \overset{\Gamma}{\mathcal{K}}{}^2_{|\mu|\rho]} \,. \tag{A.60}$$

where

$$\overset{\Gamma}{\mathcal{R}}_{\mu\nu\rho\sigma} := \overset{\gamma}{\mathcal{R}}_{\mu\nu\rho\sigma} + 2 D_{[\rho} \hat{\kappa}_{\mu\nu|\sigma]} + 2 \hat{\kappa}_\mu{}^\alpha{}_{[\rho} \hat{\kappa}_{\alpha\nu|\sigma]} \tag{A.61}$$

and $\overset{\gamma}{\mathcal{R}}_{\mu\nu\rho\sigma}$ is the spatial curvature constructed by the spatial metric $\gamma_{\mu\nu}$.

Using this result, one can find that the non-minimal couplings (5.15) introduced in §5.2.1 can be computed as,

$$\overset{\Gamma}{G}{}^{\mu\nu} n_\mu n_\nu = \frac{1}{2} \left( \overset{\Gamma}{\mathcal{R}}{}^{\mu\nu}{}_{\mu\nu} + \overset{\Gamma}{\mathcal{K}}{}^{1\mu}{}_\mu \overset{\Gamma}{\mathcal{K}}{}^{2\mu}{}_\mu - \overset{\Gamma}{\mathcal{K}}{}^{1\mu\nu} \overset{\Gamma}{\mathcal{K}}{}^2_{\nu\mu} \right) , \tag{A.62}$$

$$\begin{aligned} \overset{\Gamma}{G}{}^{\mu\alpha\nu\beta} n_\mu n_\nu \gamma^{\alpha'}_\alpha \gamma^{\beta'}_\beta = \frac{1}{2} \Bigg[ & \overset{\Gamma}{\mathcal{K}}{}^{1\mu\alpha'} \overset{\Gamma}{\mathcal{K}}{}^{2\beta'}{}_\mu + \overset{\Gamma}{\mathcal{K}}{}^{1\beta'\mu} \overset{\Gamma}{\mathcal{K}}{}^2{}_\mu{}^{\alpha'} - \overset{\Gamma}{\mathcal{K}}{}^{1\mu}{}_\mu \overset{\Gamma}{\mathcal{K}}{}^{2\beta'\alpha'} - \overset{\Gamma}{\mathcal{K}}{}^{1\beta'\alpha'} \overset{\Gamma}{\mathcal{K}}{}^{2\mu}{}_\mu + \left( \overset{\Gamma}{\mathcal{K}}{}^{1\mu}{}_\mu \overset{\Gamma}{\mathcal{K}}{}^{2\mu}{}_\mu - \overset{\Gamma}{\mathcal{K}}{}^{1\mu\nu} \overset{\Gamma}{\mathcal{K}}{}^2_{\nu\mu} \right) \gamma^{\alpha'\beta'} \\ & - \overset{\Gamma}{\mathcal{R}}_\mu{}^{\beta'\mu\alpha'} - \overset{\Gamma}{\mathcal{R}}{}^{\beta'\mu\alpha'}{}_\mu + \overset{\Gamma}{\mathcal{R}}{}^{\mu\nu}{}_{\mu\nu} \gamma^{\alpha'\beta'} \Bigg] , \end{aligned} \tag{A.63}$$

which indeed show that they do not have time derivatives of the distorsion tensor in the unitary gauge.

## A.4 Irreducible Decomposition of the Riemann Tensor

The Riemann tensor, having anti-symmetry in the last two indices, have in total 96 components. In this section, the irreducible decomposition of the Riemann tensor will be introduced.

First, consider dividing the Riemann curvature tensor with respect to symmetric and anti-symmetric indices of its first two indices as,

$$Z_{\alpha\beta\gamma\delta} := \overset{\Gamma}{R}_{(\alpha\beta)\gamma\delta} \,, \tag{A.64}$$

$$W_{\alpha\beta\gamma\delta} := \overset{\Gamma}{R}_{[\alpha\beta]\gamma\delta} \tag{A.65}$$

with $Z_{\alpha\beta\gamma\delta}$ having 60 independent components and $W_{\alpha\beta\gamma\delta}$ having 36. Both tensors can then be decomposed irreducibly, with respect to trace and traceless parts, as,

$$\begin{aligned}
{}^{(1)}Z_{\alpha\beta\gamma\delta} &= \frac{1}{2}\left(\not{Z}_{\alpha\beta\gamma\delta} + \not{Z}_{[\delta|\beta|\gamma]\alpha} + \not{Z}_{[\delta|\alpha|\gamma]\beta}\right) - \frac{1}{12}\left(g_{\beta\delta}\not{Z}^{\lambda}{}_{[\alpha\gamma]\lambda} - g_{\beta\gamma}\not{Z}^{\lambda}{}_{[\alpha\delta]\lambda} + g_{\alpha\delta}\not{Z}^{\lambda}{}_{[\beta\gamma]\lambda} - g_{\alpha\gamma}\not{Z}^{\lambda}{}_{[\beta\delta]\lambda} + 2g_{\alpha\beta}\not{Z}^{\lambda}{}_{[\delta\gamma]\lambda}\right) \\
&\quad -\frac{1}{4}\left(g_{\beta\delta}\not{Z}^{\lambda}{}_{(\alpha\gamma)\lambda} + g_{\alpha\delta}\not{Z}^{\lambda}{}_{(\beta\gamma)\lambda} - g_{\beta\gamma}\not{Z}^{\lambda}{}_{(\alpha\delta)\lambda} - g_{\alpha\gamma}\not{Z}^{\lambda}{}_{(\beta\delta)\lambda}\right), \qquad \text{(A.66)} \\
{}^{(2)}Z_{\alpha\beta\gamma\delta} &= -\frac{1}{4}\left(g_{\beta\delta}\not{Z}^{\lambda}{}_{[\alpha\gamma]\lambda} + g_{\alpha\delta}\not{Z}^{\lambda}{}_{[\beta\gamma]\lambda} - g_{\beta\gamma}\not{Z}^{\lambda}{}_{[\alpha\delta]\lambda} - g_{\alpha\gamma}\not{Z}^{\lambda}{}_{[\beta\delta]\lambda}\right) - \frac{3}{4}\left(\not{Z}_{\beta[\alpha\gamma\delta]} + \not{Z}_{\gamma[\beta\gamma\delta]}\right) + \frac{1}{2}g_{\alpha\beta}\not{Z}^{\lambda}{}_{[\delta\gamma]\lambda}, \qquad \text{(A.67)} \\
{}^{(3)}Z_{\alpha\beta\gamma\delta} &= -\frac{1}{3}\left(g_{\beta\delta}\not{Z}^{\lambda}{}_{[\gamma\alpha]\lambda} + g_{\alpha\delta}\not{Z}^{\lambda}{}_{[\gamma\beta]\lambda} - g_{\beta\gamma}\not{Z}^{\lambda}{}_{[\delta\alpha]\lambda} - g_{\alpha\gamma}\not{Z}^{\lambda}{}_{[\delta\beta]\lambda} + g_{\alpha\beta}\not{Z}^{\lambda}{}_{[\delta\gamma]\lambda}\right), \qquad \text{(A.68)} \\
{}^{(4)}Z_{\alpha\beta\gamma\delta} &= \frac{1}{4}g_{\alpha\beta}Z^{\lambda}{}_{\lambda\gamma\delta}, \qquad \text{(A.69)} \\
{}^{(5)}Z_{\alpha\beta\gamma\delta} &= -\frac{1}{4}\left(g_{\beta\gamma}Z^{\lambda}{}_{(\delta\alpha)\lambda} + g_{\alpha\gamma}Z^{\lambda}{}_{(\delta\beta)\lambda} - g_{\beta\delta}Z^{\lambda}{}_{(\gamma\alpha)\lambda} - g_{\alpha\delta}Z^{\lambda}{}_{(\gamma\beta)\lambda}\right), \qquad \text{(A.70)} \\
{}^{(2)}W_{\alpha\beta\gamma\delta} &= \frac{1}{2}(g_{\alpha\mu}\mathcal{X}^{T}_{\beta\nu} - g_{\beta\mu}\mathcal{X}^{T}_{\alpha\nu})\epsilon^{\mu\nu}{}_{\gamma\delta}, \qquad \text{(A.71)} \\
{}^{(3)}W_{\alpha\beta\gamma\delta} &= \frac{1}{12}\mathcal{X}\epsilon_{\alpha\beta\gamma\delta}, \qquad \text{(A.72)} \\
{}^{(4)}W_{\alpha\beta\gamma\delta} &= g_{\alpha[\gamma}W^{T}_{|\beta|\delta]} - g_{\beta[\gamma}W^{T}_{|\alpha|\delta]}, \qquad \text{(A.73)} \\
{}^{(5)}W_{\alpha\beta\gamma\delta} &= g_{\alpha[\gamma}{}^{*}\mathcal{X}_{|\beta|\delta]} - g_{\beta[\gamma}{}^{*}\mathcal{X}_{|\alpha|\delta]}, \qquad \text{(A.74)} \\
{}^{(6)}W_{\alpha\beta\gamma\delta} &= \frac{1}{6}\overset{\Gamma}{R}g_{\alpha[\gamma}g_{|\beta|\delta]}, \qquad \text{(A.75)} \\
{}^{(1)}W_{\alpha\beta\gamma\delta} &= W_{\alpha\beta\gamma\delta} - \sum_{n=2}^{6}{}^{(n)}W_{\alpha\beta\gamma\delta}, \qquad \text{(A.76)}
\end{aligned}$$

where it was defined,

$$\not{Z}_{\alpha\beta\gamma\delta} = Z_{\alpha\beta\gamma\delta} - \frac{1}{4}g_{\alpha\beta}Z^{\lambda}{}_{\lambda\gamma\delta}, \qquad \text{(A.77)}$$

$$\mathcal{X} = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}W_{\mu\nu\rho\sigma}, \qquad \text{(A.78)}$$

$$\mathcal{X}^{\mu\nu} = \frac{1}{2}W^{\mu}{}_{\alpha\beta\gamma}\epsilon^{\alpha\beta\gamma\nu}, \qquad \text{(A.79)}$$

$$\mathcal{X}^{T}_{\mu\nu} = \mathcal{X}_{(\mu\nu)} - \frac{1}{4}g_{\mu\nu}\mathcal{X}, \qquad \text{(A.80)}$$

$${}^{*}\mathcal{X}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\mathcal{X}^{[\alpha\beta]}, \qquad \text{(A.81)}$$

$$W^{T}_{\mu\nu} = W_{(\mu\nu)} - \frac{1}{4}g_{\mu\nu}\overset{\Gamma}{R}. \qquad \text{(A.82)}$$

# Appendix B

# Helmholtz Conditions and it's Applications

When one first learns analytical mechanics, it is common to be given a certain Lagrangian $L(x^\mu, v^\mu)$ and then computes its equation of motion $\mathcal{E}_\mu =$. A natural thought that could arise from this classical textbook method is the question *"what kind of equation has a Lagrangian?"*. In this section, the "Helmholz conditions" which precisely answers this question will be introduced, and also applied to classical field theories.

In the main part of the thesis, Lovelock and Horndeski theories approached the *most general Lagrangians* through an Eulerian approach, i.e. computing the theorems through the equation of motion. By using the Helmholz conditions and the idea thereof, one could correctly obtain the Lagrangian of its Eulerian counterpart. Thus through Helmholz conditions and the later introduced Vainberg-Tonti trick, one can easily get the best of both worlds which is why it is introduced in this part of the appendix.

## B.1 The Helmholtz Conditions

Let one start with a Lagrangian,

$$S = \int d\tau L(x^\mu, v^\mu), \tag{B.1}$$

where $v^\mu := \frac{dx^\mu}{d\tau}$ is the velocity, and the Greek indicies run up to D-dimensions ($\mu = (0, 1, 2, \cdots, D-1)$)

The equation of motion is written as

$$\tilde{\mathcal{E}}_\mu = \frac{\partial L}{\partial x^\mu} - \frac{\partial^2 L}{\partial v^\mu \partial x^\nu} v^\nu - \frac{\partial^2 L}{\partial v^\mu \partial v^\nu} a^\nu = 0, \tag{B.2}$$

with acceleration being $a^\mu := \frac{dv^\mu}{d\tau}$. Obviously, in a general Lagrangian there always exists an equation of motion. However, one can ask whether the other direction can be tread.

Thus the inverse problem of (Lagrangian) mechanics is the following question:

*Does a function $L(x, v)$ exist for a given equation $\mathcal{E}_\mu = \mathcal{E}_\mu(x, v, a) = 0$ so that $\mathcal{E}_\mu = \tilde{\mathcal{E}}_\mu$ with respect to (B.2)?*[1]

**Theorem 1.**

[1]Notice that it was assumed $\mathcal{E}_\mu = \tilde{\mathcal{E}}_\mu$, therefore this question is about whether the given equation is an EL equation, no more no less. However, this question is not the same as: can a given dynamics of the equation have a Lagrangian? For example consider $\mathcal{E}_\mu = O^\nu{}_\mu \tilde{\mathcal{E}}_\nu$ with $O^\nu{}_\mu$ being a *non-degenerate* matrix(tensor). In such case *the dynamics of* $\mathcal{E}_\mu = 0$ does indeed have a Lagrangian whereas the equation itself does not. This is why the Helmholz conditions are by themselves somewhat incomplete which leads to the Douglas' theorem that will be introduced later.

*A certain equation of $\mathcal{E}_\mu = \mathcal{E}_\mu(x, v, a) = 0$ has a Lagrangian if and only if the following three Helmholtz conditions are satisfied.*

$$(h1): 0 = \quad h_{1,\mu\nu} \quad := \frac{\partial \mathcal{E}_\mu}{\partial a^\nu} - \frac{\partial \mathcal{E}_\nu}{\partial a^\mu}. \tag{B.3a}$$

$$(h2): 0 = \quad h_{2,\mu\nu} \quad = \frac{\partial \mathcal{E}_\mu}{\partial x^\nu} - \frac{\partial \mathcal{E}_\nu}{\partial x^\mu} - \frac{1}{2}\frac{d}{d\tau}\left(\frac{\partial \mathcal{E}_\mu}{\partial v^\nu} - \frac{\partial \mathcal{E}_\nu}{\partial v^\mu}\right). \tag{B.3b}$$

$$(h3): 0 = \quad h_{3,\mu\nu} \quad = \frac{\partial \mathcal{E}_\mu}{\partial v^\nu} + \frac{\partial \mathcal{E}_\nu}{\partial v^\mu} - \frac{d}{d\tau}\left(\frac{\partial \mathcal{E}_\mu}{\partial a^\nu} + \frac{\partial \mathcal{E}_\nu}{\partial a^\mu}\right). \tag{B.3c}$$

The necessary conditions are straightforward, for the following the sufficient conditions will be outlined. Firstly, (B.3c) computes

$$h_{3,\mu\nu} = -2\frac{\partial^2 \mathcal{E}_\mu}{\partial a^\nu \partial a^\sigma}\dot{a}^\sigma \cdots .$$

Since $\mathcal{E}_\mu$ only depends up to $a^\nu$, the first term disappears when the Helmholtz conditions hold. Thus $\mathcal{E}_\mu$ is linear with respect to $a^\nu$. Such as

$$\mathcal{E}_\mu = b_{\mu\nu}a^\nu + c_\mu,$$

for some function of $b_{\mu\nu} = b_{\mu\nu}(x, v)$ and $c = c(x, v)$.

Secondly, with above, (B.3b) computes

$$h_{2,\mu\nu} = -\frac{1}{2}\left(\frac{\partial b_{\mu\sigma}}{\partial v^\nu} - \frac{\partial b_{\nu\sigma}}{\partial v^\mu}\right)\dot{a}^\sigma \cdots .$$

Therefore, noting that $b_{\mu\nu}$ is symmetric since (B.3a), there exists a function $d(x, v)$ such that

$$b_{\mu\nu} = \frac{\partial^2 d}{\partial v^\mu \partial v^\nu}. \tag{B.4}$$

Substituting the results back into (B.3c) computes

$$h_{3,\mu\nu} = \frac{\partial}{\partial v^{(\nu}}\left(c_{\mu)} - \frac{\partial^2 d}{\partial v^{\mu)}\partial x^\lambda}v^\lambda + \frac{\partial d}{\partial x^{\mu)}}\right).$$

Thus

$$c_\mu = \frac{\partial^2 d}{\partial v^\mu \partial x^\lambda}v^\lambda - \frac{\partial d}{\partial x^\mu} + e_\mu,$$

for some function $e_\mu = e_\mu(x)$
Then (B.3b) computes

$$h_{2,\mu\nu} = \frac{\partial e_\mu}{\partial x^\nu} - \frac{\partial e_\nu}{\partial x^\mu}.$$

Thus there exist a function $f = f(x)$ such that

$$e_\mu = \frac{\partial f}{\partial x^\mu}.$$

As a result, when the Helmholtz conditions are satisfied the equation of motion must have the form of

$$\begin{aligned}\mathcal{E}_\mu &= \frac{\partial^2 d}{\partial v^\mu \partial v^\nu}a^\nu + \frac{\partial^2 d}{\partial v^\mu \partial x^\lambda}v^\lambda - \frac{\partial d}{\partial x^\mu} + \frac{\partial f}{\partial x^\mu} \\ &= \left(\frac{\partial}{\partial x^\mu} - \frac{d}{d\tau}\frac{\partial}{\partial v^\mu}\right)(-d + f).\end{aligned}$$

Thus, the Lagrangian is $L = -d + f$ up to some constant, and the sufficient conditions were proved.

■

**Vainberg-Tonti trick**

*When the Helmholtz conditions are satisfied*, a second-order Lagrangian of the form,

$$L_{VT} = x^{\sigma} \int_0^1 \mathcal{E}_{\sigma}(sx, sv, sa) ds, \tag{B.5}$$

computes the Euler-Lagrangian equation that coincides with the equation $\mathcal{E}_{\sigma}(x, v, a)$. Direct computation results as,

$$\frac{\delta L_{VT}}{\delta x^{\nu}} = \mathcal{E}_{\nu}(x, v, a) - \int_0^1 \left[ a^{\mu} h_{1,\mu\nu}(sx, sv, sa) + x^{\mu} h_{2,\mu\nu}(sx, sv, sa) + \left( v^{\mu} + \frac{1}{2} x^{\mu} \frac{d}{d\tau} \right) h_{3,\mu\nu}(sx, sv, sa) \right] ds, \tag{B.6}$$

where $h_{i,\mu\nu}$ are the three Helmholtz conditions.[2]

This *second-order* Lagrangian is called the *Vainberg-Tonti Lagrangian* and has a key role in this chapter.

Furthermore, when the Helmholtz condition holds, it is always possible to reduce the second-order Vainberg-Tonti Lagrangian to a first-order Lagrangian by integration by parts. Since there exists a function $L = L(x, v)$ such that the equation of motion is the form,

$$\mathcal{E}_{\mu}(x, v, a) = \frac{\partial L}{\partial x^{\mu}} - \frac{d}{d\tau} \left( \frac{\partial L}{\partial v^{\mu}} \right)$$

The Vainberg-Tonti Lagrangian becomes

$$\begin{aligned} L_{VT} &= x^{\mu} \int_0^1 \left[ \frac{\partial L}{\partial (sx^{\mu})} - \frac{d}{d\tau} \left( \frac{\partial L}{\partial (sv^{\mu})} \right) \right] ds \\ &= \int_0^1 \left[ \frac{1}{s} \left\{ (sx^{\mu}) \frac{\partial L}{\partial (sx^{\mu})} + (sv^{\mu}) \frac{\partial L}{\partial (sv^{\mu})} \right\} - \frac{d}{d\tau} \left( x^{\nu} \frac{\partial L}{\partial (sv^{\nu})} \right) \right] ds \\ &= \int_0^1 \left[ \frac{dL}{ds} - \frac{d}{d\tau} \left( x^{\mu} \frac{\partial L}{\partial (sv^{\mu})} \right) \right] ds \\ &= L(x, v) - L(0, 0) - \frac{d}{d\tau} \int_0^1 \left( x^{\mu} \frac{\partial L}{\partial (sv^{\mu})} \right) ds \end{aligned}$$

where the arguments of functions within the s-integration is $(sx, sv, sa)$, and $L(0, 0)$ is assumed to be a constant. Therefore the Vainberg-Tonti Lagrangian could be reduced to the original first-order Lagrangian by integration by parts.

The above calculations can be extended to higher-order cases. For example, consider an nth-order equation of $\mathcal{E}_{\mu}(x^{\mu}, x'^{\mu}, x''^{\mu}, \cdots, x^{(n)\mu}) = 0$. If there exists a m-th order Lagrangian $L(x^{\mu}, x'^{\mu}, x''^{\mu}, \cdots, x^{(m)\mu})$ that computes the equation as its EL- equation, it is related to the Vainberg-Tonti Lagrangian upon integration

[2] Variational calculus makes the above clearer to calculate.

$$\begin{aligned} \frac{\delta L_{VT}}{\delta x^{\nu}} &= \int_0^1 \left[ \mathcal{E}_{\nu} + x^{\mu} \frac{\delta \mathcal{E}_{\mu}}{\delta x^{\nu}} \right] ds = \int_0^1 \left[ \mathcal{E}_{\nu} + x^{\mu} \frac{\delta \mathcal{E}^{\mu}}{\delta x^{\nu}} \right] ds \\ &= \int_0^1 \left[ \mathcal{E}_{\nu} + x^{\mu} \frac{\delta \mathcal{E}_{\nu}}{\delta x^{\mu}} + x^{\mu} \left\{ \frac{\delta \mathcal{E}_{\mu}}{\delta x^{\nu}} - \frac{\delta \mathcal{E}_{\nu}}{\delta x^{\mu}} \right\} \right] ds = \int_0^1 \left[ \frac{d(s\mathcal{E}_{\nu})}{ds} + x^{\mu} \left\{ \frac{\delta \mathcal{E}_{\mu}}{\delta x^{\nu}} - \frac{\delta \mathcal{E}_{\nu}}{\delta x^{\mu}} \right\} \right] ds = \mathcal{E}_{\nu} + \int_0^1 x^{\mu} \left\{ \frac{\delta \mathcal{E}_{\mu}}{\delta x^{\nu}} - \frac{\delta \mathcal{E}_{\nu}}{\delta x^{\mu}} \right\} ds \end{aligned}$$

by parts, i.e.

$$\begin{aligned} L_{VT} &= x^\mu \int_0^1 \mathcal{E}_\mu(sx^\mu, sx'^\mu, sx''^\mu, \cdots, sx^{(n)\mu}) ds \\ &= x^\mu \int_0^1 \left[ \frac{\partial L}{\partial(sx^\mu)} - \frac{d}{d\tau}\left(\frac{\partial L}{\partial(sx'^\mu)}\right) + \cdots + (-1)^m \frac{d^m}{d\tau^m}\left(\frac{\partial L}{\partial(sx^{(m)\mu})}\right)\right] ds \\ &= L(x^\mu, x'^\mu, x''^\mu, \cdots, x^{(m)\mu}) + L(0,0,0,\cdots,0) + \text{surface terms} \end{aligned}$$

Now, this computation actually shows the following feature.
*When $m > n$, the original mth-order Lagrangian can be reduced to nth-order through integration by parts.* Thus a seemingly higher-order Lagrangian can be reduced its order if its equation of motion has lower order in derivatives.

### B.1.1 The Douglas Theorem

The Helmholtz conditions (B.3a)-(B.3c) introduced are a powerful method to see whether a second-order differential equation $\mathcal{E}_\mu = 0$ has a Lagrangian. However, even if $\mathcal{E}_\mu = 0$ can not be computed from a Lagrangian, it is possible that there could be a different equation $\mathcal{E}'_\mu = 0$ that could have a Lagrangian, with $\mathcal{E}'_\mu$ being a one-to-one map of $\mathcal{E}_\mu$ such as $\mathcal{E}'_\mu = \Lambda_\mu{}^\nu \mathcal{E}_\nu = 0$ where $\Lambda$ being a non-degenerate matrix. If so, although the equation $\mathcal{E}_\mu = 0$ does not have a Lagrangian, *it's dynamics has a Lagrangian.*

For example, the equation with friction $\ddot{x} + b\dot{x} = 0$ does not satisfy the Helmholtz conditions and thus one expects that there is no Lagrangian for such equation. However, the equation $e^{bt}(\ddot{x} + b\dot{x}) = 0$, satisfies the conditions and indeed can be derived from the Lagrangian $L = e^{bt}\dot{x}^2$. Since $e^{bt} \neq 0$ for $t = (-\infty, \infty)$, it can be safely said that the *dynamics* of the friction equation has a Lagrangian, although the equation itself does not.

Now consider generalizing this case further. Since it was assumed that the Lagrangian is written with only respect to position and velocity, the equation of motion is always linear with respect to acceleration, i.e. $\mathcal{E}_\mu = \Lambda_{\mu\nu}(a^\mu - f^\mu(x, v))$. Substituting this to the Helmholtz conditions (B.3a)-(B.3c)the following are obtained.

$$(\text{H0})' \quad : \quad \Lambda_{\mu\nu} = \Lambda_{\nu\mu}. \tag{B.7a}$$

$$(\text{H1})'\&(\text{H2})' \quad : \quad 0 = \Lambda_{\lambda[\mu}\Phi^\lambda{}_{\nu]}. \tag{B.7b}$$

$$(\text{H2})' \quad : \quad 0 = \Lambda_{\lambda\mu}\frac{\partial f^\lambda}{\partial v_\nu} + \Lambda_{\lambda\nu}\frac{\partial f^\lambda}{\partial v_\mu} + 2\frac{d}{d\tau}\Lambda_{\mu\nu}. \tag{B.7c}$$

where $\Phi^\mu{}_\nu(x, v)$ is defined as

$$\Phi^\mu{}_\nu = \frac{1}{2}\frac{d}{d\tau}\frac{\partial f^\mu}{\partial v^\nu} - \partial_\nu f^\mu - \frac{1}{4}\frac{\partial f^\mu}{\partial v^\sigma}\frac{\partial f^\sigma}{\partial v^\nu}. \tag{B.8}$$

and is related to curvature.

Then the Helmholtz conditions can be equivalently restated into the Douglas' theorem.

**Theorem 2.** *The (tensor) equations of motion,*

$$0 = a^\mu - f^\mu(x, v) \tag{B.9}$$

*are an Euler-Lagrangian equations of a Lagrangian if and only if there exists a non-singular tensor $\Lambda_{\mu\nu}(x, v)$*

*such that the following three Helmholtz conditions hold.*

$$\textbf{(H1)} \quad : \quad \Lambda_{\mu\nu}\Phi^{\nu}{}_{\sigma} = \Lambda_{\sigma\nu}\Phi^{\nu}{}_{\mu}, \tag{B.10a}$$

$$\textbf{(H2)} \quad : \quad \frac{d}{d\tau}\Lambda_{\mu\nu} + \frac{\partial f^{\sigma}}{\partial v^{(\mu}}\Lambda_{\nu)\sigma} = 0, \tag{B.10b}$$

$$\textbf{(H3)} \quad : \quad \frac{\partial}{\partial v^{[\mu}}\Lambda_{\nu]\sigma} = 0. \tag{B.10c}$$

### B.1.2 Application: The Non-integrability of Autoparallel Equations

Using the given Douglas' theorem it can be shown that

*Autoparallel Equations can be computed from a Lagrangian if and only if it is a geodesic equation of some metric.*

First consider the autoparallel equations of

$$f^{\lambda}(x,v) = \frac{d^2x^{\lambda}}{d\tau^2} = -\Gamma^{\lambda}{}_{\mu\nu}(x)v^{\mu}v^{\nu}. \tag{B.11}$$

Once a metric $g_{\mu\nu}$ is introduced the connection could be decomposed into the symmetric and anti-symmetric components as,

$$\Gamma^{\lambda}{}_{\mu\nu} = S^{\lambda}{}_{\mu\nu} + T^{\lambda}{}_{\mu\nu}, \tag{B.12}$$

$$S^{\lambda}{}_{\mu\nu} = \left\{ \begin{array}{c} \lambda \\ \mu\nu \end{array} \right\}_g + \frac{1}{2}\left[Q_{(\mu\nu)}{}^{\lambda} - Q^{\lambda}{}_{\mu\nu} - T_{(\mu\nu)}{}^{\lambda}\right], \tag{B.13}$$

with $\left\{ \begin{array}{c} \lambda \\ \mu\nu \end{array} \right\}_g$ being the Levi-civita tensor constructed from the metric. Recall that the torsion tensor and the non-metricity tensor are defined as

$$T^{\lambda}{}_{\mu\nu} = 2\Gamma^{\lambda}{}_{[\mu\nu]}, \tag{B.14}$$

$$Q_{\lambda}{}^{\mu\nu} = \overset{\Gamma}{\nabla}_{\lambda} g^{\mu\nu}. \tag{B.15}$$

Note that although the autoparallel equations are governed by the symmetric part of the connection, *torsion does not decouple*. Keeping this in mind the autoparallel equations are equivalent to,

$$f^{\lambda}(x,v) = -S^{\lambda}{}_{\mu\nu}(x)v^{\mu}v^{\nu}. \tag{B.16}$$

On the other hand, $\Phi^{\mu}{}_{\nu}$ can be calculated as

$$\begin{aligned}\Phi^{\mu}{}_{\nu} &= \left(2\partial_{[\nu}S^{\lambda}{}_{\alpha]\beta} - S^{\mu}{}_{\alpha\sigma}S^{\sigma}_{\nu\beta}\right)v^{\alpha}v^{\beta} - f^{\sigma}S^{\mu}{}_{\sigma\nu} \\ &= \overset{S}{R}{}^{\mu}{}_{\sigma\nu\lambda}v^{\sigma}v^{\lambda},\end{aligned} \tag{B.17}$$

with $\overset{S}{R}{}^{\mu}{}_{\sigma\nu\lambda}$ being the Riemann curvature of the symmetric part of the connection, which is related to the usual metric-affine Riemann curvature as

$$\overset{\Gamma}{R}{}^{\mu}{}_{\sigma\nu\lambda} = \overset{S}{R}{}^{\mu}{}_{\sigma\nu\lambda} - 2\overset{S}{\nabla}{}_{[\nu}T^{\mu}{}_{\lambda]\sigma} - 2T^{\mu}{}_{\rho[\nu}T^{\rho}{}_{\lambda]\sigma}. \tag{B.18}$$

Calculating the second Helmholtz condition computes

$$\begin{aligned} 0 &= v^\sigma \partial_\sigma \Lambda_{\mu\nu} + f^\sigma \frac{\partial}{\partial v^\sigma} \Lambda_{\mu\nu} - 2S^\lambda{}_{\sigma(\mu}\Lambda_{\nu)\lambda} v^\sigma \\ &= v^\sigma \left( \partial_\sigma \Lambda_{\mu\nu} - 2S^\lambda{}_{\sigma(\mu}\Lambda_{\nu)\lambda} \right) - S^\sigma{}_{\alpha\beta} v^\alpha v^\beta \frac{\partial}{\partial v^\sigma} \Lambda_{\mu\nu}. \end{aligned} \tag{B.19}$$

As a next step, note that the tensor $\Lambda_{\mu\nu}$ and $S^\lambda_{\mu\nu}$ transforms under coordinate transformation of $x^\mu \to x'^a$ as

$$\Lambda_{ab} = \frac{\partial x^\mu}{\partial x'^a} \frac{\partial x^\nu}{\partial x'^b} \Lambda_{\mu\nu}, \tag{B.20a}$$

$$S^a_{\ bc} = \frac{\partial x'^a}{\partial x^\lambda} \left( \frac{\partial x^\mu}{\partial x'^b} \frac{\partial x^\nu}{\partial x'^c} S^\lambda{}_{\mu\nu} + \frac{\partial^2 x^\lambda}{\partial x'^b \partial x'^c}. \right) \tag{B.20b}$$

Therefore (B.19) transforms as

$$\begin{aligned} & v'^c \left( \frac{\partial}{\partial x'^c} \Lambda_{ab} - 2S^d{}_{c(a}\Lambda_{b)d} \right) - S^c{}_{de} v'^d v'^e \frac{\partial}{\partial v'^c} \Lambda_{ab} \\ = & \frac{\partial x^\mu}{\partial x'^a} \frac{\partial x^\nu}{\partial x'^b} \left[ v^\sigma \left( \partial_\sigma \Lambda_{\mu\nu} - 2S^\lambda{}_{\sigma(\mu}\Lambda_{\nu)\lambda} \right) - S^\sigma{}_{\alpha\beta} v^\alpha v^\beta \frac{\partial}{\partial v^\sigma} \Lambda_{\mu\nu} \right] - \frac{\partial x^\mu}{\partial x'^a} \frac{\partial x^\nu}{\partial x'^b} \frac{\partial^2 x^\lambda}{\partial x'^c \partial x'^d} v'^c v'^d \frac{\partial}{\partial v^\lambda} \Lambda_{\mu\nu} \end{aligned}$$

For the above to be a consistent tensor equation, the final non-tensorial term must vanish. Thus

$$0 = \frac{\partial}{\partial v^\sigma} \Lambda_{\mu\nu}. \tag{B.21}$$

which automatically satisfies the third Helmholtz condition.
Then (B.19) computes

$$0 = \partial_\sigma \Lambda_{\mu\nu} - 2S^\lambda{}_{\sigma(\mu}\Lambda_{\nu)\lambda}. \tag{B.22}$$

Where cyclic permutation shows that,

$$\begin{aligned} 0 &= \partial_\sigma \Lambda_{\mu\nu} + \partial_\mu \Lambda_{\sigma\nu} - \partial_\nu \Lambda_{\sigma\mu} \\ &= +2S^\lambda{}_{\sigma(\mu}\Lambda_{\nu)\lambda} + 2S^\lambda{}_{\mu(\sigma}\Lambda_{\nu)\lambda} - 2S^\lambda{}_{\nu(\sigma}\Lambda_{\mu)\lambda} \\ &= 2S^\lambda{}_{\sigma\mu}\Lambda_{\nu\lambda}. \end{aligned}$$

Since $\Lambda_{\mu\nu}$ is by definition non-singular, it is invertible. Thus the symmetric part of the connection and $\Lambda_{\mu\nu}$ has the relation of

$$S^\lambda_{\ \mu\nu} = \frac{1}{2} \left( \Lambda^{-1} \right)^{\lambda\sigma} \left( \partial_\mu \Lambda_{\sigma\nu} + \partial_\nu \Lambda_{\sigma\mu} - \partial_\sigma \Lambda_{\mu\nu} \right). \tag{B.23}$$

Therefore, for an autoparallel equations of motion to have a Lagrangian, the symmetric part of the connection must be the Levi-Civita connection of a certain symmetric and nonsingular tensor $\Lambda_{\mu\nu}$.

Finally, the Riemann curvature with respect to the Levi-Civita connection has the symmetry of

$$\Lambda_{\mu\nu} \overset{S}{R}{}^\nu{}_{\sigma\alpha\beta} = \Lambda_{\alpha\nu} \overset{S}{R}{}^\nu{}_{\beta\mu\sigma}. \tag{B.24}$$

Multiplying both sides with $v^\sigma v^\beta$ computes

$$\Lambda_{\mu\nu} \Phi^\nu{}_\alpha = \Lambda_{\alpha\nu} \Phi^\nu{}_\mu. \tag{B.25}$$

which is none other than the first Helmholtz condition. Thus all of the Helmholtz conditions are satisfied if and if only the symmetric part of the connection is a Levi-Civita connection and thus by Douglas' theorem rendering the autoparallel equations of motion to be integrable.

Obviously, when the symmetric part of the connection is a Levi-Civita connection of some metric $\Lambda_{\mu\nu}$, the action that computes this autoparallel/geodesic equation is the well known,

$$S = \int d\tau \sqrt{\Lambda_{\mu\nu} v^\mu v^\nu} \tag{B.26}$$

or

$$S = \int d\tau \Lambda_{\mu\nu} v^\mu v^\nu \tag{B.27}$$

This result implies that in order to have autoparallel equations to be the equation of motion that govern particle dynamics, one has to introduce non-Lagrangian methods. Furthermore, from a Lagrangian perspective, no matter can be deduced to follow autoparallels unless it is taken as a principle.

### B.1.3 Autoparallel equations with friction

The previous discussion can be easily extended to theories with friction.

$$f^\mu = -S^\lambda{}_{\mu\nu} v^\mu v^\nu + F^\mu{}_\lambda v^\lambda, \tag{B.28}$$

where $F^\mu{}_\lambda = F^\mu{}_\lambda(x)$ is some asymmetric tensor only dependent on position.
First of all, the second Helmholtz conditions computes

$$v^\sigma \left(\partial_\sigma \Lambda_{\mu\nu} - 2S^\lambda{}_{\sigma(\mu}\Lambda_{\nu)\lambda}\right) + \Lambda_{\lambda(\mu}F^\lambda{}_{\nu)} + f^\lambda \frac{\Lambda_{\mu\nu}}{\partial v^\lambda}. \tag{B.29}$$

Similar to the previous section, the non-tensorial parts must be zero, and this again computes

$$\frac{\partial \Lambda_{\mu\nu}}{\partial v^\lambda} = 0. \tag{B.30}$$

Noting that both $S^\lambda{}_{\mu\nu}$ and $F^\lambda{}_\mu$ are independent with respect to velocity, the first two terms and third term are independent. Thus

$$0 = \partial_\sigma \Lambda_{\mu\nu} - 2S^\lambda{}_{\sigma(\mu}\Lambda_{\nu)\lambda}, \tag{B.31a}$$

$$0 = \Lambda_{\lambda(\mu}F^\lambda{}_{\nu)}. \tag{B.31b}$$

The first condition again implies that $S^\lambda{}_{\mu\nu}$ is the Levi-Civita connection of $\Lambda_{\mu\nu}$ where as the second condition shows that $F_{\mu\nu} := \Lambda_{\lambda\mu}F^\lambda{}_\nu$ is anti-symmetric.
For the first Helmholtz condition, noting that

$$\Phi^\mu{}_\nu = \overset{S}{R}{}^\mu{}_{\alpha\nu\beta} v^\alpha v^\beta + \frac{1}{2} v^\alpha \overset{S}{\nabla}_\alpha F^\mu{}_\nu - \overset{S}{\nabla}_\nu (F^\mu{}_\alpha v^\alpha) - \frac{1}{4} F^\mu{}_\sigma F^\sigma{}_\nu, \tag{B.32}$$

and $\overset{S}{\nabla}_\mu(\Lambda_{\nu\lambda}) = 0$, the second Helmholtz condition is re-written as

$$0 = \Phi_{[\mu\nu]} = \frac{1}{2} v^\alpha \left[\overset{S}{\nabla}_\alpha F_{\mu\nu} + \overset{S}{\nabla}_\nu F_{\alpha\mu} + \overset{S}{\nabla}_\mu F_{\nu\alpha}\right] + \frac{1}{4} F^\sigma{}_{[\mu} F_{\nu]\sigma}. \tag{B.33}$$

Again using the velocity independency of ${A^\mu}_\nu$, the condition above is re-written as

$$0 = \overset{S}{\nabla}_\alpha F_{\mu\nu} + \overset{S}{\nabla}_\nu F_{\alpha\mu} + \overset{S}{\nabla}_\mu F_{\nu\alpha}, \tag{B.34a}$$

$$0 = F^{\sigma}{}_{[\mu} F_{\nu]\sigma}, \tag{B.34b}$$

where the indices are lowered with respect to the tensor $\Lambda_{\mu\nu}$.
The first condition, under general conditions, implies that $F_{\mu\nu}$ must be closed, [3]and thus there exists a vector such that

$$F_{\mu\nu} = 2\overset{S}{\nabla}_{[\mu} A_{\nu]} = 2\partial_{[\mu} A_{\nu]}, \tag{B.35}$$

where the final equality comes from the fact that ${S^\lambda}_{\mu\nu}$ is a Levi-Civita connection. When $F_{\mu\nu}$ is written as such, the second condition B.34b is automatically satisfied.

Thus, for an autoparallel equation with some friction to have an action, the connection again must be a Levi-Civita connection and the friction term $F$ must be closed. When so, the equation of motion can be computed from the following action,

$$L = \Lambda_{\mu\nu} v^\mu v^\nu - A_\mu v^\mu \tag{B.36}$$

## B.2 Extending Helmholtz Conditions to Classical Field Theories

### B.2.1 Existence of a first-order Lagrangian

In 'conventional' field theories of multi-scalars, the Lorentz invariant equation of motion are constructed up to second-order derivative.

$$\epsilon^I_\phi = \epsilon^I_\phi(\phi^J, \phi^J_\mu, \phi^J_{\mu\nu}), \tag{B.37}$$

with $\phi^I_\mu := \partial_\mu \phi^I$ and $\phi^I_{\mu\nu} := \partial_\mu \partial_\nu \phi^I$ The question at hand is whether this can be computed from a Lagrangian of

$$S = \int d^4x L(\phi^I, \phi^I_\mu). \tag{B.38}$$

In other words; Is there some function $L(\phi^I, \phi^I_\mu)$ that satisfies

$$\epsilon^I_\phi = \frac{\partial L}{\partial \phi^I} - \frac{\partial L}{\partial \phi^I_\mu \partial \phi^J} \phi^J_\mu - \frac{\partial L}{\partial \phi^I_\mu \partial \phi^J_\nu} \phi^J_{\mu\nu}, \tag{B.39}$$

for a given $\epsilon^I_\phi$?

Analogous to the previous section for the case of analytical mechanics consider a field-theoretic extended Vainberg-Tonti Lagrangian,

$$L_{VT}(\phi^I, \phi^I_\mu, \phi^I_{\mu\nu}) = \phi^I \int_0^1 \epsilon^I_\phi(s\phi^J, s\phi^J_\mu, s\phi^J_{\mu\nu}) ds. \tag{B.40}$$

which computes the Euler Lagrangian equation of [4]

$$\frac{\delta L_{VT}}{\delta \phi^I} = \epsilon^I_\phi(\phi^J, \phi^J_\mu, \phi^J_{\mu\nu}) - \int_0^1 ds \left[ \phi^J_{\mu\nu} H_1^{\mu\nu,IJ} + \phi^J H_2^{IJ} + \left( \phi^J_\mu + \frac{1}{2}\phi^J \partial_\mu \right) H_3^{\mu,IJ} \right].$$

[3]In the language of differential forms, when $dF = 0$ for a two form $F_{\mu\nu} dx^\mu \wedge dx^\nu$, $F = dA$

[4]When inside the integral, the argument of the functions are with respect to $(s\phi, s\phi_\mu, s\phi_{\mu\nu})$ which are omitted when obvious.

Naively, one would expect the *candidates* for the extended Helmholtz conditions could be,

$$0 = \quad H_1^{\mu\nu,IJ} \quad := \frac{\partial \epsilon_\phi^I}{\partial \phi_{\mu\nu}^J} - \frac{\partial \epsilon_\phi^J}{\partial \phi_{\mu\nu}^I}, \tag{B.41a}$$

$$0 = \quad H_2^{IJ} \quad := \frac{\partial \epsilon_\phi^I}{\partial \phi^J} - \frac{\partial \epsilon_\phi^J}{\partial \phi^I} - \frac{1}{2}\partial_\mu \left( \frac{\partial \epsilon_\phi^I}{\partial \phi_\mu^J} - \frac{\partial \epsilon_\phi^J}{\partial \phi_\mu^I} \right), \tag{B.41b}$$

$$0 = \quad H_3^{\mu,IJ} \quad := \frac{\partial \epsilon_\phi^I}{\partial \phi_\mu^J} + \frac{\partial \epsilon_\phi^J}{\partial \phi_\mu^I} - \partial_\nu \left( \frac{\partial \epsilon_\phi^I}{\partial \phi_{\mu\nu}^J} + \frac{\partial \epsilon_\phi^J}{\partial \phi_{\mu\nu}^I} \right). \tag{B.41c}$$

It is straightforward to see that (B.39) satisfies the conditions above. Therefore these extended conditions are indeed *necessary* conditions for the equation of motion to have a Lagrangian. The question arises whether, similar to the analytical mechanics' case, these are also *sufficient* conditions. It will soon be shown that this is not the case.

First, from (B.41c),

$$\begin{aligned} H_3^{\mu,IJ} \quad &:= \quad \frac{\partial \epsilon_\phi^I}{\partial \phi_\mu^J} + \frac{\partial \epsilon_\phi^J}{\partial \phi_\mu^I} - 2\partial_\nu \left( \frac{\partial \epsilon_\phi^I}{\partial \phi_{\mu\nu}^J} \right) \\ &= \quad -2\frac{\partial^2 \epsilon_\phi^I}{\partial \phi_{\mu\nu}^J \partial \phi_{\alpha\beta}^K} \phi_{\alpha\beta\nu}^K + \cdots . \end{aligned}$$

Noting $\epsilon_\phi^I$ only depends on $\phi^I, \phi_\mu^I, \phi_{\mu\nu}^I$ the first terms of above will vanish when (B.41c) is satisfied. Thus

$$0 \quad = \quad \frac{\partial^2 \epsilon_\phi^I}{\partial \phi_{\mu(\nu}^J \partial \phi_{\alpha)\beta}^K}. \tag{B.42}$$

which under the condition (B.41a), is fully symmetric with respect to the indices $(IJK)$.

At first glimpse, one may be tempted to conclude from (B.42) that $\epsilon_\phi^I$ is linear with respect to $\phi_{\mu\nu}^J$, just as the analytical mechanics' case, since the affinity of $\phi_{\mu\nu}^J$ is an obvious necessary condition for $\epsilon_\phi^I$ to be first-order variational. However, the term

$$\frac{\partial^2 \epsilon_\phi^I}{\partial \phi_{\mu[\nu}^J \partial \phi_{\alpha]\beta}^K}$$

decouples from (B.42) and allows the equation of motion with higher-orders of $\phi_{\mu\nu}^J$, while still satisfying the 'candidate' Helmholtz conditions. Such terms frequently appears in contexts of Galileons [108, 158]. For example a single scalar equation of motion of the form

$$\epsilon_\phi = (\overset{g}{\Box}\phi)^2 - \phi^{\mu\nu}\phi_{\mu\nu} \tag{B.43}$$

satisfies all of the 'candidate' Helmholtz conditions, but obviously 2nd order of $\phi_{\mu\nu}$. [5]

Therefore, another condition of

$$\frac{\partial^2 \epsilon_\phi^I}{\partial \phi_{\mu[\nu}^J \partial \phi_{\alpha]\beta}^K} = 0 \tag{B.44}$$

[5] This could be thought as follows: In analytical mechanics, where there is only one parameter $\tau$ that governs dynamics, such terms like $\frac{\partial^2 \epsilon_\phi^I}{\partial \phi_{\mu[\nu}^J \partial \phi_{\alpha]\beta}^K} \sim \frac{\partial^2 \epsilon_\phi^I}{\partial \ddot{\phi}^J \partial \ddot{\phi}^K} - \frac{\partial^2 \epsilon_\phi^I}{\partial \ddot{\phi}^K \partial \ddot{\phi}^J} = 0$ trivially degenerates. However, in field theory where there are $D \neq 1$ parameters for space-time variables, this degeneracy does not hold.

must be imposed so that $\epsilon_\phi^I$ is linear with respect to $\phi^J_{\mu\nu}$.

From (B.41b)

$$H_2^{IJ} := \frac{\partial \epsilon_\phi^I}{\partial \phi^J} - \frac{\partial \epsilon_\phi^J}{\partial \phi^I} - \frac{1}{2}\partial_\mu \left( \frac{\partial \epsilon_\phi^I}{\partial \phi_\mu^J} - \frac{\partial \epsilon_\phi^J}{\partial \phi_\mu^I} \right) \tag{B.45}$$

$$= -\frac{1}{2}\left[ \frac{\partial^2 \epsilon_\phi^I}{\partial \phi_\mu^J \partial \phi_{\alpha\beta}^K} - \frac{\partial^2 \epsilon_\phi^J}{\partial \phi_\mu^I \partial \phi_{\alpha\beta}^K} \right] \phi^K_{\alpha\beta\mu} + \cdots , \tag{B.46}$$

Since the first terms to disappear

$$\begin{aligned} 0 &= \frac{\partial^2 \epsilon_\phi^I}{\partial \phi_{(\mu}^J \partial \phi_{\alpha)\beta}^K} - \frac{\partial^2 \epsilon_\phi^J}{\partial \phi_{(\mu}^I \partial \phi_{\alpha)\beta}^K} \\ &= \frac{\partial^2 \epsilon_\phi^K}{\partial \phi_{(\mu}^J \partial \phi_{\alpha)\beta}^I} - \frac{\partial^2 \epsilon_\phi^K}{\partial \phi_{(\mu}^I \partial \phi_{\alpha)\beta}^J}, \end{aligned} \tag{B.47}$$

Again imposing

$$0 = \frac{\partial^2 \epsilon_\phi^K}{\partial \phi_{[\mu}^I \partial \phi_{\alpha]\beta}^J}, \tag{B.48}$$

the equation of motion must be the form of,

$$\epsilon_\phi^I = \frac{\partial^2 E(\phi^I, \phi_\mu^I)}{\partial \phi_\mu^I \partial \phi_\nu^J} \phi^J_{\mu\nu} + F^I(\phi^I, \phi^I_\mu), \tag{B.49}$$

with some function $E = E(\phi^I, \phi^I_\mu)$ and $F^I(\phi^I, \phi^I_\mu)$, where (B.44) was used. This of course satisfies the first candidate Helmholtz condition (B.41a).[6]

Substituting this back into (B.41c)

$$\begin{aligned} H_3^{\mu,IJ} &:= \frac{\partial \epsilon_\phi^I}{\partial \phi_\mu^J} + \frac{\partial \epsilon_\phi^J}{\partial \phi_\mu^I} - 2\partial_\nu \left( \frac{\partial \epsilon_\phi^I}{\partial \phi_{\mu\nu}^J} \right) \\ &= 2\frac{\partial}{\partial \phi_\mu^{(J}} \left[ F^{I)} - \frac{\partial^2 E}{\partial \phi_\nu^{I)} \partial \phi^K} \phi_\nu^K + \frac{\partial E}{\partial \phi^{I)}} \right]. \end{aligned}$$

Which computes,

$$F^I = \frac{\partial^2 E}{\partial \phi_\nu^I \partial \phi^K} \phi_\nu^K - \frac{\partial E}{\partial \phi^I} + D^I(\phi^I). \tag{B.50}$$

with some function $D^I(\phi^I)$.

Then (B.41b) becomes,

$$\frac{\partial D^I}{\partial \phi^J} - \frac{\partial D^J}{\partial \phi^I} = 0,$$

Therefore, for general cases, there exists a function $C = C(\phi^I)$ such that

$$D^I = \frac{\partial}{\partial \phi^I} C.$$

[6]For the general proof for the existence of an *exact form* in field theory, see Appendix B.2.6

As a result equation of motion that satisfies the extended Helmholtz conditions are of the form

$$\epsilon_\phi^I = \frac{\partial^2 E}{\partial\phi_\mu^I \partial\phi_\nu^J}\phi_{\mu\nu}^J + \frac{\partial E}{\partial\phi_\mu^I \partial\phi^J}\phi_\mu^J - \frac{\partial(E-C)}{\partial\phi^I}. \tag{B.51}$$

With defining the Lagrangian to be $L = -E + C$, one obtains the same equation as (B.39) and therefore there indeed is a Lagrangian.

To conclude, the 5 extended Helmholtz conditions

$$(H0a): \ 0 = \frac{\partial^2 \epsilon_\phi^I}{\partial\phi_{\mu[\nu}^J \partial\phi_{\alpha]\beta}^K}, \tag{B.52a}$$

$$(H0b): \ 0 = \frac{\partial^2 \epsilon_\phi^I}{\partial\phi_{[\mu}^J \partial\phi_{\alpha]\beta}^K}, \tag{B.52b}$$

$$(H1): \ 0 = H_1^{\mu\nu,IJ} := \frac{\partial \epsilon_\phi^I}{\partial\phi_{\mu\nu}^J} - \frac{\partial \epsilon_\phi^J}{\partial\phi_{\mu\nu}^I}, \tag{B.52c}$$

$$(H2): \ 0 = H_2^{IJ} := \frac{\partial \epsilon_\phi^I}{\partial\phi^J} - \frac{\partial \epsilon_\phi^J}{\partial\phi^I} - \frac{1}{2}\partial_\mu\left(\frac{\partial \epsilon_\phi^I}{\partial\phi_\mu^J} - \frac{\partial \epsilon_\phi^J}{\partial\phi_\mu^I}\right), \tag{B.52d}$$

$$(H3): \ 0 = H_3^{\mu,IJ} := \frac{\partial \epsilon_\phi^I}{\partial\phi_\mu^J} + \frac{\partial \epsilon_\phi^J}{\partial\phi_\mu^I} - \partial_\nu\left(\frac{\partial \epsilon_\phi^I}{\partial\phi_{\mu\nu}^J} + \frac{\partial \epsilon_\phi^J}{\partial\phi_{\mu\nu}^I}\right), \tag{B.52e}$$

are the necessary and sufficient conditions for an equation of motion $\epsilon_\phi^I(\phi^I, \phi_\mu^I, \phi_{\mu\nu}^I)$ to have a Lagrangian of $L(\phi^I, \phi_\mu^I)$.

Similar to the previous case, the extended Vainberg-Tonti Lagrangian is equivalent to the original Lagrangian up to the surface terms and a constant.[7]

$$L_{VT} = L(\phi^I, \phi_\mu^I) - L(0,0) - \partial_\mu \int_0^1 \left(\phi^I \frac{\partial L}{\partial(s\phi_\mu^I)}\right) ds \tag{B.53}$$

### B.2.2 Existence of a second-order Lagrangian

In certain theories, it is common that even though the equation of motions is second-order, the Lagrangian is also of second-order. One famous example is General Relativity since the Einstein equations are of second-order while the Einstein-Hilbert action is also of second-order. Furthermore, in the context of recent approaches of modified gravity, a Lagrangian of second-order that computes the equation of motion of second-order was widely explored, such as the Horndeski theory and multi-Horndeski theories [110, 112]. This is due to the fact that second-order equation of motion avoid Ostragradsky ghosts which appear in the context of higher derivative equations of motion, as seen in §2.5.1.

Therefore it is also important to see the criteria for the existence of second-order Lagrangian in the presence of a second-order equation of motion. Following the previous section, a weakened form of the Helmholtz conditions (or more famously the integrability conditions ) could be stated as follows.

**Theorem 3.**

[7] Through out this paper, the Lagrangian is assumed to be a constant when all of the arguments are zero, such as $L(0, 0)$, which could be considered as a vacuum limit.

*For a given second-order equation of motion $\epsilon^I_\phi(\phi^J, \phi^J_\mu, \phi^J_{\mu\nu})$ there exists a second-order Lagrangian if and only if the weak Helmholtz conditions of the following are satisfied.*

$$
\begin{aligned}
(H1):\ 0 = \quad H_1^{\mu\nu,IJ} \quad &:= \frac{\partial \epsilon^I_\phi}{\partial \phi^J_{\mu\nu}} - \frac{\partial \epsilon^J_\phi}{\partial \phi^I_{\mu\nu}}, &\text{(B.54a)}\\
(H2):\ 0 = \quad H_2^{IJ} \quad &:= \frac{\partial \epsilon^I_\phi}{\partial \phi^J} - \frac{\partial \epsilon^J_\phi}{\partial \phi^I} - \frac{1}{2}\partial_\mu \left(\frac{\partial \epsilon^I_\phi}{\partial \phi^J_\mu} - \frac{\partial \epsilon^J_\phi}{\partial \phi^I_\mu}\right), &\text{(B.54b)}\\
(H3):\ 0 = \quad H_3^{\mu,IJ} \quad &:= \frac{\partial \epsilon^I_\phi}{\partial \phi^J_\mu} + \frac{\partial \epsilon^J_\phi}{\partial \phi^I_\mu} - \partial_\nu \left(\frac{\partial \epsilon^I_\phi}{\partial \phi^J_{\mu\nu}} + \frac{\partial \epsilon^J_\phi}{\partial \phi^I_{\mu\nu}}\right), &\text{(B.54c)}
\end{aligned}
$$

The necessary condition can be proved by straightforward substitution of the second-order Lagrangian. The sufficient condition could be shown by first varying the Vainberg-Tonti Lagrangian computes

$$\frac{\delta L_{VT}}{\delta \phi^I} = \epsilon^I_\phi(\phi^J, \phi^J_\mu, \phi^J_{\mu\nu}) - \int_0^1 ds \left[\phi^J_{\mu\nu} H_1^{\mu\nu,IJ} + \phi^J H_2^{IJ} + \left(\phi^J_\mu + \frac{1}{2}\phi^J \partial_\mu\right) H_3^{\mu,IJ}\right]. \tag{B.55}$$

Thus when the weak Helmholtz conditions are satisfied, Vainberg-Tonti Lagrangian computes the same equation of motion as the given $\epsilon^I_\phi$, and since Vainberg-Tonti Lagrangian is *by definition* second-order, it is sufficient.

For example, consider the Lagrangian

$$L = \phi^\mu \phi_{\mu\nu} \phi^\nu \tag{B.56}$$

that computes the equation of motion

$$\epsilon_\phi = (\overset{g}{\Box}\phi)^2 - \phi^{\mu\nu}\phi_{\mu\nu}. \tag{B.57}$$

This satisfies all of the weak Helmholtz conditions(B.54a),(B.54b),(B.54c), but fails for the full Helmholtz conditions (B.52a), (B.52b), (B.52c), (B.52d), (B.52e). Thus, there is a second-order Lagrangian for this equation of motion and not of first-order. Indeed, the Vainberg-Lagrangian computed for this equation of motion is

$$
\begin{aligned}
L_{VT} \quad &= \quad \phi \int_0^1 \left[s^2 \left\{(\overset{g}{\Box}\phi)^2 - \phi^{\mu\nu}\phi_{\mu\nu}\right\}\right] ds\\
&= \quad \frac{1}{3}\phi \left\{(\overset{g}{\Box}\phi)^2 - \phi^{\mu\nu}\phi_{\mu\nu}\right\}\\
&= \quad \phi^\mu \phi_{\mu\nu}\phi^\nu + \frac{1}{3}\partial_\mu \left(\phi\phi^\mu \overset{g}{\Box}\phi - \phi\phi^\nu\phi_{\mu\nu} - \phi^\mu\phi_\nu\phi^\nu\right) \qquad \text{(B.58)}
\end{aligned}
$$

Which is none-other than the Lagrangian first introduced. Furthermore, there are no non-trivial integration by parts that can make this Lagrangian first-order since the full Helmholtz conditions (B.52a),(B.52b), (B.52c), (B.52d), (B.52e) do not hold.

### B.2.3 Application: On non-tensorial Lagrangians with tensorial equations of motion

*Theories* are expected to be diffeomorphism invariant and thus written with respect to tensors. However, *theories* are somewhat ambiguous since it can be either considered a Lagrangian or the equation of motion.

Usually, this is not a problem since both are assumed to be tensorial. However, even though the equation of motion is tensorial, the Lagrangian does not have to be so. [8]. Such example is the so-called $\Gamma\Gamma$ Lagrangian,

$$\mathcal{L}_{\Gamma\Gamma} = -g^{\mu\nu}\left(\left\{ \begin{array}{c} \lambda \\ \mu\sigma \end{array} \right\}\left\{ \begin{array}{c} \sigma \\ \nu\lambda \end{array} \right\} - \left\{ \begin{array}{c} \lambda \\ \mu\nu \end{array} \right\}\left\{ \begin{array}{c} \sigma \\ \sigma\lambda \end{array} \right\}\right), \tag{B.59}$$

are by definition non-tensorial, but computes tensorial equation of motion,

$$0 = \frac{1}{\sqrt{-g}}\frac{\delta\sqrt{-g}\mathcal{L}_{\Gamma\Gamma}}{\delta g^{\mu\nu}} = \overset{g}{G}_{\mu\nu}. \tag{B.60}$$

This is obvious since the $\Gamma\Gamma$ Lagrangian is actually related to the Einstein-Hilbert $\mathcal{L}_{EH} = R$ up to total derivative as,

$$\int d^4x\sqrt{-g}\mathcal{L}_{EH} = \int d^4x\sqrt{-g}\left(\mathcal{L}_{\Gamma\Gamma} + \partial_\mu B^\mu\right), \tag{B.61}$$

where,

$$B^\mu = \sqrt{-g}\left(g^{\mu\alpha}\left\{ \begin{array}{c} \lambda \\ \mu\alpha \end{array} \right\} - g^{\alpha\beta}\left\{ \begin{array}{c} \mu \\ \alpha\beta \end{array} \right\}\right). \tag{B.62}$$

Most theorems tend to implicitly assume the diffeomorphism of the action. Therefore, one could dream to explore loopholes through non-tensorial Lagrangian with tensorial equations of motion. However, thanks to the Vainberg-Tonti trick, one could show that
*any non-tensorial Lagrangian with tensorial equations of motion can be integrated by parts to obtain a tensorial Lagrangian.*
The proof is trivial since first *by construction* the tensorial equation of motion satisfies the Helmholz conditions and the Vainberg-Tonti Lagrangian through the equation of motion is also tensorial.

### B.2.4 Application: Reducibility of higher-order Lagrangians in the presence of a lower-order equation of motion.

One may wonder that if there could be an existence criterion of Lagrangians that are third-order or higher that computes second-order equation of motion. However, it can be shown that any higher-order Lagrangian that computes an equation of motion lower than its order can be reduced to the order of the equation of motion with integration by parts.

**Theorem 4.**
*A Lagrangian of nth order,*

$$L = L(\phi^I, \phi^I_{\mu_1}, \phi^I_{\mu_1\mu_2}\cdots\phi^I_{\mu_1\mu_2\cdots\mu_n}), \tag{B.63}$$

*that computes Euler-Lagrangian equations of kth order*

$$\epsilon^I_\phi(\phi^I, \phi^I_{\mu_1}, \cdots, \phi^I_{\mu_1\cdots\mu_k}) = \frac{\partial L}{\partial\phi^I} - \partial_{\mu_1}\left(\frac{\partial L}{\partial\phi^I_{\mu_1}}\right) + \cdots(-1)^n\partial_{\mu_1}\cdots\partial_{\mu_n}\left(\frac{\partial L}{\partial\phi^I_{\mu_1\cdots\mu_n}}\right) \tag{B.64}$$

*with $k < n$, can be reduced to kth order using integration by parts.*

[8] In general, if a Lagrangian is tensorial and the *variation of the independent variables* are also tensorial the resulting equation of motion is also tensorial. This is why, even though the affine connection $\Gamma$ is not a tensor, its equation of motion is, since $\delta\Gamma$ is a tensor. It will be interesting whether, a non-tensorial Lagrangian and a non-tensorial independent variable can compute tensorial equation of motion, but it is beyond the scope of this thesis.

Straightforward calculation shows that

$$
\begin{aligned}
& L(\phi^I, \phi^I_{\mu_1}, \cdots \phi^I_{\mu_1\mu_2\cdots\mu_n}) \\
= \; & \left[L(s\phi^I, s\phi^I_{\mu_1}, \cdots s\phi^I_{\mu_1\mu_2\cdots\mu_n})\right]^{s=1}_{s=0} + L(0,0,\cdots,0) \\
= \; & \int_0^1 \left[\frac{d}{ds} L(s\phi^I, s\phi^I_{\mu_1}, \cdots s\phi^I_{\mu_1\mu_2\cdots\mu_n})\right] ds + L(0,0,\cdots,0) \\
= \; & \int_0^1 \frac{1}{s}\left[\phi^I \frac{\partial}{\partial\phi^I} + \phi^I_{\mu_1}\frac{\partial}{\partial\phi^I_{\mu_1}} + \cdots \phi^I_{\mu_1\mu_2\cdots\mu_n}\frac{\partial}{\partial\phi^I_{\mu_1\mu_2\cdots\mu_n}}\right] L(s\phi^I, s\phi^I_{\mu_1}, \cdots s\phi^I_{\mu_1\mu_2\cdots\mu_n}) ds + L(0,0,\cdots,0) \\
\overset{\text{i.b.p.}}{=} \; & \int_0^1 \frac{1}{s}\phi^I \left[\frac{\partial}{\partial\phi^I} - \partial^I_{\mu_1}\frac{\partial}{\partial\phi^I_{\mu_1}} + \cdots (-1)^n \partial_{\mu_1}\cdots\partial_{\mu_n}\frac{\partial}{\partial\phi^I_{\mu_1\mu_2\cdots\mu_n}}\right] L(s\phi^I, s\phi^I_{\mu_1}, \cdots s\phi^I_{\mu_1\mu_2\cdots\mu_n}) ds + L(0,0,\cdots,0) + \text{s.t.} \\
= \; & \int_0^1 \phi^I \epsilon^I_\phi(s\phi^I, s\phi^I_{\mu_1}, \cdots, s\phi^I_{\mu_1\cdots\mu_k}) ds + L(0,0,\cdots,0) + \text{s.t.}
\end{aligned}
$$

The first line is n-th order while the final line is of k-th order, and since $n > k$ is assumed, the original n-th order Lagrangian is indeed reduced to k-th order under integration by parts.

Since this theorem states that any Lagrangian that is higher than third-order can be reduced to second-order. The existence of a Lagrangian for a second-order equation of motion can be confirmed by only seeing whether a first-order or second-order Lagrangian exists.

### B.2.5 Application: Deriving K(X) from Equation of Motions

A simple example of the effectiveness of using the Helmholz conditions and the Vainberg-Tonti trick is computing Horndeski action for the Horndeski equation of motion. For simplicity, consider a (shift-symmetric) K-essence Lagrangian,

$$\mathcal{L} = K(X)\,, \tag{B.65}$$

where $X = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$, which gives the equation of motion,

$$E'_\phi = K_X \overset{g}{\Box}\phi - K_{XX}\phi^\mu\phi_{\mu\nu}\phi^\nu\,. \tag{B.66}$$

Keeping the above in mind, consider the equation of motion of the form.

$$E_\phi = A(X)\overset{g}{\Box}\phi + B(X)\phi^\mu\phi_{\mu\nu}\phi^\nu\,. \tag{B.67}$$

For a single-scalar case, the Helmholtz conditions are given as

$$\frac{\partial E_\phi}{\partial\phi_\mu} - \overset{g}{\nabla}_\nu\left(\frac{\partial E_\phi}{\partial\phi_{\mu\nu}}\right) = 0\,, \tag{B.68}$$

which for the given equation of motion earlier, computes,

$$0 = (A_X + B)(\phi_{\mu\nu}\phi^\nu - \overset{g}{\Box}\phi\phi_\mu)\,. \tag{B.69}$$

Therefore, for $E_\phi$ to be the EL equations, the functions must satisfy,

$$B = -A_X\,. \tag{B.70}$$

Then the Vainberg-Tonti Lagrangian computes

$$
\begin{aligned}
L_{VT} &= \phi \int_0^1 A(s^2X) \times s\overset{g}{\Box}\phi - \frac{\partial A(s^2X)}{\partial s^2 X} \times s^3 \phi^\mu \phi_{\mu\nu}\phi^\nu \\
&= \phi \int_0^1 s \left[ A(s^2X)\overset{g}{\Box}\phi + \frac{\partial A(s^2X)}{\partial s^2 X} \times \overset{g}{\nabla}_\mu (s^2 X) \right] ds \\
&= \phi \int_0^1 s \overset{g}{\nabla}_\mu \left( A(s^2X) \partial^\mu \phi \right) ds \\
&= \int_0^1 A(s^2X) \times 2X s ds + \text{surface terms} \\
&= \int_0^X A(X')dX' \quad \text{(Here it was redefined: } s^2X = dX') \\
&:= K(X) \, .
\end{aligned}
$$

which is precisely the initial K-essence Lagrangian.

### B.2.6 Comment on closed and exact forms for field theories

**Theorem 5.**

*Finally, here a comment on closed and exact forms for field theories will be introduced. The focus will be to obtain an intuitive picture rather than a rigorous and mathematically complete proof.*

*First, consider D-dimensional N-vectors labeled with $I, J$, when*

$$\frac{\partial}{\partial A^{(I}_{[\mu}} E^{\nu]}_{J)} = 0 \tag{B.71a}$$

$$\frac{\partial}{\partial A^{[I}_{(\mu}} E^{\nu)}_{J]} = 0 \tag{B.71b}$$

[9] *then there exists some function $S(A^a_\mu)$ such that*

$$E^\mu_I = \frac{\partial}{\partial A^I_\mu} S \tag{B.72}$$

The proof will be done through two-folds of induction.

For D=1 and arbitrary number of vectors, (B.71a) is trivially satisfied and (B.71b) becomes,

$$\frac{\partial}{\partial A^{[I}_0} E^0_{J]} = 0.$$

Then by using Poincare's lemma, there exists a function $\mathcal{S} = \mathcal{S}(A^1_0, A^2_0, \cdots)$ such that

$$E^0_I = \frac{\partial}{\partial A^I_0} \mathcal{S}$$

[9] or equivalently

$$\frac{\partial}{\partial A^I_\mu} E^\nu_J - \frac{\partial}{\partial A^J_\nu} E^\mu_I = 0$$

Which, as one example, could be written as

$$\mathcal{S} = A_0^I \int_0^1 E_I^0 \left(sA_0^I\right) ds \tag{B.73}$$

Similarly for arbitrary dimension $D = n$ and one vector, (B.71b) is trivially satisfied and (B.71a) becomes,

$$\frac{\partial}{\partial A^1_{[\mu}} E_1^{\nu]} = 0$$

Therefore, there exists a function $\mathcal{S}' = \mathcal{S}'(A_0^1, \cdots, A_{n-1}^1)$ such that

$$E_1^\mu = \frac{\partial}{\partial A_\mu^1} \mathcal{S}'$$

Secondly, let it be assumed that for a fixed dimension $D = k$ with arbitrary number of vectors, when

$$\frac{\partial}{\partial A^{(I}_{[\mu}} E_{J)}^{\nu]} = \frac{\partial}{\partial A^{[I}_{(\mu}} E_{J]}^{\nu)} = 0$$

there exists a function $\bar{S} = \bar{S}(A_0^I, \cdots A_{k-1}^I)$ such that

$$E_I^\mu = \frac{\partial}{\partial A_\mu^I} \bar{S} \tag{B.74}$$

up to dimension $D = k$.

Consider when $(\mu, \nu) = (k, k)$, n such case, (B.71a) is trivially satisfied and (B.71b) becomes,

$$\frac{\partial}{\partial A^{[I}_{k}} E_{J]}^{k} = 0$$

This implies that, there exists a function $\tilde{S} = \tilde{S}(A_k^I)$ such that

$$E_I^k = \frac{\partial}{\partial A_k^I} \tilde{S} \tag{B.75}$$

For $(\mu, \nu) = (k, k-1)$, (B.71a) and (B.71b) is,

$$\begin{aligned} \frac{\partial}{\partial A^{(I}_{[k}} E_{J)}^{k-1]} &= 0 \\ \frac{\partial}{\partial A^{[I}_{(k}} E_{J]}^{k-1)} &= 0 \end{aligned}$$

where both become trivially satisfied under (B.74) and (B.75). Similarly for $(\mu, \nu) = (k-1, k)$.

Defining a new function $S(A_0^I, \cdots, A_{k-1}^I, A_k^I) = \bar{S} + \tilde{S}$, it could be said that, for dimension $D = k+1$, when

$$\frac{\partial}{\partial A^{(I}_{[\mu}} E_{J)}^{\nu]} = \frac{\partial}{\partial A^{[I}_{(\mu}} E_{J]}^{\nu)} = 0$$

there exists a function $S = S(A_0^I, \cdots, A_{k-1}^I, A_k^I)$ such that

$$E_I^\mu = \frac{\partial}{\partial A_\mu^I} S$$

up to dimension $D = k+1$. Similarly, one can show for a fixed number of vectors and arbitrary dimensions. Through induction, the theorem is proved.

# Appendix C

# Degrees of Freedom in General Relativity

## C.1 Hamiltonian Analysis of General Relativity

Here the Hamiltonian analysis of General Relativity will be conducted. As a result, it will be shown that indeed there are two degrees of freedom. This will follow the Appendix A of [179].
The main goal of this section is to derive the following,

$$\{\mathcal{H}_0(x), \mathcal{H}_0(y)\} = \mathcal{H}^i(x)\partial_{x^i}\delta^3(x-y) - \mathcal{H}^i(y)\partial_{y^i}\delta^3(x-y)\,, \tag{C.1}$$

$$\{\mathcal{H}_0(x), \mathcal{H}_i(y)\} = \mathcal{H}_0(y)\partial_{x^i}\delta^3(x-y)\,, \tag{C.2}$$

$$\{\mathcal{H}_i(x), \mathcal{H}_i(y)\} = \mathcal{H}_j(x)\partial_{x^i}\delta^3(x-y) - \mathcal{H}_i(y)\partial_{y^i}\delta^3(x-y)\,, \tag{C.3}$$

where the Poisson bracket is

$$\{A, B\} \equiv \int d^3z \left(\frac{\delta A}{\delta h_{ij}(z)}\frac{\delta B}{\delta \pi^{ij}(z)} - \frac{\delta A}{\delta \pi^{ij}(z)}\frac{\delta B}{\delta h_{ij}(z)}\right)\,, \tag{C.4}$$

and

$$\mathcal{H}_0 \equiv -\sqrt{h}(\overset{3}{R} - 2\Lambda) + \frac{1}{\sqrt{h}}\left(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2\right)\,, \tag{C.5}$$

$$\mathcal{H}_i \equiv -2h_{ij}D_k\pi^{jk}\,. \tag{C.6}$$

First, start off with the action of

$$S = \int dtd^3x\sqrt{-g}\overset{g}{R}\,. \tag{C.7}$$

Then consider the metric decomposed into the ADM variables as,

$$g_{\mu\nu}dx^\mu dx^\nu = -(N^0)^2dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt)\,, \tag{C.8}$$

with $N^0$ being the lapse, $N^i$ being the shift, and $h_{ij}$ being the spatial metric. The Einstein-Hilbert action could then be composed into,

$$S = \int dtd^3x\sqrt{h}N^0(K_{ij}K^{ij} - K^2 + \overset{3}{R})\,, \tag{C.9}$$

where $K_{ij}$ is the extrinsic curvature tensor which could be expressed as,

$$K_{ij} = \frac{1}{2N^0}(\dot{h}_{ij} - 2D_{(i}N_{j)})\,. \tag{C.10}$$

Here $D_i$ is the covariant derivative acting on the spatial dimension.

Now, introduce and calculate the momentum conjugate for the spatial metric as

$$\pi^{ij} \equiv \frac{\delta L}{\delta \dot{h}_{ij}} \tag{C.11}$$

$$= \sqrt{h}(K^{ij} - Kh^{ij})\,. \tag{C.12}$$

Thus, general relativity in its canonical form could be written as

$$S = \int dtd^3x(\pi^{ij}\dot{h}_{ij} - N^\mu \mathcal{H}_\mu)\,, \tag{C.13}$$

with

$$\mathcal{H}_0 \equiv -\sqrt{h}(\overset{3}{R} - 2\Lambda) + \frac{1}{\sqrt{h}}\left(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2\right)\,, \tag{C.14}$$

$$\mathcal{H}_i \equiv -2h_{ij}D_k\pi^{jk}\,. \tag{C.15}$$

The variation of the canonical form of General Relativity with respect to $N^\mu$, gives the constraint equations,

$$\mathcal{H}_\mu \sim 0\,, \tag{C.16}$$

where $\sim$ means 'weak equality', such that the relations holds when the constraints are considered.

The next step is to calculate the Poisson brackets. To simplify the calculations of the Poisson brackets of the fields, one may define functionals by using test functions, such as

$$F_0 \equiv \int d^3x f^0(x)\mathcal{H}_0(x) \tag{C.17}$$

$$F \equiv \int d^3x f^i(x)\mathcal{H}_i(x) \tag{C.18}$$

$$G_0 \equiv \int d^3y g^0(y)\mathcal{H}_0(y) \tag{C.19}$$

$$G \equiv \int d^3y g^i(y)\mathcal{H}_i(y) \tag{C.20}$$

The Poisson brackets of them are,

$$\begin{aligned}
\{F_0, G_0\} &= \int d^3xd^3y f^0(x)g^0(y)\left\{\mathcal{H}_0(x), \mathcal{H}_0(y)\right\}\,, \\
\{F_0, G\} &= \int d^3xd^3y f^0(x)g^i(y)\left\{\mathcal{H}_0(x), \mathcal{H}_i(y)\right\}\,, \\
\{F, G\} &= \int d^3xd^3y f^i(x)g^j(y)\left\{\mathcal{H}_i(x), \mathcal{H}_j(y)\right\}\,,
\end{aligned} \tag{C.21}$$

This drastically simplifies the calculation. Following [179], consider dividing the Hamiltonian constraint $\mathcal{H}_0$ into the kinetic term and potential term as

$$\mathcal{H}_0 = \mathcal{H}_K + \mathcal{H}_V\,, \tag{C.22}$$

$$\mathcal{H}_K \equiv \frac{1}{\sqrt{h}}\left(h_{ik}h_{jl} - \frac{1}{2}h_{ij}h_{kl}\right)\pi^{ij}\pi^{kl}\,, \tag{C.23}$$

$$\mathcal{H}_V \equiv -\sqrt{h}(\overset{3}{R} - 2\Lambda)\,, \tag{C.24}$$

and their functionals as

$$F_K \equiv \int d^3x f^0(x)\mathcal{H}_K(x)\,, \tag{C.25}$$

$$F_V \equiv \int d^3x f^0(x)\mathcal{H}_V(x)\,. \tag{C.26}$$

The variations of $F_K$ become,

$$\frac{\delta F_K}{\delta h_{ij}} = f^0\left(\frac{1}{\sqrt{h}}(2\pi^i{}_k\pi^{kj} - \pi^k{}_k\pi^{ij}) - \frac{1}{2}\mathcal{H}_K h^{ij}\right)\,, \tag{C.27}$$

$$\frac{\delta F_K}{\delta \pi^{ij}} = f^0\frac{1}{\sqrt{h}}(2\pi_{ij} - h_{ij}\pi^k{}_k)\,, \tag{C.28}$$

with the results of $G_K$ being similar. Likewise the variations of $F_V$ are

$$\begin{aligned}\frac{\delta F_V}{\delta h_{ij}} &= f^0\left(\frac{1}{2}\mathcal{F}_V h^{ij} + \sqrt{h}R^{ij}\right) + \sqrt{h}\left(h^{ij}D_kD^kf^0 - D^iD^jf^0\right)\\ \frac{\delta F_V}{\delta \pi^{ij}} &= 0\end{aligned} \tag{C.29}$$

Using $\mathcal{H}_i = -2h_{ij}D_k\pi^{jk}$, F could be written as

$$F = 2\int d^3x h_{ij}\pi^{jk}D_kf^i + s.t.\,, \tag{C.30}$$

the variation could be written as,

$$\frac{\delta F}{\delta h_{ij}} = 2(D_kf^l)\pi^{mk}\delta^{ij}_{lm} - D_i(f^i\pi^{ij})\,, \tag{C.31}$$

$$\frac{\delta F}{\delta \pi^{ij}} = 2(D_kf^l)h_{lm}\delta^{mk}_{ij}\,, \tag{C.32}$$

with $G$ being similar.

Now, the Poisson bracket of General Relativity is calculable.
For starters, $\{F_0, G_0\}$ could be calculated as

$$\{F_0, G_0\} = \int d^3z\left(f^0\mathcal{H}^i\partial_ig^0 - g^0\mathcal{H}^i\partial_if^0\right)\,. \tag{C.33}$$

Now, one could use the identities,

$$g^j(x) = \int d^3y\delta^3(x-y)g^j(y) \tag{C.34}$$

$$f^i(y) = \int d^3x\delta^3(x-y)f^i(x)\,, \tag{C.35}$$

and compute,

$$\{F_0, G_0\} = \int d^3xd^3yf^0(x)g^0(y)\left\{\mathcal{H}^i(x)\partial_{x^i}\delta^3(x-y) - \mathcal{H}^i(y)\partial_{y^i}\delta^3(x-y)\right\} \tag{C.36}$$

By using (C.21), one could derive the Poisson brackets as,

$$\{\mathcal{H}_0(x), \mathcal{H}_0(y)\} \quad = \quad \mathcal{H}^i(x)\partial_{x^i}\delta^3(x-y) - \mathcal{H}^i(y)\partial_{y^i}\delta^3(x-y)\,. \tag{C.37}$$

Next, $\{F_0, G\}$ could be calculated as

$$\begin{aligned}\{F_0, G\} &= f^0\partial_i(g^i\mathcal{H}_0) \\ &= \int d^3 f^0 \partial_{x^i} g^i(x)\mathcal{H}_0(x)\,.\end{aligned} \tag{C.38}$$

Similarly, using the identities,

$$g^i(x)\mathcal{H}_0(x) = \int d^3y\delta^3(x-y)g^i(y)\mathcal{H}_0(y) \tag{C.39}$$

it could be calculated as,

$$\{F_0, G\} = \int d^3x d^3y f^0(x) g^i(y)\mathcal{H}_0(y)\partial_{x^i}\delta^3(x-y) \tag{C.40}$$

So, by comparing the above with (C.21), one could calculate the Poisson brackets as,

$$\{\mathcal{H}_0(x), \mathcal{H}_i(y)\} \quad = \quad \mathcal{H}_0(y)\partial_{x^i}\delta^3(x-y)\,. \tag{C.41}$$

Finally, since $\{F, G\}$ is,

$$\begin{aligned}\{F, G\} &= \int d^3z(f^i\mathcal{H}_j\partial_i g^j - g^j\mathcal{H}_i\partial_j f^i) \\ &= \int d^3x f^i(x)\mathcal{H}_j(x)\partial_{x^i}g^j(x) - \int d^3y g^j(y)\mathcal{H}_i(y)\partial_{y^j}f^i(y)\,,\end{aligned} \tag{C.42}$$

and the identities could be used to extract the test functions, which results as,

$$\{F, G\} = \int d^3x d^3y f^i(x) g^j(y)\left\{\mathcal{H}_j(x)\partial_{x^i}\delta^3(x-y) - \mathcal{H}_i(y)\partial_{y^j}\delta^3(x-y)\right\}\,. \tag{C.43}$$

Therefore by comparing the above with (C.21), one could compute the Poisson brackets as,

$$\{\mathcal{H}_i(x), \mathcal{H}_i(y)\} \quad = \quad \mathcal{H}_j(x)\partial_{x^i}\delta^3(x-y) - \mathcal{H}_i(y)\partial_{y^i}\delta^3(x-y)\,. \tag{C.44}$$

Thus all Poisson brackets are computed.

From the calculation above, one obtains,

$$\{\mathcal{H}_\mu(x), \mathcal{H}_\nu(y)\} \sim 0\,. \tag{C.45}$$

Thus all the constraints are first class. Since there are 6 degrees of freedom in $h_{ij}$, subtracting the four degrees of freedom from the constraints, yield two physical degrees of freedom. Therefore, indeed General Relativity has two degrees of freedom.