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Doctoral Thesis

Superconformal indices of M-brane theories via the AdS/CFT correspondence

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Abstract

Superstring theory is thought to be a candidate for theory including quantum gravity. There are five types of superstring theories in ten dimensions: Type I superstring theory, Type IIA superstring theory, Type IIB superstring theory, $E_8 \times E_8$ heterotic superstring theory, and $SO(32)$ heterotic superstring theory.

A unifying theory underlying these five superstring theories is thought to exist and the theory is called M-theory. M-theory is defined in eleven-dimensional spacetime and contains two fundamental objects. One is called an M2-brane, and the other is called an M5-brane. The M2-brane is a 2+1 dimensional object, and the M5-brane is a 5+1 dimensional object. As in the case of D-branes in superstring theory, in the low energy limit, superconformal field theories (SCFTs) are realized on M2-/M5- branes. The theory realized on M2-branes is called the 3d Aharony-Bergman-Jafferis-Maldacena (ABJM) theory. On the other hand, the theory realized on M5-branes is called the 6d $\mathcal{N} = (2, 0)$ theory.

The purpose of this thesis is to investigate those SCFTs by using a quantity called the superconformal index. Here, the superconformal index is a kind of partition function defined in a supersymmetric field theory, which exhibits a spectrum of the local operators. We propose a new method of calculating the superconformal indices of theories realized on M2-/M5- branes. In our method, we calculate the index from the dual gravity theory by using the Anti-de Sitter (AdS)/Conformal Field Theory (CFT) correspondence. The AdS/CFT correspondence is a conjecture that states a certain CFT is equivalent to the corresponding gravity system. Especially for the M-brane theories, AdS/CFT argues that the 3d ABJM theory with Chern-Simon level $k = 1$ corresponds to M-theory on $\text{AdS}_4 \times S^7$, and the 6d $\mathcal{N} = (2, 0)$ theory corresponds to M-theory on $\text{AdS}_7 \times S^4$.

It is already well known that in the large-N limit, where N is the number of M-branes, the superconformal index is calculated from the bulk Kaluza-Klein modes. In our study, we calculate the superconformal indices in the finite-N region. So far, the study of AdS/CFT in the finite-N region has been thought to be difficult due to quantum gravity effects. However, if we utilize the robust nature of supersymmetry, there is a possibility to avoid this problem. We assume that at the level of the superconformal index the quantum gravity effects are not required. Further, in the finite-N region, we need additional contributions to the indices. At finite-N, the contribution of M-branes becomes effective, and to calculate finite-N corrections we have to include the contribution of M-branes wrapped on the internal space. Actually, calculating the contribution of the M-branes to the indices is the main work in this thesis.

For the ABJM theory, the finite-N corrections to the index are given by M5-branes

wrapped on a large S^5 in the internal space S^7. We confirm the validity of our formula by comparing the results of our formula with the localization results of the ABJM indices.

For the 6d $\mathcal{N} = (2, 0)$ theory, the finite-N corrections to the index are given by M2-branes wrapped on a large S^2 in S^4. We give new results of superconformal indices of the six-dimensional theories by using our method and decompose them in terms of the superconformal representations. In addition, we analyze a special limit of the superconformal index called the Schur-like index.

We also analyze the M-brane theories in the presence of the $\mathbb{Z}_k$ orbifolds. For the orbifold cases, the AdS/CFT correspondence claims that the 3d ABJM theory with Chern-Simon level k corresponds to M-theory on $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$, and the 6d $\mathcal{N} = (1, 0)$ theory corresponds to M-theory on $\mathrm{AdS}_7 \times S^4/\mathbb{Z}_k$. We analyze the indices of these SCFTs from the dual gravity theories. For 6d $\mathcal{N} = (1, 0)$ theories, in particular, we confirm strange flavor symmetries of the theories via the superconformal indices.

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Contents

Chapter 1

Introduction

Standard Model based on Quantum Field Theory (QFT) is a well-established framework in theoretical physics to describe our world. However, it is inadequate in some respects. One of the big problems is the missing of quantum gravity.

String theory is one candidate for theory including quantum gravity. There are five types of consistent superstring theories:

- Type I superstring theory
- Type IIA superstring theory
- Type IIB superstring theory
- $E_8 \times E_8$ heterotic superstring theory
- $SO(32)$ heterotic superstring theory

The idea that these five superstring theories have the same origin and there exists a unique underlying fundamental theory was proposed by Witten in 1995 [1], the theory is called M-theory. [1]

The precise definition of M-theory is still missing, but at least the theory has the following properties:

- In the low energy limit, M-theory is described by 11-dimensional supergravity.
- In a certain compactification, M-theory reproduces string-theory. Especially, S^1 compactification of M-theory gives type IIA superstring theory.

Since 11-dimensional supergravity contains a 3-form gauge field, M-theory has corresponding objects coupled to it. The electrically coupled object is called an M2-brane, and the magnetically coupled object is called an M5-brane. These M2-/M5- branes are objects expanding in 2+1/5+1 spacetime dimensions. Since these M-branes are fundamental objects

[1] The word "M-theory" was first used in [2]

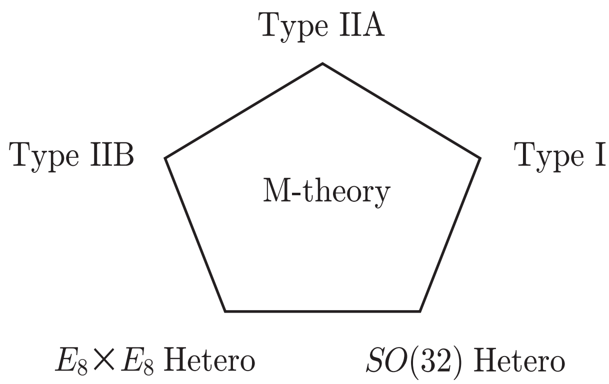

Figure 1.1: A schematic illustration of the relation between M-theory and five types of superstring theories is shown.

in M-theory, studying the nature of these branes is an important problem to understand M-theory.

Let us first recall the situation in the superstring theory. The discovery of D-branes in the era of second revolution [3] provides us great insights for understanding the superconformal field theories (SCFTs). We have learned that various SCFTs can be constructed as low energy theories realized on D-branes and we can analyze them by open string perturbations.

However, the situation is different for M-theory and analyzing the field theories on M2-/M5- branes are quite difficult. This is because we do not know how to quantize M2-/M5-branes. To avoid this problem, in this thesis, we use the AdS (Anti de Sitter) /CFT (Conformal Field Theory) correspondence [4]. The AdS/CFT correspondence, which is one of the greatest successes in string- (M-) theory, is a conjecture that claims the equivalence of a certain CFT and the corresponding gravity system. The equivalence of the two systems enables us to analyze two systems complementarily. The most famous example of the AdS/CFT is the correspondence between the $\mathcal{N} = 4$ $U(N)$ supersymmetric Yang-Mills (SYM) theory and type IIB superstring theory on $AdS_5 \times S^5$. This correspondence is understood by a N D3-branes system in type IIB superstring theory.

When N is large, the corresponding gravity theory is well approximated by classical supergravity. Therefore in this large-N region, AdS/CFT is well established and we can easily study physics of strongly coupled field theory by using AdS/CFT.

In this thesis, we rather try to analyze in the finite-N region. In general, due to quantum gravity effects, it was thought to be hard to use AdS/CFT in the finite-N region. However, there is a possibility that such quantum gravity effects might not affect supersymmetric protected quantities. From this assumption, we consider a quantity called a superconformal index. In short, the superconformal index is a kind of partition function, which includes the information of local gauge invariant operators.

The main theme of this thesis is to study theories realized on M2-/M5- branes via the superconformal index. The theory on M2-branes is called the Aharony-Bergman-Jafferis-Maldacena (ABJM) theory [5] and the theory on M5-branes is called the 6d $\mathcal{N} = (2, 0)$

theory. We propose a new method of calculating the superconformal indices of these theories from their dual gravity theories.

It is well known that, in the large-N limit, the superconformal indices are calculated from the bulk Kaluza-Klein modes of dual gravity theories. Recently, finite-N corrections to the index was calculated for some four-dimensional SCFTs [6, 7, 8, 9, 10]. In particular, in [6], the authors studied the superconformal index of the $\mathcal{N}=4$ $U(N)$ SYM theory from dual type IIB string theory on $\mathrm{AdS}_5 \times S^5$. In this case, they found that D3-branes wrapping on three-cycles in S^5 give finite-N corrections to the index and the index of the $\mathcal{N}=4$ $U(N)$ SYM theory was reproduced from the dual theory even in the finite-N region. Also, for other four-dimensional SCFTs, finite-N corrections to the indices were calculated from their dual gravity theories by introducing contributions of D3-branes in some three-cycles in the internal spaces.

We generalize their formulas to the M-brane theories and propose a method of calculating the indices which is available even in the finite-N region. For our M-theory setup, finite-N corrections to the indices are calculated from contributions of M-branes instead of D-branes. Namely, we will see finite-N corrections to the indices of ABJM theories are given by contributions of M5-branes in S^7 from dual M-theory on $\mathrm{AdS}_4 \times S^7$ and also finite-N corrections to the indices of 6d (2,0) theories are given by contributions of M2-branes in S^4 from dual M-theory on $\mathrm{AdS}_7 \times S^4$.

For M2-brane theories, the theories of multiple M2-branes are described by the ABJM theories and the indices of the ABJM theories were calculated in [11] by using the localization method. We compare the indices calculated by using our formula with ABJM indices and find a nice agreement even in the finite-N region.

For M5-brane theories, unlike the situation of M2-branes, the Lagrangians of theories on multiple M5-branes are not yet known. Thus we cannnot use the localization method to obtain the index except for $N=1$. For the 6d theories, by using our formula, we propose new results for the indices of these unknown theories. Further, for the orbifold case, we confirm strange flavor symmetries of 6d $\mathcal{N}=(1,0)$ theories via the superconformal indices.

This thesis is organized as follows.

In the rest of Chapter 1, we review the basic facts about type IIA superstring theory, M-theory, the AdS/CFT correspondence, and the superconformal index. In particular, we explain the low energy description of M-theory: eleven-dimensional supergravity, and the basics of M-theory focused on M2-/M5- branes. Then we review the low energy limit of type IIA superstring theory: ten-dimensional type IIA supergravity, the basics of type IIA superstring theory mainly focused on charged objects appearing in type IIA superstring theory, and the relation between type IIA superstring theory and M-theory. We then explain the basics of the AdS/CFT correspondence. Also, we introduce the superconformal index as a generalized Witten index and discuss the calculation method of the index. We also discuss the supersymmetric localization, which is an essential technique to perform exact analyses of supersymmetric field theories.

In Chapter 2, we review previously known superconformal indices of theories realized on M2-branes. We first explain the theory realized on a single M2-brane and calculate its index. Since the single M2-brane theory is a free theory, this can be easily done. Then, we review theories on multiple M2-branes and their superconformal indices. We show explicit results for small N. We also review the index calculation from the dual gravity side in the large-N limit. We give the explicit result of the index from the Kaluza Klein modes. In addition, we discuss the $\mathbb{Z}_k$ orbifold case and calculate the index for this case.

In Chapter 3, we review previously known superconformal indices of theories realized on M5-branes. We first explain a single M5-brane theory and calculate the index of the theory. Then, we show the large-N index, which can be calculated from the dual supergravity in $\text{AdS}_7 \times S^4$ by using AdS/CFT correspondence.

In Chapter 4, we calculate the superconformal indices of M2-brane theories from the dual gravity side at finite-N. This chapter is based on the author's and his collaborator's original work [12]. We compare our results with the indices of ABJM theories calculated in Chapter 2 and confirm the validity of our formula. We also perform the same analysis for the orbifold case.

In Chapter 5, we calculate the superconformal indices of M5-brane theories at finite-N from the dual gravity side. This analysis is based on our paper [12]. We compare our result with the $N = 1$ 6d $\mathcal{N} = (2,0)$ index and confirm our formula for the 6d case. Further, we give new results for the 6d indices with $N > 1$. We also study the 6d $\mathcal{N} = (1,0)$ case corresponding to the $\mathbb{Z}_k$ orbifold case, which is based on our paper [13].

Chapter 6 is devoted to the conclusions.

1.1 M-theory

1.1.1 11-dimensional supergravity

Let us first discuss the eleven-dimensional supergravity, which is believed to be the low energy description of M-theory. Eleven dimensions are the largest dimensions of supergravity theory with spins ≤ 2 [14]. As a field content, this theory contains a graviton g_{MN}, a rank 3 anti-symmetric tensor field A_{MNP} and their supersymmetric partner called gravitino Ψ_M, where $M, N, P = 0, 1, ..., 9, 11$ are spacetime indices in eleven dimensions. Since the the theory is supersymmetric, the bosonic degrees of freedom **44**+**84** matches fermionic degrees of freedom **128**, see Table 1.1.

field contents of 11d supergravity		
graviton	g_{MN}	**44**
3-form gauge field	A_{MNP}	**84**
gravitino	Ψ_M	**128**

Table 1.1: field contents and degree of freedom of 11d supergravity are shown.

From the requirement of the supersymmetry and invariance under general coordinate transformation and local Lorentz transformation, the action of the 11-dimensional supergravity (with two or fewer derivatives) is uniquely determined up to field redefinitions. The bosonic part of the action is given by

$$S = \frac{1}{16\pi G_{11}} \left[\int d^{11}x\sqrt{-g} \left(R - \frac{1}{2}|F_4|^2 \right) - \frac{1}{3!} \int A_3 \wedge F_4 \wedge F_4 \right], \tag{1.1}$$

where G_{11} is the 11-dimensional Newton constant, which is related to the 11-dimensional Planck length l_p by

$$16\pi G_{11} = \frac{1}{2\pi}(2\pi l_p)^9. \tag{1.2}$$

Also, R is the Ricci scalar and F_4 is a four form field strength defined by $F_4 = dA_3$ and g is the determinant of the metric.

Differential form

For a rank p anti-symmetric tensor field $A_{\mu_1 \dots \mu_p}$, we can define the p-form A_p as

$$A_p = \frac{1}{p!} A_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \cdots dx^{\mu_p}, \tag{1.3}$$

where $\wedge$ is a wedge product satisfying

$$dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu. \tag{1.4}$$

We can define the exterior derivative d as follows:

$$dA_p = \frac{1}{p!} \partial_\mu A_{\mu_1 \dots \mu_p} dx^\mu \wedge dx^{\mu_1} \wedge \cdots dx^{\mu_p}. \tag{1.5}$$

We also define

$$|F_p|^2 = \frac{1}{p!} g^{\mu_1 \nu_1} \cdots g^{\mu_p \nu_p} F_{\mu_1 \dots \mu_p} F_{\nu_1 \dots \nu_p}. \tag{1.6}$$

1.1.2 M-brane

To see the relation between the eleven-dimensional supergravity and M-theory, we first explain "M-branes", which are extended objects in M-theory. To introduce M-branes, let us first recall the ordinary 4d electromagnetism. The action of the particle of mass m with charge q is given by

$$S = -m \int_C ds + q \int_C A_1, \tag{1.7}$$

where C is a worldline of the particle and $A_1 = A_\mu dx^\mu$ is a 1-form gauge field. The second term of the action represents the coupling of the charged particle and the 1-form field. This charged particle can be regarded as the 0-brane. This term can be easily generalized to the higher form fields in other dimensions. As we saw in the previous subsection, the 11-dimensional supergravity contains 3-form gauge potential A_3. Then, in M-theory the corresponding charged objects exist. The electrically charged object is called the M2-brane. The M2-brane couples to the 3-form potential by

$$S_{M2} = q_{M2} \int_{V_{\rm M2}} A_3, \tag{1.8}$$

where q_{M2} is the charge of the M2-brane and $V_{\rm M2}$ is a three-dimensional hypersurface swept by the M2-brane. We call such hypersurfaces as worldvolume. Hence, the M2-brane is a membrane which spatially expands in two dimensions. (In general, a p-brane is an object whose spatial dimension is p.)

We can also introduce the magnetically charged object called the M5-brane. By denoting the dual field of A_3 as A_6, the coupling of the M5-brane and A_6 is given by

$$S_{M5} = q_{M5} \int_{V_{\rm M5}} A_6, \tag{1.9}$$

where q_{M5} is the charge of the M5-brane and $V_{\rm M5}$ is the worldvolume of the M5-brane whose dimension is 5+1. The detailed analysis shows that charges q_{M2} and q_{M5} are given by

$$q_{M2} = \frac{2\pi}{(2\pi l_p)^3}, \qquad q_{M5} = \frac{2\pi}{(2\pi l_p)^6}. \tag{1.10}$$

The first term of (1.7) is also easily generalized to the p-brane case. The action is called the Nambu-Goto action and is given by

$$S_{\rm NG} = -T \int d^{p+1}\sigma \sqrt{-\det G_{ab}}, \tag{1.11}$$

where σ^a $(a = 0, ..., p)$ are coordinates on the brane and G_{ab} is the induced metric defined by

$$G_{ab} = \frac{\partial x^M}{\partial \sigma^a} \frac{\partial x^N}{\partial \sigma^b} g_{MN}, \tag{1.12}$$

with g_{MN} is the metric of the eleven-dimensional target space. The integral in (1.11) now gives the worldvolume of the p-brane instead of worldline and also the coefficient T is the tension of the brane. Note that the whole action of the brane includes fermionic and gauge fields contribution which we omit here, but these terms are not necessary to the analysis in the rest of this thesis.

BPS M-brane

In the supersymmetry algebra of 11 dimensions, the commutation relation of supercharges is given by

$$\{Q, Q^\dagger\} = (\Gamma_M \Gamma_0) P^M, \tag{1.13}$$

where the supercharge Q is a 32-component spinor and each component is hermitian and $\Gamma_M (M = 0, 1, ..., 9, 11)$ are the Gamma matrices in eleven dimensions.

It is also possible to add more central charges:

$$\{Q, Q^\dagger\} = (\Gamma_M \Gamma_0) P^M + \frac{1}{2}(\Gamma_{MN}\Gamma_0) Z^{MN}_{M2} + \frac{1}{5!}(\Gamma_{M_1...M_5}\Gamma_0) Z^{M_1...M_5}_{M5}, \tag{1.14}$$

where $Z^{MN}_{\rm M2}$ and $Z^{M_1...M_5}_{\rm M5}$ are central charges associated to the M2-brane and M5-brane respectively. Note that due to the symmetric nature of the indices on the left-hand side, only these two central charges are allowed.

Let us consider a situation with only non-vanishing $Z^{12}_{\rm M2}$. In the frame $P^0 = E$, the anti-commutation relation (1.14) becomes

$$\{Q, Q^\dagger\} = E + (\Gamma_{12}\Gamma_0) Z^{12}_{\rm M2} \tag{1.15}$$

Due to the positivity of the left hand side and the fact that eigenvalues of the $\Gamma_{12}\Gamma_0$ is ± 1, we can find

$$E \geq |Z^{12}_{\rm M2}|. \tag{1.16}$$

This bound is called the Bogomol'nyi-Prasad-Sommerfield (BPS) bound. We normalize Z^{12}_{M2} such that the single M2-brane has charge 1. Then we obtain

$$E \geq q_{\rm M2} |\tilde{Z}^{12}_{M2}| \tag{1.17}$$

The prefactor declares the energy of a single M2-brane of unit volume, i.e. the tension of BPS M2-brane, thus from the first relation in (1.10) we obtain the follwing M2-brane tension:

$$T_{\rm M2} = \frac{2\pi}{(2\pi l_{\rm p})^3}. \tag{1.18}$$

Next, we consider the BPS M5-brane. similarly to the M2-brane case, we consider non-vanishing Z^{12345}_{M5} and $P^0 = E$ while other terms are zero:

$$\{Q, Q^\dagger\} = E + (\Gamma_{12345}\Gamma_0) Z^{12345}_{\rm M5} \tag{1.19}$$

This leads to the following inequality.

$$E \geq |Z^{12345}_{\rm M5}| \tag{1.20}$$

Again, we normalize $Z^{12345}_{\rm M5}$ such that the single M5-brane has charge 1 and obtain

$$E \geq q_{\rm M5} |\tilde{Z}^{12345}_{M5}|, \tag{1.21}$$

Then, from the second relation in (1.10) we obtain the follwing M5-brane tension:

$$T_{\rm M5} = \frac{2\pi}{(2\pi l_{\rm p})^6}. \tag{1.22}$$

1.2 Type IIA superstring theory

Among five types of superstring theories, the theory with $\mathcal{N} = (1,1)$ supersymmetry is called type IIA superstring theory which is non-chiral in ten dimensions. Let us first see its low energy effective theory, type IIA supergravity.

1.2.1 Type IIA supergravity

Type IIA supergravity theory is a ten-dimensional supergravity theory. This theory contains graviton $g_{\mu\nu}$ ($\mu, \nu = 0, 1, ..., 9$), 2-form field called B-field $B_{\mu\nu}$, dilaton ϕ, Ramond-Ramond fields $C_\mu, C_{\mu\nu\rho}$, and also fermionic fields. The field contents are summarized in Table 1.2.

Type IIA supergravity multiplet		
graviton	$g_{\mu\nu}$	$\mathbf{35}$
NS-NS 2-form	$B_{\mu\nu}$	$\mathbf{28}$
dilaton	Φ	$\mathbf{1}$
RR 1-form	C_μ	$\mathbf{8}$
RR 3-form	$C_{\mu\nu\rho}$	$\mathbf{56}$
gravitino	$\Psi_\mu^{(+)}$	$\mathbf{56}_+$
gravitino	$\Psi_\mu^{(-)}$	$\mathbf{56}_-$
dilatino	$\lambda^{(+)}$	$\mathbf{8}_+$
dilatino	$\lambda^{(-)}$	$\mathbf{8}_-$

Table 1.2: The field contents of type IIA supergravity theory are listed.

The bosonic part of the ten-dimensional type IIA supergravity consists of three parts.

$$S_{\mathrm{IIA}} = S_{\mathrm{NS}} + S_{\mathrm{R}} + S_{\mathrm{CS}}. \tag{1.23}$$

The first term S_{NS} is given by

$$S_{\mathrm{NS}} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}e^{-2\Phi}\left(R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}|H_3|^2\right), \tag{1.24}$$

where $H_3 = dB_2$. The second term S_{R} is given by

$$S_{\mathrm{R}} = -\frac{1}{4\kappa^2}\int d^{10}x\sqrt{-g}\left(F_2^2 + \tilde{F}_4^2\right), \tag{1.25}$$

where $F_2 = dC_1$ and $\tilde{F}_4$ is defined by

$$\tilde{F}_4 = dC_3 + C_1 \wedge H_3. \tag{1.26}$$

Finally, the last term S_{CS} is

$$S_{\mathrm{CS}} = -\frac{1}{4\kappa^2}\int d^{10}x\sqrt{-g}\left(B_2 \wedge F_4 \wedge F_4\right), \tag{1.27}$$

The relation of string length l_s and string coupling constant g_s and 10d Newton constant κ is as follows:

$$2\kappa^2 = 2\kappa_{10}^2/g_s^2 = \frac{1}{2\pi}(2\pi l_s)^8. \tag{1.28}$$

$$16\pi G_{10} = 2\kappa_{10}^2 = \frac{1}{2\pi}(2\pi l_s)^8 g_s^2. \tag{1.29}$$

1.2.2 Objects in type IIA superstring theory

Since type IIA supergravity contains various higher rank gauge fields, type IIA superstring theory has corresponding extended objects couple to those fields.

Fundamental string

The most important one is the fundamental string $F1$. The $F1$ string electrically couples to 2-form B-field. The motion of the string is described by the following Polyakov action:

$$S_{\text{Polyakov}} = -\frac{T_{F1}}{2}\int d^2\sigma\sqrt{-h}h^{ab}\partial_a X^\mu \partial_b X_\mu, \tag{1.30}$$

where the tension of the string T_{F1} is given by

$$T_{F1} = \frac{1}{2\pi l_s^2}. \tag{1.31}$$

The length l_s is called string length, a unique length of the string world sheet action.

Dp-brane

The Dp-brane is an object that electrically couples to R-R $p+1$ form (or dual $7-p$ form). Type IIA superstring theory contains Dp-branes with $p = 0, 2, 4, 6, 8$. The action of Dp-brane is given by Dirac-Born-Infeld type action:

$$S_{\text{DBI}} = -\frac{2\pi}{(2\pi l_s)^{p+1}}\int d^{p+1}\sigma e^{-\Phi}\sqrt{-\det(G_{ab} + B_{ab} + 2\pi l_s^2 F_{ab})}. \tag{1.32}$$

The vacuum expectation value of the dilaton gives the string coupling constant $g_s = e^{<\Phi>}$. Then the tension of Dp brane is given by

$$T_{Dp} = \frac{2\pi}{(2\pi l_s)^{p+1} g_s}. \tag{1.33}$$

NS5-brane

NS5-brane is an object that magnetically couples to 2-form B field. Let μ_{F1} be a unit B-field electric charge of a fundamental string. Also, let μ_{NS5} be a unit B-field magnetic charge of an NS5-brane. The Dirac quantization condition reads

$$\mu_{F1}\mu_{NS5} \cdot 2\kappa_{10}^2 \in 2\pi\mathbb{Z}. \tag{1.34}$$

Thus, the magnetic charge of a NS5-brane is given by

$$\mu_{NS5} = \frac{2\pi}{(2\pi l_s)^6 g_s^2}. \tag{1.35}$$

For supersymmetric NS5-brane, this should be equal to the tension of NS5-brane, hence we obtain the following tension of NS5-brane:

$$T_{NS5} = \frac{2\pi}{(2\pi l_s)^6 g_s^2}. \tag{1.36}$$

1.3 Relation between M-theory and type IIA superstring theory

1.3.1 11d/IIA supergravity

The 11-dimensional supergravity action is related to the type IIA supergravity action by the dimensional reduction. We compactify eleven-dimensional direction x^{11} as a circle with radius R_{11}.

Let us roughly see how the bosonic field contents of the two theories are related. The 3-form field A_{MNP} in 11 dimensions gives the RR 3-form field $C_{\mu\nu\rho}$ and the 2-from B-field $B_{\mu\nu}$ in IIA supergravity:

$$A_{\mu\nu\rho} \sim C_{\mu\nu\rho}, \qquad A_{\mu\nu 11} \sim B_{\mu\nu}. \tag{1.37}$$

The metric $g^{(11)}_{MN}$ in 11 dimension reduces to 10 d metric $g_\mu\nu^{(10)}$ and RR 1-form C_μ and dilaton Φ:

$$g^{(11)}_{\mu\nu} \sim g^{(10)}_{\mu\nu}, \qquad g^{(11)}_{\mu 11} \sim C_\mu, \qquad g^{(11)}_{11,11} \sim \Phi. \tag{1.38}$$

1.3.2 M/IIA

Let R_{11} be a radius of compactification circle in M-theory. We can relate M-theory with type IIA string theory by the following parameter identification

$$l_p^3 = l_s^3 g_s, \tag{1.39}$$

together with the relation of the Newton constant in 11 dimension and 10 dimension

$$G_{11} = 2\pi R_{11} G_{10}. \tag{1.40}$$

These two relations imply that in terms of l_s and g_s, radius R_{11} is given by

$$R_{11} = g_s l_s. \tag{1.41}$$

This means that the uncompactifying limit $R \to \infty$ corresponds to the strong coupling limit. Namely, M-theory is the strong coupling limit of the type IIA string theory.

Various objects in type IIA string theory are also obtained from M-theory. We first discuss the compactification of the M2-brane. We can consider two cases depending on whether the M2-brane is wrapped around the compactified direction S_1 or not. When an M2-brane wrapped around the S^1, it becomes a fundamental string. Actually, from the equations (1.39) and (1.41), the tension of the M2-brane and the tension of the fundamental string are related with

$$2\pi R_{11} T_{M2} = T_{F1}. \tag{1.42}$$

On the other hand, an M2-brane not spreading along S^1 direction becomes a D2-brane. We can easily find the following relation between the tension of M2-brane and M2-brane.

$$T_{\mathrm{M2}} = T_{\mathrm{D2}}. \tag{1.43}$$

Next, we consider the compactification of the M5-brane. When an M5-brane is wrapped around the S^1, it becomes a D4-brane. We can find the relation between the tensions of these objects

$$2\pi R_{11} T_{M5} = T_{D4}. \tag{1.44}$$

On the other hand, if an M5-brane does not wrap the S^1 direction, it becomes an NS5-brane. We can confirm the following relation

$$T_{\mathrm{M5}} = T_{\mathrm{NS5}}. \tag{1.45}$$

We show a list of the relation between objects in M/IIA theory in Table 1.3.

1.4 AdS/CFT correspondence

The AdS/CFT correspondence is a conjecture of a duality between a certain CFT and the corresponding gravity theory in AdS spacetime, which was proposed in 1997 [4]. Let us first see the most famous example, the correspondence between the four-dimensional $\mathcal{N} = 4$ $U(N)$ SYM theory and type IIB string theory on $\mathrm{AdS}_5 \times S^5$. This correspondence can be understood by a N D3-branes system. We consider a stack of N D3-branes in ten-dimensional spacetime. We have two points of view of the system. First, the field theory view comes from

dimension	object in M-theory	S^1	tension	object in IIA	tension
0	KK-particle		$\frac{1}{R_{11}}$	D0-brane	$\frac{1}{g_s l_s}$
1	M2-brane	○	$\frac{2\pi}{(2\pi l_p)^3}$	F1-string	$\frac{1}{2\pi l_s^2}$
2	M2-brane		$\frac{2\pi}{(2\pi l_p)^3}$	D2-brane	$\frac{2\pi}{(2\pi l_s)^3 g_s}$
4	M5-brane	○	$\frac{2\pi}{(2\pi l_p)^6}$	D4-brane	$\frac{2\pi}{(2\pi l_s)^5 g_s}$
5	M5-brane		$\frac{2\pi}{(2\pi l_p)^6}$	NS5-brane	$\frac{2\pi}{(2\pi l_s)^6 g_s^2}$
6	KK-monopole		$\frac{R_{11}^2}{(2\pi)^6 l_p^9}$	D6-brane	$\frac{2\pi}{(2\pi l_s)^7 g_s}$

Table 1.3: The relation of objects appearing in M/IIA theory

the worldvolume theory on the D3-branes. At the low energy limit, four-dimensional $\mathcal{N} = 4$ $U(N)$ SYM theory is realized on the D3-branes. On the other hand, the geometry created by the D3-branes becomes $\mathrm{AdS}_5 \times S^5$ in the near horizon limit. Thus, the N D3-branes system in the near horizon limit is described by type IIB superstring theory on $\mathrm{AdS}_5 \times S^5$. The AdS/CFT correspondence argues that these two descriptions are equivalent.

Here, we show the parameter relations of the system. In the AdS/CFT correspondence, the Yang-Mills coupling g_{YM} of the gauge theory, the string coupling g_s, the string length l_s, the AdS radius L, and the number of D3-branes N are related by the following relations.

$$g_{\mathrm{YM}}^2 \sim g_s, \qquad L^4 \sim g_s N l_s^4. \tag{1.46}$$

Note that in the region where $\lambda \equiv g_{\mathrm{YM}}^2 N \gg 1$ and $N \gg 1$, the CFT is in the strongly coupled region, and the gravity theory is approximated by the classical type IIB supergravity.

Now we consider M-theory setup. For a stack of N M2-branes in eleven-dimensional spacetime, the near horizon geometry becomes $\mathrm{AdS}_4 \times S^7$. See Subsection 2.5.1 for more detail. On the other hand, the worldvolume theory on the M2-branes is described by the ABJM theory. Therefore, the AdS/CFT correspondence argument is the following.

- M-theory on $\mathrm{AdS}_4 \times S^7$ is equivalent to the ABJM theory.

Similarly, if we consider a stack of N M5-branes, the near horizon geometry becomes $\mathrm{AdS}_7 \times S^4$. See Subsection 3.4.1 for more detail. The worldvolume theory on the M5-branes is called the 6d $\mathcal{N} = (2,0)$ theory, and AdS/CFT statement is the following.

- M-theory on $\mathrm{AdS}_7 \times S^4$ is equivalent to the 6d $\mathcal{N} = (2,0)$ theory.

1.5 Superconformal index

1.5.1 Witten index

To study quantum field theories in the strong coupling region, it is useful if we can define a quantity which does not depend on the coupling constant. Fot theories with supersymmetry, we can define such a quantity. The simplest one is called the Witten index [15]. The Witten index is defined by

$$\mathcal{I}_W = \mathrm{tr}_{\mathcal{H}} \left[(-1)^F e^{-\beta\Delta} \right], \tag{1.47}$$

where trace is taken over all states of the theory, F is the fermionic number which gives 0 for bosons and 1 for fermions. Δ is the operator obtained from the anticommutaion relation of a supercharge Q and its hermitian conjugate $Q^\dagger$

$$\Delta = \{Q, Q^\dagger\}. \tag{1.48}$$

Note that for the supersymmetric quantum field mechanics this Δ gives the Hamiltonian of the theory. These supercharges have nilpotent nature $Q^2 = (Q^\dagger)^2 = 0$. Thus, Δ and Q ($Q\dagger$) is commutative,

$$[\Delta, Q] = [\Delta, Q^\dagger] = 0. \tag{1.49}$$

By using these features, we can show that Δ is positive definite. Let be $|\psi\rangle$ an eigenstate of Δ:

$$\Delta |\psi\rangle = \Delta |\psi\rangle . \tag{1.50}$$

Sandwiching (1.48) by $|\psi\rangle$, we obtain

$$\Delta = |Q|\psi\rangle|^2 + |Q^\dagger|\psi\rangle|^2 \geq 0. \tag{1.51}$$

Here we used positivity of the norm since we are interested in unitary theories.

Next, we show that the Witten index receives contributions from only $\Delta = 0$ states. Let C_{Δ_0} be sets of states with positive energy $\Delta = \Delta_0 > 0$. If a state $|\phi\rangle$ is included in C_{Δ_0}, then we can write

$$|\phi\rangle = \frac{\Delta_0}{\Delta_0} |\phi\rangle = Q\frac{Q^\dagger}{\Delta_0} |\phi\rangle + Q^\dagger \frac{Q}{\Delta_0} |\phi\rangle . \tag{1.52}$$

Among C_{Δ_0}, we represent a set of states annihilated by Q as $C^Q_{\Delta_0}$ and a set of states annihilated by $Q^\dagger$ as $C^{Q^\dagger}_{\Delta_0}$. Then C_{Δ_0} can be written as

$$C_{\Delta_0} = C^Q_{\Delta_0} + C^{Q^\dagger}_{\Delta_0}. \tag{1.53}$$

Since the action of $Q^\dagger$ on $|\phi\rangle$ is given by

$$Q^\dagger |\phi\rangle = Q^\dagger Q\frac{Q^\dagger}{\Delta_0} |\phi\rangle + Q^\dagger Q^\dagger \frac{Q}{\Delta_0} |\phi\rangle \tag{1.54}$$

$$= Q^\dagger Q\frac{Q^\dagger}{\Delta_0} |\phi\rangle , \tag{1.55}$$

this gives the 1 to 1 map from $C^{Q}_{\Delta_0}$ to $C^{Q^\dagger}_{\Delta_0}$. Actually, if we represent a state in $C^{Q}_{\Delta_0}$ as $|\phi^Q\rangle$,

$$QQ^\dagger |\phi^Q\rangle = \left(\Delta_0 - Q^\dagger Q\right) |\phi^Q\rangle = \Delta_0 |\phi^Q\rangle \tag{1.56}$$

holds and Q/Δ_0 gives the inverse map. Hence, states of positive energy always appear with a pair of bosonic state and a fermionic state, and contribution of the index is canceled out each other due to the $(-1)^F$ factor. Thus, only the states with $\Delta = 0$ contribute to the Witten index and the Witten index is also expressed as

$$\mathcal{I}_W = (\#\text{ of bosonic states with } \Delta = 0) - (\#\text{ of fermionic states with } \Delta = 0) \tag{1.57}$$

If we change a parameter of the theory, say a coupling constant, some states may be excited and both the number of bosonic states with $\Delta = 0$ and the number of fermionic states with $\Delta = 0$ could change. However, the difference between them, i.e., the Witten index, does not change since the states with $\Delta > 0$ always form a pair of a bosonic state and a fermionic state.

So far, we have seen that the Witten index (1.47) receives contributions only from states with $\Delta = 0$. This indicates that the index is actually independent of β. We can directly check this fact by differentiating the Witten index

$$\frac{d\mathcal{I}_W}{d\beta} = -\operatorname{tr}_{\mathcal{H}}\left[(-1)^F \Delta e^{-\beta\Delta}\right] = 0. \tag{1.58}$$

This follows from the equation below obtained from cyclic nature of the trace:

$$\operatorname{tr}_{\mathcal{H}}\left[(-1)^F QQ^\dagger e^{-\beta\Delta}\right] = -\operatorname{tr}_{\mathcal{H}}\left[(-1)^F Q^\dagger Q e^{-\beta\Delta}\right]. \tag{1.59}$$

The condition $\Delta = 0$ is the necessary and sufficient condition for

$$Q|\psi\rangle = Q^\dagger |\psi\rangle = 0. \tag{1.60}$$

Hence, these states preserve the supersymmetry and calld the Bogomol'nyi-Prasad-Sommerfield (BPS) states.

If the Witten index is not zero, this ensures the existence of zero energy states, so supersymmetry is not spontaneously broken. However, if the Witten index is zero, we have two possibilities.

- Neither bosonic zero energy states nor fermionic zero energy states exist. Therefore, supersymmetry is broken.

- The equal number of bosonic zero energy states and fermionic zero energy states exist. Therefore, supersymmetry is unbroken.

Unfortunately, in this situation we cannot distinguish these two cases.

1.5.2 Example: a supersymmetric harmonic oscillator

Let us consider the Witten index of a supersymmetric harmonic oscillator as a simple example [16]. The Lagrangian of a supersymmetric harmonic oscillator which includes a bosonic corrdinate x and its fermionic partner $\psi, \bar{\psi}$, is given by

$$L = \frac{1}{2}\dot{x} - \frac{1}{2}x^2 + i\bar{\psi}\dot{\psi} - \bar{\psi}\psi. \tag{1.61}$$

The corresponding Hamiltonian is

$$H = \frac{1}{2}p^2 + \frac{1}{2}x^2 + \bar{\psi}\psi. \tag{1.62}$$

We consider the quantum mechanics described by the Hamiltonian with commutation relations $[x, p] = i$ and $\{\psi, \bar{\psi}\} = 1$. The bosonic creation and the annihilation operator are defined as

$$a_B^\dagger = \frac{1}{\sqrt{2}}(-ip + x), \qquad a_B = \frac{1}{\sqrt{2}}(ip + x), \tag{1.63}$$

which satisfy

$$[a_B, a_B^\dagger] = 1. \tag{1.64}$$

On the other hand, the fermionic creation and the annihilation operator are given by

$$a_F^\dagger = \bar{\psi}, \qquad a_F = \psi, \tag{1.65}$$

which satisfy

$$\{a_F, a_F^\dagger\} = 1. \tag{1.66}$$

In terms of these operators the Hamiltonian is given by

$$H = a_B^\dagger a_B + a_F^\dagger a_F. \tag{1.67}$$

We define supercharges as

$$Q = a_B^\dagger a_F, \qquad Q^\dagger = a_F^\dagger a_B. \tag{1.68}$$

This satisfy following algebra

$$\{Q, Q^\dagger\} = H, \qquad [H, Q] = [H, Q^\dagger] = 0. \tag{1.69}$$

The usual thermal partition function is given by

$$Z = \mathrm{tr}_{\mathcal{H}}\, x^H = \frac{1+x}{1-x}, \qquad x = e^{-\beta} \tag{1.70}$$

The Witten index is given by

$$\mathcal{I}_W = \mathrm{tr}_{\mathcal{H}}(-1)^F x^H = 1. \tag{1.71}$$

1.5.3 Superconformal index as generalized Witten index

In the presence of additional global symmetries, we can generalize the Witten index by adding fugacities associated with the symmetries. Then the generalized Witten index has following form:

$$\mathcal{I} = \mathrm{tr}\left[(-1)^F e^{-\beta\Delta}\mu_i^{\mathcal{M}_i}\right], \tag{1.72}$$

$$\Delta = \{Q, Q^\dagger\}. \tag{1.73}$$

where $\mathcal{M}_i$ are generators of global symmetries and μ_i are associated fugacities. In order to repeat the discussion in the previous subsection, the generators $\mathcal{M}_i$ must commute Q and $Q^\dagger$ (hence commute with Δ). Then, again boson/fermion cancellation occurs for $\Delta > 0$ states and the generalized Witten index receives contributions from $\Delta = 0$ states, meaning that the index (1.72) is independent of β.

In general, to remove the IR divergence, quantum field theories are defined on a compact manifold with finite volume. For Witten indices of d dimensional field theories, the torus T^{d-1} is usually used. Instead of T^{d-1}, We can also use a $d-1$ dimensional sphere S^{d-1}. In addition, if the theory has conformal symmetry, we can relate a state defined on $S^{d-1}\times\mathbb{R}$ with an operator inserted at the origin of $\mathbb{R}^d$. Therefore, in such a case the index has information on local gauge invariant operators. The index is especially called the superconformal index [17]. In this case, $\mathcal{M}_i$ can contain generators of a subalgebra of the superconformal symmetry which commute with Q and $Q^\dagger$ and also generators of other global symmetries in the theory.

This superconformal index is the main tool we use throughout this thesis. See Chapter 2 and 3 for precise definition of the superconformal index in 3 and 6 dimensions.

1.5.4 Calculation of the superconformal index for free theories

To calculate the superconformal index, it is useful to define "plethystic exponential" Pexp by

$$\mathrm{Pexp}\left[f(x_i)\right] = \exp\left[\sum_{n=1}^{\infty}\frac{1}{n}f(x_i^n)\right]. \tag{1.74}$$

This is equivalent to the replacement of monomial $c_i x_i$ to $\frac{1}{(1-x_i)^{c_i}}$:

$$\mathrm{Pexp}\left[\sum_i c_i x_i\right] = \prod_i \frac{1}{(1-x_i)^{c_i}}. \tag{1.75}$$

Let us explain how to calculate the superconformal index for free theories. For theories without gauge symmetry, the superconformal index is easily calculated by the plethystic exponential of the single-particle index i_{sp}:

$$\mathcal{I} = \mathrm{Pexp}[i_{sp}] \tag{1.76}$$

For theories with gauge symmetry, we have to pick up the gauge singlet states. This is achieved by the following integral:

$$\mathcal{I} = \int d\mu \, \mathrm{Pexp}[i_{sp}], \tag{1.77}$$

where $d\mu$ is the Haar measure and i_{sp} is a single-particle index including the gauge fugacities.

1.5.5 Localization formula

In general, it is difficult to calculate the partition function of quantum field theories exactly since it contains an infinite-dimensional integral, i.e., a path integral. However, if a theory has supersymmetry , sometimes we can reduce infinite-dimensional integral to finite-dimensional integral, and the calculation of the exact partition function becomes possible. This method is called supersymmetric localization [18]. The localization method is a very powerful tool and is widely used in supersymmetric gauge theories. For the concrete examples/review see [19] and reference there in.

In this subsection, we shortly review the mechanism of the supersymmetric localization. We consider the following partition function in a compact manifold $\mathcal{M}$:

$$Z_{\mathcal{M}} = \int D\phi e^{-S[\phi]}. \tag{1.78}$$

Of course, this is an infinite dimensional integral, and it is difficult to carry out the integral generally. We perform following deformation by adding Q-exact term with parameter t.

$$Z_{\mathcal{M}} = \int D\phi e^{-S[\phi]-tQV[\phi]}, \tag{1.79}$$

where $V[\phi]$ is a fermionic (Grassmann odd) function and we demand $Q^2V[\phi] = 0$. Note that the original action is invariant under the transformation of Q: $QS[\phi] = 0$. Also, $t \geq 0$ is a deformation parameter and $t = 0$ reproduces the original partition function $Z_{\mathcal{M}} = Z_{\mathcal{M}}(t = 0)$. It seems that $Z_{\mathcal{M}}(t)$ depends on t, but it is not true. To ses this, we derivatiate (1.79) by t and find

$$\frac{dZ_{\mathcal{M}}(t)}{dt} = -\int D\phi QVe^{-S-tQV} = -\int D\phi Q\left(Ve^{-S-tQV}\right) = 0, \tag{1.80}$$

where we used $QS[\phi] = 0$ and assumed that the integral measure is invariant. Hence, the equation (1.79) is independent of t. For the calculation we take $t \to \infty$ limit

$$Z_{\mathcal{M}} = \lim_{t\to\infty} Z_{\mathcal{M}}(t). \tag{1.81}$$

We choose $V[\phi]$ satisfying $QV[\phi] \geq 0$, then at $t \to \infty$ contributions of $QV[\phi] > 0$ are suppressed and only the saddle points configurations satisfying $QV[\phi] = 0$ contribute to the path integral. Let us denote the saddle points by ϕ_n. Now, the path integral reduces to Gauss integral around ϕ_n. To perform this, we take

$$\phi = \phi_n + t^{-\frac{1}{2}}\hat{\phi}. \tag{1.82}$$

Substituting this for the deformed action, we obtain

$$S + tQV = S[\phi_n] + \left(QV_{\phi_n}\right)^{(2)}[\hat{\phi}] + \mathcal{O}(t^{-\frac{1}{2}}), \tag{1.83}$$

where $\left(QV_{\phi_n}\right)^{(2)}[\hat{\phi}]$ is the second order term of $\hat{\phi}$ and the first order term vanishes since $QV[\phi_n]$ is a local minimum of $QV[\phi]$. Then the partition function becomes

$$Z_{\mathcal{M}} = \sum_n e^{-S(\phi_n)} Z_{1-\mathrm{loop}}(\phi_n), \qquad Z_{1-\mathrm{loop}}(\phi_n) = \int D\hat{\phi} e^{-\left(QV_{\phi_n}\right)^{(2)}[\hat{\phi}]}. \tag{1.84}$$

where $Z_{1-\mathrm{loop}}(\phi_n)$ represents the Gauss integral and often called the 1-loop determinant. Here we assumed that saddle points are discrete. If not, we need integration along the spreading direction.

Chapter 2

Superconformal indices of M2-brane theories

In this chapter, we review the superconformal indices of theories realized on M2-branes. The theory on a single M2-brane is described by free scalar fields and their supersymmetric partners. Therefore, its index calculation is relatively simple.

When the number of M2-branes N is larger than one, how theories on multiple coincident M2-branes are described has been a long-standing problem in itself. For the $N = 2$ case, Bagger, Lambert, and Gusstavson succeeded to construct a 3d $\mathcal{N} = 8$ Chern-Simons theory which describes the theory on the two M2-branes [20, 21, 22, 23, 24], and now it is called the BLG model. Unfortunately, this theory was not successful in describing an arbitrary number of M2-branes. After while, Aharony, Bergman, Jafferis, and Maldacena proposed a 3d Chern-Simons matter theory describing arbitrary number of M2-branes [5]. The theory is called the ABJM theory. The superconformal index of the ABJM theory was first calculated in [11] by using the supersymmetric localization method. We shortly review their results in this chapter.

In the large-N limit, the dual supergravity calculation is also useful. The AdS/CFT correspondence claims that the ABJM theory in the large-N limit is equivalent to the eleven-dimensional supergravity on $\mathrm{AdS}_4 \times S^7$. Hence, the calculation from the dual supergravity theory is possible. We review the basic concepts of the AdS/CFT correspondence and the index calculation from the supergravity in large-N.

The goal of this chapter is to show the explicit result of the superconformal indices of the following theories:

- the theory on a single M2-brane ($N = 1$)
- the ABJM theory (arbitrary N)
- the dual supergravity theory ($N = \infty$)

In the last two sections of this chapter, we also discuss the orbifolding case, i.e. theories realized on M2-branes on $\mathbb{C}^4/\mathbb{Z}_k$.

2.1 The definition of the superconformal indices for M2-brane theories

We consider a stack of N M2-branes expanding in $x^0 \sim x^2$ directions as in Table 2.1 or Figure 2.1.

	0	1	2	3	4	5	6	7	8	9	11
M2-branes	∘	∘	∘								

Table 2.1: The configuration of M2-branes are shown. The M2-branes are expanding in $x^0 \sim x^2$ direction, which are marked as ∘.

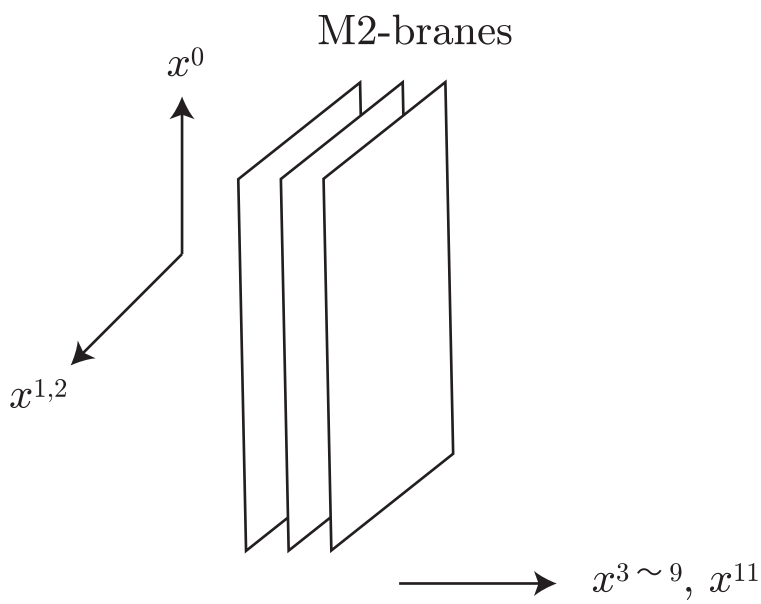

Figure 2.1: The schematic figure of the M2-branes are shown.

The system possesses $SO(1,2)$ Lorentz symmetry and $SO(8)$ internal symmetry corresponding to the rotation of transverse directions of the M2-branes. Let ϵ be a 32-component spinor in eleven dimensions, which plays the role of supersymmetry transformation parameter. The M2-branes break half of the supersymmetry and 16 of 32 supercharges survive. The corresponding parameter satisfies

$$\Gamma_0 \Gamma_1 \Gamma_2 \epsilon = \epsilon. \tag{2.1}$$

Therefore, 3d $\mathcal{N}=8$ supersymmetry is preserved. Further, as we consider the low energy limit, 3d $\mathcal{N}=8$ superconformal symmetry should be realized on the M2-branes. The 3d $\mathcal{N}=8$ superconformal algebra is $\hat{\mathcal{A}}=osp(8|4)$, whose bosonic subalgebra is $so(2,3)\times so(8)\subset\hat{\mathcal{A}}$. The generators are as follows [1]:

$$\hat{H},\quad \hat{Q}_i,\quad \hat{S}_i,\quad \hat{P}_\mu,\quad \hat{K}_\mu,\quad \hat{R}_{ab}.\quad \hat{J}_{\mu\nu}, \tag{2.2}$$

where $\hat{H}$ is Hamiltonian, $\hat{Q}_i(\hat{S}_i)$ are super (conformal) charges, $\hat{P}_\mu$ are momentum operators, $\hat{K}_\mu$ are generators of special conformal transformations, $\hat{R}_{ab}$ are R-symmetry generators ,and $\hat{J}_{\mu\nu}$ are Lorenrz generators.

Let us define the superconformal index of the M2-brane theories. There are six Cartan generators in $\hat{\mathcal{A}}$:

$$\hat{H},\quad \hat{J}_{12},\quad \hat{R}_{12},\quad \hat{R}_{34},\quad \hat{R}_{56},\quad \hat{R}_{78}. \tag{2.3}$$

To define the superconformal index we choose one complex supercharge $\hat{\mathcal{Q}}$ that carries specific Cartan charges. We take the one with the following quantum numbers:

$$\hat{\mathcal{Q}}:(\hat{H},\hat{J}_{12};\hat{R}_{12},\hat{R}_{34},\hat{R}_{56},\hat{R}_{78})=(+\tfrac{1}{2},-\tfrac{1}{2};+\tfrac{1}{2},+\tfrac{1}{2},+\tfrac{1}{2},+\tfrac{1}{2}). \tag{2.4}$$

The subalgebra of $\hat{\mathcal{A}}$ that keeps the above supercharge $\hat{\mathcal{Q}}$ is

$$\hat{\mathcal{B}}\times u(1)_{\hat{\Delta}}\subset\hat{\mathcal{A}}, \tag{2.5}$$

where $\hat{\mathcal{B}}=osp(6|2)$ is the superalgebra whose bosonic subalgebra is $sl(2,\mathbb{R})\times so(6)$. The central factor $u(1)_{\hat{\Delta}}$ is generated by

$$\hat{\Delta}\equiv\{\hat{\mathcal{Q}},\hat{\mathcal{Q}}^\dagger\}=\hat{H}-\hat{J}_{12}-\frac{1}{2}(\hat{R}_{12}+\hat{R}_{34}+\hat{R}_{56}+\hat{R}_{78}). \tag{2.6}$$

The superconformal index associated with the BPS bound $\hat{\Delta}\geq 0$ is defined as the $\hat{\mathcal{B}}$ character by [2]

$$\mathcal{I}(\hat{q},\hat{u}_i)=\mathrm{tr}[(-1)^F\hat{x}^{\hat{\Delta}}\hat{q}^{\hat{H}+\hat{J}_{12}}\hat{u}_1^{\hat{R}_{12}}\hat{u}_2^{\hat{R}_{34}}\hat{u}_3^{\hat{R}_{56}}\hat{u}_4^{\hat{R}_{78}}],\quad \hat{u}_1\hat{u}_2\hat{u}_3\hat{u}_4=1. \tag{2.7}$$

Due to the Bose-Fermi degeneracy for $\hat{\Delta}>0$ the index does not depend on $\hat{x}$.

2.2 A single M2-brane

In this section, we consider the theory realized on a single M2-brane. We first show that the theory is described by 8 free scalar fields and their fermionic partners. Then, we calculate the superconformal index of the theory.

[1] Here, we use hats (ˆ) to distinguish from similar symbols in the next chapter. (We use hats for symbols related to the 3d case and checks (ˇ) for symbols related to the 6d case.)

[2] The fugacities used here are related to those in Section 2 of [25] by $\hat{q}=x$, $\hat{u}_1=y_1^{-\frac{1}{2}}y_2^{\frac{1}{2}}y_3^{\frac{1}{2}}$, $\hat{u}_2=y_1^{\frac{1}{2}}y_2^{-\frac{1}{2}}y_3^{\frac{1}{2}}$, $\hat{u}_3=y_1^{\frac{1}{2}}y_2^{\frac{1}{2}}y_3^{-\frac{1}{2}}$, and $\hat{u}_4=y_1^{-\frac{1}{2}}y_2^{-\frac{1}{2}}y_3^{-\frac{1}{2}}$.

2.2.1 Lagrangian

Let us consider the field theory realized on a single M2-brane. The M2-brane is expanding in x^a ($a = 0, 1, 2$) and infinitesimally oscillating in transverse x^i ($i = 3, ..., 9, 11$) direction. This configuration is again given by Table 2.1. We set the background metric as $g_{MN} = \eta_{MN}$ ($M, N = 0, 1, ..., 9, 11$). For the worldvolume coordinates σ^a, we take $\sigma^a = x^a$ which is called a static gauge. In this choice the induced metric is given by

$$G_{ab} = \eta_{ab} + \partial_a x^i \partial_b x^i, \tag{2.8}$$

where $\partial_a = \frac{\partial}{\partial \sigma^a}$. Then the Nambu-Goto action (1.11) reads

$$S_{\rm NG} = -T_{\rm M2} \int d^3\sigma \left(1 + \frac{1}{2}(\partial_a x^i)^2 + \cdots \right), \tag{2.9}$$

where $\cdots$ represent infinitesimal terms of order $\mathcal{O}\left((\partial x)^4\right)$. The second term gives a kinetic term for the scalar fields x^i. We neglect constant term "1" which is not relevant to the equation of motion and define scalar fields ϕ^i as

$$\phi^i = \sqrt{T_{\rm M2}}\, x^i. \tag{2.10}$$

Then the Nambu-Goto action becomes

$$S_{\rm NG} = \int d^3\sigma \left(-\frac{1}{2}(\partial_a \phi^i)^2 + \cdots \right). \tag{2.11}$$

Further, by taking the decoupling limit $T_{\rm M2} \to \infty$, we can ignore the $\cdots$ terms because these terms have $T_{\rm M2}^{-1}$ coefficient or higher. Therefore we obtain the action of 8 free massless scalar fields:

$$S_{\rm NG} = \int d^3\sigma \left(-\frac{1}{2}(\partial_a \phi^i)^2 \right). \tag{2.12}$$

The eight scalar fields ϕ^i describe the fluctuation of the M2-brane. In addition, there exist fermion fields which are supersymmetric parters of the scalar fields:

$$\text{fields on the single M2-brane} \;=\; \begin{cases} \cdot \text{ 8 scalar fields } \phi^i \ (i = 1 \sim 8) \\ \cdot \text{ 8 Majorana fermions } \psi_m \ \ (m = 1 \sim 8) \end{cases} \tag{2.13}$$

The scalar fields belong to the $so(8)$ vector representation and the fermion fields belonging to the $so(8)$ conjugate spinor representation.

The complete Lagrangian is given by

$$S = \int d^3\sigma \left(-\frac{1}{2}(\partial_a \phi^i)^2 + \frac{1}{2}\psi_m \gamma^a \partial_a \psi_m \right). \tag{2.14}$$

2.2.2 Superconformal index

Let us explicitly calculate the superconformal index for the single M2-brane theory. We define the index as a formal power series of one of the fugacities $\hat{q}$, which is roughly a fugacity for the energy, and expand the index up to a certain order of $\hat{q}$. [3]

[3]Similary, for the 6d superconformal index we define the index as power series of $\check{q}$.

Since the theory on the single M2-brane is a free theory of scalar and fermion fields, the index is simply given by the plethystic exponential of the single-particle index

$$\mathcal{I}^{\text{M2}}_{N=1} = \text{Pexp}\, i^{\text{M2}}, \tag{2.15}$$

where i^{M2} is the single-particle index of the fields on the M2-brane. The field contents and their contributions to the index are summarized in Table 2.2. From this table, we can easily

	H	$so(3)$	$so(8)$	contribution to the index
Q	$\frac{1}{2}$	$\mathbf{2}$	$\mathbf{8}_s$	
ϕ	$\frac{1}{2}$	$\mathbf{1}$	$\mathbf{8}_v$	$\hat{q}^{\frac{1}{2}}\chi_{[1,0,0]}(\hat{u})$
$\partial^2\phi$	$\frac{5}{2}$	$\mathbf{1}$	$\mathbf{8}_v$	
ψ	1	$\mathbf{2}$	$\mathbf{8}_c$	$-\hat{q}^{\frac{3}{2}}\chi_{[0,0,1]}(\hat{u})$
$\partial\psi$	2	$\mathbf{2}$	$\mathbf{8}_c$	
∂	1	$\mathbf{3}$	$\mathbf{1}$	$\hat{q}^2$

Table 2.2: The field contents and their contributions to the superconformal index for the single M2-brane theory are shown.

read off the single-particle index

$$i^{\text{M2}} = \frac{\hat{q}^{\frac{1}{2}}\chi_{[1,0,0]}(\hat{u}) - \hat{q}^{\frac{3}{2}}\chi_{[0,0,1]}(\hat{u})}{1-\hat{q}^2}, \tag{2.16}$$

where $\chi_{[a,b,c]}(\hat{u}_a)$ is the characters of $su(4)$ representation with Dynkin label $[a,b,c]$. Since the subgroup $su(4) \sim so(6) \subset so(8)$ is manifest in the superconformal index, we use the $su(4)$ characters to write down the index. The characters of the fundamental representation and the anti-fundamental representation in our convention are given as follows:

$$\chi_{[1,0,0]}(\hat{u}) = \hat{u}_1 + \hat{u}_2 + \hat{u}_3 + \hat{u}_4, \quad \chi_{[0,0,1]}(\hat{u}) = \hat{u}_1^{-1} + \hat{u}_2^{-1} + \hat{u}_3^{-1} + \hat{u}_4^{-1}. \tag{2.17}$$

Also, note that the denominator in (2.16)

$$\frac{1}{1-\hat{q}^2} = 1 + \hat{q}^2 + (\hat{q}^2)^2 + \cdots . \tag{2.18}$$

comes from the descendant operators generated by acting on the derivative on the fields ϕ and ψ. Now, it is easy to perform the index calculation. We obtain the following result for the superconformal index of the single M2-brane theory.

$$\begin{aligned}
\mathcal{I}^{\text{M2}}_{N=1} = {} & 1 + \chi_{[1,0,0]}\hat{q}^{\frac{1}{2}} + \chi_{[2,0,0]}\hat{q} + (-\chi_{[0,0,1]} + \chi_{[3,0,0]})\hat{q}^{\frac{3}{2}} \\
& + (-1 - \chi_{[1,0,1]} + \chi_{[4,0,0]})\hat{q}^2 + (-\chi_{[2,0,1]} + \chi_{[5,0,0]})\hat{q}^{\frac{5}{2}} \\
& + (2\chi_{[0,1,0]} - \chi_{[3,0,1]} + \chi_{[6,0,0]})\hat{q}^3 + (2\chi_{[1,1,0]} - \chi_{[4,0,1]} + \chi_{[7,0,0]})\hat{q}^{\frac{7}{2}} \\
& + (-2 - \chi_{[1,0,1]} + 2\chi_{[2,1,0]} - \chi_{[5,0,1]} + \chi_{[8,0,0]})\hat{q}^4 + \mathcal{O}(\hat{q}^{\frac{9}{2}}).
\end{aligned} \tag{2.19}$$

2.3 Multiple M2-branes

2.3.1 ABJM theory

In [5], Aharony, Bergman, Jafferis, and Maldacena constructed a three-dimensional $\mathcal{N} = 6$ Chern-Simons matter theory with gauge group $U(N)_k \times U(N)_{-k}$, where integer k is called Chern-Simons level. The theory is called the ABJM theory. It is believed that the low energy description of the theory on N coincident M2-branes in flat space is given by the ABJM theory with $k = 1$. Although the manifest supersymmetry of the Lagrangian is $\mathcal{N} = 6$, it is argued that $\mathcal{N} = 8$ supersymmetry is restored non-perturbatively for the $k = 1, 2$ case [26, 27].

It has been confirmed through various evidence that this theory actually gives a low energy effective theory on M2-branes. (moduli space, supersymmetry enhancement, partition function, etc...) For example, the AdS/CFT argument claims that the degrees of freedom of the theory should scale as $N^{\frac{3}{2}}$ in the large-N limit. This characteristic behavior was found in the S^3 partition function of the ABJM theories [28, 29, 30].

The ABJM model is described by the quiver diagram shown in Figure 2.2. The circles

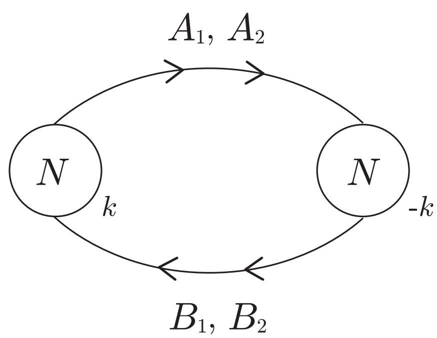

Figure 2.2: The quiver diagram of the ABJM theory is shown.

denote $U(N)$ gauge nodes and the arrows show the bi-fundamental (the fundamental representation of $U(N)_k$ and the anti-fundamental representation of $U(N)_{-k}$) chiral superfields.

The Lagrangian is given by

$$S = S_{CS} + S_{kin} + S_{pot} + S_{Yuk}, \tag{2.20}$$

where

$$S_{CS} = \frac{k}{4\pi} \int d^3x \epsilon^{\mu\nu\rho} \operatorname{tr} \left[\left(A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho \right) - \left(\tilde{A}_\mu \partial_\nu \tilde{A}_\rho - \frac{2i}{3} \tilde{A}_\mu \tilde{A}_\nu \tilde{A}_\rho \right) \right], \tag{2.21}$$

$$S_{kin} = \frac{1}{2\pi} \int d^3x \operatorname{tr} \left[-D^\mu q^i D_\mu \bar{q}_i + \bar{\psi}^i \gamma^\mu D_\mu \psi_i \right], \tag{2.22}$$

$$S_{pot} = \frac{1}{6\pi k^2}\int d^3x\, \mathrm{tr}\left[q^i\bar{q}_i q^j\bar{q}_j q^k\bar{q}_k + \bar{q}_i q^i\bar{q}_j q^j\bar{q}_k q^k + 4q^i\bar{q}_j p^k\bar{q}_i q^j\bar{p}_k - 6q^i\bar{q}_j p^j\bar{q}_i q^k\bar{q}_k\right], \quad (2.23)$$

$$S_{Yuk} = -\frac{1}{2\pi k}\int d^3x\, \mathrm{tr}\left[\bar{q}_i q^i\bar{\psi}^j\psi_j - q^i\bar{q}_i\psi_j\bar{\psi}^j + 2q^i\bar{q}_j\psi_i\bar{\psi}^k - 2\bar{q}_i q^j\bar{\psi}^i\psi_j + \epsilon^{ijkl}\bar{q}_i\psi_j\bar{q}_k\psi_l - \epsilon^{ijkl}q^i\bar{\psi}^j q^k\bar{\psi}^l\right], \quad (2.24)$$

where the spacetime indices μ,ν,ρ runs from 0 to 2 and the indices i,j,k runs from 1 to 4. The supersymmetry transformation is

$$\delta q^i = \sqrt{2}\xi^{ij}\psi_j \quad (2.25)$$

$$\delta\psi_i = -\sqrt{2}\gamma^\mu\xi_{ij}D_\mu q^j + \frac{2\sqrt{2}\pi}{k}\xi_{ij}\left(q^k\bar{q}_k q^j - q^j\bar{q}_k q^k\right) - \frac{4\sqrt{2}\pi}{k}\xi_{jk}\left(q^j\bar{q}_i q^k\right) \quad (2.26)$$

$$\delta A_\mu = -\frac{2i\sqrt{2}\pi}{k}\left[\xi_{ij}\gamma_\mu\left(q^i\bar{\psi}^j\right) + \xi^{ij}\gamma_\mu\left(\psi_i\bar{q}_j\right)\right] \quad (2.27)$$

$$\delta\tilde{A}_\mu = \frac{2i\sqrt{2}\pi}{k}\left[\xi^{ij}\gamma_\mu\left(\bar{q}_i\psi_j\right) + \xi_{ij}\gamma_\mu\left(\bar{\psi}^i q^j\right)\right] \quad (2.28)$$

The supersymmetry transformation parameter ξ_{ij} belongs to the **6** of $SU(4)$ and satisfy

$$\xi_{ij} = -\xi_{ji}, \qquad (\xi_{ij})^* = -\frac{1}{2}\epsilon^{ijkl}\xi_{kl}. \quad (2.29)$$

The Lagrangian (2.20) is invariant under the global $SU(4)$ symmetry. We show charges of the fields in Table 2.3, where $U(1)_B$ is a baryonic symmetry.

	$U(N)_k$	$U(N)_{-k}$	$SU(4)_R$	$U(1)_B$
A	adj.	1	1	0
$\tilde{A}$	1	adj.	1	0
q^i	N	$\bar{N}$	4	1
ψ_i	N	$\bar{N}$	$\bar{4}$	1

Table 2.3: Charges of the fields are shown.

2.4 Superconformal index for the ABJM theory

The superconformal index of the ABJM theory was studied in [11] by using the localization method. The analysis yields following expression for the superconformal index of the ABJM theory for each monopole charge sector.

$$\mathcal{I}_{m_\alpha,\widetilde{m}_\alpha} = \frac{1}{(N!)^2}\prod_{\alpha=1}^{N}\int\frac{d\zeta_\alpha}{2\pi i\zeta_\alpha}\prod_{\alpha=1}^{N}\int\frac{d\widetilde{\zeta}_\alpha}{2\pi i\widetilde{\zeta}_\alpha} \times \frac{\prod_{\alpha,\beta}\hat{q}^{|m_\alpha-\widetilde{m}_\beta|-\frac{1}{2}|m_\alpha-m_\beta|-\frac{1}{2}|\widetilde{m}_\alpha-\widetilde{m}_\beta|}}{\prod_{\alpha=1}^{N}\zeta_\alpha^{km_\alpha}\widetilde{\zeta}_\alpha^{-k\widetilde{m}_\alpha}}\,\mathrm{Pexp}\, i, \quad (2.30)$$

where the single-particle index i is given by

$$
\begin{aligned}
i(\hat{q},\hat{u}_i;\zeta_a,\widetilde{\zeta}_b) = &-\sum_{\alpha\neq\beta}\hat{q}^{|m_\alpha-m_\beta|}\frac{\zeta_\alpha}{\zeta_\beta}-\sum_{\alpha\neq\beta}\hat{q}^{|\widetilde{m}_\alpha-\widetilde{m}_\beta|}\frac{\widetilde{\zeta}_\alpha}{\widetilde{\zeta}_\beta}\\
&+\sum_{\alpha,\beta=1}^{N}\frac{\hat{q}^{|m_\alpha-\widetilde{m}_\beta|}}{1-\hat{q}^2}\left[\hat{q}^{\frac{1}{2}}(\hat{u}_1+\hat{u}_2)-\hat{q}^{\frac{3}{2}}(\hat{u}_3^{-1}+\hat{u}_4^{-1})\right]\frac{\zeta_\alpha}{\widetilde{\zeta}_\beta}\\
&+\sum_{\alpha,\beta=1}^{N}\frac{\hat{q}^{|m_\alpha-\widetilde{m}_\beta|}}{1-\hat{q}^2}\left[\hat{q}^{\frac{1}{2}}(\hat{u}_3+\hat{u}_4)-\hat{q}^{\frac{3}{2}}(\hat{u}_1^{-1}+\hat{u}_2^{-1})\right]\frac{\widetilde{\zeta}_\beta}{\zeta_\alpha}.
\end{aligned}
\tag{2.31}
$$

The whole superconformal index is give by the sum of the monopole charges.

$$
\mathcal{I}_{\mathrm{ABJM}} = \sum_{m_\alpha,\widetilde{m}_\alpha\in\mathbb{Z}}\mathcal{I}_{m_\alpha,\widetilde{m}_\alpha} \tag{2.32}
$$

Note that only when the monopole charges satisfy the following relation, the integral have non-zero value.

$$
m_{\mathrm{tot}} := \sum_{\alpha=1}^{N} m_\alpha = \sum_{\alpha=1}^{N}\widetilde{m}_\alpha. \tag{2.33}
$$

One comment is that this is the exact result, but the integral is difficult to carry out generically. Therefore we will calculate the index for specific N, for example N=1,2,3, and so on.

2.4.1 Result for small N with Chern-Simons level $k = 1$

Let us calculate the superconformal index of the ABJM with Chern-Simons level $k = 1$ for a small value of N. For the $k = 1$ case, the theory has $\mathcal{N} = 8$ supersymmetry and the index should be written in terms of $su(4)$ characters.

We show the result of calculation of the ABJM indices for $N = 1, 2, 3$ with Chern-Simons level $k = 1$.

$$
\begin{aligned}
\mathcal{I}^{\mathrm{ABJM}}_{(N=1)} = &\ 1+\chi_{[1,0,0]}\hat{q}^{\frac{1}{2}}+\chi_{[2,0,0]}\hat{q}+(-\chi_{[0,0,1]}+\chi_{[3,0,0]})\hat{q}^{\frac{3}{2}}\\
&+(-1-\chi_{[1,0,1]}+\chi_{[4,0,0]})\hat{q}^2+(-\chi_{[2,0,1]}+\chi_{[5,0,0]})\hat{q}^{\frac{5}{2}}\\
&+(2\chi_{[0,1,0]}-\chi_{[3,0,1]}+\chi_{[6,0,0]})\hat{q}^3+(2\chi_{[1,1,0]}-\chi_{[4,0,1]}+\chi_{[7,0,0]})\hat{q}^{\frac{7}{2}}\\
&+(-2-\chi_{[1,0,1]}+2\chi_{[2,1,0]}-\chi_{[5,0,1]}+\chi_{[8,0,0]})\hat{q}^4+\mathcal{O}(\hat{q}^{\frac{9}{2}}).
\end{aligned}
\tag{2.34}
$$

$$
\begin{aligned}
\mathcal{I}^{\mathrm{ABJM}}_{(N=2)} = {} & 1 + \chi_{[1,0,0]}\hat{q}^{\frac{1}{2}} + 2\chi_{[2,0,0]}\hat{q} + (-\chi_{[0,0,1]} + \chi_{[1,1,0]} + 2\chi_{[3,0,0]})\hat{q}^{\frac{3}{2}} \\
& + (-1 + \chi_{[0,2,0]} - 2\chi_{[1,0,1]} + \chi_{[2,1,0]} + 3\chi_{[4,0,0]})\hat{q}^{2} \\
& + (-\chi_{[0,1,1]} - 2\chi_{[1,0,0]} + \chi_{[1,2,0]} - 3\chi_{[2,0,1]} + 2\chi_{[3,1,0]} + 3\chi_{[5,0,0]})\hat{q}^{\frac{5}{2}} \\
& + (\chi_{[0,1,0]} - 2\chi_{[1,1,1]} - 3\chi_{[2,0,0]} + 2\chi_{[2,2,0]} - 4\chi_{[3,0,1]} + 2\chi_{[4,1,0]} + 4\chi_{[6,0,0]})\hat{q}^{3} \\
& + (2\chi_{[0,0,1]} - \chi_{[0,2,1]} + \chi_{[1,0,2]} + 2\chi_{[1,1,0]} + \chi_{[1,3,0]} - 3\chi_{[2,1,1]} - 4\chi_{[3,0,0]} \\
& \quad + 2\chi_{[3,2,0]} - 5\chi_{[4,0,1]} + 3\chi_{[5,1,0]} + 4\chi_{[7,0,0]})\hat{q}^{\frac{7}{2}} \\
& + (-2 + \chi_{[0,1,2]} + \chi_{[0,2,0]} + \chi_{[0,4,0]} + 5\chi_{[1,0,1]} - 2\chi_{[1,2,1]} + \chi_{[2,0,2]} + 4\chi_{[2,1,0]} \\
& \quad + \chi_{[2,3,0]} - 4\chi_{[3,1,1]} - 5\chi_{[4,0,0]} + 3\chi_{[4,2,0]} - 6\chi_{[5,0,1]} + 3\chi_{[6,1,0]} + 5\chi_{[8,0,0]})\hat{q}^{4} \\
& + (\chi_{[0,1,1]} - \chi_{[0,3,1]} - 4\chi_{[1,0,0]} + \chi_{[1,1,2]} + 3\chi_{[1,2,0]} + \chi_{[1,4,0]} \\
& \quad + 7\chi_{[2,0,1]} - 3\chi_{[2,2,1]} + 2\chi_{[3,0,2]} + 5\chi_{[3,1,0]} + 2\chi_{[3,3,0]} - 5\chi_{[4,1,1]} \\
& \quad - 6\chi_{[5,0,0]} + 3\chi_{[5,2,0]} - 7\chi_{[6,0,1]} + 4\chi_{[7,1,0]} + 5\chi_{[9,0,0]})\hat{q}^{\frac{9}{2}} \\
& + (-2\chi_{[0,0,2]} - 6\chi_{[0,1,0]} + 2\chi_{[0,3,0]} - 2\chi_{[1,3,1]} - 9\chi_{[2,0,0]} + 2\chi_{[2,1,2]} \\
& \quad + 4\chi_{[2,2,0]} + 2\chi_{[2,4,0]} + 9\chi_{[3,0,1]} - 4\chi_{[3,2,1]} + 2\chi_{[4,0,2]} + 7\chi_{[4,1,0]} \\
& \quad + 2\chi_{[4,3,0]} - 6\chi_{[5,1,1]} - 7\chi_{[6,0,0]} + 4\chi_{[6,2,0]} - 8\chi_{[7,0,1]} + 4\chi_{[8,1,0]} \\
& \quad + 6\chi_{[10,0,0]})\hat{q}^{5} + \mathcal{O}(\hat{q}^{\frac{11}{2}}).
\end{aligned}
\tag{2.35}
$$

$$\begin{aligned}
\mathcal{I}^{\mathrm{ABJM}}_{N=3} &= 1 + \chi_{[1,0,0]}\hat{q}^{\frac{1}{2}} + 2\chi_{[2,0,0]}q + (-\chi_{[0,0,1]} + \chi_{[1,1,0]} + 3\chi_{[3,0,0]})\hat{q}^{\frac{3}{2}} \\
&+ (2\chi_{[0,2,0]} - 2\chi_{[1,0,1]} + 2\chi_{[2,1,0]} + 4\chi_{[4,0,0]} - 1)\hat{q}^2 \\
&+ (-2\chi_{[1,0,0]} + 3\chi_{[1,2,0]} - 4\chi_{[2,0,1]} + 4\chi_{[3,1,0]} + 5\chi_{[5,0,0]})\hat{q}^{\frac{5}{2}} \\
&+ (\chi_{[0,0,2]} + \chi_{[0,3,0]} - 3\chi_{[1,1,1]} - 5\chi_{[2,0,0]} + 6\chi_{[2,2,0]} - 5\chi_{[3,0,1]} + 5\chi_{[4,1,0]} + 7\chi_{[6,0,0]})\hat{q}^3 \\
&+ (\chi_{[0,0,1]} - 4\chi_{[0,2,1]} + \chi_{[1,0,2]} - 3\chi_{[1,1,0]} + 4\chi_{[1,3,0]} - 4\chi_{[2,1,1]} - 8\chi_{[3,0,0]} + 8\chi_{[3,2,0]} \\
&\qquad - 8\chi_{[4,0,1]} + 8\chi_{[5,1,0]} + 8\chi_{[7,0,0]})\hat{q}^{\frac{7}{2}} \\
&+ (-4\chi_{[0,2,0]} + 4\chi_{[0,4,0]} + 4\chi_{[1,0,1]} - 6\chi_{[1,2,1]} + 2\chi_{[2,0,2]} - 4\chi_{[2,1,0]} + 6\chi_{[2,3,0]} \\
&\qquad - 8\chi_{[3,1,1]} - 13\chi_{[4,0,0]} + 12\chi_{[4,2,0]} - 10\chi_{[5,0,1]} + 10\chi_{[6,1,0]} + 10\chi_{[8,0,0]} - 2)\hat{q}^4 \\
&+ (5\chi_{[0,1,1]} - \chi_{[0,3,1]} - \chi_{[1,0,0]} + \chi_{[1,1,2]} - 4\chi_{[1,2,0]} + 6\chi_{[1,4,0]} + 10\chi_{[2,0,1]} - 12\chi_{[2,2,1]} \\
&\qquad + 3\chi_{[3,0,2]} - 8\chi_{[3,1,0]} + 11\chi_{[3,3,0]} - 11\chi_{[4,1,1]} - 18\chi_{[5,0,0]} + 15\chi_{[5,2,0]} - 13\chi_{[6,0,1]} \\
&\qquad + 13\chi_{[7,1,0]} + 12\chi_{[9,0,0]})\hat{q}^{\frac{9}{2}} \\
&+ (\chi_{[0,0,2]} + 4\chi_{[0,1,0]} + 2\chi_{[0,2,2]} + 2\chi_{[0,3,0]} + 2\chi_{[0,5,0]} + 13\chi_{[1,1,1]} - 8\chi_{[1,3,1]} \\
&\qquad - 2\chi_{[2,0,0]} + 2\chi_{[2,1,2]} - 8\chi_{[2,2,0]} + 12\chi_{[2,4,0]} + 18\chi_{[3,0,1]} - 16\chi_{[3,2,1]} + 4\chi_{[4,0,2]} \\
&\qquad - 10\chi_{[4,1,0]} + 14\chi_{[4,3,0]} - 16\chi_{[5,1,1]} - 25\chi_{[6,0,0]} + 20\chi_{[6,2,0]} - 16\chi_{[7,0,1]} \\
&\qquad + 16\chi_{[8,1,0]} + 14\chi_{[10,0,0]})\hat{q}^5 \\
&+ (2\chi_{[0,0,1]} + 8\chi_{[0,2,1]} - 8\chi_{[0,4,1]} - \chi_{[1,0,2]} - 3\chi_{[1,1,0]} + 3\chi_{[1,2,2]} - \chi_{[1,3,0]} + 8\chi_{[1,5,0]} \\
&\qquad + 23\chi_{[2,1,1]} - 12\chi_{[2,3,1]} - 2\chi_{[3,0,0]} + 4\chi_{[3,1,2]} - 8\chi_{[3,2,0]} + 16\chi_{[3,4,0]} + 29\chi_{[4,0,1]} \\
&\qquad - 24\chi_{[4,2,1]} + 5\chi_{[5,0,2]} - 15\chi_{[5,1,0]} + 20\chi_{[5,3,0]} - 20\chi_{[6,1,1]} - 32\chi_{[7,0,0]} + 24\chi_{[7,2,0]} \\
&\qquad - 20\chi_{[8,0,1]} + 20\chi_{[9,1,0]} + 16\chi_{[11,0,0]})\hat{q}^{\frac{11}{2}} \\
&+ (-6\chi_{[0,1,2]} - 14\chi_{[0,2,0]} + \chi_{[0,3,2]} - 4\chi_{[0,4,0]} + 7\chi_{[0,6,0]} - 10\chi_{[1,0,1]} + 17\chi_{[1,2,1]} \\
&\qquad - 13\chi_{[1,4,1]} - 4\chi_{[2,0,2]} - 11\chi_{[2,1,0]} + 6\chi_{[2,2,2]} + 2\chi_{[2,3,0]} + 12\chi_{[2,5,0]} + 36\chi_{[3,1,1]} \\
&\qquad - 21\chi_{[3,3,1]} - 5\chi_{[4,0,0]} + 5\chi_{[4,1,2]} - 12\chi_{[4,2,0]} + 23\chi_{[4,4,0]} + 42\chi_{[5,0,1]} - 30\chi_{[5,2,1]} \\
&\qquad + 7\chi_{[6,0,2]} - 18\chi_{[6,1,0]} + 25\chi_{[6,3,0]} - 27\chi_{[7,1,1]} - 41\chi_{[8,0,0]} + 30\chi_{[8,2,0]} - 23\chi_{[9,0,1]} \\
&\qquad + 23\chi_{[10,1,0]} + 19\chi_{[12,0,0]})\hat{q}^6 + \mathcal{O}(\hat{q}^{\frac{13}{2}}). \qquad (2.36)
\end{aligned}$$

As we anticipated the indices are written in terms of $su(4)$ characters.

2.5 Large N limit

A duality is often useful to analyze quantum field theories. In this section, we discuus the most famous duality in string- (M-) theory called AdS/CFT correspondence [4, 31, 32]. The AdS/CFT correspondence claims the equivalence of a conformal field theory and the corresponding gravity system in Anti-de-Sitter spacetime. In the context of string- (M-) theory, the gravity theory is M- (string-) theory in $\mathrm{AdS}_{d+1} \times X$, with X being a certain compact manifold.

We consider $d = 3$, $X = S^7/\mathbb{Z}_k$ case here. This is called the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondences. The precise argument for the $\mathrm{AdS}_4/\mathrm{CFT}_3$ is

- The M-theory on $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ is equivalent to three-dimensional ABJM theories with Chern-Simons level k.

In the followings, we first discuss the $k = 1$ case, which means the duality between the ABJM theory with Chern-Simons level $k = 1$ and 11d supergravity on $\mathrm{AdS}_4 \times S^7$, and calculate the large-N index from the dual gravity side. We discus generalization to the $\mathbb{Z}_k$ case in Section 2.6 and 2.7.

2.5.1 M2-brane solution

Black M2-brane solution

The solution of the eleven-dimensional supergravity describing the stack of M2-branes. flat N M5-branes is given by [33, 34]

$$ds^2 = H(r)^{-2/3}(-f dt^2 + dx \cdot dx) + H(r)^{1/3}(f^{-1}dr^2 + r^2 d\Omega_7^2), \tag{2.37}$$

with the 3-form field

$$A_3 = H(r)^{-1} dx^0 \wedge dx^1 \wedge dx^2, \qquad (F_4 = dx^0 \wedge dx^1 \wedge dx^2 \wedge dH^{-1}), \tag{2.38}$$

where $H(r)$ and $f(r)$ are harmonic functions on $\mathbb{R}_8$ defined by

$$H(r) = 1 + \frac{\hat{r}^6}{r^6}, \qquad f(r) = 1 - \frac{\hat{r}_h^6}{r^6} \tag{2.39}$$

The lengths $\hat{r}$ and $\hat{r}_h$ satisfy the following relation:

$$(\hat{r}^6 + \hat{r}_h^6)\hat{r}^6 = \left(32\pi^2 N l_p^6\right)^2. \tag{2.40}$$

The horizon is given by $r = \hat{r}_h (\geq 0)$.

The ADM energy [35] of this solution is given by

$$E = \frac{2\pi^4}{(2\pi l_p)^9}\left(\hat{r}^6 + \frac{7}{6}\hat{r}_h^6\right). \tag{2.41}$$

Here, let us recall the central charge of the M2-branes discussed in Section 1.1.2.

$$Z_{\mathrm{M2}} = \frac{2\pi}{(2\pi l_p)^3} N \tag{2.42}$$

To compare these two relation (2.41) with (2.42), it is obvious that $E \geq |Z_{\mathrm{M2}}|$ is satisfied if $\hat{r}_h \geq 0$. The equality is satisfied when $\hat{r}_h$ is 0 since, in this case, $\hat{r}^6 = 32\pi^2 N l_p^6$. Thus, this is consistent with the BPS bound discussed in Section 1.1.2.

Extremal black M2-brane solution

Let us consider the extremal (BPS) solution, satisfying $E = |Z_{\rm M2}|$ given by $\hat{r}_h = 0$ condition. This solution preserves 16 of 32 supercharges. The metric reduces to

$$ds^2 = H(r)^{-2/3}\eta_{ab}dx^a dx^b + H(r)^{1/3}(dr^2 + r^2 d\Omega_7^2), \tag{2.43}$$

where $a, b = 0, 1, 2$ run the worldvolume directions for the M2-brane and a harmonic function $H(r)$ is given by

$$H(r) = 1 + \frac{\hat{r}^6}{r^6}, \tag{2.44}$$

with $\hat{r}$ satisfying

$$\hat{r}^6 = 32\pi^2 N l_p^6. \tag{2.45}$$

Again the 3-form field is given by

$$A_3 = H(r)^{-1}dx^0 \wedge dx^1 \wedge dx^2, \qquad (F_4 = dx^0 \wedge dx^1 \wedge dx^2 \wedge dH^{-1}). \tag{2.46}$$

Near horizon geometry of M2-brane solution

Let us take the near horizon limit $r \ll \hat{r}$ of (2.43). In this limit, $H(r)$ is approximated to $\hat{r}^6/r^6$ and the metric reads

$$ds^2 = \frac{r^4}{\hat{r}^4}\eta_{ab}dx^a dx^b + \frac{\hat{r}^2}{r^2}dr^2 + \hat{r}^2 d\Omega_7^2. \tag{2.47}$$

Now we can see that $\hat{r}$ is nothing but the radius of S^7. To move on the standard convention, we perform the variable change

$$z = \frac{\hat{r}^3}{2r^2}. \tag{2.48}$$

Then, the metric becomes $\mathrm{AdS}_4 \times S^7$:

$$\begin{aligned} ds^2 &= \frac{\hat{r}^2}{4z^2}\left(\eta_{ab}dx^a dx^b + dz^2\right) + \hat{r}^2 d\Omega_7^2 \\ &= \hat{L}^2 ds^2_{\mathrm{AdS}_4} + \hat{r}^2 ds^2_{S^7}, \end{aligned} \tag{2.49}$$

where $\hat{L}$ is the AdS_4 radius and this is just one half of the radius of S^7

$$\hat{L} = \frac{\hat{r}}{2}. \tag{2.50}$$

The bosonic symmetry of the supergravity solution (2.49) is $SO(2,3) \times SO(8)$. The $SO(2,3)$ part corresponds to conformal symmetry in 3 dimension and $SO(8)$ corresponds to the 3d $\mathcal{N} = 8$ $\mathcal{R}$-symmetry. Hence, we expect that the dual superconformal field theory realized on multiple M2-branes has the $\mathcal{N} = 8$ superconformal symmetry. This is consistent with the amount of supersymmetry of the ABJM theory for $k = 1$. Furthermore, the $N^{\frac{3}{2}}$ behavior of the entropy of the black M2-brane is shown in [36]. This is also consistent with the analysis of the partition function of the ABJM theory.

2.5.2 Index from Kaluza Klein modes

Let us calculate the superconformal index of the M2-brane theory in the large-N limit. Although we can calculate the index by using the localization formula (2.30), here instead, we calculate the index from the dual supergravity theory on $\mathrm{AdS}_4 \times S^7$. On the supergravity side, the index can be calculated by the contribution of the Kaluza Klein modes in S^7. The contribution is given by plethystic exponential of the single-particle index of the Kaluza-Klein modes:

$$\mathcal{I}_{\mathrm{KK}}^{S^7} = \mathrm{Pexp}\, i_{\mathrm{KK}}^{S^7}, \tag{2.51}$$

with the single-particle index given by [25]

$$i_{\mathrm{KK}} = \frac{(1-\hat{q}^{\frac{3}{2}}\hat{u}_1^{-1})(1-\hat{q}^{\frac{3}{2}}\hat{u}_2^{-1})(1-\hat{q}^{\frac{3}{2}}\hat{u}_3^{-1})(1-\hat{q}^{\frac{3}{2}}\hat{u}_4^{-1})}{(1-\hat{q}^{\frac{1}{2}}\hat{u}_1)(1-\hat{q}^{\frac{1}{2}}\hat{u}_2)(1-\hat{q}^{\frac{1}{2}}\hat{u}_3)(1-\hat{q}^{\frac{1}{2}}\hat{u}_4)(1-\hat{q}^2)^2} - \frac{1-\hat{q}^2+\hat{q}^4}{(1-\hat{q}^2)^2}. \tag{2.52}$$

The explicit calculation of (2.51) reads the following result of the Kaluza Klein index in $\mathrm{AdS}_4 \times S^7$:

$$\begin{aligned}\mathcal{I}_{\mathrm{KK}} = 1 &+ \chi_{[1,0,0]}\hat{q}^{\frac{1}{2}} + 2\chi_{[2,0,0]}\hat{q} + (3\chi_{[3,0,0]} + \chi_{[1,1,0]} - \chi_{[0,0,1]})\hat{q}^{\frac{3}{2}} \\ &+ (5\chi_{[4,0,0]} + 2\chi_{[2,1,0]} + 2\chi_{[0,2,0]} - 2\chi_{[1,0,1]} - 1)\hat{q}^2 + \mathcal{O}(\hat{q}^{\frac{5}{2}}).\end{aligned} \tag{2.53}$$

The agreement of the KK index in (2.51) with the large-N ABJM index was confirmed in [11]. Namely,

$$\mathcal{I}_{N=\infty}^{\mathrm{ABJM}} = \mathcal{I}_{\mathrm{KK}}^{S^7}. \tag{2.54}$$

2.6 $\mathbb{Z}_k$ orbifold

Next, let us consider the $\mathbb{Z}_k$ orbifold case. The ABJM model with Chern-Simons level k corresponds to M2-branes on $\mathbb{C}^2/\mathbb{Z}_k$. We define z_i $(i = 1,2,3,4)$ coordinates by

$$z_1 = x_3 + ix_4, \qquad z_2 = x_5 + ix_6, \qquad z_3 = x_7 + ix_8, \qquad z_4 = x_9 + ix_{11}. \tag{2.55}$$

The orbifold is defined by

$$(z_1, z_2, z_3, z_4) \to (\omega_k z_1, \omega_k z_2, \omega_k^{-1} z_3, \omega_k^{-1} z_4), \quad \omega_k \equiv \exp\frac{2\pi i}{k}. \tag{2.56}$$

The orbifold with $k \geq 3$ brakes $SO(8)$ symmetry to $SO(6)$. Further, this orbifolding generally leaves the $\mathcal{N} = 6$ of $\mathcal{N} = 8$ supersymmetry unbroken. For $k = 1$ and $k = 2$, the $\mathcal{N} = 8$ supersymmetry remains unbroken. This is consistent with the supersymmetry enhancement of the ABJM theory.

2.6.1 ABJM with $k > 1$

Here, we show the indices for $N = 1, 2, 3$ with $k = 2, 3$. We can also calculate the ABJM indices with $k > 1$ by using the formula (2.30)

The $k = 2$ results are as follows:

$$\mathcal{I}^{\rm ABJM}_{(N,k)=(1,2)} = 1 + \chi_{[2,0,0]}\hat{q} + (-1 - \chi_{[1,0,1]} + \chi_{[4,0,0]})\hat{q}^2 + \mathcal{O}(\hat{q}^3). \tag{2.57}$$

$$\begin{aligned}\mathcal{I}^{\rm ABJM}_{(N,k)=(2,2)} &= 1 + \chi_{[2,0,0]}\hat{q} + (\chi_{[0,2,0]} - \chi_{[1,0,1]} + 2\chi_{[4,0,0]})\hat{q}^2 \\ &+ (-\chi_{[1,1,1]} - \chi_{[2,0,0]} + \chi_{[2,2,0]} - 2\chi_{[3,0,1]} + \chi_{[4,1,0]} + 2\chi_{[6,0,0]})\hat{q}^3 + \mathcal{O}(\hat{q}^4).\end{aligned} \tag{2.58}$$

$$\begin{aligned}\mathcal{I}^{\rm ABJM}_{(N,k)=(3,2)} &= 1 + \chi_{[2,0,0]}\hat{q} + (\chi_{[0,2,0]} - \chi_{[1,0,1]} + 2\chi_{[4,0,0]})\hat{q}^2 \\ &+ (\chi_{[0,0,2]} - \chi_{[1,1,1]} + 2\chi_{[2,2,0]} - 2\chi_{[3,0,1]} + \chi_{[4,1,0]} + 3\chi_{[6,0,0]})\hat{q}^3 \\ &+ (-1 - \chi_{[0,2,0]} + 2\chi_{[0,4,0]} - 2\chi_{[1,2,1]} + \chi_{[2,0,2]} + \chi_{[2,3,0]} - 2\chi_{[3,1,1]} \\ &\quad - 2\chi_{[4,0,0]} + 4\chi_{[4,2,0]} - 4\chi_{[5,0,1]} + 2\chi_{[6,1,0]} + 4\chi_{[8,0,0]})\hat{q}^4 + \mathcal{O}(\hat{q}^{\frac{9}{2}}).\end{aligned} \tag{2.59}$$

Since the theory still possesses the $\mathcal{N} = 8$ supersymmetry, the indices are written in terms of $su(4)$ characters.

Next, we show the $k = 3$ indices. If $k \geq 3$ the supersymmetry is $\mathcal{N} = 6$. Correspondingly, the R-symmetry is $so(6) = su(4)$, and after the choice of the complex supercharge $\hat{\mathcal{Q}}$ the manifest symmetry becomes $so(2) \times so(4) = u(1) \times su(2)_1 \times su(2)_2$. Correspondingly, we define fugacities u, u', and u'' for $u(1)$, $su(2)_1$, and $su(2)_2$ by

$$\hat{u}_1 = uu', \qquad \hat{u}_2 = uu'^{-1}, \qquad \hat{u}_1 = u^{-1}u'', \qquad \hat{u}_2 = u^{-1}u''^{-1}. \tag{2.60}$$

In the following we use the $so(4)$ characters $\chi_{a,b} \equiv \chi_a(u')\chi_b(u'')$. The results are as follows:

$$\begin{aligned}\mathcal{I}^{\rm ABJM}_{(N,k)=(1,3)} &= 1 + \chi_{1,1}\hat{q} + (u^{-3}\chi_{0,3} + u^3\chi_{3,0})\hat{q}^{\frac{3}{2}} \\ &+ (-2 - \chi_{0,2} - \chi_{2,0} + \chi_{2,2})\hat{q}^2 + \mathcal{O}(\hat{q}^{\frac{5}{2}}).\end{aligned} \tag{2.61}$$

$$\begin{aligned}\mathcal{I}^{\rm ABJM}_{(N,k)=(2,3)} &= 1 + \chi_{1,1}\hat{q} + (u^{-3}\chi_{0,3} + u^3\chi_{3,0})\hat{q}^{\frac{3}{2}} + (-\chi_{0,2} - \chi_{2,0} + 2\chi_{2,2})\hat{q}^2 \\ &+ (2u^{-3}\chi_{1,4} + 2u^3\chi_{4,1})\hat{q}^{\frac{5}{2}} \\ &+ (u^{-6}(\chi_{0,2} + 2\chi_{0,6}) - 2\chi_{1,1} - \chi_{1,3} - \chi_{3,1} + 3\chi_{3,3} \\ &\quad + u^6(\chi_{2,0} + 2\chi_{6,0}))\hat{q}^3 + \mathcal{O}(\hat{q}^{\frac{7}{2}}).\end{aligned} \tag{2.62}$$

$$\begin{aligned}\mathcal{I}^{\rm ABJM}_{(N,k)=(3,3)} &= 1 + \chi_{1,1}\hat{q} + (u^3\chi_{3,0} + u^{-3}\chi_{0,3})\hat{q}^{\frac{3}{2}} + (-\chi_{0,2} - \chi_{2,0} + 2\chi_{2,2})\hat{q}^2 \\ &+ 2(u^3\chi_{4,1} + u^{-3}\chi_{1,4})\hat{q}^{\frac{5}{2}} \\ &+ (u^6(\chi_{2,0} + 2\chi_{6,0}) - \chi_{1,3} - \chi_{3,1} + 4\chi_{3,3} + u^{-6}(\chi_{0,2} + 2\chi_{0,6}))\hat{q}^3 \\ &+ (u^3(-\chi_{1,0} + \chi_{1,2} - \chi_{3,0} - \chi_{5,0} + 4\chi_{5,2}) \\ &\quad + u^{-3}(-\chi_{0,1} - \chi_{0,3} - \chi_{0,5} + \chi_{2,1} + 4\chi_{2,5}))\hat{q}^{\frac{7}{2}} \\ &+ (u^6(\chi_{3,1} + \chi_{5,1} + 4\chi_{7,1}) - 1 + \chi_{0,4} - 3\chi_{2,2} - \chi_{2,4} + \chi_{4,0} - \chi_{4,2} \\ &\quad + 7\chi_{4,4} + u^{-6}(\chi_{1,1} + \chi_{1,1} + \chi_{1,3} + \chi_{1,5} + 4\chi_{1,7}))\hat{q}^4 + \mathcal{O}(\hat{q}^{\frac{9}{2}}).\end{aligned} \tag{2.63}$$

2.7 Large N index from supergravity on $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$

To calculate the large-N superconformal index for the $\mathbb{Z}_k$ orbifold case from the supergravity on $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$, we only have to introduce the orbifold projection on the single-particle index (2.52). We define orbifold projection operator $\mathcal{P}_k$ as follows:

$$\mathcal{P}_k g(\hat{u}_1, \hat{u}_2, \hat{u}_3, \hat{u}_4) = \frac{1}{k} \sum_{i=0}^{k-1} g(\omega_k^i \hat{u}_1, \omega_k^i \hat{u}_2, \omega_k^{-i} \hat{u}_3, \omega_k^{-i} \hat{u}_4), \tag{2.64}$$

where $g(\hat{u}_1, \hat{u}_2, \hat{u}_3, \hat{u}_4)$ is a function of $su(4)$ fugacities. Then, the formula (2.51) is replaced to

$$\mathcal{I}_{\mathrm{KK}}^{\mathbb{Z}_k} = \mathrm{Pexp}\, \mathcal{P}_k i_{\mathrm{KK}}. \tag{2.65}$$

The $k = 2$ result is given by

$$\begin{aligned}
\mathcal{I}_{\mathrm{KK}}^{\mathbb{Z}_2} = {} & 1 + \chi_{[2,0,0]}\hat{q} + (\chi_{[0,2,0]} - \chi_{[1,0,1]} + 2\chi_{[4,0,0]})\hat{q}^2 \\
& + (\chi_{[0,0,2]} - \chi_{[1,1,1]} + 2\chi_{[2,2,0]} - 2\chi_{[3,0,1]} + \chi_{[4,1,0]} + 3\chi_{[6,0,0]})\hat{q}^3 \\
& + (3\chi_{[0,4,0]} - 2\chi_{[1,2,1]} + 2\chi_{[2,0,2]} + \chi_{[2,3,0]} - 2\chi_{[3,1,1]} - \chi_{[4,0,0]} + 5\chi_{[4,2,0]} - 4\chi_{[5,0,1]} \\
& \quad + 2\chi_{[6,1,0]} + 5\chi_{[8,0,0]})\hat{q}^4 + \mathcal{O}(\hat{q}^5).
\end{aligned} \tag{2.66}$$

The index is again written in terms of $su(4)$ characters. Finally, we give the KK index for the $k = 3$ case.

$$\begin{aligned}
\mathcal{I}_{\mathrm{KK}}^{\mathbb{Z}_3} = {} & 1 + \chi_{1,1}\hat{q} + (u^3\chi_{3,0} + u^{-3}\chi_{0,3})\hat{q}^{\frac{3}{2}} + (-\chi_{0,2} - \chi_{2,0} + 2\chi_{2,2})\hat{q}^2 \\
& + 2(u^3\chi_{4,1} + u^{-3}\chi_{1,4})\hat{q}^{\frac{5}{2}} \\
& + (u^6(\chi_{2,0} + 2\chi_{6,0}) - \chi_{1,3} - \chi_{3,1} + 4\chi_{3,3} + u^{-6}(\chi_{0,2} + 2\chi_{0,6}))\hat{q}^3 \\
& + (u^3(-\chi_{1,0} + \chi_{1,2} - \chi_{3,0} - \chi_{5,0} + 4\chi_{5,2}) \\
& \quad + u^{-3}(-\chi_{0,1} - \chi_{0,3} - \chi_{0,5} + \chi_{2,1} + 4\chi_{2,5}))\hat{q}^{\frac{7}{2}} \\
& + (+u^6(\chi_{1,1} + \chi_{3,1} + \chi_{5,1} + 4\chi_{7,1}) + \chi_{0,4} - \chi_{2,2} - \chi_{2,4} + \chi_{4,0} \\
& \quad - \chi_{4,2} + 8\chi_{4,4} + 1 + u^{-6}(\chi_{1,1} + \chi_{1,3} + \chi_{1,5} + 4\chi_{1,7}))\hat{q}^4 + \mathcal{O}(\hat{q}^{\frac{9}{2}}).
\end{aligned} \tag{2.67}$$

Note that the KK index for the $\mathbb{Z}_k$ orbifold case also agrees with the large-N ABJM index with Chern-Simons level k.

$$\mathcal{I}_{(N,k)=(\infty,k)}^{\mathrm{ABJM}} = \mathcal{I}_{\mathrm{KK}}^{\mathbb{Z}_k}. \tag{2.68}$$

Summary of Chapter 2

In this chapter, we explained the superconformal index of the M2-brane theories investigated so far. We first defined the superconformal indices of the M2-brane theories in (2.7). Then, we discussed the way to calculate the superconformal index. When the number of M2-brane

N is 1, the theory is described by the free theory of scalar fields and fermions, and the index is given by (2.15). For general N, the theory is described by the ABJM theory, and the index can be calculated by the localization formula (2.30). In the large N limit, we can use the dual supergravity description. We saw that in the dual gravity theory, the superconformal index is given by the contribution of the Kaluza Klein modes on S^7. We also discussed the superconformal index for the $\mathbb{Z}_k$ orbifold case in Section 2.6 and Section 2.7.

Chapter 3

Superconformal indices of M5-brane theories

In this chapter, we review the superconformal indices of theories realized on M5-branes. Similarly to the M2-brane case, the theory on a single M5-brane is described by the free theory of a tensor multiplet and we can easily calculate the index.

However, analyzing theories on multiple M5-branes is quite difficult. The theory on M5-branes is called the 6d $\mathcal{N} = (2,0)$ theory, but the (2,0) theory is not well understood. Particularly, we have no Lagrangian description of the theory yet. Hence, we cannot calculate the index of the 6d $\mathcal{N} = (2,0)$ theory with $N > 1$ directly by the localization method.

In the large-N limit, we can use the AdS/CFT correspondence to calculate the index of the 6d (2,0) theory. The dual gravity theory is the eleven-dimensional supergravity on $\mathrm{AdS}_7 \times S^4$ and we can calculate the index from the Kaluza Klein spectrum.

The goal of this chapter is to show the superconformal indices of the following theories:

- the theory on a single M5-brane ($N = 1$)
- the dual supergravity theory ($N = \infty$)

In Section 3.5 $\sim$ 3.7, we also discuss a theory realized on M5-branes on the orbifold singularity $\mathbb{C}^2/\mathbb{Z}_k$, called the 6d $\mathcal{N} = (1,0)$ theory, and its gravity dual: The eleven-dimensional supergravity on $\mathrm{AdS}_7 \times S^4/\mathbb{Z}_k$.

3.1 6d $\mathcal{N} = (2,0)$ theory

The superconformal field theory realized on N flat M5-branes is called the 6d $\mathcal{N} = (2,0)$ theory, which was investigated in [37, 38, 39]. Unlike the ABJM case, the Lagrangian of this theory is not known yet and thus, it is difficult to study the theory directly. If we remove the center of mass (2,0) tensor multiplet, we obtain an interacting superconformal field theory which is especially called the 6d A_{N-1} theory.

There are also other stringy constructions of the theory. A construction from type IIB superstring theory compactification [37] and an F-theory construction [40, 41] are known, but we do not treat them in detail in this thesis.

A reduction to the 5d theory is also important since the theory admits Lagrangian description. S^1 compactification of the 6d $\mathcal{N} = (2,0)$ theory gives the 5d $\mathcal{N} = 2$ supersymmetric Yang-Mills theory and we can extract the information of the 6d $\mathcal{N} = (2,0)$ theory from the 5d theory. In fact, the 6d $\mathcal{N} = (2,0)$ index was calculated from the 5d SYM theory [42]. However, the method is highly complicated and their analysis was limited to a few terms of the index. See Footnote in the page 73 for more detail.

3.2 The definition of the superconformal indices for M5-brane theories

We consider M5-branes expanding spreading in $x^0 \sim x^5$ directions. See Table 3.2 and Figure 3.2. The system possesses $SO(2,6)$ and $SO(5)$ symmetry.

	0	1	2	3	4	5	6	7	8	9	11
M5-brane	∘	∘	∘	∘	∘	∘					

Table 3.1: The configuration of M5-branes is shown. The M5-branes are expanding in $x^0 \sim x^5$ directions, which are marked as ∘.

Again, the insertion of the M5-branes breaks half of the supersymmetry parametrized by ϵ satisfying

$$\Gamma_0\Gamma_1...\Gamma_5\epsilon = \epsilon. \tag{3.1}$$

This leaves only left-handed supersymmetry and 6d $\mathcal{N} = (2,0)$ supersymmetry is realized on the M5-brane. At the low energy limit, the supersymmetry together with the $SO(2,6)$ and $SO(5)$ symmetry enhanced to the 6d $\mathcal{N} = (2,0)$ superconformal symmetry. The six-dimensional $\mathcal{N} = (2,0)$ superconformal algebra is $\check{\mathcal{A}} := osp(8^*|4)$, whose bosonic subalgebra is

$$so(2,6) \times so(5) \subset \check{\mathcal{A}}. \tag{3.2}$$

The generators are

$$\check{H}, \quad \check{Q}_i, \quad \check{S}_i, \quad \check{P}_\mu, \quad \check{K}_\mu, \quad \check{R}_{ab}, \quad \check{J}_{\mu\nu}, \tag{3.3}$$

where $\check{H}$ is Hamiltonian, $\check{Q}_i(\check{S}_i)$ are (conformal) supercharges, $\check{P}_\mu$ are momentum operator, $\check{K}_\mu$ are generators of special conformal transformations, $\check{R}_{ab}$ are R-symmetry generators, and $\check{J}_{\mu\nu}$ are Lorenrz generators.

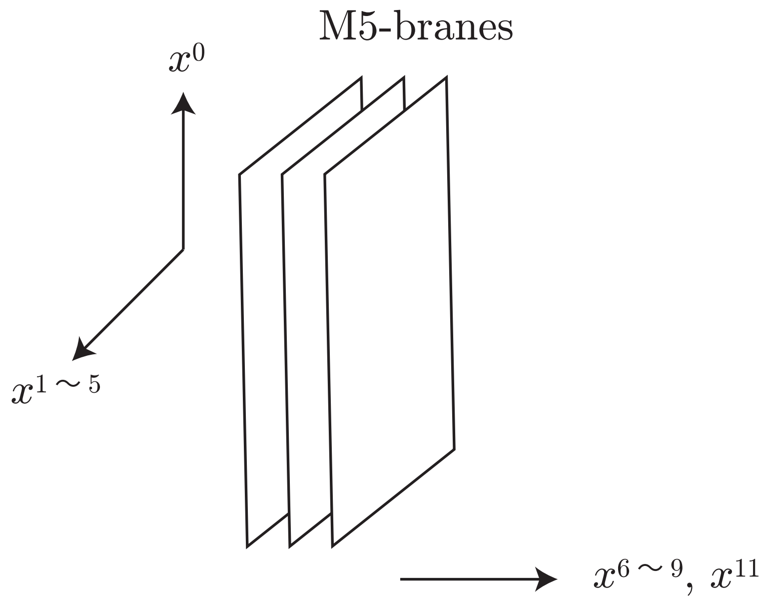

Figure 3.1: The schematic figure of the M5-branes configuration is shown.

Let us define the superconformal index of the 6d (2,0) theory. There are six Cartan generators:

$$\check{H}, \quad \check{J}_{12}, \quad \check{J}_{34}, \quad \check{J}_{56}, \quad \check{R}_{12}, \quad \check{R}_{34}. \tag{3.4}$$

To define the superconformal index, we need to choose one complex supercharge $\check{\mathcal{Q}}$ carrying specific Cartan charges. We take the one with the quantum numbers

$$\check{\mathcal{Q}} : (\check{H}, \check{J}_{12}, \check{J}_{34}, \check{J}_{56}; \check{R}_{12}, \check{R}_{34}) = (+\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}; +\tfrac{1}{2}, +\tfrac{1}{2}). \tag{3.5}$$

The subalgebra that keeps $\check{\mathcal{Q}}$ intact is

$$\check{\mathcal{B}} \times u(1)_{\check{\Delta}}, \tag{3.6}$$

where $\check{\mathcal{B}} = osp(6|2)$ is the super algebra whose bosonic subalgebra is $su(1,3) \times su(2) \subset \check{\mathcal{B}}$. The central factor $u(1)_{\check{\Delta}}$ is generated by

$$\check{\Delta} \equiv \{\check{\mathcal{Q}}, \check{\mathcal{Q}}^\dagger\} = \check{H} - (\check{J}_{12} + \check{J}_{34} + \check{J}_{56}) - 2(\check{R}_{12} + \check{R}_{34}). \tag{3.7}$$

We define the superconformal index associated with the BPS bound $\check{\Delta} \geq 0$ as the $\check{\mathcal{B}}$ character by [1]

$$\mathcal{I}(\check{q}, \check{y}_a, \check{u}) = \mathrm{tr}[(-1)^F \check{x}^{\check{\Delta}} \check{q}^{\check{H}+\frac{1}{3}(\check{J}_{12}+\check{J}_{34}+\check{J}_{56})} \check{y}_1^{\check{J}_{12}} \check{y}_2^{\check{J}_{34}} \check{y}_3^{\check{J}_{56}} \check{u}^{\check{R}_{12}-\check{R}_{34}}], \quad \check{y}_1\check{y}_2\check{y}_3 = 1. \tag{3.8}$$

Due to the Bose-Fermi degeneracy for $\check{\Delta} > 0$ The index does not depend on $\check{x}$.

3.3 A Single M5-brane

In this section, we consider the theory realized on a single M5-brane. We first show the Lagrangian of the free theory of the single M5-brane. Then, we discuss its superconformal index.

3.3.1 Lagranigan

A similar analysis to the single M2-brane case shows that, in the decoupling limit $T_{\mathrm{M5}} \to \infty$, the Nambu-Goto action of the M5-brane reduces to the action of 5 free massless scalar fields:

$$S_{\mathrm{NG}} = \int d^6\sigma \left(-\frac{1}{2}(\partial_a \phi^i)^2 \right), \tag{3.9}$$

where a runs from 0 to 5 and i runs from 1 to 5. The five scalars describe the fluctuation of the M5-brane in the five transverse directions.

Now, let us include the supersymmetry partners. Except for a gravity multiplet, only 6d $\mathcal{N} = (2,0)$ multiplet is a (2,0) tensor multiplet. The five scalar fields, one anti-self dual tensor fields, and four fermions form a 6d $\mathcal{N} = (2,0)$ tensor multiplet. The single M5-brane theory is described by the free theory of the (2,0) tensor multiplet.

$$\text{fields on the single M5-brane} = \begin{cases} \cdot \text{ 5 scalar fields } \phi^i \ (i = 1 \sim 5) \\ \cdot \text{ 1 anti-self dual tensor fields } B_{ab} \\ \cdot \text{ 4 fermions } \psi_m \ (m = 1 \sim 4) \end{cases} \tag{3.10}$$

The scalar fields belong to the $so(5)$ vector representation, the fermion fields belong to the $so(5)$ spinor representations, and the anti-self dual tensor field is an $so(5)$ singlet. Then, the complete Lagrangian for the single M5-brane theory is given by

$$S = \int d^6\sigma \left(-\frac{1}{2}(\partial_a \phi^i)^2 + \frac{1}{2}\psi_m \gamma^a \partial_a \psi_m + \frac{1}{2}|H_3|^2 \right), \tag{3.11}$$

where $H_3 = dB_2$ Note that the anti-self dual condition leads to $|H_3|^2 = 0$. Here, we use the action only to derive the equation of the motion, and impose the condition after the equation of motion is obtained.

[1]The fugacities $\check{q}$, $\check{y}_i$, and $\check{u}$ are related to those used in Section 3 of [25] by $\check{q} = x^3$, $\check{y}_1 = y_1$, $\check{y}_2 = y_1^{-1} y_2$, $\check{y}_3 = y_2^{-1}$, and $\check{u} = z^{\frac{1}{2}}$

3.3.2 Superconformal index

Let us calculate the index of the single M5-brane theory. In this case, the six-dimensional theory is the free theory of a single tensor multiplet and we can easily calculate the index. From Table 3.3.2, we can read off the single-particle index of the (2,0) tensor multiplet. It is

	H	$so(6)$	$so(5)$	contribution to the index
Q	$\frac{1}{2}$	$\overline{\mathbf{4}}$	$\mathbf{4}$	$\check{q}^2\chi_1(\check{u})$
ϕ	2	$\mathbf{1}$	$\mathbf{5}$	
$\partial^2\phi$	4	$\mathbf{1}$	$\mathbf{5}$	
ψ	$\frac{5}{2}$	$\overline{\mathbf{4}}$	$\mathbf{4}$	$-\check{q}^{\frac{8}{3}}\chi_{[0,1]}(\check{y})$
$\partial\psi$	$\frac{7}{2}$	$\mathbf{4}$	$\mathbf{4}$	
H_3	3	$\overline{\mathbf{10}}$	$\mathbf{1}$	$\check{q}^4$
∂H_3	4	$\mathbf{15}$	$\mathbf{1}$	
$\partial^2 H_3$	5	$\mathbf{6}$	$\mathbf{1}$	
$\partial^3 H_3$	6	$\mathbf{1}$	$\mathbf{1}$	
∂	1	$\mathbf{6}$	$\mathbf{1}$	$\check{q}^{\frac{4}{3}}\check{y}_{1,2,3}$

Table 3.2: The field contents and contribution to the index of the $\mathcal{N}=(2,0)$ tensor multiplet

explicitly given by [25]

$$i^{\text{M5}} = \frac{\check{q}^2\chi_1(\check{u}) - \check{q}^{\frac{8}{3}}\chi_{[0,1]}(\check{y}) + \check{q}^4}{(1-\check{q}^{\frac{4}{3}}\check{y}_1)(1-\check{q}^{\frac{4}{3}}\check{y}_2)(1-\check{q}^{\frac{4}{3}}\check{y}_3)}, \tag{3.12}$$

where $\chi_m(\check{u})$ is the $su(2)$ character of the spin $m/2$ representation

$$\chi_m(\check{u}) = \frac{\check{u}^{m+1} - \check{u}^{-m-1}}{\check{u} - \check{u}^{-1}} = \check{u}^m + \cdots + \check{u}^{-m}, \tag{3.13}$$

and $\chi_{[a,b]}(\check{y})$ is the $su(3)$ character of the representation with Dynkin labels $[a,b]$. $\chi_{[1,0]}$ for the fundamental representation and $\chi_{[0,1]}$ for the anti-fundamental representation are

$$\chi_{[1,0]}(\check{y}) = \check{y}_1 + \check{y}_2 + \check{y}_3, \quad \chi_{[0,1]}(\check{y}) = \check{y}_1^{-1} + \check{y}_2^{-1} + \check{y}_3^{-1}. \tag{3.14}$$

Let $\mathcal{I}_N^{(2,0)}$ be the superconformal index of the theory realized on the stack of N M5-branes. The superconformal index of 6d (2,0) theory with $N=1$ is simply given by $\mathcal{I}_{N=1}^{(2,0)} = \text{Pexp}\, i^{\text{M5}}$. The explicit calculation shows the following superconformal index of the 6d $\mathcal{N}=(2,0)$ theory with $N=1$:

$$
\begin{aligned}
\mathcal{I}^{(2,0)}_{N=1} &= 1+\chi^{\check{u}}_1\check{q}^2-\chi_{[0,1]}\check{q}^{\frac{8}{3}}+\chi^{\check{u}}_1\chi_{[1,0]}\check{q}^{\frac{10}{3}}+(\chi^{\check{u}}_2-\chi_{[1,1]})\check{q}^4+\chi^{\check{u}}_1(\chi_{[2,0]}-\chi_{[0,1]})\check{q}^{\frac{14}{3}}\\
&+((\chi^{\check{u}}_2+2)\chi_{[1,0]}-\chi_{[2,1]})\check{q}^{\frac{16}{3}}+(\chi^{\check{u}}_3+\chi^{\check{u}}_1(-2\chi_{[1,1]}+\chi_{[3,0]}-1))\check{q}^6\\
&+(-(\chi^{\check{u}}_2-2)\chi_{[0,1]}+\chi_{[1,2]}+2\chi^{\check{u}}_2\chi_{[2,0]}+2\chi_{[2,0]}-\chi_{[3,1]})\check{q}^{\frac{20}{3}}\\
&+(\chi^{\check{u}}_3\chi_{[1,0]}+\chi^{\check{u}}_1(-\chi_{[0,2]}-3\chi_{[2,1]}+\chi_{[4,0]}))\check{q}^{\frac{22}{3}}\\
&+(\chi^{\check{u}}_4+\chi_{[0,3]}+2\chi_{[1,1]}+\chi_{[2,2]}-\chi^{\check{u}}_2(\chi_{[1,1]}-2\chi_{[3,0]}+1)+4\chi_{[3,0]}-\chi_{[4,1]}-2)\check{q}^8\\
&+(\chi^{\check{u}}_1(2\chi_{[0,1]}-\chi_{[1,2]}+\chi_{[2,0]}-4\chi_{[3,1]}+\chi_{[5,0]})-\chi^{\check{u}}_3(\chi_{[0,1]}-2\chi_{[2,0]}))\check{q}^{\frac{26}{3}}\\
&+(-2\chi_{[0,2]}+(-\chi^{\check{u}}_2+\chi^{\check{u}}_4-3)\chi_{[1,0]}+\chi_{[1,3]}-3\chi^{\check{u}}_2\chi_{[2,1]}+2\chi_{[2,1]}+2\chi_{[3,2]}\\
&\quad+3\chi^{\check{u}}_2\chi_{[4,0]}+4\chi_{[4,0]}-\chi_{[5,1]})\check{q}^{\frac{28}{3}}\\
&+(\chi^{\check{u}}_5-\chi^{\check{u}}_3(\chi_{[1,1]}-3\chi_{[3,0]}+1)+\chi^{\check{u}}_1(\chi_{[0,3]}+6\chi_{[1,1]}-\chi_{[2,2]}+3\chi_{[3,0]}-5\chi_{[4,1]}\\
&\quad+\chi_{[6,0]}-1))\check{q}^{10}+\mathcal{O}(\check{q}^{\frac{55}{3}}).
\end{aligned}
\tag{3.15}
$$

3.4 Large N limit

Contrary to the ABJM case, we do not have a direct method of calculating the indices of 6d $\mathcal{N}=(2,0)$ theories with $N>1$. Instead, in this section, we use the AdS/CFT correspondence to calculate the large-N limit of the index. The statement for the $\mathrm{AdS}_7/\mathrm{CFT}_6$ is

- The M-theory on $\mathrm{AdS}_7\times S^4$ is equivalent to the 6d $\mathcal{N}=(2,0)$ theory

Further, in the presence of the $\mathbb{Z}_k$ orbifold, the supersymmetry is reduced from $\mathcal{N}=(2,0)$ to $\mathcal{N}=(1,0)$. The theory realized on M5-branes on $\mathbb{C}^2/\mathbb{Z}_k$ is called the 6d $\mathcal{N}=(1,0)$ theory. For the orbifold the $\mathrm{AdS}_7/\mathrm{CFT}_6$ claims

- The M-theory on $\mathrm{AdS}_7\times S^4/\mathbb{Z}_k$ is equivalent to the 6d $\mathcal{N}=(1,0)$ theory

In the followings, we first discuss the $k=1$ case and calculate the large-N index from the Kaluza Klein spectrum. Then, we discuss the $\mathbb{Z}_k$ orbifold case.

3.4.1 M5-brane solution

Extremal black M5-brane solution

Similarly to the M2-brane case, the extremal M5-brane solution preserving 16 supercharges is given by [43]

$$ds^2=H(r)^{-1/3}\eta_{ab}dx^adx^b+H(r)^{2/3}(dr^2+r^2d\Omega_4^2), \tag{3.16}$$

with the 6-form field

$$A_6=H(r)^{-1}dx^0\wedge dx^1\wedge\cdots\wedge dx^5. \tag{3.17}$$

$a, b = 0, 1, ..., 5$ run directions parallel to the M5-brane and $H(r)$ is again the harmonic function defined by

$$H(r) = 1 + \frac{\check{r}^3}{r^3}. \tag{3.18}$$

The length $\check{r}$, which becomes radius of S^4 in near horizon limit, is given by

$$\check{r}^3 = \pi N l_p^3. \tag{3.19}$$

Near horizon geometry of M5-brane solution

Again, we consider the near horizon limit of (3.16) by taking $r \ll \check{r}$. With the variable change $z = 2\check{r}^{3/2}/r^{1/2}$, the metric reduces to $\mathrm{AdS}_7 \times S^4$:

$$\begin{aligned} ds^2 &= \frac{4\check{r}^2}{z^2}\left(\eta_{ab}dx^a dx^b + dz^2\right) + \check{r}^2 d\Omega_4^2 \\ &= \check{L}^2 ds^2_{\mathrm{AdS}_7} + \check{r}^2 ds^2_{S^4}, \end{aligned} \tag{3.20}$$

where $\check{L} = 2\check{r}$, meaning that the radius of AdS_7 is two times the radius of S^4.

The bosonic symmetry of the supergravity solution (3.20) is $SO(2,6) \times SO(5)$. The $SO(2,6)$ part corresponds to the conformal symmetry in 6 dimension and $SO(5)$ corresponds to the 6d $\mathcal{N} = (2,0)$ $\mathcal{R}$-symmetry. Hence, we expect that the dual superconformal theory is a 6d $\mathcal{N} = (2,0)$ superconformal field theory. Further, the entropy of the M5-branes scales as

$$S \sim N^3. \tag{3.21}$$

This is consistent with the field theory analysis [44].

3.4.2 Index from Kaluza Klein modes

The large-N limit $\mathcal{I}^{(2,0)}_{N=\infty}$ of the superconformal index is given by the Kaluza-Klein index of $AdS_7 \times S^4$. It is given by $\mathcal{I}^{S^4}_{\mathrm{KK}} = \mathrm{Pexp}\, i^{S^4}_{\mathrm{KK}}$ with the single-particle index [25]

$$i^{S^4}_{\mathrm{KK}} = \frac{\check{q}^2\chi_1(\check{u}) - \check{q}^{\frac{8}{3}}\chi_{[0,1]}(\check{y}) + \check{q}^{\frac{16}{3}}\chi_{[1,0]}(\check{y}) - \check{q}^6\chi_1(\check{u})}{(1-\check{u}\check{q}^2)(1-\check{u}^{-1}\check{q}^2)(1-\check{y}_1\check{q}^{\frac{4}{3}})(1-\check{y}_2\check{q}^{\frac{4}{3}})(1-\check{y}_3\check{q}^{\frac{4}{3}})}. \tag{3.22}$$

We show the explicit result of the Kaluza Klein index.

$$\begin{aligned}
\mathcal{I}_{\rm KK}^{S^4} = & 1 + \chi_1^u q^2 - \chi_{[0,1]} q^{\frac{8}{3}} + \chi_{[1,0]}\chi_1^u q^{\frac{10}{3}} + \left(2\chi_2^u - \chi_{[1,1]}\right) q^4 + \left(\chi_{[2,0]}\chi_1^u - 2\chi_{[0,1]}\chi_1^u\right) q^{\frac{14}{3}} \\
& + \left(\chi_{[1,0]}\left(2\chi_2^u + 3\right) - \chi_{[2,1]}\right) q^{\frac{16}{3}} + \left(-3\chi_{[1,1]}\chi_1^u + \chi_{[3,0]}\chi_1^u - \chi_1^u + 3\chi_3^u\right) q^6 \\
& + \left(\chi_{[0,1]}\left(2 - 4\chi_2^u\right) + \chi_{[2,0]}\left(3\chi_2^u + 4\right) + \chi_{[1,2]} - \chi_{[2,0]} - \chi_{[3,1]}\right) q^{\frac{20}{3}} \\
& + \left(-4\chi_{[2,1]}\chi^u(1) + \chi_{[4,0]}\chi_1^u + \chi_{[1,0]}\left(4\chi_1^u + 4\chi_3^u\right)\right) q^{\frac{22}{3}} \\
& + \left(\chi_{[1,1]}\left(1 - 6\chi_2^u\right) + 3\chi_{[3,0]}\left(\chi_2^u + 2\right) + \chi_{[0,3]} - \chi_{[1,1]} + \chi_{[2,2]} - \chi_{[3,0]} - \chi_{[4,1]} - 3\chi_2^u + 5\chi_4^u - 4\right) q^8 \\
& + \left(\chi_{[1,2]}\chi_1^u - \chi_{[2,0]}\chi_1^u - 5\chi_{[3,1]}\chi_1^u + \chi_{[5,0]}\chi_1^u + \chi_{[0,1]}\left(5\chi_1^u - 6\chi_3^u\right) + \chi_{[2,0]}\left(9\chi_1^u + 6\chi_3^u\right)\right) q^{\frac{26}{3}} \\
& + \left(\chi_{[2,1]}\left(-10\chi_2^u - 1\right) + \chi_{[4,0]}\left(4\chi_2^u + 7\right) + \chi_{[1,0]}\left(6\chi_2^u + 7\chi_4^u - 9\right)\right. \\
& \quad \left. -3\chi_{[0,2]} + 3\chi_{[1,0]} + \chi_{[1,3]} - \chi_{[2,1]} + 2\chi_{[3,2]} - 2\chi_{[4,0]} - \chi_{[5,1]}\right) q^{\frac{28}{3}} \\
& + \left(2\chi_{[0,3]}\chi_1^u - 2\chi_{[1,1]}\chi_1^u + 2\chi_{[2,2]}\chi_1^u - 2\chi_{[3,0]}\chi_1^u - 6\chi_{[4,1]}\chi_1^u + \chi_{[6,0]}\chi_1^u + \chi_{[1,1]}\left(7\chi_1^u - 11\chi_3^u\right)\right. \\
& \quad \left. + \chi_{[3,0]}\left(15\chi_1^u + 8\chi_3^u\right) - 9\chi_1^u - 6\chi_3^u + 7\chi_5^u\right) q^{10} + \mathcal{O}\left(q^{\frac{46}{3}}\right).
\end{aligned} \tag{3.23}$$

3.5 6d $\mathcal{N} = (1,0)$ theory

Let us discuss the $\mathbb{Z}_k$ orbifold case. Generally, in the presence of the orbifold singularity $\mathbb{C}^2/\Gamma$, where Γ is a discrete subgroup of $SU(2)$, the 6d $\mathcal{N} = (1,0)$ theories are realized on the worldvoume of M5-branes. Among many types of 6d $(1,0)$ theories, in this thesis, we only discuss the $\Gamma = \mathbb{Z}_k$ case.

3.5.1 M-theory set-up

We consider M-theory in the background $\mathbb{R}^{1,5} \times \mathbb{C}^2/\mathbb{Z}_k \times \mathbb{R}_T$. Let X_μ ($\mu = 0, 1, \ldots, 5$), z_i ($i = 1, 2$), and x_5 be the coordinates of $\mathbb{R}^{1,5}$, $\mathbb{C}^2$, and $\mathbb{R}_T$, respectively. We also define x_m ($m = 1, 2, 3, 4$) by

$$z_1 = x_1 + ix_2, \quad z_2 = x_3 + ix_4. \tag{3.24}$$

Let us define the orbifold action. Let R_{ab} ($a, b = 1, \ldots, 5$) be the generators of the rotation group $SO(5)_R$ in the x_a space. We define the orbifold by $\mathbb{Z}_k$ generated by

$$\exp\left(\frac{2\pi i}{k}(R_{12} - R_{34})\right). \tag{3.25}$$

This acts on (z_1, z_2, x_5) as

$$(z_1, z_2, x_5) \to (e^{2\pi i/k} z_1, e^{-2\pi i/k} z_2, x_5). \tag{3.26}$$

We put N M5-branes at $x_1 = \cdots = x_5 = 0$. If it were not for the orbifolding, the A_{N-1}-type $\mathcal{N} = (2,0)$ theory would be realized on the worldvolume of the M5-branes. The orbifolding breaks the $\mathcal{N} = (2,0)$ supersymmetry down to $\mathcal{N} = (1,0)$. At the same time, the

$SO(5)_R$ symmetry is broken to $SU(2)_R \times U(1)_F$ for $k \geq 3$. $U(1)_F$ is replaced by $SU(2)_F$ for $k = 2$. The $SU(2)_R$ is the R-symmetry of the 6d $(1,0)$ SCFTs, while $U(1)_F$ or $SU(2)_F$ does not act on the $\mathcal{N} = (1,0)$ supercharges and is treated as a flavor symmetry. In addition, the orbifold singularity provides $SU(k)$ flavor symmetry. The singular locus $\mathbb{R}^{1,5} \times \mathbb{R}_T$ is divided by the M5-branes at $x_5 = 0$ into two parts: the $x_5 > 0$ part and the $x_5 < 0$ part. Correspondingly, we have two copies of $SU(k)$ symmetry which we denote by $SU(k)_a$ and $SU(k)_b$. In summary, the bosonic global symmetry is

$$SO(2,6)_{\text{conf}} \times SU(2)_R \times G_{\text{flavor}}, \tag{3.27}$$

where $SO(2,6)_{\text{conf}} \times SU(2)_R$ is the bosonic subgroup of the 6d $\mathcal{N} = (1,0)$ superconformal symmetry $OSp(8|2)$ and the flavor symmetry G_{flavor} is generically given by

$$G_{\text{flavor}} = U(1)_F \times SU(k)_a \times SU(k)_b. \tag{3.28}$$

A subtle point is that the symmetry may be different from what is read off from the corresponding quiver gauge theory discussed in the next subsection. It was proposed that a certain discrete symmetry of the quiver gauge theory, which is not manifest perturbatively, is gauged in the strong coupling limit and as a result the flavor symmetry is reduced [45]. For example, in the case of $N = k = 2$, although the flavor symmetry of the quiver gauge theory is $SO(8)$, that of the superconformal theory is $SO(7)$ [46]. See Table 3.3 for the flavor symmetries for different k and N.

Table 3.3: The flavor symmetries of $\mathcal{N} = (1,0)$ theories.

	$N \neq 2$	$N = 2$
$k = 2$	$SU(2)_a \times SU(2)_b \times SU(2)_F$	$SO(7)$
$k \geq 3$	$SU(k)_a \times SU(k)_b \times U(1)_F$	$SU(2k)$

On the dual AdS side, these symmetries are manifest when $N \neq 2$. For generic values of k and N the flavor symmetry is $SU(k)_a \times SU(k)_b \times U(1)_F$. In addition, we also have $SU(2)_R$ symmetry.

$U(1)_F \times SU(2)_R$ is the isometry of $S^4/\mathbb{Z}_k$, and two $SU(k)$ symmetries are associated with the two A_{k-1} singularities at the fixed points. In the case of $k = 2$ and $N \neq 2$, $U(1)_F$ is enhanced to $SU(2)_F$, and this is also understood as the isometry of $S^4/\mathbb{Z}_2$.

However, the enhancement for $N = 2$ is not manifest on the AdS side, and it is interesting to study how this is realized. This cannot be seen in the large-N limit, and to confirm such symmetry enhancement we need to include finite-N corrections. We will discuss this subject in Section 5.2.

3.5.2 Type IIA description

For the analysis of the operator spectrum and the flavor symmetry of the theory, it is convenient to consider the quiver gauge theories realized in the tensor branch. By taking $U(1)_F$

orbits as M-theory circles we can regard the system as a type IIA brane configuration [47]. N M5-branes become N NS5-branes, and the A_{k-1} singularity becomes a stack of k D6-branes.

The $(2,0)$ tensor multiplet on an M5-brane separate into a $(1,0)$ tensor multiplet and a $(1,0)$ hypermultiplet on the corresponding NS5-brane, and the scalar component in the $(1,0)$ tensor multiplet corresponds to the location of the NS5-branes in the x_5 direction. At a generic point in the tensor branch all the NS5-branes are separated one by one in the x_5 direction, and a linear quiver gauge theory is realized on the D6-branes.

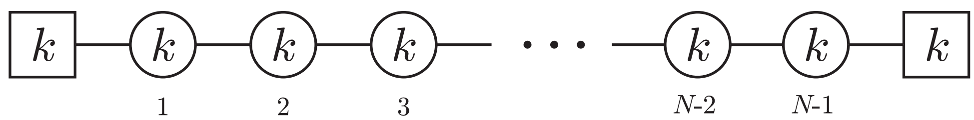

Figure 3.2: The linear quiver diagram of the gauge theory realized in the tensor branch is shown. The circles denote $SU(k)$ gauge nodes and the boxes declares the $SU(k)$ flavor symmetries.

The worldvolume of the stack of D6-branes is divided into $N+1$ parts by the NS5-branes. We label the NS5-branes by $i = 1, 2, \ldots, N$. The D6-branes suspended between two NS5-branes i and $i+1$ give $SU(k)_i$ gauge group while two semi-infinite parts of D6-branes give the flavor symmetries $SU(k)_a \equiv SU(k)_0$ and $SU(k)_b \equiv SU(k)_N$. Let $(h_i, \widetilde{h}_i)$ be the hypermultiplet arising from open strings crossing the i-th NS5-brane. h_i and $\widetilde{h}_i$ belong to the bi-fundamental representations $(k, \overline{k})$ and $(\overline{k}, k)$, respectively, of $SU(k)_{i-1} \times SU(k)_i$. The $SU(N)$ groups and the hypermultiplets are depicted as the linear quiver diagram in Figure 3.2. In addition, we also have degrees of freedom that are implicit in the diagram; in each gauge node there exists a tensor multiplet corresponding to the degrees of freedom of the NS5-brane.

Among different gauge invariant operators let us focus on two classes of operators. The first class includes operators defined by

$$S_{ij} = (h_0)_{ia}(\widetilde{h}_0)_{aj}, \quad S'_{ij} = (\widetilde{h}_N)_{ia}(h_N)_{aj}. \tag{3.29}$$

The other class includes

$$L_{ij} = (h_0)_{ia_1}(h_1)_{a_1a_2} \cdots (h_N)_{a_Nj}, \quad L'_{ij} = (\widetilde{h}_N)_{ia_N} \cdots (h_1)_{a_2a_1}(h_0)_{a_1j}. \tag{3.30}$$

These operators play an important role when we discuss the flavor symmetry.

The operators S_{ij} and S'_{ij} belong to the adjoint representations of $SU(k)_0$ and $SU(k)_N$, respectively. They have dimension 4 and are the primary operators of the current multiplets of the generic flavor symmetry

$$G_{\text{flavor}} = SU(k)_0 \times SU(k)_N \times U(1)_F. \tag{3.31}$$

(Although there are N classical $U(1)$ symmetries only one of them is anomaly free.)

The operators L_{ij} and L'_{ij}, which belong to the bi-fundamental representations $(k,\overline{k})_{+1}$ and $(\overline{k},k)_{-1}$ of $G_{\rm flavor}$ in (3.31), respectively, have dimension $2N$. These operators appear in the spectrum only when N is finite, and play a role similar to baryonic operators in four-dimensional quiver gauge theories. They are expected to correspond to wrapped M2-branes on the gravity side. Indeed, their dimension coincide with the mass of an M2-brane wrapped around a large $S^2/\mathbb{Z}_k$ in the unit of the inverse AdS radius.

The flavor symmetry (3.31) is enhanced if one of k or N becomes 2. If $k=2$ and $N\geq 3$ $U(1)_F$ is enhanced to $SU(2)_F$. Correspondingly, the index is written in terms of $SU(2)_F$ characters. This symmetry is manifest on the gravity side as the isometry of $S^4/\mathbb{Z}_2$.

The enhancement for $N=2$ is more interesting. If $N=2$ the operators L and L' have dimension 4 as well as S and S', and they give additional current multiplets. As the result the flavor symmetry (3.31) is enhanced to $SU(2k)$ for $k\geq 3$. On the gravity side, this enhancement should be realized when we include the contribution of wrapped M2-branes.

The $k=N=2$ case is most interesting. In this case, there are eight $SU(2)$ gauge symmetry doublet in the hypermultiplets, and we can write down 28 gauge invariant dimension 4 operators forming the $SO(8)$ adjoint representation. They correspond to the $SO(8)$ global symmetry of the quiver gauge theory. However, it is known that the symmetry is reduced to $SO(7)$ in a highly non-trivial way [46], and it would be nice if we can reproduce this flavor symmetry on the gravity side by the index calculation.

3.6 Supperconformal indices of 6d $\mathcal{N}=(1,0)$ theories

We define the superconformal index as follows. Let H and J_{ij} $(i,j=1,\ldots,6)$ be the generators of $SO(2)_H\times SO(6)_{\rm spin}\subset SO(2,6)_{\rm conf}$, and take H, J_{12}, J_{34}, and J_{56} as Cartan generators. H is the Hamiltonian and J_{ij} are Lorentz generators. We also take R_{12} and R_{34} as $SO(5)_R$ Cartan generators. To define the index we need to choose one component of the supercharge. We take the component with the following quantum numbers:

$$\mathcal{Q}:(H,J_{12},J_{34},J_{56};R_{12},R_{34})=(+\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2};+\tfrac{1}{2},+\tfrac{1}{2}). \tag{3.32}$$

Note that $\mathcal{Q}$ is invariant under the orbifold group $\mathbb{Z}_k$ generated by (3.25). The anticommutation relation between $\mathcal{Q}$ and its hermitian conjugate $\mathcal{Q}^\dagger$ is

$$\Delta\equiv\{\mathcal{Q},\mathcal{Q}^\dagger\}=H-(J_{12}+J_{34}+J_{56})-2(R_{12}+R_{34}). \tag{3.33}$$

Then we define the superconformal index by

$$\mathcal{I}(q,y_a,u,a_i,b_i)={\rm tr}\left[(-1)^Fx^\Delta q^{H+\frac{1}{3}(J_{12}+J_{34}+J_{56})}y_1^{J_{12}}y_2^{J_{34}}y_3^{J_{56}}u^{R_{12}-R_{34}}\prod_{i=1}^{k-1}a_i^{F_{a,i}}b_i^{F_{b,i}}\right]. \tag{3.34}$$

Due to the boson/fermion cancellation only the BPS operators with $\Delta=0$ contribute to the index, and hence the index does not depend on x. The choice of $\mathcal{Q}$ breaks the $SO(6)_{\rm spin}$ to $U(1)_{\rm spin}\times SU(3)_{\rm spin}$, and y_1, y_2, and y_3 are the $SU(3)_{\rm spin}$ fugacities constrained by $y_1y_2y_3=1$. $F_{a,i}$ and $F_{b,i}$ are respectively Cartan generators of $SU(k)_a$ and $SU(k)_b$. u is the fugacity for $U(1)_F$ generated by $R_{12}-R_{34}$. Note that for $k=1$ this index agree with the $\mathcal{N}=(2,0)$ superconformal index defined in (3.8).

3.6.1 Result for $N = 1$

The theory with $N = 1$ is the free theory consisting of the "center-of-mass" tensor multiplet and hypermultiplets belonging to the bi-fundamental representation of $SU(k)_a \times SU(k)_b$. The index with the tensor multiplet contribution removed is given by

$$\mathcal{I}^{(1,0)}_{N=1} = \text{Pexp}\left[i_{\text{hyper}}(\chi^a_{\text{fund.}}\chi^b_{\overline{\text{fund.}}}u + \chi^a_{\overline{\text{fund.}}}\chi^b_{\text{fund.}}u^{-1})\right], \tag{3.35}$$

where i_{hyper} is given by [25]

$$i_{\text{hyper}} = \frac{q^2}{(1-q^{\frac{4}{3}}y_1)(1-q^{\frac{4}{3}}y_2)(1-q^{\frac{4}{3}}y_3)}. \tag{3.36}$$

Expanding (3.35) with $k = 2$ we obtain

$$\mathcal{I}^{(1,0)}_{N=1,k=2} = 1 + \chi^a_{[1]}\chi^b_{[1]}\chi^u_{[1]}q^2 + \chi^a_{[1]}\chi^b_{[1]}\chi^u_{[1]}\chi^y_{[1,0]}q^{\frac{10}{3}} + (\chi^a_{[2]} + \chi^b_{[2]} + \chi^u_{[2]} + \chi^a_{[2]}\chi^b_{[2]}\chi^u_{[2]})q^4 + \mathcal{O}(q^{\frac{14}{3}}), \tag{3.37}$$

where $\chi^u_{[n]}$ are the $SU(2)$ characters defined by

$$\chi^u_{[n]} = \frac{u^{n+1} - u^{-(n+1)}}{u - u^{-1}}, \tag{3.38}$$

and $\chi^y_{[m_1,m_2]}$ are the $SU(3)$ characters of the representations with Dynkin labels $[m_1, m_2]$. On the otehr hand, for the $k = 3$ case we obtain

$$\begin{aligned}\mathcal{I}^{(1,0)}_{N=1,k=3} &= 1 + \left(u\chi^a_{[1,0]}\chi^b_{[0,1]} + u^{-1}\chi^a_{[0,1]}\chi^b_{[1,0]}\right)q^2 + \left(u\chi^a_{[1,0]}\chi^b_{[0,1]}\chi^y_{[1,0]} + u^{-1}\chi^a_{[0,1]}\chi^b_{[1,0]}\chi^y_{[1,0]}\right)q^{\frac{10}{3}} \\ &\quad + \left(1 + \chi^a_{[1,1]} + \chi^b_{[1,1]} + u^2\chi^a_{[0,1]}\chi^b_{[1,0]} + u^{-2}\chi^a_{[1,0]}\chi^b_{[0,1]}\right. \\ &\quad\quad \left. + u^2\chi^a_{[2,0]}\chi^b_{[0,2]} + u^{-2}\chi^a_{[0,2]}\chi^b_{[2,0]} + \chi^a_{[1,1]}\chi^b_{[1,1]}\right)q^4 + \mathcal{O}(q^{\frac{14}{3}}).\end{aligned} \tag{3.39}$$

3.7 Large N index from supergravity on AdS$_7 \times S^4/\mathbb{Z}_k$

Here, we calculate the large-N indices of the $\mathcal{N} = (1, 0)$ theories from the dual gravity theories: The eleven-dimensional supergravity on AdS$_7 \times S^4/\mathbb{Z}_k$. We denote the bulk contribution as $\mathcal{I}^{\text{bulk}}$. This is given by the plethystic exponential of the single-particle index, which is the sum of two contributions: the supergravity Kaluza-Klein modes in the internal space $\boldsymbol{S}^4/\mathbb{Z}_k$ and the vector multiplets localized at the two fixed points of $\boldsymbol{S}^4/\mathbb{Z}_k$.

The contribution of the Kaluza-Klein modes in $AdS_7 \times \boldsymbol{S}^4$ without orbifolding has already been studied in [25] and is given by Pexp i_{KK} with the single-particle index

$$i_{\text{KK}} = \frac{q^2\chi^u_{[1]} - q^{\frac{8}{3}}\chi^y_{[0,1]} + q^{\frac{16}{3}}\chi^y_{[1,0]} - q^6\chi^u_{[1]}}{(1-uq^2)(1-u^{-1}q^2)(1-y_1q^{\frac{4}{3}})(1-y_2q^{\frac{4}{3}})(1-y_3q^{\frac{4}{3}})}. \tag{3.40}$$

The Kaluza-Klein modes in the orbifold $S^4/\mathbb{Z}_k$ is obtained by picking up the $\mathbb{Z}_k$ invariant modes from the modes in S^4 [48]. Correspondingly, the single-particle index for the orbifold is given by $\mathcal{P}_k i_{\mathrm{KK}}$, where $\mathcal{P}_k$ is the projection operator associated with the $\mathbb{Z}_k$ orbifold which acts on a function of the fugacity u as

$$\mathcal{P}_k f(u) = \frac{1}{k}\sum_{l=0}^{k-1} f(e^{2\pi i l/k}u). \tag{3.41}$$

The other contribution we need to include in the single-particle index comes from two A_{k-1} singularities on $\mathbb{C}^2/\mathbb{Z}_k$ at $(z_1, z_2, x_5) = (0, 0, \pm 1)$, where the 7d $SU(k)_a \times SU(k)_b$ vector multiplets are localized. In general, a gauge field in the bulk of AdS corresponds to a flavor symmetry on the boundary, and the corresponding current multiplet contributes to the index. The corresponding single-particle index is $i_F(\chi^a_{\mathrm{adj.}}+\chi^b_{\mathrm{adj.}})$, where $\chi^{a/b}_{\mathrm{adj.}}$ are characters of adjoint representations of the global $SU(k)_{a/b}$ symmetries and i_F is given by

$$i_F = \frac{q^4}{(1-q^{\frac{4}{3}}y_1)(1-q^{\frac{4}{3}}y_2)(1-q^{\frac{4}{3}}y_3)}. \tag{3.42}$$

Note that i_F is independent of u and we do not have to perform the $\mathbb{Z}_k$ projection.

By combining two contributions, we can calculate the index for the large-N limit. For example, for $k = 2$ we obtain

$$\begin{aligned}\mathrm{Pexp}(\mathcal{P}_2 i_{\mathrm{KK}} + i_F(\chi^a_{[2]} + \chi^b_{[2]})) &= 1 - \chi^y_{[0,1]}q^{\frac{8}{3}} + (\chi^u_{[2]} + \chi^a_{[2]} + \chi^b_{[2]} - \chi^y_{[1,1]})q^4 \\ &\quad + ((2 + \chi^u_{[2]} + \chi^a_{[2]} + \chi^b_{[2]})\chi^y_{[1,0]} - \chi^y_{[2,1]})q^{\frac{16}{3}} + \mathcal{O}(q^{\frac{20}{3}}).\end{aligned} \tag{3.43}$$

This includes the contribution of the "center of mass" free tensor multiplet. The existence of such a decoupled free sector is suggested by the coefficient "2" of the term $\chi^y_{[1,0]}q^{\frac{16}{3}}$ in the above expansion, which is identified as the contribution of two copies of stress-tensor multiplets. Such a free tensor multiplet exists for all k and N, and we always remove its contribution in the following calculation. Namely, we define $\mathcal{I}^{\mathrm{bulk}}$ in (5.59) by

$$\mathcal{I}^{\mathrm{bulk}} = \mathrm{Pexp}(\mathcal{P}_k i_{KK} + i_F(\chi^a_{\mathrm{adj}} + \chi^b_{\mathrm{adj}}) - i_{\mathrm{tensor}}), \tag{3.44}$$

where the single-particle index i_{tensor} of the free tensor multiplet is given by [25]

$$i_{\mathrm{tensor}} = \frac{-q^{\frac{8}{3}}\chi^y_{[0,1]} + q^4}{(1-q^{\frac{4}{3}}y_1)(1-q^{\frac{4}{3}}y_2)(1-q^{\frac{4}{3}}y_3)}. \tag{3.45}$$

For $k = 2, 3$, the equation (3.44) gives

$$\begin{aligned}\mathcal{I}^{\mathrm{bulk}}_{k=2} &= \mathrm{Pexp}(\mathcal{P}_2 i_{KK} + i_F(\chi^a_{[2]} + \chi^b_{[2]}) - i_{\mathrm{tensor}}) \\ &= 1 + (\chi^u_{[2]} + \chi^a_{[2]} + \chi^b_{[2]})q^4 + (1 + \chi^u_{[2]} + \chi^a_{[2]} + \chi^b_{[2]})\chi^y_{[1,0]}q^{\frac{16}{3}} + \mathcal{O}(q^{\frac{20}{3}}),\end{aligned} \tag{3.46}$$

$$\begin{aligned}\mathcal{I}^{\mathrm{bulk}}_{k=3} &= \mathrm{Pexp}(\mathcal{P}_3 i_{KK} + i_F(\chi^a_{[1,1]} + \chi^b_{[1,1]}) - i_{\mathrm{tensor}}) \\ &= 1 + (1 + \chi^a_{[1,1]} + \chi^b_{[1,1]})q^4 + (2 + \chi^a_{[1,1]} + \chi^b_{[1,1]})\chi^y_{[1,0]}q^{\frac{16}{3}} + (-\chi^u_{[1]} + \chi^u_{[3]})q^6 + \mathcal{O}(q^{\frac{20}{3}}).\end{aligned} \tag{3.47}$$

These are interpreted as the indices in the large-N limit. The q^4 terms in each index are the contribution of the flavor current multiplets, and we can read off the expected flavor symmetries $SU(2)^3$ for $k = 2$ and $SU(3)^2 \times U(1)$ for $k = 3$. We also confirm that all other terms are consistent with these flavor symmetries.

Summary of Chapter 3

In this chapter, we explained the superconformal index of the M5-brane theories investigated so far. We first study the index of the $\mathcal{N} = (2,0)$ theory. The definition of the superconformal index for the $\mathcal{N} = (2,0)$ theory is given by (3.8). For the single M5-brane ($N = 1$) case, we calculated the index from the free theory of the tensor multiplet. On the other hand, when N is large, we use the dual supergravity description to calculate the index.

We also discussed the superconformal index of the 6d $\mathcal{N} = (1,0)$ theory. The definition of the index for the (1,0) theory is given by (3.34). Similarly to the (2,0) case, we calculated the index for the $N = 1$ case and the large N case. In addition, we discussed the expected flavor symmetries of the 6d (1,0) theory, which are summarized in Table 3.3.

Chapter 4

Finite N corrections to the indices of the M2-brane theories

In this chapter, we discuss a new method of calculating the superconformal indices of M2-brane theories from the dual gravity side in the finite-N region. We will see that finite-N corrections to the indices of ABJM theories are given by M5-branes wrapped on $S^5 \subset S^7$. This chapter is based on the author's and his collaborator's work [12].

4.1 Finite N corrections to the indices of the ABJM theories with $k = 1$

4.1.1 Difference appearing at finite-N

We consider the ABJM theory with Chern-Simons level $k = 1$ and its gravity dual. Let us first see the difference between the finite-N indices of ABJM theories and the Kaluza Klein index. Here, we set $\hat{u}_a$ fugacities to be 1 for simplicity and the results (2.53) read

$$\mathcal{I}_{\mathrm{KK}}|_{\hat{u}=1} = 1 + 4\hat{q}^{\frac{1}{2}} + 20\hat{q} + 76\hat{q}^{\frac{3}{2}} + 274\hat{q}^2 + 900\hat{q}^{\frac{5}{2}} + 2826\hat{q}^3 + 8400\hat{q}^{\frac{7}{2}} + 24079\hat{q}^4 + 66540\hat{q}^{\frac{9}{2}} + 178578\hat{q}^5 + 466248\hat{q}^{\frac{11}{2}} + 1188829\hat{q}^6 + \mathcal{O}(\hat{q}^{\frac{13}{2}}). \tag{4.1}$$

On the other hand, the indices of ABJM theories (2.34) $\sim$ (2.36) are

$$\mathcal{I}^{\mathrm{ABJM}}_{N=1}|_{\hat{u}=1} = 1 + 4\hat{q}^{\frac{1}{2}} + 10\hat{q} + 16\hat{q}^{\frac{3}{2}} + 19\hat{q}^2 + 20\hat{q}^{\frac{5}{2}} + 26\hat{q}^3 + 40\hat{q}^{\frac{7}{2}} + 49\hat{q}^4 + \mathcal{O}(\hat{q}^{\frac{9}{2}}), \tag{4.2}$$

$$\mathcal{I}^{\mathrm{ABJM}}_{N=2}|_{\hat{u}=1} = 1 + 4\hat{q}^{\frac{1}{2}} + 20\hat{q} + 56\hat{q}^{\frac{3}{2}} + 139\hat{q}^2 + 260\hat{q}^{\frac{5}{2}} + 436\hat{q}^3 + 640\hat{q}^{\frac{7}{2}} + 954\hat{q}^4 + 1420\hat{q}^{\frac{9}{2}} + 2076\hat{q}^5 + \mathcal{O}(\hat{q}^{\frac{11}{2}}), \tag{4.3}$$

$$\mathcal{I}^{\mathrm{ABJM}}_{N=3}|_{\hat{u}=1} = 1 + 4\hat{q}^{\frac{1}{2}} + 20\hat{q} + 76\hat{q}^{\frac{3}{2}} + 239\hat{q}^2 + 644\hat{q}^{\frac{5}{2}} + 1512\hat{q}^3 + 3100\hat{q}^{\frac{7}{2}} + 5743\hat{q}^4 + 9856\hat{q}^{\frac{9}{2}} + 16182\hat{q}^5 + 25988\hat{q}^{\frac{11}{2}} + 40764\hat{q}^6 + \mathcal{O}(\hat{q}^{\frac{13}{2}}). \tag{4.4}$$

We can easily find the following finite-N "corrections".

$$\mathcal{I}^{\rm ABJM}_{N=1}|_{\hat{u}=1} - \mathcal{I}_{\rm KK}|_{\hat{u}=1} = -10\hat{q} + \cdots , \tag{4.5}$$

$$\mathcal{I}^{\rm ABJM}_{N=2}|_{\hat{u}=1} - \mathcal{I}_{\rm KK}|_{\hat{u}=1} = -20\hat{q}^{\frac{3}{2}} + \cdots , \tag{4.6}$$

$$\mathcal{I}^{\rm ABJM}_{N=3}|_{\hat{u}=1} - \mathcal{I}_{\rm KK}|_{\hat{u}=1} = -35\hat{q}^{2} + \cdots . \tag{4.7}$$

We can see that the differences appear at the order $\hat{q}^{\frac{1}{2}(N+1)}$. If we restore the $\hat{u}$ fugacities, we obtain

$$\mathcal{I}^{\rm ABJM}_{N} - \mathcal{I}_{\rm KK} = -\chi_{[N+1,0,0]}(\hat{u})\hat{q}^{\frac{1}{2}(N+1)} + \cdots . \tag{4.8}$$

Now our question is, what is the origin of these differences appearing at finite-N, or equivalently, how we can reproduce the index of the field theory side at finite-N from the gravity side. As we will discuss in the next subsection, the answer is contributions of M5-branes wrapped on five-cycles in S^7.

4.1.2 Wrapped M5-brane

The parameter relations in (2.45) imply that

$$\frac{\hat{r}^6}{l_p^6} \sim N. \tag{4.9}$$

This suggests that if N is large, we can neglect effects of quantum gravity and classical supergravity description is valid. However, if N is finite (small) the supergravity description is no longer valid and quantum gravity effect becomes important, since the typical length of the solution $\hat{r}$ is in the same order as Planck length l_p at finite-N.

However, we may be able to overcome this difficult problem by using the robust nature of supersymmetry. Namely, we claim that there is still a possibility to avoid such quantum gravity effects by using a supersymmetry protected quantity. We think the superconformal index is one candidate for such a quantity. If this assumption is true we can calculate the index on the gravity side without considering quantum gravity corrections.

Further, at finite-N we need to consider another contribution to the index in addition to the KK modes. To see this, we rewrite the parameter relation (2.45) by using the tension of the M5-brane (1.22). The relation becomes

$$N = \pi^3 T_{\rm M5}\hat{r}^6, \tag{4.10}$$

and implies that M5-brane contribution is necessary at finite-N.

Actually, in the context of AdS/CFT, conformal dimension δ of an operator in the CFT corresponds to the mass m of corresponding object in the AdS space multiplied by AdS radius $L_{\rm AdS}$:

$$\delta = mL_{\rm AdS}. \tag{4.11}$$

The mass of the wrapped M5-brane on a great circle of S^7 (multiplied by the AdS_4 radius) is given by

$$T_{\mathrm{M5}}\hat{r}^5 V_5 \hat{L} = \frac{N}{2}, \tag{4.12}$$

where $V_5 = \pi^3$ is the volume of a unit five-sphere. The wrapped M5-brane contribution seems to reproduce the corrections discussed in the previous subsection. Note that the difference in (4.8) appears at the order $\mathcal{O}(\hat{q}^{\frac{1}{2}(N+1)})$, not at the order $\mathcal{O}(\hat{q}^{\frac{N}{2}})$. This is due to the tachyonic shift explained later. See discussion around the equation (4.23).

In the large-N limit, such heavy branes do not contribute to the index, but at finite-N, as we saw, contributions of these branes become effective. Hence, the relation in the large-N limit (2.51) should be modified at finite-N. We propose the following hypothesis formula of the index valid at finite-N. The change is simply the inclusion of contributions of wrapped M5-branes:

$$\mathcal{I}_N^{\mathrm{ABJM}} = \mathcal{I}_{\mathrm{KK}}\left(1 + \sum_C \mathcal{I}_C^{\mathrm{M5}}\right). \tag{4.13}$$

$\mathcal{I}_C^{\mathrm{M5}}$ is the contribution of M5-branes in a certain configuration C and consists of two factors; $\mathcal{I}_C^{\mathrm{M5}} = \mathcal{I}_C^{\mathrm{ground}}\mathcal{I}_C^{\mathrm{excitations}}$. The factor $\mathcal{I}_C^{\mathrm{ground}}$ gives the classical contribution from wrapped M5-branes without fluctuations. The other factor $\mathcal{I}_C^{\mathrm{excitations}}$ is the index of the theory realized on the configuration C. The sum runs over "the representative configurations" of M5-branes. In the next subsection, we will discuss this wrapped M5-brane contribution in detail.

The objects contributing to the indices at finite-N are schematically shown in Figure 4.1.

Figure 4.1: The objects contributing to the indices at finite-N are shown. The left figure shows the Kaluza Klein particles in S^7, which gives the large-N indices. The right figure shows the M5-brane wrapped on a great circle in S^7, which gives the finite-N corrections to the indices.

4.1.3 Detail of M5-branes contribution

We determine the representative configurations C by a preliminary analysis of a rigid M5-brane, an M5-brane wrapped on a large $\boldsymbol{S}^5$ in $\boldsymbol{S}^7$. Let us introduce complex coordinates z_a

($a = 1, 2, 3, 4$) to describe the $\boldsymbol{S}^7$ by $\sum_{a=1}^{4} |z_a|^2 = 1$. The R-symmetry $su(4) \subset \hat{\mathcal{B}}$ acts on these coordinates in the natural way. For a rigid M5-brane to be BPS with respect to the chosen supercharge $\hat{\mathcal{Q}}$ the worldvolume must be given by the holomorphic equation [49]

$$c_1 z_1 + c_2 z_2 + c_3 z_3 + c_4 z_4 = 0, \tag{4.14}$$

where c_a are homogeneous coordinates in $\mathbb{P}^3$. The collective motion of the M5-brane can be treated as a particle in the moduli space $\mathbb{P}^3$. By the analysis of the coupling of the brane and the background flux we find the wave function Ψ of a rigid M5-brane is a section of the line bundle $\mathcal{O}(N)$ over $\mathbb{P}^3$. We can give Ψ as a homogeneous function of the coordinates c_a of degree N. States described by such wave functions belong to the $su(4)$ representation with Dynkin labels $[N, 0, 0]$. On the gauge theory side these states are identified with baryonic type operators in the ABJM theory[5]. The corresponding index is $\hat{q}^{\frac{1}{2}N}\chi_{[N,0,0]}(\hat{u}_a)$, where $\chi_{[a,b,c]}(\hat{u}_a)$ is the $su(4)$ character of the representation $[a, b, c]$.

Now let us remember the Weyl's character formula. It gives $\hat{q}^{\frac{1}{2}N}\chi_{[N,0,0]}(\hat{u}_a)$ as the sum:

$$\begin{aligned}\hat{q}^{\frac{1}{2}N}\chi_{[N,0,0]}(\hat{u}_a) &= \frac{\hat{q}^{\frac{1}{2}N}\hat{u}_4^N}{(1-\frac{\hat{u}_1}{\hat{u}_4})(1-\frac{\hat{u}_2}{\hat{u}_4})(1-\frac{\hat{u}_3}{\hat{u}_4})} + (\text{permutations}) \\ &= \hat{q}^{\frac{1}{2}N}\hat{u}_4^N \,\mathrm{Pexp}\left(\frac{\hat{u}_1}{\hat{u}_4} + \frac{\hat{u}_2}{\hat{u}_4} + \frac{\hat{u}_3}{\hat{u}_4}\right) + (\text{permutations}),\end{aligned} \tag{4.15}$$

where "permutations" represents three terms obtained from the first term by cyclic permutations of $\hat{u}_a$. From the quantum mechanical point of view, the first term can be interpreted as the partition function of the system with the ground state $\hat{q}^{\frac{1}{2}N}\hat{u}_4^N$ and three bosonic excitations $\hat{u}_1/\hat{u}_4$, $\hat{u}_2/\hat{u}_4$, and $\hat{u}_3/\hat{u}_4$. We define the representative configuration as the M5-brane corresponding to the ground state. For the first term in (4.15) it is given by $z_4 = 0$. Corresponding to the other terms obtained by the permutations there are three more representative configurations $z_a = 0$ ($a = 1, 2, 3$).

The main idea in [6] is that we can obtain the finite-N corrections to the index by ornamenting the Weyl's formula (4.15) with all other fluctuation modes by replacing the zero-mode contribution $\hat{u}_1/\hat{u}_4 + \hat{u}_2/\hat{u}_4 + \hat{u}_3/\hat{u}_4$ by the complete single-particle index of the theory on the worldvolume of the M5-brane. In addition, to obtain the complete corrections, we need to take account of representative configurations including more than one branes [50]. Namely, the general form of C is given by

$$C : z_1^{n_1} z_2^{n_2} z_3^{n_3} z_4^{n_4} = 0, \quad n_a \in \mathbb{Z}_{\geq 0}, \quad (n_1, n_2, n_3, n_4) \neq (0, 0, 0, 0), \tag{4.16}$$

where a multiple zero is understood as coincident branes. $(n_1, n_2, n_3, n_4) = (0, 0, 0, 0)$ is excluded because it corresponds to the first term in the parentheses in (4.13). The contribution of each configuration C is factorized into two factors $\mathcal{I}_C^{\text{ground}}$ and $\mathcal{I}_C^{\text{excitations}}$. Each wrapped brane contributes $N/2$ to the energy (in the unit of $\hat{L}^{-1}$) and the ground state of C includes the factor $\hat{q}^{\frac{1}{2}nN}$ with $n = n_1 + n_2 + n_3 + n_4$. $\mathcal{I}_C^{\text{ground}}$ is given as the product of the ground state contribution of each brane:

$$\mathcal{I}_C^{\text{ground}} = \hat{q}^{\frac{1}{2}nN}\hat{u}_1^{n_1 N}\hat{u}_2^{n_2 N}\hat{u}_3^{n_3 N}\hat{u}_4^{n_4 N}. \tag{4.17}$$

$\mathcal{I}_C^{\text{excitations}}$ is the contribution of excitations on C. If $n \geq 2$ the theory on C is interacting and it is not so easy to calculate $\mathcal{I}_C^{\text{excitations}}$. In this work we only consider four configurations with $n = 1$ given by $z_a = 0$ ($a = 1, 2, 3, 4$). Then, the theory on C is free and $\mathcal{I}_C^{\text{excitations}}$ is given by

$$\mathcal{I}_{z_a=0}^{\text{excitations}} = \text{Pexp}\, i_{z_a=0}^{\text{M5}}, \tag{4.18}$$

where $i_{z_a=0}^{\text{M5}}$ is the single-particle index of the fluctuation modes on the worldvolume of an M5-brane wrapped on $z_a = 0$.

The single-particle index from the DBI action

Let us calculate the single-particle index $i_{z_a=0}^{\text{M5}}$ for each representative configuration. In the following we consider the configuration $z_4 = 0$. The other three are obtained by the permutations of the fugacities $\hat{u}_a$. We start with the analysis of the scalar modes. If we neglect the self-dual potential field and fermion fields on the worldvolume the M5-brane action is given as the sum of the Nambu-Goto action S_{NG} and the Chern-Simons term S_{CS}:

$$S_{\text{NG}} = -T_{\text{M5}} \int d^6\sigma \sqrt{-\det G_{ab}}, \quad S_{\text{CS}} = \int A_6, \tag{4.19}$$

where G_{ab} is the induced metric and A_6 is the background 6-form potential satisfying $dA_6 = (2\pi N/V_7)\text{vol}(\boldsymbol{S}^7)$. We use the following $\text{AdS}_4 \times \boldsymbol{S}^7$ metric:

$$ds^2 = \hat{L}^2(-\cosh^2\rho d\hat{t}^2 + d\rho^2 + \sinh^2\rho d\Omega_2^2) + \hat{r}^2(\cos^2\theta d\Omega_5^2 + d\theta^2 + \sin^2\theta d\phi^2). \tag{4.20}$$

We consider an M5-brane wrapped on $\mathbb{R} \times \boldsymbol{S}^5$ defined by $\rho = \theta = 0$. There are 5 scalar fields corresponding to transverse directions of the M5-brane: three in AdS_4 and two in $\boldsymbol{S}^7$. To describe fluctuations in AdS_4 we introduce a three-dimensional unit vector $\boldsymbol{n}$ and rewrite $d\Omega_2^2$ as $d\boldsymbol{n}^2$. We define fluctuation fields by

$$\boldsymbol{X} = \rho\boldsymbol{n}, \quad z = \theta e^{i\phi}. \tag{4.21}$$

By neglecting higher order terms and using the relations (2.50) and (4.10), we obtain

$$\begin{aligned} S_{\text{NG}} + S_{\text{CS}} =& \frac{N}{2\pi^3} \int d\hat{t} d\Omega_5 \Big[-1 + \frac{1}{2}(\partial_{\hat{t}}\boldsymbol{X})^2 - \frac{1}{8}(\nabla\boldsymbol{X})^2 - \frac{1}{2}\boldsymbol{X}^2 \\ & + 2|\partial_{\hat{t}} z|^2 - \frac{1}{2}|\nabla z|^2 + \frac{5}{2}|z|^2 + 3i(-z^*\partial_{\hat{t}} z + z\partial_{\hat{t}} z^*) \Big], \end{aligned} \tag{4.22}$$

where ∇ is the derivative on the unit $\boldsymbol{S}^5$. The constant term gives the energy $E = \frac{1}{2}N$ of the wrapped M5-brane. By solving the equations of motion we can easily obtain the spectrum of fluctuation modes. (See Table 4.1.3.) We have six zero-modes of z^* at $\ell = 1$ and three of them are BPS. They correspond to three excitations $\hat{u}_1/\hat{u}_4$, $\hat{u}_2/\hat{u}_4$, and $\hat{u}_3/\hat{u}_4$ appearing in

fields	$\hat{J}_{12}$	$so(6)$	$\hat{R}_{78}$	$\hat{H}$
$\boldsymbol{X}$	$0, \pm 1$	$[0, \ell, 0]$	0	$(\ell+2)/2$
z	0	$[0, \ell, 0]$	$+1$	$(\ell+5)/2$
z^*	0	$[0, \ell, 0]$	-1	$(\ell-1)/2$

Table 4.1: Scalar fluctuation modes on an M5-brane wrapped on $z_4 = 0$. $\ell = 0, 1, 2, \ldots$ is the angular momentum on $\boldsymbol{S}^5$.

the Weyl's formula (4.15). We also have one BPS mode of z^* at $\ell = 0$ which corresponds to a term $\hat{q}^{-\frac{1}{2}} \hat{u}_4^{-1}$. We call such modes with negative energy tachyonic modes.

A few comments on the tachyonic modes are in order. First, When the single-particle index includes a term proportional to the negative power of $\hat{q}$, for example $\hat{q}^{-\frac{1}{2}}$ (We set other fugacities to be 1.), the plethystic exponential of this term is given by

$$\mathrm{Pexp}(q^{-\frac{1}{2}}) = \frac{1}{1 - q^{-\frac{1}{2}}} = -\frac{q^{\frac{1}{2}}}{1 - q^{\frac{1}{2}}}. \tag{4.23}$$

We can see that this factor increases the order of the index by $\hat{q}^{\frac{1}{2}}$ and changes the overall sign of the correction. Now we find, together with the ground state contribution, the correction starts at the order $\mathcal{O}(\hat{q}^{\frac{1}{2}(N+1)})$ with the negative coefficient. This precisely agrees with the analysis in Subsection 4.1.1.

We also note that the existence of the tachyonic mode does not cause the instability of the system. The tachyonic mode carries the R-charge $\hat{R}_{78} = -1$, and a tachyonic particle is always created together with an anti-particle with $\hat{R}_{78} = +1$. As is shown in Table 4.1.3 such an anti-particle, which corresponds to the $\ell = 0$ mode of z, carries the energy $E = 5/2$, and the pair creation raises the total energy of the system. Another comment is about the consistency with the BPS bound. Ordinarily, a particle with negative energy is against the BPS bound $E \geq 0$. In the theory on the wrapped brane, however, we do not have such a bound. An M5-brane wrapped on $z_4 = 0$ breaks the half supersymmetries. Among 32 supercharges only 16 that commute with the generator

$$\hat{Z} = \hat{H} - \hat{R}_{78} \tag{4.24}$$

are preserved. The algebra of the preserved symmetry is

$$\hat{\mathcal{C}} \times u(1)_{\hat{Z}}, \quad \hat{\mathcal{C}} = su(2|4). \tag{4.25}$$

The central factor $u(1)_{\hat{Z}}$ is generated by $\hat{Z}$. The bosonic subalgebra of $\hat{\mathcal{C}}$ is $so(3) \times so(6) \times u(1)$ generated by

$$\hat{J}_{ij} \quad (i, j = 1, 2, 3), \quad \hat{R}_{ab} \quad (a, b = 1, \ldots, 6), \quad \hat{C} \equiv \hat{H} - \frac{1}{2}\hat{R}_{78}. \tag{4.26}$$

The fluctuation modes on the M5-brane form a representation of the unbroken algebra $\hat{\mathcal{C}}$. The Hamiltonian $\hat{H}$ appears in $\hat{\mathcal{C}}$ only through $\hat{C}$, and the bound obtained from the algebra is not $\hat{H} \geq 0$ but $\hat{C} \geq 0$. The tachyonic mode saturates this bound.

The single-particle index via the variable change

In principle, we can calculate the complete single-particle index $i^{\rm M5}_{z_a=0}$ by carrying out the mode expansion of the tensor and the fermion fields. However, there is an easy way to obtain the index from the known 6d superconformal index of the tensor multiplet.

We are interested in the theory of a tensor multiplet living on $\mathbb{R}\times \boldsymbol{S}^5$, the worldvolume of a wrapped M5-brane. This system is similar to the system of a tensor multiplet living on the boundary of AdS_7. In Section secsinglem5, we investigated the six-dimensional system living on the AdS boundary $\mathbb{R}\times \boldsymbol{S}^5$, on which the $(2,0)$ superconformal algebra $\check{\mathcal{A}}$ acts. The two free theories, the theory on a wrapped M5-brane in $AdS_4\times \boldsymbol{S}^7$ and the theory on the boundary of AdS_7, are in fact the same theory, at least at the linearized level, and we can obtain the index of the former from the index of the latter by a simple variable change of fugacities.

We first establish the relation between the symmetry algebras. Namely, we need to find an isomorphism between the unbroken algebra on the wrapped M5-brane (4.25) and a subalgebra of $\check{\mathcal{A}}$. There is an ambiguity of the choice of the subalgebra of $\check{\mathcal{A}}$. A convenient one is the symmetry (5.12) realized on a wrapped M2-brane studied in the next chapter. It is isomorphic to (4.25);

$$\hat{\mathcal{C}}\times u(1)_{\hat{Z}}\simeq \check{\mathcal{C}}\times u(1)_{\check{Z}}. \tag{4.27}$$

The explicit relations between the two sets of the bosonic generators are as follows.

$$\begin{aligned}&\check{J}_{ij}=\hat{R}_{ij}\quad (i,j=1,\ldots,6),\quad \check{R}_{a+2,b+2}=\hat{J}_{ab}\quad (a,b=1,2,3),\\ &\check{Z}=2\hat{Z},\quad \check{C}=2\hat{C}.\end{aligned} \tag{4.28}$$

We can relate the two systems not only at the level of the symmetry but also at the level of the Lagrangians. The boundary metric of AdS_7 is

$$ds^2\propto -d\check{t}^2+d\Omega_5^2. \tag{4.29}$$

For distinction from $\hat{t}$ used in (4.22) we use $\check{t}$ for the time coordinate. The Lagrangian of the five scalar fields ϕ_I $(I=1,\ldots,5)$ living on this background is

$$\mathcal{L}\propto\sum_{I=1}^{5}\left[(\partial_{\check{t}}\phi_I)^2-(\nabla\phi_I)^2-4\phi_I^2\right], \tag{4.30}$$

where the last term is the conformal coupling to the background curvature. We simply relate the triplet fields by $\boldsymbol{X}\propto(\phi_3,\phi_4,\phi_5)$, while in the relation between z and $\phi_{1,2}$ we need to apply the time-dependent phase rotation

$$z\propto e^{-3i\check{t}}(\phi_1+i\phi_2), \tag{4.31}$$

corresponding to the relation of two Hamiltonians $2\hat{H}=\check{H}-3\check{R}_{12}$ obtained from the last two equations in (4.28). In addition, we rescale the time coordinate by $\hat{t}=2\check{t}$ to match the background metric (4.29) and the metric on the wrapped M5-brane

$$ds^2 = \hat{r}^2 \left(-\frac{1}{4} d\hat{t}^2 + d\Omega_5^2 \right) \tag{4.32}$$

obtained from (4.20) by the restriction $\rho = \theta = 0$. Then, we obtain the Lagrangian in (4.22) from (4.30).

We can extend the relations (4.28) to fermionic generators. An important fact is that the supercharges used to define the superconformal indices on two sides are related by

$$\check{\mathcal{Q}} = \sqrt{2} \hat{\mathcal{Q}}^\dagger, \tag{4.33}$$

and the relation $\check{\Delta} = 2\hat{\Delta}$ immediately follows from this. This implies that the superconformal indices defined on two sides are essentially the same. Indeed, we can rewrite the six-dimensional index (3.8) to the three-dimensional index (2.7) by using the map (4.28) and the variable change

$$\check{q} = \hat{q}^{\frac{3}{8}} \hat{u}_4^{-\frac{1}{4}}, \quad \check{y}_1 = \hat{u}_1 \hat{u}_4^{\frac{1}{3}}, \quad \check{y}_2 = \hat{u}_2 \hat{u}_4^{\frac{1}{3}}, \quad \check{y}_3 = \hat{u}_3 \hat{u}_4^{\frac{1}{3}}, \quad \check{u} = \hat{q}^{-\frac{5}{4}} \hat{u}_4^{-\frac{1}{2}}. \tag{4.34}$$

Applying the variable change (4.34) to the index i^{M5} in (3.12) of the free tensor multiplet we obtain the following single-particle index for the excitations on an M5-brane wrapped on $z_4 = 0$:

$$\begin{aligned} i^{\mathrm{M5}}_{z_4=0} &= \frac{\hat{q}^{-\frac{1}{2}} \hat{u}_4^{-1} - \hat{q} \hat{u}_4^{-1} (\hat{u}_1^{-1} + \hat{u}_2^{-1} + \hat{u}_3^{-1}) + \hat{q}^{\frac{3}{2}} \hat{u}_4^{-1} + \hat{q}^2}{(1 - \hat{q}^{\frac{1}{2}} \hat{u}_1)(1 - \hat{q}^{\frac{1}{2}} \hat{u}_2)(1 - \hat{q}^{\frac{1}{2}} \hat{u}_3)} \\ &= \frac{1}{\hat{q}^{\frac{1}{2}} \hat{u}_4} + \frac{\hat{u}_1 + \hat{u}_2 + \hat{u}_3}{\hat{u}_4} + \cdots . \end{aligned} \tag{4.35}$$

The first few terms in the expansion correspond to the tachyonic modes and rigid motion modes obtained in the analysis of scalar fluctuations.

4.1.4 Comparison

Let us calculate the indices by using our formula and compare the results with the ABJM indices. In the last subsection, we obtained the following hypothetical formula

$$\mathcal{I}_N^{\mathrm{ABJM}} = \mathcal{I}_N^{\mathrm{grav}} + \mathcal{O}(\hat{q}^{\frac{1}{2}(2N+\delta)}), \tag{4.36}$$

where the first term in the right-hand side is defined by

$$\mathcal{I}_N^{\mathrm{grav}} := \mathcal{I}_{\mathrm{KK}} \left(1 + \sum_{a=1}^{4} \hat{q}^{\frac{1}{2}N} \hat{u}_a^N \, \mathrm{Pexp}\, i^{\mathrm{M5}}_{z_a=0} \right), \tag{4.37}$$

and the second term $\mathcal{O}(\hat{q}^{\frac{1}{2}(2N+\delta)})$ is the expected error due to the neglect of the multiple-wrapping configurations with the tachyonic shift δ. Based on the experience in the D3-brane case we expect δ is independent of N, and this is directly confirmed below for small N.

Let us see the result including the single-wrapping M5-branes by using our formula. We again set $\hat{u}_i = 1$ for readability. See Appendix A for the full expressions of the indices. For the $N = 1, 2, 3$ case, the formula (4.37) gives

$$\mathcal{I}^{\text{grav}}_{N=1}|_{\hat{u}=1} = 1 + 4\hat{q}^{\frac{1}{2}} + 10\hat{q} + 16\hat{q}^{\frac{3}{2}} + 19\hat{q}^2 + 20\hat{q}^{\frac{5}{2}} + 26\hat{q}^3 + 40\hat{q}^{\frac{7}{2}} + 5769\hat{q}^4 + \mathcal{O}(\hat{q}^{\frac{9}{2}}), \tag{4.38}$$

$$\mathcal{I}^{\text{grav}}_{N=2}|_{\hat{u}=1} = 1 + 4\hat{q}^{\frac{1}{2}} + 20\hat{q} + 56\hat{q}^{\frac{3}{2}} + 139\hat{q}^2 + 260\hat{q}^{\frac{5}{2}} + 436\hat{q}^3 + 640\hat{q}^{\frac{7}{2}} + 954\hat{q}^4 + 1420\hat{q}^{\frac{9}{2}} + 15518\hat{q}^5 + \mathcal{O}(\hat{q}^{\frac{11}{2}}), \tag{4.39}$$

$$\mathcal{I}^{\text{grav}}_{N=3}|_{\hat{u}=1} = 1 + 4\hat{q}^{\frac{1}{2}} + 20\hat{q} + 76\hat{q}^{\frac{3}{2}} + 239\hat{q}^2 + 644\hat{q}^{\frac{5}{2}} + 1512\hat{q}^3 + 3100\hat{q}^{\frac{7}{2}} + 5743\hat{q}^4 + 9856\hat{q}^{\frac{9}{2}} + 16182\hat{q}^5 + 25988\hat{q}^{\frac{11}{2}} + 70079\hat{q}^6 + \mathcal{O}(\hat{q}^{\frac{13}{2}}). \tag{4.40}$$

We find nice agreement with the CFT results (4.2),(4.3),(4.4). Now the differences appear at $\hat{q}^{\frac{1}{2}(2N+6)}$, which means $\delta = 6$. (Recall that with only the Kaluza Klein modes we have errors at $\hat{q}^{\frac{1}{2}(2N+1)}$.) Unfortunately, we have no clear explanation for this value of δ.

4.2 $\mathbb{Z}_k$ orbifold

Let us consider the case with the Chern-Simons level k larger than 1. corresponding to $\mathbb{Z}_k$ orbifold.

For the $\text{AdS}_4 \times S^7/\mathbb{Z}_k$ case, the non-trivial fifth homology $H_5(S^7/\mathbb{Z}_k, \mathbb{Z}) = \mathbb{Z}_k$ means $S^7/\mathbb{Z}_k$ have topologically non-trivial five-cycles. M5-branes can be wrapped on these cycles.

Let us calculate the mass of the wrapped M5-brane in the presence of the $\mathbb{Z}_k$ orbifold. The relation between the M5-brane tension and the radius of S^7 is modified as follows:

$$T_{M5} = \frac{2\pi}{(2\pi l_p)^6} = \frac{2\pi N}{6\hat{r}^7 V_7/k} = \frac{Nk}{\hat{r}^6\pi^3}, \tag{4.41}$$

where $V_7 = \frac{\pi^4}{3}$ is volume of unit seven-sphere, while the relation of radii of AdS_4 and S^7 is preserved:

$$\hat{L} = \frac{\hat{r}}{2}. \tag{4.42}$$

Then the mass of the wrapped M5-brane for the orbifold case is given by

$$T_{\text{M5}}\hat{r}^5\frac{V_5}{k}\hat{L} = \frac{N}{2}. \tag{4.43}$$

Hence, the M5-brane again gives the finite-N corrections to the index.

As in the Kaluza Klein index, the M5-brane contribution to the index for the orbifold case can be calculated simply by applying the orbifold projection operator $\mathcal{P}_k$ on the single-particle index:

$$\mathcal{I}^{\text{M5}}_{z_a=0} = \hat{q}^{\frac{1}{2}N}\hat{u}_a^N \,\text{Pexp}\,\mathcal{P}_k i^{\text{M5}}_{z_a=0}. \tag{4.44}$$

Because of the non-trivial five-cycle homology $H_5(\boldsymbol{S}^7/\mathbb{Z}_k) = \mathbb{Z}_k$ we can classify states by the topological wrapping number $B \in \mathbb{Z}_k$ of M5-branes, and we can calculate the index for each sector with specific B. If a configuration C is given by equation $h(z) = 0$ the function $h(z)$ must have a specific $\mathbb{Z}_k$ charge for consistency with the $\mathbb{Z}_k$ orbifolding. Namely, it must satisfy

$$h(\omega_k z_1, \omega_k z_2, \omega_k^{-1} z_3, \omega_k^{-1} z_4) = \omega_k^B h(z_1, z_2, z_3, z_4) \tag{4.45}$$

with some $B \in \mathbb{Z}_k$. Then, B is the topological wrapping number of the worldvolume. Among the four representative configurations with $n = 1$, $z_1 = 0$ and $z_2 = 0$ carry $B = +1$, and $z_3 = 0$ and $z_4 = 0$ carry $B = -1$.

On the ABJM theory side k is the Chern-Simons level, and a wrapped M5-brane with $B \neq 0$ corresponds to a baryonic operator carrying $\mathbb{Z}_k$ baryonic charge B. In the ABJM theory with the gauge group $U(N)_k \times U(N)_{-k}$ this baryonic symmetry is a part of gauge symmetry, and baryonic operators are not gauge invariant. In order to calculate the index with the contribution of baryonic operators we need to use the ABJM theory with the gauge group $(U(N)_k \times U(N)_{-k})/\mathbb{Z}_k$ where the $\mathbb{Z}_k$ quotient acts on the diagonal $U(1)$ symmetry [51, 52]. In the index calculation, this quotient changes the quantization of monopole charges.

The index of ABJM theory is calculated by summing up contribution of different monopole charges [11]. The monopole charges are labeled by $2N$ GNO charges: $(m_1. \ldots, m_N)$ for $U(N)_k$ and $(\widetilde{m}_1, \ldots, \widetilde{m}_N)$ for $U(N)_{-k}$. In the $U(N)_k \times U(N)_{-k}$ theory all charges are integers, while in the $(U(N)_k \times U(N)_{-k})/\mathbb{Z}_k$ theory the quantization condition is given by

$$m_\alpha, \widetilde{m}_\alpha \in \mathbb{Z} + \frac{B}{k}, \quad B \in \mathbb{Z}_k. \tag{4.46}$$

The index of the $B = 0$ sector is the same as the index of the $U(N)_k \times U(N)_{-k}$ ABJM theory, while $B \neq 0$ sector gives the index for baryonic operators, which corresponds to the contribution of M5-branes with topological wrapping number B on the gravity side.

In the following we calculate the indices for $k = 2$ and $k = 3$ on both sides of the duality, and confirm the agreement up to the expected order of $\hat{q}$. We use the notations $\mathcal{I}_N^{\mathrm{ABJM}(B/k)}$ and $\mathcal{I}_N^{\mathrm{grav}(B/k)}$ for the indices calculated on the two sides of the duality.

4.2.1 Comparison

$k = 2$

In the case of $k = 2$, there are two sectors labeled by $B \in \mathbb{Z}_2$.

Let us first calculate the index of the $B = 0$ sector. The indices for $N = 1, 2, 3$ are

$$\mathcal{I}_{N=1}^{\mathrm{ABJM}(0/2)}|_{\hat{u}=1} = 1 + 10\hat{q} + 19\hat{q}^2 + \mathcal{O}(\hat{q}^3), \tag{4.47}$$

$$\mathcal{I}_{N=2}^{\mathrm{ABJM}(0/2)}|_{\hat{u}=1} = 1 + 10\hat{q} + 75\hat{q}^2 + 220\hat{q}^3 + \mathcal{O}(\hat{q}^4), \tag{4.48}$$

$$\mathcal{I}_{N=3}^{\mathrm{ABJM}(0/2)}|_{\hat{u}=1} = 1 + 10\hat{q} + 75\hat{q}^2 + 450\hat{q}^3 + 1595\hat{q}^4 + \mathcal{O}(\hat{q}^5). \tag{4.49}$$

Let us compare these with the Kaluza-Klein contribution.

$$\mathcal{I}_{\rm KK}^{\mathbb{Z}_2}|_{\hat{u}=1} = 1 + 10\hat{q} + 75\hat{q}^2 + 450\hat{q}^3 + 2365\hat{q}^4 + \mathcal{O}(\hat{q}^5). \tag{4.50}$$

We find the corrections appear at order $\hat{q}^{\frac{1}{2}(2N+2)}$. They are interpreted as contributions of two-brane configurations, which belong to the $B=0$ sector. Hence, it exceeds our scope.

Next, let us consider the index of $B=1$ sector:

$$\mathcal{I}_{N=1}^{\rm ABJM(1/2)}|_{\hat{u}=1} = 4\hat{q}^{\frac{1}{2}} + 16\hat{q}^{\frac{3}{2}} + 20\hat{q}^{\frac{5}{2}} + 40\hat{q}^{\frac{7}{2}} + 40\hat{q}^{\frac{9}{2}} + \mathcal{O}(\hat{q}^{\frac{11}{2}}), \tag{4.51}$$

$$\mathcal{I}_{N=2}^{\rm ABJM(1/2)}|_{\hat{u}=1} = 10\hat{q} + 65\hat{q}^2 + 220\hat{q}^3 + 455\hat{q}^4 + 1060\hat{q}^5 + 1645\hat{q}^6 + \mathcal{O}(\hat{q}^{\frac{13}{2}}), \tag{4.52}$$

$$\mathcal{I}_{N=3}^{\rm ABJM(1/2)}|_{\hat{u}=1} = 20\hat{q}^{\frac{3}{2}} + 164\hat{q}^{\frac{5}{2}} + 780\hat{q}^{\frac{7}{2}} + 2500\hat{q}^{\frac{9}{2}} + 6300\hat{q}^{\frac{11}{2}} + 15720\hat{q}^{\frac{13}{2}} + 30496\hat{q}^{\frac{15}{2}} + \mathcal{O}(\hat{q}^8). \tag{4.53}$$

On the gravity side we need to consider wrapped M5-brane with $B=1$. Because B is $\mathbb{Z}_2$-valued $B=+1$ and $B=-1$ are identified, and all four configurations $z_a=0$ $(a=1,2,3,4)$ contribute to the index;

$$\mathcal{I}_N^{\rm grav(1/2)} = \mathcal{I}_{\rm KK}^{\mathbb{Z}_2} \sum_{a=1}^{4} \hat{q}^{\frac{1}{2}N} \hat{u}_a^N \,{\rm Pexp}\, \mathcal{P}_2 i^{\rm M5}_{z_a=0}. \tag{4.54}$$

The results for $N=1,2,3$ are

$$\mathcal{I}_{N=1}^{\rm grav(1/2)}|_{\hat{u}=1} = 4\hat{q}^{\frac{1}{2}} + 16\hat{q}^{\frac{3}{2}} + 20\hat{q}^{\frac{5}{2}} + 40\hat{q}^{\frac{7}{2}} - 1500\hat{q}^{\frac{9}{2}} + \mathcal{O}(\hat{q}^5), \tag{4.55}$$

$$\mathcal{I}_{N=2}^{\rm grav(1/2)}|_{\hat{u}=1} = 10\hat{q} + 65\hat{q}^2 + 220\hat{q}^3 + 455\hat{q}^4 + 1060\hat{q}^5 - 7210\hat{q}^6 + \mathcal{O}(\hat{q}^{\frac{13}{2}}), \tag{4.56}$$

$$\mathcal{I}_{N=3}^{\rm grav(1/2)}|_{\hat{u}=1} = 20\hat{q}^{\frac{3}{2}} + 164\hat{q}^{\frac{5}{2}} + 780\hat{q}^{\frac{7}{2}} + 2500\hat{q}^{\frac{9}{2}} + 6300\hat{q}^{\frac{11}{2}} + 15720\hat{q}^{\frac{13}{2}} - 12008\hat{q}^{\frac{15}{2}} + \mathcal{O}(\hat{q}^8). \tag{4.57}$$

In all cases the leading term is of order $\hat{q}^{\frac{1}{2}N}$, and there is no tachyonic shift. This is because the $\mathbb{Z}_2$ projection removes the tachyonic term from the single-particle index. This is consistent with the fact that the branes are wrapped on topologically non-trivial cycles. The error between the ABJM index and (4.54) appears at $\hat{q}^{\frac{1}{2}(3N+6)}$. This is consistent with the fact that only brane configuration with odd n contribute to the index of the $B=1$ sector and the error is due to $n=3$ configurations.

$k=3$

The $\mathbb{Z}_k$ orbifolding with $k \geq 3$ breaks the $\mathcal{N}=8$ supersymmetry down to $\mathcal{N}=6$.

We consider $k=3$ case and there are three sectors specified by $B \in \mathbb{Z}_3$. Let us first consider the $B=0$ sector. The ABJM index is given for $N=1,2,3$ as follows.

$$\mathcal{I}_{N=1}^{\rm ABJM(0/3)}|_{\hat{u}=1} = 1 + 4\hat{q} + 8\hat{q}^{\frac{3}{2}} + \hat{q}^2 + \mathcal{O}(\hat{q}^{\frac{5}{2}}), \tag{4.58}$$

$$\mathcal{I}_{N=2}^{\rm ABJM(0/3)}|_{\hat{u}=1} = 1 + 4\hat{q} + 8\hat{q}^{\frac{3}{2}} + 12\hat{q}^2 + 40\hat{q}^{\frac{5}{2}} + 58\hat{q}^3 + \mathcal{O}(\hat{q}^{\frac{7}{2}}), \tag{4.59}$$

$$\mathcal{I}_{N=3}^{\rm ABJM(0/3)}|_{\hat{u}=1} = 1 + 4\hat{q} + 8\hat{q}^{\frac{3}{2}} + 12\hat{q}^2 + 40\hat{q}^{\frac{5}{2}} + 82\hat{q}^3 + 132\hat{q}^{\frac{7}{2}} + 303\hat{q}^4 + \mathcal{O}(\hat{q}^{\frac{9}{2}}). \tag{4.60}$$

Let us compare these with the Kaluza-Klein index

$$\mathcal{I}_{\rm KK}^{\mathbb{Z}_3}|_{\hat{u}=1} = 1 + 4\hat{q} + 8\hat{q}^{\frac{3}{2}} + 12\hat{q}^2 + 40\hat{q}^{\frac{5}{2}} + 82\hat{q}^3 + 132\hat{q}^{\frac{7}{2}} + 348\hat{q}^4 + \mathcal{O}(\hat{q}^{\frac{9}{2}}). \tag{4.61}$$

We find the corrections at $\hat{q}^{\frac{1}{2}(2N+2)}$. We can interpret these corrections as the contributions of brane configurations with $n = 2$ consisting of a brane with $B = +1$ and another brane with $B = -1$.

Next, let us consider baryonic sectors with $B = \pm 1$. These two sectors are related by the charge conjugation symmetry $B \to -B$ we focus only on the $B = +1$ sector. The ABJM index is given as follows for $N = 1, 2, 3$.

$$\mathcal{I}_{N=1}^{\rm ABJM(1/3)}|_{\hat{u}=1} = 2\hat{q}^{\frac{1}{2}} + 3\hat{q} + 4\hat{q}^{\frac{3}{2}} + 9\hat{q}^2 + \mathcal{O}(\hat{q}^{\frac{5}{2}}), \tag{4.62}$$

$$\mathcal{I}_{N=2}^{\rm ABJM(1/3)}|_{\hat{u}=1} = 3\hat{q} + 6\hat{q}^{\frac{3}{2}} + 14\hat{q}^2 + 32\hat{q}^{\frac{5}{2}} + 51\hat{q}^3 + \mathcal{O}(\hat{q}^{\frac{7}{2}}), \tag{4.63}$$

$$\mathcal{I}_{N=3}^{\rm ABJM(1/3)}|_{\hat{u}=1} = 4\hat{q}^{\frac{3}{2}} + 9\hat{q}^2 + 24\hat{q}^{\frac{5}{2}} + 65\hat{q}^3 + 126\hat{q}^{\frac{7}{2}} + 215\hat{q}^4 + \mathcal{O}(\hat{q}^{\frac{9}{2}}). \tag{4.64}$$

On the gravity side we take only two single-wrapping configurations $z_1 = 0$ and $z_2 = 0$ into account because the other two carry $B = -1$.

$$\mathcal{I}_N^{\rm grav(1/3)} = \mathcal{I}_{\rm KK}^{\mathbb{Z}_3} \sum_{a=1}^{2} \hat{q}^{\frac{1}{2}N} \hat{u}_a^N \, {\rm Pexp}\, \mathcal{P}_3 i_{z_a=0}^{\rm M5}. \tag{4.65}$$

The results for $N = 1, 2, 3$ are

$$\mathcal{I}_{N=1}^{\rm grav(1/3)}|_{\hat{u}=1} = 2\hat{q}^{\frac{1}{2}} + 3\hat{q} + 4\hat{q}^{\frac{3}{2}} - \hat{q}^2 + \mathcal{O}(\hat{q}^{\frac{5}{2}}), \tag{4.66}$$

$$\mathcal{I}_{N=2}^{\rm grav(1/3)}|_{\hat{u}=1} = 3\hat{q} + 6\hat{q}^{\frac{3}{2}} + 14\hat{q}^2 + 32\hat{q}^{\frac{5}{2}} + 36\hat{q}^3 + \mathcal{O}(\hat{q}^{\frac{7}{2}}), \tag{4.67}$$

$$\mathcal{I}_{N=3}^{\rm grav(1/3)}|_{\hat{u}=1} = 4\hat{q}^{\frac{3}{2}} + 9\hat{q}^2 + 24\hat{q}^{\frac{5}{2}} + 65\hat{q}^3 + 126\hat{q}^{\frac{7}{2}} + 194\hat{q}^4 + \mathcal{O}(\hat{q}^{\frac{9}{2}}). \tag{4.68}$$

We find errors at $\hat{q}^{\frac{1}{2}(2N+2)}$. We can interpret them as the contribution of $n = 2$ configurations with $B = -2 \approx +1$.

Summary of Chapter 4

In this chapter, we investigated a method of calculating the superconformal index of the M2-brane theories from the dual gravity theory in the finite-N region. Our formula (4.13) includes the wrapped M5-brane contribution as the finite-N corrections.

We checked the validity of our formula by comparing the results obtained via the formula with the ABJM indices. Especially, we found the agreement of the indices up to the order $\mathcal{O}(\hat{q}^{\frac{1}{2}(2N+6)})$ for $k = 1$. The difference is thought to be the contribution of multiple-wrapping M5-branes we ignored in the analyses in this chapter. For $k > 1$, we also did a similar analysis for each baryonic charge sector and found nice agreement of the indices.

Chapter 5

Finite N corrections to the indices of the M5-brane theories

Next, we discuss the superconformal indices of 6d $\mathcal{N} = (2,0)$ theories. As we saw, this theory is realized on a stack of M5-branes, and the gravity dual is M-theory on $\mathrm{AdS}_7 \times S^4$. Since there are no Lagrangian descriptions of the theory until now, we cannot perform the direct localization analysis to calculate the superconformal index.

We propose a formula for calculating the indices from the dual gravity theory, which is very similar to the previous one (4.13). In this 6d case, finite-N corrections to the indices of 6d (2,0) theories are given by the contribution of M2-branes wrapped on $S^2 \subset S^4$. For the $N = 1$ case, we confirm the validity of our formula by comparing the result from our formula with the free theory calculation (3.15). For $N > 1$, by removing the free tensor multiplet contribution, we propose new results of the indices of 6d A_{N-1} theories. We also expand the indices of A_{N-1} theories in terms of superconformal representations.

We also discuss the superconformal index of the 6d $\mathcal{N} = (1,0)$ theory in Sec 5.2.

5.1 Finite N corrections to the indices of the 6d $\mathcal{N} = (2,0)$ theories

5.1.1 Difference at finite-N

As in the previous $\mathrm{AdS}_4/\mathrm{CFT}_3$ example, we first show the difference between the finite-N superconformal index of the 6d $\mathcal{N} = (2,0)$ theory and the Kaluza Klein index. From the equations (3.15) and (3.23), we obtain

$$\mathcal{I}^{(2,0)}_{N=1} - \mathcal{I}^{S^4}_{\mathrm{KK}} = -\chi^u_2 \check{q}^4 + \cdots . \tag{5.1}$$

Although we can perform the calculation for the $N = 1$ case only, this seems to implies

$$\mathcal{I}^{(2,0)}_{N} - \mathcal{I}^{S^4}_{\mathrm{KK}} = -\chi^u_{N+1} \check{q}^{2(N+1)} + \cdots . \tag{5.2}$$

Let us discuss the parameter relations for the $\mathrm{AdS}_7/\mathrm{CFT}_6$ case. In terms of the M2-brane tension, the relation (3.19) can be written as

$$N = 4\pi T_{\mathrm{M2}}\check{r}^3. \tag{5.3}$$

Also, the mass of the M2-brane wrapped on maximal S^4 (multiplied by AdS_7 length) is given by

$$T_{\mathrm{M2}}\check{r}^2 V_2 \check{L} = 2N, \tag{5.4}$$

with $V_2 = 4\pi$ is volume of two-sphere. This suggest that the M2-brane contribution gives the finite-N corrections to the indices of the 6d $(2,0)$ theories. Note that the appearance of the difference at the order $\mathcal{O}(\check{q}^{2(N+1)})$ in (5.2), not as the order $\mathcal{O}(\check{q}^{2(N+1)})$ is again due to the tachyonic shift.

5.1.2 Wrapped M2-brane

Similarly to the previous $\mathrm{AdS}_4/\mathrm{CFT}_3$ case, for the theory on a finite number of M5-branes we propose the following formula of the superconformal index:

$$\mathcal{I}_N^{(2,0)} = \mathcal{I}_{\mathrm{KK}}\left(1 + \sum_C \mathcal{I}_C^{\mathrm{M2}}\right). \tag{5.5}$$

$\mathcal{I}_C^{\mathrm{M2}}$ is the contribution of an M2-brane configuration C. The sum of C runs over representative configurations, which are determined shortly in a parallel way to the three-dimensional case. We again schematically show the objects contributing to the indices at finite-N in Figure 5.1.

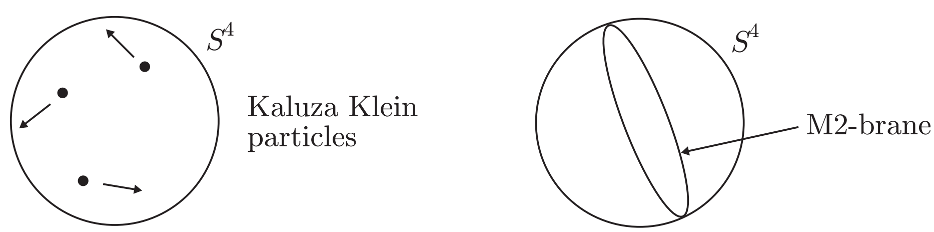

Figure 5.1: The objects contributing to the indices at finite-N are shown. The left figure shows the Kaluza Klein particles in S^4, which gives the large-N indices. The right figure shows the M2-brane wrapped on a great circle in S^4, which gives the finite-N corrections to the indices.

Let us introduce Cartesian coordinates $x_1, \ldots, x_5$ and describe $\boldsymbol{S}^4$ by $\sum_{a=1}^5 x_a^2 = 1$. We also introduce the complex coordinates

$$z_1 = x_1 + ix_2, \quad z_2 = x_3 + ix_4. \tag{5.6}$$

The subalgebra $su(2) \subset so(5)$ of the R-symmetry commuting with $\check{\mathcal{Q}}$ transforms these complex coordinates as a doublet. For a rigid M2-brane wrapped on a large $\boldsymbol{S}^2$ in $\boldsymbol{S}^4$ to preserve the supersymmetry $\check{\mathcal{Q}}$, the M2-brane worldvolume must be given by the holomorphic equation [53]

$$a_1 z_1 + a_2 z_2 = 0, \tag{5.7}$$

where (a_1, a_2) are homogeneous coordinates of the moduli space $\mathbb{P}^1$ of the rigid brane. Due to the coupling to the background flux the wave function Ψ of the rigid brane is a section of $\mathcal{O}(N)$ line bundle over $\mathbb{P}^1$. Namely, Ψ can be given as a homogeneous polynomial of (a_1, a_2) of degree N. There are $N+1$ such linearly independent polynomials belonging to the $(N+1)$-dimensional representation of $su(2)$ acting on $\mathbb{P}^1$. The corresponding index is

$$\check{q}^{2N}\chi_N(\check{u}) = \frac{\check{q}^{2N}\check{u}^N}{1-\check{u}^{-2}} + \frac{\check{q}^{2N}\check{u}^{-N}}{1-\check{u}^2}. \tag{5.8}$$

As in the case of wrapped M5-branes the two terms are interpreted as the contribution of two representative configurations of M2-brane, $z_1 = 0$ and $z_2 = 0$, respectively. The general representative configurations are given in the form

$$C : z_1^{n_1} z_2^{n_2} = 0, \quad n_1, n_2 \in \mathbb{Z}_{\geq 0}, \quad (n_1, n_2) \neq (0,0), \tag{5.9}$$

and the corresponding contribution $\mathcal{I}_C^{\text{M2}}$ is given by

$$\mathcal{I}_C^{\text{M2}} = \check{q}^{2nN}\check{u}^{(n_1-n_2)N}\mathcal{I}_C^{\text{excitations}}, \tag{5.10}$$

where $n = n_1 + n_2$. For C with $n \geq 2$ it is difficult to calculate $\mathcal{I}_C^{\text{excitations}}$, while for $n = 1$ configurations $z_a = 0$ $(a = 1, 2)$ the theory on the wrapped brane is free and given by $\mathcal{I}_{z_a=1}^{\text{excitations}} = \operatorname{Pexp} i_{z_a=0}^{\text{M2}}$, where $i_{z_a=0}^{\text{M2}}$ is the single-particle index on an M2-brane wrapped on $z_a = 0$.

Let us consider an M2-brane wrapped on $\boldsymbol{S}^2 \subset \boldsymbol{S}^4$ on $z_1 = 0$. Among 32 supercharges only 16 that commute with

$$\check{Z} = \check{H} - \check{R}_{12} \tag{5.11}$$

are preserved by the wrapped brane. The superconformal algebra $\check{\mathcal{A}}$ is broken to

$$so(2)_{\check{Z}} \times \check{\mathcal{C}}, \quad \check{\mathcal{C}} = su(4|2), \tag{5.12}$$

where $so(2)_{\check{Z}}$ is the central factor generated by $\check{Z}$. The bosonic subalgebra $su(4) \times su(2) \times u(1) \subset \check{\mathcal{C}}$ is generated by

$$\check{J}_{ij} \quad (i,j = 1,\ldots,6), \quad \check{R}_{ab} \quad (a,b = 3,4,5), \quad \check{C} \equiv \check{H} - 2\check{R}_{12}. \tag{5.13}$$

As is explained in the last section this is isomorphic to the symmetry preserved by a wrapped M5-brane in (4.25). By using the isomorphism map (4.28), we can obtain $i_{z_1=0}^{\text{M2}}$ from i^{M2} in (2.16) by a simple variable change. The inverse of (4.34) is

$$\hat{q} = \check{q}\check{u}^{-\frac{1}{2}}, \quad \hat{u}_1 = \check{q}^{\frac{5}{6}}\check{y}_1\check{u}^{\frac{1}{4}}, \quad \hat{u}_2 = \check{q}^{\frac{5}{6}}\check{y}_2\check{u}^{\frac{1}{4}}, \quad \hat{u}_3 = \check{q}^{\frac{5}{6}}\check{y}_3\check{u}^{\frac{1}{4}}, \quad \hat{u}_4 = \check{q}^{-\frac{5}{2}}\check{u}^{-\frac{3}{4}}, \tag{5.14}$$

Table 5.1: Scalar modes on an M2-brane wrapped on $z_1 = 0$. $\ell = 0, 1, 2, \dots$ is the orbital angular momentum in $\boldsymbol{S}^2$. States with $(R_{12}, R_{34}) = (-1, \ell)$ saturate the BPS bound $\check{H} \geq 2(\check{R}_{12} + \check{R}_{34})$.

$so(6)$	$\check{R}_{12}$	$\check{R}_{34}$	$\check{H}$
6	0	$-\ell \sim \ell$	$2\ell + 1$
1	$+1$	$-\ell \sim \ell$	$2\ell + 4$
1	-1	$-\ell \sim \ell$	$2\ell - 2$

and by substituting these relations into (2.16) we obtain

$$i^{\rm M2}_{z_1=0} = \frac{\check{q}^{-2}\check{u}^{-1} - \check{q}^{\frac{2}{3}}\check{u}^{-1}\chi_{[0,1]}(\check{y}) + \check{q}^{\frac{4}{3}}\chi_{[1,0]}(\check{y}) - \check{q}^4}{1 - \check{q}^2\check{u}^{-1}}. \tag{5.15}$$

The index $i^{\rm M2}_{z_2=0}$ for the other configuration $z_2 = 0$ is obtained from (5.15) by the Weyl reflection $\check{u} \to \check{u}^{-1}$.

It is of course possible to calculate the index directly by the mode expansion of fields on the wrapped brane. We show the results for scalar fields in Table 5.1. There is one BPS tachyonic mode with $\check{H} = -2$ and one BPS zero mode. These correspond to the first two terms in the $\check{q}$ expansion of $i^{\rm M2}_{z_1=0}$:

$$i^{\rm M2}_{z_1=0} = \frac{1}{\check{q}^2\check{u}} + \frac{1}{\check{u}^2} + \cdots . \tag{5.16}$$

5.1.3 Results and consistency check

The formula for the finite-N corrections is

$$\mathcal{I}^{(2,0)}_N = \mathcal{I}^{\rm grav}_N + \mathcal{O}(\check{q}^{2(2N+\delta)}), \tag{5.17}$$

where $\mathcal{I}^{\rm grav}_N$ includes the Kaluza-Klein contribution and the contribution of single wrapping M2-branes, and the second term is the contribution of multiple wrapping configurations, which we do not calculate in this paper. δ is the tachyonic shift of configurations with $n = 2$. The explicit form of $\mathcal{I}^{\rm grav}_N$ is

$$\mathcal{I}^{\rm grav}_N = \mathcal{I}_{\rm KK}\left(1 + \check{q}^{2N}\check{u}^N \,{\rm Pexp}\, i^{\rm M2}_{z_1=0} + \check{q}^{2N}\check{u}^{-N}\,{\rm Pexp}\, i^{\rm M2}_{z_2=0}\right). \tag{5.18}$$

Let us first consider the $N=1$ case. The formula (5.18) with $N=1$ gives

$$\begin{aligned}\mathcal{I}^{\text{grav}}_{N=1} &= 1+\chi^{\check{u}}_1\check{q}^2-\chi_{[0,1]}\check{q}^{\frac{8}{3}}+\chi_{[1,0]}\chi^{\check{u}}_1\check{q}^{\frac{10}{3}}+\left(\chi^{\check{u}}_2-\chi_{[1,1]}\right)\check{q}^4+\left(\chi_{[2,0]}-\chi_{[0,1]}\right)\chi^{\check{u}}_1\check{q}^{\frac{14}{3}}\\
&+\left(\chi_{[1,0]}\left(\chi^{\check{u}}_2+2\right)-\chi_{[2,1]}\right)\check{q}^{\frac{16}{3}}+\left(\left(-2\chi_{[1,1]}+\chi_{[3,0]}-1\right)\chi^{\check{u}}_1+\chi^{\check{u}}_3\right)\check{q}^6\\
&+\left(2\chi_{[2,0]}\chi^{\check{u}}_2-\chi_{[0,1]}\left(\chi^{\check{u}}_2-2\right)+\chi_{[1,2]}+2\chi_{[2,0]}-\chi_{[3,1]}\right)\check{q}^{\frac{20}{3}}\\
&+\left(-\chi_{[0,2]}\chi^{\check{u}}_1-3\chi_{[2,1]}\chi^{\check{u}}_1+\chi_{[4,0]}\chi^{\check{u}}_1+\chi_{[1,0]}\chi^{\check{u}}_3\right)\check{q}^{\frac{22}{3}}\\
&+\left(2\chi_{[3,0]}\chi^{\check{u}}_2-\chi_{[1,1]}\left(\chi^{\check{u}}_2-2\right)+\chi_{[0,3]}+\chi_{[2,2]}+4\chi_{[3,0]}-\chi_{[4,1]}-\chi^{\check{u}}_2+\chi^{\check{u}}_4-2\right)\check{q}^8\\
&+\left(-\chi_{[1,2]}\chi^{\check{u}}_1+\chi_{[2,0]}\chi^{\check{u}}_1-4\chi_{[3,1]}\chi^{\check{u}}_1+\chi_{[5,0]}\chi^{\check{u}}_1+\chi_{[0,1]}\left(2\chi^{\check{u}}_1-\chi^{\check{u}}_3\right)+2\chi_{[2,0]}\chi^{\check{u}}_3\right)\check{q}^{\frac{26}{3}}\\
&+\Big(-3\chi_{[2,1]}\chi^{\check{u}}_2+3\chi_{[4,0]}\chi^{\check{u}}_2+\chi_{[1,0]}\left(-\chi^{\check{u}}_2+\chi^{\check{u}}_4-3\right)\\
&\qquad-2\chi_{[0,2]}+\chi_{[1,3]}+2\chi_{[2,1]}+2\chi_{[3,2]}+4\chi_{[4,0]}-\chi_{[5,1]}\Big)\check{q}^{\frac{28}{3}}\\
&+\Big(\left(\chi_{[0,3]}+6\chi_{[1,1]}-\chi_{[2,2]}+3\chi_{[3,0]}-5\chi_{[4,1]}+\chi_{[6,0]}-1\right)\chi^{\check{u}}_1\\
&\qquad-\left(\chi_{[1,1]}-3\chi_{[3,0]}+1\right)\chi^{\check{u}}_3\Big)\check{q}^{10}+\mathcal{O}(\check{q}^{\frac{32}{3}}).\end{aligned}\tag{5.19}$$

On the other hand, for the $N=1$ case, the six-dimensional theory is the free theory of a single tensor multiplet and we explicitly calculated the index in (3.15). If we compare the gravity calculation (5.19) with the field theory calculation (3.15), we obtain

$$\mathcal{I}^{(2,0)}_{N=1}-\mathcal{I}^{\text{grav}}_{N=1}=\chi^{\check{u}}_5\check{q}^{10}+\mathcal{O}(\check{q}^{\frac{32}{3}}).\tag{5.20}$$

We find nice agreement. The error appears at order $\check{q}^{10}$. This means the tachyonic shift $\delta=3$. Although we have no interpretation of this value of δ, let us assume that this is N-independent as in the 3d and 4d cases.

The first few terms for $N\geq 2$ are

$$\begin{aligned}\mathcal{I}^{\text{grav}}_{N\geq 2} &= 1+\chi^{\check{u}}_1\check{q}^2-\chi_{[0,1]}\check{q}^{\frac{8}{3}}+\chi_{[1,0]}\chi^{\check{u}}_1\check{q}^{\frac{10}{3}}+\left(2\chi^{\check{u}}_2-\chi_{[1,1]}\right)\check{q}^4+\left(\chi_{[2,0]}-2\chi_{[0,1]}\right)\chi^{\check{u}}_1\check{q}^{\frac{14}{3}}\\
&+\left(\chi_{[1,0]}\left(2\chi^{\check{u}}_2+3\right)-\chi_{[2,1]}\right)\check{q}^{\frac{16}{3}}+\mathcal{O}(\check{q}^6).\end{aligned}\tag{5.21}$$

The leading finite-N correction given by (5.18) is $-\chi^{\check{u}}_{N+1}\check{q}^{2(N+1)}$,[1] and the terms in the range shown in (5.21) is the same as the supergravity approximation $\mathcal{I}_{\text{KK}}$.

The second term $\chi^{\check{u}}_1\check{q}^2$ is the contribution of the primary operators in the free tensor multiplet. The term $\chi^{\check{u}}_2\check{q}^4$ is the contribution of the stress-tensor multiplet. The coefficient 2 of the term suggests that the theory has two stress-energy tensors. Namely, the system consists of two decoupled theories. One is the free theory of the tensor multiplet, and the other is the interacting theory called the A_{N-1} theory.

By removing the contribution of the free tensor multiplet we obtain the index of the A_{N-1} theory:

$$\mathcal{I}_{A_{N-1}}=\frac{\mathcal{I}^{(2,0)}_N}{\mathcal{I}^{(2,0)}_{N=1}}.\tag{5.22}$$

[1] In [42], the finite-N index of the 6d $(2,0)$ theory was studied from 5d SYM on $\mathbb{CP}^2\times\boldsymbol{S}^1$. Especially, the authors found that the leading correction for $N=2$ is $-q^3y^3$, where the fugacities q and y are related to ours by $\check{q}=q^{\frac{3}{4}}$ and $\check{u}=q^{-\frac{1}{2}}y$. See (3.65) in [42]. This is consistent with our result: $-\chi^{\check{u}}_3\check{q}^6=-q^3y^3+\cdots$.

Explicit forms of $\mathcal{I}_{A_{N-1}}$ for small N obtained by using (5.18) are as follows.

$$\begin{aligned}\mathcal{I}_{A_1} &= 1+\chi_2^{\check{u}}\check{q}^4-\chi_{[0,1]}\chi_1^{\check{u}}\check{q}^{\frac{14}{3}}+\chi_{[1,0]}\left(\chi_2^{\check{u}}+1\right)\check{q}^{\frac{16}{3}}-(\chi_{[1,1]}+1)\chi_1^{\check{u}}\check{q}^6\\&+\left(\chi_{[2,0]}\left(\chi_2^{\check{u}}+1\right)+\chi_{[0,1]}\right)\check{q}^{\frac{20}{3}}-(\chi_{[1,0]}+\chi_{[2,1]})\chi_1^{\check{u}}\check{q}^{\frac{22}{3}}\\&+\left(\chi_{[3,0]}\left(\chi_2^{\check{u}}+1\right)+\chi_{[1,1]}+\chi_4^{\check{u}}\right)\check{q}^8+\left(-\chi_{[2,0]}\chi_1^{\check{u}}-\chi_{[3,1]}\chi_1^{\check{u}}-\chi_{[0,1]}\chi_3^{\check{u}}\right)\check{q}^{\frac{26}{3}}\\&+\left(\chi_{[4,0]}\left(\chi_2^{\check{u}}+1\right)+\chi_{[1,0]}\left(2\chi_2^{\check{u}}+\chi_4^{\check{u}}\right)+\chi_{[2,1]}\right)\check{q}^{\frac{28}{3}}\\&+\left(-\chi_{[1,1]}(\chi_1^{\check{u}}+2\chi_3^{\check{u}})-\chi_{[3,0]}\chi_1^{\check{u}}-\chi_{[4,1]}\chi_1^{\check{u}}-2\chi_1^{\check{u}}-\chi_3^{\check{u}}\right)\check{q}^{10}\\&+\mathcal{O}(\check{q}^{\frac{32}{3}}).\end{aligned}\tag{5.23}$$

$$\begin{aligned}\mathcal{I}_{A_2} &= 1+\chi_2^{\check{u}}\check{q}^4-\chi_{[0,1]}\chi_1^{\check{u}}\check{q}^{\frac{14}{3}}+\chi_{[1,0]}\left(\chi_2^{\check{u}}+1\right)\check{q}^{\frac{16}{3}}+\left(\chi_3^{\check{u}}-(\chi_{[1,1]}+1)\chi_1^{\check{u}}\right)\check{q}^6\\&+\left(\chi_{[2,0]}\left(\chi_2^{\check{u}}+1\right)-\chi_{[0,1]}\left(\chi_2^{\check{u}}-1\right)\right)\check{q}^{\frac{20}{3}}\\&+\left(\chi_{[1,0]}\chi_3^{\check{u}}-\chi_{[2,1]}\chi_1^{\check{u}}\right)\check{q}^{\frac{22}{3}}+\left(-\chi_{[1,1]}\left(\chi_2^{\check{u}}-1\right)+\chi_{[3,0]}\left(\chi_2^{\check{u}}+1\right)-\chi_2^{\check{u}}+\chi_4^{\check{u}}\right)\check{q}^8\\&+\left((\chi_{[2,0]}-\chi_{[0,1]})\chi_3^{\check{u}}-\chi_{[3,1]}\chi_1^{\check{u}}\right)\check{q}^{\frac{26}{3}}\\&+\left(-\chi_{[2,1]}\left(\chi_2^{\check{u}}-1\right)+2\chi_{[1,0]}\chi_2^{\check{u}}+\chi_{[4,0]}\chi_2^{\check{u}}+\chi_{[1,0]}\chi_4^{\check{u}}+\chi_{[0,2]}+\chi_{[4,0]}\right)\check{q}^{\frac{28}{3}}\\&+\left(-2\chi_{[1,1]}(\chi_1^{\check{u}}+\chi_3^{\check{u}})-\chi_{[4,1]}\chi_1^{\check{u}}+\chi_{[3,0]}\chi_3^{\check{u}}-3\chi_1^{\check{u}}-\chi_3^{\check{u}}+\chi_5^{\check{u}}\right)\check{q}^{10}\\&+\mathcal{O}(\check{q}^{\frac{32}{3}}).\end{aligned}\tag{5.24}$$

$$\begin{aligned}\mathcal{I}_{A_3} &= 1+\chi_2^{\check{u}}\check{q}^4-\chi_{[0,1]}\chi_1^{\check{u}}\check{q}^{\frac{14}{3}}+\chi_{[1,0]}\left(\chi_2^{\check{u}}+1\right)\check{q}^{\frac{16}{3}}+\left(\chi_3^{\check{u}}-(\chi_{[1,1]}+1)\chi_1^{\check{u}}\right)\check{q}^6\\&+\left(\chi_{[2,0]}\left(\chi_2^{\check{u}}+1\right)-\chi_{[0,1]}\left(\chi_2^{\check{u}}-1\right)\right)\check{q}^{\frac{20}{3}}\\&+\left(\chi_{[1,0]}\chi_3^{\check{u}}-\chi_{[2,1]}\chi_1^{\check{u}}\right)\check{q}^{\frac{22}{3}}+\left(-\chi_{[1,1]}\left(\chi_2^{\check{u}}-1\right)+\chi_{[3,0]}\left(\chi_2^{\check{u}}+1\right)-\chi_2^{\check{u}}+2\chi_4^{\check{u}}\right)\check{q}^8\\&+\left((\chi_{[2,0]}-2\chi_{[0,1]})\chi_3^{\check{u}}-\chi_{[3,1]}\chi_1^{\check{u}}\right)\check{q}^{\frac{26}{3}}\\&+\left(-\chi_{[2,1]}\left(\chi_2^{\check{u}}-1\right)+3\chi_{[1,0]}\chi_2^{\check{u}}+\chi_{[4,0]}\chi_2^{\check{u}}+2\chi_{[1,0]}\chi_4^{\check{u}}+\chi_{[0,2]}+\chi_{[4,0]}\right)\check{q}^{\frac{28}{3}}\\&+\left(-\chi_{[1,1]}(2\chi_1^{\check{u}}+3\chi_3^{\check{u}})-\chi_{[4,1]}\chi_1^{\check{u}}+\chi_{[3,0]}\chi_3^{\check{u}}-3\chi_1^{\check{u}}-2\chi_3^{\check{u}}+\chi_5^{\check{u}}\right)\check{q}^{10}\\&+\mathcal{O}(\check{q}^{\frac{32}{3}}).\end{aligned}\tag{5.25}$$

$$\begin{aligned}\mathcal{I}_{A_{\geq 4}} &= 1+\chi_2^{\check{u}}\check{q}^4-\chi_{[0,1]}\chi_1^{\check{u}}\check{q}^{\frac{14}{3}}+\chi_{[1,0]}\left(\chi_2^{\check{u}}+1\right)\check{q}^{\frac{16}{3}}+\left(\chi_3^{\check{u}}-(\chi_{[1,1]}+1)\chi_1^{\check{u}}\right)\check{q}^6\\&+\left(\chi_{[2,0]}\left(\chi_2^{\check{u}}+1\right)-\chi_{[0,1]}\left(\chi_2^{\check{u}}-1\right)\right)\check{q}^{\frac{20}{3}}\\&+\left(\chi_{[1,0]}\chi_3^{\check{u}}-\chi_{[2,1]}\chi_1^{\check{u}}\right)\check{q}^{\frac{22}{3}}+\left(-\chi_{[1,1]}\left(\chi_2^{\check{u}}-1\right)+\chi_{[3,0]}\left(\chi_2^{\check{u}}+1\right)-\chi_2^{\check{u}}+2\chi_4^{\check{u}}\right)\check{q}^8\\&+\left((\chi_{[2,0]}-2\chi_{[0,1]})\chi_3^{\check{u}}-\chi_{[3,1]}\chi_1^{\check{u}}\right)\check{q}^{\frac{26}{3}}\\&+\left(-\chi_{[2,1]}\left(\chi_2^{\check{u}}-1\right)+3\chi_{[1,0]}\chi_2^{\check{u}}+\chi_{[4,0]}\chi_2^{\check{u}}+2\chi_{[1,0]}\chi_4^{\check{u}}+\chi_{[0,2]}+\chi_{[4,0]}\right)\check{q}^{\frac{28}{3}}\\&+\left(-\chi_{[1,1]}(2\chi_1^{\check{u}}+3\chi_3^{\check{u}})-\chi_{[4,1]}\chi_1^{\check{u}}+\chi_{[3,0]}\chi_3^{\check{u}}-3\chi_1^{\check{u}}-2\chi_3^{\check{u}}+2\chi_5^{\check{u}}\right)\check{q}^{10}\\&+\mathcal{O}(\check{q}^{\frac{32}{3}}).\end{aligned}\tag{5.26}$$

We gave the above $\check{q}$-expansion up to $\check{q}^{10}$ terms. The error in $\mathcal{I}_{A_{N-1}}$ estimated with $\delta = 3$ is $\check{q}^{2(2N+3)}$, and all terms shown above are expected to be correct.

As far as we are aware there are no explicit results in the literature which can be compared with these results. As a consistency check, let us expand these results by indices of superconformal representations. It is guaranteed by construction that (5.18) can be expanded by characters of the bosonic subalgebra $su(3) \times su(2)$. However, it is non-trivial if it can be expanded by the indices of superconformal representations. The results are as follows.

$$\mathcal{I}_{A_1} = 1 + \mathcal{D}[2,0] + \mathcal{D}[4,0] + \mathcal{B}[2,0]_0 + \mathcal{O}(\check{q}^{\frac{32}{3}}), \tag{5.27}$$
$$\mathcal{I}_{A_2} = 1 + \mathcal{D}[2,0] + \mathcal{D}[3,0] + \mathcal{D}[4,0] + \mathcal{D}[0,4] + \mathcal{B}[2,0]_0 + \mathcal{D}[5,0] + \mathcal{D}[3,2] + \mathcal{O}(\check{q}^{\frac{32}{3}}), \tag{5.28}$$
$$\mathcal{I}_{A_3} = 1 + \mathcal{D}[2,0] + \mathcal{D}[3,0] + 2\mathcal{D}[4,0] + \mathcal{D}[0,4] + \mathcal{B}[2,0]_0 + \mathcal{D}[5,0] + \mathcal{D}[3,2] + \mathcal{D}[1,4] + \mathcal{O}(\check{q}^{\frac{32}{3}}), \tag{5.29}$$
$$\mathcal{I}_{A_{\geq 4}} = 1 + \mathcal{D}[2,0] + \mathcal{D}[3,0] + 2\mathcal{D}[4,0] + \mathcal{D}[0,4] + \mathcal{B}[2,0]_0 + 2\mathcal{D}[5,0] + \mathcal{D}[3,2] + \mathcal{D}[1,4] + \mathcal{O}(\check{q}^{\frac{32}{3}}). \tag{5.30}$$

We exploited the notation for representations used in [54] to denote the corresponding indices. See the following discussion for a detailed explanation for the index of each irreducible representation. These results support the correctness of the formula (5.18). In addition, the expansion of $\mathcal{I}_{A_1}$ seems to be exceptionally simple. In particular, as was pointed out in [54] the $\mathcal{D}[0,4]$ representation is absent in the A_1 theory.

Detail of 6d superconformal representations

To calculate the index of superconformal representations we mainly followed the procedure proposed in [55]. In the expansion in the previous calculations the D-type and B-type representations appear. We used the notations in [54]. They correspond to those used in [55] as follows.

$$\mathcal{D}[a,b] = D_1[0,0,0]^{(b,a)}_{2a+2b}, \quad \mathcal{B}[a,b]_n = B_\ell[0,n,0]^{(b,a)}_{n+2a+2b+4} \tag{5.31}$$

where ℓ is the level of the primary null state. It is $\ell = 3$ for $n = 0$ and $\ell = 1$ for $n \geq 1$.

The series of representations $\mathcal{D}[m,0]$ ($m = 1,2,3,\ldots$) appear in the Kaluza-Klein spectrum in $AdS_7 \times S^4$, and have been well studied. The superconformal index of each of them is[2]

$$\mathcal{D}[m,0] = \frac{\chi^{\check{u}}_m \check{q}^{2m} - \chi^{\check{u}}_{m-1}\chi_{(0,1)}\check{q}^{2m+\frac{2}{3}} + \chi^{\check{u}}_{m-2}\chi_{(1,0)}\check{q}^{2m+\frac{4}{3}} - \chi^{\check{u}}_{m-3}\check{q}^{2m+2}}{(1-\check{q}^{\frac{4}{3}}\check{y}_1)(1-\check{q}^{\frac{4}{3}}\check{y}_2)(1-\check{q}^{\frac{4}{3}}\check{y}_3)} \tag{5.32}$$

The index of the free tensor multiplet (3.12) is obtained by setting $m = 1$, and the single-particle index of Kaluza-Klein modes (3.22) is obtained by summing up (5.32) over $m \in \mathbb{Z}_{\geq 1}$.

[2] For $\mathcal{D}[1,0]$ and $\mathcal{D}[2,0]$ we use the definitions $\chi^{\check{u}}_{-1} = 0$ and $\chi^{\check{u}}_{-2} = -1$.

For $m = 1$ (free tensor multiplet) and $m = 2$ (stress tensor multiplet) some Racah Speiser (RS) trial states have negative coefficients. They are interpreted as equations of motion and conservation laws.

Other D-type representations appearing in the expansion are

$$\begin{aligned}\mathcal{D}[0,4] &= \check{q}^8 - \chi_1^{\check{u}}\chi_{[0,1]}\check{q}^{\frac{26}{3}} + (\chi_{[0,2]} + \chi_{[1,0]} + \chi_2^{\check{u}}\chi_{[1,0]})\check{q}^{\frac{28}{3}} \\ &\quad - (\chi_1^{\check{u}} + \chi_3^{\check{u}} + 2\chi_1^{\check{u}}\chi_{[1,1]})\check{q}^{10} + \mathcal{O}(\check{q}^{\frac{32}{3}}), \end{aligned} \tag{5.33}$$

$$\mathcal{D}[1,4] = \chi_1^{\check{u}}\check{q}^{10} + \mathcal{O}(\check{q}^{\frac{32}{3}}), \tag{5.34}$$

$$\mathcal{D}[3,2] = \chi_3^{\check{u}}\check{q}^{10} + \mathcal{O}(\check{q}^{\frac{32}{3}}). \tag{5.35}$$

For $\mathcal{D}[0,4]$ and $\mathcal{D}[1,4]$ the RS procedure works well, and we obtain no RS trial weights with negative coefficients. For $\mathcal{D}[3,2]$ we obtain many weights with negative coefficients. In [55] it is proposed that such weights should be simply eliminated. However, we found that this procedure gives $\check{x}$-dependent result. Namely, the elimination spoils the Bose-Fermi degeneracy of states with $\check{\Delta} \neq 0$. Fortunately, the elimination affects terms of order $\check{q}^{12}$ or higher, and the lowest order of the $\check{x}$-dependent terms is $\check{q}^{\frac{38}{3}}$. Therefore, we expect the term shown in (5.35) is correct.

The B-type representation appearing in the expansion is

$$\mathcal{B}[2,0]_0 = \chi_2^{\check{u}}\chi_{[1,0]}\check{q}^{\frac{28}{3}} - (\chi_1^{\check{u}} + \chi_1^{\check{u}}\chi_{[1,1]} + \chi_3^{\check{u}}\chi_{[1,1]})\check{q}^{10} + \mathcal{O}(\check{q}^{\frac{32}{3}}). \tag{5.36}$$

For this representation we obtain many weights with negative coefficients. We again found that the elimination of them causes the $\check{x}$-dependence of the result. The elimination affects the terms of order $\check{q}^{10}$ or higher, and the $\check{x}$-dependence appears at $\check{q}^{\frac{32}{3}}$. (5.36) is the index after the elimination. Fortunately, terms shown in (5.36) do not depend on $\check{x}$.

We also calculated (5.36) in another way. For $n \geq 1$ the primary null state of $\mathcal{B}[2,0]_n$ appears at level $\ell = 1$, and the procedure is much simpler than the case of $n = 0$ for which the level of the primary null state is $\ell = 3$. The RS procedure works well for such representations and all generated weights have positive coefficients. To obtain $\mathcal{B}[2,0]_0$ we simply substitute $n = 0$ in the general formula for $n \geq 1$. Although we have no justification for this "continuation," this kind of continuation reproduces correct results in many cases. Indeed, we obtained the result whose first few terms agree with (5.36), and this strongly suggests the correctness of (5.36).

5.1.4 Schur-like index

As shown in (5.9) a generic representative configuration consists of M2-branes wrapped on two cycles $z_1 = 0$ and $z_2 = 0$. We can simplify the problem by taking a special limit in which only one of these two cycles, say, $z_1 = 0$, contributes to the index. For M2-branes wrapped on $z_2 = 0$ not to contribute to the index we need to tune the fugacities so that an extra supersymmetry which is broken by the M2-brane wrapped on $z_2 = 0$ is preserved by the definition of the index (3.8).

The single-particle index $i^{\rm M2}_{z_2=0}$ includes $-\check{q}^{\frac{2}{3}}\check{u}\chi_{[0,1]}(\check{y})$, which is the Weyl reflection of the second term in the numerator of (5.15), and it consists of three terms

$$-\check{q}^{\frac{2}{3}}\check{u}\chi_{[0,1]}(\check{y})=-\check{q}^{\frac{2}{3}}\check{u}\check{y}_1^{-1}-\check{q}^{\frac{2}{3}}\check{u}\check{y}_2^{-1}-\check{q}^{\frac{2}{3}}\check{u}\check{y}_3^{-1}. \tag{5.37}$$

These three terms correspond to Nambu-Goldstone fermions associated with the breaking of supersymmetry due to the presence of the wrapped brane. Let us focus on the first term corresponding to the supercharge $\check{\mathcal{Q}}'$ with the quantum numbers [3]

$$\check{\mathcal{Q}}':(\check{H},\check{J}_{12},\check{J}_{34},\check{J}_{56},\check{R}_{12},\check{R}_{34})=(+\tfrac{1}{2},-\tfrac{1}{2},+\tfrac{1}{2},+\tfrac{1}{2},+\tfrac{1}{2},-\tfrac{1}{2}). \tag{5.38}$$

To make the definition of the index (3.8) respect this supercharge we impose the following condition on the fugacities.

$$\check{q}^{\frac{2}{3}}\check{u}\check{y}_1^{-1}=1. \tag{5.39}$$

Then the first term in (5.37) becomes -1, and its plethystic exponential vanishes. As a result, only configurations consisting of M2-branes wrapped on $z_1=0$ contribute to the index. We adopt the following parametrization of fugacities satisfying (5.39) (and $\check{y}_1\check{y}_2\check{y}_3=1$).

$$\check{q}=\check{q}'\check{x}',\quad \check{y}_1=\check{q}'^{\frac{2}{3}}\check{x}'^{-\frac{4}{3}},\quad \check{y}_2=\check{q}'^{-\frac{1}{3}}\check{x}'^{\frac{2}{3}}\check{y},\quad \check{y}_3=\check{q}'^{-\frac{1}{3}}\check{x}'^{\frac{2}{3}}\check{y}^{-1},\quad \check{u}=\check{x}'^{-2}. \tag{5.40}$$

New fugacities $\check{q}'$, $\check{x}'$, $\check{y}$ are unconstrained variables. With this specialization the index (3.8) becomes

$$\widetilde{\mathcal{I}}(\check{q}',\check{y})={\rm tr}[(-1)^F\check{x}^{\check{\Delta}}\check{x}'^{\check{\Delta}'}\check{q}'^{\check{H}+\check{J}_{12}}\check{y}^{\check{J}_{34}-\check{J}_{56}}], \tag{5.41}$$

where

$$\check{\Delta}'=\{\check{\mathcal{Q}}',\check{\mathcal{Q}}'^{\dagger}\}=\check{H}-(\check{J}_{12}-\check{J}_{34}-\check{J}_{56})-2(\check{R}_{12}-\check{R}_{34}). \tag{5.42}$$

(5.41) is nothing but the Schur-like index studied in [56]. [4] In fact, the analytic result of the index for M5-brane theories was obtained from five-dimensional $U(N)$ SYM [42, 56]:

$$\begin{aligned}\widetilde{\mathcal{I}}^{(2,0)}_N&={\rm Pexp}\left[\frac{\check{q}'^2+\check{q}'^4+\cdots+\check{q}'^{2N}}{1-\check{q}'^2}\right]=\prod_{k=1}^{N}\prod_{m=0}^{\infty}\frac{1}{1-\check{q}'^{2(k+m)}}\\&=\widetilde{\mathcal{I}}^{(2,0)}_{N=\infty}\prod_{k=0}^{\infty}\prod_{m=0}^{\infty}(1-\check{q}'^{2N}\check{q}'^{2(k+m+1)}).\end{aligned} \tag{5.43}$$

[3]The $\mathbb{Z}_k$ symmetry (2.56) acts on the first two terms and the last term in different ways and this causes inequality between the third one and the others. We should not take the third term to define the Schur-like limit because the corresponding supercharge is non-perturbative in the sense that it is not manifest in the ABJM Lagrangian and is generated dynamically.

[4]The fugacities in this paper are related to those in [56] by $\check{q}'=q^{\frac{1}{2}}$ and $\check{y}=s$.

By expanding this with respect to $\check{q}'^{2N}$ we obtain

$$\widetilde{\mathcal{I}}_N^{(2,0)} = \widetilde{\mathcal{I}}_{N=\infty}^{(2,0)} \left(1 + \sum_{n=1}^{\infty} \check{q}'^{2nN} F_n(\check{q}')\right), \tag{5.44}$$

where $F_n(\check{q}')$ are rational functions of $\check{q}'$. The functions for $n = 1, 2, 3$ are

$$F_1(\check{q}') = \frac{-\check{q}'^2}{(1-\check{q}'^2)^2} = -\check{q}'^2 - 2\check{q}'^4 - 3\check{q}'^6 - \cdots, \tag{5.45}$$

$$F_2(\check{q}') = \frac{2\check{q}'^6}{(1-\check{q}'^2)^2(1-\check{q}'^4)^2} = 2\check{q}'^6 + 4\check{q}'^8 + 10\check{q}'^{10} + \cdots, \tag{5.46}$$

$$F_3(\check{q}') = \frac{-\check{q}'^{10} - 4\check{q}'^{12} - \check{q}'^{14}}{(1-\check{q}'^2)^2(1-\check{q}'^4)^2(1-\check{q}'^6)^2} = -\check{q}'^{10} - 6\check{q}'^{12} - 14\check{q}'^{14} - \cdots. \tag{5.47}$$

Let us compare (5.44) with the hypothetical relation (5.5), which reduces in the Schur-like limit to the following relation:

$$\widetilde{\mathcal{I}}_N^{(2,0)}(\check{q}', \check{y}) = \widetilde{\mathcal{I}}_{\rm KK} \left(1 + \sum_{n=1}^{\infty} \check{q}'^{2nN} \widetilde{\mathcal{I}}_n^{\rm M2}(\check{q}', \check{y})\right), \tag{5.48}$$

where $\widetilde{\mathcal{I}}_n^{\rm M2}$ is the Schur-like index of the theory realized on a stack of n M2-branes wrapped around the cycle $z_1 = 0$. The agreement in the large-N limit is easily confirmed:

$$\widetilde{\mathcal{I}}_{N=\infty}^{(2,0)} = \mathrm{Pexp}\, \frac{\check{q}'^2}{(1-\check{q}'^2)^2} = \widetilde{\mathcal{I}}_{\rm KK}. \tag{5.49}$$

The agreement of finite-N corrections requires

$$\widetilde{\mathcal{I}}_n^{\rm M2}(\check{q}', \check{y}) = F_n(\check{q}') \quad n = 1, 2, 3, \ldots. \tag{5.50}$$

For $n = 1$, the single-wrapping contribution, we can easily confirm (5.50) by using the Schur-like limit of $i_{z_1=0}^{\rm M2}$ in (5.15)

$$\widetilde{i}_{z_1=0}^{\rm M2} = \frac{1}{\check{q}'^2} + \check{q}'^2. \tag{5.51}$$

For $n \geq 2$ we expect that F_n is the index of the ABJM theory realized on $\boldsymbol{S}^2 \subset \boldsymbol{S}^4$. It is straightforward to write down the integral form if the index. A non-trivial point is how we should choose the integration contours. Although at present, we have not completely understood it we found that with a certain prescription we can reproduce the first few terms in F_2 and F_3. See the discussion below for details.

Detail of Schur like index

Let us discuss the Schur like index in more detail. The integral form giving the Schur-like index of each monopole sector $\widetilde{\mathcal{I}}_{m_\alpha,\widetilde{m}_\alpha}$ is obtained from (2.30) by setting $k=1$ and the variable change

$$\hat{q}=\check{q}'\check{x}'^{2},\quad \hat{u}_1=\check{q}'^{\frac{3}{2}}\check{x}'^{-1},\quad \hat{u}_2=\check{q}'^{\frac{1}{2}}\check{x}'\check{y},\quad \hat{u}_3=\check{q}'^{\frac{1}{2}}\check{x}'\check{y}^{-1},\quad \hat{u}_4=\check{q}'^{-\frac{5}{2}}\check{x}'^{-1}. \tag{5.52}$$

These are compositions of (5.14) and (5.40). The single-particle index (2.31) reduces to

$$\begin{aligned}\widetilde{i}(\check{q}',\check{x}';\zeta_\alpha,\widetilde{\zeta}_\alpha)=&-\sum_{\alpha\neq\beta}\hat{q}^{|m_\alpha-m_\beta|}\frac{\zeta_\alpha}{\zeta_\beta}-\sum_{\alpha\neq\beta}\hat{q}^{|\widetilde{m}_\alpha-\widetilde{m}_\beta|}\frac{\widetilde{\zeta}_\alpha}{\widetilde{\zeta}_\beta}\\&+\sum_{\alpha,\beta=1}^{N}\hat{q}^{|m_\alpha-\widetilde{m}_\beta|}\left(\check{q}'^{2}\frac{\zeta_\alpha}{\widetilde{\zeta}_\beta}+\check{q}'^{-2}\frac{\widetilde{\zeta}_\beta}{\zeta_\alpha}\right),\end{aligned} \tag{5.53}$$

where we leave $\hat{q}$ to keep the expression simple. As expected this is $\check{y}$-independent. Although the Schur-like index must be $\check{x}'$-independent the single-particle index depends on $\check{x}'$ through $\hat{q}=\check{q}'\check{x}'^{2}$. This is because the above formula is derived by deforming the Lagrangian by $\check{\mathcal{Q}}$-exact terms, which does not respect the extra supercharge $\check{\mathcal{Q}}'$ used in the definition of the Schur-like index.

If we regard $\widetilde{\mathcal{I}}_{m_a,\widetilde{m}_a}$ as a function of $\check{q}'$ and $\hat{q}$, we can easily factor out the $\check{q}'$-dependence by the replacement

$$\zeta_\alpha\to\check{q}'^{-1}\zeta_\alpha,\quad \widetilde{\zeta}_\alpha\to\check{q}'\widetilde{\zeta}_\alpha, \tag{5.54}$$

and obtain

$$\widetilde{\mathcal{I}}_{m_\alpha,\widetilde{m}_\alpha}=\check{q}'^{2m_{\rm tot}}\times(\text{function of }\hat{q}). \tag{5.55}$$

Furthermore, the $\check{x}'$-independence of the Schur-like index guarantees that the function of $\hat{q}$ is in fact a $\hat{q}$-independent constant.

In order to carry out the gauge fugacity integrals, we need to choose integration contours. Although we have not yet completely understood how we should do it, we found a prescription that reproduces the known results after some trial and error. We express the integrand as the expansion

$$\sum_{k=0}^{\infty}\check{q}'^{k}\sum_{l}\check{x}'^{l}f_{k,l}(\zeta_\alpha,\widetilde{\zeta}_\alpha). \tag{5.56}$$

Namely, we first expand the integrand with respect to $\check{q}'$, and then expand the result with respect to $\check{x}'$. The coefficients $f_{k,l}$ are Laurant polynomials of the gauge fugacities. The integration over gauge fugacities is equivalent to picking up the terms independent of gauge fugacities form each $f_{k,l}$. For each monopole sector the gauge integral leaves only terms of the form $\check{q}'^{2m_{\rm tot}}\check{x}'^{0}$. We confirmed that by the summation over monopole sectors the first few terms of F_2 and F_3 shown in (5.46) and (5.47) are reproduced.

5.2 Finite N corrections to the indices of the 6d $\mathcal{N} = (1,0)$ theories

In this section, we calculate the superconformal indices of 6d $\mathcal{N} = (1,0)$ theories by generalizing the method discussed in the previous section. For the $N = 1$ case, we will confirm the validity of our formula by comparing with the free theory results (3.37) and (3.39). For the $N = 2$ case, we will check the non-trivial flavor symmetries of 6d (1,0) theories from the superconformal indices. We also calculate the $N = 3$ indices, which give a new prediction of the indices of 6d $\mathcal{N} = (1,0)$ theories. This section is based on the author's and his collaborator's original work [13].

5.2.1 Conjectural formula

Based on the idea explained in the previous section we propose the formula of the index for 6d $(1,0)$ theories

$$\mathcal{I}_{N,k}^{(1,0)} = \mathcal{I}^{\text{bulk}} \sum_{n_1,n_2=0}^{\infty} \mathcal{I}_{(n_1,n_2)}^{\text{M2}}. \tag{5.57}$$

This formula gives the index as the combination of contributions from objects in the dual geometry $AdS_7 \times S^4/\mathbb{Z}_k$, where the internal space $\boldsymbol{S}^4/\mathbb{Z}_k$ is defined by

$$|z_1|^2 + |z_2|^2 + x_5^2 = 1, \tag{5.58}$$

together with the identification by the $\mathbb{Z}_k$ action (3.26). $\mathcal{I}^{\text{bulk}}$ is the contribution of Kaluza-Klein modes in the bulk. We also include in $\mathcal{I}^{\text{bulk}}$ the contribution from localized modes at the fixed points of the orbifold. $\mathcal{I}_{(n_1,n_2)}^{\text{M2}}$ are contributions of wrapped M2-branes in the internal space. n_1 and n_2 are numbers of M2-brane wrapped on the two specific two-cycles $z_1 = 0$ and $z_2 = 0$, respectively. We show the schematic figure of these objects in Figure 5.2.

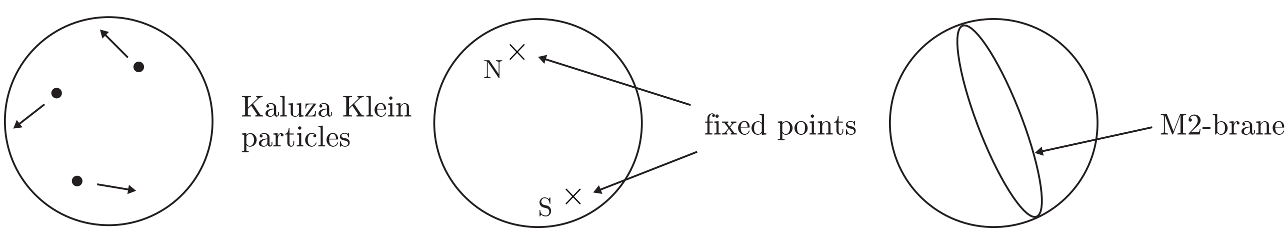

Figure 5.2: The objects contributing to the indices at finite-N are shown. The left figure shows the Kaluza Klein particles in $S^4/\mathbb{Z}_k$ and the middle figure shows the contribution of localized modes at the poles. These two contributions give the large-N index. The right figure shows the M2-brane wrapped on a non-trivial cycle in $S^4/\mathbb{Z}_k$, which gives the finite-N corrections to the indices.

As we will explain later, the q expansion of $\mathcal{I}_{(n_1,n_2)}$ starts from order $q^{2(n_1+n_2)N}$ terms. In the large-N limit all contributions but $\mathcal{I}_{(0,0)} = 1$ decouple and the formula reduces to $\mathcal{I}^{(1,0)}_{N=\infty,k} = \mathcal{I}^{\rm bulk}$. On the other hand, if N is finite, all sectors labeled by (n_1, n_2) contribute to the index. $\mathcal{I}_{(n_1,n_2)}$ for each (n_1, n_2) is calculated as the index of the theory realized on the wrapped M2-branes by the standard localization formula. If $n_1+n_2 \geq 2$ the formula includes non-trivial gauge integrals, and unfortunately, we have not yet found systematic rules for the integration contours. For this reason we leave the analysis of $n_1 + n_2 \geq 2$ for future work and in this paper we focus only on the single-wrapping sectors $(n_1, n_2) = (1, 0)$ and $(0, 1)$. Namely, we consider the formula

$$\mathcal{I}^{(1,0)}_{N,k} = \mathcal{I}^{\rm grav}_{N,k} + \mathcal{O}(q^{4N}), \tag{5.59}$$

where $\mathcal{I}^{\rm grav}_{N,k}$ is defined by

$$\mathcal{I}^{\rm grav}_{N,k} = \mathcal{I}^{\rm bulk}\left(1 + \mathcal{I}^{\rm M2}_{(1,0)} + \mathcal{I}^{\rm M2}_{(0,1)}\right). \tag{5.60}$$

With the conjectural formula (5.59), we can calculate the index for an arbitrary N and k up to the expected error of order q^{4N}.

5.2.2 Wrapped M2-branes on $S^4/\mathbb{Z}_k$

Next, we consider the contribution of M2-branes. Again the worldvolume of a BPS M2-brane is described by the intersection of $\boldsymbol{S}^4$ and a holomorphic surface [49, 57]

$$f(z_1, z_2) = 0. \tag{5.61}$$

The consistency with the $\mathbb{Z}_k$ orbifolding require the function f to satisfy

$$f(e^{2\pi i/k}z_1, e^{-2\pi i/k}z_2) = e^{2\pi i w/k} f(z_1, z_2), \tag{5.62}$$

where $w \in \mathbb{Z}/k\mathbb{Z}$ is the topological wrapping number.

In the previous section, we discussed the system without $\mathbb{Z}_k$ orbifolding and proposed that we can take only M2-brane configurations given by monomials of the form $f(z_1, z_2) = z_1^{n_1} z_2^{n_2}$ and showed that the formula passes some non-trivial checks. Let us adopt the same assumption. The function $f(z_1, z_2) = z_1^{n_1} z_2^{n_2}$ gives the system with n_1 M2-branes wrapped on $z_1 = 0$ and n_2 M2-branes wrapped on $z_2 = 0$. The total topological wrapping number is $w = n_1 - n_2 \mod k$. $\mathcal{I}^{\rm M2}_{(n_1,n_2)}$ in (5.57) is the contribution from the specific wrapping sector with (n_1, n_2). We focus on the two sectors $(1, 0)$ and $(0, 1)$. In the absence of the $\mathbb{Z}_k$ orbifolding, the contribution of the $(1, 0)$ sector, a single M2-brane wrapped on $z_1 = 0$, is

$$(q^2 u)^N \,{\rm Pexp}\, i^{\rm M2}_{z_1=0}, \tag{5.63}$$

with the single-particle index $i^{\rm M2}_{z_1=0}$ given by

$$i^{\rm M2}_{z_1=0} = \frac{q^{-2}u^{-1} - q^{\frac{2}{3}}u^{-1}\chi^y_{[0,1]} + q^{\frac{4}{3}}\chi^y_{[1,0]} - q^4}{1 - q^2 u^{-1}}. \tag{5.64}$$

To obtain the index for the $\mathbb{Z}_k$ orbifold, we need two modifications. First, we perform the $\mathbb{Z}_k$ projection on the single-particle index. Second, we insert the character of the $SU(k)_a \times SU(k)_b$ bi-fundamental representation because the wrapped M2-brane couples to the localized vector multiplets at the fixed points $(z_1, z_2, x_5) = (0, 0, \pm 1)$. As the result, the contribution of the $(1, 0)$ sector is given by

$$\mathcal{I}^{\text{M2}}_{(1,0)} = (q^2 u)^N \chi^a_{\text{fund.}} \chi^b_{\overline{\text{fund.}}} \operatorname{Pexp}\left[\mathcal{P}_k i^{\text{M2}}_{z_1=0}\right], \tag{5.65}$$

where $\chi^{a/b}_{\text{fund.}}$ ($\chi^{a/b}_{\overline{\text{fund.}}}$) are characters of (anti-) fundamental representations of the $SU(k)_{a/b}$ symmetries. The contribution of the other sector $(0, 1)$ is given from (5.65) by the replacement $u \to u^{-1}$, $a_i \to a_i^{-1}$, and $b_i \to b_i^{-1}$. $u \to u^{-1}$ is the Weyl reflection of $SO(5)_R$ exchanging z_1 and z_2. The inversion of a_i and b_i are necessary because the cycle $z_2 = 0$ has the opposite topological wrapping number to the cycle $z_1 = 0$; the former has $w = +1$ while the latter has $w = -1$. After the replacement we obtain

$$\mathcal{I}^{\text{M2}}_{(0,1)} = (q^2 u^{-1})^N \chi^a_{\overline{\text{fund.}}} \chi^b_{\text{fund.}} \operatorname{Pexp}\left[\mathcal{P}_k i^{\text{M2}}_{z_2=0}\right], \tag{5.66}$$

where $i^{\text{M2}}_{z_2=0} = i^{\text{M2}}_{z_1=0}|_{u \to u^{-1}}$.

If $k = 2$ the flavor characters appearing in (5.65) and (5.66) are the same, $\chi^a_{\text{fund.}} \chi^b_{\overline{\text{fund.}}} = \chi^a_{\overline{\text{fund.}}} \chi^b_{\text{fund.}}$, and $\mathcal{I}^{\text{M2}}_{(1,0)} + \mathcal{I}^{\text{M2}}_{(0,1)}$ is invariant under the $SU(2)_R$ Weyl reflection $u \to u^{-1}$. This is consistent with the symmetry enhancement $U(1)_F \to SU(2)_F$.

5.2.3 Results and consistency check

Now we are ready to calculate the index of the 6d $(1, 0)$ theories by using our formula (5.60) for different values of k and N.

$N = 1$

Let us compare the index on the gravity side based on the formula (5.60) with the free theory result from (3.35). As we do not include the multiple-wrapping M2-branes, the errors should start at q^4 terms and we check the agreement up to errors of this order.

Let us first consider the $k = 2$ case. On the gravity side (5.60) yields

$$\mathcal{I}^{\text{grav}}_{N=1,k=2} = 1 + \chi^a_{[1]} \chi^b_{[1]} \chi^u_{[1]} q^2 + \chi^a_{[1]} \chi^b_{[1]} \chi^u_{[1]} \chi^y_{[1,0]} q^{\frac{10}{3}} + (\chi^a_{[2]} + \chi^b_{[2]} + \chi^u_{[2]}) q^4 + \mathcal{O}(q^{\frac{14}{3}}). \tag{5.67}$$

If we compare this result with (3.37), we find

$$\mathcal{I}^{(1,0)}_{N=1,k=2} - \mathcal{I}^{\text{gr}}_{N=1,k=2} = \chi^a_{[2]} \chi^b_{[2]} \chi^u_{[2]} q^4 + \mathcal{O}(q^{\frac{14}{3}}). \tag{5.68}$$

We can see the agreement up to the error of $\mathcal{O}(q^4)$ as expected.

Then, we consider the $k = 3$ case. For $k = 3$ the result on the gravity side is

$$\begin{aligned}\mathcal{I}^{\text{grav}}_{N=1,k=3} = 1 &+ \left(u \chi^a_{[1,0]} \chi^b_{[0,1]} + u^{-1} \chi^a_{[0,1]} \chi^b_{[1,0]}\right) q^2 + \left(u \chi^a_{[1,0]} \chi^b_{[0,1]} \chi^y_{[1,0]} + u^{-1} \chi^a_{[0,1]} \chi^b_{[1,0]} \chi^y_{[1,0]}\right) q^{\frac{10}{3}} \\ &+ \left(1 + \chi^a_{[1,1]} + \chi^b_{[1,1]} + u^2 \chi^a_{[0,1]} \chi^b_{[1,0]} + u^{-2} \chi^a_{[1,0]} \chi^b_{[0,1]}\right) q^4 + \mathcal{O}(q^{\frac{14}{3}}).\end{aligned} \tag{5.69}$$

Comparing the result with the field theory calculation (3.39), we obtain

$$\mathcal{I}^{(1,0)}_{N=1,k=3} - \mathcal{I}^{\rm grav}_{N=1,k=3} = \left(u^2\chi^a_{[2,0]}\chi^b_{[0,2]} + u^{-2}\chi^a_{[0,2]}\chi^b_{[2,0]} + \chi^a_{[1,1]}\chi^b_{[1,1]}\right)q^4 + \mathcal{O}(q^{\frac{14}{3}}). \tag{5.70}$$

Again we can find agreement up to the expected error terms of order q^4.

$N = 2$

When $N = 2$, the generic flavor symmetry $SU(k)_a \times SU(k)_b \times U(1)$ is enhanced to $SU(2k)$ for $k \geq 3$ and $SO(7)$ for $k = 2$ [46, 58, 45]. Then, the indices should be written in terms of the characters of the enhanced symmetries. Let us confirm this for $k = 2$ and $k = 3$. The expected errors due to double-wrapping configurations are of order q^8, and we show the results below the order.

We consider the $k = N = 2$ case first. In this case the index should be written in terms of the $SO(7)$ character $\chi^{SO(7)}_{[l_1,l_2;l_3]}$. The last component of the Dynkin labels corresponds to the short root. The formula (5.59) gives

$$\begin{aligned}\mathcal{I}^{\rm grav}_{N=k=2} &= 1 + \chi^{SO(7)}_{[0,1;0]}q^4 + (1+\chi^{SO(7)}_{[0,1;0]})\chi^y_{[1,0]}q^{\frac{16}{3}} \\ &\quad + \left((1+\chi^{SO(7)}_{[0,1;0]})\chi^y_{[2,0]} + (1-\chi^{SO(7)}_{[1,0;0]})\chi^y_{[0,1]}\right)q^{\frac{20}{3}} + \mathcal{O}(q^8).\end{aligned} \tag{5.71}$$

This is correctly expanded in terms of $SO(7)$ characters. We also confirm that it is not written in terms of characters of $SO(8)$, the symmetry of the corresponding quiver gauge theory.

Next we consider the $k = 3$ and $N = 2$ case. The expected flavor symmetry is $SU(6)$. The formula (5.59) gives

$$\begin{aligned}\mathcal{I}^{\rm grav}_{N=2,k=3} &= 1 + \chi^{SU(6)}_{[1,0,0,0,1]}q^4 + (1+\chi^{SU(6)}_{[1,0,0,0,1]})\chi^y_{[1,0]}q^{\frac{16}{3}} + \chi^{SU(6)}_{[0,0,1,0,0]}q^6 \\ &\quad + \left(1+\chi^{SU(6)}_{[1,0,0,0,1]}\right)\chi^y_{[2,0]}q^{\frac{20}{3}} + \chi^{SU(6)}_{[0,0,1,0,0]}\chi^y_{[1,0]}q^{\frac{22}{3}} + \mathcal{O}(q^8),\end{aligned} \tag{5.72}$$

and this is correctly written in terms of $SU(6)$ characters.

$N = 3$

Here, we show the results for $N = 3$ calculated on the gravity side. Because we do not have results we can compare, we give the results simply as predictions. The expected errors are of order q^{12}, and we show the results below the order.

We first consider the $N = 3$ and $k = 2$ case. The global symmetry for $N = 3$ and $k = 2$

is $G_{\text{flavor}} = SU(2)_a \times SU(2)_b \times SU(2)_F$. The formula (5.59) gives

$$\begin{aligned}
\mathcal{I}^{\text{grav}}_{N=3,k=2} &= 1 + (\chi^a_{[2]} + \chi^b_{[2]} + \chi^u_{[2]})q^4 + (\chi^a_{[2]}\chi^y_{[1,0]} + \chi^b_{[2]}\chi^y_{[1,0]} + \chi^u_{[2]}\chi^y_{[1,0]} + \chi^y_{[1,0]})q^{\frac{16}{3}} \\
&+ \chi^a_{[1]}\chi^b_{[1]}\chi^u_{[3]}q^6 + (\chi^y_{[0,1]} + \chi^a_{[2]}\chi^y_{[2,0]} + \chi^b_{[2]}\chi^y_{[2,0]} + \chi^y_{[2,0]} + \chi^u_{[2]}(\chi^y_{[2,0]} - \chi^y_{[0,1]}))q^{\frac{20}{3}} \\
&+ \chi^a_{[1]}\chi^b_{[1]}\chi^u_{[3]}\chi^y_{[1,0]}q^{\frac{22}{3}} + (\chi^a_{[4]} + \chi^a_{[2]}\chi^b_{[2]} + \chi^b_{[4]} + 2\chi^u_{[4]} + \chi^y_{[1,1]} + \chi^a_{[2]}\chi^y_{[3,0]} + \chi^b_{[2]}\chi^y_{[3,0]} + \chi^y_{[3,0]} \\
&\qquad + \chi^u_{[2]}(\chi^a_{[2]} + \chi^b_{[2]} - \chi^y_{[1,1]} + \chi^y_{[3,0]} - 1) + 2)q^8 \\
&+ (\chi^a_{[1]}\chi^b_{[1]}\chi^u_{[3]}\chi^y_{[2,0]} - \chi^a_{[1]}\chi^b_{[1]}\chi^u_{[1]}\chi^y_{[0,1]})q^{\frac{26}{3}} \\
&+ (2\chi^a_{[2]}\chi^y_{[1,0]} + \chi^a_{[4]}\chi^y_{[1,0]} + 2\chi^a_{[2]}\chi^b_{[2]}\chi^y_{[1,0]} + 2\chi^b_{[2]}\chi^y_{[1,0]} + \chi^b_{[4]}\chi^y_{[1,0]} + 2\chi^u_{[4]}\chi^y_{[1,0]} \\
&\quad + 2\chi^y_{[1,0]} + \chi^y_{[2,1]} + \chi^a_{[2]}\chi^y_{[4,0]} + \chi^b_{[2]}\chi^y_{[4,0]} + \chi^y_{[4,0]} \\
&\quad + \chi^u_{[2]}(2\chi^a_{[2]}\chi^y_{[1,0]} + 2\chi^b_{[2]}\chi^y_{[1,0]} + 2\chi^y_{[1,0]} - \chi^y_{[2,1]} + \chi^y_{[4,0]}))q^{\frac{28}{3}} \\
&+ (\chi^a_{[1]}\chi^b_{[1]}\chi^u_{5} - \chi^a_{[1]}\chi^b_{[1]}\chi^u_{[1]}\chi^y_{[1,1]} + \chi^u_{[3]}(2\chi^a_{[1]}\chi^b_{[1]} + \chi^a_{[3]}\chi^b_{[1]} + \chi^a_{[1]}\chi^y_{[3,0]}\chi^b_{[1]} + \chi^a_{[1]}\chi^b_{[3]}))q^{10} \\
&+ (3\chi^a_{[2]}\chi^y_{[0,1]} + \chi^a_{[2]}\chi^b_{[2]}\chi^y_{[0,1]} + 3\chi^b_{[2]}\chi^y_{[0,1]} - \chi^y_{[0,1]} + 3\chi^a_{[2]}\chi^y_{[2,0]} + 2\chi^a_{[4]}\chi^y_{[2,0]} \\
&\quad + 3\chi^a_{[2]}\chi^b_{[2]}\chi^y_{[2,0]} + 3\chi^b_{[2]}\chi^y_{[2,0]} + 2\chi^b_{[4]}\chi^y_{[2,0]} + 6\chi^y_{[2,0]} + \chi^u_{[4]}(3\chi^y_{[2,0]} - 2\chi^y_{[0,1]}) \\
&\quad + \chi^y_{[3,1]} + \chi^a_{[2]}\chi^y_{[5,0]} + \chi^b_{[2]}\chi^y_{[5,0]} + \chi^y_{[5,0]} + \chi^u_{[2]}(3\chi^y_{[0,1]} + 3\chi^a_{[2]}\chi^y_{[2,0]} + 3\chi^b_{[2]}\chi^y_{[2,0]} \\
&\quad + 3\chi^y_{[2,0]} - \chi^y_{[3,1]} + \chi^y_{[5,0]}))q^{\frac{32}{3}} \\
&+ (2\chi^a_{[1]}\chi^b_{[1]}\chi^u_{[5]}\chi^y_{[1,0]} + \chi^u_{[1]}(2\chi^a_{[1]}\chi^b_{[1]}\chi^y_{[1,0]} - \chi^a_{[1]}\chi^b_{[1]}\chi^y_{[2,1]}) + \chi^u_{[3]}(6\chi^a_{[1]}\chi^b_{[1]}\chi^y_{[1,0]} \\
&\quad + 2\chi^a_{[3]}\chi^b_{[1]}\chi^y_{[1,0]} + 2\chi^a_{[1]}\chi^b_{[3]}\chi^y_{[1,0]} + \chi^a_{[1]}\chi^b_{[1]}\chi^y_{[4,0]}))q^{\frac{34}{3}} + \mathcal{O}\left(q^{12}\right).
\end{aligned} \tag{5.73}$$

Next we consider the $N=3$ and $k=3$ case. The global symmetry is $G_{\text{flavor}} = SU(3)_a \times SU(3)_b \times U(1)_F$. The formula (5.59) gives

$$
\begin{aligned}
\mathcal{I}^{\text{grav}}_{N=3,k=3} = {} & 1 + (\chi^a_{[1,1]} + \chi^b_{[1,1]} + 1)q^4 + (\chi^a_{[1,1]}\chi^y_{[1,0]} + \chi^b_{[1,1]}\chi^y_{[1,0]} + 2\chi^y_{[1,0]})q^{\frac{16}{3}} \\
& + (u^3\chi^a_{[1,0]}\chi^b_{[0,1]} + u^3 + u^{-3}\chi^a_{[0,1]}\chi^b_{[1,0]} + u^{-3})q^6 + (\chi^a_{[1,1]}\chi^y_{[2,0]} + \chi^b_{[1,1]}\chi^y_{[2,0]} + 2\chi^y_{[2,0]})q^{\frac{20}{3}} \\
& + (u^3\chi^a_{[1,0]}\chi^b_{[0,1]}\chi^y_{[1,0]} + u^3\chi^y_{[1,0]} + u^{-3}\chi^a_{[0,1]}\chi^b_{[1,0]}\chi^y_{[1,0]} + u^{-3}\chi^y_{[1,0]})q^{\frac{22}{3}} \\
& + (\chi^a_{[1,1]}\chi^b_{[1,1]} + \chi^a_{[1,1]}\chi^y_{[3,0]} + 2\chi^a_{[1,1]} + \chi^a_{[2,2]} + \chi^a_{[1,0]}\chi^b_{[0,1]} \\
& \qquad + \chi^a_{[0,1]}\chi^b_{[1,0]} + 2\chi^b_{[1,1]} + \chi^b_{[2,2]} + \chi^b_{[1,1]}\chi^y_{[3,0]} + 2\chi^y_{[3,0]} + 2)q^8 \\
& + (-u^3\chi^y_{[0,1]} + u^3\chi^a_{[1,0]}\chi^b_{[0,1]}\chi^y_{[2,0]} + u^3\chi^y_{[2,0]} - u^{-3}\chi^y_{[0,1]} + u^{-3}\chi^a_{[0,1]}\chi^b_{[1,0]}\chi^y_{[2,0]} + u^{-3}\chi^y_{[2,0]})q^{\frac{26}{3}} \\
& + (\chi^a_{[0,3]}\chi^y_{[1,0]} + 5\chi^a_{[1,1]}\chi^y_{[1,0]} + \chi^a_{[2,2]}\chi^y_{[1,0]} + \chi^a_{[3,0]}\chi^y_{[1,0]} + \chi^a_{[1,0]}\chi^b_{[0,1]}\chi^y_{[1,0]} + \chi^b_{[0,3]}\chi^y_{[1,0]} \\
& \qquad + \chi^a_{[0,1]}\chi^b_{[1,0]}\chi^y_{[1,0]} + 2\chi^a_{[1,1]}\chi^b_{[1,1]}\chi^y_{[1,0]} + 5\chi^b_{[1,1]}\chi^y_{[1,0]} + \chi^b_{[2,2]}\chi^y_{[1,0]} + \chi^b_{[3,0]}\chi^y_{[1,0]} \\
& \qquad + 4\chi^y_{[1,0]} + \chi^a_{[1,1]}\chi^y_{[4,0]} + \chi^b_{[1,1]}\chi^y_{[4,0]} + 2\chi^y_{[4,0]})q^{\frac{28}{3}} \\
& + (u^3\chi^a_{[1,1]} + u^3\chi^a_{[0,2]}\chi^b_{[0,1]} + 2u^3\chi^a_{[1,0]}\chi^b_{[0,1]} + u^3\chi^a_{[2,1]}\chi^b_{[0,1]} + u^3\chi^a_{[0,1]}\chi^b_{[1,0]} + u^3\chi^b_{[1,1]} \\
& \qquad + u^3\chi^a_{[1,0]}\chi^b_{[1,2]} + u^3\chi^a_{[1,0]}\chi^b_{[2,0]} - u^3\chi^y_{[1,1]} + u^3\chi^a_{[1,0]}\chi^b_{[0,1]}\chi^y_{[3,0]} + u^3\chi^y_{[3,0]} + u^3 \\
& \qquad + u^{-3}\chi^a_{[1,1]} + u^{-3}\chi^a_{[1,0]}\chi^b_{[0,1]} + u^{-3}\chi^a_{[0,1]}\chi^b_{[0,2]} + 2u^{-3}\chi^a_{[0,1]}\chi^b_{[1,0]} + u^{-3}\chi^a_{[1,2]}\chi^b_{[1,0]} \\
& \qquad + u^{-3}\chi^a_{[2,0]}\chi^b_{[1,0]} + u^{-3}\chi^b_{[1,1]} + u^{-3}\chi^a_{[0,1]}\chi^b_{[2,1]} - u^{-3}\chi^y_{[1,1]} + u^{-3}\chi^a_{[0,1]}\chi^b_{[1,0]}\chi^y_{[3,0]} \\
& \qquad + u^{-3}\chi^y_{[3,0]} + u^{-3})q^{10} \\
& + (\chi^a_{[0,3]}\chi^y_{[0,1]} + 3\chi^a_{[1,1]}\chi^y_{[0,1]} + \chi^a_{[3,0]}\chi^y_{[0,1]} - \chi^a_{[1,0]}\chi^b_{[0,1]}\chi^y_{[0,1]} + \chi^b_{[0,3]}\chi^y_{[0,1]} - \chi^a_{[0,1]}\chi^b_{[1,0]}\chi^y_{[0,1]} \\
& \qquad + \chi^a_{[1,1]}\chi^b_{[1,1]}\chi^y_{[0,1]} + 3\chi^b_{[1,1]}\chi^y_{[0,1]} + \chi^b_{[3,0]}\chi^y_{[0,1]} + \chi^y_{[0,1]} + \chi^a_{[0,3]}\chi^y_{[2,0]} + 8\chi^a_{[1,1]}\chi^y_{[2,0]} \\
& \qquad + 2\chi^a_{[2,2]}\chi^y_{[2,0]} + \chi^a_{[3,0]}\chi^y_{[2,0]} + \chi^a_{[1,0]}\chi^b_{[0,1]}\chi^y_{[2,0]} + \chi^b_{[0,3]}\chi^y_{[2,0]} + \chi^a_{[0,1]}\chi^b_{[1,0]}\chi^y_{[2,0]} \\
& \qquad + 3\chi^a_{[1,1]}\chi^b_{[1,1]}\chi^y_{[2,0]} + 8\chi^b_{[1,1]}\chi^y_{[2,0]} + 2\chi^b_{[2,2]}\chi^y_{[2,0]} + \chi^b_{[3,0]}\chi^y_{[2,0]} + 9\chi^y_{[2,0]} + \chi^a_{[1,1]}\chi^y_{[5,0]} \\
& \qquad + \chi^b_{[1,1]}\chi^y_{[5,0]} + 2\chi^y_{[5,0]})q^{\frac{32}{3}} \\
& + (2u^3\chi^a_{[1,1]}\chi^y_{[1,0]} + 2u^3\chi^a_{[0,2]}\chi^b_{[0,1]}\chi^y_{[1,0]} + 6u^3\chi^a_{[1,0]}\chi^b_{[0,1]}\chi^y_{[1,0]} + 2u^3\chi^a_{[2,1]}\chi^b_{[0,1]}\chi^y_{[1,0]} \\
& \qquad + u^3\chi^a_{[0,1]}\chi^b_{[1,0]}\chi^y_{[1,0]} + 2u^3\chi^b_{[1,1]}\chi^y_{[1,0]} + 2u^3\chi^a_{[1,0]}\chi^b_{[1,2]}\chi^y_{[1,0]} + 2u^3\chi^a_{[1,0]}\chi^b_{[2,0]}\chi^y_{[1,0]} \\
& \qquad + 4u^3\chi^y_{[1,0]} - u^3\chi^y_{[2,1]} + u^3\chi^a_{[1,0]}\chi^b_{[0,1]}\chi^y_{[4,0]} + u^3\chi^y_{[4,0]} + 2u^{-3}\chi^a_{[1,1]}\chi^y_{[1,0]} \\
& \qquad + u^{-3}\chi^a_{[1,0]}\chi^b_{[0,1]}\chi^y_{[1,0]} + 2u^{-3}\chi^a_{[0,1]}\chi^b_{[0,2]}\chi^y_{[1,0]} + 6u^{-3}\chi^a_{[0,1]}\chi^b_{[1,0]}\chi^y_{[1,0]} + 2u^{-3}\chi^a_{[1,2]}\chi^b_{[1,0]}\chi^y_{[1,0]} \\
& \qquad + 2u^{-3}\chi^a_{[2,0]}\chi^b_{[1,0]}\chi^y_{[1,0]} + 2u^{-3}\chi^b_{[1,1]}\chi^y_{[1,0]} + 2u^{-3}\chi^a_{[0,1]}\chi^b_{[2,1]}\chi^y_{[1,0]} + 4u^{-3}\chi^y_{[1,0]} \\
& \qquad - u^{-3}\chi^y_{[2,1]} + u^{-3}\chi^a_{[0,1]}\chi^b_{[1,0]}\chi^y_{[4,0]} + u^{-3}\chi^y_{[4,0]})q^{\frac{34}{3}} + \mathcal{O}\left(q^{12}\right).
\end{aligned}
\tag{5.74}
$$

Summary of Chapter 5

In this chapter, we studied a method of calculating the superconformal index of the M5-brane theories from the dual gravity theory in the finite-N region. We first analyzed the index of the 6d $\mathcal{N} = (2,0)$ theory and proposed the formula (5.5). The formula includes the wrapped M2-brane contribution as the finite-N corrections. For $N = 1$, we checked the validity of our formula by comparing the results obtained via the formula with the free theory result (3.15). For $N \geq 2$, we proposed the new results of the indices of the A_{N-1} theories (5.23) $\sim$ (5.26) and decomposed them in terms of the superconformal representations.

We also analyzed the superconformal index of the 6d $\mathcal{N} = (1,0)$ theory and proposed the formula (5.57). Again, for the (1,0) theory, the finite N corrections are given by the contribution of the wrapped M5-branes. For $N = 1$, we compared our results via the formula with the free theory results (3.37) and (3.39) and found nice refinement. For $N = 2$, we explicitly checked our results exhibit expected flavor symmetries. We also proposed the indices of the $N = 3$ case by using our formula.

Chapter 6

Conclusions and discussion

In this thesis, we investigated a new method of calculating the superconformal indices from dual gravity theories at finite-N. Especially, we calculated the finite-N corrections to the superconformal indices for theories realized on M2-/M5- branes. The significance of our formula is the inclusion of the contribution of M5-/M2- branes as the finite-N corrections.

In Chapter 4, we studied the indices of M2-brane theories from dual M-theory on $\mathrm{AdS}_4 \times S^7$. We compared the results from our formula (4.13) with the ABJM indices and found a nice agreement. Let us review the results in more detail. The large-N index of Kaluza Klein modes and finite-N ABJM indices differ from $\hat{q}^{\frac{N+1}{2}}$ order. Then, by using the equation (4.13) and including the single M5-brane contribution, we found the error appearing at $\hat{q}^{\frac{2N+6}{2}}$ order. Thus, we have an excellent refinement of the indices. The error is due to the lack of multiple wrapping M5-branes which we did not take into account. These terms should be restored by introducing such multiple wrapping branes, but it is beyond the scope of this thesis.

We also analyzed the $\mathbb{Z}_k$ orbifold case corresponding to ABJM theories with Chern-Simons level k. We classified sectors with baryonic charge $B \in \mathbb{Z}_k$ and calculated the indices for each sector from the dual gravity side. In this thesis, we explicitly analyzed $k = 2, 3$ cases.

The $k = 2$ case has two sectors $B = 0, 1$. For the $B = 0$ sector, the difference between the large-N Kaluza Klein index and finite-N ABJM indices appear at $\hat{q}^{\frac{2N+2}{2}}$ order. This is thought to be a contribution of double wrapping M5-branes and there is no single M5-brane contribution. For the $B = 1$ sector, we computed the single M5-brane contribution to the indices. The leading contribution starts at $\hat{q}^{\frac{N}{2}}$, meaning that there is no tachyonic shift, and we found agreement with ABJM indices up to $\hat{q}^{\frac{3N+6}{2}}$. The error is thought to be triple wrapping M5-branes contribution we ignored.

On the other hand, the $k = 3$ case has $B = 0, \pm 1$ sectors. For the $B = 0$ sector, again the difference between the Kaluza Klein index and the ABJM indices appear at $\hat{q}^{\frac{2N+2}{2}}$ order and there are no single M5-brane contributions. For the $B = \pm 1$ sectors, since these two sectors are related by the charge conjugation, we only analyzed the $B = +1$ sector. Again, we calculated the single M5-brane contribution and found agreement up to $\hat{q}^{\frac{2N+2}{2}}$, which is thought to be double wrapping M5-branes contribution.

Although all these M2-brane indices are already known results, we succeeded in showing the correctness of our formula for all examples.

In Chapter 5, we studied the indices of M5-brane theories from dual M-theory on $\mathrm{AdS}_7 \times S^4$. Especially, we propose new results of the indices of 6d $\mathcal{N} = (2,0)$ theories from our formula (5.5). Detailed results are as follows. For the 6d case, the difference between the Kaluza Klein index and the 6d index appears at $\check{q}^{2(N+1)}$ order. (We only checked this for $N = 1$, since we can calculate the 6d (2,0) indices only for the $N = 1$ case.) Then, introducing the single M2-brane contribution via the formula (5.5), for $N = 1$, we found the agreement of the indices on both duality sides up to $\check{q}^{10}$. It seems that generally the error appears at $\check{q}^{2(2N+3)}$, and this is due to the contribution of multiple M2-branes we did not take into account. For $N > 1$, by removing the free tensor multiplet contribution, we calculated the indices of A_{N-1} theories from our formula. Since, except for the highly complicated calculation from the 5d SYM theory, there has been no way to calculate the indices of 6d $\mathcal{N} = (2,0)$ theories with $N > 1$, our formula proposes new results of the indices of the A_{N-1} theories. We also rewrote the indices of the $\mathcal{A}_{N-1}$ theories in terms of the superconformal representations and found that it is consistent with the bootstrap analysis. In addition, we analyze the Schur index of the 6d $\mathcal{N} = (2,0)$ theory.

Also, we analyzed $\mathbb{Z}_k$ orbifold case corresponding to the 6d $\mathcal{N} = (1,0)$ theory. The gravity calculation consists of three parts; the Kaluza Klein modes, the localized vector multiplets at the poles, and the wrapped M2-branes, which give finite-N corrections to the indices. For the $N = 1$ case, we checked our formula reproduce the index of the free theory up to $\check{q}^4$. For the $N = 2$ case, we confirmed that our indices reproduced the non-trivial flavor symmetries studied in [46, 45]. We also propose new results of the indices of 6d $\mathcal{N} = (1,0)$ theories for the $N = 3$ case.

There are many future works. First, it is an important task to calculate the multiple wrapping M5-/M2- branes contribution and reproduce higher-order terms of the indices. To calculate the multiple wrapping contributions, not only the contribution of multiple wrapping branes itself, but also we have to consider intersection modes on the branes.

Fortunately for the indices of the M5-brane theories, we can calculate the indices of multiple wrapping M2-brane, by using the ABJM index formula together with fugacities change. However, now we have no idea about the analysis of the intersection modes.

On the other hand, for the indices of the M2-brane theories, the situation is even worse. We have no way to calculate the multiple M5-branes contributions to the indices since we have no analytic formula for the indices for the 6d $\mathcal{N} = (2,0)$ theories. Itwould be very nice if we could solve this problem and reproduce the higher-order terms of the indices.

Second, a similar analysis has been done recently by [59, 60]. They reformulated the superconformal indices of some 4-dimensional superconformal field theories and M2-brane theories by using a method called a determinant modification. Further, inspired by their results, the indices of M5-brane theories were calculated in [61]. To compare their results with our results, careful treatment for the analytic continuation is required. It is also an important problem to discuss the relation between our formula and the determinant modification method.

Finally, an application to BPS black holes may be possible. Since the superconformal index counts BPS states in the theory, it is natural to expect that the index reproduces the entropy of the dual BPS black hole. Recently many works towards this problem have been done starting from [62, 63] and the entropies of five-dimensional asymptotically AdS black holes were reproduced from the superconformal indices of dual four-dimensional field theories. Further, analyses in [64, 65] show that the q-expansion form of the index is enough to capture the behavior of the entropy of the dual BPS black hole. It is an interesting task to calculate the higher-order terms of the index of the 6d $\mathcal{N} = (2, 0)$ theory and to analyze the dual BPS AdS_7 black hole [66, 67].

Appendix A

Full expressions of the indices

A.1 $k = 1$

We show the full expression of the superconformal indices from $\mathrm{AdS}_4 \times S^7$. The index is written in $su(4)$ characters.

$$\begin{aligned}\mathcal{I}^{\text{grav}}_{N=1} = (\cdots \text{terms identical to } (2.34) \cdots) \\ + (\chi_{[0,0,4]} + \chi_{[0,2,0]} + \chi_{[0,4,0]} + \chi_{[0,6,0]} - \chi_{[1,0,1]} + \chi_{[2,0,2]} + 2\chi_{[2,1,0]} \\ + \chi_{[2,2,2]} + \chi_{[4,0,0]} + \chi_{[4,2,0]} + \chi_{[4,4,0]} - \chi_{[5,0,1]} + \chi_{[6,0,2]} + 2\chi_{[8,0,0]} \\ + \chi_{[8,2,0]} + \chi_{[12,0,0]} - 1)\hat{q}^4 + \mathcal{O}(\hat{q}^{\frac{9}{2}}).\end{aligned} \tag{A.1}$$

$$\begin{aligned}\mathcal{I}^{\text{grav}}_{N=2} = (\cdots \text{terms identical to } (2.35) \cdots) \\ + (-\chi_{[0,0,2]} - 6\chi_{[0,1,0]} + \chi_{[0,2,2]} + 2\chi_{[0,3,0]} + \chi_{[0,4,2]} - 2\chi_{[1,3,1]} - 8\chi_{[2,0,0]} \\ + \chi_{[2,0,4]} + 2\chi_{[2,1,2]} + 5\chi_{[2,2,0]} + 3\chi_{[2,4,0]} + \chi_{[2,6,0]} + 9\chi_{[3,0,1]} - 4\chi_{[3,2,1]} \\ + 3\chi_{[4,0,2]} + 7\chi_{[4,1,0]} + \chi_{[4,2,2]} + 2\chi_{[4,3,0]} - 6\chi_{[5,1,1]} - 6\chi_{[6,0,0]} + 5\chi_{[6,2,0]} \\ + \chi_{[6,4,0]} - 8\chi_{[7,0,1]} + \chi_{[8,0,2]} + 4\chi_{[8,1,0]} + 7\chi_{[10,0,0]} + \chi_{[10,2,0]} + \chi_{[14,0,0]})\hat{q}^5 \\ + \mathcal{O}(\hat{q}^{\frac{11}{2}}).\end{aligned} \tag{A.2}$$

$$\begin{aligned}\mathcal{I}^{\text{grav}}_{N=3} = (\cdots \text{terms identical with } (2.36) \cdots) \\ + (\chi_{[0,0,4]} - 6\chi_{[0,1,2]} - 13\chi_{[0,2,0]} + \chi_{[0,2,4]} + \chi_{[0,3,2]} - 3\chi_{[0,4,0]} + 8\chi_{[0,6,0]} + \chi_{[0,8,0]} \\ - 10\chi_{[1,0,1]} + 17\chi_{[1,2,1]} - 13\chi_{[1,4,1]} - 3\chi_{[2,0,2]} - 11\chi_{[2,1,0]} + 7\chi_{[2,2,2]} + 2\chi_{[2,3,0]} \\ + \chi_{[2,4,2]} + 12\chi_{[2,5,0]} + 36\chi_{[3,1,1]} - 21\chi_{[3,3,1]} - 4\chi_{[4,0,0]} + \chi_{[4,0,4]} + 5\chi_{[4,1,2]} \\ - 11\chi_{[4,2,0]} + 24\chi_{[4,4,0]} + \chi_{[4,6,0]} + 42\chi_{[5,0,1]} - 30\chi_{[5,2,1]} + 8\chi_{[6,0,2]} - 18\chi_{[6,1,0]} \\ + \chi_{[6,2,2]} + 25\chi_{[6,3,0]} - 27\chi_{[7,1,1]} - 40\chi_{[8,0,0]} + 31\chi_{[8,2,0]} + \chi_{[8,4,0]} - 23\chi_{[9,0,1]} \\ + \chi_{[10,0,2]} + 23\chi_{[10,1,0]} + 20\chi_{[12,0,0]} + \chi_{[12,2,0]} + \chi_{[16,0,0]} + 1)\hat{q}^6 + \mathcal{O}(\hat{q}^{\frac{13}{2}}).\end{aligned} \tag{A.3}$$

A.2 $k = 2$

In the case of $k = 2$ the system still has $\mathcal{N} = 8$ supersymmetry, and the index can be expanded in terms of $su(4)$ characters.

$$\mathcal{I}^{\text{ABJM}(0/2)}_{N=1} = 1 + \chi_{[2,0,0]}\hat{q} + (-1 - \chi_{[1,0,1]} + \chi_{[4,0,0]})\hat{q}^2 + \mathcal{O}(\hat{q}^3). \tag{A.4}$$

$$\begin{aligned}\mathcal{I}^{\text{ABJM}(0/2)}_{N=2} &= 1 + \chi_{[2,0,0]}\hat{q} + (\chi_{[0,2,0]} - \chi_{[1,0,1]} + 2\chi_{[4,0,0]})\hat{q}^2 \\ &+ (-\chi_{[1,1,1]} - \chi_{[2,0,0]} + \chi_{[2,2,0]} - 2\chi_{[3,0,1]} + \chi_{[4,1,0]} + 2\chi_{[6,0,0]})\hat{q}^3 + \mathcal{O}(\hat{q}^4).\end{aligned} \tag{A.5}$$

$$\begin{aligned}\mathcal{I}^{\text{ABJM}(0/2)}_{N=3} &= 1 + \chi_{[2,0,0]}\hat{q} + (\chi_{[0,2,0]} - \chi_{[1,0,1]} + 2\chi_{[4,0,0]})\hat{q}^2 \\ &+ (\chi_{[0,0,2]} - \chi_{[1,1,1]} + 2\chi_{[2,2,0]} - 2\chi_{[3,0,1]} + \chi_{[4,1,0]} + 3\chi_{[6,0,0]})\hat{q}^3 \\ &+ (-1 - \chi_{[0,2,0]} + 2\chi_{[0,4,0]} - 2\chi_{[1,2,1]} + \chi_{[2,0,2]} + \chi_{[2,3,0]} - 2\chi_{[3,1,1]} \\ &\quad - 2\chi_{[4,0,0]} + 4\chi_{[4,2,0]} - 4\chi_{[5,0,1]} + 2\chi_{[6,1,0]} + 4\chi_{[8,0,0]})\hat{q}^4 + \mathcal{O}(\hat{q}^{\frac{9}{2}}).\end{aligned} \tag{A.6}$$

$$\begin{aligned}\mathcal{I}^{\text{ABJM}(1/2)}_{N=1} &= \chi_{[1,0,0]}\hat{q}^{\frac{1}{2}} + (-\chi_{[0,0,1]} + \chi_{[3,0,0]})\hat{q}^{\frac{3}{2}} + (-\chi_{[2,0,1]} + \chi_{[5,0,0]})\hat{q}^{\frac{5}{2}} \\ &+ (2\chi_{[1,1,0]} - \chi_{[4,0,1]} + \chi_{[7,0,0]})\hat{q}^{\frac{7}{2}} \\ &+ (-\chi_{[0,1,1]} - 3\chi_{[1,0,0]} - \chi_{[2,0,1]} + 2\chi_{[3,1,0]} - \chi_{[6,0,1]} + \chi_{[9,0,0]})\hat{q}^{\frac{9}{2}} + \mathcal{O}(\hat{q}^{\frac{11}{2}}).\end{aligned} \tag{A.7}$$

$$\begin{aligned}\mathcal{I}^{\text{grav}(1/2)}_{N=1} &= (\cdots \text{terms identical with (A.7)} \cdots) \\ &+ (-\chi_{[0,1,1]} - \chi_{[0,3,1]} - 3\chi_{[1,0,0]} - \chi_{[2,0,1]} - \chi_{[3,0,2]} \\ &\quad + 2\chi_{[3,1,0]} - \chi_{[3,3,0]} - \chi_{[5,2,0]} - \chi_{[6,0,1]}])\hat{q}^{\frac{9}{2}} + \mathcal{O}(\hat{q}^{\frac{11}{2}}).\end{aligned} \tag{A.8}$$

$$\begin{aligned}\mathcal{I}^{\text{ABJM}(1/2)}_{N=2} &= \chi_{[2,0,0]}\hat{q} + (-\chi_{[1,0,1]} + \chi_{[2,1,0]} + \chi_{[4,0,0]})\hat{q}^2 \\ &+ (-\chi_{[1,1,1]} - \chi_{[2,0,0]} + \chi_{[2,2,0]} - 2\chi_{[3,0,1]} + \chi_{[4,1,0]} + 2\chi_{[6,0,0]})\hat{q}^3 \\ &+ (\chi_{[0,2,0]} + 2\chi_{[1,0,1]} - \chi_{[1,2,1]} + \chi_{[2,0,2]} + \chi_{[2,1,0]} + \chi_{[2,3,0]} - 2\chi_{[3,1,1]} \\ &\quad - 2\chi_{[4,0,0]} + \chi_{[4,2,0]} - 3\chi_{[5,0,1]} + 2\chi_{[6,1,0]} + 2\chi_{[8,0,0]})\hat{q}^4 \\ &+ (-\chi_{[0,0,2]} - 2\chi_{[0,1,0]} + \chi_{[0,3,0]} + \chi_{[1,1,1]} - \chi_{[1,3,1]} - 3\chi_{[2,0,0]} + \chi_{[2,1,2]} \\ &\quad + 2\chi_{[2,2,0]} + \chi_{[2,4,0]} + 4\chi_{[3,0,1]} - 2\chi_{[3,2,1]} + \chi_{[4,0,2]} + 3\chi_{[4,1,0]} + \chi_{[4,3,0]} \\ &\quad - 3\chi_{[5,1,1]} - 3\chi_{[6,0,0]} + 2\chi_{[6,2,0]} - 4\chi_{[7,0,1]} + 2\chi_{[8,1,0]} + 3\chi_{[10,0,0]})\hat{q}^5 \\ &+ (-\chi_{[0,1,2]} - 2\chi_{[0,2,0]} + \chi_{[0,4,0]} - \chi_{[1,2,1]} - \chi_{[1,4,1]} - 3\chi_{[2,0,2]} - 9\chi_{[2,1,0]} \\ &\quad + \chi_{[2,2,2]} + 2\chi_{[2,3,0]} + \chi_{[2,5,0]} - 2\chi_{[3,3,1]} - 7\chi_{[4,0,0]} + \chi_{[4,1,2]} + 4\chi_{[4,2,0]} \\ &\quad + \chi_{[4,4,0]} + 6\chi_{[5,0,1]} - 3\chi_{[5,2,1]} + 2\chi_{[6,0,2]} + 4\chi_{[6,1,0]} + 2\chi_{[6,3,0]} \\ &\quad - 4\chi_{[7,1,1]} - 4\chi_{[8,0,0]} + 2\chi_{[8,2,0]} - 5\chi_{[9,0,1]} + 3\chi_{[10,1,0]} + 3\chi_{[12,0,0]} + 1)\hat{q}^6 \\ &+ \mathcal{O}(\hat{q}^{\frac{13}{2}}).\end{aligned} \tag{A.9}$$

$$\begin{aligned}\mathcal{I}^{\text{grav}(1/2)}_{N=2} &= (\cdots\text{terms identical with (A.9)}\cdots)\\ &+(-\chi_{[0,0,4]}-\chi_{[0,1,2]}-2\chi_{[0,2,0]}+\chi_{[0,4,0]}-\chi_{[0,6,0]}-2\chi_{[1,2,1]}-\chi_{[1,4,1]}\\ &\quad-3\chi_{[2,0,2]}-9\chi_{[2,1,0]}+2\chi_{[2,3,0]}+\chi_{[2,5,0]}-3\chi_{[3,3,1]}-8\chi_{[4,0,0]}+4\chi_{[4,2,0]}\\ &\quad+6\chi_{[5,0,1]}-3\chi_{[5,2,1]}+\chi_{[6,0,2]}+4\chi_{[6,1,0]}+\chi_{[6,3,0]}-4\chi_{[7,1,1]}\\ &\quad-4\chi_{[8,0,0]}+\chi_{[8,2,0]}-5\chi_{[9,0,1]}+3\chi_{[10,1,0]}+2\chi_{[12,0,0]}+1)\hat{q}^{6}+\mathcal{O}(\hat{q}^{\frac{13}{2}}).\end{aligned}\tag{A.10}$$

$$\begin{aligned}\mathcal{I}^{\text{ABJM}(1/2)}_{N=3} &= \chi_{[3,0,0]}\hat{q}^{\frac{3}{2}}+(\chi_{[1,2,0]}-\chi_{[2,0,1]}+\chi_{[3,1,0]}+\chi_{[5,0,0]})\hat{q}^{\frac{5}{2}}\\ &+(-\chi_{[0,2,1]}-\chi_{[1,1,0]}+\chi_{[1,3,0]}-\chi_{[2,1,1]}-\chi_{[3,0,0]}\\ &\quad+2\chi_{[3,2,0]}-2\chi_{[4,0,1]}+2\chi_{[5,1,0]}+2\chi_{[7,0,0]})\hat{q}^{\frac{7}{2}}\\ &+(\chi_{[0,1,1]}-\chi_{[1,2,0]}+\chi_{[1,4,0]}+\chi_{[2,0,1]}-3\chi_{[2,2,1]}+\chi_{[3,0,2]}-2\chi_{[3,1,0]}\\ &\quad+3\chi_{[3,3,0]}-3\chi_{[4,1,1]}-3\chi_{[5,0,0]}+4\chi_{[5,2,0]}-3\chi_{[6,0,1]}+3\chi_{[7,1,0]}+3\chi_{[9,0,0]})\hat{q}^{\frac{9}{2}}\\ &+(\chi_{[0,0,1]}+2\chi_{[0,2,1]}-2\chi_{[0,4,1]}+\chi_{[1,0,2]}+\chi_{[1,1,0]}+\chi_{[1,2,2]}+2\chi_{[1,5,0]}\\ &\quad+5\chi_{[2,1,1]}-3\chi_{[2,3,1]}+\chi_{[3,1,2]}-2\chi_{[3,2,0]}+4\chi_{[3,4,0]}+4\chi_{[4,0,1]}\\ &\quad-6\chi_{[4,2,1]}+\chi_{[5,0,2]}-4\chi_{[5,1,0]}+5\chi_{[5,3,0]}-5\chi_{[6,1,1]}-6\chi_{[7,0,0]}\\ &\quad+6\chi_{[7,2,0]}-5\chi_{[8,0,1]}+5\chi_{[9,1,0]}+4\chi_{[11,0,0]})\hat{q}^{\frac{11}{2}}\\ &+(-\chi_{[0,0,3]}-5\chi_{[0,1,1]}+2\chi_{[0,3,1]}-\chi_{[0,5,1]}-2\chi_{[1,0,0]}-2\chi_{[1,1,2]}\\ &\quad-5\chi_{[1,2,0]}+\chi_{[1,3,2]}+3\chi_{[1,6,0]}-2\chi_{[2,0,1]}+7\chi_{[2,2,1]}-6\chi_{[2,4,1]}\\ &\quad-\chi_{[3,1,0]}+2\chi_{[3,2,2]}+5\chi_{[3,5,0]}+12\chi_{[4,1,1]}-7\chi_{[4,3,1]}+2\chi_{[5,1,2]}\\ &\quad-3\chi_{[5,2,0]}+7\chi_{[5,4,0]}+10\chi_{[6,0,1]}-10\chi_{[6,2,1]}+2\chi_{[7,0,2]}\\ &\quad-6\chi_{[7,1,0]}+8\chi_{[7,3,0]}-8\chi_{[8,1,1]}-10\chi_{[9,0,0]}+9\chi_{[9,2,0]}\\ &\quad-7\chi_{[10,0,1]}+7\chi_{[11,1,0]}+5\chi_{[13,0,0]})\hat{q}^{\frac{13}{2}}\\ &+(2\chi_{[0,0,1]}+3\chi_{[0,4,1]}-3\chi_{[0,6,1]}+\chi_{[1,0,2]}+3\chi_{[1,1,0]}-4\chi_{[1,2,2]}-9\chi_{[1,3,0]}\\ &\quad+\chi_{[1,4,2]}+\chi_{[1,5,0]}+3\chi_{[1,7,0]}-2\chi_{[2,0,3]}-13\chi_{[2,1,1]}+8\chi_{[2,3,1]}-6\chi_{[2,5,1]}\\ &\quad-4\chi_{[3,0,0]}-7\chi_{[3,1,2]}-17\chi_{[3,2,0]}+3\chi_{[3,3,2]}+\chi_{[3,4,0]}+7\chi_{[3,6,0]}-11\chi_{[4,0,1]}\\ &\quad+14\chi_{[4,2,1]}-12\chi_{[4,4,1]}-3\chi_{[5,0,2]}-10\chi_{[5,1,0]}+4\chi_{[5,2,2]}+\chi_{[5,3,0]}+9\chi_{[5,5,0]}\\ &\quad+21\chi_{[6,1,1]}-12\chi_{[6,3,1]}-2\chi_{[7,0,0]}+3\chi_{[7,1,2]}-4\chi_{[7,2,0]}+11\chi_{[7,4,0]}\\ &\quad+18\chi_{[8,0,1]}-15\chi_{[8,2,1]}+3\chi_{[9,0,2]}-9\chi_{[9,1,0]}+12\chi_{[9,3,0]}-12\chi_{[10,1,1]}\\ &\quad-15\chi_{[11,0,0]}+12\chi_{[11,2,0]}-9\chi_{[12,0,1]}+9\chi_{[13,1,0]}+7\chi_{[15,0,0]})\hat{q}^{\frac{15}{2}}+\mathcal{O}(\hat{q}^{8}).\end{aligned}\tag{A.11}$$

$$
\begin{aligned}
\mathcal{I}^{\text{grav}(1/2)}_{N=3} &= (\cdots\text{terms identical with (A.11)}\cdots) \\
&+ (2\chi_{[0,0,1]} - \chi_{[0,3,3]} + 3\chi_{[0,4,1]} - 3\chi_{[0,6,1]} + 3\chi_{[1,1,0]} - 4\chi_{[1,2,2]} \\
&\quad - 10\chi_{[1,3,0]} + \chi_{[1,5,0]} + 3\chi_{[1,7,0]} - 3\chi_{[2,0,3]} - 13\chi_{[2,1,1]} + 7\chi_{[2,3,1]} \\
&\quad - 7\chi_{[2,5,1]} - 4\chi_{[3,0,0]} - \chi_{[3,0,4]} - 8\chi_{[3,1,2]} - 18\chi_{[3,2,0]} + 2\chi_{[3,3,2]} \\
&\quad + \chi_{[3,4,0]} + 6\chi_{[3,6,0]} - 11\chi_{[4,0,1]} - \chi_{[4,1,3]} + 13\chi_{[4,2,1]} - 12\chi_{[4,4,1]} \\
&\quad - 3\chi_{[5,0,2]} - 11\chi_{[5,1,0]} + 2\chi_{[5,2,2]} + \chi_{[5,3,0]} + 8\chi_{[5,5,0]} + 21\chi_{[6,1,1]} \\
&\quad - 13\chi_{[6,3,1]} - 3\chi_{[7,0,0]} + 2\chi_{[7,1,2]} - 4\chi_{[7,2,0]} + 10\chi_{[7,4,0]} + 18\chi_{[8,0,1]} \\
&\quad - 15\chi_{[8,2,1]} + 2\chi_{[9,0,2]} - 9\chi_{[9,1,0]} + 11\chi_{[9,3,0]} - 12\chi_{[10,1,1]} - 15\chi_{[11,0,0]} \\
&\quad + 11\chi_{[11,2,0]} - 9\chi_{[12,0,1]} + 9\chi_{[13,1,0]} + 6\chi_{[15,0,0]})\hat{q}^{\frac{15}{2}} + \mathcal{O}(\hat{q}^{\frac{17}{2}}). \qquad \text{(A.12)}
\end{aligned}
$$

A.3 $k = 3$

$$
\begin{aligned}
\mathcal{I}^{\text{ABJM}(1/3)}_{N=1} &= u\chi_{1,0}\hat{q}^{\frac{1}{2}} + u^{-2}\chi_{0,2}\hat{q} + u(\chi_{2,1} - \chi_{0,1})\hat{q}^{\frac{3}{2}} \\
&+ \left(u^4\chi_{4,0} - u^{-2}\chi_{1,1} + u^{-2}\chi_{1,3}\right)\hat{q}^2 + \mathcal{O}(\hat{q}^{\frac{5}{2}}). \qquad \text{(A.13)}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{I}^{\text{grav}(1/3)}_{N=1} &= (\cdots\text{terms identical with (A.13)}\cdots) \\
&+ \left(u^4\chi_{4,0} - u^{-2}\chi_{1,1} + u^{-2}\chi_{1,3} - u^{-8}\chi_{0,2} - u^{-8}\chi_{0,6}\right)\hat{q}^2 + \mathcal{O}(\hat{q}^{\frac{5}{2}}). \qquad \text{(A.14)}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{I}^{\text{ABJM}(1/3)}_{N=2} &= u^2\chi_{2,0}\hat{q} + u^{-1}\chi_{1,2}\hat{q}^{\frac{3}{2}} + \left(u^2\chi_{3,1} + u^{-4}\chi_{0,4} + u^{-4}\right)\hat{q}^2 \\
&+ \left(u^5\chi_{3,0} + u^5\chi_{5,0} - u^{-1}\chi_{0,1} + 2u^{-1}\chi_{2,3}\right)\hat{q}^{\frac{5}{2}} \\
&+ \left(2u^{-4}\chi_{1,5} - u^2\chi_{2,0} - u^2\chi_{2,2} - u^2\chi_{4,0} + 3u^2\chi_{4,2} - u^2\right)\hat{q}^3 + \mathcal{O}(\hat{q}^{\frac{7}{2}}). \qquad \text{(A.15)}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{I}^{\text{grav}(1/3)}_{N=2} &= (\cdots\text{terms identical with (A.15)}\cdots) \\
&+ (2u^{-4}\chi_{1,5} - u^2\chi_{2,0} - u^2\chi_{2,2} - u^2\chi_{4,0} + 3u^2\chi_{4,2} - u^2 \\
&\quad - u^{-10}\chi_{0,4} - u^{-10}\chi_{0,8} - u^{-10})\hat{q}^3 + \mathcal{O}(\hat{q}^{\frac{7}{2}}). \qquad \text{(A.16)}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{I}^{\text{ABJM}(1/3)}_{N=3} &= u^3\chi_{3,0}\hat{q}^{\frac{3}{2}} + \chi_{2,2}\hat{q}^2 + \left(u^3\chi_{0,1} + u^3\chi_{4,1} + u^{-3}\chi_{1,0} + u^{-3}\chi_{1,4}\right)\hat{q}^{\frac{5}{2}} \\
&+ \left(u^6\chi_{2,0} + u^6\chi_{4,0} + u^6\chi_{6,0} + u^{-6}\chi_{0,2} + u^{-6}\chi_{0,6} + \chi_{1,3} + 2\chi_{3,3}\right)\hat{q}^3 \\
&+ \big(-u^3\chi_{1,0} - u^3\chi_{3,0} + u^3\chi_{3,2} - u^3\chi_{5,0} + 3u^3\chi_{5,2} \\
&\quad + u^{-3}\chi_{2,1} + u^{-3}\chi_{2,3} + 3u^{-3}\chi_{2,5}\big)\hat{q}^{\frac{7}{2}} \\
&+ \big(u^6\chi_{3,1} + u^6\chi_{5,1} + 2u^6\chi_{7,1} + 2u^{-6}\chi_{1,3} + u^{-6}\chi_{1,5} + 2u^{-6}\chi_{1,7} - 2\chi_{0,2} \\
&\quad + \chi_{0,4} - \chi_{2,0} - 3\chi_{2,2} + 2\chi_{4,0} + 5\chi_{4,4} - 1\big)\hat{q}^4 + \mathcal{O}(\hat{q}^{\frac{9}{2}}). \qquad \text{(A.17)}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{I}_{N=3}^{\text{grav}(1/3)} = &(\cdots \text{terms identical with (A.17)} \cdots) \\
&+ \big(u^6\chi_{3,1} + u^6\chi_{5,1} + 2u^6\chi_{7,1} + 2u^{-6}\chi_{1,3} + u^{-6}\chi_{1,5} + 2u^{-6}\chi_{1,7} - 2\chi_{0,2} \\
&\quad + \chi_{0,4} - \chi_{2,0} - 3\chi_{2,2} + 2\chi_{4,0} + 5\chi_{4,4} - 1 \\
&\quad -u^{-12}\chi_{0,2} - u^{-12}\chi_{0,6} - u^{-12}\chi_{0,10}\big)\,\hat{q}^4 + \mathcal{O}(\hat{q}^{\frac{9}{2}}).
\end{aligned} \tag{A.18}
$$

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