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Control of Self-Interested Agents in Noncooperative Dynamical Systems

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Abstract

Coordination between individual interests and social interests have become essential for studying multiagent systems in the future smart society. This thesis provides a line of work on control problems of self-interested agents in pseudo-gradient-based noncooperative dynamical systems. In the first part, we focus on developing several utility-transfer frameworks for pseudo-gradient-based noncooperative dynamical systems to remodel agents' dynamical decision-making. Specifically, a zero-sum tax/subsidy approach, a hierarchical incentive framework, and a Pareto-improving incentive mechanism are constructed to deal with the control problem in the face of agents' private information, large-scale system, and Pareto improvement. In the second part, we investigate the influence of psychological considerations in noncooperative systems, including the loss-aversion phenomena and the incorporation of cognitive predictions of agents into pseudo-gradient dynamics.

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Summary

Coordination between individual interests and social interests have become essential for studying multiagent systems in the future smart society. This thesis provides a line of work on control problems of self-interested agents in pseudo-gradient-based noncooperative dynamical systems. In the first part, we focus on developing several utility-transfer frameworks for pseudo-gradient-based noncooperative dynamical systems to remodel agents' dynamical decision-making. Specifically, a zero-sum tax/subsidy approach, a hierarchical incentive framework, and a Pareto-improving incentive mechanism are constructed to deal with the control problems in the face of agents' private information, large-scale system, and Pareto improvement. In the second part, we investigate the influence of psychological considerations in noncooperative systems, including the loss-aversion phenomena and the incorporation of cognitive predictions of agents into pseudo-gradient dynamics.

First of all, to deal with the control problem of noncooperative dynamical system where the sensitivity parameters of the pseudo-gradient dynamics are uncertain to the system manager, a zero-sum tax/subsidy approach is constructed to stabilize a possibly unstable Nash equilibrium. We first characterize the stability of the Nash equilibrium for *arbitrary* values of sensitivity and then investigate the zero-sum tax/subsidy framework without knowing the sensitivity parameters. In the proposed framework, the system manager defines the utility-transfer structure dividing the agents into subgroups so that the utility transfers are completed within the subgroups in a zero-sum and distributed manner. The amounts of tax (negative incentive) and subsidy (positive incentive) for each agent are determined by quadratic incentive functions with well-chosen control parameters.

For a noncooperative system with a large number of agents, the requirement for a single system manager to know all agents' payoff functions is extremely stringent. To handle this issue, in light of the hierarchical government structures in real society, we develop a hierarchical incentive framework for large-scale noncooperative dynamical systems to achieve social welfare improvement. In the proposed framework, the agents

in the noncooperative system are divided into several groups and are influenced by the corresponding group managers via some intra-group incentives. We characterize the situation where group managers try to enhance the welfare of their groups by continually updating their own intra-group incentives to the group members. We explore the stability of group Nash equilibrium of the hierarchical noncooperative systems and derive conditions where the trajectory of agents' states converges to the group Nash equilibrium under group managers' intra-group incentives. Furthermore, the inter-group incentive mechanism for a system governor is proposed to reconstruct the group utility functions at the group managers level to move the group Nash equilibrium so that the social (entire) welfare is improved. To deal with the situation where the system governor may not know all the agents' individual payoff functions and all the agents' states, we present sufficient conditions to guarantee the convergence of agents' states towards a target (suboptimal but not optimal due to the lack of enough information) equilibrium using some macroscopic data.

Usually, the constructed incentive mechanisms are designed as coercion policies under which the agents cannot escape once in place. However, the agents may have the freedom to break away from the mechanism when they come across some undesired situations (e.g., when their payoffs decrease after the mechanism is executed). To address this problem, we develop a Pareto-improving incentive mechanism to remodel agents' dynamical decision-making to guarantee that all the agents are Pareto improving and their state converges to a Pareto-efficient Nash equilibrium. Considering the priorities among the agents, we construct a weighted social welfare function for the incentive mechanism and hence derive the socially maximum state as the target Nash equilibrium. With the well-designed incentive functions associated with the weighted social welfare function, the socially maximum state is ensured to be a Pareto-efficient Nash equilibrium in the incentivized noncooperative system. Several sufficient stability conditions are presented to guarantee that the agents are Pareto improving under the pseudo-gradient dynamics and their state converges to the socially maximum state with known or unknown sensitivity parameters. We reveal the fact that the Pareto improvement and potentialization do not have an inclusive relation with each other.

In light of psychological game theory and cognitive hierarchy theory, the conventional pseudo-gradient dynamics with static sensitivity by ignoring all psychological considerations and predictions about the likely actions of other agents seem unnatural to describe agents' behavior in real society. We connect the phenomenon of loss-aversion in prospect theory with the pseudo-gradient dynamics and focus on the stability problem of 2-agent noncooperative switched systems, which are characterized as payoff-driven piecewise linear systems for describing agents' dynamic decision-making with the quadratic payoffs and loss-aversion phenomena. Based on the transition analysis and mode analysis, the sufficient and necessary conditions under which agents' state converge to the Nash equilibrium are derived in accordance with the location of the Nash equilibrium. In the analysis, we observe an interesting phenomenon that we call a flash switching instant where a single agent's sensitivity transition makes the other agent immediately switch its sensitivity almost at the same time instant.

Finally, we connect cognitive hierarchy theory with the pseudo-gradient dynamics in noncooperative systems to extend the pseudo-gradient dynamics with some prediction behaviors under Level-k thinking. The modified pseudo-gradient dynamics under Level-k thinking are presented according to the knowledge network of the payoff functions so that the agents are allowed to base their decisions on the predictions about the likely actions (best-response states) of other agents with a bounded depth of reasoning. To deal with the uncertainties on the knowledge network of the payoff functions and sensitivity parameters, we characterize stability property with arbitrary knowledge network of the payoff functions for the cases with a pure population of the agents in the same level and the mixed population of the agents in different levels. In addition, we present the applications of the results in optical communication systems, homogeneous oligopoly markets, and differentiated oligopoly markets. It is observed that to ensure asymptotic stability of the differentiated oligopoly markets with Cournot competition, a larger market with more firms requires more differentiated products. In contrast, this phenomenon does not happen in Bertrand competition.

Chapter 1

Introduction

1.1 Control Problem in Noncooperative Systems: The Needs of Incentive Design

Coordination issues between the individual interests and social interest have become essentially important for studying multiagent systems in the coming smart society. In order investigate the coordination issues, game theory has been used as one of the disciplines concerning the relations between human decision making and resulting phenomena as a whole [1, 2]. In noncooperative systems, each agent is presumed to be fully rational and selfish, and hence aims to increase its own payoff by adjusting its individual state in the system. Under this presupposition, Agents in the noncooperative systems mutually affect the selfish decision making of the other agents through the interconnected relations of their utilities or payoffs. Many applications are found in both engineering and economics, e.g., wireless sensor networks [1], communication channel allocation [3], signal interference avoidance [4], data security in intelligent transportation systems [5], electricity market [6], to name but a few.

It is common knowledge that in noncooperative systems, the agents' selfish decision making may degrade the social welfare [7, 8]. For example, the *tragedy of the commons* describes a social trap involving the conflict between the individual interests and the public interest in the allocation of resources [9]. In such a situation, without a person who is entitled to control the entire noncooperative system, every agent expands its demand independently according to his own self-interest, and the limited resources are destined to be over-exploited by the unrestricted demands, which eventually harms the common good of all agents in the common resource systems.

For the aggregation of such self-interested agents, it has turned out that the imposition of external policies or explicit incentive mechanisms changes agents' decision making tendencies and hence results in the endogenously cooperative behaviors in the noncooperative systems [10–12]. In such a case, the imposition of explicit incentive mechanisms is regarded as the control behavior for the noncooperative systems. For example, as a coercion policy which agents cannot escape once in place, a tax/subsidy approach was proposed by [13] to reward or penalize the deviations from the average contribution of the other competitors to the public goods. In contrast to the coercion policy, the authors in [14] investigated a compensation mechanism where agents are allowed to voluntarily subsidize the other agents in the pre-stage when the other agents' decisions are not made yet. The compensation mechanisms are understood as a liberal solution as agents have freedom to escape the mechanism. In usual, the liberal solution works as a weak external rule to the noncooperative system and is expected to be less efficient than the coercion solution.

In order to describe the state change of noncooperative systems, several models are proposed in the literature. Specifically, agents' dynamic decision behaviors are typically characterized by the best-response dynamics (or named as dynamic fictitious play) [15, 16] and myopic pseudo-gradient dynamics (or named as better-response dynamics, or dynamic gradient play) [17–19] for discrete-time and continuous-time systems, respectively. In the pseudo-gradient dynamics setup, the agents continuously change their state according to the pseudo-gradient projection onto their own local state space without having foresight. For example, the authors in [20] analyzed agents' behaviors in a noncooperative system with two agents and quadratic payoff functions. The authors in [21] investigated the agents' behaviors with a variable learning rate for the case where an agent wins (possesses higher utility than the opponent) in the two-agent noncooperative system. The paper [22] proposed a congestion control framework for data traffic with the pseudo-gradient dynamics for the users on the internet while [23] discussed the relationship between the positively invariant set and the set of positive externalities for a pseudo-gradient-based noncooperative system with two agents and quadratic payoffs. The related works of dynamic agents' behavior characterized by pseudo-gradient dynamics are found in [24–34] and the references therein, which include the applications of game theoretic approach inspired by the pseudo-gradient dynamics in communication networks, smart grid, pricing mechanisms, to name but a few.

To improve the social utility level, it is preferable to develop a compensation mechanism that collects taxes from some agents and gives subsidies to some other agents. Specifically, the authors in [35] modified agents' original payoff functions in order to reach the highest social welfare by adding a pricing term among the agents. For stabilizing minimum latency flows in the Braess graphs, [36] considered the capitation tax and subsidy. The authors in [37] imposed a subsidy mechanism to achieve stabilization for heterogeneous replicator dynamics. It is necessary to emphasize that in the above works the existence of a system manager is assumed and he/she is characterized as a resource owner or distributor who is able to give additional subsidies. However, the system manger in many economic applications serves merely as a mediator and does not have productivity to pay the additional profits to the agents. In such a case, every subsidy has to be financed by taxes taken from the others [38] and hence the tax/subsidy mechanism ought to be designed in a zero-sum fashion, e.g., [39].

During designing the incentive mechanism, there exist some important problems with respect to uncertainty, large-scale system, and Pareto improvement. First of all, the system manager, ideally, has all the knowledge about the noncooperative system including the payoff functions and the decision dynamics of the agents. In reality, it is often difficult to observe perfect information about the activities of the noncooperative agents. This hidden information is termed as private information in economics [40] and this uncertainty can be obstructive for designing the incentive mechanisms. Even though in the existing gradient-based Nash equilibrium seeking problems [35, 41–43], the seeking speed is predetermined, the rational agents in a noncooperative dynamical system in general change their states according to their own inherent sensitivities which may not be observed by the system manager. The work in [44] provided an explicit mechanism by side payments with the idea of transferring the utility in a two agent system, which induces cooperation and drives the noncooperative system to the socially maximum welfare state, but unfortunately, the case with more agents and the sensitivity parameters are not considered. Indeed, even though for a two-agent noncooperative system, the sensitivity parameters do not change the stability property of Nash equilibria [45], they may change the stability property in the system with more than two agents and bring agents' state to a worse utility state.

Secondly, for a noncooperative system with a large number of agents, the requirement for a single system manager knowing all agents' payoff functions is extremely stringent. To deal with this problem, hierarchical structures consisting of a system governor (e.g., president) and multiple managers (e.g., mayors) often exist in our society, where the agents are divided into several groups controlled by the corresponding group managers. In those structures, the system governor usually observe only limited information from each of the groups but the group mangers know more specific information in their own groups. In the literature, some hierarchical structures of incentive mechanisms or noncooperative systems can be found in [46–49]. For example, Ng *et al.* considered a two-level incentive mechanism design problem in [47] to mitigate the straggler effects in the federated learning training tasks. Mukaidani and Xu studied incentive Stackelberg games with multiple leaders and followers for a class of stochastic linear systems with external disturbance in [48], where several agents take the position as leaders and the rest of the agents take the position as followers so that the outcome of entire systems depends on the state of both the leaders and the followers. Alternatively, in the literature of economics, delegation games describe a different situation in which some principals choose a compensation scheme for their agents while the latter play a game on behalf of the principals [49]. In such a case, the payoffs of all players (i.e., principals and agents) are determined by the actions chosen by the agents. However, to our knowledge, the theoretical analysis of pseudo-gradient-based noncooperative dynamic systems with hierarchical incentives is not considered yet in the literature.

Thirdly, the constructed incentive mechanisms are often designed as coercion policies under which the agents cannot escape once in place. However, since the agents may have freedom to break away from the mechanism when the agents come across some undesired situations (e.g., when their payoffs decrease after the mechanism is executed), it is significant to develop incentive mechanisms enhancing the payoff values of *all* the agents at the same time guaranteeing Pareto improvements [50] under the imposed incentives. In such a case, it is essential to ensure that the desired state is Pareto efficient [51–53]. In the literature of economics, Pareto-efficient states capture the strategy profiles where no individual agent can be better off without making the others worse off by deviating from the characterized state [54–56] so that there is no space for further Pareto improvement. If the Nash equilibrium of the noncooperative system is not Pareto efficient, then there is still some room to increase the payoffs for some of the agents without decreasing any other agents' payoffs [57, 58]. In this case, some agents may seek private agreement (negotiation) with each other outside the incentive mechanism so that the incentive mechanism constructed by the system manager collapses. To avoid such a case, the incentive mechanism needs to guarantee Pareto efficiency on the target (desired) Nash equilibrium [59, 60].

Moreover, depending on the specific goal of the government, the government in real society usually gives more preferential treatments to some of the companies/individuals when the performance of those companies/individuals is crucial in achieving the government's goal. For example, tackling extreme poverty was set to be an essential policy goal by developing countries and hence their governments are likely to provide more resources (e.g., job opportunities or common resources) to the poorer people than the others for enhancing the poor people's lives. Another example is that industry-oriented countries have given more preferential treatments to the NEVs (new energy vehicles) companies to improve international competitiveness under the challenge of global climate issue [61]. Therefore, while designing the incentive mechanism, the system manager may evaluate the priority among the agents for constructing a social welfare function [62].

1.2 Psychological Consideration in Noncooperative Dynamical Systems

As mentioned in Section 1.1, the agents' selfish dynamic decision behaviors are typically modeled by the pseudo-gradient dynamics for continuous-time systems [17–19]. In such setup, agents' decision depends on the projection of the agents' payoff functions onto their own state proportioned by their own sensitivity parameters without having foresight. Some issues in pseudo-gradient dynamics are discussed for different scenarios. For example, the impact of quantized communication [24], leader-following consensus [42], augmented gradient-play dynamics [25], external disturbance [63], and redistributive side payments [26] were investigated.

However, psychological game theory shows by experimental research that it is inaccurate to simply assume that all the agents are fully rational and selfish because the agents may have some social and psychological considerations such as the influence of fairness, guilt aversion, hesitation, and inequality aversion in the decision making [64]. On the basis of various psychological considerations, agents make their decisions in significantly different ways [65, 66]. Therefore, the pseudo-gradient dynamics with static sensitivity by ignoring all psychological considerations seem unnatural to describe agents' behavior in the real society. To our knowledge, the paper [67] is the first work to characterize the pseudo-gradient dynamics with variable sensitivity. The essence of [67] is to consider the situation where each agent makes a decision quickly when losing and cautiously when winning in a two-agent iterated matrix game. However, for describing agents' different decision making when they are facing losses and gains, *loss-aversion* in cognitive psychology and decision theory [68] tells the completely opposite scenario, that is, agents' decision is more *cautious* when they are expected to lose utilities. In light of the difference between *loss-aversion* phenomena and psychological consideration in [67], it is significant to consider the pseudo-gradient dynamics under the loss-aversion scenario.

In the last decade, switched systems, which are characterized by a signal specifying the mode transition among a finite number of subsystems, have widely applied to numerous areas such as servomechanism systems [69], formation flying [70], stochastic systems [71], to name but a few. As the most important issues in control systems, stability properties of an equilibrium in such switched systems has been extensively characterized [72–76]. In terms of piecewise linear systems, Iwatani and Hara characterized the stability problem based on poles and zeros of the subsystems [77]. Nishiyama and Hayakawa provided a series of sufficient conditions to determine stability for 2-dimensional switched linear systems and piecewise nonlinear homogeneous systems [78–80]. An integral function approach based on normalized growth rate was formulated as a tool for judging whether the trajectory is coming closer to the equilibrium or not in [78]. In the above works, the triggers of mode transitions in the switched systems are usually understood as event-driven but the events are assumed to be independent of the systems' dynamics. The fundamental problems on stability and switching behaviors for the special class of switched systems with correlative dynamics and switching events (conditions) get few attentions.

On the other hand, on the basis of behavioural economics and cognitive hierarchy theory, it may be inaccurate to assume that the agents are simply myopic decision-makers without making any prediction or reasoning about the likely actions of other agents [81]. In fact the agents usually have the tend to estimate or predict how their opponents act in the noncooperative system based on the information of the other agents' payoff functions. Furthermore, more complicated behaviors may happen when agents are conscious of the opponent's estimation or prediction. This is because the agents' decision behavior may be totally different when they know or do not know whether the other agents' are making prediction of its own future state. The framework of those cognitive operations is referred to as Level-k framework in cognitive hierarchy theory.

Roughly speaking, the Level-k framework [82, 83] categorises the agents of the noncooperative systems into several types (levels) according to the depth of the agent's strategic thought (reasoning). First of all, Level-k framework begins with the first level called Level-0 in which agents make the decision non-strategically. Except for the agents in Level-0, each agent in Level-k firmly believes that he/she is the most sophisticated person in the system because all the other agents are in Level-$(k-1)$. For this reason, the agent make the decision according to some strategic reasoning of the other agents' likely actions. For example, Level-1 agents (e.g., the agents in conventional best-response dynamics or pseudo-gradient dynamics) make the decision strategically according to its own payoff functions and the other agents' current state because they firmly believe that their opponents are making non-strategic decisions [84]. Some related works in cognitive hierarchy theory, predictive control, nonequilibrium dynamic game and cyber-physical security are found in [85–88]. However, to our best knowledge, the theoretical analysis of noncooperative systems with pseudo-gradient dynamics under Level-k thinking is not considered yet in the literature.

1.3 Overview

The thesis is organized as follows. In Chapter 2, we develop a utility-transfer framework for pseudo-gradient-based noncooperative dynamical systems to remodel agents' dynamical decision making in the face of agents' private information. We assume that the sensitivity parameters in the pseudo-gradient dynamics are uncertain to the system manager. Under this uncertainty, the system manager is expected to construct a zero-sum tax/subsidy mechanism to (globally) stabilize a Nash equilibrium. In particular, we first present several sufficient conditions for guaranteeing stability of a possibly unstable Nash equilibrium in the face of uncertainty, and then we construct a zero-sum tax/subsidy incentive structure by collecting taxes from some agents and giving the same amount of subsidy in total to other agents so that the agents' payoff structure is properly modified.

In Chapter 3, we focus on the social welfare improvement problem for large-scale hierarchical noncooperative dynamical systems driven by the pseudo-gradient dynamics. A framework for hierarchical noncooperative systems with dynamic agents is proposed. In the characterized framework, agents in each group are incentivized by a corresponding group manager who represents the benefits of group utility via an intra-group incentive mechanism. Furthermore, to improve the social welfare of the entire system, we propose an inter-group incentive scheme in the group managers level for a system governor to bring agents' state to a target equilibrium. In this chapter, to deal with the uncertain information on agents' personal payoff functions for the system governor, sufficient conditions are presented to guarantee the convergence of agents' state to the target equilibrium.

In Chapter 4, we propose a Pareto-improving incentive mechanism to improve the weighted social welfare and achieve continual Pareto improvement for pseudo-gradient-based noncooperative dynamical systems. The proposed explicit incentive mechanism remodels agents' dynamical decision making for guaranteeing that all the agents are Pareto improving and their state converges to a Pareto-efficient Nash equilibrium. Similar to Chapter 2, we consider the situation where the system manager remodels agents' dynamical decision making by collecting taxes from some agents and giving some of the collected taxes to other agents as subsidies with a sustainable budget constraint. Sufficient conditions are derived under which agents' state converges towards the socially maximum state associated with a weighted social welfare function depending on the priority ratio of the agents and the initial state. We discuss the connection between Pareto improvement and potentialization and reveal the fact that the Pateto improvement and potentialization do not have an inclusive relation with each other.

In Chapter 5, we focus on the stability problem for 2-agent noncooperative switched systems, which are characterized as *payoff-driven* piecewise linear systems for describing agents' dynamic decision making with the quadratic payoffs and loss-aversion phenomena. In particular, we assume that each agent adopts lower sensitivity in the pseudo-gradient dynamics for the case of losing utility than gaining utility and hence both the system dynamics and the switching instants depend on the agents' payoff functions. Based on the transition analysis and mode analysis, the sufficient and necessary conditions under which agents' state converge to the Nash equilibrium are derived in accordance with the location of the Nash equilibrium. In the analysis, the mode transition sequence and interesting phenomena which we call flash switching are characterized. It is found that the loss-aversion behaviors may destabilize the Nash equilibrium. A sufficient condition of robust stability under which the loss-aversion behaviors never destabilize the Nash equilibrium for any sensitivity parameters is presented. The result indicates that by well defining (modifying) the agents' payoff functions, it is possible to avoid destabilization of a Nash equilibrium caused by the agents' loss-aversion consideration.

In Chapter 6, we we connect cognitive hierarchy theory with the pseudo-gradient dynamics in noncooperative systems to extend the pseudo-gradient dynamics with some prediction behaviors under Level-k thinking.In the characterized system, all the agents are allowed to base their decisions on the predictions about the likely actions (best-response states) of other agents with a bounded depth of reasoning. We suppose that those predictions are made according to the information of the payoff functions that the agents know from a knowledge network of the payoff functions. The modified pseudo-gradient dynamics under Level-k thinking are presented according to the knowledge network of the payoff functions. We suppose that the sensitivity parameters and the knowledge network of the payoff functions in the pseudo-gradient dynamics are uncertain to a system manager who wishes to ensure stabilization of a Nash equilibrium. To deal with the uncertainties, we first characterize stability property with arbitrary knowledge network of the payoff functions for the cases with pure population of the agents in the same level and mixed population of the agents in different levels, and then investigate a stabilization method via zero-sum tax/subsidy approach to ensure stability of a Nash equilibrium without using the information of sensitivity parameters nor the knowledge network of payoff functions. In addition, we present the applications of the results in optical communication systems, homogeneous oligopoly markets and differentiated oligopoly markets. It is observed that to ensure asymptotic stability of the differentiated oligopoly markets with Cournot competition, a larger market with more firms requires more differentiated products, whereas this phenomena does not happen in Bertrand competition.

1.4 Notations

We use the following notations in this thesis. We write $\mathbb{Z}_0$ for the set of nonnegative integers, $\mathbb{Z}_+$ for the set of positive integers, $\mathbb{Z}_\mathrm{o}$ for the set of positive odd integers, $\mathbb{Z}_\mathrm{e}$ for the set of positive even integers. $\mathbb{R}$ for the set of real numbers, $\mathbb{R}_+$ for the set of positive real numbers, $\mathbb{R}^n$ for the set of $n \times 1$ real column vectors, $\mathbb{R}^{n\times m}$ for the set of $n \times m$ real matrices, $\wedge$ for the logical conjunction, and $\vee$ for the logical disjunction. Moreover, $\det(\cdot)$ denotes determinant, $(\cdot)^\mathrm{T}$ denotes transpose, (block-)diag$[\cdot]$ denotes a (block-)diagonal matrix, $f'(\cdot)$ denotes the gradient of function $f(\cdot)$, I_n and 1_n denote the identity matrix and the ones vector of dimension n, respectively. Finally, $[\mathrm{row}_i(A)]$ denotes a matrix with entries same as ith row of matrix A, $\circ$ denotes Schur product, $\|x\| = \sqrt{x^\mathrm{T} x}$ denotes the Euclidean norm of a vector x, $\|A\|$ denotes the matrix norm of a matrix A, and $\mathrm{He}(\cdot)$ denotes the Hermitian part of a matrix.

Chapter 2

Control of Uncertain Noncooperative Dynamical Systems: A Tax/Subsidy Approach

2.1 Introduction

In this chapter, we develop a utility-transfer framework for pseudo-gradient-based noncooperative dynamical systems to remodel agents' dynamical decision making in the face of agents' private information. Specifically, we assume that that the sensitivity parameters in the pseudo-gradient dynamics are uncertain to the system manager. Under this uncertainty, the system manager is expected to construct a zero-sum tax/subsidy mechanism to (globally) stabilize a Nash equilibrium. To deal with the uncertainty, we first characterize the stability of the Nash equilibrium for *arbitrary* values of sensitivity and then investigate the zero-sum tax/subsidy framework without knowing the sensitivity parameters. In the proposed tax/subsidy approach, the system manager defines the utility-transfer structure dividing the agents into subgroups so that the utility transfers are completed within the subgroups in a zero-sum and distributed manner. The amounts of tax (negative incentive) and subsidy (positive incentive) for each agent are determined by quadratic incentive functions with well-chosen control parameters. It turns out from the numerical examples that the proposed framework can guarantee *global* asymptotic stabilizability for some noncooperative systems with non-quadratic payoff functions.

This chapter is organized as follows. In Section 2.2, we characterize the pseudo-gradient-based noncooperative dynamical systems and present the main problem along with motivations. In Section 2.3, we discuss the stability of a Nash equilibrium for multi-agent noncooperative systems without knowing agents' sensitivity parameter. In Section 2.4, we first introduce our zero-sum tax/subsidy mechanism for two-agent noncooperative systems, and then extend it to more general multi-agent systems. Furthermore, in Section 2.5, we present a couple of illustrative numerical examples. Finally, Section 2.6 concludes this chapter.

2.2 Problem Formulation

2.2.1 System Description

Consider the noncooperative system with payoff functions $J_i : \mathbb{R}^n \to \mathbb{R}$ for agent $i \in \mathcal{N}$, where $\mathcal{N} \triangleq \{1, \ldots, n\}$ denotes the set of agents. Each agent $i \in \mathcal{N}$ controls its state (strategy) $x_i \in \mathbb{R}$, $i \in \mathcal{N}$. Let $x = (x_i, x_{-i}) \in \mathbb{R}^n$ denote all agents' state (strategy) profile, where $x_{-i} \in \mathbb{R}^{n-1}$ denotes the agents' state profile except agent i. In this chapter, we suppose that each agent i aims to increase its own payoff $J_i(x_i, x_{-i})$, where J_i may depend on all the agents' state. We denote the noncooperative system by $\mathcal{G}(J)$ with $J \triangleq \{J_i\}_{i\in\mathcal{N}}$.

Definition 2.1. [89] For the noncooperative system $\mathcal{G}(J)$, the state profile $x^* \in \mathbb{R}^n$ is called a Nash equilibrium of $\mathcal{G}(J)$ if

$$J_i(x_i^*, x_{-i}^*) \geq J_i(x_i, x_{-i}^*), \quad x_i \in \mathbb{R}, \quad i \in \mathcal{N}. \tag{2.1}$$

The best-response state x_i for agent i defined as the state x_i yielding the largest value of J_i given the state profile x_{-i} of the other agents is expressed by the mapping $\mathrm{BR}_i : \mathbb{R}^{n-1} \to \mathbb{R}$ given by

$$x_i = \mathrm{BR}_i(x_{-i}) \triangleq \mathrm{argmax}_{x_i\in\mathbb{R}} J_i(x_i, x_{-i}). \tag{2.2}$$

It is worth noting that the Nash equilibrium x^* is understood as an intersecting point of the best-response curves/planes (2.2), i.e., $x^* = [x_1^*, \ldots, x_n^*]^{\mathrm{T}}$ satisfies

$$\mathrm{BR}_i(x_{-i}^*) = x_i^*, \quad i \in \mathcal{N}. \tag{2.3}$$

Therefore, at a Nash equilibrium x^* no agent has any intension to deviate unilaterally from the equilibrium state.

Assumption 2.1. The payoff functions $J_i(x)$, $i \in \mathcal{N}$, are twice continuously differentiable.

Note that the noncooperative system $\mathcal{G}(J)$ may not possess any Nash equilibrium. Some sufficient conditions for existence of a Nash equilibrium with the closed convex domain can be found in [17], [90, Chapter 2]. However, in general, guaranteeing the existence of a Nash equilibrium for an unbounded state space is a complicated problem. In this chapter, we suppose that there exists at least one Nash equilibrium. In this case, under Assumption 2.1, since the Nash equilibrium x^* satisfies $x_i^* = \arg\max_{x_i \in \mathbb{R}} J_i(x_i, x_{-i}^*)$ for all $i \in \mathcal{N}$, it follows that

$$\frac{\partial J_i(x^*)}{\partial x_i} = 0, \quad i \in \mathcal{N}. \tag{2.4}$$

Moreover, it is important to note that the Nash equilibrium is characterized independent of the underlying dynamics.

2.2.2 Myopic Pseudo-Gradient Dynamics

In this chapter, we suppose that each agent continuously changes its state (strategy) of the noncooperative system $\mathcal{G}(J)$ in the unbounded state space $\mathbb{R}^n$ in order to increase its own payoff. Specifically, we assume that the state profile $x(\cdot)$ is available for all the agents and each agent follows the pseudo-gradient dynamics given by

$$\dot{x}_i(t) = \alpha_i \frac{\partial J_i(x(t))}{\partial x_i}, \quad i \in \mathcal{N}, \tag{2.5}$$

where α_i, $i \in \mathcal{N}$, are agent-dependent positive constant parameters representing sensitivity to the increasing/decreasing payoff per unit state change [17]. In this case, agents selfishly concern their own payoffs and myopically change their states (strategies) according to the current information without any foresight on the future state of the other agents. The pseudo-gradient dynamics are widely used as the dynamics for rational and selfish agents [20–22, 44, 25, 45, 42]. The agents' moving rates given by (2.5) are characterized to be proportional to the projection of the gradient of $J_i(x)$ onto x_i-axis, which is termed as the *pseudo-gradient*, but the sensitivity parameters α_i, $i \in \mathcal{N}$, which decide how fast the agents move, are in many cases private so that they are not observed. It is important to note that at the Nash equilibrium x^*, $\dot{x}(t) = 0$ since (2.4) holds.

2.2.3 Motivations and Problem Statement

Motivation: Some of the Nash equilibria may be unstable in the noncooperative system $\mathcal{G}(J)$, since agents' payoff functions are generally different from each other. For instance, Fig. 2.1 shows the payoff functions of each agent in a two-agent noncooperative system with an unstable Nash equilibrium. Assume there is a system manager, e.g., the governor of the markets, who controls the amount of tax and subsidy (negative and positive incentives, respectively) and demands to stabilize around a Nash equilibrium for encouraging agents to converge to it. Assuming all the information of the payoff functions $J_i(x)$, $i \in \mathcal{N}$, is known, we suppose that the system manager chooses the Nash equilibrium possessing the largest social utility from the set of Nash equilibria of $\mathcal{G}(J)$ as the *target* Nash equilibrium. A fundamental question is how the system manager designs an incentive mechanism to stabilize the possibly unstable target Nash equilibrium with uncertain sensitivity parameters α_i, $i \in \mathcal{N}$.

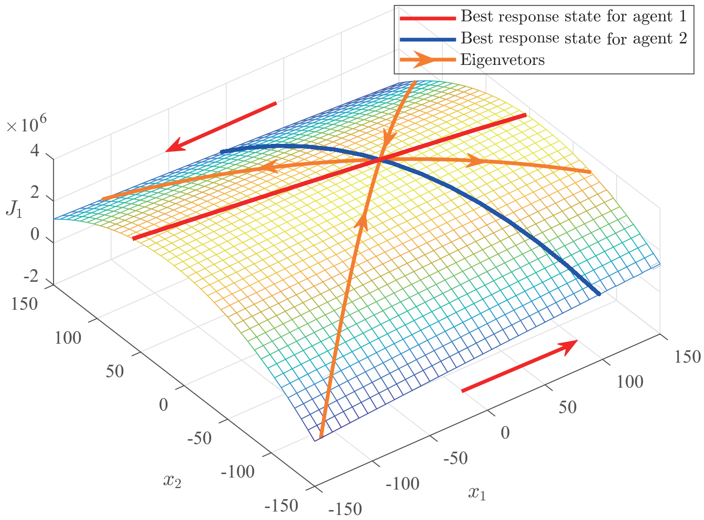
Assumption 2.2. There exists a known Nash equilibrium x^* satisfying $\frac{\partial^2 J_i(x^*)}{\partial x_i^2} < 0$, $i \in \mathcal{N}$, which is the target equilibrium such that the system manager wishes to guarantee stability around x^*.

Note that the computation of the Nash equilibrium for the noncooperative system $\mathcal{G}(J)$ is beyond the scope of this paper. The relevant methods for calculating Nash equilibria can be found in [25, 91–94] and the references therein.

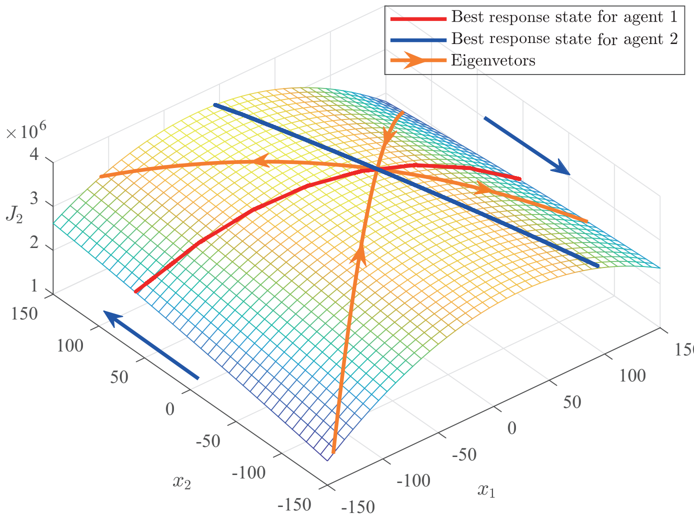
Problem: Consider the the target Nash equilibrium x^* with uncertain sensitivity parameters α_i, $i \in \mathcal{N}$, for the system manager. Our main objectives are two folds: (i) Find the condition for determining the stability property of the Nash equilibrium x^* with *arbitrary* α_i, $i \in \mathcal{N}$; (ii) Design an explicit incentive mechanism to stabilize the possibly unstable Nash equilibrium x^* with the *unknown* sensitivity parameters α_i, $i \in \mathcal{N}$.

2.3 Stability Analysis of Nash Equilibrium with Unknown Sensitivity Parameters

In this section, we characterize stability properties of the Nash equilibrium of the noncooperative system $\mathcal{G}(J)$. Specifically, we first present the results for the general n-agent case, and then specialize the results to 3-agent and 2-agent cases. For the



(a) Payoff function of agent 1



(b) Payoff function of agent 2

Figure. 2.1 Payoff functions for an unstable Nash equilibrium.

statement of the following results, let $\alpha \triangleq (\alpha_1, \dots, \alpha_n)$ and define

$$\mathcal{A}(J, \alpha, x) \triangleq \begin{bmatrix} \alpha_1 \frac{\partial^2 J_1(x)}{\partial x_1^2} & \cdots & \alpha_1 \frac{\partial^2 J_1(x)}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \alpha_n \frac{\partial^2 J_n(x)}{\partial x_n \partial x_1} & \cdots & \alpha_n \frac{\partial^2 J_n(x)}{\partial x_n^2} \end{bmatrix}. \tag{2.6}$$

Note that under Assumption 2.1, since the functions $J_i(x)$, $i \in \mathcal{N}$, are twice continuously differentiable, the matrix (2.6) is a continuous function with respect to x. Moreover, under Assumption 2.2, the diagonal terms $\alpha_i \frac{\partial^2 J_i(x^*)}{\partial x_i^2}$, $i \in \mathcal{N}$, in $\mathcal{A}(J, \alpha, x^*)$ are all negative. This fact is used in the analysis of the following results.

Stability Analysis for n-Agent Noncooperative Systems

The sensitivity parameters α_i, $i \in \mathcal{N}$, are inherent to each of the agents and are not exactly observed. *Without knowing* the value of α for the n-agent noncooperative system, the following results provide several ways to determine stability of the Nash equilibrium.

Corollary 2.1. Consider the Nash equilibrium $x^* \in \mathbb{R}^n$ for the n-agent noncooperative system $\mathcal{G}(J)$ with myopic pseudo-gradient dynamics (2.5). If the payoff functions $J_i(x)$, $i \in \mathcal{N}$, satisfy

$$(-1)^n \det \mathcal{A}(J, 1_n, x^*) < 0, \tag{2.7}$$

then the Nash equilibrium x^* is unstable for any positive constants α_i, $i \in \mathcal{N}$.

Proof First, let $\tilde{x} \triangleq x - x^*$. Note that linearizing the system dynamics (2.5) around x^* yields

$$\dot{\tilde{x}}(t) = \mathcal{A}(J, \alpha, x^*)\tilde{x}(t). \tag{2.8}$$

The result is a direct consequence of the Lyapunov's indirect method. Specifically, consider the characteristic equation $\det(sI - \mathcal{A}(J, \alpha, x^*)) = s^n + a_{N-1}s^{N-1} + \cdots + a_1 s + a_0 = 0$ of $\mathcal{A}(J, \alpha, x^*)$, where $a_0, \dots, a_{n-1}$ are appropriate constants. In particular, $a_0 = (-1)^n \det \mathcal{A}(J, \alpha, x^*) = (-1)^n \det \mathcal{A}(J, 1_n, x^*) \times \prod_{i \in \mathcal{N}} \alpha_i$. Now, since $\alpha_i > 0, i \in \mathcal{N}$, it follows from (2.7) that $a_0 < 0$. Hence, it follows from Routh or Hurwitz criterion that the Nash equilibrium x^* is unstable. □

The fictitious sensitivity 1_n in (2.7) can be replaced by any $\hat{\alpha} \in \mathbb{R}^n_+$ to determine instability because it does not change the sign of the determinant of $\mathcal{A}(J, \cdot, x^*)$.

Relation of payoff dependency between the agents can be characterized by defining a graph. For specific graph structures, we can specialize the condition (2.7) as shown in the following examples.

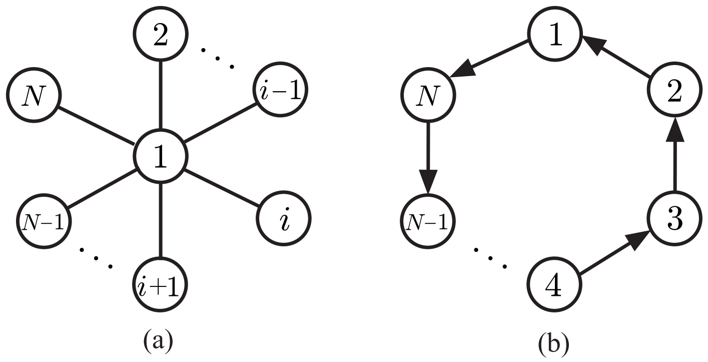


Figure. 2.2 Network topologies of payoff dependency. (a) n-agent center-sponsored star network where agent 1 is the center, (b) directed ring network: the arrows of the graph indicate that $J_i(x) = J_i(x_i, x_{i+1})$ where x_{N+1} is understood as x_1.

Example 2.1. Consider the noncooperative system with the payoff dependency given by the center-sponsored star network illustrated in Fig. 2.2(a), where agent 1 is the center of the network. In this case, note that since

$$\mathcal{A}(J, 1_n, x^*) = \begin{bmatrix} \frac{\partial^2 J_1(x^*)}{\partial x_1^2} & \frac{\partial^2 J_1(x^*)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 J_1(x^*)}{\partial x_1 \partial x_n} \\ \frac{\partial^2 J_2(x^*)}{\partial x_2 \partial x_1} & \frac{\partial^2 J_2(x^*)}{\partial x_2^2} & & 0 \\ \vdots & & \ddots & \\ \frac{\partial^2 J_n(x^*)}{\partial x_n \partial x_1} & 0 & & \frac{\partial^2 J_n(x^*)}{\partial x_n^2} \end{bmatrix}, \tag{2.9}$$

the left-hand side of (2.7) is given by

$$(-1)^n \Big(\frac{\partial^2 J_1(x^*)}{\partial x_1^2} - \sum_{i=2}^{n} \frac{\frac{\partial^2 J_1(x^*)}{\partial x_1 \partial x_i} \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_1}}{\frac{\partial^2 J_i(x^*)}{\partial x_i^2}} \Big) \prod_{i=2}^{n} \frac{\partial^2 J_i(x^*)}{\partial x_i^2}. \tag{2.10}$$

Noting that Assumption 2.2 implies that $(-1)^n \prod_{i=2}^{n} \frac{\partial^2 J_i(x^*)}{\partial x_i^2}$ is negative, it follows from Corollary 2.1 that if the payoff functions $J_i(x)$, $i \in \mathcal{N}$, satisfy

$$\frac{\partial^2 J_1(x^*)}{\partial x_1^2} - \sum_{i=2}^{n} \frac{\frac{\partial^2 J_1(x^*)}{\partial x_1 \partial x_i} \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_1}}{\frac{\partial^2 J_i(x^*)}{\partial x_i^2}} > 0, \tag{2.11}$$

then the Nash equilibrium x^* is unstable for any positive constants α_i, $i \in \mathcal{N}$.

Example 2.2. Consider the noncooperative system with the payoff dependency given by the directed ring network illustrated in Fig. 2.2(b). In this case, note that since

$$\mathcal{A}(J, 1_n, x^*) = \begin{bmatrix} \frac{\partial^2 J_1(x^*)}{\partial x_1^2} & \frac{\partial^2 J_1(x^*)}{\partial x_1 \partial x_2} & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & \frac{\partial^2 J_{N-1}(x^*)}{\partial x_{N-1}^2} & \frac{\partial^2 J_{N-1}(x^*)}{\partial x_{N-1} \partial x_n} \\ \frac{\partial^2 J_n(x^*)}{\partial x_n \partial x_1} & 0 & 0 & \frac{\partial^2 J_n(x^*)}{\partial x_n^2} \end{bmatrix}, \tag{2.12}$$

the left-hand side of (2.7) is $\prod_{i=1}^n \left(-\frac{\partial^2 J_i(x^*)}{\partial x_i^2} \right) - \prod_{i=1}^n \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_{i+1}}$, where x_{n+1} is understood as x_1. Thus, it follows from Corollary 2.1 that if the payoff functions $J_i(x)$, $i \in \mathcal{N}$, satisfy

$$\prod_{i=1}^n \Big(-\frac{\partial^2 J_i(x^*)}{\partial x_i^2} \Big) < \prod_{i=1}^n \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_{i+1}}, \tag{2.13}$$

then the Nash equilibrium x^* is unstable for any positive constants α_i, $i \in \mathcal{N}$.

Now a sufficient condition is provided to guarantee stability *without knowing* α_i, $i \in \mathcal{N}$, in the following theorem.

Theorem 2.1. Consider the Nash equilibrium $x^* \in \mathbb{R}^n$ for the n-agent noncooperative system $\mathcal{G}(J)$ with pseudo-gradient dynamics (2.5). If there exists $\hat{\alpha} \in \mathbb{R}_+^n$ such that

$$\mathcal{A}^{\mathrm{T}}(J, \hat{\alpha}, x^*) + \mathcal{A}(J, \hat{\alpha}, x^*) < 0, \tag{2.14}$$

then the Nash equilibrium x^* is locally asymptotically stable for any positive constants α_i, $i \in \mathcal{N}$.

Proof Letting $\tilde{x} = x - x^*$, consider the Lyapunov function candidate $V(\tilde{x}) = \tilde{x}^{\mathrm{T}} P \tilde{x}$ with the positive-definite matrix $P \triangleq \operatorname{diag}\left[\frac{\hat{\alpha}_1}{\alpha_1}, \ldots, \frac{\hat{\alpha}_n}{\alpha_n}\right] > 0$. Since

$$\mathcal{A}^{\mathrm{T}}(J, \alpha, x^*) P + P \mathcal{A}(J, \alpha, x^*) = \mathcal{A}^{\mathrm{T}}(J, \hat{\alpha}, x^*) + \mathcal{A}(J, \hat{\alpha}, x^*) < 0,$$

is satisfied, it follows using the linearized dynamics (2.8) that

$$\dot{V}(\tilde{x}(t)) = \tilde{x}^{\mathrm{T}}(t)(\mathcal{A}^{\mathrm{T}}(J, \hat{\alpha}, x^*) + \mathcal{A}(J, \hat{\alpha}, x^*))\tilde{x}(t) < 0, \tag{2.15}$$

around x^* and hence the Nash equilibrium x^* is asymptotically stable for all positive sensitivity parameters α_i, $i \in \mathcal{N}$. □

Remark 2.1. The result in Theorem 2.1 appears to be similar to Theorems 8 and 9 of [17] but it is certainly different in that Theorem 2.1 guarantees asymptotic stability for *arbitrary* α by evaluating the sign-definiteness of $\mathcal{A}^{\mathrm{T}}(J,\hat{\alpha},x^*)+\mathcal{A}(J,\hat{\alpha},x^*)$ for a *particular* $\hat{\alpha}$. To determine whether such $\hat{\alpha}$ exists, we can address the linear matrix inequality (LMI) feasibility problem given by

$$\mathrm{diag}[\hat{\alpha}]\mathcal{A}(J,1_n,x^*)+\mathcal{A}^{\mathrm{T}}(J,1_n,x^*)\mathrm{diag}[\hat{\alpha}]<0, \tag{2.16}$$

assuming that all the information of J is known.

Remark 2.2. Because of the continuity of $\mathcal{A}(J,\hat{\alpha},x)$ with respect to x, (2.14) implies that there exists a connected set

$$\mathcal{D}_1^{\hat{\alpha}} \triangleq \{x\in\mathbb{R}^n : \mathcal{A}^{\mathrm{T}}(J,\hat{\alpha},x)+\mathcal{A}(J,\hat{\alpha},x)<0\} \tag{2.17}$$

containing x^*. Let $f(x) \triangleq [\alpha_1\frac{\partial J_1(x)}{\partial x_1},\ldots,\alpha_n\frac{\partial J_n(x)}{\partial x_n}]^{\mathrm{T}}$ denote the vector field of the pseudo-gradient dynamics and let $V(x)\triangleq f^{\mathrm{T}}(x)Pf(x)$. It is important to note that a subset of the region of attraction can be characterized by

$$\mathcal{D}_2^{\delta} \triangleq \{x\in\mathbb{R}^n : V(x)<\delta\}, \tag{2.18}$$

with the maximum attainable $\delta\in\mathbb{R}_+$ such that $\mathcal{D}_2^{\delta}\subseteq\mathcal{D}_1^{\hat{\alpha}}$ and $\mathcal{D}_2^{\tilde{\delta}}$ is connected in the neighborhood of x^* for all $\tilde{\delta}<\delta$. This is because $V(x)$ is understood as a Lyapunov function and it satisfies $\dot{V}(x(t))=f^{\mathrm{T}}(x(t))(\mathcal{A}^{\mathrm{T}}(J,\hat{\alpha},x(t))+\mathcal{A}(J,\hat{\alpha},x(t)))f(x(t))<0$ for all $x(t)\in\mathcal{D}_2^{\delta}\setminus\{x^*\}$. It is important to note that the estimated region of attraction $\mathcal{D}_2^{\delta}$ depends on the choice of $\hat{\alpha}$ in $\mathcal{A}(J,\hat{\alpha},x^*)$ and can be substantially smaller than the actual region of attraction. But for the special case where $\mathcal{A}^{\mathrm{T}}(J,\hat{\alpha},x)+\mathcal{A}(J,\hat{\alpha},x)<0$ holds for all $x\in\mathbb{R}^n$, since it can be shown that $f(x)=0$ only when $x=x^*$ in $\mathbb{R}^n$, it follows that the Nash equilibrium x^* is *globally* asymptotically stable for *arbitrary* α. For instance, if the payoff functions are *quadratic*, then (2.14) guarantees *global* asymptotic stability as (2.6) is a constant matrix.

Remark 2.3. For the noncooperative system with the payoff functions satisfying $\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_j}\geq 0$, $i,j\in\mathcal{N}$, $i\neq j$, it follows from the properties of Metzler matrix that (2.14) in Theorem 2.1 is also a necessary condition for the Nash equilibrium x^* to be asymptotically stable for *arbitrary* α.

Example 2.3. Consider the n-agent noncooperative system with the payoff dependency given by the center-sponsored star network illustrated in Fig. 2.2(a). To investigate the conditions for the payoff functions $J_i(x)$, $i\in\mathcal{N}$, such that $\hat{\alpha}\in\mathbb{R}_+^n$ exists to satisfy (2.14), note that the kth-order leading principal minor of $\mathcal{A}^{\mathrm{T}}(J,\hat{\alpha},x^*)+\mathcal{A}(J,\hat{\alpha},x^*)$

with $\hat{\alpha}_1 = 1$ is given by $L_k \triangleq \left(2\frac{\partial^2 J_1(x^*)}{\partial x_1^2} - \sum_{i=2}^{k} \frac{(\frac{\partial^2 J_1(x^*)}{\partial x_1 \partial x_i} + \hat{\alpha}_i \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_1})^2}{2\hat{\alpha}_i \frac{\partial^2 J_i(x^*)}{\partial x_i^2}}\right) \prod_{i=2}^{k} \left(2\hat{\alpha}_i \frac{\partial^2 J_i(x^*)}{\partial x_i^2}\right)$ for $k = 2, \ldots, n$. Since, by Assumption 2.2, $\frac{\partial^2 J_i(x^*)}{\partial x_i^2} < 0$, $i = 2, \ldots, n$, and hence $(-1)^k \prod_{i=2}^{k} \left(2\hat{\alpha}_i \frac{\partial^2 J_i(x^*)}{\partial x_i^2}\right) < 0$, $k = 2, \ldots, n$, the inequality $(-1)^k L_k > 0$ for guaranteeing (2.14) is equivalent to

$$\frac{\partial^2 J_1(x^*)}{\partial x_1^2} < \frac{1}{2} \sum_{i=2}^{k} \frac{\left(\frac{\partial^2 J_1(x^*)}{\partial x_1 \partial x_i} + \hat{\alpha}_i \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_1}\right)^2}{2\hat{\alpha}_i \frac{\partial^2 J_i(x^*)}{\partial x_i^2}}, \tag{2.19}$$

for $k = 2, \ldots, n$. Therefore, since all the terms in the right-hand side are negative, the existence problem of $\hat{\alpha}$ in satisfying (2.14) is equivalent to finding a solution $\hat{\alpha} = (1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n)$ for (2.19) with $k = n$. Now, such $\hat{\alpha}$ exists if and only if the simple condition

$$\begin{aligned} \frac{\partial^2 J_1(x^*)}{\partial x_1^2} &< \frac{1}{2} \sum_{i=2}^{n} \max_{\hat{\alpha}_i \in \mathbb{R}_+} \frac{\left(\frac{\partial^2 J_1(x^*)}{\partial x_1 \partial x_i} + \hat{\alpha}_i \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_1}\right)^2}{2\hat{\alpha}_i \frac{\partial^2 J_i(x^*)}{\partial x_i^2}} \\ &= \frac{1}{2} \sum_{i \in \mathcal{N}_0} \max_{\hat{\alpha}_i \in \mathbb{R}_+} \frac{\left(\frac{\partial^2 J_1(x^*)}{\partial x_1 \partial x_i} + \hat{\alpha}_i \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_1}\right)^2}{2\hat{\alpha}_i \frac{\partial^2 J_i(x^*)}{\partial x_i^2}} = \sum_{i \in \mathcal{N}_0} \frac{\frac{\partial^2 J_1(x^*)}{\partial x_1 \partial x_i} \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_1}}{\frac{\partial^2 J_i(x^*)}{\partial x_i^2}}, \end{aligned} \tag{2.20}$$

is satisfied for $\mathcal{N}_0 \triangleq \{i \in \mathcal{N} : \frac{\partial^2 J_1(x^*)}{\partial x_1 \partial x_i} \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_1} > 0\}$, where in (2.20) we used $\max_{\alpha \in \mathbb{R}_+} \frac{(A+\alpha B)^2}{2\alpha C} = \frac{2AB}{C}$ for $AB > 0$ and $C < 0$.

Remark 2.4. Note that the local stability of the Nash equilibrium x^* under the dynamics (2.5) can also be directly derived if the matrix $\mathcal{A}(J, \alpha, x^*)$ (or, equivalently, $\mathcal{A}(J, 1_n, x^*)$) is strictly diagonally dominant (i.e., $\frac{\partial^2 J_i(x^*)}{\partial x_i^2} < -\sum_{j \neq i} |\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_j}|$ for all $i \in \mathcal{N}$) [42]. The proof is based on Gershgorin's circle theorem [95].

Stability Analysis for 3-Agent Noncooperative Systems

Recall that based on the Lyapunov's stability method, Theorem 2.1 requires us to look for $\hat{\alpha}$ to make the symmetric part of $\mathcal{A}(J, \hat{\alpha}, x^*)$ negative definite to guarantee stability. For the case of $n = 3$, it is possible to characterize a different set of stability conditions on the payoff functions based on the Hurwitz criterion.

Proposition 2.1. Consider the Nash equilibrium $x^* \in \mathbb{R}^3$ for the 3-agent noncooperative system $\mathcal{G}(\{J_1, J_2, J_3\})$ with pseudo-gradient dynamics (2.5). If the payoff funstions

$J_i(x)$, $i \in \{1,2,3\}$, satisfy

$$\det \mathcal{A}(J, 1_3, x^*) < 0, \tag{2.21}$$

$$\frac{\partial^2 J_i(x^*)}{\partial x_i^2}\frac{\partial^2 J_j(x^*)}{\partial x_j^2} - \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_j}\frac{\partial^2 J_j(x^*)}{\partial x_j \partial x_i} > 0, \quad i,j \in \{1,2,3\}, \quad i \neq j, \tag{2.22}$$

$$2\frac{\partial^2 J_1(x^*)}{\partial x_1^2}\frac{\partial^2 J_2(x^*)}{\partial x_2^2}\frac{\partial^2 J_3(x^*)}{\partial x_3^2} - \frac{\partial^2 J_1(x^*)}{\partial x_1 \partial x_2}\frac{\partial^2 J_2(x^*)}{\partial x_2 \partial x_3}\frac{\partial^2 J_3(x^*)}{\partial x_3 \partial x_1} - \frac{\partial^2 J_1(x^*)}{\partial x_1 \partial x_3}\frac{\partial^2 J_2(x^*)}{\partial x_2 \partial x_1}\frac{\partial^2 J_3(x^*)}{\partial x_3 \partial x_2} < 0, \tag{2.23}$$

then the Nash equilibrium x^* is asymptotically stable for any positive constants $\alpha_1, \alpha_2, \alpha_3$.

Proof Consider the characteristic polynomial $s^3 + a_2 s^2 + a_1 s + a_0$ of $\mathcal{A}(J, \alpha, x^*)$, where

$$a_2 = -\sum_{i \in \mathcal{N}} \alpha_i \frac{\partial^2 J_i(x^*)}{\partial x_i^2}, \tag{2.24}$$

$$a_1 = \sum_{i \neq j}\Big(\alpha_i \alpha_j \big(\frac{\partial^2 J_i(x^*)}{\partial x_i^2}\frac{\partial^2 J_j(x^*)}{\partial x_j^2} - \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_j}\frac{\partial^2 J_j(x^*)}{\partial x_j \partial x_i}\big)\Big), \tag{2.25}$$

$$a_0 = -\det \mathcal{A}(J, \alpha, x^*) = -\alpha_1\alpha_2\alpha_3 \det \mathcal{A}(J, 1_3, x^*). \tag{2.26}$$

Note that Assumption 2.2 implies $a_2 > 0$ and (2.21), (2.22) imply $a_0 > 0$, $a_1 > 0$, respectively. Furthermore, it follows from (2.22), (2.23) that

$$\begin{aligned} a_2 a_1 - a_0 = &-\sum_{i \neq j}\Big(\alpha_i^2 \alpha_j \frac{\partial^2 J_i(x^*)}{\partial x_i^2}\big(\frac{\partial^2 J_i(x^*)}{\partial x_i^2}\frac{\partial^2 J_j(x^*)}{\partial x_j^2} - \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_j}\frac{\partial^2 J_j(x^*)}{\partial x_j \partial x_i}\big)\Big) \\ &- \alpha_1\alpha_2\alpha_3\big(2\frac{\partial^2 J_1(x^*)}{\partial x_1^2}\frac{\partial^2 J_2(x^*)}{\partial x_2^2}\frac{\partial^2 J_3(x^*)}{\partial x_3^2} - \frac{\partial^2 J_1(x^*)}{\partial x_1 \partial x_2}\frac{\partial^2 J_2(x^*)}{\partial x_2 \partial x_3}\frac{\partial^2 J_3(x^*)}{\partial x_3 \partial x_1} \\ &- \frac{\partial^2 J_1(x^*)}{\partial x_1 \partial x_3}\frac{\partial^2 J_2(x^*)}{\partial x_2 \partial x_1}\frac{\partial^2 J_3(x^*)}{\partial x_3 \partial x_2}\big) > 0. \end{aligned} \tag{2.27}$$

Hence, it follows from the Hurwitz criterion that the Nash equilibrium x^* is stable for any positive constants $\alpha_1, \alpha_2, \alpha_3$. □

Remark 2.5. The conditions in Proposition 2.1 provide different sufficient conditions from the one in Theorem 2.1. For example, $\mathcal{A}(J, 1_3, x^*) = \begin{bmatrix} -1 & 0 & 50 \\ -1 & -1 & 0 \\ -1 & -1 & -1 \end{bmatrix}$ satisfies (2.21)–(2.23), but there does not exist $\hat{\alpha} \in \mathbb{R}_+^3$ such that $\mathcal{A}^{\mathrm{T}}(J, \hat{\alpha}, x^*) + \mathcal{A}(J, \hat{\alpha}, x^*) < 0$. On the contrary, $\mathcal{A}^{\mathrm{T}}(J, 1_3, x^*) + \mathcal{A}(J, 1_3, x^*) < 0$ for $\mathcal{A}(J, 1_3, x^*) = \begin{bmatrix} -6 & -5 & 1 \\ -2 & -2 & -5 \\ -5 & 3 & -1 \end{bmatrix}$, but in this case, the condition in (2.23) is false.

For a special case of the payoff dependency, it is interesting to observe that the conditions in Proposition 2.1 are equivalent to (2.14) in Theorem 2.1. In such a case, (2.21)–(2.23) guarantee the existence of $\hat{\alpha}$ for $\mathcal{A}^{\mathrm{T}}(J,\hat{\alpha},x^*)+\mathcal{A}(J,\hat{\alpha},x^*)<0$ as shown in the following remark.

Remark 2.6. Consider the 3-agent noncooperative system with the payoff dependency given by the undirected serial graph, which is a special case of the center-sponsored star network discussed in Example 2.3. Note that $\frac{\partial^2 J_2(x^*)}{\partial x_2 \partial x_3}=\frac{\partial^2 J_3(x^*)}{\partial x_3 \partial x_2}=0$ because $J_2(x)$ and $J_3(x)$ are not the functions of x_3 and x_2, respectively. In this case, inequality (2.23) is automatically satisfied. Furthermore, note that $\det \mathcal{A}(J,1_3,x^*)=-\frac{\partial^2 J_3(x^*)}{\partial x_3^2}\frac{\partial^2 J_1(x^*)}{\partial x_1 \partial x_2}\frac{\partial^2 J_2(x^*)}{\partial x_2 \partial x_1}-\frac{\partial^2 J_2(x^*)}{\partial x_2^2}\frac{\partial^2 J_1(x^*)}{\partial x_1 \partial x_3}\frac{\partial^2 J_3(x^*)}{\partial x_3 \partial x_1}+\frac{\partial^2 J_1(x^*)}{\partial x_1^2}\frac{\partial^2 J_2(x^*)}{\partial x_2^2}\frac{\partial^2 J_3(x^*)}{\partial x_3^2}$. Hence, the conditions (2.21)–(2.23) are satisfied if and only if

$$\frac{\partial^2 J_1(x^*)}{\partial x_1^2}<\min\left\{\frac{\frac{\partial^2 J_1(x^*)}{\partial x_1 \partial x_2}\frac{\partial^2 J_2(x^*)}{\partial x_2 \partial x_1}}{\frac{\partial^2 J_2(x^*)}{\partial x_2^2}}+\frac{\frac{\partial^2 J_1(x^*)}{\partial x_1 \partial x_3}\frac{\partial^2 J_3(x^*)}{\partial x_3 \partial x_1}}{\frac{\partial^2 J_3(x^*)}{\partial x_3^2}},\frac{\frac{\partial^2 J_1(x^*)}{\partial x_1 \partial x_2}\frac{\partial^2 J_2(x^*)}{\partial x_2 \partial x_1}}{\frac{\partial^2 J_2(x^*)}{\partial x_2^2}},\frac{\frac{\partial^2 J_1(x^*)}{\partial x_1 \partial x_3}\frac{\partial^2 J_3(x^*)}{\partial x_3 \partial x_1}}{\frac{\partial^2 J_3(x^*)}{\partial x_3^2}}\right\}, \tag{2.28}$$

where the right-hand side is same as (2.20). Therefore, for this special case of the payoff dependency, Proposition 2.1 provides exactly the same sufficient conditions as the one given in Theorem 2.1.

Stability Analysis for 2-Agent Noncooperative Systems

Now, we assume $n=2$ for the noncooperative system $\mathcal{G}(\{J_1,J_2\})$. The following results are investigated in [45] and fundamental in constructing the incentive function that we develop in Section 2.4. First, we note that stability can be determined by the sign of the determinant of $\mathcal{A}$.

Proposition 2.2. [45] Consider the Nash equilibrium $x^*\in\mathbb{R}^2$ for the 2-agent noncooperative system $\mathcal{G}(\{J_1,J_2\})$ with pseudo-gradient dynamics (2.5). If the payoff functions $J_1(x)$, $J_2(x)$ satisfy

$$\det \mathcal{A}(\{J_1,J_2\},1_2,x^*)>0, \tag{2.29}$$

then the Nash equilibrium x^* is asymptotically stable for any positive constants $\alpha_1,\alpha_2>0$.

Remark 2.7. The undirected graph topology of the payoff dependency for the 2-agent system is a special case of the center-sponsored star network discussed in Example 2.3. Note that (2.29) is equivalent to (2.20) by letting $n=2$, and hence (2.29) represents the necessary and sufficient condition for the existence of $\hat{\alpha}$ in Theorem 2.1.

It follows from Corollary 1 (for $N = 2$) and Proposition 2 that if $\det \mathcal{A}(\{J_1, J_2\}, 1_2, x^*) > 0$ (resp., < 0), then the Nash equilibrium x^* is asymptotically stable (resp., unstable). This fact implies that the existence of $\hat{\alpha}$ for (2.14) is in fact the necessary and sufficient condition for stability of x^* assuming that there is no eigenvalue of $\mathcal{A}(\{J_1, J_2\}, \{\alpha_1, \alpha_2\}, x^*)$ on the imaginary axis. In the case where $\det \mathcal{A}(\{J_1, J_2\}, 1_2, x^*) = 0$ implying that at least one of the eigenvalues of $\mathcal{A}(\{J_1, J_2\}, \{\alpha_1, \alpha_2\}, x^*)$ is zero, the Nash equilibrium x^* of (2.5) may be stable or unstable depending on the payoff functions that the agents are associated with. For an example of addressing the center manifold to determine stability, see [45].

The next result shows the fact that the eigenvalues of the 2×2 Jacobian matrix of an *unstable* Nash equilibrium does not possess complex conjugate eigenvalues.

Proposition 2.3. [45] Consider the 2-agent noncooperative system $\mathcal{G}(\{J_1, J_2\})$. If the Nash equilibrium x^* is unstable under the pseudo-gradient dynamics (2.5), then it is a saddle point.

Here we define a noncooperative system $\mathcal{G}(\{J_1, J_2\})$ with the *quadratic* payoff functions given by

$$J_i(x) = -x^{\mathrm{T}} A_i x + b_i^{\mathrm{T}} x + c_i, \quad i = 1, 2, \tag{2.30}$$

where $A_i \triangleq \begin{bmatrix} a_{11}^i & a_{12}^i \\ a_{12}^i & a_{22}^i \end{bmatrix} \in \mathbb{R}^{2\times 2}$ is symmetric with $a_{ii}^i > 0$, $b_i \triangleq [b_1^i, b_2^i]^{\mathrm{T}} \in \mathbb{R}^2$, and $c_i \in \mathbb{R}$, $i = 1, 2$. Note that different from the noncooperative system with non-quadratic payoff functions, if the Jacobian matrix $\mathcal{A}(\{J_1, J_2\}, \{\alpha_1, \alpha_2\}, x^*)$ is non-singular, then the Nash equilibrium is unique. Alternatively, if $\mathcal{A}(\{J_1, J_2\}, \{\alpha_1, \alpha_2\}, x^*)$ is singular, then there may exist infinitely many Nash equilibria.

Example 2.4. Consider the 2-agent noncooperative system $\mathcal{G}(\{J_1, J_2\})$ with the *quadratic* payoff functions (2.30). Since $\det \mathcal{A}(\{J_1, J_2\}, \{1, 1\}, x^*) = 4(a_{11}^1 a_{22}^2 - a_{12}^1 a_{12}^2)$, it follows from Proposition 2.2 that if the payoff functions $J_1(x)$, $J_2(x)$ satisfy $a_{11}^1 a_{22}^2 < a_{12}^1 a_{12}^2$ (resp., $a_{11}^1 a_{22}^2 > a_{12}^1 a_{12}^2$), then the Nash equilibrium x^* is unstable (resp., asymptotically stable). Three typical examples showing the vector fields with different combinations of eigenvalues are given in Fig. 2.3, and the payoff functions of each agent for the unstable case (Fig. 2.3(a)) are shown in Fig. 2.1 above. Notice that when $a_{11}^1 a_{22}^2 = a_{12}^1 a_{12}^2$, the red and the blue lines in Fig. 2.3, which represent the best response state of agents 1 and 2, respectively, coincide with each other, and the Nash equilibrium x^* is Lyapunov stable (all the trajectories converge to the line in this case).

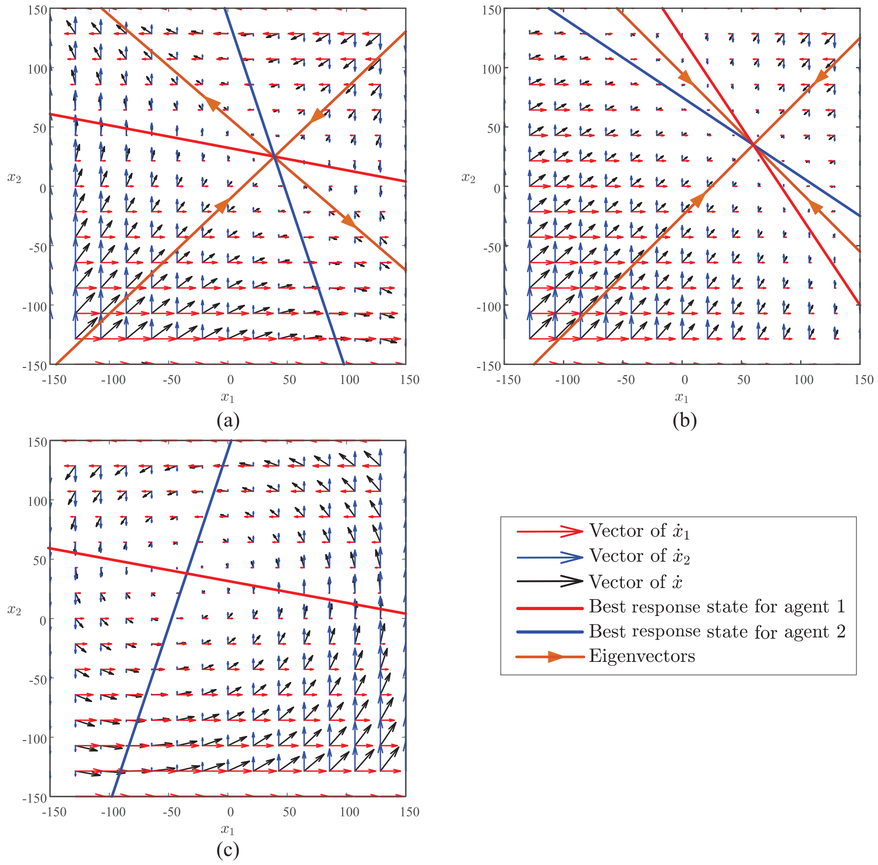


Figure. 2.3 Vector fields of a 2-agent noncooperative system $\mathcal{G}(\{J_1, J_2\})$ with quadratic payoffs (2.30). (a) $a^1_{11}a^2_{22} < a^1_{12}a^2_{12}$ (Real eigenvalues: positive and negative), (b) $a^1_{11}a^2_{22} > a^1_{12}a^2_{12}$ (Negative real eigenvalues), (c) $a^1_{11}a^2_{22} > a^1_{12}a^2_{12}$ (Complex conjugate eigenvalues, real part: negative).

2.4 Stabilization of Existing Nash Equilibrium with Zero-Sum Tax/Subsidy Approach

In this section, we characterize the stabilization method which is called a *tax/subsidy approach* around the target Nash equilibrium x^* for the noncooperative system *without* the knowledge of the sensitivity parameters α_i, $i \in \mathcal{N}$. In this framework, the system manager imposes an incentive mechanism so that the possibly unstable Nash equilibrium state x^* is stabilized by transferring the utility between the agents in a zero-sum fashion, i.e., the payoff functions of agents are changed to $\tilde{J} \triangleq \{\tilde{J}_i\}_{i\in\mathcal{N}}$ such that

$$\sum\nolimits_{i\in\mathcal{N}} \tilde{J}_i(x) = \sum\nolimits_{i\in\mathcal{N}} J_i(x). \tag{2.31}$$

In this case, the pseudo-gradient dynamics (2.5) are consequently changed to

$$\dot{x}(t) = \Big[\alpha_1 \frac{\partial \tilde{J}_1(x(t))}{\partial x_1}, \ldots, \alpha_n \frac{\partial \tilde{J}_n(x(t))}{\partial x_n}\Big]^{\mathrm{T}}, \quad x(0) = x_0 \in \mathbb{R}^n, \quad t \geq 0, \tag{2.32}$$

and the corresponding Jacobian matrix (2.6) at the Nash equilibrium is given by $\mathcal{A}(\tilde{J}, \alpha, x^*)$. Here we suppose that the amount of tax/subsidy affects the agents' utility in the additive way. We begin by characterizing the tax/subsidy approach for the simple 2-agent noncooperative systems, and then extend the approach to more general n-agent systems.

Tax/Subsidy Approach for 2-Agent Case

In this section, we discuss the *tax/subsidy approach* for the 2-agent noncooperative system $\mathcal{G}(\{J_1, J_2\})$. Specifically, consider the noncooperative system $\mathcal{G}(\{\tilde{J}_1, \tilde{J}_2\})$ with the adjusted payoff functions $\tilde{J}_1(x)$, $\tilde{J}_2(x)$ given by

$$\tilde{J}_1(x) \triangleq J_1(x) + p^k(x), \tag{2.33}$$

$$\tilde{J}_2(x) \triangleq J_2(x) - p^k(x), \tag{2.34}$$

where $p^k : \mathbb{R}^2 \to \mathbb{R}$ denotes an incentive function which is twice continuously differentiable, k is a scalar parameter, and $J_1(x), J_2(x)$ are the original payoff functions satisfying Assumption 2.2.

The incentive function $p^k(x)$ can be considered to be a feedback that is designed by the system manager. Note that $p^k(x)$ should be determined in such a way that x^* remains the Nash equilibrium of $\mathcal{G}(\{\tilde{J}_1, \tilde{J}_2\})$ and $\tilde{J}_1(x)$, $\tilde{J}_2(x)$ should be still partially strictly concave at the desired Nash equilibrium x^*, i.e.,

$$\frac{\partial^2 \tilde{J}_i(x^*)}{\partial x_i^2} < 0, \quad i = 1, 2. \tag{2.35}$$

Furthermore, $p^k(x)$ should satisfy

$$p^k(x^*) = 0, \quad \frac{\partial p^k(x^*)}{\partial x_i} = 0, \quad i = 1, 2, \tag{2.36}$$

for all $k \in \mathbb{R}$, which guarantee $\tilde{J}_i(x^*) = J_i(x^*)$ and $\partial \tilde{J}_i(x^*)/\partial x_i = 0$, $i = 1, 2$. This framework indicates that the system manager collects tax $p^k(x)$ from one agent and gives the same amount to the other agent as subsidy, so that the respective payoff functions are accordingly changed to stabilize the possibly desirable Nash equilibrium. Note that (2.36) implies that there is no compensation once the agents reach the target Nash equilibrium.

Corollary 2.2. Consider the 2-agent noncooperative system $\mathcal{G}(\{J_1, J_2\})$ with tax/subsidy approach (2.33) and the pseudo-gradient dynamics (2.32). If $p^k(x)$ in (2.33) satisfies

$$\det \mathcal{A}(\{\tilde{J}_1, \tilde{J}_2\}, 1_2, x^*) > 0, \tag{2.37}$$

then the Nash equilibrium x^* is stabilized for any positive constants α_1 and α_2.

Proof The result is a direct consequence of Proposition 2.2. □

As a typical form of the tax/subsidy approach, we consider the case with a simple quadratic incentive function given by

$$p^k(x) \triangleq k(x_1 - x_1^*)(x_2 - x_2^*), \tag{2.38}$$

which satisfies (2.35)–(2.36) for all $k \in \mathbb{R}$. In this case, since $\tilde{J}_i(x_i, x_{-i}^*) = J_i(x_i, x_{-i}^*)$ implies $\arg\max_{x_i \in \mathbb{R}} \tilde{J}_i(x_i, x_{-i}^*) = \arg\max_{x_i \in \mathbb{R}} J_i(x_i, x_{-i}^*) = x_i^*$ for each $i = 1, 2$, the state profile x^* remains the Nash equilibrium of $\mathcal{G}(\{\tilde{J}_1, \tilde{J}_2\})$. Moreover, since $\mathcal{A}(\{\tilde{J}_1, \tilde{J}_2\}, 1_2, x^*)$ $= \mathcal{A}(\{J_1, J_2\}, 1_2, x^*) + k \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, the condition (2.37) for k to stabilize the Nash equilibrium is given by

$$k \in (-\infty, \gamma_1) \cup (\gamma_2, \infty), \tag{2.39}$$

where

$$\gamma_1 =\frac{1}{2}\left(\frac{\partial^2 J_2(x^*)}{\partial x_2 \partial x_1} - \frac{\partial^2 J_1(x^*)}{\partial x_1 \partial x_2}\right) - \frac{1}{2}\sqrt{\left(\frac{\partial^2 J_1(x^*)}{\partial x_1 \partial x_2} + \frac{\partial^2 J_2(x^*)}{\partial x_2 \partial x_1}\right)^2 - 4\frac{\partial^2 J_1(x^*)}{\partial x_1^2}\frac{\partial^2 J_2(x^*)}{\partial x_2^2}} (< 0), \quad (2.40)$$

$$\gamma_2 =\frac{1}{2}\left(\frac{\partial^2 J_2(x^*)}{\partial x_2 \partial x_1} - \frac{\partial^2 J_1(x^*)}{\partial x_1 \partial x_2}\right) + \frac{1}{2}\sqrt{\left(\frac{\partial^2 J_1(x^*)}{\partial x_1 \partial x_2} + \frac{\partial^2 J_2(x^*)}{\partial x_2 \partial x_1}\right)^2 - 4\frac{\partial^2 J_1(x^*)}{\partial x_1^2}\frac{\partial^2 J_2(x^*)}{\partial x_2^2}} (> 0). \quad (2.41)$$

Similarly, consider the case with a simple quadratic incentive function

$$p^k(x) \triangleq \frac{1}{2}k\Big[(x_1 - x_1^*)^2 - (x_2 - x_2^*)^2\Big], \quad k \leq 0, \qquad (2.42)$$

which satisfies (2.35)–(2.36) for all $k \leq 0$. In this case, since (2.42) implies $\tilde{J}_1(x_1, x_2^*) = J_1(x_1, x_2^*) + \frac{1}{2}k(x_1 - x_1^*)^2$ and $\tilde{J}_2(x_2, x_1^*) = J_2(x_2, x_1^*) + \frac{1}{2}k(x_2 - x_2^*)^2$, it follows from $\arg\max_{x_i \in \mathbb{R}} J_i(x_i, x_{-i}^*) = x_i^*$, $i = 1, 2$, that $\arg\max_{x_i \in \mathbb{R}} \tilde{J}_i(x_i, x_{-i}^*) = x_i^*$, $i = 1, 2$, and hence the state profile x^* remains the Nash equilibrium of $\mathcal{G}(\tilde{J})$. Moreover, since $\mathcal{A}(\{\tilde{J}_1, \tilde{J}_2\}, \{1, 1\}, x^*) = \mathcal{A}(\{J_1, J_2\}, \{1, 1\}, x^*) + kI_2$, the condition (2.37) for k to stabilize the Nash equilibrium is given by

$$\begin{aligned} k < \gamma \triangleq &- \frac{1}{2}\left(\frac{\partial^2 J_1(x^*)}{\partial x_1^2} + \frac{\partial^2 J_2(x^*)}{\partial x_2^2}\right) - \frac{1}{2}\sqrt{\left(\frac{\partial^2 J_1(x^*)}{\partial x_1^2} + \frac{\partial^2 J_2(x^*)}{\partial x_2^2}\right)^2 - 4\det(\Psi)} \\ = &- \frac{1}{2}\left(\frac{\partial^2 J_1(x^*)}{\partial x_1^2} + \frac{\partial^2 J_2(x^*)}{\partial x_2^2}\right) - \frac{1}{2}\sqrt{\left(\frac{\partial^2 J_1(x^*)}{\partial x_1^2} - \frac{\partial^2 J_2(x^*)}{\partial x_2^2}\right)^2 + 4\frac{\partial^2 J_1(x^*)}{\partial x_1 \partial x_2}\frac{\partial^2 J_2(x^*)}{\partial x_2 \partial x_1}} (\leq 0) \end{aligned} \qquad (2.43)$$

where $\Psi = \mathcal{A}(\{J_1, J_2\}, \{1, 1\}, x^*)$.

For the case where the original payoff functions are quadratic as given in (2.30), the stabilizing condition of k for the incentive function (2.38) (resp., (2.42)) is given by (2.39) with $\gamma_1 = a_{12}^1 - a_{12}^2 - \sqrt{(a_{12}^1 + a_{12}^2)^2 - 4a_{11}^1 a_{22}^2}$, $\gamma_2 = a_{12}^1 - a_{12}^2 + \sqrt{(a_{12}^1 + a_{12}^2)^2 - 4a_{11}^1 a_{22}^2}$ (resp., $k < a_{11}^1 + a_{22}^2 - \sqrt{(a_{11}^1 - a_{22}^2)^2 + 4a_{12}^1 a_{12}^2}$).

Distributed Tax/Subsidy Approach for n-Agent Case

In the following, we extend the tax/subsidy approach characterized in the previous section to a higher-dimensional system $\mathcal{G}(J)$ with $\mathcal{N} = \{1, \ldots, n\}$. In particular, we suppose that the system manager decomposes the agents into several subgroups and installs *distributed* controllers (computers) for each of the subgroups. Each of the distributed controllers defines a utility transfer structure represented by a graph within the subgroup, which we call the *tax/subsidy adjustment graph*, such that the graph is weakly connected. Even though the controllers work in a distributed manner, the system manager needs to know, *a priori*, the information of the payoff functions of all the agents before the operation.

We suppose that the number of subgroups is c and the tax/subsidy adjustment graphs $\mathbb{G}_1, \ldots, \mathbb{G}_c$ are chosen as undirected graphs in such a way that there is no isolated agent that is free from the compensation mechanism. It is important to note that each distributed controller $\eta \in \{1, \ldots, c\}$ transfers the utilities between the agents consisting of $\mathbb{G}_\eta$ with the information from the same set of the agents, i.e., x_i, $i \in \mathcal{V}_\eta$, where $\mathcal{V}_\eta$ denotes the set of nodes constituting the tax/subsidy adjustment graph $\mathbb{G}_\eta$. Henceforth, let $\mathcal{N}_i$ be the set of neighbor agents for agent i.

Now, consider the adjusted payoff functions given by

$$\tilde{J}_i(x) \triangleq J_i(x) + p_i^K(x), \quad i \in \mathcal{N}, \tag{2.44}$$

with the quadratic incentive functions

$$\begin{aligned} p_i^K(x) \triangleq & \frac{1}{2} k_{ii}(x_i - x_i^*)^2 - \frac{1}{2} \sum_{j \in \mathcal{N}_i} k_{jj}(x_j - x_j^*)^2 / N_j \\ & + \sum_{j \in \mathcal{N}_i} k_{ij}(x_i - x_i^*)(x_j - x_j^*), \quad i \in \mathcal{V}_\eta, \end{aligned} \tag{2.45}$$

where $K = \{k_{ij}\}_{i,j \in \mathcal{N}} \in \mathcal{K} \triangleq \{K \in \mathbb{R}^{N \times N} : k_{ii} \leq 0, i \in \mathcal{N}, k_{ij} = -k_{ji}, i, j \in \mathcal{N}, i \neq j, k_{ij} = 0, j \notin \mathcal{N}_i, i \in \mathcal{N}\}$ and N_i is the number of the agents in $\mathcal{N}_i$. Note that $p_i^K(x)$ depends only on part of the agents' state x_i, $i \in \mathcal{V}_\eta$, in the subgraph $\mathbb{G}_\eta$. Furthermore, if there are multiple subgroups, then K can be transformed to a block-diagonal matrix by re-ordering the labels of the agents. Notice that the incentive functions given by (2.45) are a generalization of the combined functions of (2.38) and (2.42). Furthermore, (2.45) implies

$$\sum_{i \in \mathcal{N}} p_i^K(x) = 0, \quad x \in \mathbb{R}^n, \quad K \in \mathcal{K}, \tag{2.46}$$

$$p_i^K(x^*) = 0, \quad \frac{\partial p_i^K(x^*)}{\partial x_i} = 0, \quad \frac{\partial^2 \tilde{J}_i(x^*)}{\partial x_i^2} < 0, \tag{2.47}$$

for all $i \in \mathcal{N}$. In this case, since (2.45) implies $\tilde{J}_i(x_i, x_{-i}^*) = J_i(x_i, x_{-i}^*) + \frac{1}{2} k_{ii}(x_i - x_i^*)^2$, $i \in \mathcal{N}$, it follows from $\arg\max_{x_i \in \mathbb{R}} J_i(x_i, x_{-i}^*) = x_i^*$, $i \in \mathcal{N}$, that $\arg\max_{x_i \in \mathbb{R}} \tilde{J}_i(x_i, x_{-i}^*) = x_i^*$, $i \in \mathcal{N}$, and hence the state profile x^* remains the Nash equilibrium of $\mathcal{G}(\tilde{J})$. Consequently, the Jacobian matrix of the adjusted pseudo-gradient dynamics is written as $\mathcal{A}(\tilde{J}, \alpha, x^*) = \operatorname{diag}[\alpha] (\mathcal{A}(J, 1_n, x^*) + K)$.

The following result provides a way to determine $K \in \mathcal{K}$ in the incentive functions given by (2.45) for the n-agent noncooperative system.

Corollary 2.3. Consider the n-agent noncooperative system $\mathcal{G}(J)$ and the pseudo-gradient dynamics (2.32). If the matrix $K \in \mathcal{K}$ in (2.45) satisfies one of the following two sets of conditions:
(i) $\frac{\partial^2 J_i(x^*)}{\partial x_i^2} + k_{ii} < -\sum_{j \notin \mathcal{V}_\eta} \left|\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_j}\right| - \sum_{j \in \mathcal{V}_\eta \setminus \{i\}} \left|\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_j} + k_{ij}\right|$, $i \in \mathcal{V}_\eta$, $\eta \in \{1, \ldots, c\}$.
(ii) $\mathrm{diag}[\hat{\alpha}]K + K^{\mathrm{T}}\mathrm{diag}[\hat{\alpha}] + \mathcal{A}^{\mathrm{T}}(J, \hat{\alpha}, x^*) + \mathcal{A}(J, \hat{\alpha}, x^*) < 0$ for some $\hat{\alpha} \in \mathbb{R}_+^n$,
then the Nash equilibrium x^* is stabilized by the tax/subsidy approach (2.44), (2.45) for any positive constants α_i, $i \in \mathcal{N}$.

Proof Note that since $k_{ij} = 0$, $j \notin \mathcal{V}_\eta$, n number of inequalities characterized by (i) make $\mathcal{A}(\tilde{J}, 1_n, x^*)$ strictly diagonally dominant (i.e., $\frac{\partial^2 \tilde{J}_i(x^*)}{\partial x_i^2} < -\sum_{j \neq i} |\frac{\partial^2 \tilde{J}_i(x^*)}{\partial x_i \partial x_j}|$ for all $i \in \mathcal{N}$), and the inequality in (ii) makes $\mathcal{A}^{\mathrm{T}}(\tilde{J}, \hat{\alpha}, x^*) + \mathcal{A}(\tilde{J}, \hat{\alpha}, x^*) = \mathrm{diag}[\hat{\alpha}]K + K^{\mathrm{T}}\mathrm{diag}[\hat{\alpha}] + \mathcal{A}^{\mathrm{T}}(J, \hat{\alpha}, x^*) + \mathcal{A}(J, \hat{\alpha}, x^*)$ negative definite. Hence, the two results are direct consequences of Gershgorin's circle theorem and Theorem 2.1, respectively. □

Remark 2.8. Corollary 2.3 indicates that with the information of agents' original payoff functions $J_1, \ldots, J_n$, the system manager can command the distributed controllers to process the tax/subsidy framework (2.44), (2.45) by transmitting the information of corresponding elements of a well-chosen matrix K to the distributed controllers. As such, the system manager can stabilize the target Nash equilibrium x^* for arbitrary α_i, $i \in \mathcal{N}$, even though the sensitivity parameters α_i, $i \in \mathcal{N}$, are *unknown* to him/her.

It can be easily found that n number of inequalities characterized by (i) are always solvable for $K \in \mathcal{K}$ such that $\mathcal{A}(\tilde{J}, 1_n, x^*)$ is strictly diagonally dominant, because k_{ii}, $i \in \mathcal{N}$, can be taken to be sufficiently small so that each agent's own utility is dominant compared to the effect by the other agents. Moreover, even though the inequality characterized in (ii) is a special linear matrix inequality with the constraint $K \in \mathcal{K}$, it is possible to make (ii) (i.e., $\mathcal{A}(\tilde{J}, \hat{\alpha}, x^*) + \mathcal{A}^{\mathrm{T}}(\tilde{J}, \hat{\alpha}, x^*)$) strictly diagonally dominant to make sure that it is negative definite, i.e.,

$$\sum_{j \neq i} \left| \hat{\alpha}_i \frac{\partial^2 J_i(x)}{\partial x_i \partial x_j} + \hat{\alpha}_j \frac{\partial^2 J_j(x)}{\partial x_j \partial x_i} + (\hat{\alpha}_i - \hat{\alpha}_j) k_{ij} \right| < -2\hat{\alpha}_i \Big(\frac{\partial^2 J_i(x)}{\partial x_i^2} + k_{ii} \Big), \quad i \in \mathcal{N}, \tag{2.48}$$

for $x = x^*$. It is interesting to see that (2.48) can determine $\{k_{ij}\}_{i \in \mathcal{N}, j \in \{i+1, \ldots, n\}}$ with a given $\hat{\alpha}$ satisfying $\hat{\alpha}_i - \hat{\alpha}_j \neq 0, i \in \mathcal{N}_j$, $j \in \mathcal{N}_i$, and $k_{ii} \leq 0, i \in \mathcal{N}$. Furthermore, when (2.48) is satisfied for all $x \in \mathbb{R}^n$, it can be shown that the possibly unstable Nash equilibrium x^* is *globally* asymptotically stabilized.

Remark 2.9. The conditions (i) in Corollary 2.3 also indicate that each distributed controller $\eta \in \{1, \ldots, c\}$ can independently choose parameters $\{k_{ij}\}_{i,j \in \mathcal{V}_\eta}$, if the information of $\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_j}$, $j \in \mathcal{N}$, $i \in \mathcal{V}_\eta$, is given. In other words, each distributed controller η can work in a decentralized way even for the case where the number of the agents is large.

Remark 2.10. In the case where the number n of the agents is so large that the calculation of the target Nash equilibrium x^* is infeasible, our proposed framework can be similarly implemented without calculating the Nash equilibria for $\mathcal{G}(J)$. Specifically, by setting $\hat{x}^*$ as the target state, the incentive functions for the subgroup η, $\eta \in \{1, \ldots, c\}$, are given by

$$\begin{aligned} p_i^K(x) \triangleq \; & \frac{1}{2} k_{ii}(x_i - \hat{x}_i^*)^2 - \frac{1}{2} \sum_{j \in \mathcal{N}_i} k_{jj}(x_j - \hat{x}_j^*)^2 / N_j \\ & + \sum_{j \in \mathcal{N}_i} k_{ij}(x_i - \hat{x}_i^*)(x_j - \hat{x}_j^*) + \beta_i (x_i - \hat{x}_i^*) \\ & - \sum_{j \in \mathcal{N}_i} \beta_j (x_j - \hat{x}_j^*)/N_j, \quad i \in \mathcal{V}_\eta, \end{aligned} \tag{2.49}$$

with $\beta_i \in \mathbb{R}$, $i \in \mathcal{V}_\eta$, satisfying

$$\arg\max \tilde{J}_i(x_i, \hat{x}_{-i}^*) = \hat{x}_i^*, \quad i \in \mathcal{V}_\eta, \tag{2.50}$$

and $\{k_{ij}\}_{i,j \in \mathcal{V}_\eta}$ satisfying the condition (i) in Corollary 2.3 with x^* replaced by $\hat{x}^*$. Note that when the target state $\hat{x}^*$ is not the original Nash equilibrium x^* in $\mathcal{G}(J)$, the linear terms $\beta_i(x_i - \hat{x}_i^*) - \sum_{j \in \mathcal{N}_i} \beta_j (x_j - \hat{x}_j^*)/N_j$ of the incentive functions (2.49) with $\beta_i \in \mathbb{R}$, $i \in \mathcal{V}_\eta$, satisfying (2.50), contribute to make the target state $\hat{x}^*$ a Nash equilibrium in $\mathcal{G}(\tilde{J})$. In such a case, it is understood that the original Nash equilibrium x^* in $\mathcal{G}(J)$ is moved to the target state $\hat{x}^*$ in $\mathcal{G}(\tilde{J})$ under the proposed tax/subsidy approach. Alternatively, when the target state $\hat{x}^*$ happens to be the same as the original Nash equilibrium x^* in $\mathcal{G}(J)$, the condition (2.50), which is met by the distributed controller, requires $\beta_i = 0$ in order for (2.49) to reduce to (2.45). It is worth noting that the establishment of (2.49) does *not* force the system manager to collect global information of the payoff functions $J_i(x)$, $i \in \mathcal{N}$, since the target state $\hat{x}^*$ is *not* necessary to be the original Nash equilibrium x^* in $\mathcal{G}(J)$.

2.5 Illustrative Numerical Examples

In this section, a couple of numerical examples are presented for illustrating the results and the conditions concerning the proposed zero-sum tax/subsidy mechanism. The first example exhibits diverging trajectory whereas the trajectory of the second example converges to one of the Nash equilibria which is not the target one without the tax/subsidy mechanism.

Example 2.5. Consider a wireless communication system being composed of n senders who compete with each other on quality of service characterized by signal-to-interference-plus-noise-ratio for a unique receiver [96]. Each sender (agent) adjusts its transmission power $x_i \in \mathbb{R}_+$ to maximize its profit given by

$$J_i(x) = \beta_0 \log_{10}(1 + \frac{g_i x_i L}{\sum_{j \neq i} g_i x_j + \sigma}) - \beta_i x_i, \quad i \in \mathcal{N}, \tag{2.51}$$

where $\beta_0 \in \mathbb{R}_+$ denotes the earning rate for service quality, $\sigma \in \mathbb{R}_+$ denotes the additive white noise, $L \in \mathbb{R}_+$ denotes the spreading gain, $g_i \in \mathbb{R}_+$, $i \in \mathcal{N}$, denote the channel gain, and $\beta_i \in \mathbb{R}_+$, $i \in \mathcal{N}$, denote the price per unit power. Suppose $n = 2$, $\beta_0 = 1$, $\sigma = 0.1$, $L = 0.5$, $g_1 = 1$, $g_2 = 2$, $\beta_1 = 0.1$, $\beta_2 = 0.2$, so that there exists a unique Nash equilibrium $x^* = [1.3810, 0.6905]^{\mathrm{T}}$. It follows from Corollary 2.1 that x^* is unstable under the pseudo-gradient dynamics (2.5) for any $\alpha \in \mathbb{R}_+^2$ since $\det \mathcal{A}(J, 1_2, x^*) = -0.0064 < 0$.

Now, it follows from Corollary 2.2 that the tax/subsidy approach (2.33) along with the incentive function (2.42) with $k = -0.3 < \gamma = -0.0408$ satisfying (2.43) guarantees that the target Nash equilibrium x^* is asymptotically stabilized for any $\alpha \in \mathbb{R}_+^2$. (In fact, the choice of $k = -0.3$ also satisfies (42) for all $x \in \mathbb{R}_+^2$ with $k_{11} = k_{22} = k$, $k_{12}(= -k_{12}) = 0$, and $\hat{\alpha}_1 = \hat{\alpha}_2 = 1$ so that global asymptotic stabilization is guaranteed.) The initial state is set to $x(0) = [1, 0]^{\mathrm{T}}$ in the simulation. Figure 2.4 shows the trajectories of agents' states under the pseudo-gradient dynamics (2.5) with 10 different values of α satisfying $\alpha_1 \in [20, 50]$ and $\alpha_2 \in [30, 85]$. It can be seen from the figure that the agents' state converges to x^* with the tax/subsidy approach for all those various sensitivity parameters.

Example 2.6. Consider the noncooperative system being composed of five agents with *non-quadratic* payoff functions given by $J_1(x) = -(x_1 + \sin x_2 - 0.5 \sin x_3)^2 + e^{(-x_1^2 - x_2^2 - x_3^2)}$, $J_2(x) = -\frac{1}{2}(2x_2 - \sin x_1 + 2\sin x_3)^2 + e^{(-x_1^2 - x_2^2 - x_3^2)}$, $J_3(x) = -\frac{1}{3}(3x_3 + 3\sin x_1 - \sin x_2 - \sin x_4 + \sin x_5)^2 + e^{(-x_1^2 - x_2^2 - x_3^2 - x_4^2 - x_5^2)}$, $J_4(x) = -(x_4 - 2\sin x_3 + \sin x_5)^2 + e^{(-x_3^2 - x_4^2 - x_5^2)}$, $J_5(x) = -(x_5 + 3\sin x_3 + 2\sin x_4)^2 + e^{(-x_3^2 - x_4^2 - x_5^2)}$, where the payoff dependency network

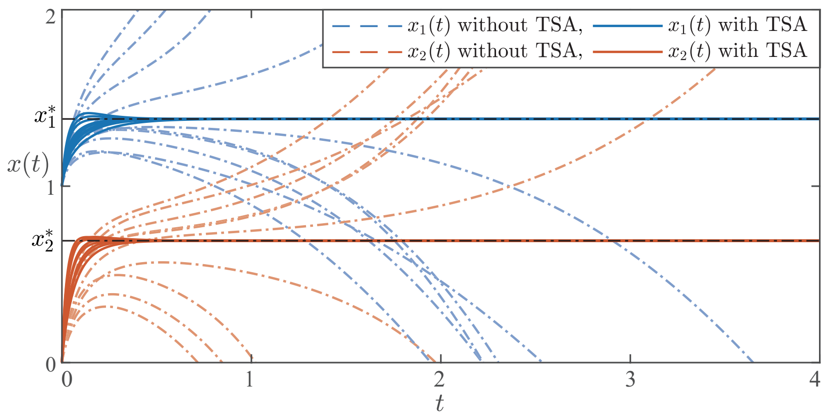


Figure. 2.4 Trajectories of the states with and without the zero-sum tax/subsidy approach (TSA) under 10 different sets of sensitivity parameters $\alpha_1 \in [20, 50]$ and $\alpha_2 \in [30, 85]$. The trajectories of agents' states diverge without the tax/subsidy approach but converge to the target Nash equilibrium x^* with the proposed tax/subsidy approach for the same set of sensitivity parameters.

topology is shown in Fig. 2.5. Note that the noncooperative system possesses multiple Nash equilibria and $x^* = [0, 0, 0, 0, 0]^{\mathrm{T}}$ is one of the Nash equilibria which maximizes every agent's payoff. In this example, since $\det \mathcal{A}(J, 1_5, x^*) = 482.67 > 0$, it follows from Corollary 2.1 that the Nash equilibrium is unstable under the pseudo-gradient dynamics (2.5) for any $\alpha \in \mathbb{R}_+^5$.

To achieve stabilization of the Nash equilibrium x^* by employing (2.44), (2.45), we decompose the agents into 2 subgroups and and install distributed controllers for each of the subgroups. We let the distributed controllers' tax/subsidy adjustment graphs $\mathbb{G}_1, \mathbb{G}_2$ be given by Fig. 2.5 so that agents' payoffs are transferred between agents 1 and 2 in $\mathcal{V}_1 = \{1, 2\}$ and between agents 3 and 5 as well as between agents 4 and 5 in $\mathcal{V}_2 = \{3, 4, 5\}$. In this case, only the parameters $\{k_{11}, k_{22}, k_{12}\}$ and $\{k_{33}, k_{44}, k_{55}, k_{35}, k_{45}\}$ should be designed because $K = \{k_{ij}\}_{i,j \in \{1,\ldots,5\}}$ should belong to the class $\mathcal{K}$. Specifically, suppose that the system manager provides the information of $\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_j}$, $j \in \{1, \ldots, 5\}$, $i \in \mathcal{V}_1$ (resp., $i \in \mathcal{V}_2$) to the distributed controller for $\mathbb{G}_1$ (resp., $\mathbb{G}_2$). Then it follows from conditions (i) of Corollary 2.3 and Remark 2.9 that the tax/subsidy approach (2.44) along with the incentive functions (2.45) with the choice $\{k_{11} = -2,\ k_{22} = 0,\ k_{12} = 1\}$ for $\mathbb{G}_1$ and $\{k_{33} = -4.2,\ k_{44} = -2,\ k_{55} = -10,\ k_{35} = k_{45} = 1\}$ for $\mathbb{G}_2$ guarantees that the target Nash equilibrium x^* is asymptotically stabilized for arbitrary $\alpha \in \mathbb{R}_+^5$. Furthermore, since these parameters in K happen to satisfy (2.48) for all $x \in \mathbb{R}^5$ with $\hat{\alpha}_1 = 0.6$, $\hat{\alpha}_2 = 0.2$, $\hat{\alpha}_3 = 0.3$, $\hat{\alpha}_4 = 0.2$, $\hat{\alpha}_5 = 0.05$, we can further guarantee (with the global knowledge of the payoff functions) that the target Nash equilibrium x^* is *globally* asymptotically

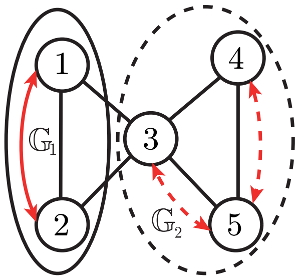


Figure. 2.5 Network topology of payoff dependency (black) and tax/subsidy adjustment graphs $\mathbb{G}_1$ and $\mathbb{G}_2$ (red). Agents' payoffs are transferred between agents 1 and 2 in subgroup 1 and between agents 3 and 5 as well as between agents 4 and 5 in subgroup 2.

stabilized for arbitrary $\alpha \in \mathbb{R}^5_+$. In this case, the incentive functions (2.45) are given by $\{p_1^K(x) = -p_2^K(x) = -x_1^2 + x_1x_2\}$ for $\mathbb{G}_1$ and $\{p_3^K(x) = -2.1x_3^2 + x_3x_5 + 2.5x_5^2,$ $p_4^K(x) = -x_4^2 + x_4x_5 + 2.5x_5^2,$ $p_5^K(x) = -5x_5^2 - x_3x_5 - x_4x_5 + 2.1x_3^2 + x_4^2\}$ for $\mathbb{G}_2$. The initial state is set to $x(0) = [2, 1, 0, -1, 2]^{\mathrm{T}}$ in the simulation. Figure 2.6 shows the trajectories of agents' states under the pseudo-gradient dynamics (2.5) with 8 different values of α satisfying $\alpha_1 \in [1, 4]$, $\alpha_2 \in [2, 4]$, $\alpha_3 \in [1, 4]$, $\alpha_4 \in [2, 3]$ and $\alpha_5 \in [2, 3]$. It can be seen from the figure that without tax/subsidy approach, the agents' state converges to another Nash equilibrium $\tilde{x}^* = [-0.1356, 0.1146, -0.2884, -1.075, 2.611]^{\mathrm{T}}$ instead of the target Nash equilibrium x^* at the origin for all those various sensitivity parameters. However, the agents' state converges to x^* when we apply the tax/subsidy approach for the same set of sensitivity parameters.

2.6 Chapter Conclusion

In this chapter, we investigated the Nash equilibrium stabilization problem for noncooperative dynamical systems through a tax/subsidy approach. In the proposed tax/subsidy approach, a system manager collects some taxes from some of the agents and gives the same amount in total as subsidies to the neighbor agents in the tax/subsidy adjustment graphs. To deal with the uncertainty in terms of the private information, we explored the stability conditions of Nash equilibria without knowing the private information, and also obtained the conditions under which the state trajectory converges to the originally unstable Nash equilibrium using incentive functions. Finally, we provided the numerical examples for demonstrating stabilization of unstable Nash equilibrium for two-agent and five-agent noncooperative systems.

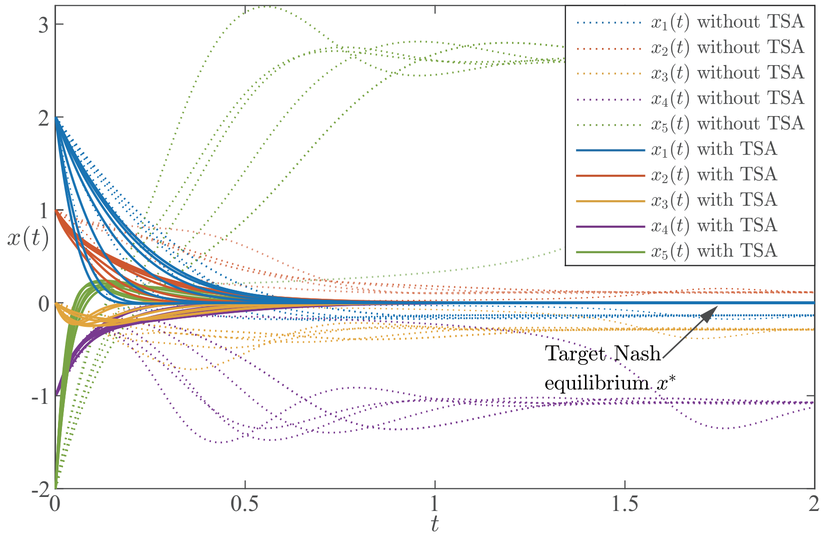


Figure. 2.6 Trajectories of the states with and without the tax/subsidy approach (TSA) under 8 different sets of sensitivity parameters $\alpha_1 \in [1,4]$, $\alpha_2 \in [2,4]$, $\alpha_3 \in [1,4]$, $\alpha_4 \in [2,3]$ and $\alpha_5 \in [2,3]$. The trajectories of agents' states converge to another Nash equilibrium $\tilde{x}^*$ without the tax/subsidy approach but converge to the target Nash equilibrium x^* with the proposed tax/subsidy approach for the same set of sensitivity parameters.

Chapter 3

Control of Large-Scale Noncooperative Dynamical Systems: Hierarchical Incentive Framework

3.1 Introduction

In this chapter, we focus on the social welfare improvement problem for large-scale hierarchical noncooperative dynamical systems driven by the pseudo-gradient dynamics. Specifically, we assume that the agents in the noncooperative system belong to one of the several groups and are influenced by the corresponding group managers via some intra-group incentives. We characterize the situation where group managers try to enhance the welfare of their own groups by continually updating their own intra-group incentives to the group members. We explore the stability of group Nash equilibrium of the hierarchical noncooperative systems, and derive conditions where the trajectory of agents' state converges to the group Nash equilibrium under group managers' intra-group incentives. Furthermore, we propose the inter-group incentive mechanism for a system governor in order to reconstruct the group utility functions in the group managers level to move the group Nash equilibrium so that the social (entire) welfare is improved. To deal with the situation where the system governor may not know all the agents' individual payoff functions and all the agents' state, we present sufficient conditions to guarantee the convergence of agents' state towards a target (suboptimal but not optimal due to the lack of enough information) equilibrium using some macroscopic data.

The rest of this chapter is organized as follows. We explain hierarchical noncooperative systems with dynamic agents under intra-group incentives in Section 3.2. In Section 3.3, we propose a couple of update rules for the group managers to update their intra-group incentives. Furthermore, in Section 3.4, we characterize the inter-group incentive mechanisms in the manager layer to increase the social welfare of the entire multiagent system. A couple of illustrative numerical examples are presented in Section 3.5. Conclusions are given in Section 3.6.

3.2 Problem Formulation

3.2.1 System Description

Consider the hierarchical noncooperative system consisting of an agent layer and a manager layer, where n number of agents belong to one of the m number of groups in the agent layer and are influenced by the corresponding group managers with some intra-group incentives. Let $\mathbb{M} = \{1, \ldots, m\}$ denote the set of groups and let n_k denote the number of agents in group $k \in \mathbb{M}$, where $\sum_{k\in\mathbb{M}} n_k = n$. The set of overall agents is denoted by $\mathcal{N} = \{1, \ldots, n\} = \{\mathbb{N}_1, \ldots, \mathbb{N}_m\}$, where $\mathbb{N}_k$ denotes the set of members (agents) in group $k \in \mathbb{M}$ satisfying $\mathbb{N}_k \cap \mathbb{N}_j = \emptyset$, $j \in \mathbb{M}$, $j \neq k$. Let $x = [x_1, \ldots, x_n]^{\mathrm{T}} = [(x^1)^{\mathrm{T}}, \ldots, (x^m)^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^n$ denote the state profile of all the agents, where $x_i \in \mathbb{R}$ denotes the state of agent i, and $x^k \in \mathbb{R}^{n_k}$ denotes the state profile of the agents in group $k \in \mathbb{M}$. The payoff function of agent $i \in \mathcal{N}$ without incentive is denoted by $J_i : \mathbb{R}^n \to \mathbb{R}$, which may depend on all the agents' state and is supposed to be continuously differentiable and strictly concave with respect to x_i.

In this chapter, we assume that the m number of the group managers try to enhance the welfare of their own groups, which they evaluate by the individual group utility functions, by imposing *intra-group* incentive mechanism to the agents in their own groups. The group utility functions $U^k : \mathbb{R}^n \to \mathbb{R}$, $k \in \mathbb{M}$, are defined as the weighted sum of the payoff functions of their own group members, i.e.,

$$U^k(x) \triangleq \sum_{i\in\mathbb{N}_k} \eta_i J_i(x), \quad k \in \mathbb{M}, \tag{3.1}$$

where $\eta_i \in \mathbb{R}_+$, $i \in \mathbb{N}_k$, denote the weights (priorities) of the agents evaluated by the group manager $k \in \mathbb{M}$. Furthermore, we assume that there is the system governor who also imposes a similar *inter-group* incentive mechanism on the manager layer so that

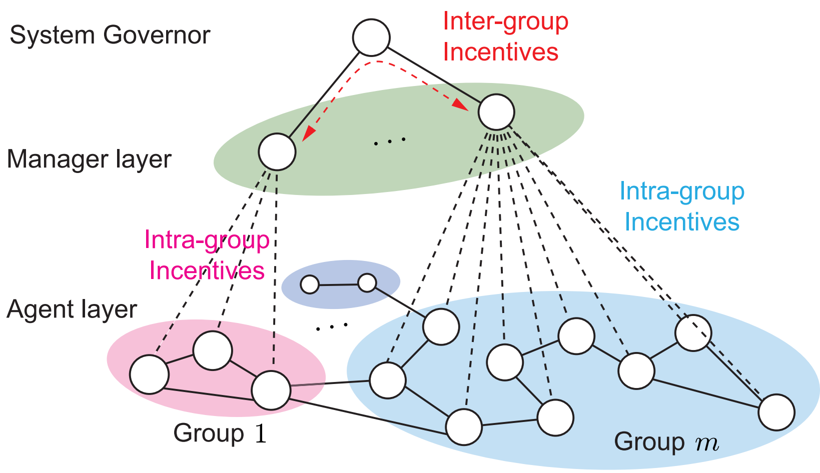


Figure. 3.1 Hierarchical noncooperative system with m groups of agents. Agents are incentivized by group managers for the benefit of group utility. The network in the agent layer represents payoff dependencies. A system governor (e.g., president) appears at the top of the hierarchy and constructs some incentive mechanism among the group managers (e.g., mayors), which we call *inter-group* incentive mechanism. The detailed discussion of the inter-group incentives imposed by the system governor for improving the social welfare with limited information are given in Section 3.4 below.

the welfare of the entire agents defined by

$$\Pi(x) \triangleq \sum\nolimits_{k\in\mathbb{M}} \xi_k U^k(x), \tag{3.2}$$

for some weights $\xi_k \in \mathbb{R}_+$, $k \in \mathbb{M}$, of the groups is improved (see the structure of the hierarchical noncooperative system illustrated in Fig. 3.1).

In order to improve the group utility, the group managers shift the Nash equilibrium (defined in Definition 3.1 below) of the group through the intra-group incentive mechanism. Specifically, the incentivized payoff functions for each agent to increase are given by

$$\tilde{J}_i(u^k, x) \triangleq J_i(x) + p_i^k(u^k, x^k), \quad i \in \mathbb{N}_k, \quad k \in \mathbb{M}, \tag{3.3}$$

where p_i^k denotes the intra-group incentive function imposed by group manager $k \in \mathbb{M}$ to the agents in $\mathbb{N}_k$ under its control given by

$$p_i^k(u^k, x^k) = u_i^k x_i - \sum\nolimits_{j\in\mathbb{N}_k\setminus\{i\}} \frac{u_j^k x_j}{n_k - 1}, \quad i \in \mathbb{N}_k, \tag{3.4}$$

and $u^k = \{u_i^k\}_{i\in\mathbb{N}_k} \in \mathbb{R}^{n_k}$ denotes the strategy of the group manager k. Note that the group managers serve merely as mediators transferring payoffs among the agents so that the sum of the incentive functions is zero, i.e., $\sum_{i\in\mathbb{N}_k} p_i^k(u^k, x^k) = 0$, $k \in \mathbb{M}$.

Definition 3.1. Given the strategy $u = [(u^1)^{\mathrm{T}}, \dots, (u^m)^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^n$ of the group managers, the profile $x^*(u) \in \mathbb{R}^n$ is called a Nash equilibrium with respect to $\{\tilde{J}_i(u^k, x)\}_{i\in\mathcal{N}}$ given by (3.3) if

$$\tilde{J}_i(u^k, x_i^*(u), x_{-i}^*(u)) \geq \tilde{J}_i(u^k, x_i, x_{-i}^*(u)), \quad x_i \in \mathbb{R}, \tag{3.5}$$

holds for all $i \in \mathbb{N}_k$ and $k \in \mathbb{M}$, where x_{-i} is the agents' state profile except agent i.

With a given u, since $\tilde{J}_i(u^k, x)$ is strictly concave with respect to x_i for all $i \in \mathcal{N}$ under (3.4), the Nash equilibrium $x^*(u)$ satisfies

$$0 = \frac{\partial \tilde{J}_i^k(u^k, x^*(u))}{\partial x_i} = \frac{\partial J_i^k(x^*(u))}{\partial x_i} + u_i^k, \quad i \in \mathbb{N}_k, \tag{3.6}$$

for all $k \in \mathbb{M}$. On the other hand, at the Nash equilibrium $x^*(u)$, the group manager k may wish to unilaterally change its strategy u^k to benefit its own group when $\arg\max_{x^k\in\mathbb{R}^{n_k}} U^k(x^k, x^{-k^*}(u)) \neq x^{k^*}(u)$ holds, where $x^{k^*}(u) \triangleq \{x_i^*(u)\}_{i\in\mathbb{N}_k} \in \mathbb{R}^{n_k}$ and $x^{-k^*}(u) \triangleq \{x_i^*(u)\}_{i\notin\mathbb{N}_k} \in \mathbb{R}^{n-n_k}$. This observation induces another concept of equilibrium at which no group manager can benefit its own group more by unilaterally changing its strategy for the intra-group incentives.

Definition 3.2. For the group utility functions $U^k(x)$, $k \in \mathbb{M}$, the profile $x_\triangle \in \mathbb{R}^n$ is called a group Nash equilibrium if

$$U^k(x_\triangle^k, x_\triangle^{-k}) \geq U^k(x^k, x_\triangle^{-k}), \quad x^k \in \mathbb{R}^{n_k}, \quad k \in \mathbb{M}, \tag{3.7}$$

where x^{-k} is the agents' state profile except group k.

It is worth mentioning that both of the Nash equilibrium and the group Nash equilibrium are characterized independently of the agents' underlying dynamics. Since $U^k(x)$ is continuously differentiable for all $k \in \mathbb{M}$, the existing group Nash equilibrium $x_\triangle$ satisfies

$$0 = \left[\frac{\partial U^1(x_\triangle)}{\partial x^1}, \dots, \frac{\partial U^m(x_\triangle)}{\partial x^m}\right] \in \mathbb{R}^{1\times n}. \tag{3.8}$$

Definition 3.3. The strategy $u_\triangle = [(u^1_\triangle)^{\mathrm{T}}, \ldots, (u^m_\triangle)^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^n$ is called a subgame perfect equilibrium for intra-group incentives if the corresponding Nash equilibrium $x^*(u_\triangle)$ coincides with the group Nash equilibrium $x_\triangle$.

In this chapter, we consider the situation where each agent is a selfish and dynamic decision maker continually changing its own state by following the pseudo-gradient dynamics [17] in terms of the *incentivized* payoff functions, i.e.,

$$\dot{x}_i(t) = \alpha_i \frac{\partial \tilde{J}_i(u^k(t), x(t))}{\partial x_i}, \quad i \in \mathbb{N}_k, \quad k \in \mathbb{M}, \tag{3.9}$$

where $\alpha_1, \ldots, \alpha_n$ denote the agent-dependent sensitivity parameters. The pseudo-gradient dynamics (3.9) capture the fact that the agents concern their own incentivized payoffs and myopically change their states according to the current information without any foresight on the future state [20, 21, 44, 25, 42]. Consequently, the agents' state dynamics (3.9) with the intra-group incentive functions (3.4) are described by the dynamics given by

$$\dot{x}(t) = \mathrm{diag}[\alpha](f(x(t)) + u(t)), \quad x(0) = x_0, \quad t \geq 0, \tag{3.10}$$

where $f(x) \triangleq [\frac{\partial J_1(x)}{\partial x_1}, \ldots, \frac{\partial J_n(x)}{\partial x_n}]^{\mathrm{T}}$ denotes the pseudo-gradient function characterized by the agents' individual payoff functions, and $\alpha \triangleq (\alpha_1, \ldots, \alpha_n)$.

It is important that for a given $u(t) \equiv \bar{u} \in \mathbb{R}^n$, all the Nash equilibria of the noncooperative system are the equilibria of the dynamics (3.10) since $\dot{x}(t) \equiv 0$ holds under (3.6) with u replaced by $\bar{u}$. In general, there may be multiple Nash equilibria in the noncooperative system. Some sufficient conditions for existence of a unique Nash equilibrium can be found in [17] and [90, Chapter 2], which can also guarantee global stability of the pseudo-gradient dynamics with $u(t) \equiv 0$. For example, supposing that the Jacobian matrix $\nabla f(x)$ of the pseudo-gradient function $f(x)$ satisfies $(\nabla f(x))^{\mathrm{T}}\mathrm{diag}[\hat{\alpha}] + \mathrm{diag}[\hat{\alpha}]\nabla f(x) < 0$, $x \in \mathbb{R}^n$, for some $\hat{\alpha} \in \mathbb{R}^n_+$, it can be shown that the nonincentivized system exhibits a unique and globally asymptotically stable Nash equilibrium under the pseudo-gradient dynamics (with $u(t) \equiv 0$). Alternatively, supposing that the nonincentivized system is a strictly monotone game (i.e., the pseudo-gradient function $f(x)$ satisfies $(f(x) - f(x'))^{\mathrm{T}}(x - x') < 0$ for all x, $x' \in \mathbb{R}^n$, $x \neq x'$) [90], it can be also shown that the nonincentivized system exhibits a unique and globally asymptotically stable Nash equilibrium under the pseudo-gradient dynamics (with $u(t) \equiv 0$). In these two cases, for a given $u(t) \equiv \bar{u} \in \mathbb{R}^n$, noticing that the matrix $(\nabla(f(x) + \bar{u}))^{\mathrm{T}}\mathrm{diag}[\hat{\alpha}] + \mathrm{diag}[\hat{\alpha}]\nabla(f(x) + \bar{u})$ remains as a negative-definite matrix or

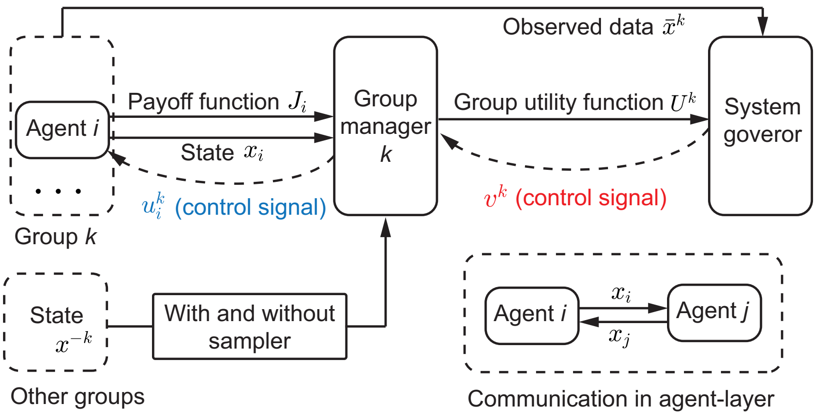


Figure. 3.2 Block diagram of signal flows between the layers with solid (resp., dashed) arrows representing available signals of information (resp., control signals). The strategy u_i^k of group manager k and the inter-group incentive coefficient v^k (introduced in Section 3.4 below) are understood as the control signals to the agents in $\mathbb{N}_k$ and the group managers, respectively.

the noncooperative system remains as a strictly monotone game, the Nash equilibrium $x^*(\bar{u})$ is the unique and globally asymptotically stable equilibrium of the dynamics (3.10) satisfying $\dot{x}(t) \equiv 0$. Therefore, by well designing the strategies u^k, $k \in \mathbb{M}$, for the intra-group incentive schemes, the group managers may be able to move the Nash equilibrium to a state possessing a better group utility than the nonincentivized $(u(t) \equiv 0)$ case.

3.2.2 Motivations, Information Hierarchy, and Problems

Motivation 1: In general, the group manager k is not able to obtain the group utility functions $U^{-k}(\cdot)$ from the other groups. The group managers may continually change their own strategy $u^k(t)$, $k \in \mathbb{M}$, $t \geq 0$, in order to change the Nash equilibrium to a state associated with a better group utility.

Motivation 2: Given the subgame perfect equilibrium $u_\triangle$, even though the agents' state may reach the group Nash equilibrium $x_\triangle$, the entire social welfare may still not be maximized because the group managers do not cooperate with each other. Since the fact that the system governor may not know the full information of the agents' state and payoff functions makes it difficult to control the entire system, a fundamental question is how to design the inter-group incentive mechanism among the group managers to improve the social welfare only using some *low dimensional (macroscopic) data* observed by the system governor.

Consequently, the information hierarchy among the three layers of the hierarchical noncooperative system is summarized below.

Available information for group managers: In this paper, we assume that group manager k has access to the payoff functions $J_i(\cdot)$, $i \in \mathbb{N}_k$, and the state $x_i(t)$, $i \in \mathbb{N}_k$, in its own group. The state profile $x^{-k}(t)$ of other groups can be continually or intermittently observed by group manager k. No communication between the group mangers is assumed, i.e., the strategies of the other group managers is unavailable. The block diagram of information hierarchy is illustrated in Fig. 3.2.

Available information for agents: The state profile $x(\cdot)$ is available for all the agents. No information of payoff functions is exchanged among the agents. The signal $u_i^k(t)$ from group manager k is available only for agent i.

Available information for system governor: We suppose that the system governor does not know the full information of the agents' state and payoff functions, but have access to the group utility functions $U^k(\cdot)$, $k \in \mathbb{M}$, and the *low dimensional (macroscopic) data* $\bar{x}^k$, $k \in \mathbb{M}$ (defined in Section 3.4 later) from the groups.

Now, we present the main problem of this chapter.

Problem: Considering the hierarchical noncooperative dynamical system, our main objectives in the paper are two folds: i) Design some update rules for the group manager $k \in \mathbb{M}$ to continually update its strategy $u^k(t)$ only using the information on the agents' state $x(t)$ and payoff functions $J_i(\cdot)$, $i \in \mathbb{N}_k$, without the knowledge of the strategies of the other group managers; ii) Design the inter-group incentive mechanism among the group managers to stabilize a target equilibrium for improving the entire social welfare using limited information.

3.3 Update Rules for Group Managers' Intra-group Incentives

In this section, we propose our update rule for group manager k to update $u^k(t)$ for its intra-group incentive mechanism under the scenarios with 2 types of observations, i.e., continual and intermittent observations, on $x^{-k}(t)$ whereas the state information $x^k(t)$ of its own group is available for all t. Furthermore, we assume that group manager k has access to the payoff functions $J_i(\cdot)$, $i \in \mathbb{N}_k$, in its own group and no communication between the group mangers is assumed in this chapter.

3.3.1 Update Rule with Continual Observation

In this section, we consider the situation where the value of $x^{-k}(t)$ is fully observed by group manager k for every time instant $t \geq 0$. Note that our main idea in constructing the update rule of $u^k(t)$ for group manager k is to make the best-response state for group k coincide with the individual best-response state for all the group members in $\mathbb{N}_k$. Specifically, we consider the update rule for the group managers given by

$$u_i^k(t) = -\frac{\partial J_i(\hat{x}^k(t), x^{-k}(t))}{\partial x_i}, \quad i \in \mathbb{N}_k, \quad k \in \mathbb{M}, \tag{3.11}$$

where

$$\hat{x}^k(t) = \gamma^k(x^{-k}(t)) \triangleq \arg\max_{x^k \in \mathbb{R}^{n_k}} U^k(x^k, x^{-k}(t)), \tag{3.12}$$

represents the best-response state of group k given the other groups' state $x^{-k}(t)$. The update rule (3.11) captures the fact that the group managers concern their own group utilities and myopically change their strategies according to the current information without foresight on the other groups' future state. Note that the best-response state $\hat{x}^k(t)$ of group k is invariant under the same priority ratio, i.e., $\eta_{i+1} : \eta_{i+2} : \cdots : \eta_{i+n_k}$ with $i = n_1 + n_2 + \ldots + n_{k-1}$.

Assumption 3.1. The group utility function $U^k(x)$ is strictly concave with respect to x^k for each group $k \in \mathbb{M}$.

Assumption 3.1 ensures that there is a unique $\hat{x}^k$ for given x^{-k}. Recalling that the group utility function $U^k(x)$ is continuously differentiable, the mapping $\gamma^k : \mathbb{R}^{n-n_k} \to \mathbb{R}^{n_k}$ in (3.12) is understood as a *continuously differentiable* function with respect to x^{-k} for each group $k \in \mathbb{M}$. For the statement of the following results, we let the state profile $x \in \mathbb{R}^n$ be partitioned by $x = \left[\begin{array}{c:c:c} (x_-^k)^{\mathrm{T}} & (x^k)^{\mathrm{T}} & (x_+^k)^{\mathrm{T}} \end{array}\right]^{\mathrm{T}}$, where $x_-^k \in \mathbb{R}^{\sum_{i=1}^{k-1} n_i}$ and $x_+^k \in \mathbb{R}^{n-\sum_{i=1}^{k} n_i}$. Considering the Jacobian matrix $\nabla f(x) = [[\mathrm{row}_i(\nabla f(x))]_{i\in\mathbb{N}_1}^{\mathrm{T}}, \ldots, [\mathrm{row}_i(\nabla f(x))]_{i\in\mathbb{N}_m}^{\mathrm{T}}]^{\mathrm{T}}$ of $f(x)$ given in (3.10), we partition the matrix $[\mathrm{row}_i(\nabla f(x))]_{i\in\mathbb{N}_k} \in \mathbb{R}^{n_k \times n}$ by

$$[\mathrm{row}_i(\nabla f(x))]_{i\in\mathbb{N}_k} = \left[\begin{array}{c:c:c} \nabla f_-^k(x) & \nabla f^k(x) & \nabla f_+^k(x) \end{array}\right], \quad k \in \mathbb{M}, \tag{3.13}$$

where $\nabla f^k_-(x) \in \mathbb{R}^{n_k \times \sum_{i=1}^{k-1} n_i}$, $\nabla f^k_+(x) \in \mathbb{R}^{n_k \times (n-\sum_{i=1}^{k} n_i)}$, and $\nabla f^k(x) \in \mathbb{R}^{n_k \times n_k}$. Furthermore, we denote

$$\nabla \gamma^k_+(x^{-k}) \triangleq \frac{\partial \gamma^k(x^k_-, x^k_+)}{\partial x^k_+} = -\left(\left[\frac{\partial^2 U^k(x)}{\partial x^k \partial x^k} \right]^{-1} \frac{\partial^2 U^k(x)}{\partial x^k \partial x^{-k}_+} \right)\Bigg|_{x=(\gamma^k, x^{-k})} \in \mathbb{R}^{n_k \times (n-\sum_{i=1}^{k} n_i)}, \tag{3.14}$$

$$\nabla \gamma^k_-(x^{-k}) \triangleq \frac{\partial \gamma^k(x^k_-, x^k_+)}{\partial x^k_-} = -\left(\left[\frac{\partial^2 U^k(x)}{\partial x^k \partial x^k} \right]^{-1} \frac{\partial^2 U^k(x)}{\partial x^k \partial x^{-k}_-} \right)\Bigg|_{x=(\gamma^k, x^{-k})} \in \mathbb{R}^{n_k \times \sum_{i=1}^{k-1} n_i}, \tag{3.15}$$

where we used the fact that

$$\frac{\partial g(x)}{\partial x} = -\left[\frac{\partial^2 f(x, g(x))}{\partial y^2} \right]^{-1} \frac{\partial^2 f(x, g(x))}{\partial x \partial y} \in \mathbb{R}^{m \times n} \tag{3.16}$$

holds for $g(x) = \arg\max_y f(x, y) \in \mathbb{R}^m$ with a continuously differentiable function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$. With a slight abuse of notation, we write $\nabla \gamma^k_+(x)$ for $\nabla \gamma^k_+(x^{-k})$, and $\nabla \gamma^k_-(x)$ for $\nabla \gamma^k_-(x^{-k})$. Before we present a theorem, we define an $n \times n$ matrix

$$\mathcal{A}(\gamma, x) = \left[\begin{array}{c|c|c|c} \multicolumn{2}{c|}{\nabla f^1(x)} & \multicolumn{2}{c}{-\nabla f^1(x) \nabla \gamma^1_+(x)} \\ \hline -\nabla f^2(x) \nabla \gamma^2_-(x) & \nabla f^2(x) & \multicolumn{2}{c}{-\nabla f^2(x) \nabla \gamma^2_+(x)} \\ \hline \vdots & & \ddots & \vdots \\ \hline \multicolumn{3}{c|}{-\nabla f^m(x) \nabla \gamma^m_-(x)} & \nabla f^m(x) \end{array} \right], \tag{3.17}$$

for a group Nash equilibrium $x_\vartriangle \in \mathbb{R}^n$.

Theorem 3.1. Consider a group Nash equilibrium $x_\vartriangle \in \mathbb{R}^n$ of the noncooperative system with the pseudo-gradient dynamics (3.9) and the intra-group incentive function (3.4) under Assumption 3.1. Let the group managers' strategy u^k be updated by (3.11) and (3.12). If the matrix $\mathcal{A}_\mathrm{s} \triangleq \mathrm{diag}[\alpha] \mathcal{A}(\gamma, x_\vartriangle)$ is Hurwitz, then the group Nash equilibrium $x_\vartriangle$ is locally asymptotically stable and the group managers' strategy $u(t)$ converges to the corresponding subgame perfect equilibrium as $t \to \infty$.

Proof First, note that the group Nash equilibrium is an equilibrium of the closed-loop dynamics of (3.10)–(3.12). Recalling that u is a function of x under the update rule (3.11), it follows that $\nabla u(x) = \begin{bmatrix} -\frac{\partial f^1(\hat{x}^1, x^{-1})}{\partial x^1} \frac{\partial \gamma^1(x^{-1})}{\partial x} - \frac{\partial f^1(\hat{x}^1, x^{-1})}{\partial x^{-1}} \frac{\partial x^{-1}}{\partial x} \\ \vdots \\ -\frac{\partial f^m(\hat{x}^m, x^{-m})}{\partial x^m} \frac{\partial \gamma^m(x^{-m})}{\partial x} - \frac{\partial f^m(\hat{x}^m, x^{-m})}{\partial x^{-m}} \frac{\partial x^{-m}}{\partial x} \end{bmatrix}$ can be expressed by

$$\nabla u(x) = \begin{bmatrix} 0^{n_1 \times n_1} & \multicolumn{2}{c}{-\nabla f^1(\hat{x}^1, x^{-1})\nabla\gamma_+^1(x^{-1}) - \nabla f_+^1(\hat{x}^1, x^{-1})} \\ -\nabla f^2(\hat{x}^2, x^{-2})\nabla\gamma_-^2(x^{-2}) - \nabla f_-^2(\hat{x}^2, x^{-2}) & 0^{n_2 \times n_2} & -\nabla f^2(\hat{x}^2, x^{-2})\nabla\gamma_+^2(x^{-2}) - \nabla f_+^2(\hat{x}^2, x^{-2}) \\ \vdots & \ddots & \vdots \\ -\nabla f^m(\hat{x}^m, x^{-m})\nabla\gamma_-^m(x^{-m}) - \nabla f_-^m(\hat{x}^m, x^{-m}) & & 0^{n_m \times n_m} \end{bmatrix}. \tag{3.18}$$

Therefore, the Jacobian matrix of the closed-loop dynamics of (3.10)–(3.12) at the group Nash equilibrium $x_\triangle$ is given by $\mathrm{diag}[\alpha](\nabla u(x_\triangle) + \nabla f(x_\triangle)) = \mathcal{A}_\mathrm{s}$. Then, it follows from Lyapunov's indirect method that the result is immediate. □

Remark 3.1. To construct the update rule (3.11), each manager k only needs to observe the state profile $x^{-k}(t) \in \mathbb{R}^{n-n_k}$ from the other groups instead of observing the other mangers' strategy $u^{-k}(t)$ and hence the proposed update rule in the manager layer is *certainly different* from the existing Nash equilibrium seeking dynamics. But note that the state profile $x^k(t) \in \mathbb{R}^{n_k}$ is also required for constructing the intra-group incentive functions (3.4) within group $\mathbb{N}_k$.

Remark 3.2. Implementing the update rule (3.11) is understood as a reasonable and intuitive but myopic try for the group managers. None of those group managers can know stability beforehand because they never know the exact expression of the matrix $\mathcal{A}_\mathrm{s}$ as the information $x_\triangle$, $\nabla f^{-k}(x)$, $\nabla\gamma_-^{-k}(x)$, and $\nabla\gamma_+^{-k}(x)$ are undisclosed to them. To guarantee stability of the hierarchical noncooperative system, the behavior of a system governor who imposes *inter-group* incentive mechanism among the group managers is explored in Section 3.4.

The next result provides a sufficient stability condition without the information of agents' personal sensitivity parameters $\alpha_1, \ldots, \alpha_N$.

Proposition 3.1. Consider a group Nash equilibrium $x_\triangle \in \mathbb{R}^n$ of the noncooperative system with the pseudo-gradient dynamics (3.9) and the intra-group incentive function (3.4) under Assumption 3.1. Let the group managers' strategy u^k be updated by (3.11) and (3.12). If there exists $\hat{\alpha} \in \mathbb{R}_+^n$ such that $\mathcal{A}^\mathrm{T}(\gamma, x_\triangle)\mathrm{diag}[\hat{\alpha}] + \mathrm{diag}[\hat{\alpha}]\mathcal{A}(\gamma, x_\triangle) < 0$ holds, then the group Nash equilibrium $x_\triangle$ is locally asymptotically stable and the group managers' strategy $u(t)$ converges to the corresponding subgame perfect equilibrium as $t \to \infty$ for any $\alpha \in \mathbb{R}_+^n$.

Proof First, letting $\tilde{x} \triangleq x - x_\triangle$. Recall that linearizing the system dynamics (3.9) around $x_\triangle$ yields

$$\dot{\tilde{x}}(t) = \mathcal{A}_\mathrm{s}\tilde{x}(t). \tag{3.19}$$

Consider the Lyapunov function candidate $V(\tilde{x}) = \tilde{x}^{\mathrm{T}} P \tilde{x}$ with the matrix $P \triangleq \mathrm{diag}\left[\frac{\hat{\alpha}_1}{\alpha_1}, \ldots, \frac{\hat{\alpha}_N}{\alpha_N}\right] > 0$. Since

$$\mathcal{A}_{\mathrm{s}}^{\mathrm{T}} P + P\mathcal{A}_{\mathrm{s}} = \mathcal{A}^{\mathrm{T}}(\gamma, x_\triangle)\mathrm{diag}[\hat{\alpha}] + \mathrm{diag}[\hat{\alpha}]\mathcal{A}(\gamma, x_\triangle) < 0,$$

is satisfied, it follows using the linearized dynamics (3.19) that

$$\dot{V}(\tilde{x}(t)) = \tilde{x}^{\mathrm{T}}(t)(\mathcal{A}_{\mathrm{s}}^{\mathrm{T}} P + P\mathcal{A}_{\mathrm{s}})\tilde{x}(t) < 0, \tag{3.20}$$

around $x_\triangle$ and hence the group Nash equilibrium $x_\triangle$ is locally asymptotically stable for any $\alpha \in \mathbb{R}_+^n$. □

Now, we specialize the results to the noncooperative systems with quadratic payoff functions $J_i(x)$, $i \in \mathcal{N}$, given by

$$J_i(x) = \frac{1}{2} x^{\mathrm{T}} A_i x + b_i^{\mathrm{T}} x + c_i, \quad i \in \mathbb{N}, \tag{3.21}$$

where $A_i \triangleq \begin{bmatrix} a_{11}^i & \cdots & a_{1n}^i \\ \vdots & \ddots & \vdots \\ a_{1n}^i & \cdots & a_{nn}^i \end{bmatrix} \in \mathbb{R}^{n\times n}$ with $a_{ii}^i < 0$ (indicating that $J_i(x)$ is strictly concave with respect to x_i), $b_i \triangleq [b_1^i, \ldots, b_n^i]^{\mathrm{T}} \in \mathbb{R}^n$, and $c_i \in \mathbb{R}$, $i \in \mathcal{N}$. Supposing that $A \triangleq \begin{bmatrix} \mathrm{row}_1(A_1) \\ \vdots \\ \mathrm{row}_n(A_n) \end{bmatrix} \in \mathbb{R}^{n\times n}$ is nonsingular, for the given u, it follows that there exists a unique Nash equilibrium $x^*(u)$ given by $x^*(u) = -A^{-1}(b+u)$, where $b \triangleq [b_1^1, \ldots, b_n^n]^{\mathrm{T}} \in \mathbb{R}^n$. Hence, for a group Nash equilibrium $x_\triangle$, the subgame perfect equilibrium $u_\triangle$ is given by $u_\triangle = -Ax_\triangle - b$.

Consequently, the agents' state dynamics (3.9) with the quadratic payoff functions (3.21) and the intra-group incentive functions (3.4) are described by the affine dynamics given by

$$\dot{x}(t) = \mathrm{diag}[\alpha](Ax(t) + b + u(t)), \quad x(0) = x_0, \quad t \geq 0. \tag{3.22}$$

For the following statements, for each $k \in \mathbb{M}$, we let $\mathbb{A}^k \triangleq \sum_{i\in\mathbb{N}_k} \eta_i A_i$ and $\mathbb{B}^k \triangleq \sum_{i\in\mathbb{N}_k} \eta_i b_i$ be partitioned by

$$\mathbb{A}^k = \underbrace{\begin{pmatrix} * & (\mathbb{A}^k_-)^{\mathrm{T}} & * \\ \mathbb{A}^k_- & \mathbb{A}^k_k & \mathbb{A}^k_+ \\ * & (\mathbb{A}^k_+)^{\mathrm{T}} & * \end{pmatrix}}_{\sum_{i=1}^{k-1} n_k \;\; n_k \;\; n-\sum_{i=1}^{k} n_k} \in \mathbb{R}^{n\times n}, \tag{3.23}$$

$$\mathbb{B}^k = \Big[\underbrace{*}_{\sum_{i=1}^{k-1} n_k} \;\vdots\; \underbrace{(\mathbb{B}^k_k)^{\mathrm{T}}}_{n_k} \;\vdots\; \underbrace{*}_{n-\sum_{i=1}^{k} n_k} \Big]^{\mathrm{T}} \in \mathbb{R}^{n}, \tag{3.24}$$

which are used in (3.1) to be rewritten as $U^k(x) = \frac{1}{2}x^{\mathrm{T}}\mathbb{A}^k x + {\mathbb{B}^k}^{\mathrm{T}} x + \sum_{i\in\mathbb{N}_k} \eta_i c_i$, $k \in \mathbb{M}$. Note that "*" represents some matrices with consistent orders. Here, we note that $\mathbb{A}^k_k \in \mathbb{R}^{n_k\times n_k}$, $k \in \mathbb{M}$, are symmetric so that $\mathbb{A}_k$, $k \in \mathbb{M}$, are symmetric. Furthermore, we define $P_k = \{a^i_{ij}\}_{i,j\in\mathbb{N}_k} \in \mathbb{R}^{n_k\times n_k}$, $P^-_k = \{a^i_{ij}\}_{i\in\mathbb{N}_k, j\in\{\mathbb{N}_1,\ldots,\mathbb{N}_{k-1}\}} \in \mathbb{R}^{n_k\times \sum_{i=1}^{k-1} n_i}$, and $P^+_k = \{a^i_{ij}\}_{i\in\mathbb{N}_k, j\in\{\mathbb{N}_{k+1},\ldots,\mathbb{N}_m\}} \in \mathbb{R}^{n_k\times(n-\sum_{i=1}^{k} n_i)}$, so that

$$[\mathrm{row}_i(A)]_{i\in\mathbb{N}_k} = \Big[\; P^-_k \;\vdots\; P_k \;\vdots\; P^+_k \;\Big] \in \mathbb{R}^{n_k\times n}, \quad k \in \mathbb{M}. \tag{3.25}$$

In this case, notice that P_k is equivalent to the matrix $\nabla f^k(x)$ defined in (3.13).

Assumption 3.2. The group utility functions $U^k(x)$, $k \in \mathbb{M}$, are concave with respect to x^k, i.e., $\mathbb{A}^k_k < 0$, $k \in \mathbb{M}$. Furthermore, there exists a unique group Nash equilibrium $x_\triangle \in \mathbb{R}^n$.

Remark 3.3. Note that the assumption of $\mathbb{A}^k_k < 0$ in Assumption 3.2 guarantees the existence and uniqueness of $\hat{x}^k(t)$ in (3.12) given by

$$\hat{x}^k(t) = -(\mathbb{A}^k_k)^{-1}[\mathbb{A}^k_- x^k_-(t) + \mathbb{A}^k_+ x^k_+(t) + \mathbb{B}^k_k], \tag{3.26}$$

which implies that the matrices defined in (3.14) and (3.15) are simply given by $\nabla\gamma^k_+(x^{-k}) = -(\mathbb{A}^k_k)^{-1}\mathbb{A}^k_+$, $\nabla\gamma^k_+(x^{-k}) = -(\mathbb{A}^k_k)^{-1}\mathbb{A}^k_-$, $k \in \mathbb{M}$. Furthermore, it follows from (3.8) that the unique group Nash equilibrium $x_\triangle$ satisfies $Gx_\triangle + \rho = 0$ with $G \triangleq \begin{bmatrix} \mathbb{A}^1_1 & & & \mathbb{A}^1_+ \\ \mathbb{A}^2_- & \mathbb{A}^2_2 & & \mathbb{A}^2_+ \\ & & \ddots & \\ \mathbb{A}^m_- & & & \mathbb{A}^m_m \end{bmatrix} \in \mathbb{R}^{n\times n}$, $\rho \triangleq \begin{bmatrix} \mathbb{B}^1_1 \\ \vdots \\ \mathbb{B}^m_m \end{bmatrix} \in \mathbb{R}^n$. Those facts are used in the proof of the following theorem.

Theorem 3.2. Consider the noncooperative system with the pseudo-gradient dynamics (3.9), quadratic payoff functions (3.21), and the intra-group incentive function (3.4) under Assumption 3.2. Let the group managers' strategy u^k be updated by (3.11) and (3.12). Then, the group Nash equilibrium $x_\triangle$ is globally asymptotically stable and the group managers' strategy $u(t)$ converges to the corresponding subgame perfect equilibrium $u_\triangle = -Ax_\triangle - b$ as $t \to \infty$, if and only if the matrix $\mathcal{A}_\mathrm{s} = \mathrm{diag}[\alpha]A_\mathrm{s}$ is Hurwitz with

$$A_\mathrm{s} = \text{block-diag}[P_1(\mathbb{A}_1^1)^{-1}, \ldots, P_m(\mathbb{A}_m^m)^{-1}]G. \tag{3.27}$$

Proof First, note that the sufficiency is a direct consequence from Theorem 3.1. For necessity, it follows from $f^k(x) \triangleq \{f_i(x)\}_{i\in\mathbb{N}_k} = [\mathrm{row}_i(A)]_{i\in\mathbb{N}_k}x + b^k$ with $b^k = \{b_i^i\}_{i\in\mathbb{N}_k}$ that (3.11), (3.25) and (3.26) yield

$$\begin{aligned} u^k(t) &= -f^k(\hat{x}^k, x^{-k}(t)) = -[P_k^- \quad P_k \quad P_k^+]\begin{bmatrix} x_-^k(t) \\ \hat{x}^k(t) \\ x_+^k(t) \end{bmatrix} - b^k \\ &= (P_k(\mathbb{A}_k^k)^{-1}\mathbb{A}_-^k - P_k^-)x_-^k(t) + (P_k(\mathbb{A}_k^k)^{-1}\mathbb{A}_+^k - P_k^+)x_+^k + P_k(\mathbb{A}_k^k)^{-1}\mathbb{B}_k^k - b^k. \end{aligned}$$

Then, it follows that

$$u(t) = Kx(t) + l - b, \tag{3.28}$$

where

$$K = \left[\begin{array}{c:c:c:c} \multicolumn{2}{c:}{0_{n_1\times n_1}} & \multicolumn{2}{c}{P_1(\mathbb{A}_1^1)^{-1}\mathbb{A}_+^1 - P_1^+} \\ \hdashline P_2(\mathbb{A}_2^2)^{-1}\mathbb{A}_-^2 - P_2^- & 0_{n_2\times n_2} & \multicolumn{2}{c}{P_2(\mathbb{A}_2^2)^{-1}\mathbb{A}_+^2 - P_2^+} \\ \hdashline \multicolumn{2}{c:}{\vdots} & \ddots & \vdots \\ \hdashline \multicolumn{3}{c:}{P_m(\mathbb{A}_m^m)^{-1}\mathbb{A}_-^m - P_m^-} & 0_{n_m\times n_m} \end{array}\right], \tag{3.29}$$

$$\begin{aligned} l &= \left[\; (P_1(\mathbb{A}_1^1)^{-1}\mathbb{B}_1^1)^\mathrm{T}, \quad \ldots, \quad (P_m(\mathbb{A}_m^m)^{-1}\mathbb{B}_m^m)^\mathrm{T} \;\right]^\mathrm{T} \\ &= \text{block-diag}[P_1(\mathbb{A}_1^1)^{-1}, \ldots, P_m(\mathbb{A}_m^m)^{-1}]\rho \in \mathbb{R}^n. \end{aligned} \tag{3.30}$$

Now, the closed-loop dynamics of (3.11) and (3.22) are given by

$$\dot{x}(t) = \mathrm{diag}[\alpha]((A+K)x(t) + l) = \mathcal{A}_\mathrm{s}(x(t) - x_\triangle), \tag{3.31}$$

where we used $A + K = A_\mathrm{s}$ and $A_\mathrm{s}x(t) + l = \text{block-diag}[P_1(\mathbb{A}_1^1)^{-1}, \ldots, P_m(\mathbb{A}_m^m)^{-1}] \cdot (Gx + \rho) = A_\mathrm{s}(x(t) - x_\triangle)$. Since the group Nash equilibrium $x_\triangle$ is a unique equilibrium of the closed-loop dynamics under Assumption 3.2, it follows that $x_\triangle$ is globally asymptotically stable if only if A_s is Hurwitz. The convergence result for $u(t)$ is also immediate since (3.28) holds. □

Remark 3.4. The feedback matrix K is understood as the matrix ∇u defined in (3.18), where $\nabla f_+^k(\hat{x}^k, x^{-k}) = P_k^+$ and $\nabla f_-^k(\hat{x}^k, x^{-k}) = P_k^-$, $k \in \mathbb{M}$, hold for any $x \in \mathbb{R}^n$.

Remark 3.5. If $\mathbb{A}_-^k = 0$ and $\mathbb{A}_+^k = 0$ hold for all $k \in \mathbb{M}$, then the pseudo-gradient dynamics of the agents in $\mathbb{N}_k$ are not mutually affected by the agents in the other groups, and hence $\hat{x}^k(t) = \arg\max_{x^k \in \mathbb{R}^{n_k}} U^k(x^k, x^{-k}(t)) = -(\mathbb{A}_k^k)^{-1}\mathbb{B}_k^k$ in (3.26) is in fact constant being independent of the values of $x^{-k}(t)$ for all $k \in \mathbb{M}$. Hence, the system is understood as a combination of m number of independent noncooperative systems with the sets of agents $\mathbb{N}_1$, $\mathbb{N}_2$, ..., $\mathbb{N}_m$ incentivized by the corresponding group managers.

3.3.2 Update Rule With Intermittent Observation

It is not always the case where the group managers are able to observe the state profile $x^{-k}(t)$ from the other groups for every time instant $t \geq 0$. In this section, we characterize the situation where group manager k only has intermittent access to $x^{-k}(t)$ at some specific time instants, whereas the state information $x^k(t)$ of its own group is available for all t to process the intra-group incentive function (3.4). It is observed from real society that the governments/public may have intermittent access to realize the financial status of local companies because those local companies usually have termly financial reports to the public or they may need to go through a temporary inspection for some specific time instants required by the financial department of the government.

Therefore, we consider the sampled-data-based, piecewise constant update rule (3.11) and (3.12) with $(\hat{x}^k(t), x^{-k}(t))$ replaced by $(\hat{x}^k(t_s), x^{-k}(t_s))$ for $t_s \leq t < t_{s+1}$, where $\{t_s\}_{s=0,1,2,\ldots}$ denotes the sequence of sampling instants with $t_0 = 0$ and $\lim_{s\to\infty} t_s = \infty$. The sampling intervals between two sampling instants are defined by $T_s \triangleq t_{s+1} - t_s \in \mathbb{R}_+$ for $s \in \mathbb{Z}_0$, which may be constant or time-varying depending on the information disclosure structure. In this case, linearizing the pseudo-gradient $f(x)$ and the update rule $u(x)$ around the group Nash equilibrium $x_\triangle$, the linearized closed-loop dynamics of (3.10) and (3.11) with the shifted $\tilde{x} = x - x_\triangle$ state are given by

$$\dot{\tilde{x}}(t) = \mathrm{diag}[\alpha](\nabla f(x_\triangle)\tilde{x}(t) + \nabla u(x_\triangle)\tilde{x}(t_s)), \quad t \in [t_s, t_{s+1}), \tag{3.32}$$

where $\nabla u(x)$ is defined in (3.18).

Suppose that the payoff functions $J_i(x)$, $i \in \mathcal{N}$, are given as quadratic functions (3.21). Then, it follows that the entire profile u in pseudo-gradient dynamics (3.22) characterized from managers' intra-group incentive schemes is given by $u(t) = Kx(t_s) + l - b$, $t \in [t_s, t_{s+1})$, where the matrix K and the vector l are defined in (3.29) and (3.30). In this case, the closed-loop dynamics of (3.11) and (3.22) are given by

$$\begin{aligned}\dot{\tilde{x}}(t) &= \dot{x}(t) = \mathrm{diag}[\alpha](Ax(t) + Kx(t_s) + l) \\ &= \mathrm{diag}[\alpha](A(\tilde{x}(t) + x_\vartriangle) + K(\tilde{x}(t_s) + x_\vartriangle) + l) \\ &= \mathrm{diag}[\alpha](A\tilde{x}(t) + K\tilde{x}(t_s)), \quad t \in [t_s, t_{s+1}),\end{aligned} \tag{3.33}$$

where we used the fact $(A + K)x_\vartriangle + l = 0$.

The next result provides a sufficient stability condition for the proposed sampled-data-based update rule of (3.11), (3.12) with quadratic payoff functions. But note that as long as local stability is concerned around a group Nash equilibrium $x_\vartriangle$, the result can be generalized to nonquadratic cases considering the linearized dynamics (3.32) (i.e., using $\nabla f(x_\vartriangle)$ as A and using $\nabla u(x_\vartriangle)$ as K).

For the statements of the following results, let $\Phi(t) \triangleq e^{\mathrm{diag}[\alpha]At}(I_n + A^{-1}K) - A^{-1}K$.

Proposition 3.2. Consider the noncooperative system with the pseudo-gradient dynamics (3.9), intra-group incentive function (3.4), and quadratic payoff functions (3.21) under Assumption 3.2. Let the group managers' strategy u^k be updated by the sampled-data-based update rule of (3.11), (3.12) with $(\hat{x}^k(t), x^{-k}(t))$ replaced by $(\hat{x}^k(t_s), x^{-k}(t_s))$. If there exists a positive-definite matrix $P \in \mathbb{R}^{n\times n}$ such that

$$\Phi^{\mathrm{T}}(T_s)P\Phi(T_s) - P < 0, \tag{3.34}$$

for all $s \in \mathbb{Z}_0$, then the group Nash equilibrium $x_\vartriangle$ is globally asymptotically stable and the group managers' strategy $u(t)$ converges to the corresponding subgame perfect equilibrium as $t \to \infty$.

Proof First, it follows from (3.33) that

$$\dot{\tilde{x}}(t) = \mathrm{diag}[\alpha]A(\tilde{x}(t) + A^{-1}K\tilde{x}(t_s)), \quad t \in [t_s, t_{s+1}). \tag{3.35}$$

Then, the solution of the continuous-time dynamics (3.35) satisfies

$$\tilde{x}(t) + A^{-1}K\tilde{x}(t_s) = e^{\mathrm{diag}[\alpha]A\tau}(\tilde{x}(t_s) + A^{-1}K\tilde{x}(t_s)), \tag{3.36}$$

for $t \in [t_s, t_{s+1})$ with $\tau \triangleq t - t_s \in [0, T_s)$, which indicates

$$\begin{aligned}\tilde{x}(t_{s+1}) &= e^{\mathrm{diag}[\alpha]AT_s}(\tilde{x}(t_s) + A^{-1}K\tilde{x}(t_s)) - A^{-1}K\tilde{x}(t_s) \\ &= (e^{\mathrm{diag}[\alpha]AT_s}(I_n + A^{-1}K) - A^{-1}K)\tilde{x}(t_s) = \Phi(T_s)\tilde{x}(t_s), \quad s \in \mathbb{Z}_0. \end{aligned} \tag{3.37}$$

Consider the Lyapunov function candidate $V(\tilde{x}) = \tilde{x}^{\mathrm{T}}P\tilde{x}$ for the discrete-time dynamics (3.37) with P satisfying (3.34). Then, it follows that

$$\begin{aligned}\Delta V(\tilde{x}(t_s)) &= V(\tilde{x}(t_{s+1})) - V(\tilde{x}(t_s)) = \tilde{x}^{\mathrm{T}}(t_{s+1})P\tilde{x}(t_{s+1}) - \tilde{x}^{\mathrm{T}}(t_s)P\tilde{x}(t_s) \\ &= \tilde{x}^{\mathrm{T}}(t_s)(\Phi^{\mathrm{T}}(T_s)P\Phi(T_s) - P)\tilde{x}(t_s) < 0, \end{aligned} \tag{3.38}$$

for all $s = 0, 1, 2, \ldots$, and hence the proof is complete. The convergence result for $u(t)$ is also immediate since (3.28) and $\lim_{s\to\infty} t_s = \infty$ hold. □

Proposition 3.2 indicates that the choice of the sampling instants $\{t_s\}_{s=0,1,2,\ldots}$ is essential in the sampled-data-based update rule of (3.11), (3.12). The next result shows that sufficiently small sampling intervals should preserve the asymptotic stability when the group Nash equilibrium is asymptotically stable under the continual update rule (3.11), (3.12).

Theorem 3.3. Consider the noncooperative system with the pseudo-gradient dynamics (3.9), intra-group incentive function (3.4), and quadratic payoff functions (3.21) under Assumption 3.2. Let the group managers' strategy u^k be updated by the sampled-data-based update rule of (3.11), (3.12) with $(\hat{x}^k(t), x^{-k}(t))$ replaced by $(\hat{x}^k(t_s), x^{-k}(t_s))$. If the matrix $\mathcal{A}_{\mathrm{s}} = \mathrm{diag}[\alpha]A_{\mathrm{s}}$ is Hurwitz with A_{s} defined in (3.27), then there exists a positive scalar $\sigma \in \mathbb{R}_+$ such that the group Nash equilibrium $x_\triangle$ is asymptotically stable for any sampling instants $t_s, s \in \mathbb{Z}_0$, satisfying $T_s < \sigma$, $s \in \mathbb{Z}_0$.

Proof Since $\mathcal{A}_{\mathrm{s}} = \mathrm{diag}[\alpha]A_{\mathrm{s}} = \mathrm{diag}[\alpha](A + K)$ is Hurwitz, there exists a positive-definite matrix $P \in \mathbb{R}^{n\times n}$ such that

$$0 = (\mathrm{diag}[\alpha](A + K))^{\mathrm{T}}P + P\mathrm{diag}[\alpha](A + K) + Q, \tag{3.39}$$

for any positive-definite matrix $Q \in \mathbb{R}^{\mathbb{R}^{n\times n}}$. It follows from (3.36) that the state $x(t)$ can be expressed as $\tilde{x}(t) = \Phi(\tau)\tilde{x}(t_s)$, $t \in [t_s, t_{s+1})$ with $\tau \triangleq t - t_s \in [0, T_s)$. Since $\Phi(t)$ is continuous and $\Phi(0) = I_n$ holds, there exists a $T \in \mathbb{R}_+$ such that $\Phi(\tau)$ is invertible for all $t < T$. Hence, it follows from $\Phi^{-1}(\tau)\tilde{x}(t) = \tilde{x}(t_s)$ that

$$\|\tilde{x}(t_s) - \tilde{x}(t)\| = \|(\Phi^{-1}(\tau) - I_n)\tilde{x}(t)\| \leq \|\Phi^{-1}(\tau) - I_n\|\|\tilde{x}(t)\|. \tag{3.40}$$

Since $\Phi(\tau) - I_n \to 0$ as $\tau \to 0$, there exists $\zeta \in \mathbb{R}_+$ such that

$$\|\Phi^{-1}(\tau) - I_n\| < \frac{w}{\|P\mathrm{diag}[\alpha]K\|}, \tag{3.41}$$

holds for all $\tau < \zeta$, where $w = \lambda_{\min}(Q) > 0$ denotes the minimum eigenvalue of Q. Now, let $\sigma = \min(\zeta, T)$ so that $T_s < \zeta$ and $T_s < T$ for all $s \in \mathbb{Z}_0$.

Now, consider the Lyapunov function candidate $V(\tilde{x}) = \frac{1}{2}\tilde{x}^{\mathrm{T}}P\tilde{x}$. Then, it follows from $\tilde{x}^{\mathrm{T}}Q\tilde{x} > w\|\tilde{x}\|^2$, (3.39), (3.40), and (3.41) that the time derivative of $V(\tilde{x})$ along the system trajectories of (3.33) is given by

$$\begin{aligned}
\dot{V}(t) &= \tilde{x}^{\mathrm{T}}(t)P\mathrm{diag}[\alpha](A\tilde{x}(t) + K\tilde{x}(t) + K\tilde{x}(t_s) - K\tilde{x}(t)) \\
&= -\tilde{x}^{\mathrm{T}}(t)Q\tilde{x}(t) + \tilde{x}^{\mathrm{T}}(t)P\mathrm{diag}[\alpha]K(\tilde{x}(t_s) - \tilde{x}(t)) \\
&\leq -w\|\tilde{x}(t)\|^2 + \|\tilde{x}(t)\|\|P\mathrm{diag}[\alpha]K\|\|\tilde{x}(t_s) - \tilde{x}(t)\| \\
&\leq -w\|\tilde{x}(t)\|^2 + \|P\mathrm{diag}[\alpha]K\|\|\Phi^{-1}(\tau) - I_n\|\|\tilde{x}(t)\|^2 \\
&= -(w - \|P\mathrm{diag}[\alpha]K\|\|\Phi^{-1}(\tau) - I_n\|)\|\tilde{x}(t)\|^2 < 0,
\end{aligned}$$

and hence the group Nash equilibrium $x_\vartriangle$ is asymptotically stable. □

Even though the conditions shown in Proposition 3.2 and Theorem 3.3 require the knowledge of agents' sensitivity $\alpha \in \mathbb{R}^n_+$, it is worth noting that Theorem 3.3 along with Proposition 3.1 in Section 3.3.1 suggests a sufficient stability condition for unknown agents' sensitivity parameters $\alpha \in \mathbb{R}^n_+$.

Corollary 3.1. Consider the noncooperative system with the pseudo-gradient dynamics (3.9), intra-group incentive function (3.4), and quadratic payoff functions (3.21) under Assumption 3.2. Let the group managers' strategy u^k be update by the sampled-data-based update rule of (3.11), (3.12) with $(\hat{x}^k(t), x^{-k}(t))$ replaced by $(\hat{x}^k(t_s), x^{-k}(t_s))$. Suppose that there exists $\hat{\alpha} \in \mathbb{R}^n_+$ such that $A_{\mathrm{s}}^{\mathrm{T}}\mathrm{diag}[\hat{\alpha}] + \mathrm{diag}[\hat{\alpha}]A_{\mathrm{s}} < 0$ for the matrix A_{s} defined in (3.27). Then, there exists a positive scalar $\sigma \in \mathbb{R}_+$ such that the group Nash equilibrium $x_\vartriangle$ is asymptotically stable for any $\alpha \in \mathbb{R}^n_+$ and any sampling instants $t_s, s \in \mathbb{Z}_0$, satisfying $T_s < \sigma$, $s \in \mathbb{Z}_0$.

Proof The result is direct consequence of Theorem 3.3 by noting from Proposition 3.1 that the matrix $\mathcal{A}_{\mathrm{s}} = \mathrm{diag}[\alpha]A_{\mathrm{s}}$ is Hurwitz for any $\alpha \in \mathbb{R}^n_+$. □

3.4 Social Welfare Improvement Via Inter-group Incentives

In this section, we characterize the inter-group incentive mechanism in the manager layer to improve the weighted social welfare of the entire hierarchical system defined in (3.2). Similar to the process for the group managers who are capable to control the amounts of tax and subsidy in the agent layer, we assume that the system governor is able to control the amount of tax and subsidy in the manager layer under limited information in order to to stabilize a target equilibrium increasing the weighted social welfare function as much as possible[1]. To this end, the system governor is supposed to impose an explicit inter-group incentive mechanism to change the group Nash equilibrium by reconstructing the group utility functions and hence affect the group managers' behavior.

As the system governor in many economic applications serves merely as a mediator and does not have productivity to pay the additional profits to the agents [38, 39], we consider the hierarchical noncooperative system under inter-group incentives with the reconstructed group utility functions $\tilde{U}^k(x)$ given by

$$\tilde{U}^k(x) \triangleq U^k(x) + g^k(\bar{x}), \quad k \in \mathbb{M}, \tag{3.42}$$

where $g^k(\bar{x})$ denotes the inter-group incentive function[2] for group k satisfying

$$\sum_{k \in \mathbb{M}} g^k(\bar{x}) = 0 \tag{3.43}$$

and $\bar{x}$ denotes limited information of the state profile x which is precisely defined below. This constraint, once again, represents the case where the system governor serves merely as a mediator transferring payoffs among the agents.

In general, the system governor may not know the specific values of agents' state $x_1(t), \ldots, x_n(t)$ especially when n is large. In this chapter, we suppose that the system governor observes some kind of macroscopic data (e.g., average of the state values) from each of the groups, and the inter-group incentive function $g^k(x)$ is a simple function mapping from $\mathbb{R}^m$ to $\mathbb{R}$ (instead of $\mathbb{R}^n \to \mathbb{R}$). Those observed data can be considered

[1] The system governor may not achieve the maximum point because of the lack of enough information as discussed later.

[2] Those inter-group incentives are equally distributed to (or collected from) the group members in the agent layer irrespective of the state so that they do not affect the behavior of the pseudo-gradient dynamics.

as a linear mapping from the agents' state given by

$$\bar{x}(t) \triangleq \begin{bmatrix} \bar{x}^1(t) \\ \vdots \\ \bar{x}^m(t) \end{bmatrix} = \begin{bmatrix} C_1 x^1(t) \\ \vdots \\ C_m x^m(t) \end{bmatrix} = \begin{bmatrix} C_1 & & 0 \\ & \ddots & \\ 0 & & C_m \end{bmatrix} x(t) \triangleq Cx(t),$$

where $C_k \in \mathbb{R}^{1\times n_k}$, $k \in \mathbb{M}$, and $C \in \mathbb{R}^{m\times n}$. For example, if the observed data $\bar{x}^k$ is simply given as the average of the state values by $\bar{x}^k = 1/n_k \sum_{i\in\mathbb{N}_k} x_i$, then it implies $C^k = 1_{n_k}^{\mathrm{T}}/n_k$.

Now, consider the inter-group incentive function given by

$$g^k(\bar{x}) = v^k \bar{x}^k - \sum\nolimits_{j\neq k} \left(\frac{v_j}{m-1}\bar{x}^j\right), \tag{3.44}$$

where $v \triangleq [v_1, \ldots, v_m]^{\mathrm{T}} \in \mathbb{R}^m$ represents the inter-group incentive coefficient, so that the parameter $\hat{x}^k(t)$ in the group manager's strategy update rule (3.11) is remodeled from (3.12) to

$$\hat{x}^k(t) = \tilde{\gamma}^k(x^{-k}(t), v^k) \triangleq \arg\max_{x^k\in\mathbb{R}^{n_k}} \tilde{U}^k(x^k, x^{-k}(t)). \tag{3.45}$$

Note that the group Nash equilibrium under the inter-group incentive mechanism depends on v and is denoted by $x_\triangle(v) \in \mathbb{R}^n$. For a given $v \in \mathbb{R}^m$, we suppose that there exists a unique group Nash equilibrium $x_\triangle(v)$ satisfying $x_\triangle^k(v) = \tilde{\gamma}^k(x_\triangle^{-k}(v), v^k)$, $k \in \mathbb{M}$, under Assumption 3.1 (i.e., $0 = \left[\frac{\partial \tilde{U}^1(x_\triangle(v))}{\partial x^1}, \ldots, \frac{\partial \tilde{U}^m(x_\triangle(v))}{\partial x^m}\right] = \left[\frac{\partial U^1(x_\triangle(v))}{\partial x^1} + v^1 C_1, \ldots, \frac{\partial U^m(x_\triangle(v))}{\partial x^m} + v^m C_m\right]$). Note that there may not exist a coefficient v such that $x_\triangle(v)$ coincide with the maximum point of $\Pi(x)$ because the inter-group incentive function (3.44) under observed (limited) data restricts feasibility on changing the partial derivatives of $\frac{\partial \tilde{U}^k(x_\triangle(v))}{\partial x^k}$, $k \in \mathbb{M}$, but there may exist a best inter-group incentive coefficient $v^\star$ maximizing the weighted social welfare $\Pi(x_\triangle(v))$ given by (3.2) at $v = v^\star \in \mathbb{R}^m$. For the following statements, we denote the best inter-group incentive coefficient by $v^\star \triangleq \arg\max_{v\in\mathbb{R}^m} \Pi(x_\triangle(v))$ and use the corresponding group Nash equilibrium $x^\star \triangleq x_\triangle(v^\star)$ as a target equilibrium[3]. Moreover, we denote

$$\begin{aligned}\nabla\tilde{\gamma}_+^k(x^{-k},v^k) &\triangleq \frac{\partial\tilde{\gamma}^k(x_-^k,x_+^k,v^k)}{\partial x_+^k} = -\left(\left[\frac{\partial^2\tilde{U}^k(x)}{\partial x^k\partial x^k}\right]^{-1}\frac{\partial^2\tilde{U}^k(x)}{\partial x^k\partial x_+^{-k}}\right)\Bigg|_{x=(\tilde{\gamma}^k(x^{-k},v^k),x^{-k})}\\ &= -\left(\left[\frac{\partial^2 U^k(x)}{\partial x^k\partial x^k}\right]^{-1}\frac{\partial^2 U^k(x)}{\partial x^k\partial x_+^{-k}}\right)\Bigg|_{x=(\tilde{\gamma}^k(x^{-k},v^k),x^{-k})} \in \mathbb{R}^{n_k\times(n-\sum_{i=1}^k n_i)},\end{aligned} \tag{3.46}$$

$$\begin{aligned}\nabla\tilde{\gamma}_-^k(x^{-k},v^k) &\triangleq \frac{\partial\tilde{\gamma}^k(x_-^k,x_+^k,v^k)}{\partial x_-^k} = -\left(\left[\frac{\partial^2\tilde{U}^k(x)}{\partial x^k\partial x^k}\right]^{-1}\frac{\partial^2\tilde{U}^k(x)}{\partial x^k\partial x_-^{-k}}\right)\Bigg|_{x=(\tilde{\gamma}^k(x^{-k},v^k),x^{-k})}\\ &= -\left(\left[\frac{\partial^2 U^k(x)}{\partial x^k\partial x^k}\right]^{-1}\frac{\partial^2 U^k(x)}{\partial x^k\partial x_-^{-k}}\right)\Bigg|_{x=(\tilde{\gamma}^k(x^{-k},v^k),x^{-k})} \in \mathbb{R}^{n_k\times\sum_{i=1}^{k-1} n_i},\end{aligned} \tag{3.47}$$

which are known to the system governor because the group utility functions $U^1,\ldots,U^m$ are supposed to be known to him. In addition, we denote

$$\begin{aligned}\gamma_k(x,v) &\triangleq \frac{\partial\tilde{\gamma}^k(x^k,v^k)}{\partial v^k}\\ &= -\left(\left[\frac{\partial^2\tilde{U}^k(x)}{\partial x^k\partial x^k}\right]^{-1}\frac{\partial^2\tilde{U}^k(x)}{\partial x^k\partial v^k}\right)\Bigg|_{x=(\tilde{\gamma}^k(x^{-k},v^k),x^{-k})}\\ &= -\left[\frac{\partial^2 U^k(\tilde{\gamma}^k(x^{-k},v^k),x^{-k})}{\partial x^k\partial x^k}\right]^{-1} C_k^{\mathrm{T}} \in \mathbb{R}^{n_k}, \quad k\in\mathbb{M},\end{aligned} \tag{3.48}$$

and define $\Gamma(x,v) \triangleq \begin{bmatrix}\gamma_1(x,v) & & 0\\ & \ddots & \\ 0 & & \gamma_m(x,v)\end{bmatrix} \in \mathbb{R}^{n\times m}$.

Remark 3.6. If $n_k = 1$ for all $k\in\mathbb{M}$, the considered problem is reduced to an incentive designing problem for m-agent systems (which has been addressed in Chapter 2).

Now, we propose a framework of how the system governor appropriately designs $v(t)\in\mathbb{R}^m$ for the inter-group incentives in the manager layer to encourage the trajectory of agents' state converge towards the target equilibrium $x^\star$. In the beginning, let us suppose that $v(t)\equiv v^\star\in\mathbb{R}^m$.

Corollary 3.2. Consider the hierarchical noncooperative system with pseudo-gradient dynamics (3.9). Let the intra-group incentive function (3.4) be updated by (3.11)

[3] Here, note that if $\frac{\partial U^k(x)}{\partial x^k}$, $k\in\mathbb{M}$, at the the maximum point of $\Pi(x)$ are coincidently equal to $\delta_k C_k$, $k\in\mathbb{M}$, for some scaling factors $\delta_k\in\mathbb{R}$, $k\in\mathbb{M}$, then the target equilibrium $x^\star$ is understood the maximum point of $\Pi(x)$ whereas the best inter-group incentive coefficient is given by $v^\star = [\delta_1,\cdots,\delta_m]^{\mathrm{T}}$.

with (3.45). If the matrix $\mathcal{A}_\mathrm{s} \triangleq \mathrm{diag}[\alpha]\mathcal{A}(\tilde{\gamma}, x^\star)$ is Hurwitz, then the inter-group incentive functions (3.42), (3.44) along with $v(t) \equiv v^\star \in \mathbb{R}^m$ guarantee that the target equilibrium $x^\star$ is locally asymptotically stable.

Proof The proof is a direct consequence of Theorem 3.2 since the Jacobian matrix at $x^\star$ is $\mathcal{A}_\mathrm{s}$. □

Remark 3.7. For the case of quadratic payoff functions $J_i(x)$, $i \in \mathcal{N}$, as defined in (3.21), it follows that the parameter $\hat{x}^k(t)$ in (3.45) is given by

$$\hat{x}^k(t) = -(\mathbb{A}_k^k)^{-1}[\mathbb{A}_-^k x_-^k(t) + \mathbb{A}_+^k x_+^k(t) + \mathbb{B}_k^k + C_k^\mathrm{T} v^k(t)]. \tag{3.49}$$

Hence, the matrices $\nabla\tilde{\gamma}_+^k(\cdot)$, $\nabla\tilde{\gamma}_-^k(\cdot)$ are given by

$$\nabla\tilde{\gamma}_+^k(x^{-k}, v^k) = \nabla\gamma_+^k(x^{-k}) = -(\mathbb{A}_k^k)^{-1}\mathbb{A}_+^k, \tag{3.50}$$

$$\nabla\tilde{\gamma}_-^k(x^{-k}, v^k) = \nabla\gamma_-^k(x^{-k}) = -(\mathbb{A}_k^k)^{-1}\mathbb{A}_-^k, \tag{3.51}$$

so that they do not depend on v^k and x. As a direct consequence of Theorem 3.2, it can be shown that the inter-group incentive function (3.44) along with $v(t) \equiv v^\star \in \mathbb{R}^m$ guarantees that the target equilibrium $x^\star$ is globally asymptotically stable if and only if $\mathcal{A}_\mathrm{s} \triangleq \mathrm{diag}[\alpha]\mathcal{A}(\tilde{\gamma}, x^\star) = \mathrm{diag}[\alpha]\mathcal{A}(\gamma, x^\star) = \mathrm{diag}[\alpha]A_\mathrm{s}$ is Hurwitz with A_s defined in (3.27).

It is intuitive that only letting v be a constant vector does not guarantee the convergence when the matrix $\mathcal{A}_\mathrm{s}$ is not Hurwitz. Hence, it is natural to consider a feedback controller for the inter-group incentive mechanism for the system governor based on the observed data $\bar{x}(t)$. Specifically, consider a linear feedback controller

$$v(t) = v^\star + \tilde{K}(\bar{x}(t) - \bar{x}^\star) \in \mathbb{R}^m, \tag{3.52}$$

with $\bar{x}^\star \triangleq Cx^\star$ and $\tilde{K} = \{k_{ij}\}_{i,j\in\mathbb{M}} \in \mathbb{R}^{m\times m}$. Note that the linear feedback controller (3.52) ensures that the target equilibrium $x^\star$ is an equilibrium of the closed-loop dynamics of (3.10), (3.11), (3.45) given by

$$\dot{x}(t) = \mathrm{diag}[\alpha][f(x(t)) + u(x(t), v(\bar{x}(t)))], \tag{3.53}$$

$$\bar{x}(t) = Cx(t), \tag{3.54}$$

where the group managers' strategy profile $u(x, v(\bar{x})) = [(u^1(x^{-1}, v^1(\bar{x})))^\mathrm{T}, \ldots, (u^m(x^{-m}, v^m(\bar{x})))^\mathrm{T}]^\mathrm{T} \in \mathbb{R}^n$ is understood as a function solely depending on x. Now, linearizing the pseudo-gradient $f(x)$ and the update rule $u(x, v)$ around the target equilibrium $x^\star$, the linearized closed-loop dynamics with the shifted $\tilde{x} = x - x^\star$ state are given by

$$\dot{\tilde{x}}(t) = \tilde{\mathcal{A}}_{\mathrm{s}} \tilde{x}(t), \tag{3.55}$$

where

$$\tilde{\mathcal{A}}_{\mathrm{s}} \triangleq \mathrm{diag}[\alpha](\nabla f(x^\star) + \nabla \tilde{u}(x^\star, v^\star)), \tag{3.56}$$

and $\nabla \tilde{u}(x^\star, v^\star)$ is the Jacobian matrix of the function $u(x, v(\bar{x}))$ with respect to x at $(x^\star, v^\star) = (x^\star, v(x^\star))$ given by

$$\nabla \tilde{u}(x^\star, v^\star) = \left(\frac{\partial u(x,v)}{\partial x} + \frac{\partial u(x,v)}{\partial v} \frac{\partial v}{\partial x} \right) \bigg|_{x=x^\star, v=v^\star}. \tag{3.57}$$

Note that the expression of $\frac{\partial u(x,v)}{\partial x}$ is given by

$$\frac{\partial u(x,v)}{\partial x} = \left[\begin{array}{c:c:c:c} 0^{n_1 \times n_1} & \multicolumn{3}{c}{-\nabla f^1(\hat{x}^1, x^{-1}) \nabla \tilde{\gamma}^1_+(x^{-1}, v^1) - \nabla f^1_+(\hat{x}^1, x^{-1})} \\ \hdashline -\nabla f^2(\hat{x}^2, x^{-2}) \nabla \tilde{\gamma}^2_-(x^{-2}, v^2) - \nabla f^2_-(\hat{x}^2, x^{-2}) & 0^{n_2 \times n_2} & \multicolumn{2}{c}{-\nabla f^2(\hat{x}^2, x^{-2}) \nabla \tilde{\gamma}^2_+(x^{-2}, v^2) - \nabla f^2_+(\hat{x}^2, x^{-2})} \\ \hdashline \vdots & & \ddots & \vdots \\ \hdashline \multicolumn{3}{c:}{-\nabla f^m(\hat{x}^m, x^{-m}) \nabla \tilde{\gamma}^m_-(x^{-m}, v^m) - \nabla f^m_-(\hat{x}^m, x^{-m})} & 0^{n_m \times n_m} \end{array} \right]. \tag{3.58}$$

Then, it follows from

$$\begin{aligned} \frac{\partial u(x,v)}{\partial v} &= - \begin{bmatrix} \frac{\partial f^1(\tilde{\gamma}^1(x^{-1}, v^1), x^{-1})}{\partial x^1} \frac{\partial \tilde{\gamma}^1(x^{-1}, v^1)}{\partial v} \\ \vdots \\ \frac{\partial f^m(\tilde{\gamma}^m(x^{-m}, v^m), x^{-m})}{\partial x^m} \frac{\partial \tilde{\gamma}^m(x^{-m}, v^m)}{\partial v} \end{bmatrix} \\ &= -\,\text{block-diag}[\nabla f^1(x), \dots, \nabla f^m(x)] \Gamma(x, v) \in \mathbb{R}^{n \times m}, \end{aligned} \tag{3.59}$$

and

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial \bar{x}} \frac{\partial \bar{x}}{\partial x} = \tilde{K} C \in \mathbb{R}^{m \times n}, \tag{3.60}$$

that the Jacobian matrix in the linearized closed-loop dynamics (3.55) is given by $\tilde{\mathcal{A}}_{\mathrm{s}} = \mathrm{diag}[\alpha]\, \text{block-diag}[\nabla f^1(x^\star), \dots, \nabla f^m(x^\star)](S - \Gamma(x^\star, v^\star) K C)$, where

$$S \triangleq \left[\begin{array}{c:c:c:c} I_{n_1} & \multicolumn{3}{c}{\nabla \tilde{\gamma}^1_+(x^{-1}, v^1)} \\ \hdashline \nabla \tilde{\gamma}^2_-(x^{-2}, v^2) & I_{n_2} & \multicolumn{2}{c}{\nabla \tilde{\gamma}^2_+(x^{-2}, v^2)} \\ \hdashline \vdots & & \ddots & \vdots \\ \hdashline \multicolumn{3}{c:}{\nabla \tilde{\gamma}^m_-(x^{-m}, v^m)} & I_{n_m} \end{array} \right] \Bigg|_{x=x^\star, v=v^\star}. \tag{3.61}$$

Remark 3.8. Note that when $\tilde{K} = 0$ (i.e., $v(t) \equiv v^\star$), the matrix $\tilde{\mathcal{A}}_\mathrm{s}$ is same as the matrix $\mathcal{A}_\mathrm{s} \triangleq \mathrm{diag}[\alpha]\mathcal{A}(\tilde{\gamma}, x^\star)$ that is used in Corollary 3.2 because $\Gamma(x, v) = 0$.

However, it is necessary to point out that the matrices $\nabla f^k(x^\star)$, $k \in \mathbb{M}$, are yielded from the agents' individual payoff functions and hence may be unknown to the system governor. To deal with the uncertanty in $\nabla f^k(x^\star)$, $k \in \mathbb{M}$, we present the following result for guaranteeing asymptotic stabilization.

Theorem 3.4. Consider the hierarchical noncooperative system with pseudo-gradient dynamics (3.9). Let the intra-group incentive function (3.4) be updated by (3.11) with (3.45). Suppose that there exists $\hat{\alpha} \in \mathbb{R}^n_+$ such that the matrix

$$\mathrm{He}(\mathrm{diag}[\alpha]\,\text{block-diag}[\nabla f^1(x^\star), \ldots, \nabla f^m(x^\star)]\mathrm{diag}[\hat{\alpha}]) \tag{3.62}$$

is negative definite. Then, the inter-group incentives (3.42), (3.44), (3.52) with the matrix $\tilde{K} \in \mathbb{R}^{m \times m}$ satisfying

$$R \triangleq \mathrm{He}\big(\mathrm{diag}[\hat{\alpha}]^{-1}(S - \Gamma(x^\star, v^\star)\tilde{K}C)\big) > 0, \tag{3.63}$$

guarantee that the solution $x(t) \equiv x^\star$ of the closed-loop dynamics given by (3.53), (3.54) is locally asymptotically stable.

Proof Consider the Lyapunov function candidate $V(x) = (x - x^\star)^\mathrm{T} P (x - x^\star)$ with the positive-definite matrix $P \triangleq -(\mathrm{diag}[\alpha]\mathrm{diag}[\nabla f^1(x^\star), \ldots, \nabla f^m(x^\star)]\mathrm{diag}[\hat{\alpha}])^{-1} > 0$. Since the Lyapunov inequality $(D(S - \Gamma(x^\star, v^\star)\tilde{K}C))^\mathrm{T} P^\mathrm{T} + PD(S - \Gamma(x^\star, v^\star)\tilde{K}C) = -2R < 0$ is satisfied with $D = \mathrm{diag}[\alpha]\,\text{block-diag}[\nabla f^1(x^\star), \ldots, \nabla f^m(x^\star)]$, it follows using the linearized dynamics (3.55) that

$$\dot{V}(x(t)) = -2(x(t) - x^\star)^\mathrm{T} R (x(t) - x^\star) < 0, \tag{3.64}$$

around $x^\star$ and hence the target equilibrium $x^\star$ is asymptotically stable for any matrices $\nabla f^k(x^\star)$, $k \in \mathbb{M}$, and any $\alpha \in \mathbb{R}^n_+$. □

Remark 3.9. The conditions in Theorem 3.4 can be simplified for the case where the payoff functions are quadratic and given by (3.21). Specifically, supposing that there exists $\hat{\alpha} \in \mathbb{R}^n_+$ such that $\mathrm{He}(\mathrm{diag}[\alpha]\,\text{block-diag}[P_1, \ldots, P_n]\mathrm{diag}[\hat{\alpha}]) < 0$ holds, the condition (3.63) reduces to

$$\mathrm{He}\Big(\mathrm{diag}[\hat{\alpha}]^{-1}\mathrm{block} - \mathrm{diag}[(\mathbb{A}^1_1)^{-1}, \ldots, (\mathbb{A}^n_n)^{-n}](G + C^\mathrm{T}\tilde{K}C)\Big) > 0. \tag{3.65}$$

This is because $S = \text{block-diag}[(\mathbb{A}_1^1)^{-1}, \ldots, (\mathbb{A}_n^n)^{-n}]G$ and

$$\Gamma(x^\star, v^\star) = -\text{diag}[(\mathbb{A}_1^1)^{-1}, \ldots, (\mathbb{A}_n^n)^{-n}]C^{\mathrm{T}}, \tag{3.66}$$

hold under $\gamma_k = -(\mathbb{A}_k^k)^{-1}C_k^{\mathrm{T}}$ and (3.51). An interesting discussion is on controllability and observability analysis for the noncooperative system. It follows that the open-loop dynamics in terms of the shifted state $\tilde{x} = x - x^\star$ are given by

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + Bv(t), \quad y(t) \triangleq \tilde{\tilde{x}}(t) = C\tilde{x}(t), \tag{3.67}$$

where $\tilde{A} \triangleq \text{diag}[\alpha]A_{\mathrm{s}} \in \mathbb{R}^{n\times n}$ with A_{s} defined in (3.27) and

$$B \triangleq -\text{diag}[\alpha]\,\text{block-diag}[\nabla f^1(x^\star), \ldots, \nabla f^m(x^\star)]\Gamma(x^\star, v^\star) \tag{3.68}$$
$$= -\text{diag}[\alpha]\,\text{block-diag}[P_1(\mathbb{A}_1^1)^{-1}, \ldots, P_m(\mathbb{A}_m^m)^{-1}]C^{\mathrm{T}} \in \mathbb{R}^{n\times m}. \tag{3.69}$$

It is important to note that the open-loop dynamics (3.67) are understood as a continuous-time, linear time-invariant system with $v(t) \in \mathbb{R}^m$ being the control input and $y(t) \in \mathbb{R}^m$ being the output. Hence, this system is controllable if

$$\text{rank}[B\ \tilde{A}B\ \tilde{A}^2B\ \cdots\ \tilde{A}^{n-1}B] = n, \tag{3.70}$$

whereas the system is observable if

$$\text{rank}[C^{\mathrm{T}}\ (C\tilde{A})^{\mathrm{T}}\ (C\tilde{A}^2)^{\mathrm{T}}\ \cdots\ (C\tilde{A}^{n-1})^{\mathrm{T}}]^{\mathrm{T}} = n. \tag{3.71}$$

Note that the matrix C appears in both of the controllablility and observablility conditions, which also indicates why some of state (e.g., the maximum point of $\Pi(x)$) may be unreachable in the hierarchical noncooperative system for a given matrix C.

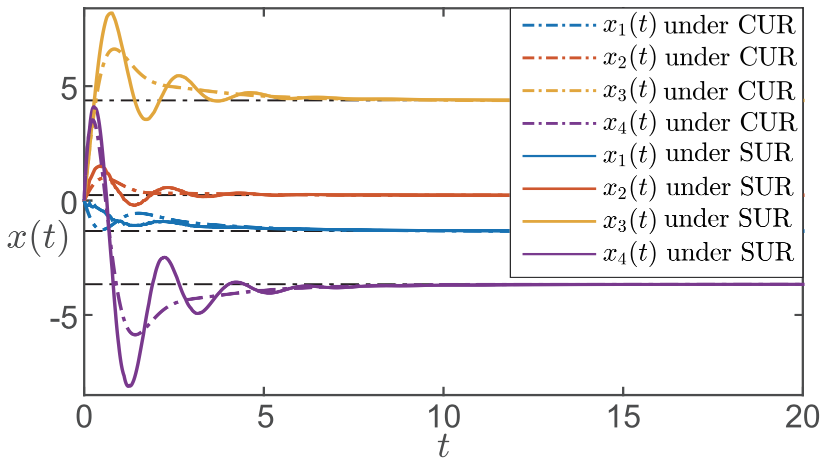
3.5 Illustrative Numerical Examples

In this section, a couple of numerical examples are provided to demonstrate the efficacy of our proposed approach.

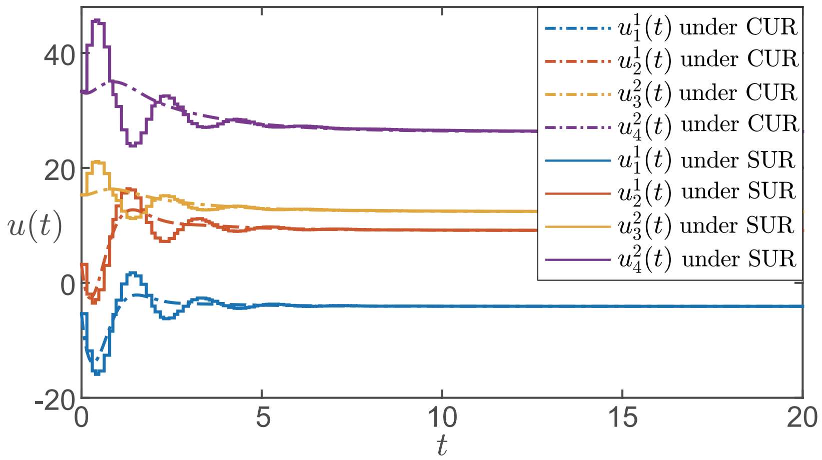
Example 3.1. Consider the 4-agent hierarchical noncooperative market with the agents' sets $\mathbb{N}_1 = \{1, 2\}$, $\mathbb{N}_2 = \{3, 4\}$, and the payoff functions (3.21) with $a_{11}^1 = -2$, $a_{12}^1 = 3$, $a_{14}^1 = -8$, $a_{22}^1 = -8$, $a_{44}^1 = -2$, $a_{11}^2 = -8$, $a_{12}^2 = 1$, $a_{22}^2 = -2$, $a_{24}^2 = 2$, $a_{44}^2 = -3$, $a_{33}^3 = -2$, $a_{34}^3 = 1$, $a_{44}^3 = -6$, $a_{11}^4 = -2$, $a_{13}^4 = 1$, $a_{14}^4 = -1$, $a_{23}^4 = 3$, $a_{33}^4 = -6$, $a_{34}^4 = -8$,

$a^4_{44} = -2$, $b^1_2 = 15$, $b^2_1 = -15$, $b^4_3 = 10$, and the other unmentioned parameters being zero. Suppose that there is no system governor coordinating the two subgroups. Let the priorities evaluated by the group managers be equal, e.g., $\eta_1 = \eta_2 = 1$ and $\eta_3 = \eta_4 = 1$. letting the sensitivity parameters be given by $\alpha = (1, 1, 1, 1)$, the group Nash equilibrium *without* inter-group incentive is given by $x_\triangle = [-1.3350, 0.2341, 4.3729, -3.6594]^{\mathrm{T}}$ and the matrix $\mathcal{A}_{\mathrm{s}} = A_{\mathrm{s}} = \begin{bmatrix} -2 & 3 & 0.0952 & -0.4286 \\ 1 & -2 & -0.0238 & 0.3571 \\ 3 & 4.6 & -2 & 1 \\ 6 & 10 & -8 & -2 \end{bmatrix}$ is Hurwitz. Then, it follows from Theorem 3.2 that the group Nash equilibrium $x_\triangle$ is globally asymptotically stable under the pseudo-gradient dynamics (3.9) incentivized by the intra-group incentive scheme (3.4), (3.11). On the other hand, let the smapling instants $t_s, s \in \mathbb{Z}_0$, satisfy that $T_s = t_{s+1} - t_s \in \{0.15, 0.09\}$ for the sampled-data-based update rule. In this case, $\Phi(T_s) = e^{T_s A}(I_4 + A^{-1}K) - A^{-1}K$ with K given by (3.29) for $T_s = 0.15$ and 0.09 satisfies (3.34) for $P = \begin{bmatrix} 29.9538 & 48.0453 & -12.0335 & -4.0689 \\ 48.0453 & 82.7924 & -21.1742 & -7.3734 \\ -12.0335 & -21.1742 & 14.8137 & 1.7948 \\ -4.0689 & -7.3734 & 1.7948 & 3.9462 \end{bmatrix} > 0$. It follows from Proposition 3.2 that the group Nash equilibrium $x_\triangle$ is globally asymptotically stable under the sampled-data-based update rule. Those results can be verified by the trajectories of the agents' state $x(t)$ and the group managers' strategy $u(t)$ shown in Figs. 3.3a and 3.3b.

Example 3.2. Consider the 4-agent hierarchical noncooperative market with agents' sets $\mathbb{N}_1 = \{1, 2\}$, $\mathbb{N}_2 = \{3, 4\}$, and the payoff functions (3.21) with $a^1_{11} = -2$, $a^1_{12} = 1$, $a^1_{13} = 1$, $a^1_{14} = 8$, $a^1_{22} = -8$, $a^1_{44} = -3$, $a^2_{11} = -8$, $a^2_{12} = 1$, $a^2_{22} = -2$, $a^2_{24} = 2$, $a^2_{33} = -9$, $a^2_{44} = -3$, $a^3_{11} = -1$, $a^3_{14} = -5$, $a^3_{33} = -9$, $a^3_{34} = -8$, $a^3_{44} = -8.1$, $a^4_{11} = -2$, $a^4_{13} = 1$, $a^4_{14} = -6$, $a^4_{23} = 9$, $a^4_{33} = -6$, $a^4_{34} = -8$, $a^4_{44} = -9$, $b^1_2 = 4$, $b^2_1 = -1$, $b^2_3 = 10$, $b^3_1 = 10$, $b^4_1 = -40$, $b^4_3 = 1$, and the other unmentioned parameters being zero. Let the priorities evaluated by the group managers be given by $\eta_1 = 2\eta_2 = 2$ and $\eta_3 = \eta_4 = 1$. Letting the sensitivity parameters be given by $\alpha = (1, 1, 1, 1)$, the group Nash equilibrium *without* inter-group incentive is given by $x_\triangle = [-0.3652, 0.349, 0.5834, -0.3109]^{\mathrm{T}}$, and the matrix $\mathcal{A}_{\mathrm{s}} = \begin{bmatrix} -2 & 1 & 0.3188 & 2.8116 \\ 1 & -2 & -0.1159 & -0.8406 \\ 579.8 & 466.2 & -9 & -8 \\ -168.4 & -129.6 & -8 & -9 \end{bmatrix}$ is *not* Hurwitz. Then, it follows from Theorem 3.2 that the group Nash equilibrium $x_\triangle$ under the pseudo-gradient dynamics (3.9) incentivized by the intra-group incentive scheme (3.4), (3.11)



(a) Trajectories of agents' state $x(t)$



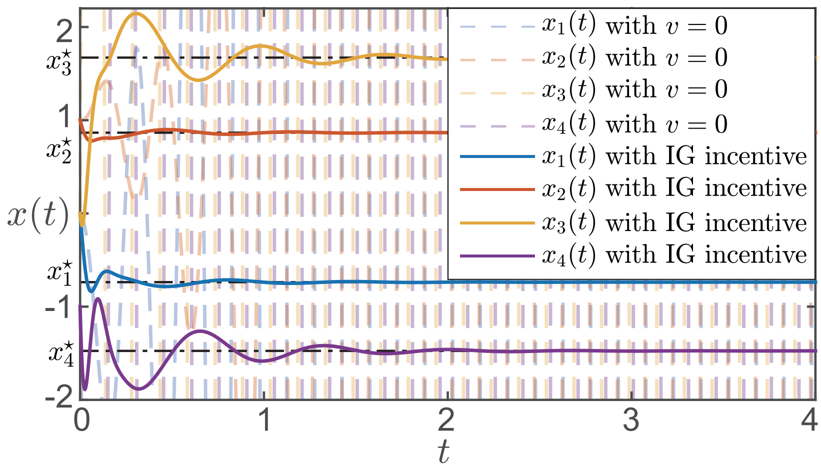
(b) Trajectories of group managers' strategy $u(t)$

Figure. 3.3 Trajectories of $x(t)$ and $u(t)$ influenced by the managers' intra-group incentives and update rule (3.11). Dash-dotted: under the continual update rule (CUR), solid: under the sampled-data-based update rule (SUR). The black dash-dotted lines in (a) represent the group Nash equilibrium $x_\triangle$.

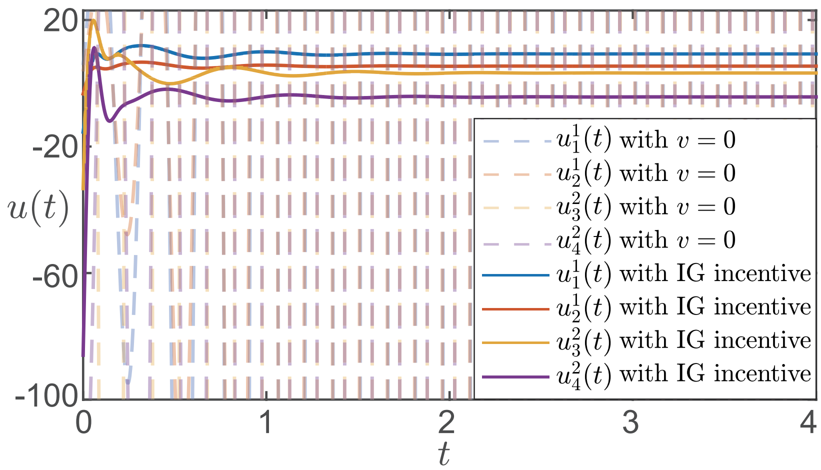
is unstable, which can be verified by the oscillatory trajectories of $x(t)$ and $u(t)$ shown as transparent dashed lines in Figs. 3.4a and 3.4b.

Now, letting the group priority evaluated by the system governor be equal, e.g., $\xi_1 = \xi_2 = 1$, we construct the inter-group incentive scheme (3.42), (3.44) in the manager layer to achieve social welfare improvement. We suppose that the observed data from the 2 groups are simply given by $\bar{x}^1 = x_1 + x_2$ and $\bar{x}^2 = x_3 + x_4$ so that $C^k = 1_{n_k}^{\mathrm{T}}$, $k = 1, 2$, hold. In this case, the best inter-group incentive coefficient is found as $v^\star \triangleq \arg\max_{v \in \mathbb{R}^2}(U^1(x_\triangle(v)) + U^2(x_\triangle(v))) = [12.7543, -6.5762]^{\mathrm{T}}$ and hence the corresponding group Nash equilibrium given by $x^\star = [-0.7372, 0.8663, 1.6718, -1.4746,]^{\mathrm{T}}$ is considered as the target equilibrium. Then, it follows from Theorem 3.4 that the inter-group incentive scheme (3.42) along with the inter-group incentive function (3.44) updated by (3.52) with $\tilde{K} = \begin{bmatrix} -136.8393 & 46.9339 \\ -159.7815 & -0.5447 \end{bmatrix}$ satisfying the linear matrix inequality (3.65) with $\hat{\alpha} = (1, 1, 1, 1)$ guarantees that the target equilibrium $x^\star$ is asymptotically stabilized, which can be verified by the trajectories of $x(t)$ and $u(t)$ shown as solid lines in Figs. 3.4a and 3.4b.

Example 3.3. Consider a market economic country being composed of n firms (agents) located in m cities (groups) selling homogeneous products produced by themselves with the market price function [97] given by $\lambda = \lambda_0 - \sum_{i=1}^{n} \beta_i x_i$, where $x_i \in \mathbb{R}_+$ denotes the quantity of the produced products, $\beta_i \in \mathbb{R}_+$ denotes the market power of the firm-i, and $\lambda_0 \in \mathbb{R}_+$ is a market specific parameter representing the cap price. In this country (Cournot game), firms compete in quantities rather than prices according to the payoff functions given by $J_i(x) = \lambda x_i - C_i(x_i)$, $i \in \mathcal{N}$, where $C_i(\cdot)$ is the production cost of firm-i given by $C_i(x_i) = a_i x_i^2 + b_i x_i$, $i \in \mathcal{N}$, with $a_i \geq 0$ and $b_i > 0$. The gross sales value of production in city $k \in \mathbb{M}$ is given as the group utility function $U^k(x)$ defined in (3.1) with $\eta_i = 1$, $i \in \mathbb{N}_k$, whereas the gross domestic product is given as the social welfare function $\Pi(x)$ defined in (3.2) with $\xi_k = 1$, $k \in \mathbb{M}$. In terms of incentives, each firm in city k is influenced by the production taxes/subsidies (intra-group incentive) linearly depending on the firm's production quantity given by (3.4) administered by a mayor (group manager). Each city $k \in \mathbb{M}$ is influenced by the transaction taxes/subsidies (inter-group incentive) linearly depending on the sum of the firms' production quantities $\bar{x}^k = 1_{n_k}^{\mathrm{T}} x^k$ in city k given by (3.44) administered by the national economic administration (system governor), i.e., $C^k = 1_{n_k}^{\mathrm{T}}$, $k \in \mathbb{M}$. Different from the objective of mayor k on maximizing the incentivized group utility $\tilde{U}^k$, the objective of the national economic administration is to maximize the gross domestic product $\Pi(x)$ using the observed data $\bar{x}^k$, $k \in \mathbb{M}$, and the gross



(a) Trajectories of agents' state $x(t)$



(b) Trajectories of group managers' strategy $u(t)$

Figure. 3.4 Trajectories of $x(t)$ and $u(t)$ of the hierarchical noncooperative system under intra-group incentives with and without the inter-group (IG) incentives. Agents' state diverges without inter-group incentives (i.e., $v = 0$) but converges to the target equilibrium $x^\star$ with inter-group incentives.

sales value of production $U^k(x)$, $k \in \mathbb{M}$. Now, let $n = 60$ and suppose that the amount of firms in each city is equal to each other satisfying $n_k = n/m$, $k \in \mathbb{M}$. Figure 3.5(a) shows the gross domestic product (social welfare) $\Pi(x)$ at the (unique) group Nash equilibrium $x_{\triangle}(0)$ without inter-group incentive for $a_i = 10$, $i \in \mathcal{N}$, $b_i = 3$, $i \in \mathcal{N}$, $\lambda_0 = 8$, and $\beta_i \in (0, 0.2)$, $i \in \mathcal{N}$, satisfying Assumption 3.1 with $m = 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30$, and 60, where the number of firms in each city $k \in \mathbb{M}$ is given by $n_k = 60, 30, 20, 15, 12, 10, 6, 5, 4, 3, 2$, and 1, respectively. Figure 3.5(b) captures the difference value between the gross domestic product $\Pi(x)$ at the group Nash equilibrium $x_{\triangle}(v)$ with $v = 0$ and $v = v^{\star} \triangleq \arg\max_{v \in \mathbb{R}^m} \Pi(x_{\triangle}(v))$, which is understood as the improvement made by the national economic administration via constructing the inter-group incentives.

Note that when $m = 1$ (i.e., $n_k = 60$), the system governor and inter-group incentives vanish so that the mayor is the unique institution constructing the incentive mechanism for the entire society with complete information from the agent layer and hence the social maximum is achieved. Alternatively, when $m = 60$ (i.e., $n_k = 1$), the mayors and intra-group incentives vanish so that the national economic administration is the unique institution constructing the incentive mechanism for the entire society with complete information from the agent layer and hence the social maximum is achieved. In either of the two cases, the three-layer hierarchical incentive structure reduces to the two-layer incentive structure characterized in Chapter 2. However, the full information of all the 60 agents' payoff functions and states can be hardly known to an individual and hence the three-layer incentive structure has to be established. Even though degeneration from the social maximum happens in the three-layer incentive structure (i.e., $m \neq 1, m \neq 60$) since $\frac{\partial U^k(x)}{\partial x^k} = \delta_k {n_k}^{\mathrm{T}}$ is not true for any scaling factor $\delta_k \in \mathbb{R}$ at the maximum point of $\Pi(x)$, the orange line in Fig. 3.5(a) is understood as the maximum value of social welfare that the national economic administration can help to reach. It is interesting to note from Fig. 3.5(a) that a larger number of groups m indicates a larger improvement by constructing the inter-group incentive, but does not indicate a larger social welfare $\Pi(x_{\triangle}(v^{\star}))$ at the group Nash equilibrium with the best inter-group incentive coefficient. This is because increasing m results in decreasing n_k and may decrease the social welfare $\Pi(x_{\triangle}(0))$ with only intra-group incentives.

3.6 Chapter Conclusion

In this chapter, we investigated the stability and stabilization problem for the noncooperative systems. In the characterized framework of hierarchical noncooperative

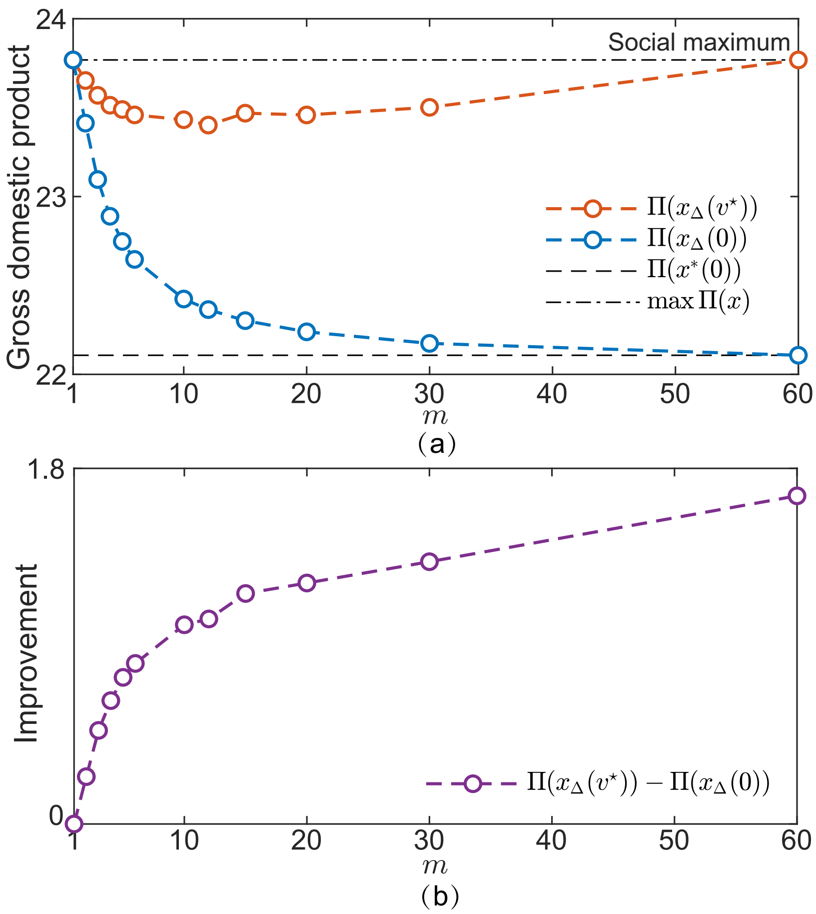


Figure. 3.5 Gross domestic product (a) at $x_\Delta(0)$ and $x_\Delta(v^\star)$ and improved gross domestic product (b) versus the number of groups m of a 60-agent system with $n_k = n/m$, $k \in \mathbb{M}$. Note that when m increases, the system governor possesses more information from the agent-layer. The improved gross domestic product is given by $\Pi(x_\Delta(v^\star)) - \Pi(x_\Delta(0))$.

systems, agents selfishly make their decision under some intra-group incentives, which are controlled by the group managers and updated by our proposed update rules. We explored the stability of group Nash equilibrium of the hierarchical noncooperative systems with dynamic agents, and derived conditions where the trajectory of agents' state converges to the group Nash equilibrium under group managers' intra-group incentives. Furthermore, we proposed the inter-group incentive mechanism for a system governor in order to reconstruct the group utility functions in the group managers level to move the group Nash equilibrium so that the social welfare is improved. To deal with the situation where the system governor may not know all the agents' individual payoff functions and all the agents' state, we presented sufficient conditions to guarantee the convergence of agents' state towards a target equilibrium using some macroscopic data. In this chapter, even though we assumed that the system governor is able to obtain 1-dimensional data from each group, the case where richer (higher-dimensional) information is available for the system governor is expected to have higher welfare state when we evaluate the target equilibrium. Finally, we provided three numerical examples for demonstrating stability and stabilization of group Nash equilibrium for 4-agent hierarchical noncooperative systems and 60-agent hierarchical noncooperative systems.

Chapter 4

Control of Noncooperative Dynamical Systems With Pareto Improvement: Pareto-Improving Incentive Mechanism

4.1 Introduction

In this chapter, we develop an explicit incentive mechanism for noncooperative systems to remodel agents' dynamical decision making for guaranteeing that all the agents are Pareto improving and their state converges to a Pareto-efficient Nash equilibrium. Specifically, we suppose that the system manger collects taxes from some agents and gives some of the collected taxes to other agents as subsidies with a sustainable budget constraint. Considering the priorities among the agents, we construct a weighted social welfare function for the incentive mechanism and hence derive the socially maximum state as the target Nash equilibrium. With the well-designed incentive functions associated with the weighted social welfare function, the socially maximum state is ensured to be a Pareto-efficient Nash equilibrium in the incentivized noncooperative system. Several sufficient stability conditions are presented to guarantee that the agents are Pareto improving under the pseudo-gradient dynamics and their state converges to the socially maximum state with known or unknown sensitivity parameters. As a result, it turns out that the initial state plays an important role on constructing the Pareto-improving incentive mechanism under sustainable budget constraint. For the case with equal priority between the agents, a balanced budget constraint is guaranteed and the connection between Pareto improvement and potentialization is explored. Our numerical examples exhibit a direct evidence that the Pateto improvement and potentialization do not have an inclusive relation with each other.

The rest of this chapter is organized as follows. We explain the incentivized noncooperative system and introduce the problem of this paper in Section 4.2. In Section 4.3, we design the incentive mechanisms to achieve Pareto improvements with arbitrary priorities for the agents under sustainable budget constraint for a given initial state. In Section 4.4, we specialize the result to the case where the priorities of the agents are all the same. Several numerical examples are shown in those two sections. Finally, we conclude this chapter in Section 4.5.

4.2 Problem Formulation

4.2.1 System Description

Consider a noncooperative system with n number of agents adjusting their state (strategy) in an unbounded state space $\mathbb{R}^n$. Let $\mathcal{N} \triangleq \{1, \ldots, n\}$ denote the set of all agents. The payoff function of agent i is denoted by $J_i : \mathbb{R}^n \to \mathbb{R} : x \mapsto J_i(x)$ and the profile of all agents' state is denoted by $x = [x_1, \ldots, x_n]^{\mathrm{T}} \in \mathbb{R}^n$, where $x_i \in \mathbb{R}$ is agent i's individual state. We assume that there is a system manager who imposes some incentive mechanisms among the agents to reconstruct the agents' payoff functions and hence alters agents' decision for improving the welfare of the entire system. (The precise definition of the welfare of the entire system is given as the weighted social welfare function in Section 4.3 considering the priory of the agents.) Specifically, let agents' incentivized payoff functions be given by

$$\tilde{J}_i(x) \triangleq J_i(x) + p_i(x), \quad i \in \mathcal{N}, \tag{4.1}$$

where $p_i : \mathbb{R}^n \to \mathbb{R}$ is the incentive function for agent $i \in \mathcal{N}$. We denote the incentivized noncooperative system by $\mathcal{G}(\tilde{J})$ and the original (un-incentivized) noncooperative system by $\mathcal{G}(J)$ with $\tilde{J} \triangleq \{\tilde{J}_i\}_{i\in\mathcal{N}}$ and $J \triangleq \{J_i\}_{i\in\mathcal{N}}$. In order to establish the pseudo-gradient dynamics for the agents, we assume that the payoff functions $J_i(x)$, $i \in \mathcal{N}$, and the incentive functions $p_i(x)$, $i \in \mathcal{N}$, are continuously differentiable.

It is worth noting that at a Nash equilibrium (defined in Definition 4.1 below) no agent has any intension to deviate unilaterally from the equilibrium state. Therefore, the Nash equilibrium is often working as an operating point in noncooperative systems. Furthermore, we note that Pareto efficiency is an important notion in economics for indicating efficiency of a society. For the convenience of readers, the notions of the Nash equilibrium and a Pareto-efficient state are given as follows.

Definition 4.1. For the incentivized noncooperative system $\mathcal{G}(\tilde{J})$, the state profile $\tilde{x}^* \in \mathbb{R}^n$ is called a Nash equilibrium if

$$\tilde{J}_i(\tilde{x}_i^*, \tilde{x}_{-i}^*) \geq \tilde{J}_i(x_i, \tilde{x}_{-i}^*), \quad x_i \in \mathbb{R}, \quad i \in \mathcal{N}. \tag{4.2}$$

Definition 4.2. For the incentivized noncooperative system $\mathcal{G}(\tilde{J})$, the state profile $\tilde{x}^* \in \mathbb{R}^n$ is Pareto efficient (optimal) if there is no other state $x \in \mathbb{R}^n$ such that $\tilde{J}_i(x) \geq \tilde{J}_i(\tilde{x}^*)$ for all $i \in \mathcal{N}$ with strict inequalities for some $i \in \mathcal{N}$.

Note that the state profile $\hat{x}^* \in \mathbb{R}^n$ which maximizes the function $\sum_{i\in\mathcal{N}} \tilde{J}_i(x)$ is always Pareto efficient in $\mathcal{G}(\tilde{J})$ because no agent can further increase $\tilde{J}_i(x)$ without decreasing others' payoffs from $\hat{x}^*$. Furthermore, since $J_i(x)$, $i \in \mathcal{N}$, and $p_i(x)$, $i \in \mathcal{N}$, are continuously differentiable, the Nash equilibrium $\tilde{x}^*$ satisfies

$$\frac{\partial \tilde{J}_i(\tilde{x}^*)}{\partial x_i} = 0, \quad i \in \mathcal{N}. \tag{4.3}$$

In general, the Nash equilibrium x^* in the original noncooperative system $\mathcal{G}(J)$ is not Pareto efficient. Pareto improvement [55] can actually be achieved under private agreements made by some agents who are able to communicate (negotiate) with each other. However, those private agreements are hardly observed by the system manager and hence bring difficulties on properly incentivizing the agents. To avoid the case where agents seek private negotiation from the basis of the incentivized (reconstructed) payoff functions given by the system manager, the system manager should properly design the incentive functions to make the Nash equilibrium $\tilde{x}^*$ of $\mathcal{G}(\tilde{J})$ be Pareto efficient.

4.2.2 Motivation and Problem

Before we present the main problem of this paper, we give some motivations of this work. Considering the case where the agents (companies) may leave the market when their payoffs decrease after the incentive mechanism is executed, it is important to discuss how to design a special incentive mechanism where every agent's payoff is monotonically increasing over time in the incentivized noncooperative dynamical system $\mathcal{G}(\tilde{J})$. In other words, not only may the system manager wish to guarantee the Pareto efficiency at the Nash equilibrium $\tilde{x}^*$ of $\mathcal{G}(\tilde{J})$, but also $\dot{\tilde{J}}_i(x(t)) \geq 0$, $t \geq 0$, for all $i \in \mathcal{N}$ along the system trajectories of (2.32).

Definition 4.3. Given the system trajectory $x(t)$, $t \geq 0$, with $x(0) = x_0$, the agents in the incentivized noncooperative system $\mathcal{G}(\tilde{J})$ are Pareto improving if

$$\tilde{J}_i(x_0) = J_i(x_0), \quad i \in \mathcal{N}, \tag{4.4}$$

$$\dot{\tilde{J}}_i(x(t)) \geq 0, \quad t \geq 0, \quad i \in \mathcal{N}, \tag{4.5}$$

where $J_i(x_0)$ denotes the payoff value of agent i at the initial time.

Note that the condition (4.4) is equivalent to

$$p_i(x_0) = 0, \quad i \in \mathcal{N}, \tag{4.6}$$

representing the assumption that there is no change in the payoff levels when we start to impose the incentive mechanism. On the other hand, the system manager in many economic applications serves merely as a mediator (or a tax collector) and does not have productivity to pay the additional profits to the agents. In such a case, it is worth asking whether it is possible to achieve (4.4) and (4.5) by using some well-designed incentive functions $p_i(x)$, $i \in \mathcal{N}$, satisfying

$$\sum\nolimits_{i \in \mathcal{N}} p_i(x(t)) \leq 0, \quad t \geq 0. \tag{4.7}$$

Note that the condition (4.7) imposes some sustainable budget constraint representing the fact that the system manager collects taxes from some agents and gives some of the collected taxes to other agents as subsidies. When the equality holds, the system manager is understood as a mediator who collects taxes from some agents and gives the same amount of subsidy in total to other agents.

Now, we present the problem of this paper as follows.

Problem: Consider the incentivized noncooperative system $\mathcal{G}(\tilde{J})$ with the pseudo-gradient dynamics (2.32). Suppose that the system manager knows all the agents' payoff functions $J_i(x)$, $i \in \mathcal{N}$. Our objective is to design the incentive functions $p_i(x)$, $i \in \mathcal{N}$, satisfying (4.7) for the incentive mechanism guaranteeing that the agents are Pareto improving and their state converges to a Pareto-efficient Nash equilibrium in $\mathcal{G}(\tilde{J})$.

4.3 Achieving Pareto Improvements with Sustainable Budget Constraint

In this section, we characterize the incentive mechanisms for the noncooperative system. It is necessary to emphasize that the system manager may evaluate the *priority* among the agents. In real society, the policies given by a government are often constructed according to the specific goal of the government considering the priority. For example, the government may give more preferential treatments to the semiconductor companies when the government wishes to raise the international competitiveness of the semiconductor industry in its country. Another example is that the government may provide more resources (e.g., job opportunity or common resource) to the poorer people than the others in its country for enhancing the poor people's income and hence for tackling extreme poverty.

In light of this observation, we suppose that the priority ratio of the agents evaluated by the system manager is given by

$$\eta_1 : \cdots : \eta_n, \tag{4.8}$$

for some $\eta_i \in \mathbb{R}_+$, $i \in \mathcal{N}$. Without loss of generality, η_1 is taken as 1. Then, we consider the weighted social welfare function $U : \mathbb{R}^n \to \mathbb{R}$ given by

$$U(x) \triangleq \sum\nolimits_{i\in\mathcal{N}} \eta_i J_i(x). \tag{4.9}$$

Furthermore, we define the target state as the socially maximum state with respect to $U(x)$ given by

$$\hat{x}^* \triangleq \arg\max_{x\in\mathbb{R}^n} U(x). \tag{4.10}$$

Now, we consider the situation where the incentive functions $p_i(x)$, $i \in \mathcal{N}$, in (4.1) satisfy

$$\sum\nolimits_{i\in\mathcal{N}} \tilde{J}_i(x) = \sigma U(x), \quad x \in \mathbb{R}^n, \tag{4.11}$$

$$\arg\max_{x_i\in\mathbb{R}} \tilde{J}_i(x_i, \hat{x}^*_{-i}) = \hat{x}_i, \quad i \in \mathcal{N}, \tag{4.12}$$

with $\sigma > 0$ being a scaling factor characterized later. Obviously, the variable σ does not affect the maximum state of $\sum_{i\in\mathcal{N}} \tilde{J}_i(x)$, but we keep the notation for further characterization of some requirements below. As a result, the constraint (4.11) guarantees that the target state $\hat{x}^*$ is Pareto efficient in $\mathcal{G}(\tilde{J})$, whereas the constraints

(4.12) make $\hat{x}^*$ to be a Nash equilibrium. In other words, the target state $\hat{x}^*$ maximizing the social welfare function $U(x)$ is a Pareto-efficient Nash equilibrium in the incentivized noncooperative system $\mathcal{G}(\tilde{J})$ under (4.11) and (4.12). Note that the condition (4.11) along with (4.1) and (4.9) is equivalent to

$$\sum_{i\in\mathcal{N}} p_i(x) = \sum_{i\in\mathcal{N}} \tilde{J}_i(x) - \sum_{i\in\mathcal{N}} J_i(x) = \sum_{i\in\mathcal{N}} (\sigma\eta_i - 1)J_i(x). \tag{4.13}$$

Hence, the incentive functions should be designed in such a way that the system trajectories of (2.32) remain in the domain

$$\mathcal{D}_{\text{bud}}(\sigma) \triangleq \left\{ x \in \mathbb{R}^n : \sum_{i\in\mathcal{N}} (\sigma\eta_i - 1)J_i(x) \leq 0 \right\}, \tag{4.14}$$

in order to maintain the sustainable budget constraint (4.7). For the given priority ratio (4.8), it turns out that the initial state x_0 plays an important role during designing the incentive functions. In the following statements, we explore two requirements on the initial state x_0 for constructing our incentive mechanism to allow the system trajectories of (2.32) to remain in the domain $\mathcal{D}_{\text{bud}}$.

Requirement 1:

Since (4.6) holds at the initial state x_0 and hence $\sum_{i\in\mathcal{N}} p_i(x_0) = 0$, (4.13) implies that the scaling factor σ in (4.9) should be determined to satisfy

$$\sum_{i\in\mathcal{N}} (\sigma\eta_i - 1)J_i(x_0) = 0. \tag{4.15}$$

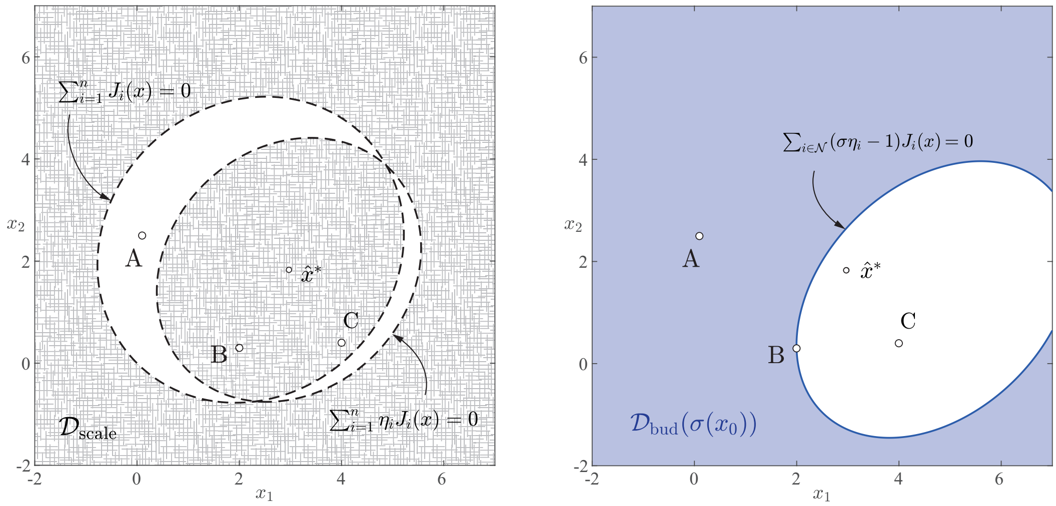
Note that the solution σ of (4.15) is unique as given by

$$\sigma(x_0) = \frac{\sum_{i=1}^n J_i(x_0)}{\sum_{i=1}^n \eta_i J_i(x_0)}. \tag{4.16}$$

In order for $\sigma(x_0)$ to be positive, our framework requires the initial state x_0 to satisfy

$$x_0 \in \mathcal{D}_{\text{scale}} \triangleq \left\{ x \in \mathbb{R}^n : \sum_{i=1}^n J_i(x) \Big/ \sum_{i=1}^n \eta_i J_i(x) > 0 \right\}. \tag{4.17}$$

We emphasize that the condition (4.17) may not hold for some initial state x_0 (and hence we cannot find a positive scaling factor $\sigma(x_0)$). Figure 4.1 shows an example



(a) (b)

Figure. 4.1 An example of (a) the domain $\mathcal{D}_{\text{scale}}$ and (b) the domain $\mathcal{D}_{\text{bud}}(\sigma(x_0))$. The boundary of $\mathcal{D}_{\text{scale}}$ are the two dashed curves elaborated by $\sum_{i=1}^n J_i(x) = 0$ and $\sum_{i=1}^n \eta_i J_i(x) = 0$ in (a). The domain $\mathcal{D}_{\text{bud}}(\sigma(x_0))$ is characterized by the initial state indicated by the point B in (b). In this example, the socially maximum state $\hat{x}^*$ is not contained in $\mathcal{D}_{\text{bud}}(\sigma(x_0))$. Another domain $\mathcal{D}_{\text{bud}}(\sigma(x_0))$ characterized by the initial state on the point C in (a) is depicted in Fig. 4.2 where $\hat{x}^* \in \mathcal{D}_{\text{bud}}(\sigma(x_0))$ holds. The initial state is likely to be on the boundary of $\mathcal{D}_{\text{bud}}(\sigma(x_0))$.

of an infeasible initial state (e.g., point A) outside the domain $\mathcal{D}_{\text{scale}}$ indicated by the striated region given the priority ratio $\eta_1 : \eta_2$ for a two-agent noncooperative system $\mathcal{G}(J)$. Note that $\mathcal{D}_{\text{scale}}$ is invariant with respect to $\sigma(x_0)$. Specifically, it follows from (4.17) that $\mathcal{D}_{\text{scale}}$ is characterized as the union of the domains $\{x \in \mathbb{R}^n : \sum_{i=1}^n J_i(x) > 0 \cap \sum_{i=1}^n \eta_i J_i(x) > 0\}$ and $\{x \in \mathbb{R}^n : \sum_{i=1}^n J_i(x) < 0 \cap \sum_{i=1}^n \eta_i J_i(x) < 0\}$ so that the boundary of $\mathcal{D}_{\text{scale}}$ is given by $\sum_{i=1}^n J_i(x) = 0$ and $\sum_{i=1}^n \eta_i J_i(x) = 0$ irrespective of the agents' individual payoff functions. When the priority ratio (4.8) changes, the domain $\mathcal{D}_{\text{scale}}$ alters along with the changes of the level set of the weighted social welfare function. But when all the agents have the equal priority (i.e., $\eta_1 = \cdots = \eta_n$), those two boundaries coincide with each other and the domain $\mathcal{D}_{\text{scale}}$ is understood as the entire space $\mathbb{R}^n$ because $\sum_{i=1}^n J_i(x) / \sum_{i=1}^n \eta_i J_i(x)$ is constant and positive in (4.17) for all $x \in \mathbb{R}^n$. This special case is elaborated in Section 4.4 below.

Requirement 2:

It is important to notice from (4.16) that since the value of $\sigma(x_0)$ depends on the initial state x_0, the domain $\mathcal{D}_{\text{bud}}(\sigma(x_0))$ given by (4.14) also depends on the initial state x_0. Recalling that the system trajectories of (2.32) should remain in the domain $\mathcal{D}_{\text{bud}}(\sigma(x_0))$

for maintaining the sustainable budget constraint (4.7), some initial state may not be allowed for the existence of the incentive functions that meets this requirement. For instance, when the target state $\hat{x}^*$, which does not depend on the initial state as given by (4.10), does not belong to the domain $\mathcal{D}_{\mathrm{bud}}(\sigma(x_0))$, there is *no possibility* to establish incentive functions satisfying (4.7) around the target state $\hat{x}^*$. An example of the initial state where $\hat{x}^* \notin \mathcal{D}_{\mathrm{bud}}(\sigma(x_0))$ holds is shown as the point B in Fig. 4.1, where the domain $\mathcal{D}_{\mathrm{bud}}(\sigma(x_0))$ is indicated by the blue region. Therefore, in order to make the socially maximum state $\hat{x}^*$ be the target state for the incentive mechanisms, we further suppose that the initial state x_0 yields the domain $\mathcal{D}_{\mathrm{bud}}(\sigma(x_0))$ satisfying $\hat{x}^* \in \operatorname{int} \mathcal{D}_{\mathrm{bud}}(\sigma(x_0))$. In the case where there is a incentive supply from outside the system and its supply rate is given by $c \in \mathbb{R}_+$, the right-hand side of (4.7) should be replaced by c. In this case, the characterization of $\mathcal{D}_{\mathrm{bud}}(\sigma(x_0))$ can be similarly established.

Now, we design the incentive functions $p_i(x)$, $i \in \mathcal{N}$, to satisfy (4.5), (4.6), (4.11) and (4.12). Specifically, we consider the incentive functions used in (4.1) given by

$$p_i(x) \triangleq \zeta_i \sigma(x_0) U(x) - J_i(x) + \sum_{j \neq i} b_{ij}(x_i - \hat{x}_i^*)(x_j - \hat{x}_j^*) + w_i(x_0), \tag{4.18}$$

for each agent $i \in \mathcal{N}$, where $\zeta_i \in (0, 1)$, $i \in \mathcal{N}$, satisfying $\sum_{i \in \mathcal{N}} \zeta_i = 1$, $b_{ij} = -b_{ji}$, $i, j \in \mathcal{N}$, $w_i(x_0) = J_i(x_0) - \zeta_i \sigma(x_0) U(x_0) - \sum_{j \neq i} b_{ij}(x_{0i} - \hat{x}_i^*)(x_{0j} - \hat{x}_j^*)$, $i \in \mathcal{N}$, so that (4.6) holds. Then, the agent's incentivized payoff functions (4.1) are given by

$$\tilde{J}_i(x) = \zeta_i \sigma(x_0) U(x) + \sum_{j \neq i} b_{ij}(x_i - \hat{x}_i^*)(x_j - \hat{x}_j^*) + w_i(x_0). \tag{4.19}$$

Proposition 4.1. If the incentive functions are constructed by (4.18), then the socially maximum state $\hat{x}^*$ associated with the weighted social welfare function $U(x)$ is a Pareto-efficient Nash equilibrium in $\mathcal{G}(\tilde{J})$.

Proof Note that (4.11) holds because $\sum_{i \in \mathcal{N}} \zeta_i = 1$, $b_{ij} = -b_{ji}$, $i, j \in \mathcal{N}$. The proof is immediate by noting from (4.19) that $\tilde{J}_i(x_i, \hat{x}_{-i}^*) = \zeta_i \sigma(x_0) U(x_i, \hat{x}_{-i}^*) + c_i$, $i \in \mathcal{N}$, imply (4.12) holds. □

Consequently, it follows from (2.32) and (4.19) that the pseudo-gradient dynamics are given by

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0 \in \mathbb{R}^n, \quad t \geq 0, \tag{4.20}$$

where $f(x) \triangleq \operatorname{diag}[\alpha](\mathcal{Z} g(x) + B(x - \hat{x}^*))$ with $\mathcal{Z} = \operatorname{diag}[\zeta_1, \ldots, \zeta_n] \in \mathbb{R}^{n \times n}$, $g(x) \triangleq \sigma(x_0) U'(x) \in \mathbb{R}^n$, and $B \triangleq \{b_{ij}\}_{i,j \in \mathcal{N}} = -B^{\mathrm{T}} \in \mathbb{R}^{n \times n}$. For the statement of the following results, we let $B_i \triangleq \begin{bmatrix} 0_{(i-1)\times(i-1)} & \begin{matrix} b_{i1} \\ \vdots \end{matrix} & 0_{(i-1)\times(n-i)} \\ b_{i1} \quad \cdots & b_{ii} & \cdots \quad b_{in} \\ 0_{(n-i)\times(i-1)} & \begin{matrix} \vdots \\ b_{in} \end{matrix} & 0_{(n-i)\times(n-i)} \end{bmatrix} \in \mathbb{R}^{n\times n}$ with $b_{ii} = 0$. Note that $B_1 = \begin{bmatrix} 0 & b_{12} & \cdots & b_{1n} \\ b_{12} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{1n} & 0 & \cdots & 0 \end{bmatrix}$, $B_n = \begin{bmatrix} 0 & \cdots & 0 & b_{n1} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & b_{n(n-1)} \\ b_{n1} & \cdots & b_{n(n-1)} & 0 \end{bmatrix}$, and $\sum_{i\in\mathcal{N}} B_i = 0$. Furthermore, we let $\mathcal{D}_i \triangleq \{x \in \mathbb{R}^n : \tilde{J}_i'(x)f(x) \geq 0\} = \{x \in \mathbb{R}^n : [\zeta_i g(x) + B_i(x - \hat{x}^*)]^{\mathrm{T}} f(x) \geq 0\}$, $i \in \mathcal{N}$. Note that $\dot{\tilde{J}}_i(x(t)) \geq 0$ when the agents' state $x(t)$ belongs to $\mathcal{D}_i$.

Theorem 4.1. Consider the n-agent noncooperative system $\mathcal{G}(J)$ with the incentive mechanism (4.1) and the pseudo-gradient dynamics (4.20). If the parameters $\zeta_i \in (0, 1)$, $i \in \mathcal{N}$, and $b_{ij} = -b_{ji} \in \mathbb{R}$, $i, j \in \mathcal{N}$, are chosen in such a way that there exists a function $V : \mathbb{R}^n \to \mathbb{R}$ such that

$$V(\hat{x}^*) = 0, \tag{4.21}$$

$$V(x) > 0, \tag{4.22}$$

$$V'(x)f(x) < 0, \tag{4.23}$$

for all $x \in \mathcal{D} \triangleq \{x \in \mathbb{R}^n : V(x) \leq V(x_0)\} \setminus \{\hat{x}^*\}$ satisfying $\mathcal{D} \subseteq \mathcal{D}_1 \cap \cdots \cap \mathcal{D}_n \cap \mathcal{D}_{\mathrm{bud}}(\sigma(x_0))$ for the given initial state x_0, then the incentive functions $p_i(x)$, $i \in \mathcal{N}$, given by (4.18) guarantee that the socially maximum state $\hat{x}^*$ is an asymptotically stable equilibrium point and all the agents are Pareto improving with the sustainable budget constraint (4.7).

Proof It follows from (4.21)–(4.23) that $\hat{x}^*$ is an asymptotically stable equilibrium point. Furthermore, since the trajectory remains in the domain $\mathcal{D}$ (and hence $\mathcal{D}_i$), it follows that $\dot{\tilde{J}}_i(x(t)) = \tilde{J}_i'(x(t))f(x(t)) \geq 0$ for all $i \in \mathcal{N}$ and $t \geq 0$. Moreover, since the trajectory remains in the domain $\mathcal{D}_{\mathrm{bud}}(\sigma(x_0))$, it follows from (4.13) and (4.14) that (4.7) holds. The proof is complete. □

Example 4.1. Consider the two-agent noncooperative system with

$$J_1(x) = -0.5x_1^2 + 0.3x_1x_2 - 0.5x_2^2 + 4x_1 - 5.8, \tag{4.24}$$

$$J_2(x) = -0.5x_1^2 - 0.1x_1x_2 - 0.5x_2^2 + 4x_2 + 5.8. \tag{4.25}$$

Even though the constant terms -5.8 and 5.8 in the payoff functions above do not affect the behavior of the agents (these constants are included in $w_1(x_0), \ldots, w_n(x_0)$ in (4.18) so that its time derivative vanishes in the calculation of pseudo-gradient), we keep the constant terms to effectively illustrate the domains in the figures. Let the priority evaluated by the system manager be given by $\eta_1 = 1$ and $\eta_2 = 0.5$. Note that the domain $\mathcal{D}_{\text{scale}}$ is already indicated by the striated domain in Fig. 4.1 and the socially maximum state is given by $\hat{x}^* = [2.9714, 1.8286]^{\mathrm{T}}$. Supposing that the initial state is given by $x_0 = [4, 0.4]^{\mathrm{T}}$, which is exactly the point C in Fig. 4.1 satisfying $x_0 \in \mathcal{D}_{\text{scale}}$, the scaling factor is obtained by (4.16) as $\sigma(x_0) = 0.8074$. In this case, the domain $\mathcal{D}_{\text{bud}}(\sigma(x_0))$ satisfying $\hat{x}^* \in \operatorname{int} \mathcal{D}_{\text{bud}}(\sigma(x_0))$ is illustrated as the red region in Fig. 4.2. Let the sensitivity parameters be given by $\alpha = (1, 1)$ so that the vector field of the incentivized pseudo-gradient dynamics is given by $f(x) = \mathcal{Z}g(x) + B(x - \hat{x}^*)$ with $g(x) = 0.5\sigma(x_0)[-3x_1 + 0.5x_2 + 8, 0.5x_1 - 3x_2 + 4]^{\mathrm{T}}$. It follows from Theorem 4.1 that the incentive mechanism (4.1) along with the incentive function (4.18) with $\zeta_1 = 1 - \zeta_2 = 0.4$, $b_{12} = -b_{21} = 0.3$ satisfying (4.21)–(4.23), $\mathcal{D}_1 = \mathcal{D}_2 = \mathbb{R}^2$, and $\mathcal{D} \subseteq \mathcal{D}_{\text{bud}}(\sigma(x_0))$ with $V(x) = (x - \hat{x}^*)^{\mathrm{T}} \begin{bmatrix} 2 & 1.2 \\ 1.2 & 1.08 \end{bmatrix} (x - \hat{x}^*)$ guarantees that the agents' state $x(t)$ converges to the socially maximum state $\hat{x}^*$ and both of the agents are Pareto improving with the sustainable budget constraint (4.7). Figure 4.3 shows the trajectories of the agents' payoffs and incentives versus time. It can be seen from Figs. 4.2 and 4.3 that the agents' state indeed converges to the socially maximum state $\hat{x}^*$ with monotonically increasing $\tilde{J}_1(x(t))$ and $\tilde{J}_2(x(t))$ even though the sum of the incentive functions $p_1(x(t))$ and $p_2(x(t))$ are nonpositive for all $t \geq 0$ (see the red solid curve in Fig. 4.3(b)).

Example 4.2. Consider the two-agent noncooperative system with

$$J_1(x) = -x_1^2 - x_2^2 + 0.5x_1x_2 - \sin(x_1x_2) + 3.8x_1 - 4, \tag{4.26}$$

$$J_2(x) = -x_1^2 - x_2^2 + 8x_1 + 3.8x_2 - 20. \tag{4.27}$$

Let the priority evaluated by the system manager be given by $\eta_1 = 1$ and $\eta_2 = 2$. Note that the domain $\mathcal{D}_{\text{scale}}$ is given by $\mathbb{R}^2$ because $J_1(x) < 0$ and $J_2(x) < 0$ for all

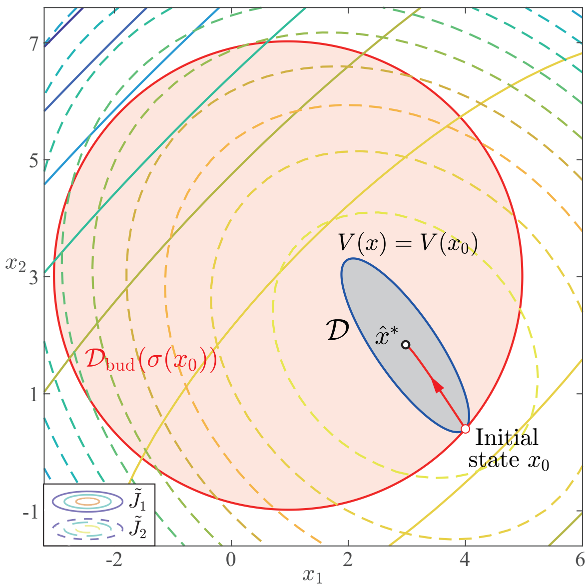


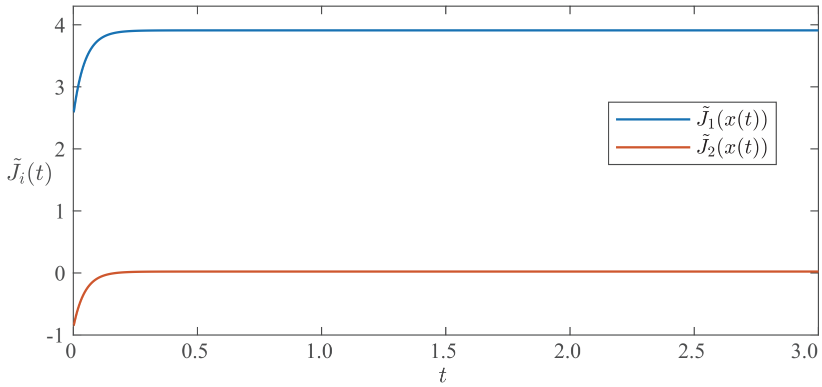
Figure. 4.2 Level sets of $\tilde{J}_1(x)$ and $\tilde{J}_2(x)$ with the domain $\mathcal{D}_{\text{bud}}(\sigma(x_0))$ and the trajectory of $x(t)$ under the incentive functions (4.18) with $\zeta_1 = 1 - \zeta_2 = 0.4$, $b_{12} = -b_{21} = 0.3$ in Example 4.1. The state converges to the socially maximum state $\hat{x}^*$ and its trajectory is contained in the domain $\mathcal{D}_{\text{bud}}(\sigma(x_0))$.

$x \in \mathbb{R}^2$. Furthermore, the socially maximum state is given by $\hat{x}^* = [3.3779, 1.4480]^{\mathrm{T}}$. Supposing that the initial state is given by $x_0 = [3.6720, 1.5360]^{\mathrm{T}}$, the scaling factor is obtained by (4.16) as $\sigma(x_0) = 0.8311$. In this case, the domain $\mathcal{D}_{\text{bud}}(\sigma(x_0))$ satisfying $\hat{x}^* \in \operatorname{int} \mathcal{D}_{\text{bud}}(\sigma(x_0))$ is illustrated as the red region in Fig. 4.4. Let the sensitivity parameters be given by $\alpha = (1, 1.5)$ so that the vector field of the incentivized pseudo-gradient dynamics is given by $f(x) = \operatorname{diag}[\alpha](\mathcal{Z}g(x) + B(x - \hat{x}^*))$ with

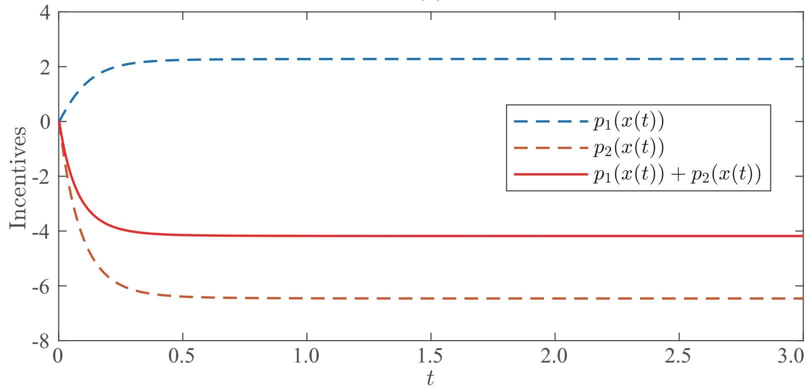
$$g(x) = \sigma(x_0) \begin{bmatrix} -6x_1 + 0.5x_2 - x_2 \cos(x_1 x_2) + 19.8 \\ 0.5x_1 - 6x_2 - x_1 \cos(x_1 x_2) + 7.6 \end{bmatrix}.$$

It follows from Theorem 4.1 that the incentive mechanism (4.1) along with the incentive function (4.18) with $\zeta_1 = 1 - \zeta_2 = 0.5$ and $b_{12} = -b_{21} = 2$ satisfying (4.21)–(4.23) and $\mathcal{D} \subseteq \mathcal{D}_1 \cap \mathcal{D}_2 \cap \mathcal{D}_{\text{bud}}(\sigma(x_0))$ with $V(x) = -U(x) + U(\hat{x}^*)$ guarantees that the agents' state $x(t)$ converges to the socially maximum state $\hat{x}^*$ and both of the agents are Pareto improving with the sustainable budget constraint (4.7).

In general, it may be hard to examine the existence of the domain $\mathcal{D}_1 \cap \cdots \cap \mathcal{D}_n$ when the number of the agents is large. However, the next result deals with the case where the sensitivity parameters of the agents are uncertain and suggests that the domain $\mathcal{D}_1 \cap \cdots \cap \mathcal{D}_n$ exists as long as b_{ij} is taken to be sufficiently lose to 0.



(a)



(b)

Figure. 4.3 Trajectories of the amount of incentives and agents' payoffs under the incentive functions (4.18) with $\zeta_1 = 1 - \zeta_2 = 0.4$, $b_{12} = -b_{21} = 0.3$ in Example 4.1. The agents' payoffs are monotonically increasing under the incentives satisfying $p_1(x_0) = p_2(x_0) = 0$ and $p_1(x(t)) + p_2(x(t)) < 0$ for all $t > 0$.

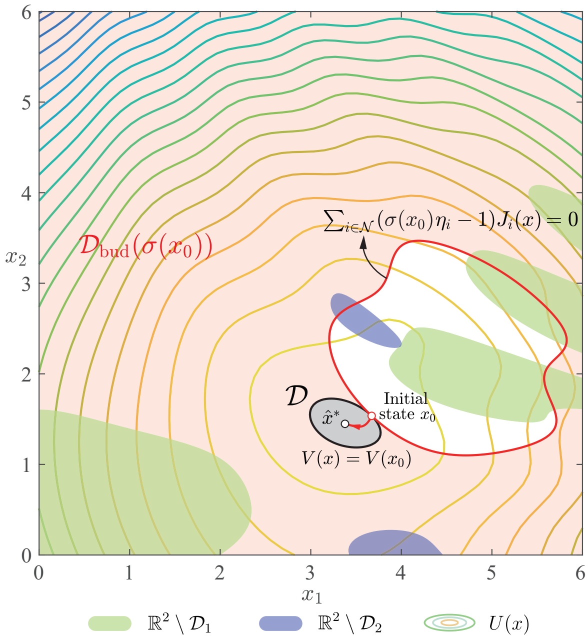


Figure. 4.4 Level sets of $U(x)$ with the domain $\mathcal{D}_{\text{bud}}(\sigma(x_0))$ and the trajectory of $x(t)$ under the incentive functions (4.18) with $\zeta_1 = 1 - \zeta_2 = 0.5$, $b_{12} = -b_{21} = 2$ in Example 4.2. The state converges to the socially maximum state $\hat{x}^*$ and its trajectory is contained in the domains $\mathcal{D}_{\text{bud}}(\sigma(x_0))$, $\mathcal{D}_1$ and $\mathcal{D}_2$.

Corollary 4.1. Consider the n-agent noncooperative system $\mathcal{G}(J)$ with the incentive mechanism (4.1) and the pseudo-gradient dynamics (4.20). If the domain $\mathcal{D} \triangleq \{x \in \mathbb{R}^n : U(x) \geq U(x_0)\}$ satisfies $\mathcal{D} \subseteq \mathcal{D}_{\text{bud}}(\sigma(x_0))$ for the given initial state x_0, then the incentive functions $p_i(x)$, $i \in \mathcal{N}$, given by (4.18) with $b_{ij} = 0$, $i, j \in \mathcal{N}$, guarantee that the socially maximum state $\hat{x}^*$ is an asymptotically stable equilibrium point and all the agents are Pareto improving with the sustainable budget constraint (4.7) for any positive constants α_i, $i \in \mathcal{N}$.

Proof The result is a direct consequence of Theorem 4.1 with $V(x) \triangleq -U(x) + U(\hat{x}^*)$ and $\mathcal{D}_i = \{x \in \mathbb{R}^n : [\zeta_i g(x) + B_i(x - \hat{x}^*)]^{\mathrm{T}} f(x) \geq 0\} = \{x \in \mathbb{R}^n : \zeta_i g^{\mathrm{T}}(x)\mathrm{diag}[\alpha]\mathcal{Z}g(x) \geq 0\} = \mathbb{R}^n$, $i \in \mathcal{N}$, for the case of $b_{ij} = 0$, $i, j \in \mathcal{N}$. □

Now, we specialize the result of Theorem 4.1 with quadratic payoff functions given by

$$J_i(x) = \frac{1}{2}x^{\mathrm{T}} A_i x + b_i^{\mathrm{T}} x + c_i, \quad i \in \mathcal{N}, \tag{4.28}$$

where $A_i \triangleq \{a^i_{j,k}\}_{(j,k)\in\mathcal{N}\times\mathcal{N}} \in \mathbb{R}^{n\times n}$, $b_i \triangleq [b^i_1, \ldots, b^i_n]^{\mathrm{T}} \in \mathbb{R}^n$, and $c_i \in \mathbb{R}$, $i \in \mathcal{N}$. The social welfare function (4.9) is hence given by

$$U(x) = \frac{1}{2}x^{\mathrm{T}}\mathbb{A}x + \mathbb{B}^{\mathrm{T}}x + c_0, \tag{4.29}$$

with $c_0 \triangleq \sum_{i\in\mathcal{N}} \eta_i c_i \in \mathbb{R}$, $\mathbb{A} \triangleq \sum_{i\in\mathcal{N}} \eta_i A_i \in \mathbb{R}^{n\times n}$ and $\mathbb{B} \triangleq \sum_{i\in\mathcal{N}} \eta_i b_i \in \mathbb{R}^n$. Supposing that $U(x)$ is concave (i.e., $\mathbb{A} < 0$), it follows that the unique socially maximum state $\hat{x}^*$ is given by $\hat{x}^* = -\mathbb{A}^{-1}\mathbb{B} \in \mathbb{R}^n$ and the social welfare function can be rewritten as

$$U(x) = \frac{1}{2}\tilde{x}^{\mathrm{T}}\mathbb{A}\tilde{x} - \frac{1}{2}(\hat{x}^*)^{\mathrm{T}}\mathbb{A}\hat{x}^* + c_0, \tag{4.30}$$

with $\tilde{x} \triangleq x - \hat{x}^*$. For the statement of the following results, let

$$\mathcal{A} \triangleq \mathrm{diag}[\alpha](\sigma(x_0)\mathcal{Z}\mathbb{A} + B) \in \mathbb{R}^{n\times n}. \tag{4.31}$$

Corollary 4.2. Consider the n-agent noncooperative system $\mathcal{G}(J)$ with the incentive mechanism (4.1), the pseudo-gradient dynamics (4.20), and the quadratic payoff functions (4.28). Let $P_i \triangleq \zeta_i\sigma(x_0)\mathbb{A} + B_i \in \mathbb{R}^{n\times n}$, $i \in \mathcal{N}$. If the parameters $\zeta_i \in (0, 1)$, $i \in \mathcal{N}$, and $b_{ij} = -b_{ji} \in \mathbb{R}$, $i, j \in \mathcal{N}$, are chosen in such a way that

$$0 < \mathcal{A}^{\mathrm{T}}P_i + P_i\mathcal{A}, \quad i \in \mathcal{N}, \tag{4.32}$$

then the incentive functions $p_i(x)$, $i \in \mathcal{N}$, given by (4.18) guarantee that the socially maximum state $\hat{x}^*$ is globally asymptotically stable. Furthermore, all the agents are Pareto improving with the sustainable budget constraint (4.7) for the given initial state x_0 satisfying $\mathcal{D} \subseteq \mathcal{D}_{\mathrm{bud}}(\sigma(x_0))$, where $\mathcal{D} \triangleq \{x \in \mathbb{R}^n : V(x) \le V(x_0)\} \setminus \{\hat{x}^*\}$ with $V(x)$ satisfying (4.21)–(4.23).

Proof First, note that the vector field $f(x)$ of the pseudo-gradient dynamics (4.20) becomes $f(x) = \mathrm{diag}[\alpha](\sigma(x_0)\mathcal{Z}\mathbb{A} + B)\tilde{x}$. Furthermore, note from (4.19) and (4.30) that the agents' incentivized payoff functions are given by

$$\tilde{J}_i(x) = \frac{1}{2}\tilde{x}^{\mathrm{T}}P_i\tilde{x} + \zeta_i(-\frac{1}{2}(\hat{x}^*)^{\mathrm{T}}\mathbb{A}\hat{x}^* + c_0) + w_i(x_0), \quad i \in \mathcal{N}. \tag{4.33}$$

Therefore, it follows that

$$\tilde{J}_i'(x)f(x) = (P_i\tilde{x})^{\mathrm{T}}f(x) = (P_i\tilde{x})^{\mathrm{T}}\mathrm{diag}[\alpha](\sigma(x_0)\mathcal{Z}\mathbb{A} + B)\tilde{x}$$
$$= \frac{1}{2}\tilde{x}^{\mathrm{T}}(\mathcal{A}^{\mathrm{T}}P_i + P_i\mathcal{A})\tilde{x} > 0, \quad i \in \mathcal{N}, \tag{4.34}$$

for all $x \in \mathbb{R}^n \setminus \{\hat{x}^*\}$ and hence

$$\mathcal{D}_i = \{x \in \mathbb{R}^n : \tilde{J}_i'(x)f(x) \geq 0\} = \mathbb{R}^n, \quad i \in \mathcal{N}. \tag{4.35}$$

Then, the result is a direct consequence of Theorem 4.1 using the Lyapunov function candidate $V(x) = -U(x) + U(\hat{x}^*)$ satisfying (4.21)–(4.23) since

$$V'(x)f(x) = -\frac{1}{\sigma(x_0)}\sum_{i\in\mathcal{N}}\tilde{J}_i'(x)f(x) < 0, \tag{4.36}$$

holds for all $x \in \mathbb{R}^n \setminus \{\hat{x}^*\}$. □

Note that it may be hard to determine the parameters ζ_i, $i \in \mathcal{N}$, and b_{ij}, $i, j \in \mathcal{N}$, to guarantee $\mathcal{D} \subseteq \mathcal{D}_{\mathrm{bud}}(\sigma(x_0))$ when the number of the agents is large because we cannot easily find the function $V(x)$. The following result provides different conditions without looking for a function $V(x)$ guaranteeing $\mathcal{D} \subseteq \mathcal{D}_{\mathrm{bud}}(\sigma(x_0))$ for the noncooperative system $\mathcal{G}(J)$ with quadratic payoff functions when $\mathcal{A}$ possesses a real eigenvalue in its spectrum.

Proposition 4.2. Consider the n-agent noncooperative system $\mathcal{G}(J)$ with the incentive mechanism (4.1), the pseudo-gradient dynamics (4.20), and the quadratic payoff functions (4.28). If the parameters $\zeta_i \in (0, 1)$, $i \in \mathcal{N}$, and $b_{ij} = -b_{ji} \in \mathbb{R}$, $i, j \in \mathcal{N}$, are chosen in such a way that (4.32) holds along with

$$x_0 - \hat{x}^* \in \mathrm{null}(\mathcal{A} - \lambda I_n), \tag{4.37}$$

where $\lambda \in \mathbb{R}$ is a real eigenvalue of the matrix $\mathcal{A}$, then all the agents are Pareto improving with the sustainable budget constraint (4.7) for the given initial state x_0 satisfying that the straight segment from x_0 to $\hat{x}^*$ is contained in the domain $\mathcal{D}_{\mathrm{bud}}(\sigma(x_0))$.

Proof The proof is immediate since (4.37) indicates that the vector $x_0 - \hat{x}^*$ is the eigenvector of the matrix $\mathcal{A}$ associated with the eigenvalue λ and hence the system trajectory $x(t)$ is a straight line starting at the initial state x_0 and ending at the target state $\hat{x}^*$. □

For a given vector $\hat{x} \triangleq [\hat{x}_1, \ldots, \hat{x}_n]^{\mathrm{T}} = x_0 - \hat{x}^* \in \mathbb{R}^n$, even though it appears to be hard to find the parameters ζ_i, $i \in \mathcal{N}$, and b_{ij}, $i, j \in \mathcal{N}$, such that the condition (4.37) is satisfied, it is possible to solve some linear equations to derive such parameters by constructing a special form for the matrix $\mathcal{A}$ when $n \geq 5$ and $\hat{x}_i \neq 0$, $i \in \mathcal{N}$. For example, let $b_{1j} = -\zeta_1 \sigma(x_0) \mathbb{A}_1^j$, $j \in \mathcal{N}$, where $\mathbb{A}_i^j \triangleq \mathbb{A}(i,j)$, so that $\mathcal{A}$ is given by a special matrix shown in (B.4) in Appendix B with $\tilde{\mathbb{A}}_i^j \triangleq \sigma(x_0)\mathbb{A}_i^j$. Note that $(1,1)$-entry of $\mathcal{A}$, which is $\alpha_1 \zeta_1 \tilde{\mathbb{A}}_1^1 < 0$, is one of its eigenvalues because $\mathcal{A}$ is a lower block-triangular matrix. Now, taking $\lambda = \alpha_1 \zeta_1 \tilde{\mathbb{A}}_1^1$, it follows from (B.4) that the condition (4.37) is equivalent to

$$0 = (\mathcal{A} - \alpha_1 \zeta_1 \tilde{\mathbb{A}}_1^1 I_n)\hat{x}, \tag{4.38}$$

which is essentially a system of $n-1$ number of linear equations shown in (B.5) in Appendix B with $n-1$ variables given by $b_{34}, b_{23}, b_{24}, \ldots, b_{2n}$ for the given $\zeta_i \in (0,1)$, $i \in \mathcal{N}$, satisfying $\sum_{i=1}^{n} \zeta_i = 1$, and $b_{35}, b_{36}, \ldots, b_{(n-1)n}$, because $\mathrm{row}_1(\mathcal{A} - \alpha_1 \zeta_1 \sigma(x_0) \mathbb{A}_1^1 I_n) = 0_n^{\mathrm{T}}$ holds. Now, note that the matrix Π in (B.5) satisfies $\det(\Pi) = 0$ and hence there are infinitely many solutions of $(b_{34}, b_{23}, b_{24}, \ldots, b_{2n})$ when the parameters $\zeta_i \in (0,1)$, $i \in \mathcal{N}$, and $b_{35}, b_{36}, \ldots, b_{(n-1)n}$ satisfy $\mathrm{rank}(\Pi) = \mathrm{rank}([\Pi, \xi])$. Note that $\hat{x}_i \neq 0$, $i = 2, 3, 4$, imply $\mathrm{rank}(\Pi) = n-2$ and hence the condition $\mathrm{rank}(\Pi) = \mathrm{rank}([\Pi, \xi])$ is equivalent to

$$\frac{\hat{x}_2}{\alpha_2}\xi_1 + \frac{\hat{x}_3}{\alpha_3}\xi_2 + \frac{\hat{x}_4}{\alpha_4}\zeta_3 + \ldots + \frac{\hat{x}_n}{\alpha_n}\zeta_{n-1} = 0. \tag{4.39}$$

This is because $\mathrm{rank}([\Pi, \xi]) = \mathrm{rank}(\Gamma[\Pi, \xi]) = n-2$ hold with

$$\Gamma[\Pi, \xi] = \left[\begin{array}{ccccc|c} 0 & 0 & 0 & \cdots & 0 & \tilde{\xi}_1 \\ \alpha_3 \hat{x}_4 & -\alpha_3 \hat{x}_2 & 0 & \cdots & 0 & \xi_2 \\ -\alpha_4 \hat{x}_3 & 0 & -\alpha_4 \hat{x}_2 & \cdots & 0 & \xi_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\alpha_n \hat{x}_2 & \xi_{n-1} \end{array}\right], \tag{4.40}$$

for a nonsingular $\Gamma \triangleq \begin{bmatrix} 1 & \frac{\alpha_2 \hat{x}_3}{\hat{x}_2 \alpha_3} & \cdots & \frac{\alpha_2 \hat{x}_n}{\hat{x}_2 \alpha_n} \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \in \mathbb{R}^{(n-1)\times(n-1)}$ if and only if $\tilde{\xi}_1 \triangleq \xi_1 + \frac{\alpha_2 \hat{x}_3}{\hat{x}_2 \alpha_3}\xi_2 + \frac{\alpha_2 \hat{x}_4}{\hat{x}_2 \alpha_4}\zeta_3 + \ldots + \frac{\alpha_2 \hat{x}_n}{\hat{x}_2 \alpha_n}\zeta_{n-1} = 0$. Therefore, since $\xi_2 \in \mathbb{R}$ is an affine function of b_{35}, we can always find b_{35} to satisfy (4.39) (i.e., $\mathrm{rank}(\Pi) = \mathrm{rank}([\Pi, \xi])$) and hence there exist parameters ζ_i, $i \in \mathcal{N}$, and b_{ij}, $i, j \in \mathcal{N}$, satisfying (4.37) when $n \geq 5$ and $\hat{x}_i \neq 0$, $i \in \mathcal{N}$.

However, when $n < 5$, the conditions of $\hat{x}_i \neq 0$, $i \in \mathcal{N}$, may no longer be able to guarantee the existence of parameters ζ_i, $i \in \mathcal{N}$, and b_{ij}, $i, j \in \mathcal{N}$, such that the condition (4.37) holds. For example, suppose $n = 2$. Given an arbitrary ζ_1, letting $b_{12} = -\zeta_1 \sigma(x_0) \mathbb{A}_1^2$, the condition (4.38) yields a 1-dimensional linear equation of ζ_2 given by

$$\alpha_2 \tilde{\mathbb{A}}_1^2 \hat{x}_1 + (\alpha_2 \zeta_2 \tilde{\mathbb{A}}_2^2 - \alpha_1 \zeta_1 \tilde{\mathbb{A}}_1^1) \hat{x}_2 = 0. \tag{4.41}$$

Recalling that $\zeta_1 + \zeta_2 = 1$, it follows that

$$\zeta_2 = \frac{\alpha_2 \tilde{\mathbb{A}}_2^2 \hat{x}_1 - \alpha_1 \tilde{\mathbb{A}}_1^1 \hat{x}_2}{(\alpha_1 \tilde{\mathbb{A}}_1^1 + \alpha_2 \tilde{\mathbb{A}}_2^2) \hat{x}_2}. \tag{4.42}$$

Therefore, there exist parameters $\zeta_1 \in (0, 1)$, $\zeta_2 \in (0, 1)$, and b_{12}, such that the condition (4.37) is satisfied when the initial state $x_0 = [x_{01}, x_{02}]^{\mathrm{T}}$ satisfy

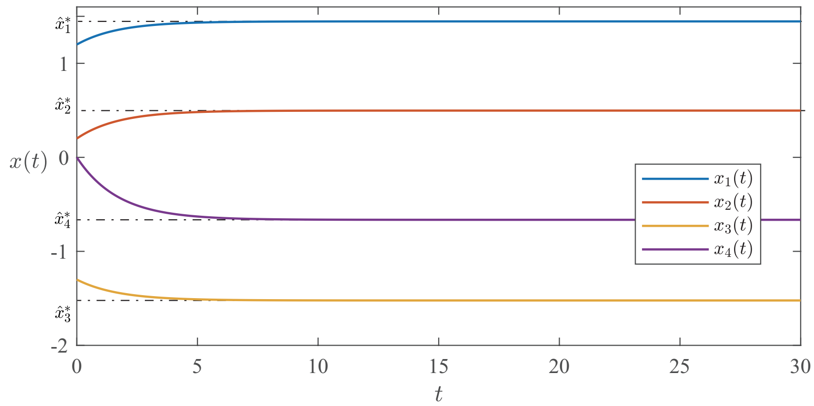
$$\frac{\alpha_2 \tilde{\mathbb{A}}_2^2 (x_{01} - \hat{x}_1^*) - \alpha_1 \tilde{\mathbb{A}}_1^1 (x_{02} - \hat{x}_2^*)}{(\alpha_1 \tilde{\mathbb{A}}_1^1 + \alpha_2 \tilde{\mathbb{A}}_2^2)(x_{02} - \hat{x}_2^*)} \in (0, 1). \tag{4.43}$$

Example 4.3. Consider the four-agent noncooperative system with the quadratic payoff functions (4.28) with

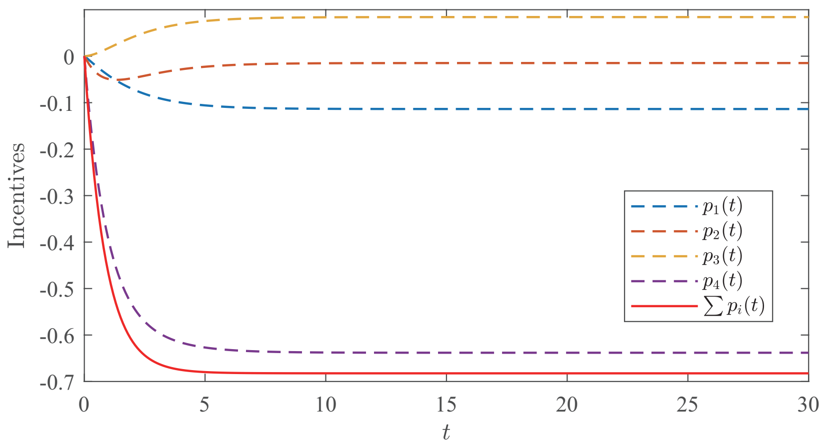
$$A_1 = \begin{bmatrix} -1 & 0.1 & 0.1 & 0.1 \\ 0.1 & -2 & 0.1 & 0 \\ 0.1 & 0.1 & 0 & 0 \\ 0.1 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & -0.2 & 0 & 0 \\ -0.2 & -1 & 0 & 0.1 \\ 0 & 0 & 0 & 0.1 \\ 0 & 0.1 & 0.1 & -1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & -0.1 & 0 & 0.1 \\ -0.1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0.2 \\ 0.1 & 0 & 0.2 & -1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} -1 & -0.2 & 0 & 0 \\ -0.2 & -1 & 0 & 0.1 \\ 0 & 0 & -1 & -0.5 \\ 0 & 0.1 & -0.5 & -2 \end{bmatrix},$$

$$b_1 = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 0 \\ 4 \\ 10 \\ 0 \end{bmatrix}, \quad b_3 = \begin{bmatrix} 0 \\ 0 \\ -4 \\ 0 \end{bmatrix}, \quad b_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -4 \end{bmatrix},$$



(a)



(b)

Figure. 4.5 Trajectories of the agents' state and the amount of incentives under the incentive functions (4.18) in Example 4.3. The dash-dot lines in figure (a) indicate the socially maximum state.

and $c_1 = c_2 = c_3 = c_4 = 0$. Let the priority evaluated by the system manager be given by $\eta_1 = 1$, $\eta_2 = 0.5$, $\eta_3 = 1$, and $\eta_4 = 0.5$. Note that, in this case, the socially maximum state is given by $\hat{x}^* = [1.4462, 0.4974, -1.5223, -0.6644]^{\mathrm{T}}$. Supposing that the initial state is given by $x_0 = [1.2, 0.2, -1.3, 0]^{\mathrm{T}}$, the scaling factor is obtained by (4.17) as $\sigma(x_0) = 0.8011$. In this case, it can be verified that the straight segment from x_0 to $\hat{x}^*$ is contained in the domain $\mathcal{D}_{\mathrm{bud}}(\sigma(x_0))$ since $\sum_{i \in \mathcal{N}} (\sigma\eta_i - 1) J_i(x) \leq 0$ holds with $x = \gamma(x_0 - \hat{x}^*) + \hat{x}^*$ for all $\gamma \in (0, 1)$. Let the sensitivity parameters be given by $\alpha = (1, 1, 1, 1)$ so that the matrix $\mathcal{A}$ is given by $\mathcal{A} = \sigma(x_0)\mathcal{Z}\mathbb{A} + B$. Let $\zeta_i = 0.25$, $i = 1, 2, 3, 4$, $b_{12} = -0.1007$, $b_{13} = 0$, $b_{14} = -0.0868$, $b_{23} = -0.0641$, $b_{24} = 0$, and $b_{34} = 0.0238$, so that the condition (4.32) holds and the condition (4.37) holds for the real eigenvalue of $\mathcal{A}$ given by $\lambda = -0.5712$. It follows from Proposition 4.2 that the incentive mechanism (4.1) along with the incentive functions (4.18) guarantee that all the agents are Pareto improving with the sustainable budget constraint (4.7). Figure 4.5 shows the trajectories of the agents' state and incentives versus time. It can be seen from those figures that the agents' state indeed converges to the socially maximum state $\hat{x}^*$ under the sustainable budget constraint (4.7) (see the red solid curve in Fig. 4.5(b)).

4.4 Connection Between Pareto Improvement and Potentialization Under Equal Priority

In general, the domains $\mathcal{D}_{\text{bud}}(\sigma(x_0))$ and $\mathcal{D}_{\text{scale}}$ characterized in Section 4.3 are not the entire state space and hence we may not be able to construct a Pareto-improving incentive functions for some initial state x_0 with unequal priority. But for a special situation where the agents have the equal priority in (4.8), i.e., $\eta_i = 1 \in \mathbb{R}_+$ for all $i \in \mathcal{N}$, recall from (4.17) that $\mathcal{D}_{\text{scale}} = \mathbb{R}^n$ holds. In this case, since the scaling factor is simply obtained from (4.16) as $\sigma(x_0) = 1$ irrespective of the initial state, it is worth noting from (4.13) and (4.14) that the incentive functions $p_i(x)$, $i \in \mathcal{N}$, satisfy

$$\sum_{i \in \mathcal{N}} p_i(x) = \sum_{i \in \mathcal{N}} (\sigma(x_0)\eta_i - 1) J_i(x) = 0, \quad x \in \mathbb{R}^n, \tag{4.44}$$

i.e., the system manager exactly works as a mediator transferring the payoff values among the n agents, and hence the domain $\mathcal{D}_{\text{bud}}(\sigma(x_0))$ becomes $\mathbb{R}^n$ for all $x_0 \in \mathbb{R}^n$. Furthermore, the social welfare function (4.9) simply becomes

$$U(x) = \sum_{i \in \mathcal{N}} J_i(x). \tag{4.45}$$

Therefore, in this section, we specialize the incentive mechanism characterized in Section 4.3 to this special situation and show the fact that the Pareto-improving incentive mechanism can be constructed for *any* initial state x_0 in $\mathbb{R}^n$.

Theorem 4.2. Consider the n-agent noncooperative system $\mathcal{G}(J)$ with the incentive mechanism (4.1) and the pseudo-gradient dynamics (4.20). Suppose that the agents have the equal priority in (4.8) with $\eta_i = 1$ for all $i \in \mathcal{N}$. If the parameters $\zeta_i \in (0, 1)$, $i \in \mathcal{N}$, and $b_{ij} = -b_{ji} \in \mathbb{R}$, $i, j \in \mathcal{N}$, are chosen in such a way that the socially maximum state $\hat{x}^*$ belongs to the interior of $\mathcal{D}_1 \cap \cdots \cap \mathcal{D}_n$, then the incentive functions $p_i(x)$, $i \in \mathcal{N}$, given by (4.18) guarantee that the socially maximum state $\hat{x}^*$ is asymptotically stable. Furthermore, all the agents are Pareto improving with the

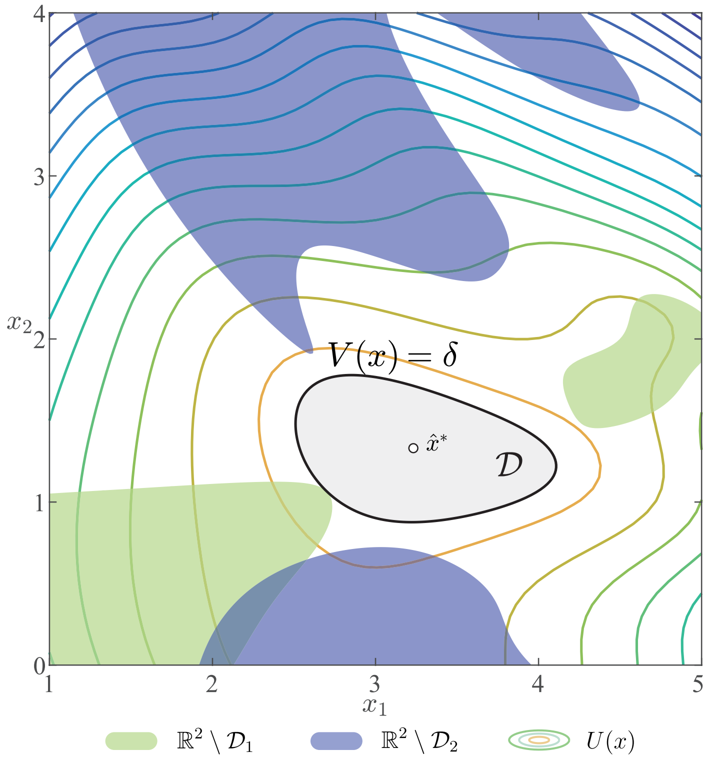


Figure. 4.6 Level sets of $U(x)$ with the guaranteed region of attraction under the incentive functions (4.18) with $\zeta_1 = 1 - \zeta_2 = 0.4$, $b_{12} = -b_{21} = 1.9$ in Example 4.4.

sustainable (balanced) budget constraint (4.7) holding with equality for any initial state $x_0 \in \mathcal{D}$, where $\mathcal{D} \triangleq \{x \in \mathbb{R}^n : V(x) \le \delta\}$ with the maximum attainable $\delta \in \mathbb{R}_+$ such that $\mathcal{D} \subseteq \mathcal{D}_1 \cap \cdots \cap \mathcal{D}_n$ with $V(x)$ satisfying (4.21)–(4.23).

Proof Consider the Lyapunov function candidate defined by $V(x) = -U(x) + U(\hat{x}^*)$. Since $\hat{x}^* \in \text{int}\,(\mathcal{D}_1 \cap \cdots \cap \mathcal{D}_n)$ indicates $\tilde{J}_i'(x) f(x) > 0$, $i \in \mathcal{N}$, around the socially maximum state $\hat{x}^*$, it follows that $V'(x)f(x) = -\sum_{i \in \mathcal{N}} \tilde{J}_i'(x) f(x) > 0$ holds around $\hat{x}^*$ and hence $\hat{x}^*$ is asymptotically stable. Now, recalling that $\mathcal{D}_{\text{bud}}(\sigma(x_0)) = \mathbb{R}^n$ for any initial state $x_0 \in \mathbb{R}^n$, the result is immediate. □

Example 4.4. Consider the two-agent noncooperative system with

$$J_1(x) = -\sin(x_1 x_2) - 0.2x_1 - 6e^{-(x_1-5)^2-(x_2-2)^2}, \tag{4.46}$$

$$J_2(x) = -2x_1^2 - 2x_2^2 + 12x_1 + 3.8x_2 - 24. \tag{4.47}$$

Let the priority evaluated by the system manager be given by $\eta_1 : \eta_2 = 1 : 1$. Note that the socially maximum state is given by $\hat{x}^* = [3.2321, 1.3303]^{\mathrm{T}}$. Let the sensitivity parameters be given by $\alpha = (3, 1)$. In this case, it follows from Theorem 4.2 that the incentive mechanism (4.1) along with the incentive functions (4.18) with $\zeta_1 = 1 - \zeta_2 = 0.4$ and $b_{12} = -b_{21} = 1.9$ satisfying $\hat{x}^* \in \mathrm{int}\,(\mathcal{D}_1 \cap \mathcal{D}_2)$ (see the white region representing the domain $\mathcal{D}_1 \cap \mathcal{D}_2$ in Fig. 4.6) guarantees that the socially maximum state $\hat{x}^*$ is asymptotically stable. Furthermore, both of the agents are Pareto improving with the sustainable budget constraint (4.7) holding with equality for all $x_0 \in \mathcal{D} \triangleq \{x \in \mathbb{R}^n : V(x) \leq \delta\}$ with $V(x) = -U(x) + U(\hat{x}^*)$ where the maximum attainable δ is given by $\delta = 1.0354$.

The following result provides one of the ways to achieve Pareto improvements without the information of agents' personal sensitivity parameters $\alpha_1, \ldots, \alpha_N$. We let

$$\mathcal{D} \triangleq \{x \in \mathbb{R}^n : V(x) \triangleq -U(x) + U(\hat{x}^*) \leq \delta\}, \tag{4.48}$$

with the maximum attainable $\delta \in \mathbb{R}_+$ such that $U'(x) = 0$ holds only at $\hat{x}^*$ in $\mathcal{D}$.

Corollary 4.3. Consider the n-agent noncooperative system $\mathcal{G}(J)$ with the incentive mechanism (4.1) and the pseudo-gradient dynamics (4.20). Suppose that the agents have the equal priority in (4.8) with $\eta_i = 1$ for all $i \in \mathcal{N}$. Then the incentive functions $p_i(x)$, $i \in \mathcal{N}$, given by (4.18) with $b_{ij} = 0$, $i, j \in \mathcal{N}$, guarantee that the socially maximum state $\hat{x}^*$ is asymptotically stable and all the agents are Pareto improving with the sustainable (balanced) budget constraint (4.7) holding with equality for all x_0 in $\mathcal{D}$ given by (4.48) for any positive constants α_i, $i \in \mathcal{N}$.

Proof First, let $g(x) \triangleq U'(x)$. Note that the vector field $f(x)$ of the pseudo-gradient dynamics (4.20) becomes $f(x) = \mathrm{diag}[\alpha]\mathcal{Z}g(x)$ and hence

$$V'(x)f(x) = -\sum\nolimits_{i\in\mathcal{N}} \tilde{J}_i'(x)f(x) = -\sum\nolimits_{i\in\mathcal{N}} \zeta_i g^{\mathrm{T}}(x)\mathrm{diag}[\alpha]\mathcal{Z}g(x) < 0, \tag{4.49}$$

for all $x \in \mathbb{R}^n$ except for the state x satisfying $g(x) = 0$. Furthermore, since $\mathcal{D}_i = \{x \in \mathbb{R}^n : \tilde{J}_i'(x)f(x) \geq 0\} = \{x \in \mathbb{R}^n : \zeta_i g^{\mathrm{T}}(x)\mathrm{diag}[\alpha]\mathcal{Z}g(x) \geq 0\} = \mathbb{R}^n$, $i \in \mathcal{N}$, the result is a direct consequence of Theorem 4.2 using the Lyapunov function candidate $V(x) = -U(x) + U(\hat{x}^*)$ satisfying (4.21)–(4.23). □

Remark 4.1. The incentive mechanism in Corollary 4.3 (i.e., $b_{ij} = 0$, $i, j \in \mathcal{N}$) potentializes the agents' payoff functions in $\mathcal{G}(J)$, i.e., $\mathcal{G}(\tilde{J})$ reduces to a special class of potential games by noting that each agent's payoff function is characterized as $\tilde{J}_i(x) = \zeta_i U(x) + w_i$ by the common function $U(x)$ in (4.45). Furthermore, since the domain $\mathcal{D}$ is understood as an invariant set for arbitrary sensitivity parameters α_i, $i \in \mathcal{N}$, they do not have to be known.

Example 4.5. Consider the two-agent noncooperative system with

$$J_1(x) = -\sin(x_1x_2) - 0.2x_1 - 9e^{-(x_1-5)^2-(x_2-2)^2} - 24, \quad (4.50)$$

$$J_2(x) = -2x_1^2 - 2x_2^2 + 12x_1 + 3.8x_2 - 4e^{-(x_1-2)^2-(x_2-5)^2}. \quad (4.51)$$

Let the priority evaluated by the system manager be given by $\eta_1 : \eta_2 = 1 : 1$. Note that $U'(x) = 0$ holds at the socially maximum state $\hat{x}^* = [3.3524, 1.3187]^{\mathrm{T}}$, the state $x^1 = [4.6971, 2.0236]^{\mathrm{T}}$, and the locally maximum state $x^2 = [4.496, 1.715]^{\mathrm{T}}$. Figure 4.7 shows the domain $\mathcal{D}$ of (4.48) indicated by the grey region with $\delta = 1.24$. It follows from Corollary 4.3 that the incentive mechanism (4.1) along with the incentive functions (4.18) with $\zeta_1 = 1 - \zeta_2 = 0.4$ and $b_{12} = -b_{21} = 0$ guarantees that the socially maximum state $\hat{x}^*$ is asymptotically stable and both of the agents are Pareto improving with the sustainable budget constraint (4.7) holding with equality for all $x_0 \in \mathcal{D}$ for any sensitivity parameters α_1 and α_2. With the sensitivity parameters be given by $\alpha = (2, 1)$, the vector field of the pseudo-gradient dynamics (4.20) is shown in Fig. 4.7. It can be seen from the figure that the socially maximum state $\hat{x}^*$, the state x^1, and the locally maximum state x^2 are asymptotically stable, unstable, and asymptotically stable, respectively. Note that the state x^1 is a saddle point of the pseudo-gradient dynamics and the domain $\mathcal{D}$ is an invariant set for arbitrary sensitivity parameters α_1 and α_2.

Now, we specialize the result of Theorem 4.2 with the quadratic payoff functions given by (4.28).

Corollary 4.4. Consider the n-agent noncooperative system $\mathcal{G}(J)$ with the incentive mechanism (4.1), the pseudo-gradient dynamics (4.20), and the quadratic payoff functions (4.28). Suppose that the agents have the equal priority in (4.8) with $\eta_i = 1$ for all $i \in \mathcal{N}$. Let $\mathcal{A} \triangleq \mathrm{diag}[\alpha](\mathcal{Z}\mathbb{A} + B) \in \mathbb{R}^{n\times n}$ and $P_i = \zeta_i\mathbb{A} + B_i \in \mathbb{R}^{n\times n}$, $i \in \mathcal{N}$. If the parameters $\zeta_i \in (0, 1)$, $i \in \mathcal{N}$, and $b_{ij} = -b_{ji} \in \mathbb{R}$, $i, j \in \mathcal{N}$, are chosen in such a way that (4.32) holds, then the incentive functions $p_i(x)$, $i \in \mathcal{N}$, given by (4.18) guarantee that the socially maximum state $\hat{x}^*$ is globally asymptotically stable and all the agents are Pareto improving for any initial state $x_0 \in \mathbb{R}^n$.

Proof Recalling (4.33)–(4.36), the result is a direct consequence of Theorem 4.2. □

Note that the selection of the parameters $\zeta_i, i \in \mathcal{N}$, and $b_{ij}, i, j \in \mathcal{N}$, may potentialize the agents' payoff functions in the incentivized noncooperative system. For example, it is straightforward to see that if $b_{ij} = 0$ for all $i, j \in \mathcal{N}$, then the agents are all Pateto improving (because of $\mathcal{A}^{\mathrm{T}}P_i + P_i\mathcal{A} = 2\zeta_i\mathbb{A}\mathrm{diag}[\alpha]\mathcal{Z}\mathbb{A} > 0$, $i \in \mathcal{N}$) and the incentivized noncooperative system $\mathcal{G}(\tilde{J})$ is exactly a weighted potential game (see the definitions of

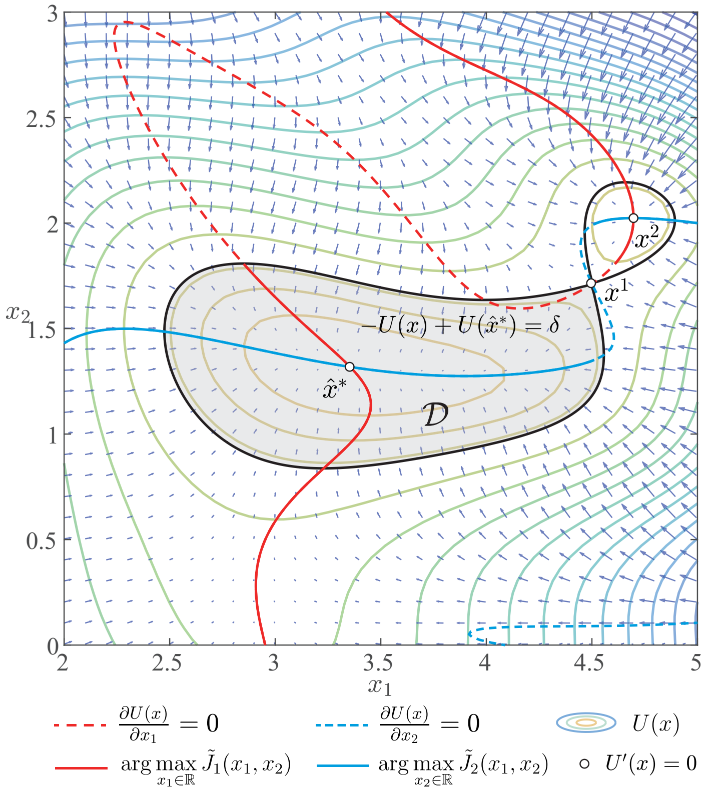


Figure. 4.7 Level sets of $U(x)$ with the vector field of the pseudo-gradient dynamics (4.20) under the incentive functions (4.18) with $\zeta_1 = 1 - \zeta_2 = 0.4$, $b_{12} = -b_{21} = 0$ in Example 4.5. The state x^1 is a saddle point of the dynamics. The guaranteed region of attraction $\mathcal{D}$ (grey region) is understood as the invariant set for arbitrary α_1 and α_2.

various types of potential games in Appendix B). But the connection between Pateto improvement and potentialization is obscure when b_{ij} is nonzero for some $i, j \in \mathcal{N}$. Does Pareto improvement always imply potentialization or potentialization always indicate Pateto improvement? To clarify the connections between Pareto improvement and potentialization, we present two numerical examples below. It turns out from those numerical examples that the Pateto improvement and potentialization do not have an inclusive relation with each other.

Example 4.6. Consider the two-agent noncooperative system with quadratic payoff functions $J_1(x)$ and $J_2(x)$ such that the social welfare function is given by (4.30) with $\mathbb{A} = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}$ and $\hat{x}^* = [0, 0]^{\mathrm{T}}$. Now, supposing $\alpha = (1, 3)$ and letting $\zeta = \zeta_1 = 1 - \zeta_2 \in (0, 1)$, $b = b_{12} = -b_{21} \in \mathbb{R}$, the feasible ζ-b region satisfying the condition (4.32) in Corollary 4.4 is shown in Fig. 4.8. It can be seen from the figure that the feasible ζ-b region is closed and bounded. Figure 4.9 shows the level sets of $\{\tilde{J}_i\}_{i=1,2}$ and the agents' trajectories under the values of $\zeta = 0.6$ and $b = 0.6$ satisfying the condition (4.32). It is interesting to see that even $b \neq 0$ (where agents' incentivized payoff functions are not simple proportion of the social welfare function $U(x)$), the agents' state still converges to the socially maximum state with a monotonically increasing payoff (in other words, agents are driven by a noncooperative way but result in a cooperative benefit). In fact, in this example, it can be shown that the incentivized noncooperative system $\mathcal{G}(\tilde{J})$ is never an ordinal potential (nor a weighted potential) game when b is non-zero. Hence, our example numerically shows the fact that Pareto improvements do not indicate potentialization.

Next, we show an example to reveal that the agents in the incentivized noncooperative system $\mathcal{G}(\tilde{J})$ possessing an ordinal potential may not be Pareto improving.

Example 4.7. Consider the two-agent noncooperative system with quadratic payoff functions $J_1(x)$ and $J_2(x)$ such that the social welfare function is given by (4.30) with $\mathbb{A} = \begin{bmatrix} -2 & -2 \\ -2 & -4 \end{bmatrix}$ and $\hat{x}^* = [0, 0]^{\mathrm{T}}$. Now, supposing $\alpha = (3, 1)$ and letting $\zeta = \zeta_1 = 1 - \zeta_2 \in (0, 1)$, $b = b_{12} = -b_{21} \in \mathbb{R}$, the feasible ζ-b region satisfying the condition (4.32) in Corollary 4.4 is shown in Fig. 4.10. Similar to Example 4.6, it can be seen from the figure that the feasible ζ-b region is bounded and closed. Moreover, we illustrate the ζ-b region under which the incentivized noncooperative system $\mathcal{G}(\tilde{J})$ is working as an ordinal potential game as the grey region in Fig. 4.10, where we used the fact that $\mathcal{G}(\tilde{J})$ possesses an ordinal potential if and only if $(-2\zeta + b)(-2(1 - \zeta) - b) > 0$

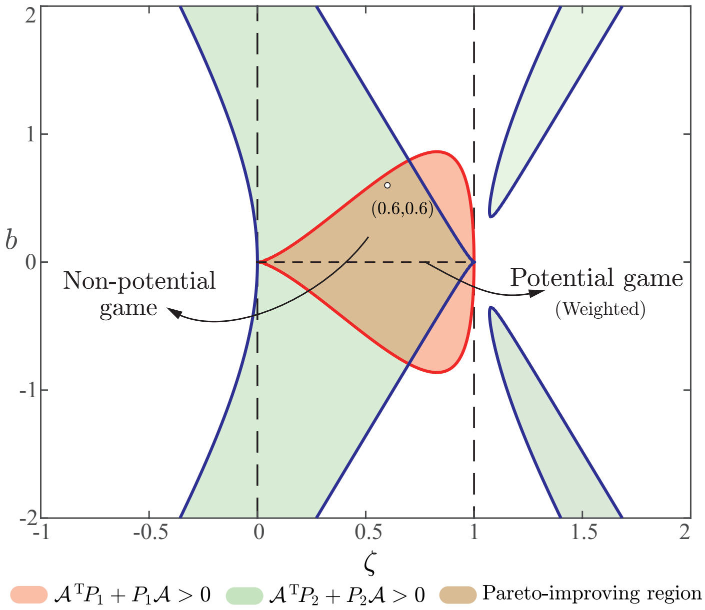


Figure. 4.8 Feasible solutions in ζ-b region for achieving Pareto improvement in Example 4.6. The overlapped (brown) region of the red and the green regions denotes the region under which the agents are Pareto improving. In this example, the incentivized noncooperative system $\mathcal{G}(\tilde{J})$ possesses an ordinal potential (or weighted potential) only when $b = 0$.

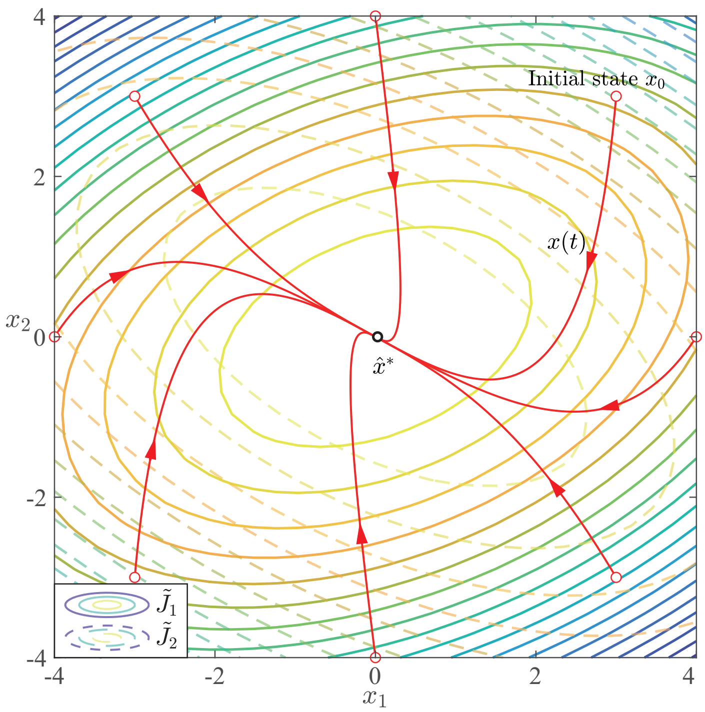


Figure. 4.9 Level sets and trajectories under $k = 0.6, b = 0.6$ in Example 4.6.

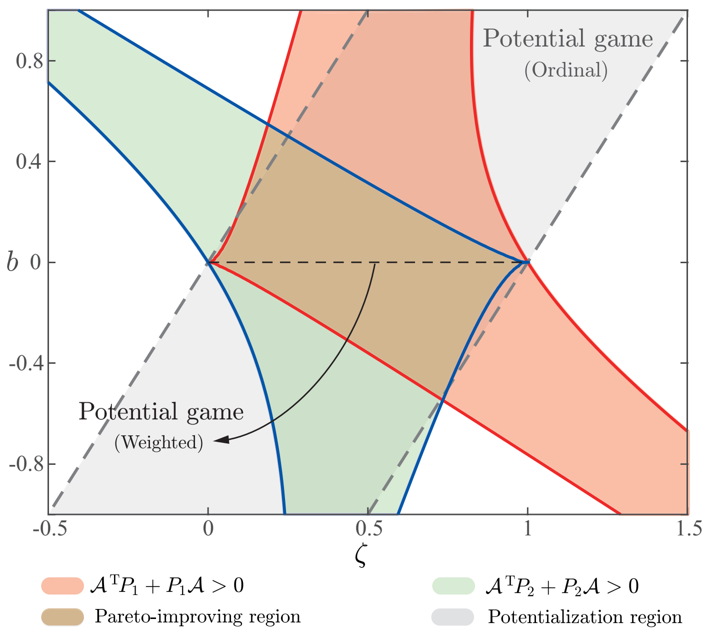


Figure. 4.10 Feasible solutions in ζ-b domain for achieving Pareto improvement in Example 4.7. The overlapped (brown) region of the red and the green regions denotes the region under which the agents are Pareto improving. The grey region denotes the potentialization region under which the incentivized noncooperative system $\mathcal{G}(\tilde{J})$ possesses an ordinal potential. Obviously, the brown region is not contained in the grey region, and vice versa.

(see Lemma B.1 in Appendix B below). It can be seen from the figure that the feasible ζ-b region is not contained in the grey region (the strip bounded by the dashed lines), and vice versa. Hence, our example numerically shows the fact that the agents in the incentivized noncooperative system $\mathcal{G}(\tilde{J})$ possessing an ordinal potential may not be Pareto improving.

4.5 Chapter Conclusion

In this chapter, we investigated the social welfare improvement problem for the noncooperative dynamical systems through a Pareto-improving incentive mechanism under sustainable budget constraint, where a system manager collects taxes from some agents and gives some of the collected taxes to other agents as subsidies in order to remodel agents' dynamical decision making. Sufficient stability conditions for our incentive functions were proposed to guarantee that the agents are Pareto improving under the pseudo-gradient dynamics and their state converges to a Pareto-efficient Nash equilibrium associated with a weighted social welfare function depending on the priority ratio of the agents. It was found that the initial state plays an important role on constructing our incentive mechanism to satisfy the sustainable budget constraint. Furthermore, we revealed the connection between Pareto improvement and potentialization with equal priority between the agents. Our numerical examples give a direct evidence that the Pareto improvement is not the same as potentialization.

Chapter 5

Stability Analysis of Loss-Aversion-Based Noncooperative Switched Systems

5.1 Introduction

In this chapter, we focus on the stability problem for 2-agent noncooperative switched systems, which are characterized as *payoff-driven* piecewise linear systems for describing agents' dynamic decision making with the quadratic payoffs and loss-aversion phenomena. Specifically, we assume that each agent adopts lower sensitivity in the pseudo-gradient dynamics for the case of losing utility than gaining utility and hence both the systems' dynamics and the switching instants depend on agents' payoff functions. To determine stability property of the loss-aversion-based noncooperative switched systems, we characterize the domains in which agents' payoffs are either increasing or decreasing, and use the normalized radial growth rate for the Nash equilibrium. By assuming that the agents keep on rotating, we reveal an interesting property of agents' decision behaviors in terms of the consistent rotational direction of the trajectories in the state space. This chapter categorizes the loss-aversion-based non-cooperative systems to 3 cases in accordance with the location of the Nash equilibrium relative to the 2 payoff functions and comprehensively analyze the differences between the 3 cases in terms of mode transition and normalized radial growth rates. Observing the fact that the Nash equilibrium is always on the boundaries of the aforementioned domains, by making the approximation for the domains around the Nash equilibrium, we characterize the partition of the state space and the mode transitions as a piecewise linear system. Moreover, we observe an interesting phenomenon that we call a flash switching instant where a single agents' sensitivity transition makes the other agent immediately switch its sensitivity almost at the same time instant, and we characterize the necessary condition for a switching instant holding such a phenomenon.

5.2 Problem Formulation

5.2.1 Noncooperative Systems with Quadratic Payoffs

Consider the noncooperative system with 2 agents selfishly controlling their individual state $x_i \in \mathbb{R}$, $i \in \{1, 2\}$. Let $x = [x_1, x_2]^{\mathrm{T}} \in \mathbb{R}^2$ denote the agents' state profile. In this chapter, we consider the situation where each agent i aims to increase its own payoff function $J_i(x_i, x_j)$, where $J_i : \mathbb{R}^2 \to \mathbb{R}$ and j is the opponent of agent $i \neq j$. We denote the noncooperative system by $\mathcal{G}(J)$ with $J \triangleq \{J_1, J_2\}$.

In this chapter, we consider the noncooperative system $\mathcal{G}(J)$ with *quadratic* payoff functions $J_i : \mathbb{R}^2 \to \mathbb{R}$ given by

$$J_i(x) = \frac{1}{2} x^{\mathrm{T}} A_i x + b_i^{\mathrm{T}} x + c_i, \quad i \in \{1, 2\}, \tag{5.1}$$

where $A_i \triangleq \begin{bmatrix} a_{11}^i & a_{12}^i \\ a_{12}^i & a_{22}^i \end{bmatrix} \in \mathbb{R}^{2\times 2}$ with $a_{ii}^i < 0$ (indicating that $J_i(x)$ is strictly concave with respect to x_i) and $a_{11}^1 a_{22}^2 \neq a_{12}^1 a_{12}^2$, $b_i \triangleq [b_1^i, b_2^i]^{\mathrm{T}} \in \mathbb{R}^2$, and $c_i \in \mathbb{R}$, $i \in \{1, 2\}$. It is important to note that there exists a *unique* Nash equilibrium x^* in $\mathcal{G}(J)$ in the unbounded state space satisfying

$$0 = \frac{\partial J_1(x)}{\partial x_1} = a_{11}^1 x_1 + a_{12}^1 x_2 + b_1^1, \tag{5.2}$$

$$0 = \frac{\partial J_2(x)}{\partial x_2} = a_{12}^2 x_1 + a_{22}^2 x_2 + b_2^2, \tag{5.3}$$

for $x = x^*$. Specifically, the *unique* Nash equilibrium is given by

$$x^* = -\begin{bmatrix} a_{11}^1 & a_{12}^1 \\ a_{12}^2 & a_{22}^2 \end{bmatrix}^{-1} \begin{bmatrix} b_1^1 \\ b_2^2 \end{bmatrix}, \tag{5.4}$$

because the condition $a_{11}^1 a_{22}^2 \neq a_{12}^1 a_{12}^2$ implies the inverse exists. Notice that the straight lines (5.2) and (5.3) are understood as the *best-response lines* for agents 1 and 2, respectively.

5.2.2 Loss-Aversion-Based Pseudo-Gradient Dynamics

In this chapter, we consider the situation where each agent selfishly and continually changes its state in the noncooperative system $\mathcal{G}(J)$. We suppose the state profile $x(\cdot)$ is available for both the agents. In addition, associated with agents' payoff functions J_1, J_2, the pseudo-gradient dynamics are used to describe agents' myopic selfish behaviors given by

$$\dot{x}_i(t) = \alpha_i(t)\frac{\partial J_i(x(t))}{\partial x_i}, \quad i \in \{1, 2\}, \tag{5.5}$$

where $\alpha_1(t)$, $\alpha_2(t) \in \mathbb{R}_+$ are agents' personal (private) sensitivity parameters. Note that the pseudo-gradient dynamics capture the fact that the agents concern their own payoffs and myopically change their states according to the current information without any foresight on the future state.

Different from the models in [42, 25], where $\alpha_1(t), \alpha_2(t)$ are constant, in this chapter we suppose that each agent directly observes its own payoff level $J_i(t)$ for agent i, i.e., the payoff level J_i is *not calculated* by (5.1) through the knowledge of x. As such, each agent is supposed to be able to evaluate $\dot{J}_i$ at infinitesimally small previous time instant t^-. Furthermore, the agents' sensitivity parameters α_1, α_2 are *piecewise constant* between 2 values following the loss-aversion-based psychological consideration defined by

$$\alpha_i(t) \triangleq \begin{cases} \alpha_i^{\mathrm{L}}, & \text{if } \dot{J}_i(t^-) < 0, \\ \alpha_i^{\mathrm{H}}, & \text{if } \dot{J}_i(t^-) > 0, \end{cases} \quad i \in \{1, 2\}, \tag{5.6}$$

where $\alpha_i^{\mathrm{L}}, \alpha_i^{\mathrm{H}} \in \mathbb{R}_+$ capture the sensitivity of the change of agent i's state per unit time against losing and gaining payoff environment, respectively, for $i \in \{1, 2\}$. As soon as agent i reaches the state observing $\dot{J}_i(t) = 0$, it switches its $\alpha_i(t)$ to the other value. An interesting observation that this sensitivity parameter change by one agent may give rise to the parameter change of the other agent is elaborated in Section 5.4 below.

We connect the phenomenon of *loss-aversion* in *prospect theory* [65] with the noncooperative behaviors in $\mathcal{G}(J)$. It is well known that humans are more cautious to make the decision when they face losing payoff than gaining payoff. As a typical example, in the stock investment market, investors (agents) have the tendency to hold losing investments very long and sell winning investments very soon [98]. In light of this observation, in this chapter we suppose that the sensitivity parameters satisfy $\alpha_i^{\mathrm{L}} \leq \alpha_i^{\mathrm{H}}$,

$i \in \{1, 2\}$, to describe agents' slower behavior for the case where their corresponding $\dot{J}_i(t)$ is negative.

It is important to note that there are 4 possibly different combinations (modes) of agents' sensitivities depending on the signs of $\dot{J}_1$ and $\dot{J}_2$. Henceforth, we let

$$\alpha^{\mathrm{LL}} \triangleq \mathrm{diag}[\alpha_1^{\mathrm{L}}, \alpha_2^{\mathrm{L}}], \quad \alpha^{\mathrm{HL}} \triangleq \mathrm{diag}[\alpha_1^{\mathrm{H}}, \alpha_2^{\mathrm{L}}], \tag{5.7}$$

$$\alpha^{\mathrm{LH}} \triangleq \mathrm{diag}[\alpha_1^{\mathrm{L}}, \alpha_2^{\mathrm{H}}], \quad \alpha^{\mathrm{HH}} \triangleq \mathrm{diag}[\alpha_1^{\mathrm{H}}, \alpha_2^{\mathrm{H}}], \tag{5.8}$$

to denote the entire sensitivity profile of the 2 agents. Consequently, agents' decision behaviors (5.5) with the loss-aversion-based sensitivity (5.6) and the quadratic payoff functions (5.1) under mode $k \in \mathcal{K} \triangleq \{\mathrm{LL}, \mathrm{HL}, \mathrm{LH}, \mathrm{HH}\}$ are described as

$$\dot{x}(t) = \alpha^{k(t)} \left[\frac{\partial J_1(x(t))}{\partial x_1}, \frac{\partial J_2(x(t))}{\partial x_2} \right]^{\mathrm{T}} = \mathbb{A}_{k(t)}(x(t) - x^*), \tag{5.9}$$

where $\mathbb{A}_k \triangleq \alpha^k \begin{bmatrix} a_{11}^1 & a_{12}^1 \\ a_{12}^2 & a_{22}^2 \end{bmatrix}$ denotes the system matrix under mode $k \in \mathcal{K}$ and x^* is given by (5.4). As discussed in the following sections, it turns out that which mode is active can be characterized by some domains in the state space.

5.3 Hyperbolic/Elliptic Domains Characterizing Utility Trends

In this section, we characterize the 4 domains associated with the 4 modes in $\mathcal{K}$ depending on the utility trends (increasing or decreasing) of the 2 players. Specifically, we define the 4 domains in which the signs of $\dot{J}_1$ and $\dot{J}_2$ associated with (5.1) remain the same to be positive/negative along the system trajectories of (5.5). With a slight abuse of notation, let the functions $\dot{J}_i^k : \mathbb{R}^2 \to \mathbb{R}$ represent the time rate of change $\dot{J}_i$ of J_i as a function of the state x for agent $i \in \{1, 2\}$ with mode $k \in \mathcal{K}$ given by

$$\begin{aligned} \dot{J}_i^k(x) &\triangleq \left[\frac{\partial J_i(x)}{\partial x_1}, \frac{\partial J_i(x)}{\partial x_2} \right] \mathbb{A}_k(x - x^*) = \frac{1}{2} x^{\mathrm{T}} Q_i^k x + \left(\mathbb{A}_k^{\mathrm{T}} b_i - A_i \mathbb{A}_k x^* \right)^{\mathrm{T}} x - b_i^{\mathrm{T}} \mathbb{A}_k x^* \\ &= \frac{1}{2}(x - x^*)^{\mathrm{T}} Q_i^k (x - x^*) + {\beta_i^k}^{\mathrm{T}} (x - x^*), \end{aligned} \tag{5.10}$$

with $Q_i^k \triangleq A_i\mathbb{A}_k + \mathbb{A}_k^{\mathrm{T}}A_i \in \mathbb{R}^{2\times 2}$ and $\beta_i^k \triangleq \mathbb{A}_k^{\mathrm{T}}(A_i x^* + b_i) \in \mathbb{R}^2$, $i \in \{1,2\}, k \in \mathcal{K}$. The function $\dot{J}_i^k(x)$ is reminiscent of the time rate of change of $J_i(\cdot)$ along the system trajectories given by $\dot{J}_i(t) = \frac{\partial J_i(x(t))}{\partial x}\dot{x}(t)$ with mode k being active at state x.

We define the domains $\mathcal{D}_k$, $k \in \mathcal{K}$, in which each of the agents keeps either the high sensitivity α_i^{H} or the low sensitivity α_i^{L} as

$$\mathcal{D}_{\mathrm{LL}} \triangleq \{x \in \mathbb{R}^2 : \dot{J}_1^{\mathrm{LL}}(x) \le 0, \dot{J}_2^{\mathrm{LL}}(x) \le 0\}, \tag{5.11}$$

$$\mathcal{D}_{\mathrm{HL}} \triangleq \{x \in \mathbb{R}^2 : \dot{J}_1^{\mathrm{HL}}(x) \ge 0, \dot{J}_2^{\mathrm{HL}}(x) \le 0\}, \tag{5.12}$$

$$\mathcal{D}_{\mathrm{LH}} \triangleq \{x \in \mathbb{R}^2 : \dot{J}_1^{\mathrm{LH}}(x) \le 0, \dot{J}_2^{\mathrm{LH}}(x) \ge 0\}, \tag{5.13}$$

$$\mathcal{D}_{\mathrm{HH}} \triangleq \{x \in \mathbb{R}^2 : \dot{J}_1^{\mathrm{HH}}(x) \ge 0, \dot{J}_2^{\mathrm{HH}}(x) \ge 0\}. \tag{5.14}$$

Note that some of these 4 domains may *not* exist (as explained in Remark 5.1 below). Furthermore, the Nash equilibrium x^* belongs to all the existing domains, since $\dot{J}_i^k(x^*) = 0$ for all $i \in \{1,2\}$ and $k \in \mathcal{K}$.

It is important to note that the boundaries of $\mathcal{D}_k$, $k \in \mathcal{K}$, may be either straight lines or quadratic curves depending on whether β_i^k in (5.10) vanishes or not. Specifically, since $\mathbb{A}_k$, $k \in \mathcal{K}$, are nonsingular, $A_i x^* + b_i \neq 0$ (resp., $A_i x^* + b_i = 0$) if and only if $\beta_i^k = \mathbb{A}_k^{\mathrm{T}}(A_i x^* + b_i) \neq 0$ (resp., $\beta_i^k = 0$), $k \in \mathcal{K}$, so that the boundaries associated with $\dot{J}_i^k(x) = 0$ are quadratic (hyperbolic/elliptic) curves (resp., straight lines intersected at x^* when Q_i^k is sign-indefinite). Since the domains $\mathcal{D}_k$, $k \in \mathcal{K}$, are characterized by the two equations $\dot{J}_1^k(x) = 0$ and $\dot{J}_2^k(x) = 0$, we categorize 3 cases as shown in Fig. 5.1, that is, $A_i x^* + b_i \neq 0$ for $i \in \{1,2\}$ (Case 1); $A_i x^* + b_i = 0$ for $i \in \{1,2\}$ (Case 2); and $A_1 x^* + b_1 \neq 0$, $A_2 x^* + b_2 = 0$ (Case 3). In any case, x^* is always on the cusp of $\mathcal{D}_k$ for mode k that exists (except for the domain where $\beta_i^k = 0$ and Q_i^k is positive definite (see Remark 5.1 for an example)). Here we note that because $A_i x^* + b_i$ is equal to $\frac{\partial J_i(x^*)}{\partial x}$, the above 3 cases are categorized according to whether x^* coincides with the maximum (or saddle) point of $J_i(x)$ for agent i (i.e., $\frac{\partial J_i(x^*)}{\partial x} = 0$) or not.

Remark 5.1. Some of the domains $\operatorname{int}\mathcal{D}_k$, $k \in \mathcal{K}$, may not exist. For example, consider Case 2 where there exists $\lambda > 0$ such that $J_1(x) = \lambda J_2(x)$. In this case, since $Q_1^k = \lambda A_2\alpha^k \mathrm{diag}[\lambda, 1]A_2 + \lambda A_2 \mathrm{diag}[\lambda, 1]\alpha^k A_2 > 0$ and $Q_2^k = A_2\alpha^k\mathrm{diag}[\lambda, 1]A_2 + A_2\mathrm{diag}[\lambda, 1]\alpha^k A_2 > 0$ hold for all $k \in \mathcal{K}$, it follows that $\operatorname{int}\mathcal{D}_{\mathrm{LL}}, \operatorname{int}\mathcal{D}_{\mathrm{HL}}, \operatorname{int}\mathcal{D}_{\mathrm{LH}} = \emptyset$ and $\mathcal{D}_{\mathrm{HH}} = \mathbb{R}^2$. Alternatively, consider the case with the zero-sum payoffs, where $J_1(x) = -J_2(x)$. In this case, since $\dot{J}_1^k(x) = -\dot{J}_2^k(x)$ holds for all $x \in \mathbb{R}^2$ and $k \in \mathcal{K}$, we have $\operatorname{int}\mathcal{D}_{\mathrm{LL}} = \operatorname{int}\mathcal{D}_{\mathrm{HH}} = \emptyset$ and $\mathcal{D}_{\mathrm{HL}} \bigcup \mathcal{D}_{\mathrm{LH}} = \mathbb{R}^2$.

Remark 5.2. There may exist some overlaps between $\mathcal{D}_k$, $k \in \mathcal{K}$. Figure 5.2 shows a typical example of the 4 domains indicated by the orange regions. Note that point A

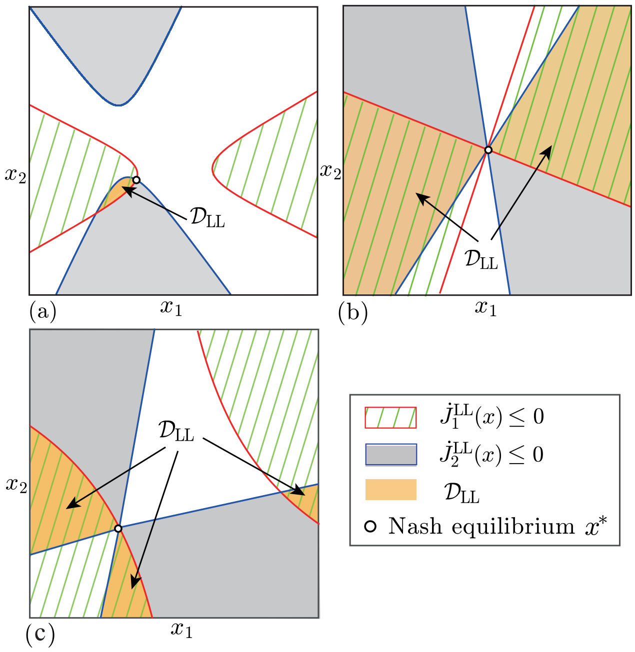


Figure. 5.1 Examples of the domain $\mathcal{D}_{\rm LL}$. (a): Case 1, (b): Case 2, (c): Case 3.

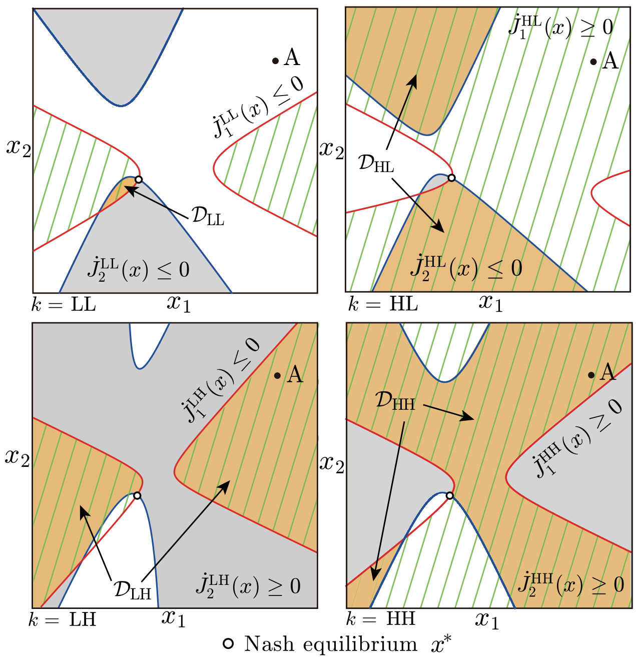


Figure. 5.2 An example of the 4 domains $\mathcal{D}_{\rm LL}$, $\mathcal{D}_{\rm HL}$, $\mathcal{D}_{\rm LH}$, $\mathcal{D}_{\rm HH}$ for Case 1. The figure for the case of k = LL is the copy of Fig. 5.1(a).

belongs to the 2 domains $\mathcal{D}_{\mathrm{LH}}$ and $\mathcal{D}_{\mathrm{HH}}$ but not $\mathcal{D}_{\mathrm{LL}}$ nor $\mathcal{D}_{\mathrm{HL}}$. In defining the mode of the system dynamics (5.9) in the overlapped regions, the agents keep mode k at time t^+ if $x(t) \in \operatorname{int} \mathcal{D}_k$, given an active mode k at time t.

Lemma 5.1. Consider the loss-aversion-based noncooperative system $\mathcal{G}(J)$ with the pseudo-gradient dynamics (5.5), (5.6). Then, it follows that

$$\bigcup_{k \in \mathcal{K}} \mathcal{D}_k = \mathbb{R}^2, \tag{5.15}$$

for any $\alpha_i^{\mathrm{H}} \geq \alpha_i^{\mathrm{L}}$, $i = 1, 2$.

Proof First, by defining

$$\Delta_1^1(x) \triangleq (a_{11}^1 x_1 + a_{12}^1 x_2 + b_1^1)^2 \geq 0, \tag{5.16}$$
$$\Delta_1^2(x) \triangleq (a_{12}^1 x_1 + a_{22}^1 x_2 + b_2^1)(a_{12}^2 x_1 + a_{22}^2 x_2 + b_2^2), \tag{5.17}$$
$$\Delta_2^1(x) \triangleq (a_{11}^2 x_1 + a_{12}^2 x_2 + b_1^2)(a_{11}^1 x_1 + a_{12}^1 x_2 + b_1^1), \tag{5.18}$$
$$\Delta_2^2(x) \triangleq (a_{12}^2 x_1 + a_{22}^2 x_2 + b_2^2)^2 \geq 0, \tag{5.19}$$

the functions in (5.10) can be calculated with (5.4) as

$$\dot{J}_i^k(x) = \alpha_1^k \Delta_i^1(x) + \alpha_2^k \Delta_i^2(x), \quad i \in \{1, 2\}, \quad k \in \mathcal{K}. \tag{5.20}$$

Let $\delta_i \triangleq \alpha_i^{\mathrm{H}} - \alpha_i^{\mathrm{L}} \geq 0$, $i = 1, 2$. Now, we suppose $\mathcal{D}_{\mathrm{LL}} \neq \mathbb{R}^2$ so that there exists $\bar{x} \in \mathbb{R}^2$ such that $\bar{x} \notin \mathcal{D}_{\mathrm{LL}}$. In this case, there are three cases in terms of $\bar{x}$ that may happen: $\dot{J}_1^{\mathrm{LL}}(\bar{x}) > 0 \wedge \dot{J}_2^{\mathrm{LL}}(\bar{x}) \leq 0$; $\dot{J}_1^{\mathrm{LL}}(\bar{x}) \leq 0 \wedge \dot{J}_2^{\mathrm{LL}}(\bar{x}) > 0$; $\dot{J}_1^{\mathrm{LL}}(\bar{x}) > 0 \wedge \dot{J}_2^{\mathrm{LL}}(\bar{x}) > 0$. For the case of $\dot{J}_1^{\mathrm{LL}}(\bar{x}) > 0 \wedge \dot{J}_2^{\mathrm{LL}}(\bar{x}) \leq 0$, since $\Delta_1^1(\bar{x}) \geq 0$, we have $\dot{J}_1^{\mathrm{HL}}(\bar{x}) = \dot{J}_1^{\mathrm{LL}}(\bar{x}) + \delta_1 \Delta_1^1(\bar{x}) > 0$. Moreover, since $\dot{J}_2^{\mathrm{LL}}(\bar{x}) = \alpha_1^{\mathrm{L}} \Delta_2^1(x) + \alpha_2^{\mathrm{L}} \Delta_2^2(x) \leq 0$ and $\Delta_2^2(\bar{x}) \geq 0$ imply $\Delta_2^1(\bar{x}) \leq 0$, we have $\dot{J}_2^{\mathrm{HL}}(\bar{x}) = \dot{J}_2^{\mathrm{LL}}(\bar{x}) + \delta_1 \Delta_2^1(\bar{x}) \leq 0$. Hence, $\bar{x} \in \mathcal{D}_{\mathrm{HL}}$. For the case of $\dot{J}_1^{\mathrm{LL}}(\bar{x}) \leq 0 \wedge \dot{J}_2^{\mathrm{LL}}(\bar{x}) > 0$, since $\dot{J}_1^{\mathrm{LL}}(\bar{x}) = \alpha_1^{\mathrm{L}} \Delta_1^1(x) + \alpha_2^{\mathrm{L}} \Delta_1^2(x) \leq 0$ and $\Delta_1^1(\bar{x}) \geq 0$ imply $\Delta_1^2(\bar{x}) \leq 0$, we have $\dot{J}_1^{\mathrm{LH}}(\bar{x}) = \dot{J}_1^{\mathrm{LL}}(\bar{x}) + \delta_2 \Delta_1^2(\bar{x}) \leq 0$. Moreover, since $\Delta_2^2(\bar{x}) \geq 0$, we have $\dot{J}_2^{\mathrm{LH}}(\bar{x}) = \dot{J}_2^{\mathrm{LL}}(\bar{x}) + \delta_2 \Delta_2^2(\bar{x}) > 0$. Hence, $\bar{x} \in \mathcal{D}_{\mathrm{LH}}$. For the case of $\dot{J}_1^{\mathrm{LL}}(\bar{x}) > 0 \wedge \dot{J}_2^{\mathrm{LL}}(\bar{x}) > 0$, note that since $\Delta_1^1(\bar{x}) \geq 0$ and $\Delta_2^2(\bar{x}) \geq 0$, the inequalities $\dot{J}_1^{\mathrm{HL}}(\bar{x}) = \dot{J}_1^{\mathrm{LL}}(\bar{x}) + \delta_1 \Delta_1^1(\bar{x}) > 0$ and $\dot{J}_2^{\mathrm{LH}}(\bar{x}) = \dot{J}_2^{\mathrm{LL}}(\bar{x}) + \delta_2 \Delta_2^2(\bar{x}) > 0$ must hold. Now, we further suppose that $\bar{x} \notin \mathcal{D}_{\mathrm{HL}}$ and $\bar{x} \notin \mathcal{D}_{\mathrm{LH}}$ hold, i.e., we suppose that $\dot{J}_2^{\mathrm{HL}}(\bar{x}) > 0 \wedge \dot{J}_1^{\mathrm{LH}}(\bar{x}) > 0$ holds. Then, since $\dot{J}_1^{\mathrm{HH}}(\bar{x}) = \dot{J}_1^{\mathrm{LH}}(\bar{x}) + \delta_1 \Delta_1^1(\bar{x}) > 0 \wedge \dot{J}_2^{\mathrm{HH}}(\bar{x}) = \dot{J}_2^{\mathrm{HL}}(\bar{x}) + \delta_2 \Delta_2^2(\bar{x}) > 0$, we have $\bar{x} \in \mathcal{D}_{\mathrm{HH}}$.

Thus, for any $\bar{x} \in \mathbb{R}^2$, there exist $k \in \mathcal{K}$ s.t. $\bar{x} \in \mathcal{D}_k$, which completes the proof. □

Remark 5.3. Note that if the agents' loss-averse behavior is characterized by $\alpha_i^{\mathrm{H}} < \alpha_i^{\mathrm{L}}$, $i = 1, 2$, then $\bigcup_{k\in\mathcal{K}} \mathcal{D}_k = \mathbb{R}^2$ may not hold even though (5.11)–(5.14) are a complete enumeration of all possible cases.

Different from the standard piecewise linear system with conewise partitions [77, 78], the main problem in investigating stability property in this chapter is to appropriately deal with the overlaps of the domains (Remark 5.2) and non-conewise domains. In the following section, we introduce how to appropriately partition the state space depending on the rotational directions of the system trajectories and how to characterize stability according to a piecewise linearized system of (5.5), (5.6) whose state is traveling over the partitioned domains.

5.4 Stability Analysis With Complex Conjugate Eigenvalues

In this section, we characterize stability properties of the Nash equilibrium x^* for the loss-aversion-based noncooperative system $\mathcal{G}(J)$. Specifically, we first present the properties of agents' behavior under (5.5), (5.6) in terms of the rotational direction of the trajectories. We let $\tilde{x} \triangleq x - x^*$ and consider the polar form (r, θ) of the coordinate $(\tilde{x}_1, \tilde{x}_2)$. Note that the rotational direction of the trajectories at phase θ under mode $k \in \mathcal{K}$ can be determined by the sign of

$$\begin{aligned}\dot{\theta}_k &= \frac{\mathrm{d}}{\mathrm{d}t}(\tan^{-1}\frac{\tilde{x}_2}{\tilde{x}_1}) = \frac{-\dot{\tilde{x}}_1\tilde{x}_2 + \tilde{x}_1\dot{\tilde{x}}_2}{\tilde{x}_1^2 + \tilde{x}_2^2} = \frac{1}{r^2}\det\begin{bmatrix}\tilde{x}_1 & \dot{\tilde{x}}_1\\ \tilde{x}_2 & \dot{\tilde{x}}_2\end{bmatrix}\\ &= \det[\eta(\theta), \mathbb{A}_k\eta(\theta)] = \eta^{\mathrm{T}}(\theta)P_k\eta(\theta),\end{aligned} \tag{5.21}$$

where $\eta(\theta) = [\cos\theta, \sin\theta]^{\mathrm{T}}$ and

$$P_k \triangleq \begin{bmatrix}\alpha_2^k a_{12}^2 & \frac{-\alpha_1^k a_{11}^1 + \alpha_2^k a_{22}^2}{2}\\ \frac{-\alpha_1^k a_{11}^1 + \alpha_2^k a_{22}^2}{2} & -\alpha_1^k a_{12}^1\end{bmatrix}, \quad k \in \mathcal{K}, \tag{5.22}$$

with $\alpha_1^{\mathrm{XY}} \triangleq \alpha_1^{\mathrm{X}}$, $\alpha_2^{\mathrm{XY}} \triangleq \alpha_2^{\mathrm{Y}}$, $\mathrm{X}, \mathrm{Y} \in \{\mathrm{L}, \mathrm{H}\}$. In particular, the trajectories under mode $k \in \mathcal{K}$ are moving in the counterclockwise (resp., clockwise) direction when $\dot{\theta}_k > 0$ (resp., $\dot{\theta}_k < 0$).

To focus on the case where there exist infinitely many mode transitions for the agents, we assume that the eigenvalues of $\mathbb{A}_k$ are all complex conjugate in our stability analysis. The case where there are finite number of mode transitions can be handled by simply investigating the stability property of the possible final modes.

Assumption 5.1. The system matrix $\mathbb{A}_k$ has a pair of complex conjugate eigenvalues for all the modes $k \in \mathcal{K}$.

Under Assumption 5.1, the eigenvalues of the system matrix $\mathbb{A}_k$ are computed as $\psi_k \pm \sqrt{\psi_k^2 - \alpha_1^k \alpha_2^k (a_{11}^1 a_{22}^2 - a_{12}^1 a_{12}^2)}$, where $\psi_k \triangleq \frac{1}{2}(\alpha_1^k a_{11}^1 + \alpha_2^k a_{22}^2) < 0$, which implies that the complex conjugate eigenvalues of $\mathbb{A}_k$ have negative real part for all $k \in \mathcal{K}$. Note that the expression in the square root satisfies $0 > \psi_k^2 - \alpha_1^k \alpha_2^k (a_{11}^1 a_{22}^2 - a_{12}^1 a_{12}^2) = \frac{1}{4}(\alpha_1^k a_{11}^1 - \alpha_2^k a_{22}^2)^2 + \alpha_1^k \alpha_2^k a_{12}^1 a_{12}^2$, which implies that $a_{12}^1 a_{12}^2 < 0$ (i.e., $a_{12}^1 < 0 \wedge a_{12}^2 > 0$ or $a_{12}^1 > 0 \wedge a_{12}^2 < 0$) and $\det P_k = -\frac{1}{4}(\alpha_1^k a_{11}^1 - \alpha_2^k a_{22}^2)^2 - \alpha_1^k \alpha_2^k a_{12}^1 a_{12}^2 > 0$, $k \in \mathcal{K}$. These facts are used in the following lemma and its proof. Note that the case where $\mathbb{A}_k$ possesses real eigenvalues may also exhibit infinitely many mode transitions. This complicated case is addressed in Section 5.5 below.

Lemma 5.2. Consider the loss-aversion-based noncooperative system $\mathcal{G}(J)$ with the pseudo-gradient dynamics (5.5), (5.6) under Assumption 5.1. Then, the rotational directions of the trajectories are consistently the same in the entire state space $\mathbb{R}^2$. Specifically, if $a_{12}^1 < 0$ and $a_{12}^2 > 0$ (resp., $a_{12}^1 > 0$ and $a_{12}^2 < 0$), then the trajectory of (5.5), (5.6), keeps the counterclockwise (resp., clockwise) direction for any $\alpha_i^{\mathrm{H}} \geq \alpha_i^{\mathrm{L}}$, $i = 1, 2$.

Proof Note that $a_{12}^1 < 0 \wedge a_{12}^2 > 0$ and $a_{12}^1 > 0 \wedge a_{12}^2 < 0$ imply that the diagonal elements of P_k are all positive and negative, respectively, and hence $P_k > 0$ (resp., $P_k < 0$), $k \in \mathcal{K}$, because Assumption 5.1 implies $\det P_k > 0$. Thus, the result is immediate since $\dot{\theta}_k = \eta^{\mathrm{T}}(\theta) P_k \eta(\theta)$. □

This result is used in the following sections to partition the state space and to define a piecewise linearized system of (5.5), (5.6).

Case 1: $A_i x^* + b_i \neq 0$, $i \in \{1, 2\}$

In this section, we characterize the local stability property of the Nash equilibrium x^* for $A_i x^* + b_i \neq 0$ for $i \in \{1, 2\}$. Recall that x^* is located on the cusp of the domains $\mathcal{D}_k$, $k \in \mathcal{K}$ (see Fig. 5.2). In the beginning, we approximate the domain $\mathcal{D}_k$ around x^* to the convex cone $\hat{\mathcal{D}}_k$ by linearizing the quadratic curves characterized by $\dot{J}_1^k(x) = 0$ and $\dot{J}_2^k(x) = 0$ around x^* for all $k \in \mathcal{K}$. In particular, since x^* corresponds to the origin in the shifted space $\tilde{x}$, we denote the linearized straight lines of the curves $\dot{J}_i^k(x) = 0$,

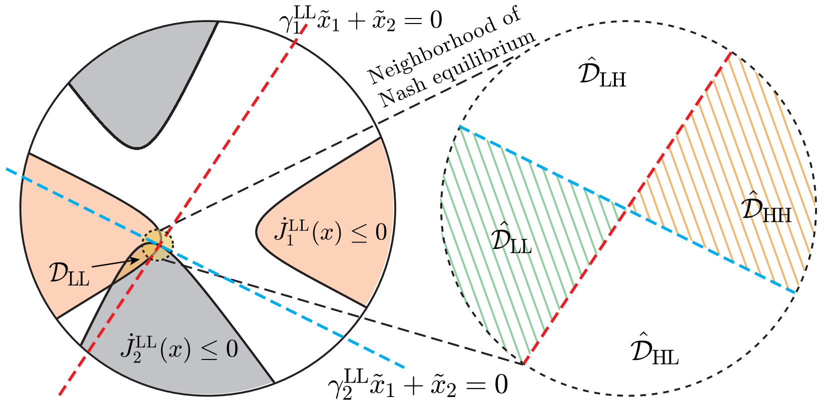


Figure. 5.3 Approximated domain where $a_{12}^2(a_{12}^1x_1^*+a_{22}^1x_2^*+b_2^1)>0$, $a_{11}^1(a_{11}^2x_1^*+a_{12}^2x_2^*+b_1^2)>0$. (a): $\mathcal{D}_{\mathrm{LL}}$, (b): the approximated domains $\hat{\mathcal{D}}_k$, $k\in\mathcal{K}$, around the neighborhood of Nash equilibrium. The rotational direction is counterclockwise since $\gamma_1^{\mathrm{LL}}<0\wedge\gamma_2^{\mathrm{LL}}>0$ implies $a_{12}^1<0\wedge a_{12}^2>0$.

$i\in\{1,2\}$, $k\in\mathcal{K}$, at x^* as

$$\gamma_i^k\tilde{x}_1+\tilde{x}_2=0,\quad i\in\{1,2\},\quad k\in\mathcal{K},\tag{5.23}$$

where $\gamma_i^k\triangleq\left(\frac{\partial\dot{J}_i^k(x)}{\partial x_1}\Big/\frac{\partial\dot{J}_i^k(x)}{\partial x_2}\right)\Big|_{x=x^*}\in\mathbb{R}$, $i\in\{1,2\}$, $k\in\mathcal{K}$. For example, Fig. 5.3 shows the domain $\mathcal{D}_{\mathrm{LL}}$ and its approximated cone $\hat{\mathcal{D}}_{\mathrm{LL}}$ in the neighborhood of x^*.

For the statement of the following result, note that $a_{12}^1\neq 0$ and $a_{12}^2\neq 0$ since $a_{12}^1a_{12}^2<0$ under Assumption 5.1.

Proposition 5.1. If $A_ix^*+b_i\neq 0$ for $i=1$ (resp., $i=2$), then $\gamma_1^k=\frac{a_{12}^2}{a_{22}^2}$ (resp., $\gamma_2^k=\frac{a_{11}^1}{a_{12}^1}$), $k\in\mathcal{K}$, for any $\alpha_1^{\mathrm{H}},\alpha_1^{\mathrm{L}},\alpha_2^{\mathrm{H}},\alpha_2^{\mathrm{L}}\in\mathbb{R}_+$.

Proof First, recall (5.20). Then, for each mode $k\in\mathcal{K}$, we have

$$\begin{aligned}\frac{\partial\dot{J}_1^k(x)}{\partial x_1}=&2\alpha_1^ka_{11}^1(a_{11}^1x_1+a_{12}^1x_2+b_1^1)+\alpha_2^ka_{12}^1(a_{12}^2x_1+a_{22}^2x_2+b_2^2)\\&+\alpha_2^ka_{12}^2(a_{12}^1x_1+a_{22}^1x_2+b_2^1),\end{aligned}\tag{5.24}$$

$$\begin{aligned}\frac{\partial\dot{J}_1^k(x)}{\partial x_2}=&2\alpha_1^ka_{12}^1(a_{11}^1x_1+a_{12}^1x_2+b_1^1)+\alpha_2^ka_{22}^1(a_{12}^2x_1+a_{22}^2x_2+b_2^2)\\&+\alpha_2^ka_{22}^2(a_{12}^1x_1+a_{22}^1x_2+b_2^1),\end{aligned}\tag{5.25}$$

$$\begin{aligned}\frac{\partial \dot{J}_2^k(x)}{\partial x_1} =& 2\alpha_2^k a_{12}^2(a_{12}^2x_1 + a_{22}^2x_2 + b_2^2) + \alpha_1^k a_{11}^2(a_{11}^1x_1 + a_{12}^1x_2 + b_1^1)\\ &+ \alpha_1^k a_{11}^1(a_{11}^2x_1 + a_{12}^2x_2 + b_1^2), \qquad (5.26)\\ \frac{\partial \dot{J}_2^k(x)}{\partial x_2} =& 2\alpha_2^k a_{22}^2(a_{12}^2x_1 + a_{22}^2x_2 + b_2^2) + \alpha_1^k a_{12}^2(a_{11}^1x_1 + a_{12}^1x_2 + b_1^1)\\ &+ \alpha_1^k a_{12}^1(a_{11}^2x_1 + a_{12}^2x_2 + b_1^2). \qquad (5.27)\end{aligned}$$

By noting that the Nash equilibrium x^* satisfies (5.2) and (5.3) and hence $a_{11}^1x_1^* + a_{12}^1x_2^* + b_1^1 = 0$ and $a_{12}^2x_1^* + a_{22}^2x_2^* + b_2^2 = 0$, for each mode $k \in \mathcal{K}$, we have

$$\left.\frac{\partial \dot{J}_1^k(x)}{\partial x_1}\right|_{x=x^*} = \alpha_2^k a_{12}^2(a_{12}^1x_1^* + a_{22}^1x_2^* + b_2^1), \qquad (5.28)$$

$$\left.\frac{\partial \dot{J}_1^k(x)}{\partial x_2}\right|_{x=x^*} = \alpha_2^k a_{22}^2(a_{12}^1x_1^* + a_{22}^1x_2^* + b_2^1), \qquad (5.29)$$

$$\left.\frac{\partial \dot{J}_2^k(x)}{\partial x_1}\right|_{x=x^*} = \alpha_1^k a_{11}^1(a_{11}^2x_1^* + a_{12}^2x_2^* + b_1^2), \qquad (5.30)$$

$$\left.\frac{\partial \dot{J}_2^k(x)}{\partial x_2}\right|_{x=x^*} = \alpha_1^k a_{12}^1(a_{11}^2x_1^* + a_{12}^2x_2^* + b_1^2). \qquad (5.31)$$

Consequently, since $A_1x^* + b_1 \neq 0$ and $A_2x^* + b_2 \neq 0$ imply $a_{12}^1x_1^* + a_{22}^1x_2^* + b_2^1 \neq 0$ and $a_{11}^2x_1^* + a_{12}^2x_2^* + b_1^2 \neq 0$, respectively, it follows that

$$\gamma_1^k = \frac{\left.\frac{\partial \dot{J}_1^k(x)}{\partial x_1}\right|_{x=x^*}}{\left.\frac{\partial \dot{J}_1^k(x)}{\partial x_2}\right|_{x=x^*}} = \frac{a_{12}^2}{a_{22}^2}, \quad \gamma_2^k = \frac{\left.\frac{\partial \dot{J}_2^k(x)}{\partial x_1}\right|_{x=x^*}}{\left.\frac{\partial \dot{J}_2^k(x)}{\partial x_2}\right|_{x=x^*}} = \frac{a_{11}^1}{a_{12}^1}, \quad k \in \mathcal{K}. \qquad (5.32)$$

Thus, the proof is complete.

Remark 5.4. It is interesting to note from Proposition 1 that the linearized line (5.23) of $\dot{J}_1^k(x) = 0$ coincides with the best-response line (5.3) for agent 2 (instead of agent 1). The similar observations hold for the linearized lines of $\dot{J}_2^k(x) = 0$.

Remark 5.5. Since $a_{11}^1a_{22}^2 \neq a_{12}^1a_{12}^2$ holds in (5.1), it follows that $\gamma_1^k \neq \gamma_2^k$, $k \in \mathcal{K}$, and hence the boundaries of $\hat{\mathcal{D}}_k$, $k \in \mathcal{K}$, are simply characterized by the two *intersected* straight lines (5.2) and (5.3). Consequently, since $\operatorname{int}\hat{\mathcal{D}}_k = \emptyset$ holds only for $\gamma_1^k = \gamma_2^k$, $k \in \mathcal{K}$, all of the 4 approximated cones must exist with $\operatorname{int}\mathcal{D}_k$, $k \in \mathcal{K}$, being non-empty.

Lemma 5.3. The approximated domains $\hat{\mathcal{D}}_k$, $k \in \mathcal{K}$, are identified to be the 4 convex cones partitioned by the best-response lines (5.2) and (5.3), and satisfy $(\operatorname{int}\hat{\mathcal{D}}_i) \cap$

(int $\hat{\mathcal{D}}_j) = \emptyset$ for $i, j \in \mathcal{K}$, $i \neq j$, int $\hat{\mathcal{D}}_k \neq \emptyset$, $k \in \mathcal{K}$, for any $\alpha_i^{\mathrm{H}} \geq \alpha_i^{\mathrm{L}}$, $i = 1, 2$. Moreover, the domain $\hat{\mathcal{D}}_{\mathrm{LL}}$ (resp., $\hat{\mathcal{D}}_{\mathrm{HL}}$) is centrally symmetric about the Nash equilibrium x^* to $\hat{\mathcal{D}}_{\mathrm{HH}}$ (resp., $\hat{\mathcal{D}}_{\mathrm{LH}}$).

Proof As the curve $\dot{J}_1^k(x) = 0$ (resp., $\dot{J}_2^k(x) = 0$) is linearized by the straight line (5.3) (resp., (5.2)) for all $k \in \mathcal{K}$ (Proposition 5.1), the proof is immediate by checking whether the 4 domains $\{x \in \mathbb{R}^2 : \dot{J}_i^k(x) \geq 0\}$, $k \in \mathcal{K}$, share exactly the same half plane in the neighborhood of x^*, which is proved by the fact that

$$\begin{aligned}
\dot{J}_1^k(\hat{x}) =& \alpha_1^k \Delta_1^1(\hat{x}) + \alpha_2^k \Delta_1^2(\hat{x}) = (\alpha_1^k a_{11}^1 a_{11}^1 + \alpha_2^k a_{12}^1 a_{12}^2)\varepsilon^2 + \varepsilon \alpha_2^k a_{12}^2 (a_{12}^1 x_1^* + a_{22}^1 x_2^* + b_2^1) \\
\approx& \varepsilon \alpha_2^k a_{12}^2 (a_{12}^1 x_1^* + a_{22}^1 x_2^* + b_2^1), \quad k \in \mathcal{K}, \qquad (5.33) \\
\dot{J}_2^k(\hat{x}) =& \alpha_1^k \Delta_2^1(\hat{x}) + \alpha_2^k \Delta_2^2(\hat{x}) = (\alpha_2^k a_{12}^2 a_{12}^2 + \alpha_1^k a_{11}^1 a_{11}^2)\varepsilon^2 + \varepsilon \alpha_1^k a_{11}^1 (a_{11}^2 x_1^* + a_{12}^2 x_2^* + b_1^2) \\
\approx& \varepsilon \alpha_1^k a_{11}^1 (a_{11}^2 x_1^* + a_{12}^2 x_2^* + b_1^2), \quad k \in \mathcal{K}. \qquad (5.34)
\end{aligned}$$

hold for $\hat{x} \triangleq [x_1^* + \varepsilon, x_2^*]^{\mathrm{T}}$ with an infinitesimal number ε. □

Remark 5.6. Lemma 5.3 implies that the best-response lines (5.2) and (5.3) coincide with the switching phases (see a typical example of the approximated domains $\hat{\mathcal{D}}_k$, $k \in \mathcal{K}$, shown in Fig. 5.3(b)) and hence the switching phases at which agents switch the modes around the Nash equilibrium x^* are given by $\theta = \arctan(-\frac{a_{12}^2}{a_{22}^2})$, $\arctan(-\frac{a_{11}^1}{a_{12}^1})$, $\arctan(-\frac{a_{12}^2}{a_{22}^2}) + \pi$, $\arctan(-\frac{a_{11}^1}{a_{12}^1}) + \pi$. Recalling the fact shown in Lemma 5.2, the transition of agents' modes around x^* includes only two possibilities depending on the rotational directions, which are

$$\cdots \to \mathrm{HH} \to \mathrm{LH} \to \mathrm{LL} \to \mathrm{HL} \to \mathrm{HH} \to \cdots, \qquad (5.35)$$

$$\cdots \to \mathrm{HH} \to \mathrm{HL} \to \mathrm{LL} \to \mathrm{LH} \to \mathrm{HH} \to \cdots. \qquad (5.36)$$

The transition sequence of the noncooperative system $\mathcal{G}(J)$ used for Fig. 5.3(b) is depicted in Fig. 5.4, where the sequence is given by the former one since the rotational direction of the trajectories is counterclockwise.

Now, the local stability property of Nash equilibrium x^* of the pseudo-gradient dynamics (5.5), (5.6) is equivalent to the stability property of the piecewise linearized system given by

$$\dot{x}(t) = \mathbb{A}_k(x(t) - x^*), \quad x(t) \in \hat{\mathcal{D}}_k. \qquad (5.37)$$

Recalling that $\hat{\mathcal{D}}_k, k \in \mathcal{K}$, satisfy $\bigcup_{k\in\mathcal{K}} \hat{\mathcal{D}}_k = \mathbb{R}^2$ (Lemma 5.3) and (int $\hat{\mathcal{D}}_i) \cap$ (int $\hat{\mathcal{D}}_j) = \emptyset$ for $i, j \in \mathcal{K}$, $i \neq j$, we use the method shown in [78] to determine stability of the

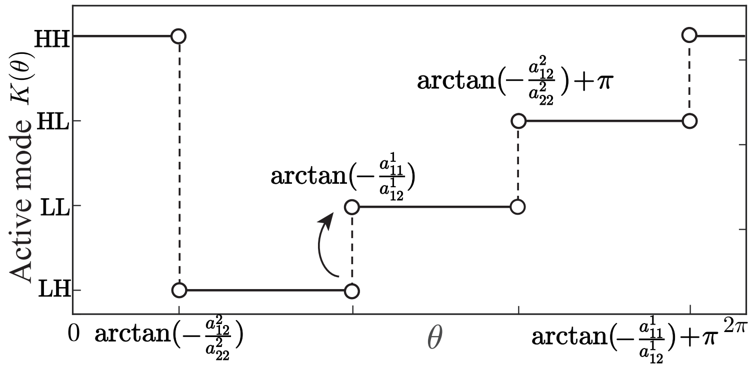


Figure. 5.4 Mode transition in (5.5), (5.6) around x^* in the same $\mathcal{G}(J)$ as Fig. 5.3 where $A_i x^* + b_i \neq 0$, $i \in \{1, 2\}$.

piecewise linear system (5.37). Specifically, define the normalized radial growth rate for each mode $k \in \mathcal{K}$ by

$$\rho_k(\theta) \triangleq \frac{1}{r}\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{\eta^{\mathrm{T}}(\theta)\mathbb{A}_k\eta(\theta)}{\det[\eta(\theta), \mathbb{A}_k\eta(\theta)]} = \frac{\eta^{\mathrm{T}}(\theta)\mathbb{A}_k\eta(\theta)}{\eta^{\mathrm{T}}(\theta)P_k\eta(\theta)}, \tag{5.38}$$

where P_k is defined in (5.22). Note that $\rho_k(\theta)$, $k \in \mathcal{K}$, are continuous in θ. Then, the integral of the normalized radial growth rate is given by

$$\gamma_{\mathrm{rg}} \triangleq \int_{\theta_0}^{\theta_0+2\pi} \rho_{K(\theta)}(\theta)\mathrm{d}\theta, \tag{5.39}$$

where $\theta_0 \in \mathbb{R}$ and $K(\theta) \in \mathcal{K}$ is a function of the phase θ representing which mode is active for (5.37) around the Nash equilibrium x^*. Note that γ_{rg} in (5.39) is invariant under θ_0 because $\rho_K(\theta)$ is a periodic function of θ of period 2π. The value of γ_{rg} is numerically evaluated once the active mode $K(\theta)$ is determined.

Theorem 5.1. Consider the loss-aversion-based noncooperative system $\mathcal{G}(J)$ with the pseudo-gradient dynamics (5.5), (5.6) under Assumption 5.1 for $A_i x^* + b_i \neq 0$, $i \in \{1, 2\}$. If $a^1_{12}\gamma_{\mathrm{rg}} > 0$ and $a^2_{12}\gamma_{\mathrm{rg}} < 0$ (resp., $a^1_{12}\gamma_{\mathrm{rg}} < 0$ and $a^2_{12}\gamma_{\mathrm{rg}} > 0$), then the Nash equilibrium x^* in (5.5), (5.6) hold, is asymptotically stable (resp., unstable).

Proof First, note that $\gamma_{\mathrm{rg}} = \int_{\theta_0}^{\theta_0+2\pi} \rho_{K(\theta)}(\theta)\mathrm{d}\theta = \int_{\theta_0}^{\theta_0+2\pi} \frac{1}{r}\frac{\mathrm{d}r}{\mathrm{d}\theta}\mathrm{d}\theta = \log\frac{r_{\theta_0+2\pi}}{r_{\theta_0}}$, where $\frac{r_{\theta_0+2\pi}}{r_{\theta_0}}$ represents the ratio of the distances between the Nash equilibrium x^* and the states when the state travels for one round from the phase θ_0 to $\theta_0 + 2\pi$. For the counterclockwise case (i.e., $a^1_{12} < 0 \wedge a^2_{12} > 0$), $\gamma_{\mathrm{rg}} < 0$ (resp., $\gamma_{\mathrm{rg}} > 0$) implies that the state is coming closer to (resp., farther from) x^* under (5.37) after it travels for one round. For the clockwise case (i.e., $a^1_{12} > 0 \wedge a^2_{12} < 0$), the opposite is true. Hence, if

$a^1_{12}\gamma > 0 \wedge a^2_{12}\gamma < 0$ (resp., $a^1_{12}\gamma < 0 \wedge a^2_{12}\gamma > 0$), then noting $a^1_{12}a^2_{12} < 0$, the Nash equilibrium x^* is asymptotically stable (resp., unstable). □

Remark 5.7. Even though it follows from Lemma 5.3 that $\alpha_1^{\mathrm{H}}, \alpha_1^{\mathrm{L}}, \alpha_2^{\mathrm{H}}$, α_2^{L} do not change the partition of $\hat{\mathcal{D}}_k$, $k \in \mathcal{K}$, they affect the normalized radial growth rates ρ_k, $k \in \mathcal{K}$, in (5.38) by altering P_k, $\mathbb{A}_k$, $k \in \mathcal{K}$, and hence may change the stability property.

Remark 5.8. The parameters $a^1_{22}, a^2_{11}, b^1_2, b^2_1$ neither change the normalized radial growth rates $\rho_k(\theta)$, $k \in \mathcal{K}$, nor the switching phases $\theta = \arctan(-\frac{a^2_{12}}{a^2_{22}})$, $\arctan(-\frac{a^1_{11}}{a^1_{12}})$, $\arctan(-\frac{a^2_{12}}{a^2_{22}}) + \pi$, $\arctan(-\frac{a^1_{11}}{a^1_{12}}) + \pi$, but they affect the active mode $K(\theta)$ due to a permutation of the locations of $\hat{\mathcal{D}}_k$, $k \in \mathcal{K}$, among the 4 convex cones partitioned by (5.2) and (5.3), and hence may change the stability property.

Case 2: $A_i x^* + b_i = 0$, $i \in \{1, 2\}$

In this section, we characterize the stability property of the Nash equilibrium x^* for $A_i x^* + b_i = 0$ for $i \in \{1, 2\}$. In such a case, recall that the domains $\mathcal{D}_k, k \in \mathcal{K}$, are convex cones with x^* being the center since $\dot{J}_i^k(x) = (x - x^*)^{\mathrm{T}} Q_i^k (x - x^*)$, $i \in \{1, 2\}$, $k \in \mathcal{K}$, in (5.10).

Note that if $Q_i^k > 0$, $i \in \{1, 2\}$, $k \in \mathcal{K}$, then it follows that $\operatorname{int}\mathcal{D}_{\mathrm{LL}}, \operatorname{int}\mathcal{D}_{\mathrm{HL}}, \operatorname{int}\mathcal{D}_{\mathrm{LH}} = \emptyset$, and $\mathcal{D}_{\mathrm{HH}} = \mathbb{R}^2$ (Remark 5.1) so that there is no mode transition. Henceforth, in this section for *Case 2*, suppose that the matrices $Q_i^k = Q_i^{k\,\mathrm{T}}$, $i \in \{1, 2\}$, $k \in \mathcal{K}$, are all sign-indefinite. Under this condition, each of the domains $\mathcal{D}_k$, $k \in \mathcal{K}$, satisfies $\mathcal{D}_k \neq \mathbb{R}^2$ and the boundaries of the existing convex cones $\mathcal{D}_k$ characterized by $\dot{J}_1^k(x) = 0$ and/or $\dot{J}_2^k(x) = 0$ are given by the 2 lines out of the 4 lines

$$\tilde{\gamma}_1^{k+}\tilde{x}_1 + \tilde{x}_2 = 0, \quad \tilde{\gamma}_1^{k-}\tilde{x}_1 + \tilde{x}_2 = 0, \tag{5.40}$$

$$\tilde{\gamma}_2^{k+}\tilde{x}_1 + \tilde{x}_2 = 0, \quad \tilde{\gamma}_2^{k-}\tilde{x}_1 + \tilde{x}_2 = 0, \tag{5.41}$$

where

$$\tilde{\gamma}_i^{k\pm} = \frac{Q^k_{i\,(1,2)} \pm \sqrt{Q^k_{i\,(1,2)} Q^k_{i\,(1,2)} - Q^k_{i\,(1,1)} Q^k_{i\,(2,2)}}}{Q^k_{i\,(2,2)}} \in \mathbb{R}, \tag{5.42}$$

and $Q_i^k(a, b)$ denotes the (a, b)th entry of Q_i^k.

In general, it turns out that there may be overlapped regions between $\mathcal{D}_k$, $k \in \mathcal{K}$. Depending on the rotational direction characterized in Lemma 5.2, we define the

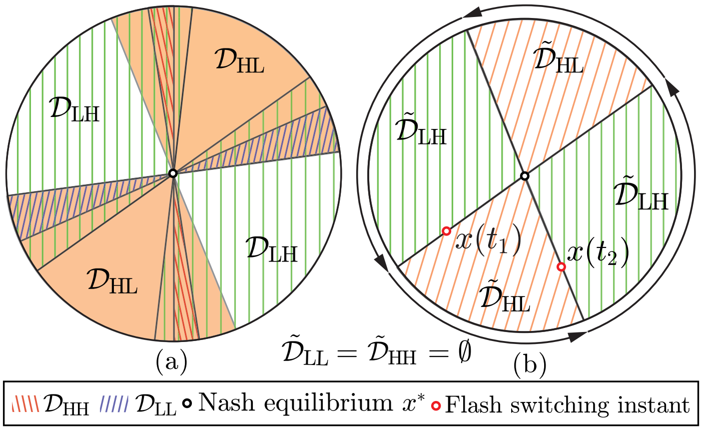


Figure. 5.5 An example of the partition of $\tilde{\mathcal{D}}_k, k \in \mathcal{K}$, from the domains $\mathcal{D}_k, k \in \mathcal{K}$, for $A_i x^* + b_i = 0, i \in \{1, 2\}$. (a): $\mathcal{D}_k, k \in \mathcal{K}$, (b): effective domains $\tilde{\mathcal{D}}_k, k \in \mathcal{K}$, with counterclockwise trajectories.

effective domains $\tilde{\mathcal{D}}_k$, $k \in \mathcal{K}$, indicating that which mode is active in the overlapped regions by properly partitioning the state space. Specifically, we assume that the modes do not change until increasing/decresing property of J_i changes so that agent i switches its sensitivity parameter $\alpha_i(\cdot)$ when agent i reaches the boundary of the current mode (see the effective domains for a trajectory moving in the counterclockwise direction in Fig. 5.5(b) yielded from the domains $\mathcal{D}_k$, $k \in \mathcal{K}$, given in Fig. 5.5(a)). Note as a direct consequence of Lemma 5.1 that $\tilde{\mathcal{D}}_k$, $k \in \mathcal{K}$, satisfy $\bigcup_{k \in \mathcal{K}} \tilde{\mathcal{D}}_k = \mathbb{R}^2$.

Consequently, the stability property of the Nash equilibrium x^* of the pseudo-gradient dynamics (5.5), (5.6) is equivalent to the stability property in the piecewise linear system given by (5.37) with $\hat{\mathcal{D}}_k$ replaced by $\tilde{\mathcal{D}}_k$. Similar to the previous section, we use the integral of the normalized radial growth rate $\gamma_{\rm rg}$ to determine stability of the Nash equilibrium x^*. Note that since the active mode of (5.37) at phase $\theta + \pi$ is exactly same as the active mode at phase θ (i.e., $K(\theta + \pi) = K(\theta)$), we have $\gamma_{\rm rg} = 2 \int_{\theta_0}^{\theta_0 + \pi} \rho_{K(\theta)}(\theta) \mathrm{d}\theta$.

Theorem 5.2. Consider the loss-aversion-based noncooperative system $\mathcal{G}(J)$ with the pseudo-gradient dynamics (5.5), (5.6) under Assumption 5.1 for $A_i x^* + b_i = 0, i \in \{1, 2\}$. Then the following statements hold:

1) If $a_{12}^1 \gamma_{\rm rg} > 0$ and $a_{12}^2 \gamma_{\rm rg} < 0$, then the Nash equilibrium x^* in (5.5), (5.6) is globally asymptotically stable;

2) If $\gamma_{\rm rg} = 0$, then (5.5), (5.6) are marginally stable and the trajectory of (5.5), (5.6) constitutes a closed orbit;

3) If $a_{12}^1 \gamma_{\rm rg} < 0$ and $a_{12}^2 \gamma_{\rm rg} > 0$, then the Nash equilibrium x^* in (5.5), (5.6) is unstable.

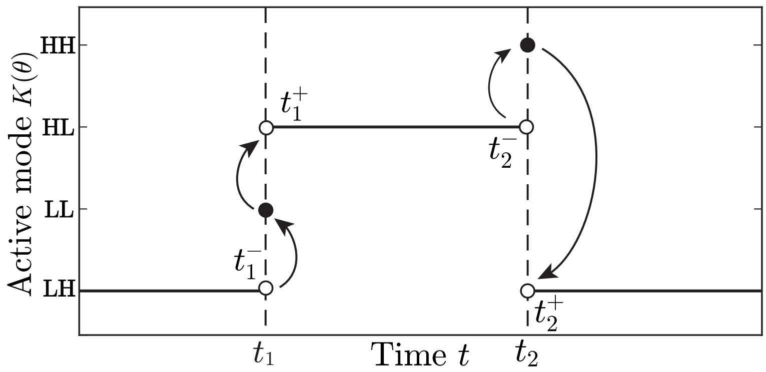


Figure. 5.6 Mode transition sequence in (5.5), (5.6) around the flash switching instants t_1 and t_2 in Fig. 5.5. At those flash switching instants, agent 2 first switches its sensitivity parameter and agent 1 further switches its sensitivity right after agent 2's switch.

Proof The proof for *1)* and *3)* is similar to the proof of Theorem 5.1. For both cases of counterclockwise and clockwise trajectories, $\gamma_{\text{rg}} = 0$ implies that the trajectory goes back to the same point when it travels for one round from the phase θ_0 to $\theta_0 + 2\pi$. Now, *2)* is immediate. □

Next, we present several interesting observations on agents' behavior in the following statements. In terms of the mode transition sequence, there may exist some time instant t at phase θ at which the agents switch the sensitivity parameters such that the active mode $K(\theta(t))$ experiences

$$K(\theta(t^-)) \neq K(\theta(t)) \neq K(\theta(t^+)). \tag{5.43}$$

We call such a switching instant t as a *flash switching instant*. Figure 5.6 shows an example of the mode transition around a flash switching instant t_1 used for Fig. 5.5(b), where agents' state enters into $\mathcal{D}_{\text{HL}}$ after leaving $\mathcal{D}_{\text{LH}}$ at time t_1. In this example, when the 2 agents are in the domain $\mathcal{D}_{\text{LH}}$ and agent 2 reaches its boundary at t_1, agent 2 switches the sensitivity from α_2^{H} to α_2^{L} since $\dot{J}_2(t_1)$ becomes 0 from $\dot{J}_2(t_1^-) > 0$. However, since agent 2's switching behavior results in $\dot{J}_1(t_1) > 0$ from $\dot{J}_1(t_1^-) < 0$, agent 1 further switches its sensitivity from α_1^{L} to α_1^{H} right after the agent 2's switch (t_1^+). After time t_1^+, since agents' state successfully enters into the domain $\mathcal{D}_{\text{HL}}$, the agents keep the mode HL. In the example of Fig. 5.5, the next switching instant t_2 (and all the switching instants) shown to be flash switching instants as well. In short, the reason why there may exist a flash switching instant is that a single agents' sensitivity transition can be a trigger to make the other agent immediately switch its sensitivity almost at the same time instant.

Under the following assumption, the next results show that the effective domain $\tilde{\mathcal{D}}_{\mathrm{LL}}$ can never be adjacent to $\tilde{\mathcal{D}}_{\mathrm{HH}}$ and a *flash switching instant* t exists *only if* the sequence of the active modes satisfies $(K(\theta(t^-)), K(\theta(t^+))) \in \{(\mathrm{LH}, \mathrm{HL}), (\mathrm{HL}, \mathrm{LH})\}$.

Assumption 5.2. The straight lines characterized by $\dot{J}_1^k(x) = 0$ do not coincide with the lines characterized by $\dot{J}_2^k(x) = 0$ for any modes $k \in \mathcal{K}$. In other words, $\tilde{\gamma}_1^{k+}, \tilde{\gamma}_1^{k-}, \tilde{\gamma}_2^{k+}, \tilde{\gamma}_2^{k-}$ are all different in (5.40), (5.41) when k is fixed.

Before we present a theorem, we give the following lemma.

Lemma 5.4. If both A_1 and A_2 in (5.1) are sign-indefinite under Assumption 5.1 for $A_i x^* + b_i = 0, i \in \{1, 2\}$, then $\operatorname{int}\mathcal{D}_{\mathrm{HL}}$ and $\operatorname{int}\mathcal{D}_{\mathrm{LH}}$ are non-empty for any $\alpha_1^{\mathrm{H}}, \alpha_1^{\mathrm{L}}, \alpha_2^{\mathrm{H}}, \alpha_2^{\mathrm{L}} \in \mathbb{R}_+$. Furthermore, the best response line $a_{11}^1 x_1 + a_{12}^1 x_2 + b_1^1 = 0$ for agent 1 (resp., $a_{12}^2 x_1 + a_{22}^2 x_2 + b_2^2 = 0$ for agent 2) belongs only to $\operatorname{int}\mathcal{D}_{\mathrm{LH}}$ (resp., $\operatorname{int}\mathcal{D}_{\mathrm{HL}}$).

Proof Note that $a_{12}^1 a_{12}^2 < 0$, $a_{ii}^i < 0, i = 1, 2$, and since A_1 and A_2 are sign-indefinite, $\det A_i = a_{11}^i a_{22}^i - (a_{12}^i)^2 < 0$, $i = \{1, 2\}$. Furthermore, on the line $a_{11}^1 x_1 + a_{12}^1 x_2 + b_1^1 = a_{11}^1 \tilde{x}_1 + a_{12}^1 \tilde{x}_2 = 0$, since $\Delta_1^1(x) = 0, \Delta_2^1(x) = 0$ in (5.16) and (5.18), it follows from (5.20) that

$$
\begin{aligned}
\dot{J}_1^k(x) =& \alpha_1^k \Delta_1^1(x) + \alpha_2^k \Delta_1^2(x) = \alpha_2^k \Delta_1^2(x) = \alpha_2^k (a_{12}^1 \tilde{x}_1 + a_{22}^1 \tilde{x}_2)(a_{12}^2 \tilde{x}_1 + a_{22}^2 \tilde{x}_2) \\
=& \alpha_2^k \frac{a_{12}^1 a_{12}^2 - a_{11}^1 a_{22}^2}{a_{12}^1 a_{12}^1} (-a_{11}^1 a_{22}^1 + (a_{12}^1)^2) \tilde{x}_1^2 < 0, \qquad (5.44) \\
\dot{J}_2^k(x) =& \alpha_1^k \Delta_2^1(x) + \alpha_2^k \Delta_2^2(x) = \alpha_2^k \Delta_2^2(x) = \alpha_2^k (a_{12}^2 x_1 + a_{22}^2 x_2 + b_2^2)^2 > 0, \qquad (5.45)
\end{aligned}
$$

hold for all $k \in \mathcal{K}$ and any $\alpha_1^{\mathrm{H}}, \alpha_1^{\mathrm{L}}, \alpha_2^{\mathrm{H}}, \alpha_2^{\mathrm{L}} \in \mathbb{R}_+$. Thus, the best response line $a_{11}^1 x_1 + a_{12}^1 x_2 + b_1^1 = 0$ for agent 1 belongs only to $\operatorname{int}\mathcal{D}_{\mathrm{LH}}$ for any $\alpha_1^{\mathrm{H}}, \alpha_1^{\mathrm{L}}, \alpha_2^{\mathrm{H}}, \alpha_2^{\mathrm{L}} \in \mathbb{R}_+$ and hence $\operatorname{int}\mathcal{D}_{\mathrm{LH}}$ is non-empty.

The proof for the other case can be similarly handled. □

Theorem 5.3. Let t_1, t_2 be two consecutive switching instants for the noncooperative system $\mathcal{G}(J)$ under Assumption 5.2 for $A_i x^* + b_i = 0, i \in \{1, 2\}$. If $K(\theta(t)) = \mathrm{LL}$ or HH for $t_1 < t < t_2$, then neither the switching instant t_1 nor t_2 is a flash switching instant and the mode transition satisfies $K(\theta(t_1^-)), K(\theta(t_2^+)) \in \{\mathrm{LH}, \mathrm{HL}\}$ for any $\alpha_i^{\mathrm{H}} \geq \alpha_i^{\mathrm{L}}$, $i = 1, 2$. If, in addition, both A_1 and A_2 are sign-indefinite with Assumption 5.1, then $(K(\theta(t_1^-)), K(\theta(t_2^+))) \in \{(\mathrm{LH}, \mathrm{HL}), (\mathrm{HL}, \mathrm{LH})\}$ for any $\alpha_i^{\mathrm{H}} \geq \alpha_i^{\mathrm{L}}$, $i = 1, 2$.

Proof First, we prove $K(\theta(t_2)) = K(\theta(t_2^+)) \in \{\mathrm{LH}, \mathrm{HL}\}$ (implying that t_2 is not a flash switching instant). To this end, let the state at t_2 as $\bar{x}$ and characterize cases in terms of $K(\theta(t_2^-))$ and $\bar{x}$. For example, consider $K(\theta(t_2^-)) = \mathrm{LL}$ (i.e., $K(\theta(t)) = \mathrm{LL}$, $t_1 < t < t_2$) and $\dot{J}_1^{\mathrm{LL}}(\bar{x}) = 0 \wedge \dot{J}_2^{\mathrm{LL}}(\bar{x}) < 0$ (i.e., $K(\theta(t_2)) = \mathrm{HL}$). In this case, Since $\Delta_2^2(\bar{x}) \geq 0$ and $\dot{J}_2^{\mathrm{LL}}(\bar{x}) = \alpha_1^{\mathrm{L}}\Delta_2^1(\bar{x}) + \alpha_2^{\mathrm{L}}\Delta_2^2(\bar{x}) < 0$ imply $\Delta_2^1(\bar{x}) < 0$, we have $\dot{J}_2^{\mathrm{HL}}(t_2^+) \approx \dot{J}_2^{\mathrm{HL}}(\bar{x}) = \dot{J}_2^{\mathrm{LL}}(\bar{x}) + \delta_1\Delta_2^1(\bar{x}) < 0$. Note that when $\Delta_1^1(\bar{x}) > 0$, we have $\dot{J}_1^{\mathrm{HL}}(t_2^+) \approx \dot{J}_1^{\mathrm{HL}}(\bar{x}) = \dot{J}_1^{\mathrm{LL}}(\bar{x}) + \delta_1\Delta_1^1(\bar{x}) > 0$. Alternatively, when $\Delta_1^1(\bar{x}) = \dot{J}_1^{\mathrm{LL}}(\bar{x}) = 0$, we have $\Delta_1^2(\bar{x}) = \dot{J}_1^{\mathrm{HL}}(\bar{x}) = 0$. By neglecting the second-order infinitesimal in $\Delta_1^1(x(t_2^+))$ in (5.20), it follows that $\dot{J}_1^k(t_2^+) \approx \alpha_2^k\Delta_1^2(x(t_2^+))$ holds for $k \in \mathcal{K}$. Hence, since $\dot{J}_1^{\mathrm{LL}}(t_2^+) > 0$, it follows that $\dot{J}_1^{\mathrm{HL}}(t_2^+) \approx \alpha_2^{\mathrm{L}}\Delta_1^2(x(t_2^+)) \approx \dot{J}_1^{\mathrm{LL}}(t_2^+) > 0$. Consequently, $x(t_2^+) \in \operatorname{int}\mathcal{D}_{\mathrm{HL}}$ holds for both the two cases above in terms of $\Delta_1^1(\bar{x})$ and hence $K(\theta(t_2^+)) = K(\theta(t_2)) = \mathrm{HL}$ holds. The proof for the other cases can be similarly handled with the conclusion of $K(\theta(t_2)) = K(\theta(t_2^+)) \in \{\mathrm{LH}, \mathrm{HL}\}$. Thus, $K(\theta(t_2^+)) = K(\theta(t_2)) \in \{\mathrm{LH}, \mathrm{HL}\}$ for $K(\theta(t_2^-)) \in \{\mathrm{LL}, \mathrm{HH}\}$. Furthermore, since $K(\theta(t_1^+)) \in \{\mathrm{LL}, \mathrm{HH}\}$, it follows that $K(\theta(t_1^-)) \notin \{\mathrm{LL}, \mathrm{HH}\}$, i.e., $K(\theta(t_1^-)) \in \{\mathrm{LH}, \mathrm{HL}\}$, which implies that t_1 is not a flash switching instant, either.

Next, we prove $K(\theta(t_1^-)) \neq K(\theta(t_2^+))$ for sign-indefinite A_1, A_2. To this end, we show that $\mathcal{D}_{\mathrm{LH}}$, $\mathcal{D}_{\mathrm{HL}}$ are never composed of 4 convex cones. Suppose, *ad absurdum*, $\mathcal{D}_{\mathrm{LH}} = \{x \in \mathbb{R}^2 : \dot{J}_1^{\mathrm{LH}}(x) \leq 0\} \cap \{x \in \mathbb{R}^2 : \dot{J}_2^{\mathrm{LH}}(x) \geq 0\}$ is composed of 4 convex cones in $\mathcal{G}(J)$ with a certain set of sensitivity parameters $(\alpha_1^{\mathrm{H}}, \alpha_1^{\mathrm{L}}, \alpha_2^{\mathrm{H}}, \alpha_2^{\mathrm{L}}) = (\tilde{\alpha}_1^{\mathrm{H}}, \tilde{\alpha}_1^{\mathrm{L}}, \tilde{\alpha}_2^{\mathrm{H}}, \tilde{\alpha}_2^{\mathrm{L}})$. In this case, from a geometric consideration of the domains, it follows that $\{x \in \mathbb{R}^2 : \dot{J}_1^{\mathrm{LH}}(x) \leq 0\} \cup \{x \in \mathbb{R}^2 : \dot{J}_2^{\mathrm{LH}}(x) \geq 0\} = \mathbb{R}^2$. Next, consider a set of sensitivity parameters $(\alpha_1^{\mathrm{H}}, \alpha_1^{\mathrm{L}}, \alpha_2^{\mathrm{H}}, \alpha_2^{\mathrm{L}})$ with $\alpha_1^{\mathrm{H}} = \tilde{\alpha}_1^{\mathrm{L}}$ and $\alpha_2^{\mathrm{L}} = \tilde{\alpha}_2^{\mathrm{H}}$. Note that since the sensitivity profile $\alpha^{\mathrm{HL}} = \operatorname{diag}[\tilde{\alpha}_1^{\mathrm{L}}, \tilde{\alpha}_2^{\mathrm{H}}]$ for the second set is the same as the value of $\alpha^{\mathrm{LH}} = \operatorname{diag}[\tilde{\alpha}_1^{\mathrm{L}}, \tilde{\alpha}_2^{\mathrm{H}}]$ in the first set, the domain

$$\begin{aligned}
\operatorname{int}\mathcal{D}_{\mathrm{HL}} &= \{x \in \mathbb{R}^2 : \dot{J}_1^{\mathrm{HL}}(x) > 0\} \cap \{x \in \mathbb{R}^2 : \dot{J}_2^{\mathrm{HL}}(x) < 0\} \\
&= \{x \in \mathbb{R}^2 : \dot{J}_1^{\mathrm{LH}}(x) > 0\} \cap \{x \in \mathbb{R}^2 : \dot{J}_2^{\mathrm{LH}}(x) < 0\} \\
&= (\mathbb{R}^2 \setminus \{x \in \mathbb{R}^2 : \dot{J}_1^{\mathrm{LH}}(x) \leq 0\}) \cap \{x \in \mathbb{R}^2 : \dot{J}_2^{\mathrm{LH}}(x) < 0\} \\
&= (\{x \in \mathbb{R}^2 : \dot{J}_2^{\mathrm{LH}}(x) \geq 0\} \setminus \mathcal{D}_{\mathrm{LH}}) \cap \{x \in \mathbb{R}^2 : \dot{J}_2^{\mathrm{LH}}(x) < 0\}
\end{aligned}$$

is empty, which contradicts with Lemma 5.4. Thus, $\mathcal{D}_{\mathrm{LH}}$ is never composed of 4 convex cones. The proof for $\mathcal{D}_{\mathrm{HL}}$ can be similarly handled. Now, suppose, *ad absurdum*, that $K(\theta(t_1^-)) = K(\theta(t_2^+)) \in \{\mathrm{LH}, \mathrm{HL}\}$. Since the rotational direction of the trajectories are consistently the same in $\mathbb{R}^2$, it follows that $\tilde{\mathcal{D}}_{\mathrm{LL}} \cup \tilde{\mathcal{D}}_{\mathrm{LH}} = \mathbb{R}^2$ or $\tilde{\mathcal{D}}_{\mathrm{LL}} \cup \tilde{\mathcal{D}}_{\mathrm{HL}} = \mathbb{R}^2$ must hold for $K(\theta(t)) = \mathrm{LL}$, $t_1 < t < t_2$, which also contradicts with Lemma 5.4. (The case of $K(\theta(t)) = \mathrm{HH}$, $t_1 < t < t_2$, can be similarly handled.) Thus, the proof is complete. $\square$

Remark 5.9. Theorem 5.3 implies that if $\tilde{\mathcal{D}}_{\text{LL}}$ or $\tilde{\mathcal{D}}_{\text{HH}}$ exists for sign-indefinite A_1, A_2, then $\tilde{\mathcal{D}}_{\text{LH}}$ and $\tilde{\mathcal{D}}_{\text{HL}}$ are adjacent to $\tilde{\mathcal{D}}_{\text{LL}}$ and/or $\tilde{\mathcal{D}}_{\text{HH}}$ and hence the mode transition sequence around $\tilde{\mathcal{D}}_{\text{LL}}$ and $\tilde{\mathcal{D}}_{\text{HH}}$ is respectively given by $\cdots \rightleftharpoons \text{LH} \rightleftharpoons \text{LL} \rightleftharpoons \text{HL} \rightleftharpoons \cdots$ and $\cdots \rightleftharpoons \text{LH} \rightleftharpoons \text{HH} \rightleftharpoons \text{HL} \rightleftharpoons \cdots$. Alternatively, in the case where $\tilde{\mathcal{D}}_{\text{LL}} = \tilde{\mathcal{D}}_{\text{HH}} = \emptyset$ (as in Fig. 5.5), both $\tilde{\mathcal{D}}_{\text{LH}}$ and $\tilde{\mathcal{D}}_{\text{HL}}$ must exist, since $\tilde{\mathcal{D}}_{\text{LH}} \cup \tilde{\mathcal{D}}_{\text{HL}} = \mathbb{R}^2$ and $\tilde{\mathcal{D}}_{\text{LH}}, \tilde{\mathcal{D}}_{\text{HL}} \neq \mathbb{R}^2$. In such a case, the mode transition sequence is given by $\cdots \rightleftharpoons \text{LH} \rightleftharpoons \text{HL} \rightleftharpoons \cdots$. As a result, the modes LH and HL always exist when A_1 and A_2 are sign-indefinite.

Remark 5.10. Theorem 5.3 does not imply that there always exists a flash switching instant when the mode transition LH $\to$ HL or HL $\to$ LH happens. For instance, consider the case with zero-sum payoffs. In this case, the overall mode transition sequence is composed of only modes LH and HL and the agents always simultaneously switch the sensitivity parameters at the same switching instants since the straight lines $\dot{J}_1^{\text{LH}}(x) = 0$ and $\dot{J}_2^{\text{LH}}(x) = 0$ (or, $\dot{J}_1^{\text{HL}}(x) = 0$ and $\dot{J}_2^{\text{HL}}(x) = 0$) coincide with each other. As a result, the switching instants in such a system are not flash switching instants.

Note that the case where Q_1^k, $k \in \mathcal{K}$, are positive definite and Q_2^k, $k \in \mathcal{K}$, are sign-indefinite can be similarly handled by evaluating the sign of γ_{rg} in (5.39) with possibly fewer number of domains.

Case 3: $A_1 x^* + b_1 \neq 0$**,** $A_2 x^* + b_2 = 0$

In this section, we characterize the stability property of the Nash equilibrium x^* for $A_1 x^* + b_1 \neq 0$, $A_2 x^* + b_2 = 0$ with all sign-indefinite matrices Q_2^k, $k \in \mathcal{K}$, in (5.10). In such a case, each of the domains $\mathcal{D}_k$, $k \in \mathcal{K}$, is understood as the overlap of convex cones and the regions whose boundaries are characterized by hyperbolic (or elliptic) functions (see the example shown in Fig. 5.1(c) for $k = \text{LL}$).

Similar to *Case 1* in Section 5.4, we approximate the domain $\mathcal{D}_k$ around x^* to the convex cone $\hat{\mathcal{D}}_k$ by linearizing the quadratic curve characterized by $\dot{J}_1^k(x) = 0$ around x^* to the straight line (5.3) for all $k \in \mathcal{K}$. It can be similarly shown that $\hat{\mathcal{D}}_{\text{LL}} \cup \hat{\mathcal{D}}_{\text{LH}}$ and $\hat{\mathcal{D}}_{\text{HH}} \cup \hat{\mathcal{D}}_{\text{HL}}$ are the two half planes partitioned by (5.3) (see Fig. 5.7(b)). Then, considering the overlapped regions, similar to *Case 2* (Section 5.4), we define the effective domains $\tilde{\mathcal{D}}_k$, $k \in \mathcal{K}$, by partitioning the state space according to the rotational direction (see Fig. 5.7(c)). Different from *Case 2* where some of the effective domains $\tilde{\mathcal{D}}_k, k \in \mathcal{K}$, may be empty, none of $\tilde{\mathcal{D}}_k$, $k \in \mathcal{K}$, is empty in *Case 3* and hence all of the 4 modes exist. Then, the stability property of the Nash equilibrium x^* of the pseudo-gradient dynamics (5.5), (5.6) is equivalent to the stability property in the piecewise linearized system given by (5.37) with $\hat{\mathcal{D}}_k$ replaced by $\tilde{\mathcal{D}}_k$.

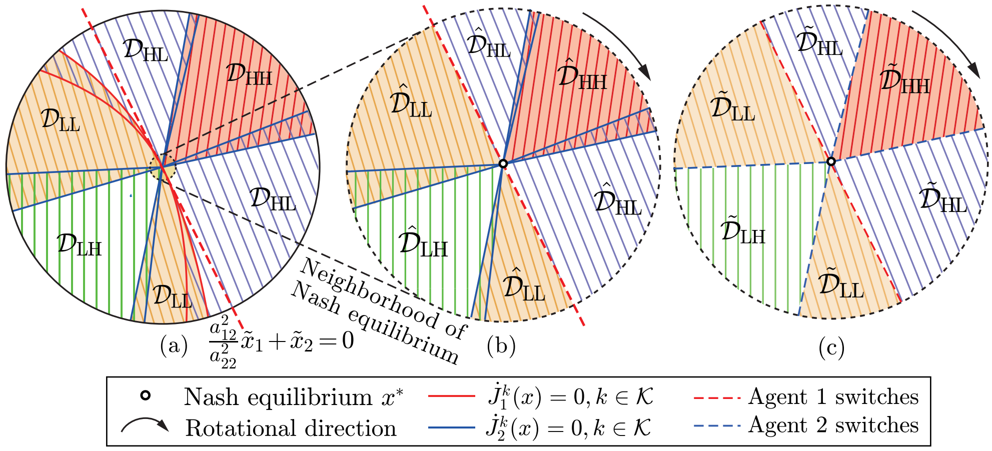


Figure. 5.7 An example of the partitions of $\mathcal{D}_k, k \in \mathcal{K}, \hat{\mathcal{D}}_k, k \in \mathcal{K}$, and $\tilde{\mathcal{D}}_k, k \in \mathcal{K}$, under Assumption 5.1 for $A_1x^* + b_1 \neq 0, A_2x^* + b_2 = 0$. (a): $\mathcal{D}_k, k \in \mathcal{K}$, (b): approximated domains $\hat{\mathcal{D}}_k, k \in \mathcal{K}$, (c): effective domains $\tilde{\mathcal{D}}_k, k \in \mathcal{K}$, determined from (b) with trajectories moving in the clockwise direction.

Theorem 5.4. Consider the loss-aversion-based noncooperative system $\mathcal{G}(J)$ with the pseudo-gradient dynamics (5.5), (5.6) under Assumption 5.1 for $A_1x^* + b_1 \neq 0$, $A_2x^* + b_2 = 0$. If $a_{12}^1\gamma_{\rm rg} > 0$ and $a_{12}^2\gamma_{\rm rg} < 0$ (resp., $a_{12}^1\gamma_{\rm rg} < 0$ and $a_{12}^2\gamma_{\rm rg} > 0$), then the Nash equilibrium x^* in (5.5), (5.6) is asymptotically stable (resp., unstable), where $\gamma_{\rm rg}$ is defined in (5.39).

Proof The proof is similar to the proof of Theorems 5.1 and 5.2. □

Proposition 5.2. Assume that x^* satisfies $A_1x^* + b_1 \neq 0$, $A_2x^* + b_2 = 0$. Then there is no flash switching instant for any $\alpha_i^{\rm H} \geq \alpha_i^{\rm L}$, $i = 1, 2$, in the neighborhood of x^*.

Proof Let t_1, t_2 be two consecutive switching instants. Note that if $K(\theta(t)) = \text{LL}$ or HH (resp., LH or HL) for $t_1 < t < t_2$, then $K(\theta(t_1^-))$, $K(\theta(t_2^+)) \in \{\text{LH}, \text{HL}\}$ (resp., $\{\text{LL}, \text{HH}\}$). Moreover, since $\dot{J}_2^k(x) = \alpha_1^k\Delta_2^1(x) + \alpha_2^k\Delta_2^2(x)$, $k \in \mathcal{K}$, and $\Delta_2^2(x) = 0$ on the best-response line (5.3), the sensitivity change $\alpha_1^{\rm L} \to \alpha_1^{\rm H}$ or $\alpha_1^{\rm H} \to \alpha_1^{\rm L}$ does not change the sign of $\dot{J}_2^k(x)$, $k \in \mathcal{K}$, on the best-response line (5.3) and hence agents' state can never enter $\tilde{\mathcal{D}}_{\rm HL}$ after leaving $\tilde{\mathcal{D}}_{\rm LH}$. In other words, around the Nash equilibrium x^*, if $K(\theta(t)) = \text{LH}$ or HL for $t_1 < t < t_2$ where t_1, t_2 are two consecutive switching instants, then $K(\theta(t_1^-))$, $K(\theta(t_2^+)) \in \{\text{LL}, \text{HH}\}$ for any $\alpha_i^{\rm H} \geq \alpha_i^{\rm L}$, $i = 1, 2$. Therefore, there does not exist any flash switching instant in the loss-aversion-based noncooperative system around the Nash equilibrium x^* for $A_1x^* + b_1 \neq 0$, $A_2x^* + b_2 = 0$. □

Note that the case where Q_2^k, $k \in \mathcal{K}$, are positive definite can be similarly handled by evaluating the sign of $\gamma_{\rm rg}$ in (5.39) with possibly fewer number of domains. In this case, the modes LH and HH always exist. As a result, there exist at least 2 modes in the loss-aversion-based noncooperative system for $A_1x^* + b_1 \neq 0$, $A_2x^* + b_2 = 0$.

Discussion

In this section, we further compare the *Cases 1–3* characterized in the previous sections in terms of the normalized radial growth rate and extend the proposed framework for the case where the order of the payoff functions is greater than 2.

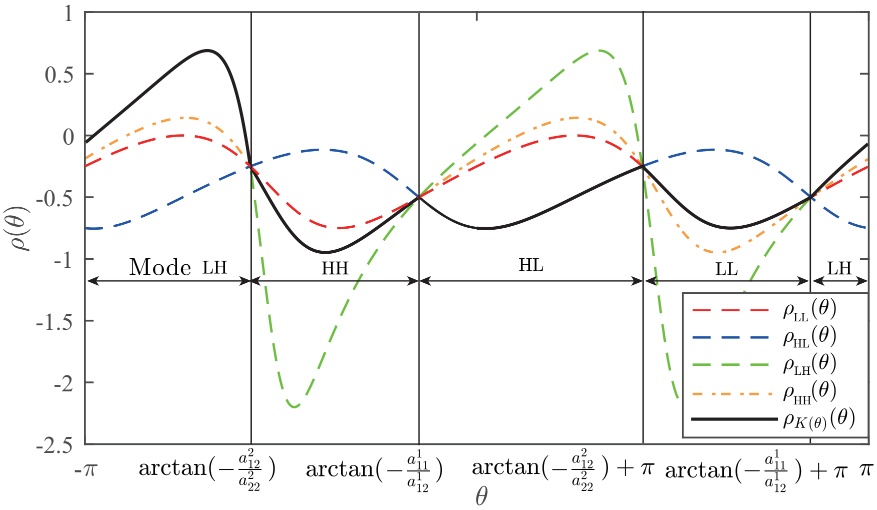
The following result shows a special property of the normalized radial growth rates $\rho_k(\theta)$, $k \in \mathcal{K}$, defined in (5.38).

Proposition 5.3. The normalized radial growth rates $\rho_k(\theta)$, $k \in \mathcal{K}$, possess the common values at the 4 phases $\theta = \arctan(-\frac{a^2_{12}}{a^2_{22}})$, $\arctan(-\frac{a^1_{11}}{a^1_{12}})$, $\arctan(-\frac{a^2_{12}}{a^2_{22}}) + \pi$, $\arctan(-\frac{a^1_{11}}{a^1_{12}}) + \pi$ characterized as the switching phases for Case 1 (Remark 5.6). Specifically, $\rho_k(\arctan(-\frac{a^2_{12}}{a^2_{22}})) = \rho_k(\arctan(-\frac{a^2_{12}}{a^2_{22}}) + \pi) = \frac{a^2_{22}}{a^2_{12}}$, $\rho_k(\arctan(-\frac{a^1_{11}}{a^1_{12}})) = \rho_k(\arctan(-\frac{a^1_{11}}{a^1_{12}}) + \pi) = -\frac{a^1_{11}}{a^1_{12}}$ for all $k \in \mathcal{K}$ with any $\alpha^{\mathrm{H}}_1, \alpha^{\mathrm{L}}_1, \alpha^{\mathrm{H}}_2, \alpha^{\mathrm{L}}_2 \in \mathbb{R}_+$.

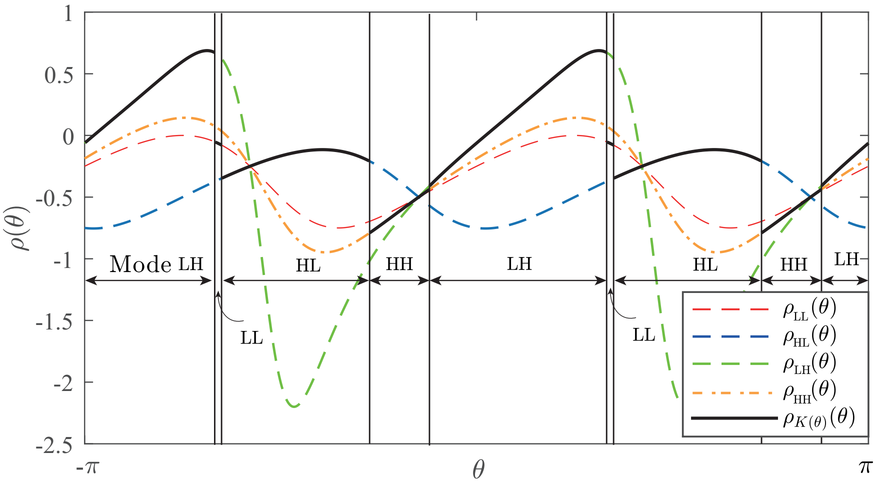
Proof The proof is immediate by checking the values of $\rho_k(\theta)$ at the specified phases. □

Remark 5.11. Proposition 5.3 implies that the normalized radial growth rate $\rho_{K(\theta)}(\theta)$ in *Case 1* is continuous on θ, since $\rho_k(\theta)$, $k \in \mathcal{K}$, possess the same values at the 4 switching phases (see Fig. 5.8(a)). However, in *Cases 2* and *3*, since agents may switch the sensitivity parameters at a phase $\theta \notin \{\arctan(-\frac{a^2_{12}}{a^2_{22}}), \arctan(-\frac{a^1_{11}}{a^1_{12}}), \arctan(-\frac{a^2_{12}}{a^2_{22}})+\pi, \arctan(-\frac{a^1_{11}}{a^1_{12}}) + \pi\}$, $\rho_{K(\theta)}(\theta)$ is most likely to be discontinuous at the switching phases (see Fig. 5.8(b)).

To discuss how a small perturbation on the parameters in A_1, A_2, b_1, b_2 affect the stability of the Nash equilibrium x^*, since from (5.4) the small perturbations on $a^1_{11}, a^1_{12}, b^1_1, a^2_{12}, a^2_{22}, b^2_2$ change the location of the Nash equilibrium in the state space, we focus only on the parameters $a^1_{22}, a^2_{11}, b^2_1, b^1_2$ which do not affect the value of x^*. Specifically, consider *Case 2* ($A_ix^* + b_i = 0, i \in \{1, 2\}$). Then even a small change in a^1_{22} or b^1_2 yields $A_1x^* + b_1 \neq 0$ so that *Case 2* changes to *Case 3* however small the perturbation is. Moreover, if there further exists a small perturbation on a^2_{11} or b^2_1, then $A_2x^* + b_2$ also becomes nonzero and hence *Case 3* changes to *Case 1*. For example, it can be seen from Fig. 5.8 that since the small perturbations on b^2_1, b^1_2 for $A_ix^* + b_i = 0, i \in \{1, 2\}$, change the noncooperative system $\mathcal{G}(J)$ from *Case 2* to *Case 1*, the active mode $K(\theta)$ may drastically change depending on the phase θ and hence the stability property of the Nash equilibrium x^* may also be affected.



(a)



(b)

Figure. 5.8 Typical normalized radial growth rates $\rho_{K(\theta)}(\theta), \rho_k(\theta), k \in \mathcal{K}, \theta \in [-\pi, \pi]$, with the same A_1, A_2, b_1^1, b_2^2 but different b_1^2, b_2^1. (a): $A_i x^* + b_i \neq 0, i \in \{1, 2\}$ (*Case 1*), (b): $A_i x^* + b_i = 0, i \in \{1, 2\}$ (*Case 2*). The parameters b_1^2, b_2^1 in (a) are obtained by giving small perturbations on b_1^2 and b_2^1 in (b).

It is worth noting that as long as local stability is concerned around a Nash equilibrium x^*, the similar results can be drawn for the case of non-quadratic payoff functions which yield nonlinear pseudo-gradient dynamics. Specifically, for a (not necessarily quadratic) payoff function $J_i(x)$, it can be expressed in the form of

$$\begin{aligned} J_i(x) &= J_i(x^*) + \big(\tfrac{\partial J_i(x^*)}{\partial x}\big)^{\mathrm{T}}\tilde{x} + \tfrac{1}{2}\tilde{x}^{\mathrm{T}} A_i \tilde{x} + \varepsilon_i(x) \\ &= \tfrac{1}{2} x^{\mathrm{T}} A_i x + b_i^{\mathrm{T}} x + c_i + \varepsilon_i(x), \end{aligned} \tag{5.46}$$

where $\varepsilon_i(x)$ includes 3rd- or higher-order terms, $A_i \in \mathbb{R}^{2\times 2}$ is the Hessian matrix of $J_i(x)$ evaluated at x^*, $b_i = \frac{\partial J_i(x^*)}{\partial x} - A_i x^* \in \mathbb{R}^2$, and $c_i = J_i(x^*) - (\frac{\partial J_i(x^*)}{\partial x})^{\mathrm{T}} x^* + \frac{1}{2} x^{*\mathrm{T}} A_i x^* \in \mathbb{R}$. Noting that A_i in (5.46) plays a similar role as the one in (5.1), stability analysis around the Nash equilibrium can be similarly conducted as in the theorems and the propositions given in this section.

5.5 Stability Analysis With Real Eigenvalues

In this section, we generalize the stability results in Section 5.4 by relaxing the restriction on complex conjugate eigenvalues. For simplicity, we suppose that $A_i x^* + b_i \neq 0$, $i \in \{1, 2\}$, hold with $a^1_{11}a^2_{22} \neq a^1_{12}a^2_{12} \neq 0$. In this case, the approximated domains $\hat{\mathcal{D}}_k$, $k \in \mathcal{K}$, of the piecewise linearized system (5.37) are partitioned by the best-response lines (5.2) and (5.3) (Lemma 5.3). Moreover, since the eigenvalues of $\mathbb{A}_k$ are given by

$$\begin{aligned} \lambda^k_1 &\triangleq \psi_k - \sqrt{\psi_k^2 - \alpha^k_1\alpha^k_2(a^1_{11}a^2_{22} - a^1_{12}a^2_{12})}, \\ \lambda^k_2 &\triangleq \psi_k + \sqrt{\psi_k^2 - \alpha^k_1\alpha^k_2(a^1_{11}a^2_{22} - a^1_{12}a^2_{12})}, \end{aligned} \tag{5.47}$$

with $\psi_k \triangleq \frac{1}{2}(\alpha^k_1 a^1_{11} + \alpha^k_2 a^2_{22}) < 0$, all of the matrices $\mathbb{A}_k$, $k \in \mathcal{K}$, are stable (resp., unstable) for $a^1_{11}a^2_{22} - a^1_{12}a^2_{12} > 0$ (resp., < 0). The eigenvectors of the system matrix $\mathbb{A}_k$ ($\neq \sigma I$ for all $\sigma < 0$ under $a^1_{12}a^2_{12} \neq 0$) are denoted by υ^k_1 and υ^k_2 satisfying

$$\mathbb{A}_k \upsilon^k_i = \lambda^k_i \upsilon^k_i, \quad i \in \{1, 2\}. \tag{5.48}$$

In the following subsections, we first handle the stable subsystems case ($a^1_{11}a^2_{22} - a^1_{12}a^2_{12} > 0$), then we give the results for the unstable subsystems case ($a^1_{11}a^2_{22} - a^1_{12}a^2_{12} < 0$).

Stable Subsystems

In this section, we consider the case with stable subsystems, i.e., $a^1_{11}a^2_{22} - a^1_{12}a^2_{12} > 0$.

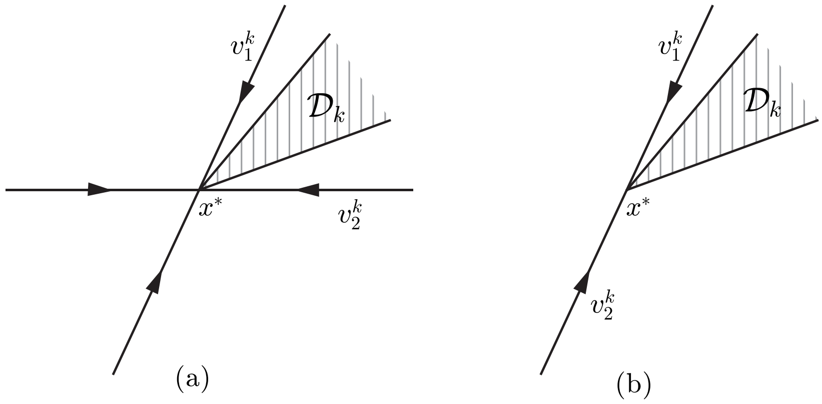


Figure. 5.9 Examples of the noncooperative system with a strongly transitive mode $k \in \mathcal{K}$ where $\lambda_1^k, \lambda_2^k \in \mathbb{R}$, $a_{11}^1 a_{22}^2 > a_{12}^1 a_{12}^2$. Arrows: eigenvectors. (a): $\lambda_1^k \neq \lambda_2^k$ (two independent eigenvectors), (b): $\lambda_1^k = \lambda_2^k$ with the improper node x^*. The mode k is strongly transitive since the rotational direction of the trajectories keeps counterclockwise or clockwise in $\hat{\mathcal{D}}_k$.

Definition 5.1. [77] The mode $k \in \mathcal{K}$ is strongly transitive if there exists a time instant t such that $x(t) \notin \hat{\mathcal{D}}_k$ for any x_0 satisfying $x_0 \in \hat{\mathcal{D}}_k \setminus \{x^*\}$.

Definition 5.2. [77] The mode $k \in \mathcal{K}$ is weakly transitive if one of the following statements is true for any x_0 satisfying $x_0 \in \hat{\mathcal{D}}_k \setminus \{x^*\}$:
1) there exists a time instant t such that $x(t) \notin \hat{\mathcal{D}}_k$.
2) $x(t) \in \hat{\mathcal{D}}_k$ for all $t \geq 0$ and $\lim_{t\to\infty} x(t) = x^*$.

To reveal the generalized results on stability property, we use the above notions for the rest of this paper. Here, we note that every strongly transitive mode is also a weakly transitive mode [77]; if all of the modes are strongly transitive, then there exist infinitely many mode transitions in the noncooperative system. Section 5.4 only handles the simplest case with 4 strongly transitive modes and infinitely many mode transitions since $\lambda_1^k, \lambda_2^k \in \mathbb{C}$ hold for all modes $k \in \mathcal{K}$. However for the case with some real eigenvalues, there may still exist 4 strongly transitive modes and infinitely many mode transitions.

Remark 5.12. Recalling $\alpha^k \mathcal{A} \neq \sigma I$ for all $\sigma < 0$ under $a_{12}^1 a_{12}^2 \neq 0$, the noncooperative system traces a straight-line trajectory only when the initial state x_0 is on the eigenvectors. Hence, the mode $k \in \mathcal{K}$ satisfying $\lambda_1^k, \lambda_2^k \in \mathbb{R}$ is strongly transitive if and only if no eigenvectors are containing in $\hat{\mathcal{D}}_k$ (see Fig. 5.9).

Now, we begin to identify the transitivity of the 4 modes. In the following results, we first present a necessary condition for a mode to be strongly transitive, and then we present the detail conditions to identify the transitivity.

Proposition 5.4. Let $\mathbb{K}_{\mathrm{ST}} \subseteq \mathcal{K}$ be the set of strongly transitive modes for $A_i x^* + b_i \neq 0$, $i \in \{1, 2\}$. If $\mathbb{K}_{\mathrm{ST}}$ is non-empty, then $a^1_{12} a^2_{12} < 0$ holds and the rotational directions of the system trajectories never change in the domains $\mathcal{D}_k$, $k \in \mathbb{K}_{\mathrm{ST}}$, around the Nash equilibrium x^* for any $\alpha^{\mathrm{H}}_i \geq \alpha^{\mathrm{L}}_i$, $i = 1, 2$. Specifically, if $a^1_{12} < 0 \wedge a^2_{12} > 0$ (resp., $a^1_{12} > 0 \wedge a^2_{12} < 0$) holds, then the rotational directions of the system trajectories are counterclockwise (resp., clockwise).

Proof: Note that on the best-response lines (5.3) and (5.2), we have

$$\begin{aligned}\tilde{x}^{\mathrm{T}} P_k \tilde{x} &= \alpha^k_2 a^2_{12} \tilde{x}^2_1 - \alpha^k_1 a^1_{12} \tilde{x}^2_2 + (-\alpha^k_1 a^1_{11} + \alpha^k_2 a^2_{22}) \tilde{x}_1 \tilde{x}_2 \\ &= (\alpha^k_2 a^2_{12} - \alpha^k_1 \frac{a^1_{12} (a^2_{12})^2}{(a^2_{22})^2} + \frac{a^2_{12} (\alpha^k_1 a^1_{11} - \alpha^k_2 a^2_{22})}{a^2_{22}}) \tilde{x}^2_1 = \alpha^k_1 a^2_{12} \frac{-a^2_{12} a^1_{12} + a^1_{11} a^2_{22}}{(a^2_{22})^2} \tilde{x}^2_1,\end{aligned} \tag{5.49}$$

$$\begin{aligned}\tilde{x}^{\mathrm{T}} P_k \tilde{x} &= \alpha^k_2 a^2_{12} \tilde{x}^2_1 - \alpha^k_1 a^1_{12} \tilde{x}^2_2 + (-\alpha^k_1 a^1_{11} + \alpha^k_2 a^2_{22}) \tilde{x}_1 \tilde{x}_2 \\ &= (\alpha^k_2 \frac{a^2_{12} (a^1_{12})^2}{(a^1_{11})^2} - \alpha^k_1 a^1_{12} + \frac{a^1_{12} (\alpha^k_1 a^1_{11} - \alpha^k_2 a^2_{22})}{a^1_{11}}) \tilde{x}^2_2 = -\alpha^k_2 a^1_{12} \frac{-a^2_{12} a^1_{12} + a^1_{11} a^2_{22}}{(a^1_{11})^2} \tilde{x}^2_2,\end{aligned} \tag{5.50}$$

respectively, which can be used in (5.21) for determining the rotational direction at the states on best-response lines. Now, note that the rotation directions of the trajectories in a strongly transitive domain $\hat{\mathcal{D}}_k$ (including the boundaries (5.2) and (5.3)) with $k \in \mathbb{K}_{\mathrm{ST}}$ must be the same and hence it follows from (5.49) and (5.50) that the non-empty set $\mathbb{K}_{\mathrm{ST}}$ indicates $a^1_{12} a^2_{12} < 0$. Noting that the signs of (5.49) and (5.50) do not change when α^k_1 or α^k_2 changes, it follows that the rotational directions of the system trajectories are the same in the domains $\hat{\mathcal{D}}_k$, $k \in \mathbb{K}_{\mathrm{ST}}$. Thus, the result is immediate since $a^1_{12} < 0 \wedge a^2_{12} > 0$ (resp., $a^1_{12} > 0 \wedge a^2_{12} < 0$) under $-a^2_{12} a^1_{12} + a^1_{11} a^2_{22} > 0$ indicates that (5.49), (5.50), and $\dot{\theta}$ defined in (5.21) are positive (resp., negative) and hence the rotational directions of the system trajectories are counterclockwise (resp., clockwise). □

Remark 5.13. Note that Proposition 5.4 is a generalized result of Lemma 5.2. Lemma 5.2 characterizes a special case with $\mathbb{K}_{\mathrm{ST}} = \mathcal{K}$ so that the rotational directions of the trajectories are consistently the same in the entire state space $\mathbb{R}^2$.

In general, all modes being strongly transitive is only a sufficient condition for a piecewise linear system possessing infinitely many mode transitions [77]. For example, a sliding mode may exist in the boundary of two convex cones if the rotational directions of the system trajectories changes are opposite among two strongly transitive modes (see Example 13 of [77]). However, as the mode transition does not affect the rotational direction of the trajectories at the boundaries of the domains $\hat{\mathcal{D}}_{\mathrm{LL}}$ (see (5.49) and (5.50)), it follows that the sliding mode never happen in the loss-aversion-based noncooperative system and hence that all modes being strongly transitive is also a necessary condition for the piecewise linearized system (5.37) possessing infinitely many mode transitions. Hence, the following result is immediate.

Lemma 5.5. Consider the loss-aversion-based noncooperative system $\mathcal{G}(J)$ with the pseudo-gradient dynamics (5.5), (5.6) for $A_i x^* + b_i \neq 0$, $i \in \{1, 2\}$. Then, there exists infinitely many transitions around the Nash equilibrium x^* if and only if all of the modes are strongly transitive, i.e., $\mathbb{K}_{\mathrm{ST}} = \mathcal{K}$. Moreover, in that case, the active mode experiences all of the 4 modes and the transition sequence only includes two possibilities, which are expressed in (5.35) and (5.36).

Next, we present the conditions to identify the transitivity of mode $k \in \mathcal{K}$ to derive the set $\mathbb{K}_{\mathrm{ST}}$.

Proposition 5.5. Consider the loss-aversion-based noncooperative system $\mathcal{G}(J)$ with the pseudo-gradient dynamics (5.5), (5.6) for $A_i x^* + b_i \neq 0$, $i \in \{1, 2\}$. Let k be a mode satisfying $\lambda_1^k, \lambda_2^k \in \mathbb{R}$. If $a_{11}^1 a_{22}^2 > a_{12}^1 a_{12}^2 \geq 0$ holds, then all of the modes $\mathrm{LL}, \mathrm{HL}, \mathrm{LH}, \mathrm{HH}$ are weakly but not strongly transitive around the Nash equilibrium x^* for any $\alpha_i^{\mathrm{H}} \geq \alpha_i^{\mathrm{L}}$, $i = 1, 2$. If $a_{12}^1 a_{12}^2 < 0$ holds and $a_{12}^2(a_{12}^1 x_1^* + a_{22}^1 x_2^* + b_2^1)$, $(\alpha_1^k a_{11}^1 - \alpha_2^k a_{22}^2)(a_{11}^2 x_1^* + a_{12}^2 x_2^* + b_1^2)$ possess different signs for $k \in \{\mathrm{LL}, \mathrm{HH}\}$ (resp., $\{\mathrm{LH}, \mathrm{HL}\}$) (or, same signs for $k \in \{\mathrm{LH}, \mathrm{HL}\}$ (resp., $\{\mathrm{LL}, \mathrm{HH}\}$)), then k is strongly (resp., weakly but not strongly) transitive around x^*.

Proof: Since $a_{12}^1 a_{12}^2 < 0$ is the necessary condition for strongly transitive modes, the result for the case $a_{12}^1 a_{12}^2 \geq 0$ is a direct consequence of Proposition 5.4. Now, consider $a_{12}^1 a_{12}^2 < 0$ and suppose that η is an infinitesimal real number. In this case, a state on the eigenvector v_i^k sufficiently close to the Nash equilibrium x^* can be expressed by $\eta v_i^k - x^*$. Recalling the definition of eigenvectors in (5.48), it follows from $\mathbb{A}^k(\eta v_i^k - x^*) = \eta \lambda_i^k v_i^k$ that

$$\begin{aligned} \dot{J}_1^k(\eta v_i^k - x^*) =& \lambda_i^k \eta^2 {v_i^k}^{\mathrm{T}} A_1 v_i^k + \lambda_i^k \eta (A_1 x^* + b_1)^{\mathrm{T}} v_i^k \\ \approx& \lambda_i^k \eta (A_1 x^* + b_1)^{\mathrm{T}} v_i^k = \lambda_i^k \eta \alpha_2^k a_{12}^2 (a_{12}^1 x_1^* + a_{22}^1 x_2^* + b_2^1), \end{aligned} \tag{5.51}$$

$$\begin{aligned} \dot{J}_2^k(\eta v_i^k - x^*) =& \lambda_i^k \eta^2 {v_i^k}^{\mathrm{T}} A_2 v_i^k + \lambda_i^k \eta (A_2 x^* + b_2)^{\mathrm{T}} v_i^k \\ \approx& \lambda_i^k \eta (A_2 x^* + b_2)^{\mathrm{T}} v_i^k = \lambda_i^k \eta (\lambda_i^k - \alpha_2^k a_{22}^2)(a_{11}^2 x_1^* + a_{12}^2 x_2^* + b_1^2), \end{aligned} \tag{5.52}$$

hold for $i = 1, 2$. Note from (5.47) that

$$\lambda_1^k - \alpha_2^k a_{22}^2 = \tilde{\psi}_k - \sqrt{\psi_k^2 - \alpha_1^k \alpha_2^k (a_{11}^1 a_{22}^2 - a_{12}^1 a_{12}^2)} = \tilde{\psi}_k - \sqrt{\tilde{\psi}_k^2 + \alpha_1^k \alpha_2^k a_{12}^1 a_{12}^2}, \quad (5.53)$$

$$\lambda_2^k - \alpha_2^k a_{22}^2 = \tilde{\psi}_k + \sqrt{\psi_k^2 - \alpha_1^k \alpha_2^k (a_{11}^1 a_{22}^2 - a_{12}^1 a_{12}^2)} = \tilde{\psi}_k + \sqrt{\tilde{\psi}_k^2 + \alpha_1^k \alpha_2^k a_{12}^1 a_{12}^2}, \quad (5.54)$$

for $\tilde{\psi}_k \triangleq \frac{1}{2}(\alpha_1^k a_{11}^1 - \alpha_2^k a_{22}^2)$ and hence both the signs of (5.53) and (5.54) depend on the sign of $\tilde{\psi}_k$ with $a_{12}^1 a_{12}^2 < 0$. Hence, if $a_{12}^2(a_{12}^1 x_1^* + a_{22}^1 x_2^* + b_2^1)$ and $(\alpha_1^k a_{11}^1 - \alpha_2^k a_{22}^2)(a_{11}^2 x_1^* + a_{12}^2 x_2^* + b_1^2)$ have the same sign for $k =$ LL or HH, then there exists $\eta \neq 0$ such that (5.51), (5.52) possess the same signs for each $i = 1, 2$, which implies that both the eigenvectors v_1^k, v_2^k are containing in $\hat{\mathcal{D}}_k$ and hence mode k is weakly but not strongly transitive. Alternatively, if $a_{12}^2(a_{12}^1 x_1^* + a_{22}^1 x_2^* + b_2^1)$ and $(\alpha_1^k a_{11}^1 - \alpha_2^k a_{22}^2)(a_{11}^2 x_1^* + a_{12}^2 x_2^* + b_1^2)$ have different signs, then there is no η such that (5.51) and (5.52) possess the same signs and hence none of eigenvectors v_1^k, v_2^k is containing in $\hat{\mathcal{D}}_k$, i.e., mode k is strongly transitive. The proof for $k =$ HL or LH can be similarly handled. Thus, the proof is complete. $\square$

Remark 5.14. Noticing that λ_1^k, λ_2^k are complex conjugate for all the modes when $a_{12}^1 a_{12}^2$ is negatively small enough, it is interesting to observe that as $a_{12}^1 a_{12}^2$ increases from a negatively infinite small value to a positive value $a_{11}^1 a_{22}^2$, the transitivity of the 4 modes experiences from full strongly transitive to full non-strongly transitive.

Now, recalling the definition of normalized radial growth rate γ_{rg} in (5.39), we are ready to present the generalized result for stability.

Theorem 5.5. Consider the noncooperative system with dynamics (5.5), (5.6) for $A_i x^* + b_i \neq 0$, $i \in \{1, 2\}$ with stable subsystems (i.e., $a_{11}^1 a_{22}^2 > a_{12}^1 a_{12}^2$). If some of the modes are weakly but not strongly transitive, i.e., $\mathbb{K}_{\text{ST}} \subset \mathcal{K}$, then the Nash equilibrium x^* is asymptotically stable. Alternatively, if all the modes are strongly transitive, i.e., $\mathbb{K}_{\text{ST}} = \mathcal{K}$, and in addition, if $a_{12}^1 \gamma_{\text{rg}} > 0$ and $a_{12}^2 \gamma_{\text{rg}} < 0$ (resp., $a_{12}^1 \gamma_{\text{rg}} < 0$ and $a_{12}^2 \gamma_{\text{rg}} > 0$) hold, then the Nash equilibrium x^* in dynamics (5.5), (5.6) is asymptotically stable (resp., unstable).

Proof: The result is a direct consequence of Proposition 5.4 and Proposition 5.5. $\square$

We present the robust stability condition for the uncertain loss-averse parameters $\alpha_i^{\text{H}} \geq \alpha_i^{\text{L}}$ as follows.

Corollary 5.1. Consider the noncooperative system with dynamics (5.5), (5.6) for $A_i x^* + b_i \neq 0$, $i \in \{1, 2\}$. If $a_{11}^1 a_{22}^2 > a_{12}^1 a_{12}^2 \geq 0$ holds, then the Nash equilibrium x^* is asymptotically stable for any $\alpha_i^{\text{H}} \geq \alpha_i^{\text{L}}$, $i = 1, 2$.

Proof: The result is a direct consequence of Theorem 5.5 and Proposition 5.4 by noting that $\mathbb{K}_{\mathrm{ST}} = \emptyset$ under $a_{12}^1 a_{12}^2 \geq 0$. □

Unstable Subsystems

In this section, we consider the case with unstable subsystems, i.e., $a_{11}^1 a_{22}^2 - a_{12}^1 a_{12}^2 < 0$. In this case, the eigenvalues λ_1^k, λ_2^k must be real and satisfy $\lambda_1^k < 0$ and $\lambda_2^k > 0$ (see Proposition 2.3 in Chapter 2) and hence for the subsystem under mode k there exists a stable manifold and an unstable manifold characterized by the eigenvectors v_1^k and v_2^k respectively. Here, we notice that the analysis in terms of the approximated domains in Proposition 5.3 still hold. Similar to the previous section, we present the condition to determine the transitivity properties of the 4 modes and then show the stability.

Before we present a proposition, we note that the mode $k \in \mathcal{K}$ is strongly transitive if no eigenvectors are containing in $\hat{\mathcal{D}}_k$; the mode k is weakly but not strongly transitive if only the stable eigenvector (i.e., v_1^k) is containing in $\hat{\mathcal{D}}_k$; the mode k is non-weakly transitive if the unstable eigenvector (i.e., v_2^k) is containing in $\hat{\mathcal{D}}_k$.

Proposition 5.6. Consider the loss-aversion-based noncooperative system $\mathcal{G}(J)$ with the pseudo-gradient dynamics (5.5), (5.6) for $A_i x^* + b_i \neq 0$, $i \in \{1, 2\}$. If $a_{11}^1 a_{22}^2 < a_{12}^1 a_{12}^2$ holds, then none of the modes LL, HL, LH, HH is strongly transitive around the Nash equilibrium x^*. If, in addition, $a_{12}^2(a_{12}^1 x_1^* + a_{22}^1 x_2^* + b_2^1)$ and $a_{11}^2 x_1^* + a_{12}^2 x_2^* + b_1^2$ possess different (resp., same) signs, then the modes LL, HH are weakly transitive but LH, HL are non-weakly transitive (resp., LH, HL are weakly transitive but LL, HH are non-weakly transitive).

Proof: First, the result that one of the modes LL, HL, LH, HH is strongly transitive around the Nash equilibrium x^* is a direct consequence of Proposition 5.4 since $a_{12}^1 a_{12}^2 > a_{11}^1 a_{22}^2 > 0$. Next, it follows from $\lambda_2^k - \alpha_2^k a_{22}^2 > 0$, $k \in \mathcal{K}$, that the variables (5.51) and (5.52) with $i = 2$ possess the same (resp., different) signs for all modes $k \in \mathcal{K}$ when $a_{12}^2(a_{12}^1 x_1^\star + a_{22}^1 x_2^\star + b_2^1)$ and $a_{11}^2 x_1^\star + a_{12}^2 x_2^\star + b_1^2$ have the same (resp., different) sign. Thus, when $a_{12}^2(a_{12}^1 x_1^\star + a_{22}^1 x_2^\star + b_2^1)$ and $a_{11}^2 x_1^\star + a_{12}^2 x_2^\star + b_1^2$ possess the different signs, it follows that the unstable manifold characterized by the unstable eigenvector v_2^k is containing in $\hat{\mathcal{D}}_k$ only for $k =$ LH, HL and hence the modes LH, HL (resp., LL, HH) are non-weakly transitive and (resp., weakly transitive). Alternatively, when $a_{12}^2(a_{12}^1 x_1^\star + a_{22}^1 x_2^\star + b_2^1)$ and $a_{11}^2 x_1^\star + a_{12}^2 x_2^\star + b_1^2$ possess the same signs, it follows that the unstable manifold characterized by the unstable eigenvector v_2^k is containing in $\hat{\mathcal{D}}_k$ only for the modes $k =$ LL, HH and hence the modes LL, HH (resp., LH, HL) are non-weakly transitive and (resp., weakly transitive). □

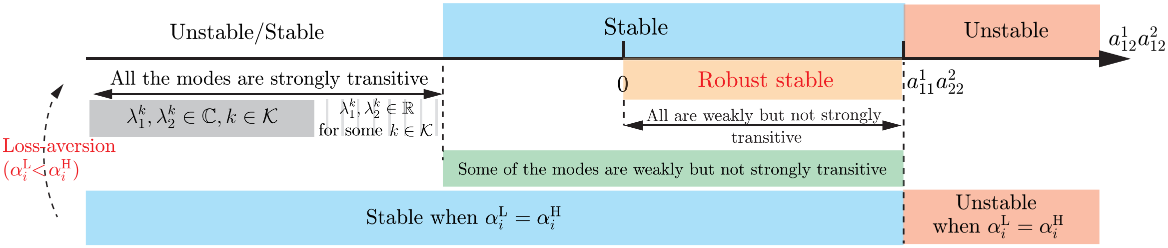


Figure. 5.10 A diagram showing the stability change and mode transitivity change along with the change of $a^1_{12}a^2_{12}$.

Then, the following statement is immediate.

Theorem 5.6. Consider the noncooperative system with dynamics (5.5), (5.6) for $A_i x^* + b_i \neq 0$, $i \in \{1, 2\}$, with unstable subsystems (i.e., $a^1_{11}a^2_{22} < a^1_{12}a^2_{12}$). Then, the Nash equilibrium x^* is unstable for any $\alpha^{\rm H}_i \geq \alpha^{\rm L}_i$, $i = 1, 2$.

Proof: The result is a direct consequence of Proposition 5.6. □

Remark 5.15. Theorem 5.6 reveals the fact that loss-aversion behavior are *not able to stabilize* the unstable two-agent noncooperative system.

Discussion

In this subsection, we show a diagram summarizing the stability results derived from the above 2 subsections in Fig. 5.10, which can be seen as a bifurcation diagram illustrating the stability change and the mode transitivity change along with the change of the bifurcation parameter $a^1_{12}a^2_{12}$. It turns out that with the change of the value $a^1_{12}a^2_{12}$, loss-aversion behavior may or may not destabilize the Nash equilibrium x^*.

First, recall that the eigenvalues λ^k_1, λ^k_2 defined in (5.47) are complex conjugate and hence all of the 4 modes are strongly transitive for all the modes when $a^1_{12}a^2_{12}$ is negatively small enough. In this case, loss-aversion behavior destabilizes the Nash equilibrium x^* when the integral of normalized radial growth rate $\gamma_{\rm rg}$ defined in (5.39) satisfies $a^1_{12}\gamma < 0$ and $a^2_{12}\gamma > 0$ (see Theorem 5.5). Then, as the value of $a^1_{12}a^2_{12}$ increases, the eigenvalues λ^k_1, λ^k_2 tune to real for some modes $k \in \mathcal{K}$ and some of the modes change to be weakly (but not strongly) transitive. In this case, loss-aversion behavior does not destabilize the Nash equilibrium x^*. When $a^1_{12}a^2_{12}$ increases to a positive value, i.e., $a^1_{12}a^2_{12} > 0$, then all of the 4 modes tune to weakly but not strongly transitive and hence the Nash equilibrium x^* is robust stability for any loss-averse parameters $\alpha^{\rm H}_i \geq \alpha^{\rm L}_i$ (see Corollary 5.1). In this case, loss-aversion behavior never destabilize the Nash equilibrium x^* for any loss-averse parameters $\alpha^{\rm H}_i \geq \alpha^{\rm L}_i$. Next, as

$a_{12}^1 a_{12}^2$ increases to a positive value larger than $a_{11}^1 a_{22}^2$ so that all of the subsystems are unstable, it turns out that two of the four modes are non-weakly transitive and hence the Nash equilibrium x^* in unstable for any $\alpha_i^{\mathrm{H}} \geq \alpha_i^{\mathrm{L}}$ (see Theorem 5.6). In this case, it can be seen from the result that loss-aversion behavior can not bring stabilization.

5.6 Illustrative Numerical Examples

In this section, we provide a couple of numerical examples in order to validate the results in the paper.

Example 5.1. Consider the noncooperative system $\mathcal{G}(J)$ with $A_1 = \begin{bmatrix} -2 & -4 \\ -4 & -9 \end{bmatrix}$, $A_2 = \begin{bmatrix} -6 & 3 \\ 3 & -2 \end{bmatrix}$, $b_1 = [-10, -5]^{\mathrm{T}}$, $b_2 = [30, -25]^{\mathrm{T}}$, $c_1 = 162.47, c_2 = 0$, where the Nash equilibrium $x^* = [5, -5]^{\mathrm{T}}$ satisfies $A_i x^* + b_i \neq 0, i \in \{1, 2\}$ (*Case 1*). Letting $\alpha_1^{\mathrm{L}} = \alpha_2^{\mathrm{L}} = 1$, $\alpha_1^{\mathrm{H}} = 2$, $\alpha_2^{\mathrm{H}} = 3$, Assumption 5.1 is satisfied. Figure 5.11 shows the curves of $\dot{J}_i^k(x) = 0, i = 1, 2$, for all the modes $k \in \mathcal{K}$. In this case, $\mathbb{A}_k^{\mathrm{T}} + \mathbb{A}_k < 0$ for all $k \in \mathcal{K}$, $\theta \in [0, 2\pi]$, and hence the normalized radial growth rates $\rho_k(\theta) < 0, k \in \mathcal{K}$, imply $\gamma_{\mathrm{rg}} < 0$. Hence, it follows from Theorem 5.5 that the Nash equilibrium x^* is asymptotically stable, which can be verified by the trajectories of states and payoffs shown in Figs. 5.12 and 5.13.

Example 5.2. Consider the noncooperative system $\mathcal{G}(J)$ with $A_1 = \begin{bmatrix} -2 & 4 \\ 4 & -10 \end{bmatrix}$, $A_2 = \begin{bmatrix} -10 & -4 \\ -4 & -2 \end{bmatrix}$, $b_1 = b_2 = [0, 0]^{\mathrm{T}}$, $c_1 = c_2 = 0$, where the Nash equilibrium $x^* = [0, 0]^{\mathrm{T}}$ satisfies $A_i x^* + b_i = 0, i \in \{1, 2\}$ (*Case 2*). Suppose that $\alpha_1^{\mathrm{L}} = \alpha_1^{\mathrm{H}} = 6$, $\alpha_2^{\mathrm{L}} = \alpha_2^{\mathrm{H}} = 9$ for representing the case where the agents are not loss-averse. Then, the eigenvalues of $\mathbb{A}_{\mathrm{LL}} = \mathbb{A}_{\mathrm{HL}} = \mathbb{A}_{\mathrm{LH}} = \mathbb{A}_{\mathrm{HH}}$ are given by $-15.0 \pm 29.2i$, which imply that the Nash equilibrium is stable with the identical subsystem dynamics for all the modes. Now, suppose that both agents are loss-averse and let $\alpha_1^{\mathrm{L}} = \alpha_2^{\mathrm{L}} = 1$, then the eigenvalues of $\mathbb{A}_k$, $k = \mathrm{LL}, \mathrm{HL}, \mathrm{LH}, \mathrm{HH}$, are respectively given by $-1.0 \pm 4.0i$, $-7.0 \pm 8.4i$, $-10.0 \pm 8.9i$, and $-15.0 \pm 29.2i$ so that $\mathbb{A}_k, k \in \mathcal{K}$, are still all stable matrices. Figure 5.14(a) shows the domains of $\hat{\mathcal{D}}_k, k \in \mathcal{K}$. Noting that $a_{12}^1 > 0$ and $a_{12}^2 < 0$, it follows from Lemma 5.2 that the rotational direction is clockwise. Hence, we re-partition the state space from $\mathcal{D}_k$, $k \in \mathcal{K}$, to identify the effective domains $\tilde{\mathcal{D}}_k$, $k \in \mathcal{K}$, as shown in Fig. 5.14(b), and hence derive the function of $K(\theta)$. Note that the integral of the normalized radial growth rate is $\gamma_{\mathrm{rg}} = \int_0^{2\pi} \rho_{K(\theta)}(\theta) \mathrm{d}\theta = 2 \int_0^{\pi} \rho_{K(\theta)}(\theta) \mathrm{d}\theta = -0.3224 < 0$. Hence, it follows

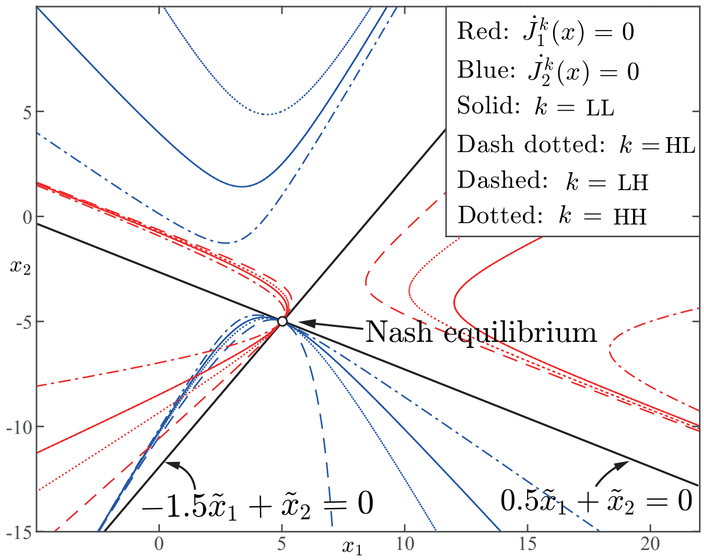


Figure. 5.11 The curves of $\dot{J}_i^k(x) = 0$, $i \in \{1, 2\}$, $k \in \mathcal{K}$, in Example 5.1.

from Theorem 5.2 that the Nash equilibrium is unstable since $a_{12}^1\gamma_{\rm rg} < 0$ and $a_{12}^2\gamma_{\rm rg} > 0$ even though all the subsystem matrices are stable. The result of Lemma 5.2 and Theorem 5.2 can be verified from the trajectories of states and payoff values shown in Figs. 5.15 and 5.16.

Example 5.3. Consider the noncooperative system $\mathcal{G}(J)$ with $A_1 = \begin{bmatrix} -2 & 4 \\ 4 & 14 \end{bmatrix}^{\rm T}$, $A_2 = \begin{bmatrix} 14 & -4 \\ -4 & -2 \end{bmatrix}$, $b_1 = [0, -20]^{\rm T}$, $b_2 = [0, 0]^{\rm T}$, $c_1 = c_2 = 0$, where the Nash equilibrium $x^* = [0, 0]^{\rm T}$ satisfies $A_1x^* + b_1 \neq 0, A_2x^* + b_2 = 0$ (*Case 3*). Letting $\alpha_1^{\rm L} = 2$, $\alpha_2^{\rm L} = 1$, $\alpha_1^{\rm H} = 4$, $\alpha_2^{\rm H} = 3$, the eigenvalues of $\mathcal{A}_k$, $k = $ LL, HL, LH, HH, are respectively given by $-3.0 \pm 5.6i$, $-5.0 \pm 7.4i$, $-5.0 \pm 9.4i$, and $-7.0 \pm 13.8i$. The domains $\mathcal{D}_k, k \in \mathcal{K}$, the approximated domains $\hat{\mathcal{D}}_k, k \in \mathcal{K}$, and the effective domains $\tilde{\mathcal{D}}_k$, $k \in \mathcal{K}$, are already shown in Fig. 5.7(a), (b), (c), respectively. Figure 5.17 shows the the normalized radial growth rate $\rho_{K(\theta)}(\theta)$, $\theta \in [0, 2\pi]$. Note that the integral of the normalized radial growth rate is

$$\gamma_{\rm rg} = \int_0^{2\pi} \rho_{K(\theta)}(\theta) {\rm d}\theta = 0.9520 > 0. \tag{5.55}$$

Hence, it follows from Theorem 5.4 that the Nash equilibrium is stable since $a_{12}^1\gamma_{\rm rg} > 0$ and $a_{12}^2\gamma_{\rm rg} < 0$. The result of Theorem 5.4 can be verified from the trajectories of states and payoff values shown in Figs. 5.18 and 5.19.

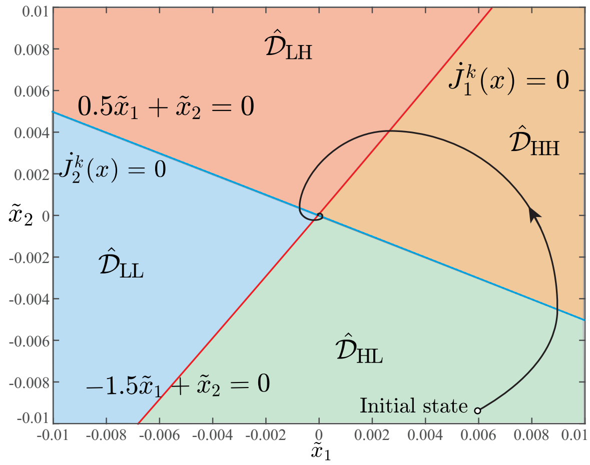


Figure. 5.12 The approximated domains $\hat{\mathcal{D}}_k$, $k \in \mathcal{K}$, and an orbit with $\tilde{x} = x - x^*$, in Example 5.1.

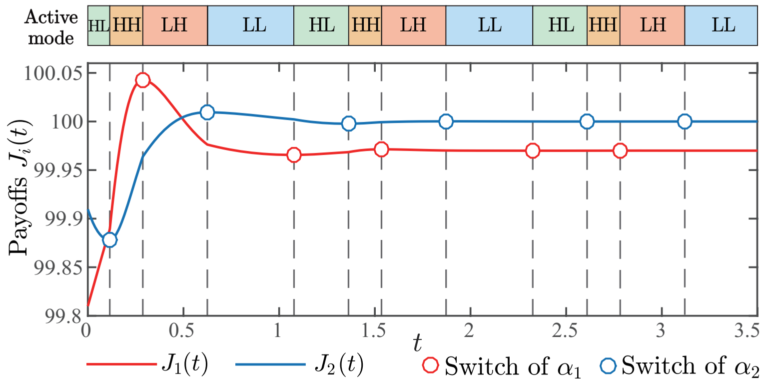


Figure. 5.13 Agents' payoffs versus time in Example 5.1.

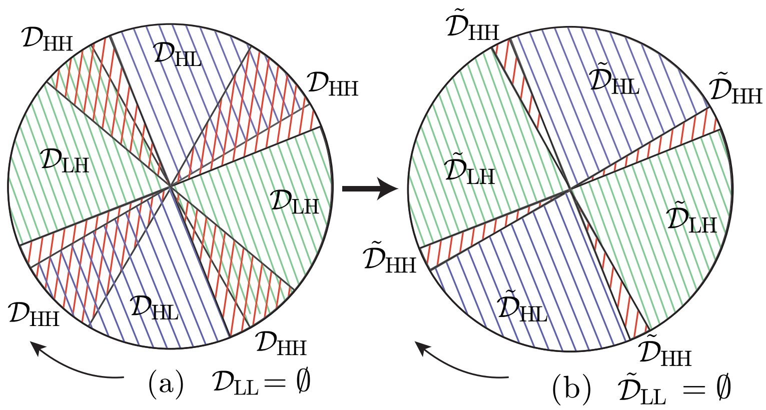


Figure. 5.14 The domains of $\mathcal{D}_k$ and $\tilde{\mathcal{D}}_k$, $k \in \mathcal{K}$, in Example 5.2. (a): $\mathcal{D}_k$, $k \in \mathcal{K}$, (b): $\tilde{\mathcal{D}}_k$, $k \in \mathcal{K}$ (obtained from (a) with clockwise rotational direction) from which $K(\theta)$ is determined.

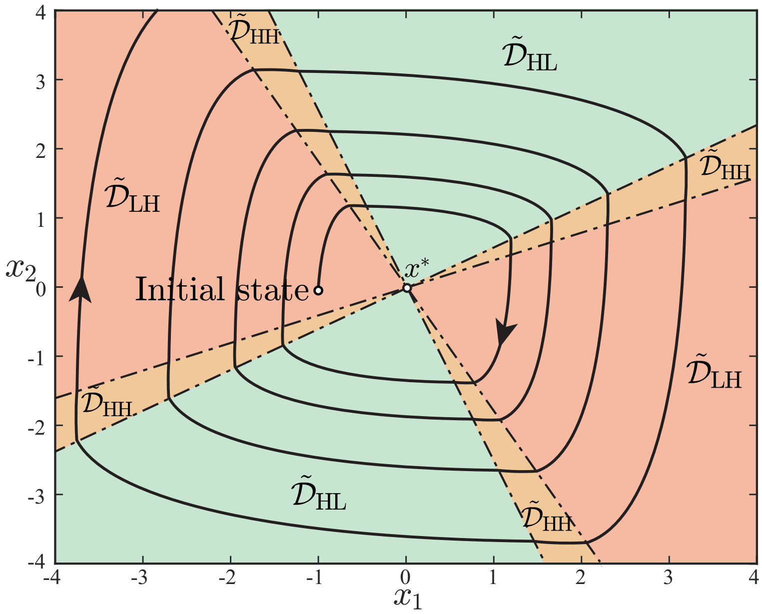


Figure. 5.15 The effective domains $\tilde{\mathcal{D}}_k$, $k \in \mathcal{K}$, and an orbit in Example 5.2.

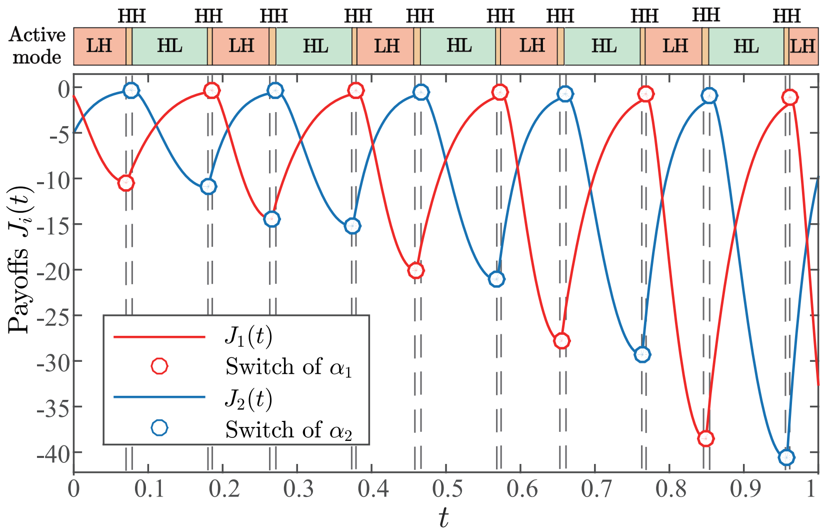


Figure. 5.16 Agents' payoffs versus time in Example 5.2.

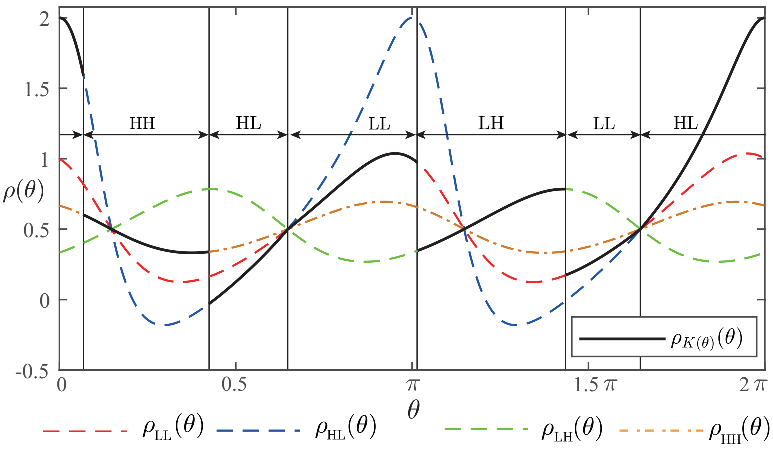


Figure. 5.17 Normalized radial growth rates $\rho_k(\theta)$, $k \in \mathcal{K}$, and $\rho_{K(\theta)}(\theta)$, $\theta \in [0, 2\pi]$, in Example 5.3.

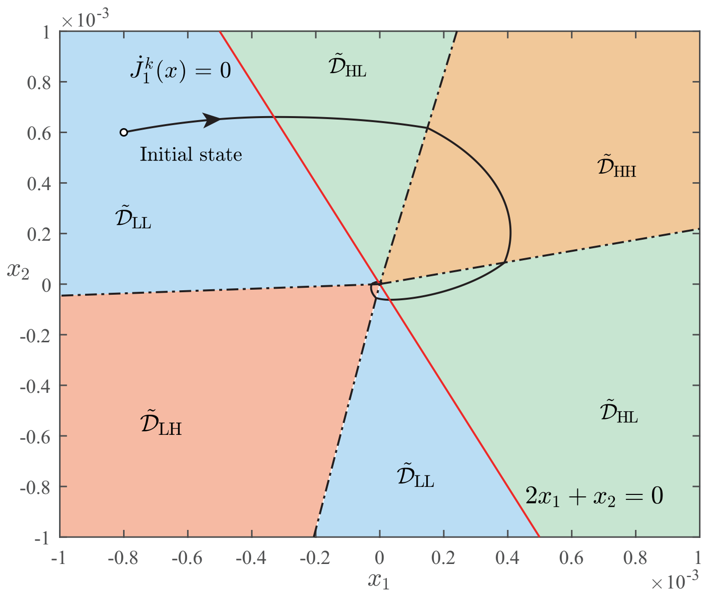


Figure. 5.18 The effective domains $\tilde{\mathcal{D}}_k$, $k \in \mathcal{K}$, and an orbit in Example 5.3.

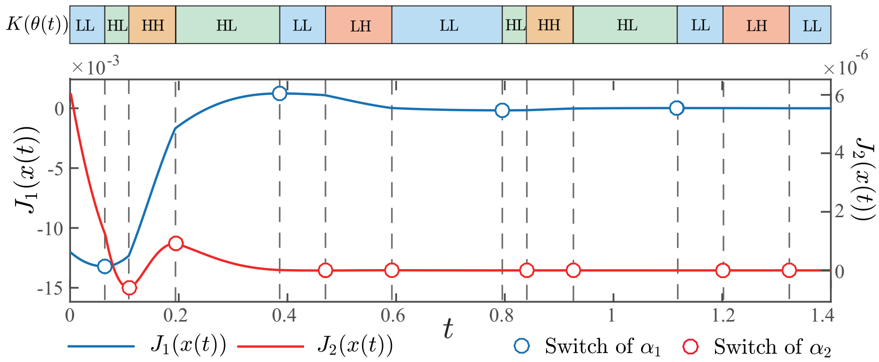


Figure. 5.19 Agents' payoffs versus time in Example 5.3.

Example 5.4. Consider a noncooperative system with $A_1 = \begin{bmatrix} -4 & 0.2 \\ 0.2 & 10 \end{bmatrix}$, $A_2 = \begin{bmatrix} -4.2 & -0.2 \\ -0.2 & -4 \end{bmatrix}$, $b_1 = [0, 2005]^{\mathrm{T}}$, $b_2 = [2000, 0]^{\mathrm{T}}$, $c_1 = c_2 = 0$, where $a^1_{11}a^2_{22} > a^1_{12}a^2_{12}$ holds and the Nash equilibrium $x^* = [0, 0]^{\mathrm{T}}$ satisfies $A_i x^* + b_i \neq 0$, $i \in \{1, 2\}$.

Let $\alpha_1^{\mathrm{L}} = 3$, $\alpha_1^{\mathrm{H}} = 5$, $\alpha_2^{\mathrm{L}} = 4$, $\alpha_2^{\mathrm{H}} = 6$ so that $\lambda_1^k, \lambda_2^k \in \mathbb{R}$ hold for $k \in \mathcal{K}$. In this case, it follows from Proposition 5.4 that at least one of the 4 modes is weakly but not strongly transitive and the Nash equilibrium x^* is asymptotically stable. Furthermore, it follows from Theorem 5.5 that the modes $\mathrm{HH}, \mathrm{LL}, \mathrm{HL}$ are strongly transitive but mode LH is weakly but not strongly transitive. Figure 5.15 shows the approximated cones $\hat{\mathcal{D}}_k$, $k \in \mathcal{K}$. The eigenvectors of the 4 modes are shown as colored lines in Fig. 5.15 where the dashed (resp., solid) lines denote the eigenvectors satisfying $v_i^k \notin \hat{\mathcal{D}}_k$ (resp., $v_i^k \in \hat{\mathcal{D}}_k$). It can be seen from the figure that only the mode LH is weakly but not strongly transitive since the eigenvectors v_1^k, v_2^k are containing in the domain $\hat{\mathcal{D}}_k$ only for $k = \mathrm{LH}$. The trajectories of agents' sensitivity parameters and payoff values under the initial state $x_0 = [4, 1]^{\mathrm{T}}$ are shown in Fig. 5.21. It can be seen from the figure that the mode is changed from LL to HL, HH, LH and never changed after LH, and hence the modes $\mathrm{HH}, \mathrm{LL}, \mathrm{HL}$ are strongly transitive but mode LH is weakly but not strongly transitive, which verifies Theorems 5.5, 5.5 and Proposition 5.4.

5.7 Chapter Conclusion

In this chapter, we investigated the stability conditions of the noncooperative switched systems with loss-averse agents, where each agent under pseudo-gradient dynamics exhibits lower sensitivity for the cases of losing payoffs. We characterized the notion

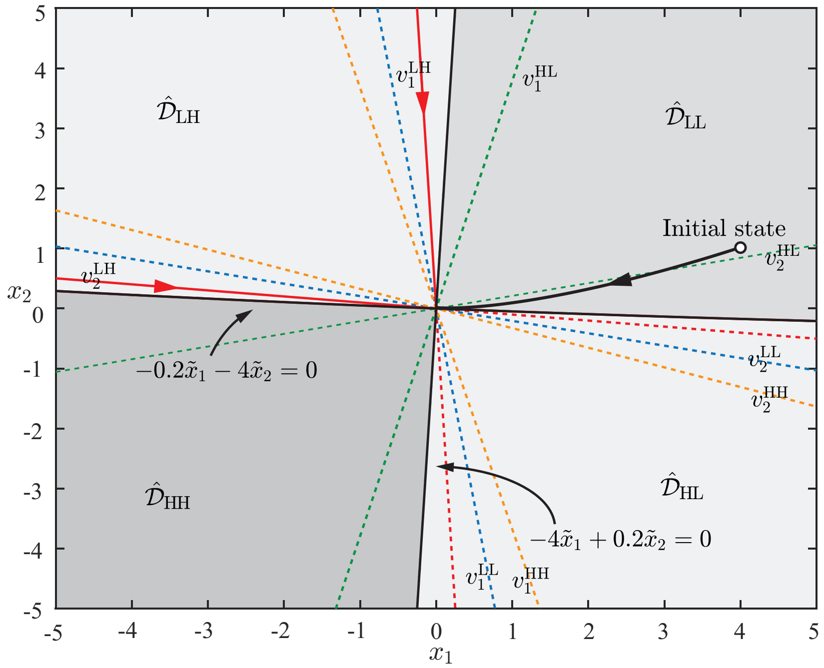


Figure. 5.20 Phase portrait with an orbit where $-0.2\tilde{x}_1 - 4\tilde{x}_2 = 0$ and $-4\tilde{x}_1 + 0.2\tilde{x}_2 = 0$ are the best response lines for agents 2 and 1.

of the flash switching phenomenon and examined stability properties in accordance with the location of the Nash equilibrium for 3 cases. We revealed how the sensitivity parameters influence the stability property of the system in terms of the dynamics, partition of the state space, mode transition, and the normalized radial growth rate for each of the 3 cases. One of the illustrative examples indicates that loss-aversion behavior inspired by psychological consideration in prospect theory may result in changing the stability property of the Nash equilibrium from stable to unstable.

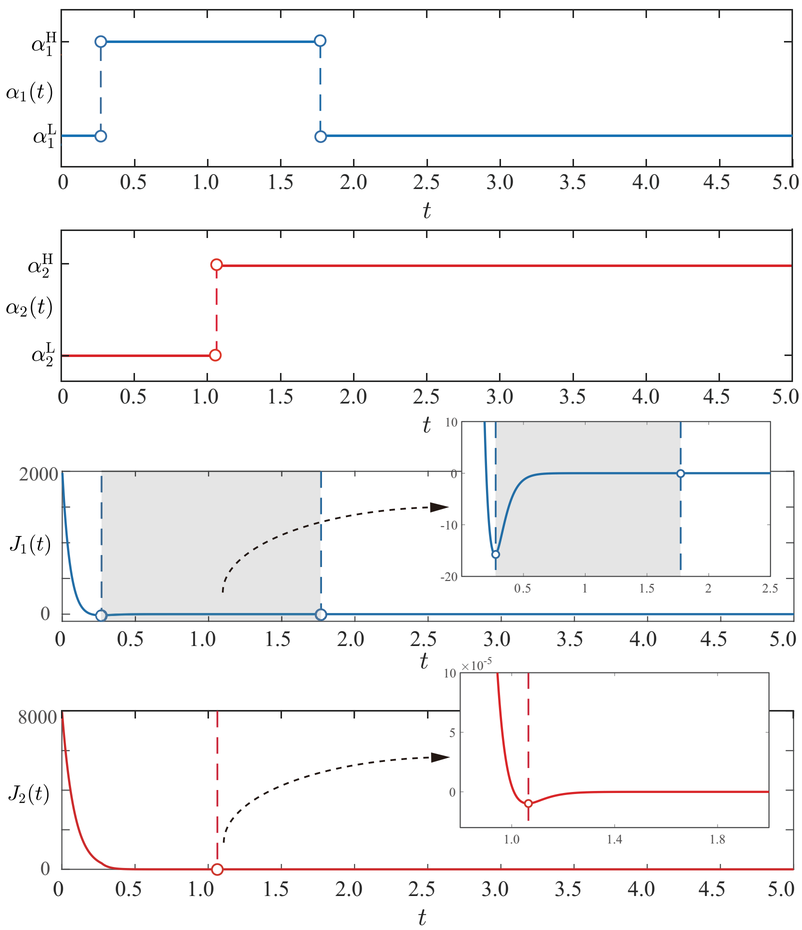


Figure. 5.21 Agents' sensitivities and payoff values versus time.

Chapter 6

Incorporation of Predictions of Other Agents' Behavior into Pseudo-Gradient Dynamics

6.1 Introduction

In this chapter, we we connect cognitive hierarchy theory with the pseudo-gradient dynamics in noncooperative systems to extend the pseudo-gradient dynamics with some prediction behaviors under Level-k thinking. In the characterized framework, all the agents are allowed to base their decisions on the predictions about the likely actions of other agents with a bounded depth of reasoning. Each agent believes that he/she is the most sophisticated person in the noncooperative system and makes the decision according to some strategic reasoning of the other agents' likely action. Depending on a knowledge network of payoff functions, the modified pseudo-gradient dynamics are presented under the assumption that the agents may be able to reasoning the other agents' best-response states and use these predicted states in the pseudo-gradient dynamics. Some sufficient conditions are presented to guarantee stability of a Nash equilibrium with uncertain sensitivity parameters or uncertain knowledge network. The transition of the agents' target state while increasing the depth of reasoning for a two-agent noncooperative system with quadratic payoff functions is characterized. We present the applications of our results to optical communication systems, homogeneous oligopoly markets and differentiated oligopoly markets. Our result indicates that to ensure asymptotic stability of the differentiated oligopoly markets with Cournot competition, a larger market with more firms requires more differentiated products.

6.2 Problem Formulation

6.2.1 Conventional Pseudo-Gradient Dynamics

Consider the noncooperative system $\mathcal{G}(J)$ defined in Chapter 2 with n number of agents, where the set of overall agents is denoted by $\mathcal{N} = \{1, \ldots, n\}$ and the agents are playing noncooperative games. Let $x = [x_1, \ldots, x_n]^{\mathrm{T}} = (x_i, x_{-i}) \in \mathbb{R}^n$ denote the state profile of all the agents, where $x_i \in \mathbb{R}$ and $x_{-i} \in \mathbb{R}^{n-1}$ denote the state of agent i and the state profile except agent i, respectively. Recall from Chapter 2 that the conventional pseudo-gradient dynamics given by

$$\dot{x}_i(t) = \alpha_i \frac{\partial J_i(x(t))}{\partial x_i}, \quad i \in \mathcal{N}, \tag{6.1}$$

with $\alpha_i \in \mathbb{R}_+$, $i \in \mathcal{N}$, capture the fact that the agents concern their own payoffs (without taking into account the other agents' payoffs) and myopically change their states without any foresight on the future state. In this case, each agent's best-response state $\mathrm{BR}_i(x_{-i}(t))$ corresponds to the largest value of J_i given the state profile $x_{-i}(t)$ and satisfies $\frac{\partial J_i(\mathrm{BR}_i(x_{-i}(t)), x_{-i}(t))}{\partial x_i} = 0$ under Assumption 2.1. Therefore, the best-response state $\mathrm{BR}_i(x_{-i}(t))$ for agent i, which does not depend on $x_i(t)$, is *understood* as the target state of agent i at time t. For example, Fig. 6.1(a) shows the case of 2 agents where the targeted best-response state $\mathrm{BR}_2(x_1(t))$ for agent 2 is greater than $x_2(t)$ so that agent 2 tries to improve its payoff function $J_2(x(t))$ by moving upward in the phase space. Under Assumption 2.1, since the Nash equilibrium x^* satisfies (2.3), it follows that

$$\frac{\partial J_i(x^*)}{\partial x_i} = 0, \quad i \in \mathcal{N}, \tag{6.2}$$

which imply $\dot{x}(t) = 0$ at the Nash equilibrium x^* for the conventional pseudo-gradient dynamics (6.1).

6.2.2 Prediction-Incorporated Pseudo-Gradient Dynamics

Involving *Level-k thinking* from cognitive hierarchy theory into the noncooperative system $\mathcal{G}(J)$, we consider a scenario where some of the agents may base their decisions on the predictions about the likely actions of other agents. To establish predictions of such likely actions, it is important to notice that the information of the payoff functions of these agents are essential for the agents. In this paper, we suppose that the agents know the payoff functions of *part* of the overall agents so that the agents can predict the behavior of these agents. Here, we characterize the relation of the possession of

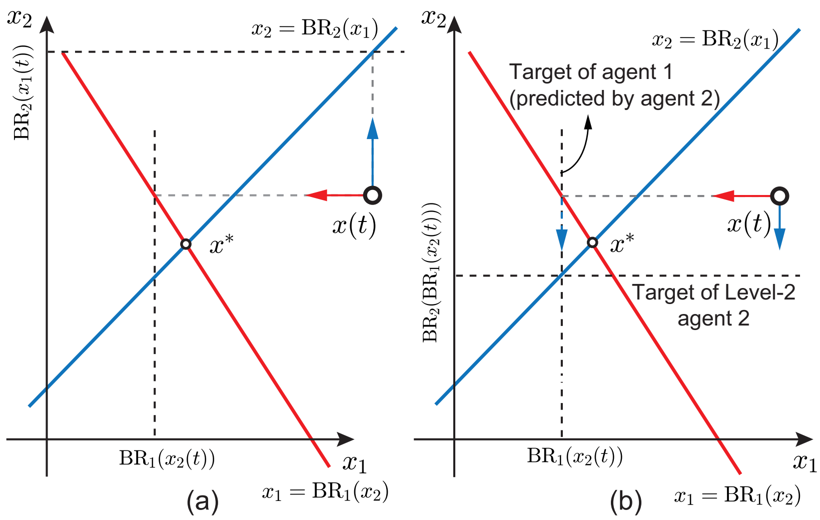


Figure. 6.1 Moving directions of x_2 of a two-agent noncooperative system with red (resp., blue) arrow representing the moving direction of agent 1 (resp., agent 2). (a): both agents are Level-1, (b): agent 1 is Level-1 but agent 2 is Level-2. Agent 2 believes that agent 1 is targeting on $\mathrm{BR}_1(x_2(t))$ and hence the moving direction of its state is opposite in comparison to Case (a).

the payoff information among the agents by defining a directed graph (termed as the knowledge network) as explained below.

Knowledge network of payoff functions: Let the knowledge network be represented by a directed graph $G(\mathcal{N}, E)$, where $E \subseteq \{(j,i) \in \mathcal{N}^2 : i \neq j\}$ denotes the set of edges of the graph. The edge (j,i) directed from agent j to i indicates that agent i can obtain the information of the payoff function $J_j(\cdot)$. The neighbor set of agent i representing the set of agents whose payoff functions are known to agent i is denoted by $\mathcal{N}_i \triangleq \{j \in \mathcal{N} : (j,i) \in E\}$. Among these agents in $\mathcal{N}_i$, the set of agents with the edges also directed from agent i is denoted by $\mathbb{N}_i^{\mathrm{pr1}} \triangleq \{j \in \mathcal{N}_i : (i,j) \in E\}$, whereas the set of the rest of the other neighbor agents is denoted by $\mathbb{N}_i^{\mathrm{pr2}} \triangleq \{j \in \mathcal{N}_i : (i,j) \notin E\}$ satisfying $\mathcal{N}_i = \mathbb{N}_i^{\mathrm{pr1}} \cup \mathbb{N}_i^{\mathrm{pr2}}$. For example, for the knowledge network $G(\mathcal{N}, E)$ shown in Fig. 6.2, $\mathcal{N}_2 = \{1, 4\}$ holds with $\mathbb{N}_2^{\mathrm{pr1}} = \{1\}$ and $\mathbb{N}_2^{\mathrm{pr2}} = \{4\}$. This decomposition of $\mathbb{N}_i$ is necessary when agents become sophisticated as explained later. Furthermore, the adjacency matrix for $G(\mathcal{N}, E)$ is defined as $A_{\mathrm{ad}} = [a_{ij}] \in \mathbb{R}^{n \times n}$, where $a_{ij} = 1$ if $j \in \mathcal{N}_i$, and $a_{ij} = 0$ otherwise.

Assumption 6.1. The knowledge network is not a public information, i.e., only the connections of $(j,i) \in E$, $j \in \mathcal{N}$, and $(i,j) \in E$, $j \in \mathcal{N}$, associated with agent i can be known to agent i.

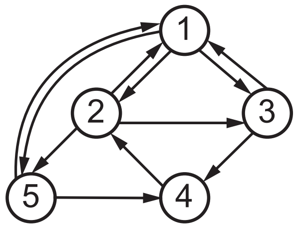


Figure. 6.2 A knowledge network of payoff functions $G(\mathcal{N}, E)$. The arrows represent the fact that the agent at the destination node knows the payoff function of the agent at the start node.

For the following statements, we denote the set of Level-k agents that we define below by $\mathbb{L}_k \subseteq \mathcal{N}$, $k = 1, 2, \ldots$.

Level-1 agent: Without any prediction, agent $i \in \mathbb{L}_1$ who tries to improve $J_i(x_i, x_{-i})$ by adjusting its state x_i towards the targeted best-response state $\mathrm{BR}_i(x_{-i})$ based on the current state of the other agents x_{-i} is referred to as a Level-1 agent. The dynamic decision process of Level-1 agents is given by

$$\dot{x}_i(t) = \alpha_i \frac{\partial J_i(x_i(t), x_{-i}(t))}{\partial x_i}, \quad i \in \mathbb{L}_1, \tag{6.3}$$

which is equivalent to the conventional pseudo-gradient dynamics (6.1).

Level-2 agent: A more sophisticated agent $i \in \mathbb{L}_2$ believes in the hypothesis that all the other agents $j \in \mathcal{N} \setminus \{i\}$ are Level-1 following the pseudo-gradient dynamics (6.3) and hence targeting on their own best-response states $\mathrm{BR}_j(x_{-j})$ (even though this hypothesis is not true in reality). Therefore, agent i tries to adjust its state x_i following the pseudo-gradient dynamics as if its neighbor agents $j \in \mathcal{N}_i$ were already at their targeted best-response states $\mathrm{BR}_j(x_{-j})$ instead of being based on the current states x_j. Those targeted best-response states $\mathrm{BR}_j(x_{-j})$, $j \in \mathcal{N}_i$, are regarded as the predicted states of the agents from agent i's point of view. For the other agents $j \in \mathcal{N} \setminus \{i, \mathcal{N}_i\}$, since agent i does not possess the knowledge of their payoff functions, agent i cannot predict their targeted best-response states and hence relies on the current states $x_{-i}^{\mathrm{un}} \triangleq \{x_j\}_{j \in \mathcal{N} \setminus \{i, \mathcal{N}_i\}}$. Consequently, the pseudo-gradient dynamics for Level-2 agents are given by

$$\dot{x}_i(t) = \alpha_i \frac{\partial J_i(x_i(t), x_{-i}^{\mathrm{pr}}(t), x_{-i}^{\mathrm{un}}(t))}{\partial x_i}, \quad i \in \mathbb{L}_2, \tag{6.4}$$

where $x_{-i}^{\mathrm{pr}} \triangleq \{\mathrm{BR}_j(x_{-j})\}_{j \in \mathcal{N}_i}$ denotes the predicted states of the neighbor agents calculated by agent i. In the case where agent $i \in \mathbb{L}_2$ has no access to the information of any other agent's payoff function, i.e., $\mathcal{N}_i = \emptyset$, it follows that (6.4) reduces to (6.3) because no prediction can be made. For this reason, agent i can act as a Level-2 agent only if at least one of the other agents' payoff functions is known (i.e., $\mathcal{N}_i \neq \emptyset$).

An example of the moving direction of the state $x(t)$ with $n = 2$, $\mathbb{L}_1 = \{1\}$, $\mathbb{L}_2 = \{2\}$ and $E = \{(1,2),(2,1)\}$ is elaborated in Fig. 6.1(b), whereas the case with $\mathbb{L}_1 = \{1,2\}$ (which corresponds to the conventional pseudo-gradient dynamics) is shown in Fig. 6.1(a).

Predictions under cognitive reasoning: It is important to notice that the calculation of the predicted states x_{-i}^{pr} of the neighbor agents for agent $i \in \mathbb{L}_k$ with $k \geq 3$ is not as simple as the one defined in (6.4) because the neighbor agents in $\mathbb{N}_i^{\mathrm{pr1}}$ and $\mathbb{N}_i^{\mathrm{pr2}}$ in $\mathcal{N}_i$ should be separately treated with different times of iterations of cognitive reasoning. Specifically, agent i believes that the neighbor agents in $\mathbb{N}_i^{\mathrm{pr1}}$ are making cognitive operations (predictions) about the likely targeted best-response state of agent i and hence tries to be more sophisticated than these agents with multiple cognitive operations for calculating the predicted state. However, agent i does not need to make multiple cognitive operations to the rest of the neighbor agents in $\mathbb{N}_i^{\mathrm{pr2}}$ since they are impossible to predict the targeted best-response state of agent i. In this paper, the predicted state of agent $j \in \mathbb{N}_i^{\mathrm{pr2}}$ calculated by agent $i \in \mathbb{L}_k$ is defined with only one iteration of $\mathrm{BR}_j(x_{-j})$, which has the same expression as in x_{-i}^{pr} of (6.4).

Denoting the profile of the predicted states of the neighbor agents in $\mathbb{N}_i^{\mathrm{pr1}}$ and $\mathbb{N}_i^{\mathrm{pr2}}$ for agent $i \in \mathbb{L}_k$ by x_{-i}^{pr1} and $x_{-i}^{\mathrm{pr2}} = \{\mathrm{BR}_j(x_{-j})\}_{j \in \mathbb{N}_i^{\mathrm{pr2}}}$, respectively, we emphasize that the actual targeted best-response state of agent i (following the pseudo-gradient dynamics as if its neighbor agents were already at the predicted states) is given by $\mathrm{BR}_i(x_{-i}^{\mathrm{pr1}}, x_{-i}^{\mathrm{pr2}}, x_{-i}^{\mathrm{un}})$. Before we give the detailed expression of the predicted states of the neighbor agents $j \in \mathbb{N}_i^{\mathrm{pr1}}$ for agent $i \in \mathbb{L}_k$, we note that the predicted state of agent j is in general different from the actual targeted best-response state of agent j (even when $j \in \mathbb{L}_{k-1}$ is true in reality) for noncooperative systems with more than 2 agents. Recalling the fact from (6.4) that x_{-j}^{pr1} is given by $\{\mathrm{BR}_s(x_{-s})\}_{s \in \mathbb{N}_j^{\mathrm{pr1}}}$ for agent $j \in \mathbb{L}_2$, it follows that the actual targeted best-response state of agent $j \in \mathbb{L}_2$ is given by

$$\begin{aligned} &\mathrm{BR}_j(x_{-j}^{\mathrm{pr1}}, x_{-j}^{\mathrm{pr2}}, x_{-j}^{\mathrm{un}}) = \mathrm{BR}_j(x_{-j}^{\mathrm{pr}}, x_{-j}^{\mathrm{un}}) \\ &= \mathrm{BR}_j(\{\mathrm{BR}_s(x_{-s})\}_{s \in \mathcal{N}_j}, \{x_s\}_{s \in \mathcal{N} \setminus \{j, \mathcal{N}_j\}}), \quad j \in \mathbb{L}_2. \end{aligned} \tag{6.5}$$

However, since $\mathcal{N}_j$ is unknown for agent i by Assumption 6.1, agent $i \in \mathbb{L}_3$ may hardly predict the actual targeted best-response state of neighbor agent $j \in \mathbb{L}_2$ with uncertain $\mathcal{N}_j \setminus \{i\}$ even with full knowledge of the other agents' payoff functions when $n > 2$.

In this paper, we assume that agent $i \in \mathbb{L}_k$ evaluates the profile x_{-i}^{pr1} of the targeted best-response states of the neighbor agents $j \in \mathbb{N}_i^{\text{pr1}}$ under the naive hypothesis that agent j is aware of the payoff function of only agent i, i.e., $\mathcal{N}_j = \mathbb{N}_j^{\text{pr1}} = \{i\}$, $\mathbb{N}_j^{\text{pr2}} = \emptyset$, because agent i does not have the information of $\mathcal{N}_j$. In this case, it follows from (6.5) with $\mathcal{N}_j = \{i\}$ that the predicted states of the neighbor agents $j \in \mathbb{N}_i^{\text{pr1}}$ for agent $i \in \mathbb{L}_3$ is evaluated as $\text{BR}_j(\text{BR}_i(x_{-i}), x_{-(j,i)})$ with $x_{-(j,i)} \in \mathbb{R}^{n-2}$ denoting the state profile except agents j and i (i.e., $x_{-(j,i)} = \{x_s\}_{s \in \mathcal{N} \setminus \{j,i\}}$), and hence the actual targeted best-response state of agent $i \in \mathbb{L}_3$ is given by

$$
\begin{aligned}
\text{BR}_i(x_{-i}^{\text{pr1}}, x_{-i}^{\text{pr2}}, x_{-i}^{\text{un}}) &= \text{BR}_i(\{\varphi_j^i\}_{j \in \mathbb{N}_i^{\text{pr1}}}, \{\phi_j\}_{j \in \mathbb{N}_i^{\text{pr2}}}, x_{-i}^{\text{un}}) \\
&= \text{BR}_i(\{\varphi_j^i\}_{j \in \mathbb{N}_i^{\text{pr1}}}, \{\phi_j\}_{j \in \mathbb{N}_i^{\text{pr2}}}, \{x_s\}_{s \in \mathcal{N} \setminus \{j, \mathcal{N}_j\}}),
\end{aligned} \tag{6.6}
$$

with $\varphi_j^i = \text{BR}_j(\text{BR}_i(x_{-i}), x_{-(j,i)})$ and $\phi_j = \text{BR}_j(x_{-j})$. Likewise, for agent $i \in \mathbb{L}_4$, with the hypothesis of $\mathbb{N}_j^{\text{pr1}} = \{i\}$, $\mathbb{N}_j^{\text{pr2}} = \emptyset$, it follows from (6.6) that the predicted states of the neighbor agents $j \in \mathbb{N}_i^{\text{pr1}}$ are evaluated as

$$
\text{BR}_j(\text{BR}_i(\text{BR}_j(x_{-j}), x_{-(i,j)}), x_{-(j,i)}), \tag{6.7}
$$

which can be further used in characterizing the actual targeted best-response state (6.6) of agent $i \in \mathbb{L}_4$ with φ_j^i replaced by the predicted states (6.7). Subsequently, the above procedure continues for higher-level agents.

Based on the above discussion, we define the functions $\text{BR}_{j,i}^k(x)$ to characterize the profile $x_{-i}^{\text{pr1}} \triangleq \{\text{BR}_{j,i}^k(x)\}_{j \in \mathbb{N}_i^{\text{pr1}}}$ of the predicted states of the neighbor agent j in $\mathbb{N}_i^{\text{pr1}}$ evaluated by agent $i \in \mathbb{L}_k$ given by

$$
\text{BR}_{j,i}^2(x) = \text{BR}_j(x_{-j}), \tag{6.8}
$$

$$
\text{BR}_{j,i}^3(x) = \text{BR}_j(\text{BR}_i(x_{-i}), x_{-(j,i)}), \tag{6.9}
$$

$$
\text{BR}_{j,i}^4(x) = \text{BR}_j(\text{BR}_i(\text{BR}_j(x_{-j}), x_{-(i,j)}), x_{-(j,i)}). \tag{6.10}
$$

For the higher-level agents, $\text{BR}_{j,i}^k(x)$ can be recursively expressed by

$$
\text{BR}_{j,i}^k(x) = \text{BR}_{j,i}^3(\text{BR}_{j,i}^{k-2}(x), x_{-j}), \quad k \geq 4. \tag{6.11}
$$

Note that the function $\text{BR}_{j,i}^k(x)$ evaluated by agent i is defined as the mapping with $k-1$ times of iterations of $\text{BR}_j(x_{-j})$ along with $\text{BR}_i(x_{-i})$ itself. The predicted state $\text{BR}_{j,i}^k(x)$ coincides with the actual targeted best-response state of agent $j \in \mathbb{L}_{k-1}$ when $n = 2$ as the hypotheses of $\mathcal{N}_j = \mathbb{N}_j^{\text{pr1}} = \{i\}$, $\mathbb{N}_j^{\text{pr2}} = \emptyset$ hold in reality. An example showing the predicted state and the actual targeted best-response state of agent 2 coincides is shown in Fig. 6.3(b) with $\mathbb{L}_2 = \{2\}$ and $\mathbb{L}_3 = \{1\}$.

Level-k agent: The prediction-incorporated pseudo-gradient dynamics for Level-k agents with $k \geq 3$ are given by

$$\dot{x}_i(t) = \alpha_i \frac{\partial J_i(x_i(t), x_{-i}^{\mathrm{pr1}}(t), x_{-i}^{\mathrm{pr2}}(t), x_{-i}^{\mathrm{un}}(t))}{\partial x_i}, \quad i \in \mathbb{L}_k. \tag{6.12}$$

recalling $x_{-i}^{\mathrm{pr1}} = \{\mathrm{BR}_{j,i}^k(x)\}_{j \in \mathbb{N}_i^{\mathrm{pr1}}}$ and $x_{-i}^{\mathrm{pr2}} = \{\mathrm{BR}_j(x_{-j})\}_{j \in \mathbb{N}_i^{\mathrm{pr2}}}$. Once again, the essence of the general form (6.12) is that agent $i \in \mathbb{L}_k$ tries to adjust its state x_i following the pseudo-gradient dynamics as if its neighbor agents $j \in \mathbb{N}_i^{\mathrm{pr1}}$ (resp., $\mathbb{N}_i^{\mathrm{pr2}}$) were already at the predicted states $\mathrm{BR}_{j,i}^k(x)$ (resp., $\mathrm{BR}_j(x_{-j})$) instead of the current states x_j.

Remark 6.1. Note that the dynamics (6.12) for $k = 2$ is compatible with (6.4) since $(x_{-i}^{\mathrm{pr1}}, x_{-i}^{\mathrm{pr2}})$ reduces to x_{-i}^{pr} for agent $i \in \mathbb{L}_2$. Moreover, (6.12) for $k = 1$ is also compatible with (6.3) for agent $i \in \mathbb{L}_1$ with $\mathbb{N}_i^{\mathrm{pr1}} = \mathbb{N}_i^{\mathrm{pr2}} = \emptyset$ since x_{-i}^{un} reduces to x_{-i} in (6.3). In the case where no agent is the destination of agent $i \in \mathbb{L}_k$ in $G(\mathcal{N}, E)$ with $k \geq 3$, i.e., $\mathbb{N}_i^{\mathrm{pr1}} = \emptyset$ or $\mathbb{N}_i^{\mathrm{pr2}} = \mathcal{N}_i$, (6.12) reduces to (6.4). Therefore, agent i can act as a Level-k agent with $k \geq 3$ only if there is at least one edge directed from agent i to its neighbor $\mathcal{N}_i$, i.e., $\mathbb{N}_i^{\mathrm{pr1}} \neq \emptyset$ (e.g., agent 4 of the knowledge network shown in Fig. 6.2 is never a Level-k agent with $k \geq 3$).

In this paper, since the agents usually have only a finite depth of reasoning, we suppose that there is a limit $\xi \in \mathbb{Z}_+$ to the depth to which the agents can reason strategically, i.e., $k \leq \xi$. Note that the Nash equilibrium x^* is the equilibrium of the pseudo-gradient dynamics (6.3), (6.4), and (6.12) for arbitrary set of $\mathbb{L}_1, \ldots, \mathbb{L}_\xi$, because $\frac{\partial J_i(x_i, x_{-i})}{\partial x_i} = 0$, $\frac{\partial J_i(x_i, x_{-i}^{\mathrm{pr}}, x_{-i}^{\mathrm{un}})}{\partial x_i} = 0$, and $\frac{\partial J_i(x_i, x_{-i}^{\mathrm{pr1}}, x_{-i}^{\mathrm{pr2}}, x_{-i}^{\mathrm{un}})}{\partial x_i} = 0$ at x^*. Henceforth, we focus on the discussion about this Nash equilibrium under the prediction-incorporated pseudo-gradient dynamics.

6.2.3 Motivating Example and Problem Statement

In this section, we first show a numerical example where the Level-k thinking significantly changes the behavior of the dynamical system, and then present the main problems of this paper. Specifically, consider a 5-agent noncooperative system with the knowledge network of payoff functions $G(\mathcal{N}, E)$ given by Fig. 6.2, where agent 4 is never a Level-3 agent under $\mathbb{N}_4^{\mathrm{pr1}} = \emptyset$ and all the other agents are able to be a Level-k agent with $k \geq 3$. In this case, it follows from $\mathcal{N}_3 = \{1, 2\}$ with $\mathbb{N}_3^{\mathrm{pr1}} = \{1\}$

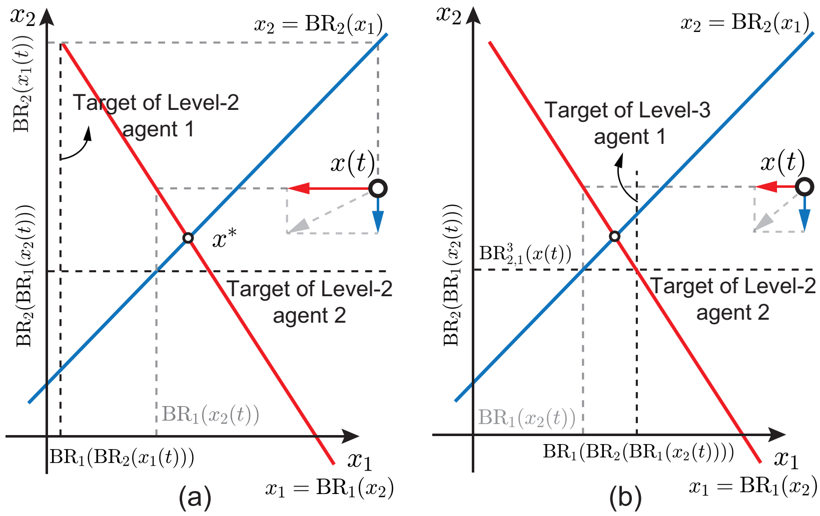


Figure. 6.3 Target states of the agents of a two-agent noncooperative system under $E = \{(1,2),(2,1)\}$ with red (resp., blue) arrow representing the moving direction of agent 1 (resp., agent 2). (a): $\mathbb{L}_2 = \{1,2\}$, (b): $\mathbb{L}_2 = \{2\}$, $\mathbb{L}_3 = \{1\}$. The predicted state $\mathrm{BR}^3_{2,1}(x)$ predicted by agent 1 in (b) is the same as the actual targeted best-response state of agent 2 given by $\mathrm{BR}_2(\mathrm{BR}_1(x_2(t)))$.

and $\mathbb{N}_3^{\mathrm{pr}2} = \{2\}$ that the prediction-incorporated pseudo-gradient dynamics (6.12) for agent 3 being a Level-k agent with $k \geq 2$ is given by

$$\dot{x}_3(t) = \alpha_3 \frac{\partial J_3(\mathrm{BR}^{k-1}_{1,3}(x(t)), \mathrm{BR}_2(x_{-2}(t)), x_{-(1,2)}(t))}{\partial x_3}, \tag{6.13}$$

where the current state x_4 and x_5 are used because the information of $J_4(\cdot)$ and $J_5(\cdot)$ are unknown (i.e., agents 4 and 5 are not included in $\mathcal{N}_3$). The trajectories of the agents' state under the prediction-incorporated pseudo-gradient dynamics with $\mathbb{L}_3 = \mathcal{N} \setminus \{4\}$ and $\mathbb{L}_2 = \{4\}$ are shown as dashed lines in Fig. 6.4, whereas the trajectories under the conventional pseudo-gradient dynamics (6.3) with $\mathbb{L}_1 = \mathcal{N}$ are shown as solid lines. It can be seen from this example that Level-k thinking may destabilize the noncooperative dynamical system.

Motivation: The information of the agents' sensitivity parameters and the knowledge network of the payoff functions may not be precisely observed by anybody. Assume that there is a system manager who is authorized to control the amount of incentives in order to stabilize a Nash equilibrium x^* by encouraging agents to converge to it. A fundamental question is how to ensure stability of the Nash equilibrium with those uncertain information.

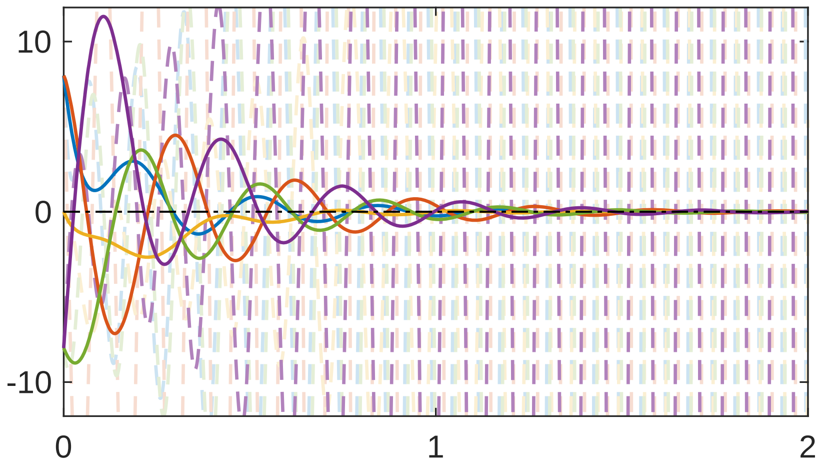


Figure. 6.4 Trajectories of the agents' state $x(t)$. Solid lines: conventional pseudo-gradient dynamics with $\mathbb{L}_1 = \mathcal{N} = \{1, 2, 3, 4, 5\}$, dashed lines: $\mathbb{L}_3 = \{1, 2, 3, 5\}$ and $\mathbb{L}_2 = \{4\}$ under the knowledge network of payoff functions given by Fig. 6.2.

Problem: Consider the noncooperative system $\mathcal{G}(J)$ with the pseudo-gradient dynamics under predictions. Suppose that the agents have only bounded rationality on reasoning with Level-$k \leq \xi \in \mathbb{Z}_+$. Our main objectives are two folds: (i) find the stability conditions of the Nash equilibrium x^* with *arbitrary* sensitivity parameters α_i, $i \in \mathcal{N}$, with the knowledge network $G(\mathcal{N}, E)$; (ii) develop a framework to guarantee stability of the Nash equilibrium x^* under the unknown sensitivity parameters α_i, $i \in \mathcal{N}$, with uncertain cognitive hierarchy levels of the agents.

6.3 Stability Analysis of Prediction-Incorporated Pseudo-Gradient Dynamics

In this section, we characterize stability properties of the Nash equilibrium for the noncooperative system $\mathcal{G}(J)$ with pseudo-gradient dynamics (6.3), (6.4), and (6.12). Specifically, we first assume that the agents are at Level-$k \leq \xi = 2$, and then extend the results for the cases with $\xi = 3$ and $\xi > 3$. The reason why we present the results for $\xi = 2$ and $\xi = 3$ in separate subsections comes from the fact that the neighbor agents of each agent should be decomposed into 2 groups as we did in Section 6.2.2 where we characterized the prediction-incorporated pseudo-gradient dynamics for $\xi > 2$. For the statement of the following results, let $\alpha \triangleq (\alpha_1, \ldots, \alpha_n)$

and $\mathcal{A}(J,\alpha,x) \triangleq \begin{bmatrix} \alpha_1 \frac{\partial^2 J_1(x)}{\partial x_1^2} & \cdots & \alpha_1 \frac{\partial^2 J_1(x)}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \alpha_n \frac{\partial^2 J_N(x)}{\partial x_n \partial x_1} & \cdots & \alpha_n \frac{\partial^2 J_n(x)}{\partial x_n^2} \end{bmatrix}$, which is exactly the Jacobian matrix of the (conventional) pseudo-gradient dynamics (6.3) for $\mathbb{L}_1 = \mathcal{N}$ (all the agents are at Level-1). Since this case is addressed in Chapter 2, we consider the case where at least one agent is in $\mathbb{L}_2, \ldots, \mathbb{L}_\xi$ in this paper.

Noncooperative Systems with Mixed Level-1 and Level-2 Agents

In this section, we present the stability conditions of the Nash equilibrium for the case where the agents have bounded rationality on reasoning with Level-$k \leq 2$, i.e., $\mathbb{L}_1 \cup \mathbb{L}_2 = \mathcal{N}$ with $\mathbb{L}_2 \neq \emptyset$. First, we present a sufficient condition for determining stability of the Nash equilibrium x^* with *arbitrary* α_i, $i \in \mathcal{N}$. For the statement of the following results, we define $\Lambda(x) \triangleq \begin{bmatrix} \frac{\partial \mathrm{BR}_1(x_{-1})}{\partial x} \\ \vdots \\ \frac{\partial \mathrm{BR}_n(x_{-n})}{\partial x} \end{bmatrix} \in \mathbb{R}^{n \times n}$. Note that the diagonal terms of $\Lambda(x)$ are all zero because $\mathrm{BR}_i(x_{-i})$ does not depend on x_i. The vector fields of a two-agent noncooperative system are introduced later for comparisons with systems with higher cognitive hierarchy level agents (see Section 6.3 below).

Proposition 6.1. Consider the noncooperative system $\mathcal{G}(J)$ with the agents either at Level-1 or Level-2 satisfying $\mathbb{L}_1 \cup \mathbb{L}_2 = \mathcal{N}$ so that the agents follow the pseudo-gradient dynamics (6.3) and (6.4) depending on their cognitive hierarchy levels. Let

$$\Pi_2(J,\alpha,x) = [\mathrm{row}_i(\Pi_2(J,\alpha,x))]_{i \in \mathcal{N}} \in \mathbb{R}^{n \times n}, \tag{6.14}$$

with

$$\mathrm{row}_i(\Pi_2(J,\alpha,x)) \triangleq \begin{cases} \mathrm{row}_i(\mathcal{A}(J,\alpha,x)), & i \in \mathbb{L}_1; \\ \mathrm{row}_i(R_2(J,\alpha,x)), & i \in \mathbb{L}_2, \end{cases} \tag{6.15}$$

where

$$\begin{aligned} R_2(J,\alpha,x) \triangleq & \mathcal{A}(J,\alpha,x) \circ (1_n 1_n^{\mathrm{T}} - A_{\mathrm{ad}}) \\ & + (\mathcal{A}(J,\alpha,x) \circ A_{\mathrm{ad}}) \Lambda(x) \in \mathbb{R}^{n \times n}. \end{aligned} \tag{6.16}$$

If there exists $\hat{\alpha} \in \mathbb{R}_+^N$ such that

$$\Pi_2^{\mathrm{T}}(J,\hat{\alpha},x^*) + \Pi_2(J,\hat{\alpha},x^*) < 0, \tag{6.17}$$

then the Nash equilibrium x^* satisfying (2.1) is locally asymptotically stable for any sensitivity parameters $\alpha_i \in \mathbb{R}_+$, $i \in \mathcal{N}$.

Proof First, it follows from

$$\begin{aligned}
&\sum_{i\in\mathcal{N}\backslash\mathcal{N}_j} \frac{\partial^2 J_j(x^*)}{\partial x_j \partial x_i}\frac{\partial x_i}{\partial x} = \sum_{i\in\mathcal{N}} (1-a_{ji})\frac{\partial^2 J_j(x^*)}{\partial x_j \partial x_i}\frac{\partial x_i}{\partial x} \\
&= \mathrm{row}_j(\mathcal{A}(J, 1_n, x^*)) \circ (1_n^{\mathrm{T}} - \mathrm{row}_j(A_{\mathrm{ad}})) \in \mathbb{R}^{1\times n}, \qquad (6.18)\\
&\sum_{i\in\mathcal{N}_j} \frac{\partial^2 J_j(x^*)}{\partial x_j \partial x_i}\frac{\partial \mathrm{BR}_i(x_{-i}^*)}{\partial x} = \sum_{i\in\mathcal{N}} a_{ji}\frac{\partial^2 J_j(x^*)}{\partial x_j \partial x_i}\frac{\partial \mathrm{BR}_i(x_{-i}^*)}{\partial x} \\
&= \Big(\mathrm{row}_j(\mathcal{A}(J, 1_n, x^*)) \circ \mathrm{row}_j(A_{\mathrm{ad}})\Big)\Lambda(x^*) \in \mathbb{R}^{1\times n}, \qquad (6.19)
\end{aligned}$$

for $j \in \mathcal{N}$ that the Jacobian matrix of the pseudo-gradient dynamics (6.4) at the equilibrium x^* with $\mathbb{L}_2 = \mathcal{N}$ is given by

$$\begin{aligned}
&\mathrm{diag}[\alpha]\begin{bmatrix} \sum_{i\in\mathcal{N}\backslash\mathcal{N}_1} \frac{\partial^2 J_1(x^*)}{\partial x_1 \partial x_i}\frac{\partial x_i}{\partial x} + \sum_{i\in\mathcal{N}_1} \frac{\partial^2 J_1(x^*)}{\partial x_1 \partial x_i}\frac{\partial \mathrm{BR}_i(x_{-i}^*)}{\partial x} \\ \vdots \\ \sum_{i\in\mathcal{N}\backslash\mathcal{N}_n} \frac{\partial^2 J_n(x^*)}{\partial x_n \partial x_i}\frac{\partial x_i}{\partial x} + \sum_{i\in\mathcal{N}_n} \frac{\partial^2 J_n(x^*)}{\partial x_n \partial x_i}\frac{\partial \mathrm{BR}_i(x_{-i}^*)}{\partial x} \end{bmatrix} \\
&= R_2(J, \alpha, x^*). \qquad (6.20)
\end{aligned}$$

Recalling that $\mathcal{A}(J, \alpha, x)$ is the Jacobian matrix of the pseudo-gradient dynamics (6.3) with $\mathbb{L}_1 = \mathcal{N}$, it follows that the Jacobian matrix of the pseudo-gradient dynamics (6.3), (6.4) at x^* given $\mathbb{L}_1$ and $\mathbb{L}_2$ is given by $\Pi_2(J, \alpha, x)$. Hence, linearizing the system dynamics (6.4) with $\tilde{x} \triangleq x - x^*$ around x^* yields

$$\dot{\tilde{x}}(t) = \Pi_2(J, \alpha, x^*)\tilde{x}(t). \qquad (6.21)$$

Consider the Lyapunov function candidate $V(\tilde{x}) = \tilde{x}^{\mathrm{T}} P \tilde{x}$ with a positive-definite matrix $P \triangleq \mathrm{diag}[\frac{\hat{\alpha}_1}{\alpha_1}, \ldots, \frac{\hat{\alpha}_N}{\alpha_N}]$. Since

$$\Pi_2^{\mathrm{T}}(J, \alpha, x^*)P + P\Pi_2(J, \alpha, x^*) = \Pi_2^{\mathrm{T}}(J, \hat{\alpha}, x^*) + \Pi_2(J, \hat{\alpha}, x^*) < 0,$$

is satisfied, it follows using the linearized dynamics (6.21) that

$$\dot{V}(\tilde{x}(t)) = \tilde{x}^{\mathrm{T}}(t)(\Pi_2^{\mathrm{T}}(J, \hat{\alpha}, x^*) + \Pi_2(J, \hat{\alpha}, x^*))\tilde{x}(t) < 0, \qquad (6.22)$$

around x^* and hence the Nash equilibrium x^* is asymptotically stable for all $\alpha_i \in \mathbb{R}_+$, $i \in \mathcal{N}$. □

Remark 6.2. Note that $\Pi_2(J, \hat{\alpha}, x^*)$ is the Jacobian matrix of the prediction-incorporated pseudo-gradient dynamics consisting of (6.3), (6.4) depending on the adjacency matrix A_{ad} of the knowledge network $G(\mathcal{N}, E)$ of the payoff functions. In the case where none of the agents has the access to the information of the other agents' payoff functions, i.e., $\mathcal{N}_i = \emptyset$, $i \in \mathcal{N}$, the matrices $R_2(J, \hat{\alpha}, x^*)$ and $\Pi_2(J, \hat{\alpha}, x^*)$ reduce to $\mathcal{A}(J, \alpha, x^*)$, which is exactly the Jacobian matrix of the conventional pseudo-gradient dynamics (6.1).

Remark 6.3. Note that the (i, j)th element of $\Lambda(x^*)$ is given by

$$\frac{\partial \mathrm{BR}_i(x^*_{-i})}{\partial x_j} = -\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_j} \Big/ \frac{\partial^2 J_i(x^*)}{\partial x_i^2}, \quad j \neq i, \tag{6.23}$$

where we used the fact that

$$\frac{\partial g(x)}{\partial x} = -\frac{\partial^2 f(x, g(x))}{\partial x \partial y} \Big/ \frac{\partial^2 f(x, g(x))}{\partial y^2}, \tag{6.24}$$

holds for $g(x) = \arg\max_y f(x, y)$ with a continuous function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ [99]. Thus, the matrix $\Lambda(x^*)$ can be written as

$$\Lambda(x^*) = -\mathrm{diag}[\psi]\mathcal{A}(J, 1_n, x^*) + I, \tag{6.25}$$

with $\psi = [1/\frac{\partial^2 J_1(x^*)}{\partial x_1^2}, \ldots, 1/\frac{\partial^2 J_n(x^*)}{\partial x_n^2}]$, which implies that

$$\begin{aligned} R_2(J, \alpha, x^*) &= \mathcal{A}(J, \alpha, x^*) + (\mathcal{A}(J, \alpha, x^*) \circ A_{\mathrm{ad}})(\Lambda(x^*) - I) \\ &= \mathrm{diag}[\alpha]\hat{A} - \mathrm{diag}[\alpha](\hat{A} \circ A_{\mathrm{ad}})\mathrm{diag}[\psi]\hat{A}, \end{aligned} \tag{6.26}$$

with $\hat{A} = \mathcal{A}(J, 1_n, x^*)$. For example, supposing that $G(\mathcal{N}, E)$ is a complete graph, it follows from $\hat{A} \circ A_{\mathrm{ad}} = \hat{A} - \mathrm{diag}^{-1}[\psi]$ that $R_2(J, \alpha, x^*) = 2\mathrm{diag}[\alpha]\hat{A} - \mathrm{diag}[\alpha]\hat{A}\mathrm{diag}[\psi]\hat{A}$.

Remark 6.4. For the noncooperative system satisfying $[\Pi_2]_{ij} \geq 0$, $i, j \in \mathcal{N}$, $i \neq j$, it follows from the properties of Metzler matrices that the condition (6.17) in Proposition 6.1 is also a necessary condition for the Nash equilibrium x^* to be locally asymptotically stable for *arbitrary* α. Note that the typical numerical examples satisfying $[\Pi_2]_{ij} = [R_2]_{ij} \geq 0$, $i, j \in \mathcal{N}$, $i \neq j$, can be found in oligopoly markets given in Section 6.5 below with $\mathbb{L}_1 = \emptyset$ and $\mathbb{L}_2 = \mathcal{N}$.

Now, we characterize the stability conditions for arbitrary cognitive hierarchy levels of the agents. In this case, $\Pi_2(J, \alpha, x)$ cannot be constructed to determine stability as in Proposition 6.1. The following result provides some sufficient conditions to guarantee stability with arbitrary $\mathbb{L}_1$ and $\mathbb{L}_2$.

Proposition 6.2. Consider the noncooperative system $\mathcal{G}(J)$ with the agents either at Level-1 or Level-2 satisfying $\mathbb{L}_1 \cup \mathbb{L}_2 = \mathcal{N}$ so that the agents follow the pseudo-gradient dynamics (6.3) and (6.4) depending on their cognitive hierarchy levels. If the payoff functions $J_i(x)$, $i \in \mathcal{N}$, satisfy

$$\frac{\partial^2 J_i(x^*)}{\partial x_i^2} < \delta_i, \quad i \in \mathcal{N}, \tag{6.27}$$

with $\delta_i \triangleq \min(\delta_i^1, \delta_i^2)$, $\delta_i^1 \triangleq -\sum_{j\neq i} |\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_j}|$, and

$$\begin{aligned}\delta_i^2 \triangleq -\sum_{j\neq i}\Big|(1-a_{ij})\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_j} + \sum_{s\in\mathcal{N}_i}\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s}\frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_j}\Big| \\ -\sum_{s\in\mathcal{N}_i}\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s}\frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_i},\end{aligned} \tag{6.28}$$

then the Nash equilibrium x^* satisfying (2.1) is locally asymptotically stable for any cognitive level sets $\mathbb{L}_1$ and $\mathbb{L}_2$ and any sensitivity parameters $\alpha_i \in \mathbb{R}_+$, $i \in \mathcal{N}$. If, in addition, (6.27) holds with

$$\delta_i = \underline{\delta}_i^2 \triangleq -\sum_{j\neq i}\Big|\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_j}\Big| - \sum_{j\neq s}\sum_{s\neq i}\Big|\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s}\frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_j}\Big|, \tag{6.29}$$

then the Nash equilibrium x^* is locally asymptotically stable for any knowledge network $G(\mathcal{N}, E)$ of the payoff functions.

Proof First, note from the expression (6.20) that the (i,i)th element of $R_2(J, 1_n, x^*)$ is given by

$$\frac{\partial^2 J_i(x^*)}{\partial x_i^2} + \sum_{s\in\mathcal{N}_i}\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s}\frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_i}, \quad i \in \mathcal{N}, \tag{6.30}$$

whereas the (i,j)th element is given by

$$(1-a_{ij})\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_j} + \sum_{s\in\mathcal{N}_i}\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s}\frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_j}, \quad j \neq i,$$

recalling that $a_{ij}=1$ if $j \in \mathcal{N}_i$, and $a_{ij}=0$ otherwise. Then, the condition (6.27) with $\delta_i \triangleq \min(\delta_i^1, \delta_i^2)$ indicates that the matrix $\Pi_2(J, 1_n, x^*)$ (or, equivalently, $\Pi_2(J, \alpha, x^*)$) is strictly diagonally dominant because the matrices $R_2(J,\alpha,x^*)$ and $\mathcal{A}(J,\alpha,x^*)$ are all strictly diagonally dominant. Now, it follows from Gershgorin's circle theorem [95] that the matrix $\Pi(J,\alpha,x^*)$ is Hurwitz and hence the Nash equilibrium x^* is asymptotically stable for any $\mathbb{L}_1$ and $\mathbb{L}_2$ and any α_i, $i \in \mathcal{N}$. Next, it follows from

$$\begin{aligned}
\underline{\delta}_i^2 \leq & -\sum_{j\neq i}\Big|\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_j}\Big| - \sum_{j\in\mathcal{N}}\sum_{s\in\mathcal{N}_i}\Big|\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s}\frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_j}\Big| \\
= & -\sum_{j\neq i}\Big|\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_j}\Big| - \sum_{j\neq i}\sum_{s\in\mathcal{N}_i}\Big|\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s}\frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_j}\Big| \\
& -\sum_{s\in\mathcal{N}_i}\Big|\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s}\frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_i}\Big| \\
\leq & -\sum_{j\neq i}\Big|(1-a_{ij})\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_j} + \sum_{s\in\mathcal{N}_i}\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s}\frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_j}\Big| \\
& -\sum_{s\in\mathcal{N}_i}\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s}\frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_i} = \delta_i^2, \quad i \in \mathcal{N},
\end{aligned} \tag{6.31}$$

and $\underline{\delta}_i^2 \leq \delta_i^1$, $i \in \mathcal{N}$, that $\underline{\delta}_i^2 \leq \min(\delta_i^1, \delta_i^2)$ holds for $i \in \mathcal{N}$, i.e., the conditions (6.27) along with (6.29) indicate that (6.27) holds with $\delta_i = \min(\delta_i^1, \delta_i^2)$, $i \in \mathcal{N}$. The proof is complete. □

Remark 6.5. The sufficient conditions in Propositions 6.1 and 6.2 have an inclusive relation since a strictly diagonally dominant matrix $\Pi_2(J, \hat{\alpha}, x^*)$ [100, Theorem 3] in Proposition 6.2 indicates that there must exists $\hat{\alpha} \in \mathbb{R}_+^N$ satisfying the condition (6.17) in Proposition 6.1. This is consistent with common sense that guaranteeing stability for some arbitrary parameters may may require stringent stability conditions.

Noncooperative Systems with Mixed Level-1, Level-2, and Level-3 Agents

In this section, we present the stability conditions of the Nash equilibrium for the case where the agents have bounded rationality on reasoning with Level-$k \leq 3$, i.e., $\mathbb{L}_1 \cup \mathbb{L}_2 \cup \mathbb{L}_3 = \mathcal{N}$. For the statement of the following results, we decompose the knowledge network $G(\mathcal{N}, E)$ into an undirected network $G_{\mathrm{ud}}(\mathcal{N}, E_{\mathrm{ud}})$ and a directed network $G_{\mathrm{d}}(\mathcal{N}, E_{\mathrm{d}})$ with

$$E_{\mathrm{ud}} \triangleq \{(j,i) \in E : (i,j) \in E\}, \tag{6.32}$$
$$E_{\mathrm{d}} \triangleq \{(j,i) \in E : (i,j) \notin E\}. \tag{6.33}$$

It is immediate that $E = E_{\mathrm{ud}} \cup E_{\mathrm{d}}$ and $E_{\mathrm{ud}} \cap E_{\mathrm{d}} = \emptyset$. Let the adjacency matrices of $G_{\mathrm{ud}}(\mathcal{N}, E_{\mathrm{ud}})$ and $G_{\mathrm{d}}(\mathcal{N}, E_{\mathrm{d}})$ be denoted by $B_{\mathrm{ud}} = [b_{ij}^{\mathrm{ud}}] \in \mathbb{R}^{n\times n}$ and $B_{\mathrm{d}} = [b_{ij}^{\mathrm{d}}] \in \mathbb{R}^{n\times n}$, respectively. Here, note that B_{ud} is symmetric and satisfies

$$B_{\mathrm{ud}} + B_{\mathrm{d}} = A_{\mathrm{ad}}. \tag{6.34}$$

Depending on the adjacency matrix B_1, we define a matrix

$$W_3(J,\alpha,x) = \begin{bmatrix} (\mathrm{row}_1(\mathcal{A}(J,\alpha,x)) \circ \mathrm{row}_1(B_{\mathrm{ud}}))F_1^3(x) \\ \vdots \\ (\mathrm{row}_n(\mathcal{A}(J,\alpha,x)) \circ \mathrm{row}_n(B_{\mathrm{ud}}))F_n^3(x) \end{bmatrix}, \tag{6.35}$$

with $F_i^3(x) \triangleq \begin{bmatrix} \frac{\partial \mathrm{BR}_1(\mathrm{BR}_i(x_{-i}), x_{-(1,i)})}{\partial x} \\ \vdots \\ \frac{\partial \mathrm{BR}_n(\mathrm{BR}_i(x_{-i}), x_{-(n,i)})}{\partial x} \end{bmatrix} \in \mathbb{R}^{n\times n}$, $i \in \mathcal{N}$, where $\frac{\partial \mathrm{BR}_i(\mathrm{BR}_i(x_{-i}), x_{-(i,i)})}{\partial x}$ is defined as 0.

Now, a sufficient condition is provided in the following theorem to determine the stability of the Nash equilibrium x^* with *arbitrary* α_i, $i \in \mathcal{N}$.

Proposition 6.3. Consider the noncooperative system $\mathcal{G}(J)$ with the agents at Level-$k \le 3$ satisfying $\mathbb{L}_1 \cup \mathbb{L}_2 \cup \mathbb{L}_3 = \mathcal{N}$ so that the agents follow the pseudo-gradient dynamics (6.3), (6.4), and (6.12) depending on their cognitive hierarchy levels. Let

$$\Pi_3(J,\alpha,x) = [\mathrm{row}_i(\Pi_3(J,\alpha,x))]_{i\in\mathcal{N}} \in \mathbb{R}^{n\times n}, \tag{6.36}$$

with

$$\mathrm{row}_i(\Pi_3(J,\alpha,x)) \triangleq \begin{cases} \mathrm{row}_i(\mathcal{A}(J,\alpha,x)), & i \in \mathbb{L}_1; \\ \mathrm{row}_i(R_2(J,\alpha,x)), & i \in \mathbb{L}_2; \\ \mathrm{row}_i(R_3(J,\alpha,x)), & i \in \mathbb{L}_3, \end{cases} \tag{6.37}$$

where $R_2(J,\alpha,x) \in \mathbb{R}^{n\times n}$ is defined in (6.16) and

$$\begin{aligned} R_3(J,\alpha,x) \triangleq{} & \mathcal{A}(J,\alpha,x) \circ (1_n 1_n^{\mathrm{T}} - A_{\mathrm{ad}}) + W_3(J,\alpha,x) \\ & + (\mathcal{A}(J,\alpha,x) \circ B_{\mathrm{d}})\Lambda(x) \in \mathbb{R}^{n\times n}. \end{aligned} \tag{6.38}$$

If there exists $\hat{\alpha} \in \mathbb{R}_+^N$ such that

$$\Pi_3^{\mathrm{T}}(J, \hat{\alpha}, x^*) + \Pi_3(J, \hat{\alpha}, x^*) < 0, \tag{6.39}$$

then the Nash equilibrium x^* satisfying (2.1) is locally asymptotically stable for any sensitivity parameters $\alpha_i \in \mathbb{R}_+$, $i \in \mathcal{N}$.

Proof First, it follows from (6.18) and

$$\begin{aligned}
&\sum_{i\in\mathbb{N}_j^{\mathrm{pr1}}} \frac{\partial^2 J_j(x^*)}{\partial x_j \partial x_i} \frac{\partial \mathrm{BR}_i(\mathrm{BR}_j(x_{-j}), x_{-(i,j)})}{\partial x} \\
&= \sum_{i\in\mathcal{N}} b_{ji}^{\mathrm{ud}} \frac{\partial^2 J_j(x^*)}{\partial x_j \partial x_i} \frac{\partial \mathrm{BR}_i(\mathrm{BR}_j(x_{-j}^*), x_{-(i,j)})}{\partial x} \\
&= \Big(\mathrm{row}_j(\mathcal{A}(J, 1_n, x^*)) \circ \mathrm{row}_j(B_{\mathrm{ud}})\Big) F_j^3(x^*) \in \mathbb{R}^{1\times n}, \\
&\sum_{i\in\mathbb{N}_j^{\mathrm{pr2}}} \frac{\partial^2 J_j(x^*)}{\partial x_j \partial x_i} \frac{\partial \mathrm{BR}_i(x_{-i}^*)}{\partial x} = \sum_{i\in\mathcal{N}} b_{ji}^{\mathrm{d}} \frac{\partial^2 J_j(x^*)}{\partial x_j \partial x_i} \frac{\partial \mathrm{BR}_i(x_{-i}^*)}{\partial x} \\
&= \Big(\mathrm{row}_j(\mathcal{A}(J, 1_n, x^*)) \circ \mathrm{row}_j(B_{\mathrm{d}})\Big) \Lambda(x^*) \in \mathbb{R}^{1\times n},
\end{aligned} \tag{6.40}$$

hold for $j \in \mathcal{N}$ that the Jacobian matrix of the pseudo-gradient dynamics (6.12) at x^* with $\mathbb{L}_3 = \mathcal{N}$ is given by

$$\begin{aligned}
&\mathrm{diag}[\alpha] \begin{bmatrix}
\sum_{i\in\mathcal{N}\backslash\mathcal{N}_1} \frac{\partial^2 J_1(x^*)}{\partial x_1 \partial x_i} \frac{\partial x_i}{\partial x} + \sum_{i\in\mathbb{N}_1^{\mathrm{pr2}}} \frac{\partial^2 J_1(x^*)}{\partial x_1 \partial x_i} \frac{\partial \mathrm{BR}_i(x_{-i}^*)}{\partial x} + \sum_{i\in\mathbb{N}_1^{\mathrm{pr1}}} \frac{\partial^2 J_1(x^*)}{\partial x_1 \partial x_i} \frac{\partial \mathrm{BR}_i(\mathrm{BR}_1(x_{-1}^*), x_{-(i,1)}^*)}{\partial x} \\
\vdots \\
\sum_{i\in\mathcal{N}\backslash\mathcal{N}_n} \frac{\partial^2 J_n(x^*)}{\partial x_n \partial x_i} \frac{\partial x_i}{\partial x} + \sum_{i\in\mathbb{N}_n^{\mathrm{pr2}}} \frac{\partial^2 J_n(x^*)}{\partial x_n \partial x_i} \frac{\partial \mathrm{BR}_i(x_{-i}^*)}{\partial x} + \sum_{i\in\mathbb{N}_n^{\mathrm{pr1}}} \frac{\partial^2 J_n(x^*)}{\partial x_n \partial x_i} \frac{\partial \mathrm{BR}_i(\mathrm{BR}_n(x_{-n}^*), x_{-(i,n)}^*)}{\partial x}
\end{bmatrix} \\
&= \mathcal{A}(J, \alpha, x^*) \circ (1_n 1_n^{\mathrm{T}} - A_{\mathrm{ad}}) + (\mathcal{A}(J, \alpha, x^*) \circ B_{\mathrm{d}}) \Lambda(x^*) + W_3(J, \alpha, x^*) = R_3(J, \alpha, x^*).
\end{aligned} \tag{6.41}$$

Recalling that $\mathcal{A}(J, \alpha, x)$ (resp., $R_2(J, \alpha, x)$) is the Jacobian matrix of the pseudo-gradient dynamics (6.3), (6.4) with $\mathbb{L}_1 = \mathcal{N}$ (resp., $\mathbb{L}_2 = \mathcal{N}$), it follows that the Jacobian matrix of the pseudo-gradient dynamics (6.3), (6.4), and (6.12) at x^* given $\mathbb{L}_1$, $\mathbb{L}_2$, and $\mathbb{L}_3$ is given by $\Pi_3(J, \alpha, x)$. The rest of the proof can be similarly obtained as in the proof of Proposition 6.1. □

Now, we characterize the stability conditions for arbitrary cognitive hierarchy levels of the agents. In this case, $\Pi_3(J, \alpha, x)$ cannot be constructed to determine stability as in Proposition 6.3. The following result provides some sufficient conditions to guarantee stability with arbitrary $\mathbb{L}_1$, $\mathbb{L}_2$, and $\mathbb{L}_3$.

Proposition 6.4. Consider the noncooperative system $\mathcal{G}(J)$ with the agents at Level-$k \leq 3$ satisfying $\mathbb{L}_1 \cup \mathbb{L}_2 \cup \mathbb{L}_3 = \mathcal{N}$ so that the agents follow the pseudo-gradient dynamics (6.3), (6.4), and (6.12) depending on their cognitive hierarchy levels. If the payoff functions $J_i(x)$, $i \in \mathcal{N}$, satisfy (6.27) with $\delta_i \triangleq \min(\delta_i^1, \delta_i^2, \delta_i^3)$ for

$$\begin{aligned}\delta_i^3 \triangleq -\sum_{j\neq i}\Big|(1-a_{ij})\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_j} + \sum_{s\in\mathcal{N}_i}\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s}\frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_j} \\ + \sum_{s\in\mathbb{N}_i^{\mathrm{pr1}}}\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s}\frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_i}\frac{\partial \mathrm{BR}_i(x^*_{-i})}{\partial x_j}\Big| \\ - \sum_{s\in\mathbb{N}_i^{\mathrm{pr2}}}\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s}\frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_i},\end{aligned} \tag{6.42}$$

then the Nash equilibrium x^* is locally asymptotically stable for any cognitive level sets $\mathbb{L}_1$, $\mathbb{L}_2$, and $\mathbb{L}_3$ and any sensitivity parameters $\alpha_i \in \mathbb{R}_+$, $i \in \mathcal{N}$. If, in addition, (6.27) holds with

$$\delta_i = \underline{\delta}_i^3 \triangleq \underline{\delta}_i^2 - \sum_{j\neq i}\sum_{s\neq i}\left|\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s}\frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_i}\frac{\partial \mathrm{BR}_i(x^*_{-i})}{\partial x_j}\right|, \tag{6.43}$$

then the Nash equilibrium x^* is locally asymptotically stable for any knowledge network $G(\mathcal{N}, E)$ of the payoff functions.

Proof First, it is worth noting that $\frac{\partial \mathrm{BR}_s(\mathrm{BR}_i(x^*_{-i}), x^*_{-(s,i)})}{\partial x_j} = \frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_i}\frac{\partial \mathrm{BR}_i(x^*_{-i})}{\partial x_j} + \frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_j}$ for $j \neq i$, $s \neq i$, whereas $\frac{\partial \mathrm{BR}_s(\mathrm{BR}_i(x^*_{-i}), x^*_{-(s,i)})}{\partial x_j} = 0$ for $j = i$, $s \neq i$. Now, note from (6.41) that the (i,j)th element of $R_3(J, 1_n, x^*)$ is given by $[R_3]_{ij} = (1-a_{ij})\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_j} + \sum_{s\in\mathcal{N}_i}\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s}\frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_j} + \sum_{s\in\mathcal{N}_i^{\mathrm{pr1}}}\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s}\frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_i}\frac{\partial \mathrm{BR}_i(x^*_{-i})}{\partial x_j}$, whereas $[R_3]_{ii} = \frac{\partial^2 J_i(x^*)}{\partial x_i^2} + \sum_{s\in\mathcal{N}_i^{\mathrm{pr2}}}\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s}\frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_i}$ recalling that $a_{ij} = 1$ if $j \in \mathcal{N}_i$, and $a_{ij} = 0$ otherwise. Then, the condition (6.27) with $\delta_i \triangleq \min(\delta_i^1, \delta_i^2, \delta_i^3)$ indicates that the matrix $\Pi_3(J, 1_n, x^*)$ (or, equivalently, $\Pi_3(J, \alpha, x^*)$) is strictly diagonally dominant because $R_3(J, \alpha, x^*)$, $R_2(J, \alpha, x^*)$, and $\mathcal{A}(J, \alpha, x^*)$ are strictly diagonally dominant. Then, it follows from Gershgorin's circle theorem [95] that the matrix $\Pi_3(J, \alpha, x^*)$ is Hurwitz and hence the Nash equilibrium x^* is asymptotically stable for any $\mathbb{L}_1$, $\mathbb{L}_2$, $\mathbb{L}_3$, and any α_i, $i \in \mathcal{N}$.

Next it follows from

$$
\begin{aligned}
\underline{\delta}_i^3 \leq & -\sum_{j\neq i}\Big|\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_j}\Big| - \sum_{j\in\mathcal{N}}\sum_{s\in\mathcal{N}_i}\Big|\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s}\frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_j}\Big| \\
& -\sum_{j\neq i}\sum_{s\in\mathbb{N}_i^{\mathrm{pr1}}}\Big|\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s}\frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_i}\frac{\partial \mathrm{BR}_i(x^*_{-i})}{\partial x_j}\Big| \\
\leq & -\sum_{j\neq i}\Big|\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_j}\Big| - \sum_{j\neq i}\sum_{s\in\mathcal{N}_i}\Big|\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s}\frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_j}\Big| \\
& -\sum_{j\neq i}\sum_{s\in\mathbb{N}_i^{\mathrm{pr1}}}\Big|\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s}\frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_i}\frac{\partial \mathrm{BR}_i(x^*_{-i})}{\partial x_j}\Big| \\
& -\sum_{s\in\mathbb{N}_i^{\mathrm{pr2}}}\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s}\frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_i} \\
\leq & -\sum_{j\neq i}\Big|[R_3]_{ij}\Big| - \sum_{s\in\mathbb{N}_i^{\mathrm{pr2}}}\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s}\frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_i} = \delta_i^3,
\end{aligned}
\tag{6.44}
$$

and $\underline{\delta}_i^3 \leq \underline{\delta}_i^2 \leq \min(\delta_i^1, \delta_i^2)$, $i \in \mathcal{N}$, that $\underline{\delta}_i^3 \leq \min(\delta_i^1, \delta_i^2, \delta_i^3)$ holds for $i \in \mathcal{N}$, i.e., the conditions (6.27) along with (6.43) imply that (6.27) holds with $\delta_i = \min(\delta_i^1, \delta_i^2, \delta_i^3)$, $i \in \mathcal{N}$. Thus, the proof is complete. □

Noncooperative Systems with Higher Cognitive Hierarchy Level Agents

In this section, we generalize the results in Sections 6.3 and 6.3 to the case where the agents have bounded rationality on reasoning with Level-$k \leq \xi$ with $\xi \geq 3$, i.e., $\mathbb{L}_1 \cup \cdots \cup \mathbb{L}_\xi = \mathcal{N}$. For the statement of following results, we define a matrix $W_k(J,\alpha,x) = \begin{bmatrix} (\mathrm{row}_1(\mathcal{A}(J,\alpha,x)) \circ \mathrm{row}_1(B_{\mathrm{ud}}))F_1^k(x) \\ \vdots \\ (\mathrm{row}_n(\mathcal{A}(J,\alpha,x)) \circ \mathrm{row}_n(B_{\mathrm{ud}}))F_n^k(x) \end{bmatrix}$ with $F_i^k(x) \triangleq \begin{bmatrix} \frac{\partial \mathrm{BR}_{1,i}^k(x)}{\partial x} \\ \vdots \\ \frac{\partial \mathrm{BR}_{n,i}^k(x)}{\partial x} \end{bmatrix} \in \mathbb{R}^{n\times n}$, $i \in \mathcal{N}$, where $\mathrm{BR}_{j,i}^k(\cdot)$ is defined in (6.11). Now, a sufficient condition is provided in the following theorem to guarantee stability without knowing α_i, $i \in \mathcal{N}$.

Theorem 6.1. Consider the noncooperative system $\mathcal{G}(J)$ with the agents at Level-$k \leq \xi$ with $\xi \geq 3$ satisfying $\mathbb{L}_1 \cup \cdots \cup \mathbb{L}_\xi = \mathcal{N}$ so that the agents follow the pseudo-gradient dynamics (6.3), (6.4), and (6.12) depending on their cognitive hierarchy levels. Let

$$
\Pi_\xi(J,\alpha,x) = [\mathrm{row}_i(\Pi_\xi(J,\alpha,x))]_{i\in\mathcal{N}} \in \mathbb{R}^{n\times n}, \tag{6.45}
$$

with

$$\mathrm{row}_i(\Pi_\xi(J,\alpha,x)) \triangleq \begin{cases} \mathrm{row}_i(\mathcal{A}(J,\alpha,x)), & i \in \mathbb{L}_1; \\ \mathrm{row}_i(R_2(J,\alpha,x)), & i \in \mathbb{L}_2; \\ \vdots & \vdots \\ \mathrm{row}_i(R_\xi(J,\alpha,x)), & i \in \mathbb{L}_\xi, \end{cases} \tag{6.46}$$

where $R_2(J,\alpha,x) \in \mathbb{R}^{n\times n}$ is defined in (6.16) and

$$R_k(J,\alpha,x) \triangleq \mathcal{A}(J,\alpha,x) \circ (1_n 1_n^{\mathrm{T}} - A_{\mathrm{ad}}) + W_k(J,\alpha,x) + (\mathcal{A}(J,\alpha,x) \circ B_{\mathrm{d}})\Lambda(x), \quad k \geq 3. \tag{6.47}$$

If there exists $\hat{\alpha} \in \mathbb{R}_+^N$ such that

$$\Pi_\xi^{\mathrm{T}}(J,\hat{\alpha},x^*) + \Pi_\xi(J,\hat{\alpha},x^*) < 0, \tag{6.48}$$

then the Nash equilibrium x^* satisfying (2.1) is locally asymptotically stable for any sensitivity parameters $\alpha_i \in \mathbb{R}_+$, $i \in \mathcal{N}$.

Proof The proof is similar to the proof of Theorem 6.3. □

Now, we characterize the stability conditions for arbitrary cognitive hierarchy levels of the agents $\mathbb{L}_1, \cdots, \mathbb{L}_\xi$.

Theorem 6.2. Consider the noncooperative system $\mathcal{G}(J)$ with the agents at Level-$k \leq \xi$ with $\xi > 3$ satisfying $\mathbb{L}_1 \cup \cdots \cup \mathbb{L}_\xi = \mathcal{N}$ so that the agents follow the pseudo-gradient dynamics (6.3), (6.4), and (6.12) depending on their cognitive hierarchy levels. Let $\Psi_{si}^\alpha = \epsilon_{si}^0 + \cdots + \epsilon_{si}^\alpha$ and $\psi_{si}^\alpha = |\epsilon_{si}|^0 + \cdots + |\epsilon_{si}|^\alpha$ with $\epsilon_{si} = \frac{\partial \mathrm{BR}_s(x_{-s}^*)}{\partial x_i}\frac{\partial \mathrm{BR}_i(x_{-i}^*)}{\partial x_s}$ and $\alpha \in \mathbb{Z}_+$. If the payoff functions $J_i(x)$, $i \in \mathcal{N}$, satisfy (6.27) with $\delta_i = \min(\delta_i^1, \delta_i^2, \ldots \delta_i^\xi)$ and

$$\begin{aligned} \delta_i^k \triangleq &- \sum_{s\in\mathbb{N}_i^{\mathrm{pr1}}} \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s} m_{si}^{ik} - \sum_{s\in\mathbb{N}_i^{\mathrm{pr2}}} \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s}\frac{\partial \mathrm{BR}_s(x_{-s}^*)}{\partial x_i} \\ &- \sum_{j\neq i}\left|(1-a_{ij})\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_j} + \sum_{s\in\mathbb{N}_i^{\mathrm{pr1}}} \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s} m_{si}^{jk} + \sum_{s\in\mathcal{N}_i^{\mathrm{pr2}}} \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s}\frac{\partial \mathrm{BR}_s(x_{-s}^*)}{\partial x_j}\right| \end{aligned} \tag{6.49}$$

for $3 < k \le \xi$ with

$$m_{si}^{jk} \triangleq \frac{\partial \mathrm{BR}_{s,i}^k(x^*)}{\partial x_j} = \begin{cases} \frac{\partial \mathrm{BR}_s(x_{-s}^*)}{\partial x_i}\frac{\partial \mathrm{BR}_i(x_{-i}^*)}{\partial x_j}\Psi_{si}^{\frac{k-3}{2}} + \frac{\partial \mathrm{BR}_s(x_{-s}^*)}{\partial x_j}\Psi_{si}^{\frac{k-3}{2}}, & k \in \mathbb{Z}_{\mathrm{o}}, \quad j \neq i, \quad j \neq s; \\ \frac{\partial \mathrm{BR}_s(x_{-s}^*)}{\partial x_i}\frac{\partial \mathrm{BR}_i(x_{-i}^*)}{\partial x_j}\Psi_{si}^{\frac{k-4}{2}} + \frac{\partial \mathrm{BR}_s(x_{-s}^*)}{\partial x_j}\Psi_{si}^{\frac{k-2}{2}}, & k \in \mathbb{Z}_{\mathrm{e}}, \quad j \neq i, \quad j \neq s; \\ 0, & k \in \mathbb{Z}_{\mathrm{o}}, \quad j = i; \\ \frac{\partial \mathrm{BR}_s(x_{-s}^*)}{\partial x_i}\epsilon_{si}^{\frac{k-2}{2}}, & k \in \mathbb{Z}_{\mathrm{e}}, \quad j = i; \\ \epsilon_{si}^{\frac{k-1}{2}}, & k \in \mathbb{Z}_{\mathrm{o}}, \quad j = s; \\ 0, & k \in \mathbb{Z}_{\mathrm{e}}, \quad j = s, \end{cases} , \tag{6.50}$$

then the Nash equilibrium x^* satisfying (2.1) is locally asymptotically stable for any cognitive level sets $\mathbb{L}_1, \cdots, \mathbb{L}_\xi$ and any sensitivity parameters $\alpha_i \in \mathbb{R}_+$, $i \in \mathcal{N}$. If, in addition, (6.27) holds with $\delta_i = \underline{\delta}_i^\xi$ with

$$\underline{\delta}_i^k = \begin{cases} -\sum_{j\neq i}\left|\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_j}\right| - \sum_{j\neq s}\sum_{s\neq i}\left|\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s}\frac{\partial \mathrm{BR}_s(x_{-s}^*)}{\partial x_j}\right|\psi_{si}^{\frac{k-3}{2}} \\ \qquad -\sum_{j\neq i}\sum_{s\neq i}\left|\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s}\frac{\partial \mathrm{BR}_s(x_{-s}^*)}{\partial x_i}\frac{\partial \mathrm{BR}_i(x_{-i}^*)}{\partial x_j}\right|\psi_{si}^{\frac{k-3}{2}}, \quad k \in \mathbb{Z}_{\mathrm{o}}; \\ -\sum_{j\neq i}\left|\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_j}\right| - \sum_{j\neq s}\sum_{s\neq i}\left|\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s}\frac{\partial \mathrm{BR}_s(x_{-s}^*)}{\partial x_j}\right|\psi_{si}^{\frac{k-2}{2}} \\ \qquad -\sum_{j\neq i}\sum_{s\neq i}\left|\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s}\frac{\partial \mathrm{BR}_s(x_{-s}^*)}{\partial x_i}\frac{\partial \mathrm{BR}_i(x_{-i}^*)}{\partial x_j}\right|\psi_{si}^{\frac{k-4}{2}}, \quad k \in \mathbb{Z}_{\mathrm{e}}, \end{cases} \tag{6.51}$$

then the Nash equilibrium x^* is locally asymptotically stable for any knowledge network $G(\mathcal{N}, E)$ of the payoff functions.

Proof First, note from (6.8)–(6.11) that $\frac{\partial \mathrm{BR}_{s,i}^k(x^*)}{\partial x_j}$ is understood as 0 when $j = i$ (resp., $j = s$) for an odd (resp., even) number $k \in \mathbb{Z}_+$. Furthermore, $\frac{\partial \mathrm{BR}_{s,i}^k(x^*)}{\partial x_j}$ is understood by $\frac{\partial \mathrm{BR}_s(x_{-s}^*)}{\partial x_i}\epsilon_{si}^{\frac{k-2}{2}}$ (resp., $\epsilon_{si}^{\frac{k-1}{2}}$) when $j = i$ (resp., $j = s$) for an even (resp., odd) $k \in \mathbb{Z}_+$. For the other cases (i.e., $j \neq i$, $j \neq s$), $\frac{\partial \mathrm{BR}_{s,i}^k(x^*)}{\partial x_j} = \frac{\partial \mathrm{BR}_s(\mathrm{BR}_i(\mathrm{BR}_{s,i}^{k-2}(x^*), x_{-(i,s)}^*), x_{-(s,i)}^*)}{\partial x_j}$ is given by $\frac{\partial \mathrm{BR}_s(x_{-s}^*)}{\partial x_i}(\frac{\partial \mathrm{BR}_i(x_{-i}^*)}{\partial x_s}\frac{\partial \mathrm{BR}_{s,i}^{k-2}(x^*)}{\partial x_j} + \frac{\partial \mathrm{BR}_i(x_{-i}^*)}{\partial x_j}) + \frac{\partial \mathrm{BR}_s(x_{-s}^*)}{\partial x_j} = \epsilon_{si}\frac{\partial \mathrm{BR}_{s,i}^{k-2}(x^*)}{\partial x_j} + \frac{\partial \mathrm{BR}_s(x_{-s}^*)}{\partial x_i}\frac{\partial \mathrm{BR}_i(x_{-i}^*)}{\partial x_j}) + \frac{\partial \mathrm{BR}_s(x_{-s}^*)}{\partial x_j}$. Now, it follows from $\frac{\partial \mathrm{BR}_{s,i}^2(x^*)}{\partial x_j} = \frac{\partial \mathrm{BR}_s(x_{-s}^*)}{\partial x_j}$ and $\frac{\partial \mathrm{BR}_{s,i}^3(x^*)}{\partial x_j} = \frac{\partial \mathrm{BR}_s(x_{-s}^*)}{\partial x_i}\frac{\partial \mathrm{BR}_i(x_{-i}^*)}{\partial x_j} + \frac{\partial \mathrm{BR}_s(x_{-s}^*)}{\partial x_j}$ for $j \neq i$ and $j \neq s$ that $\frac{\partial \mathrm{BR}_{s,i}^k(x^*)}{\partial x_j}$ can be recursively expressed by (6.50). It follows that the (i, j)th element of $R_k(J, 1_n, x^*)$ given by $[R_k]_{ij} = (1 - a_{ij})\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_j} + \sum_{s\in\mathcal{N}_i^{\mathrm{pr2}}}\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s}\frac{\partial \mathrm{BR}_s(x_{-s}^*)}{\partial x_j} + \sum_{s\in\mathcal{N}_i^{\mathrm{pr1}}}\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s}m_{si}^{jk}$, $j \neq i$, whereas $[R_k]_{ii} = \frac{\partial^2 J_i(x^*)}{\partial x_i^2} + \sum_{s\in\mathcal{N}_i^{\mathrm{pr2}}}\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s}\frac{\partial \mathrm{BR}_s(x_{-s}^*)}{\partial x_i} + \sum_{s\in\mathcal{N}_i^{\mathrm{pr1}}}\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s}m_{si}^{ik}$. Hence, the condition (6.27) with $\delta_i = \min(\delta_i^1, \ldots, \delta_i^\xi)$ indicates that the matrices $\Pi_\xi(J, 1_n, x^*)$ (or, equivalently $\Pi_\xi(J, \alpha, x^*)$)

is strictly diagonally dominant because $R_\xi(J,\alpha,x^*),\dots,R_2(J,\alpha,x^*)$, $\mathcal{A}(J,\alpha,x^*)$ are strictly diagonally dominant. Now, it follows from Gershgorin's circle theorem [95] that the matrix $\Pi_\xi(J,\alpha,x^*)$ is Hurwitz and hence the Nash equilibrium x^* is locally asymptotically stable for any $\mathbb{L}_1,\dots,\mathbb{L}_\xi$, and any α_i, $i\in\mathcal{N}$. Next, using $m_{si}^{ik}=0$ and $m_{si}^{sk}=\epsilon_{si}^{\frac{k-1}{2}}$ for an odd k and using $m_{si}^{ik}=\frac{\partial \mathrm{BR}_s(x_{-s}^*)}{\partial x_i}\epsilon_{si}^{\frac{k-2}{2}}$ and $m_{si}^{sk}=0$ for an even k, it follows from the inequalities (B.6) and (B.7) in Appendix B that $\underline{\delta}_i^k\le\delta_i^k$ holds for any $k\ge 3$. Recalling $\underline{\delta}_i^3\le\min(\delta_i^1,\delta_i^2,\delta_i^3)$ and noting from (6.51) that $\underline{\delta}_i^{k+1}\le\underline{\delta}_i^k$ holds for any $k\ge 3$, it follows that $\underline{\delta}_i^\xi\le\min(\delta_i^1,\dots\delta_i^\xi)$, i.e., the conditions (6.27) along with $\delta_i=\underline{\delta}_i^\xi$ imply that (6.27) holds with $\delta_i=\min(\delta_i^1,\dots,\delta_i^\xi)$, $i\in\mathcal{N}$. The proof is complete. □

Remark 6.6. Note that the accumulation ψ_{si}^α or $\Psi_{si}^\alpha=0$ for $\alpha<0$ and $\alpha=0$ can be understood as 0 and 1, respectively. In this case, the expression (6.51) for $\underline{\delta}_i^k$ is compatible with (6.29) and (6.43) for $k=2$ and $k=3$, respectively. Furthermore, the expression (6.49) for δ_i^k is compatible with δ_i^2 and δ_i^3 defined in (6.28) and (6.28) for $k=2$ and $k=3$, respectively. Therefore, the results in Theorem 6.2 are understood as a synthesis of the ones in Propositions 6.2 and 6.4.

Remark 6.7. Consider the noncooperative systems with the quadratic payoff functions $J_i(x)$, $i\in\mathcal{N}$, given by

$$J_i(x)=\frac{1}{2}x^{\mathrm{T}}A_ix+b_i^{\mathrm{T}}x+c_i,\quad i\in\mathcal{N},\tag{6.52}$$

where $A_i\triangleq\begin{bmatrix}a_{11}^i & \cdots & a_{1n}^i\\ \vdots & \ddots & \vdots\\ a_{n1}^i & \cdots & a_{nn}^i\end{bmatrix}\in\mathbb{R}^{n\times n}$ with $a_{ii}^i<0$ (indicating that $J_i(x)$ is strictly concave with respect to x_i) and $a_{ij}^i=a_{ji}^i$, $b_i\triangleq[b_1^i,\dots,b_n^i]^{\mathrm{T}}\in\mathbb{R}^n$, and $c_i\in\mathbb{R}$, $i\in\mathcal{N}$. In this case, the best-response state $\mathrm{BR}_i(x_{-i})$ is given by

$$\mathrm{BR}_i(x_{-i})=-\frac{\sum_{j\ne i}a_{ij}^ix_j+b_i^i}{a_{ii}^i},\quad i\in\mathcal{N},\tag{6.53}$$

so that $\frac{\partial \mathrm{BR}_i(x_{-i}^*)}{\partial x_j}=-\frac{a_{ij}^i}{a_{ii}^i}$ holds for $i\ne j$. Supposing that $\hat{A}=\begin{bmatrix}\mathrm{row}_1(A_1)\\ \vdots\\ \mathrm{row}_n(A_n)\end{bmatrix}$ is nonsingular, it follows that there exists a unique Nash equilibrium x^* given by $x^*=-\hat{A}^{-1}b\in\mathbb{R}^n$ for $b=[b_1^1,\dots,b_n^n]^{\mathrm{T}}$. In this case, the conditions in Theorems 6.1 and 6.2 can further guarantee globally asymptotic stability of x^*. Moreover, the condition

(6.51) can be explicitly expressed by

$$\underline{\delta}_i^k = -\sum_{j\neq i}\left|a_{ij}^i\right| - \sum_{j\neq i}\sum_{s\neq i}\left|a_{is}^i\frac{a_{si}^s}{a_{ss}^s}\frac{a_{ij}^i}{a_{ii}^i}\right|(\sigma_{si}^0+\ldots+\sigma_{si}^{\frac{k-3}{2}})$$
$$-\sum_{j\neq s}\sum_{s\neq i}\left|a_{is}^i\frac{a_{sj}^s}{a_{ss}^s}\right|(\sigma_{si}^0+\ldots+\sigma_{si}^{\frac{k-3}{2}}),\quad k\in\mathbb{Z}_{\mathrm{o}}; \tag{6.54}$$
$$\underline{\delta}_i^k = -\sum_{j\neq i}\left|a_{ij}^i\right| - \sum_{j\neq i}\sum_{s\neq i}\left|a_{is}^i\frac{a_{si}^s}{a_{ss}^s}\frac{a_{ij}^i}{a_{ii}^i}\right|(\sigma_{si}^0+\ldots+\sigma_{si}^{\frac{k-4}{2}})$$
$$-\sum_{j\neq s}\sum_{s\neq i}\left|a_{is}^i\frac{a_{sj}^s}{a_{ss}^s}\right|(\sigma^0+\ldots+\sigma^{\frac{k-2}{2}}),\quad k\in\mathbb{Z}_{\mathrm{e}}. \tag{6.55}$$

with $\sigma_{si}=|\epsilon_{si}|=|\frac{a_{si}^s}{a_{ss}^s}\frac{a_{is}^i}{a_{ii}^i}|\geq 0$. In the case where the payoff functions are nonquadratic, it follows from (6.23) that $\underline{\delta}_i^k$ can also be explicitly expressed by (6.54) and (6.55) with a_{ij}^i replaced by $\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_j}$, $i,j\in\mathcal{N}$.

Remark 6.8. It follows from Chapter 2 that the condition (6.27) with $\delta_i=\underline{\delta}_i^1\triangleq\delta_i^1$ guarantees asymptotic stability of the Nash equilibrium for the conventional pseudo-gradient dynamics for any sensitivity parameters. Furthermore, note that $\underline{\delta}_i^{k+1}\leq\underline{\delta}_i^k$ holds for any $k\in\mathbb{Z}_+$, which implies that the condition $\delta_i=\underline{\delta}_i^\xi$ in Theorem 6.2 requires a smaller $\underline{\delta}_i^\xi$ for a noncooperative system with higher cognitive hierarchy levels.

The following proposition reveals the fact that, compared to the conventional pseudo-gradient dynamics, Level-ξ thinking may destabilize the Nash equilibrium x^* for a two-agent noncooperative system with $\xi=4,8,12,\ldots$, but never change the stability of x^* for other cases. An example showing the destabilized vector fields when $\xi=4$ are illustrated in Fig. 6.5, which also indicates that agents' Level-k thinking may bring more equilibria in the pseudo-gradient dynamics. In such a case, even though the Nash equilibria are still the equilibria of the prediction-incorporated pseudo-gradient dynamics, the trajectory of the agents' state may be attracted to other equilibria when we change ξ.

Proposition 6.5. Consider the two-agent noncooperative system $\mathcal{G}(\{J_1,J_2\})$ with both the agents at Level-ξ following either the pseudo-gradient dynamics (6.3), (6.4), or (6.12) under the complete knowledge network $G(\mathcal{N},E)$ with $\mathbb{L}_\xi=\{1,2\}$ and $E=\{(1,2),(2,1)\}$. If the payoff functions $J_1(x)$ and $J_2(x)$ satisfy $\det\mathcal{A}(J,1_2,x^*)>0$ (resp., $\det\mathcal{A}(J,1_2,x^*)<0$) for $\xi\in\mathbb{Z}_+$ satisfying $\xi\neq 4m$ for any $m\in\mathbb{Z}_+$, then the Nash equilibrium x^* satisfying (2.1) is locally asymptotically stable (resp., unstable) for

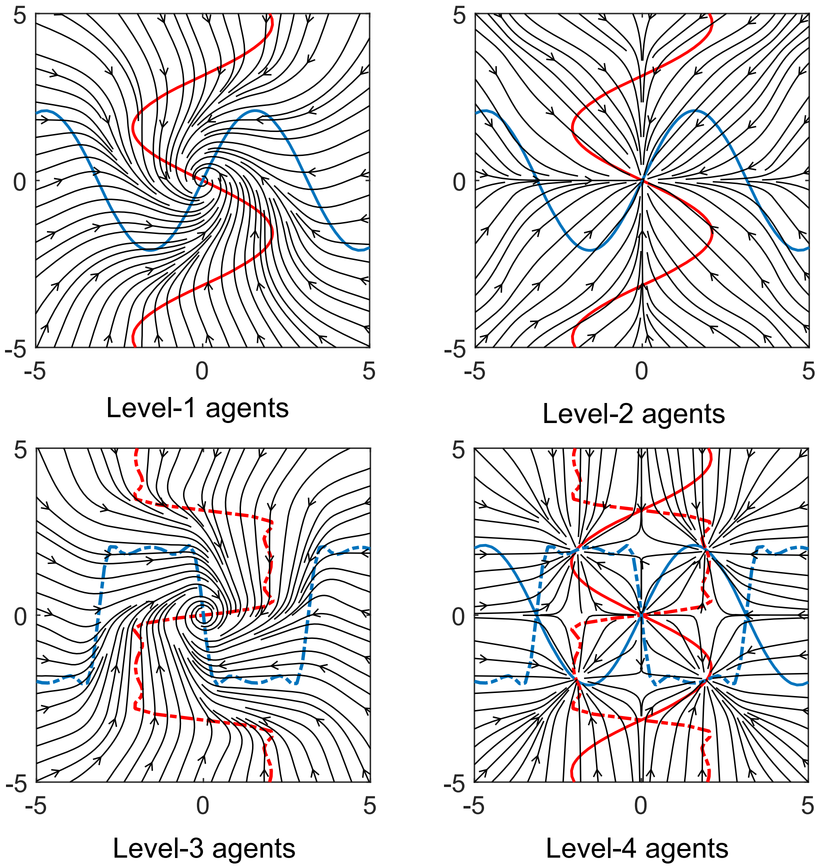


Figure. 6.5 Vector fields of the prediction-incorporated pseudo-gradient dynamics of a two-agent noncooperative system with a unique Nash equilibrium $x^* = [0, 0]^{\mathrm{T}}$ and Level-ξ agents for $\xi = 1, 2, 3, 4$. The red solid lines: $x_1 = \mathrm{BR}_1(x_2)$; blue solid lines: $x_2 = \mathrm{BR}_2(x_1)$; red dash-doted lines: $x_1 = \mathrm{BR}_1(\mathrm{BR}_2(\mathrm{BR}_1(x_2)))$; blue dash-doted lines: $x_2 = \mathrm{BR}_2(\mathrm{BR}_1(\mathrm{BR}_2(x_1)))$. When $\xi = 4$, there exists 9 equilibria in the pseudo-gradient dynamics with x^* being destabilized.

any sensitivity parameters $\alpha_1, \alpha_2 \in \mathbb{R}_+$. Alternatively, if the payoff functions $J_1(x)$ and $J_2(x)$ satisfy $\left|\frac{\partial^2 J_1(x^*)}{\partial x_1 \partial x_2} \frac{\partial^2 J_2(x^*)}{\partial x_2 \partial x_1}\right| - \frac{\partial^2 J_1(x^*)}{\partial x_1^2} \frac{\partial^2 J_2(x^*)}{\partial x_2^2} < 0$ (resp., > 0) for $\xi \in \mathbb{Z}_+$ satisfying $\xi = 4m$ with some $m \in \mathbb{Z}_+$, then the Nash equilibrium x^* is locally asymptotically stable (resp., unstable) for any sensitivity parameters $\alpha_1, \alpha_2 \in \mathbb{R}_+$.

Proof Note that when ξ is even (i.e., $\xi = 2m$ for some $m \in \mathbb{Z}_+$), it follows that $\Pi_\xi(J, \alpha, x^*) = \mathrm{diag}[\alpha_1\tau_1, \alpha_2\tau_2]$ with

$$\tau_1 = \frac{\left(\frac{\partial^2 J_1(x^*)}{\partial x_1^2}\frac{\partial^2 J_2(x^*)}{\partial x_2^2}\right)^{\xi/2} - \left(\frac{\partial^2 J_1(x^*)}{\partial x_1 \partial x_2}\frac{\partial^2 J_2(x^*)}{\partial x_2 \partial x_1}\right)^{\xi/2}}{\frac{\partial^2 J_2(x^*)}{\partial x_2^2}\left(\frac{\partial^2 J_1(x^*)}{\partial x_1^2}\frac{\partial^2 J_2(x^*)}{\partial x_2^2}\right)^{\xi/2-1}}, \tag{6.56}$$

$$\tau_2 = \frac{\left(\frac{\partial^2 J_1(x^*)}{\partial x_1^2}\frac{\partial^2 J_2(x^*)}{\partial x_2^2}\right)^{\xi/2} - \left(\frac{\partial^2 J_1(x^*)}{\partial x_1 \partial x_2}\frac{\partial^2 J_2(x^*)}{\partial x_2 \partial x_1}\right)^{\xi/2}}{\frac{\partial^2 J_1(x^*)}{\partial x_1^2}\left(\frac{\partial^2 J_1(x^*)}{\partial x_1^2}\frac{\partial^2 J_2(x^*)}{\partial x_2^2}\right)^{\xi/2-1}}. \tag{6.57}$$

In this case, when $\xi/2$ is odd (i.e., $\xi = 4m + 2$ for some $m \in \mathbb{Z}_0$), it follows from the monotonically increasing function $f(x) = x^{\xi/2}$ that $\det \mathcal{A}(J, 1_2, x^*) = \frac{\partial^2 J_1(x^*)}{\partial x_1^2}\frac{\partial^2 J_2(x^*)}{\partial x_2^2} - \frac{\partial^2 J_1(x^*)}{\partial x_1 \partial x_2}\frac{\partial^2 J_2(x^*)}{\partial x_2 \partial x_1} > 0$ (resp., < 0) indicates $\tau_1, \tau_2 < 0$ (resp., > 0). Alternatively, when $\xi/2$ is even (i.e., $\xi = 4m$ for some $m \in \mathbb{Z}_+$), it follows that $\left|\frac{\partial^2 J_1(x^*)}{\partial x_1 \partial x_2}\frac{\partial^2 J_2(x^*)}{\partial x_2 \partial x_1}\right| - \frac{\partial^2 J_1(x^*)}{\partial x_1^2}\frac{\partial^2 J_2(x^*)}{\partial x_2^2} < 0$ (resp., > 0) indicates $\tau_1, \tau_2 < 0$ (resp., > 0). Therefore, the results for the case where ξ is even is immediate since $\tau_1, \tau_2 < 0$ (resp., > 0) implies the Jacobian matrix $\Pi_\xi(J, \alpha, x^*) = \mathrm{diag}[\alpha_1\tau_1, \alpha_2\tau_2]$ of the system dynamics (6.12) is stable (resp., unstable). Next, consider the case with odd ξ (i.e., $\xi = 2m + 1$ for a $m \in \mathbb{Z}_0$). In this case, $\Pi_\xi(J, \alpha, x^*) = \begin{bmatrix} \alpha_1 \frac{\partial^2 J_1(x^*)}{\partial x_1^2} & \alpha_1 \frac{\partial^2 J_1(x^*)}{\partial x_1 \partial x_2}\tau \\ \alpha_2 \frac{\partial^2 J_2(x^*)}{\partial x_2 \partial x_1}\tau & \alpha_2 \frac{\partial^2 J_2(x^*)}{\partial x_2^2} \end{bmatrix}$ where $\tau = \left(\frac{\partial^2 J_1(x^*)}{\partial x_1 \partial x_2}\frac{\partial^2 J_2(x^*)}{\partial x_2 \partial x_1}\right)^{\frac{\xi-1}{2}} / \left(\frac{\partial^2 J_1(x^*)}{\partial x_1^2}\frac{\partial^2 J_2(x^*)}{\partial x_2^2}\right)^{\frac{\xi-1}{2}}$. Furthermore, $\det \Pi_\xi(J, \alpha, x^*)$ is given by

$$\frac{\alpha_1\alpha_2\left[\left(\frac{\partial^2 J_1(x^*)}{\partial x_1^2}\frac{\partial^2 J_2(x^*)}{\partial x_2^2}\right)^{\xi} - \left(\frac{\partial^2 J_1(x^*)}{\partial x_1 \partial x_2}\frac{\partial^2 J_2(x^*)}{\partial x_2 \partial x_1}\right)^{\xi}\right]}{\left(\frac{\partial^2 J_1(x^*)}{\partial x_2 \partial x_1}\frac{\partial^2 J_2(x^*)}{\partial x_2^2}\right)^{\xi-1}}, \tag{6.58}$$

which possesses the opposite sign as $\det \mathcal{A}(J, 1_2, x^*)$ for an odd number ξ. Therefore, the results are immediate since $\det \mathcal{A}(J, 1_2, x^*) > 0$ (resp., < 0) implies the Jacobian matrix $\Pi_\xi(J, \alpha, x^*)$ of the system dynamics (6.12) is stable (resp., unstable) with negative diagonal terms. □

Next, we characterize the transition of the agents' targeted best-response state with respect to the cognitive level $\xi \in \mathbb{Z}_+$ in a two-agent noncooperative system with $\mathbb{L}_\xi = \{1, 2\}$ and quadratic payoff functions (6.52). Specifically, denote the targeted best-response state of agents $\mathbb{L}_\xi = \{1, 2\}$ for $\xi \in \mathbb{Z}_+$ at time t as $x^\xi_{\mathrm{tgt}}(t) = [x^\xi_{\mathrm{tgt}1}(t), x^\xi_{\mathrm{tgt}2}(t)]^{\mathrm{T}} \in \mathbb{R}^2$, where $x^\xi_{\mathrm{tgt}i}(t)$ is the targeted best-response state of agent $i \in \{1, 2\}$. Recalling that the predicted state $\mathrm{BR}^\xi_{j,i}(x)$ evaluated by agent $i \in \mathbb{L}_\xi$ coincides with the targeted best-response state of agent $j \in \mathbb{L}_{\xi-1}$ for $n = 2$ (i.e., $\mathrm{BR}^\xi_{j,i}(x(t)) = x^{\xi-1}_{\mathrm{tgt}j}(t)$, $i, j \in \{1, 2\}$, $i \neq j$), it follows that the targeted best-response state $x_{\text{tgt}}^{\xi}(t)$ with $\xi \geq 2$ is given by

$$x_{\text{tgt}}^{\xi}(t) = \begin{bmatrix} \text{BR}_1(\text{BR}_{2,1}^{\xi}(x(t))) \\ \text{BR}_2(\text{BR}_{1,2}^{\xi}(x(t))) \end{bmatrix} = \begin{bmatrix} \text{BR}_1(x_{\text{tgt}2}^{\xi-1}(t)) \\ \text{BR}_2(x_{\text{tgt}1}^{\xi-1}(t)) \end{bmatrix},$$

whereas $x_{\text{tgt}}^{1}(t) = [\text{BR}_1(x_2(t)), \text{BR}_2(x_1(t))]^{\text{T}}$ is understood as the targeted best-response state state in the conventional pseudo-gradient dynamics. Now, it follows from the best-response mapping (6.53) that the transition of the targeted best-response state with respect to the cognitive level follows the recursive relation given by

$$x_{\text{tgt}}^{\xi+1}(t) = Bx_{\text{tgt}}^{\xi}(t) + C, \quad \xi = 1, 2, 3, \ldots, \tag{6.59}$$

with $x_{\text{tgt}}^{1}(t) = Bx(t) + C$, where

$$B \triangleq \Lambda(\cdot) = \begin{bmatrix} 0 & -\frac{a_{12}^1}{a_{11}^1} \\ -\frac{a_{12}^2}{a_{22}^2} & 0 \end{bmatrix}, \quad C = \begin{bmatrix} -\frac{b_1^1}{a_{11}^1} \\ -\frac{b_2^2}{a_{22}^2} \end{bmatrix}. \tag{6.60}$$

Noticing that the recursive relation (6.59) possesses a similar expression as the best-response dynamics [16] given with slight abuse of notation by

$$x(t+1) = Bx(t) + C, \quad t = 0, 1, 2, \ldots, \tag{6.61}$$

where $Bx^* + C = x^*$ holds and hence the Nash equilibrium x^* is the fixed point of (6.61). Here, since both of the eigenvalues λ_1, λ_2 of B satisfy $\lambda_1^2 = \lambda_2^2 = a_{12}^1 a_{12}^2 / a_{11}^1 a_{22}^2$, it follows from the property of discrete-time linear systems that the agents' targeted best-response state $x_{\text{tgt}}^{\xi}(t)$ converges to the Nash equilibrium x^* as $\xi \to \infty$ for any $x(t) \in \mathbb{R}^2$ when $\left| a_{12}^1 a_{12}^2 / a_{11}^1 a_{22}^2 \right| < 1$. An example showing Level-ξ agents' targeted best-response state $x_{\text{tgt}}^{\xi}(t)$ with $\xi = 1, \ldots, 8$ is illustrated in Fig. 6.6(b) where the agents' targeted best-response state $x_{\text{tgt}}^{\xi}(t)$ converges to the Nash equilibrium x^* as $\xi \to \infty$.

6.4 Incentive-Based Stabilization by a System Manager

In this section, assuming the existence of the system manager who has all the information of the payoff functions $J_i(x)$, $i \in \mathcal{N}$, and is authorized to design an incentive rule, we generalize the stabilization method via zero-sum tax/subsidy approach in Chapter 2

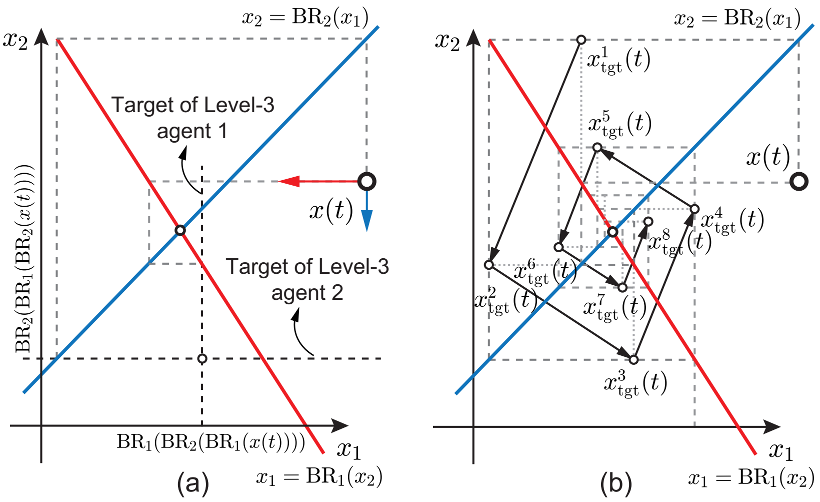


Figure. 6.6 Target states of the agents of a two-agent noncooperative system with the agents at different hierarchy levels. (a): both agents are Level 3; (b): both the agents are Level-ξ with $\xi = 1, 2, 3, 4, \ldots$. The knowledge network of payoff functions are considered with $E = \{(1, 2), (2, 1)\}$.

to ensure stability of a Nash equilibrium for the agents at Level-$k \leq \xi$ with $\xi > 1$. Consider the incentivized payoff functions

$$\tilde{J}_i(x) \triangleq J_i(x) + p_i^K(x), \quad i \in \mathcal{N}, \tag{6.62}$$

with the quadratic incentive functions

$$p_i^K(x) \triangleq \frac{1}{2}k_{ii}(x_i - x_i^*)^2 - \frac{1}{2}\sum\nolimits_{j \neq i} k_{jj}(x_j - x_j^*)^2/(n-1) \\ + \sum\nolimits_{j \neq i} k_{ij}(x_i - x_i^*)(x_j - x_j^*), \quad i \in \mathcal{N}, \tag{6.63}$$

where $K = \{k_{ij}\}_{i,j \in \mathcal{N}} \in \mathcal{K} \triangleq \{K \in \mathbb{R}^{n \times n} : k_{ii} \leq 0, i \in \mathcal{N}, k_{ij} = -k_{ji}, i, j \in \mathcal{N}, i \neq j\}$. In this case, the sum of all the incentive functions satisfy $\sum_{i \in \mathcal{N}} p_i^K(x) = 0$ for all $x \in \mathbb{R}^n$ and hence the system manager serves merely as a mediator in the noncooeprative system to assure that every subsidy is financed by taxes taken from the others, i.e., $\sum_{i \in \mathcal{N}} \tilde{J}_i(x) = \sum_{i \in \mathcal{N}} J_i(x)$. Furthermore, the Nash equilibrium x^* of $\mathcal{G}(J)$ remains the Nash equilibrium of $\mathcal{G}(\tilde{J})$ (see Chapter 2).

Assuming that the cognitive hierarchy levels for each of the agents are known, we present a sufficient condition to ensure stabilization for the Nash equilibrium x^*.

Corollary 6.1. Consider the incentivized noncooperative system $\mathcal{G}(\tilde{J})$ with the agents at Level-$k \leq \xi$ with $\xi \geq 3$ satisfying $\mathbb{L}_1 \cup \cdots \cup \mathbb{L}_\xi = \mathcal{N}$ so that the agents follow the pseudo-gradient dynamics (6.3), (6.4), and (6.12) with $J_i(x)$ replaced by incentivized $\tilde{J}_i(x)$, depending on their cognitive hierarchy levels. If there exists $\hat{\alpha} \in \mathbb{R}_+^N$ such that

$$\Pi_\xi^{\mathrm{T}}(\tilde{J}, \hat{\alpha}, x^*) + \Pi_\xi(\tilde{J}, \hat{\alpha}, x^*) < 0, \tag{6.64}$$

then the incentive functions (6.63) guarantees that the Nash equilibrium x^* is asymptotically stabilized for any sensitivity parameters $\alpha_i \in \mathbb{R}_+$, $i \in \mathcal{N}$.

Proof The result is a direct consequence of Theorem 6.1. □

Now, supposing that the cognitive hierarchy levels for each of the agents are uncertain, the following result provides some sufficient conditions to guarantee stability *without knowing* $\mathbb{L}_1, \cdots, \mathbb{L}_\xi$.

Corollary 6.2. Consider the incentivized noncooperative system $\mathcal{G}(\tilde{J})$ with the agents at Level-$k \leq \xi$ with $\xi \geq 3$ satisfying $\mathbb{L}_1 \cup \cdots \cup \mathbb{L}_\xi = \mathcal{N}$ so that the agents follow the pseudo-gradient dynamics (6.3), (6.4), and (6.12) with $J_i(x)$ replaced by incentivized $\tilde{J}_i(x)$, depending on their cognitive hierarchy levels. If the matrix $K \in \mathcal{K}$ in (6.63) satisfies

$$k_{ii} < \delta_i - \frac{\partial^2 J_i(x^*)}{\partial {x_i}^2}, \quad i \in \mathcal{N}, \tag{6.65}$$

with $\delta_i = \min(\delta_i^1, \ldots, \delta_i^\xi)$ for δ_i^k, $k = 1, \ldots, \xi$, defined in (6.28) and (6.49) with $\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_j}$ replaced by $\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_j} + k_{ij}$, and $\frac{\partial \mathrm{BR}_i(x^*_{-i})}{\partial x_j}$ replaced by $-(\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_j} + k_{ij})/(\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_i} + k_{ii})$, $i, j \in \mathcal{N}$, then the incentive functions (6.63) guarantees that the Nash equilibrium x^* is asymptotically stabilized for any cognitive level sets $\mathbb{L}_1, \cdots, \mathbb{L}_\xi$ and any sensitivity parameters $\alpha_i \in \mathbb{R}_+$, $i \in \mathcal{N}$. If, in addition, (6.65) holds with $\delta_i = \underline{\delta}_i^\xi$ defined in (6.54), (6.55) with a_{ij}^i replaced by $\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_j} + k_{ij}$, $i, j \in \mathcal{N}$, then the Nash equilibrium x^* is locally asymptotically stable for any knowledge network $G(\mathcal{N}, E)$ of the payoff functions.

Proof The result is a direct consequence of Theorem 6.2 by noting from (6.23) that $\frac{\partial \widetilde{\mathrm{BR}}_i(x^*_{-i})}{\partial x_j} = -(\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_j} + k_{ij})/(\frac{\partial^2 J_i(x^*)}{\partial x_i^2} + k_{ii})$ for $j \neq i$. □

It can be easily found that n number of inequalities characterized by (6.65) are always solvable for $K \in \mathcal{K}$, because k_{ii}, $i \in \mathcal{N}$, can be taken to be sufficiently small so that each agent's own payoff is dominant compared to the effect by the other agents. Different from Corollary 6.1, Corollary 6.2 gives contribution to find the explicit lower boundary for k_{ii}, $i \in \mathcal{N}$, guaranteeing asymptotic stabilization without knowing $G(\mathcal{N}, E)$.

6.5 Applications With Numerical Examples

In this section, a couple of numerical examples are presented for illustrating the results and the conditions concerning the proposed stabilization method by incentive functions.

6.5.1 Application to Optical Communication System

Consider a power control problem in optical communication system with n channels who compete with each other on quality of service characterized by channel optical signal-to-noise ratio (OSNR). Each channel adjusts its input power $x_i \in \mathbb{R}_+$ to maximize its profit [101] given by

$$J_i(x) = \beta_i \ln(1 + \frac{a_i x_i}{\sum_{j \neq i} \Gamma_{ij} x_j + \sigma}) - \gamma_i x_i, \quad i \in \mathcal{N}, \tag{6.66}$$

where $\beta_i \in \mathbb{R}_+$ is the earning rate for optical communication quality, $\sigma \in \mathbb{R}_+$ is the constant noise power, $a_i \in \mathbb{R}_+$ is a channel specific parameter, $\Gamma_{ij} \in \mathbb{R}_+$, $j \neq i$, are the channel gains, and $\gamma_i \in \mathbb{R}_+$, $i \in \mathcal{N}$, denote the price per unit power. It follows from $\partial J_i(x)/\partial x_i = \frac{a_i \beta_i}{\sum_{j \neq i} \Gamma_{ij} x_j + \sigma + a_i x_i} - \gamma_i$ that the Nash equilibrium x^* satisfying $a_i x_i^* + \sum_{j \neq i} \Gamma_{ij} x_j^* = \frac{a_i \beta_i}{\gamma_i} - \sigma$, $i \in \mathcal{N}$, is unique and given by $\tilde{\Gamma}^{-1}\tilde{b}$ where $\tilde{\Gamma}_{ij} = \Gamma_{ij}$, $j \neq i$, $\tilde{\Gamma}_{ii} = a_i$, $i \in \mathcal{N}$, and $\tilde{b}_i = \frac{a_i \beta_i}{\gamma_i} - \sigma$, $i \in \mathcal{N}$, when the matrix $\tilde{\Gamma}$ is non-singular. Furthermore, it is obtained that

$$\frac{\partial^2 J_i(x)}{\partial x_i \partial x_j} = \begin{cases} \frac{-a_i \beta_i \Gamma_{ij}}{(\sum_{j \neq i} \Gamma_{ij} x_j + \sigma + a_i x_i)^2}, & j \neq i; \\ \frac{-a_i^2 \beta_i}{(\sum_{j \neq i} \Gamma_{ij} x_j + \sigma + a_i x_i)^2}, & j = i. \end{cases} \tag{6.67}$$

Example 6.1. Consider $n = 3$ for the optical communication system and let $a_1 = 0.74$, $a_2 = 0.79$, $a_3 = 0.52$, $\beta_1 = 3.656$, $\beta_2 = 4.28$, $\beta_3 = 7$, $\Gamma_{12} = 2.5$, $\Gamma_{13} = 1.4$, $\Gamma_{21} = 1.8$, $\Gamma_{23} = 1.8$, $\Gamma_{31} = 3.7$, $\Gamma_{32} = 1.0$, $\gamma_1 = 4$, $\gamma_2 = 4$, $\gamma_3 = 1$, and $\sigma = 0.0043$, so that there exists a unique Nash equilibrium given by $x^* = [1.7568, 2.7188, 4.2384]^{\mathrm{T}}$. In this example, since $\mathcal{A}^{\mathrm{T}}(J, 1_3, x^*) + \mathcal{A}(J, 1_3, x^*) < 0$ holds, it follows that the Nash equilibrium x^* is asymptotically stable in the conventional pseudo-gradient dynamics for any sensitivity parameters. Figure 6.7 shows the trajectories of agents' states under the pseudo-gradient dynamics (6.3), (6.4), (6.12) with $\mathbb{L}_3 = \{1, 2\}$ and agent 3 at Level-$k \leq 3$ under 8 different knowledge graphs satisfying $\{(2,1), (1,2), (3,1)\} \subseteq E$, where agent 3 is understood as a Level-1 agent if $\{(1,3)\} \not\subset E$ and $\{(2,3)\} \not\subset E$, a Level-2 agent if $\{(2,3)\} \subset E$ but $\{(1,3), (3,2)\} \not\subset E$, and a Level-3 agent otherwise. In the simulation, the initial state is set to $x(0) = [1, 4, 2]^{\mathrm{T}}$ with random α satisfying

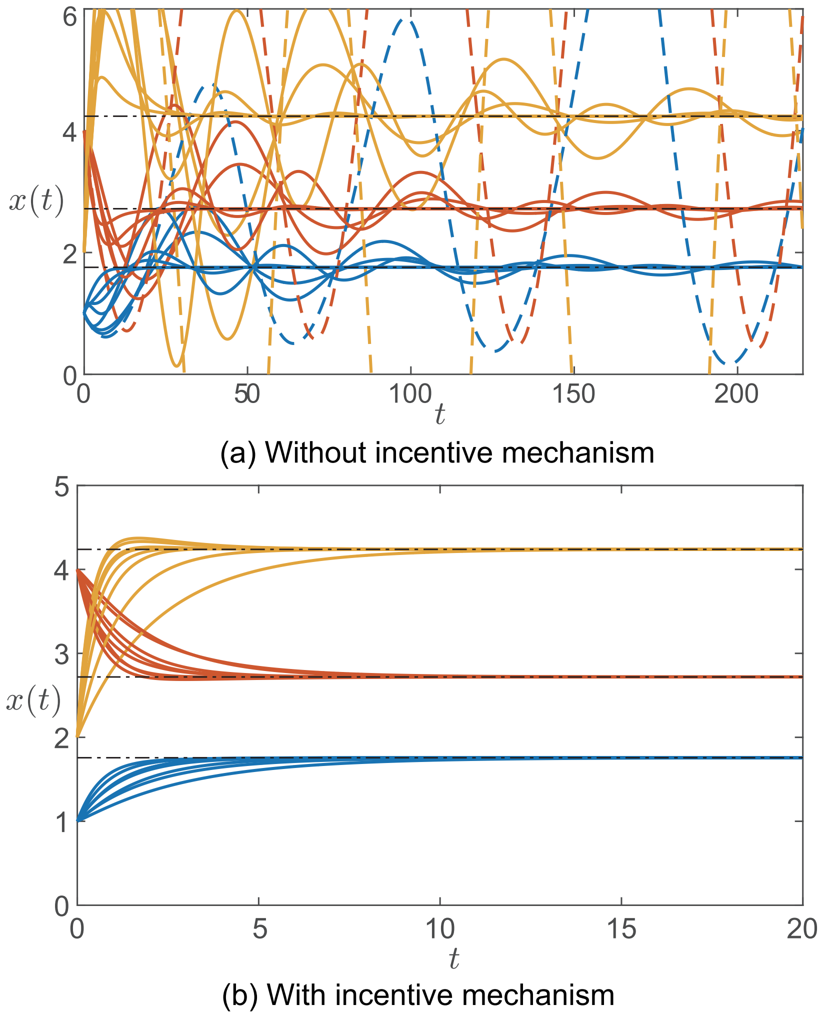


Figure. 6.7 Trajectories of $x(t)$ under the incentive function (6.63) with 8 different knowledge graphs satisfying $\{(2,1),(1,2),(3,1)\} \subseteq E$. All the agents are Level-3, i.e., $\mathbb{L}_3 = \{1,2,3\}$. The diverged dashed lines in (a) are simulated under $E = \{(2,1),(1,2),(3,1),(3,2)\}$.

$\alpha_1, \alpha_2 \in [2,4]$ and $\alpha_3 \in [4,6]$. It can be seen from Fig. 6.7(a) that the Nash equilibrium x^* may be unstable under Level-3 thinking for some knowledge graphs (see the diverged dashed lines which correspond to the case where $\mathbb{L}_3 = \{1,2\}$ and $\mathbb{L}_1 = \{3\}$ under $E = \{(2,1),(1,2),(3,1),(3,2)\}$).

Now, it follows from Corollary 6.2 that the incentive mechanism (6.62) along with the incentive function (6.63) with $k_{11} = k_{22} = -0.12$ and $k_{33} = -0.262$ satisfying (6.65) with $\delta_i = \underline{\delta}_i^{\xi}$ defined in (6.54) for $\xi = 3$ guarantees that the Nash equilibrium x^* is asymptotically stabilized for any cognitive level sets $\mathbb{L}_1, \mathbb{L}_2, \mathbb{L}_3$, any sensitivity parameters $\alpha_i \in \mathbb{R}_+$, $i \in \mathcal{N}$, and any knowledge network $G(\mathcal{N}, E)$. This result can be verified by the trajectories of the agents' state shown in Fig. 6.7(b).

6.5.2 Application to Cournot Games in Homogeneous Oligopoly

Consider a market being composed of n firms selling homogeneous products produced by themselves, where the market price (inverse demand) function [97] is given by

$$\lambda = \lambda_0 - \sum_{i=1}^{n} \beta_i x_i, \tag{6.68}$$

where $x_i \in \mathbb{R}_+$ denotes the quantity of the produced products, $\beta_i \in \mathbb{R}_+$ denotes the market power of the firm-i, and $\lambda_0 \in \mathbb{R}_+$ is a market specific parameter representing the cap price. In this market, firms compete in quantities rather than prices according to the payoff functions given by

$$J_i(x) = \lambda x_i - C_i(x_i), \quad i \in \mathcal{N}, \tag{6.69}$$

where $C(\cdot)$ denotes the production cost of firm-i given by

$$C_i(x_i) = a_i x_i^2 + b_i x_i, \quad i \in \mathcal{N}, \tag{6.70}$$

with $a_i \geq 0$ and $b_i > 0$. Here, it is straightforward to see that increasing the production quantity x_i may result in decreasing the market price λ significantly for a large market power β_i. Therefore, the market powers β_i, $i \in \mathcal{N}$, are understood as the parameters representing the sensitivity of the market in terms of the influence of individual firms by manipulating the supply of the product. It follows from $\partial J_i(x)/\partial x_i = -2(a_i + \beta_i)x_i - \sum_{j \neq i} \alpha_j x_j + \lambda_0 - b_i$ that the Nash equilibrium x^* satisfying $-2(a_i + \beta_i)x_i^* - \sum_{j \neq i} \beta_j x_j^* + \lambda_0 - b_i = 0$, $i \in \mathcal{N}$, is unique and given by $-\hat{A}^{-1}b$ with $\hat{A}_{ij} = -\beta_j$, $j \neq i$, $\hat{A}_{ii} = -2(a_i + \beta_i)$, $i \in \mathcal{N}$, and $b_i = \lambda_0 - b_i$, $i \in \mathcal{N}$, when the matrix $\hat{A}$ is non-singular. Moreover, since $\frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_j} = \hat{A}_{ij} < 0$, it follows from the properties of Metzler matrices that the Nash equilibrium is asymptotically stable (for arbitrary α) under the conventional pseudo-gradient dynamics if and only if there exists $\hat{\alpha} \in \mathbb{R}_+^N$ such that $\mathcal{A}^{\mathrm{T}}(J, \hat{\alpha}, x^*) + \mathcal{A}(J, \hat{\alpha}, x^*) = \hat{A}^{\mathrm{T}}\mathrm{diag}[\hat{\alpha}] + \mathrm{diag}[\hat{\alpha}]\hat{A} < 0$. Supposing that the knowledge network is a complete graph, the following result is immediate.

Theorem 6.3. Suppose that the knowledge network is a complete graph. Then, it follows that the Nash equilibrium x^* of the Cournot game (6.68)–(6.70) under the pseudo-gradient dynamics (6.4) with $\mathbb{L}_2 = \mathcal{N}$ is asymptotically stable for any sensitivity parameters $\alpha_i \in \mathbb{R}_+$, $i \in \mathcal{N}$, if and only if there exists $\hat{\alpha} \in \mathbb{R}_+^N$ such that $\hat{R}^{\mathrm{T}}\mathrm{diag}[\hat{\alpha}] + \mathrm{diag}[\hat{\alpha}]\hat{R} < 0$ for $\hat{R} = [\hat{R}_{ij}]$ defined with $\hat{R}_{ij} = \sum_{s \notin \{i,j\}} \frac{\beta_j \beta_s}{2a_s + 2\beta_s} > 0$, $j \neq i$, $\hat{R}_{ii} = -2a_i - 2\beta_i + \sum_{s \neq i} \frac{\beta_i \beta_s}{2a_s + 2\beta_s}$, $i \in \mathcal{N}$.

Proof First, recall the fact from Remark 6.3 that $R(J, 1_n, x^*) = 2\hat{A} - \hat{A}\mathrm{diag}[\psi]\hat{A}$ for the knowledge network being a complete graph, where $P = [P_{ij}] =$ satisfies $P_{ij} = -2\beta_j - \sum_{s\notin\{i,j\}} \frac{\beta_j\beta_s}{2a_s+2\beta_s}$, $j \neq i$, $P_{ii} = -2a_i - 2\beta_i - \sum_{s\neq i} \frac{\beta_i\beta_s}{2a_s+2\beta_s}$, $i \in \mathcal{N}$. Recalling $\hat{A}_{ij} = -\beta_j$, $j \neq i$, $\hat{A}_{ii} = -2(a_i + \beta_i)$, it follows that $\hat{R} = R_2(J, 1_n, x^*) = \Pi_2(J, 1_n, x^*)$ and hence the sufficiency result is a direct consequence of Proposition 6.1. Furthermore, since $\Pi_2(J, \alpha, x^*) = R_2(J, \alpha, x^*) = \mathrm{diag}[\alpha]\hat{R}$ is a Metzler matrix, it follows that $\Pi_2(J, \alpha, x^*)$ is Hurwitz only if there exists $\hat{\alpha} \in \mathbb{R}_+^N$ such that $\hat{R}^{\mathrm{T}}\mathrm{diag}[\hat{\alpha}] + \mathrm{diag}[\hat{\alpha}]\hat{R} < 0$. Thus, the necessity is immediate. □

Example 6.2. Consider $n = 5$ for Cournot game (market) and let $a_1 = 0.23$, $a_2 = 0.35$, $a_3 = 0.46$, $a_4 = 0.18$, $a_5 = 0.05$, $\beta_1 = 1.09$, $\beta_2 = 1.42$, $\beta_3 = 1.99$, $\beta_4 = 1.19$, $\beta_5 = 1.54$, $b_1 = 5.2$, $b_2 = 3.6$, $b_3 = 6.6$, $b_4 = 3.2$, $b_5 = 5.2$, and $\lambda_0 = 15$, so that there exists a unique Nash equilibrium given by $x^* = [1.1038, 1.5618, 0.1068, 2.3941, 1.0432]^{\mathrm{T}}$. In this example, since $\mathcal{A}^{\mathrm{T}}(J, 1_5, x^*) + \mathcal{A}(J, 1_5, x^*) < 0$ holds, it follows that the Nash equilibrium x^* is asymptotically stable in the conventional pseudo-gradient dynamics for any sensitivity parameters. However, since there is no feasible $\hat{\alpha}$ in the linear matrix inequality (LMI) feasibility problem $\hat{R}^{\mathrm{T}}\mathrm{diag}[\hat{\alpha}] + \mathrm{diag}[\hat{\alpha}]\hat{R} < 0$, it follows from Theorem 6.3 that the Nash equilibrium is unstable under the pseudo-gradient dynamics (6.4) with $\mathbb{L}_2 = \mathcal{N}$ for any sensitivity parameters. This result can be verified by the dashed trajectories of agents' state shown in Fig. 6.8(a), where the sensitivity parameter α are set to $\alpha_1, \alpha_3, \alpha_4 \in [0.5, 2.5]$, $\alpha_2, \alpha_5 \in [1, 3]$, and the initial state is $x(0) = [2, 1, 1, 1, 1]^{\mathrm{T}}$.

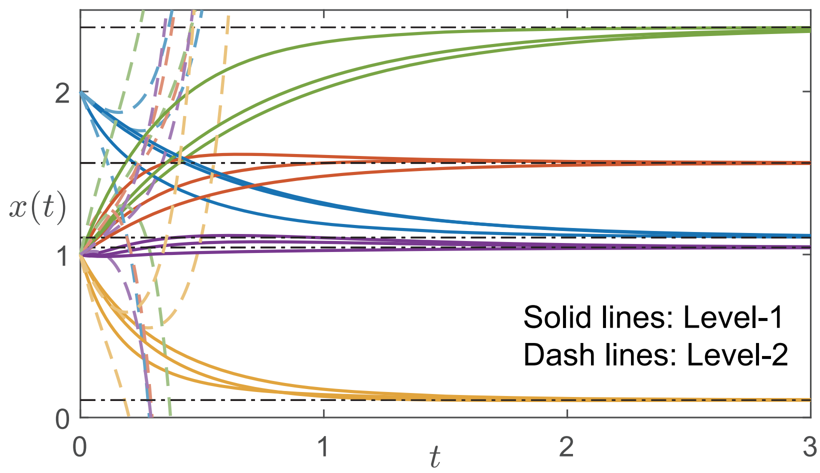
Now it follows from Corollary 6.1 that the incentive mechanism (6.62) along with the incentive function (6.63) with $k_{11} = k_{22} = -3$, $k_{33} = k_{44} = -2$, and $k_{44} = -4$ satisfying (6.64) with $\hat{\alpha} = 1_n$ guarantees that the Nash equilibrium x^* is asymptotically stabilized under the pseudo-gradient dynamics (6.4) with $\mathbb{L}_2 = \mathcal{N}$ for any sensitivity parameters $\alpha_i \in \mathbb{R}_+$, $i \in \mathcal{N}$. This result can be verified by the trajectories of the agents' state shown in Fig. 6.8(b).

6.5.3 Application to Differentiated Oligopoly

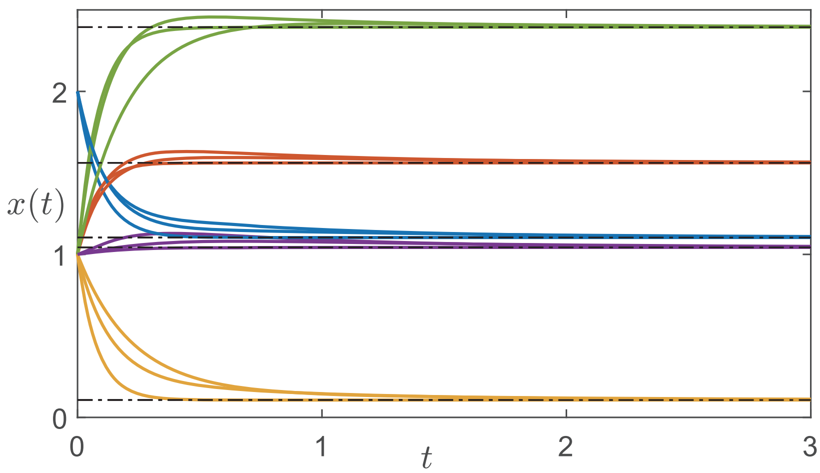
Consider a market being composed of n firms selling different products, where the market price (inverse demand) function [102] is given by

$$p_i = \lambda_0 - \beta q_i - \beta\delta \sum_{j\neq i} q_j, \quad i \in \mathcal{N}, \tag{6.71}$$

where $p_i \in \mathbb{R}_+$ denotes the price of the produced products, $\lambda_0 \in \mathbb{R}_+$ denotes the cap price, $q_i \in \mathbb{R}_+$ denotes the quantity of the produced products, $\beta \in \mathbb{R}_+$ denotes a market



(a) Without incentive mechanism



(b) With incentive mechanism

Figure. 6.8 Trajectories of $x(t)$ under the pseudo-gradient dynamic with Level-1 and Level-2 agents. In (a), solid line: $\mathbb{L}_1 = \mathcal{N}$; dashed: $\mathbb{L}_2 = \mathcal{N}$. In (b), $\mathbb{L}_2 = \mathcal{N}$. In both (a) and (b), blue: $x_1(t)$, orange: $x_2(t)$, yellow: $x_3(t)$, green: $x_4(t)$, purple: $x_5(t)$.

power, and $\delta \in [0, 1)$ denotes the degree of product differentiation. In this market, firms compete in either quantities or prices according to the payoff functions given by

$$\Pi_i = (p_i - c_i)q_i, \quad i \in \mathcal{N}, \tag{6.72}$$

where $c_i \in \mathbb{R}_+$ denotes the marginal cost of firm-i satisfying $c_i < \lambda_0$. Here, it is worth noting that a lager δ indicates a smaller differentiation among the products. That is to say, if δ is extremely close to 1, then it is understood that the n firms are selling almost homogeneous products in the market, whereas the n firms are selling almost totally different types of products in the market if $\delta = 0$. This is because the price of firm's product is closely related to existing of replaceable products. In terms of the (dynamic) strategy of the firms, there are two different competitions named Cournot and Bertrand competition for the case when the firms compete in quantities and prices respectively.

Cournot competition

Consider the quantities as the decision variables of the firms (i.e., $x_i = q_i$) so that the payoff functions from (6.72) are given by

$$J_i(x) = (\lambda_0 - \beta x_i - \sum\nolimits_{j \neq i} \beta\delta x_j - c_i)x_i, \quad i \in \mathcal{N}. \tag{6.73}$$

In this case, it follows from $\partial J_i(x)/\partial x_i = -2\beta x_i - \beta\delta \sum_{j\neq i} x_j + \lambda_0 - c_i$ that the Nash equilibrium x^* satisfying $-2\beta x_i^* - \beta\delta \sum_{j\neq i} x_j^* + \lambda_0 - c_i = 0$, $i \in \mathcal{N}$, is unique and given by $-\hat{A}^{-1}b$ with $\hat{A}_{ij} = -\beta\delta < 0$, $j \neq i$, $\hat{A}_{ii} = -2\beta < 0$, $i \in \mathcal{N}$, and $b_i = \lambda_0 - c_i$, $i \in \mathcal{N}$.

Lemma 6.1. The Nash equilibrium x^* of the n-firms differentiated oligopoly market (6.71), (6.72) with Cournot competition is asymptotically stable under the conventional pseudo-gradient dynamics (6.1) for any sensitivity parameters $\alpha_i \in \mathbb{R}_+$, $i \in \mathcal{N}$.

Proof First, recall that $\mathcal{A}(J, 1_n, x^*) = \hat{A}$ is symmetric matrix. The result is immediate since the eigenvalues of $\frac{1}{2}(\hat{A}^{\mathrm{T}} + \hat{A}) = \hat{A}$ given by $\lambda_1 = \cdots = \lambda_{n-1} = -\beta(2-\delta) < 0$, $\lambda_n = -\beta(2 + (n-1)\delta) < 0$, imply $\hat{A}^{\mathrm{T}} + \hat{A} < 0$. □

Proposition 6.6. Suppose that the knowledge network is a complete graph. Then, the Nash equilibrium x^* of the n-firms differentiated oligopoly market (6.71), (6.72) with Cournot competition is asymptotically stable under the pseudo-gradient dynamics (6.4) with $\mathbb{L}_2 = \mathcal{N}$ for any sensitivity parameters $\alpha_i \in \mathbb{R}_+$, $i \in \mathcal{N}$, if and only if the degree of product differentiation $\delta \in [0, 1)$ of the market satisfies $\delta < 2/(n-1)$.

Furthermore, if the degree of product differentiation $\delta \in [0,1)$ satisfies $\delta = 2/(n-1)$, then the Nash equilibrium x^* is Lyapunov stable under the pseudo-gradient dynamics (6.4) with $\mathbb{L}_2 = \mathcal{N}$ for any sensitivity parameters $\alpha_i \in \mathbb{R}_+$, $i \in \mathcal{N}$.

Proof First, recall the fact from Remark 6.3 that $R_2(J, 1_n, x^*) = 2\hat{A} - \hat{A}\mathrm{diag}[\psi]\hat{A}$ for the knowledge network being a complete graph, where $P = [P_{ij}] = \hat{A}\mathrm{diag}[\psi]\hat{A}$ satisfies $P_{ij} = -2\beta\delta - (n-2)\beta\delta^2/2$, $j \neq i$, $P_{ii} = -2\beta - (n-1)\beta\delta^2/2$, $i \in \mathcal{N}$. Thus, it follows from $\hat{R} = R_2(J, 1_n, x^*) = 2\hat{A} - P$ that $\hat{R}_{ij} = (n-2)\beta\delta^2/2 \geq 0$, $j \neq i$, $\hat{R}_{ii} = -2\beta + (n-1)\beta\delta^2/2$,$i \in \mathcal{N}$, which imply $\hat{R}$ and $\mathrm{diag}[\alpha]\hat{R}$ to be Metzler matrices. Thus, $\mathrm{diag}[\alpha]\hat{R}$ is Hurwitz for any $\alpha_i \in \mathbb{R}_+$, $i \in \mathcal{N}$, if and only if the symmetric matrix $\hat{R}$ is Hurwitz. Now, note that the eigenvalues of $\hat{R}$ is given by

$$\lambda_1 = \cdots = \lambda_{n-1} = -2\beta + (n-1)\beta\delta^2/2 - (n-2)\beta\delta^2/2 = \beta(-2 + \delta^2/2) < 0, \quad (6.74)$$

$$\lambda_n = -2\beta + (n-1)\beta\delta^2/2 + (n-1)(n-2)\beta\delta^2/2 = \beta(-2 + (n-1)^2\delta^2/2). \quad (6.75)$$

Thus, the Nash equilibrium x^* is asymptotically stable if and only if $-2 + (n-1)^2\delta^2/2 < 0$. For the case $\delta = 2/(n-1)$, consider the Lyapunov function candidate $V(\tilde{x}) = \tilde{x}^{\mathrm{T}} P \tilde{x}$ with the positive-definite matrix $P \triangleq \mathrm{diag}[\frac{1}{\alpha_1}, \cdots, \frac{1}{\alpha_N}] > 0$. Since $\hat{R}^{\mathrm{T}}\mathrm{diag}[\alpha]P + P\mathrm{diag}[\alpha]\hat{R} = \hat{R}^{\mathrm{T}} + \hat{R} = 2\hat{R} \leq 0$ is satisfied, it follows that the Nash equilibrium x^* is Lyapunov stable for any $\alpha_i \in \mathbb{R}_+$, $i \in \mathcal{N}$. □

Theorem 6.6 indicates an interesting observation that to ensure asymptotic stability, a larger market (with bigger n) requires more differentiated products (i.e., with smaller δ) when firms compete in product quantities instead of product prices (see the curve in Fig. 6.9). When $n = 2$, the Nash equilibrium is always asymptotically stable for any degree of product differentiation $\delta \in [0,1)$. Moreover, the market power $\beta \in \mathbb{R}_+$ does not give any contribution on affecting the stability of the Nash equilibrium.

Bertrand competition

Consider the prices as the decision variables of the firms (i.e., $x_i = p_i$). It follows from the demand function of (6.71) given by [102]

$$q_i = \frac{\lambda_0(1-\delta) - (1+\delta(n-1))p_i + \delta\sum_{j\neq i} p_j}{\beta(1-\delta)(1+n\delta)}, \quad (6.76)$$

that the payoff functions from (6.72) are given by

$$J_i(x) = (x_i - c_i)\frac{\lambda_0(1-\delta) - (1+\delta(n-1))x_i + \delta\sum_{j\neq i} x_j}{\beta(1-\delta)(1+n\delta)}. \quad (6.77)$$

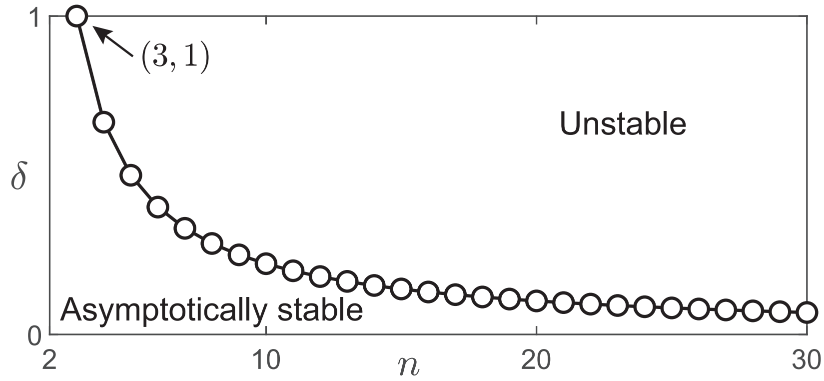


Figure. 6.9 The n-δ region of the n-firms differentiated oligopoly market with Cournot competition and Level-2 thinking. The nodes on the solid curve are understood the case where the Nash equilibrum is Lyapunov stable.

In this case, it follows from

$$\frac{\partial J_i(x)}{\partial x_i} = -2\beta x_i - \beta\delta \sum_{j\neq i} x_j + \lambda_0 - c_i \tag{6.78}$$

that the Nash equilibrium x^* satisfying $-2\beta x_i^* - \beta\delta \sum_{j\neq i} x_j^* + \lambda_0 - c_i = 0$, $i \in \mathcal{N}$, is unique and given by $-\hat{A}^{-1}b$ with $\hat{A}_{ij} = \frac{\delta}{\beta(1-\delta)(1+n\delta)} > 0$, $j \neq i$, $\hat{A}_{ii} = -\frac{2+2\delta(n-1)}{\beta(1-\delta)(1+n\delta)} < 0$, $i \in \mathcal{N}$, and $b_i = \frac{\lambda_0(1-\delta)+c_i(1+\delta(n-1))}{\beta(1-\delta)(1+n\delta)}$, $i \in \mathcal{N}$.

Corollary 6.3. Consider the the n-firms differentiated oligopoly market (6.71), (6.72) with the firms at Level-$k \leq \xi$ satisfying $\mathbb{L}_1 \cup \cdots \cup \mathbb{L}_\xi = \mathcal{N}$ so that the firms follow the pseudo-gradient dynamics (6.3), (6.4), (6.12) depending on their cognitive hierarchy levels. If the firms follow Bertrand competition, then the Nash equilibrium x^* is asymptotically stable for any degree of product differentiation δ, any cognitive level sets $\mathbb{L}_1, \ldots, \mathbb{L}_\xi$, any sensitivity parameters $\alpha_i \in \mathbb{R}_+$, $i \in \mathcal{N}$, and any knowledge network $G(\mathcal{N}, E)$ of payoff functions.

Proof First, note that $a_{ij}^i = \hat{A}_{ij} = \frac{\delta}{\beta(1-\delta)(1+n\delta)}$, $a_{ii}^i = \hat{A}_{ii} = -\frac{2+2\delta(n-1)}{\beta(1-\delta)(1+n\delta)}$, $\sigma_{si} = |\frac{a_{si}^s}{a_{ss}^s}\frac{a_{is}^i}{a_{ii}^i}| = \frac{\delta^2}{(2+2\delta(n-1))^2} < 1$ imply that

$$y = \sigma_{si}^0 + \ldots + \sigma_{si}^\infty = \frac{1}{1 - \frac{\delta^2}{(2+2\delta(n-1))^2}}. \tag{6.79}$$

Hence, it follows from (6.54) and (6.55) that

$$\phi\underline{\delta}_i^\infty = -(n-1)\delta - \frac{(n-1)^2\delta^3 y}{(2+2\delta(n-1))^2} - \frac{(n-1)^2\delta^2 y}{2+2\delta(n-1)}, \tag{6.80}$$

with $\phi = \frac{1}{\beta(1-\delta)(1+n\delta)} > 0$, which implies that that

$$\phi(a_{ii}^i - \underline{\delta}_i^\infty) = -(2 + 2\delta(n-1)) - \phi\delta_i^\infty < 0. \tag{6.81}$$

Then, the result is a direct consequence of Theorem 6.2 as (6.27) holds with $\delta_i = \underline{\delta}_i^\xi$ for $\xi = \infty$. □

It is interesting to observe that the cognitive operations of Level-2 agents never destabilize the Nash equilibrium of the n-firms differentiated oligopoly market with Bertrand competition for any degree of product differentiation, but may destabilize the Nash equilibrium with Cournot competition for a too large degree of product differentiation when the number of firms is larger than 3 (see Fig. 6.9). Recalling that the matrix $\hat{A}$ is given by $\hat{A}_{ij} = -\beta\delta$, $j \neq i$, $\hat{A}_{ii} = -2\beta$, $i \in \mathcal{N}$, in Cournot competition, and $\hat{A}_{ij} = \frac{\delta}{\beta(1-\delta)(1+n\delta)}$, $j \neq i$, $\hat{A}_{ii} = -\frac{2+2\delta(n-1)}{\beta(1-\delta)(1+n\delta)}$, $i \in \mathcal{N}$, in Bertrand competition, both of the matrices $\hat{A}$ in Cournot and Bertrand competitions belong to the same class of matrices where all the off-diagonal terms are the same and all the diagonal terms are the same. Noticing that the condition (6.27) in Theorem 6.2 requires sufficiently small $\hat{A}_{ii}$ to ensure stability, the reason why the difference in terms of stability comes between Cournot and Bertrand competitions is because the absolute value of the ratio $\hat{A}_{ii}/\hat{A}_{ij}$ given by $2/\delta$ in Cournot competition is not big enough as the one given by $2/\delta + 2(n-1)$ in Bertrand competition for any $j \neq i$.

6.6 Chapter Conclusion

We investigated the stability problem for noncooperative dynamical systems with Level-k thinking under bounded depth of reasoning. In the characterized noncooperative system, the agents are allowed to base their decisions on the predictions about the likely actions of other agents. Depending on a knowledge network of the payoff functions, the prediction-incorporated pseudo-gradient dynamics are proposed. We presented sufficient conditions to guarantee stability of a Nash equilibrium with uncertain sensitivity parameters and uncertain knowledge network of the payoff functions in order to characterize a stabilization method with incentives. The applications of our results in optical communication systems, homogeneous oligopoly markets and differentiated oligopoly markets were considered. We observed that to ensure asymptotic stability of the differentiated oligopoly markets with Cournot competition, larger market requires more differentiated products. But this phenomena does not happen in Bertrand competition because the cognitive operations in Level-k thinking never destabilize the n-firms differentiated oligopoly market with Bertrand competition.

Chapter 7

Concluding Remarks and Future Research Recommendations

7.1 Conclusion

In this thesis, we provided a line of work on control problems of self-interested agents in pseudo-gradient-based noncooperative dynamical systems.

First, in Chapter 2, we investigated the Nash equilibrium stabilization problem for noncooperative dynamical systems through a tax/subsidy approach. In the proposed tax/subsidy approach, the system manager defines the utility-transfer structure dividing the agents into subgroups so that the utility transfers are completed within the subgroups in a zero-sum and distributed manner. To deal with the uncertainty, we first characterized the stability of the Nash equilibrium for *arbitrary* values of sensitivity and then investigated the zero-sum tax/subsidy framework without knowing the sensitivity parameters.

In Chapter 3, we developed a hierarchical incentive framework for large-scale noncooperative dynamical systems to achieve social welfare improvement. In the proposed framework, the agents in the noncooperative system are divided into several groups and are influenced by the corresponding group managers via some intra-group incentives. We explored the stability of group Nash equilibrium of the hierarchical noncooperative systems and derive conditions where the trajectory of agents' states converges to the group Nash equilibrium under group managers' intra-group incentives. Furthermore, the inter-group incentive mechanism for a system governor is proposed to reconstruct the group utility functions at the group managers level to move the group Nash equilibrium so that the social (entire) welfare is improved. To deal with the situation where the system governor may not know all the agents' individual payoff functions and all the agents' states, we presented sufficient conditions to guarantee the convergence of agents' states towards a target (suboptimal but not optimal due to the lack of enough information) equilibrium using some macroscopic data.

In Chapter 4, we investigated the social welfare improvement problem for the noncooperative dynamical systems through a Pareto-improving incentive mechanism under sustainable budget constraint, where a system manager collects taxes from some agents and gives some of the collected taxes to other agents as subsidies in order to remodel agents' dynamical decision making. We presented sufficient stability conditions for our incentive functions were proposed to guarantee that the agents are Pareto improving under the pseudo-gradient dynamics and their state converges to a Pareto-efficient Nash equilibrium associated with a weighted social welfare function depending on the priority ratio of the agents.

In Chapter 5, we investigated the stability conditions of the noncooperative switched systems with loss-averse agents, where each agent under pseudo-gradient dynamics exhibits lower sensitivity for the cases of losing payoffs. We characterized the notion of the flash switching phenomenon and examined stability properties in accordance with the location of the Nash equilibrium for 3 cases. We revealed how the sensitivity parameters influence the stability property of the system in terms of the dynamics, partition of the state space, mode transition, and the normalized radial growth rate for each of the 3 cases.

In Chapter 6, we investigated the stability problem for noncooperative dynamical systems with Level-k thinking under bounded depth of reasoning. We characterized the transition of the agents' target state while increasing the depth of reasoning for a two-agent noncooperative system with quadratic payoff functions. We presented sufficient conditions to guarantee stability of a Nash equilibrium with uncertain sensitivity parameters and uncertain knowledge network of the payoff functions in order to characterize a stabilization method with incentives. The applications of our results in optical communication systems, homogeneous oligopoly markets and differentiated oligopoly markets were considered.

7.2 Future Research Recommendations

There still remain several open problems on the analysis and stabilization of agent's selfish behaviors in the noncooperative dynamical systems. For the hierarchical noncooperative system in Chapter 3, allowing the agents to switch the membership may be an interesting future direction. In real society, many cities construct some special subsidies to attract young people, talents, and potential firms to move to their city so that the vitality and development prospects of the city can be guaranteed. Therefore, with the switching of the agents, the social welfare may be significatively influenced with the changing of the group topology. It is nature to ask what is the best grouping topology for the hierarchical noncooperative system. Moreover, Chapter 6 showed an example where the cognitive hierarchy levels may destabilize the Nash equilibrium of the noncooperative system and generate some new equilibria in the dynamical system. From the system manager's point of view, letting the agents change their cognitive hierarchy levels can be a essential method to improve the social welfare. The future research direction may includes the investigation of cognitive hierarchy level switching framework.

The emerging problems in intelligent transportation system and smart grid market with game theoretic approach are expected in the future research directions. The security problems in engineering systems with game theoretic approach are also expected. Furthermore, the payoff-value based learning dynamics is important in the future research direction. The agents' behavioral dynamics in the most of the literature require the exact payoff function. However, the agents may not really know the explicit form of their payoff functions but the value of payoff functions. In such a case, how to construct the behavioral dynamics for the agents is an important question in the future research.

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Appendix A

List of Publications

Journal Papers

- Y. Yan and T. Hayakawa, "Stability and stabilization of Nash equilibrium for uncertain noncooperative dynamical systems with zero-sum tax/subsidy approach", *IEEE Transactions on Cybernetics*, 2022, accepted for publication.
- Y. Yan and T. Hayakawa, "Stability analysis of Nash equilibrium for 2-agent loss-aversion-based noncooperative switched systems", *IEEE Transactions on Automatic Control*, vol. 67, no. 5, pp. 2505-2513, 2022. DOI: 10.1109/TAC.2021.3079276
- Y. Yan and T. Hayakawa, "Pareto-improving incentive mechanism for noncooperative dynamical systems under sustainable budget constraint", *IEEE Transactions on Automatic Control*, under revision (Provisionally rejected, may be resubmitted as Paper).
- Y. Yan and T. Hayakawa, "Hierarchical noncooperative dynamical systems under intra-group and inter-group incentives", *IEEE Transactions on Control of Network Systems*, under revision (Provisionally rejected, may be resubmitted as Paper).

Conference Papers

- Y. Yan and T. Hayakawa, "Pseudo-gradient dynamics with Level-k predictions in noncooperative dynamical systems," in *Proc. IEEE Conference on Decision and Control*, 2022, accepted.

- Y. Yan and T. Hayakawa, "Incentive design for noncooperative dynamical systems under sustainable budget constraint for Pareto improvement," in *Proc. American Control Conference*, 2022, pp. 580–585.

- Y. Yan and T. Hayakawa, "Hierarchical noncooperative systems with dynamic agents under intra-group and inter-group incentives," in *Proc. IEEE Conference on Decision and Control*, 2021, pp. 1644–1649.

- Y. Yan and T. Hayakawa, "Loss-averse behavior may destabilize Nash equilibrium: Generalized stability results for noncooperative agents," in *Proc. IEEE Conference on Decision and Control*, 2021, pp. 3824–3829.

- Y. Yan and T. Hayakawa, "Existence of feasible provisional transfer-based tax/subsidy approach for stabilizing noncooperative dynamical systems: graph analysis," in *Proc. European Control Conference*, 2020, pp.142–147.

- Y. Yan and T. Hayakawa, "Social welfare improvement for noncooperative dynamical systems with tax/subsidy approach," in *Proc. IEEE Conference on Decision and Control*, 2019, pp. 3116–3121.

- Y. Yan, T. Hayakawa and N. Thanomvajamun, "Stability analysis of Nash equilibrium in loss-aversion-based noncooperative dynamical systems," in *Proc. IEEE Conference on Decision and Control*, 2019, pp. 3122–3127.

Appendix B

Supplemental Information

Several classes of potential games are found in the literature [103]. Specifically, a noncooperative game with the payoff functions $J_i(x)$, $i \in \mathcal{N}$, is called an (exact) potential game if there exists a function $f : \mathbb{R}^n \to \mathbb{R}$ such that

$$J_i(x_i, x_{-i}) - J_i(\hat{x}_i, x_{-i}) = f(x_i, x_{-i}) - f(\hat{x}_i, x_{-i}), \tag{B.1}$$

for any $i \in \mathcal{N}$, $x_i \in \mathbb{R}$, $\hat{x}_i \in \mathbb{R}$, and $x_{-i} \in \mathbb{R}^{n-1}$. This notion can be generalized to the notion of weighted potential game when there exists a positive weight vector $(w_i)_{i\in\mathcal{N}}$ such that $J_i(x_i, x_{-i}) - J_i(\hat{x}_i, x_{-i}) = w_i(f(x_i, x_{-i}) - f(\hat{x}_i, x_{-i}))$ for any $i \in \mathcal{N}$, $x_i \in \mathbb{R}$, $\hat{x}_i \in \mathbb{R}$, and $x_{-i} \in \mathbb{R}^{n-1}$. Furthermore, the notion of weighted potential game can be generalized to the notion of ordinal potential game when

$$J_i(x_i, x_{-i}) > J_i(\hat{x}_i, x_{-i}) \Leftrightarrow f(x_i, x_{-i}) > f(\hat{x}_i, x_{-i}), \tag{B.2}$$

for any $i \in \mathcal{N}$, $x_i \in \mathbb{R}$, $\hat{x}_i \in \mathbb{R}$, and $x_{-i} \in \mathbb{R}^{n-1}$. The weighted potential game (and hence the exact potential game) is a special class of ordinal potential game.

Lemma B.1. Consider the two-agent noncooperative system $\mathcal{G}$ with quadratic payoff functions (2.30) satisfying $a^1_{11} < 0$ and $a^2_{22} < 0$. Then, the game $\mathcal{G}$ admits a ordinal potential if and only if $a^1_{12}a^2_{12} > 0$.
Proof: The necessity is proved by Theorem 1 of [103]. For sufficiency, when $a^1_{12}a^2_{12} > 0$, the function given by

$$f(x) = \frac{1}{2}(x - x^*)^{\mathrm{T}} \begin{bmatrix} a^1_{11}a^2_{12} & a^1_{12}a^2_{12} \\ a^1_{12}a^2_{12} & a^1_{12}a^2_{22} \end{bmatrix} (x - x^*), \tag{B.3}$$

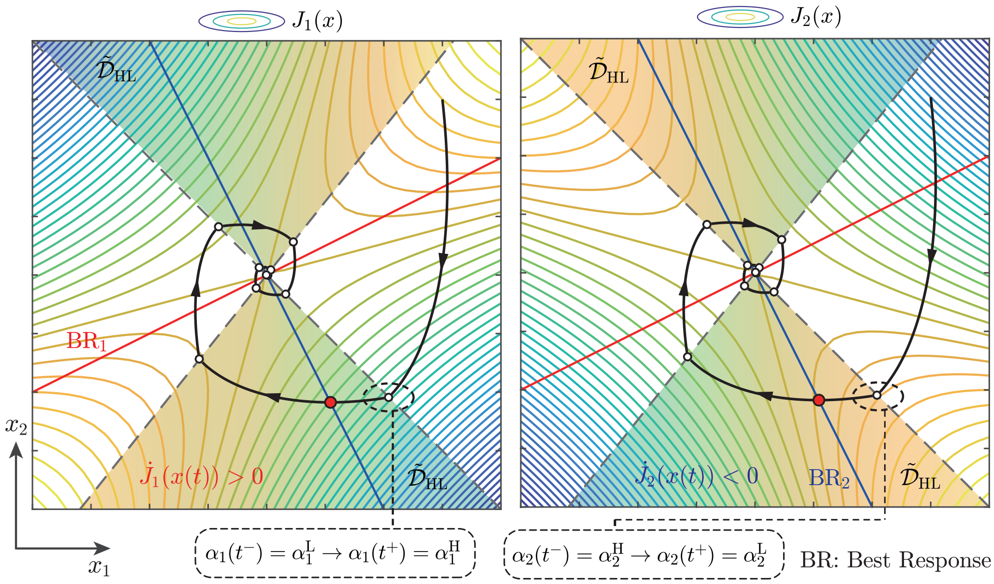

Figure. B.1 Graphic abstract of two-agent loss-aversion-based noncooperative system with zero-sum payoff functions.

is an ordinal potential for $\mathcal{G}$ because the function $f(x)$ satisfies $\arg\max_{x_i \in \mathbb{R}} J_i(x_i, x_{-i}) = \arg\max_{x_i \in \mathbb{R}} f(x_i, x_{-i})$, $i = 1, 2$, and hence (B.2) holds. $\square$

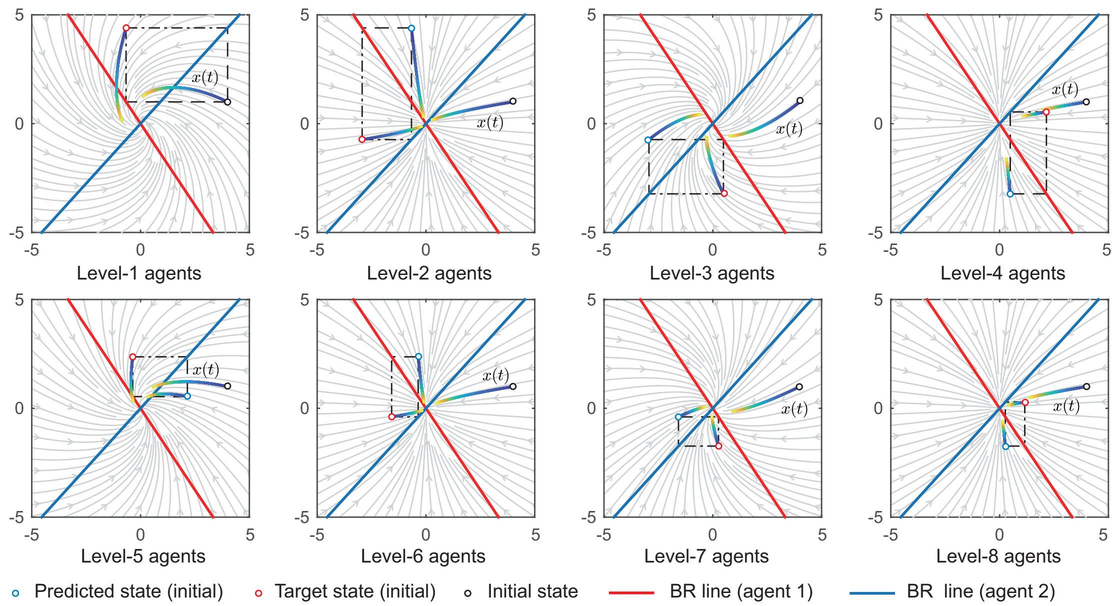

Figure. B.2 Trajectories of the agents' state, predicted state, and targeted best-response state in a two-agent noncooperative system with prediction-incorporated pseudo-gradient dynamics.

$$\mathcal{A} = \mathrm{diag}[\alpha] \begin{bmatrix} \zeta_1 \tilde{\mathbb{A}}_1^1 & 0 & 0 & 0 & \cdots & 0 \\ (\zeta_1+\zeta_2)\tilde{\mathbb{A}}_1^2 & \zeta_2 \tilde{\mathbb{A}}_2^2 & \zeta_2 \tilde{\mathbb{A}}_2^3 + b_{23} & \zeta_2 \tilde{\mathbb{A}}_2^4 + b_{24} & \cdots & \zeta_2 \tilde{\mathbb{A}}_2^n + b_{2n} \\ (\zeta_1+\zeta_3)\tilde{\mathbb{A}}_1^3 & \zeta_3 \tilde{\mathbb{A}}_2^3 - b_{23} & \zeta_3 \tilde{\mathbb{A}}_3^3 & \zeta_3 \tilde{\mathbb{A}}_3^4 + b_{34} & \cdots & \zeta_3 \tilde{\mathbb{A}}_3^n + b_{3n} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ (\zeta_1+\zeta_{n-1})\tilde{\mathbb{A}}_1^{n-1} & \zeta_{n-1} \tilde{\mathbb{A}}_2^{n-1} - b_{2(n-1)} & \cdots & \zeta_{n-1} \tilde{\mathbb{A}}_3^n - b_{(n-2)(n-1)} & \zeta_{n-1} \tilde{\mathbb{A}}_{n-1}^{n-1} & \zeta_2 \tilde{\mathbb{A}}_{n-1}^n + b_{(n-1)n} \\ (\zeta_1+\zeta_n)\tilde{\mathbb{A}}_1^n & \zeta_n \tilde{\mathbb{A}}_2^n - b_{2n} & \cdots & \zeta_n \tilde{\mathbb{A}}_{n-1}^n - b_{(n-2)n} & \zeta_n \tilde{\mathbb{A}}_{n-1}^n - b_{(n-1)n} & \zeta_n \tilde{\mathbb{A}}_n^n \end{bmatrix} \tag{B.4}$$

$$0 = \underbrace{\begin{bmatrix} 0 & \alpha_2 \hat{x}_3 & \alpha_2 \hat{x}_4 & \alpha_2 \hat{x}_5 & \cdots & \alpha_2 \hat{x}_n \\ \alpha_3 \hat{x}_4 & -\alpha_3 \hat{x}_2 & 0 & 0 & \cdots & 0 \\ -\alpha_4 \hat{x}_3 & 0 & -\alpha_4 \hat{x}_2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -\alpha_5 \hat{x}_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\alpha_n \hat{x}_2 \end{bmatrix}}_{\Pi \in \mathbb{R}^{(n-1)\times(n-1)}} \begin{bmatrix} b_{34} \\ b_{23} \\ b_{24} \\ b_{25} \\ \vdots \\ b_{2n} \end{bmatrix} + \underbrace{\begin{bmatrix} \alpha_2 \zeta_2 \sum_{i=1}^n \tilde{\mathbb{A}}_2^i \hat{x}_i + \alpha_2 \zeta_1 \tilde{\mathbb{A}}_1^2 \hat{x}_1 - \lambda \hat{x}_2 \\ \alpha_3 \zeta_3 \sum_{i=1}^n \tilde{\mathbb{A}}_3^i \hat{x}_i + \alpha_3 \zeta_1 \tilde{\mathbb{A}}_1^3 \hat{x}_1 - \lambda \hat{x}_3 + \alpha_3 \sum_{i=5}^n b_{3i} \hat{x}_i \\ \alpha_4 \zeta_4 \sum_{i=1}^n \tilde{\mathbb{A}}_4^i \hat{x}_i + \alpha_4 \zeta_1 \tilde{\mathbb{A}}_1^4 \hat{x}_1 - \lambda \hat{x}_4 + \alpha_4 \sum_{i=5}^n b_{4i} \hat{x}_i \\ \alpha_5 \zeta_5 \sum_{i=1}^n \tilde{\mathbb{A}}_5^i \hat{x}_i + \alpha_5 \zeta_1 \tilde{\mathbb{A}}_1^5 \hat{x}_1 - \lambda \hat{x}_5 + \alpha_5 \sum_{i=3}^n b_{5i} \hat{x}_i \\ \vdots \\ \alpha_n \zeta_n \sum_{i=1}^n \tilde{\mathbb{A}}_n^i \hat{x}_i + \alpha_n \zeta_1 \tilde{\mathbb{A}}_1^n \hat{x}_1 - \lambda \hat{x}_n + \alpha_n \sum_{i=3}^n b_{ni} \hat{x}_i \end{bmatrix}}_{\xi = [\xi_1, \ldots, \xi_{n-1}]^{\mathrm{T}} \in \mathbb{R}^{n-1}} \tag{B.5}$$

$$
\begin{aligned}
\underline{\delta}_i^k \leq & \underline{\delta}_i^2 - \sum_{j\neq i} \Big[\sum_{s\neq i}^{s\neq j} \Big| \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s} \Big| \Big(\Big| \frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_j} \Big| (\psi_{si}^{\frac{k-3}{2}} - 1) + \Big| \frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_i} \frac{\partial \mathrm{BR}_i(x^*_{-i})}{\partial x_j} \Big| \psi_{si}^{\frac{k-3}{2}} \Big) + \Big| \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_j} \Big| (\psi_{ji}^{\frac{k-1}{2}} - 1) \Big] \\
\leq & -\sum_{j\neq i} \Big[\sum_{s\in \mathbb{N}_i^{\mathrm{pr1}}}^{s\neq j} \Big| \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s} \Big| \Big(\Big| \frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_j} \Big| (\psi_{si}^{\frac{k-3}{2}} - 1) + \Big| \frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_i} \frac{\partial \mathrm{BR}_i(x^*_{-i})}{\partial x_j} \Big| \psi_{si}^{\frac{k-3}{2}} \Big) + \Big| \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_j} \epsilon_{ji}^{\frac{k-1}{2}} \Big| \Big] \\
& - \sum_{j\neq i} \Big| (1 - a_{ij}) \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_j} \Big| - \sum_{j\neq i} \sum_{s\in \mathbb{N}_i^{\mathrm{pr1}}} \Big| \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s} \frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_j} \Big| - \sum_{j\in \mathcal{N}} \sum_{s\in \mathbb{N}_i^{\mathrm{pr2}}} \Big| \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s} \frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_j} \Big| \\
\leq & -\sum_{j\neq i} \Big[\sum_{s\in \mathbb{N}_i^{\mathrm{pr1}}}^{s\neq j} \Big| \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s} \Big| \Big(\Big| \frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_j} \Psi_{si}^{\frac{k-3}{2}} \Big| + \Big| \frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_i} \frac{\partial \mathrm{BR}_i(x^*_{-i})}{\partial x_j} \Psi_{si}^{\frac{k-3}{2}} \Big| \Big) + \Big| \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_j} \epsilon_{ji}^{\frac{k-1}{2}} \Big| \Big] \\
& - \sum_{j\neq i} \Big| (1 - a_{ij}) \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_j} \Big| - \sum_{j\neq i} \sum_{s\in \mathbb{N}_i^{\mathrm{pr2}}} \Big| \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s} \frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_j} \Big| - \sum_{s\in \mathbb{N}_i^{\mathrm{pr2}}} \Big| \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s} \frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_i} \Big| \\
\leq & -\sum_{j\neq i} \Big(\sum_{s\in \mathbb{N}_i^{\mathrm{pr1}}} \Big| \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s} m_{si}^{jk} \Big| + \Big| (1 - a_{ij}) \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_j} \Big| + \sum_{s\in \mathbb{N}_i^{\mathrm{pr2}}} \Big| \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s} \frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_j} \Big| \Big) \\
& - \sum_{s\in \mathbb{N}_i^{\mathrm{pr2}}} \Big| \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s} \frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_i} \Big| - \sum_{s\in \mathbb{N}_i^{\mathrm{pr1}}} \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s} m_{si}^{ik} \leq \delta_i^k, \quad k = 3, 5, 7, \ldots .
\end{aligned} \tag{B.6}
$$

$$
\begin{aligned}
\underline{\delta}_i^k \leq & \underline{\delta}_i^2 - \sum_{j\neq s} \sum_{s\neq i} \Big| \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s} \frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_j} \Big| (\psi_{si}^{\frac{k-2}{2}} - 1) - \sum_{j\neq i} \sum_{s\neq i}^{s\neq j} \Big| \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s} \frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_i} \frac{\partial \mathrm{BR}_i(x^*_{-i})}{\partial x_j} \Big| \psi_{si}^{\frac{k-4}{2}} \\
\leq & -\sum_{j\neq i} \sum_{s\in \mathbb{N}_i^{\mathrm{pr1}}} \Big| \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s} \frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_j} \Big| (\psi_{si}^{\frac{k-2}{2}} - 1) - \sum_{s\in \mathbb{N}_i^{\mathrm{pr1}}} \Big| \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s} \frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_i} \epsilon_{ji}^{\frac{k-2}{2}} \Big| - \sum_{j\neq i} \Big| (1 - a_{ij}) \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_j} \Big| \\
& -\sum_{j\neq i} \sum_{s\in \mathbb{N}_i^{\mathrm{pr1}}}^{s\neq j} \Big| \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s} \frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_i} \frac{\partial \mathrm{BR}_i(x^*_{-i})}{\partial x_j} \Big| \psi_{si}^{\frac{k-4}{2}} - \sum_{j\neq i} \sum_{s\in \mathcal{N}_i} \Big| \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s} \frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_j} \Big| \\
\leq & -\sum_{j\neq i} \sum_{s\in \mathbb{N}_i^{\mathrm{pr1}}} \Big| \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s} \frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_j} \Psi_{si}^{\frac{k-2}{2}} \Big| - \sum_{s\in \mathbb{N}_i^{\mathrm{pr1}}} \Big| \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s} \frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_i} \epsilon_{ji}^{\frac{k-2}{2}} \Big| - \sum_{j\neq i} \Big| (1 - a_{ij}) \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_j} \Big| \\
& -\sum_{j\neq i} \sum_{s\in \mathbb{N}_i^{\mathrm{pr1}}}^{s\neq j} \Big| \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s} \frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_i} \frac{\partial \mathrm{BR}_i(x^*_{-i})}{\partial x_j} \Psi_{si}^{\frac{k-4}{2}} \Big| - \sum_{j\neq i} \sum_{s\in \mathbb{N}_i^{\mathrm{pr2}}} \Big| \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s} \frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_j} \Big| \\
\leq & -\sum_{j\neq i} \Big(\sum_{s\in \mathbb{N}_i^{\mathrm{pr1}}} \Big| \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s} m_{si}^{jk} \Big| + \Big| (1 - a_{ij}) \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_j} \Big| + \sum_{s\in \mathbb{N}_i^{\mathrm{pr2}}} \Big| \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s} \frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_j} \Big| \Big) \\
& - \sum_{s\in \mathbb{N}_i^{\mathrm{pr2}}} \Big| \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s} \frac{\partial \mathrm{BR}_s(x^*_{-s})}{\partial x_i} \Big| - \sum_{s\in \mathbb{N}_i^{\mathrm{pr1}}} \frac{\partial^2 J_i(x^*)}{\partial x_i \partial x_s} m_{si}^{ik} \leq \delta_i^k, \quad k = 4, 6, 8, \ldots .
\end{aligned} \tag{B.7}
$$