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Studies on Propositional Connectives in Classical and Intuitionistic Logics

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Abstract

A great number of logical systems have been studied in mathematical logic. Among those systems, classical logic and intuitionistic logic have extensively been studied as the two most fundamental logical systems. Furthermore, the logical system called the logic of constant domains is an important system between classical and intuitionistic predicate logics.

We study these logics in the very general setting where they treat general connectives defined by truth tables, including usual connectives such as conjunction, disjunction, and implication. We comprehensively analyze how the choice of connectives affects the relations between classical propositional logic and intuitionistic propositional logic and the relations between classical predicate logic, intuitionistic predicate logic, and the logic of constant domains. For each pair of these logics, we give a simple necessary and sufficient condition on connectives for one logic to coincide with the other logic.

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Contents

Chapter 1

Introduction

A great number of logical systems have been studied in mathematical logic. Among those systems, classical logic and intuitionistic logic are certainly the two most fundamental logical systems. The relations between these logics have been extensively studied as one of the most important subjects in mathematical logic. For instance, it is one of the most important results that classical logic can be embedded into intuitionistic logic by Glivenko translation and Gödel-Gentzen translation. This dissertation adds a new perspective to this area. We give comprehensive and exhaustive results on how the choice of connectives affects the relations between the strengths of classical and intuitionistic logics in a very general setting.

This chapter introduces the background of our research and presents our results. In § 1.1, we introduce the background of classical and intuitionistic propositional logics with general connectives. In § 1.2, we present our results on propositional logic. In § 1.3, we introduce the background concerning first-order predicate logics. In § 1.4, we present our results on first-order logics. § 1.5 gives a brief overview of the following chapters.

1.1 Background on propositional logic with general connectives

Propositional logic

Propositional logic deals with statements and connections of statements. Those which connect statements are called propositional connectives. For example, for given statements α and β, using connectives such as negation ($\neg$), conjunction ($\wedge$), disjunction ($\vee$), and implication ($\to$), we can construct new statements such as $\neg\alpha$, $\alpha \wedge \beta$, $\alpha \vee \beta$, and $\alpha \to \beta$. For the time being, we only use these four standard connectives. The most basic propositions are called propositional variables, and expressions built up from propositional variables using propositional connectives are called (propositional) formulas. Thus, formulas are formal expressions that are interpreted as statements. Besides, in studies of logic, we also deal with formal expressions that are interpreted as expressing the consequence relation between the hypotheses and the conclusions. Such expressions are called sequents. Formally, sequents are those expressions of the form $\Gamma \Rightarrow \Delta$, where Γ is a set of formulas called antecedents and Δ is a set of formulas called succedents. Informally, Sequent $\Gamma \Rightarrow \Delta$ means one of the succedents Δ follows from the antecedents Γ.

There are various types of propositional logics, and so formulas and sequents are interpreted in different ways depending on logics. Here, we review the semantics of the two most fundamental propositional logics—classical propositional logic and intuitionistic propositional logic. For more detailed presentations of classical and intuitionistic logics (both propositional and predicate), including proof theory, the reader is referred to standard textbooks such as [1, 18, 19].

Classical logic and classical semantics

In classical logic, statements are absolutely interpreted as either true or false. Thus, in classical semantics, a model assigns one of the truth values, 1 (truth) and 0 (falsity), to each propositional variable, and then the truth value of a complex formula is determined by the truth values of its constituents and the meaning of the connective that connects those constituents.

Thus, the (truth) value $[\![\alpha]\!]_{\mathscr{M}}$ of a formula α in a classical model $\mathscr{M}$ is defined as follows:

- $[\![\neg\alpha_1]\!]_{\mathscr{M}} = 1$ if and only if $[\![\alpha_1]\!]_{\mathscr{M}} = 0$;
- $[\![\alpha_1 \wedge \alpha_2]\!]_{\mathscr{M}} = 1$ if and only if $[\![\alpha_1]\!]_{\mathscr{M}} = 1$ and $[\![\alpha_2]\!]_{\mathscr{M}} = 0$;
- $[\![\alpha_1 \vee \alpha_2]\!]_{\mathscr{M}} = 1$ if and only if $[\![\alpha_1]\!]_{\mathscr{M}} = 1$ or $[\![\alpha_2]\!]_{\mathscr{M}} = 1$;
- $[\![\alpha_1 \to \alpha_2]\!]_{\mathscr{M}} = 1$ if and only if $[\![\alpha_1]\!]_{\mathscr{M}} = 1$ implies $[\![\alpha_2]\!]_{\mathscr{M}} = 1$.

The notion of classical validity is defined based on this interpretation. A formula is said to be classically valid if it is interpreted as 1 (truth) in all classical models. Also, the notion of validity is extended to sequents based on this interpretation. A sequent $\Gamma \Rightarrow \Delta$ is said to be classically valid if for any classical model, at least one of the formulas in Δ is interpreted as 1 whenever all formulas in Γ are interpreted as 1.

Intuitionistic logic and Kripke semantics

In contrast to classical logic, intuitionistic logic is based on the notion of construction, not truth. Kripke [9] gives semantics for intuitionistic logic which reflects the constructive interpretation of the meaning.

In Kripke semantics, a model, called a Kripke model, consists of states of knowledge, or states of information. At each state, we can only assert that a proposition is true if the proposition turns out to be true from the information we have at the current state. As we proceed to a next state, we acquire more knowledge or information; that is, more propositions turn out to be true. Thus, a Kripke model is mathematically defined to be a tuple $\langle W, \preceq, I\rangle$, where W is a set of states; $\preceq$ is a pre-order which represents the precedence relation between states; I is an assignment which assigns one of the truth values to each propositional variable *at each state*. The (truth) value $\|\alpha\|_w$ of a formula α at a state w is defined inductively as follows:

- $\|\neg\alpha_1\|_w = 1$ if and only if $\|\alpha_1\|_v = 0$ for all states $v \succeq w$;
- $\|\alpha_1 \wedge \alpha_2\|_w = 1$ if and only if $\|\alpha_1\|_w = 1$ and $\|\alpha_2\|_w = 1$;
- $\|\alpha_1 \vee \alpha_2\|_w = 1$ if and only if $\|\alpha_1\|_w = 1$ or $\|\alpha_2\|_w = 1$;
- $\|\alpha_1 \to \alpha_2\|_w = 1$ if and only if for all states $v \succeq w$, $\|\alpha_1\|_v = 1$ implies $\|\alpha_2\|_v = 1$.

The notion of validity in intuitionistic logic may be defined with this interpretation. A formula is said to be intuitionistically valid, or Kripke-valid, if it is interpreted as 1 at every state of all Kripke models. Furthermore, a sequent $\Gamma \Rightarrow \Delta$ is said to be intuitionistically valid, or Kripke-valid, if, for every state of all Kripke models, at least one of the formulas in Δ is interpreted as 1 at the state whenever so do all formulas in Γ at the state.

Since classical models can be viewed as one-state Kripke models, Kripke-valid formulas and sequents are also classically valid. However, the converse does not hold. This fact is discussed in more detail later.

Truth-functional connectives

So far, we have only considered the usual connectives $\neg$, $\wedge$, $\vee$, and $\to$. However, many other interesting connectives are known, and especially, the class of truth-functional connectives has been extensively studied. Truth-functional connectives are those connectives that are assigned with truth functions. For a truth-functional connective c, we denote its truth function by t_c. Let us see some interesting examples of truth-functional connectives.

NAND is the truth-functional connective that means "not both," so that the truth function of NAND is defined by $\mathrm{t}_{\mathrm{NAND}} = \mathrm{t}_\neg \circ \mathrm{t}_\wedge$. It is a well-known fact that NAND is functionally complete just by itself; that is, all other truth-functional connectives can be defined by NAND. Also, exclusive disjunction ($\oplus$) is an example of a truth-functional connective. While the usual disjunction $\vee$ is inclusive so that $\alpha \vee \beta$ means that at least one of α or β is true, $\alpha \oplus \beta$ means either α or β is true, but not both. Furthermore, we can consider connectives of any arity. If we define ternary truth function f by

$$f(x,y,z) = \begin{cases} y & \text{if } x = 1 \\ z & \text{if } x = 0, \end{cases}$$

then the connective whose truth function is f represents "if then else."

Here, let us consider the interpretation of formulas with general truth-functional connectives. First, we consider the classical interpretation. We can see that in the definition of the value $[\![\alpha]\!]_{\mathscr{M}}$ of formula α with the usual connectives, the meanings of connectives are determined by their truth functions. And so, we can integrate the clauses for $\neg, \wedge, \vee, \to$ in the definition of $[\![\alpha]\!]_{\mathscr{M}}$ in page 2 into one format by using the truth function:

$$[\![c(\alpha_1,\dots,\alpha_n)]\!]_{\mathscr{M}} = 1 \text{ if and only if } \mathrm{t}_c([\![\alpha_1]\!]_{\mathscr{M}},\dots,[\![\alpha_n]\!]_{\mathscr{M}}) = 1, \tag{1.1}$$

where $c \in \{\neg, \wedge, \vee, \to\}$ and n is the arity of c. Furthermore, formulas with general truth-functional connectives can be interpreted based on (1.1).

Rousseau [17] and Geuvers and Hurkens [4] extended Kripke interpretation to formulas with general truth-functional connectives.[1] For a connective c with truth function t_c, the value $\|c(\alpha_1,\dots,\alpha_n)\|_w$ of $c(\alpha_1,\dots,\alpha_n)$ at state w is given by

$$\|c(\alpha_1,\dots,\alpha_n)\|_w = 1 \iff \mathrm{t}_c(\|\alpha_1\|_v,\dots,\|\alpha_n\|_v) = 1 \text{ for all } v \succeq w. \tag{1.2}$$

At a glance, this condition does not seem to coincide with the clauses in the definition of $\|\alpha\|_w$ in page 2 in the cases $\alpha = \alpha_1 \wedge \alpha_2$ and $\alpha = \alpha_1 \vee \alpha_2$. However, it can easily be verified that these conditions give an equivalent definition. Thus, (1.2) generalizes the Kripke interpretation of formulas with the usual connectives as (1.1) generalizes the classical interpretation of them. Based on these generalized interpretations, the notions of classical validity and Kripke-validity are also extended to formulas and sequents with truth-functional connectives.

Although we shall not go into it in this paper, we remark that the proof theory of generalized propositional logic has widely been studied. Rousseau [17] gave sequent calculi for classical and intuitionistic many-valued logic with truth-functional connectives by generalizing Gentzen's **LK** and **LJ**. As to natural deduction, Geuvers and Hurkens [4] introduced classical and intuitionistic natural deduction systems with truth-functional connectives. For a set $\mathscr{C}$ of truth-functional connectives, they denote classical natural deduction with $\mathscr{C}$ by $\mathrm{CPC}_{\mathscr{C}}$ and intuitionistic natural deduction with $\mathscr{C}$ by $\mathrm{IPC}_{\mathscr{C}}$. Furthermore, Geuvers and Hurkens [5, 3] extended the Curry-Howard isomorphism to truth-functional connectives and analyzed conversions of derivations of $\mathrm{IPC}_{\mathscr{C}}$ as reductions of proof terms.

[1] Furthermore, Rousseau [17] introduced Kripke semantics for many-valued intuitionistic logic, though we shall only treat two-valued logic in this dissertation.

Relation between classical logic and intuitionistic logic

It is known that the relations between intuitionistic and classical logics change depending on what kinds of propositional connectives are chosen. Here, we observe two kinds of fundamental relations: the inclusion relation between the set of classically valid formulas and that of Kripke-valid formulas and the inclusion relation between the set of classically valid sequents and that of Kripke-valid sequents.

First, we observe the formula-level inclusion relation. For a given set $\mathscr{C}$ of truth-functional connectives, we denote by $\mathrm{IL}(\mathscr{C})$ the set of Kripke-valid formulas built out of connectives in $\mathscr{C}$ and by $\mathrm{CL}(\mathscr{C})$ the set of classically valid formulas built out of connectives in $\mathscr{C}$. Here, IL stands for Intuitionistic Logic and CL stands for Classical Logic. Since classical models can be regarded as one-state Kripke models, it can immediately be seen that $\mathrm{IL}(\mathscr{C}) \subseteq \mathrm{CL}(\mathscr{C})$ holds for any $\mathscr{C}$. For example, Figure 1.1 illustrates the relation between $\mathrm{IL}(\{\neg, \wedge, \vee\})$ and $\mathrm{CL}(\{\neg, \wedge, \vee\})$. However, it depends on $\mathscr{C}$ whether the converse inclusion $\mathrm{CL}(\mathscr{C}) \subseteq \mathrm{IL}(\mathscr{C})$ holds. As to the usual

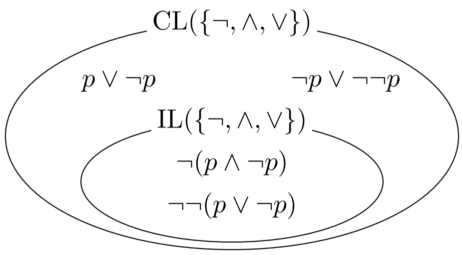

Figure 1.1: The relation between $\mathrm{IL}(\{\neg, \wedge, \vee\})$ and $\mathrm{CL}(\{\neg, \wedge, \vee\})$

connectives, for example, the following facts are known.

- If we choose $\{\wedge, \vee\}$ or $\{\neg\}$ as $\mathscr{C}$, then $\emptyset = \mathrm{IL}(\mathscr{C}) = \mathrm{CL}(\mathscr{C})$.
- If we choose $\{\neg, \wedge\}$ as $\mathscr{C}$, then $\emptyset \subsetneq \mathrm{IL}(\mathscr{C}) = \mathrm{CL}(\mathscr{C})$; we can construct a formula $\neg(p \wedge \neg p) \in \mathrm{IL}(\{\neg, \wedge\})$.
- If we choose $\{\neg, \vee\}$ or $\{\to\}$ as $\mathscr{C}$, then $\emptyset \subsetneq \mathrm{IL}(\mathscr{C}) \subsetneq \mathrm{CL}(\mathscr{C})$; we can construct formulas such as $\neg\neg(p \vee \neg p) \in \mathrm{IL}(\{\neg, \vee\})$, $p \to p \in \mathrm{IL}(\{\to\})$, $p \vee \neg p \in \mathrm{CL}(\{\neg, \vee\}) \setminus \mathrm{IL}(\{\neg, \vee\})$ and $((p \to q) \to p) \to p \in \mathrm{CL}(\{\to\}) \setminus \mathrm{IL}(\{\to\})$.

Table 1.1 gives a complete classification of the usual connectives with respect to the relation between $\mathrm{IL}(\mathscr{C})$ and $\mathrm{CL}(\mathscr{C})$.

Table 1.1: The relation between $\mathrm{IL}(\mathscr{C})$ and $\mathrm{CL}(\mathscr{C})$ for usual connectives $\mathscr{C}$

The choice of $\mathscr{C}(\subseteq \{\neg, \wedge, \vee, \to\})$	The relation between $\mathrm{IL}(\mathscr{C})$ and $\mathrm{CL}(\mathscr{C})$
$\emptyset, \{\neg\}, \{\wedge\}, \{\vee\}, \{\wedge, \vee\}$	$\emptyset = \mathrm{IL}(\mathscr{C}) = \mathrm{CL}(\mathscr{C})$
$\{\neg, \wedge\}$	$\emptyset \subsetneq \mathrm{IL}(\mathscr{C}) = \mathrm{CL}(\mathscr{C})$
$\{\to\}, \{\neg, \to\}, \{\neg, \vee\}, \{\to, \vee\}$, $\{\to, \wedge\}, \{\neg, \to, \wedge\}, \{\neg, \to, \vee\}$ $\{\neg, \wedge, \vee\}, \{\to, \wedge, \vee\}, \{\neg, \to, \wedge, \vee\}$	$\emptyset \subsetneq \mathrm{IL}(\mathscr{C}) \subsetneq \mathrm{CL}(\mathscr{C})$

Next, we observe the sequent-level inclusion relation. For a given set $\mathscr{C}$ of truth-functional connectives, we denote by $\mathrm{ILS}(\mathscr{C})$ the set of Kripke-valid sequents built out of connectives in

$\mathscr{C}$ and by CLS($\mathscr{C}$) the set of classically valid sequents built out of connectives in $\mathscr{C}$. Here, the letter S in ILS($\mathscr{C}$) and CLS($\mathscr{C}$) indicates that these sets are sets of valid Sequents. For the same reason as IL($\mathscr{C}$) $\subseteq$ CL($\mathscr{C}$) holds for any $\mathscr{C}$, ILS($\mathscr{C}$) $\subseteq$ CLS($\mathscr{C}$) holds for any $\mathscr{C}$. For example, Figure 1.2 illustrates the relation between ILS($\{\neg\}$) and CLS($\{\neg\}$).

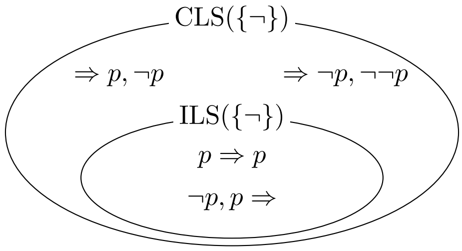

Figure 1.2: The relation between ILS($\{\neg\}$) and CLS($\{\neg\}$)

Note that, contrary to the case of the formula-level relation, the set of Kripke-valid sequents and the set of classically valid sequents are always non-empty because a sequent $p \Rightarrow p$, which includes no connectives, is Kripke-valid. Thus, ILS($\mathscr{C}$), and hence CLS($\mathscr{C}$) are always non-empty. As in the case of the formula-level relation, it depends on $\mathscr{C}$ whether the converse inclusion CLS($\mathscr{C}$) $\subseteq$ ILS($\mathscr{C}$) holds. As to the usual connectives, the following facts are known.

- If we only use $\wedge$ and $\vee$, then a sequent is Kripke-valid if and only if it is classically valid, i.e., ILS($\{\wedge, \vee\}$) = CLS($\{\wedge, \vee\}$).
- If we use $\neg$, then we can construct a sequent $\neg\neg p \Rightarrow p$, which is classically valid but not Kripke-valid. Hence, ILS($\{\neg\}$) $\subsetneq$ CLS($\{\neg\}$).
- If we use $\to$, then we can construct a sequent $\Rightarrow (((p \to q)) \to p) \to p$, which is classically valid but not Kripke-valid. Hence, ILS($\{\to\}$) $\subsetneq$ CLS($\{\to\}$).

Thus, it can be seen that, in the case of the usual connectives, i.e., in the case where $\mathscr{C} \subseteq \{\neg, \wedge, \vee, \to\}$, ILS($\mathscr{C}$) = CLS($\mathscr{C}$) if and only if $\mathscr{C} \subseteq \{\wedge, \vee\}$. Table 1.2 gives a complete classification of the usual connectives with respect to the relation between ILS($\mathscr{C}$) and CLS($\mathscr{C}$).

Table 1.2: The relation between ILS($\mathscr{C}$) and CLS($\mathscr{C}$) for usual connectives $\mathscr{C}$

The choice of $\mathscr{C}(\subseteq \{\neg, \wedge, \vee, \to\})$	The relation between ILS($\mathscr{C}$) and CLS($\mathscr{C}$)
$\emptyset, \{\wedge\}, \{\vee\}, \{\wedge, \vee\}$	$\emptyset \subsetneq$ ILS($\mathscr{C}$) = CLS($\mathscr{C}$)
$\{\neg\}, \{\to\}, \{\neg, \wedge\}, \{\neg, \vee\}, \{\to, \wedge\}, \{\to, \vee\}$, $\{\neg, \to\}, \{\neg, \to, \wedge\}, \{\neg, \to, \vee\}$ $\{\neg, \wedge, \vee\}, \{\to, \wedge, \vee\}, \{\neg, \to, \wedge, \vee\}$	$\emptyset \subsetneq$ ILS($\mathscr{C}$) $\subsetneq$ CLS($\mathscr{C}$)

Note that in the case $\mathscr{C} = \{\neg\}$, ILS($\mathscr{C}$) coincides with CLS($\mathscr{C}$) but IL($\mathscr{C}$) does not coincide with CL($\mathscr{C}$). This difference arises because, for sequents, we can use $\Rightarrow$ as implication between the antecedents and the succedents. In the case of sequents, in some sense, we can use $\wedge$'s at the outermost places in the left to $\Rightarrow$; $\vee$'s at the outermost places in the right to $\Rightarrow$; and $\to$ between the antecedents and the succedents.

1.2 Results on propositional logic

We have seen how the formula-level and sequent-level relations between classical logic and intuitionistic logic change depending on the set $\mathscr{C}$ of connectives we choose, in the case where $\mathscr{C}$ consists of usual connectives, that is, $\mathscr{C} \subseteq \{\neg, \wedge, \vee, \to\}$. Then, what about general $\mathscr{C}$?

First, let us consider the formula-level relation. As mentioned before, since classical models can be regarded as one-state Kripke models, for any $\mathscr{C}$, it holds that

$$\emptyset \subseteq \mathrm{IL}(\mathscr{C}) \subseteq \mathrm{CL}(\mathscr{C}).$$

Thus, there are 4 $(=2^2)$ possible cases according to whether each of the two inclusion relations is actually an equality or a proper inclusion relation. We found out for what kinds of connectives $\mathscr{C}$ each of the four cases holds. Table 1.3 summarizes the result. This result tells us what *properties* of connectives cause the difference between classical logic and intuitionistic logic, while Table 1.1, which provides the classification only when $\mathscr{C}$ consists of usual connectives, doesn't. Conditions $(\star 1)$, $(\star 2)$, (M), and $(\sqcap\text{-}\sqsubseteq\text{-}1)$ in Table 1.3 are defined as follows. (For the definitions of the symbols such as $\bar{\mathbf{a}}$, $\sqcap$, and $\sqsubseteq$, see § 2.2.1.)

$(\star 1)$ For any $c \in \mathscr{C}$, $\mathrm{t}_c(\mathbf{0}) = 0$.

$(\star 2)$ For any $c \in \mathscr{C}$ and any $\mathbf{a} \in \{0,1\}^{\mathrm{ar}(c)}$, $\mathrm{t}_c(\mathbf{a}) \neq \mathrm{t}_c(\bar{\mathbf{a}})$.

(M) For any $c \in \mathscr{C}$ and any $\mathbf{a}, \mathbf{b} \in \{0,1\}^{\mathrm{ar}(c)}$, if $\mathbf{a} \sqsubseteq \mathbf{b}$, then $\mathrm{t}_c(\mathbf{a}) \leq \mathrm{t}_c(\mathbf{b})$.

$(\sqcap\text{-}\sqsubseteq\text{-}1)$ Both of the following conditions hold:

- $(\sqcap\text{-}1)$ For any $c \in \mathscr{C}$ and any $\mathbf{a}, \mathbf{b} \in \{0,1\}^{\mathrm{ar}(c)}$, if $\mathrm{t}_c(\mathbf{a}) = \mathrm{t}_c(\mathbf{b}) = 1$, then $\mathrm{t}_c(\mathbf{a} \sqcap \mathbf{b}) = 1$.
- $(\sqsubseteq\text{-}1)$ For any $c \in \mathscr{C}$ and any $\mathbf{a}, \mathbf{b} \in \{0,1\}^{\mathrm{ar}(c)}$, if $\mathrm{t}_c(\mathbf{a}) = \mathrm{t}_c(\mathbf{b}) = 1$ and $\mathbf{a} \sqsubseteq \mathbf{c} \sqsubseteq \mathbf{b}$, then $\mathrm{t}_c(\mathbf{c}) = 1$.

Here, (M) stands for the Monotonicity of the connectives.

Table 1.3: The relation between $\mathrm{IL}(\mathscr{C})$ and $\mathrm{CL}(\mathscr{C})$

The conditions for $\mathscr{C}$	The relation between $\mathrm{IL}(\mathscr{C})$ and $\mathrm{CL}(\mathscr{C})$
Either $(\star 1)$ or $(\star 2)$ holds.	$\emptyset = \mathrm{IL}(\mathscr{C}) = \mathrm{CL}(\mathscr{C})$
This case cannot happen.	$\emptyset = \mathrm{IL}(\mathscr{C}) \subsetneq \mathrm{CL}(\mathscr{C})$
Neither $(\star 1)$ nor $(\star 2)$ holds and either (M) or $(\sqcap\text{-}\sqsubseteq\text{-}1)$ holds.	$\emptyset \subsetneq \mathrm{IL}(\mathscr{C}) = \mathrm{CL}(\mathscr{C})$
None of $(\star 1)$, $(\star 2)$, (M), and $(\sqcap\text{-}\sqsubseteq\text{-}1)$ hold.	$\emptyset \subsetneq \mathrm{IL}(\mathscr{C}) \subsetneq \mathrm{CL}(\mathscr{C})$

Secondly, let us consider the sequent-level relation. Similarly to the case of the formula-level relation, for any $\mathscr{C}$, it holds that

$$\emptyset \subseteq \mathrm{ILS}(\mathscr{C}) \subseteq \mathrm{CLS}(\mathscr{C}),$$

and hence there are 4 $(=2^2)$ possible cases according to whether each of the two inclusion relations is actually an equality or a proper inclusion relation. We found out for what kinds of connectives $\mathscr{C}$ each of the four cases holds. Table 1.4 summarizes the result. It generalizes Table 1.2. Furthermore, we show that the same result holds if we only use sequents with a single succedent instead of those with multi-succedents. This result gives an affirmative answer to van der Giessen's conjecture in [6], where she showed that intuitionistic natural deduction system $\mathrm{IPC}_{\mathscr{C}}$ with connectives in $\mathscr{C}$ and classical natural deduction system $\mathrm{CPC}_{\mathscr{C}}$ with connectives in $\mathscr{C}$ are equivalent if all connectives in $\mathscr{C}$ are monotonic, and conjectured the converse holds.[2]

Table 1.4: The relation between ILS($\mathscr{C}$) and CLS($\mathscr{C}$)

The conditions for $\mathscr{C}$	The relation between ILS($\mathscr{C}$) and CLS($\mathscr{C}$)
This case cannot happen.	$\emptyset = \mathrm{ILS}(\mathscr{C}) \subsetneq \mathrm{CLS}(\mathscr{C})$
This case cannot happen.	$\emptyset = \mathrm{ILS}(\mathscr{C}) = \mathrm{CLS}(\mathscr{C})$
(M) holds.	$\emptyset \subsetneq \mathrm{ILS}(\mathscr{C}) = \mathrm{CLS}(\mathscr{C})$
(M) doesn't hold.	$\emptyset \subsetneq \mathrm{ILS}(\mathscr{C}) \subsetneq \mathrm{CLS}(\mathscr{C})$

1.3 Background on first-order logic

First-order (predicate) logic deals with reasoning concerning individual objects and quantification over individual objects, as well as reasoning concerning the connection of statements as in propositional logic. Thus, the most basic statements in first-order logic are expressions that represent predicate phrases. These expressions are called atomic formulas, and are of the form $p(t_1, \ldots, t_n)$, where p is a predicate symbol and $t_1, \ldots, t_n$ are terms. Formulas in first-order predicate logic, called predicate formulas, are built out from atomic formulas using universal quantifier $\forall$ and existential quantifier $\exists$ in addition to propositional connectives. $\forall x\alpha$ is interpreted to mean that α is true for all x and $\exists x\alpha$ is interpreted to mean there exits some x such that α is true.

In first-order logic, classical models are equipped with an individual domain so that individuals and quantifications over individuals are interpreted as individual elements and quantifications over individual domains. Also, Kripke models are equipped with an individual domain at each state so that the constructive interpretation can be extended to predicate formulas. With these interpretations, the notions of classical validity and the Kripke-validity are extended to predicate formulas and sequents. So, as in the case of propositional logic, given a set $\mathscr{C}$ of connectives, we can define the set of valid formulas and that of valid sequents in each logic. Let FOCL($\mathscr{C}$) denote the set of classically valid predicate formulas built out of connectives in $\mathscr{C}$; FOIL($\mathscr{C}$) the set of Kripke-valid predicate formulas built out of connectives in $\mathscr{C}$; FOCLS($\mathscr{C}$) the set of classically valid predicate sequents built out of connectives in $\mathscr{C}$; and FOILS($\mathscr{C}$) the set of Kripke-valid predicate sequents built out of connectives in $\mathscr{C}$. Recall that CL and IL stand for Classical Logic and Intuitionistic Logic, respectively, and the letter S in FOCLS and FOILS indicates the sets are those of valid Sequents. Furthermore, FO stands for First-Order.

Then, as in the case of propositional logic, every classical model can be regarded as a one-state Kripke model, and hence Kripke-valid formulas and sequents are also classically valid. Thus, $\mathrm{FOIL}(\mathscr{C}) \subseteq \mathrm{FOCL}(\mathscr{C})$ and $\mathrm{FOILS}(\mathscr{C}) \subseteq \mathrm{FOCLS}(\mathscr{C})$ hold for any $\mathscr{C}$. On the other hand, there arise various new formulas and sequents that are classically valid but not Kripke-valid. Among such expressions, formulas of the form $\forall x(\alpha \vee \beta) \to \forall x\alpha \vee \beta$ with $x \notin \mathrm{FV}(\beta)$, called **D**-axioms, and equivalent sequents of the form $\forall x(\alpha \vee \beta) \Rightarrow \forall x\alpha \vee \beta$ with $x \notin \mathrm{FV}(\beta)$ are especially important. The logical system obtained from intuitionistic predicate logic by adding this axiom schema is known as a representative system of intermediate strength between intuitionistic predicate logic and classical predicate logic. It is called the logic of constant domains, or CD, because it is semantically characterized by Kripke models with constant domains. For a detailed presentation

[2]Precisely, she showed that the rules of the two systems coincide if they are optimized.

of CD, including the completeness theorem, see, e.g., [2]. Also, for the proof-theoretical properties of CD, see, e.g., [7]. As in the cases of classical and intuitionistic first-order logics, we denote by FOCD($\mathscr{C}$) the set of CD-valid predicate formulas built out of connectives in $\mathscr{C}$ and FOCDS($\mathscr{C}$) the set of CD-valid predicate sequents built out of connectives in $\mathscr{C}$. Then, since classical models can be regarded as one-state Kripke models, which are by definition constant domain, and constant domain Kripke models are of course Kripke models, for any $\mathscr{C}$, it holds that

$$\emptyset \subseteq \mathrm{FOIL}(\mathscr{C}) \subseteq \mathrm{FOCD}(\mathscr{C}) \subseteq \mathrm{FOCL}(\mathscr{C}), \tag{1.3}$$

$$\emptyset \subseteq \mathrm{FOILS}(\mathscr{C}) \subseteq \mathrm{FOCDS}(\mathscr{C}) \subseteq \mathrm{FOCLS}(\mathscr{C}). \tag{1.4}$$

As in the case of propositional logic, whether each of these inclusion relations is actually an equality or a proper inclusion depends on the choice of connectives. For example, Figure 1.3 illustrates the inclusion relations between FOIL($\{\vee, \to\}$), FOCD($\{\vee, \to\}$), and FOCL($\{\vee, \to\}$) and Figure 1.4 illustrates the inclusion relation between FOIL($\{\neg, \wedge, \to\}$), FOCD($\{\neg, \wedge, \to\}$), and FOCL($\{\neg, \wedge, \to\}$).

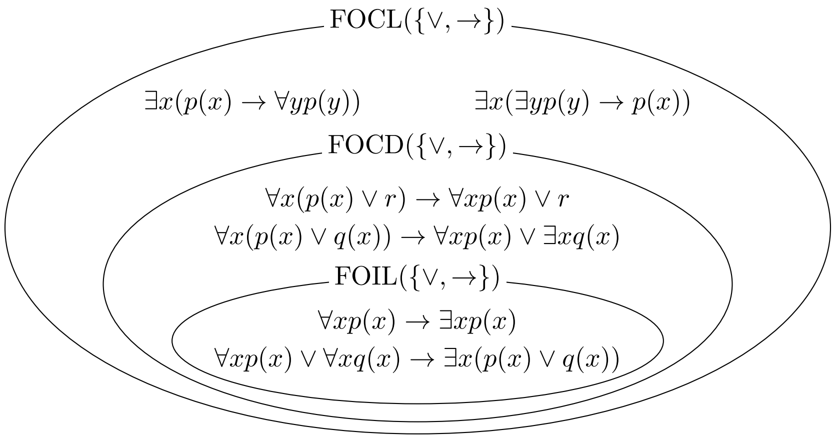

Figure 1.3: The relation between FOIL($\{\vee, \to\}$), FOCD($\{\vee, \to\}$) and FOCL($\{\vee, \to\}$)

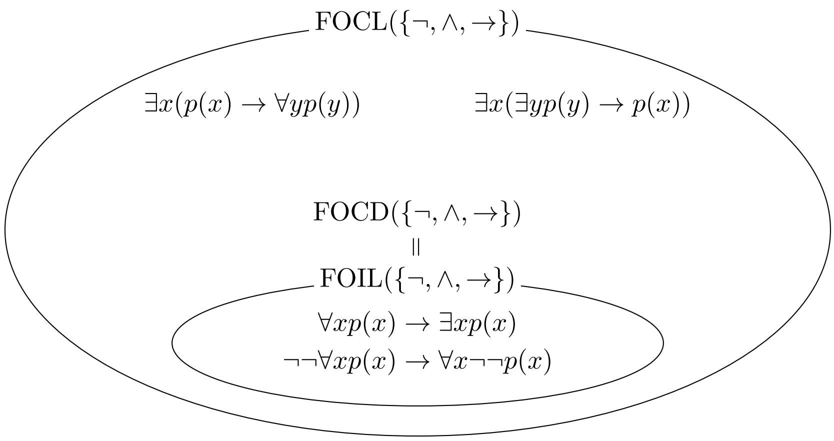

Figure 1.4: The relation between FOIL($\{\neg, \wedge, \to\}$), FOCD($\{\neg, \wedge, \to\}$), and FOCL($\{\neg, \wedge, \to\}$)

1.4 Results on first-order logic

For the sequence of inclusion relations (1.3), $\emptyset \subseteq \mathrm{FOIL}(\mathscr{C}) \subseteq \mathrm{FOCD}(\mathscr{C}) \subseteq \mathrm{FOCL}(\mathscr{C})$, there are 8 ($= 2^3$) possible cases according to whether each of the three inclusion relations is actually an equality or a proper inclusion relation; and for the sequence of inclusion relations (1.4), $\emptyset \subseteq \mathrm{FOILS}(\mathscr{C}) \subseteq \mathrm{FOCDS}(\mathscr{C}) \subseteq \mathrm{FOCLS}(\mathscr{C})$, the same holds. For a general set $\mathscr{C}$ of connectives, we characterize these possible cases by simple properties of connectives in $\mathscr{C}$.

First, let us consider the sequent-level relation. Table 1.5 summarizes the result. Conditions (M) and ($\sqcap$-1) were defined in § 1.2 as follows.

(M) For any $c \in \mathscr{C}$ and any $\mathbf{a} \in \{0,1\}^{\mathrm{ar}(c)}$, if $\mathbf{a} \sqsubseteq \mathbf{b}$, then $\mathrm{t}_c(\mathbf{a}) \leq \mathrm{t}_c(\mathbf{b})$.

($\sqcap$-1) For any $c \in \mathscr{C}$ and any $\mathbf{a}, \mathbf{b} \in \{0,1\}^{\mathrm{ar}(c)}$, if $\mathrm{t}_c(\mathbf{a}) = \mathrm{t}_c(\mathbf{b}) = 1$, then $\mathrm{t}_c(\mathbf{a} \sqcap \mathbf{b}) = 1$.

Table 1.5: The relation between FOILS($\mathscr{C}$), FOCDS($\mathscr{C}$), and FOCLS($\mathscr{C}$)

The conditions for $\mathscr{C}$	The relation between FOILS($\mathscr{C}$), FOCDS($\mathscr{C}$), and FOCLS($\mathscr{C}$)
This case cannot happen.	$\emptyset = \mathrm{FOILS}(\mathscr{C}) = \mathrm{FOCDS}(\mathscr{C}) = \mathrm{FOCLS}(\mathscr{C})$
This case cannot happen.	$\emptyset = \mathrm{FOILS}(\mathscr{C}) \subsetneq \mathrm{FOCDS}(\mathscr{C}) = \mathrm{FOCLS}(\mathscr{C})$
This case cannot happen.	$\emptyset = \mathrm{FOILS}(\mathscr{C}) = \mathrm{FOCDS}(\mathscr{C}) \subsetneq \mathrm{FOCLS}(\mathscr{C})$
This case cannot happen.	$\emptyset = \mathrm{FOILS}(\mathscr{C}) \subsetneq \mathrm{FOCDS}(\mathscr{C}) \subsetneq \mathrm{FOCLS}(\mathscr{C})$
Both (M) and ($\sqcap$-1) hold.	$\emptyset \subsetneq \mathrm{FOILS}(\mathscr{C}) = \mathrm{FOCDS}(\mathscr{C}) = \mathrm{FOCLS}(\mathscr{C})$
(M) holds and ($\sqcap$-1) doesn't hold.	$\emptyset \subsetneq \mathrm{FOILS}(\mathscr{C}) \subsetneq \mathrm{FOCDS}(\mathscr{C}) = \mathrm{FOCLS}(\mathscr{C})$
(M) doesn't hold and ($\sqcap$-1) holds.	$\emptyset \subsetneq \mathrm{FOILS}(\mathscr{C}) = \mathrm{FOCDS}(\mathscr{C}) \subsetneq \mathrm{FOCLS}(\mathscr{C})$
Neither (M) nor ($\sqcap$-1) holds.	$\emptyset \subsetneq \mathrm{FOILS}(\mathscr{C}) \subsetneq \mathrm{FOCDS}(\mathscr{C}) \subsetneq \mathrm{FOCLS}(\mathscr{C})$

First, as in the case of propositional logic, FOILS($\mathscr{C}$) is always non-empty, so that the inclusion relation between $\emptyset$ and FOILS($\mathscr{C}$) is always proper. Regarding the inclusion relation between FOCDS($\mathscr{C}$) and FOCLS($\mathscr{C}$), we can extend the result on the necessary and sufficient condition for ILS($\mathscr{C}$) to coincide with CLS($\mathscr{C}$). (Recall that ILS($\mathscr{C}$) = CLS($\mathscr{C}$) if and only if (M) holds, that is, all connectives in $\mathscr{C}$ are monotonic. See Table 1.4.) Thus, we find out that the condition (M) is also a necessary and sufficient condition for FOCDS($\mathscr{C}$) to coincide with FOCLS($\mathscr{C}$). As to the inclusion relation between FOILS($\mathscr{C}$) and FOCDS($\mathscr{C}$), we remark that it is known that in the case of the usual connectives, the presence of disjunction causes the difference between FOILS($\mathscr{C}$) and FOCDS($\mathscr{C}$) (see, e.g., [2]); that is, for $\mathscr{C} \subseteq \{\neg, \wedge, \vee, \rightarrow\}$, FOILS($\mathscr{C}$) = FOCDS($\mathscr{C}$) if and only if $\vee \notin \mathscr{C}$. We find out that, for a general set $\mathscr{C}$ of connectives, the condition ($\sqcap$-1), which means the supermultiplicativity of the connectives $\mathscr{C}$ (cf. § 2.2.1), is a necessary and sufficient condition for FOILS($\mathscr{C}$) to coincide with FOCDS($\mathscr{C}$).

Next, let us consider the formula-level relation. Table 1.6 summarizes the result. The new condition ($\sqcap$-$\sqcup$-$\sqsubseteq$) is defined as follows. (The other conditions were introduced in § 2.2.1. For the definitions of the symbols $\sqcap$ and $\sqcup$, see § 2.2.1.)

($\sqcap$-$\sqcup$-$\sqsubseteq$) All of the following six conditions hold.

- ($\sqcap$-1) For any $c \in \mathscr{C}$ and any $\mathbf{a}, \mathbf{b} \in \{0,1\}^{\mathrm{ar}(c)}$, if $\mathrm{t}_c(\mathbf{a}) = \mathrm{t}_c(\mathbf{b}) = 1$, then $\mathrm{t}_c(\mathbf{a} \sqcap \mathbf{b}) = 1$.
- ($\sqcap$-0) For any $c \in \mathscr{C}$ and any $\mathbf{a}, \mathbf{b} \in \{0,1\}^{\mathrm{ar}(c)}$, if $\mathrm{t}_c(\mathbf{a}) = \mathrm{t}_c(\mathbf{b}) = 0$, then $\mathrm{t}_c(\mathbf{a} \sqcap \mathbf{b}) = 0$.

($\sqcup$-1) For any $c \in \mathscr{C}$ and any $\mathbf{a}, \mathbf{b} \in \{0,1\}^{\mathrm{ar}(c)}$, if $\mathrm{t}_c(\mathbf{a}) = \mathrm{t}_c(\mathbf{b}) = 1$, then $\mathrm{t}_c(\mathbf{a} \sqcup \mathbf{b}) = 1$.

($\sqcup$-0) For any $c \in \mathscr{C}$ and any $\mathbf{a}, \mathbf{b} \in \{0,1\}^{\mathrm{ar}(c)}$, if $\mathrm{t}_c(\mathbf{a}) = \mathrm{t}_c(\mathbf{b}) = 0$, then $\mathrm{t}_c(\mathbf{a} \sqcup \mathbf{b}) = 0$.

($\sqsubseteq$-1) For any $c \in \mathscr{C}$ and any $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \{0,1\}^{\mathrm{ar}(c)}$, if $\mathrm{t}_c(\mathbf{a}) = \mathrm{t}_c(\mathbf{b}) = 1$ and $\mathbf{a} \sqsubseteq \mathbf{c} \sqsubseteq \mathbf{b}$, then $\mathrm{t}_c(\mathbf{c}) = 1$.

($\sqsubseteq$-0) For any $c \in \mathscr{C}$ and any $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \{0,1\}^{\mathrm{ar}(c)}$, if $\mathrm{t}_c(\mathbf{a}) = \mathrm{t}_c(\mathbf{b}) = 0$ and $\mathbf{a} \sqsubseteq \mathbf{c} \sqsubseteq \mathbf{b}$, then $\mathrm{t}_c(\mathbf{c}) = 0$.

Apparently, condition ($\sqcap$-$\sqcup$-$\sqsubseteq$) may not seem simple. However, we have found out that condition ($\sqcap$-$\sqcup$-$\sqsubseteq$) represents that any $c \in \mathscr{C}$ is one of the following three kinds of connectives: a connective whose truth function is constant; a connective whose truth function is a projection function; and a connective whose truth function is the composition of the truth function of $\neg$ and a projection function.

Table 1.6: The relation between FOIL($\mathscr{C}$), FOCD($\mathscr{C}$), and FOCL($\mathscr{C}$)

The conditions for $\mathscr{C}$	The relation between FOIL($\mathscr{C}$), FOCD($\mathscr{C}$), and FOCL($\mathscr{C}$)
Either ($\star$1) or ($\star$2) holds.	$\emptyset = \mathrm{FOIL}(\mathscr{C}) = \mathrm{FOCD}(\mathscr{C}) = \mathrm{FOCL}(\mathscr{C})$
This case cannot happen.	$\emptyset = \mathrm{FOIL}(\mathscr{C}) \subsetneq \mathrm{FOCD}(\mathscr{C}) = \mathrm{FOCL}(\mathscr{C})$
This case cannot happen.	$\emptyset = \mathrm{FOIL}(\mathscr{C}) = \mathrm{FOCD}(\mathscr{C}) \subsetneq \mathrm{FOCL}(\mathscr{C})$
This case cannot happen.	$\emptyset = \mathrm{FOIL}(\mathscr{C}) \subsetneq \mathrm{FOCD}(\mathscr{C}) \subsetneq \mathrm{FOCL}(\mathscr{C})$
Neither ($\star$1) nor ($\star$2) holds and either (M) or ($\sqcap$-$\sqcup$-$\sqsubseteq$) holds.	$\emptyset \subsetneq \mathrm{FOIL}(\mathscr{C}) = \mathrm{FOCD}(\mathscr{C}) = \mathrm{FOCL}(\mathscr{C})$
This case cannot happen.	$\emptyset \subsetneq \mathrm{FOIL}(\mathscr{C}) \subsetneq \mathrm{FOCD}(\mathscr{C}) = \mathrm{FOCL}(\mathscr{C})$
None of ($\star$1), ($\star$2), (M), and ($\sqcap$-$\sqcup$-$\sqsubseteq$) hold.	$\emptyset \subsetneq \mathrm{FOIL}(\mathscr{C}) = \mathrm{FOCD}(\mathscr{C}) \subsetneq \mathrm{FOCL}(\mathscr{C})$ or $\emptyset \subsetneq \mathrm{FOIL}(\mathscr{C}) \subsetneq \mathrm{FOCD}(\mathscr{C}) \subsetneq \mathrm{FOCL}(\mathscr{C})$

So far, we have not found out what conditions characterize $\emptyset \subsetneq \mathrm{FOIL}(\mathscr{C}) = \mathrm{FOCD}(\mathscr{C}) \subsetneq \mathrm{FOCL}(\mathscr{C})$ and $\emptyset \subsetneq \mathrm{FOIL}(\mathscr{C}) \subsetneq \mathrm{FOCD}(\mathscr{C}) \subsetneq \mathrm{FOCL}(\mathscr{C})$, respectively, but we have found out that $\mathscr{C}$ satisfies none of ($\star$1), ($\star$2), (M), and ($\sqcap$-$\sqcup$-$\sqsubseteq$) if and only if either $\emptyset \subsetneq \mathrm{FOIL}(\mathscr{C}) = \mathrm{FOCD}(\mathscr{C}) \subsetneq \mathrm{FOCL}(\mathscr{C})$ or $\emptyset \subsetneq \mathrm{FOIL}(\mathscr{C}) \subsetneq \mathrm{FOCD}(\mathscr{C}) \subsetneq \mathrm{FOIL}(\mathscr{C})$ holds. Here, we comment on the relation of the new condition ($\sqcap$-$\sqcup$-$\sqsubseteq$) to condition ($\sqcap$-$\sqsubseteq$-1) in Table 1.3. ($\sqcap$-$\sqcup$-$\sqsubseteq$) is stronger than ($\sqcap$-$\sqsubseteq$-1) by ($\sqcap$-0), ($\sqcup$-1), ($\sqcup$-0), and ($\sqsubseteq$-0). Thus, in the case where $\mathscr{C}$ satisfies ($\sqcap$-$\sqsubseteq$-1) but does not any of ($\star$1), ($\star$2), and ($\sqcap$-$\sqcup$-$\sqsubseteq$), while IL($\mathscr{C}$) coincides with CL($\mathscr{C}$) by Table 1.3, FOCD($\mathscr{C}$) and a fortiori FOIL($\mathscr{C}$) do not coincide with FOCL($\mathscr{C}$) by Table 1.6. For example, in the case where $\mathscr{C}$ is $\{\neg, \wedge\}$, which satisfies ($\sqcap$-$\sqsubseteq$-1) but does not any of ($\star$1), ($\star$2), and ($\sqcap$-$\sqcup$-$\sqsubseteq$), we can show that formula $\exists x(\neg p(x) \wedge \neg\forall y p(y))$ is classically valid and not CD-valid as in the proof of Theorem 3.5.2.

1.5 Overview

In Chapter 2, we analyze how the relations between classical propositional logic and intuitionistic propositional logic change depending on the choice of connectives. Our results provide a necessary and sufficient condition for $\emptyset = \mathrm{IL}(\mathscr{C})$ and $\emptyset = \mathrm{CL}(\mathscr{C})$, a necessary and sufficient condition for $\mathrm{IL}(\mathscr{C}) = \mathrm{CL}(\mathscr{C})$ and a necessary and sufficient condition for $\mathrm{ILS}(\mathscr{C}) = \mathrm{CLS}(\mathscr{C})$. Table 1.3 and Table 1.4 are derived from these results. In Chapter 3, we analyze how the relations between intuitionistic first-order logic, classical first-order logic and the logic of constant domains change depending on the choice of connectives. There, our results provide a necessary and sufficient condition for each of the following equalities: $\emptyset = \mathrm{FOIL}(\mathscr{C})$, $\emptyset = \mathrm{FOCD}(\mathscr{C})$, $\emptyset = \mathrm{FOCL}(\mathscr{C})$, $\mathrm{FOCD}(\mathscr{C}) = \mathrm{FOCL}(\mathscr{C})$, $\mathrm{FOIL}(\mathscr{C}) = \mathrm{FOCL}(\mathscr{C})$, $\mathrm{FOILS}(\mathscr{C}) = \mathrm{FOCDS}(\mathscr{C})$, $\mathrm{FOCDS}(\mathscr{C}) = \mathrm{FOCLS}(\mathscr{C})$, and $\mathrm{FOILS}(\mathscr{C}) = \mathrm{FOCLS}(\mathscr{C})$. Table 1.5 and Table 1.6 are derived from these results. In Chapter 4, we conclude our results and discuss some future research.

The following table summarizes the results concerning equalities between sets of valid formulas and between sets of valid sequents. For example, this table indicates that a necessary and suffcient condition for $\mathrm{FOILS}(\mathscr{C}) = \mathrm{FOCDS}(\mathscr{C})$ is discussed in § 3.4 and given in Theorem 3.4.1.

	Propositional logic	First-order logic		
	IL and CL	FOIL and FOCD	FOCD and FOCL	FOIL and FOCL
Sequent	§ 2.3, Theorem 2.3.1	§ 3.3, Theorem 3.3.1	§ 3.4, Theorem 3.4.1	§ 3.4, Corollary 3.4.2
Formula	§ 2.5, Theorem 2.5.1	Open	§ 3.5, Theorem 3.5.4	§ 3.5, Theorem 3.5.4

Chapter 2

Propositional logic

2.1 Overview

In this chapter, we analyze how the choice of connectives affects the relations between intuitionistic and classical propositional logics. We consider the inclusion relation between the sets of valid sequents and that between the sets of formulas. In concrete, we give answers to the following questions:

(i) For what $\mathscr{C}$, does $\mathrm{ILS}(\mathscr{C}) = \mathrm{CLS}(\mathscr{C})$ hold?

(ii) For what $\mathscr{C}$, does $\mathrm{IL}(\mathscr{C}) = \mathrm{CL}(\mathscr{C})$ hold?

As to (ii), as we mentioned in Chapter 1, we further analyze when $\mathrm{IL}(\mathscr{C})$ and $\mathrm{CL}(\mathscr{C})$ become empty.

In § 2.2, we introduce basic concepts concerning truth functions and truth-functional connectives and classical semantics and intuitionistic Kripke semantics with general truth-functional connectives, although most of the contents are already explained informally in Chapter 1. In § 2.3, we show $\mathrm{ILS}(\mathscr{C})$ and $\mathrm{CLS}(\mathscr{C})$ coincide if and only if all connectives in $\mathscr{C}$ are monotonic. In § 2.4, we provide a concise condition which is equivalent to $\emptyset = \mathrm{IL}(\mathscr{C})$, and show that the condition is also equivalent to $\emptyset = \mathrm{CL}(\mathscr{C})$. In § 2.5, we provide a necessary and sufficient condition for $\mathrm{IL}(\mathscr{C}) = \mathrm{CL}(\mathscr{C})$. The contents in this chapter are based on [8].

2.2 Preliminaries

2.2.1 Connectives and truth functions

The elements of a set $\{0, 1\}$ are called the *truth values*. $\{0, 1\}^n$ denotes the set of sequences of truth values of length n. We shall use letters $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ to denote arbitrary finite sequences of truth values. We denote by $\mathbf{0}_n$ and $\mathbf{1}_n$ the sequence $\langle 0, \ldots, 0\rangle \in \{0, 1\}^n$ and $\langle 1, \ldots, 1\rangle \in \{0, 1\}^n$, respectively. For $\mathbf{a} \in \{0, 1\}^n$, we denote by $\mathbf{a}[i]$ the i-th value of $\mathbf{a}$. For example, $\langle 0, 1, 0\rangle[1] = \langle 0, 1, 0\rangle[3] = 0$ and $\langle 0, 1, 0\rangle[2] = 1$. For $\mathbf{a} \in \{0, 1\}^n$, $\overline{\mathbf{a}}$ denotes the sequence obtained from $\mathbf{a}$ by inverting 0 and 1. For example, $\overline{\langle 0, 1, 0\rangle} = \langle 1, 0, 1\rangle$. An n-ary *truth function* is a function from $\{0, 1\}^n$ to $\{0, 1\}$.

The natural order $\sqsubseteq_n$ on $\{0, 1\}^n$ is defined as follows: for $\mathbf{a} \in \{0, 1\}^n$ and $\mathbf{b} \in \{0, 1\}^n$, $\mathbf{a} \sqsubseteq_n \mathbf{b}$ if and only if $\mathbf{a}[i] \leq \mathbf{b}[i]$ for all $i = 1, \ldots, n$. Here, $\leq$ denotes the usual order on $\{0, 1\}$ defined by $0 \leq 0$, $1 \leq 1$, $0 \leq 1$, and $1 \not\leq 0$. In what follows, we shall omit the subscript n of $\mathbf{0}_n$, $\mathbf{1}_n$, and

$\sqsubseteq_n$, since it is clear from the context. For $\mathbf{a}, \mathbf{b} \in \{0,1\}^n$, $\mathbf{a} \sqcap \mathbf{b}$ denotes the infimum of $\{\mathbf{a}, \mathbf{b}\}$ and $\mathbf{a} \sqcup \mathbf{b}$ denotes the supremum of $\{\mathbf{a}, \mathbf{b}\}$. It is obvious that $\mathbf{a} \sqcap \mathbf{b}$ and $\mathbf{a} \sqcup \mathbf{b}$ can be calculated as follows:

$$(\mathbf{a} \sqcap \mathbf{b})[i] = \begin{cases} 1 & \text{if } \mathbf{a}[i] = 1 \text{ and } \mathbf{b}[i] = 1 \\ 0 & \text{if } \mathbf{a}[i] = 0 \text{ or } \mathbf{b}[i] = 0, \end{cases}$$

$$(\mathbf{a} \sqcup \mathbf{b})[i] = \begin{cases} 0 & \text{if } \mathbf{a}[i] = 0 \text{ and } \mathbf{b}[i] = 0 \\ 1 & \text{if } \mathbf{a}[i] = 1 \text{ or } \mathbf{b}[i] = 1. \end{cases}$$

An n-ary truth function f is said to be *monotonic* if for all $\mathbf{a}, \mathbf{b} \in \{0,1\}^n$, $\mathbf{a} \sqsubseteq \mathbf{b}$ implies $f(\mathbf{a}) \leq f(\mathbf{b})$. An n-ary truth function f is said to be *supermultiplicative* if for all $\mathbf{a}, \mathbf{b} \in \{0,1\}^n$, $f(\mathbf{a}) = f(\mathbf{b}) = 1$ implies $f(\mathbf{a} \sqcap \mathbf{b}) = 1$.[1]

2.2.2 Truth-functional connectives and propositional formulas

A *truth-functional connective*, or *propositional connective* is a symbol with a truth function. For a truth-functional connective c, we denote by t_c the truth function associated with c, and $\mathrm{ar}(c)$ is the arity of t_c. We use letters c and d to denote arbitrary truth-functional connectives. We say that a truth-functional connective c is *monotonic* if its truth function is monotonic and is *supermultiplicative* if its truth function is supermultiplicative. Among the usual connectives $\neg$, $\wedge$, $\vee$, and $\to$, the monotonic ones are $\vee$ and $\wedge$, and the supuermultiplicative ones are $\neg$, $\wedge$, and $\to$.

Given a set $\mathscr{C}$ of truth-functional connectives and a set PV of propositional variables, the set $\mathrm{Fml}(\mathscr{C})$ of *(propositional) formulas* is defined inductively as follows:

- $\mathrm{PV} \subseteq \mathrm{Fml}(\mathscr{C})$,
- if $c \in \mathscr{C}$ and $\alpha_1, \ldots, \alpha_{\mathrm{ar}(c)} \in \mathrm{Fml}(\mathscr{C})$, then $c(\alpha_1, \ldots, \alpha_{\mathrm{ar}(c)}) \in \mathrm{Fml}(\mathscr{C})$.

We use letters p, q, r to denote arbitrary propositional variables, and $\alpha, \beta, \gamma, \varphi, \psi, \tau$ are arbitrary formulas. Moreover, $\mathrm{PV}(\alpha)$ denotes the set of propositional variables occurring in α. A *sequent* is an expression of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are sets of formulas. We denote by $\mathrm{Sqt}(\mathscr{C})$ the set $\{\Gamma \Rightarrow \Delta \mid \Gamma, \Delta \subseteq \mathrm{Fml}(\mathscr{C})\}$. If $\Gamma = \{\alpha_1, \ldots, \alpha_n\}$ and $\Delta = \{\beta_1, \ldots, \beta_m\}$, we often omit the braces and simply write $\alpha_1, \ldots, \alpha_n \Rightarrow \beta_1, \ldots, \beta_m$ for $\{\alpha_1, \ldots, \alpha_n\} \Rightarrow \{\beta_1, \ldots, \beta_m\}$. Moreover, we denote by $\mathrm{Sqt}_1(\mathscr{C})$ the set $\{\Gamma \Rightarrow \{\alpha\} \mid \Gamma \subseteq \mathrm{Fml}(\mathscr{C}), \alpha \in \mathrm{Fml}(\mathscr{C})\}$.

2.2.3 Classical semantics

A *(classical) model* $\mathscr{M}$ is a function from PV into $\{0,1\}$ For a model $\mathscr{M}$ and a formula $\alpha \in \mathrm{Fml}(\mathscr{C})$, we define the *value* $[\![\alpha]\!]_{\mathscr{M}} \in \{0,1\}$ of α in $\mathscr{M}$ inductively as follows:

- $[\![p]\!]_{\mathscr{M}} = \mathscr{M}(p)$,
- $[\![c(\alpha_1, \ldots, \alpha_{\mathrm{ar}(c)})]\!]_{\mathscr{M}} = \mathrm{t}_c([\![\alpha_1]\!]_{\mathscr{M}}, \ldots, [\![\alpha_{\mathrm{ar}(c)}]\!]_{\mathscr{M}})$.

If $\vec{\alpha}$ denotes a sequence of formulas $\alpha_1, \ldots, \alpha_n$, then we denote by $[\![\vec{\alpha}]\!]_{\mathscr{M}}$ the sequence $\langle [\![\alpha_1]\!]_{\mathscr{M}}, \ldots, [\![\alpha_n]\!]_{\mathscr{M}} \rangle$. For example, if $\alpha \equiv c(\beta_1, \ldots, \beta_{\mathrm{ar}(c)})$ and $\vec{\beta} = \beta_1, \ldots, \beta_{\mathrm{ar}(c)}$, then $[\![\alpha]\!]_{\mathscr{M}} = \mathrm{t}_c([\![\vec{\beta}]\!]_{\mathscr{M}})$.

[1] Note that $f(\mathbf{a}) = f(\mathbf{b}) = 1 \implies f(\mathbf{a} \sqcap \mathbf{b}) = 1$ if and only if $f(\mathbf{a}) \sqcap f(\mathbf{b}) \leq f(\mathbf{a} \sqcap \mathbf{b})$.

When $[\![\alpha]\!]_{\mathscr{M}} = 1$ for every $\mathscr{M}$, we say that α is *classically valid*. The set $\{\alpha \in \mathrm{Fml}(\mathscr{C}) \mid \alpha \text{ is valid}\}$ is denoted by $\mathrm{CL}(\mathscr{C})$.

For a sequent $\Gamma \Rightarrow \Delta \in \mathrm{Sqt}(\mathscr{C})$, the value $[\![\Gamma \Rightarrow \Delta]\!]_{\mathscr{M}} \in \{0,1\}$ of $\Gamma \Rightarrow \Delta$ in $\mathscr{M}$ is defined by

$$[\![\Gamma \Rightarrow \Delta]\!]_{\mathscr{M}} = \begin{cases} 0 & \text{if } [\![\alpha]\!]_{\mathscr{M}} = 1 \text{ for all } \alpha \in \Gamma \text{ and } [\![\beta]\!]_{\mathscr{M}} = 0 \text{ for all } \beta \in \Delta \\ 1 & \text{otherwise.} \end{cases}$$

A sequent $\Gamma \Rightarrow \Delta \in \mathrm{Sqt}(\mathscr{C})$ is *(classically) valid* if $[\![\Gamma \Rightarrow \Delta]\!]_{\mathscr{M}} = 1$ for all classical models $\mathscr{M}$. We denote by $\mathrm{CLS}(\mathscr{C})$ the set $\{\Gamma \Rightarrow \Delta \in \mathrm{Sqt}(\mathscr{C}) \mid \Gamma \Rightarrow \Delta \text{ is classically valid}\}$. Moreover, we denote by $\mathrm{CLS}_1(\mathscr{C})$ the set $\{\Gamma \Rightarrow \{\alpha\} \mid \Gamma \Rightarrow \{\alpha\} \text{ is classically valid}\}$.

2.2.4 Kripke semantics

A *Kripke model* is a tuple of the form $\langle W, \preceq, I \rangle$, where

- W is a non-empty set of *states*, or *possible worlds*;
- $\preceq$ is a pre-order on W;
- I is a mapping from $W \times \mathrm{PV}$ in to $\{0,1\}$ that satisfies the *hereditary condition*; that is, $w \preceq v$ implies $I(w,p) \leq I(v,p)$.

For a Kripke model $\mathscr{K} = \langle W, \preceq, I \rangle$, a state $w \in W$ and a formula $\alpha \in \mathrm{Fml}(\mathscr{C})$, we define the *value* $\|\alpha\|_{\mathscr{K},w}$ of α at w in $\mathscr{K}$ as follows:

- $\|p\|_{\mathscr{K},w} = I(w,p)$;
- $\|c(\alpha_1, \ldots, \alpha_{\mathrm{ar}(c)})\|_{\mathscr{K},w} = 1$ if and only if $\mathrm{t}_c(\|\alpha_1\|_{\mathscr{K},v}, \ldots, \|\alpha_{\mathrm{ar}(c)}\|_{\mathscr{K},v}) = 1$ for each $v \succeq w$.

As in the case of the usual connectives, the hereditary condition easily extends to any formula:

Lemma 2.2.1. *For every Kripke model $\mathscr{K} = \langle W, \preceq, I \rangle$, every $w, v \in W$ and every $\alpha \in \mathrm{Fml}(\mathscr{C})$, $w \preceq v$ implies $\|\alpha\|_{\mathscr{K},w} \leq \|\alpha\|_{\mathscr{K},v}$.*

A formula α is said to be *Kripke-valid* if for each $\mathscr{K} = \langle W, \preceq, I \rangle$ and $w \in W$, $\|\alpha\|_{\mathscr{K},w} = 1$. $\mathrm{IL}(\mathscr{C})$ denotes the set $\{\alpha \in \mathrm{Fml}(\mathscr{C}) \mid \alpha \text{ is Kripke-valid}\}$. For a Kripke model $\mathscr{K} = \langle W, \preceq, I \rangle$, a state $w \in W$ and a sequent $\Gamma \Rightarrow \Delta \in \mathrm{Sqt}(\mathscr{C})$, the *value* $\|\Gamma \Rightarrow \Delta\|_{\mathscr{K},w} \in \{0,1\}$ of $\Gamma \Rightarrow \Delta$ at w in $\mathscr{K}$ is defined by

$$\|\Gamma \Rightarrow \Delta\|_{\mathscr{K},w} = \begin{cases} 0 & \text{if } \|\alpha\|_{\mathscr{K},w} = 1 \text{ for all } \alpha \in \Gamma \text{ and } \|\beta\|_{\mathscr{K},w} = 0 \text{ for all } \beta \in \Delta \\ 1 & \text{otherwise.} \end{cases}$$

A sequent $\Gamma \Rightarrow \Delta \in \mathrm{Sqt}(\mathscr{C})$ is *Kripke-valid* if $\|\Gamma \Rightarrow \Delta\|_{\mathscr{K},w} = 1$ for all models $\mathscr{K}$ and all states w of $\mathscr{K}$. We denote by $\mathrm{ILS}(\mathscr{C})$ the set $\{\Gamma \Rightarrow \Delta \in \mathrm{Sqt}(\mathscr{C}) \mid \Gamma \Rightarrow \Delta \text{ is Kripke-valid}\}$. Moreover, we denote by $\mathrm{ILS}_1(\mathscr{C})$ the set $\{\Gamma \Rightarrow \{\alpha\} \in \mathrm{Sqt}_1(\mathscr{C}) \mid \Gamma \Rightarrow \{\alpha\} \text{ is Kripke-valid}\}$.

Remark 2.2.2 (semantical consequence relations)**.** $\mathrm{ILS}_1(\mathscr{C})$ and $\mathrm{CLS}_1(\mathscr{C})$ are defined to be the sets of Kripke-valid sequents with a single succedent and the set of classically valid sequents with a single succedent, respectively. We can regard them as defining the familiar notions so-called *semantical consequence relations* since the validity of $\Gamma \Rightarrow \alpha$ in each semantics means the relation usually denoted by $\Gamma \vDash \alpha$ holds in that semantics.

The following lemmas play important roles in this study.

Lemma 2.2.3. *(i)* $\mathrm{ILS}(\mathscr{C}) \subseteq \mathrm{CLS}(\mathscr{C})$.

(ii) $\mathrm{IL}(\mathscr{C}) \subseteq \mathrm{CL}(\mathscr{C})$.

Proof. Obvious. □

Lemma 2.2.4. *Let $\mathscr{K} = \langle W, \preceq, I \rangle$ be a Kripke model, and let P be a finite set of propositional variables. Then, for each $w \in W$, there exists $v \succeq w$ such that for each $u \succeq v$ and each $p \in P$, $I(v, p) = I(u, p)$.*

Proof. We denote by $P[w]$ the set $\{p \in P \mid I(w, p) = 0\}$. If the lemma does not hold, then there exists an infinite sequence $w = w_1 \preceq w_2 \preceq w_2 \preceq w_3 \preceq \ldots$ such that $P[w_i] \neq P[w_{i+1}]$ for every i. Then, because of the heredity condition of $\mathscr{K}$, we obtain $P[w_1] \supsetneq P[w_2] \supsetneq \ldots$. However, this contradicts the fact that P is finite. □

Lemma 2.2.5. *Let $\mathscr{K} = \langle W, \preceq, I \rangle$ be a Kripke model, let $w \in W$ and let $\alpha \in \mathrm{Fml}(\mathscr{C})$. If $I(w, p) = I(v, p)$ for each $v \succeq w$ and each $p \in \mathrm{PV}(\alpha)$, then $\|\alpha\|_{\mathscr{K}, w} = [\![\alpha]\!]_{\lambda x.\, I(w,x)}$. Here $\lambda x.\, I(w, x)$ is the classical model such that $(\lambda x.\, I(w, x))(p) = I(w, p)$.*

Proof. By an easy induction on the size of α. □

Lemma 2.2.6.

(1) *Suppose that $\alpha \in \mathrm{Fml}(\mathscr{C})$ satisfies $[\![\alpha]\!]_{\mathscr{M}} = 0$ for every model $\mathscr{M}$. Then, $\|\alpha\|_{\mathscr{K}, w} = 0$ for every Kripke model $\mathscr{K} = \langle W, \preceq, I \rangle$ and every $w \in W$.*

(2) *Let $F_1, \ldots, F_n$ be finite subsets of $\mathrm{Fml}(\mathscr{C})$. Suppose that for any classical model $\mathscr{M}$, there exists some $i \in \{1, \ldots, n\}$ such that $[\![\alpha]\!]_{\mathscr{M}} = 0$ for all $\alpha \in F_i$. Then, for any Kripke model $\mathscr{K} = \langle W, \preceq, I \rangle$ and any $w \in W$, there exists some $i \in \{1, \ldots, n\}$ such that $\|\alpha\|_{\mathscr{K}, w} = 0$ for all $\alpha \in F_i$.*

Proof. As (1) is a special case of (2), we prove only (2). Define P to be the set $\bigcup\{\mathrm{PV}(\alpha) \mid \alpha \in F_1 \cup \cdots \cup F_n\}$. Then, by Lemma 2.2.4, there is some $v \succeq w$ such that for any $u \succeq v$ and any $p \in P$, $I(v, p) = I(u, p)$ holds. Then, by Lemma 2.2.5, it holds that $\|\alpha\|_{\mathscr{K}, v} = [\![\alpha]\!]_{\lambda x.\, I(v,x)}$ for any $\alpha \in F_1 \cup \cdots \cup F_n$. Then, by the assumption, there exists some $i \in \{1, \ldots, n\}$ such that $[\![\alpha]\!]_{\lambda x.\, I(v,x)} = 0$ for all $\alpha \in F_i$. Thus, we have $\|\alpha\|_{\mathscr{K}, v} = [\![\alpha]\!]_{\lambda x.\, I(v,x)} = 0$ for all $\alpha \in F_i$. □

2.3 Condition for $\mathrm{ILS}(\mathscr{C}) = \mathrm{CLS}(\mathscr{C})$

In this section, we show the following theorem.

Theorem 2.3.1. *The following conditions are equivalent:*

(α) *any $c \in \mathscr{C}$ is monotonic;*

(β) $\mathrm{ILS}(\mathscr{C}) = \mathrm{CLS}(\mathscr{C})$*;*

(γ) $\mathrm{ILS}_1(\mathscr{C}) = \mathrm{CLS}_1(\mathscr{C})$.

First, we remark (β) immediately leads to (γ). We prove (α) $\Longrightarrow$ (β) in § 2.3.1 and (γ) $\Longrightarrow$ (α) in § 2.3.2.

2.3.1 Proof of $(\alpha) \implies (\beta)$

In order to prove $(\alpha) \implies (\beta)$, we need the following lemma:

Lemma 2.3.2. *Let $\mathscr{C}$ be a set of monotonic connectives. Then, for any $\alpha \in \mathrm{Fml}(\mathscr{C})$, any Kripke model $\mathscr{K} = \langle W, \preceq, I \rangle$ and any $w \in W$, $[\![\alpha]\!]_{\lambda x.\, I(w,x)} = \|\alpha\|_{\mathscr{K},w}$.*

Proof. The proof proceeds by induction on α. The base case, where α is a propositional variable, immediately follows from the definition of the classical model $\lambda x.\, I(w, x)$. So, now we consider the case where α is of the form $c(\beta_1, \dots, \beta_{\mathrm{ar}(c)})$. Put $\vec{\beta} = \beta_1, \dots, \beta_{\mathrm{ar}(c)}$. By the hereditary, we have $\|\vec{\beta}\|_{\mathscr{K},w} \sqsubseteq \|\vec{\beta}\|_{\mathscr{K},v}$ for all $v \succeq w$. Hence, by the monotonicity of c, we have $\mathrm{t}_c(\|\vec{\beta}\|_{\mathscr{K},w}) \leq \mathrm{t}_c(\|\vec{\beta}\|_{\mathscr{K},v})$ for all $v \succeq w$, so that $\|\alpha\|_{\mathscr{K},w} = \mathrm{t}_c(\|\vec{\beta}\|_{\mathscr{K},w})$ holds. On the other hand, by the induction hypothesis, we have $\mathrm{t}_c(\|\vec{\beta}\|_{\mathscr{K},w}) = \mathrm{t}_c([\![\vec{\beta}]\!]_{\lambda x.\, I(w,x)}) = [\![\alpha]\!]_{\lambda x.\, I(w,x)}$. □

Now we show $(\alpha) \implies (\beta)$, that is, if all $c \in \mathscr{C}$ are monotonic, then $\mathrm{ILS}(\mathscr{C}) = \mathrm{CLS}(\mathscr{C})$. Suppose $\mathscr{C}$ be a set of monotonic connectives. By Lemma 2.2.3, we only have to show that $\Gamma \Rightarrow \Delta \in \mathrm{ILS}(\mathscr{C})$ holds for any $\Gamma \Rightarrow \Delta \in \mathrm{CLS}(\mathscr{C})$. Let $\Gamma \Rightarrow \Delta \in \mathrm{CLS}(\mathscr{C})$. Then, by Lemma 2.3.2, for any Kripke model $\mathscr{K}$ and any $w \in W$, $\|\Gamma \Rightarrow \Delta\|_{\mathscr{K},w} = [\![\Gamma \Rightarrow \Delta]\!]_{\lambda x.\, I(w,x)} = 1$. This completes the proof. □

2.3.2 Proof of $(\gamma) \implies (\alpha)$

Here, we show that if $\mathscr{C}$ includes a non-monotonic connective, then $\mathrm{ILS}_1(\mathscr{C}) \neq \mathrm{CLS}_1(\mathscr{C})$.

Let c be a non-monotonic connective in $\mathscr{C}$. Then, we have the following four cases: (a) $\mathrm{t}_c(\mathbf{0}) = \mathrm{t}_c(\mathbf{1}) = 0$; (b) $\mathrm{t}_c(\mathbf{0}) = 0$ and $\mathrm{t}_c(\mathbf{1}) = 1$; (c) $\mathrm{t}_c(\mathbf{0}) = 1$ and $\mathrm{t}_c(\mathbf{1}) = 0$; and (d) $\mathrm{t}_c(\mathbf{0}) = \mathrm{t}_c(\mathbf{1}) = 1$.

2.3.2.1 (d) $\implies \mathrm{ILS}_1(\mathscr{C}) \neq \mathrm{CLS}_1(\mathscr{C})$

To prove (d) $\implies \mathrm{ILS}_1(\mathscr{C}) \neq \mathrm{CLS}_1(\mathscr{C})$, it suffices to show the following two lemmas. For, if we have proved the lemmas, then since the hypotheses of the two lemmas are satisfied in case (d), and hence, there is some $\alpha \in \mathrm{CL}(\mathscr{C}) \setminus \mathrm{IL}(\mathscr{C})$, and thus $\emptyset \Rightarrow \{\alpha\} \in \mathrm{CLS}_1(\mathscr{C}) \setminus \mathrm{ILS}_1(\mathscr{C})$ holds.

Lemma 2.3.3. *If $\mathrm{t}_c(\mathbf{0}) = \mathrm{t}_c(\mathbf{1}) = 1$ for some $c \in \mathscr{C}$, then $\mathrm{IL}(\mathscr{C}) \neq \emptyset$.*

Proof. Suppose $\mathrm{t}_c(\mathbf{0}) = \mathrm{t}_c(\mathbf{1}) = 1$ for some $c \in \mathscr{C}$. We consider an arbitrary propositional variable p and define $\varphi^R \in \mathrm{Fml}(\mathscr{C})$ as follows:

$$\varphi^R \equiv c(p, \dots, p).$$

Then, obviously, $\|\varphi^R\|_{\mathscr{K},w} = 1$ for every Kripke model $\mathscr{K} = \langle W, \preceq, I \rangle$ and every $w \in W$. Hence, $\mathrm{IL}(\mathscr{C}) \neq \emptyset$. □

Remark 2.3.4. Let $\mathscr{C} = \{\to\}$, then $\mathscr{C}$ satisfies the assumption of Lemma 2.3.3. In this case, we obtain

$$\varphi^R \equiv p \to p \in \mathrm{IL}(\{\to\})$$

according to the above procedure.

Lemma 2.3.5. *If there exists a non-monotonic connective $c \in \mathscr{C}$ such that $\mathrm{t}_c(\mathbf{1}) = 1$ and $\mathrm{IL}(\mathscr{C}) \neq \emptyset$, then $\mathrm{IL}(\mathscr{C}) \subsetneq \mathrm{CL}(\mathscr{C})$.*

Proof. Suppose that the following conditions are satisfied:

- there exists a non-monotonic connective $c \in \mathscr{C}$ such that $\mathrm{t}_c(\mathbf{1}) = 1$;
- $\mathrm{IL}(\mathscr{C}) \neq \emptyset$.

Since c is non-monotonic, there exist $\mathbf{a}, \mathbf{b} \in \{0,1\}^{\mathrm{ar}(c)}$ such that $\mathbf{a} \sqsubseteq \mathbf{b}$, $\mathrm{t}_c(\mathbf{a}) = 1$, and $\mathrm{t}_c(\mathbf{b}) = 0$. Let $\overline{\mathbf{b}}^{\mathbf{a}}$ be the sequence in $\{0,1\}^{\mathrm{ar}(c)}$ defined by

$$\overline{\mathbf{b}}^{\mathbf{a}}[i] = \begin{cases} 0 & \text{if } \mathbf{a}[i] = 0 \text{ and } \mathbf{b}[i] = 1 \\ 1 & \text{if } \mathbf{a}[i] = 1 \text{ or } \mathbf{b}[i] = 0. \end{cases}$$

Fix distinct propositional variables $p, q \in \mathrm{PV}$ and $\tau \in \mathrm{IL}(\mathscr{C})$. Subsequently, we divide the analysis into two cases: (Case 1) $\mathrm{t}_c(\overline{\mathbf{b}}^{\mathbf{a}}) = 1$; and (Case 2) $\mathrm{t}_c(\overline{\mathbf{b}}^{\mathbf{a}}) = 0$.

Case 1: $\mathrm{t}_c(\overline{\mathbf{b}}^{\mathbf{a}}) = 1$. Define formulas $\sigma^P_1, \ldots, \sigma^P_{\mathrm{ar}(c)}, \sigma^P$ by

$$\sigma^P_i \equiv \begin{cases} q & \text{if } \mathbf{a}[i] = 0 \text{ and } \mathbf{b}[i] = 0 \\ p & \text{if } \mathbf{a}[i] = 0 \text{ and } \mathbf{b}[i] = 1 \\ \tau & \text{if } \mathbf{a}[i] = 1, \end{cases}$$
$$\sigma^P \equiv c(\sigma^P_1, \ldots, \sigma^P_{\mathrm{ar}(c)}).$$

Moreover, formulas $\psi^P_1, \ldots, \psi^P_{\mathrm{ar}(c)}, \psi^P$ are defined as follows:

$$\psi^P_i \equiv \begin{cases} p & \text{if } \mathbf{a}[i] = 0 \text{ and } \mathbf{b}[i] = 0 \\ \sigma^P & \text{if } \mathbf{a}[i] = 0 \text{ and } \mathbf{b}[i] = 1 \\ \tau & \text{if } \mathbf{a}[i] = 1, \end{cases}$$
$$\psi^P \equiv c(\psi^P_1, \ldots, \psi^P_{\mathrm{ar}(c)}).$$

Furthermore, formulas $\varphi^P_1, \ldots, \varphi^P_{\mathrm{ar}(c)}, \varphi^P$ are defined as follows:

$$\varphi^P_i \equiv \begin{cases} p & \text{if } \mathbf{a}[i] = 0 \text{ and } \mathbf{b}[i] = 0 \\ \psi^P & \text{if } \mathbf{a}[i] = 0 \text{ and } \mathbf{b}[i] = 1 \\ \tau & \text{if } \mathbf{a}[i] = 1, \end{cases}$$
$$\varphi^P \equiv c(\varphi^P_1, \ldots, \varphi^P_{\mathrm{ar}(c)}).$$

Then, we obtain $\varphi^P \in \mathrm{CL}(\mathscr{C})$ from the following table. For example, that the element in the third row and third column is $\overline{\mathbf{b}}^{\mathbf{a}}$ means that for any classical model $\mathscr{M}$ with $\mathscr{M}(p) = 0$ and $\mathscr{M}(q) = 1$,

$$\langle [\![\sigma^P_1]\!]_{\mathscr{M}}, \ldots, [\![\sigma^P_{\mathrm{ar}(c)}]\!]_{\mathscr{M}} \rangle = \overline{\mathbf{b}}^{\mathbf{a}};$$

and that the element in the fourth row and sixth column is 1 means that for any classical model $\mathscr{M}$ with $\mathscr{M}(p) = 1$ and $\mathscr{M}(q) = 0$,

$$[\![\psi^P]\!]_{\mathscr{M}} = 1.$$

Note that since $\tau \in \mathrm{IL}(\mathscr{C}) \subseteq \mathrm{CL}(\mathscr{C})$, $[\![\tau]\!]_{\mathscr{M}} = 1$ holds for any $\mathscr{M}$.

p	q	$\langle \sigma_1^P, \ldots, \sigma_{\mathrm{ar}(c)}^P \rangle$	σ^P	$\langle \psi_1^P, \ldots, \psi_{\mathrm{ar}(c)}^P \rangle$	ψ^P	$\langle \varphi_1^P, \ldots, \varphi_{\mathrm{ar}(c)}^P \rangle$	φ^P
0	0	$\mathbf{a}$	1	$\mathbf{b}$	0	$\mathbf{a}$	1
0	1	$\overline{\mathbf{b}}^{\mathbf{a}}$	1	$\mathbf{b}$	0	$\mathbf{a}$	1
1	0	$\mathbf{b}$	0	$\overline{\mathbf{b}}^{\mathbf{a}}$	1	$\mathbf{1}$	1
1	1	$\mathbf{1}$	1	$\mathbf{1}$	1	$\mathbf{1}$	1

Now consider the Kripke model $\mathscr{K}^* = \langle \{w_0, w_1\}, \preceq^*, I^* \rangle$, where

$w_i \preceq^* w_j \iff i \leq j$;
$I^*(w_0, p) = 0$, $I^*(w_0, q) = 0$, $I^*(w_1, p) = 1$, and $I^*(w_1, q) = 0$.

Then, we obtain $\|\varphi^P\|_{\mathscr{K}^*, w_0} = 0$ from the following table. For example, that the element in the second row and fourth column is $\overline{\mathbf{b}}^{\mathbf{a}}$ means that

$$\langle \|\psi_1^P\|_{\mathscr{K}^*, w_1}, \ldots, \|\psi_{\mathrm{ar}(c)}^P\|_{\mathscr{K}^*, w_1} \rangle = \overline{\mathbf{b}}^{\mathbf{a}}.$$

	$\langle \sigma_1^P, \ldots, \sigma_{\mathrm{ar}(c)}^P \rangle$	σ^P	$\langle \psi_1^P, \ldots, \psi_{\mathrm{ar}(c)}^P \rangle$	ψ^P	$\langle \varphi_1^P, \ldots, \varphi_{\mathrm{ar}(c)}^P \rangle$	φ^P
$\|\cdot\|_{\mathscr{K}^*, w_1}$	$\mathbf{b}$	0	$\overline{\mathbf{b}}^{\mathbf{a}}$	1	$\mathbf{1}$	1
$\|\cdot\|_{\mathscr{K}^*, w_0}$	$\mathbf{a}$	0	$\mathbf{a}$	1	$\mathbf{b}$	0

Hence, $\mathrm{IL}(\mathscr{C}) \subsetneq \mathrm{CL}(\mathscr{C})$.

Case 2: $\mathrm{t}_c(\overline{\mathbf{b}}^{\mathbf{a}}) = 0$. We define formulas $\sigma_1^Q, \ldots, \sigma_{\mathrm{ar}(c)}^Q, \sigma^Q$ as follows:

$$\sigma_i^Q \equiv \begin{cases} q & \text{if } \mathbf{a}[i] = 0 \text{ and } \mathbf{b}[i] = 0 \\ p & \text{if } \mathbf{a}[i] = 0 \text{ and } \mathbf{b}[i] = 1 \\ \tau & \text{if } \mathbf{a}[i] = 1, \end{cases}$$
$$\sigma^Q \equiv c(\sigma_1^Q, \ldots, \sigma_{\mathrm{ar}(c)}^Q).$$

Then, we define formulas $\psi_1^Q, \ldots, \psi_{\mathrm{ar}(c)}^Q, \psi^Q$ as follows:

$$\psi_i^Q \equiv \begin{cases} \sigma^Q & \text{if } \mathbf{a}[i] = 0 \text{ and } \mathbf{b}[i] = 0 \\ q & \text{if } \mathbf{a}[i] = 0 \text{ and } \mathbf{b}[i] = 1 \\ \tau & \text{if } \mathbf{a}[i] = 1, \end{cases}$$
$$\psi^Q \equiv c(\psi_1^Q, \ldots, \psi_{\mathrm{ar}(c)}^Q).$$

Furthermore, we define formulas $\varphi_1^Q, \ldots, \varphi_{\mathrm{ar}(c)}^Q, \varphi^Q$ as follows:

$$\varphi_i^Q \equiv \begin{cases} \psi^Q & \text{if } \mathbf{a}[i] = 0 \text{ and } \mathbf{b}[i] = 0 \\ p & \text{if } \mathbf{a}[i] = 0 \text{ and } \mathbf{a}[i] = 1 \\ \tau & \text{if } \mathbf{a}[i] = 1, \end{cases}$$
$$\varphi^Q \equiv c(\varphi_1^Q, \ldots, \varphi_{\mathrm{ar}(c)}^Q).$$

Then, we obtain $\varphi^Q \in \mathrm{CL}(\mathscr{C})$ from the following table.

p	q	$\langle\sigma_1^Q,\ldots,\sigma_{\mathrm{ar}(c)}^Q\rangle$	σ^Q	$\langle\psi_1^Q,\ldots,\psi_{\mathrm{ar}(c)}^Q\rangle$	ψ^Q	$\langle\varphi_1^Q,\ldots,\varphi_{\mathrm{ar}(c)}^Q\rangle$	φ^Q
0	0	$\mathbf{a}$	1	$\overline{\mathbf{b}}^{\mathbf{a}}$	0	$\mathbf{a}$	1
0	1	$\overline{\mathbf{b}}^{\mathbf{a}}$	0	$\mathbf{b}$	0	$\mathbf{a}$	1
1	0	$\mathbf{b}$	0	$\mathbf{a}$	1	$\mathbf{1}$	1
1	1	$\mathbf{1}$	1	$\mathbf{1}$	1	$\mathbf{1}$	1

On the other hand, we obtain $\|\varphi^Q\|_{\mathscr{K}^*,w_0} = 0$ from the following table.

	$\langle\sigma_1^Q,\ldots,\sigma_{\mathrm{ar}(c)}^Q\rangle$	σ^Q	$\langle\psi_1^Q,\ldots,\psi_{\mathrm{ar}(c)}^Q\rangle$	ψ^Q	$\langle\varphi_1^Q,\ldots,\varphi_{\mathrm{ar}(c)}^Q\rangle$	φ^Q
$\|\cdot\|_{\mathscr{K}^*,w_1}$	$\mathbf{b}$	0	$\mathbf{a}$	1	$\mathbf{1}$	1
$\|\cdot\|_{\mathscr{K}^*,w_0}$	$\mathbf{a}$	0	$\mathbf{a}$	1	$\overline{\mathbf{b}}^{\mathbf{a}}$	0

Hence, $\mathrm{IL}(\mathscr{C}) \subsetneq \mathrm{CL}(\mathscr{C})$.

□

Remark 2.3.6. If $\mathscr{C}$ includes $\to$, then $\mathscr{C}$ satisfies the assumption of Lemma 2.3.5. In this case, we obtain

$$\varphi^P \equiv ((p \to q) \to p) \to p \in CL(\mathscr{C}) \setminus \mathrm{IL}(\mathscr{C})$$

according to the above procedure in CASE 1.

If $\mathscr{C}$ includes $\leftrightarrow$ (biconditional), then $\mathscr{C}$ satisfies the assumption of Lemma 2.3.5. In this case, we obtain

$$\varphi^Q \equiv ((q \leftrightarrow p) \leftrightarrow q) \leftrightarrow p \in \mathrm{CL}(\mathscr{C}) \setminus \mathrm{IL}(\mathscr{C})$$

according to the above procedure in CASE 2.

2.3.2.2 (c) $\implies \mathrm{ILS}_1(\mathscr{C}) \neq \mathrm{CLS}_1(\mathscr{C})$

Suppose $\mathrm{t}_c(\mathbf{0}) = 1$ and $\mathrm{t}_c(\mathbf{1}) = 0$ hold. First, for each $\alpha \in \mathrm{Fml}(\mathscr{C})$, we define formula $\neg_c\alpha$ by $\neg_c\alpha \equiv c(\alpha,\ldots,\alpha)$. Then, we can easily see that $\neg_c\alpha$ has the same meaning as $\neg\alpha$, that is, for any Kripke model $\mathscr{K} = \langle W, \preceq, I\rangle$ and any $w \in W$, $\|\neg_c\alpha\|_{\mathscr{K},w} = 1$ if and only if for any $v \succeq w$, $\|\alpha\|_{\mathscr{K},v} = 0$; and for any classical model $\mathscr{M}$, $[\![\neg_c\alpha]\!]_{\mathscr{M}} = 1$ if and only if $[\![\alpha]\!]_{\mathscr{M}} = 0$.

Now, it is easy to show that $\neg_c\neg_c p \Rightarrow p \in \mathrm{CLS}_1(\mathscr{C}) \setminus \mathrm{ILS}_1(\mathscr{C})$. Hence, $\mathrm{ILS}_1(\mathscr{C}) \neq \mathrm{CLS}_1(\mathscr{C})$.

2.3.2.3 (b) $\implies \mathrm{ILS}_1(\mathscr{C}) \neq \mathrm{CLS}_1(\mathscr{C})$

Suppose $\mathrm{t}_c(\mathbf{0}) = 0$ and $\mathrm{t}_c(\mathbf{1}) = 1$. Since c is non-monotonic, there exist $\mathbf{a},\mathbf{b} \in \{0,1\}^{\mathrm{ar}(c)}$ such that $\mathbf{a} \sqsubseteq \mathbf{b}$, $\mathrm{t}_c(\mathbf{a}) = 1$, and $\mathrm{t}_c(\mathbf{b}) = 0$. Fix distinct propositional variables $p, q, r \in \mathrm{PV}$. We divide into two cases: (CASE 1) $\mathrm{t}_c(\overline{\mathbf{a}}) = 1$ and (CASE 2) $\mathrm{t}_c(\overline{\mathbf{a}}) = 0$.

CASE 1: $\mathrm{t}_c(\overline{\mathbf{a}}) = 1$. We define formulas $\chi_1,\ldots,\chi_{\mathrm{ar}(c)},\chi$ as follows:

$$\chi_i \equiv \begin{cases} q & \text{if } \mathbf{a}[i] = 0 \\ p & \text{if } \mathbf{a}[i] = 1, \end{cases}$$

$$\chi \equiv c(\chi_1,\ldots,\chi_{\mathrm{ar}(c)}).$$

Then, we can easily verify that, for any model $\mathscr{M}$, if $\mathscr{M}(p) = 1$ or $\mathscr{M}(q) = 1$, then $[\![\chi]\!]_{\mathscr{M}} = 1$.

Now, we define formulas $\psi_1, \ldots, \psi_{\mathrm{ar}(c)}, \psi$ as follows:

$$\psi_i \equiv \begin{cases} q & \text{if } \mathbf{a}[i] = 0 \text{ and } \mathbf{b}[i] = 0 \\ p & \text{if } \mathbf{a}[i] = 0 \text{ and } \mathbf{b}[i] = 1 \\ r & \text{if } \mathbf{a}[i] = 1 \text{ and } \mathbf{b}[i] = 1, \end{cases}$$
$$\psi \equiv c(\psi_1, \ldots, \psi_{\mathrm{ar}(c)}).$$

Then, we can easily verify that, for any model $\mathscr{M}$, $\mathscr{M}(p) = \mathscr{M}(q) = 0$ implies that $[\![\psi]\!]_{\mathscr{M}} = \mathscr{M}(r)$.

Next, we define formulas $\varphi_1, \ldots, \varphi_{\mathrm{ar}(c)}, \varphi$ as follows:

$$\varphi_i \equiv \begin{cases} q & \text{if } \mathbf{a}[i] = 0 \text{ and } \mathbf{b}[i] = 0 \\ \psi & \text{if } \mathbf{a}[i] = 0 \text{ and } \mathbf{b}[i] = 1 \\ r & \text{if } \mathbf{a}[i] = 1 \text{ and } \mathbf{b}[i] = 1, \end{cases}$$
$$\varphi \equiv c(\varphi_1, \ldots, \varphi_{\mathrm{ar}(c)}).$$

Then, we can see that, for any model $\mathscr{M}$, if $\mathscr{M}(p) = \mathscr{M}(q) = 0$ then $[\![\varphi]\!]_{\mathscr{M}} = 0$.

From the above observation, we obtain $\varphi \Rightarrow \chi \in \mathrm{CLS}_1(\mathscr{C})$. Now, let $\mathscr{K}^+ = \langle \{w_0, w_1\}, \preceq^+, I^+ \rangle$ be the Kripke model defined as follows:

- $w_i \preceq^+ w_j$ if and only if $i \leq j$;
- $I^+(w_0, p) = 0$, $I^+(w_0, q) = 0$, $I^+(w_0, r) = 1$, $I^+(w_1, p) = 1$, $I^+(w_1, q) = 0$, $I^+(w_1, r) = 1$.

Then, from the following table, we obtain $\|\varphi\|_{\mathscr{K}^+, w_0} = 1$ and $\|\chi\|_{\mathscr{K}^+, w_0} = 0$. Hence, $\varphi \Rightarrow \chi \notin \mathrm{ILS}_1(\mathscr{C})$.

	$\langle \chi_1, \ldots, \chi_{\mathrm{ar}(c)} \rangle$	χ	$\langle \psi_1, \ldots, \psi_{\mathrm{ar}(c)} \rangle$	ψ	$\langle \varphi_1, \ldots, \varphi_{\mathrm{ar}(c)} \rangle$	φ
$\|\cdot\|_{\mathscr{K}^+, w_1}$	$\mathbf{a}$	1	$\mathbf{b}$	0	$\mathbf{a}$	1
$\|\cdot\|_{\mathscr{K}^+, w_0}$	$\mathbf{0}$	0	$\mathbf{a}$	0	$\mathbf{a}$	1

Therefore, $\mathrm{ILS}_1(\mathscr{C}) \neq \mathrm{CLS}_1(\mathscr{C})$

Case 2: $\mathrm{t}_c(\overline{\mathbf{a}}) = 0$. First, we define $\psi_1, \ldots, \psi_{\mathrm{ar}(c)}, \psi$ by

$$\psi_i \equiv \begin{cases} q & \text{if } \mathbf{a}[i] = 0 \\ r & \text{if } \mathbf{a}[i] = 1, \end{cases}$$
$$\psi \equiv c(\psi_1, \ldots, \psi_{\mathrm{ar}(c)}).$$

Then, it is easy to check that $[\![\psi]\!]_{\mathscr{M}} = 1$ holds for any $\mathscr{M}$ with $\mathscr{M}(r) = 1$.

Now, let φ^{PP} be the formula obtained from φ^P, which was given in Lemma 2.3.5, by replacing every occurrence of τ by r. Moreover, let φ^{QQ} be the formula obtained from φ^Q by replacing every occurrence of τ by r. Then, similarly to Lemma 2.3.5, we obtain either $\psi \Rightarrow \varphi^{PP} \in \mathrm{CLS}_1(\mathscr{C}) \setminus \mathrm{ILS}_1(\mathscr{C})$ or $\psi \Rightarrow \varphi^{QQ} \in \mathrm{CLS}_1(\mathscr{C}) \setminus \mathrm{ILS}_1(\mathscr{C})$.

Hence, $\mathrm{ILS}_1(\mathscr{C}) \neq \mathrm{CLS}_1(\mathscr{C})$.

2.3.2.4 (a) $\implies \mathrm{ILS}_1(\mathscr{C}) \neq \mathrm{CLS}_1(\mathscr{C})$

Suppose $\mathrm{t}_c(\mathbf{0}) = \mathrm{t}_c(\mathbf{1}) = 0$. Since c is non-monotonic, there exists some $\mathbf{a} \in \{0,1\}^{\mathrm{ar}(c)}$ such that $\mathrm{t}_c(\mathbf{a}) = 1$. Fix distinct propositional variables $p, r \in \mathrm{PV}$.

We define formulas $\psi_1, \ldots, \psi_{\mathrm{ar}(c)}, \psi$ as follows:

$$\psi_i \equiv \begin{cases} p & \text{if } \mathbf{a}[i] = 0 \\ r & \text{if } \mathbf{b}[i] = 1, \end{cases}$$
$$\psi \equiv c(\psi_1, \ldots, \psi_{\mathrm{ar}(c)}).$$

Then, we can easily verify that, for any model $\mathscr{M}$, if $\mathscr{M}(p) = 0$ then $[\![\psi]\!]_{\mathscr{M}} = \mathscr{M}(r)$.

Now, we define formulas $\varphi_1, \ldots, \varphi_{\mathrm{ar}(c)}, \varphi$ as follows:

$$\varphi_i \equiv \begin{cases} \psi & \text{if } \mathbf{a}[i] = 0 \\ r & \text{if } \mathbf{a}[i] = 1, \end{cases}$$
$$\varphi \equiv c(\varphi_1, \ldots, \varphi_{\mathrm{ar}(c)}).$$

Then, we can easily verify that, for any model $\mathscr{M}$, if $\mathscr{M}(p) = 0$ then $[\![\varphi]\!]_{\mathscr{M}} = 0$. Hence, we obtain $\varphi \Rightarrow p \in \mathrm{CLS}_1(\mathscr{C})$.

On the other hand, for the Kripke model $\mathscr{K}^+$ given in case (b), we have $\|p\|_{\mathscr{K}^+, w_0} = 0$, and we obtain $\|\varphi\|_{\mathscr{K}^+, w_0} = 1$ from the following table. Hence, we have $\varphi \Rightarrow p \notin \mathrm{ILS}_1(\mathscr{C})$.

	$\langle \psi_1, \ldots, \psi_{\mathrm{ar}(c)} \rangle$	ψ	$\langle \varphi_1, \ldots, \varphi_{\mathrm{ar}(c)} \rangle$	φ
$\|\cdot\|_{\mathscr{K}^+, w_1}$	$\mathbf{1}$	0	$\mathbf{a}$	1
$\|\cdot\|_{\mathscr{K}^+, w_0}$	$\mathbf{a}$	0	$\mathbf{a}$	1

Therefore, $\mathrm{ILS}_1(\mathscr{C}) \neq \mathrm{CLS}_1(\mathscr{C})$.

Thus we have completed the proof of $(\gamma) \implies (\alpha)$.

2.4 Condition for $\emptyset = \mathrm{IL}(\mathscr{C})$ and $\emptyset = \mathrm{CL}(\mathscr{C})$

In this section, we show the following theorem.

Theorem 2.4.1. *The following three conditions are equivalent:*

(A) $\mathrm{CL}(\mathscr{C}) = \emptyset$;

(B) $\mathrm{IL}(\mathscr{C}) = \emptyset$;

(C) $\mathscr{C}$ *satisfies either of the following conditions:*

- ($\star$1) *for any* $c \in \mathscr{C}$, $\mathrm{t}_c(\mathbf{0}) = 0$;
- ($\star$2) *for any* $c \in \mathscr{C}$ *and any* $\mathbf{a} \in \{0,1\}^{\mathrm{ar}(c)}$, $\mathrm{t}_c(\mathbf{a}) \neq \mathrm{t}_c(\bar{\mathbf{a}})$.

First, we remark that Lemma 2.2.3 immediately leads to (A) $\implies$ (B). In § 2.4.1, we show (C) $\implies$ (A). Furthermore, in § 2.4.2, we show (B) $\implies$ (C).

2.4.1 Proof of (C) $\implies$ (A)

Here, we show that if $\mathscr{C}$ satisfies either ($\star$1) or ($\star$2), then $\mathrm{CL}(\mathscr{C}) = \emptyset$.

First, we can prove the following lemmas by an easy induction on the size of α.

Lemma 2.4.2. *Suppose $\mathscr{C}$ satisfies ($\star$1); that is, all $c \in \mathscr{C}$ satisfy $\mathrm{t}_c(\mathbf{0}) = 0$. Let $\mathscr{M}$ be the classical model such that $\mathscr{M}(p) = 0$ for all $p \in \mathrm{PV}$. Then, for any $\alpha \in \mathrm{Fml}(\mathscr{C})$, $[\![\alpha]\!]_{\mathscr{M}} = 0$.*

Lemma 2.4.3. *Suppose $\mathscr{C}$ satisfies ($\star$2); that is, all $c \in \mathscr{C}$ and all $\mathbf{a} \in \{0,1\}^{\mathrm{ar}(c)}$ satisfy $\mathrm{t}_c(\mathbf{a}) \neq \mathrm{t}_c(\overline{\mathbf{a}})$. For an arbitrary classical model $\mathscr{M}$, we define a model $\overline{\mathscr{M}}$ as follows:*

$$\overline{\mathscr{M}}(p) = \begin{cases} 1 & \textit{if } \mathscr{M}(p) = 0 \\ 0 & \textit{if } \mathscr{M}(p) = 1. \end{cases}$$

Then, for any $\alpha \in \mathrm{Fml}(\mathscr{C})$, $[\![\alpha]\!]_{\mathscr{M}} \neq [\![\alpha]\!]_{\overline{\mathscr{M}}}$.

Now, suppose that $\mathscr{C}$ satisfies either ($\star$1) or ($\star$2) and α is a formula in $\mathrm{Fml}(\mathscr{C})$. Let $\mathscr{M}$ be the model such that $\mathscr{M}(p) = 0$ for all $p \in \mathrm{PV}$.

If ($\star$1) holds, then by Lemma 2.4.2, $[\![\alpha]\!]_{\mathscr{M}} = 0$ for all $\alpha \in \mathrm{Fml}(\mathscr{C})$. Hence, $\mathrm{CL}(\mathscr{C}) = 0$.

On the other hand, if ($\star$2) holds, then from Lemma 2.4.3, we obtain either $[\![\alpha]\!]_{\mathscr{M}} = 0$ or $[\![\alpha]\!]_{\overline{\mathscr{M}}} = 0$. Therefore, $\mathrm{CL}(\mathscr{C}) = \emptyset$. This completes the proof.

2.4.2 Proof of (B) $\implies$ (C)

Here, we show that if $\mathscr{C}$ satisfies neither ($\star$1) nor ($\star$2), then $\mathrm{IL}(\mathscr{C}) \neq \emptyset$.

First, we remark that if $\mathscr{C}$ satisfies neither ($\star$1) nor ($\star$2), then either of the following conditions holds.

(i) $\mathrm{t}_c(\mathbf{0}) = \mathrm{t}_c(\mathbf{1}) = 1$ for some $c \in \mathscr{C}$.

(ii) $\mathrm{t}_c(\mathbf{0}) = 1$ and $\mathrm{t}_c(\mathbf{1}) = 0$ for some $c \in \mathscr{C}$. Furthermore, $\mathrm{t}_d(\mathbf{a}) = \mathrm{t}_d(\overline{\mathbf{a}}) = 0$ for some $d \in \mathscr{C}$ and some $\mathbf{a} \in \{0,1\}^{\mathrm{ar}(d)}$.

(iii) $\mathrm{t}_c(\mathbf{0}) = 1$ and $\mathrm{t}_c(\mathbf{1}) = 0$ for some $c \in \mathscr{C}$. Furthermore, $\mathrm{t}_d(\mathbf{a}) = \mathrm{t}_d(\overline{\mathbf{a}}) = 1$ for some $d \in \mathscr{C}$ and some $\mathbf{a} \in \{0,1\}^{\mathrm{ar}(d)}$.

Hence, to prove $\mathrm{IL}(\mathscr{C}) \neq \emptyset$, it is sufficient to prove the following lemmas, the first of which is already proved in the last section as Lemma 2.3.3. Thus, we only prove the last two here.

Lemma 2.3.3 (repeated)**.** *If $\mathscr{C}$ satisfies* (i), *then* $\mathrm{IL}(\mathscr{C}) \neq \emptyset$.

Lemma 2.4.4. *If $\mathscr{C}$ satisfies* (ii), *then* $\mathrm{IL}(\mathscr{C}) \neq \emptyset$.

Lemma 2.4.5. *If $\mathscr{C}$ satisfies* (iii), *then* $\mathrm{IL}(\mathscr{C}) \neq \emptyset$.

2.4.2.1 Proof of Lemma 2.4.4

Suppose $\mathscr{C}$ satisfies (*ii*); that is, there exist $c, d \in \mathscr{C}$ and $\mathbf{a} \in \{0,1\}^{\mathrm{ar}(c)}$ such that $\mathrm{t}_c(\mathbf{0}) = 1$, $\mathrm{t}_c(\mathbf{1}) = 0$ and $\mathrm{t}_d(\mathbf{a}) = \mathrm{t}_d(\overline{\mathbf{a}}) = 0$.

First, for each $\alpha \in \mathrm{Fml}(\mathscr{C})$, we define formula $\neg_c\alpha$ in the same way as § 2.3.2.2; that is, by $\neg_c\alpha \equiv c(\alpha, \ldots, \alpha)$, so that $\neg_c\alpha$ has the same meaning as $\neg\alpha$.

Now, we consider an arbitrary propositional variable p and define formulas $\psi_1^{NC}, \ldots, \psi_{\mathrm{ar}(d)}^{NC}$, ψ^{NC}, φ^{NC} as follows:

$$\psi_i^{NC} \equiv \begin{cases} p & \text{if } \mathbf{a}[i] = 1 \\ \neg_c p & \text{if } \mathbf{a}[i] = 0, \end{cases}$$
$$\psi^{NC} \equiv d(\psi_1^{NC}, \ldots, \psi_{\mathrm{ar}(d)}^{NC}),$$
$$\varphi^{NC} \equiv \neg_c \psi^{NC}.$$

By the definition of ψ_i^{NC}, for any classical model $\mathscr{M}$, the tuple

$$\langle [\![\psi_1^{NC}]\!]_{\mathscr{M}}, \ldots, [\![\psi_{\mathrm{ar}(d)}^{NC}]\!]_{\mathscr{M}} \rangle$$

coincides with either $\mathbf{a}$ or $\bar{\mathbf{a}}$. Hence, $[\![\psi^{NC}]\!]_{\mathscr{M}} = 0$ for every $\mathscr{M}$. then, from Lemma 2.2.6, $\|\psi^{NC}\|_{\mathscr{K},w} = 0$ for every Kripke model $\mathscr{K} = \langle W, \preceq, I \rangle$ and every $w \in W$. Hence, $\|\varphi^{NC}\|_{\mathscr{K},w} = 1$ for every $\mathscr{K}$ and every w. Therefore, $\mathrm{IL}(\mathscr{C}) \neq \emptyset$.

Remark 2.4.6. Let $\mathscr{C} = \{\neg, \wedge\}$. Then, $\mathscr{C}$ satisfies (ii). In this case, we obtain

$$\varphi^{NC} \equiv \neg(p \wedge \neg p) \in \mathrm{IL}(\mathscr{C})$$

according to the above procedure.

2.4.2.2 Proof of Lemma 2.4.5

Suppose $\mathscr{C}$ satisfies (iii); that is, there exist $c, d \in \mathscr{C}$ and $\mathbf{a} \in \{0,1\}^{\mathrm{ar}(d)}$ such that $\mathrm{t}_c(\mathbf{0}) = 1, \mathrm{t}_c(\mathbf{1}) = 0$ and $\mathrm{t}_d(\mathbf{a}) = \mathrm{t}_d(\bar{\mathbf{a}}) = 1$. In this case, we can define $\neg_c \alpha$ in the same way as § 2.4.2.1. Now we consider an arbitrary propositional variable p and define formulas $\psi_1^{NNEM}, \ldots, \psi_{\mathrm{ar}(d)}^{NNEM}$, ψ^{NNEM}, φ^{NNEM} as follows:

$$\psi_i^{NNEM} \equiv \begin{cases} p & \text{if } \mathbf{a}[i] = 1 \\ \neg_c p & \text{if } \mathbf{a}[i] = 0, \end{cases}$$
$$\psi^{NNEM} \equiv \neg_c d(\psi_1^{NNEM}, \ldots, \psi_{\mathrm{ar}(d)}^{NNEM}),$$
$$\varphi^{NNEM} \equiv \neg_c \psi^{NNEM}.$$

Then, in a similar way to § 2.4.2.1, we can verify $\varphi \in \mathrm{IL}(\mathscr{C})$. Hence, $\mathrm{IL}(\mathscr{C}) \neq \emptyset$. This compteles the proof.

Remark 2.4.7. Let $\mathscr{C} = \{\neg, \vee\}$. Then, $\mathscr{C}$ satisfies (iii). In this case, we obtain

$$\varphi^{NNEM} \equiv \neg\neg(p \vee \neg p) \in \mathrm{IL}(\mathscr{C})$$

according to the above procedure.

2.5 Condition for $\mathrm{IL}(\mathscr{C}) = \mathrm{CL}(\mathscr{C})$

In this section, we prove the following theorem.

Theorem 2.5.1. $\mathrm{IL}(\mathscr{C}) = \mathrm{CL}(\mathscr{C})$ *if and only if* $\mathscr{C}$ *satisfies either of the following conditions:*

($\star$1) *for any* $c \in \mathscr{C}$, $\mathrm{t}_c(\mathbf{0}) = 0$*;*

($\star$2) *for any* $c \in \mathscr{C}$ *and any* $\mathbf{a}, \mathbf{b} \in \{0,1\}^{\mathrm{ar}(c)}$, $\mathrm{t}_c(\mathbf{a}) \neq \mathrm{t}_c(\bar{\mathbf{a}})$*;*

(M) *any connective in* $\mathscr{C}$ *is monotonic;*

($\sqcap$-$\sqsubseteq$-1) *any connective* $c \in \mathscr{C}$ *satisfies the following conditions:*

($\sqcap$-1) *for any* $\mathbf{a}, \mathbf{b} \in \{0,1\}^{\mathrm{ar}(c)}$*, if* $\mathrm{t}_c(\mathbf{a}) = \mathrm{t}_c(\mathbf{b}) = 1$*, then* $\mathrm{t}_c(\mathbf{a} \sqcap \mathbf{b}) = 1$*;*

($\sqsubseteq$-1) *for any* $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \{0,1\}^{\mathrm{ar}(c)}$*, if* $\mathrm{t}_c(\mathbf{a}) = \mathrm{t}_c(\mathbf{b}) = 1$ *and* $\mathbf{a} \sqsubseteq \mathbf{c} \sqsubseteq \mathbf{b}$*, then* $\mathrm{t}_c(\mathbf{c}) = 1$.

We show the "if" part in § 2.5.1 and the "only if" part in § 2.5.2.

2.5.1 Proof of "if" part

Here, we show the "if" part of Theorem 2.5.1; that is, if $\mathscr{C}$ satisfies either ($\star$1), ($\star$2), (M), or ($\sqcap$-$\sqsubseteq$-1), then $\mathrm{IL}(\mathscr{C}) = \mathrm{CL}(\mathscr{C})$.

First, by Threorem 2.4.1, if $\mathscr{C}$ satisfies either ($\star$1) or ($\star$2), then $\mathrm{IL}(\mathscr{C}) = \mathrm{CL}(\mathscr{C}) = \emptyset$. Hence, to prove the "if" part of Theorem 2.5.1, it is sufficent to prove the following two lemmas.

Lemma 2.5.2. *If* $\mathscr{C}$ *satitsfies* (M)*, then* $\mathrm{IL}(\mathscr{C}) = \mathrm{CL}(\mathscr{C})$*.*

Lemma 2.5.3. *If* $\mathscr{C}$ *satisfies* ($\sqcap$-$\sqsubseteq$-1)*, then* $\mathrm{IL}(\mathscr{C}) = \mathrm{CL}(\mathscr{C})$*.*

Since $\mathrm{IL}(\mathscr{C}) = \{\alpha \in \mathrm{Fml}(\mathscr{C}) \mid \emptyset \Rightarrow \alpha \in \mathrm{ILS}(\mathscr{C})\}$ and $\mathrm{CL}(\mathscr{C}) = \{\alpha \in \mathrm{Fml}(\mathscr{C}) \mid \emptyset \Rightarrow \alpha \in \mathrm{CLS}(\mathscr{C})\}$, Lemma 2.5.2 follows from $(\alpha) \implies (\beta)$ of Theorem 2.3.1. Thus, we only prove Lemma 2.5.3.

2.5.1.1 Proof of Lemma 2.5.3

First, we show the following lemma.

Lemma 2.5.4. *Suppose* $\mathscr{C}$ *satisfies* ($\sqcap$-$\sqsubseteq$-1)*. Then, for each* $c \in \mathscr{C}$*, there exist* $n \in \mathbb{N}$ *and* $\mathbf{x}, \mathbf{y}_1, \ldots, \mathbf{y}_n \in \{0,1\}^{\mathrm{ar}(c)}$ *such that for any* $\mathbf{a} \in \{0,1\}^{\mathrm{ar}(c)}$*,*

$$\mathrm{t}_c(\mathbf{a}) = 1 \iff \textit{there is some } i \textit{ such that } \mathbf{x} \sqsubseteq \mathbf{a} \sqsubseteq \mathbf{y}_i.$$

Proof. Let $\mathbf{x} = \bigsqcap\{\mathbf{a} \mid \mathrm{t}_c(\mathbf{a}) = 1\}$ and $\mathbf{y}_1, \ldots, \mathbf{y}_n$ be the maximal elements in $\{\mathbf{a} \mid \mathrm{t}_c(\mathbf{a}) = 1\}$. Then, they satisfy the claimed property. □

Now we return to the proof of Lemma 2.5.3. By Lemma 2.5.4, there exist $\mathbf{x}, \mathbf{y}_1, \ldots, \mathbf{y}_n$ such that for any $\mathbf{a} \in \{0,1\}^{\mathrm{ar}(c)}$,

$$\mathrm{t}_c(\mathbf{a}) = 1 \iff \text{there is some } i \text{ such that } \mathbf{x} \sqsubseteq \mathbf{a} \sqsubseteq \mathbf{y}_i. \tag{2.1}$$

Since $\mathrm{IL}(\mathscr{C}) \subseteq \mathrm{CL}(\mathscr{C})$, it suffices to show the converse inclusion. We show that $\alpha \in \mathrm{IL}(\mathscr{C})$ holds for any $\alpha \in \mathrm{CL}(\mathscr{C})$, by induction on the size of α. The case where α is a propositional variable is vacuously true. So, we consider the case α is of the form $c(\vec{\beta})$, where $\beta = \beta_1, \ldots, \beta_{\mathrm{ar}(c)}$. We suppose $\alpha \in \mathrm{CL}(\mathscr{C})$, and show $\alpha \in \mathrm{IL}(\mathscr{C})$. Let $\mathscr{K} = \langle W, \preceq, I \rangle$ be any Kripke model and w any state in W. In order to show $\|\alpha\|_{\mathscr{K},w} = 1$, by (2.1), it suffices to show for any $v \succeq w$, there exists some i such that $\mathbf{x} \sqsubseteq \|\vec{\beta}\|_{\mathscr{K},v} \sqsubseteq \mathbf{y}_i$. Let $v \succeq w$.

By $\alpha \in \mathrm{CL}(\mathscr{C})$ and (2.1), for any classical model $\mathscr{M}$, there exists some i such that $\mathbf{x} \sqsubseteq [\![\vec{\beta}]\!]_{\mathscr{M}} \sqsubseteq \mathbf{y}_i$. By the left-hand side inequality, $\beta_j \in \mathrm{CL}(\mathscr{C})$ holds for any j with $\mathbf{x}[j] = 1$. Thus, by the induction hypothesis, $\beta_j \in \mathrm{IL}(\mathscr{C})$ holds for any j with $\mathbf{x}[j] = 1$, and hence, $\mathbf{x} \sqsubseteq \|\vec{\beta}\|_{\mathscr{K},v}$

holds. On the other hand, since the right-hand side inequality $[\![\vec{\beta}]\!]_{\mathscr{M}} \sqsubseteq \mathbf{y}_i$ means that $[\![\beta_j]\!]_{\mathscr{M}} = 0$ for any j with $\mathbf{y}_i[j] = 0$, if we put $F_i = \{\beta_j \mid \mathbf{y}_i[j] = 0\}$ for each i, then the hypothesis of Lemma 2.2.6 (ii) is satisfied, and hence, there exists some i such that $\|\gamma\|_{\mathscr{K},v} = 0$ for all $\gamma \in F_i$, which means $\|\vec{\beta}\|_{\mathscr{K},v} \sqsubseteq \mathbf{y}_i$. Thus we have completed the proof.

Remark 2.5.5. Let $\mathscr{B}$ be either $\{\wedge, \neg\}$, $\{\text{NAND}\}$, or $\{\text{NOR}\}$. Then, for any $\mathscr{C}$, it is well-known that there exists a natural embedding from $\mathrm{CL}(\mathscr{C})$ into $\mathrm{CL}(\mathscr{B})$ (cf., e.g, [16, 1]). Thta is, there exists a mapping E from $\mathrm{Fml}(\mathscr{C})$ into $\mathrm{Fml}(\mathscr{B})$ such that for each $\varphi \in \mathrm{Fml}(\mathscr{C})$, $\varphi \in \mathrm{CL}(\mathscr{C})$ if and only if $E(\varphi) \in \mathrm{CL}(\mathscr{B})$.

On the other hand, our result shows $\mathrm{IL}(\mathscr{B}) = \mathrm{CL}(\mathscr{B})$. This fac shows that the above E defines an embedding from $\mathrm{CL}(\mathscr{C})$ into $\mathrm{IL}(\mathscr{B})$, that is,

$$\varphi \in \mathrm{CL}(\mathscr{C}) \iff E(\varphi) \in \mathrm{IL}(\mathscr{B})$$

for each $\varphi \in \mathrm{Fml}(\mathscr{C})$.

2.5.2 Proof of "only if" part

Here, we show the "only if" part of Theorem 2.5.1; that is, if $\mathscr{C}$ satisfies neither ($\star$1), ($\star$2), (M) nor ($\sqcap$-$\sqsubseteq$-1), then $\mathrm{IL}(\mathscr{C}) \neq \mathrm{CL}(\mathscr{C})$.

First, we remark that if $\mathscr{C}$ satisfies neither (M) nor ($\sqcap$-$\sqsubseteq$-1), then either of the following conditions holds.

(I) There exists a non-monotonic function $c \in \mathscr{C}$ such that $\mathrm{t}_c(\mathbf{1}) = 1$.

(II) There exists a non-monoatonic function $c \in \mathscr{C}$ such that $\mathrm{t}_c(\mathbf{1}) = 0$. Furthermore, there exist $d \in \mathscr{C}$ and $\mathbf{a}, \mathbf{b} \in \{0,1\}^{\mathrm{ar}(d)}$ such that $\mathrm{t}_d(\mathbf{a}) = \mathrm{t}_d(\mathbf{b}) = 1$ and $\mathrm{t}_d(\mathbf{a} \sqcap \mathbf{b}) = 0$.

(III) There exists a non-monotonic function $c \in \mathscr{C}$ such that $\mathrm{t}_c(\mathbf{1}) = 0$. Furthermore, there exist $d \in \mathscr{C}$ and $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \{0,1\}^{\mathrm{ar}(d)}$ such that $\mathrm{t}_d(\mathbf{a}) = \mathrm{t}_d(\mathbf{b}) = 1$, $\mathbf{a} \sqsubseteq \mathbf{c} \sqsubseteq \mathbf{b}$ and $\mathrm{t}_d(\mathbf{c}) = 0$.

Furthermore, by Theorem 2.4.1, if $\mathscr{C}$ satisfies neither ($\star$1) nor ($\star$2), then $\mathrm{IL}(\mathscr{C}) \neq \emptyset$. Hence, it is sufficient to prove the following lemmas, the first of which is already proved in the last section as Lemma 2.3.5. Here we only prove the last two.

Lemma 2.3.5 (repeated)**.** *If* (I) *and* $\mathrm{IL}(\mathscr{C}) \neq \emptyset$ *hold, then* $\mathrm{IL}(\mathscr{C}) \subsetneq \mathrm{CL}(\mathscr{C})$*.*

Lemma 2.5.6. *If* (II) *and* $\mathrm{IL}(\mathscr{C}) \neq \emptyset$ *hold, then* $\mathrm{IL}(\mathscr{C}) \subsetneq \mathrm{CL}(\mathscr{C})$*.*

Lemma 2.5.7. *If* (III) *and* $\mathrm{IL}(\mathscr{C}) \neq \emptyset$ *hold, then* $\mathrm{IL}(\mathscr{C}) \subsetneq \mathrm{CL}(\mathscr{C})$*.*

2.5.2.1 Proof of Lemma 2.5.6

Suppose the following conditions are satisfied:

- there exists a non-monotonic connective $c \in \mathscr{C}$ such that $\mathrm{t}_c(\mathbf{1}) = 0$;
- there exist $d \in \mathscr{C}$ and $\mathbf{a}, \mathbf{b} \in \{0,1\}^{\mathrm{ar}(d)}$ such that $\mathrm{t}_d(\mathbf{a}) = \mathrm{t}_d(\mathbf{b}) = 1$ and $\mathrm{t}_d(\mathbf{a} \sqcap \mathbf{b}) = 0$;
- $\mathrm{IL}(\mathscr{C}) \neq \emptyset$.

First, we show, by the first and third conditions, negation can be defined by c. Since c is non-monotonic, $\mathrm{t}_c(\mathbf{x}) = 1$ for some $\mathbf{x}$. Furthermore, since $\mathrm{IL}(\mathscr{C}) \neq \emptyset$, there exists $\tau \in \mathrm{IL}(\mathscr{C})$. For each $\alpha \in \mathrm{Fml}(\mathscr{C})$, we define a formula $\neg'_c\alpha$ by

$$\neg'_c\alpha \equiv c(\alpha_1, \ldots, \alpha_{\mathrm{ar}(c)}),$$

where α_i is given by

$$\alpha_i \equiv \begin{cases} \alpha & \text{if } \mathbf{x}[i] = 0 \\ \tau & \text{if } \mathbf{x}[i] = 1. \end{cases}$$

Then, $\neg'_c\alpha$ has the same meaning as $\neg\alpha$ in the same sense as in § 2.3.2.2.

Now, we fix $p \in \mathrm{PV}$ and define formulas $\varphi_1^{EM}, \ldots, \varphi_{\mathrm{ar}(d)}^{EM}, \varphi^{EM}$ by

$$\varphi_i^{EM} \equiv \begin{cases} \neg'_c\tau & \text{if } \mathbf{a}[i] = 0 \text{ and } \mathbf{b}[i] = 0 \\ \neg'_c p & \text{if } \mathbf{a}[i] = 0 \text{ and } \mathbf{b}[i] = 1 \\ p & \text{if } \mathbf{a}[i] = 1 \text{ and } \mathbf{b}[i] = 0 \\ \tau & \text{if } \mathbf{a}[i] = 1 \text{ and } \mathbf{b}[i] = 1, \end{cases}$$
$$\varphi^{EM} \equiv c(\varphi_1^{EM}, \ldots, \varphi_{\mathrm{ar}(d)}^{EM}).$$

Then, for every classical model $\mathscr{M}$, the sequence $\langle [\![\varphi_1^{EM}]\!]_{\mathscr{M}}, \ldots, [\![\varphi_{\mathrm{ar}(d)}^{EM}]\!]_{\mathscr{M}}\rangle$ coincides with either $\mathbf{a}$ or $\mathbf{b}$, and hence, $[\![\varphi^{EM}]\!]_{\mathscr{M}} = 1$. Therefore, $\varphi^{EM} \in \mathrm{CL}(\mathscr{C})$.

On the other hand, for the Kripke model $\mathscr{K}^*$ given in the proof of Lemma 2.3.5, we obtain $\|\varphi^{EM}\|_{\mathscr{K}^*,w_0} = 0$ (see the table below). Hence, $\varphi^{EM} \notin \mathrm{IL}(\mathscr{C})$.

	p	$\neg'_c p$	τ	$\neg'_c\tau$	$\langle \varphi_1^{EM}, \ldots, \varphi_{\mathrm{ar}(d)}^{EM}\rangle$	φ^{EM}
$\|\cdot\|_{\mathscr{K}^+,w_1}$	1	0	1	0	$\mathbf{a}$	1
$\|\cdot\|_{\mathscr{K}^+,w_0}$	0	0	1	0	$\mathbf{a} \sqcap \mathbf{b}$	0

Therefore, $\mathrm{IL}(\mathscr{C}) \subsetneq \mathrm{CL}(\mathscr{C})$. This completes the proof.

Remark 2.5.8. If $\mathscr{C}$ include $\{\neg, \vee\}$, then $\mathscr{C}$ satisfies (II). In this case, we obtain

$$\varphi^{EM} \equiv p \vee \neg p \in \mathrm{CL}(\mathscr{C}) \setminus \mathrm{IL}(\mathscr{C})$$

according to the above procedure.

2.5.2.2 Proof of Lemma 2.5.7

Suppose the following conditions are satisfied:

- there exists a non-monotonic connective $c \in \mathscr{C}$ such that $\mathrm{t}_c(\mathbf{1}) = 0$;
- there exists $d \in \mathscr{C}$ and $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \{0,1\}^{\mathrm{ar}(d)}$ such that $\mathrm{t}_d(\mathbf{a}) = \mathrm{t}_d(\mathbf{b}) = 1$, $\mathbf{a} \sqsubseteq \mathbf{c} \sqsubseteq \mathbf{b}$ and $\mathrm{t}_d(\mathbf{c}) = 0$;
- $\mathrm{IL}(\mathscr{C}) \neq \emptyset$.

In this case, we define $\neg'_c$ similarly to § 2.5.2.1. Now, we define formulas $\varphi_1^{DNE}, \ldots, \varphi_{\mathrm{ar}(d)}^{DNE}, \varphi^{DNE}$ as follows:

$$\varphi_i^{DNE} \equiv \begin{cases} \neg'_c\tau & \text{if } \mathbf{a}[i] = \mathbf{c}[i] = \mathbf{b}[i] = 0 \\ p & \text{if } \mathbf{a}[i] = \mathbf{c}[i] = 0 \text{ and } \mathbf{b}[i] = 1 \\ \neg'_c\neg'_c p & \text{if } \mathbf{a}[i] = 0 \text{ and } \mathbf{c}[i] = \mathbf{b}[i] = 1 \\ \tau & \text{if } \mathbf{a}[i] = \mathbf{c}[i] = \mathbf{b}[i] = 1, \end{cases}$$
$$\varphi^{DNE} \equiv c(\varphi_1^{DNE}, \ldots, \varphi_{\mathrm{ar}(c)}^{DNE}),$$

where $\tau \in \mathrm{IL}(\mathscr{C})$ and $p \in \mathrm{PV}$. Then, for every model $\mathscr{M}$, the sequence

$$\langle [\![\varphi_1^{DNE}]\!]_{\mathscr{M}}, \ldots, [\![\varphi_{\mathrm{ar}(d)}^{DNE}]\!]_{\mathscr{M}} \rangle$$

coincides with either **a** or **b**, and hence, $[\![\varphi^{DNE}]\!]_{\mathscr{M}} = 1$. Therefore, $\varphi^{DNE} \in \mathrm{CL}(\mathscr{C})$.

On the other hand, for the Kripke model $\mathscr{K}^*$ given in the proof of Lemma 2.3.5, we obtain $\|\varphi^{DNE}\|_{\mathscr{K}^*,w_0} = 0$ (see the table below). Hence, $\varphi^{DNE} \notin \mathrm{IL}(\mathscr{C})$.

	p	$\neg'_c p$	$\neg'_c \neg'_c p$	τ	$\neg'_c \tau$	$\langle \varphi_1^{EM}, \ldots, \varphi_{\mathrm{ar}(d)}^{EM} \rangle$	φ^{EM}
$\|\cdot\|_{\mathscr{K}^+,w_1}$	1	0	1	1	0	**b**	1
$\|\cdot\|_{\mathscr{K}^+,w_0}$	0	0	1	1	0	**c**	0

Hence, $\mathrm{IL}(\mathscr{C}) \subsetneq \mathrm{CL}(\mathscr{C})$. This completes the proof.

Remark 2.5.9. If $\mathscr{C}$ includes $\{\neg, \to\}$, then $\mathscr{C}$ satisfies (III). In this case, we obtain

$$\varphi^{DNE} \equiv \neg\neg p \to p \in \mathrm{CL}(\mathscr{C}) \setminus \mathrm{IL}(\mathscr{C})$$

according to the above procedure.

Chapter 3

First-order logic

3.1 Overview

In this chapter, we extend the classical and Kripke interpretations given in § 2.2.3 and § 2.2.4 to predicate formulas with general truth-functional connectives, and analyze how the choice of connectives affects the relations between intuitionistic first-order logic, classical first-order logic, and the logic of constant domains. As in the case of propositional logic, we consider both sequent-level and formula-level relations and, in concrete, give answers to the following questions except (vi), which is left open:

(iii) For what $\mathscr{C}$, $\mathrm{FOILS}(\mathscr{C}) = \mathrm{FOCDS}(\mathscr{C})$ hold?

(iv) For what $\mathscr{C}$, $\mathrm{FOCDS}(\mathscr{C}) = \mathrm{FOCLS}(\mathscr{C})$ hold?

(v) For what $\mathscr{C}$, $\mathrm{FOILS}(\mathscr{C}) = \mathrm{FOCLS}(\mathscr{C})$ hold?

(vi) For what $\mathscr{C}$, $\mathrm{FOIL}(\mathscr{C}) = \mathrm{FOCD}(\mathscr{C})$ hold?

(vii) For what $\mathscr{C}$, $\mathrm{FOCD}(\mathscr{C}) = \mathrm{FOCL}(\mathscr{C})$ hold?

(viii) For what $\mathscr{C}$, $\mathrm{FOIL}(\mathscr{C}) = \mathrm{FOCL}(\mathscr{C})$ hold?

Regarding (vi), (vii), and (viii), as in the case of propositional logic, we further analyze when $\mathrm{FOIL}(\mathscr{C})$, $\mathrm{FOCD}(\mathscr{C})$, and $\mathrm{FOCL}(\mathscr{C})$ become empty.

In § 3.2, we introduce a basic definitions concerning classical and Kripke semantics for first-order logic with general propositional connectives. In § 3.3, we give a necessary and sufficient condition for $\mathrm{FOILS}(\mathscr{C}) = \mathrm{FOCDS}(\mathscr{C})$. In § 3.4, we give a necessary and sufficient condition for $\mathrm{FOCDS}(\mathscr{C}) = \mathrm{FOCLS}(\mathscr{C})$ and that for $\mathrm{FOILS}(\mathscr{C}) = \mathrm{FOCLS}(\mathscr{C})$. In § 3.5, first, we see that the necessary and sufficient condition for $\emptyset = \mathrm{IL}(\mathscr{C})$, $\emptyset = \mathrm{FOCD}(\mathscr{C})$ and $\emptyset = \mathrm{CL}(\mathscr{C})$ in Theorem 2.4.1 is also a necessary and suffcient condition for $\emptyset = \mathrm{FOIL}(\mathscr{C})$ and $\emptyset = \mathrm{FOCL}(\mathscr{C})$, and then, we give a necessary and sufficient condition for $\mathrm{FOCD}(\mathscr{C}) = \mathrm{FOCL}(\mathscr{C})$ and shows that it is also a necessary and sufficient condition for $\mathrm{FOIL}(\mathscr{C}) = \mathrm{FOCL}(\mathscr{C})$. The contents in § 3.2 and § 3.3 are based on [11, 13, 14], the contents in § 3.4 are based on [12], and the contents in § 3.5 are based on [10].

3.2 Preliminaries

3.2.1 Formulas and sequents

Assume a set $\mathscr{C}$ of truth-functional connectives is given. We define the first-order language with truth-functional connectives in $\mathscr{C}$. It consists of the following symbols: countably infinitely many individual variables; countably infinitely many n-ary predicate symbols for each $n \in \mathbb{N}$;[1] truth-functional connectives in $\mathscr{C}$; and quantifiers $\forall$ and $\exists$. 0-ary predicate symbols are also called *propositional symbols* or *propositional variables*.[2] Although all arguments in this dissertation work with trivial modifications if the language has function symbols and constant symbols, we assume that the language has no function symbols and no constant symbols for simplicity. We shall use x, y, and z as metavariables for individual variables; p, q, and r for predicate symbols. On some occasions, we use R and T as metavariables for propositional symbols. An *atomic formula* is an expression of the form $p(x_1, \ldots, x_n)$, where p is an n-ary predicate symbol. The set $\mathrm{FOFml}(\mathscr{C})$ of *(predicate) formulas* is defined inductively as follows:

- if α is an atomic formula, then $\alpha \in \mathrm{FOFml}(\mathscr{C})$;
- if $c \in \mathscr{C}$ and $\alpha_1, \ldots, \alpha_{\mathrm{ar}(c)} \in \mathrm{FOFml}(\mathscr{C})$, then $c(\alpha_1, \ldots, \alpha_{\mathrm{ar}(c)}) \in \mathrm{FOFml}(\mathscr{C})$;
- if $\alpha \in \mathrm{FOFml}(\mathscr{C})$ and x is an individual variable, then $\forall x\alpha \in \mathrm{FOFml}(\mathscr{C})$ and $\exists x\alpha \in \mathrm{FOFml}(\mathscr{C})$.

We shall use α, β, γ, φ, and ψ as metavariables for formulas. The set $\mathrm{FV}(\alpha)$ of free variables of α is defined inductively as follows:

$$\begin{aligned}
\mathrm{FV}(p(x_1, \ldots, x_n)) &= \{x_1, \ldots, x_n\};\\
\mathrm{FV}(c(\alpha_1, \ldots, \alpha_{\mathrm{ar}(c)})) &= \mathrm{FV}(\alpha_1) \cup \cdots \cup \mathrm{FV}(\alpha_{\mathrm{ar}(c)});\\
\mathrm{FV}(\forall x\alpha) = \mathrm{FV}(\exists x\alpha) &= \mathrm{FV}(\alpha) \setminus \{x\}.
\end{aligned}$$

For a set Γ of formulas, $\mathrm{FV}(\Gamma)$ denotes the set of free variables of formulas in Γ.

A *(predicate) sequent* is an expression $\Gamma \Rightarrow \Delta$, where Γ and Δ are sets of formulas. We denote by $\mathrm{FOSqt}(\mathscr{C})$ the set $\{\Gamma \Rightarrow \Delta \mid \Gamma, \Delta \subseteq \mathrm{FOFml}(\mathscr{C})\}$. If $\Gamma = \{\alpha_1, \ldots, \alpha_n\}$ and $\Delta = \{\beta_1, \ldots, \beta_m\}$, we often omit the braces and simply write $\alpha_1, \ldots, \alpha_n \Rightarrow \beta_1, \ldots, \beta_m$ for $\Gamma \Rightarrow \Delta$.

3.2.2 Classical semantics

A *(classical) model* $\mathscr{M}$ is a tuple $\langle D, I\rangle$ in which

- D is a non-empty set, called the *individual domain*;
- I is a function, called the *interpretation function*, which assigns to each n-ary predicate symbol a function from D^n to $\{0, 1\}$.

An *assignment* in D is a function which assigns to each individual variable an element of D. For an assignment ρ in D, an individual variable x and an element $a \in D$, we write $\rho[x \mapsto a]$ for the assignment in D which maps x to a and is equal to ρ everywhere else. For a model $\mathscr{M} = \langle D, I\rangle$, a formula $\alpha \in \mathrm{Fml}(\mathscr{C})$ and an assignment in D, we define the *value* $[\![\alpha]\!]^{\rho}_{\mathscr{M}}$ of α with respect to ρ inductively as follows:

[1] As we can see from the proofs in this dissertation, only a small number of supplies of predicate symbols suffice actually.

[2] As usual, by regarding 0-ary predicate symbols as propositional variables, we assume that $\mathrm{Fml}(\mathscr{C}) \subseteq \mathrm{FOFml}(\mathscr{C})$ and $\mathrm{Sqt}(\mathscr{C}) \subseteq \mathrm{FOSqt}(\mathscr{C})$, where $\mathrm{Fml}(\mathscr{C})$ and $\mathrm{Sqt}(\mathscr{C})$ are defined in § 2.2.2 and $\mathrm{FOFml}(\mathscr{C})$ and $\mathrm{FOSqt}(\mathscr{C})$ are defined below.

- $[\![p(x_1,\dots,x_n)]\!]^{\rho}_{\mathscr{M}} = I(p)(\rho(x_1),\dots,\rho(x_n))$;
- $[\![c(\alpha_1,\dots,\alpha_{\mathrm{ar}(c)})]\!]^{\rho}_{\mathscr{M}} = \mathrm{t}_c([\![\alpha_1]\!]^{\rho}_{\mathscr{M}},\dots,[\![\alpha_{\mathrm{ar}(c)}]\!]^{\rho}_{\mathscr{M}})$;
- $[\![\forall x\alpha]\!]^{\rho}_{\mathscr{M}} = 1$ if and only if $[\![\alpha]\!]^{\rho[x\mapsto a]}_{\mathscr{M}} = 1$ for all $a \in D$;
- $[\![\exists x\alpha]\!]^{\rho}_{\mathscr{M}} = 1$ if and only if $[\![\alpha]\!]^{\rho[x\mapsto a]}_{\mathscr{M}} = 1$ for some $a \in D$.

A formula $\alpha \in \mathrm{FOFml}(\mathscr{C})$ is *valid* in a classical model $\mathscr{M} = \langle D, I\rangle$ if $[\![\alpha]\!]^{\rho}_{\mathscr{M}} = 1$ holds for all assignments ρ in D. A formula $\alpha \in \mathrm{FOFml}(\mathscr{C})$ is *classically valid* if it is valid in all classical models. We denote by $\mathrm{FOCL}(\mathscr{C})$ the set $\{\alpha \in \mathrm{FOFml}(\mathscr{C}) \mid \alpha \text{ is classically valid}\}$. A formula $\alpha \in \mathrm{FOFml}(\mathscr{C})$ is *classically satisfiable* if there exists a classical model $\mathscr{M} = \langle D, I\rangle$ and an assignment ρ in D such that $[\![\alpha]\!]^{\rho}_{\mathscr{M}} = 1$. We denote by $\mathrm{DFOCL}(\mathscr{C})$ the set $\{\alpha \in \mathrm{FOFml} \mid \alpha \text{ is not classically satisfiable}\}$.

For a sequent $\Gamma \Rightarrow \Delta \in \mathrm{FOSqt}(\mathscr{C})$, the *value* $[\![\Gamma \Rightarrow \Delta]\!]^{\rho}_{\mathscr{M}} \in \{0,1\}$ of $\Gamma \Rightarrow \Delta$ with respect to ρ is defined by

$$[\![\Gamma \Rightarrow \Delta]\!]^{\rho}_{\mathscr{M}} = \begin{cases} 0 & \text{if } [\![\alpha]\!]^{\rho}_{\mathscr{M}} = 1 \text{ for all } \alpha \in \Gamma \text{ and } [\![\beta]\!]^{\rho}_{\mathscr{M}} = 0 \text{ for all } \beta \in \Delta \\ 1 & \text{otherwise.} \end{cases}$$

A sequent $\Gamma \Rightarrow \Delta \in \mathrm{FOSqt}(\mathscr{C})$ is *valid* in a classical model $\mathscr{M} = \langle D, I\rangle$ if $[\![\Gamma \Rightarrow \Delta]\!]^{\rho}_{\mathscr{M}} = 1$ holds for all assignments ρ in D. A sequent $\Gamma \Rightarrow \Delta \in \mathrm{FOSqt}(\mathscr{C})$ is *classically valid* if it is valid in all classical models. We denote by $\mathrm{FOCLS}(\mathscr{C})$ the set $\{\Gamma \Rightarrow \Delta \in \mathrm{FOSqt}(\mathscr{C}) \mid \Gamma \Rightarrow \Delta \text{ is classically valid}\}$.

3.2.3 Kripke semantics

A *Kripke model* is a tuple $\langle W, \preceq, D, I\rangle$ in which

- W is a non-empty set, called a set of *states* or a set of *possible worlds*;
- $\preceq$ is a pre-order on W;
- D is a function that assigns to each $w \in W$ a non-empty set $D(w)$, which is called the *individual domain* at w. Furthermore, D satisfies the monotonicity: for all $w, v \in W$, if $w \preceq v$ then $D(w) \subseteq D(v)$.
- I is a function, called an *interpretation function*, that assigns to each pair $\langle w, p\rangle$ of a state and an n-ary predicate symbol a function $I(w,p)$ from $D(w)^n$ to $\{0,1\}$. Furthermore, I satisfies the *hereditary condition*: for all n-ary predicate symbols p and all $w, v \in W$, if $w \preceq v$ then $I(w,p)(a_1,\dots,a_n) \leq I(v,p)(a_1,\dots,a_n)$ holds for all $a_1,\dots,a_n \in D(w)$.

An *assignment* in $D(w)$ is a function which assigns to each individual variable an element of $D(w)$. For an assignment ρ in $D(w)$, an individual variable x and an element $a \in D(w)$, we write $\rho[x \mapsto a]$ for the assignment in $D(w)$ which maps x to a and is equal to ρ everywhere else. For a Kripke model $\mathscr{K} = \langle W, \preceq, D, I\rangle$, a state $w \in W$, an assignment ρ in $D(w)$ and a formula $\alpha \in \mathrm{FOFml}(\mathscr{C})$, we define the *value* $\|\alpha\|^{\rho}_{\mathscr{K},w} \in \{0,1\}$ of α at w with respect to ρ as follows:

- $\|p(x_1,\dots,x_n)\|^{\rho}_{\mathscr{K},w} = I(w,p)(\rho(x_1),\dots,\rho(x_n))$;
- $\|c(\alpha_1,\dots,\alpha_n)\|^{\rho}_{\mathscr{K},w} = 1$ if and only if $\mathrm{t}_c(\|\alpha_1\|^{\rho}_{\mathscr{K},v},\dots,\|\alpha_n\|^{\rho}_{\mathscr{K},v}) = 1$ for all $v \succeq w$;
- $\|\forall x\alpha\|^{\rho}_{\mathscr{K},w} = 1$ if and only if $\|\alpha\|^{\rho[x\mapsto a]}_{\mathscr{K},v} = 1$ for all $v \succeq w$ and all $a \in D(v)$;

- $\|\exists x\alpha\|^{\rho}_{\mathscr{K},w} = 1$ if and only if $\|\alpha\|^{\rho[x\mapsto a]}_{\mathscr{K},w} = 1$ for some $a \in D(w)$.

Note that, in case $c = \wedge$ or $c = \vee$, the statement of the definition of $\|c(\alpha_1, \alpha_2)\|^{\rho}_{\mathscr{K},w}$ differs from the usual one, in which the value is defined by the values of α_1 and α_2 only at w, but we can easily verify that this definition is equivalent to the usual one.

As in the case of the usual connectives, the hereditary condition easily extends to any formula:

Lemma 3.2.1. *For any formula $\alpha \in \mathrm{FOFml}(\mathscr{C})$, any Kripke model $\mathscr{K} = \langle W, \preceq, D, I\rangle$, any $w, v \in W$ and any assignment ρ in $D(w)$, if $w \preceq v$ then $\|\alpha\|^{\rho}_{\mathscr{K},w} \leq \|\alpha\|^{\rho}_{\mathscr{K},w}$.*

A formula $\alpha \in \mathrm{FOFml}(\mathscr{C})$ is *valid* in a Kripke model $\mathscr{K} = \langle W, \preceq, D, I\rangle$ if $\|\alpha\|^{\rho}_{\mathscr{K},w} = 1$ for any $w \in W$ and any assignment ρ in $D(w)$. A formula $\alpha \in \mathrm{FOFml}(\mathscr{C})$ is *Kripke-valid* if it is valid in all Kripke models. We denote by $\mathrm{FOIL}(\mathscr{C})$ the set $\{\alpha \in \mathrm{FOFml}(\mathscr{C}) \mid \alpha \text{ is Kripke-valid}\}$. A formula $\alpha \in \mathrm{FOFml}(\mathscr{C})$ is *Kripke-satisfiable* if there are a Kripke model $\mathscr{K} = \langle W, \preceq, D, I\rangle$, a state $w \in W$ and an assignment ρ in $D(w)$ such that $\|\alpha\|^{\rho}_{\mathscr{K},w} = 1$. We denote by $\mathrm{DFOIL}(\mathscr{C})$ the set $\{\alpha \in \mathrm{FOFml}(\mathscr{C}) \mid \alpha \text{ is not Kripke-satisfiable}\}$.

For a Kripke model $\mathscr{K} = \langle W, \preceq, D, I\rangle$, a state $w \in W$, an assignment ρ in $D(w)$ and a sequent $\Gamma \Rightarrow \Delta \in \mathrm{FOSqt}(\mathscr{C})$, the *value* $\|\Gamma \Rightarrow \Delta\|^{\rho}_{\mathscr{K},w} \in \{0, 1\}$ of $\Gamma \Rightarrow \Delta$ at w with respect to ρ is defined by

$$\|\Gamma \Rightarrow \Delta\|^{\rho}_{\mathscr{K},w} = \begin{cases} 0 & \text{if } \|\alpha\|^{\rho}_{\mathscr{K},w} = 1 \text{ for all } \alpha \in \Gamma \text{ and } \|\beta\|^{\rho}_{\mathscr{K},w} = 0 \text{ for all } \beta \in \Delta \\ 1 & \text{otherwise.} \end{cases}$$

For a Kripke model $\mathscr{K} = \langle W, \preceq, D, I\rangle$, a sequent $\Gamma \Rightarrow \Delta \in \mathrm{FOSqt}(\mathscr{C})$ is *valid* in $\mathscr{K}$ (notation: $\mathscr{K} \vDash \Gamma \Rightarrow \Delta$) if $\|\Gamma \Rightarrow \Delta\|^{\rho}_{\mathscr{K},w} = 1$ for all $w \in W$ and all assignment ρ in $D(w)$. A sequent $\Gamma \Rightarrow \Delta \in \mathrm{FOSqt}(\mathscr{C})$ is *Kripke-valid* if it is valid in all Kripke models. We denote by $\mathrm{FOILS}(\mathscr{C})$ the set $\{\Gamma \Rightarrow \Delta \in \mathrm{FOSqt}(\mathscr{C}) \mid \Gamma \Rightarrow \Delta \text{ is Kripke-valid}\}$.

A Kripke model $\mathscr{K} = \langle W, \preceq, D, I\rangle$ is said to be *constant domain* if $D(w) = D(v)$ for all $w, v \in W$. In this case, we simply write D for $D(w)$ for any $w \in W$. A formula $\alpha \in \mathrm{FOFml}(\mathscr{C})$ is *CD-valid* if it is valid in all constant domain Kripke models. We denote by $\mathrm{FOCD}(\mathscr{C})$ the set $\{\alpha \in \mathrm{FOFml}(\mathscr{C}) \mid \alpha \text{ is CD-valid}\}$. A sequent $\Gamma \Rightarrow \Delta \in \mathrm{Sqt}(\mathscr{C})$ is *CD-valid* if it is valid in all constant domain Kripke models. We denote by $\mathrm{FOCDS}(\mathscr{C})$ the set $\{\Gamma \Rightarrow \Delta \in \mathrm{FOSqt} \mid \Gamma \Rightarrow \Delta \text{ is CD-valid}\}$.

Here, for later use, we establish some notations. Although we describe the notations only for the case of Kripke semantics here, we shall adopt similar notations for the case of classical semantics. The value $\|\alpha\|^{\rho}_{\mathscr{K},w}$ of a formula α depends only on the values of an assignment ρ on $\mathrm{FV}(\alpha)$. Hence, for a *partial* function ρ from the set of individual variables to $D(w)$, $\|\alpha\|^{\rho}_{\mathscr{K},w}$ can be defined if $\rho(x)$ is defined for all $x \in \mathrm{FV}(\alpha)$. For such (partial) assignments, we define $\rho[x \mapsto a]$ to be the function which maps x to a and is equal to ρ on $\mathrm{dom}(\rho) \setminus \{x\}$. We use $\varnothing$ to denote the empty assignment $\emptyset \to D(w)$. For example, for a Kripke model $\mathscr{K} = \langle W, \preceq, D, I\rangle$, $w \in W$ and $a, b \in D(w)$, we have $\|p(x, y)\|^{\varnothing[x\mapsto a][y\mapsto b]}_{\mathscr{K},w} = I(w, p)(a, b)$.

Next, we introduce a notational convention with respect to sequences of truth values. If $\vec{\alpha}$ denotes a sequence of formulas $\alpha_1, \ldots, \alpha_n$, then we denote by $\|\vec{\alpha}\|^{\rho}_{\mathscr{K},w}$ the sequence $\langle \|\alpha_1\|^{\rho}_{\mathscr{K},w}, \ldots, \|\alpha_n\|^{\rho}_{\mathscr{K},w}\rangle$. For example, if $\vec{\beta} = \beta_1, \ldots, \beta_n$, then $\mathrm{t}_c(\|\vec{\beta}\|^{\rho}_{\mathscr{K},w}) = \mathrm{t}_c(\|\beta_1\|^{\rho}_{\mathscr{K},w}, \ldots, \|\beta_n\|^{\rho}_{\mathscr{K},w})$.

Remark 3.2.2. Although we shall not describe it explicitly, we can easily see from the proofs that Theorem 3.3.1 and 3.4.1 and Corollary 3.4.2 also hold if we only use sequents with a single succedent, as in the case of Theorem 2.3.1. For example, the proof of Thoerem 3.3.1 shows that $\mathrm{FOILS}_1(\mathscr{C}) = \mathrm{FOCDS}_1(\mathscr{C})$ if and only if all connectives in $\mathscr{C}$ are supermultiplicative, where $\mathrm{FOILS}_1(\mathscr{C})$ and $\mathrm{FOCDS}_1(\mathscr{C})$ are the set of Kripke-valid sequents with a single succedent and the set of CD-valid sequents with a single succedent, respectively.

The following lemmas follow easily from the definition of each kind of set of valid formulas and each kind of set of valid sequents.

Lemma 3.2.3. *Let $\mathscr{C}$ be a set of connectives. Then, the following hold.*

(i) $\mathrm{FOILS}(\mathscr{C}) \subseteq \mathrm{FOCDS}(\mathscr{C}) \subseteq \mathrm{FOCLS}(\mathscr{C})$.

(ii) $\mathrm{FOIL}(\mathscr{C}) \subseteq \mathrm{FOCD}(\mathscr{C}) \subseteq \mathrm{FOCL}(\mathscr{C})$.

Lemma 3.2.4. *Let $\mathscr{C}$ be a set of connectives. Then, the following hold (cf. the second footnote in § 3.2.1).*

- $\mathrm{FOCLS}(\mathscr{C}) \cap \mathrm{Sqt}(\mathscr{C}) = \mathrm{CLS}(\mathscr{C})$.
- $\mathrm{FOILS}(\mathscr{C}) \cap \mathrm{Sqt}(\mathscr{C}) = \mathrm{FOCDS}(\mathscr{C}) \cap \mathrm{Sqt}(\mathscr{C}) = \mathrm{ILS}(\mathscr{C})$.
- $\mathrm{FOCL}(\mathscr{C}) \cap \mathrm{Fml}(\mathscr{C}) = \mathrm{CL}(\mathscr{C})$.
- $\mathrm{FOIL}(\mathscr{C}) \cap \mathrm{Fml}(\mathscr{C}) = \mathrm{FOCD}(\mathscr{C}) \cap \mathrm{Fml}(\mathscr{C}) = \mathrm{IL}(\mathscr{C})$.

3.3 Condition for $\mathrm{FOILS}(\mathscr{C}) = \mathrm{FOCLS}(\mathscr{C})$

In this section, we show the following main theorem:

Theorem 3.3.1. $\mathrm{FOILS}(\mathscr{C}) = \mathrm{FOCDS}(\mathscr{C})$ *if and only if all connectives in $\mathscr{C}$ are supermultiplicative.*

We show the "if" part in § 3.3.1 and the "only if" part in § 3.3.2.

3.3.1 The "if" part

Here, we show that if every connective in $\mathscr{C}$ is supermultiplicative, then $\mathrm{FOILS}(\mathscr{C}) = \mathrm{FOCDS}(\mathscr{C})$. First, for later use, we remark that the supermultiplicativity can be extended to any number of arguments:

Lemma 3.3.2. *If a connective c is supermultiplicative, then c satisfies the following condition: for all $n \geq 1$ and all $\mathbf{a}_1, \dots, \mathbf{a}_n \in \{0,1\}^{\mathrm{ar}(c)}$, if $\mathrm{t}_c(\mathbf{a}_1) = \cdots = \mathrm{t}_c(\mathbf{a}_n) = 1$ then $\mathrm{t}_c(\mathbf{a}_1 \sqcap \cdots \sqcap \mathbf{a}_n) = 1$.*

Proof. This lemma can be easily proved by induction on n. □

Assume that every connective in $\mathscr{C}$ is supermultiplicative. Since $\mathrm{FOILS}(\mathscr{C}) \subseteq \mathrm{FOCDS}(\mathscr{C})$ holds, in order to prove $\mathrm{FOILS}(\mathscr{C}) = \mathrm{FOCDS}(\mathscr{C})$, it suffices to show the converse inclusion, and hence it suffices to show the following claim: if $\mathscr{K} \nvDash \Gamma \Rightarrow \Delta$ for some $\Gamma \Rightarrow \Delta \in \mathrm{FOSqt}(\mathscr{C})$ and some Kripke model $\mathscr{K} = \langle W, \preceq, D, I \rangle$, then there is a constant domain Kripke model $\mathscr{K}''$ such that $\mathscr{K}'' \nvDash \Gamma \Rightarrow \Delta$. We show this claim by generalizing the method in [2], which is used to prove the claim for the usual connectives: for $\mathscr{C} \subseteq \{\neg, \wedge, \rightarrow\}$, $\mathrm{FOILS}(\mathscr{C}) = \mathrm{FOCDS}(\mathscr{C})$ holds.

Before describing the proof, we introduce some definitions. Let $\langle A, \preceq \rangle$ be a pre-ordered set. A *path* from $a \in A$ is a maximal linear subset of $\{b \in A \mid b \succeq a\}$. For $a \in A$ and $B \subseteq A$, we say B *bars* a (notation: $a \dashv B$) if, for any path $\mathscr{P}$ from a, $B \cap \mathscr{P} \neq \emptyset$ holds.

Here, before formal discussions, we give an overview of how to transform $\mathscr{K}$ into $\mathscr{K}''$. The transformation is divided into two steps:

(1) Transform the Kripke model $\mathscr{K} = \langle W, \preceq, D, I \rangle$ into a Kripke model $\mathscr{K}' = \langle W', \preceq', D', I' \rangle$ such that

- $\mathscr{K}' \not\models \Gamma \Rightarrow \Delta$;
- $\mathscr{K}'$ is a tree;
- for any $\alpha \in \mathrm{FOFml}(\mathscr{C})$, any $w' \in W'$, and any assignment ρ' in $D'(w')$, $\|\alpha\|^{\rho'}_{\mathscr{K}',w'} = 1$ if and only if $w' \dashv \{v' \succeq' w' \mid \|\alpha\|^{\rho'}_{\mathscr{K}',v'} = 1\}$.

Step (1) is the same as the first step in the transformation of a Kripke model into a Beth model in [9].

(2) Transform the Kripke model $\mathscr{K}' = \langle W', \preceq', D', I' \rangle$ into a constant domain Kripke model $\mathscr{K}'' = \langle W', \preceq', D'', I'' \rangle$ in which the values of formulas in $\mathscr{K}'$ are preserved in some sense (that is, in the sense explained in the next paragraph), so that $\mathscr{K}'' \not\models \Gamma \Rightarrow \Delta$ holds. We define the domain D'' of $\mathscr{K}''$ to be a certain set of partial functions from W' into the union of all domains of $\mathscr{K}'$. By this definition, quantifications over individuals in the variable domains of $\mathscr{K}'$ can be translated into quantifications over individuals in the constant domains.

Here, we outline a proof of $\mathscr{K}'' \not\models \Gamma \Rightarrow \Delta$. Just for the sake of simplicity, we consider the case in which $\Gamma = \emptyset$ and $\Delta = \{\alpha\}$, where α has only one free variable, say, x. Since $\mathscr{K}' \not\models \Gamma \Rightarrow \Delta$, that is, $\mathscr{K}' \not\models\ \Rightarrow \alpha$, if we denote the root of W' by $w'_\star$, then $\|\alpha\|^{\varnothing[x \mapsto a]}_{\mathscr{K}',w'_\star} = 0$ for some $a \in D'(w'_\star)$. Hence, in order to prove that $\mathscr{K}'' \not\models \Gamma \Rightarrow \Delta$, it suffices to show that there is some $\mathcal{F}$ in D'' such that $\|\alpha\|^{\varnothing[x \mapsto \mathcal{F}]}_{\mathscr{K}'',w'_\star} = \|\alpha\|^{\varnothing[x \mapsto a]}_{\mathscr{K}',w'_\star}$. To prove this, we shall show that the values of formulas in $\mathscr{K}'$ are preserved in $\mathscr{K}''$ in the following sense: α is true at w' in $\mathscr{K}''$ with x interpreted as a function $\mathcal{F} \in D''$ if and only if for any $v' \succeq' w'$ such that $v' \in \mathrm{dom}(\mathcal{F})$ (where $\mathrm{dom}(\mathcal{F})$ denotes the domain of $\mathcal{F}$), α is true at v' in $\mathscr{K}'$ with x interpreted as $\mathcal{F}(v') \in D'(v')$. (Lemma 3.3.6 below states this preservation formally.) The interpretation function I'' of $\mathscr{K}''$ is defined so that this preservation property holds in the case where α is atomic. Using the second and third properties of $\mathscr{K}'$ in step (1) and the assumption that every connective is supermultiplicative, we can extend the preservation property to any formula.

Step (2) is a simplification of the method in [2] for transforming a Beth model into a constant domain Kripke model.[3]

Now, we describe how to transform $\mathscr{K}$ into $\mathscr{K}''$ formally. Since $\mathscr{K} \not\models \Gamma \Rightarrow \Delta$, there exist some $w_\star \in W$ and some assignment $\rho_\star$ in $D(w_\star)$ such that $\|\Gamma \Rightarrow \Delta\|^{\rho_\star}_{\mathscr{K},w_\star} = 0$. First, we transform $\mathscr{K}$ into a tree Kripke model $\mathscr{K}' = \langle W', \preceq', D', I' \rangle$.

Definition 3.3.3. Let Last denote the function which assigns to each non-empty finite sequence of elements of W its last component, so that $\mathrm{Last}(w_0, \ldots, w_n) = w_n$. Let the Kripke model $\mathscr{K}' = \langle W', \preceq', D', I' \rangle$ be such that

- $W' = \{\langle w_\star, w_1, \ldots, w_n \rangle \mid n \geq 0, w_1, \ldots, w_n \in W, w_\star \preceq w_1 \preceq \cdots \preceq w_n\}$;
- $w' \preceq' v'$ if and only if w' is an initial segment of v', that is, $\langle w_\star, w_1, \ldots, w_n \rangle \preceq' \langle w_\star, v_1, \ldots, v_m \rangle$ if and only if $n \leq m$ and $w_i = v_i$ for all $i = 1, \ldots, n$;
- $D'(w') = D(\mathrm{Last}(w'))$;
- $I'(w') = I(\mathrm{Last}(w'), p)$.

We denote by $w'_\star$ the minimum element $\langle w_\star \rangle$.

Then, it can be shown that $\mathscr{K}'$ has the following two properties (cf. [9]):

[3]In [2], $\mathscr{K}$ is transformed into a Beth model, and then the Beth model is transformed into a constant domain Kripke model. In contrast, we do not introduce a Beth model because it is not necessary for the proof, and instead transform $\mathscr{K}$ into Kripke model $\mathscr{K}'$, which plays essentially the same role as the Beth model.

(I) for any $\alpha \in \mathrm{FOFml}(\mathscr{C})$, any $w' \in W'$, and any assignment ρ' in $D'(w') = D(\mathrm{Last}(w'))$, $\|\alpha\|^{\rho'}_{\mathscr{K}',w'} = \|\alpha\|^{\rho'}_{\mathscr{K},\mathrm{Last}(w')}$;

(II) for any $\alpha \in \mathrm{FOFml}(\mathscr{C})$, any $w' \in W'$, and any assignment ρ' in $D'(w')$, $\|\alpha\|^{\rho'}_{\mathscr{K}',w'} = 1$ if and only if $w' \dashv \{v' \succeq' w' \mid \|\alpha\|^{\rho'}_{\mathscr{K}',v'} = 1\}$.

In particular, by (I), it holds that $\|\Gamma \Rightarrow \Delta\|^{\rho_\star}_{\mathscr{K}',w'_\star} = \|\Gamma \Rightarrow \Delta\|^{\rho_\star}_{\mathscr{K},w_\star} = 0$.

Next, we transform $\mathscr{K}'$ into the constant domain Kripke model $\mathscr{K}''$.

Definition 3.3.4. Let the constant domain Kripke model $\mathscr{K}'' = \langle W', \preceq', D'', I''\rangle$ be such that

- the pre-ordered set $\langle W', \preceq'\rangle$ is the same as that of $\mathscr{K}'$;
- D'' is the set of those partial functions $\mathcal{F}$ from W' to $\bigcup_{w' \in W'} D'(w')$ which satisfy the following conditions:
  - $w'_\star \dashv \mathrm{dom}(\mathcal{F})$ (where $\mathrm{dom}(\mathcal{F})$ denots the domain of $\mathcal{F}$);
  - $\mathrm{dom}(\mathcal{F})$ is an upward-closed subset of W';
  - $\mathcal{F}(w') \in D'(w')$ for any $w' \in \mathrm{dom}(\mathcal{F})$;
  - if $w' \in \mathrm{dom}(\mathcal{F})$ and $w' \preceq' v'$ then $\mathcal{F}(w') = \mathcal{F}(v')$.
- $I''(w', p)(\mathcal{F}_1, \ldots, \mathcal{F}_n) = 1$ if and only if for any $v' \succeq' w'$ such that $v' \in \bigcap_{i=1,\ldots,n} \mathrm{dom}(\mathcal{F}_i)$, $I'(v', p)(\mathcal{F}_1(v'), \ldots, \mathcal{F}_n(v')) = 1$.

Then, the following lemma can be easily shown:

Lemma 3.3.5. *Let $\mathcal{F}_1, \ldots, \mathcal{F}_n \in D''$. Then, the following statements hold.*

(i) $\bigcap_{i=1,\ldots,n} \mathrm{dom}(\mathcal{F}_i)$ *is an upward-closed subset of W'.*

(ii) *For any $w' \in W'$, $w' \dashv \bigcap_{i=1,\ldots,n} \mathrm{dom}(\mathcal{F}_i)$.*

The following lemma ensures that the values of formulas in $\mathscr{K}'$ are preserved in $\mathscr{K}''$ in some sense. Before describing the lemma, we establish the λ-notation. Let V be a set of individual variables and $\mathfrak{E}(x)$ an expression of our meta-language that denotes some value for each $x \in V$. Then $\lambda x \in V.\, \mathfrak{E}(x)$ denotes the function whose domain is V and whose value at each argument x is $\mathfrak{E}(x)$.

Lemma 3.3.6. *Let $\alpha \in \mathrm{FOFml}(\mathscr{C})$. Then, the following conditions are equivalent:*

(i) $\|\alpha\|^{\rho''}_{\mathscr{K}'',w'} = 1$.

(ii) *For any $v' \succeq' w'$ such that $v' \in \bigcap_{x \in \mathrm{FV}(\alpha)} \mathrm{dom}(\rho''(x))$, $\|\alpha\|^{\lambda x \in \mathrm{FV}(\alpha).\, \rho''(x)(v')}_{\mathscr{K}',v'} = 1$.*

Note that if $w' \in \bigcap_{x \in \mathrm{FV}(\alpha)} \mathrm{dom}(\rho''(x))$, then (ii) is equivalent to $\|\alpha\|^{\lambda x \in \mathrm{FV}(\alpha).\, \rho''(x)(w')}_{\mathscr{K}',w'} = 1$.

Proof. The proof proceeds by induction on α. Since the other cases can be proved in the same way as in [2], we only prove the claim in the case where α is of the form $c(\beta_1, \ldots, \beta_{\mathrm{ar}(c)})$. For the whole proof, see [13]. Suppose $\alpha \equiv c(\beta_1, \ldots, \beta_{\mathrm{ar}(c)})$. Put $\vec{\beta} = \beta_1, \ldots, \beta_{\mathrm{ar}(c)}$ and $n = \mathrm{ar}(c)$.

(i) $\Rightarrow$ (ii): Suppose (i) holds. Let $v' \succeq' w'$ and $v' \in \bigcap_{x\in\mathrm{FV}(\alpha)} \mathrm{dom}(\rho''(x))$. In order to show $\|c(\vec{\beta})\|^{\lambda x\in\mathrm{FV}(\alpha).\,\rho''(x)(v')}_{\mathscr{K}',v'} = 1$, we take an arbitrary $u' \succeq' v'$, and show that $\mathrm{t}_c(\|\vec{\beta}\|^{\lambda x\in\mathrm{FV}(\alpha).\,\rho''(x)(v')}_{\mathscr{K}',u'}) = 1$. For each $1 \leq i \leq n$, by the induction hypothesis, we have $\|\beta_i\|^{\rho''}_{\mathscr{K}'',u'} = \|\beta_i\|^{\lambda x\in\mathrm{FV}(\beta_i).\,\rho''(x)(u')}_{\mathscr{K}',u'} = \|\beta_i\|^{\lambda x\in\mathrm{FV}(\beta).\,\rho''(x)(v')}_{\mathscr{K}',u'}$. Thus, we have $\mathrm{t}_c(\|\vec{\beta}\|^{\rho''}_{\mathscr{K}'',u'}) = \mathrm{t}_c(\|\vec{\beta}\|^{\lambda x\in\mathrm{FV}(\alpha).\,\rho''(x)(v')}_{\mathscr{K}',u'})$, the left hand side of which equals to 1 by the supposition (i).

(ii) $\Rightarrow$ (i): Suppose (ii) holds. We show that $\|c(\vec{\beta})\|^{\rho''}_{\mathscr{K}'',w'} = 1$. In order to prove this, we suppose $v' \succeq' w'$ and show that $\mathrm{t}_c(\|\vec{\beta}\|^{\rho''}_{\mathscr{K}'',v'}) = 1$. First, we consider the case $\|\vec{\beta}\|^{\rho''}_{\mathscr{K}'',v'} = \mathbf{1}$. Then, since $v' \dashv \bigcap_{x\in\mathrm{FV}(\alpha)} \mathrm{dom}(\rho''(x))$, there is some $u' \succeq' v'$ such that $u' \in \bigcap_{x\in\mathrm{FV}(\alpha)} \mathrm{dom}(\rho''(x))$. For any $1 \leq i \leq n$, by the hereditary property we have $\|\beta_i\|^{\rho''}_{\mathscr{K}'',u'} \geq \|\beta_i\|^{\rho''}_{\mathscr{K}'',v'} = 1$, and hence, by the induction hypothesis, we have $\|\beta_i\|^{\lambda x\in\mathrm{FV}(\beta_i).\,\rho''(x)(u')}_{\mathscr{K}',u'} = 1$. Thus, we have $\|\vec{\beta}\|^{\rho''}_{\mathscr{K}'',v'} = \mathbf{1} = \|\vec{\beta}\|^{\lambda x\in\mathrm{FV}(\alpha).\,\rho''(x)(u')}_{\mathscr{K}',u'}$. On the other hand, since $\|\alpha\|^{\lambda x\in\mathrm{FV}(\alpha).\,\rho''(x)(u')}_{\mathscr{K}',u'} = 1$ holds by assumption (ii), we have $\mathrm{t}_c(\|\vec{\beta}\|^{\lambda x\in\mathrm{FV}(\alpha).\,\rho''(x)(u')}_{\mathscr{K}',u'}) = 1$. Combining these results, we have $\mathrm{t}_c(\|\vec{\beta}\|^{\rho''}_{\mathscr{K}'',v'}) = 1$.

Secondly, we consider the case $\|\vec{\beta}\|^{\rho''}_{\mathscr{K}'',v'} \neq \mathbf{1}$. Put $\mathcal{I} = \{1 \leq i \leq n \mid \|\beta_i\|^{\rho''}_{\mathscr{K}'',v'} = 0\}$ and $\mathcal{J} = \{1 \leq i \leq n \mid \|\beta_i\|^{\rho''}_{\mathscr{K}'',v'} = 1\}$. Note that $\mathcal{I} \neq \emptyset$. Now, it suffices to show that for each $i \in \mathcal{I}$, there exists a $t'_i \in W'$ such that $t'_i \in \bigcap_{x\in\mathrm{FV}(\alpha)} \mathrm{dom}(\rho''(x))$; $\|\beta_i\|^{\lambda x\in\mathrm{FV}(\alpha).\,\rho''(x)(t'_i)}_{\mathscr{K}',t'_i} = 0$; $\|\beta_j\|^{\lambda x\in\mathrm{FV}(\alpha).\,\rho''(x)(t'_i)}_{\mathscr{K}',t'_i} = 1$ for all $j \in \mathcal{J}$; and $\mathrm{t}_c(\|\vec{\beta}\|^{\lambda x\in\mathrm{FV}(\alpha).\,\rho''(x)(t'_i)}_{\mathscr{K}',t'_i}) = 1$. This is because, for such t'_i's, if we take $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ in Lemma 3.3.2 to be the set $\{\|\vec{\beta}\|^{\lambda x\in\mathrm{FV}(\alpha).\,\rho''(x)}_{\mathscr{K}',t'_i} \mid i \in \mathcal{I}\}$, then it holds that

$$\mathbf{a}_1 \sqcap \cdots \sqcap \mathbf{a}_n = \bigsqcap_{i\in\mathcal{I}} \|\vec{\beta}\|^{\lambda x\in\mathrm{FV}(\alpha).\,\rho''(x)}_{\mathscr{K}',t'_i} = \|\vec{\beta}\|^{\rho''}_{\mathscr{K}'',v'},$$

and hence, $\mathrm{t}_c(\|\vec{\beta}\|^{\rho''}_{\mathscr{K}'',v'}) = 1$ follows from Lemma 3.3.2. So, we fix an arbitrary $i \in \mathcal{I}$ and show that such $t'_i \in W'$ exists. By the induction hypothesis for β_i, there is some $u' \succeq' v'$ such that $u' \in \bigcap_{x\in\mathrm{FV}(\beta_i)} \mathrm{dom}(\rho''(x))$ and $\|\beta_i\|^{\lambda x\in\mathrm{FV}(\beta_i).\,\rho''(x)(u')}_{\mathscr{K}',u'} = 0$. By the property (II) of $\mathscr{K}'$, $u' \not\dashv \{r' \succeq' u' \mid \|\beta_i\|^{\lambda x\in\mathrm{FV}(\beta_i).\,\rho''(x)(u')}_{\mathscr{K}',r'} = 1\}$ holds. Hence, there is some path $\mathscr{P}$ from u' such that $\mathscr{P} \cap \{r' \succeq' u' \mid \|\beta_i\|^{\lambda x\in\mathrm{FV}(\beta_i).\,\rho''(x)(u')}_{\mathscr{K}',r'} = 1\} = \emptyset$. On the other hand, since $u' \dashv \bigcap_{x\in\mathrm{FV}(\alpha)} \mathrm{dom}(\rho''(x))$, $\mathscr{P}$ and $\bigcap_{x\in\mathrm{FV}(\alpha)} \rho''(x)$ intersect at some point, say, $t'_i \in W'$. Then, we have $t'_i \succeq' u' \succeq' v'$, $t'_i \in \bigcap_{x\in\mathrm{FV}(\alpha)} \mathrm{dom}(\rho''(x))$ and

$$\|\beta_i\|^{\lambda x\in\mathrm{FV}(\alpha).\,\rho''(x)(t'_i)}_{\mathscr{K}',t'_i} = \|\beta_i\|^{\lambda x\in\mathrm{FV}(\alpha).\,\rho''(x)(u')}_{\mathscr{K}',t'_i} = 0.$$

By the hereditary property, we have $\|\beta_j\|^{\rho''}_{\mathscr{K}'',t'_i} \geq \|\beta_j\|^{\rho''}_{\mathscr{K}'',v'} = 1$ for every $j \in \mathcal{J}$. Hence, by the induction hypothesis we have $\|\beta_j\|^{\lambda x\in\mathrm{FV}(\alpha).\,\rho''(x)(t'_i)}_{\mathscr{K}',t'_i} = 1$ for every $j \in \mathcal{J}$. Finally, since $\|c(\vec{\beta})\|^{\lambda x\in\mathrm{FV}(\alpha).\,\rho''(x)(t'_i)}_{\mathscr{K}',t'_i} = 1$ holds by the assumption (ii), we have $\mathrm{t}_c(\|\vec{\beta}\|^{\lambda x\in\mathrm{FV}(\alpha).\,\rho''(x)(t'_i)}_{\mathscr{K}',t'_i}) = 1$. Thus, we have proved that t'_i satisfies the desired conditions.

□

From Lemma 3.3.6, it follows that $\|\Gamma \Rightarrow \Delta\|^{\rho''}_{\mathscr{K}'',w'_\star} = \|\Gamma \Rightarrow \Delta\|^{\rho_\star}_{\mathscr{K}',w'_\star} = 0$, where ρ'' is the assignment in D'' such that for each free variables x in Γ and Δ, $\rho''(x)$ is the function on W' whose value is constantly $\rho_\star(x) \in D'(w'_\star)$. Thus, we have finished the proof of the "if" part of Theorem 3.3.1.

3.3.2 The "only if" part

Here, we show the "only if" part of Theorem 3.3.1 by showing its contraposition:

Proposition 3.3.7. *Suppose there is a non-supermultiplicative $c \in \mathscr{C}$. Then, it holds that* $\mathrm{FOILS}(\mathscr{C}) \neq \mathrm{FOCDS}(\mathscr{C})$.

First, let us consider the case in which $\mathrm{ar}(c) \leq 2$. It can be easily seen that non-supermultiplicative connectives whose arity is less than or equal to 2 are only $\vee$ (disjunction) and $\oplus$ (exclusive disjunction). Thus, we only have to consider the cases $c = \vee$ and $c = \oplus$. Regarding disjunction, it is known that the sequents of the form $\forall x(p(x) \vee q(x)) \Rightarrow \forall xp(x) \vee \exists xq(x)$ are CD-valid but not Kripke-valid (cf., e.g., [15]). As to exclusive disjunction, we can verify that the corresponding sequents of the form $\forall x(p(x) \oplus q(x)) \Rightarrow \forall xp(x) \oplus \exists xq(x)$ are also CD-valid but not Kripke-valid (cf. case (A) in the proof below).[4] For c of general airty, we construct a sequent in $\mathrm{FOCDS}(\mathscr{C}) \setminus \mathrm{FOILS}(\mathscr{C})$ which plays the same role as $\forall x(p(x) \vee q(x)) \Rightarrow \forall xp(x) \vee \exists xq(x)$ and $\forall x(p(x) \oplus q(x)) \Rightarrow \forall xp(x) \oplus \exists xq(x)$. This construction requires an elaborate case analysis.

Proof. Suppose there is a non-supermultiplicative connective $c \in \mathscr{C}$. Then, there are $\mathbf{a}, \mathbf{b} \in \{0,1\}^{\mathrm{ar}(c)}$ such that $\mathrm{t}_c(\mathbf{a}) = \mathrm{t}_c(\mathbf{b}) = 1$ and $\mathrm{t}_c(\mathbf{a} \sqcap \mathbf{b}) = 0$. Let p and q be distinct unary predicate symbols. Fix two 0-ary predicate symbols T and R, which shall play particular roles later.

First, we define a Kripke model $\mathscr{K}^\star = \langle W^\star, \preceq^\star, D^\star, I^\star \rangle$, which is used to show the constructed sequents are not Kripke-valid.

- $W^\star = \{w_1, w_2\}$;
- $w_i \preceq^\star w_j$ if and only if $i \leq j$;
- $D^\star(w_1) = \{a_1\}$, $D^\star(w_2) = \{a_1, a_2\}$;
-
  - $I(w_1, p)(a_1) = 1$, $I(w_1, q)(a_1) = 0$, $I(w_1, T) = 1$, $I(w_1, R) = 0$;
  - $I(w_2, p)(a_1) = 1$, $I(w_2, q)(a_1) = 0$, $I(w_2, T) = 1$, $I(w_2, R) = 0$, $I(w_2, p)(a_2) = 0$, $I(w_2, q)(a_2) = 1$.

We define $\mathbf{a}^*, \mathbf{b}^* \in \{0,1\}^{\mathrm{ar}(c)}$ as follows:

$$\mathbf{a}^*[i] = \begin{cases} \mathbf{a}[i] & \text{if } \mathbf{a}[i] = 1 \text{ or } \mathbf{b}[i] = 1 \\ 1 & \text{if } \mathbf{a}[i] = 0 \text{ and } \mathbf{b}[i] = 0, \end{cases}$$

$$\mathbf{b}^*[i] = \begin{cases} \mathbf{b}[i] & \text{if } \mathbf{a}[i] = 1 \text{ or } \mathbf{b}[i] = 1 \\ 1 & \text{if } \mathbf{a}[i] = 0 \text{ and } \mathbf{b}[i] = 0. \end{cases}$$

Then, we have $\mathbf{a} \sqcap \mathbf{b}^* = \mathbf{a}^* \sqcap \mathbf{b} = \mathbf{a} \sqcap \mathbf{b}$ and $\mathbf{a} \sqcup \mathbf{b}^* = \mathbf{a}^* \sqcup \mathbf{b} = \mathbf{a}^* \sqcup \mathbf{b}^* = \mathbf{1}$. We divide into five cases: (A) $\mathrm{t}_c(\mathbf{1}) = 0$; (B) $\mathrm{t}_c(\mathbf{a}^*) = 1$; (C) $\mathrm{t}_c(\mathbf{b}^*) = 1$; (D) $\mathrm{t}_c(\mathbf{a}^*) = \mathrm{t}_c(\mathbf{b}^*) = \mathrm{t}_c(\mathbf{a}^* \sqcap \mathbf{b}^*) = 0$ and $\mathrm{t}_c(\mathbf{1}) = 1$; (E) $\mathrm{t}_c(\mathbf{a}^*) = \mathrm{t}_c(\mathbf{b}^*) = 0$ and $\mathrm{t}_c(\mathbf{a}^* \sqcap \mathbf{b}^*) = \mathrm{t}_c(\mathbf{1}) = 1$.

Case (A): $\mathrm{t}_c(\mathbf{1}) = 0$. Define $F \in \mathrm{FOFml}(\mathscr{C})$ by $F \equiv c(T, \ldots, T)$. Note that, for any Kripke model $\mathscr{K} = \langle W, \preceq, D, I \rangle$ and any $w \in W$, if $\|T\|^{\varnothing}_{\mathscr{K},w} = 1$ then $\|F\|^{\varnothing}_{\mathscr{K},v} = 0$ for all $v \succeq w$.

[4] In contrast, we can verify that the sequents corresponding to **D**-axioms, $\forall x(p(x) \oplus r) \Rightarrow \forall p(x) \oplus r$, where r is a 0-ary predicate symbol, are Kripke-valid.

We define formulas $\varphi_1, \ldots, \varphi_{\mathrm{ar}(c)}, \varphi, \psi_1, \ldots, \psi_{\mathrm{ar}(c)}, \psi \in \mathrm{FOFml}(\mathscr{C})$ as follows:

$$\varphi_i \equiv \begin{cases} F & \text{if } \mathbf{a}[i] = 0 \text{ and } \mathbf{b}[i] = 0 \\ p(x) & \text{if } \mathbf{a}[i] = 0 \text{ and } \mathbf{b}[i] = 1 \\ q(x) & \text{if } \mathbf{a}[i] = 1 \text{ and } \mathbf{b}[i] = 0 \\ T & \text{if } \mathbf{a}[i] = 1 \text{ and } \mathbf{b}[i] = 1, \end{cases}$$

$$\varphi \equiv \forall x c(\varphi_1, \ldots, \varphi_{\mathrm{ar}(c)}),$$

$$\psi_i \equiv \begin{cases} F & \text{if } \mathbf{a}[i] = 0 \text{ and } \mathbf{b}[i] = 0 \\ \forall x p(x) & \text{if } \mathbf{a}[i] = 0 \text{ and } \mathbf{b}[i] = 1 \\ \exists x q(x) & \text{if } \mathbf{a}[i] = 1 \text{ and } \mathbf{b}[i] = 0 \\ T & \text{if } \mathbf{a}[i] = 1 \text{ and } \mathbf{b}[i] = 1, \end{cases}$$

$$\psi \equiv c(\psi_1, \ldots, \psi_{\mathrm{ar}(c)}).$$

Put $\vec{\varphi} = \varphi_1, \ldots, \varphi_{\mathrm{ar}(c)}$ and $\vec{\psi} = \psi_1, \ldots, \psi_{\mathrm{ar}(c)}$. We show $T, \varphi \Rightarrow \psi \in \mathrm{FOCDS}(\mathscr{C}) \setminus \mathrm{FOILS}(\mathscr{C})$.

First, we show $T, \varphi \Rightarrow \psi \in \mathrm{FOCDS}(\mathscr{C})$. Let $\mathscr{K} = \langle W, \preceq, D, I \rangle$ be a constant domain Kripke model and $w \in W$. We suppose $\|T\|^{\varnothing}_{\mathscr{K},w} = \|\varphi\|^{\varnothing}_{\mathscr{K},w} = 1$, in order to show $\|\psi\|^{\varnothing}_{\mathscr{K},w} = 1$. To show $\|\psi\|^{\varnothing}_{\mathscr{K},w} = 1$, it suffices to show $\mathrm{t}_c(\|\vec{\psi}\|^{\varnothing}_{\mathscr{K},v}) = 1$ for any $v \succeq w$. Let $v \succeq w$. Then, we can see $\|\forall x p(x)\|^{\varnothing}_{\mathscr{K},v} = 1$ or $\|\exists x q(x)\|^{\varnothing}_{\mathscr{K},v} = 1$ holds. For otherwise there exists some $a \in D$ such that $\|p(x)\|^{\varnothing[x \mapsto a]}_{\mathscr{K},v} = \|q(x)\|^{\varnothing[x \mapsto a]}_{\mathscr{K},v} = 0$, and hence, $\|\vec{\varphi}\|^{\varnothing[x \mapsto a]}_{\mathscr{K},v} = \mathbf{a} \sqcap \mathbf{b}$, and thus, $\mathrm{t}_c(\|\vec{\varphi}\|^{\varnothing[x \mapsto a]}_{\mathscr{K},v}) = 0$, which contradicts $\|\varphi\|^{\varnothing}_{\mathscr{K},w} = 1$. First, we consider the case $\|\forall x p(x)\|^{\varnothing}_{\mathscr{K},v} = \|\exists x q(x)\|^{\varnothing}_{\mathscr{K},v} = 1$. Then we have $\|\vec{\psi}\|^{\varnothing}_{\mathscr{K},v} = \mathbf{a} \sqcup \mathbf{b}$ and there exists some $a \in D$ such that $\|p(x)\|^{\varnothing[x \mapsto a]}_{\mathscr{K},v} = \|q(x)\|^{\varnothing[x \mapsto a]}_{\mathscr{K},v} = 1$, so that $\|\vec{\varphi}\|^{\varnothing[x \mapsto a]}_{\mathscr{K},v} = \mathbf{a} \sqcup \mathbf{b}$. Hence, we have $\mathrm{t}_c(\|\vec{\psi}\|^{\varnothing}_{\mathscr{K},v}) = \mathrm{t}_c(\mathbf{a} \sqcup \mathbf{b}) = \mathrm{t}_c(\|\vec{\varphi}\|^{\varnothing[x \mapsto a]}_{\mathscr{K},v})$ the right hand side of which equals to 1 by $\|\varphi\|^{\varnothing}_{\mathscr{K},w} = 1$. Thus, $\mathrm{t}_c(\|\vec{\psi}\|^{\varnothing}_{\mathscr{K},v}) = 1$. Next, we consider the case that one of $\|\forall x p(x)\|^{\varnothing}_{\mathscr{K},v}$ and $\|\exists x q(x)\|^{\varnothing}_{\mathscr{K},v}$ is 1 and the other is 0. Then, either $\|\vec{\psi}\|^{\varnothing}_{\mathscr{K},v} = \mathbf{a}$ or $\|\vec{\psi}\|^{\varnothing}_{\mathscr{K},v} = \mathbf{b}$, and hence, we have $\mathrm{t}_c(\|\vec{\psi}\|^{\varnothing}_{\mathscr{K},v}) = 1$.

Secondly, we show $T, \varphi \Rightarrow \psi \notin \mathrm{FOILS}(\mathscr{C})$. In order to do so, we verify $\|T, \varphi \Rightarrow \psi\|^{\varnothing}_{\mathscr{K}^\star,w_1} = 0$. First, we can easily see the followings: $\|\vec{\varphi}\|^{\varnothing[x \mapsto a_1]}_{\mathscr{K}^\star,w_1} = \|\vec{\varphi}\|^{\varnothing[x \mapsto a_1]}_{\mathscr{K}^\star,w_2} = \mathbf{b}$; $\|\vec{\varphi}\|^{\varnothing[x \mapsto a_2]}_{\mathscr{K}^\star,w_2} = \mathbf{a}$; $\|\forall x p(x)\|^{\varnothing}_{\mathscr{K}^\star,w_1} = \|\exists x q(x)\|^{\varnothing}_{\mathscr{K}^\star,w_1} = 0$; and $\|\vec{\psi}\|^{\varnothing}_{\mathscr{K}^\star,w_1} = \mathbf{a} \sqcap \mathbf{b}$. From these it follows that $\|\varphi\|^{\varnothing}_{\mathscr{K}^\star,w_1} = 1$ and $\|\psi\|^{\varnothing}_{\mathscr{K}^\star,w_1} = 0$.

Case (B): $\mathrm{t}_c(\mathbf{a}^*) = 1$. Note that, in this case, there is no i such that $\mathbf{a}^*[i] = \mathbf{b}[i] = 0$. We define formulas $\varphi_1, \dots, \varphi_{\mathrm{ar}(c)}, \varphi, \psi_1, \dots, \psi_{\mathrm{ar}(c)}, \psi \in \mathrm{FOFml}(\mathscr{C})$ as follows:

$$\varphi_i \equiv \begin{cases} p(x) & \text{if } \mathbf{a}^*[i] = 0 \text{ and } \mathbf{b}[i] = 1 \\ q(x) & \text{if } \mathbf{a}^*[i] = 1 \text{ and } \mathbf{b}[i] = 0 \\ T & \text{if } \mathbf{a}^*[i] = 1 \text{ and } \mathbf{b}^*[i] = 1 \end{cases}$$

$$\varphi \equiv \forall x c(\varphi_1, \dots, \varphi_{\mathrm{ar}(c)})$$

$$\psi_i \equiv \begin{cases} \forall x p(x) & \text{if } \mathbf{a}^*[i] = 0 \text{ and } \mathbf{b}[i] = 1 \\ \exists x q(x) & \text{if } \mathbf{a}^*[i] = 1 \text{ and } \mathbf{b}[i] = 0 \\ T & \text{if } \mathbf{a}^*[i] = 1 \text{ and } \mathbf{b}^*[i] = 1 \end{cases}$$

$$\psi \equiv c(\psi_1, \dots, \psi_{\mathrm{ar}(c)})$$

Put $\vec{\varphi} = \varphi_1, \dots, \varphi_{\mathrm{ar}(c)}$ and $\vec{\psi} = \psi_1, \dots, \psi_{\mathrm{ar}(c)}$. We show $T, \varphi \Rightarrow \psi \in \mathrm{FOCDS}(\mathscr{C}) \setminus \mathrm{FOILS}(\mathscr{C})$.

First, $T, \varphi \Rightarrow \psi \in \mathrm{FOCDS}(\mathscr{C})$ can be proved similarly to case (A); in fact, if we replace every $\mathbf{a}$ in the proof in case (A) by $\mathbf{a}^*$ (and thus, $\mathbf{a} \sqcap \mathbf{b}$ by $\mathbf{a}^* \sqcap \mathbf{b}$ and $\mathbf{a} \sqcup \mathbf{b}$ by $\mathbf{a}^* \sqcup \mathbf{b}$), then we obtain a proof for case (B).

$T, \varphi \Rightarrow \psi \notin \mathrm{FOILS}(\mathscr{C})$ can also be proved similarly to case (A); in fact, if we replace every $\mathbf{a}$ in the proof in (A) by $\mathbf{a}^*$ (and thus, $\mathbf{a} \sqcap \mathbf{b}$ by $\mathbf{a}^* \sqcap \mathbf{b}$), then we obtain a proof for case (B).

Case (C): $\mathrm{t}_c(\mathbf{b}^*) = 1$. This case can be shown similarly to case (B).

Case (D): $\mathrm{t}_c(\mathbf{a}^*) = \mathrm{t}_c(\mathbf{b}^*) = \mathrm{t}_c(\mathbf{a}^* \sqcap \mathbf{b}^*) = 0$ and $\mathrm{t}_c(\mathbf{1}) = 1$. We define formulas $\varphi_1, \dots, \varphi_{\mathrm{ar}(c)}, \varphi, \psi_1, \dots, \psi_{\mathrm{ar}(c)}, \psi \in \mathrm{FOFml}(\mathscr{C})$ as follows:

$$\varphi_i \equiv \begin{cases} R & \text{if } \mathbf{a}[i] = 0 \text{ and } \mathbf{b}[i] = 0 \\ p(x) & \text{if } \mathbf{a}[i] = 0 \text{ and } \mathbf{b}[i] = 1 \\ q(x) & \text{if } \mathbf{a}[i] = 1 \text{ and } \mathbf{b}[i] = 0 \\ T & \text{if } \mathbf{a}[i] = 1 \text{ and } \mathbf{b}[i] = 1 \end{cases}$$

$$\varphi \equiv \forall x c(\varphi_1, \dots, \varphi_{\mathrm{ar}(c)})$$

$$\psi_i \equiv \begin{cases} R & \text{if } \mathbf{a}[i] = 0 \text{ and } \mathbf{b}[i] = 0 \\ \forall x p(x) & \text{if } \mathbf{a}[i] = 0 \text{ and } \mathbf{b}[i] = 1 \\ \exists x q(x) & \text{if } \mathbf{a}[i] = 1 \text{ and } \mathbf{b}[i] = 0 \\ T & \text{if } \mathbf{a}[i] = 1 \text{ and } \mathbf{b}[i] = 1 \end{cases}$$

$$\psi \equiv c(\psi_1, \dots, \psi_{\mathrm{ar}(c)})$$

Put $\vec{\varphi} = \varphi_1, \dots, \varphi_{\mathrm{ar}(c)}$ and $\vec{\psi} = \psi_1, \dots, \psi_{\mathrm{ar}(c)}$. We show $T, \varphi \Rightarrow \psi \in \mathrm{FOCDS}(\mathscr{C}) \setminus \mathrm{FOILS}(\mathscr{C})$.

First, we show $T, \varphi \Rightarrow \psi \in \mathrm{FOCDS}(\mathscr{C})$. Let $\mathscr{K} = \langle W, \preceq, D, I \rangle$ be a constant domain Kripke model and $w \in W$. We suppose $\|T\|^{\varnothing}_{\mathscr{K},w} = \|\varphi\|^{\varnothing}_{\mathscr{K},w} = 1$, in order to show $\|\psi\|^{\varnothing}_{\mathscr{K},w} = 1$. To show $\|\psi\|^{\varnothing}_{\mathscr{K},w} = 1$, it suffices to show $\mathrm{t}_c(\|\vec{\psi}\|^{\varnothing}_{\mathscr{K},v}) = 1$ for any $v \succeq w$. Let $v \succeq w$. Then, we divide into two subcases according to the value of $\|R\|^{\varnothing}_{\mathscr{K},v}$.

Subcase (i): $\|R\|^{\varnothing}_{\mathscr{K},v} = 0$. In this case, $\mathrm{t}_c(\|\vec{\psi}\|^{\varnothing}_{\mathscr{K},v}) = 1$ can be shown similarly to case (A) because R plays the same role as F in case (A).

SUBCASE (ii): $\|R\|^{\varnothing}_{\mathscr{K},v} = 1$. In this case, $\|\forall xp(x)\|^{\varnothing}_{\mathscr{K},v} = \|\exists xq(x)\|^{\varnothing}_{\mathscr{K},v} = 1$ holds. For, otherwise, there is some $a \in D$ such that $\|\vec{\varphi}\|^{\varnothing[x\mapsto a]}_{\mathscr{K},v}$ equals to either $\mathbf{a}^*$, $\mathbf{b}^*$, or $\mathbf{a}^* \sqcap \mathbf{b}^*$, and hence, $\mathrm{t}_c(\|\vec{\varphi}\|^{\varnothing[x\mapsto a]}_{\mathscr{K},v}) = 0$, which contradicts $\|\varphi\|^{\varnothing}_{\mathscr{K},w} = 1$. Thus, we have $\|\vec{\psi}\|^{\varnothing}_{\mathscr{K},v} = \mathbf{1}$. Hence, $\mathrm{t}_c(\|\vec{\psi}\|^{\varnothing}_{\mathscr{K},v}) = 1$.

Secondly, we show $T, \varphi \Rightarrow \psi \notin \mathrm{FOILS}(\mathscr{C})$. Since $\|R\|^{\varnothing}_{\mathscr{K}^\star,w_1} = \|R\|^{\varnothing}_{\mathscr{K}^\star,w_2} = 0$, R plays the same role as F in case (A), and hence $T, \varphi \Rightarrow \psi \notin \mathrm{FOILS}(\mathscr{C})$ can be shown similarly to case (A).

CASE (E): $\mathrm{t}_c(\mathbf{a}^*) = \mathrm{t}_c(\mathbf{b}^*) = 0$ and $\mathrm{t}_c(\mathbf{a}^* \sqcap \mathbf{b}^*) = \mathrm{t}_c(\mathbf{1}) = 1$. Note that, in this case, there is no i such that $\mathbf{a}^*[i] = \mathbf{b}^*[i] = 0$. First, in order to construct the desired sequent, we define biconditional $\leftrightarrow$ ($\mathrm{t}_{\leftrightarrow}(x, y) = 1$ if and only if $x = y$) using c. That is, for any $\alpha \in \mathrm{FOFml}(\mathscr{C})$ and any $\beta \in \mathrm{FOFml}(\mathscr{C})$, we define a formula $\alpha \leftrightarrow_c \beta \in \mathrm{FOFml}(\mathscr{C})$. First, we define $\theta^{\alpha,\beta}_1, \ldots, \theta^{\alpha,\beta}_{\mathrm{ar}(c)}$ for $\alpha, \beta \in \mathrm{FOFml}(\mathscr{C})$ by

$$\theta^{\alpha,\beta}_i \equiv \begin{cases} \alpha & \text{if } \mathbf{a}^*[i] = 0,\ \mathbf{b}^*[i] = 1 \\ \beta & \text{if } \mathbf{a}^*[i] = 1,\ \mathbf{b}^*[i] = 0 \\ T & \text{if } \mathbf{a}^*[i] = 1,\ \mathbf{b}^*[i] = 1. \end{cases}$$

Put $\overrightarrow{\theta^{\alpha,\beta}} \equiv \theta^{\alpha,\beta}_1, \ldots, \theta^{\alpha,\beta}_{\mathrm{ar}(c)}$. We define $\alpha \leftrightarrow_c \beta$ by $\alpha \leftrightarrow_c \beta \equiv c(\theta^{\alpha,\beta}_1, \ldots, \theta^{\alpha,\beta}_{\mathrm{ar}(c)})$. Then, $\leftrightarrow_c$ has the same meaning as biconditional $\leftrightarrow$ whenever the value of T is interpreted as 1, that is, for any Kripke model $\mathscr{K} = \langle W, \preceq, D, I\rangle$, any $w \in W$ and any assignment ρ in $D(w)$, if $\|T\|^{\rho}_{\mathscr{K},w} = 1$, then $\|\alpha \leftrightarrow_c \beta\|^{\rho}_{\mathscr{K},w} = 1$ holds if and only if $\|\alpha\|^{\rho}_{\mathscr{K},v} = \|\beta\|^{\rho}_{\mathscr{K},v}$ holds for all $v \succeq w$. In order to show the "if" part, let $\mathscr{K} = \langle W, \preceq, D, I\rangle$ be a Kripke model and $w \in W$, and suppose $\|T\|^{\rho}_{\mathscr{K},w} = 1$ and $\|\alpha\|^{\rho}_{\mathscr{K},v} = \|\beta\|^{\rho}_{\mathscr{K},v}$ for all $v \succeq w$. Then, for any $v \succeq w$, $\|\overrightarrow{\theta^{\alpha,\beta}}\|^{\rho}_{\mathscr{K},v}$ is either $\mathbf{a}^* \sqcap \mathbf{b}^*$ or $\mathbf{1}$, and hence, $\mathrm{t}_c(\|\overrightarrow{\theta^{\alpha,\beta}}\|^{\rho}_{\mathscr{K},v}) = 1$ for any $v \succeq w$. Thus, we have $\|\alpha \leftrightarrow_c \beta\|^{\rho}_{\mathscr{K},w} = 1$. In order to show the (contraposition of) "only if" part, let $\mathscr{K} = \langle W, \preceq, D, I\rangle$ be a Kripke model and $w \in W$, and suppose $\|T\|^{\rho}_{\mathscr{K},w} = 1$ and $\|\alpha\|^{\rho}_{\mathscr{K},v} \neq \|\beta\|^{\rho}_{\mathscr{K},v}$ for some $v \succeq w$. Then, $\|\overrightarrow{\theta^{\alpha,\beta}}\|^{\rho}_{\mathscr{K},v}$ is either $\mathbf{a}^*$ or $\mathbf{b}^*$, and hence, we have $\mathrm{t}_c(\|\overrightarrow{\theta^{\alpha,\beta}}\|^{\rho}_{\mathscr{K},v}) = 0$. Thus, we have $\|\alpha \leftrightarrow_c \beta\|^{\rho}_{\mathscr{K},w} = 0$.

Now, we define $\varphi_1, \ldots, \varphi_{\mathrm{ar}(c)}, \varphi, \psi_1, \ldots, \psi_{\mathrm{ar}(c)}.\psi \in \mathrm{FOFml}(\mathscr{C})$ as in case (D). Put $\vec{\varphi} = \varphi_1, \ldots, \varphi_{\mathrm{ar}(c)}$ and $\vec{\psi} = \psi_1, \ldots, \psi_{\mathrm{ar}(c)}$. We show $T, R \leftrightarrow_c \forall xp(x), R \leftrightarrow_c \exists xq(x), \varphi \Rightarrow \psi \in \mathrm{FOCDS}(\mathscr{C}) \setminus \mathrm{FOILS}(\mathscr{C})$.

First, we show $T, R \leftrightarrow_c \forall xp(x), R \leftrightarrow_c \exists xq(x), \varphi \Rightarrow \psi \in \mathrm{FOCDS}(\mathscr{C})$. Let $\mathscr{K} = \langle W, \preceq, D, I\rangle$ be a constant domain Kripke and $w \in W$. We suppose $\|T\|^{\varnothing}_{\mathscr{K},w} = \|R \leftrightarrow_c \forall xp(x)\|^{\varnothing}_{\mathscr{K},w} = \|R \leftrightarrow_c \exists xq(x)\|^{\varnothing}_{\mathscr{K},w} = \|\varphi\|^{\varnothing}_{\mathscr{K},w} = 1$, in order to show $\|\psi\|^{\varnothing}_{\mathscr{K},w} = 1$. To show $\|\psi\|^{\varnothing}_{\mathscr{K},w} = 1$, it suffices to show $\mathrm{t}_c(\|\vec{\psi}\|^{\varnothing}_{\mathscr{K},v}) = 1$ for any $v \succeq w$. Let $v \succeq w$. Then, we can see $\|R\|^{\varnothing}_{\mathscr{K},v} = 1$ holds. For if $\|R\|^{\varnothing}_{\mathscr{K},v} = 0$, then we would have either $\|\forall xp(x)\|^{\varnothing}_{\mathscr{K},v} = 1$ or $\|\exists xq(x)\|^{\varnothing}_{\mathscr{K},v} = 1$ similarly to case (A), whereas $\|R\|^{\varnothing}_{\mathscr{K},v} = 0$ and $\|R \leftrightarrow_c \forall xp(x)\|^{\varnothing}_{\mathscr{K},w} = \|R \leftrightarrow_c \exists xq(x)\|^{\varnothing}_{\mathscr{K},v} = \|T\|^{\varnothing}_{\mathscr{K},w} = 1$ would imply $\|\forall xp(x)\|^{\varnothing}_{\mathscr{K},v} = \|\exists xq(x)\|^{\varnothing}_{\mathscr{K},v} = 0$. Now, from $\|R\|^{\varnothing}_{\mathscr{K},v} = 1$, $\|T\|^{\varnothing}_{\mathscr{K},w} = 1$ and $\|R \leftrightarrow_c \forall xp(x)\|^{\varnothing}_{\mathscr{K},w} = \|R \leftrightarrow_c \exists xq(x)\|^{\varnothing}_{\mathscr{K},w} = 1$, it follows that $\|\forall xp(x)\|^{\varnothing}_{\mathscr{K},v} = \|\exists xq(x)\|^{\varnothing}_{\mathscr{K},v} = 1$. Thus, we have $\|\vec{\psi}\|^{\varnothing}_{\mathscr{K},v} = \mathbf{1}$. Hence, we have $\mathrm{t}_c(\|\vec{\psi}\|^{\varnothing}_{\mathscr{K},v}) = 1$.

Secondly, $T, R \leftrightarrow_c \forall xp(x), R \leftrightarrow_c \exists xq(x), \varphi \Rightarrow \psi \notin \mathrm{FOILS}(\mathscr{C})$ can be shown similarly to case (D).

□

3.4 Condition for $\mathrm{FOCDS}(\mathscr{C}) = \mathrm{FOCLS}(\mathscr{C})$ and a condition for $\mathrm{FOILS}(\mathscr{C}) = \mathrm{FOCLS}(\mathscr{C})$

In this section, we show the following theorem:

Theorem 3.4.1. $\mathrm{FOCDS}(\mathscr{C}) = \mathrm{FOCLS}(\mathscr{C})$ *if and only if all connectives* $c \in \mathscr{C}$ *are monotonic.*

Then, the following corollary immediately follows from Theorem 3.3.1 and Theorem 3.4.1:

Corollary 3.4.2. $\mathrm{FOILS}(\mathscr{C}) = \mathrm{FOCLS}(\mathscr{C})$ *if and only if all connectives* $c \in \mathscr{C}$ *are monotonic and supermultiplicative.*

Here we show Theorem 3.4.1. The "only if" part immediately follows from the "only if" part of Theorem 2.3.1 by Lemma 3.2.4. The "if" part can be proved by extending the proof of the "if" part of Theorem 2.3.1 to first-order logic. Now we prepare Lemma 3.4.3, the counterpart of Lemma 2.3.2, and the remaining part can be proved similarly to Theorem 2.3.1.

Lemma 3.4.3. *Suppose all connectives in* $\mathscr{C}$ *are monotonic. Let* $\mathscr{K} = \langle W, \preceq, D, I\rangle$ *be a constant domain Kripke model and* $w \in W$*. Let* $\mathscr{M}_{\mathscr{K},w} = \langle D, J_{\mathscr{K},w}\rangle$ *be the classical model defined by* $J_{\mathscr{K},w}(p) = I(p, w)$*. Then, for any formula* $\alpha \in \mathrm{FOFml}(\mathscr{C})$ *and any assignment* ρ *in* D*,* $\|\alpha\|^{\rho}_{\mathscr{K},w} = [\![\alpha]\!]^{\rho}_{\mathscr{M}_{\mathscr{K},w}}$ *holds.*

Proof. The proof proceeds by induction on α. The base case, in which α is atomic, immediately follows from the definition of $J_{\mathscr{K},w}$. Now, we show the inductive step by cases on the form of α.

CASE 1: α is of the form $c(\beta_1, \ldots, \beta_{\mathrm{ar}(c)})$. Put $\vec{\beta} = \beta_1, \ldots, \beta_{\mathrm{ar}(c)}$. By the hereditary, we have $\|\vec{\beta}\|^{\rho}_{\mathscr{K},w} \sqsubseteq \|\vec{\beta}\|^{\rho}_{\mathscr{K},v}$ for all $v \succeq w$. Hence, since c is monotonic, we have $\mathrm{t}_c(\|\vec{\beta}\|^{\rho}_{\mathscr{K},w}) \leq \mathrm{t}_c(\|\vec{\beta}\|^{\rho}_{\mathscr{K},v})$ for all $v \succeq w$, so that $\|\alpha\|^{\rho}_{\mathscr{K},w} = \mathrm{t}_c(\|\vec{\beta}\|^{\rho}_{\mathscr{K},w})$ holds. On the other hand, by the induction hypothesis, we have $\mathrm{t}_c(\|\vec{\beta}\|^{\rho}_{\mathscr{K},w}) = \mathrm{t}_c(\|\vec{\beta}\|^{\rho}_{\mathscr{M}_{\mathscr{K},w}}) = [\![\alpha]\!]^{\rho}_{\mathscr{M}_{\mathscr{K},w}}$.

CASE 2: α is of the form $\forall x\beta$. In this case, we have

$$\begin{aligned}
\|\alpha\|^{\rho}_{\mathscr{K},w} &= \min_{a\in D} \|\beta\|^{\rho[x\mapsto a]}_{\mathscr{K},w} \\
&= \min_{a\in D} [\![\beta]\!]^{\rho[x\mapsto a]}_{\mathscr{M}_{\mathscr{K},w}} \qquad \text{(by the induction hypothesis)} \\
&= [\![\alpha]\!]^{\rho}_{\mathscr{M}_{\mathscr{K},w}}.
\end{aligned}$$

CASE 3: α is of the form $\exists x\beta$. In this case, we have

$$\begin{aligned}
\|\alpha\|^{\rho}_{\mathscr{K},w} &= \max_{a\in D} \|\beta\|^{\rho[x\mapsto a]}_{\mathscr{K},w} \\
&= \max_{a\in D} [\![\beta]\!]^{\rho[x\mapsto a]}_{\mathscr{M}_{\mathscr{K},w}} \qquad \text{(by the induction hypothesis)} \\
&= [\![\alpha]\!]^{\rho}_{\mathscr{M}_{\mathscr{K},w}}.
\end{aligned}$$

□

3.5 Condition for $\mathrm{FOCD}(\mathscr{C}) = \mathrm{FOCL}(\mathscr{C})$ and $\mathrm{FOIL}(\mathscr{C}) = \mathrm{FOCL}(\mathscr{C})$

3.5.1 Condition for $\mathrm{FOIL}(\mathscr{C}) = \emptyset$, $\mathrm{FOCD}(\mathscr{C}) = \emptyset$ and $\mathrm{FOCL}(\mathscr{C}) = \emptyset$

Here, we show the following theorem, which claims that the condition for $\emptyset = \mathrm{IL}(\mathscr{C})$ and $\emptyset = \mathrm{CL}(\mathscr{C})$ in Theorem 2.4.1 is also a condition for $\emptyset = \mathrm{FOIL}(\mathscr{C})$, $\emptyset = \mathrm{FOCD}(\mathscr{C})$, and $\emptyset = \mathrm{FOCL}(\mathscr{C})$.

Theorem 3.5.1. *The following conditions are equivalent:*

(D) $\mathrm{FOCL}(\mathscr{C}) = \emptyset$;

(E) $\mathrm{FOCD}(\mathscr{C}) = \emptyset$;

(F) $\mathrm{FOIL}(\mathscr{C}) = \emptyset$;

(C) $\mathscr{C}$ *satisfies either of the following conditions:*

($\star$1) *for any* $c \in \mathscr{C}$, $\mathrm{t}_c(\mathbf{0}) = 0$;

($\star$2) *for any* $c \in \mathscr{C}$ *and any* $\mathbf{a} \in \{0,1\}^{\mathrm{ar}(c)}$, $\mathrm{t}_c(\mathbf{a}) \neq \mathrm{t}_c(\overline{\mathbf{a}})$.

(D) $\Longrightarrow$ (E) and (E) $\Longrightarrow$ (F) are obvious. (F) $\Longrightarrow$ (C) follows from (B) $\Longrightarrow$ (C) in Theorem 2.4.1. Thus, we only have to show (C) $\Longrightarrow$ (D). However, by (C) $\Longrightarrow$ (A) in Theorem 2.4.1, it suffices to show (A) $\Longrightarrow$ (D), which immediately follows from the following lemma:

Lemma 3.5.2. *Fix an arbitrary propositional variable* q. *For each* $\alpha \in \mathrm{FOFml}(\mathscr{C})$, *we define* $\alpha^\circ \in \mathrm{Fml}(\mathscr{C})$ *inductively as follows:*

- $(p(x_1, \ldots, x_n))^\circ \equiv q$;
- $(c(\alpha_1, \ldots, \alpha_{\mathrm{ar}(c)}))^\circ \equiv c(\alpha_1^\circ, \ldots, \alpha_{\mathrm{ar}(c)}^\circ)$;
- $(\forall x \alpha)^\circ \equiv \alpha^\circ$;
- $(\exists x \alpha)^\circ \equiv \alpha^\circ$.

Then, $\alpha^\circ \in \mathrm{CL}(\mathscr{C})$ *for any* $\alpha \in \mathrm{FOCL}(\mathscr{C})$.

Proof. Given a classical propositional model $\mathscr{M}$, we define a classical first-order model $\mathscr{M}^\circ = \langle D^\circ, I^\circ \rangle$ by

- $D^\circ = \{a\}$;
- $I^\circ(p)(a, \ldots, a) = \mathscr{M}(q)$.

Note that the variable assignment that assigns a to every variable is the only possible assignment. We denote this assignment by $\rho_{\mathscr{M}^\circ}$. Then, we can show the following claim by easy induction: for any $\alpha \in \mathrm{FOFml}(\mathscr{C})$, $[\![\alpha]\!]_{\mathscr{M}^\circ}^{\rho_{\mathscr{M}^\circ}} = [\![\alpha^\circ]\!]_{\mathscr{M}}$. From this we can easily see that $\alpha \in \mathrm{FOCL}(\mathscr{C})$ implies $\alpha^\circ \in \mathrm{CL}(\mathscr{C})$. □

3.5.2 Condition for $\mathrm{FOCD}(\mathscr{C}) = \mathrm{FOCL}(\mathscr{C})$ and $\mathrm{FOIL}(\mathscr{C}) = \mathrm{FOCL}(\mathscr{C})$

Here, we describe a theorem which gives necessary and sufficient conditions for $\mathrm{FOCD}(\mathscr{C}) = \mathrm{FOCL}(\mathscr{C})$ and $\mathrm{FOIL}(\mathscr{C}) = \mathrm{FOCL}(\mathscr{C})$. In order to describe the theorem, we establish some notations.

For a set of connective $\mathscr{C}$, we denote by $\mathrm{t}[\mathscr{C}]$ the set $\{\mathrm{t}_c \mid c \in \mathscr{C}\}$.

Definition 3.5.3. We define constant functions and projection functions as follows.

- For $n \geq 0$, we denote by 0^n the n-ary constant function of value 0 and by 1^n the constant function of value 1. That is, $0^n(x_1, \ldots, x_n) = 0$ and $1^n(x_1, \ldots, x_n) = 1$ for all $\langle x_1, \ldots, x_n \rangle \in \{0,1\}^n$.

- For $n \geq 1$ and $1 \leq j \leq n$, we denote by π^n_j the n-ary i-th projection function; that is, $\pi^n_j(x_1, \ldots, x_n) = x_j$ for all $\langle x_1, \ldots, x_n \rangle \in \{0,1\}^n$.

Furthermore, we denote by $\mathcal{B}$ the set of all constant functions, projection functions and compositions of $\mathrm{t}_\neg$ and projection functions. That is, $\mathcal{B} = \bigcup_{n \geq 0} B_n$, where $B_0 = \{0^0, 1^0\}$ and $B_n = \{0^n, 1^n\} \cup \{\pi^n_j \mid j = 1, \ldots, n\} \cup \{\mathrm{t}_\neg \circ \pi^n_j \mid j = 1, \ldots, n\}$ for $n \geq 1$.

Before the theorem, we give names for five conditions:

($\star$1) For any $c \in \mathscr{C}$, $\mathrm{t}_c(\mathbf{0}) = 0$.

($\star$2) For any $c \in \mathscr{C}$ and any $\mathbf{a} \in \{0,1\}^{\mathrm{ar}(c)}$, $\mathrm{t}_c(\mathbf{a}) \neq \mathrm{t}_c(\overline{\mathbf{a}})$.

(M) Any connective in $\mathscr{C}$ is monotonic.

($\sqcap$-$\sqcup$-$\sqsubseteq$) All of the following six conditions hold.

- ($\sqcap$-1) For any $c \in \mathscr{C}$ and any $\mathbf{a}, \mathbf{b} \in \{0,1\}^{\mathrm{ar}(c)}$, if $\mathrm{t}_c(\mathbf{a}) = \mathrm{t}_c(\mathbf{b}) = 1$, then $\mathrm{t}_c(\mathbf{a} \sqcap \mathbf{b}) = 1$.
- ($\sqcap$-0) For any $c \in \mathscr{C}$ and any $\mathbf{a}, \mathbf{b} \in \{0,1\}^{\mathrm{ar}(c)}$, if $\mathrm{t}_c(\mathbf{a}) = \mathrm{t}_c(\mathbf{b}) = 0$, then $\mathrm{t}_c(\mathbf{a} \sqcap \mathbf{b}) = 0$.
- ($\sqcup$-1) For any $c \in \mathscr{C}$ and any $\mathbf{a}, \mathbf{b} \in \{0,1\}^{\mathrm{ar}(c)}$, if $\mathrm{t}_c(\mathbf{a}) = \mathrm{t}_c(\mathbf{b}) = 1$, then $\mathrm{t}_c(\mathbf{a} \sqcup \mathbf{b}) = 1$.
- ($\sqcup$-0) For any $c \in \mathscr{C}$ and any $\mathbf{a}, \mathbf{b} \in \{0,1\}^{\mathrm{ar}(c)}$, if $\mathrm{t}_c(\mathbf{a}) = \mathrm{t}_c(\mathbf{b}) = 0$, then $\mathrm{t}_c(\mathbf{a} \sqcup \mathbf{b}) = 0$.
- ($\sqsubseteq$-1) For any $c \in \mathscr{C}$ and any $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \{0,1\}^{\mathrm{ar}(c)}$, if $\mathrm{t}_c(\mathbf{a}) = \mathrm{t}_c(\mathbf{b}) = 1$ and $\mathbf{a} \sqsubseteq \mathbf{c} \sqsubseteq \mathbf{b}$, then $\mathrm{t}_c(\mathbf{c}) = 1$.
- ($\sqsubseteq$-0) For any $c \in \mathscr{C}$ and any $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \{0,1\}^{\mathrm{ar}(c)}$, if $\mathrm{t}_c(\mathbf{a}) = \mathrm{t}_c(\mathbf{b}) = 0$ and $\mathbf{a} \sqsubseteq \mathbf{c} \sqsubseteq \mathbf{b}$, then $\mathrm{t}_c(\mathbf{c}) = 0$.

($\dagger$) $\mathrm{t}[\mathscr{C}] \subseteq \mathcal{B}$.

Note that, the first three conditions ($\star$1), ($\star$2), and (M) are already introduced in Theorem 2.5.1, and ($\sqcap$-$\sqcup$-$\sqsubseteq$) is a stronger condition than ($\sqcap$-$\sqsubseteq$-1) in Theorem 2.5.1 by ($\sqcap$-0), ($\sqcup$-1), ($\sqcup$-0), and ($\sqsubseteq$-0). Then, the theorem is described as follows:

Theorem 3.5.4. *The following conditions are equivalent:*

(I) $\mathrm{FOIL}(\mathscr{C}) = \mathrm{FOCL}(\mathscr{C})$*;*

(II) $\mathrm{FOCD}(\mathscr{C}) = \mathrm{FOCL}(\mathscr{C})$*;*

(III) $\mathscr{C}$ *satisfies either* ($\star$1), ($\star$2), (M), *or* ($\sqcap$-$\sqcup$-$\sqsubseteq$).

(IV) $\mathscr{C}$ *satisfies either* ($\star$1), ($\star$2), (M), *or* ($\dagger$).

First, we show that ($\sqcap$-$\sqcup$-$\sqsubseteq$) and ($\dagger$) are actually equivalent.

Lemma 3.5.5. *Let $\mathscr{C}$ be a set of connectives. Then, $\mathscr{C}$ satisfies* ($\dagger$) *if and only if $\mathscr{C}$ satisfies* ($\sqcap$-$\sqcup$-$\sqsubseteq$).

For the proof, we introduce a notation for unit sequences.

Definition 3.5.6. Let $n \geq 1$ and $1 \leq i \leq n$. Then, the *unit sequence* $\mathbf{e}^n_i$ is defined by

$$\mathbf{e}^n_i[j] = \begin{cases} 0 & \text{if } j \neq i \\ 1 & \text{if } j = i. \end{cases}$$

Proof of Lemma 3.5.5. (†) $\Longrightarrow$ ($\sqcap$-$\sqcup$-$\sqsubseteq$): We can straightforwardly verify that a connective c satisfies the six conditions of ($\sqcap$-$\sqcup$-$\sqsubseteq$) in each of the following four cases: $\mathrm{t}_c = 0^{\mathrm{ar}(c)}$, $\mathrm{t}_c = 1^{\mathrm{ar}(c)}$, $\mathrm{t}_c = \pi_j^{\mathrm{ar}(c)}$, and $\mathrm{t}_c = \mathrm{t}_\neg \circ \pi_j^{\mathrm{ar}(c)}$.

($\sqcap$-$\sqcup$-$\sqsubseteq$) $\Longrightarrow$ (†): Suppose ($\sqcap$-$\sqcup$-$\sqsubseteq$) holds. First, we consider the case where $\mathrm{t}_c(\mathbf{0}) = \mathrm{t}_c(\mathbf{1})$. In this case, by ($\sqsubseteq$-0) and ($\sqsubseteq$-1), we have $\mathrm{t}_c = 0^{\mathrm{ar}(c)}$ or $\mathrm{t}_c = 1^{\mathrm{ar}(c)}$, and hence, $\mathrm{t}_c \in \mathcal{B}$ holds. Secondly, we consider the case where $\mathrm{t}_c(\mathbf{0}) \neq \mathrm{t}_c(\mathbf{1})$. In this case, there is some $1 \leq i \leq \mathrm{ar}(c)$ such that $\mathrm{t}_c(\mathbf{e}_i^{\mathrm{ar}(c)}) = \mathrm{t}_c(\mathbf{1})$. For, if not, $\mathrm{t}_c(\mathbf{e}_i^{\mathrm{ar}(c)}) = \mathrm{t}_c(\mathbf{0})$ holds for all $1 \leq i \leq \mathrm{ar}(c)$, and hence, $\mathrm{t}_c(\mathbf{1}) = \mathrm{t}_c(\bigsqcup_{1 \leq i \leq \mathrm{ar}(c)} \mathbf{e}_i^{\mathrm{ar}(c)}) = \mathrm{t}_c(\mathbf{0})$ follows by ($\sqcup$-0) and ($\sqcup$-1), which contradicts the assumption $\mathrm{t}_c(\mathbf{1}) \neq \mathrm{t}_c(\mathbf{0})$. Furthermore, we can see $\mathrm{t}_c(\bigsqcup_{j \neq i} \mathbf{e}_j^{\mathrm{ar}(c)}) = \mathrm{t}_c(\mathbf{0})$ holds. For, if not, $\mathrm{t}_c(\bigsqcup_{j \neq i} \mathbf{e}_j^{\mathrm{ar}(c)}) = \mathrm{t}_c(\mathbf{1})$ holds, and hence, we have $\mathrm{t}_c(\mathbf{0}) = \mathrm{t}_c(\mathbf{e}_i^{\mathrm{ar}(c)} \sqcap \bigsqcup_{j \neq i} \mathbf{e}_j^{\mathrm{ar}(c)}) = \mathrm{t}_c(\mathbf{1})$ by ($\sqcap$-0) and ($\sqcap$-1), which contradicts the assumption $\mathrm{t}_c(\mathbf{0}) \neq \mathrm{t}_c(\mathbf{1})$. Now, by $\mathrm{t}_c(\mathbf{e}_i^{\mathrm{ar}(c)}) = \mathrm{t}_c(\mathbf{1})$, $\mathrm{t}_c(\bigsqcup_{j \neq i} \mathbf{e}_j^{\mathrm{ar}(c)}) = \mathrm{t}_c(\mathbf{0})$ and the conditions ($\sqsubseteq$-0) and ($\sqsubseteq$-1), we can see

- if $\mathbf{e}_i^{\mathrm{ar}(c)} \sqsubseteq \mathbf{x} \sqsubseteq \mathbf{1}$, that is, $\mathbf{x}[i] = 1$, then $\mathrm{t}_c(\mathbf{x}) = \mathrm{t}_c(\mathbf{1})$;
- if $\mathbf{0} \sqsubseteq \mathbf{x} \sqsubseteq \bigsqcup_{j \neq i} \mathbf{e}_j^{\mathrm{ar}(c)}$, that is, $\mathbf{x}[i] = 0$, then $\mathrm{t}_c(\mathbf{x}) = \mathrm{t}_c(\mathbf{0})$.

From these it follows that either $\mathrm{t}_c = \pi_i^{\mathrm{ar}(c)}$ or $\mathrm{t}_c = \mathrm{t}_\neg \circ \pi_i^{\mathrm{ar}(c)}$.

□

Now, let us begin to prove Theorem 3.5.4. It is clear that (I) immediately leads to (II). We prove (II) $\Longrightarrow$ (III) in § 3.5.3 and (IV) $\Longrightarrow$ (I) in § 3.5.4. So here we consider (III) $\Longrightarrow$ (IV). In order to prove (III) $\Longrightarrow$ (IV), it is sufficient to prove ($\sqcap$-$\sqcup$-$\sqsubseteq$) $\Longrightarrow$ (†). Actually, ($\sqcap$-$\sqcup$-$\sqsubseteq$) $\Longleftrightarrow$ (†) holds:

Here, for later use, we introduce the notion of each kind of equivalence between formulas.

Definition 3.5.7. Let $\mathscr{C}$ be a set of truth-functional connectives. Let $\alpha, \beta \in \mathrm{FOFml}(\mathscr{C})$.

- α is said to be *intuitionistically equivalent* to β (notation: $\alpha \sim_{\mathrm{in}} \beta$) if and only if for any Kripke model $\mathscr{K} = \langle W, \preceq, D, I \rangle$, any $w \in W$ and any assignment ρ in $D(w)$, $\|\alpha\|^{\rho}_{\mathscr{K},w} = \|\beta\|^{\rho}_{\mathscr{K},w}$ holds.
- α is said to be *CD equivalent* to β (notation: $\alpha \sim_{\mathrm{cd}} \beta$) if and only if for any constant domain Kripke model $\mathscr{K} = \langle W, \preceq, D, I \rangle$, any $w \in W$ and any assignment ρ in D, $\|\alpha\|^{\rho}_{\mathscr{K},w} = \|\beta\|^{\rho}_{\mathscr{K},w}$ holds.
- α is said to be *classically equivalent* to β (notation: $\alpha \sim_{\mathrm{cl}} \beta$) if and only if for any classical model $\mathscr{M} = \langle D, I \rangle$ and any assignment ρ in D, $[\![\alpha]\!]^{\rho}_{\mathscr{M}} = [\![\beta]\!]^{\rho}_{\mathscr{M}}$ holds.

Note that, $\alpha \sim_{\mathrm{in}} \beta$ implies $\alpha \sim_{\mathrm{cd}} \beta$, and $\alpha \sim_{\mathrm{cd}} \beta$ implies $\alpha \sim_{\mathrm{cl}} \beta$.

3.5.3 Proof of (II) $\Longrightarrow$ (III)

Here, we show the contraposition of (II) $\Longrightarrow$ (III), that is, if $\mathscr{C}$ satisfies neither ($\star$1), ($\star$2), (M) nor ($\sqcap$-$\sqcup$-$\sqsubseteq$), then $\mathrm{FOCD}(\mathscr{C}) \neq \mathrm{FOCL}(\mathscr{C})$. In this proof, we can borrow part of the proof of Theorem 2.5.1 with Lemma 3.2.4, and what we have to show is essentially the following lemma.

Lemma 3.5.8. *Suppose* $\mathscr{C}$ *does not satisfy* ($\sqcap$-$\sqcup$-$\sqsubseteq$). *Then* $\mathrm{FOCD}(\mathscr{C} \cup \{\top, \bot, \neg\}) \subsetneq \mathrm{FOCL}(\mathscr{C} \cup \{\top, \bot, \neg\})$.

Proof. Fix an arbitrary unary predicate symbol p. Let $\mathscr{K}^\triangledown = \langle W^\triangledown, \preceq^\triangledown, D^\triangledown, I^\triangledown\rangle$ be the constant domain Kripke model defined as follows:

- $W^\triangledown = \{w_0, w_1, w_2\}$;
- $\preceq^\triangledown = \{\langle w_0, w_0\rangle, \langle w_1, w_1\rangle, \langle w_2, w_2\rangle, \langle w_0, w_1\rangle, \langle w_0, w_2\rangle\}$;
- $D^\triangledown = \{a_1, a_2\}$;
- $I^\triangledown(w_i, p)(a_j) = 1 \iff i = j$.

Here, we establish one notation. For a formula φ and a truth value $z \in \{0,1\}$, we define a formula $\delta_z(\varphi)$ as follows:

$$\delta_z(\varphi) \equiv \begin{cases} \neg\varphi & \text{if } z = 0 \\ \varphi & \text{if } z = 1. \end{cases}$$

We divide into two cases according to which type of conditions fails: Case 1: ($\sqsubseteq$-0) or ($\sqsubseteq$-1) fails, and Case 2: ($\sqcap$-0), ($\sqcap$-1), ($\sqcup$-0) or ($\sqcup$-1) fails.

Case 1: ($\sqsubseteq$-0) or ($\sqsubseteq$-1) fails. Then, there are $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \{0,1\}^{\mathrm{ar}(c)}$ and $z \in \{0,1\}$ such that $\mathbf{a} \sqsubseteq \mathbf{c} \sqsubseteq \mathbf{b}$, $\mathrm{t}_c(\mathbf{a}) = \mathrm{t}_c(\mathbf{b}) = z$, and $\mathrm{t}_c(\mathbf{c}) = \overline{z}$. Define $\varphi_1, \dots, \varphi_{\mathrm{ar}(c)} \in \mathrm{FOFml}(\mathscr{C} \cup \{\top, \bot, \neg\})$ as follows

$$\varphi_i \equiv \begin{cases} \bot & \text{if } \mathbf{a}[i] = \mathbf{b}[i] = 0 \\ \top & \text{if } \mathbf{a}[i] = \mathbf{b}[i] = 1 \\ p(x) & \text{if } \mathbf{a}[i] \neq \mathbf{b}[i] \text{ and } \mathbf{c}[i] = 1 \\ \forall y p(y) & \text{if } \mathbf{a}[i] \neq \mathbf{b}[i] \text{ and } \mathbf{c}[i] = 0 \end{cases}$$

Then, put $\vec{\varphi} = \varphi_1, \dots, \varphi_{\mathrm{ar}(c)}$, and define $\varphi \in \mathrm{FOFml}(\mathscr{C} \cup \{\top, \bot, \neg\})$ by $\varphi \equiv \exists x \delta_z(c(\vec{\varphi}))$.

First, we show $\varphi \in \mathrm{FOCL}(\mathscr{C} \cup \{\top, \bot, \neg\})$. Let $\mathscr{M} = \langle D, I\rangle$ be a classical model. We show $\|\varphi\|^{\varnothing}_{\mathscr{M}} = 1$ by cases according to the value $\|\forall y p(y)\|_{\mathscr{M}}$.

Case (i): $[\![\forall y p(y)]\!]^{\varnothing}_{\mathscr{M}} = 1$. In this case, we can easily see that for all $a \in D$, $[\![\vec{\varphi}]\!]^{\varnothing[x\mapsto a]}_{\mathscr{M}} = \mathbf{b}$. Hence, for all $a \in D$, $[\![c(\vec{\varphi})]\!]^{\varnothing[x\mapsto a]}_{\mathscr{M}} = \mathrm{t}_c(\mathbf{b}) = z$ and $[\![\delta_z(c(\vec{\varphi}))]\!]^{\varnothing[x\mapsto a]}_{\mathscr{M}} = 1$. Thus, we have $[\![\varphi]\!]^{\varnothing}_{\mathscr{M}} = 1$.

Case (ii): $[\![\forall y p(y)]\!]^{\varnothing}_{\mathscr{M}} = 0$. In this case, there exists some $a \in D$ such that $[\![p(y)]\!]^{\varnothing[x\mapsto a]}_{\mathscr{M}} = 0$, and we can easily see that $[\![\vec{\varphi}]\!]^{\varnothing[x\mapsto a]}_{\mathscr{M}} = \mathbf{a}$. Hence, $[\![c(\vec{\varphi})]\!]^{\varnothing[x\mapsto a]}_{\mathscr{M}} = \mathrm{t}_c(\mathbf{a}) = z$ and $[\![\delta_z(c(\vec{\varphi}))]\!]^{\varnothing[x\mapsto a]}_{\mathscr{M}} = 1$. Thus, we have $[\![\varphi]\!]^{\varnothing}_{\mathscr{M}} = 1$.

Secondly, we show $\varphi \notin \mathrm{FOCD}(\mathscr{C} \cup \{\top, \bot, \neg\})$ by showing $\|\varphi\|^{\varnothing}_{\mathscr{K}^\triangledown, w_0} = 0$. It can be easily verified that $\|\vec{\varphi}\|^{\varnothing[x\mapsto a_1]}_{\mathscr{K}^\triangledown, w_1} = \|\vec{\varphi}\|^{\varnothing[x\mapsto a_2]}_{\mathscr{K}^\triangledown, w_2} = \mathbf{c}$. Hence, we have $\|c(\vec{\varphi})\|^{\varnothing[x\mapsto a_1]}_{\mathscr{K}^\triangledown, w_1} = \|c(\vec{\varphi})\|^{\varnothing[x\mapsto a_2]}_{\mathscr{K}^\triangledown, w_2} = \mathrm{t}_c(\mathbf{c}) = \overline{z}$. Then, it follows that $\|\delta_z(c(\vec{\varphi}))\|^{\varnothing[x\mapsto a_1]}_{\mathscr{K}^\triangledown, w_0} = \|\delta_z(c(\vec{\varphi}))\|^{\varnothing[x\mapsto a_2]}_{\mathscr{K}^\triangledown, w_0} = 0$. Therefore, we have $\|\varphi\|^{\varnothing}_{\mathscr{K}^\triangledown, w_0} = \|\exists x \delta_z(c(\vec{\varphi}))\|^{\varnothing}_{\mathscr{K}^\triangledown, w_0} = 0$.

Case 2: ($\sqcap$-0), ($\sqcap$-1), ($\sqcup$-0), or ($\sqcup$-1) fails. Then, there are $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \{0,1\}^{\mathrm{ar}(c)}$ and $z \in \{0,1\}$ such that either $\mathbf{c} = \mathbf{a} \sqcap \mathbf{b}$ or $\mathbf{c} = \mathbf{a} \sqcup \mathbf{b}$; $\mathrm{t}_c(\mathbf{a}) = \mathrm{t}_c(\mathbf{b}) = z$; and $\mathrm{t}_c(\mathbf{c}) = \overline{z}$. We define $u \in \{0,1\}$ by

$$u = \begin{cases} 1 & \text{if } \mathbf{c} = \mathbf{a} \sqcap \mathbf{b} \\ 0 & \text{if } \mathbf{c} = \mathbf{a} \sqcup \mathbf{b}. \end{cases}$$

Define $\varphi_1, \ldots, \varphi_{\mathrm{ar}(c)} \in \mathrm{FOFml}(\mathscr{C} \cup \{\top, \bot, \neg\})$ as follows.

$$\varphi_i \equiv \begin{cases} \bot & \text{if } \mathbf{a}[i] = \mathbf{b}[i] = 0 \\ \top & \text{if } \mathbf{a}[i] = \mathbf{b}[i] = 1 \\ \delta_{\overline{u}}(p(x)) & \text{if } \mathbf{a}[i] = 0 \text{ and } \mathbf{b}[i] = 1 \\ \delta_u(\forall y p(y)) & \text{if } \mathbf{a}[i] = 1 \text{ and } \mathbf{b}[i] = 0. \end{cases}$$

Then, put $\vec{\varphi} = \varphi_1, \ldots, \varphi_{\mathrm{ar}(c)}$, and define $\varphi \in \mathrm{FOFml}(\mathscr{C} \cup \{\top, \bot, \neg\})$ by $\varphi \equiv \exists x \delta_z(c(\vec{\varphi}))$.

First, we show $\varphi \in \mathrm{FOCL}(\mathscr{C} \cup \{\top, \bot, \neg\})$. Let $\mathscr{M} = \langle D, I\rangle$ be a classical model. We show $[\![\varphi]\!]^{\varnothing}_{\mathscr{M}} = 1$ by cases according to the value $[\![\forall y p(y)]\!]_{\mathscr{M}}$.

Case (i): $[\![\forall y p(y)]\!]^{\varnothing}_{\mathscr{M}} = 1$. In this case, we can easily see that for all $a \in D$,

$$[\![\vec{\varphi}]\!]^{\varnothing[x\mapsto a]}_{\mathscr{M}} = \begin{cases} \mathbf{a} & \text{if } u = 1 \\ \mathbf{b} & \text{if } u = 0. \end{cases}$$

Hence, for all $a \in D$, we have $[\![c(\vec{\varphi})]\!]^{\varnothing[x\mapsto a]}_{\mathscr{M}} = z$, and $[\![\delta_z c(\vec{\varphi})]\!]^{\varnothing[x\mapsto a]}_{\mathscr{M}} = 1$. Thus, we have $[\![\varphi]\!]^{\varnothing}_{\mathscr{M}} = 1$.

Case (ii): $[\![\forall y p(y)]\!]^{\varnothing}_{\mathscr{M}} = 0$. In this case, there exists some $a \in D$ such that $[\![p(x)]\!]^{\varnothing[x\mapsto a]}_{\mathscr{M}} = 0$. We can easily see that

$$[\![\vec{\varphi}]\!]^{\varnothing[x\mapsto a]}_{\mathscr{M}} = \begin{cases} \mathbf{b} & \text{if } u = 1 \\ \mathbf{a} & \text{if } u = 0. \end{cases}$$

Hence, we have $[\![c(\vec{\varphi})]\!]^{\varnothing[x\mapsto a]}_{\mathscr{M}} = z$, and $[\![\delta_z(c(\vec{\varphi}))]\!]^{\varnothing[x\mapsto a]}_{\mathscr{M}} = 1$. Thus, we have $[\![\varphi]\!]^{\varnothing}_{\mathscr{M}} = 1$.

Secondly, we show $\varphi \notin \mathrm{FOCD}(\mathscr{C} \cup \{\top, \bot, \neg\})$ by showing $\|\varphi\|^{\varnothing}_{\mathscr{K}^{\triangledown}, w_0} = 0$. We can easily verify that, for each $i = 1, 2$,

$$\|\vec{\varphi}\|^{\varnothing[x\mapsto a_i]}_{w_i} = \begin{cases} \mathbf{a} \sqcap \mathbf{b} & \text{if } u = 1 \\ \mathbf{a} \sqcup \mathbf{b} & \text{if } u = 0, \end{cases}$$

that is, $\|\vec{\varphi}\|^{\varnothing[x\mapsto a_i]}_{w_i} = \mathbf{c}$ for $i = 1, 2$. Hence, we have $\|c(\vec{\varphi})\|^{\varnothing[x\mapsto a_1]}_{\mathscr{K}^{\triangledown}, w_1} = \|c(\varphi)\|^{\varnothing[x\mapsto a_2]}_{\mathscr{K}^{\triangledown}, w_2} = \mathrm{t}_c(\mathbf{c}) = \overline{z}$. Then, it follows that $\|\delta_z(c(\vec{\varphi}))\|^{\varnothing[x\mapsto a_1]}_{\mathscr{K}^{\triangledown}, w_0} = \|\delta_z(c(\vec{\varphi}))\|^{\varnothing[x\mapsto a_2]}_{\mathscr{K}^{\triangledown}, w_0} = 0$. Therefore, we have $\|\varphi\|^{\varnothing}_{\mathscr{K}^{\triangledown}, w_0} = \|\exists x \delta_z(c(\vec{\varphi}))\|^{\varnothing}_{\mathscr{K}^{\triangledown}, w_0} = 0$.

□

Next, we prepare one easy lemma for the proof of (II) $\Longrightarrow$ (III).

Lemma 3.5.9. *Suppose* $\top' \in \mathrm{FOIL}(\mathscr{C})$. *Suppose a formula* $\neg'_c\alpha \in \mathrm{FOFml}(\mathscr{C})$ *is assigned to each* $\alpha \in \mathrm{FOFml}(\mathscr{C})$ *and they satisfy the relation* $\neg'_c\alpha \sim_{\mathrm{in}} \neg\alpha$*; that is, for any Kripke model* $\mathscr{K} = \langle W, \preceq, D, I\rangle$*, any* $w \in W$ *and any assignment* ρ *in* $D(w)$*,* $\|\neg'_c\alpha\|^{\rho}_{\mathscr{K}, w} = 1$ *if and only if* $\|\alpha\|^{\rho}_{\mathscr{K}, v} = 0$ *for all* $v \succeq w$*. For each* $\varphi \in \mathrm{FOFml}(\mathscr{C} \cup \{\top, \bot, \neg\})$*, define* $\varphi^* \in \mathrm{FOFml}(\mathscr{C})$ *as* *follows.*

$$\begin{aligned}
\top^* &\equiv \top' \\
\bot^* &\equiv \neg'_c\top' \\
p(x_1,\dots,x_n)^* &\equiv p(x_1,\dots,x_n) \\
(\neg\alpha)^* &\equiv \neg'_c\alpha^* \\
c(\alpha_1,\dots,\alpha_n)^* &\equiv c(\alpha_1^*,\dots,\alpha_n^*) && \textit{for } c \in \mathscr{C}\setminus\{\neg\} \\
(\mathcal{Q}x\alpha)^* &\equiv \mathcal{Q}x\alpha^* && \textit{for } \mathcal{Q}\in\{\forall,\exists\}
\end{aligned}$$

Then, $\varphi^* \sim_{\mathrm{in}} \varphi$ *for all* $\varphi \in \mathrm{FOFml}(\mathscr{C}\cup\{\top,\bot,\neg\})$*.*

Proof. Obvious. ◻

Now, we return to the main proof, the proof of (II) $\Longrightarrow$ (III).

Proof of (II) $\Longrightarrow$ (III). We suppose $\mathscr{C}$ satisfies neither ($\star$1), ($\star$2), (M) nor ($\sqcap$-$\sqcup$-$\sqsubseteq$), and show $\mathrm{FOCD}(\mathscr{C}) \subsetneq \mathrm{FOCL}(\mathscr{C})$. By Theorem 2.4.1, $\mathrm{IL}(\mathscr{C})$ is non-empty. Fix an arbitrary $\top' \in \mathrm{IL}(\mathscr{C}) \subseteq \mathrm{FOIL}(\mathscr{C})$. Since $\mathscr{C}$ does not satisfy (M), there is some non-monotonic connective c in $\mathscr{C}$. If $\mathrm{t}_c(\mathbf{1}) = 1$, then by Lemma 2.3.5, $\mathrm{IL}(\mathscr{C}) \subsetneq \mathrm{CL}(\mathscr{C})$ holds, and hence, $\mathrm{FOCD}(\mathscr{C}) \subsetneq \mathrm{FOCL}(\mathscr{C})$ holds by Lemma 3.2.4. Now, we consider the case $\mathrm{t}_c(\mathbf{1}) = 0$. In this case, $\neg$ is definable by using c as in Lemma 2.5.6. That is, for each $\alpha \in \mathrm{FOFml}(\mathscr{C})$, there is a formula $\neg'_c\alpha \in \mathrm{FOFml}(\mathscr{C})$ such that $\neg'_c\alpha \sim_{\mathrm{in}} \neg\alpha$.

By Lemma 3.5.8, there is some $\varphi \in \mathrm{FOCL}(\mathscr{C}\cup\{\top,\bot,\neg\}) \setminus \mathrm{FOCD}(\mathscr{C}\cup\{\top,\bot,\neg\})$. Let $\varphi^* \in \mathrm{FOFml}(\mathscr{C})$ be as defined in Lemma 3.5.9. Then, by Lemma 3.5.9, $\varphi^* \in \mathrm{FOCL}(\mathscr{C}) \setminus \mathrm{FOCD}(\mathscr{C})$ follows from $\varphi \in \mathrm{FOCL}(\mathscr{C}\cup\{\top,\bot,\neg\}) \setminus \mathrm{FOCD}(\mathscr{C}\cup\{\top,\bot,\neg\})$. ◻

3.5.4 Proof of (IV) $\Longrightarrow$ (I)

In order to prove (IV) $\Longrightarrow$ (I), it suffices to show the following four implications: ($\star$1) $\Longrightarrow$ (I), ($\star$2) $\Longrightarrow$ (I), (M) $\Longrightarrow$ (I), and (†) $\Longrightarrow$ (I). Since the first two hold by Theorem 3.5.1, we show the last two. We show (M) $\Longrightarrow$ (I) in 3.5.4.1 and (†) $\Longrightarrow$ (I) in 3.5.4.2.

3.5.4.1 (M) $\Longrightarrow$ (I)

In the case of propositional logic, we have already shown that (M) implies $\mathrm{IL}(\mathscr{C}) = \mathrm{CL}(\mathscr{C})$ (cf. Theorem 2.5.1). We reduce (M) $\Longrightarrow$ (I) to this propositional counterpart by the lemma below, Lemma 3.5.11 .

Definition 3.5.10. For each formula $\alpha \in \mathrm{FOFml}(\mathscr{C})$, we define a propositional formula $\alpha^{\mathrm{B}} \in \mathrm{Fml}(\mathscr{C}\cup\{\bot\})$ as follows:

- $(p(x_1,\dots,x_n))^{\mathrm{B}} \equiv \bot$;
- $(c(\alpha_1,\dots,\alpha_{\mathrm{ar}(c)}))^{\mathrm{B}} \equiv c(\alpha_1^{\mathrm{B}},\dots,\alpha_{\mathrm{ar}(c)}^{\mathrm{B}})$;
- $(\forall x\alpha)^{\mathrm{B}} \equiv \alpha^{\mathrm{B}}$;
- $(\exists x\alpha)^{\mathrm{B}} \equiv \alpha^{\mathrm{B}}$.

Lemma 3.5.11. *If* $\mathscr{C}$ *satisfies* (M)*, that is, all* $c \in \mathscr{C}$ *are monotonic, then the following hold.*

(i) *For any* $\alpha \in \mathrm{FOFml}(\mathscr{C})$*,* $\alpha \in \mathrm{FOIL}(\mathscr{C})$ *if and only if* $\alpha^{\mathrm{B}} \in \mathrm{IL}(\mathscr{C}\cup\{\bot\})$*.*

(ii) *For any* $\alpha \in \mathrm{FOFml}(\mathscr{C})$*,* $\alpha \in \mathrm{FOCL}(\mathscr{C})$ *if and only if* $\alpha^{\mathrm{B}} \in \mathrm{CL}(\mathscr{C} \cup \{\bot\})$*.*

Proof. Suppose $\mathscr{C}$ satisfies (M). Then, since $\mathscr{C} \cup \{\bot\}$ also satisfies (M), $\mathrm{IL}(\mathscr{C} \cup \{\bot\}) = \mathrm{CL}(\mathscr{C} \cup \{\bot\})$ holds by Theorem 2.5.1. By $\mathrm{IL}(\mathscr{C} \cup \{\bot\}) = \mathrm{CL}(\mathscr{C} \cup \{\bot\})$ and $\mathrm{FOIL}(\mathscr{C}) \subseteq \mathrm{FOCL}(\mathscr{C})$, we can see that the "if" part of (ii) immediately follows from that of (i), and the "only if" part of (i) immediately follows from that of (ii). Thus, we only prove the "if" part of (i) and the "only if" part of (ii).

First, we can easily show the following claim by induction on α: for any $\alpha \in \mathrm{FOFml}(\mathscr{C})$, any Kripke model $\mathscr{K} = \langle W, \preceq, D, I\rangle$, any $w, v \in V$ and any assignment ρ in $D(w)$, it holds that $\|\alpha\|^{\rho}_{\mathscr{K},w} \geq \|\alpha^{\mathrm{B}}\|^{\varnothing}_{\mathscr{K},v}$. By this claim, the "if" part of (i) immediately follows.

Now, we show the "only if" part of (ii). Suppose $\alpha^{\mathrm{B}} \notin \mathrm{CL}(\mathscr{C} \cup \{\bot\})$, in order to show $\alpha \notin \mathrm{FOCL}(\mathscr{C} \cup \{\bot\})$. Then, since the value $[\![\alpha^{\mathrm{B}}]\!]^{\varnothing}_{\mathscr{M}}$ in a classical model $\mathscr{M}$ does not depend on $\mathscr{M}$ due to the definition of α^{B}, $[\![\alpha^{\mathrm{B}}]\!]^{\varnothing}_{\mathscr{M}_\star} = 0$ for the classical model $\mathscr{M}_\star = \langle D_\star, I_\star\rangle$ defined by

- $D_\star = \{a\}$;
- $I_\star(p)(a, \ldots, a) = 0$ for any predicate symbol p.

Note that the variable assignment that assigns a to every variable is the only possible assignment. We denote this assignment by $\rho_\star$. We can easily see that $[\![\alpha]\!]^{\rho_\star}_{\mathscr{M}_\star} = [\![\alpha^{\mathrm{B}}]\!]^{\varnothing}_{\mathscr{M}_\star}$. Hence, $\alpha \notin \mathrm{FOCL}(\mathscr{C})$. □

Now, we return to the proof of (M) $\Longrightarrow$ (I).

Proof of (M) $\Longrightarrow$ (I). Suppose all $c \in \mathscr{C}$ are monotonic. For the sake of contradiction, suppose (I) fails, that is, $\mathrm{FOIL}(\mathscr{C}) \neq \mathrm{FOCL}(\mathscr{C})$. Then, there exists some $\alpha \in \mathrm{FOCL}(\mathscr{C}) \setminus \mathrm{FOIL}(\mathscr{C})$. By Lemma 3.5.11, $\alpha^{\mathrm{B}} \in \mathrm{CL}(\mathscr{C} \cup \{\bot\}) \setminus \mathrm{IL}(\mathscr{C} \cup \{\bot\})$. However, since all connectives in $\mathscr{C}$, and hence all connectives in $\mathscr{C} \cup \{\bot\}$, are monotonic, $\mathrm{IL}(\mathscr{C} \cup \{\bot\}) = \mathrm{CL}(\mathscr{C} \cup \{\bot\})$ holds by Theorem 2.5.1. Thus, we have a contradiction. □

3.5.4.2 (†) $\Longrightarrow$ (I)

Here, we prove the following proposition:

Proposition 3.5.12. *If* $\mathscr{C}$ *satisfies* (†)*, that is,* $\mathrm{t}[\mathscr{C}] \subseteq \mathcal{B}$*, then* $\mathrm{FOIL}(\mathscr{C}) = \mathrm{FOCL}(\mathscr{C})$*.*

We prove this proposition by proving a stronger claim, Lemma 3.5.16, which shows the equality between the sets of unsatisfiable formulas in addition to that between the sets of valid formulas. Recall that $\mathrm{DFOIL}(\mathscr{C})$ and $\mathrm{DFOCL}(\mathscr{C})$ denote the sets of Kripke-unsatisfiable formulas and that of classically unsatisfieble formulas, respectively (cf. §§ 3.2.2-3.2.3). Before the proof of Lemma 3.5.16, we prepare some lemmas.

Lemma 3.5.13. *Suppose a set* $\mathscr{C}$ *of connectives satisfies* (†)*; that is,* $\mathrm{t}[\mathscr{C}] \subseteq \mathcal{B}$*. Let* $\alpha \in \mathrm{FOFml}(\mathscr{C})$*. Then, there exists some* $\varphi_\alpha \in \mathrm{FOFml}(\{\top, \bot, \neg\})$ *such that*

- $\varphi_\alpha \sim_{\mathrm{cl}} \alpha$*;*
- φ_α *is one of the following forms:*
  (i) $\varphi_\alpha \equiv \mathcal{Q}_1 y_1 \cdots \mathcal{Q}_m y_m p(x_1, \ldots, x_n)$*, where* $\mathcal{Q}_1, \ldots, \mathcal{Q}_m$ *are quantifiers;*
  (ii) $\varphi_\alpha \equiv \mathcal{Q}_1 y_1 \cdots \mathcal{Q}_m y_m \neg p(x_1, \ldots, x_n)$*, where* $\mathcal{Q}_1, \ldots, \mathcal{Q}_m$ *are quantifiers;*
  (iii) $\varphi_\alpha \equiv \top$*;*
  (iv) $\varphi_\alpha \equiv \bot$*.*

Proof. The proof proceeds by induction on α.

Case 1: α is atomic. Then, we can take φ_α of the form (i) with $m = 0$.

Case 2: $\alpha \equiv c(\beta_1, \dots, \beta_k)$ with $\mathrm{t}_c = 1^{\mathrm{ar}(c)}$. In this case, we can take φ_α of the form (iii).

Case 3: $\alpha \equiv c(\beta_1, \dots, \beta_k)$ with $\mathrm{t}_c = 0^{\mathrm{ar}(c)}$. In this case, we can take φ_α of the form (iv).

Case 4: $\alpha \equiv c(\beta_1, \dots, \beta_k)$ with $\mathrm{t}_c = \pi_i^{\mathrm{ar}(c)}$. Then, we have $\alpha \sim_{\mathrm{cl}} \beta_i$, and so, we only have to apply the induction hypothesis for β_i.

Case 5: $\alpha \equiv c(\beta_1, \dots, \beta_k)$ with $\mathrm{t}_c = \mathrm{t}_\neg \circ \pi_i^{\mathrm{ar}(c)}$. Then, we have $\alpha \sim_{\mathrm{cl}} \neg\beta_i$. By the induction hypothesis for β_i, there exists $\varphi_\beta \in \mathrm{FOFml}(\{\top, \bot, \neg\})$ such that $\varphi_\beta \sim_{\mathrm{cl}} \beta_i$ and φ_β is one of the four forms (i), (ii), (iii), and (iv). We divide into cases according to the form of φ_β.

In case (iii), where $\varphi_\beta \equiv \bot$, we can take $\top$ as φ_α. In case (iv), where $\varphi_\beta \equiv \top$, we can take $\bot$ as φ_α. In case (i), where $\varphi \equiv \mathcal{Q}_1 y_1 \cdots \mathcal{Q}_m y_m p(x_1, \dots, x_m)$, we have $\alpha \sim_{\mathrm{cl}} \neg\beta_i \sim_{\mathrm{cl}} \mathcal{Q}'_1 y_1 \cdots \mathcal{Q}'_m y_m \neg p(x_1, \dots, x_n)$, where $\mathcal{Q}'_j = \forall$ if $\mathcal{Q}_j = \exists$ and $\mathcal{Q}'_j = \exists$ if $\mathcal{Q}_j = \forall$. Hence, we can take $\mathcal{Q}'_1 y_1 \cdots \mathcal{Q}'_m y_m \neg p(x_1, \dots, x_n)$ as φ_α. Case (ii) can be treated in the same way as case (i) except that, in addition, we use $\neg\neg p(x_1, \dots, x_n) \sim_{\mathrm{cl}} p(x_1, \dots, x_n)$.

Case 6: $\alpha \equiv \exists x\beta$ or $\alpha \equiv \forall x\beta$. Apply the induction hypothesis for β. Note that $\forall x\bot \sim_{\mathrm{cl}} \exists x\bot \sim_{\mathrm{cl}} \bot$ and $\forall x\top \sim_{\mathrm{cl}} \exists x\top \sim_{\mathrm{cl}} \top$.

□

Lemma 3.5.14. *Let $\mathcal{Q}_1, \dots, \mathcal{Q}_m$ be quantifiers.*

(i) *$\mathcal{Q}_1 y_1 \cdots \mathcal{Q}_m y_m p(x_1, \dots, x_n)$ is neither classically valid nor classically unsatisfiable.*

(ii) *$\mathcal{Q}_1 y_1 \cdots \mathcal{Q}_m y_m \neg p(x_1, \dots, x_n)$ is neither classically valid nor classically unsatisfiable.*

Proof. Let $\mathscr{M}_1 = \langle \{a\}, J_1 \rangle$ be an one-point classical model with $J_1(p)(a, \dots, a) = 0$ and $\mathscr{M}_2 = \langle \{a\}, J_2 \rangle$ be an one-point classical model with $J_2(p)(a, \dots, a) = 0$. Let ρ be the assignment in $\{a\}$ defined by $\rho(x) = a$ for all x. Then, we have

$$[\![\mathcal{Q}_1 y_1 \cdots \mathcal{Q}_m y_m p(x_1, \dots, x_n)]\!]^\rho_{\mathscr{M}_1} = [\![\mathcal{Q}_1 y_1 \cdots \mathcal{Q}_m y_m \neg p(x_1, \dots, x_n)]\!]^\rho_{\mathscr{M}_2} = 0$$

and

$$[\![\mathcal{Q}_1 y_1 \cdots \mathcal{Q}_m y_m p(x_1, \dots, x_n)]\!]^\rho_{\mathscr{M}_2} = [\![\mathcal{Q}_1 y_1 \cdots \mathcal{Q}_m y_m \neg p(x_1, \dots, x_n)]\!]^\rho_{\mathscr{M}_2} = 1.$$

□

Lemma 3.5.15. *Suppose $\mathrm{t}[\mathscr{C}] \subseteq \mathcal{B}$. Then, the following hold:*

(a) *If $\exists x\alpha \in \mathrm{FOCL}(\mathscr{C})$, then $\alpha \in \mathrm{FOCL}(\mathscr{C})$.*

(b) *If $\forall x\alpha \in \mathrm{DFOCL}(\mathscr{C})$, then $\alpha \in \mathrm{DFOCL}(\mathscr{C})$.*

Proof. (a): Suppose $\exists x\alpha \in \mathrm{FOCL}(\mathscr{C})$. By Lemma 3.5.13, there exists a $\varphi_\alpha \in \mathrm{FOFml}(\{\top, \bot, \neg\})$ such that $\alpha \sim_{\mathrm{cl}} \varphi_\alpha$ and φ_α is one of the four forms (i), (ii), (iii), and (iv) in Lemma 3.5.13. However, none of (i), (ii), and (iv) is possible. For, in case (i) and case (ii), we have $\exists x\alpha \notin \mathrm{FOCL}(\mathscr{C})$ by Lemma 3.5.14, and in case (iv), we have $\exists x\alpha \notin \mathrm{FOCL}(\mathscr{C})$ by $\exists x\alpha \sim_{\mathrm{cl}} \exists x\bot \sim_{\mathrm{cl}} \bot$. Thus, φ_α is of the form (iii), and hence, $\alpha \sim_{\mathrm{cl}} \top$, from which $\alpha \in \mathrm{FOCL}(\mathscr{C})$ follows immediately.

(b): This case can be shown similarly to case (a).

□

Lemma 3.5.16. *Suppose* $\mathrm{t}[\mathscr{C}] \subseteq \mathcal{B}$*. Then, the following hold:*

(a) *If* $\alpha \in \mathrm{FOCL}(\mathscr{C})$ *then* $\alpha \in \mathrm{FOIL}(\mathscr{C})$*.*

(b) *If* $\alpha \in \mathrm{DFOCL}(\mathscr{C})$ *then* $\alpha \in \mathrm{DFOIL}(\mathscr{C})$*.*

Proof. We prove (a) and (b) simultaneously by induction on α.

Case 1: $\alpha \equiv p(x_1, \ldots, x_n)$. In this case, the claims hold vacuously.

Case 2: $\alpha \equiv \exists x\beta$.

(a) Suppose $\alpha \in \mathrm{FOCL}(\mathscr{C})$. Then, by Lemma 3.5.15, we have $\beta \in \mathrm{FOCL}(\mathscr{C})$. By the induction hypothesis, we have $\beta \in \mathrm{FOIL}(\mathscr{C})$, from which it easily follows that $\alpha \equiv \exists x\beta \in \mathrm{FOIL}(\mathscr{C})$.

(b) Suppose $\alpha \in \mathrm{DFOCL}(\mathscr{C})$. Then, it easily follows that $\beta \in \mathrm{DFOCL}(\mathscr{C})$. By the induction hypothesis, we have $\beta \in \mathrm{DFOIL}(\mathscr{C})$, from which it easily follows that $\alpha \equiv \exists x\beta \in \mathrm{DFOIL}(\mathscr{C})$.

Case 3: $\alpha \equiv \forall x\beta$. This case can be shown similarly to case 2.

Case 4: $\alpha \equiv c(\beta_1, \ldots, \beta_{\mathrm{ar}(c)})$. Since $\mathrm{t}[\mathscr{C}] \subseteq \mathcal{B}$, there are four cases: $\mathrm{t}_c = 0^{\mathrm{ar}(c)}$, $\mathrm{t}_c = 1^{\mathrm{ar}(c)}$, $\mathrm{t}_c = \pi_j^{\mathrm{ar}(c)}$ and $\mathrm{t}_c = \mathrm{t}_\neg \circ \pi_j^{\mathrm{ar}(c)}$, where $j \in \{1, \ldots, \mathrm{ar}(c)\}$. Since the first two cases are easy, we show the last two cases.

Subcase 4-1: $\mathrm{t}_c = \pi_j^{\mathrm{ar}(c)}$. In this case, we can easily see that the following hold.

- If $\alpha \in \mathrm{FOCL}(\mathscr{C})$ then $\beta_j \in \mathrm{FOCL}(\mathscr{C})$.
- If $\alpha \in \mathrm{DFOCL}(\mathscr{C})$ then $\beta_j \in \mathrm{DFOCL}(\mathscr{C})$.
- If $\beta_j \in \mathrm{FOIL}(\mathscr{C})$ then $\alpha \in \mathrm{FOIL}(\mathscr{C})$.
- If $\beta_j \in \mathrm{DFOIL}(\mathscr{C})$ then $\alpha \in \mathrm{DFOIL}(\mathscr{C})$.

For example, the last two hold because, for any Kripke model $\mathscr{K} = \langle W, \preceq, D, I\rangle$, any $w \in W$ and any assignment ρ in $D(w)$, $\|\alpha\|^{\rho}_{\mathscr{K},w} = \min_{v \succeq w} \|\beta_j\|^{\rho}_{\mathscr{K},v} (= \|\beta_j\|^{\rho}_{\mathscr{K},w})$ holds. Then, claims (a) and (b) can be shown by simply using these four properties and the induction hypothesis for β_j.

Subcase 4-2: $\mathrm{t}_c = \mathrm{t}_\neg \circ \pi_j^{\mathrm{ar}(c)}$. In this case, we can easily see that the following hold.

- If $\alpha \in \mathrm{FOCL}(\mathscr{C})$ then $\beta_j \in \mathrm{DFOCL}(\mathscr{C})$.
- If $\alpha \in \mathrm{DFOCL}(\mathscr{C})$ then $\beta_j \in \mathrm{FOCL}(\mathscr{C})$.
- If $\beta_j \in \mathrm{FOIL}(\mathscr{C})$ then $\alpha \in \mathrm{DFOIL}(\mathscr{C})$.
- If $\beta_j \in \mathrm{DFOIL}(\mathscr{C})$ then $\alpha \in \mathrm{FOIL}(\mathscr{C})$.

For example, the last two hold because, for any Kripke model $\mathscr{K} = \langle W, \preceq, D, I\rangle$, any $w \in W$ and any assignment ρ in $D(w)$, $\|\alpha\|^{\rho}_{\mathscr{K},w} = \min_{v \succeq w} \mathrm{t}_\neg(\|\beta_j\|^{\rho}_{\mathscr{K},v})$ holds. Then, claims (a) and (b) can be shown by simply using these four properties and the induction hypothesis for β_j.

□

Chapter 4

Conclusion and future work

We have analyzed how the relations between two propositional logics and the relations between the three first-order logics change depending on the choice of connectives. The following table summarizes the results concerning equalities between logics:

Table 4.1: Necessary and sufficient conditions for equalities between logics

	Propositional logic	First-order logic		
	IL and CL	FOIL and FOCD	FOCD and FOCL	FOIL and FOCL
Sequent	(M)	($\sqcap$-1)	(M)	(M) & ($\sqcap$-1)
Formula	($\star$1) or ($\star$2) or (M) or ($\sqcap$-$\sqsubseteq$-1)	Open	($\star$1) or ($\star$2) or (M) or ($\sqcap$-$\sqcup$-$\sqsubseteq$)	($\star$1) or ($\star$2) or (M) or ($\sqcap$-$\sqcup$-$\sqsubseteq$)

The conditions were defined as follows.

($\star$1) For any $c \in \mathscr{C}$, $\mathrm{t}_c(\mathbf{0}) = 0$.

($\star$2) For any $c \in \mathscr{C}$ and any $\mathbf{a} \in \{0,1\}^{\mathrm{ar}(c)}$, $\mathrm{t}_c(\mathbf{a}) \neq \mathrm{t}_c(\overline{\mathbf{a}})$.

(M) For any $c \in \mathscr{C}$ and any $\mathbf{a}, \mathbf{b} \in \{0,1\}^{\mathrm{ar}(c)}$, if $\mathrm{t}_c(\mathbf{a}) = 1$ and $\mathbf{a} \sqsubseteq \mathbf{b}$, then $\mathrm{t}_c(\mathbf{b}) = 1$.

($\sqcap$-1) For any $c \in \mathscr{C}$ and any $\mathbf{a}, \mathbf{b} \in \{0,1\}^{\mathrm{ar}(c)}$, if $\mathrm{t}_c(\mathbf{a}) = \mathrm{t}_c(\mathbf{b}) = 1$, then $\mathrm{t}_c(\mathbf{a} \sqcap \mathbf{b}) = 1$.

($\sqcap$-0) For any $c \in \mathscr{C}$ and any $\mathbf{a}, \mathbf{b} \in \{0,1\}^{\mathrm{ar}(c)}$, if $\mathrm{t}_c(\mathbf{a}) = \mathrm{t}_c(\mathbf{b}) = 0$, then $\mathrm{t}_c(\mathbf{a} \sqcap \mathbf{b}) = 0$.

($\sqcup$-1) For any $c \in \mathscr{C}$ and any $\mathbf{a}, \mathbf{b} \in \{0,1\}^{\mathrm{ar}(c)}$, if $\mathrm{t}_c(\mathbf{a}) = \mathrm{t}_c(\mathbf{b}) = 1$, then $\mathrm{t}_c(\mathbf{a} \sqcup \mathbf{b}) = 1$.

($\sqcup$-0) For any $c \in \mathscr{C}$ and any $\mathbf{a}, \mathbf{b} \in \{0,1\}^{\mathrm{ar}(c)}$, if $\mathrm{t}_c(\mathbf{a}) = \mathrm{t}_c(\mathbf{b}) = 0$, then $\mathrm{t}_c(\mathbf{a} \sqcup \mathbf{b}) = 0$.

($\sqsubseteq$-1) For any $c \in \mathscr{C}$ and any $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \{0,1\}^{\mathrm{ar}(c)}$, if $\mathrm{t}_c(\mathbf{a}) = \mathrm{t}_c(\mathbf{b}) = 1$ and $\mathbf{a} \sqsubseteq \mathbf{c} \sqsubseteq \mathbf{b}$, then $\mathrm{t}_c(\mathbf{c}) = 1$.

($\sqsubseteq$-0) For any $c \in \mathscr{C}$ and any $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \{0,1\}^{\mathrm{ar}(c)}$, if $\mathrm{t}_c(\mathbf{a}) = \mathrm{t}_c(\mathbf{b}) = 0$ and $\mathbf{a} \sqsubseteq \mathbf{c} \sqsubseteq \mathbf{b}$, then $\mathrm{t}_c(\mathbf{c}) = 0$.

($\sqcap$-$\sqsubseteq$-1) Both ($\sqcap$-1) and ($\sqsubseteq$-1) hold.

($\sqcap$-$\sqcup$-$\sqsubseteq$) All of the following six conditions hold: ($\sqcap$-1), ($\sqcap$-0), ($\sqcup$-1), ($\sqcup$-0), ($\sqsubseteq$-1), and ($\sqsubseteq$-0).

Here, (M) means that every $c \in \mathscr{C}$ is monotonic, and ($\sqcap$-1) means that every $c \in \mathscr{C}$ is super-multiplicative. ($\sqcap$-$\sqcup$-$\sqsubseteq$) is equivalent to the condition (†) $\mathrm{t}[\mathscr{C}] \subseteq \mathcal{B}$; that is, the truth functions of all connectives in $\mathscr{C}$ are either constant functions, projection functions or compositions of $\mathrm{t}_\neg$ and projection functions.

Furthermore, the five kinds of sets of valid formulas, IL($\mathscr{C}$), CL($\mathscr{C}$), FOIL($\mathscr{C}$), FOCD($\mathscr{C}$), and FOCL($\mathscr{C}$), become empty under the same condition, that is, the following are equivalent (c.f., Theorem 2.4.1 and Theorem 3.5.1):

- either ($\star$1) or ($\star$2) holds;
- IL($\mathscr{C}$) $= \emptyset$;
- CL($\mathscr{C}$) $= \emptyset$;
- FOIL($\mathscr{C}$) $= \emptyset$;
- FOCD($\mathscr{C}$) $= \emptyset$;
- FOCL($\mathscr{C}$) $= \emptyset$.

Thus, considering logics with general truth-functional connectives, we have shown what properties of connectives cause coincidences, or conversely differences, between logics.

Finally, we discuss further research.

A condition for FOIL($\mathscr{C}$) = FOCD($\mathscr{C}$) We have not given a necessary and sufficient condition for FOIL($\mathscr{C}$) = FOCD($\mathscr{C}$). As a starting point, if we consider the case $\mathscr{C} \subseteq \{\neg, \wedge, \vee, \to\}$, we can tell whether FOIL($\mathscr{C}$) = FOCD($\mathscr{C}$) or FOIL($\mathscr{C}$) $\neq$ FOCD($\mathscr{C}$) from the results obtained so far (cf. Table 4.1) unless $\mathscr{C} = \{\neg, \vee\}$ or $\mathscr{C} = \{\neg, \wedge, \vee\}$. However, these two cases seem to require some new idea.

Syntactical proof In this dissertation, we only provide a semantical investigation of general connectives. On the other hand, syntactical properties of those connectives have been widely studied. For example, Rousseau [17] gave sequent calculi for IL($\mathscr{C}$) and CL($\mathscr{C}$). Furthermore, Geuvers and Hurkens [4] gave natural deduction systems for those logics. Then, it is natural to ask whether our results can be proved with their proof systems.

Many-valued logic Rousseau [17] introduced Kripke semantics for intuitionistic many-valued logic with connectives characterized by many-valued truth functions. His idea is as follows: for a connective c that is assigned a many-valued truth function t_c, the value $\|c(\alpha_1, \ldots, \alpha_n)\|_{\mathscr{K},w}$ is given by

$$\|c(\alpha_1, \ldots, \alpha_n)\|_{\mathscr{K},w} = \inf_{v \succeq w} \mathrm{t}_c(\|\alpha_1\|_{\mathscr{K},v}, \ldots, \|\alpha_n\|_{\mathscr{K},v}).$$

It may be interesting to consider whether our results can be extended to many-valued logic.

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