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著者(和文)	森達也
Author(English)	Tatsuya Mori
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Doctoral Thesis

Schur index of the $\mathcal{N} = 4$ $U(N)$ supersymmetric Yang-Mills theory via the AdS/CFT correspondence

Tatsuya Mori

Department of Physics, Tokyo Institute of Technology

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Abstract

In the thesis, we propose a new formula for the Schur index of the $\mathcal{N}=4$ $U(N)$ supersymmetric Yang-Mills theory (SYM) via the AdS/CFT correspondence.

The AdS/CFT correspondence is a conjectural relation between a superstring theory in the anti-de sitter (AdS) space and a conformal field theory (CFT). The simplest example which is discussed in this thesis is the correspondence between the Type IIB string theory in $AdS_5 \times \boldsymbol{S}^5$ and the four-dimensional $\mathcal{N}=4$ $U(N)$ SYM. In the large-N limit, the Type IIB string theory can be described by a classical supergravity, while the $\mathcal{N}=4$ SYM is in a strongly coupled region. Then, we can analyze quantities of the $\mathcal{N}=4$ SYM in the strongly coupled region via the corresponding classical supergravity. The AdS/CFT correspondence in the large-N limit is useful to study strongly coupled CFTs.

How to tackle the AdS/CFT correspondence in the finite-N region is also an important question because the rank of CFT that we are interested in is usually not large N but finite N. If N is finite, quantum gravity corrections are not negligible in the Type IIB string theory, and this is a difficulty in the finite-N AdS/CFT correspondence. Fortunately, in recent years, there has been progress in the study of the finite-N AdS/CFT correspondence. We can study the finite-N AdS/CFT correspondence by using quantities that are protected from quantum gravity corrections. One of such quantities is the superconformal index, which is a kind of the supersymmetric partition function. The superconformal index can be calculated in Lagrangian gauge theories for the arbitrary rank N and arbitrary coupling constant by using the localization method. The agreement of the index calculated on the gravity (AdS) side and that on the gauge theory (CFT) side has been confirmed at large N in different AdS/CFT examples. For example, the large-N index of $\mathcal{N}=4$ $U(N)$ SYM is the same as the index of the contribution from the

Kaluza-Klein modes in $AdS_5 \times \boldsymbol{S}^5$.

The index on the gauge theory side can be calculated in principle as long as the Lagrangian is known, while contributions to the index on the gravity side are non-trivial. It was found by Arai and Imamura that on the gravity side not only the Kaluza-Klein modes in $AdS_5 \times \boldsymbol{S}^5$ but also giant gravitons, which are objects wrapped around three-cycles in $\boldsymbol{S}^5$, contribute to the index. The contribution from giant gravitons is expected to be expressed as the sum over wrapping number $m = 1, 2, \ldots$. In previous works with the author and the collaborators only $m = 1$ contribution was taken into account. To obtain the complete index we need to include $m \geq 2$ contributions. To calculate $m \geq 2$ contributions we need to carry out certain contour integrals, and it has not yet been well understood how we should choose contours in the integrals.

The difficulty of the choice of the integration contours is caused by an unusual pole structure of the integrand. In particular, the existence of intersection strings makes the problem complicated. An intersection string is an open string appearing in the system of multiple-wrapping giant gravitons and stretches between two giant gravitons wrapped on different cycles.

In the thesis, we show that the problem of the intersection string can be avoided by taking the Schur limit, which is a specialization limit of the superconformal index. In the Schur limit, we find that the intersection string contribution in the integrand becomes a simple form and can be factor out of the integrals. Thanks to this, the multiple integrals factorize into integrals each of which is associated with coincident giant gravitons wrapped on a single cycle. This factorization makes the problem much simpler.

We also show a prescription for the choice of the integration contours. By using the prescription, we can calculate the contributions of the multiple-wrapping giant gravitons up to an arbitrary wrapping number in principle. As a consistency check, we confirm that the formula reproduces the correct index on the gauge theory side in the small N cases.

Although we use the Schur index to simplify the problem, it is desirable to use the superconformal index because the superconformal index has more information rather than the Schur index. Even so, the Schur index itself has been attracted and great interesting, and plays an important role in the analysis of superconformal field theories whose Lagrangian has not been known. We leave the analysis using such properties of the Schur index for future works.

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Contents

Chapter 1

Introduction

One of the most significant achievements of research of superstring theory is the AdS/CFT correspondence [1]. The AdS/CFT correspondence is a conjectural relation between a superstring theory in the anti-de sitter (AdS) space and a conformal field theory (CFT). It is also called the gauge/gravity correspondence because conformal field theories often are described by gauge theories and superstring theories are approximated by gravity theories. Many examples of the AdS/CFT correspondence have been studied and we focus on the simplest one, which is the correspondence between the Type IIB superstring theory in $AdS_5 \times \boldsymbol{S}^5$ and the four-dimensional $\mathcal{N} = 4$ $U(N)$ [1] supersymmetric Yang-Mills (SYM) theory [2].

An important parameter in the AdS/CFT correspondence that we are interested in is the rank of the gauge group N, which is associated with the scale of the gravity L as follows:

$$N \sim \frac{L^4}{l_p^4}, \tag{1.0.1}$$

where l_p is the Planck length. In the large-N limit, gauge theories have the rank $N = \infty$ while the scale of the gravity is much larger than the Planck length and the Type IIB superstring theory can be described by the classical supergravity. In addition, gauge theories are often in a strongly coupled region. It follows that one can analyze quantities of gauge theories

[1] In some contexts one often considers an $SU(N)$ gauge theory because the diagonal $U(1)$ in $U(N)$ is decoupled in the IR. However, we consider a $U(N)$ gauge theory to simplify some formulas. This is a matter of convention and does not affect the conclusion of the thesis.

in the strongly coupled region via the corresponding classical supergravity. For example, in [3], the Wilson loop of the strongly coupled $\mathcal{N} = 4$ SYM was calculated via the corresponding supergravity. Therefore, the AdS/CFT correspondence is useful to study strongly coupled gauge theories. Furthermore, there are theories that are only defined in the strongly coupled region and have no Lagrangian description. It is difficult to directly analyze such non-Lagrangian theories and the AdS/CFT correspondence is useful to study such theories.

The AdS/CFT correspondence has been tested with respect to various quantities. In this thesis, we focus on the superconformal index [4], which is defined as a supersymmetric partition function on $\boldsymbol{S}^1 \times \boldsymbol{S}^3$ and has information of Bogomolnyi-Prasad-Sommerfield (BPS) states in the Hilbert space on $\boldsymbol{S}^3$. The superconformal index can be calculated in principle in arbitrary Lagrangian gauge theories. In the simplest AdS/CFT example the agreement of the superconformal index in the large N limit was confirmed in [4]. The superconformal index on the gauge theory side is calculated in the $\mathcal{N} = 4$ $U(\infty)$ SYM by using the localization method. On the gravity side, Kaluza-Klein modes of massless fields in $AdS_5 \times \boldsymbol{S}^5$ contribute to the superconformal index. Such an agreement has been confirmed in different examples in the large-N limit [5, 6, 7, 8, 9].

We are interested in not only the large-N limit but also the finite N case because gauge theories we are interested in usually have finite ranks. To tackle the finite-N AdS/CFT correspondence plays an important role also for understanding quantum gravity.

The purpose of this thesis is to establish a new formula in the finite-N AdS/CFT correspondence. In this thesis, we focus on the $\mathcal{N} = 4$ $U(N)$ SYM because the superconformal index on the gauge theory side is known and we can confirm the correctness of the formula. It would be nice if we can extend it to other examples and use it to predict the index of theories that can not be analyzed directly. We leave such analysis for future works.

In general, it is difficult to study the finite-N AdS/CFT correspondence because, if N is finite, quantum gravity corrections are not negligible in the Type IIB string theory. Fortunately, in recent years, there has been progress in the study of the finite-N AdS/CFT correspondence. We can study the finite-N AdS/CFT correspondence by using quantities that are protected from quantum gravity corrections. We assume that quantum gravity corrections give continuous deformations to the background geometry with the Planck scale. The superconformal index is independent of deformations of continuous parameters of the background geometry and is not affected by quantum gravity corrections. In fact, D3-brane corrections must be taken into account even if the superconformal index is protected from quantum gravity corrections.

An importance of extended branes in the context of the AdS/CFT correspondence was first pointed out in [10]. [10] discussed the correspondence between $\mathcal{N} = 4$ $SO(N)$ SYM and the Type IIB superstring theory in $AdS_5 \times \boldsymbol{S}^5/\mathbb{Z}_2$. D3-branes wrapped on non-trivial three-cycles in $\boldsymbol{S}^5/\mathbb{Z}_2$ correspond to certain BPS operators, which must contribute to the index. A BPS D3-brane solution in $AdS_5 \times \boldsymbol{S}^5$ was proposed in [11], and it is called the giant graviton. More general BPS configurations were found in [12]. In [13] they confirmed that a BPS partition function, which counts only the operators consisting of the scalar fields, in the $\mathcal{N} = 4$ $U(N)$ SYM is reproduced by the geometric quantization of the giant gravitons.

We are interested in the superconformal index at finite N. As we mentioned above we need to include the contribution from the giant gravitons. It is expected to be expressed as the sum over wrapping number $m = 1, 2, \ldots$. The $m = 1$ case has already been studied in previous works [14, 15, 16, 17, 18], and we study the $m \geq 2$ case of multiple-wrapping giant gravitons to obtain the complete index . The calculation of the $m \geq 2$ contributions includes certain contour integrals, and the choice of integration contours has not yet been well understood.

There is a problem caused by an unusual pole structure of the integrand. In addition, the integrand is more complicated by the existence of intersection strings, which are open strings stretching between two giant gravitons wrapped on different cycles. In the thesis, we show that the complication of the integrand caused by the intersection string is resolved by taking the Schur limit, which is a specialization limit of the superconformal index. In the Schur limit, we find that the contribution of the intersection string in the integrand becomes a tractable form and can be factor out of the integrals. Due to this property, the multiple integrals factorize into integrals each of which is associated with coincident giant gravitons wrapped on a single cycle. This factorization makes the problem much simpler.

There remains the problem of an unusual pole structure of the integrand, and we give a prescription for the choice of the integration contours. By using the prescription, we can calculate the contributions of the multiple-wrapping giant gravitons up to an arbitrary wrapping number in principle. We propose a formula to calculate the Schur index of the $\mathcal{N} = 4$ $U(N)$

SYM from the contributions of the multiple-wrapping giant gravitons. As a consistency check, we numerically confirm that the formula reproduces the correct index on the gauge theory side in the small N cases. We mean by "numerical confirmation" that we expand indices and confirm the agreement of the coefficients up to the order we computed.

Although in the thesis we use the Schur index instead of the superconformal index to simplify the problem, the Schur index itself has various interesting properties and applications.

In the context of the M-theory, which is a eleven-dimensional theory that comprehensively describes all string theories, the six-dimensional $(2,0)$ superconformal field theory (SCFT) is an important theory that provides clues to elucidate the M-theory. Although the $(2,0)$ theory itself has not been well understood, there is a rich class, called the class S, of four-dimensional SCFTs obtained by compactifying the $(2,0)$ theory on a two-dimensional Riemann surface Σ [19] and the relations between four-dimensional SCFTs and two-dimensional theories realized on Σ are widely studied. Such relations has been studied with respect to various quantities. For an $SU(2)$ SCFT with flavors, the superconformal index is interpreted as a correlator in a corresponding two-dimensional topological QFT [20]. This relation was applied to several limits of the $\mathcal{N}=2$ superconformal index [21]. The Schur limit case has been also studied and it was found in [22] that the Schur index of an $\mathcal{N}=2$ generalized $SU(N)$ quiver theory is captured by the structure constants and the metric of the two-dimensional q-deformed Yang-Mills theory [23] in the zero-area limit. This relation enables us to investigate interacting field theories with no Lagrangian description.

The Schur index also appears in the context of the relation between the four-dimensional SCFT and the two-dimensional chiral algebra. For an arbitrary four-dimensional SCFT with the extended supersymmetry, a two-dimensional chiral algebra can be constructed as a sub-sector of the operators algebra [24]. In the literature, it was found that the Schur index of a corresponding SCFT is identified to the torus partition function called the vacuum character in the chiral algebra . Several application of the chiral algebra has been studied for the class S [25, 26], Argyres-Douglas theories [27, 28, 29], and $\mathcal{N}=3$ SCFTs [30, 31].

In the Coulomb branch of a four-dimensional $\mathcal{N}=2$ SCFT a gauge group is spontaneously broken to $U(1)^r$, where r is the rank of the gauge group, and BPS states are characterized by the electric and magnetic charges. By using the information of the BPS spectrum in the Coulomb branch, the procedure to calculate the Schur index was proposed in [27]. This procedure is called the infrared (IR) formula. In the literature, by using the IR formula the Schur index of non-Lagrangian theories such as Argyres-Douglas theories [32, 33] was calculated. Furthermore, in [34, 35], the IR formula for the Schur index of an $\mathcal{N} = 2$ SCFT with surface defects was developed and the Schur index including surface defects contributions is identified to a character of a non-vacuum module in the corresponding chiral algebra.

The Schur index has a wide range of applications and we leave the relation between our works and such applications of the Schur index for future works.

This thesis is organized as follows. In Chapter 2, we explain the superconformal index. In 2.1 we first introduce the Witten index and explain fundamental properties of the index. In order to extend the Witten index to the superconformal index we introduce the superconformal algebra in 2.2 and Bogomolnyi-Prasad-Sommerfield states in 2.3. In 2.4 we define the superconformal index. We also explain the localization formula, which is obtained by reducing the trace over the Hilbert space in the definition of the superconformal index to a tractable form. In 2.5 we define the Schur index and explain the relation between the superconformal index and the Schur index.

In Chapter 3, we review the $\mathcal{N} = 4$ $U(N)$ SYM. First, in 3.1, we introduce the $\mathcal{N} = 4$ vector multiplet. In 3.2 we define the Lagrangian of the $\mathcal{N} = 4$ SYM. In 3.3 and 3.4 we define the superconformal index and the Schur index of the $\mathcal{N} = 4$ $U(N)$ SYM, respectively. In particular, in 3.3, we write down the method to derive the single-particle index of the $\mathcal{N} = 4$ $U(N)$ SYM. In 3.5 we give some numerical results of the superconformal index and the Schur index for small N and $N = \infty$ cases.

In Chapter 4, we review the AdS/CFT correspondence. We first consider a system of N coincident D3-branes. In 4.1 we explain the gauge theory description of the system and explain that the $\mathcal{N} = 4$ $U(N)$ SYM appears. In 4.2 we explain the gravity description of the system and show that the metric of the supergravity solution given by the branes is expressed as $AdS_5 \times \boldsymbol{S}^5$ in the near horizon region. In 4.3 we explain the claim of the AdS/CFT correspondence in [1] and explain some properties. In 4.4 we discuss the superconformal index in the large-N limit on both the gauge theory and gravity sides and explain that the contribution from Kaluza-Klein modes is identified to the large-N index on the gauge theory side. In 4.4 we discuss the superconformal index in the finite-N case and suggest that the difference between the finite-N and large-N indices should be reproduced by the contribution from giant gravitons.

In Chapter 5, we explain the main topic of the thesis, which is based on the author's and the collaborators's paper [36]. The purpose of this chapter is to construct a formula to calculate the index by using the contributions on the gravity side, especially giant graviton contributions, and to confirm the correctness of the formula. In preparation for calculating the contributions of multiple-wrapping giant gravitons, in 5.1, we define BPS configurations of giant gravitons according to [12]. Following [14] we introduce the index of single-wrapping giant gravitons. In 5.2 we expect that the superconformal index of the $\mathcal{N} = 4$ SYM includes all contributions from giant gravitons, and give a formula by generalizing the index of single-wrapping giant gravitons. In the formula, the multiple-wrapping contributions are given by certain contour integrals. We explain the difficulty of this contour integrals and give a procedure in the case for the system of giant gravitons wrapped on a single cycle. In 5.3 we show that, by taking the Schur limit, the multiple integrals factorize into the integrals for the system of giant gravitons wrapped on a single cycle. Following the factorization, we propose a new formula to calculate the Schur index of the $\mathcal{N} = 4$ SYM from the contributions of multiple-wrapping giant gravitons. In 5.4 we explain the supergravity contribution in the Schur limit. In 5.5 we explain the procedure to calculate the integrals for the system of giant gravitons wrapped on a single cycle. In 5.6 we compare the numerical results of the multiple-wrapping contributions with the known results of the $\mathcal{N} = 4$ SYM and numerically confirm the agreement of these results.

In Chapter 6, we conclude this thesis.

Technical details are shown in the Appendix.

Chapter 2

Superconformal index

In Chapter 2, we define several kinds of indices. We first introduce the Witten index and discuss some properties of the index. We also give the general definition of the superconformal index and Schur index by extensions of the Witten index.

2.1 Witten index

The Witten index [37] is a kind of partition functions defined in a system with supersymmetry. The supersymmetry is a symmetry between bosons and fermions. The generators of the supersymmetry are called the supercharge, which is often denoted by Q. The generators Q act as

$$Q\,|\text{boson}\rangle = |\text{fermion}\rangle \quad \text{and} \quad Q\,|\text{fermion}\rangle = |\text{boson}\rangle\,. \tag{2.1.1}$$

Let us consider a system with the Hamiltonian H, a supercharge Q and its hermitian conjugate $Q^\dagger$ satisfying

$$H = \{Q, Q^\dagger\}, \quad [Q, H] = [Q^\dagger, H] = 0. \tag{2.1.2}$$

The Witten index is defined by

$$\mathcal{I}_{\text{W}} = \text{Tr}(-1)^F x^H, \tag{2.1.3}$$

where the trace is taken over the Hilbert space. x is a complex parameter and F is a fermion number which distinguish bosonic and fermionic states. The important property of the Witten index is that it receive only the contribution from ground states with $H = 0$.

Let us calculate the Witten index in a simple quantum mechanics and explicitly show that the index pick up only the contribution of the ground states. We consider a supersymmetric quantum mechanics of bosonic and fermionic harmonic oscillators. (See [38]). The Hamiltonian of the supersymmetric quantum mechanics is given by

$$H = \frac{p^2}{2} + \frac{x^2}{2} + \frac{1}{2}\left[\psi^\dagger, \psi\right], \tag{2.1.4}$$

where x is a position and p is canonical conjugate to x. ψ is a complex fermion. We define creation and annihilation operators,

$$a_B = \frac{1}{\sqrt{2}}(x + ip), \quad a_B^\dagger = \frac{1}{\sqrt{2}}(x - ip), \quad a_F = \psi, \quad a_F^\dagger = \psi^\dagger. \tag{2.1.5}$$

These operators satisfy the following commutation and anti-commutation relations,

$$[a_B, a_B^\dagger] = 1, \qquad \{a_F, a_F^\dagger\} = 1, \tag{2.1.6}$$

where the other commutation relations vanish. The Hamiltonian (2.1.4) is rewritten as

$$H = a_B^\dagger a_B + a_F^\dagger a_F. \tag{2.1.7}$$

The supercharges are given by

$$\begin{aligned} Q &= a_B^\dagger a_F = \frac{1}{\sqrt{2}}\left(x - ip\right)\psi, \\ Q^\dagger &= a_F^\dagger a_B = \frac{1}{\sqrt{2}}\left(x + ip\right)\psi^\dagger. \end{aligned} \tag{2.1.8}$$

These operators satisfy the relations in (2.1.2). We investigate the spectrum of the model. Let $|0\rangle$ be the unique bosonic ground state, which satisfies $a_B\,|0\rangle = a_F\,|0\rangle = 0$. The excitations by the creation operators are given by

$$|n\rangle_B = (a_B^\dagger)^n\,|0\rangle\,, \qquad |n\rangle_F = (a_B^\dagger)^{n-1} a_F^\dagger\,|0\rangle\,, \qquad (n \geq 1). \tag{2.1.9}$$

The excited states $|n\rangle_B$ and $|n\rangle_F$ are eigenstates for the H with the eigenvalues n. The fermionic number F counts the number of the fermionic annihilation operator $a_F^\dagger$ and F is even for the state $|n\rangle_B$ while F is odd for the state $|n\rangle_F$. The supercharges act on the excited states and transform them as the following forms,

$$Q^\dagger |n\rangle_B = n\,|n\rangle_F\,, \quad Q\,|n\rangle_F = |n\rangle_B\,, \quad Q\,|n\rangle_B = Q^\dagger |n\rangle_F = 0. \qquad (2.1.10)$$

The structure of the Hilbert space is shown in Figure 2.1.

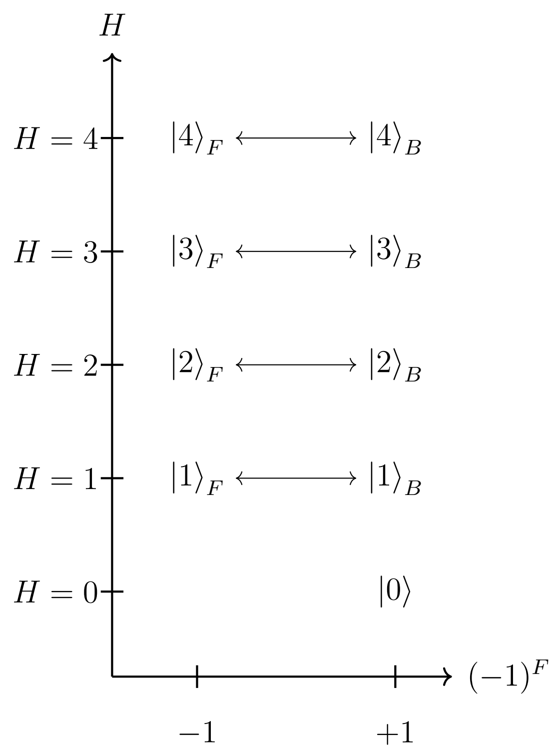

Figure 2.1: The energy spectrum of the supersymmetric harmonic oscillator. Two states connected by the arrow construct supersymmetric multiplets.

Let us calculate the Witten index. The trace counts the spectrum in Figure 2.1 and we obtain

$$\mathcal{I}_{\mathrm{W}} = \mathrm{Tr}(-1)^F x^H = 1 + (x + x^2 + \cdots) - (x + x^2 + \cdots) = 1. \qquad (2.1.11)$$

We see that only the state with $H = 0$ contributes to the Witten index and the other states are cancelled out with the bosons and the fermions respectively at each energy level.

An important property of the index is that it is invariant under a continuous deformation of the system. For example, we can deform the system considered above by replacing the supercharges in (2.1.8) by

$$Q = \frac{1}{\sqrt{2}} \left(f(x) - ip \right) \psi, \quad Q^\dagger = \frac{1}{\sqrt{2}} \left(f(x) + ip \right) \psi^\dagger. \tag{2.1.12}$$

$f(x)$ is a function given by

$$f(x) = x + gh(x), \tag{2.1.13}$$

where g is a coupling constant and $h(x)$ is a function that goes to zero at infinity. Then, the Hamiltonian $H = \{Q, Q^\dagger\}$ is given by

$$H = \frac{1}{2} f(x)^2 + \frac{1}{2} p^2 + \frac{1}{2} f(x)' \left[\psi^\dagger, \psi \right]. \tag{2.1.14}$$

The system is a particle in the potential

$$V(x) = \frac{1}{2} f(x)^2 = \frac{1}{2} \left(x + gh(x) \right)^2, \tag{2.1.15}$$

coupling to the fermionic degrees of freedom.

If g changes, the Hilbert space also changes and it is in general difficult to calculate the eigenvalues of H. Even so, the degeneracy of the excited states is preserved, and excited states always appear in a pair of a boson and a fermion. Let $|\psi\rangle$ be an eigenstate of the Hamiltonian with non-zero energy,

$$H |\psi\rangle = E |\psi\rangle, \quad E \neq 0. \tag{2.1.16}$$

By using the relations (2.1.2) we can show that $E > 0$ and $Q|\psi\rangle$ or $Q^\dagger |\psi\rangle$ gives another state $|\psi'\rangle$ with the same energy and the opposite the statistics. This means that an excited state with $E > 0$ always appears with the pair of $|\psi\rangle$ and $|\psi'\rangle$ and the contributions to the Witten index cancel with each other. Therefore, the Witten index is given by

$$\mathcal{I}_{\mathrm{W}} = \mathrm{Tr}_{H=0} (-1)^F = n_B - n_F, \tag{2.1.17}$$

where n_B and n_F are the numbers of the bosonic and fermionic zero-energy states, respectively. Although each of n_B and n_F change, the difference $n_B - n_F$ is preserved under continuous deformations of the system because of the degeneracy of excited states. (See Figure 2.2). Therefore, the Witten index is independent of the coupling constant.

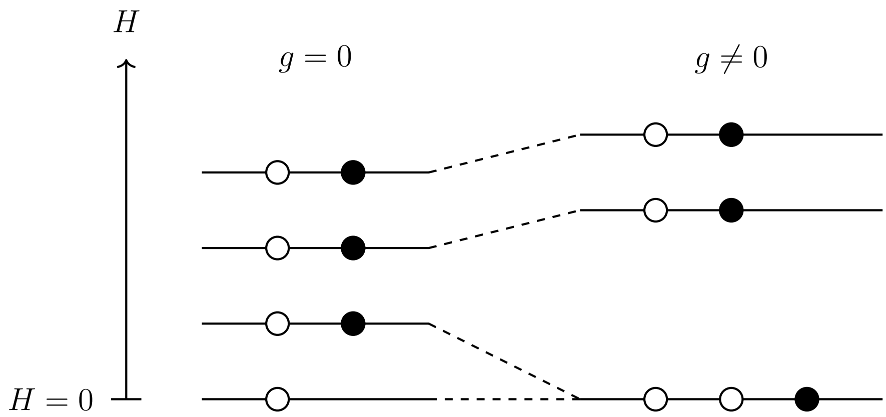

Figure 2.2: The Hilbert spaces with $g = 0$ and $g \neq 0$. The white and black circles represent respectively bosons and fermions. The transition of states always occur in the pair of bosons and fermions.

2.2 Superconformal algebra

We want to define an index in a superconformal field theory. Let us first summarize the symmetry algebra. In this thesis we use a cylinder $\mathbb{R}\times\boldsymbol{S}^3$. It is more convenient for construction of the irreducible unitary representations of the superconformal algebra [39]. Furthermore, we define the superconformal index as the partition function on the cylinder $\mathbb{R}\times\boldsymbol{S}^3$ as the background spacetime.

The superconformal algebra is an extension of the conformal algebra by the supersymmetry. The four-dimensional superconformal algebra is generated by

$$H, \quad J^{\alpha}{}_{\beta}, \quad \overline{J}^{\dot\alpha}{}_{\dot\beta}, \quad P^{\dot\alpha}{}_{\beta}, \quad K^{\alpha}{}_{\dot\beta}, \quad Q^I_{\alpha}, \quad \overline{Q}^{\dot\alpha}_I, \quad S^{\alpha}_I, \quad \overline{S}^I_{\dot\alpha}, \quad R^I{}_J, \quad r. \tag{2.2.1}$$

The first five generate the conformal algebra $so(2,4)$. In $\mathbb{R}\times\boldsymbol{S}^3$ H is the Hamiltonian generating the translation along $\mathbb{R}$. The rotation group on $\boldsymbol{S}^3$ is $SO(4) = SU(2)_{\text{left}}\times SU(2)_{\text{right}}$. $J^{\alpha}{}_{\beta}$ and $\overline{J}^{\dot\alpha}{}_{\dot\beta}$ are respectively the generators of $SU(2)_{\text{left}}$ and $SU(2)_{\text{right}}$. Indices α and $\dot\alpha$ are respectively spin indices of $SU(2)_{\text{left}}$ and $SU(2)_{\text{right}}$.

In the literature the Minkowski spacetime $\mathbb{R}^{1,3}$ is also used. It is related to $\mathbb{R}\times\boldsymbol{S}^3$ by the Weyl transformation and the Wick rotation. For $\mathbb{R}^{1,3}$ H

is the dilatation, $J^{\alpha}{}_{\beta}$, $\overline{J}^{\dot{\alpha}}{}_{\dot{\beta}}$ are the Lorentz spins, $P^{\dot{\alpha}}{}_{\beta}$ are the translations generator, and $K^{\alpha}{}_{\dot{\beta}}$ are the generators of special conformal transformations.

The commutation relations for the generators of the conformal algebra are

$$\begin{aligned}
[J^{\alpha}{}_{\beta}, J^{\gamma}{}_{\delta}] &= \delta^{\gamma}_{\beta} J^{\alpha}{}_{\delta} - \delta^{\alpha}_{\delta} J^{\gamma}{}_{\beta}, \\
[\overline{J}^{\dot{\alpha}}{}_{\dot{\beta}}, \overline{J}^{\dot{\gamma}}{}_{\dot{\delta}}] &= \delta^{\dot{\gamma}}_{\dot{\beta}} \overline{J}^{\dot{\alpha}}{}_{\dot{\delta}} - \delta^{\dot{\alpha}}_{\dot{\delta}} \overline{J}^{\dot{\gamma}}{}_{\dot{\beta}}, \\
[J^{\alpha}{}_{\beta}, P^{\dot{\gamma}}{}_{\delta}] &= \delta^{\alpha}_{\delta} P^{\dot{\gamma}}{}_{\beta} - \frac{1}{2}\delta^{\alpha}_{\beta} P^{\dot{\gamma}}{}_{\delta}, \\
[\overline{J}^{\dot{\alpha}}{}_{\dot{\beta}}, P^{\dot{\gamma}}{}_{\delta}] &= -\delta^{\dot{\gamma}}_{\dot{\beta}} P^{\dot{\alpha}}{}_{\delta} + \frac{1}{2}\delta^{\dot{\alpha}}_{\dot{\beta}} P^{\dot{\gamma}}{}_{\delta}, \\
[J^{\alpha}{}_{\beta}, K^{\gamma}{}_{\dot{\delta}}] &= -\delta^{\gamma}_{\beta} K^{\alpha}{}_{\dot{\delta}} + \frac{1}{2}\delta^{\alpha}_{\beta} K^{\gamma}{}_{\dot{\delta}}, \\
[\overline{J}^{\dot{\alpha}}{}_{\dot{\beta}}, K^{\gamma}{}_{\dot{\delta}}] &= \delta^{\dot{\alpha}}_{\dot{\delta}} K^{\gamma}{}_{\dot{\beta}} - \frac{1}{2}\delta^{\dot{\alpha}}_{\dot{\beta}} K^{\gamma}{}_{\dot{\delta}}, \\
[H, P^{\dot{\alpha}}{}_{\beta}] &= P^{\dot{\alpha}}{}_{\beta}, \\
[H, K^{\alpha}{}_{\dot{\beta}}] &= -K^{\alpha}{}_{\dot{\beta}}, \\
[K^{\alpha}{}_{\dot{\beta}}, P^{\dot{\gamma}}{}_{\delta}] &= \delta^{\alpha}_{\delta}\delta^{\dot{\gamma}}_{\dot{\beta}} H + \delta^{\dot{\gamma}}_{\dot{\beta}} J^{\alpha}{}_{\delta} - \delta^{\alpha}_{\delta} \overline{J}^{\dot{\gamma}}{}_{\dot{\beta}}.
\end{aligned}$$

The hermitian conjugates of the conformal generators are given by

$$H^{\dagger} = H, \quad (J^{\alpha}{}_{\beta})^{\dagger} = J^{\beta}{}_{\alpha}, \quad (\overline{J}^{\dot{\alpha}}{}_{\dot{\beta}})^{\dagger} = \overline{J}^{\dot{\beta}}{}_{\dot{\alpha}}, \quad (P^{\dot{\alpha}}{}_{\beta})^{\dagger} = K^{\alpha}{}_{\dot{\beta}}. \tag{2.2.2}$$

The generators P and K on the cylinder $\mathbb{R}\times\boldsymbol{S}^3$ are swapped by the hermitian conjugate. P and K change the energy by $+1$ and -1, respectively. In the construction of irreducible unitary representations we treat P and K as the raising and lowing operator for the energy.

The supercharges Q^I_{α}, $\overline{Q}^{\dot{\alpha}}_I$, S^{α}_I, and $\overline{S}^I_{\dot{\alpha}}$ are fermionic generators with $SU(2)$ spin indices α and $\dot{\alpha}$. The indices I, J run from 1 to $\mathcal{N}$, where $\mathcal{N}$ represents the number of supersymmetries. In four-dimensional field theories without gravity the maximum number of $\mathcal{N}$ is four. The anti-commutation relations for the supercharges are given by

$$\{Q^I_{\alpha}, \overline{Q}^{\dot{\beta}}_J\} = \delta^I_J P^{\dot{\beta}}{}_{\alpha}, \tag{2.2.3}$$

$$\{S^{\alpha}_I, \overline{S}^J_{\dot{\beta}}\} = \delta^J_I K^{\alpha}{}_{\dot{\beta}}, \tag{2.2.4}$$

$$\{S^{\alpha}_I, Q^J_{\beta}\} = \delta^J_I \delta^{\alpha}_{\beta}\left(\frac{1}{2}H + \frac{4-\mathcal{N}}{4\mathcal{N}} r\right) + \delta^J_I J^{\alpha}{}_{\beta} + \delta^{\alpha}_{\beta} R^J{}_I, \tag{2.2.5}$$

$$\{\overline{Q}^{\dot{\alpha}}_I, \overline{S}^J_{\dot{\beta}}\} = \delta^J_I \delta^{\dot{\alpha}}_{\dot{\beta}} \left(\frac{1}{2} H - \frac{4-\mathcal{N}}{4\mathcal{N}} r \right) - \delta^J_I \overline{J}^{\dot{\alpha}}{}_{\dot{\beta}} - \delta^{\dot{\alpha}}_{\dot{\beta}} R^J{}_I, \tag{2.2.6}$$

and the other anti-commutation relations for $Q, \overline{Q}, S, \overline{S}$ vanish. The $U(\mathcal{N})$ (or $SU(\mathcal{N})$) global symmetry acting on the index I is called the R-symmetry. $R^I{}_J$ are the generators of $SU(\mathcal{N})$ and r is the generator of $U(1) \subset U(\mathcal{N})$. In the case of $\mathcal{N} = 4$ the coefficients of r in (2.2.5) and (2.2.6) are zero and the generator r appears in no (anti-)commutation relations. This means the $U(1)$ symmetry can be removed and the R-symmetry for $\mathcal{N} = 4$ is $SU(4)$. $R^I{}_J$ obey the following relations:

$$[R^I{}_J, Q^K_\alpha] = -\delta^K_J Q^I_\alpha + \frac{1}{\mathcal{N}} \delta^I_J Q^K_\alpha, \tag{2.2.7}$$

$$[R^I{}_J, R^K{}_L] = \delta^I_L R^K{}_J - \delta^K_J R^I{}_L. \tag{2.2.8}$$

The $U(1)$ charge r is normalized so that

$$[r, Q^I_\alpha] = -Q^I_\alpha, \quad [r, \overline{Q}^{\dot{\alpha}}_I] = \overline{Q}^{\dot{\alpha}}_I, \quad [r, S^\alpha_I] = -S^\alpha_I, \quad [r, \overline{S}^I_{\dot{\alpha}}] = \overline{S}^I_{\dot{\alpha}}. \tag{2.2.9}$$

The hermiticity of the supercharges and R-symmetry generators is

$$(Q^I_\alpha)^\dagger = S^\alpha_I, \quad (\overline{Q}^{\dot{\alpha}}_I)^\dagger = \overline{S}^I_{\dot{\alpha}}, \quad (R^I{}_J)^\dagger = R^J{}_I, \quad r^\dagger = r. \tag{2.2.10}$$

The commutation relations between the Hamiltonian and the supercharges are given by

$$[H, Q] = \frac{1}{2} Q, \quad [H, \overline{Q}] = \frac{1}{2} \overline{Q}, \quad [H, S] = -\frac{1}{2} S, \quad [H, \overline{S}] = -\frac{1}{2} \overline{S}. \tag{2.2.11}$$

The other commutation relations between fermionic generators and conformal generators are

$$\begin{aligned}
&[J^\alpha{}_\beta, Q^I_\gamma] = \delta^\alpha_\gamma Q^I_\beta - \frac{1}{2} \delta^\alpha_\beta Q^I_\gamma, && [J^\alpha{}_\beta, S^\gamma_I] = -\delta^\gamma_\beta S^\alpha_I + \frac{1}{2} \delta^\alpha_\beta S^\gamma_I, \\
&[\overline{J}^{\dot{\alpha}}{}_{\dot{\beta}}, \overline{Q}^{\dot{\gamma}}_I] = -\delta^{\dot{\gamma}}_{\dot{\beta}} \overline{Q}^{\dot{\alpha}}_I + \frac{1}{2} \delta^{\dot{\alpha}}_{\dot{\beta}} \overline{Q}^{\dot{\gamma}}_I, && [\overline{J}^{\dot{\alpha}}{}_{\dot{\beta}}, \overline{S}^I_{\dot{\gamma}}] = \delta^{\dot{\alpha}}_{\dot{\gamma}} \overline{S}^I_{\dot{\beta}} - \frac{1}{2} \delta^{\dot{\alpha}}_{\dot{\beta}} \overline{S}^I_{\dot{\gamma}}, \\
&[K^\alpha{}_{\dot{\beta}}, Q^I_\gamma] = \delta^\alpha_\gamma \overline{S}^I_{\dot{\beta}}, && [P^{\dot{\alpha}}{}_\beta, S^\gamma_I] = -\delta^\gamma_\beta \overline{Q}^{\dot{\alpha}}_I, \\
&[K^\alpha{}_{\dot{\beta}}, \overline{Q}^{\dot{\gamma}}_I] = \delta^{\dot{\gamma}}_{\dot{\beta}} S^\alpha_I, && [P^{\dot{\alpha}}{}_\beta, \overline{S}^I_{\dot{\gamma}}] = -\delta^{\dot{\alpha}}_{\dot{\gamma}} Q^I_\beta.
\end{aligned}$$

The maximal compact subgroup of the superconformal symmetry with $\mathcal{N} \leq 3$ is $SO(2) \times SU(2)^2 \times U(\mathcal{N})$ and the Cartan charges of it are given by

$$H, \quad J \equiv J^1{}_1 = -J^2{}_2, \quad \overline{J} \equiv \overline{J}^{\dot{1}}{}_{\dot{1}} = -\overline{J}^{\dot{2}}{}_{\dot{2}}, \quad R^n_n - R^{n+1}_{n+1}, \quad r. \quad (2.2.12)$$

There are $\mathcal{N}+3$ Cartan generators. In the case of $\mathcal{N}=4$ the $U(1)$ Cartan generator r is removed and there are six Cartan generators.

2.3 Bogomolnyi-Prasad-Sommerfield states

In the definition of the Witten index in 2.1 the relation $H = \{Q, Q^\dagger\}$ plays an important role and states satisfying $Q\,|\psi\rangle = Q^\dagger\,|\psi\rangle = 0$ contribute to the index. Although there is no supercharge satisfying $H = \{Q, Q^\dagger\}$ in the superconformal algebra, we can define an operator Δ which plays a similar role to H. We focus on the anti-commutation relations in (2.2.5) and (2.2.6). Because of the hermiticity $S = Q^\dagger$ and $\overline{S} = \overline{Q}^\dagger$ in (2.2.10),

$$\{Q^I_\alpha, S^\alpha_I\} = \{Q^I_\alpha, (Q^I_\alpha)^\dagger\} \geq 0, \quad (2.3.1)$$

$$\{\overline{Q}^{\dot{\alpha}}_I, \overline{S}^I_{\dot{\alpha}}\} = \{\overline{Q}^{\dot{\alpha}}_I, (\overline{Q}^{\dot{\alpha}}_I)^\dagger\} \geq 0. \quad (2.3.2)$$

These bounds come from the unitarity. [1] These bounds give conditions on the Cartan charges for each choice of $\alpha, \dot{\alpha}, I$. Namely, the Cartan charges appearing in the (2.2.5) and (2.2.6) have the following bounds:

$$\frac{1}{2}H + \frac{4-\mathcal{N}}{4\mathcal{N}}r + J^\alpha{}_\alpha + R^I{}_I \geq 0, \quad (2.3.4)$$

$$\frac{1}{2}H - \frac{4-\mathcal{N}}{4\mathcal{N}}r - \overline{J}^{\dot{\alpha}}{}_{\dot{\alpha}} - R^I{}_I \geq 0. \quad (2.3.5)$$

Because the total number of the supercharges Q^I_α and $\overline{Q}^{\dot{\alpha}}_I$ is $4\mathcal{N}$, there are $4\mathcal{N}$ conditions corresponding to the respective supercharges.

[1] We act an operator $\mathcal{O}$ satisfying $\{\mathcal{O}, \mathcal{O}^\dagger\} = c$, where c is a constant parameter, on a ket vector,

$$|\mathcal{O}\,|\psi\rangle\,|^2 + |\mathcal{O}^\dagger\,|\psi\rangle\,|^2 = \langle\psi|\,\{\mathcal{O}, \mathcal{O}^\dagger\}\,|\psi\rangle = c\,\langle\psi|\psi\rangle\,. \quad (2.3.3)$$

If a theory is unitary, the norm of the ket vector is positive. In such a case, the left hand side on the (2.3.3) is positive, and it follows that the constant c must be positive, namely $c \geq 0$.

Let us consider a state saturating the bound for a supercharge $Q^{I'}_{\alpha'}$ with the fixed indices I' and α'. Such a saturating state is called the Bogomolnyi-Prasad-Sommerfield (BPS) state. A BPS state is invariant under the transformation generated by $Q^{I'}_{\alpha'}$ while it is generically transformed by the other generators. It follows that a supersymmetric multiplet is lack for one part regarding the generator $Q^{I'}_{\alpha'}$. Such a lacking multiplet is called the short multiplet and may have a manageable property in terms of non-perturbation theories. We define a quantity for the $Q^{I'}_{\alpha'}$,

$$\Delta \equiv \{Q^{I'}_{\alpha'}, (Q^{I'}_{\alpha'})^\dagger\}. \tag{2.3.6}$$

An eigenstate for Δ has a non-negative eigenvalue and Δ plays a similar role to H in the Witten index defined in 2.1. All states with $\Delta > 0$ appear in a pair of bosonic and fermionic states and do not contribute to the index. Therefore, the index dose not change under deformations of continuous parameters.

2.4 Superconformal index

Superconformal index [4, 8] is an extension of the Witten index defined in an $\mathcal{N} = 1$ superconformal field theory. Let Q be one of the supercharges. The general definition of the superconformal index is

$$\mathcal{I} = \mathrm{Tr}_{\mathcal{H}_{\mathrm{GI}}}\left[(-1)^F x^\Delta \prod_a y_a^{M_a}\right], \quad \Delta \equiv 2\{Q, Q^\dagger\}, \tag{2.4.1}$$

where the trace is taken over the Hilbert space $\mathcal{H}_{\mathrm{GI}}$ on the cylinder $\mathbb{R} \times \boldsymbol{S}^3$. $\mathcal{H}_{\mathrm{GI}}$ includes only physical (gauge invariant) states. F is the fermion number and the factor $(-1)^F$ gives $+1$ for bosons and -1 for fermions. M_a are linear combination of the Cartan generators of the global symmetry of the theory which satisfy

$$[M_a, M_b] = [M_a, Q] = 0. \tag{2.4.2}$$

The variables x, y_a are complex free parameters called "fugacities".

This index is invariant under continuous deformations [4]. The reason is essentially the same as the coupling independence of the Witten index. As we explained in the previous section, non-BPS states with $\Delta > 0$ do not contribute to the the index because all non-BPS states appear in pairs of bosons and fermions.

It follows that $\mathcal{I}$ is independent of the fugacity x. We can easily check the x-independence of $\mathcal{I}$ from the definition (2.4.1) as follows. The derivative of $\mathcal{I}$ with respect to x is

$$\frac{\partial \mathcal{I}}{\partial x} = \mathrm{Tr}\left[(-1)^F \Delta x^{\Delta-1} \prod_a y_a^{M_a}\right] = 0, \tag{2.4.3}$$

where in the last equality we use the following relation:

$$\begin{aligned} \mathrm{Tr}\left[(-1)^F Q Q^\dagger C\right] &= -\,\mathrm{Tr}\left[Q(-1)^F Q^\dagger C\right] \\ &= -\,\mathrm{Tr}\left[(-1)^F Q^\dagger C Q\right] \\ &= -\,\mathrm{Tr}\left[(-1)^F Q^\dagger Q C\right], \end{aligned} \tag{2.4.4}$$

where the first equality uses the relation $(-1)^f Q = -Q(-1)^F$, the second equality uses the property that the trace is invariant under the cyclic permutations, and the third quality uses the relation $CQ = QC$.

Now, let us give M_a explicitly for the $\mathcal{N} = 1$ superconformal index case. First, we choose a supercharge $Q = \overline{Q}^{\dot{\alpha}=\dot{1}}_{I=1}$, whose quantum numbers are shown in Table 2.1. There are four Cartan generators of the $\mathcal{N} = 1$ superconformal symmetry given in (2.2.12):

$$H, \quad J, \quad \overline{J}, \quad r_1, \tag{2.4.5}$$

where we denote the $U(1)_R$ charge r by r_1 for distinction from the $U(1)$ charge in $\mathcal{N} = 2$ theory discussed in the next section. From the relation (2.2.6) Δ is given by

$$\Delta = H - \frac{3}{2} r_1 - 2\overline{J}. \tag{2.4.6}$$

There are three independent linear combinations of (2.4.5) satisfying (2.4.2). One of three is Δ because $[\Delta, Q] = 0$. The remaining two, M_1 and M_2, can be chosen as follows:

$$M_1 = H + J + \overline{J}, \quad M_2 = J. \tag{2.4.7}$$

In addition, we may also have flavor charges

$$M_i = F_i \tag{2.4.8}$$

where F_i are the generators of flavor symmetries. Because the index is independent of the fugacity x corresponding to Δ, the $\mathcal{N}=1$ superconformal index is written as

$$\mathcal{I}(y_1, y_2, y_i) = \mathrm{Tr}\left[(-1)^F x^{\Delta} y_1^{H+J+\overline{J}} y_2^J \prod_i y_i^{F_i}\right]. \tag{2.4.9}$$

Table 2.1: The Cartan charges for the $\mathcal{N}=1$ supercharges.

Q^I, $\overline{Q}_I$	H	J	$\overline{J}$	r_1
$Q^{I=1}_{\alpha=1}$	$\frac{1}{2}$	$\frac{1}{2}$	0	-1
$Q^{I=1}_{\alpha=2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	-1
$\overline{Q}^{\dot{\alpha}=\dot{1}}_{I=1}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	1
$\overline{Q}^{\dot{\alpha}=\dot{2}}_{I=1}$	$\frac{1}{2}$	0	$\frac{1}{2}$	1

Localization formula

Thanks to the property that the index is independent of coupling constants, the index can be calculated in the weak coupling limit $g=0$. In the limit all fields become free fields and we can construct the Hilbert space explicitly. As a simple example let us consider a free real scalar field ϕ with mass m on $\mathbb{R}\times\boldsymbol{S}^3$. The equation of motion for ϕ is

$$\left(-\partial_t^2 + \Delta - m^2\right)\phi = 0, \tag{2.4.10}$$

where Δ is the Laplacian on $\boldsymbol{S}^3$. We first expand ϕ by spherical harmonics in $\boldsymbol{S}^3$. Let Y_n be the eigenfunctions of $m^2-\Delta$ and ω_n^2 ($\omega_n>0$) be the associated eigenvalues:

$$\left(m^2-\Delta\right)Y_n = -\omega_n^2 Y_n. \tag{2.4.11}$$

By using Y_n, the mode expansion of ϕ is expressed as

$$\phi = \sum_n \left(A_n e^{-i\omega_n t} Y_n + A_n^\dagger e^{i\omega_n t} Y_n\right), \tag{2.4.12}$$

where A_n and $A_n^\dagger$ are annihilation and creation operators, respectively, and they satisfy

$$\left[A_n, A_m^\dagger\right] = \delta_{nm}. \tag{2.4.13}$$

The Hilbert space $\mathcal{H}$ for ϕ is

$$\mathcal{H}: \quad \left\{\prod_n \left(A_n^\dagger\right)^{a_n} |0\rangle\right\}_{a_n}, \quad a_n \in \mathbb{Z}_{\geq 0}, \tag{2.4.14}$$

where $|0\rangle$ is the vacuum state annihilated by A_n.

Even in more general theories including fermions and gauge fields, we can construct the Hilbert space explicitly in a similar way by using the mode expansion. The index is given by

$$\begin{aligned}\mathcal{I}_G &= \mathrm{Tr}_{\mathcal{H}}\left[(-1)^F x^\Delta \prod_a y_a^{M_a} \prod_i \zeta_i^{\alpha_i}\right] \\ &= \sum_{\{a_n\}} \prod_n \langle 0| (A_n)^{a_n} \cdot (-1)^F x^\Delta \prod_a y_a^{M_a} \prod_i \zeta_i^{\alpha_i} \cdot (A_n^\dagger)^{a_n} |0\rangle . \end{aligned} \tag{2.4.15}$$

Because we are considering the free theory with $g = 0$, we can treat the gauge symmetry as the global symmetry. In (2.4.15) we introduced fugacities ζ_i for Cartan charges α_i of the gauge symmetry G. To obtain the index $\mathcal{I}$ in (2.4.1) defined as the trace over $\mathcal{H}_{\mathrm{GI}}$, we need to extract the contributions of only physical states from (2.4.15). This can be done by extracting gauge invariant contributions by ζ_i integrals:

$$\mathcal{I} = \int d\mu \mathcal{I}_G, \tag{2.4.16}$$

where $d\mu$ is the Haar measure of the gauge group G. If $G = U(N)$, $d\mu$ is defined by

$$\int d\mu_N = \frac{1}{N!} \prod_{i=1}^N \oint_{|\zeta_i|=1} \frac{d\zeta_i}{2\pi i \zeta_i} \prod_{\substack{i,j=1 \\ i\neq j}}^N \left(1 - \frac{\zeta_i}{\zeta_j}\right). \tag{2.4.17}$$

Because we have constructed the Hilbert space (2.4.14), we can calculate the summation over the states in (2.4.15) straightforwardly. There is a convenient way to do it. We first define the sub-space $\mathcal{H}_1 \subset \mathcal{H}$ of single-particle states:

$$\mathcal{H}_1 : \quad \left\{ A_n^\dagger \left|0\right\rangle \right\}_n . \tag{2.4.18}$$

We define the index i_G for this sub-space

$$\begin{aligned} i_G &= \mathrm{Tr}_{\mathcal{H}_1} \left[(-1)^F x^\Delta \prod_a y_a^{M_a} \prod_i \zeta_i^{\alpha_i} \right] \\ &= \sum_n \left\langle 0 \right| A_n (-1)^F x^\Delta \prod_a y_a^{M_a} \prod_i \zeta_i^{\alpha_i} A_n^\dagger \left|0\right\rangle . \end{aligned} \tag{2.4.19}$$

This is called the single-particle index. Once we obtain i_G, we can calculate $\mathcal{I}_G$ by the following formula:

$$\mathcal{I}_G = \mathrm{Pexp}\left(i_G\right). \tag{2.4.20}$$

Pexp is the plethystic exponential defined by

$$\mathrm{Pexp}\left(\sum_i c_i x_i \right) = \prod_i \frac{1}{(1-x_i)^{c_i}}, \tag{2.4.21}$$

where c_i are numerical coefficients and x_i are fugacities. Pexp converts single-particle contributions to multiple-particle contributions. By combining (2.4.16) and (2.4.20) we obtain the localization formula

$$\mathcal{I} = \int d\mu \mathrm{Pexp}\left(i_G\right). \tag{2.4.22}$$

2.5 Schur index

The Schur index is a specialization of the superconformal index. In addition to Q used in the definition of the superconformal index (2.4.1), we choose another supercharge $\widetilde{Q}$ which has opposite chirality to Q and anti-commutes with Q. By using the supercharge $\widetilde{Q}$ we define

$$\widetilde{\Delta} \equiv 2\{\widetilde{Q}, \widetilde{Q}^\dagger\}. \tag{2.5.1}$$

The Schur index is defined by

$$\mathcal{I} = \mathrm{Tr}\left[(-1)^F x^\Delta \widetilde{x}^{\widetilde{\Delta}} \prod_a u_a^{M_a} \right], \tag{2.5.2}$$

where the trace is taken over the Hilbert space on the cylinder. The fugacities $x, \widetilde{x}, u_a$ are complex parameters. M_a are linear combination of the Cartan generators satisfying

$$[M_a, M_b] = [M_a, Q] = [M_a, \widetilde{Q}] = 0. \tag{2.5.3}$$

Only states satisfying $\Delta = \widetilde{\Delta} = 0$ contribute to the Schur index, and the Schur index is independent of the fugacities $x, \widetilde{x}$. The Schur index inherits the properties of the superconformal index that it is independent of coupling constants, so that the Schur index can be calculated exactly even in strong-coupled theories.

Table 2.2: The Cartan charges for the $\mathcal{N} = 2$ supercharges.

Q^I, $\overline{Q}_I$	H	J	$\overline{J}$	(R, r_2)
$Q^{I=1}_{\alpha=1}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\left(-\frac{1}{2}, -1\right)$
$Q^{I=1}_{\alpha=2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\left(-\frac{1}{2}, -1\right)$
$Q^{I=2}_{\alpha=1}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\left(\frac{1}{2}, -1\right)$
$Q^{I=2}_{\alpha=2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\left(\frac{1}{2}, -1\right)$
$\overline{Q}^{\dot{\alpha}=\dot{1}}_{I=1}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\left(\frac{1}{2}, 1\right)$
$\overline{Q}^{\dot{\alpha}=\dot{2}}_{I=1}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\left(\frac{1}{2}, 1\right)$
$\overline{Q}^{\dot{\alpha}=\dot{1}}_{I=2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\left(-\frac{1}{2}, 1\right)$
$\overline{Q}^{\dot{\alpha}=\dot{2}}_{I=2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\left(-\frac{1}{2}, 1\right)$

In this thesis, we use the convention with $Q = \overline{Q}^{\dot{\alpha}=\dot{1}}_{I=1}$ and $\widetilde{Q} = Q^{I=2}_{\alpha=2}$. Their quantum numbers are shown in Table 2.2. There are five Cartan generators of the $\mathcal{N} = 2$ superconformal symmetry given in (2.2.12):

$$H, \quad J, \quad \overline{J}, \quad R, \quad r_2, \tag{2.5.4}$$

where $R \equiv R^1_1$ and $r_2 \equiv r$. From the relations (2.2.5) and (2.2.6) Δ and $\widetilde{\Delta}$ are given by

$$\Delta = H - \frac{1}{2} r_2 - 2\overline{J} - 2R, \tag{2.5.5}$$

$$\widetilde{\Delta} = H + \frac{1}{2} r_2 - 2J - 2R. \tag{2.5.6}$$

There are three independent linear combinations of (2.5.4) satisfying (2.5.3) and two of three are Δ and $\widetilde{\Delta}$. The remaining one, M_1, can be chosen as follows:

$$M_1 = H + J + \overline{J}. \tag{2.5.7}$$

We may also have flavor generators

$$M_i = F_i. \tag{2.5.8}$$

The $\mathcal{N} = 2$ Schur index is written as

$$\mathcal{I}(y_1, y_i) = \mathrm{Tr}\left[(-1)^F x^{\Delta} \widetilde{x}^{\widetilde{\Delta}} y_1^{H+J+\overline{J}} \prod_i y_i^{F_i}\right]. \tag{2.5.9}$$

We can regard the $\mathcal{N} = 2$ theory as a special $\mathcal{N} = 1$ theory. Let us use the $\mathcal{N} = 1$ sub-algebra with $Q = \overline{Q}^{\dot{\alpha}=\dot{1}}_{I=1}$ to define the superconformal index of the $\mathcal{N} = 2$ theory. Cartan generators commuting with Q are

$$\Delta, \quad M_0 = H + \frac{1}{2}r_2 - 2J - 2R, \quad M_1 = H + J + \overline{J}, \quad M_2 = J, \quad M_i = F_i. \tag{2.5.10}$$

The $\mathcal{N} = 2$ superconformal index is written as

$$\mathcal{I}(y_0, y_1, y_2, y_i) = \mathrm{Tr}\left[(-1)^F x^{\Delta} y_0^{H+\frac{1}{2}r_2-2J-2R} y_1^{H+J+\overline{J}} y_2^{J} \prod_i y_i^{F_i}\right]. \tag{2.5.11}$$

Now, we consider the limit $y_2 \to 1$ in the $\mathcal{N} = 2$ superconformal index (2.5.11). In this limit M_2 in (2.5.10) corresponding to y_2 does not contribute to the index. All generators in (2.5.10) except for M_2 satisfy the condition (2.5.3) and M_0 in (2.5.10) is nothing but $\widetilde{\Delta}$ in (2.5.6). It follows that the $\mathcal{N} = 2$ superconformal index (2.5.11) at $y_2 = 1$ is independent of y_0 corresponding to $M_0 = \widetilde{\Delta}$. Namely, in taking the limit $y_2 \to 1$, the limit $y_0 \to 1$ is automatically taken and the $\mathcal{N} = 2$ superconformal index reduces to the $\mathcal{N} = 2$ Schur index.

We can relate the $\mathcal{N} = 1$ superconformal index (2.4.9) and the $\mathcal{N} = 2$ superconformal index (2.5.11) by using the relations

$$3r_1 = r_2 + 4R, \quad F = 2R - r_2, \tag{2.5.12}$$

where F commutes with $Q^{I=2}_{\alpha=2}$ and we can regard F as a flavor symmetry from the view point of the $\mathcal{N} = 1$ theory.

Chapter 3

$\mathcal{N}=4$ supersymmetric Yang-Mills theory

In Chapter 3, we review the $\mathcal{N}=4$ $U(N)$ supersymmetric Yang-Mills theory in four-dimensions and define the superconformal index of the theory. This theory has the $\mathcal{N}=4$ superconformal symmetry.

We consider the theory in $\mathbb{R}\times\boldsymbol{S}^3$, because it is the boundary of the five-dimensional anti-de Sitter space AdS_5 in the global coordinate. We will discuss the AdS/CFT correspondence in detail in Chapter 4. On the gauge theory side the superconformal index is the partition function in $\mathbb{R}\times\boldsymbol{S}^3$, and is calculated by the localization method from the Lagrangian on the cylinder [40]. See also the Lagrangians and the conformal Killing spinors in [41, 42].

3.1 $\mathcal{N}=4$ vector multiplet

The four-dimensional $\mathcal{N}=4$ supersymmetric Yang-Mills theory consists of an $\mathcal{N}=4$ vector multiplet, whose components are

$$\begin{aligned} A_\mu &: \text{ four-dimensional vector,} \\ \phi_{IJ} &: \text{ six real scalars,} \\ \chi_I, \overline{\chi}^I &: \text{ four Weyl fermions and their conjugates.} \end{aligned} \tag{3.1.1}$$

I, J are indices of $SU(4)$ R-symmetry. χ_I and $\overline{\chi}^I$ belong to $\mathbf{4}$ and $\overline{\mathbf{4}}$ of $SU(4)$, respectively. All component fields of the vector multiplet belong to the adjoint representation of the gauge group G and they are expanded by hermitian generators T^α labeled by $\alpha = 1, \dots, \dim G$,

$$\Phi = \sum_{\alpha=1}^{\dim G} \Phi^\alpha T^\alpha. \tag{3.1.2}$$

The scalar ϕ_{IJ} belongs to the anti-symmetric representation $\mathbf{6}$ of $SU(4)$ and satisfy

$$\phi_{IJ} = -\phi_{JI}, \quad \phi^{IJ} = \frac{1}{2}\epsilon^{IJKL}\phi_{KL} = (\phi_{IJ})^\dagger. \tag{3.1.3}$$

ϕ_{IJ} consists of three independent complex scalars and we denote them by

$$X = \phi_{12}, \quad Y = \phi_{13}, \quad Z = \phi_{14}. \tag{3.1.4}$$

In the following we only consider $G = U(N)$. Then, T_α are $N \times N$ hermitian matrices and α runs from 1 to N^2. The commutation relation and the trace for Φs are expressed as

$$[\Phi, \Phi] = \sum_{\alpha,\beta} \Phi^\alpha \Phi^\beta [T^\alpha, T^\beta], \tag{3.1.5}$$

$$\mathrm{Tr}[\Phi\Phi\cdots\Phi] = \sum_{\alpha,\beta,\gamma} \Phi^\alpha\Phi^\beta\cdots\Phi^\gamma \, \mathrm{Tr}\big[T^\alpha T^\beta \cdots T^\gamma\big]. \tag{3.1.6}$$

The $\mathcal{N} = 4$ supersymmetric transformations on $\mathbb{R} \times \boldsymbol{S}^3$ are generated by the conformal Killing spinors $\epsilon_I, \bar{\epsilon}^I$ satisfying the Killing spinor equations on the cylinder $\mathbb{R} \times \boldsymbol{S}^3$,

$$\nabla_\mu \epsilon_I = \gamma_\mu \bar{\kappa}_I, \quad \nabla_\mu \bar{\epsilon}^I = \gamma_\mu \kappa^I, \tag{3.1.7}$$

where κ^I and $\bar{\kappa}_I$ are arbitrary spinor parameters. γ^μ are gamma matrices and satisfy the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. ∇_μ are the covariant derivatives for spinors, which are expressed as

$$\nabla_\mu \chi = \partial_\mu \chi + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}\chi. \tag{3.1.8}$$

ω_μ^{ab} are the spin-connections satisfying $de^a + \omega^a{}_b \wedge e^b = 0$. The $\mathcal{N} = 4$

supersymmetric transformations for the vector multiplet are

$$\begin{aligned}
\delta A_\mu &= i\left(\overline{\epsilon}^I \gamma_\mu \chi_I + \epsilon_I \gamma_\mu \overline{\chi}^I\right), \\
\delta \chi_I &= \frac{1}{2}\gamma_{\mu\nu}F^{\mu\nu}\epsilon_I + 2\gamma^\mu \left(D_\mu \phi_{IJ}\right)\overline{\epsilon}^J + 4\kappa^J \phi_{IJ} + 2i[\phi_{IK}, \phi^{KJ}]\epsilon_J, \\
\delta \overline{\chi}^I &= \frac{1}{2}\gamma_{\mu\nu}F^{\mu\nu}\overline{\epsilon}^I + 2\gamma^\mu \left(D_\mu \phi^{IJ}\right)\epsilon_J + 4\overline{\kappa}_J \phi^{IJ} + 2i[\phi^{IK}, \phi_{KJ}]\overline{\epsilon}^J, \\
\delta \phi_{IJ} &= i\left(\epsilon_I \chi_J - \epsilon_J \chi_I + \epsilon_{IJKL}\overline{\epsilon}^K \overline{\chi}^L\right), \\
\delta \phi^{IJ} &= i\left(\overline{\epsilon}^I \overline{\chi}^J - \overline{\epsilon}^J \overline{\chi}^I + \epsilon^{IJKL}\epsilon_K \chi_L\right).
\end{aligned} \tag{3.1.9}$$

The covariant derivatives D_μ acting on the scalar and spinor fields are

$$D_\mu \phi_{IJ} = \partial_\mu \phi_{IJ} + i[A_\mu, \phi_{IJ}], \qquad D_\mu \chi_I = \nabla_\mu \chi_I + i[A_\mu, \chi_I]. \tag{3.1.10}$$

3.2 Lagrangian

The Lagrangian of the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory on $\mathbb{R} \times \boldsymbol{S}^3$ is given by

$$\begin{aligned}
\mathcal{L}_{\mathcal{N}=4} = \frac{1}{g_{\mathrm{YM}}^2}\,\mathrm{Tr}\Big[&-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\overline{\chi}^I \gamma^\mu D_\mu \chi_I - \frac{1}{2}D_\mu \phi_{IJ} D^\mu \phi^{IJ} - \frac{1}{2l^2}\phi_{IJ}\phi^{IJ} \\
&+ \chi_I[\chi_J, \phi^{IJ}] + \overline{\chi}^I[\overline{\chi}^J, \phi_{IJ}] + \frac{1}{4}[\phi_{IJ}, \phi_{KL}][\phi^{IJ}, \phi^{KL}]\Big],
\end{aligned} \tag{3.2.1}$$

where g_{YM} is a coupling constant and the trace is taken over the $N \times N$ matrix representation. In the large-N limit, 't Hooft coupling

$$\lambda \equiv g_{\mathrm{YM}}^2 N \tag{3.2.2}$$

is also defined. The potential $-\frac{1}{2l^2}\phi_{IJ}\phi^{IJ}$ comes from the coupling of the scalars to the scalar curvature $R = 6/l^2$, where l is the radius of $\boldsymbol{S}^3$.

In the weak coupling limit $g_{\mathrm{YM}} \to 0$ the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with the Lagrangian (3.2.1) becomes a free theory with the potential $-\frac{1}{2l^2}\phi_{IJ}\phi^{IJ}$:

$$\begin{aligned}
\mathcal{L}_{\mathcal{N}=4}^{\mathrm{free}} = \mathrm{Tr}\Big[&-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\overline{\chi}^I \gamma^\mu \nabla_\mu \chi_I - \frac{1}{2}\left(|\partial_\mu X|^2 + |\partial_\mu Y|^2 + |\partial_\mu Z|^2\right) \\
&- \frac{1}{2l^2}\left(|X|^2 + |Y|^2 + |Z|^2\right)\Big],
\end{aligned} \tag{3.2.3}$$

where we rescaled all fields by $\Phi \to g_{\rm YM}\Phi$. The Lagrangian (3.2.3) play a role in calculating the index via the localization method, which was explained in 2.4.

3.3 Definition of superconformal index

Let us define the superconformal index for the $\mathcal{N}=4$ supersymmetric Yang-Mills theory explicitly following the general definition in 2.4. The $\mathcal{N}=4$ SYM has the $\mathcal{N}=4$ superconformal symmetry, which was introduced in 2.2. The four-dimensional $\mathcal{N}=4$ superconformal algebra has six Cartan generators

$$H, \quad J, \quad \overline{J}, \quad R_X, \quad R_Y, \quad R_Z. \tag{3.3.1}$$

The Hamiltonian H and the spins J and $\overline{J}$ are Cartan generators for the four-dimensional conformal algebra $so(2,4)$, and R_X, R_Y, and R_Z are Cartan generators of the R-symmetry $so(6)$, which are related to $R^I{}_J$ defined in 2.2 by

$$\begin{aligned} R^1{}_1 &= \frac{1}{2}\left(R_X + R_Y + R_Z\right), \\ R^2{}_2 &= \frac{1}{2}\left(R_X - R_Y - R_Z\right), \\ R^3{}_3 &= \frac{1}{2}\left(-R_X + R_Y - R_Z\right), \\ R^4{}_4 &= \frac{1}{2}\left(-R_X - R_Y + R_Z\right). \end{aligned} \tag{3.3.2}$$

The Cartan charges for the supercharges Q^I and $\overline{Q}_I$ are listed in Table 3.1. Q^I and $\overline{Q}_I$ belong to $\overline{\mathbf{4}}$ and $\mathbf{4}$ of $SU(4)$ representations, respectively.

The Cartan charges for the component fields in the vector multiplet are listed in Table 3.2.

In order to define the superconformal index we choose the supercharge with the following Cartan charges:

$$Q = \overline{Q}_{I=1}^{\dot\alpha=\dot 1} : (H, J, \overline{J}, R_X, R_Y, R_Z) = (\tfrac{1}{2}, 0, -\tfrac{1}{2}, +\tfrac{1}{2}, +\tfrac{1}{2}, +\tfrac{1}{2}). \tag{3.3.3}$$

Then, Δ is given by

$$\Delta \equiv 2\{(\overline{Q}_1^{\dot 1})^\dagger, \overline{Q}_1^{\dot 1}\} = H - 2\overline{J} - R_X - R_Y - R_Z. \tag{3.3.4}$$

Table 3.1: The Cartan charges for the supercharges. $[\frac{1}{2}]$ means the spin-half representation for $SU(2)$.

Q^I, $\overline{Q}_I$	H	J	$\overline{J}$	(R_X, R_Y, R_Z)
Q^1	$\frac{1}{2}$	$[\frac{1}{2}]$	0	$\left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$
Q^2	$\frac{1}{2}$	$[\frac{1}{2}]$	0	$\left(-\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2}\right)$
Q^3	$\frac{1}{2}$	$[\frac{1}{2}]$	0	$\left(+\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2}\right)$
Q^4	$\frac{1}{2}$	$[\frac{1}{2}]$	0	$\left(+\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2}\right)$
$\overline{Q}_1$	$\frac{1}{2}$	0	$[\frac{1}{2}]$	$\left(+\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2}\right)$
$\overline{Q}_2$	$\frac{1}{2}$	0	$[\frac{1}{2}]$	$\left(+\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$
$\overline{Q}_3$	$\frac{1}{2}$	0	$[\frac{1}{2}]$	$\left(-\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2}\right)$
$\overline{Q}_4$	$\frac{1}{2}$	0	$[\frac{1}{2}]$	$\left(-\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2}\right)$

Table 3.2: The Cartan charges of the $\mathcal{N} = 4$ vector multiplet.

FIelds	H	J	$\overline{J}$	$SO(6)$ rep.
$F_{ab}, F_{\dot{a}\dot{b}}$	2	$\pm\frac{1}{2}$	$\pm\frac{1}{2}$	**1**
χ_I	$\frac{3}{2}$	$\pm\frac{1}{2}$	0	**4**
$\overline{\chi}^I$	$\frac{3}{2}$	0	$\pm\frac{1}{2}$	$\overline{\mathbf{4}}$
ϕ_{IJ}	1	0	0	**6**

There are five independent linear combinations for Cartan generators commuting with Q:

$$\Delta, \quad H + \overline{J}, \quad J, \quad R_X - R_Y, \quad R_Y - R_Z. \tag{3.3.5}$$

We define the superconformal index by

$$\mathcal{I}(q, y, u, v) = \mathrm{Tr}\left[(-1)^F x^{\Delta} q^{H+\overline{J}} y^{2J} u^{R_X - R_Y} v^{R_Y - R_Z}\right]. \tag{3.3.6}$$

Because the index $\mathcal{I}$ counts only the BPS states with $\Delta = 0$, $\mathcal{I}$ is independent of x. We also use the following notation:

$$u_1 = u, \quad u_2 = \frac{v}{u}, \quad u_3 = \frac{1}{v}, \qquad (u_1 u_2 u_3 = 1). \tag{3.3.7}$$

In this notation the corresponding part of the index becomes

$$u_1^{R_X} u_2^{R_Y} u_3^{R_Z} = u^{R_X - R_Y} v^{R_Y - R_Z}. \tag{3.3.8}$$

The Cartan generators $R_X - R_Y$ and $R_Y - R_Z$ are the ones of the R-symmetry subgroup $SU(3) \subset SU(4)$.

By using the localization method we rewrite the superconformal index into a tractable form,

$$\mathcal{I}_{U(N)}(q, y, u, v) = \int d\mu_N \mathrm{Pexp}\left(i_{\mathrm{vec}}(q, y, u, v)\chi_N^{\mathrm{adj}}\right), \tag{3.3.9}$$

where χ_N^{adj} is the character for the adjoint representation of $U(N)$ given by

$$\chi_N^{\mathrm{adj}} = \sum_{i,j=1}^{N} \frac{\zeta_i}{\zeta_j}. \tag{3.3.10}$$

Now, let us calculate the single-particle index $i_{\mathrm{vec}}(q, y, u, v)$. The procedure is as follows: Since we are considering the weakly coupled limit, the fields that appear in the theory are free fields. We calculate the mode expansions of free fields and construct the single-particle states as given in (2.4.18). The single-particle states are composed by a single action of the creation operators $A_n^\dagger$ on the vacuum $|0\rangle$: $A_n^\dagger |0\rangle$. The single-particle index can be obtained by examining the quantum numbers of such single-particle states. A convenient way to investigate the quantum numbers is to use local operators corresponding to the single-particle states. In general CFT, there is a one-to-one correspondence between states $|n\rangle$ on $\mathbb{R} \times \boldsymbol{S}^3$ and local operators $\mathcal{O}_n$ inserted at the origin on $\mathbb{R}^4$:

$$|n\rangle \longleftrightarrow \mathcal{O}_n. \tag{3.3.11}$$

It is called the state/operator correspondence. $\mathbb{R} \times \boldsymbol{S}^3$ is related to $\mathbb{R}^4$ by the Weyl transformation and Wick rotation, and the origin on $\mathbb{R}^4$ corresponds to the infinite past $t = -\infty$ on $\mathbb{R} \times \boldsymbol{S}^3$. It follows that any state at $t = t_0$ can be generated by an operator inserted in the infinite past given as the origin. (See Figure 3.1). If $|n\rangle$ is a k-particle state, the corresponding operator consists of k elementary fields and derivatives on $\mathbb{R}^4$. Now, we consider the $k = 1$ case. The local operator consists of one elementary field and derivatives. As a simple example, let us consider the case for the scalar field ϕ_{12} given in (3.1.4). The general form of the local operator is given by

$$\left(\partial_{1\dot{1}}\right)^{l_1} \left(\partial_{1\dot{2}}\right)^{l_2} \left(\partial_{2\dot{1}}\right)^{l_3} \left(\partial_{2\dot{2}}\right)^{l_4} \phi_{12}, \tag{3.3.12}$$

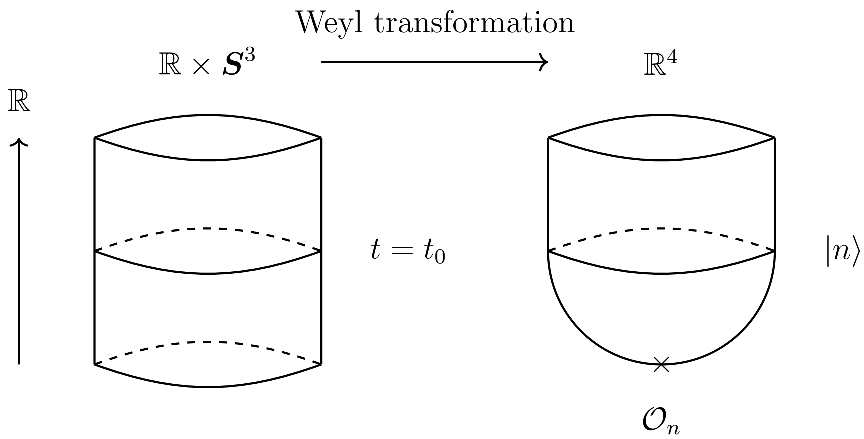

Figure 3.1: A state/operator correspondence. The infinite past $t = -\infty$ on $\mathbb{R} \times \boldsymbol{S}^3$ is moved onto the origin on $\mathbb{R}^4$ by the Weyl transformation. A state $|n\rangle$ at $t = t_0$ corresponds to a local operator inserted at the origin.

where $l_{1,2,3,4} \in \mathbb{Z}_{\geq 0}$ and the derivatives $\partial_{\alpha\dot{\alpha}}$ have the quantum numbers listed in Table 3.3. By summing over $l_{1,2,3,4}$ we obtain the contributions of ϕ_{12} to the single-particle index:

$$\sum_{l_1,l_2,l_3,l_4=0}^{\infty} \left(q^{\frac{3}{2}}y\right)^{l_1} \left(q^{\frac{1}{2}}y\right)^{l_2} \left(q^{\frac{3}{2}}y^{-1}\right)^{l_3} \left(q^{\frac{1}{2}}y^{-1}\right)^{l_4} qu. \tag{3.3.13}$$

Note that operators in (3.3.12) are not all linearly independent. Among the contributions we need to subtract the degree of freedom in the equation of

Table 3.3: The quantum numbers of the derivatives $\partial_{\alpha\dot{\alpha}}$ and the elementary field ϕ_{12}.

$Q^I, \overline{Q}_I$	H	J	$\overline{J}$	(R_X, R_Y, R_Z)
$\partial_{1\dot{1}}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$(0,0,0)$
$\partial_{1\dot{2}}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	$(0,0,0)$
$\partial_{2\dot{1}}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	$(0,0,0)$
$\partial_{2\dot{2}}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$(0,0,0)$
ϕ_{12}	1	0	0	$(1,0,0)$

motion for ϕ_{12}:

$$\partial \cdots \partial \partial^2 \phi_{12} = 0. \tag{3.3.14}$$

The subtracted contributions are

$$\sum_{l_1,l_2,l_3,l_4=0}^{\infty} \left(q^{\frac{3}{2}} y\right)^{l_1} \left(q^{\frac{1}{2}} y\right)^{l_2} \left(q^{\frac{3}{2}} y^{-1}\right)^{l_3} \left(q^{\frac{1}{2}} y^{-1}\right)^{l_4} q^3 u. \tag{3.3.15}$$

Performing the above operations for all elementary fields in (3.1.1) yields the single-particle index. In fact, since the cancellation between the contributions from bosons and fermions, in the construction of the local operator in (3.3.12) we do not have to take account of derivatives and fields with $\Delta > 0$.

By summing up all the contributions from each component fields in the free vector multiplet listed in Table 3.4, we obtain the single-particle index

$$i_{\rm vec}(q,y,u,v) = \frac{q\chi_{[1,0]} - q^{\frac{3}{2}}(y+y^{-1}) - q^2\chi_{[0,1]} + 2q^3}{(1-q^{\frac{3}{2}}y)(1-q^{\frac{3}{2}}y^{-1})}, \tag{3.3.16}$$

where $\chi_{[a,b]}$ is the $SU(3)$ Weyl character for the representation with the Dynkin label $[a,b]$ given by

$$\chi_{[a,b]} = \begin{vmatrix} u^{a+1} & 1 & u^{-b-1} \\ (v/u)^{a+1} & 1 & (v/u)^{-b-1} \\ (1/v)^{a+1} & 1 & (1/v)^{-b-1} \end{vmatrix} \Bigg/ \begin{vmatrix} u & 1 & u^{-1} \\ (v/u) & 1 & (v/u)^{-1} \\ (1/v) & 1 & (1/v)^{-1} \end{vmatrix}, \tag{3.3.17}$$

and we give simple examples of the $SU(3)$ Weyl characters:

$$\chi_{[1,0]} = u + \frac{v}{u} + \frac{1}{v}, \quad \chi_{[0,1]} = \frac{1}{u} + \frac{u}{v} + v. \tag{3.3.18}$$

3.4 Definition of Schur index

Let us define the Schur index for the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory explicitly following the general definition in 2.5. We need to choose another supercharge $\widetilde{Q}$ which has opposite chirality to Q and anti-commutes with Q. We use the one with the following Cartan charges:

$$\widetilde{Q} = Q^{I=4}_{\alpha=2} : (H, J, \overline{J}, R_X, R_Y, R_Z) = (\tfrac{1}{2}, -\tfrac{1}{2}, 0, +\tfrac{1}{2}, +\tfrac{1}{2}, -\tfrac{1}{2}). \tag{3.4.1}$$

Table 3.4: $\Delta = 0$ states in the $\mathcal{N} = 4$ vector multiplet.

Field	$[J, \overline{J}]_E^{(R_X, R_Y, R_Z)}$	index
X	$[0,0]_1^{(1,0,0)}$	$+qu$
Y	$[0,0]_1^{(0,1,0)}$	$+qu^{-1}v$
Z	$[0,0]_1^{(0,0,1)}$	$+qv^{-1}$
$\lambda_{I=1,\alpha=1}$	$[+\frac{1}{2},0]_{\frac{3}{2}}^{(\frac{1}{2},\frac{1}{2},\frac{1}{2})}$	$-q^{\frac{3}{2}}y$
$\lambda_{I=1,\alpha=2}$	$[-\frac{1}{2},0]_{\frac{3}{2}}^{(\frac{1}{2},\frac{1}{2},\frac{1}{2})}$	$-q^{\frac{3}{2}}y^{-1}$
$\overline{\lambda}^{I=2}_{\dot{\alpha}=\dot{1}}$	$[0,+\frac{1}{2}]_{\frac{3}{2}}^{(\frac{1}{2},\frac{1}{2},-\frac{1}{2})}$	$-q^2u^{-1}$
$\overline{\lambda}^{I=3}_{\dot{\alpha}=\dot{1}}$	$[0,+\frac{1}{2}]_{\frac{3}{2}}^{(\frac{1}{2},\frac{1}{2},-\frac{1}{2})}$	$-q^2uv^{-1}$
$\overline{\lambda}^{I=4}_{\dot{\alpha}=\dot{1}}$	$[0,+\frac{1}{2}]_{\frac{3}{2}}^{(\frac{1}{2},\frac{1}{2},-\frac{1}{2})}$	$-q^2v$
$\partial\lambda_{I=1,\dot{\alpha}=\dot{1}}$	$[0,+\frac{1}{2}]_{\frac{5}{2}}^{(\frac{1}{2},\frac{1}{2},\frac{1}{2})}$	$+q^3$
$F_{\dot{1}\dot{1}}$	$[0,+1]_2^{(0,0,0)}$	$+q^3$
$\partial_{1\dot{1}}$	$[+\frac{1}{2},+\frac{1}{2}]_1^{(0,0,0)}$	$+q^{\frac{3}{2}}y$
$\partial_{2\dot{1}}$	$[-\frac{1}{2},+\frac{1}{2}]_1^{(0,0,0)}$	$+q^{\frac{3}{2}}y^{-1}$

The anti-commutation relation for the supercharge is

$$\widetilde{\Delta} \equiv 2\{(Q_2^4)^\dagger, Q_2^4\} = H - 2J - R_X - R_Y + R_Z. \tag{3.4.2}$$

Δ and $\widetilde{\Delta}$ have bounds $\Delta, \widetilde{\Delta} \geq 0$ and states saturating the bounds reproduce a short multiplet. There are four linear combinations of the Cartan charges which commute with Q and $\widetilde{Q}$:

$$\Delta, \quad \widetilde{\Delta}, \quad H + J + \overline{J}, \quad R_X - R_Y. \tag{3.4.3}$$

We define the Schur index,

$$\mathcal{I}(q,u) = \mathrm{Tr}\left[(-1)^F x^{\Delta} \widetilde{x}^{\widetilde{\Delta}} q^{H+J+\overline{J}} u^{R_X - R_Y}\right]. \tag{3.4.4}$$

Because only the BPS states with $\Delta = \widetilde{\Delta} = 0$ contribute to the Schur index, $\mathcal{I}$ is independent of x and $\widetilde{x}$. The Schur index is a specialization of the superconformal index, and the Schur index is obtained from the superconformal index by taking the limit $y = q^{\frac{1}{2}}$, $v = 1$. The survived R-symmetry is $SU(2)$ and the corresponding Cartan charge is $R_X - R_Y$.

The localization method can be used for the Schur index:

$$\mathcal{I}_{U(N)}(q,u) = \int d\mu_N \mathrm{Pexp}\left(i_{\mathrm{vec}}(q,u)\chi_N^{\mathrm{adj}}\right). \tag{3.4.5}$$

The contribution from each component field in the free vector multiplet is listed in Table 3.5. The single-particle index is given by

$$i_{\mathrm{vec}}(q,u) = \frac{q\chi_1 - 2q^2}{1-q^2}, \tag{3.4.6}$$

where χ_a is the $SU(2)$ Weyl character given by

$$\chi_a(u) = \frac{u^{a+1} - u^{-(a+1)}}{u - u^{-1}}. \tag{3.4.7}$$

In [43], Bourdier, Drukker, and Felix analyzed the Schur index (3.4.5) at $u = 1$ and derived the exact solution of $\mathcal{I}_{U(N)}(q, u = 1)$. The following general formula is known:

$$\left.\frac{\mathcal{I}_{U(N)}}{\mathcal{I}_{U(\infty)}}\right|_{u=1} = \sum_{n=0}^{\infty} \mathcal{I}_n^{\mathrm{BDF}}, \qquad \mathcal{I}_n^{\mathrm{BDF}} = (-1)^n ({}_{N+n}C_N + {}_{N+n-1}C_N) q^{nN+n^2}, \tag{3.4.8}$$

where ${}_nC_k$ is the binomial coefficient.

Table 3.5: $\Delta = \widetilde{\Delta} = 0$ states in the $\mathcal{N} = 4$ vector multiplet.

Field	$[J, \overline{J}]_E^{(R_X,R_Y,R_Z)}$	index
X	$[0,0]_1^{(1,0,0)}$	$+qu$
Y	$[0,0]_1^{(0,1,0)}$	$+qu^{-1}$
λ_{1+}	$[+\frac{1}{2},0]_{\frac{3}{2}}^{(\frac{1}{2},\frac{1}{2},\frac{1}{2})}$	$-q^2$
$\overline{\lambda}^4_{\dot{+}}$	$[0,+\frac{1}{2}]_{\frac{3}{2}}^{(\frac{1}{2},\frac{1}{2},-\frac{1}{2})}$	$-q^2$
$\partial_{+\dot{+}}$	$[+\frac{1}{2},+\frac{1}{2}]_1^{(0,0,0)}$	$+q^2$

3.5 Numerical results

3.5.1 Superconformal index

We give numerical results of $\mathcal{I}_{U(N)}(q,y,u,v)$ for small N cases:

$$\begin{aligned}
\mathcal{I}_{U(0)} &= 1,\\
\mathcal{I}_{U(1)} &= 1 + q\chi_{[1,0]} - q^{\frac{3}{2}}\chi_1(y) + q^2(\chi_{[2,0]} - \chi_{[0,1]}) + q^3(-\chi_{[1,1]} + \chi_{[3,0]} - \chi_2(y)\\
&\quad + 1) + q^{\frac{7}{2}}\chi_1(y)\chi_{[0,1]} + \cdots,\\
\mathcal{I}_{U(2)} &= 1 + q\chi_{[1,0]} - q^{\frac{3}{2}}\chi_1(y) + q^2(2\chi_{[2,0]} - \chi_{[0,1]}) - q^{\frac{5}{2}}\chi_1(y)\chi_{[1,0]} + q^3(-\chi_{[1,1]}\\
&\quad + 2\chi_{[3,0]} - \chi_2(y) + 2) + q^{\frac{7}{2}}\chi_1(y)(\chi_{[0,1]} - \chi_{[2,0]}) + \cdots,\\
\mathcal{I}_{U(3)} &= 1 + q\chi_{[1,0]} - q^{\frac{3}{2}}\chi_1(y) + q^2(2\chi_{[2,0]} - \chi_{[0,1]}) - q^{\frac{5}{2}}\chi_1(y)\chi_{[1,0]} + q^3(-\chi_{[1,1]}\\
&\quad + 3\chi_{[3,0]} - \chi_2(y) + 2) + q^{\frac{7}{2}}\chi_1(y)(\chi_{[0,1]} - 2\chi_{[2,0]}) + \cdots,\\
\mathcal{I}_{U(4)} &= 1 + q\chi_{[1,0]} - q^{\frac{3}{2}}\chi_1(y) + q^2(2\chi_{[2,0]} - \chi_{[0,1]}) - q^{\frac{5}{2}}\chi_1(y)\chi_{[1,0]} + q^3(-\chi_{[1,1]}\\
&\quad + 3\chi_{[3,0]} - \chi_2(y) + 2) + q^{\frac{7}{2}}\chi_1(y)(\chi_{[0,1]} - 2\chi_{[2,0]}) + \cdots.
\end{aligned} \tag{3.5.1}$$

The above numerical results (3.5.1) show that each term in the q-expansion converge to some values as the rank N increases. In fact, $\mathcal{I}_{U(N)}$ converge in the large-N limit to the large-N index $\mathcal{I}_{U(\infty)}$. $\mathcal{I}_{U(\infty)}$ can be obtain from the saddle point analysis [4] given by

$$\mathcal{I}_{U(\infty)} = \mathrm{Pexp}\left(\frac{qu}{1-qu} + \frac{qu^{-1}v}{1-qu^{-1}v} + \frac{qv^{-1}}{1-qv^{-1}} - \frac{q^{\frac{3}{2}}y}{1-q^{\frac{3}{2}}y} - \frac{q^{\frac{3}{2}}y^{-1}}{1-q^{\frac{3}{2}}y^{-1}}\right). \tag{3.5.2}$$

The numerical result of $\mathcal{I}_{U(\infty)}$ is

$$\begin{aligned}\mathcal{I}_{U(\infty)} = 1 &+ q\chi_{[1,0]} - q^{\frac{3}{2}}\chi_1(y) + q^2(2\chi_{[2,0]} - \chi_{[0,1]}) - q^{\frac{5}{2}}\chi_1(y)\chi_{[1,0]} + q^3(-\chi_{[1,1]} \\ &+ 3\chi_{[3,0]} - \chi_2(y) + 2) + q^{7/2}\chi_1(y)(\chi_{[0,1]} - 2\chi_{[2,0]}) + \cdots \end{aligned} \tag{3.5.3}$$

3.5.2 Schur index

We give numerical results of $\mathcal{I}_{U(N)}(q,u)$ for small N cases:

$$\begin{aligned}
&\mathcal{I}_{U(0)} = 1, \\
&\mathcal{I}_{U(1)} = 1 + q\chi_1 + q^2(\chi_2 - 2) + q^3(\chi_3 - \chi_1) + q^4(\chi_4 - \chi_2) + \cdots, \\
&\mathcal{I}_{U(2)} = 1 + q\chi_1 + q^2(2\chi_2 - 2) + q^3(2\chi_3 - 2\chi_1) + q^4(3\chi_4 - 3\chi_2) + \cdots, \\
&\mathcal{I}_{U(3)} = 1 + q\chi_1 + q^2(2\chi_2 - 2) + q^3(3\chi_3 - 2\chi_1) + q^4(-4\chi_2 + 4\chi_4 + 1) + \cdots, \\
&\mathcal{I}_{U(4)} = 1 + q\chi_1 + q^2(2\chi_2 - 2) + q^3(3\chi_3 - 2\chi_1) + q^4(-4\chi_2 + 5\chi_4 + 1) + \cdots.
\end{aligned} \tag{3.5.4}$$

As in the case of the superconformal index, $\mathcal{I}_{U(N)}(q,u)$ converge in the large-N limit to the large-N index $\mathcal{I}_{U(\infty)}(q,u)$ given by

$$\mathcal{I}_{U(\infty)} = \mathrm{Pexp}\left(\frac{qu}{1-qu} + \frac{qu^{-1}}{1-qu^{-1}} - \frac{q^2}{1-q^2}\right). \tag{3.5.5}$$

The numerical result of $\mathcal{I}_{U(\infty)}$ is

$$\mathcal{I}_{U(\infty)} = 1 + q\chi_1 + q^2(2\chi_2 - 2) + q^3(3\chi_3 - 2\chi_1) + q^4(-4\chi_2 + 5\chi_4 + 1) + \cdots. \tag{3.5.6}$$

We are interested in the difference between $\mathcal{I}_{U(N)}$ and $\mathcal{I}_{U(\infty)}$, and it is useful to take the form $\mathcal{I}_{U(N)}/\mathcal{I}_{U(\infty)}$. We give the numerical results of $\mathcal{I}_{U(N)}/\mathcal{I}_{U(\infty)}$:

$$\begin{aligned}
&\mathcal{I}_{U(0)}/\mathcal{I}_{U(\infty)} = 1 - q\chi_1 + q^2(3 - \chi_2) + q^4(5 - \chi_2) + q^5(-\chi_1 - \chi_3 + \chi_5) + \cdots, \\
&\mathcal{I}_{U(1)}/\mathcal{I}_{U(\infty)} = 1 - q^2\chi_2 + q^3(2\chi_1 - \chi_3) + q^4(2\chi_2 - \chi_4 - 1) + q^6(2\chi_2 - 1) + \cdots, \\
&\mathcal{I}_{U(2)}/\mathcal{I}_{U(\infty)} = 1 - q^3\chi_3 + q^4(2\chi_2 - \chi_4 - 1) + q^5(\chi_1 + \chi_3 - \chi_5) + \cdots, \\
&\mathcal{I}_{U(3)}/\mathcal{I}_{U(\infty)} = 1 - q^4\chi_4 + q^5(-\chi_1 + 2\chi_3 - \chi_5) + q^6(\chi_4 - \chi_6 + 2) + \cdots, \\
&\mathcal{I}_{U(4)}/\mathcal{I}_{U(\infty)} = 1 - q^5\chi_5 + q^6(-\chi_2 + 2\chi_4 - \chi_6) + q^7(\chi_1 + \chi_5 - \chi_7) + \cdots.
\end{aligned} \tag{3.5.7}$$

For $u = 1$ case most of the terms in the numerical results in (3.5.7) vanish, and the numerical results with $u = 1$ become simpler:

$$\begin{aligned}
\mathcal{I}_{U(0)}/\mathcal{I}_{U(\infty)}\big|_{u=1} &= 1 - 2q + 2q^4 - 2q^9 + 2q^{16} - 2q^{25} + \cdots \,, \\
\mathcal{I}_{U(1)}/\mathcal{I}_{U(\infty)}\big|_{u=1} &= 1 - 3q^2 + 5q^6 - 7q^{12} + 9q^{20} - 11q^{30} + \cdots \,, \\
\mathcal{I}_{U(2)}/\mathcal{I}_{U(\infty)}\big|_{u=1} &= 1 - 4q^3 + 9q^8 - 16q^{15} + 25q^{24} + \cdots \,, \\
\mathcal{I}_{U(3)}/\mathcal{I}_{U(\infty)}\big|_{u=1} &= 1 - 5q^4 + 14q^{10} - 30q^{18} + 55q^{28} + \cdots \,, \\
\mathcal{I}_{U(4)}/\mathcal{I}_{U(\infty)}\big|_{u=1} &= 1 - 6q^5 + 20q^{12} - 50q^{21} + \cdots \,.
\end{aligned} \tag{3.5.8}$$

These results are consistent with the general formula (3.4.8).

Chapter 4

AdS/CFT correspondence

In Chapter 4, we review the AdS/CFT correspondence, which is a conjectural relationship between a superstring theory on the anti-de Sitter space and a conformal field theory. Especially, we introduce the original model [1] of the AdS/CFT correspondence, which is a relationship between the four-dimensional $\mathcal{N} = 4$ $U(N)$ supersymmetric Yang-Mills theory and the Type IIB string theory in the $AdS_5 \times \boldsymbol{S}^5$ spacetime.

We first consider a system of N coincident D3-branes in the ten-dimensional flat spacetime $\mathbb{R}^{1,9}$. There are two descriptions of the system, the gauge theory description and the gravity description. We first explain the gauge theory description in 4.1 and also explain the gravity description in 4.2.

After that, we focus on the superconformal index in the AdS/CFT correspondence and discuss the case for large N and finite N, respectively.

4.1 $\mathcal{N} = 4$ SYM from open strings

An open string is a string with endpoints. The endpoints are attached to objects, so-called D-branes. A D-brane is a dynamical object with different dimensions. D-branes with spatial dimension p are called Dp-branes. The motion of the open string attached to the D-branes generates gauge theories in various dimensions.

We discuss a system of coincident N Dp-branes labeled by $i = 1. \dots, N$. Because an open string has two endpoints, there are N^2 ways for the two endpoints to be attached to the Dp-branes. The degrees of freedom are described by the Chan-Paton factor [44] specified by (i, j).

Table 4.1: Massless states of the open string on the Dp-brane. The index a running from one to N^2 labels the adjoint representation of the $U(N)$ gauge group.

Sector	Field	
NS	A^a_μ	$(p+1)$-dimensional vector
	ϕ^a_I	$(9-p)$ real scalars
R	λ^a	fermion

Let us quantize the open string with the Chan-Paton factor (i,j) and obtain states described by the fluctuation modes of the open string. The states have also the Chan-Paton factor (i,j) and belong to the adjoint representation of the $U(N)$ gauge group. We denote the parallel and transverse directions to the Dp-brane as $\mu = 0,\dots,p$ and $I = p+1,\dots,9$, respectively. We obtain the massless states listed in Table 4.1. The index a in Table 4.1 labels the adjoint representation of the $U(N)$ gauge group. In summary, the worldvolume theory on the coincident N Dp-branes is the supersymmetric $U(N)$ Yang-Mills theory with the maximal supersymmetry.

The Lorentz symmetry $SO(1,9)$ in the target space is broken down to the following symmetry:

$$SO(1,p) \times SO(9-p) \subset SO(1,9). \tag{4.1.1}$$

The $SO(1,p)$ is the Lorentz symmetry in the worldvolume theory on the Dp-brane. The $SO(9-p)$ is a rotation of the transverse coordinates to the Dp-brane, and it is an internal symmetry in the worldvolume theory.

In particular, the worldvolume theory on the N coincident D3-branes is the $\mathcal{N}=4$ $U(N)$ supersymmetric Yang-Mills theory introduced in Chapter 3. The internal symmetry $SO(6) \simeq SU(4)$ is interpreted as the R-symmetry of the $\mathcal{N}=4$ supersymmetry. The fields on the worldvolume form an $\mathcal{N}=4$ vector multiplet, which contains a four-dimensional vector field, six real scalars, and four Weyl fermions.

4.2 Near horizon geometry of N coincident D3-branes

We consider a system of the N coincident D3-branes along x^0, x^1, x^2, x^3. In the gravity description, the branes are given as a supergravity solution. The metric is given by

$$ds^2 = f^{-\frac{1}{2}} ds^2_{\mathbb{R}^{1,3}} + f^{\frac{1}{2}} \left(dr^2 + r^2 d\Omega_5^2 \right),$$
$$f \equiv 1 + \frac{L^4}{r^4}, \quad L \equiv (4\pi g_s N)^{\frac{1}{4}} l_s, \tag{4.2.1}$$

where $ds^2_{\mathbb{R}^{1,3}} = \eta_{\mu\nu} dx^\mu dx^\nu$ and $d\Omega_5^2$ is the metric on the unit five-dimensional sphere. The classical solution is almost flat in the asymptotic region $r \gg L$.

Degrees of freedom propagating in the asymptotic region are decoupled from localized degrees of freedom near the horizon at $r = 0$ in the limit $l_s \to 0$. We are interested in the localized degrees of freedom. In the near horizon region $r \ll L$ the metric (4.2.1) becomes

$$ds^2 = L^2 \left(\frac{r^2}{L^4} ds^2_{\mathbb{R}^{1,3}} + \frac{dr^2}{r^2} \right) + L^2 d\Omega_5^2. \tag{4.2.2}$$

The first term represents the five-dimensional anti-de Sitter (AdS_5) spacetime with the radius L and the second term is the five-dimensional sphere ($\boldsymbol{S}^5$) with the radius L. Therefore, localized degrees of freedom around the event horizon are described by the Type IIB string theory in $AdS_5 \times \boldsymbol{S}^5$.

In the gravity description, there is a background gauge field, which is induced by the charge of the N coincident D3-branes. The N coincident D3-branes couple to the four-form gauge field C_4 in the Type IIB supergravity. The flux of the gauge field is given by

$$\int_{\boldsymbol{S}^5} dC_4 = 2\pi N. \tag{4.2.3}$$

In order to obtain a concrete form for C_4 we introduce the metric for $\boldsymbol{S}^5$,

$$L^2 d\Omega_5^2 = L^2 \left(\sin^2\theta d\phi^2 + d\theta^2 + \cos^2\theta d\Omega_3^2 \right), \tag{4.2.4}$$

and the volume element on $\boldsymbol{S}^5$,

$$\omega_5 = L^5 \sin\theta \cos^3\theta d\theta \wedge d\phi \wedge d\Omega_3, \tag{4.2.5}$$

where $0 \leq \theta < \frac{\pi}{2}$ and $0 \leq \phi < 2\pi$. Note that the volume of $\boldsymbol{S}^5$ is given by $V_5 = \int_{\boldsymbol{S}^5} \omega_5 = \pi^3 L^5$. By using the volume element, the flux is written by

$$dC_4 = \frac{2\pi N}{V_5}\omega_5 = \frac{4T_{\rm D3}}{L}\omega_5. \tag{4.2.6}$$

We integrate (4.2.6) for θ and C_4 is given by

$$C_4 = \frac{2\pi N}{V_5}\frac{r^5}{4}\left(1-\cos^4\theta\right)\frac{\partial\phi}{\partial t}dt\wedge d\Omega_3, \tag{4.2.7}$$

where the integration constant is determined so that $C_4 = 0$ at $\theta = 0$.

Global coordinate system

The five-dimensional AdS space can be given as the hypersurface in $\mathbb{R}^{2,4}$,

$$X_{-1}^2 + X_0^2 - X_1^2 - \cdots - X_4^2 = L^2. \tag{4.2.8}$$

The coordinates x^μ and r in (4.2.2) are related to X_I by

$$\begin{aligned} X_{-1} &= \frac{L^2}{2r}\left(1+\frac{r^2}{L^4}\left(L^2+\eta_{\mu\nu}x^\mu x^\nu\right)\right), \\ X_\mu &= \frac{r}{L}x^\mu, \\ X_4 &= \frac{L^2}{2r}\left(1-\frac{r^2}{L^4}\left(L^2-\eta_{\mu\nu}x^\mu x^\nu\right)\right), \end{aligned} \tag{4.2.9}$$

and the metric of AdS_5 is given by

$$ds^2 = -dX_{-1}^2 - dX_0^2 + \sum_{i=1}^4 dX_i^2. \tag{4.2.10}$$

We introduce the global coordinates (ρ, τ, ω_i) in AdS_5 by

$$\begin{aligned} X_{-1} &= L\cosh\rho\cos\tau, \\ X_0 &= L\cosh\rho\sin\tau, \\ X_i &= L\omega_i\sinh\rho, \quad \text{for } i = 1,2,3,4, \end{aligned} \tag{4.2.11}$$

where $0 \leq \tau < 2\pi$ and $\rho \in \mathbb{R}_{\geq 0}$. ω_i are the coordinates of $\boldsymbol{S}^3$ satisfying $\sum_i \omega_i^2 = 1$. The metric of AdS_5 is

$$ds^2 = L^2\left(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega_3^2\right). \tag{4.2.12}$$

The boundary is given by the limit $\rho \to \infty$. For large ρ, the metric is approximately

$$ds^2 \simeq \frac{L^2 e^{2\rho}}{4} \left(-d\tau^2 + d\Omega_3^2\right) \tag{4.2.13}$$

and we see that the boundary of AdS_5 is $\mathbb{R} \times \boldsymbol{S}^3$.

4.3 AdS/CFT correspondence

The AdS/CFT correspondence claims that a superstring theory in $d+1$-dimensional AdS spacetime (AdS_{d+1}) with a compact internal manifold X is equivalent to a d-dimensional CFT on the boundary of AdS_{d+1}. The more precise statement in [1] for the simplest example is that

the Type IIB string theory with the string length l_s and the string coupling g_s
in $AdS_5 \times \boldsymbol{S}^5$ with the AdS radius L and N units of five-form flux on $\boldsymbol{S}^5$
is equivalent to
the $\mathcal{N} = 4$ $U(N)$ SYM on $\mathbb{R} \times \boldsymbol{S}^3$ with the coupling constant g_{YM}.

We can see from the metric (4.2.12) that the topology of AdS_5 is $\mathbb{R}\times B_4$, where B_4 is the four-dimensional open ball. It has the boundary $\mathbb{R}\times\boldsymbol{S}^3$ and the CFT realizes on the boundary. See Figure 4.1. The AdS/CFT correspondence is an example for the holography. As a simple consistency check of the AdS/CFT correspondence, let us compare the symmetries of two theories. From the metric (4.2.10) we see that the AdS_5 space has the isometry $SO(2,4)$. The five-dimensional sphere $\boldsymbol{S}^5$ has the isometry $SO(6)$. Therefore, the Type IIB string theory in $AdS_5\times\boldsymbol{S}^5$ has the symmetry $SO(2,4)\times SO(6)$. On the other hand, the $\mathcal{N} = 4$ SYM has $\mathcal{N} = 4$ superconformal symmetry and its bosonic part is $SO(2,4) \times SO(6)$, where $SO(2,4)$ is the four-dimensional conformal group and $SO(6)$ is the R-symmetry. We see that two theories have the same symmetry. This fact is important because it enable us to define the superconformal index on both sides of the correspondence. The AdS/CFT correspondence also claim that their Hilbert spaces of two systems should be the same. This means that two indices calculated on each side must be identical.

There are many different ways to choose a compact space X and various CFTs or gauge theories appear depending on how X is chosen. Several examples are shown in Table 4.2.

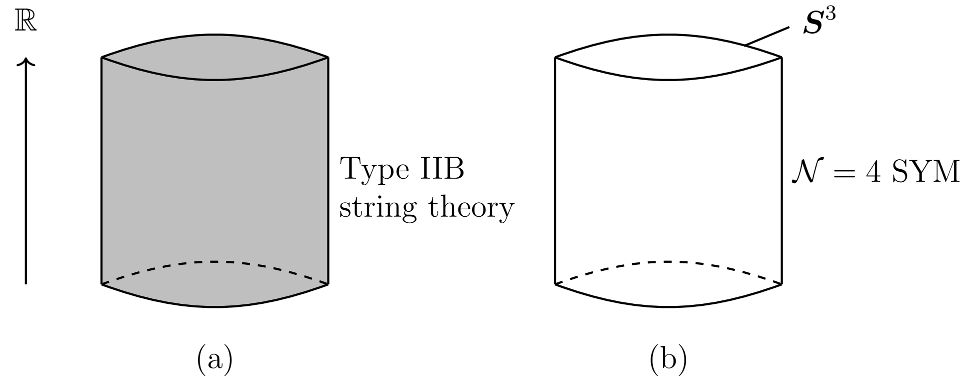

Figure 4.1: (a) The Type IIB string theory in $\mathbb{R} \times B^4$. (b) The $\mathcal{N} = 4$ SYM in $\mathbb{R} \times \boldsymbol{S}^3$.

We can give a more quantitative relation called the Gubser-Klebanov-Polyakov-Witten (GKP-Witten) relation [45, 46]:

$$\left\langle e^{\int d^d x \sum_i J_i \mathcal{O}_i} \right\rangle_{\text{CFT}} = \int \prod_i \mathcal{D}\Phi_i e^{-S[\Phi_i]}. \tag{4.3.1}$$

The left hand side is the generating function of correlation functions in the CFT. $\mathcal{O}_i$ are operators in the CFT and J_i are corresponding source fields. The left hand side is given by the following path integral with respect to the fields in the CFT:

$$\left\langle e^{\int d^d x \sum_i J_i \mathcal{O}_i} \right\rangle_{\text{CFT}} = \int \mathcal{D}[\text{fields}] e^{-S_{\text{CFT}} + \int d^d x \sum_i J_i \mathcal{O}_i}. \tag{4.3.2}$$

The right hand side in (4.3.1) is defined as the path integral with respect to the fields Φ_i in AdS_{d+1} satisfying the boundary condition imposed on the AdS boundary:

$$\Phi_i|_{\text{boundary}} \sim J_i. \tag{4.3.3}$$

Both the left and right hand sides in (4.3.1) are the functional of the source field J_i and (4.3.1) claim that they are the same. With this relation we can calculate the correlation function in the CFT by using the supergravity or superstring theory in AdS_{d+1}. For this relation to hold for each operator $\mathcal{O}_i$

Table 4.2: Several examples of the AdS/CFT correspondence. SE_5 is the five-dimensional Sasaki-Einstein manifold.

	boundary CFT	compact internal space X
$d = 4$	$\mathcal{N} = 4$ SO, Sp SYMs	$X = \boldsymbol{S}^5/\mathbb{Z}_2$
	$\mathcal{N} = 1, 2$ orbifold quiver gauge theories	$X = \boldsymbol{S}^5/\Gamma$
	$\mathcal{N} = 1, 2$ toric quiver gauge theories	$X = SE_5$
	$\mathcal{N} = 3$ S-fold SCFTs	$X = \boldsymbol{S}^5/\mathbb{Z}_k$
	$\mathcal{N} = 2$ SCFTs	$X = \boldsymbol{S}^5/\mathbb{Z}_{\Delta_7}$
$d = 6$	$\mathcal{N} = (2, 0)$ SCFTs	$X = \boldsymbol{S}^4$
	$\mathcal{N} = (1, 0)$ SCFTs	$X = \boldsymbol{S}^4/\mathbb{Z}_k$
$d = 3$	$\mathcal{N} = 6, 8$ ABJM theories	$X = \boldsymbol{S}^7$

in the CFT there must be a corresponding field Φ_i. For a special class of operators the fields Φ_i are the massless supergravity fields, while in general they may be stringy excited states or fields associated with expanded branes.

For a four-dimensional $\mathcal{N} = 4$ CFT the operators $\mathcal{O}_i$ carry the quantum numbers related to the $SU(4)$ R-symmetry. The corresponding fields Φ_i are Kaluza-Klein modes in $\boldsymbol{S}^5$ and carry the charges of the isometry $SO(6) \simeq SU(4)$. In addition to the quantum numbers, the dimension of the operators corresponds to the energy of the fields. Especially, an operator with the dimension Δ corresponds to a field with the energy of the order Δ/L.

Parameter relation

The four-dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills theory is characterized by two parameters; the gauge coupling g_{YM} and the rank N of the gauge group. On the other hand the Type IIB string theory in $AdS_5 \times \boldsymbol{S}^5$ is also characterized by two dimensionless parameters L/l_s and g_s, where l_s, L, and g_s are the string length, the AdS radius, and the string coupling constant, respectively. [1] The Planck length l_p is given by $l_p \equiv g_s^{\frac{1}{4}} l_s$. There are two parameter relations:

$$N \sim \frac{L^4}{l_p^4}, \qquad \lambda_{\text{YM}} \sim \frac{L^4}{l_s^4}. \tag{4.3.4}$$

[1] In both the $\mathcal{N} = 4$ SYM and the Type IIB string theory, there is one more parameter: θ-angle. It does not play any role in the following arguments and we ignore it.

Note that we ignore the numerical coefficients.

There are two conditions for the string theory to be approximated accurately by the classical supergravity. One is that a correction from quantum gravity is small:

$$1 \gg \frac{G_{\rm N}}{L^8} \sim \left(\frac{l_p}{L}\right)^8 \sim N^{-2}. \tag{4.3.5}$$

This means N must be sufficiently greater than one. The other condition is that stringy excitations can be ignored. This requires

$$1 \gg \frac{l_s}{L} \sim (4\pi g_s N)^{-\frac{1}{4}} \sim \lambda^{-\frac{1}{4}}. \tag{4.3.6}$$

This means λ must be sufficiently greater than one. In the large-N limit and $\lambda \gg 1$ the Type IIB string theory becomes a weakly coupled supergravity, while the corresponding $\mathcal{N} = 4$ SYM is strongly coupled. On the other hand, in the large-N limit and $\lambda \ll 1$ the $\mathcal{N} = 4$ SYM is weakly coupled, and we need to consider the stringy excitations in the Type IIB string theory.

4.4 Large-N index and supergravity Kaluza-Klein modes

It is difficult to prove the AdS/CFT correspondence because when one side of the correspondence is weakly coupled, the other side is strongly coupled and we cannot calculate quantities on the both sides. However, in the parameter region of a weakly coupled string theory and a strongly coupled gauge theory, we can calculate the superconformal index in a strongly coupled gauge theory by using the localization method and we can obtain strong evidence of the AdS/CFT correspondence by comparing the indices on the both sides of the duality.

First of all, we consider the large N case. The superconformal index $\mathcal{I}_{U(\infty)}$ on the gauge theory side can be calculated by the localization method and is given by (3.5.2). On the gravity side, the Kaluza-Klein modes, which we will explain shortly, contribute to the superconformal index $\mathcal{I}_{\rm KK}$. It was found in [4] that the superconformal indices on the both sides, $\mathcal{I}_{U(\infty)}$ and $\mathcal{I}_{\rm KK}$, agree

$$\mathcal{I}_{U(\infty)} = \mathcal{I}_{\rm KK}. \tag{4.4.1}$$

This relation also holds in the Schur limit.

In the large-N limit and $\lambda \gg 1$ the dynamical degree of freedom is a free supergravity multiplet in $AdS_5 \times \boldsymbol{S}^5$. The mode expansion of the supergravity multiplet in $AdS_5 \times \boldsymbol{S}^5$ has been studied in [47, 48]. The Kaluza-Klein modes satisfying the BPS condition $\Delta = 0$ are labeled by the Cartan charges; the energy E, $SU(2)$ spins J and $\overline{J}$, and R-charges R_X, R_Y, and R_Z. We denote the BPS states as $[J, \overline{J}]_E^{(R_X, R_Y, R_Z)}$. The Kaluza-Klein modes with $\Delta = 0$ are listed in Table 4.3. For the Kaluza-Klein modes in Table 4.3 we consider the

Table 4.3: The Kaluza-Klein modes in $AdS_5 \times \boldsymbol{S}^5$ with $\Delta = 0$ and their contribution to the superconformal index.

	$[J, \overline{J}]_E^{(R_X, R_Y, R_Z)}$	index
$n \geq 1$	$[0,0]_n^{(n,0,0)}$	$+q^n \chi_{[n,0]}$
	$[\frac{1}{2},0]_{n+\frac{1}{2}}^{(n-\frac{1}{2},\frac{1}{2},\frac{1}{2})}$	$-q^{n+1}(y+y^{-1})\chi_{[n-1,0]}$
	$[0,\frac{1}{2}]_{n+\frac{1}{2}}^{(n-\frac{1}{2},\frac{1}{2},-\frac{1}{2})}$	$-q^{n+1}\chi_{[n-1,1]}$
	$[0,1]_{n+1}^{(n-1,0,0)}$	$+q^{n+2}\chi_{[n-1,0]}$
$n \geq 2$	$[0,0]_{n+1}^{(n-1,1,1)}$	$+q^{n+1}\chi_{[n-2,0]}$
	$[\frac{1}{2},\frac{1}{2}]_{n+1}^{(n-1,1,0)}$	$+q^{n+\frac{3}{2}}(y+y^{-1})\chi_{[n-2,1]}$
	$[\frac{1}{2},1]_{n+\frac{3}{2}}^{(n-\frac{3}{2},\frac{1}{2},\frac{1}{2})}$	$-q^{n+\frac{5}{2}}(y+y^{-1})\chi_{[n-2,0]}$
$n \geq 3$	$[0,\frac{1}{2}]_{n+\frac{3}{2}}^{(n-\frac{3}{2},\frac{3}{2},\frac{1}{2})}$	$-q^{n+2}\chi_{[n-3,1]}$
	$[0,1]_{n+2}^{(n-1,1,1)}$	$+q^{n+3}\chi_{[n-3,0]}$

excitation by the following differential operators satisfying $\Delta = 0$,

$$\partial_{+\dot{+}} : (H, J, \overline{J}, R_X, R_Y, R_Z) = (1, +\tfrac{1}{2}, +\tfrac{1}{2}, 0, 0, 0), \tag{4.4.2}$$

$$\partial_{-\dot{+}} : (H, J, \overline{J}, R_X, R_Y, R_Z) = (1, -\tfrac{1}{2}, +\tfrac{1}{2}, 0, 0, 0). \tag{4.4.3}$$

The contribution of single $\partial_{+\dot{+}}$, $\partial_{-\dot{+}}$ to the superconformal index is $q^{\frac{3}{2}} y^{\pm 1}$ and the multiple excitations give the factor

$$\text{Pexp}\left(q^{\frac{3}{2}}(y + y^{-1})\right) = \frac{1}{(1 - q^{\frac{3}{2}} y)(1 - q^{\frac{3}{2}} y^{-1})}. \tag{4.4.4}$$

The superconformal index of the single-particle state of the Kaluza-Klein modes in Table 4.3 with specific n is given by

$$i_{s_n} = \frac{1}{(1-q^{\frac{3}{2}}y)(1-q^{\frac{3}{2}}y^{-1})}\Big(q^n\chi_{[n,0]} - q^{n+1}(y+y^{-1})\chi_{[n-1,0]} - q^{n+1}\chi_{[n-1,1]} + q^{n+2}\chi_{[n-1,0]} + q^{n+1}\chi_{[n-2,0]} + q^{n+\frac{3}{2}}(y+y^{-1})\chi_{[n-2,1]} - q^{n+\frac{5}{2}}(y+y^{-1})\chi_{[n-2,0]} - q^{n+2}\chi_{[n-3,1]} + q^{n+3}\chi_{[n-3,0]}\Big), \quad (4.4.5)$$

where some $SU(3)$ Weyl characters are formally calculated according to (3.3.17) as follows:

$$\chi_{[-1,0]} = \chi_{[-1,1]} = \chi_{[-2,0]} = 0, \quad \chi_{[-2,1]} = -1. \quad (4.4.6)$$

In summing over n, we obtain the single-particle index [2]

$$i_{\rm KK} = \sum_{n=1}^{\infty} i_{s_n} = \frac{qu}{1-qu} + \frac{qu^{-1}v}{1-qu^{-1}v} + \frac{qv^{-1}}{1-qv^{-1}} - \frac{q^{\frac{3}{2}}y}{1-q^{\frac{3}{2}}y} - \frac{q^{\frac{3}{2}}y^{-1}}{1-q^{\frac{3}{2}}y^{-1}}. \quad (4.4.8)$$

The contribution of the Kaluza-Klein modes including multiple-particle states is given by

$$\mathcal{I}_{\rm KK} \equiv {\rm Pexp}\,(i_{\rm KK}) = 1 + q\chi_{[1,0]} + q^2(2\chi_{[2,0]} - \chi_{[0,1]}) + q^3(-\chi_{[1,1]} + 3\chi_{[3,0]} - \chi_2(y) + 2) + \cdots, \quad (4.4.9)$$

and this agrees with $\mathcal{I}_{U(\infty)}(q,y,u,v)$ in (3.5.3).

4.5 Finite-N index and giant gravitons

In the previous section, we confirmed the large-N index $\mathcal{I}_{U(\infty)}$ is the same with the index of the supergravity Kaluza-Klein modes $\mathcal{I}_{\rm KK}$. Now, we consider the case when N is finite.

[2] The following formula is useful in calculating the summation over n:

$$\sum_{n=0}^{\infty} q^n\chi_{[n,0]} = {\rm Pexp}\,(q\chi_{[1,0]}) = \frac{1}{(1-qu)(1-qu^{-1}v)(1-qv^{-1})},$$
$$\sum_{n=0}^{\infty} q^{n+1}\chi_{[n-1,1]} = \sum_{n=0}^{\infty} q^{n+1}\left(\chi_{[n-1,0]}\chi_{[0,1]} - \chi_{[n-2,0]}\right) = \frac{q^2\chi_{[0,1]} - q^3}{(1-qu)(1-qu^{-1}v)(1-qv^{-1})}. \quad (4.4.7)$$

The difference between the superconformal indices in the large N and finite N cases is

$$\mathcal{I}_{U(N)} - \mathcal{I}_{\text{KK}} = -q^{N+1}\chi_{[N+1,0]} + \cdots . \tag{4.5.1}$$

This suggests that the corresponding object has the energy in the order of N/L. In fact, an object called a giant graviton can have this energy.

A giant graviton is a D3-brane expanded in $\boldsymbol{S}^5$. [3] Let us consider a giant graviton wrapped around a large $\boldsymbol{S}^3$ in $\boldsymbol{S}^5$. Its energy is the product of the volume $2\pi^2 L^3$ and the tension T_{D3},

$$2\pi^2 L^3 \times T_{\text{D3}} = \frac{N}{L}, \tag{4.5.2}$$

and this reproduce the order of the difference (4.5.1) up to the constant shift +1.

In fact, a giant graviton can take general configurations [11]. Let σ^a ($a = 0, 1, 2, 3$) be the coordinates on the worldvolume of the D3-brane and X^M ($M = 0, 1, \ldots, 9$) be the coordinates of the background spacetime. The motion of the D3-brane is described by maps $X^M(\sigma^a)$ from the worldvolume of the D3-brane to the background spacetime. The action of a D3-brane in a curved spacetime with non-vanishing C_4 is

$$S_{\text{D3}} = -T_{\text{D3}} \int d^4\sigma \sqrt{-\det G_{ab}} - \int C_4. \tag{4.5.3}$$

We set the gauge field and the fermion fields on the D3-brane, which we are not interested in here, to be zero. The first term is the Nambu-Goto action and the second term is the minimal coupling to the background gauge field C_4. T_{D3} is the tension of the D3-brane given in terms of the string length l_s and the string coupling g_s by

$$T_{\text{D3}} = \frac{2\pi}{(2\pi l_s)^4 g_s}. \tag{4.5.4}$$

[3] In the Type IIB string theory in $AdS_5 \times \boldsymbol{S}^5$ there are two types of giant gravitons; one wraps around $\boldsymbol{S}^3$ in $\boldsymbol{S}^5$ called the sphere giant [11] and the other wraps around $\boldsymbol{S}^3$ in AdS_5 called the AdS giant [49, 50]. These giant gravitons are complementary in the sense that they reproduce the same BPS partition function of the $\mathcal{N} = 4$ $U(N)$ SYM [13, 51]. We focus on only the sphere giant. The role of the AdS giant in the finite N index has not been known. Henceforth, we often call a sphere giant a giant graviton.

G_{ab} is the induced metric defined by

$$G_{ab} = \frac{\partial X^M}{\partial \sigma^a}\frac{\partial X^N}{\partial \sigma^b} g_{MN}. \tag{4.5.5}$$

We use the coordinate system in $\boldsymbol{S}^5$ (4.2.4). Let us consider a D3-brane with the worldvolume (4.5.3) given by

$$\theta = \text{const.}, \quad \phi = \phi(t). \tag{4.5.6}$$

This ansatz represents a giant graviton moving along the ϕ direction. (See Figure 4.2). For this giant graviton the action (4.5.3) reduces to

$$\int dt \left(-M\sqrt{1-(L\sin\theta\dot{\phi})^2} + ML\cos\theta\dot{\phi} \right), \tag{4.5.7}$$

where $M \equiv T_{\mathrm{D3}}(L\cos\theta)^3\Omega_3$.

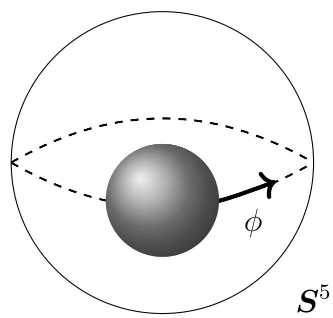

Figure 4.2: The giant graviton moving along the ϕ direction in $\boldsymbol{S}^5$.

The angular momentum P_ϕ conjugate to ϕ and the energy E are given by

$$E = \frac{N}{L}\sqrt{\cos^6\theta + \left(\frac{\frac{P_\phi}{N} - \cos^4\theta}{\sin\theta}\right)^2}, \tag{4.5.8}$$

$$P_\phi = \frac{ML^2\sin^2\theta\dot{\phi}}{\sqrt{1-(L\sin\theta\dot{\phi})^2}} + ML\cos\theta. \tag{4.5.9}$$

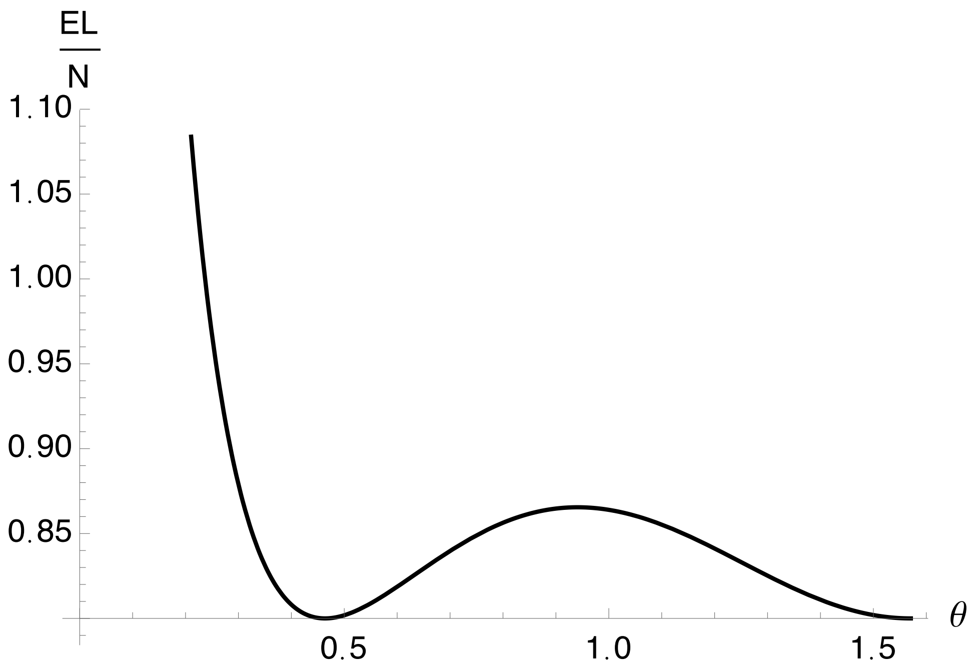

Figure 4.3: The energy E in (4.5.8). We set $P_\phi/N = 0.8$. The left minimum point at $\cos^2\theta = P_\phi/N$ represents the giant graviton. The right minimum point at $\cos^2\theta = 0$ represents the point-like Kaluza-Klein graviton.

The energy has a minimum value $E_{\min} = P_\phi/L$ at two points $\cos^2\theta = 0, P_\phi/N$ (Figure 4.3). The point $\cos^2\theta = 0$ represents the point-like Kaluza-Klein graviton and the point $\cos^2\theta = P_\phi/N$ represents the giant graviton with the finite radius.

In the following chapter, we show that we can reproduce the finite-N Schur index by taking account of the contribution of the degrees of freedom on systems consisting of different numbers of giant gravitons.

Chapter 5

Finite-N Schur index from giant gravitons

In this chapter we show a new method for calculating the Schur index of the $\mathcal{N} = 4$ $U(N)$ supersymmetric Yang-Mills theory on the gravity side via the AdS/CFT correspondence.

In [14] in calculating the superconformal index on the gravity side they take account of the contribution from configurations with a single giant graviton because there is a difficulty in the calculation for multiple giant gravitons. It is about the choice of contours in the integrals appearing in the calculation for multiple giant gravitons. We point out that this difficulty can be partially avoided if we take the Schur limit, and propose a rule for the contour determination.

5.1 BPS configuration and Rigid motion

In order to describe the configuration of giant gravitons, we introduce $\mathbb{C}^3$ with the coordinates (z_X, z_Y, z_Z) and embed the five-dimensional sphere $\boldsymbol{S}^5$ in $\mathbb{C}^3$. The coordinates of $\mathbb{C}^3$ satisfy

$$|z_X|^2 + |z_Y|^2 + |z_Z|^2 = 1 \quad \text{on} \quad \boldsymbol{S}^5. \tag{5.1.1}$$

We are interested in BPS configurations of giant gravitons wrapped on $\boldsymbol{S}^5$. According to [12], a BPS configuration can be given by the intersection of a holomorphic surface $f(z_X, z_Y, z_Z) = 0$ in $\mathbb{C}^3$ and the $\boldsymbol{S}^5$. The holomorphic function is expanded by the holomorphic coordinates as

$$f(z_X, z_Y, z_Z) = \sum_{n_X, n_Y, n_Z=0}^{\infty} a_{n_X n_Y n_Z} z_X^{n_X} z_Y^{n_Y} z_Z^{n_Z}. \tag{5.1.2}$$

The overall factor in (5.1.2) is irrelevant to determine the holomorphic surface $f = 0$.

We consider the case that f is a homogeneous polynomial with degree m, and we call $m = n_X + n_Y + n_Z$ the wrapping number. Let us consider a single giant graviton with $m = 1$. The corresponding holomorphic surface is represented by

$$a_X z_X + a_Y z_Y + a_Z z_Z = 0, \tag{5.1.3}$$

where we set $a_X \equiv a_{100}$, $a_Y \equiv a_{010}$, and $a_Z \equiv a_{001}$. Let us focus on the giant graviton wrapped around $z_X = 0$, which carries

$$H = R_X = N, \quad R_Y = R_Z = 0. \tag{5.1.4}$$

Note that we use the same notations of the Cartan generators in (3.3.1). The system of the single giant graviton has the $SU(3)$ R-symmetry that can be interpreted as rotations in $\boldsymbol{S}^5$. The $SU(3)$ rotations give the rigid motion of the giant graviton and change the R-charges (R_X, R_Y, R_Z) as the BPS condition

$$H - R_X - R_Y - R_Z = 0, \tag{5.1.5}$$

and $R_X, R_Y, R_Z \geq 0$ are satisfied. Let the state with $(R_X, R_Y, R_Z) = (N, 0, 0)$ be a highest weight for the $SU(3)$ symmetry. The general weights are composed by reducing $R_X = N$ of the highest weight and have the charge

$$(R_X, R_Y, R_Z) = (N - a - b, a, b), \tag{5.1.6}$$

where a and b run from 0 to N and satisfy $a + b \leq N$. There are two lowest weights, $(R_X, R_Y, R_Z) = (0, N, 0)$ and $(0, 0, N)$. We see that these weights construct the $[N, 0]$ representation of $SU(3)$. Their contribution to the index is

$$
\begin{aligned}
&q^N \chi^{SU(3)}_{[N,0]}(u_1,u_2,u_3)\\
&= q^N \left(\frac{u_1^N}{\left(1-\frac{u_2}{u_1}\right)\left(1-\frac{u_3}{u_1}\right)} + \frac{u_2^N}{\left(1-\frac{u_3}{u_2}\right)\left(1-\frac{u_1}{u_2}\right)} + \frac{u_3^N}{\left(1-\frac{u_1}{u_3}\right)\left(1-\frac{u_2}{u_3}\right)} \right)\\
&= (qu_1)^N \operatorname{Pexp}\left(\frac{u_2}{u_1}+\frac{u_3}{u_1}\right) + (qu_2)^N \operatorname{Pexp}\left(\frac{u_3}{u_2}+\frac{u_1}{u_2}\right) + (qu_3)^N \operatorname{Pexp}\left(\frac{u_1}{u_3}+\frac{u_2}{u_3}\right),
\end{aligned}
\tag{5.1.7}
$$

where $u_{1,2,3}$ are given in (3.3.7). Each of three terms in (5.1.7) can be interpreted as the contribution from one highest weight and two lowest weights.

The giant graviton wrapped around $z_X = 0$ carries the charges (5.1.4). This gives the factor $(qu_1)^N$ in the first term of (5.1.7). The factor $\operatorname{Pexp}(\frac{u_2}{u_1} + \frac{u_3}{u_1})$ can be regarded as the contribution from rigid motion around $z_X = 0$. Combining these two factors we obtain the first term in (5.1.7).

It was found in [14] that the correct index for single-wrapping giant gravitons can be reproduced by replacing the Pexp factors in (5.1.7) by the index of the field theory realized on the giant graviton. Namely, the contribution from the single-wrapping configurations is

$$
(qu_1)^N \operatorname{Pexp}[i_X] + (qu_2)^N \operatorname{Pexp}[i_Y] + (qu_3)^N \operatorname{Pexp}[i_Z], \tag{5.1.8}
$$

where i_X is given by [14]

$$
i_X = 1 - \frac{(1-q^{-1}u_1^{-1})(1-q^{\frac{3}{2}}y)(1-q^{\frac{3}{2}}y^{-1})}{(1-qu_2)(1-qu_3)}, \tag{5.1.9}
$$

and i_Y and i_Z, the index for the giant graviton on $z_Y = 0$ and $z_Z = 0$, respectively, are given by cyclic permutations of $u_{1,2,3}$. The q-expansion of i_X is

$$
i_X = \frac{1}{u_1 q} + \frac{u_2}{u_1} + \frac{u_3}{u_1} + \cdots, \tag{5.1.10}
$$

and it includes negative and zero power of q. We define the plethystic exponential of the negative power term, called tachyonic term, according to the formula (2.4.21),

$$
\operatorname{Pexp}\left(\frac{1}{u_1 q}\right) = \frac{1}{1-\frac{1}{u_1 q}} = -\frac{u_1 q}{1-u_1 q}. \tag{5.1.11}
$$

This means the tachyonic term raises the power of q. The terms of the zero power of q correspond to the rigid motion associated with the $SU(3)$ symmetry. These zero modes from i_X, i_Y, and i_Z generates $SU(3)$ characters.

5.2 Multiple giant gravitons

By generalizing (5.1.8) we expect that the superconformal index of the $\mathcal{N}=4$ SYM including all contributions from giant gravitons is given by

$$\frac{\mathcal{I}_{U(N)}}{\mathcal{I}_{U(\infty)}} = \sum_{n_X,n_Y,n_Z=0}^{\infty} \mathcal{I}_{(n_X,n_Y,n_Z)}, \tag{5.2.1}$$

where

$$\mathcal{I}_{(n_X,n_Y,n_Z)} = (qu_1)^{n_X N}(qu_2)^{n_Y N}(qu_3)^{n_Z N} H_{(n_X,n_Y,n_Z)}. \tag{5.2.2}$$

$H_{(n_X,n_Y,n_Z)}$ is the index of the field theory realized on the brane configuration specified by (n_X,n_Y,n_Z). It consists of n_X giant gravitons on $z_X=0$, n_Y giant gravitons on $z_Y=0$, and n_Z giant gravitons on $z_Z=0$. On each cycle $U(n_I)$ gauge theory is realized. The whole theory is the gauge theory with $U(n_X)\times U(n_Y)\times U(n_Z)$ gauge group. In addition, when there exist D3-branes wrapped around different $\boldsymbol{S}^3$ cycles in $\boldsymbol{S}^5$ at the same time, we need to include the contribution from open strings stretching between these D3-branes. We call them intersection strings. For example, an intersection string attached on two cycles $z_X=0$ and $z_Y=0$ belong to the bi-fundamental representation of $U(n_X)\times U(n_Y)$. Namely, the system consists of fields belonging to the adjoint representations and the bi-fundamental representations.

The index of the theory is

$$H_{(n_X,n_Y,n_Z)} = \int \prod_{I=X,Y,Z} d\mu_{n_I} \mathrm{Pexp}\left(\sum_{I=X,Y,Z} i_I \chi^{\mathrm{adj}}_{n_I} + \frac{1}{2}\sum_{I\neq J} i^{\mathrm{int}}_{I,J}\chi^{\text{bi-fund}}_{n_I,n_J}\right). \tag{5.2.3}$$

The first term in the parentheses is the contributions of the fields belonging to the adjoint representations and i_I are given in (5.1.9) and $\chi^{\mathrm{adj}}_{n_I}$ are given in (3.3.10).

The second term is the the contributions of the fields belonging to the bi-fundamental representations. The contribution of the intersection string between the D3-branes on $z_X = 0$ and $z_Y = 0$ to the single-particle index is

$$i^{\text{int}}_{X,Y} = \frac{u_3^{\frac{1}{2}}}{q} \frac{(1 - q^{\frac{3}{2}}y)(1 - q^{\frac{3}{2}}y^{-1})}{1 - qu_3}, \tag{5.2.4}$$

and

$$\chi^{\text{bi-fund}}_{n_I,n_J} = \sum_{a=1}^{n_I} \sum_{b=1}^{n_J} \left(\frac{\zeta_a}{\zeta'_b} + \frac{\zeta'_b}{\zeta_a} \right). \tag{5.2.5}$$

The fugacities ζ_a and ζ'_a in (5.2.5) correspond to the D3-branes wrapped around $z_I = 0$ and $z_J = 0$, respectively.

Integration contour problem

In the integration for the gauge fugacities ζ_a in (5.2.3), we need to determine contours and there is a problem about the choice of contours. The rest of this section will discuss the problem of contours.

Let us remember how we choose integration contours in standard index calculation. The q-expansion of the single-particle index is denoted by

$$i = \sum_k \alpha_k(q, y, u, v) - \sum_j \beta_j(q, y, u, v), \tag{5.2.6}$$

where α_k and β_j have the positive and negative sign, respectively. Usually, we assume $|q| < 1$ and the absolute value of the other fugacities is one, and all α_k and β_j consist of positive power of q and $|\alpha_k| < 1$ and $|\beta_j| < 1$. In such a case, we use integration contours $|\zeta_a| = 1$ and poles in $|\zeta_a| < 1$ are picked up.

However, it is not the case for the system on the giant gravitons. As we saw in (5.1.10), α_k appearing in the expansion of i_X include negative power of q or q^0 and if we assume $|q| < 1$, such α_k do not satisfy $|\alpha_k| < 1$.

Let us consider $(n_X, n_Y, n_Z) = (2, 0, 0)$ as a simple example. $H_{(n_X,n_Y,n_Z)}$ is simply written by

$$H_{(2,0,0)} = \int d\mu_2 \text{Pexp} \left(i_X \chi_2^{\text{adj}} \right). \tag{5.2.7}$$

The integrand depends on gauge fugacities ζ_1 and ζ_2 only through $\zeta \equiv \zeta_1/\zeta_2$ and the Haar measure is

$$\int d\mu_2 = \frac{1}{2} \int \frac{d\zeta}{2\pi i \zeta} (1 - \zeta)(1 - \zeta^{-1}). \tag{5.2.8}$$

$U(2)$ character is given by $\chi_2^{\text{adj}} = 2 + \zeta + \zeta^{-1}$. By using the expansion of i_X in the form of (5.2.6), the above integrand is given by

$$\text{Pexp}\left(i_X \chi_2^{\text{adj}}\right) = \prod_{j,k} \frac{(1-\beta_j)^2(1-\beta_j\zeta)(1-\beta_j\zeta^{-1})}{(1-\alpha_k)^2(1-\alpha_k\zeta)(1-\alpha_k\zeta^{-1})}. \tag{5.2.9}$$

The integrand has the poles $\zeta = \alpha_k^{\pm 1}$. If we consider the standard parameter region,

$$|q| < 1, \quad |u| = |v| = |y| = 1, \tag{5.2.10}$$

the following three α_k appear in $|\zeta| \geq 1$:

$$\alpha_1 = \frac{1}{uq}, \quad \alpha_2 = \frac{v}{u^2}, \quad \alpha_3 = \frac{1}{uv}. \tag{5.2.11}$$

The structure of the poles is shown in Figure 5.1 (a). If we use the contour $|\zeta| = 1$, the pole $\zeta = \alpha_1$ is excluded in the contour integration and for the poles $\zeta = \alpha_{2,3}$ some special treatment is required because they are on the contour. In fact, the integration with $|\zeta| = 1$ under the condition (5.2.10) does not give the correct result.

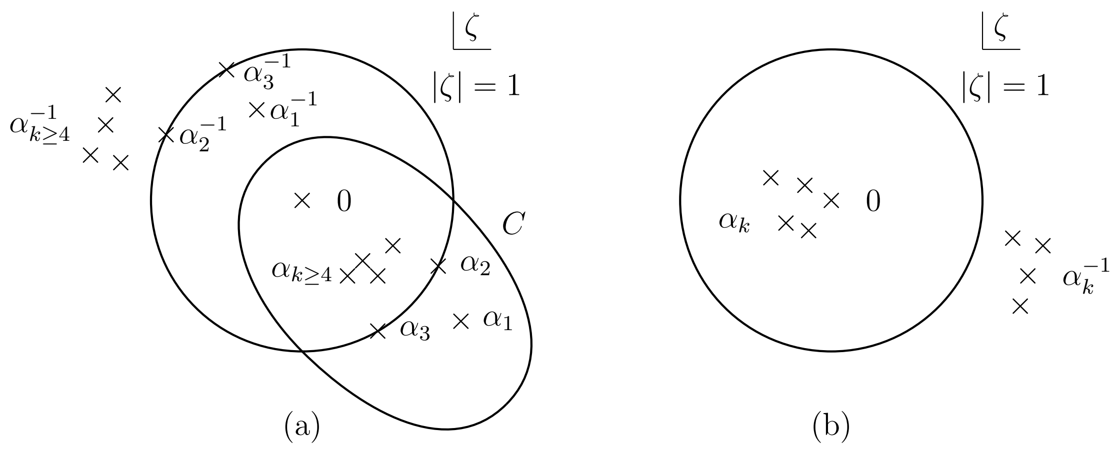

Figure 5.1: The structure of the poles in the ζ plane for i_X (a) in (5.2.10) and (b) in (5.2.14).

How do we obtain the correct result? A natural way is to use a contour that picks up all poles at α_k and excludes all poles at α_k^{-1}. See a contour C in

Figure 5.1 (a). We find that the integration with such a contour works well. This is equivalent to the following prescription. We adjust the fugacities so that all α_k satisfy $|\alpha_k| < 1$ and use the contour $|\zeta| = 1$. In fact, such adjustments are possible in the case of i_X because i_X is written by changing the fugacities from $i_{\rm vec}$ in (3.3.16) in [14],

$$i_X(q,y,u,v) = i_{\rm vec}(q^{\frac{2}{3}}u^{-\frac{1}{3}}, u^{-\frac{1}{2}}v, q^{-\frac{5}{3}}u^{-\frac{2}{3}}, q^{-\frac{5}{6}}yu^{-\frac{1}{3}}). \tag{5.2.12}$$

We know that all α_k in the expansion of $i_{\rm vec}(q,y,u,v)$ satisfy $|\alpha_k| < 1$ under the condition (5.2.10). According to the analytically continuation from $i_{\rm vec}$ to i_X all α_k of i_X also satisfy $|\alpha_k| < 1$ by using the standard parameter region for $i_{\rm vec}$ given by

$$|q^{\frac{2}{3}}u^{-\frac{1}{3}}| < 1, \quad |u^{-\frac{1}{2}}v| = |q^{-\frac{5}{3}}u^{-\frac{2}{3}}| = |q^{-\frac{5}{6}}yu^{-\frac{1}{3}}| = 1. \tag{5.2.13}$$

We rewrite the above condition into the following form:

$$|q| < 1, \quad |u| = |v^2| > 1, \quad |y| = 1. \tag{5.2.14}$$

Under the condition (5.2.14), all α_k in the expansion of i_X satisfy $|\alpha_k| < 1$ and we can obtain the correct result with the contour $|\zeta| = 1$. See Figure 5.1 (b). In a natural extension, as long as we consider one cycle, $H_{(n_X,0,0)}$, $H_{(0,n_Y,0)}$, and $H_{(0,0,n_Z)}$ can be calculated in appropriate adjustments of the fugacities and the contours $|\zeta_a| = 1$.

However, in the case for multiple contributions, the single-particle index includes the contributions from more than one cycle,

$$i = i_X + i_Y + i_Z + \cdots, \tag{5.2.15}$$

and we can not make all α_k satisfy $|\alpha_k| < 1$ no matter how we adjust the values of the fugacities. Namely, $H_{(n_X,n_Y,n_Z)}$ can not be calculated well for in the contours $|\zeta_a| = 1$. A procedure for the integration contours was proposed in [52] after some trial errors, but no obvious way to determine the integration contours based in the first principle has yet been found.

As we will show in the next section, taking the Schur limit solves the problem of contours.

5.3 Schur limit

In order to simplify the formula (5.2.3), let us consider the Schur limit

$$y = q^{\frac{1}{2}}, v = 1. \tag{5.3.1}$$

The giant graviton on $z_Z = 0$ does not contribute to the Schur index. It is because i_Z is given by

$$\begin{aligned} i_Z &= 1 - \frac{(1-q^{-1}u_3^{-1})(1-q^{\frac{3}{2}}y)(1-q^{\frac{3}{2}}y^{-1})}{(1-qu_1)(1-qu_2)} \\ &= \frac{v}{q} + uv + \frac{v^2}{u} - vyq^{\frac{1}{2}} - \frac{v}{y}q^{\frac{1}{2}} + \cdots , \end{aligned} \tag{5.3.2}$$

and in the Schur limit this becomes

$$i_Z = \frac{1}{q} + u + \frac{1}{u} - q - 1 + \cdots . \tag{5.3.3}$$

Due to the term -1 appearing in (5.3.3), $\mathrm{Pexp}(i_Z)$ vanishes and we can set $n_Z = 0$ in (5.2.3). i_X in (5.1.9) reduces

$$i_X(q,u) = \frac{\frac{1}{uq} - \frac{2q}{u} + q^2}{1 - \frac{q}{u}}, \qquad i_Y(q,u) = i_X(q,u^{-1}), \tag{5.3.4}$$

and the superconformal index of the intersection string (5.2.4) reduces to

$$i^{\mathrm{int}}_{X,Y} = \frac{1}{q} - q. \tag{5.3.5}$$

We find that $\mathrm{Pexp}(i^{\mathrm{int}}_{X,Y}\chi^{\text{bi-fund}}_{n_X,n_Y})$ becomes

$$\mathrm{Pexp}(i^{\mathrm{int}}_{X,Y}\chi^{\text{bi-fund}}_{n_X,n_Y}) = \prod_{a=1}^{n_X}\prod_{b=1}^{n_Y} \frac{\left(1 - q\frac{\zeta_a}{\zeta'_b}\right)}{\left(1 - \frac{1}{q}\frac{\zeta'_b}{\zeta_a}\right)} \frac{\left(1 - q\frac{\zeta'_b}{\zeta_a}\right)}{\left(1 - \frac{1}{q}\frac{\zeta_a}{\zeta'_b}\right)} = q^{2n_X n_Y}. \tag{5.3.6}$$

Because the contribution from the intersection string becomes the simple form (5.3.6), we can divide the integration in (5.2.3) into two independent parts; the D3-branes wrapped around $z_X = 0$ and $z_Y = 0$. The formula in (5.2.3) becomes

$$H_{(n_X,n_Y,n_Z=0)} = q^{2n_X n_Y} F_{n_X}(q,u) F_{n_Y}(q,u^{-1}), \tag{5.3.7}$$

where

$$F_n(q,u) \equiv H_{(n,0,0)} = \int d\mu_n \mathrm{Pexp}\left(i_X \chi^{\mathrm{adj}}_n\right). \tag{5.3.8}$$

In order to calculate $\mathcal{I}_{(n_X,n_Y,n_Z)}$ in (5.2.1) in the Schur limit, only F_n in (5.3.8) need to be calculated and it is possible by using the rule for integration contours explained in the previous section.

By substituting (5.3.7) into (5.2.1) we obtain

$$\frac{\mathcal{I}_{U(N)}}{\mathcal{I}_{U(\infty)}}(q,u) = \sum_{n_X,n_Y=0}^{\infty} (qu)^{n_X N} \left(qu^{-1}\right)^{n_Y N} q^{2n_X n_Y} F_{n_X}(q,u) F_{n_Y}(q,u^{-1}). \tag{5.3.9}$$

5.4 Supergravity contribution

$\mathcal{I}_{U(\infty)}$ on the left hand side in (5.3.9) is the supergravity contribution. The Kaluza-Klein modes contributing the Schur index are shown in Table 5.1. The contributions of the modes with specific n are given by

$$i_{s_n} = \frac{q^n \chi_n}{1-q^2} - \frac{2q^{n+1}\chi_{n-1}}{1-q^2} + \frac{q^{n+2}\chi_{n-2}}{1-q^2} \quad (n \geq 1), \tag{5.4.1}$$

where $\chi_{-1} = 0$. We take the sum over n and obtain the single-particle index [1]

$$i_{\mathrm{KK}} = \sum_{n=1}^{\infty} i_{s_n} = \frac{qu}{1-qu} + \frac{qu^{-1}}{1-qu^{-1}} - \frac{q^2}{1-q^2}. \tag{5.4.3}$$

Of course, we obtain this result by applying the Schur limit (5.3.1) to (4.4.8). The contribution of the Kaluza-Klein modes is given by

$$\mathcal{I}_{\mathrm{KK}} \equiv \mathrm{Pexp}\,(i_{\mathrm{KK}}) = 1 + q\chi_1 + q^2(2\chi_2 - 2) + q^3(3\chi_3 - 2\chi_1) + q^4(5\chi_4 - 4\chi_2 + 1) + \cdots \tag{5.4.4}$$

and this agrees with $\mathcal{I}_{U(\infty)}$ in (3.5.6).

[1] The following formula is useful in calculating the summation over n,

$$\sum_{n=0}^{\infty} q^n \chi_n = \mathrm{Pexp}\,(q\chi_1) = \frac{1}{(1-qu)(1-qu^{-1})}. \tag{5.4.2}$$

Table 5.1: The Kaluza-Klein modes in $AdS_5 \times \boldsymbol{S}^5$ with $\Delta = \widetilde{\Delta} = 0$ and their contribution to the Schur index.

	$[J, \overline{J}]_E^{(R_X, R_Y, R_Z)}$	index
$n \geq 1$	$[0, 0]_n^{(n,0,0)}$	$+q^n \chi_n$
	$[\frac{1}{2}, 0]_{n+\frac{1}{2}}^{(n-\frac{1}{2}, \frac{1}{2}, \frac{1}{2})}$	$-q^{n+1} \chi_{n-1}$
	$[0, \frac{1}{2}]_{n+\frac{1}{2}}^{(n-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})}$	$-q^{n+1} \chi_{n-1}$
$n \geq 2$	$[\frac{1}{2}, \frac{1}{2}]_{n+1}^{(n-1,1,1)}$	$+q^{n+2} \chi_{n-2}$

5.5 Contour integrals

We show procedure to calculate F_n in (5.3.8) and give an analytic expression for the leading term of F_n in q-expansion. We also give numerical results for $F_{n\leq 4}$ and compare them with gauge theory side results.

5.5.1 Pole selection rule

We will find that F_n is an integral of a function with infinite poles, and we need to select appropriate poles to obtain correct result. First of all, we investigate the poles appearing in the integrand. The series expansion of i_X with respect to q is

$$i_X = \frac{1}{uq} + \frac{1}{u^2} + \left(\frac{1}{u^3} - \frac{2}{u}\right) q + \sum_{k=2} \left(\frac{1}{u^{k+2}} - \frac{2}{u^k} + \frac{1}{u^{k-2}}\right) q^k. \tag{5.5.1}$$

We classify the terms in (5.5.1) into the two sets

$$A_M \equiv \left\{\frac{q^k}{u^{k+2}}, \frac{q^l}{u^{l-2}}\right\}_{\substack{-1\leq k\leq M \\ 2\leq l\leq M}}, \quad B_M \equiv \left\{\frac{q^k}{u^k}\right\}_{1\leq k\leq M}, \tag{5.5.2}$$

where A includes the positive terms and B includes the negative ones with negative sign excluded. M is cut-off for the order of q, and A and B include the monomial up to the order q^M. If M is infinity A and B include all the monomials in (5.5.1). Let α_k, β_j be elements belonging to the sets A_M, B_M, respectively:

$$\alpha_k \in A_M, \quad \beta_j \in B_M, \tag{5.5.3}$$

where k runs $1, 2, \ldots, 2M+1$ and j runs $1, 2, \ldots, M$. With the series expansion (5.5.1) and the adjoint character (3.3.10), F_n is written as

$$F_n(q,u) = \lim_{M\to\infty} \int d\mu_n \prod_{a,b=1}^{n} \frac{\prod_{j=1}^{M}\left(1-\beta_j\frac{\zeta_b}{\zeta_a}\right)^2}{\prod_{k=1}^{2M+1}\left(1-\alpha_k\frac{\zeta_b}{\zeta_a}\right)}. \tag{5.5.4}$$

F_n is the integral for the gauge fugacities $\{\zeta_1, \ldots, \zeta_n\}$ included in $d\mu_n$. Since the integrand of F_n is invariant under any permutation of gauge fugacities, the ordering of the integration on $\{\zeta_1, \ldots, \zeta_n\}$ does not affect the result. Thus, we fix the order of integration as $\zeta_1, \zeta_2, \ldots, \zeta_n$. The part of integrand related to poles with $\zeta_a \neq 0$ is

$$\lim_{M\to\infty} \prod_{k=1}^{2M+1} \prod_{\substack{a,b=1\\ a\neq b}}^{n} \frac{1}{\left(1-\alpha_k\frac{\zeta_b}{\zeta_a}\right)}. \tag{5.5.5}$$

If the cut-off M is taken to the infinity limit there are infinite poles. When we compute F_n numerically we need to determine the cut-off M as an appropriate value. The cut-off is discussed in the next subsection.

We call the pole at $\zeta_a = 0$ the "zero" pole and a pole at $\zeta_a \neq 0$ a "non-zero" pole.

Now we explain the selection rule for the poles. We supposes that $|\alpha_k| < 1$ and take $|\zeta_a| = 1$ for the contour of the ζ_a integral. We sum up residue of the poles inside the contour. A non-zero pole on ζ_a-plane takes one of two forms $\zeta_a = \alpha_k\zeta_b$ and $\zeta_a = \alpha_k^{-1}\zeta_b$ if $a < b$. The former pole is inside the contour $|\zeta_a| = 1$ and the latter is outside, and we select the inside one $\zeta_a = \alpha_k\zeta_b$ in the ζ_a integral. A pole appearing on the ζ_a-plane depends on values of poles of $\{\zeta_1, \ldots, \zeta_{a-1}\}$.

We explain how to determine the poles in order from ζ_1 to ζ_n ($n \geq 2$) in three steps. Note that if $n = 1$ there is no integral and the following steps are not required.

1. ζ_1 integration: there are three types of poles. One is zero pole $\zeta_1 = 0$ and the other two types are non-zero poles $\zeta_1 = \alpha_{k_1}^{\pm}\zeta_a$ ($a \geq 2$). We select the ones inside the contour:

$$\zeta_1 = 0 \quad \text{and} \quad \zeta_1 = \alpha_{k_1}\zeta_a, \tag{5.5.6}$$

where α_{k_1} belongs to A_M.

2. ζ_2 integration: in the $n = 2$ case there is no non-zero poles and only the $\zeta_2 = 0$ pole is selected.

   In the case $n \geq 3$ we select following poles:

$$\zeta_2 = 0, \quad \zeta_2 = \alpha_{k_2}\zeta_a, \quad \text{and} \quad \zeta_2 = \alpha_{k_1}\alpha_{k_2}\zeta_a \quad (a \geq 3), \tag{5.5.7}$$

   where a_{k_1} is given by the step 1 and a_{k_2} belongs to A_M. We explain how the third type pole appears. In the step 1 when we take $\zeta_1 = \alpha_{k_1}\zeta_a$ with $a \geq 3$, the factors $(1 - \alpha_{k_2}\frac{\zeta_2}{\zeta_1})^{-1}$ and $(1 - \alpha_{k_2}\frac{\zeta_1}{\zeta_2})^{-1}$ in (5.5.5) give poles

$$\zeta_2 = \alpha_{k_2}^{\pm}\zeta_1 = \alpha_{k_1}\alpha_{k_2}^{\pm 1}\zeta_a. \tag{5.5.8}$$

   The pole with the positive power of α_{k_2} is just the third type pole in (5.5.7). If we take $\zeta_1 = \alpha_{k_1}\zeta_2$ in the step 1 the gauge fugacity ζ_2 vanishes in the factors and no poles appear from the factors.

3. ζ_a integration ($3 \leq a \leq n-1$): We select only the pole for the positive power of α_k:

$$\zeta_a = 0 \quad \text{and} \quad \zeta_a = \alpha_{k_1}^{t_1}\alpha_{k_2}^{t_2}\cdots\alpha_{a-1}^{t_{a-1}}\alpha_{k_a}\zeta_b \quad (a < b), \tag{5.5.9}$$

   where $t_{1,\dots,a-1}$ takes the value of 0 or 1. In the integral of ζ_n the pole is only $\zeta_n = 0$ because all the ratio of the two gauge fugacities is determined by some values.

5.5.2 Excluded poles

In this subsection we discuss excluded poles in the pole selection rule. In the pole selection rule all the selected poles takes the form (5.5.9), and it has only the positive power of α_k. In other words a pole with the negative power of α_k is excluded in the pole selection rule.

There are two types of the excluded pole. One is inverse type with the negative power of all α_k,

$$\zeta_a = \frac{1}{\alpha_{k_1}\alpha_{k_2}\cdots\alpha_{k_m}}\zeta_b. \tag{5.5.10}$$

A pole of this type always is outside of the contour $|\zeta_a| = 1$ because of the condition $|\alpha_k| < 1$, and it is not considered in the ζ_a integral. The other type is mixed one, which have both the positive and negative power of α_k,

$$\zeta_a = \frac{\delta'}{\delta}\zeta_b \quad (a < b)$$
$$\text{with} \quad \delta = \alpha_{k_1}\alpha_{k_2}\cdots\alpha_{k_l}, \quad \delta' = \alpha_{k_{l+1}}\alpha_{k_{l+2}}\cdots\alpha_{k_m}. \tag{5.5.11}$$

When $|\delta'/\delta| < 1$ it seems that we have to take account of the contribution of this pole. However, in summing up all the combinations for the poles inside the contour, we will see that the mixed type contribution is cancelled out with another mixed type contribution. Therefore, we need not to consider the mixed type poles.

For example, we can show the cancellation for the mixed type poles for the simple pole case. Let the poles $\zeta_p = \delta\zeta_a$ and $\zeta_p = \delta'\zeta_b$ $(p < a < b)$ be simple poles and the integral takes the form

$$\int d\zeta_a d\zeta_b d\zeta_p \left(1 - \delta\frac{\zeta_a}{\zeta_p}\right)^{-1} \left(1 - \delta'\frac{\zeta_b}{\zeta_p}\right)^{-1} f, \tag{5.5.12}$$

where we pick up the part related to the poles of $\zeta_p = \delta\zeta_a$ and $\zeta_p = \delta'\zeta_b$ and the function f is the remaining part. There are two ways to reproduce the pole (5.5.11) on the ζ_a-plane. One way is to start from the pole $\zeta_p = \delta\zeta_a$ and the other is to start from the pole $\zeta_p = \delta'\zeta_b$. Let us sum residues at the poles $\zeta_p = \delta\zeta_a$ and $\zeta_p = \delta'\zeta_b$. After the residue integral of ζ_p the summation is

$$\lim_{\zeta_p \to \delta\zeta_a} \int d\zeta_a d\zeta_b \left(1 - \frac{\delta'}{\delta}\frac{\zeta_b}{\zeta_a}\right)^{-1} \zeta_p f + \lim_{\zeta_p \to \delta'\zeta_b} \int d\zeta_a d\zeta_b \left(1 - \frac{\delta}{\delta'}\frac{\zeta_a}{\zeta_b}\right)^{-1} \zeta_p f, \tag{5.5.13}$$

where the former term is given by the pole $\zeta_p = \delta\zeta_a$ and the latter is by $\zeta_p = \delta'\zeta_b$. We see that the mixed type pole (5.5.11) appears in ζ_a integral. We perform the residue integral for the mixed type pole

$$\lim_{\zeta_a \to \frac{\delta'}{\delta}\zeta_b} \left(\lim_{\zeta_p \to \delta\zeta_a} \int d\zeta_b \zeta_a \zeta_p f\right) + \lim_{\zeta_a \to \frac{\delta'}{\delta}\zeta_b} \left(\lim_{\zeta_p \to \delta'\zeta_b} \int d\zeta_b (-\zeta_a \zeta_p f)\right) = 0, \tag{5.5.14}$$

and we see the cancellation occurs. Unfortunately, we have no analytically explanation for the cancellation in the case of multiple poles, but some numerical results indicate such cancellations occur.

5.5.3 Cut-off

We consider a cut-off M, which means the q-expansion of i_X up to the order q^M is taken into account. The following factor in the integrand of F_n is related to the order q^M terms in i_X,

$$\mathrm{Pexp}\left[\left(\frac{1}{u^{M+2}}-\frac{2}{u^M}+\frac{1}{u^{M-2}}\right)q^M\chi_n^{\mathrm{adj}}\right]=\prod_{a,b=1}^{n}\frac{\left(1-\frac{q^M}{u^M}\frac{\zeta_b}{\zeta_a}\right)^2}{\left(1-\frac{q^M}{u^{M+2}}\frac{\zeta_b}{\zeta_a}\right)\left(1-\frac{q^M}{u^{M-2}}\frac{\zeta_b}{\zeta_a}\right)}. \tag{5.5.15}$$

Although this factor seems to have the order q^M ignoring the gauge fugacities, a pole choice related for the tachyonic term $(uq)^{-1}$ decreases the order of q. Let us select a following set for the poles,

$$\zeta_a=\frac{\zeta_{a+1}}{uq}\ (a=1,2,\dots,n-1),\quad \zeta_n=0. \tag{5.5.16}$$

These poles decreases the order of q^M by the residue integrals

$$\begin{aligned} q^M\frac{\zeta_1}{\zeta_n}\ \to\ q^M\frac{1}{uq}\frac{\zeta_2}{\zeta_n}\ \to\ \cdots \to q^M\left(\frac{1}{uq}\right)^{n-1}\frac{\zeta_n}{\zeta_n} \\ = q^{M-(n-1)}u^{-(n-1)}, \end{aligned} \tag{5.5.17}$$

where the right arrows mean the substitution for the gauge fugacity associated with the gauge integration. We see that the factor (5.5.15) has the lowest order q^{M-n+1} after all the integrations. For the order of q^{M-n+1} to be positive, $M\geq n-1$ should hold. Because F_n has the leading term q^{n^2} the contribution of the cut-off M appears from the order q^{M+n^2-n+1} in the q-expansion of F_n.

5.5.4 Leading term of F_n

In this subsection we compute F_n for $n=1,2,3,4$ in the pole selection rule and give explicit results for leading terms of F_n in q-expansion. We also predict leading term of F_n for an arbitrary n.

$\boldsymbol{F_1}$ In the case of $n=1$ the character is $\chi_1 = 1$ and the integrand of F_1 has no gauge fugacity. F_1 is given by

$$F_1(q,u) = \prod_{k=1}^{\infty} \frac{\left(1-\frac{q^k}{u^k}\right)^2}{\left(1-\frac{q^{k-2}}{u^k}\right)\left(1-\frac{q^{k+1}}{u^{k-1}}\right)}. \tag{5.5.18}$$

F_1 corresponds to $U(1)$ part of F_n, which means F_n is proportional to $(F_1)^n$. In expanding F_1 with respect to q we should be careful for the power of q on product factors in (5.5.18). The factor $(1-\frac{q^{k-2}}{u^k})$ with $k=1$ has the negative power of q, and we need to rewrite it to expand with the positive power of q,

$$\frac{1}{1-\frac{1}{uq}} = \frac{-uq}{1-uq}. \tag{5.5.19}$$

We see that the overall factor $-uq$ raises the order of q by one, and the q-expansion of F_1 starts from the order of q^1. Although we supposed the condition $1/|uq| < 1$ in the pole selection rule, we formally write down the right-hand side in (5.5.19) as the series expansion with the positive power of q. The factor $(1-\frac{q^{k-2}}{u^k})$ includes zero power of q with $k=2$, which is a factor $(1-\frac{1}{u^2})^{-1}$ and does not shift the order of q. Thus, the leading term of the q-expansion of F_1 is the product of (5.5.19) and $(1-\frac{1}{u^2})^{-1}$, namely $\widetilde{F_1} \equiv qu^3/(1-u^2)$.

$\boldsymbol{F_2}$ In the case of $n=2$ the formula of F_2 is

$$F_2(q,u) = \frac{1}{2}(F_1)^2 \int \frac{d\zeta_1}{2\pi i\zeta_1}\frac{d\zeta_2}{2\pi i\zeta_2}\left(1-\frac{\zeta_2}{\zeta_1}\right)\left(1-\frac{\zeta_1}{\zeta_2}\right)\frac{\prod_j\left(1-\beta_j\frac{\zeta_2}{\zeta_1}\right)^2\left(1-\beta_j\frac{\zeta_1}{\zeta_2}\right)^2}{\prod_k\left(1-\alpha_k\frac{\zeta_2}{\zeta_1}\right)\left(1-\alpha_k\frac{\zeta_1}{\zeta_2}\right)}. \tag{5.5.20}$$

The poles on the ζ_1-plane we have to pick up are

$$\zeta_1 = 0,\ \ \alpha_k\zeta_2. \tag{5.5.21}$$

In the calculation of F_2 we perform the residue integral for the pole $\zeta_2 = 0$ after the ζ_1 integral. To obtain the complete result of F_2 we should sum up residues for all the above poles. In computing numerical result we introduce appropriate cut-off.

Let us look into the leading term of the q-expansion of F_2 for the poles (5.5.21). First, we consider the pole $\zeta_1 = 0$. In the procedure for the residue calculation the ratio ζ_1/ζ_2 can be regarded as zero while the ratio ζ_2/ζ_1 diverges. We pick up ζ_2/ζ_1 as a pre-factor

$$\left(1 - \alpha_k \frac{\zeta_2}{\zeta_1}\right) = -\alpha_k \frac{\zeta_2}{\zeta_1}\left(1 - \frac{\zeta_1}{\zeta_2}\right). \tag{5.5.22}$$

We perform the above operation on all the factors in the integrand and find all the ratios ζ_2/ζ_1 in the pre-factors are cancelled out each other in the denominator and numerator. After the limit $\zeta_1 \to 0$ the remaining factor is given by

$$\frac{\left(-\frac{q}{u}\right)^2}{\left(-\frac{1}{uq}\right)\left(-\frac{1}{u^2}\right)\left(-\frac{q}{u^3}\right)} = q^2 u^4. \tag{5.5.23}$$

We obtain the leading term for $\zeta_1 = 0$ as the product $\frac{1}{2}(\widetilde{F}_1)^2 \times q^2 u^4$.

Next, we consider the non-zero poles $\zeta_1/\zeta_2 = q^m/u^l$, where $l = m+2$ for $m \geq -1$ and $l = m-2$ for $m \geq 2$. Similarly to F_1 we focus on the product factors with the negative power of q in the integrand. Since all the poles appearing on F_2 is simple, there is no derivative of $\zeta_{1,2}$ in the residue calculation and we just remove the factor which gives the pole and substitute associated value. The residue of the pole $\zeta_1/\zeta_2 = q^m/u^l$ is formally expanded with respect to q for any m as

$$\begin{aligned}\left(1 - \frac{u^l}{q^m}\right)\left(1 - \frac{q^m}{u^l}\right)\prod_{k=1}\frac{\left(1 - \frac{q^k}{u^k}\frac{u^l}{q^m}\right)^2\left(1 - \frac{q^k}{u^k}\frac{q^m}{u^l}\right)^2}{\left(1 - \frac{q^{k-2}}{u^k}\frac{u^l}{q^m}\right)\left(1 - \frac{q^{k-2}}{u^k}\frac{q^m}{u^l}\right)\left(1 - \frac{q^{k+1}}{u^{k-1}}\frac{u^l}{q^m}\right)\left(1 - \frac{q^{k+1}}{u^{k-1}}\frac{q^m}{u^l}\right)}&\\ = q^2 u^4 \frac{(1-u^\gamma)^2}{(1-u^{\gamma-2})(1-u^{\gamma+2})} + \cdots,&\end{aligned} \tag{5.5.24}$$

where we define the parameter $\gamma \equiv l - m$. We see that the leading term is classified by the value γ. In order to obtain the leading term of F_2 we need to sum up with respect to m and l. Note that this formula (5.5.24) includes divergence because the pole factor is not removed yet. In determining the value γ we need to eliminate the associated pole factor. The pole listed in (5.5.21) has the value with $\gamma = \pm 2$. When the pole is $\zeta_1/\zeta_2 = q^m/u^{m+2}$, which has $\gamma = +2$, the factor $1 - u^{\gamma-2}$ is eliminated and the leading term is given by

$$\frac{1}{2}(\widetilde{F}_1)^2 \times q^2 u^4 \frac{(1-u^2)^2}{1-u^4} = \frac{q^2 u^{10}}{2} \frac{1}{1-u^4}. \tag{5.5.25}$$

The case when the pole is $\zeta_1/\zeta_2 = q^{m+3}/u^{m+1}$ with $\gamma = -2$ makes the factor $1 - u^{\gamma+2}$ eliminated. For the poles (5.5.21) the leading terms of F_2 are given by

$$\begin{aligned} \zeta_1 = 0 \quad &: \quad \frac{q^4 u^{10}}{2} \frac{1}{(1-u^2)^3}, \\ \frac{\zeta_1}{\zeta_2} = \frac{q^m}{u^{m+2}} \quad &: \quad \frac{q^4 u^{10}}{2} \frac{1}{1-u^4}, \\ \frac{\zeta_1}{\zeta_2} = \frac{q^{m+3}}{u^{m+1}} \quad &: \quad -\frac{q^4 u^{10}}{2} \frac{1}{1-u^4}, \end{aligned} \tag{5.5.26}$$

where $m \geq -1$. We see that the last two leading terms are the same absolute value but the opposite sign, and most of them are cancelled out in summing up with $m \geq -1$. The survived number in sum over m is three, which corresponds to three poles $\zeta_1/\zeta_2 = (qu)^{-1}, u^{-2}, qu^{-3}$ (Figure 5.2). We denote such a number as R. Namely, $\frac{q^4 u^{10}}{2} \frac{1}{1-u^4}$ has $R = 3$. The leading term of F_2 is given by

$$F_2 = \frac{q^4 u^{10}}{2} \left(\frac{1}{(1-u^2)^3} + \frac{3}{1-u^4} \right) + \cdots . \tag{5.5.27}$$

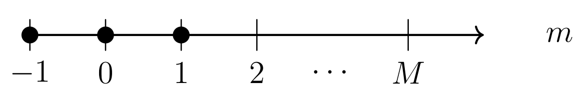

Figure 5.2: The structure of the residue. Only $m = 0, \pm 1$ contribute to the leading term of F_2.

F_3 In the case of F_3 the candidates for the pole are

$$\begin{aligned} (\zeta_1, \zeta_2) = (0,0), (0, \alpha_{k_2}\zeta_3), (\alpha_{k_1}\zeta_2, 0), (\alpha_{k_1}\zeta_2, \alpha_{k_2}\zeta_3), \\ (\alpha_{k_1}\zeta_3, 0), (\alpha_{k_1}\zeta_3, \alpha_{k_2}\zeta_3), (\alpha_{k_1}\zeta_3, \alpha_{k_1}\alpha_{k_2}\zeta_3). \end{aligned} \tag{5.5.28}$$

F_3 has no multiple poles and only simple poles, so the procedure of the residue computation is simple similarly with F_2. The adjoint character $\chi^{\mathrm{adj}}_{n=3}$ has three ratios of gauge fugacities,

$$\frac{\zeta_1}{\zeta_2}, \quad \frac{\zeta_1}{\zeta_3}, \quad \frac{\zeta_2}{\zeta_3} \tag{5.5.29}$$

and their inverses. The integrand of F_3 can be regarded as a separated form with respect to the above three gauge ratios and their inverses. A part related with each ratios is calculated by the same method with F_2. To classify leading terms of F_3 for the poles, we just consider whether the three ratios (5.5.29) are zero or non-zero in the pole selection rule, and if a ratio is non-zero and takes a value q^m/u^l, we can classify a leading term by the parameter $\gamma = l - m$ according to (5.5.24). In the case with the poles $\zeta_1 = \zeta_2 = 0$ each three parts related with (5.5.29) produce the factor q^2u^4 respectively, and the leading term is given by $\frac{1}{3!}(\widetilde{F}_1)^3 \times (q^2u^4)^3$. In the case when one of the poles for ζ_1, ζ_2 is zero and the other is non-zero, one of the three ratios (5.5.29) is non-zero. One non-zero ratio takes the value α_k, which has $\gamma = \pm 2$. The other two ratios produce the value q^2u^4 respectively. The leading term is

$$\frac{1}{3!}(\widetilde{F}_1)^3 \times (q^2u^4)^2 \times \left(\pm q^2u^4 \frac{(1-u^2)^2}{(1-u^4)}\right) = \pm \frac{q^9u^{21}}{3!}\frac{1}{(1-u^2)(1-u^4)}, \tag{5.5.30}$$

where $\pm$ comes from the value $\gamma = \pm 2$ for α_k. Similarly as F_2 there exist only three value with positive sign in summing up to k. Because there are three choices which ratio is non-zero, the remaining number of the value (5.5.30) with positive sign is nine, namely $R = 9$.

We consider the case when the three ratios (5.5.29) are non-zero. Note that there is no case when two ratios are non-zero because if two ratios among (5.5.29) are non-zero the last ratio is automatically non-zero. According to the pole selection rule, pole candidates are listed in Table 5.2. The values of γ of $\alpha_{k_1}, \alpha_{k_2}$ are ± 2 and $\alpha_{k_1}\alpha^{\pm}_{k_2}$ has $\gamma = 0, \pm 4$. If $\gamma = 0$ the leading term (5.5.24) vanishes. In the case for $\gamma = \pm 4$ the leading term is given by

$$\begin{aligned} &\pm \frac{1}{3!}(\widetilde{F}_1)^3 \times q^2u^4\frac{(1-u^2)^2}{(1-u^4)} \times q^2u^4\frac{(1-u^2)^2}{(1-u^4)} \times q^2u^4\frac{(1-u^4)^2}{(1-u^2)(1-u^6)} \\ &= \pm \frac{q^9u^{21}}{3!}\frac{1}{(1-u^6)}, \end{aligned} \tag{5.5.31}$$

Table 5.2: Non-zero poles for F_3. Non-bracket terms are selected as the poles and bracket terms are automatically determined by the product of the non-bracket terms.

ζ_1/ζ_2	ζ_1/ζ_3	ζ_2/ζ_3
α_{k_1}	$(\alpha_{k_1}\alpha_{k_2})$	α_{k_2}
$(\alpha_{k_2}^{-1})$	α_{k_1}	$\alpha_{k_1}\alpha_{k_2}$
$(\alpha_{k_1}\alpha_{k_2}^{-1})$	α_{k_1}	α_{k_2}

where the overall sign depends on two factors; one is a kind of the poles listed in Table 5.2 and the other is the value γ for α_{k_1} and α_{k_2}. The signs are shown in Table 5.3. Although the values (5.5.31) are generated infinitely, most of

Table 5.3: The values of γ and the signs of $\frac{q^9u^{21}}{3!}\frac{1}{(1-u^6)}$.

	ζ_1/ζ_2	ζ_1/ζ_3	ζ_2/ζ_3	sign of $\frac{q^9u^{21}}{3!}\frac{1}{(1-u^6)}$
γ	$+2$	$+4$	$+2$	$+$
	-2	-4	-2	$+$
γ	-2	$+2$	$+4$	$+$
	$+2$	-2	-4	$+$
γ	$+4$	$+2$	-2	$-$
	-4	-2	$+2$	$-$

them are cancelled out and only a finite number R of them remains. In order to investigate R, let m_1 and m_2 be the exponents of α_{k_1} and α_{k_2} for q. We find that R is determined by the number of (m_1, m_2) satisfying $m_1 + m_2 \leq 1$ and $-1 \leq m_{1,2}$ (Figure 5.3) and $R = {}_{2\cdot3-1}C_3 = 10$. There are two cases with both α_{k_1} and α_{k_2} having the value $\gamma = +2$ and the numerical coefficient of (5.5.31) is $2R = 20$. In summary the leading term of F_3 is given by

$$F_3 = \frac{q^9u^{21}}{3!}\left(\frac{1}{(1-u^2)^3} + \frac{9}{(1-u^2)(1-u^4)} + \frac{20}{1-u^6}\right) + \cdots . \qquad (5.5.32)$$

There is a natural generalization in the case for n non-zero ratios. The remaining number R is determined by the number of the set $(m_1, \ldots, m_n)$

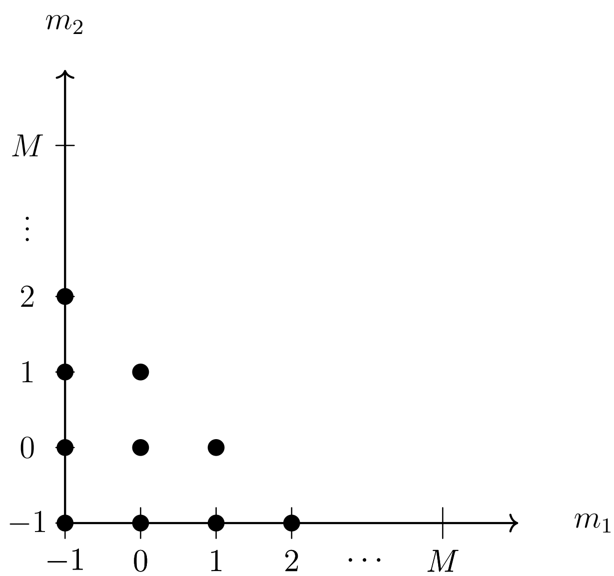

Figure 5.3: The structure of the residue contributing to the leading term of F_3.

satisfying

$$\sum_{i=1}^{n} m_i \leq 1 \quad \text{and} \quad -1 \leq m_i \quad (\text{for any } i), \tag{5.5.33}$$

and $R = {}_{2n-1}C_n$.

F_4 F_4 has double-poles and it is not simple to analyze the leading term. From the results of some numerical calculation we guess a residue from a double pole does not contribute to a leading term of F_4, in other words a double pole residue is cancelled out with a simple pole residue.

Let us calculate the leading term of F_4 only for the simple poles. The adjoint character χ_4^{adj} has six ratios for the gauge fugacities,

$$\frac{\zeta_1}{\zeta_2}, \quad \frac{\zeta_1}{\zeta_3}, \quad \frac{\zeta_1}{\zeta_4}, \quad \frac{\zeta_2}{\zeta_3}, \quad \frac{\zeta_2}{\zeta_4}, \quad \frac{\zeta_3}{\zeta_4}. \tag{5.5.34}$$

and their inverse. In the case for no non-zero pole the leading term is given by $\frac{1}{4!}(\widetilde{F}_1)^4(q^2u^4)^6$. In the case for one non-zero pole the leading term is

$$\frac{1}{4!}(\widetilde{F}_1)^4(q^2u^4)^6\left(\pm\frac{(1-u^2)^2}{1-u^4}\right)=\pm\frac{q^{16}u^{36}}{4!}\frac{1}{(1-u^2)^2(1-u^4)}. \tag{5.5.35}$$

The choices of the ratios of the gauge fugacities (5.5.34) are six. Each remaining number is three and, namely the numerical coefficient of the leading term is $6\times 3=18$.

In the case for two non-zero poles the leading term is

$$\frac{1}{4!}(\widetilde{F}_1)^4(q^2u^4)^6\left(\pm\frac{(1-u^2)^2}{1-u^4}\right)^2=\pm\frac{q^{16}u^{36}}{4!}\frac{1}{(1-u^4)^2}. \tag{5.5.36}$$

The choice of the ratios is three sets, $(\zeta_1/\zeta_2,\zeta_3/\zeta_4),(\zeta_1/\zeta_3,\zeta_2/\zeta_4),(\zeta_1/\zeta_4,\zeta_2/\zeta_3)$. Each remaining number is nine and the numerical coefficient $3\times 9=27$.

In the case for three non-zero poles the leading term is

$$\frac{1}{4!}(\widetilde{F}_1)^4(q^2u^4)^6\left(\pm\frac{(1-u^2)^2}{1-u^4}\right)^2\frac{(1-u^4)^2}{(1-u^2)(1-u^6)}=\pm\frac{q^{16}u^{36}}{4!}\frac{1}{(1-u^2)(1-u^6)}. \tag{5.5.37}$$

The choices of the ratios are four,

$$\left(\frac{\zeta_1}{\zeta_2},\frac{\zeta_1}{\zeta_3},\frac{\zeta_2}{\zeta_3}\right),\left(\frac{\zeta_1}{\zeta_2},\frac{\zeta_1}{\zeta_4},\frac{\zeta_2}{\zeta_4}\right),\left(\frac{\zeta_1}{\zeta_3},\frac{\zeta_1}{\zeta_4},\frac{\zeta_3}{\zeta_4}\right),\left(\frac{\zeta_2}{\zeta_3},\frac{\zeta_2}{\zeta_4},\frac{\zeta_3}{\zeta_4}\right). \tag{5.5.38}$$

Each remaining number is twenty and the numerical coefficient is $4\times 20=80$.

Four and five non-zero poles are automatically six non-zero poles. In the case for six non-zero poles the leading term is

$$\begin{aligned}&\frac{1}{4!}(\widetilde{F}_1)^4(q^2u^4)^6\left(\pm\frac{(1-u^2)^2}{1-u^4}\right)^3\left(\frac{(1-u^4)^2}{(1-u^2)(1-u^6)}\right)^2\frac{(1-u^6)^2}{(1-u^4)(1-u^8)}\\&=\pm\frac{q^{16}u^{36}}{4!}\frac{1}{1-u^8}\end{aligned} \tag{5.5.39}$$

The choice of the ratios is 3! and each remaining number is ${}_{2\times 4-1}C_4=35$. The numerical coefficient is 210.

In summary the leading term of F_4 is given by

$$\begin{aligned}F_4=\frac{q^{16}u^{36}}{4!}&\left(\frac{1}{(1-u^2)^4}+\frac{18}{(1-u^2)^2(1-u^4)}+\frac{27}{(1-u^4)^2}\right.\\&\left.+\frac{80}{(1-u^2)(1-u^6)}+\frac{210}{1-u^8}\right)+\cdots.\end{aligned} \tag{5.5.40}$$

F_n From the results of the leading terms of $F_{n\leq 4}$ we anticipate the leading term of F_n for arbitrary n.

First, we focus on the fractional terms $1/(1-u^{2k})$. These terms appear according to the partition of the number n. Let $\{a_k\}_{k=1,\dots,n}$ be sets satisfying $n=\sum_{k=1}^n ka_k$ and $a_k\geq 0$. For example, the partition of $n=3$ is represented as $(a_1,a_2,a_3)=(3,0,0),(1,1,0),(0,0,1)$. By using the set $\{a_k\}$ the fractional terms in the leading term is written by

$$\sum_{\{a_k\}}\prod_{k=1}^n \frac{1}{(1-u^{2k})^{a_k}}, \tag{5.5.41}$$

where the summation is taken over all the sets $\{a_k\}$.

Next, the coefficient for $1/(1-u^{2k})$ consists of the product of two factors. One is the remaining number $R={}_{2k-1}C_k$. The other is the number of elements in the conjugacy class for S_n. With the partition $n=\sum_{k=1}^n ka_k$ the number of the elements is written by

$$\frac{n!}{\prod_{k=1}^n k^{a_k}a_k!}. \tag{5.5.42}$$

We predict that the leading term in F_n takes the following form:

$$\begin{aligned} F_n =& q^{n^2}u^{2n^2+n}\sum_{\{a_k\}}\prod_{k=1}^n \frac{({}_{2k-1}C_k)^{a_k}}{k^{a_k}a_k!(1-u^{2k})^{a_k}}+\cdots \\ \equiv& q^{n^2}u^{2n^2}G_n+\cdots . \end{aligned} \tag{5.5.43}$$

The function $G_n(u)$ has a following generating function

$$\sum_{n=0}^\infty G_n(u)t^n=\exp\left(\sum_{p=1}^\infty \frac{{}_{2p-1}C_p}{p}\frac{t^p}{u^{-p}-u^p}\right). \tag{5.5.44}$$

5.5.5 Numerical results

We compute F_n for q-expansion for $n\leq 4$ by using the Mathematica and show several terms. (More details are shown in Appendix A.2.)

$$F_0=1,$$

$$F_1=qu^3\frac{1}{1-u^2}+q^2\left(1-u^2\right)+q^3\left(\frac{1}{u^3}-u^3\right)+q^4\left(\frac{1}{u^6}-u^4-\frac{1}{u^2}+1\right)+\cdots,$$

$$F_2 = \frac{q^4u^{10}}{2}\left(\frac{1}{(1-u^2)^2}+\frac{3}{1-u^4}\right) - q^5u^5\left(u^4-2\right) + q^6\left(-u^{12}+2u^6+2\right) + \cdots ,$$
$$F_3 = \frac{q^9u^{21}}{3!}\left(\frac{1}{(1-u^2)^3}+\frac{9}{(1-u^2)(1-u^4)}+\frac{20}{1-u^6}\right) + \cdots ,$$
$$F_4 = \frac{q^{16}u^{36}}{4!}\left(\frac{1}{(1-u^2)^4}+\frac{18}{(1-u^2)^2(1-u^4)}+\frac{27}{(1-u^4)^2} + \frac{80}{(1-u^2)(1-u^6)}+\frac{210}{1-u^8}\right) + \cdots . \tag{5.5.45}$$

Note that for $n = 0$ there is no wrapped D3-brane and its excitation also vanishes, which means the value of the index is one. The orders of q in the numerical results for $F_{1,2,3,4}$ we computed are shown in Table 5.4 [2] and they are determined by the limit of the computational resources for the ordinary laptop. In computing F_n numerically we introduce the cut-off M, which determines the highest order of q in the q-expansion.

Table 5.4: The orders of q in the numerical result for $F_{n\leq 4}$. We omit the coefficients of u and describe only the order of q.

	the order we computed	cut-off M
F_1	$q+\cdots+q^{22}$	21
F_2	$q^4+\cdots+q^{20}$	17
F_3	$q^9+\cdots+q^{19}$	12
F_4	$q^{16}+\cdots+q^{19}$	6

5.6 Comparison

In this section, we will confirm that (5.3.9) holds. We rewrite (5.3.9) to emphasize the wrapping number $m = n_X + n_Y$ as follows:

$$\frac{\mathcal{I}_{U(N)}}{\mathcal{I}_{U(\infty)}}(q,u) = \sum_{m=0}^{\infty} I_m, \tag{5.6.1}$$

[2] The highest orders of F_n are updated in the thesis in comparing the author's and the collaborator's paper [36]. In [36] the highest orders of F_n were q^{20-n} for $n \leq 4$.

where

$$I_m \equiv \sum_{m=n_X+n_Y} (qu)^{n_X N} \left(qu^{-1}\right)^{n_Y N} q^{2n_X n_Y} F_{n_X}(q,u) F_{n_Y}(q,u^{-1}). \qquad (5.6.2)$$

The numerical results of the left hand side in (5.6.1) are shown in Appendix A.1. In the following we confirm that they are reproduced by the right hand side in (5.6.1).

Let us take $N = 2$ case for an example. The left hand side in (5.6.1) is

$$\begin{aligned}
\mathcal{I}_{U(2)}/\mathcal{I}_{U(\infty)} = {} & 1 - q^3\chi_3 + q^4\left(2\chi_2 - \chi_4 - 1\right) + q^5\left(\chi_1 + \chi_3 - \chi_5\right) \\
& + q^6\left(2\chi_4 - \chi_6 - 3\right) + q^8\left(\chi_2 + \chi_4 + 1\right) + q^9\left(-3\chi_1 + \chi_3 - \chi_7 + \chi_9\right) \\
& + q^{10}\left(-2\chi_2 + \chi_4 + \chi_6 - 2\chi_8 + \chi_{10} + 1\right) \\
& + q^{11}\left(\chi_1 + 2\chi_3 - \chi_5 - \chi_7 - 2\chi_9 + 2\chi_{11}\right) \\
& + q^{12}\left(-5\chi_2 + \chi_4 + 2\chi_6 - 2\chi_{10} + \chi_{12} + 5\right) \\
& + q^{13}\left(-2\chi_1 + 3\chi_5 - \chi_7 - \chi_9 - 2\chi_{11} + 2\chi_{13}\right) \\
& + q^{14}\left(2\chi_2 + \chi_4 + \chi_8 - \chi_{10} - 2\chi_{12} + \chi_{14} + 2\right) \\
& + q^{15}\left(4\chi_1 - 9\chi_3 + 4\chi_5 - 2\chi_7 - \chi_{11} + \chi_{15}\right) \\
& + q^{16}\left(2\chi_2 - \chi_4 + 3\chi_6 + \chi_8 - \chi_{10} - \chi_{14} - 5\right) \\
& + q^{17}\left(\chi_1 + 2\chi_3 - \chi_5 + 2\chi_7 - \chi_9 - \chi_{11} - \chi_{13} + \chi_{15}\right) \\
& + q^{18}\left(2\chi_2 - 5\chi_4 + 3\chi_6 + 2\chi_8 - 2\chi_{10} + \chi_{16} - \chi_{18} + 4\right) \\
& + q^{19}\left(-2\chi_1 + 3\chi_3 - 3\chi_5 + 4\chi_7 - 4\chi_9 - \chi_{13} + \chi_{15} + 2\chi_{17} - \chi_{19}\right) \\
& + q^{20}\left(3\chi_6 + 2\chi_8 + \chi_{10} - \chi_{12} - 2\chi_{14} + 2\chi_{18} - 2\chi_{20} - 3\right) \\
& + q^{21}\left(\chi_1 + 2\chi_3 - 9\chi_5 + 4\chi_7 - 2\chi_{11} - \chi_{13} + \chi_{15} + \chi_{17} + 3\chi_{19}\right. \\
& \qquad \left. -2\chi_{21}\right) \\
& + q^{22}\left(5\chi_2 - 2\chi_6 + 3\chi_8 - 2\chi_{12} - \chi_{16} + \chi_{18} + 2\chi_{20} - 2\chi_{22}\right) \\
& + q^{23}\left(-3\chi_3 + 2\chi_5 + 2\chi_7 + \chi_9 - 2\chi_{11} - 3\chi_{13} - \chi_{15} + \chi_{17} + \chi_{19}\right. \\
& \qquad \left. +3\chi_{21} - 2\chi_{23}\right) \\
& + q^{24}\left(5\chi_2 + 9\chi_4 - 8\chi_6 + 2\chi_8 + 5\chi_{10} - 3\chi_{12} - \chi_{18} + \chi_{20} + 2\chi_{22}\right. \\
& \qquad \left. -2\chi_{24} - 11\right) \\
& + \mathcal{O}\left(q^{25}\right). \qquad (5.6.3)
\end{aligned}$$

Let $D_{m_{\max}}$ be the difference of the left hand side in (5.6.1) and the right hand side with cut-off $m \leq m_{\max}$:

$$D_{m_{\max}} \equiv \frac{\mathcal{I}_{U(N)}}{\mathcal{I}_{U(\infty)}} - \sum_{m=0}^{m_{\max}} I_m. \tag{5.6.4}$$

We want to check that this difference. The order of $D_{m_{\max}}$ becomes larger as we increase $m_{\max}$. The results of numerical analysis are as follows:

$$D_1 = q^8 \left(-\chi_2 + 2\chi_4 - \chi_6 + 2\chi_8 - \chi_{10} + 2\right) + \cdots , \tag{5.6.5}$$

$$D_2 = q^{15} \left(3\chi_1 - 4\chi_3 - \chi_5 + 2\chi_7 - 3\chi_9 - \chi_{11} + 4\chi_{13} - 4\chi_{15} + \chi_{17} + 3\chi_{19} - 2\chi_{21}\right) + \cdots , \tag{5.6.6}$$

$$D_3 = q^{24} \left(-\chi_2 - \chi_4 - 5\chi_6 + 10\chi_8 - 2\chi_{10} + 3\chi_{12} - 7\chi_{14} + 3\chi_{16} + 5\chi_{18} - 5\chi_{20} - 6\chi_{22} + 6\chi_{24} + 4\chi_{26} - 7\chi_{28} + \chi_{32} + 7\chi_{34} - 5\chi_{36} + 15\right) + \cdots , \tag{5.6.7}$$

$$D_4 = q^{25}(-\chi_9 + \chi_{11} - 8\chi_{23} + 8\chi_{25} - 7\chi_{33} + 7\chi_{35} - 2\chi_{69} + 2\chi_{71}) + \cdots . \tag{5.6.8}$$

By summing up the contributions with $m \leq 4$, all terms in (5.6.3) are reproduced. This strongly suggests that the formula (5.6.1) correctly reproduces the Schur index. We also carry out similar analysis for $N \leq 4$.

The left hand side in (5.6.1) can be calculated by the localization method. We calculate the left hand side up to q^{19} for $N = 0$, q^{22} for $N = 1$, q^{24} for $N = 2$, q^{25} for $N = 3$, and q^{22} for $N = 4$. For $N = 0, 1, 2, 3$ cases the calculation is performed to match the highest order on the result for the right hand side in (5.6.1) shown in Figure 5.4. For $N = 4$ case, due to the limitations of computer performance, the calculations are made to the order q^{22} smaller than the highest order q^{26} on the right-hand side in (5.6.1).

We have found that the numerical results of the left hand side in (5.6.1) are completely reproduced by the terms in the order in the shaded region in Figure 5.4.

For the $N = -1$ case the gravity side in (5.3.9) is formally zero, and we see that $\mathcal{I}_{U(-1)} = 0$.

In the relation (5.6.1), let us take the limit $u \to 1$. The left hand side in (5.6.1) reduces to the BDF formula in (3.4.8) and we obtain a relation

$$\mathcal{I}_m^{\rm BDF}(q) = \lim_{u\to 1} I_m(q, u). \tag{5.6.9}$$

Note that although F_n has a divergence at $u = 1$, the divergence is cancelled each other in the summation for n_X and n_Y.

Let us consider $m = 2$ case for an example. Following (5.6.2) I_2 is given by

$$\begin{aligned} I_2 &= (qu)^{2N} F_2(q,u)F_0(q,u^{-1}) + (qu)^N \left(qu^{-1}\right)^N q^2 F_1(q,u)F_1(q,u^{-1}) \\ &\quad + \left(qu^{-1}\right)^{2N} F_0(q,u)F_2(q,u^{-1}) \\ &= q^{2N+4}U_1(u) + q^{2N+5}U_2(u) + \cdots , \end{aligned} \tag{5.6.10}$$

where

$$\begin{aligned} U_1 &\equiv \frac{u^{2N+10}(2-u^2)}{(1-u^2)^2(1+u^2)} + \frac{1}{(1-u^2)(1-u^{-2})} + \frac{u^{-2N-10}(2-u^{-2})}{(1-u^{-2})^2(1+u^{-2})}, \\ U_2 &\equiv u^{2N+5}(2-u^4) + \frac{u^3}{1-u^2}(1-u^{-2}) + \frac{u^{-3}}{1-u^{-2}}(1-u^2) + u^{-2N-5}(2-u^{-4}). \end{aligned} \tag{5.6.11}$$

We calculate U_1 and U_2 in the limit $u \to 1$ [3] and we find

$$\lim_{u\to 1} U_1 = \frac{1}{2}(N^2 + 5N + 4), \qquad \lim_{u\to 1} U_2 = 0. \tag{5.6.13}$$

We see that the leading term $q^{2N+4}U_1$ of I_2 reproduces $\mathcal{I}_2^{\mathrm{BDF}}$ and the sub-leading term vanishes in the limit as expected. We also carry out similar analysis for $m \leq 4$.

We find that the leading term in the numerical result in the right hand side (5.6.9) reproduces the result in left hand side and the sub-leading terms in the right-hand side vanish. We confirm that (5.6.9) holds up to $q^{n(N-1)+20}$ in the q-expansion for arbitrary N and $m = 1, 2, 3, 4$.

[3] The following formula is useful for the calculation:

$$\lim_{u\to 1} \frac{1-u^b}{1-u^a} = \frac{b}{a}. \tag{5.6.12}$$

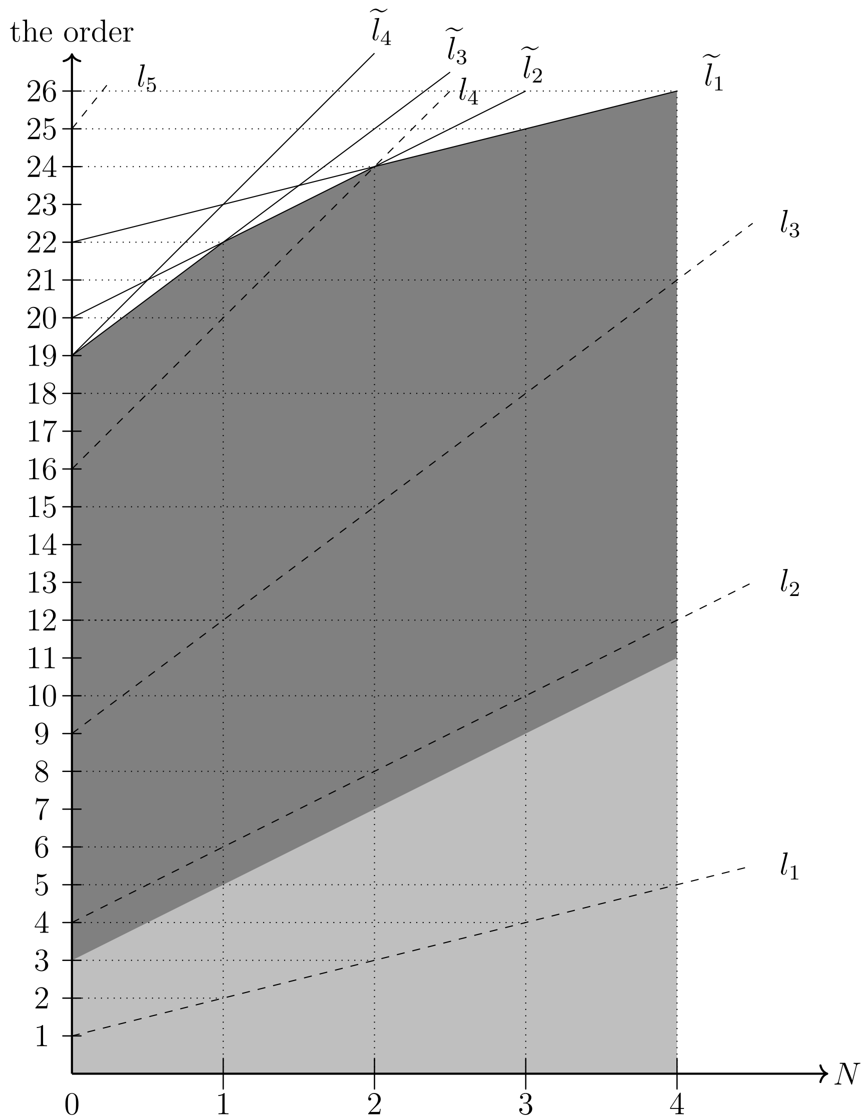

Figure 5.4: The orders we computed in the right-hand side in (5.6.1). The previous works for the supergravity [4] and the single-wrapping contributions [14] give the light gray region. Our works for the multi-wrapping contributions give the dark gray region. The dashed lines l_m labeled by the wrapping numbers m represent the lowest order of the contribution from that wrapping number, and the lines $\widetilde{l}_m$ represent the highest order corresponding the wrapping numbers m.

Chapter 6

Conclusions and discussion

In the thesis, we studied the Schur index of the $\mathcal{N}=4$ $U(N)$ supersymmetric Yang-Mills theory via the AdS/CFT correspondence. While, in the previous work [14], only the contributions of the single-wrapping giant gravitons were considered, we succeeded to develop the method to calculate the contributions of multiple-wrapping giant gravitons. We proposed the formula (5.3.9) to calculate the Schur index of the $\mathcal{N}=4$ $U(N)$ SYM from the contributions of multiple-wrapping giant gravitons and numerically confirmed the correctness of the formula. In general, it is expected that it would be difficult to analyze the finite-N AdS/CFT correspondence because quantum gravity corrections are not negligible in the Type IIB string theory. In such a situation, we succeeded to establish the formula (5.3.9) that gives the complete finite-N index via the AdS/CFT correspondence without taking account of quantum gravity corrections.

In Chapter 2, we first introduced the Witten index (2.1.3) for a theory with the relations (2.1.2) and explained the property that only the ground states with $H=0$ contribute the Witten index. We also explained the property that the Witten index is independent of the coupling constant. As the extension of the Witten index, we defined the superconformal index (2.4.1) and the Schur index (2.5.2). We explained the localization formula and, thanks to the localization formula, we saw that the superconformal index can be expressed as the tractable form (2.4.22).

In Chapter 3, we reviewed the $\mathcal{N}=4$ $U(N)$ SYM. The fields in the $\mathcal{N}=4$ $U(N)$ SYM belong to the $\mathcal{N}=4$ vector multiplet (3.1.1) and are described by the Lagrangian (3.2.1). We defined the superconformal index (3.3.6) and the Schur index (3.4.4) of the $\mathcal{N}=4$ $U(N)$ SYM, respectively. In particular, we wrote down the method to derive the single-particle index (3.3.16) of the $\mathcal{N}=4$ $U(N)$ SYM. We gave some numerical results of the superconformal index and the Schur index for small N and $N=\infty$ cases.

In Chapter 4, we reviewed the AdS/CFT correspondence. We first considered a system of N coincident D3-branes and explained the gauge theory and gravity descriptions of the system, which give the $\mathcal{N}=4$ $U(N)$ SYM in $\mathbb{R}\times\boldsymbol{S}^3$ and the Type IIB superstring theory in $AdS_5\times\boldsymbol{S}^5$, respectively. The AdS/CFT correspondence claims that the above two theories are equivalent. We explained some properties; the symmetry matching, GKP-Witten relation (4.3.1), and parameter relations (4.3.4). Especially, thanks to the symmetry matching, the two theories have the same symmetry $SO(2,4)\times SO(6)$. Following this, we can define the index on the gravity side and, following the claim of the AdS/CFT correspondence, the indices on both the gauge theory and gravity sides should match. We discussed the superconformal index in the large-N limit on both the gauge theory and gravity sides and explained in (4.4.1) that the contribution from Kaluza-Klein modes is identified to the large-N index on the gauge theory side. We also discussed the superconformal index in the finite-N case and suggested that the difference (4.5.1) between the finite-N and large-N indices should be reproduced by the contribution from giant gravitons.

In Chapter 5, we explained the main topic of the thesis, which is based on the author's and the collaborators's paper [36]. The purpose of this chapter is to construct a formula to calculate the index by using the contributions on the gravity side, especially giant graviton contributions, and to confirm the correctness of the formula. First, we defined BPS configurations of giant gravitons according to [12] and, following [14], we introduced the index (5.1.8) of single-wrapping giant gravitons. We expected that the superconformal index of the $\mathcal{N}=4$ SYM includes all the contributions of multiple-wrapping giant gravitons, and gave the formula (5.2.1) by generalizing the index of single-wrapping giant gravitons. In the formula, the multiple-wrapping contributions are given by certain contour integrals $H_{(n_X,n_Y,n_Z)}$ in (5.2.3). We explained the difficulty of this contour integrals and gave the procedure in the case for the system of giant gravitons wrapped on a single cycle: $H_{(n_X,0,0)}$, $H_{(0,n_Y,0)}$, and $H_{(0,0,n_Z)}$. We showed that, by taking the Schur limit, the multiple integrals factorize into the integrals for the system of giant gravitons wrapped on a single cycle. Following the factorization, we proposed the new formula (5.3.9) to calculate the Schur index of the $\mathcal{N}=4$ SYM from the contributions of multiple-wrapping giant gravitons F_n in (5.3.8). We explained the procedure to calculate the integrals in F_n and compared the formula with the known results and numerically confirmed the agreement of these results. This suggests the correctness of the formula (5.3.9).

There are several future directions. The formula (5.3.9) of which we confirmed the correctness is for the Schur index, and it would be nice if it is extended to the superconformal index. The version of the formula for the superconformal index is the formula (5.2.1) and there is a difficult calculation problem: we have not yet understood well the calculation of $H_{(n_X,n_Y,n_Z)}$ with arbitral wrapping numbers (n_X, n_Y, n_Z). Although a procedure for the integration contours was proposed in [52] after some trial errors, no obvious way to determine the integration contours based in the first principle has yet been found.

A Similar formula to (5.2.1) was found in [53]. In their formula, the left hand side in (5.2.1) is given by the summation over not the three wrapping numbers $n_{X,Y,Z}$ but only the single wrapping number n_X. This study is based on the analysis of determinant operators on the gauge theory side corresponding to the giant gravitons. More general analysis for determinant operators was studied in [54]. The formulas in (5.2.1) and in [53] are related to each other via the analytic continuation [55] of the fugacities and the authors claim essentially the same statement.

We can extend the works of the thesis to other examples of the AdS/CFT correspondence. The author and the collaborators studied the single-wrapping case for several examples; the four-dimensional orbifold quiver gauge theories [15] and the four-dimensional toric quiver gauge theories [16]. The gauge theories of these examples have the Lagrangian and the index can be calculated. If we can establish a similar formula to calculate the index from the giant graviton contributions, we can confirm the correctness of the formula.

The works of the thesis will be useful to analyze non-Lagrangian theories. Examples of the AdS/CFT correspondence which have non-Lagrangian theories on the CFT side are as follows; the four-dimensional $\mathcal{N} = 3$ S-fold theories [56, 57], the six-dimensional $\mathcal{N} = (2, 0)$ theories, the six-dimensional $\mathcal{N} = (1, 0)$ theories, the four-dimensional $\mathcal{N} = 2$ Argyres-Douglas (AD) theories [32, 33], and the four-dimensional $\mathcal{N} = 2$ Minahan-Nemeschansky (MN) theories [58, 59]. The single-wrapping case for the above non-Lagrangian examples was studied in [14, 17, 18, 60]. The index of the non-Lagrangian theories can not be calculated directly. If there is a formula similar to (5.3.9) to calculate the index from the dual gravity side, we can analyze the non-Lagrangian theories by using the index.

There is another interesting direction for the Schur index. Among the non-Lagrangian theories listed above, the S-fold theories, the AD theories, and the MN theories are the four-dimensional SCFT with the extended supersymmetry. It follows that these theories have the corresponding chiral algebras. In fact, serval papers studied the chiral algebra for the S-fold theories [30, 31] and for the AD theories [27, 28, 29]. It would be nice if we can derive the Schur index of these theories via the AdS/CFT correspondence and can compare the Schur index with the vacuum character of the chiral algebra.

Appendix A

Numerical results

A.1 $\mathcal{I}_{U(N)}/\mathcal{I}_{U(\infty)}(q,u)$

We give numerical results of the Schur index of $\mathcal{I}_{U(N)}/\mathcal{I}_{U(\infty)}(q,u)$ for $N = 0,1,2,3,4$.

$$\begin{aligned}
\mathcal{I}_{U(0)}/\mathcal{I}_{U(\infty)} = {} & 1 - q\chi_1 + q^2(3-\chi_2) + q^4(5-\chi_2) + q^5(-\chi_1-\chi_3+\chi_5) \\
& + q^6(-\chi_2-\chi_4+8) + q^7(\chi_7-2\chi_3) + q^8(-2\chi_2-\chi_6+13) \\
& + q^9(-\chi_1-2\chi_3+\chi_7) + q^{10}(-3\chi_2-\chi_4-\chi_6+21) + q^{11}(2\chi_7-4\chi_3) \\
& + q^{12}(-3\chi_2-\chi_4-2\chi_6+\chi_{10}-\chi_{12}+30) \\
& + q^{13}(-\chi_1-6\chi_3+3\chi_7-\chi_9+\chi_{11}) \\
& + q^{14}(-5\chi_2-2\chi_4-3\chi_6-\chi_8+2\chi_{10}-\chi_{12}+46) \\
& + q^{15}(\chi_1-9\chi_3-\chi_5+5\chi_7-\chi_9+\chi_{11}+\chi_{13}-\chi_{15}) \\
& + q^{16}(-6\chi_2-2\chi_4-5\chi_6-\chi_8+3\chi_{10}-3\chi_{12}+\chi_{14}+65) \\
& + q^{17}(-13\chi_3+6\chi_7-2\chi_9+\chi_{11}+2\chi_{13}-\chi_{15}) \\
& + q^{18}(-8\chi_2-4\chi_4-7\chi_6-2\chi_8+5\chi_{10}-4\chi_{12}+\chi_{14}+93) \\
& + q^{19}(3\chi_1-18\chi_3-3\chi_5+10\chi_7-3\chi_9+2\chi_{11}+3\chi_{13}-2\chi_{15}) \\
& + q^{20}(-10\chi_2-4\chi_4-11\chi_6-2\chi_8+7\chi_{10}-7\chi_{12}+2\chi_{14}+129) + \mathcal{O}\left(q^{20}\right)
\end{aligned} \tag{A.1.1}$$

$$\begin{aligned}
\mathcal{I}_{U(1)}/\mathcal{I}_{U(\infty)} = {} & 1 - q^2\chi_2 + q^3(2\chi_1-\chi_3) + q^4(2\chi_2-\chi_4-1) \\
& + q^6(2\chi_2-1) + q^7(\chi_3-2\chi_5+\chi_7) + q^8(-\chi_2-\chi_6+\chi_8+1)
\end{aligned}$$

$$
\begin{aligned}
&+ q^{9}\left(-2\chi_1+4\chi_3-\chi_5-2\chi_7+\chi_9\right)\\
&+ q^{10}\left(\chi_2-\chi_6-\chi_8+\chi_{10}+2\right)\\
&+ q^{11}\left(\chi_5-\chi_7-\chi_9+\chi_{11}\right)+q^{12}\left(2\chi_4-\chi_6-\chi_8-1\right)\\
&+ q^{13}\left(\chi_1+\chi_5-\chi_7\right)+q^{14}\left(2\chi_4-\chi_6-2\chi_8+\chi_{12}+2\right)\\
&+ q^{15}\left(2\chi_1-\chi_3+3\chi_5-2\chi_7+\chi_{13}-\chi_{15}\right)\\
&+ q^{16}\left(2\chi_4-\chi_8-\chi_{10}+\chi_{12}+\chi_{14}-\chi_{16}-1\right)\\
&+ q^{17}\left(2\chi_1+\chi_3-\chi_5-\chi_7-\chi_9-\chi_{11}+\chi_{13}+2\chi_{15}-\chi_{17}\right)\\
&+ q^{18}\left(\chi_4+2\chi_6-\chi_8-2\chi_{10}+\chi_{12}+\chi_{16}-\chi_{18}+1\right)\\
&+ q^{19}\left(2\chi_3+\chi_5-4\chi_9+\chi_{11}+\chi_{15}+\chi_{17}-\chi_{19}\right)\\
&+ q^{20}\left(4\chi_2+\chi_4-\chi_6-\chi_8-2\chi_{10}+\chi_{14}+\chi_{16}+\chi_{18}-\chi_{20}\right)\\
&+ q^{21}(3\chi_1+3\chi_7-3\chi_9-\chi_{11}+\chi_{13}+\chi_{19}-\chi_{21})\\
&+ q^{22}(\chi_4+2\chi_6-3\chi_8-2\chi_{12}+\chi_{14}+\chi_{16}+2)+\mathcal{O}\left(q^{23}\right)
\end{aligned}
\tag{A.1.2}
$$

$$
\begin{aligned}
\mathcal{I}_{U(2)}/\mathcal{I}_{U(\infty)} = {}& 1-q^3\chi_3+q^4\left(2\chi_2-\chi_4-1\right)+q^5\left(\chi_1+\chi_3-\chi_5\right)\\
&+ q^6\left(2\chi_4-\chi_6-3\right)+q^8\left(\chi_2+\chi_4+1\right)+q^9\left(-3\chi_1+\chi_3-\chi_7+\chi_9\right)\\
&+ q^{10}\left(-2\chi_2+\chi_4+\chi_6-2\chi_8+\chi_{10}+1\right)\\
&+ q^{11}\left(\chi_1+2\chi_3-\chi_5-\chi_7-2\chi_9+2\chi_{11}\right)\\
&+ q^{12}\left(-5\chi_2+\chi_4+2\chi_6-2\chi_{10}+\chi_{12}+5\right)\\
&+ q^{13}\left(-2\chi_1+3\chi_5-\chi_7-\chi_9-2\chi_{11}+2\chi_{13}\right)\\
&+ q^{14}\left(2\chi_2+\chi_4+\chi_8-\chi_{10}-2\chi_{12}+\chi_{14}+2\right)\\
&+ q^{15}\left(4\chi_1-9\chi_3+4\chi_5-2\chi_7-\chi_{11}+\chi_{15}\right)\\
&+ q^{16}\left(2\chi_2-\chi_4+3\chi_6+\chi_8-\chi_{10}-\chi_{14}-5\right)\\
&+ q^{17}\left(\chi_1+2\chi_3-\chi_5+2\chi_7-\chi_9-\chi_{11}-\chi_{13}+\chi_{15}\right)\\
&+ q^{18}\left(2\chi_2-5\chi_4+3\chi_6+2\chi_8-2\chi_{10}+\chi_{16}-\chi_{18}+4\right)\\
&+ q^{19}\left(-2\chi_1+3\chi_3-3\chi_5+4\chi_7-4\chi_9-\chi_{13}+\chi_{15}+2\chi_{17}-\chi_{19}\right)\\
&+ q^{20}\left(3\chi_6+2\chi_8+\chi_{10}-\chi_{12}-2\chi_{14}+2\chi_{18}-2\chi_{20}-3\right)\\
&+ q^{21}\left(\chi_1+2\chi_3-9\chi_5+4\chi_7-2\chi_{11}-\chi_{13}+\chi_{15}+\chi_{17}+3\chi_{19}\right.\\
&\qquad\left.-2\chi_{21}\right)\\
&+ q^{22}\left(5\chi_2-2\chi_6+3\chi_8-2\chi_{12}-\chi_{16}+\chi_{18}+2\chi_{20}-2\chi_{22}\right)\\
&+ q^{23}\left(-3\chi_3+2\chi_5+2\chi_7+\chi_9-2\chi_{11}-3\chi_{13}-\chi_{15}+\chi_{17}+\chi_{19}\right.
\end{aligned}
$$

$$
\begin{aligned}
&+3\chi_{21} - 2\chi_{23}) \\
&+ q^{24}\left(5\chi_2 + 9\chi_4 - 8\chi_6 + 2\chi_8 + 5\chi_{10} - 3\chi_{12} - \chi_{18} + \chi_{20} + 2\chi_{22}\right. \\
&\left.\quad -2\chi_{24} - 11\right) \\
&+ \mathcal{O}\left(q^{25}\right) \qquad \text{(A.1.3)}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{I}_{U(3)}/\mathcal{I}_{U(\infty)} = {} & 1 - q^4\chi_4 + q^5\left(-\chi_1 + 2\chi_3 - \chi_5\right) + q^6\left(\chi_4 - \chi_6 + 2\right) \\
&+ q^7\left(-3\chi_1 + 2\chi_3 + \chi_5 - \chi_7\right) + q^8\left(-\chi_4 + 2\chi_6 - \chi_8\right) \\
&+ q^{10}\left(3\chi_2 - \chi_4 + 2\chi_6 - 4\right) + q^{11}\left(-2\chi_1 + \chi_3 + \chi_5 - \chi_7 - \chi_9 + \chi_{11}\right) \\
&+ q^{12}\left(-\chi_2 - 3\chi_4 + 2\chi_6 - \chi_{10} + \chi_{12} + 2\right) \\
&+ q^{13}\left(-\chi_1 + 3\chi_3 - 2\chi_5 + \chi_9 - 3\chi_{11} + 2\chi_{13}\right) \\
&+ q^{14}\left(-5\chi_2 + \chi_4 + 3\chi_6 - \chi_8 - \chi_{10} - 2\chi_{12} + 2\chi_{14} + 5\right) \\
&+ q^{15}\left(-5\chi_1 + 3\chi_3 + \chi_5 - 2\chi_7 + 3\chi_9 - \chi_{11} - 3\chi_{13} + 2\chi_{15}\right) \\
&+ q^{16}\left(-\chi_2 - 5\chi_4 + 3\chi_6 + \chi_8 - 2\chi_{10} - 2\chi_{14} + 2\chi_{16} + 16\right) \\
&+ q^{17}\left(-3\chi_1 + \chi_3 + 3\chi_5 - 4\chi_7 + 3\chi_9 + \chi_{11} - \chi_{13} - 3\chi_{15} + 2\chi_{17}\right) \\
&+ q^{18}\left(-3\chi_2 - 7\chi_4 + 6\chi_6 + \chi_8 - \chi_{10} - \chi_{12} - \chi_{14} - \chi_{16} + \chi_{18}\right) \\
&+ q^{19}\left(7\chi_1 - 3\chi_3 + 4\chi_5 - 4\chi_7 + 3\chi_9 - 2\chi_{11} + \chi_{13} - \chi_{15} - \chi_{17} + \chi_{19}\right) \\
&+ q^{20}\left(7\chi_2 - 9\chi_4 + 6\chi_6 - 2\chi_8 + 2\chi_{12} - 2\chi_{14} + 4\right) \\
&+ q^{21}\left(-10\chi_1 + 4\chi_3 - 3\chi_5 + 4\chi_9 + \chi_{11} - 2\chi_{13} + \chi_{19} - \chi_{21}\right) \\
&+ q^{22}\left(6\chi_2 - 2\chi_4 - \chi_6 + 7\chi_8 - 8\chi_{10} + 5\chi_{12} - 2\chi_{14} - \chi_{16} - \chi_{18} + 2\chi_{20}\right. \\
&\left.\quad -\chi_{22} + 6\right) \\
&+ q^{23}\left(-6\chi_1 + 10\chi_3 - 10\chi_5 + 7\chi_9 - \chi_{11} - 3\chi_{13} + \chi_{19} + 2\chi_{21} - 2\chi_{23}\right) \\
&+ q^{24}\left(-5\chi_2 - 7\chi_4 + 5\chi_6 + 2\chi_8 - 5\chi_{10} + 6\chi_{12} - 2\chi_{14} - 2\chi_{18} + \chi_{20}\right. \\
&\left.\quad +4\chi_{22} - 3\chi_{24} + 4\right) \\
&+ q^{25}\left(-23\chi_1 + 35\chi_3 - 12\chi_5 - 10\chi_7 + 11\chi_9 - 5\chi_{13} + \chi_{15} - 2\chi_{17}\right. \\
&\left.\quad +\chi_{19} + 4\chi_{23} - 3\chi_{25}\right) \\
&+ \mathcal{O}\left(q^{26}\right) \qquad \text{(A.1.4)}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{I}_{U(4)}/\mathcal{I}_{U(\infty)} = {} & 1 - q^5\chi_5 + q^6\left(-\chi_2 + 2\chi_4 - \chi_6\right) + q^7\left(\chi_1 + \chi_5 - \chi_7\right) \\
&+ q^8\left(-\chi_2 + \chi_4 + \chi_6 - \chi_8\right) + q^9\left(-\chi_3 + \chi_5 + \chi_7 - \chi_9\right) \\
&+ q^{10}\left(-2\chi_2 + \chi_4 - \chi_6 + 2\chi_8 - \chi_{10} + 1\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ q^{12}\left(-2\chi_2 + 4\chi_4 - 2\chi_6 + 2\chi_8 + 2\right) + q^{13}\left(-\chi_1 - \chi_{11} + \chi_{13}\right) \\
&+ q^{14}\left(-2\chi_2 - \chi_4 - \chi_6 + 3\chi_8 - \chi_{10} - \chi_{12} + \chi_{14}\right) \\
&+ q^{15}\left(-4\chi_1 + 6\chi_3 - 2\chi_5 - \chi_7 - 2\chi_{13} + 2\chi_{15}\right) \\
&+ q^{16}\left(-\chi_2 + \chi_4 - 2\chi_6 + 2\chi_8 - \chi_{10} + \chi_{12} - 3\chi_{14} + 2\chi_{16} + 3\right) \\
&+ q^{17}\left(-4\chi_1 + 2\chi_7 - \chi_{13} - 3\chi_{15} + 3\chi_{17}\right) \\
&+ q^{18}\left(3\chi_2 - \chi_4 - 3\chi_6 + 4\chi_8 - 3\chi_{10} + 3\chi_{12} - \chi_{14} - 3\chi_{16} + 2\chi_{18} + 3\right) \\
&+ q^{19}\left(2\chi_1 - 3\chi_3 + 4\chi_5 - 4\chi_7 + 4\chi_9 - \chi_{13} - \chi_{15} - 3\chi_{17} + 3\chi_{19}\right) \\
&+ q^{20}\left(-8\chi_2 + 5\chi_4 - \chi_6 + 3\chi_8 - 3\chi_{10} + 3\chi_{12} - \chi_{14} - 3\chi_{18} + 2\chi_{20} + 3\right) \\
&+ q^{21}\left(10\chi_1 - 7\chi_3 - 2\chi_5 + 3\chi_7 - 4\chi_9 - \chi_{17} - 2\chi_{19} + 2\chi_{21}\right) \\
&+ q^{22}\left(-8\chi_2 + 7\chi_4 - 10\chi_6 + 5\chi_8 + 5\chi_{12} - 3\chi_{14} + \chi_{16} - \chi_{18} - 2\chi_{20}\right. \\
&\qquad \left. + \chi_{22} + 15\right) \\
&+ \mathcal{O}\left(q^{24}\right) \qquad \text{(A.1.5)}
\end{aligned}
$$

A.2 F_n

We give numerical results F_n for $n = 1, 2, 3, 4$.

$$
\begin{aligned}
F_1 = {} & \frac{qu^3}{1-u^2} + q^2\left(1-u^2\right) + q^3\left(\frac{1}{u^3} - u^3\right) + q^4\left(\frac{1}{u^6} - u^4 - \frac{1}{u^2} + 1\right) \\
& + q^5\left(\frac{1}{u^9} - u^5\right) + q^6\left(\frac{1}{u^{12}} - u^6 - \frac{1}{u^6} + 1\right) + q^7\left(\frac{1}{u^{15}} - u^7 + \frac{1}{u^5} - \frac{1}{u^3}\right) \\
& + q^8\left(\frac{1}{u^{18}} - \frac{1}{u^{10}} - u^8 + 1\right) + q^9\left(\frac{1}{u^{21}} - u^9\right) \\
& + q^{10}\left(\frac{1}{u^{24}} - \frac{1}{u^{14}} - u^{10} + \frac{1}{u^{10}} - \frac{1}{u^4} + 1\right) + q^{11}\left(\frac{1}{u^{27}} - u^{11} - \frac{1}{u^9} + \frac{1}{u^7}\right) \\
& + q^{12}\left(\frac{1}{u^{30}} - \frac{1}{u^{18}} - u^{12} + 1\right) + q^{13}\left(\frac{1}{u^{33}} + \frac{1}{u^{15}} - u^{13} - \frac{1}{u^5}\right) \\
& + q^{14}\left(\frac{1}{u^{36}} - \frac{1}{u^{22}} - u^{14} + 1\right) + q^{15}\left(\frac{1}{u^{39}} - u^{15} - \frac{1}{u^{15}} + \frac{1}{u^9}\right) \\
& + q^{16}\left(\frac{1}{u^{42}} - \frac{1}{u^{26}} + \frac{1}{u^{20}} - u^{16} + \frac{1}{u^{14}} - \frac{1}{u^{12}} - \frac{1}{u^6} + 1\right) \\
& + q^{17}\left(\frac{1}{u^{45}} - u^{17}\right) + q^{18}\left(\frac{1}{u^{48}} - \frac{1}{u^{30}} - u^{18} + 1\right)
\end{aligned}
$$

$$+ q^{19}\left(-u^{19} - \frac{1}{u^{7}} + \frac{1}{u^{11}} - \frac{1}{u^{21}} + \frac{1}{u^{25}} + \frac{1}{u^{-51}}\right)$$
$$+ q^{20}\left(\frac{1}{u^{54}} - \frac{1}{u^{34}} - u^{20} + 1\right) + q^{21}\left(-u^{21} - \frac{1}{u^{15}} + \frac{1}{u^{21}} + \frac{1}{u^{57}}\right)$$
$$+ q^{22}\left(\frac{1}{u^{60}} - \frac{1}{u^{38}} + \frac{1}{u^{30}} - u^{22} - \frac{1}{u^{20}} + \frac{1}{u^{18}} - \frac{1}{u^{8}} + 1\right) + \mathcal{O}\left(q^{23}\right) \quad \text{(A.2.1)}$$
$$F_2 = \frac{q^4 u^{10}}{2}\left(\frac{1}{(1-u^2)^2} + \frac{3}{1-u^4}\right) - q^5 u^5\left(u^4 - 2\right) + q^6\left(-u^{12} + 2u^6 + 2\right)$$
$$+ q^{7}\left(-u^{15} + 3u^{7} + \frac{2}{u^{5}}\right) + q^{8}\left(-u^{18} + \frac{2}{u^{10}} + 3u^{8} + 3\right)$$
$$+ q^{9}\left(-u^{21} + \frac{2}{u^{15}} + 4u^{9} + \frac{1}{u^{3}}\right) + q^{10}\left(-u^{24} + \frac{2}{u^{20}} + 4u^{10} + \frac{4}{u^{6}} + 3\right)$$
$$+ q^{11}\left(-u^{27} + 5u^{11} + \frac{2}{u^{9}} + \frac{2}{u^{25}}\right) + q^{12}\left(-u^{30} + \frac{2}{u^{30}} + 5u^{12} + \frac{5}{u^{12}} + 4\right)$$
$$+ q^{13}\left(-u^{33} + 6u^{13} + \frac{3}{u^{7}} + \frac{3}{u^{15}} + \frac{2}{u^{35}}\right) + q^{14}\left(\frac{2}{u^{40}} - u^{36} + \frac{6}{u^{18}} + 6u^{14} + \frac{1}{u^{4}} + 4\right)$$
$$+ q^{15}\left(-u^{39} + 7u^{15} + \frac{4}{u^{21}} + \frac{2}{u^{45}}\right) + q^{16}\left(\frac{2}{u^{50}} - u^{42} + \frac{7}{u^{24}} + 7u^{16} + \frac{3}{u^{14}} + \frac{5}{u^{8}} + 5\right)$$
$$+ q^{17}\left(-u^{45} + 8u^{17} + \frac{5}{u^{27}} + \frac{2}{u^{55}}\right) + q^{18}\left(\frac{2}{u^{60}} - u^{48} + \frac{8}{u^{30}} + 8u^{18} + \frac{2}{u^{12}} + 5\right)$$
$$+ q^{19}\left(-u^{51} + 9u^{19} + \frac{1}{u^{5}} + \frac{4}{u^{9}} + \frac{3}{u^{21}} + \frac{6}{u^{33}} + \frac{2}{u^{65}}\right)$$
$$+ q^{20}\left(\frac{2}{u^{70}} - u^{54} + \frac{9}{u^{36}} + 9u^{20} + \frac{6}{u^{16}} + 6\right) + \mathcal{O}\left(q^{21}\right) \quad \text{(A.2.2)}$$
$$F_3 = \frac{q^9 u^{21}}{3!}\left(\frac{1}{(1-u^2)^3} + \frac{9}{(1-u^2)(1-u^4)} + \frac{20}{1-u^6}\right)$$
$$+ \frac{q^{10}u^{14}}{u^4 - 1}\left(-2u^{10} + 3u^6 + 3u^4 - 5\right)$$
$$+ \frac{q^{11}u^{7}}{u^4 - 1}\left(-2u^{22} + 2u^{18} + u^{16} + 2u^{14} - 4u^8 - 2u^6 + 5u^4 - 5\right)$$
$$+ \frac{q^{12}}{u^2 - 1}\left(-2u^{32} + 2u^{30} + u^{24} + u^{22} - 9u^{14} + 9u^{12} - 2u^{10} - 2u^8 + 5u^2 - 5\right)$$
$$+ \frac{q^{13}}{u^7\left(u^4 - 1\right)}\left(-2u^{46} + 2u^{42} + u^{36} + 2u^{34} + u^{28} - 2u^{26} - 7u^{24} + 2u^{22} + 4u^{20}\right.$$

$$-2u^{18}-4u^{12}-2u^{10}+5u^{4}-5\big)$$

$$+\frac{q^{14}}{u^{14}\left(u^{4}-1\right)}\big(-2u^{58}+2u^{54}+u^{46}+2u^{44}+u^{40}+u^{38}-3u^{36}+u^{34}-8u^{32}$$
$$-4u^{30}+8u^{28}+u^{26}-2u^{24}-3u^{22}-2u^{20}+7u^{18}-4u^{14}$$
$$-2u^{12}+5u^{4}-5\big)$$

$$+\frac{q^{15}}{u^{21}-u^{23}}\big(2u^{68}-2u^{66}-u^{54}-u^{52}+u^{50}-u^{48}+2u^{44}-3u^{42}+u^{40}$$
$$+14u^{38}-14u^{36}+u^{34}+u^{32}+2u^{30}-2u^{28}+6u^{26}-4u^{24}$$
$$+2u^{20}-2u^{18}+2u^{16}+2u^{14}-5u^{2}+5\big)$$

$$+\frac{q^{16}}{u^{28}-u^{30}}\big(2u^{80}-2u^{78}-u^{64}-u^{62}+u^{60}-u^{58}-u^{56}+3u^{54}-3u^{52}$$
$$+u^{50}+2u^{48}+12u^{46}-13u^{44}+u^{42}+u^{40}+u^{38}+2u^{36}$$
$$-4u^{32}-2u^{30}+6u^{28}+3u^{26}-7u^{24}+2u^{22}-2u^{20}+2u^{18}$$
$$+2u^{16}-5u^{2}+5\big)$$

$$+\frac{q^{17}}{u^{35}\left(u^{4}-1\right)}\big(-2u^{94}+2u^{90}+u^{76}+2u^{74}+u^{68}-2u^{66}+u^{64}+2u^{62}$$
$$-3u^{60}-14u^{56}+14u^{52}-2u^{50}-3u^{48}+u^{44}-4u^{42}-3u^{40}$$
$$-2u^{38}+6u^{34}-4u^{32}-2u^{30}+4u^{28}-2u^{26}-4u^{20}-2u^{18}$$
$$+5u^{4}-5\big)$$

$$+\frac{q^{18}}{u^{42}\left(u^{4}-1\right)}\big(-2u^{106}+2u^{102}+u^{86}+2u^{84}+u^{78}-u^{76}+u^{74}+u^{72}-3u^{70}$$
$$+2u^{66}-19u^{64}-2u^{62}+17u^{60}-2u^{58}-2u^{56}+2u^{54}-4u^{52}$$
$$-2u^{50}-2u^{48}+8u^{46}+4u^{44}-7u^{42}-4u^{40}+6u^{36}-11u^{34}$$
$$-2u^{32}+11u^{30}-2u^{28}-4u^{22}-2u^{20}+5u^{4}-5\big)$$

$$+\frac{q^{19}}{u^{49}\left(u^{4}-1\right)}\big(-2u^{118}+2u^{114}+u^{96}+2u^{94}+u^{88}-2u^{86}+u^{84}+2u^{82}$$
$$-3u^{80}+2u^{76}-22u^{72}+19u^{68}-2u^{66}+2u^{64}-2u^{62}-5u^{60}$$
$$-2u^{58}-2u^{56}+2u^{54}+u^{52}-2u^{50}-3u^{48}+2u^{46}-4u^{42}$$
$$+6u^{38}-4u^{36}-2u^{34}+4u^{32}-2u^{30}-4u^{24}-2u^{22}+5u^{4}-5\big)$$
$$+\mathcal{O}\left(q^{20}\right) \tag{A.2.3}$$

$$F_4=\frac{q^{16}u^{36}}{4!}\Bigg(\frac{1}{(1-u^2)^4}+\frac{18}{(1-u^2)^2(1-u^4)}+\frac{27}{(1-u^4)^2}+\frac{80}{(1-u^2)(1-u^6)}$$

$$
\begin{aligned}
&\qquad\qquad +\frac{210}{1-u^{8}}\Bigg)+\\
&+\frac{q^{17}u^{27}}{u^{10}-u^{6}-u^{4}+1}\left(-5u^{18}+7u^{14}+7u^{12}+6u^{10}-10u^{8}-9u^{6}-9u^{4}+14\right)\\
&-\frac{q^{18}u^{18}}{\left(u^{2}-1\right)^{2}\left(u^{2}+1\right)}\big(5u^{30}-5u^{28}-5u^{26}+3u^{24}+2u^{18}+u^{16}+u^{14}+2u^{12}\\
&\qquad -19u^{6}+14u^{4}+14u^{2}-14\big)\\
&+\frac{q^{19}u^{9}}{u^{10}-u^{6}-u^{4}+1}\big(-5u^{50}+5u^{46}+5u^{44}+2u^{42}-3u^{40}+2u^{38}-4u^{36}\\
&\qquad +3u^{34}-u^{32}-5u^{30}-11u^{28}-8u^{26}+8u^{24}+5u^{22}\\
&\qquad +20u^{20}+2u^{18}-16u^{16}-10u^{14}+5u^{12}+19u^{10}+5u^{8}\\
&\qquad -14u^{6}-14u^{4}+14\big)\\
&+\mathcal{O}\left(q^{20}\right) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(A.2.4)}
\end{aligned}
$$

Appendix B

Intersection string

We discuss fluctuation modes of the intersection string.

In order to calculate the fluctuation of the intersection string we introduce a flat space $\mathbb{R}^{1,9}$ with coordinates $(x_0, x_1, \ldots, x_9)$ and consider a quantization of the intersection string on the flat space $\mathbb{R}^{1,9}$. AdS_5 space becomes a flat spacetime in five-dimension in region far enough from its center, and we associate the flat spacetime for AdS_5 with $(x_0, x_1, \ldots, x_4)$. We also denote the coordinates in $\boldsymbol{S}^5$ as $(x_5, x_6, \ldots, x_9)$ and the complex coordinates as (z_X, z_Y, z_Z), which have the following relation:

$$z_X = x_5 + ix_6, \quad z_Y = x_7 + ix_8. \tag{B.0.1}$$

We set a configuration of the D3-branes shown in Table B.1.

Table B.1: The D3-branes configuration on $\mathbb{R}^{1,9}$. D3$_{X,Y}$ means the D3-brane wrapped around $z_{X,Y} = 0$ respectively. D3-branes extend to the coordinates with the circle signs.

	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
D3$_X$	○							○	○	○
D3$_Y$	○					○	○			○

We are interested in the strings stretching on the D3-branes wrapped around $z_X = 0$ and $z_Y = 0$. Let the coordinates (x_0, x_9) be the light cone coordinate, and we perform the light cone quantization. The bosonic string for $z_{X,Y}$ direction on the worldsheet has the Dirichlet-Neumann boundary condition and satisfies anti-periodic condition. Thus the fluctuation mode of the bosonic string for $z_{X,Y}$ direction can be expanded with half-integer. The fermionic string on the world sheet has two sectors associated with two boundary conditions. Ramond (R) sector has the same anti-periodic condition with the bosonic string and Neveu-Schwarz (NS) sector has the opposite periodic condition. Therefore, the fermionic string on R-sector is expanded with half-integer while the fermionic string on NS-sector is expanded with integer. (See Table B.2).

Table B.2: The parameter of the expansion for the fluctuation mode.

	$1 \sim 4$	$5 \sim 8$
bosonic string	integer	half-integer
NS-sector	half-integer	integer
R-sector	integer	half-nteger

We investigate massless modes of the intersection string. On the NS-sector we find the ground state is massless by using the regularization of a zeta function. The ground state has four real fermionic modes, which produce $2^{4/2} = 4$ degenerate ground states. We label the four degenerate ground states by spins (R_X, R_Y) related for the rotations each on z_X-plane and z_Y-plane. These spins are the Cartan charges for the internal symmetry $SO(4)_{XY}$and the values are taken as $R_X, R_Y = \pm 1/2$. After the Gliozi-Scherk-Olive (GSO) projection two states with $R_X = R_Y = \pm 1/2$ survive, and they are the doublet of $SO(4)_{XY}$ denoted by $\mathbf{2}$ while they are invariant under the little group $SO(4)_{1234}$. Thus the two states behave two scalar fields on the AdS_5 space, and we denote them as ϕ with $R_X = R_Y = -1/2$ and ϕ' with $R_X = R_Y = +1/2$. The $SO(4)_{1234}$ is divied by two group $SU(2)_J \times SU(2)_{\overline{J}}$ and the charge relations are given by

$$J = \frac{1}{2}\left(R_{12} + R_{34}\right), \quad \overline{J} = \frac{1}{2}\left(R_{12} - R_{34}\right). \tag{B.0.2}$$

We denote the $SO(4)$ charges that a field holds as $[J, \overline{J}; R_X, R_Y]$. The two scalars are written as

$$\phi : [0, 0; -\tfrac{1}{2}, -\tfrac{1}{2}] \quad \text{and} \quad \phi' : [0, 0; +\tfrac{1}{2}, +\tfrac{1}{2}]. \tag{B.0.3}$$

On the R-sector the ground state is also massless, and it has four real fermionic modes. Due to the GSO projection the spin charges R_{12}, R_{34} of the rotation for 12-, 34-plane satisfy $R_{12} = \pm R_{34}$, where the plus-minus sign comes from the on-shell condition $\partial_0 = \pm\partial_9$. With both signs the chirality Γ on $SO(1,5)$ is $\Gamma = +1$. In the frame $\partial_0 = +\partial_9$ the charges of the states are $R_{12} = R_{34} = \pm 1/2$ and the two states are the double by $\mathbf{2}$ for the little group $SO(4)_{1234}$, while in the frame $\partial_0 = -\partial_9$ the two states with the charges $R_{12} = -R_{34} = \pm 1/2$ are the double by $\mathbf{2}'$. In each frame the two states behave a Weyl fermion on the AdS_5 space, and we denote the Weyl fermion as $\psi_{a(\dot{a})}$, where an index $a(\dot{a})$ is related for the $SU(2)_{J(\overline{J})}$. The charges that the Weyl fermion holds are written by

$$\psi_\pm : [\pm\tfrac{1}{2}, 0; 0, 0] \quad \text{or} \quad \psi_\pm : [0, \pm\tfrac{1}{2}; 0, 0]. \tag{B.0.4}$$

The fields (B.0.3) and (B.0.4) are given by one of orientations of the intersection string. The intersection string with a reversed orientation has the same charges except for gauge charges. There are totally four real scalars and two Weyl fermions, and they compose a hypermultiplet.

In order to investigate a structure of the hypermultiplet we consider preserved supercharges on the brane configuration in Table B.1. The insertion of the D3-brane wrapped around $z_X = 0$ breaks the R-symmetry $SO(6)$ into $SO(4)_{XZ} \times SO(2)_Y$. Accordingly, the four-dimensional representations [1] of $SO(6)$ are divided as

$$\mathbf{4} \to \mathbf{2}_{+\frac{1}{2}} + \mathbf{2}'_{-\frac{1}{2}}, \quad \overline{\mathbf{4}} \to \mathbf{2}_{-\frac{1}{2}} + \mathbf{2}'_{+\frac{1}{2}}, \tag{B.0.5}$$

where the subscript in the lower right corner denotes the charge for $SO(2)_Y = U(1)_Y$. Let $\mathbf{2}'_{+\frac{1}{2}}$ be left for the supercharge Q with $\overline{\mathbf{4}}$. For the $\overline{Q}$ with $\mathbf{4}$, $\mathbf{2}_{+\frac{1}{2}}$ is left so that the commutation relation with Q and $\overline{Q}$ has no momentum operator. The Hermitian conjugate $S, \overline{S}$ have the $\mathbf{2}'_{-\frac{1}{2}}, \mathbf{2}_{-\frac{1}{2}}$ each other. The conditions satisfied by the charges of $SO(4)_{XZ}$ for the supercharges are

$$Q : R_X R_Z < 0, \quad \overline{Q} : R_X R_Z > 0, \quad S : R_X R_Z < 0, \quad \overline{S} : R_X R_Z > 0. \tag{B.0.6}$$

The supercharge preserved on the the D3-brane wrapped around $z_Y = 0$ is similarly discussed, and the conditions satisfied by the supercharges are

[1] The four-dimensional representation $\mathbf{4}$ has the highest weight with the Dynkin label $[0, 0, 1]$ and the charge $(+\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2})$. The other four-dimensional representation $\overline{\mathbf{4}}$ has the highest weight with the Dynkin label $[0, 1, 0]$ and the charge $(+\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2})$.

obtained by replacing X and Y on (B.0.6),

$$Q : R_Y R_Z < 0, \quad \overline{Q} : R_Y R_Z > 0, \quad S : R_Y R_Z < 0, \quad \overline{S} : R_Y R_Z > 0. \tag{B.0.7}$$

The preserved supercharge on the D3-branes wrapped around $z_X = 0$ and $z_Y = 0$ are shown in Table B.3. The preserved supercharges on the D3-branes

Table B.3: The supercharges remain on the D3-branes wrapped aroud $z_X = 0$ and $z_Y = 0$. $[\frac{1}{2}]$ denotes the half-spin representation of $SU(2)_{J,\overline{J}}$.

	H	J	$\overline{J}$	R_X	R_Y	R_Z
Q	$+\frac{1}{2}$	$[\frac{1}{2}]$	0	$+\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$
S	$-\frac{1}{2}$	$[\frac{1}{2}]$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$
$\overline{Q}$	$+\frac{1}{2}$	0	$[\frac{1}{2}]$	$+\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$
$\overline{S}$	$-\frac{1}{2}$	0	$[\frac{1}{2}]$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$

have the following commutation relations

$$\begin{aligned} \widetilde{\Delta} &= 2\{Q, S\} = H - 2J - (R_X + R_Y - R_Z)\,, \\ \Delta &= 2\{\overline{Q}, \overline{S}\} = H - 2\overline{J} - (R_X + R_Y + R_Z)\,. \end{aligned} \tag{B.0.8}$$

Since the x_9 direction is regarded as a $\boldsymbol{S}^1$ coordinate with a radius r in $\boldsymbol{S}^5$, the fields are periodic for $x_9 \sim x_9 + 2\pi r$. A momentum R_Z corresponding to the x_9 direction has discrete value. The fields are Fourier expanded for x_9 direction,

$$\phi = \sum_{n \in \mathbb{Z}} \phi_n e^{\frac{inx_9}{r}}, \quad \phi' = \sum_{n' \in \mathbb{Z}} \phi'_{n'} e^{\frac{in'x_9}{r}}, \quad \psi = \sum_{m \in \mathbb{Z}+\frac{1}{2}} \psi_m e^{\frac{imx_9}{r}}. \tag{B.0.9}$$

The fields (ϕ, ϕ', ψ) belong to a short multiplet. Let them be trivial for the supercharges $(\overline{Q}, \overline{S})$ and ϕ be a lowest weight in a supermultiplet for the supercharges (Q, S). The relations between the fields and the supercharges are

$$[Q_a, \phi] = \psi_a, \qquad \{Q_a, \psi_b\} = \epsilon_{ab}\phi'. \tag{B.0.10}$$

Table B.4: $[\frac{1}{2}]$ denotes the half-spin representation of $SU(2)_{J,\overline{J}}$.

Fields	H	J	$\overline{J}$	R_X	R_Y	R_Z
ϕ_n	$\lvert n\rvert-1$	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$n\in\mathbb{Z}$
$\psi_{m>0}$	$\lvert m\rvert$	$[\frac{1}{2}]$	0	0	0	$m\in\mathbb{Z}+\frac{1}{2}$
$\psi_{m<0}$		0	$[\frac{1}{2}]$			
$\phi'_{n'}$	$\lvert n'\rvert+1$	0	0	$+\frac{1}{2}$	$+\frac{1}{2}$	$n'\in\mathbb{Z}$

The fields satisfy the BPS condition $2\{\overline{Q},\overline{S}\}=0$ and the charges of them are determined in Table B.4.

We assume that the norm of the lowest weight ϕ is one. Noting that the R_z charge for the supercharges is non-zero, the norms of the field are calculated as

$$\langle\phi_n|\phi_n\rangle=1,\quad \left\langle\psi^{a,n-\frac{1}{2}}\middle|\psi_{b,n-\frac{1}{2}}\right\rangle=\delta^a_b n,\quad \left\langle\phi'_{n-1}\middle|\phi'_{n-1}\right\rangle=n-1. \quad \text{(B.0.11)}$$

The supermultiplet components vary with the value of n and the supermultiplet exists in the case for $n\geq 0$. If $n=0$ the field ϕ_0 is the singlet and if $n=1$ the fields $(\phi_1,\psi_{\frac{1}{2}})$ are doublet.

We can similarly discuss a supermultiplet for $(\overline{Q},\overline{S})$ while it is trivial for the (Q,S). A short multiplet for the supercharges $(\overline{Q},\overline{S})$ exists in the case $\phi_{n\leq 0}$ and the structure of the short multiplet is shown in Figure B.1.

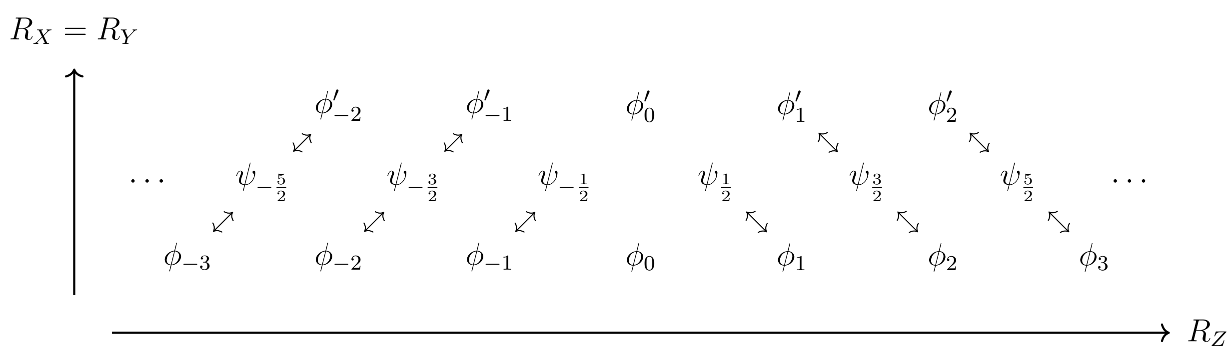

Figure B.1: The short multiplet on the intersection string.

The supermultiplet for (Q,S) contributes to the superconformal index and the single-particle index is given by

$$
\begin{aligned}
i^{\text{int}}_{X,Y} &= \sum_{n\in\mathbb{Z}_{\geq 0}} q^{n-1} v^{-n-\frac{1}{2}} - \sum_{m\in\mathbb{Z}_{\geq 0}+\frac{1}{2}} q^{m}\left(y+y^{-1}\right) v^{-m} + \sum_{n'\in\mathbb{Z}_{\geq 0}+1} q^{n'+1} v^{-n'+\frac{1}{2}} \\
&= \frac{1}{qv^{\frac{1}{2}}} \frac{\left(1-q^{\frac{3}{2}}y\right)\left(1-q^{\frac{3}{2}}y^{-1}\right)}{1-qv^{-\frac{1}{2}}}.
\end{aligned}
\tag{B.0.12}
$$

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