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T H E S I S

Twisted Reidemeister torsions of closed 3-manifolds

by

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Abstract

A Reidemeister torsion is a topological invariant of a differential manifold M, and is defined from a linear representation of the fundamental group of M. The torsion has a long history and has been studied from several viewpoints of geometry and topology with its applications. In a situation such that M has a boundary, there are many existing studies to compute the torsions of some 3-manifolds. However, if M has no boundary, there have not since been so many procedures to compute the torsions, except for some surgery formulas. Here, the difficulty of the computation is derived from a complexity of the cellular structure of the universal covering of M.

In contrast, this paper establishes a new procedure for computing the twisted Reidemeister torsions of orientable closed 3-manifolds with respect to SL_n-representations. The procedure is derived from some ideas in group cohomology. We further suggest three normalizations of the torsions and compare the normalized torsions with the preceding torsions. Moreover, we compute some twisted torsions of some Seifert 3-manifolds over the 2-sphere. These results appear in [Wak21].

As a special case, we focus on the adjoint representations and determine the adjoint Reidemeister torsion of a 3-manifold obtained by some Dehn surgery along K, where K is either the figure-eight knot or the 5_2-knot. As in a vanishing conjecture from mathematical physics, we consider a similar conjecture and obtain a result, which claims that the conjecture holds for the 3-manifold. Here, the point is that we apply Jacobi's residue theorem to a sum of the torsions, and carefully checked the condition available to the application. This result gives the first supporting evidence in the case where M is closed and hyperbolic. Furthermore, we also discuss the sum of the n-th powers of the adjoint torsions, and we show an integrality of the sum in some cases. These results are shown in another paper [Wak23].

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Contents

Chapter 1

Introduction

The (combinatorial) Reidemeister torsion is a topological invariant with respect to a differential manifold M and a representation of its fundamental group. The definition of the torsion uses basic notations of linear algebra and combinatorial topology, as in triangulations, coverings, cellular chain complexes, etc.

The torsion was originally defined for a 3-dimensional manifold by K. Reidemeister [Rei35] with respect to a commutative representation, and the torsion gave an application of the homeomorphism classification of 3-dimensional lens spaces. There are some relationships and applications of the Reidemeister torsion to other invariants, and the torsions of many M's are computed; see [Kod07, Mil62, Nic03, Tur01] and references therein. More generally, given a representation $\varphi : \pi_1(M) \to \mathrm{GL}_n(\mathbb{F})$ for some field $\mathbb{F}$, we can define the torsion in the local coefficients; the torsion is referred to as the twisted (Reidemeister) torsion. Analogously, there are some applications of the twisted torsion to, e.g., the Casson-invariant, Casson-Gordon invariant, and hyperbolic volumes; see [Joh, KL99, MFP14].

While the computation of the twisted torsion is slightly more complicated, there are some examples of computing the twisted torsions of M in the case where M is a 3-manifold with a toroidal boundary as in a knot complement. Meanwhile, if M is a closed 3-manifold, the previous computations of the torsions of M often follow from Mayer-Vietoris arguments (see [KN20, Yam17]); there are not so many examples of resulting computations of the twisted torsions.

In the paper [Wak21], we study a new procedure for computing the twisted Reidemeister torsions of closed 3-manifolds M with respect to $\mathrm{SL}_n(\mathbb{F})$-representations φ. The key is here that, in terms of the Heegaard diagram and "identity", the cellular chain complex of the universal covering space $\widetilde{M}$ can be almost described from $\pi_1(M)$ (Theorems 3.2.1 and 3.2.2). According to the description, we define a torsion and show that it is a topological invariant of a representation φ (Theorems 3.3.2 and 3.3.4). In addition, we show that the invariant recovers the original torsion above up to signs (Theorem 3.3.5); while the original definition had ambiguity of signs, our definition suggests a normalization of the torsion. Furthermore, we define another normalized torsion if the Betti number of M is larger than zero, and we show the invariance (Theorem 3.3.8).

As examples of the computation, we focus on two kinds of manifolds. In Section 5.1, we compute the torsions of specific 3-manifolds $M_{n,m,\ell}$ defined for $n, m, \ell \in \mathbb{N}_{\geq 2}$, which are the Seifert manifolds over 2-sphere. In fact, as we show in [Wak21], we compute all the twisted Reidemeister torsions of $M_{n,m,\ell}$ with respect to all the $\mathrm{SL}_2(\mathbb{C})$-representations (Theorems 5.1.5 and 5.1.6). We compare the resulting computations with the result of Kitano [Kit94] and the Brieskorn manifolds (Proposition 5.1.1 and Remark 5.1.7). In Section 5.2, we apply our procedure to compute all the adjoint Reidemeister torsions of some 3-manifolds obtained by some Dehn-surgeries on the figure-eight knot and the 5_2-knot with respect to all the irreducible $\mathrm{SL}_2(\mathbb{C})$-representations.

Let us explain the adjoint Reidemeister torsion in detail. Let $\mathfrak{g}$ be the Lie algebra of a semisimple complex Lie group G, and M be a connected compact oriented manifold. Let $R_G^{\mathrm{irr}}(M)$ be the (irreducible) character variety, that is, the set of conjugacy classes of irreducible representations $\pi_1(M) \to G$. Given a homomorphism $\varphi : \pi_1(M) \to G$, we can define the *adjoint (Reidemeister) torsion* $\tau_\varphi^{\mathrm{ad}}(M)$ via the adjoint action $G \to \mathrm{Aut}(\mathfrak{g})$, which lies in $\mathbb{C}^\times$ and is determined by the conjugacy class of φ; see [Tur01] or Section 2 for details. When $\dim M = 2$, the torsion plays an interesting role as a volume form on the space $R_G^{\mathrm{irr}}(M)$, see [Por97, Wit91]. In addition, if M is 3-dimensional and $G = \mathrm{SL}_2(\mathbb{C})$, then some attitudes of the torsions in $R_G^{\mathrm{irr}}(M)$ are physically observed from the viewpoint of a 3D-3D correspondence, and some conjectures on the torsions are mathematically proposed [BGPZ20, GKPZ20, GKY21].

The conjectures can be roughly described as follows. Suppose that $\dim M = 3$ and M has a tori-boundary. For $z \in \mathbb{C}$, the authors of [BGPZ20, GKPZ20, GKY21] introduced a finite subset "$\mathrm{tr}_\gamma^{-1}(z)$" of $R_G^{\mathrm{irr}}(M)$ which is defined from a boundary condition, and discussed the sum of the n-th powers of the twice torsions, that is, $\sum_{\varphi \in \mathrm{tr}_\gamma^{-1}(z)} (2\tau_\varphi^{\mathrm{ad}}(M))^n \in \mathbb{C}$ for $n \in \mathbb{Z}$ with $n \geq -1$. Then, it is conjectured [BGPZ20, GKPZ20, GKY21] that the sum lies in $\mathbb{Z}$ and, that if M is hyperbolic and $n = -1$, then the sum is zero. This conjecture is sometimes called *the vanishing identity*; see [PY, TY, Yoo22] and references therein for supporting evidence of this conjecture.

In this paper, we focus on the adjoint torsions in the case where M has no boundary. According to [BGPZ20, GKPZ20, GKY21], it is seemingly reasonable to consider the following conjecture:

Conjecture 1.0.1 (cf. [BGPZ20, GKPZ20, GKY21, Yoo23])**.** Take $n \in \mathbb{Z}$ with $n \geq -1$. Suppose that M is closed, and the set $R_G^{\mathrm{irr}}(M)$ is of finite order. Then, the following sum lies in the ring of integers $\mathbb{Z}$:

$$\sum_{\varphi \in R^{\mathrm{irr}}_G(M)} (2\tau^{\mathrm{ad}}_{\varphi}(M))^n. \tag{1.1}$$

Furthermore, if $G = \mathrm{SL}_2(\mathbb{C})$, M is a hyperbolic 3-manifold, and $n = -1$, then the sum (1.1) is zero.

This paper provides supporting evidence on Conjecture 1.0.1.

Theorem 1.0.2 [Wak23]**.** *Let G be $\mathrm{SL}_2(\mathbb{C})$, and $K = 4_1$ be the figure-eight knot. Let $n = -1$. Then, for any integers p and $q \neq 0$, Conjecture 1.0.1 is true when $M = S^3_{p/1}(4_1)$ and $M = S^3_{1/q}(4_1)$.*

The outline of the proof is as follows. As mentioned above, Chapter 3 gives some procedures for computing the adjoint torsions of 3-manifolds with boundary are established (see, e.g., [DHY09, TY, Yoo22]), this paper employs a procedure of computing the adjoint torsions of closed 3-manifolds, which is established in [Wak21], and determine all the adjoint torsion (see Theorem 5.2.2). As in the proof of the above supporting evidence [BGPZ20, GKPZ20, GKY21], we apply Jacobi's residue theorem (see Lemma 6.1.2) to the sum (1.1) and demonstrate Theorem 5.2.2. Since it is complicated to check the condition for applying the residue theorem, we need some careful discussion (see Sections 6.1.1–6.1.2)[1]. In addition, in Section 6.2, we also discuss the conjecture with $n > 0$ and see that some properties are needed to be addressed in future studies. Here, we analyze the 2^{2n+1}-multiple of the conjecture with $M = S^3_{2m/1}(4_1)$; see Proposition 6.2.4.

To summarize, our procedure for computing the torsion is expected to be applicable to other 3-manifolds and representations. In doing so, one may hope that the procedure produces many examples of twisted torsions in a similar way, which is a future problem.

This paper is organized as follows. In Chapter 2, we review Reidemeister torsions of manifolds with linear representations. Chapter 3 reviews basic facts of Heegaard decompositions and a description of $\pi_1(M)$ from a Heegaard diagram. In Chapter 4, we see that the cellular chain complex of $\widetilde{M}$ can be described from a Heegaard splitting and an identity. Chapter 5 computes concretely some torsions of some manifolds. Chapter 6 gives the proofs of giving supporting evidences of the above conjectures. Chapter A is an appendix to explain the (taut) identity in details.

Conventional terminology. Throughout this paper, for a commutative field $\mathbb{F}$, we use $\mathbb{F}^\times = \mathbb{F} \setminus \{0\}$ as a multiplicative group. We mean the group ring of a group

[1] As a private communication with S. Yoon [Yoo23], he tells us another proof of Conjecture 1.0.1 with $M = S^3_{p/q}(K)$ in generic condition. Here, we emphasize that, while the condition does not contain the case $(p, q) = (4m, 1)$ for some $m \in \mathbb{Z}$, Theorem 1.0.2 deals with all p.

G by $\mathbb{Z}[G]$. The universal covering space of a connected CW complex X will be denoted by $\widetilde{X}$.

Chapter 2

Review; twisted Reidemeister torsions

In this chapter, we review algebraic torsions in Section 2.1, and the twisted Reidemeister torsion in Section 2.2. In Section 2.3, we recall the adjoint Reidemeister torsion. Throughout this paper, we assume that any basis of a vector space is ordered and of finite dimension.

2.1 Algebraic torsions

To review algebraic torsions, we need terminology. Let V be a d-dimensional vector space over a field $\mathbb{F}$, and let $\mathbf{b} = (b_1, \ldots, b_d), \mathbf{c} = (c_1, \ldots, c_d)$ be two bases of V. Then, there is a transition matrix $P \in \mathrm{GL}_d(\mathbb{F})$, satisfying $b_i = \sum_{j=1}^{d} P_{(j,i)} c_j$, where $P_{(j,i)} \in \mathbb{F}$ is the (j,i)-th entry of P. We write $[\mathbf{b}/\mathbf{c}]$ for the determinant of P.

Let us define torsions. Consider a bounded chain complex

$$C_* : 0 \longrightarrow C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} \cdots \xrightarrow{\partial_1} C_0 \longrightarrow 0$$

consisting of finite dimensional vector spaces over $\mathbb{F}$. Assume that a basis $\mathbf{c}_i$ of C_i is given for each i. In a usual way, we use the common notation $Z_i = \mathrm{Ker}(\partial_i)$, $B_i = \mathrm{Im}(\partial_{i+1})$, and $H_i = Z_i/B_i$. The chain complex C_* is said to be *acyclic* if H_i is trivial for all i. We consider canonically the following exact sequences:

$$\begin{aligned} 0 \longrightarrow Z_i \longrightarrow C_i \xrightarrow{\partial_i} B_{i-1} \longrightarrow 0 \quad (\text{exact}), \\ 0 \longrightarrow B_i \longrightarrow Z_i \longrightarrow H_i \longrightarrow 0 \quad (\text{exact}). \end{aligned} \tag{2.1}$$

Let us fix $\widetilde{B}_{i-1}$ and $\widetilde{H}_i$ which are the lifts of B_{i-1} to C_i and H_i to Z_i, respectively. The above exact sequences give the following identifications:

$$C_i = Z_i \oplus \widetilde{B}_{i-1} = B_i \oplus \widetilde{H}_i \oplus \widetilde{B}_{i-1}.$$

For each i, choose a basis $\mathbf{b}_i$ of B_i, and fix its lift to C_{i+1}, and denote the lift by $\widetilde{\mathbf{b}}_i$. Similarly, choose a basis $\mathbf{h}_i$ of H_i, and fix its lift to Z_i, and denote the lift by $\widetilde{\mathbf{h}}_i$. Then, the union $\mathbf{b}_i \cup \widetilde{\mathbf{h}}_i \cup \widetilde{\mathbf{b}}_{i-1}$ is a basis of C_i. Set $\mathbf{c}_* = (\mathbf{c}_0, \mathbf{c}_1, \dots, \mathbf{c}_m), \mathbf{h}_* = (\mathbf{h}_0, \mathbf{h}_1, \dots, \mathbf{h}_m)$. *The torsion* of C_* is defined by

$$\tau(C_*, \mathbf{c}_*, \mathbf{h}_*) \coloneqq \prod_{i=0}^{m} [\mathbf{b}_i \cup \widetilde{\mathbf{h}}_i \cup \widetilde{\mathbf{b}}_{i-1}/\mathbf{c}_i]^{(-1)^{i+1}} \in \mathbb{F}^{\times}.$$

It is known that the torsion $\tau(C_*, \mathbf{c}_*, \mathbf{h}_*)$ does not depend on the choices of the basis $(\mathbf{b}_0, \mathbf{b}_1, \dots, \mathbf{b}_m)$ and the lifts $(\widetilde{\mathbf{b}}_{-1}, \widetilde{\mathbf{b}}_0, \dots, \widetilde{\mathbf{b}}_{m-1})$, $(\widetilde{\mathbf{h}}_0, \widetilde{\mathbf{h}}_1, \dots, \widetilde{\mathbf{h}}_m)$.

Remark 2.1.1. (i) The torsion $\tau(C_*, \mathbf{c}_*, \mathbf{h}_*)$ depends on the choices of $\mathbf{c}_*$ and $\mathbf{h}_*$. In fact, if we choose such other bases $\mathbf{c}'_* = (\mathbf{c}'_0, \mathbf{c}'_1, \dots, \mathbf{c}'_m)$ and $\mathbf{h}'_* = (\mathbf{h}'_0, \mathbf{h}'_1, \dots, \mathbf{h}'_m)$, the following holds [Tur01, Remark 1.4.1]:

$$\tau(C_*, \mathbf{c}_*, \mathbf{h}_*) = \tau(C_*, \mathbf{c}'_*, \mathbf{h}'_*) \prod_{i=0}^{m} ([\mathbf{c}_i/\mathbf{c}'_i][\mathbf{h}'_i/\mathbf{h}_i])^{(-1)^{i+1}}.$$

(ii) Supposing an acyclic complex C_*, we consider another chain complex of the form

$$E_*(\mathbb{F}, i) : \cdots \longrightarrow 0 \longrightarrow \mathbb{F} \xrightarrow{\mathrm{id}} \mathbb{F} \longrightarrow 0 \longrightarrow \cdots.$$

Here, the left and right modules of $\mathbb{F}$ are of degree i and $i+1$, and the modules of other degrees are trivial. Let $\mathbf{f}$ be an arbitrary basis of $\mathbb{F}$, and let $\mathbf{c}'_i$ be $\mathbf{c}_i \cup \mathbf{f}$ and $\mathbf{c}'_{i+1}$ be $\mathbf{c}_{i+1} \cup \mathbf{f}$. As seen in [Tur01, p.13–14], if we set $\mathbf{c}'_* = (\mathbf{c}_0, \dots, \mathbf{c}_{i-1}, \mathbf{c}'_i, \mathbf{c}'_{i+1}, \mathbf{c}_{i+2}, \dots, \mathbf{c}_m)$, then

$$\tau(C_*, \mathbf{c}_*) = \tau(C_* \oplus E_*(\mathbb{F}, i), \mathbf{c}'_*).$$

2.2 Twisted Reidemeister torsions of CW complexes

To define the twisted Reidemeister torsions, we prepare some terminology. Let X be a connected finite CW complex, and $\widetilde{X}$ be the universal covering space. When we regard the covering transformation of $\pi_1(X)$ on $\widetilde{X}$ as a left action, the cellular complex $(C_*(\widetilde{X}; \mathbb{Z}), \widetilde{\partial}_*)$ is made into a left $\mathbb{Z}[\pi_1(X)]$-module. Let $\varphi : \pi_1(X) \to \mathrm{SL}_n(\mathbb{F})$ be a representation, which yields the right action on $\mathbb{F}^n$ defined by

$$\mathbb{F}^n \times \pi_1(X) \longrightarrow \mathbb{F}^n; \qquad (v, g) \longmapsto \varphi(g^{-1}) \cdot v.$$

This action gives rise to the right $\mathbb{Z}[\pi_1(X)]$-module structure on $\mathbb{F}^n$. Define the chain complex $(C^{\varphi}_*(X), \partial_*)$ by

$$(C^{\varphi}_*(X), \partial_*) \coloneqq (\mathbb{F}^n \otimes_{\mathbb{Z}[\pi_1(X)]} C_*(\widetilde{X}; \mathbb{Z}), \mathrm{id}_{\mathbb{F}^n} \otimes_{\mathbb{Z}[\pi_1(X)]} \widetilde{\partial}_*).$$

Next, we establish a basis of each $C_i^{\varphi}(X)$ as follows. Let $(e_1, \dots, e_n)$ be the canonical basis of $\mathbb{F}^n$, $\boldsymbol{\sigma}^i = \{\sigma_1^i, \dots, \sigma^i_{\operatorname{rank} C_i(X;\mathbb{Z})}\}$ be the set of all oriented i-cells of X, and $\{\widetilde{\sigma}_1^i, \dots, \widetilde{\sigma}^i_{\operatorname{rank} C_i(X;\mathbb{Z})}\}$ be the set of lifts of σ_j^i to $\widetilde{X}$. Then, the tuple

$$\begin{aligned}\mathbf{c}_i \coloneqq (e_1 \otimes \widetilde{\sigma}_1^i, &e_2 \otimes \widetilde{\sigma}_1^i, \dots, e_n \otimes \widetilde{\sigma}_1^i, e_1 \otimes \widetilde{\sigma}_2^i, e_2 \otimes \widetilde{\sigma}_2^i, \dots, e_n \otimes \widetilde{\sigma}_2^i, \\ &\dots, e_1 \otimes \widetilde{\sigma}^i_{\operatorname{rank} C_i(X;\mathbb{Z})}, e_2 \otimes \widetilde{\sigma}^i_{\operatorname{rank} C_i(X;\mathbb{Z})}, \dots, e_n \otimes \widetilde{\sigma}^i_{\operatorname{rank} C_i(X;\mathbb{Z})})\end{aligned}$$

provides a basis of $C_i^{\varphi}(X)$.

When we fix a basis, $\mathbf{h}_i$, of $H_i^{\varphi}(X)$ and $\mathbf{h}_* = (\mathbf{h}_0, \mathbf{h}_1, \dots, \mathbf{h}_{\dim X})$, *the ($\varphi$-twisted) Reidemeister torsion*, $\tau_\varphi(X, \mathbf{h}_*)$, is defined by

$$\tau_\varphi(X, \mathbf{h}_*) \coloneqq \tau(C_*^{\varphi}(X), \mathbf{c}_*, \mathbf{h}_*) \in \mathbb{F}^{\times}/\{\pm 1\}.$$

Remark 2.2.1. (i) If $C_*^{\varphi}(X)$ is acyclic, and $\mathbf{h}_* = \emptyset$, then the torsion $\tau_\varphi(X, \mathbf{h}_*)$ is denoted by $\tau_\varphi(X)$.

(ii) Give a triangulation on a smooth closed manifold M, and let us regard it as a CW complex X. It is known that if $C_*^{\varphi}(X)$ is acyclic, then the Reidemeister torsion $\tau_\varphi(M) \coloneqq \tau_\varphi(X)$ does not depend on the choices of the triangulation of M, the orientation of σ_j^i, and the lift $\widetilde{\sigma}_j^i$; see Theorem 6.1 of [Tur01, Theorem 6.1].

2.3 Adjoint Reidemeister torsions

Let M be a connected oriented closed 3-manifold and G be a semisimple Lie group with Lie algebra $\mathfrak{g}$. Let $\varphi : \pi_1(M) \to G$ be a representation, that is, a group homomorphism. Suppose that G injects $\mathrm{SL}_n(\mathbb{C})$ for some $n \in \mathbb{N}$.

First, we introduce the chain complex below (2.2). Choose a finite cellular decomposition of M, and consider the universal covering space $\widetilde{M}$ with a structure of a lift of the decomposition of M. Since $\mathfrak{g}$ is a $\mathbb{Z}[\pi_1(M)]$-module via the composite of φ and the adjoint action $G \to \mathrm{Aut}(\mathfrak{g})$, we have the chain complex of the form

$$(C_*^{\varphi}(M;\mathfrak{g}), \partial_*) \coloneqq (\mathfrak{g} \otimes_{\mathbb{Z}[\pi_1(M)]} C_*(\widetilde{M};\mathbb{Z}), \mathrm{id}_{\mathfrak{g}} \otimes_{\mathbb{Z}[\pi_1(M)]} \widetilde{\partial}_*). \tag{2.2}$$

Next, we define an ordered basis of $C_i^{\varphi}(M;\mathfrak{g})$. Let $\boldsymbol{\sigma}_i = (\sigma_{i,1}, \sigma_{i,2}, \dots, \sigma_{i,\operatorname{rank} C_i(M;\mathbb{Z})})$ be a basis of $C_i(M;\mathbb{Z})$ derived from the i-cells. Then, $\widetilde{\boldsymbol{\sigma}}_i = (\widetilde{\sigma}_{i,1}, \widetilde{\sigma}_{i,2}, \dots, \widetilde{\sigma}_{i,\operatorname{rank} C_i(M;\mathbb{Z})})$ is a basis of the free $\mathbb{Z}[\pi_1(M)]$-module $C_i(\widetilde{M};\mathbb{Z})$. Here, $\widetilde{\sigma}_{i,j}$ is a lift of $\sigma_{i,j}$ to $\widetilde{M}$. Since $\mathfrak{g}$ is semisimple, the killing form B is non-degenerate, and we can fix an ordered basis $\mathcal{B} = (e_1, e_2, \dots, e_{\dim \mathcal{B}})$ of $\mathfrak{g}$, which is orthogonal with respect to B. Then the tuple

$$\begin{aligned}\mathbf{c}_i \coloneqq (e_1 \otimes \widetilde{\sigma}_{i,1}, &e_2 \otimes \widetilde{\sigma}_{i,1}, \dots, e_{\dim \mathcal{B}} \otimes \widetilde{\sigma}_{i,1}, e_1 \otimes \widetilde{\sigma}_{i,2}, e_2 \otimes \widetilde{\sigma}_{i,2}, \dots, e_{\dim \mathcal{B}} \otimes \widetilde{\sigma}_{i,2}, \\ &\dots, e_1 \otimes \widetilde{\sigma}_{i,\operatorname{rank}_{\mathbb{Z}} C_i(M;\mathbb{Z})}, e_2 \otimes \widetilde{\sigma}_{i,\operatorname{rank}_{\mathbb{Z}} C_i(M;\mathbb{Z})}, \dots, e_{\dim \mathcal{B}} \otimes \widetilde{\sigma}_{i,\operatorname{rank}_{\mathbb{Z}} C_i(M;\mathbb{Z})})\end{aligned}$$

provides an ordered basis of $C_i^{\varphi}(M;\mathfrak{g})$ as desired.

We next consider the ordinal cellular chain complex $C_*(M;\mathbb{R})$ with the real coefficient. From the Poincaré duality, we can naturally fix a *homology orientation* o_M of $H_*(M;\mathbb{R}) = \bigoplus_{i=0}^{3} H_i(M;\mathbb{R})$. Let $\mathbf{h}_* = (\mathbf{h}_0, \mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3)$ be a basis of $H_*(M;\mathbb{R})$ such that the exterior product of $\mathbf{h}_*$ coincides with o_M. The sign-refined Reidemeister torsion of $C_*(M;\mathbb{R})$ associated with $\boldsymbol{\sigma}_*$ and $\mathbf{h}_*$ lies in $\mathbb{R}^\times$. Let us define the sign

$$\tau_M \coloneqq \operatorname{sgn}(\tau(C_*(M;\mathbb{R}), \boldsymbol{\sigma}_*, \mathbf{h}_*)) \in \{\pm 1\}.$$

Then, the *adjoint Reidemeister torsion* of M associated with φ is defined to be

$$\tau_{\varphi}^{\mathrm{ad}}(M) \coloneqq (\tau_M)^{\dim \mathfrak{g}} \cdot \tau(C_*^{\varphi}(M;\mathfrak{g}), \mathbf{c}_*) \in \mathbb{C}^\times,$$

if $C_*^{\varphi}(M;\mathfrak{g})$ is acyclic. If $C_*^{\varphi}(M;\mathfrak{g})$ is not acyclic, then we define $\tau_{\varphi}^{\mathrm{ad}}(M)$ to be zero. As is known, the definition of $\tau_{\varphi}^{\mathrm{ad}}(M)$ does not depend on the choices of the orthogonal basis $\mathcal{B}$, a finite triangulation of M, $\widetilde{\boldsymbol{\sigma}_i}$, and $\mathbf{h}_i$, but depends only on M and the conjugacy class of φ.

Let $R_G^{\mathrm{irr}}(M)$ be the irreducible character variety, that is, the set of conjugacy classes of irreducible representations $\pi_1(M) \to G$. Finally, we give a criteria for the acyclicity, which might be known:

Lemma 2.3.1. *Assume that $R_G^{\mathrm{irr}}(M)$ is of finite order. Then, for any irreducible $\varphi : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$, the associated cohomology $H_{\varphi}^i(M;\mathfrak{g})$ vanishes.*

Proof. Since it is classically known [Wit91] that the first cohomology $H_{\varphi}^1(M;\mathfrak{g})$ is identified with the tangent space of the variety $R_G^{\mathrm{irr}}(M)$, it vanishes by assumption; by Poincaré duality, the second one does. Meanwhile, by definition, the zeroth cohomology $H_{\varphi}^0(M;\mathfrak{g})$ equals the invariant part $\{a \in \mathfrak{g} \mid a \cdot \varphi(g) = a \text{ for any } g \in \pi_1(M)\}$, which is zero by the irreducibility. Hence, the third one also vanishes by Poincaré duality again. $\square$

In summary, on the condition of Conjecture 1.0.1, $\tau_{\varphi}^{\mathrm{ad}}(M)$ is not zero, and we can consider the inverse $\tau_{\varphi}^{\mathrm{ad}}(M)^{-1}$.

Chapter 3

Results; invariants of 3-manifolds

We define the invariants of closed 3-manifolds in this chapter. In Section 3.1, we recall Heegaard splittings, and presentations of fundamental groups. In Section 3.2, we observe a description of the cellular chain complex of the universal covering space. Finally, in Section 3.3, we introduce other definitions of invariants in terms of twisted torsions. These results are new things and obtained in [Wak21].

3.1 Heegaard splittings, and the presentation of the fundamental group

In this section, let us review basic facts of Heegaard splittings and Heegaard diagrams. Hereafter, M means an orientable, connected, closed smooth 3-manifold.

A closed tubular neighborhood of a wedge product of g circles in $\mathbb{R}^3$ is called *a handlebody* of genus g. We say $\mathcal{H}_1 \cup_f \mathcal{H}_2$ which is homeomorphic to M to be *a Heegaard splitting* of M, if both $\mathcal{H}_1$ and $\mathcal{H}_2$ are handlebodies, and f is an orientation-reversing homeomorphism from $\partial\mathcal{H}_1$ to $\partial\mathcal{H}_2$. As is known, every M admits a Heegaard splitting.

The Heegaard diagram of M is given as follows. Consider a given Heegaard splitting $\mathcal{H}_1 \cup_f \mathcal{H}_2$ and take simple closed curves $\alpha_1, \alpha_2, \dots, \alpha_g \subset \partial\mathcal{H}_1$ as illustrated in Fig. 3.1 (here, g is the genus of $\mathcal{H}_1$ and $\mathcal{H}_2$). Letting $\boldsymbol{\beta} = \{f(\alpha_1), \dots, f(\alpha_g)\}$, we call the pair $(\mathcal{H}_2, \boldsymbol{\beta})$ *the Heegaard diagram.*

Let us observe a presentation of $\pi_1(M)$ from the Heegaard splitting $\mathcal{H}_1 \cup_f \mathcal{H}_2$ as follows. *A spine* of the handlebody $\mathcal{H}$ is a graph K embedded in $\mathcal{H}$ such that $\mathcal{H} \backslash K$ is homeomorphic to $\partial\mathcal{H} \times (0, 1]$. Consider a spine K of $\mathcal{H}_2$ which is composed of a single vertex and oriented edges $e_1, \dots, e_g$. For each e_i, choose an orientation-preserving loop $\ell_i : [0, 1] \to \mathcal{H}_2$ whose image is the edge e_i. Suppose that $\ell_i(0) =$

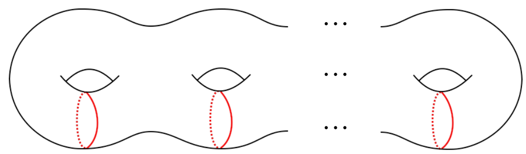

Figure 3.1: Simple closed curves $\alpha_1, \alpha_2, \dots, \alpha_g$

$\ell_i(1) = v_0$. If we view v_0 as a basepoint for the fundamental group, each ℓ_i descends to $x_i \in \pi_1(\mathcal{H}_2)$. Since $\mathcal{H}_2$ is homotopy equivalent to K, the fundamental group $\pi_1(\mathcal{H}_2)$ is a free group $\langle x_1, \dots, x_g \mid \rangle$. Let $(\mathcal{H}_2, \boldsymbol{\beta})$, $\boldsymbol{\beta} = \{\beta_1, \dots, \beta_g\}$ be a Heegaard diagram given from a Heegaard splitting $\mathcal{H}_1 \cup_f \mathcal{H}_2$. Let us fix an orientation of β_i and a point $b_i \in \beta_i$, and choose an orientation-preserving loop $k_i : [0,1] \to \beta_i$ such that $k_i(0) = k_i(1) = b_i$. Let $j_i : [0,1] \to \mathcal{H}_2$ be a path from v_0 to b_i. Since the path composite $j_i^{-1} k_i j_i$ is a closed curve starting from v_0 in $\mathcal{H}_2$, it determines an element r_i of $\pi_1(\mathcal{H}_2)$. In summary, the van Kampen theorem can lead to the following lemma:

Lemma 3.1.1 (see, e.g., [Sch02, §5])**.** *For the above x_i and r_i, the fundamental group $\pi_1(M)$ is isomorphic to the group presented by $\langle x_1, x_2, \dots, x_g \mid r_1, r_2, \dots, r_g\rangle$.*

3.2 Cellular chain complexes of covering spaces

Take a Heegaard diagram $(\mathcal{H}_2, \boldsymbol{\beta})$ of genus g. In this section, we observe the description of the cellular chain complex of the universal covering space $\widetilde{M}$ in detail.

Recall the group presentation $\pi_1(M) \cong \langle x_1, \dots, x_g \mid r_1, \dots, r_g\rangle$ from Section 3.1. By thoughtfully considering the dual handles of M, the 0- and the 3-cell of M are single, and the numbers of the 1- and the 2-cells of M are g. Therefore, the cellular complex $C_*(\widetilde{M}; \mathbb{Z})$ of $\widetilde{M}$ can be written as

$$0 \longrightarrow \mathbb{Z}[\pi_1(M)] \xrightarrow{\partial_3} \mathbb{Z}[\pi_1(M)]^{\oplus g} \xrightarrow{\partial_2} \mathbb{Z}[\pi_1(M)]^{\oplus g} \xrightarrow{\partial_1} \mathbb{Z}[\pi_1(M)] \longrightarrow 0. \quad (3.1)$$

Let $\{a_1, \dots, a_g\}$ and $\{b_1, \dots, b_g\}$ denote the canonical bases of $C_1(\widetilde{M}; \mathbb{Z})$ and $C_2(\widetilde{M}; \mathbb{Z})$ as left $\mathbb{Z}[\pi_1(M)]$-modules, respectively.

Let us explain a detailed description of the boundary maps ∂_*. For this, let us review the Fox derivative and state Theorems 3.2.1 and 3.2.2. Let F_I be a free group generated by an index set I. For $k \in I$, we write x_k for the generator, and we define *the Fox derivative* [Fox53] to be a map $\frac{\partial}{\partial x_k} : F_I \to \mathbb{Z}[F_I]$ satisfying the following rules:

$$\frac{\partial x_i}{\partial x_k} = \delta_{i,k}, \qquad \frac{\partial x_i^{-1}}{\partial x_i} = -x_i^{-1}, \qquad \frac{\partial (uv)}{\partial x_k} = u\frac{\partial v}{\partial x_k} + \frac{\partial u}{\partial x_k}, \qquad \text{for all } u, v \in F_I.$$

We also write $\frac{\partial}{\partial x_k} : \mathbb{Z}[F_I] \to \mathbb{Z}[F_I]$ for the homomorphism obtained by linearly extending $\frac{\partial}{\partial x_k}$.

Then, ∂_2 can be written as the Jacobian matrix of Fox derivatives. More precisely,

Theorem 3.2.1 [Lyn50, §4 and §5]**.** *$\partial_1(a_i) = 1 - x_i$ and $\partial_2(b_i) = \sum_j [\frac{\partial r_i}{\partial x_j}] a_j$.*

Next, we will focus on ∂_3. Let F be a free group $\langle x_1, \ldots, x_g \mid \rangle$ and P be a free group $\langle \rho_1, \ldots, \rho_g \mid \rangle$. We define the homomorphism

$$\psi : P * F \to F; \qquad \psi(\rho_j) = r_j, \ \psi(x_i) = x_i. \tag{3.2}$$

An element $s \in P * F$ is called *an identity* if $s \in \mathrm{Ker}(\psi)$ and s can be written as $\prod_{m=1}^{r} w_m \rho_{j_m}^{\epsilon_m} w_m^{-1}$, where $w_m \in F$, $\epsilon_m \in \{\pm 1\}$, and $\rho_{j_m} \in P$. We also denote $w_m \rho_{j_m}^{\epsilon_m} w_m^{-1}$ by $(r_{j_m}, w_m)^{\epsilon_m}$.

Theorem 3.2.2 [Sie80, Sie86]**.** *For any M, there exists an identity W_M such that $\partial_3(a) = \sum_j a\mu\Big([\psi(\frac{\partial W_M}{\partial \rho_j})]\Big) b_j$ establishes. Here, μ is the natural surjection from F to $\pi_1(M)$.*

The proof appears in Appendix A. We give two remarks:

Remark 3.2.3. (i) W_M is not uniquely determined from $\pi_1(M)$, and it is not easy to find W_M. (However, an algorithm to find W_M is discussed in [HAMS93].) For instance, when M is a lens space $L(p,q)$, $\pi_1(M)$ is isomorphic to $\langle x_1 \mid x_1^p \rangle \cong \mathbb{Z}/p$. Let $r := x_1^p$. Then, W_M is given by $W_M = (r,1)(r,x_1^q)^{-1}$, which depends on q [Sie80, Sie86].

(ii) Let us roughly explain the main result of [Sie86]. It is known that the identity W_M satisfies a "taut" condition. Conversely, if a group presentation $\langle x_1, \ldots, x_g \mid r_1, \ldots, r_g \rangle$ and a taut identity W are given, then there is a 3-manifold M whose boundary map in (3.1) coincides with ∂_* explained in the above theorems.

3.3 Definitions; invariants with representations

In this section, we define invariants of 3-manifolds with representations. (We defer the proofs of the theorems in this section into Chapter 4.)

Hereinafter, we fix a representation $\varphi : \pi_1(M) \to \mathrm{SL}_n(\mathbb{F})$. We write $C_*(\mathcal{H}_2, \boldsymbol{\beta})$ for the chain complex (3.1) in Section 3.2, and let d_i be $\dim_{\mathbb{Z}[\pi_1(M)]} C_i(\mathcal{H}_2, \boldsymbol{\beta})$. Let $\boldsymbol{\sigma}_i = (\sigma^i_1, \ldots, \sigma^i_{d_i})$ be the ordered standard basis of $C_i(\mathcal{H}_2, \boldsymbol{\beta})$, and $\{e_1, \ldots, e_n\}$ be the ordered standard basis of $\mathbb{F}^n$. Define $c^i_{j,k}$ to be $e_j \otimes \sigma^i_k$ for arbitrary $1 \le j \le n, 1 \le k \le d_i$. Then,

$$\mathbf{c}_i \coloneqq (c^i_{1,1}, c^i_{2,1}, \ldots, c^i_{n,1}, \ldots, c^i_{1,d_i}, c^i_{2,d_i}, \ldots, c^i_{n,d_i}) \tag{3.3}$$

provides an ordered basis of $\mathbb{F}^n \otimes_{\mathbb{Z}[\pi_1(M)]} C_i(\mathcal{H}_2, \boldsymbol{\beta})$. Let $\mathbf{c}$ be the union $(\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3)$.

3.3.1 The even case of n

First, we consider the case where n is even.

Definition 3.3.1 [Wak21, Definition 5.1]**.** Let $n \in \mathbb{N}$ be even. If the chain complex $\mathbb{F}^n \otimes_{\mathbb{Z}[\pi_1(M)]} C_*(\mathcal{H}_2, \boldsymbol{\beta})$ is acyclic, we define *the twisted torsion* $T_\varphi(M)$ to be

$$T_\varphi(M) = \tau(\mathbb{F}^n \otimes_{\mathbb{Z}[\pi_1(M)]} C_*(\mathcal{H}_2, \boldsymbol{\beta}), \mathbf{c}) \in \mathbb{F}^\times.$$

Here, the right-hand side is the algebraic torsion in Section 2.1. If the chain complex $\mathbb{F}^n \otimes_{\mathbb{Z}[\pi_1(M)]} C_*(\mathcal{H}_2, \boldsymbol{\beta})$ is not acyclic, the twisted torsion is defined to be zero.

Theorem 3.3.2 [Wak21, Theorem 5.2]**.** *$T_\varphi(M)$ does not depend on the choice of the Heegaard diagram $(\mathcal{H}_2, \boldsymbol{\beta})$, but only depends on M and φ.*

3.3.2 The odd case of n

Next, we consider the case where n is odd. According to [Tur01, Chapter III], let us define (normalized) twisted torsions. For this, we consider the following vector space over $\mathbb{R}$:

$$H_*(M; \mathbb{R}) \coloneqq \bigoplus_{i=0}^{3} H_i(M; \mathbb{R}).$$

From the Poincaré duality, we can naturally fix an orientation of $H_*(M; \mathbb{R})$. If we choose a basis $\mathbf{h} = (\mathbf{h}_0, \mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3)$ of $H_*(M; \mathbb{R})$, the torsion $\check{\tau}(C_*(M; \mathbb{R}), \boldsymbol{\sigma}, \mathbf{h}) \in \mathbb{R}^\times$ is defined as follows: Let us denote $C_*(M; \mathbb{R})$ by C_*. We consider the residues (mod2)

$$\beta_i(C_*) \coloneqq \sum_{i=0}^{3} (-1)^i \dim H_i(C_*), \quad \gamma_i(C_*) \coloneqq (-1)^i \sum_{i=0}^{3} \dim C_i,$$

$$N(C_*) \coloneqq \sum_{i=0}^{3} \beta_i(C_*)\gamma(C_*), \in \mathbb{Z}/2\mathbb{Z}.$$

Then, we set

$$\check{\tau}(C_*(M;\mathbb{R}), \boldsymbol{\sigma}, \mathbf{h}) \coloneqq (-1)^{N(C_*)}\tau(C_*, \boldsymbol{\sigma}, \mathbf{h}) \in \mathbb{R}^\times.$$

Definition 3.3.3 [Wak21, Definition 5.3]**.** Let $n \in \mathbb{N}$ be odd. If the chain complex $\mathbb{F}^n \otimes_{\mathbb{Z}[\pi_1(M)]} C_*(\mathcal{H}_2, \boldsymbol{\beta})$ is acyclic, we define *the (normalized) twisted torsion* to be

$$T_\varphi(M) = \operatorname{sgn}\bigl(\check{\tau}(C_*(\mathcal{H}_2, \boldsymbol{\beta}), \boldsymbol{\sigma}, \mathbf{h})\bigr) \cdot \tau(\mathbb{F}^n \otimes_{\mathbb{Z}[\pi_1(X)]} C_*(\mathcal{H}_2, \boldsymbol{\beta}), \mathbf{c}) \in \mathbb{F}^\times$$

If the chain complex $\mathbb{F}^n \otimes_{\mathbb{Z}[\pi_1(M)]} C_*(\mathcal{H}_2, \boldsymbol{\beta})$ is not acyclic, the twisted torsion is defined to be zero.

Theorem 3.3.4 [Wak21, Theorem 5.4]**.** *$T_\varphi(M)$ does not depend on the choice of the Heegaard diagram $(\mathcal{H}_2, \boldsymbol{\beta})$, but only depends on M and φ.*

Our torsion $T_\varphi(M)$ recovers the Reidemeister torsion $\tau_\varphi(M)$. To be precise,

Theorem 3.3.5 [Wak21, Theorem 5.5]**.** *Suppose $n \in \mathbb{N}$. Let X be a cellular structure of M obtained by a triangulation of M. For the representation $\varphi : \pi_1(M) \to \mathrm{SL}_n(\mathbb{F})$, if $\mathbb{F}^n \otimes_{\mathbb{Z}[\pi_1(M)]} C_*(\mathcal{H}_2, \boldsymbol{\beta})$ is acyclic, then $C_*^\varphi(X)$ is also acyclic and*

$$T_\varphi(M) = \tau_\varphi(M) \in \mathbb{F}^\times/\{\pm 1\}.$$

3.3.3 Case of the Betti number $b_1(M) \geq 1$

In this subsection, we suppose $b_1(M) \geq 1$. Let us fix an $\mathrm{SL}_n(\mathbb{F})$-representation $\varphi : \pi_1(M) \to \mathrm{SL}_n(\mathbb{F})$ and a surjective homomorphism $\alpha : \pi_1(M) \to \mathbb{Z} = \{t^m\}_{m\in\mathbb{Z}}$. We write C_* for the chain complex $C_*(\mathcal{H}_2, \boldsymbol{\beta})$ of (3.1). Under this condition, we will define the normalized torsion. The essential idea is based on [Kit15b, §5].

For $p, q \in \mathbb{F}[t^{\pm 1}]$, let us define the degrees of $f(t) = p(t)/q(t)$ in the quotient field $\mathbb{F}(t)$ as follows:

$$\deg f \coloneqq \deg p - \deg q,$$

$$\text{h-deg} f \coloneqq (\text{the highest degree of } p) - (\text{the highest degree of } q),$$

$$\text{l-deg} f \coloneqq (\text{the lowest degree of } p) - (\text{the lowest degree of } q) \in \mathbb{Z}.$$

Considering the tensor representation $\alpha\otimes\varphi : \pi_1(M) \to \mathrm{GL}_n(\mathbb{F}[t^{\pm 1}])$, we introduce the following terminologies:

$$\tau_0 \coloneqq \tau(\mathbb{F}^n \otimes_{\mathbb{Z}[\pi_1(M)]} C_*, \mathbf{c}) \in \mathbb{F}^*,$$

$$d \coloneqq \frac{1}{2}\big((\text{h-deg}\tau(\mathbb{F}(t)^n \otimes_{\mathbb{Z}[\pi_1(M)]} C_*, \widehat{\mathbf{c}}) - (\text{l-deg}\tau(\mathbb{F}(t)^n \otimes_{\mathbb{Z}[\pi_1(M)]} C_*, \widehat{\mathbf{c}})\big) \in \frac{1}{2}\mathbb{Z}.$$

Here, as in (3.3) $\mathbf{c}$ and $\widehat{\mathbf{c}}$ are the canonical bases of $\mathbb{F}^n \otimes_{\mathbb{Z}[\pi_1(M)]} C_*$ and $\mathbb{F}(t)^n \otimes_{\mathbb{Z}[\pi_1(M)]} C_*$, respectively.

Definition 3.3.6. Let $\widetilde{\mathbf{c}}$ be the canonical basis of $\mathbb{F}(t^{1/2})^n \otimes_{\mathbb{Z}[\pi_1(M)]} C_*$. If the chain complex $\mathbb{F}(t^{1/2})^n \otimes_{\mathbb{Z}[\pi_1(M)]} C_*$ is acyclic, we define *the normalized twisted torsion* to be

$$\widetilde{T}_{\varphi,\alpha}(M) \coloneqq \frac{\tau_0^n}{t^{nd}} \tau(\mathbb{F}(t^{1/2})^n \otimes_{\mathbb{Z}[\pi_1(M)]} C_*, \mathbf{c}) \in \mathbb{F}(t^{1/2})^*.$$

If the chain complex $\mathbb{F}(t^{1/2})^n \otimes_{\mathbb{Z}[\pi_1(M)]} C_*$ is not acyclic, the torsion is defined to be zero.

Remark 3.3.7. As seen in [Mil66, Tur01, Mil62], the original definition was in the quotient set $\mathbb{F}(t^{1/2})^*/\{\pm 1, t^{m/2}\}_{m\in\mathbb{Z}}$; thus the above definition can be considered as a normalization of the original one.

Theorem 3.3.8 [Wak21, Theorem 5.8] (cf. Theorem 4.5 and Lemma 5.3 in [Kit15b]). *$\widetilde{T}_{\varphi,\alpha}(M)$ does not depend on the choice of the Heegaard diagram $(\mathcal{H}_2, \boldsymbol{\beta})$ but depends only on M, φ, and α.*

Chapter 4

Proofs of the statements in Chapter 3

To prove the theorems in Chapter 3, we will review Heegaard moves and extended Andrews-Curtis moves in Section 4.1. In Section 4.2, we accomplish the proofs. This chapter is essentially based on Section 6 in [Wak21]; however, the author noticed many minor typo misses in the section 6 and did not take appropriate references; we here revise them.

4.1 Heegaard moves and Andrews-Curtis moves

Let us review Theorem 4.1.1 below. Let $(\mathcal{H}_2, \boldsymbol{\beta})$ with $\boldsymbol{\beta} = \{\beta_1, \dots, \beta_g\}$ be a Heegaard diagram. The following four transformations and the inverse operations are called *Heegaard moves* of $(\mathcal{H}_2, \boldsymbol{\beta})$; see Fig. 4.1:

(A) Replace $\boldsymbol{\beta} = \{\beta_1, \dots, \beta_g\}$ with $\boldsymbol{\beta}' = \{\beta'_1, \dots, \beta'_g\}$ where $\boldsymbol{\beta}$ and $\boldsymbol{\beta}'$ are isotopic to each other.

(B) Suppose that β_1 and β_2 is connected by an arc δ in $\partial\mathcal{H}_2 \setminus (\beta_1 \cup \beta_2 \cup \cdots \cup \beta_g)$. Along a neighborhood of δ, take the connected sum β'_1 of β_1 with a parallel copy of β_2. For such β'_1, replace $\boldsymbol{\beta} = \{\beta_1, \dots, \beta_g\}$ with $\boldsymbol{\beta}' = \{\beta'_1, \beta_2, \dots, \beta_g\}$. This operation is called *a handleslide*.

(C) Another diagram $(\mathcal{H}'_2, \boldsymbol{\beta}')$ is called *a stabilization* of $(\mathcal{H}_2, \boldsymbol{\beta})$, and $(\mathcal{H}_2, \boldsymbol{\beta})$ is called *a destabilization* of $(\mathcal{H}'_2, \boldsymbol{\beta}')$ if

- $\mathcal{H}'_2 = \mathcal{H}_2 \cup_h T$, where $T = S^1 \times D^2$, and the attaching map $h : D^2_1 \to D^2_2$ is a homeomorphism from a disk $D^2_1 \subset \partial\mathcal{H}_2$ to a disk $D^2_2 \subset \partial T$,
- $\boldsymbol{\beta}' = \boldsymbol{\beta} \cup \{\beta_{g+1}\}$, where the simple closed curve β_{g+1} is contained in ∂T and is a generator of $\pi_1(T)$.

Replace $(\mathcal{H}_2, \boldsymbol{\beta})$ with such a $(\mathcal{H}_2', \boldsymbol{\beta}')$.

(D) If there is a homeomorphism $\mathcal{H}_2 \to \mathcal{H}_2'$, which maps $\boldsymbol{\beta}$ to $\boldsymbol{\beta}'$, replace $(\mathcal{H}_2, \boldsymbol{\beta})$ with $(\mathcal{H}_2', \boldsymbol{\beta}')$.

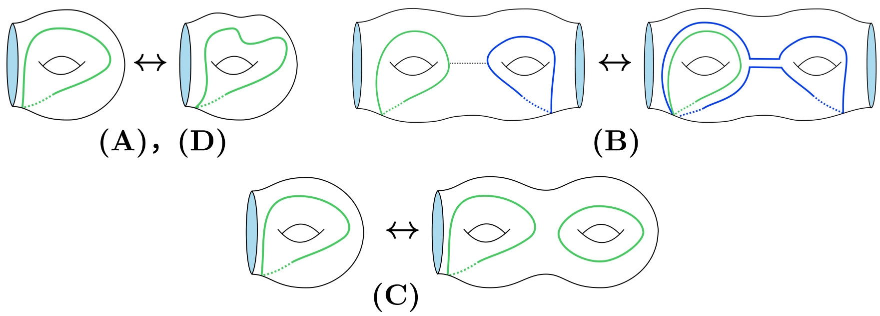

Figure 4.1: The Heegaard moves

The following theorem is well known as a theorem of Reidemeister-Singer:

Theorem 4.1.1 [Rei33] [OS04, Proposition 2.2]**.** *Let $(\mathcal{H}_2, \boldsymbol{\beta})$ and $(\mathcal{H}_2', \boldsymbol{\beta}')$ be Heegaard diagrams of M. These two diagrams are related by a finite sequence of Heegaard moves.*

Next, we turn to discussing changes in the group presentation according to the Heegaard moves. For this, we shall review extended Andrews-Curtis moves and show Lemma 4.1.2. Suppose that a group presentation $\langle x_1, \dots, x_m \mid r_1, \dots r_n \rangle$ of the group G is given. for some $n, m \in \mathbb{N}$ and some words $r_1, \dots, r_n$ of x_i's. The *Andrews-Curtis moves* [AC65] of the presentation are the following four operations on the generators and relators:

(Ia) Replace a relator r_i with r_i^{-1}.

(Ib) Replace a relator r_i with $w r_i w^{-1}$ or $r_i x_j x_j^{-1}$. Here, w is an arbitrary word of $x_1, \dots, x_m$.

(Ic) Replace a relator r_i with $r_i r_j$ where $j \neq i$.

(II) Add a new word y and a new relator y (That is, replace the presentation with $\langle x_1, \dots, x_m, y \mid r_1, \dots, r_n, y \rangle$.)

Furthermore, the *extended Andrews-Curtis moves* (EAC-moves, for short) are these moves together with the following:

(R) Replace x_i with either $x_i x_j$ or $x_i x_j^{-1}$ in all the relators for some $i \neq j$.

If two group presentations are related by a finite sequence of EAC-moves, then the associated groups are isomorphic. However, the converse is not true. In fact, the deficiency $m - n$ is constant with respect to EAC-moves.

We now see that each Heegaard move corresponds to an EAC-move in the sense of presentations of $\pi_1(M)$. More precisely, the following lemma is widely known (see, e.g., [Bag21, §1.2]):

Lemma 4.1.2. *Fix two Heegaard diagrams $(\mathcal{H}_2, \boldsymbol{\beta}), (\mathcal{H}_2', \boldsymbol{\beta}')$ of M and the associated presentations of $\pi_1(M)$ as in Section 3.1. Then, these presentations are related by a finite sequence of some EAC-moves.*

Proof. By Theorem 4.1.1, it is sufficient to consider the case where the transformation of the presentation of $\pi_1(M)$ is each Heegaard move, corresponding to an EAC-move.

First, we should check that the presentation of $\pi_1(M)$ up to the moves (R),(Ib) is independent of the choice of the order and the orientations of $\{\alpha_1, \ldots, \alpha_g\}$. Regarding the order, if we replace α_i by α_j, then we may consider

$$(x_i, x_j) \xmapsto{\text{(R)}} (x_i x_j^{-1}, x_j) \xmapsto{\text{(R)}} (x_i x_j^{-1}, x_j(x_i x_j^{-1})) \xmapsto{\text{(Ib)}} (x_j x_i, x_i) \xmapsto{\text{(R)}} (x_j x_i x_i^{-1}, x_i),$$

which is equal to (x_j, x_i). On the other hand, regarding the orientations, if we replace α_i by α_i^{-1}, then we may consider

$$(x_i, x_j) \xmapsto{\text{(R)}} (x_i x_j, x_j) \xmapsto{\text{(R)}} (x_i x_j, x_j(x_i x_j)^{-1}) = (x_i x_j, x_i^{-1}) \xmapsto{\text{(Ib)}}$$

$$\mapsto (x_j x_i, x_i^{-1}) \xmapsto{\text{(R)}} (x_j x_i x_i^{-1}, x_i^{-1}) = (x_j, x_i^{-1}),$$

which is equal to (x_i^{-1}, x_j) by the former discussion.

Next, suppose that $(\mathcal{H}_2, \boldsymbol{\beta})$ is transformed into $(\mathcal{H}_2', \boldsymbol{\beta}')$ by (A) or (D). For simplicity, we can assume $\mathcal{H}_2 = \mathcal{H}_2'$ and $\boldsymbol{\beta} = \boldsymbol{\beta}'$.

Let us observe the difference of the presentations obtained from $(\mathcal{H}_2, \boldsymbol{\beta})$. For each oriented edge e_i, choose an orientation-preserving loop $\ell_i : [0,1] \to e_i$ as in Section 3.1. Each ℓ_i determines a generator x_i of $\pi_1(\mathcal{H}_2, v_0)$. For each β_i, choose $b_i \in \beta_i$ and let $k_i : [0,1] \to \beta_i$ and $j_i : [0,1] \to \mathcal{H}_2$ be the paths in Section 3.1. Since r_i is given by the composite $j_i^{-1} k_i j_i$, r_i does not depend on the choice of j_i. If we take another orientation of β_i, r_i is replaced with r_i^{-1}. Therefore, this corresponds to (Ia).

In addition, suppose that β_1 is transformed into β_1' by the handleslide (B). Let $\langle x_1, \ldots, x_g \mid r_1, \ldots, r_g \rangle$ be the group presentation given from the Heegaard diagram before the move. After the move, r_1 is transformed into one of $r_2 r_1$, $r_2^{-1} r_1$, $r_1 r_2$, or $r_1^{-1} r_2$. Therefore, the presentation after the move can be obtained by applying (Ia) and (Ic).

Finally, if $(\mathcal{H}_2, \boldsymbol{\beta})$ is transformed into $(\mathcal{H}_2 \cup_h T^2, \boldsymbol{\beta} \cup \{\beta_{g+1}\})$ by a stabilization (C), the corresponding transformation of the group presentation can be regarded as (II). To summarize, we complete the proof. $\square$

4.2 Proofs of the theorems

For the proof of the theorems 3.3.2, 3.3.4, 3.3.5, and 3.3.8, we often use the chain rule (4.1) of the Fox derivative [Fox53]. Let F' be a free group with basis $\{y_1, \ldots, y_g\}$, and let $\lambda : F' \to F$ be a group homomorphism. Then, the following holds for any $f \in \mathbb{Z}[F']$:

$$\frac{\partial(\lambda(f))}{\partial x_j} = \sum_{k \leq g} \Big(\lambda\Big(\frac{\partial f}{\partial y_k}\Big)\Big) \frac{\partial(\lambda(y_k))}{\partial x_j}. \tag{4.1}$$

Proof of Theorem 3.3.2. It is sufficient to prove that even if we change a Heegaard diagram $(\mathcal{H}_2, \boldsymbol{\beta})$ by a Heegaard move, $T_\varphi(M)$ does not change where $C_* = \mathbb{F}^n \otimes_{\mathbb{Z}[\pi_1(M)]} C_*(\mathcal{H}_2, \boldsymbol{\beta})$ is acyclic. Let $\langle x_1, \ldots, x_g \mid r_1, \ldots, r_g \rangle$ be the group presentation before applying the move. Let $W_M = \prod_{m=1}^r w_m \rho_{j_m}^{\epsilon_m} w_m^{-1} = \prod_{m=1}^r (r_{j_m}, w_m)^{\epsilon_m}$ be an identity in Theorem 3.2.2. For $s = a_0 g_0 + \cdots + a_\ell g_\ell, a_i \in \mathbb{Z}, g_i \in \pi_1(M)$, we write s^{-1} for $a_0 g_0^{-1} + \cdots + a_\ell g_\ell^{-1}$.

If we regard $C_i = \mathbb{F}^n \otimes_{\mathbb{Z}[\pi_1(M)]} C_i(\mathcal{H}_2, \boldsymbol{\beta})$ as a vector space over $\mathbb{F}$ with the basis $\mathbf{c}_i$, the boundary maps are represented by the following matrices:

$$\partial_1 = \big(E - \varphi(x_j^{-1})\big), \; \partial_2 = \Big(\varphi([\tfrac{\partial r_i}{\partial x_j}]^{-1})\Big), \; \partial_3 = \Big(\varphi(\mu\psi([\tfrac{\partial W_M}{\partial \rho_i}]))^{-1}\Big).$$

Here, E is the $(n \times n)$-identity matrix. We write $[\partial_2]_{(i,j)}$ for the submatrix of ∂_2 that consists of the entries collecting from the $(ni+1)$-th row to the $(ni+n)$-th row and from the $(nj+1)$-th column to the $(nj+n)$-th column. Similarly, we write $[\partial_3]_i$ for the submatrix that consists of entries collecting from the $(ni+1)$-th row to the $(ni+n)$-th row. Throughout the proof, the chain complex (3.1), the identity, and the torsion, corresponding to the diagram after applying the move, will be denoted by C'_*, W'_M, $T'_\varphi(M)$, respectively.

First, we show the invariance on the choice of orientations of β_i, where r_d is replaced with r_d^{-1}. Note the following equation:

$$\frac{\partial(r_d^{-1})}{\partial x_i} = -r_d^{-1} \frac{\partial r_d}{\partial x_i}.$$

From Proposition $A.0.1$, W'_M can be obtained by replacing all r_d in W_M with r_d^{-1}. Therefore, the boundary maps $\partial'_1, \partial'_2, \partial'_3$ of C'_* are represented by the following matrices:

$$\begin{aligned}
\partial'_1 &= \partial_1, \\
[\partial'_2]_{(i,j)} &= \begin{cases} [\partial_2]_{(i,j)} & \text{if} \;\; j \neq d, \\ -\varphi([\frac{\partial r_d}{\partial x_i}]^{-1})\varphi(r_d) = -[\partial_2]_{(i,d)} & \text{if} \;\; j = d, \end{cases} \\
[\partial'_3]_i &= \begin{cases} [\partial_3]_i & \text{if} \;\; i \neq d, \\ -[\partial_3]_d & \text{if} \;\; i = d. \end{cases}
\end{aligned}$$

Let $E_{i,j}(\gamma)$ be the elementary $(n \times n)$-matrix over $\mathbb{Z}[\pi_1(M)]$ whose (i,j)-th entry is $\gamma \in \mathbb{Z}[\pi_1(M)]$. We can represent the boundary maps of C'_* by the following matrices:

$$\partial'_2 = \partial_2 E_{d,d}(-1), \quad \partial'_3 = E_{d,d}(-1)\partial_3.$$

Therefore, if we identify C'_2 with C_2 by the isomorphism corresponding to the basis transformation represented by $E_{d,d}(-1)$, then C'_* coincides with C_*. Noticing that $\det(\varphi(E_{d,d}(-1))) = (-1)^n$ and Remark 2.1.1(i), we have

$$\begin{aligned} T'_\varphi(M) &= \tau(C'_*, \mathbf{c}') = \tau(C_*, \mathbf{c}) \prod_{i=0}^{3} [\mathbf{c}'_i/\mathbf{c}_i]^{(-1)^{i+1}} \\ &= (\det(\varphi(E_{d,d}(-1))))^{-1}\tau(C_*, \mathbf{c}) = (-1)^n \tau(C_*, \mathbf{c}) = T_\varphi(M). \end{aligned}$$

The invariance for (Ib) is obvious.

Now, regarding a handleslide (B), we may focus on only the case where r_1 is changed to r_2r_1. Note the following equality:

$$\frac{\partial(r_2r_1)}{\partial x_i} = \frac{\partial r_2}{\partial x_i} + r_2\frac{\partial r_1}{\partial x_i}.$$

It follows from Proposition A.0.1 that W'_M can be obtained by replacing all the terms (r_{j_m}, w_m) in W_M satisfying $r_{j_m} = r_1$ with $(r_2, w_m)^{-1}(r_2r_1, w_m)$. Therefore, by (4.1), the boundary maps $\partial'_1, \partial'_2, \partial'_3$ of C'_* are regarded as the following matrices, respectively:

$$\begin{aligned} \partial'_1 &= \partial_1, \\ [\partial'_2]_{(i,j)} &= \begin{cases} [\partial_2]_{(i,j)} & \text{if } j \neq 1, \\ \varphi([\frac{\partial r_2}{\partial x_i}]^{-1}) + \varphi(r_2^{-1})\varphi([\frac{\partial r_1}{\partial x_i}]^{-1}) = [\partial_2]_{(i,2)} + [\partial_2]_{(i,1)} & \text{if } j = 1, \end{cases} \\ [\partial'_3]_i &= \begin{cases} [\partial_3]_i & \text{if } i \neq 1, \\ -[\partial_3]_2 + [\partial_3]_1 & \text{if } i = 1. \end{cases} \end{aligned}$$

If we identify C'_* with the chain complex $C'_*(\mathcal{H}_2, \boldsymbol{\beta})$, then the boundary maps are considered to be the following matrices:

$$\partial'_2 = \partial_2 E_{1,2}(1), \quad \partial'_3 = (E_{1,2}(1))^{-1}\partial_3. \tag{4.2}$$

Therefore, if we identify C'_2 and C_2 by the isomorphism corresponding to the basis transformation represented by $E_{1,2}(1)$, then C'_* coincides with C_*. Since φ is an $\mathrm{SL}_n(\mathbb{F})$-representation, $\det(\varphi(E_{1,2}(1))) = 1$. By Remark 2.1.1(i), we get

$$\begin{aligned} T'_\varphi(M) &= \tau(C'_*, \mathbf{c}') = \tau(C_*, \mathbf{c}) \prod_{i=0}^{3} [\mathbf{c}'_i/\mathbf{c}_i]^{(-1)^{i+1}} \\ &= (\det(\varphi(E_{1,2}(1))))^{-1}\tau(C_*, \mathbf{c}) = \tau(C_*, \mathbf{c}) = T_\varphi(M). \end{aligned}$$

Next, we show the invariance for the stabilization (C). The stabilization adds the new generator y and the new relator y. Let σ_y^1 be a 1-cell corresponding to the generator y, and σ_y^2 be a 2-cell corresponding to the relator y. Fix the lifts of σ_y^1 and σ_y^2 to $\widetilde{M}$, and denote those by $\widetilde{\sigma}_y^1$ and $\widetilde{\sigma}_y^2$. Then, $\mathbf{c}_1' := \mathbf{c}_1 \cup \{e_1 \otimes \widetilde{\sigma}_y^1, e_2 \otimes \widetilde{\sigma}_y^1, \ldots, e_n \otimes \widetilde{\sigma}_y^1\}$ is an (ordered) basis of C_1'. Similarly, we can take $\mathbf{c}_2' := \mathbf{c}_2 \cup \{e_1 \otimes \widetilde{\sigma}_y^2, e_2 \otimes \widetilde{\sigma}_y^2, \ldots, e_n \otimes \widetilde{\sigma}_y^2\}$ as an basis of C_2'. Proposition A.0.1 implies $W_M' = W_M(y,y)(y,1)^{-1}$. Thus, the boundary maps $\partial_1', \partial_2', \partial_3'$ of C_*' are represented by the following matrices:

$$\partial_1' = \begin{pmatrix} E - \varphi(x_1^{-1}) & \cdots & E - \varphi(x_g^{-1}) & E - \varphi(y^{-1}) \end{pmatrix} = \begin{pmatrix} \partial_1 & O \end{pmatrix}, \tag{4.3}$$

$$\partial_2' = \begin{pmatrix} \partial_2 & O \\ O & E \end{pmatrix}, \tag{4.4}$$

$$\partial_3' = \begin{pmatrix} \partial_3 \\ O \end{pmatrix}. \tag{4.5}$$

Therefore, $C' = C \oplus \left(\bigoplus_{i=1}^n E_*(\mathbb{F}, 1)\right)$. Hence, by Remark 2.1.1(ii), $T_\varphi(M) = \tau(C_*, \mathbf{c}) = \tau(C_*', \mathbf{c}') = T_\varphi'(M)$.

Finally, we show the invariance on (R). In this case, the generators (x_i, x_j) are replaced to $(x_i x_j, x_j)$ or $(x_i x_j^{-1}, x_j)$. Notice from (4.1) the following equality:

$$\frac{\partial r_j}{\partial (x_i x_j^{\pm 1})} = \frac{\partial r_j}{\partial x_i} \pm \frac{\partial r_j}{\partial x_j}.$$

Here, we use the chain rule (4.1). Since W_M' is obtained from W_M following the replacement, as the boundary maps ∂_2', ∂_3' of C_*' are represented by the following equalities (similar to the case (Ic)) or (4.2):

$$\partial_1' = \partial_1 \varphi(E_{i,j}(\pm 1)), \quad \partial_2' = (\varphi(E_{i,j}(\pm 1)))^{-1} \partial_2, \quad \partial_3' = \partial_3.$$

Hence, similar to (4.5), we obtain the invariance of the torsion as required. $\square$

Similarly, the proofs of Theorems 3.3.4 and 3.3.8 are given as follows:

Proof of Theorem 3.3.4. Let n be odd. We denote the term $\operatorname{sgn}\left(\check{\tau}(C_*(M;\mathbb{R}), \boldsymbol{\sigma}, \mathbf{h})\right)$ by $\check{\tau}$ and denote the torsion $\tau(\mathbb{F}^n \otimes C_*(\mathcal{H}_2, \boldsymbol{\beta}), \mathbf{c})$ by T for short. The changes of the values T, $\check{\tau}$ by reversing the orientations of β_d, e_d and by Heegaard moves are as follows (the details can be easily checked as in the proof of Theorem 3.3.2). By (A) and (D), it is clear that T and $\check{\tau}$ do not change. By reversing the orientation of β_d, we have $T \mapsto (-1)^n T$ and $\check{\tau} \mapsto -\check{\tau}$. By reversing the orientation of e_d, we have $T \mapsto (-1)^n T$ and $\check{\tau} \mapsto -\check{\tau}$. By (B) and (C), T and $\check{\tau}$ do not change. In summary, by the definition of $T_\varphi(M)$, the invariance is proved. $\square$

Proof of Theorem 3.3.8. First of all, the twisted torsion does not depend on $\mathbf{h}_i$ by Remark 2.1.1(i). We denote the torsion $\tau(\mathbb{F}(t^{1/2})^n \otimes_{\mathbb{Z}[\pi_1(M)]} C_*, \mathbf{c})$ by T. The changes of T, τ_0, d by reversing the orientations of β_d, e_d and by Heegaard moves are as follows (the details can be easily checked as in the proof of Theorem 3.3.2). By (A) and (D), it is clear that T, τ_0, and d do not change. By reversing the orientation of β_d, we have $T \mapsto (-1)^n T$, $\tau_0 \mapsto -\tau_0$, and $d \mapsto d$. By reversing the orientation of e_d, we have $T \mapsto t^{n \deg \alpha(-x_d)} T$, $\tau_0 \mapsto \tau_0$, and $d \mapsto d + \deg \alpha(-x_d)$. By (B) and (C), T, τ_0, and d do not change. In summary, by the definition of $\widetilde{T}_{\varphi,\alpha}(M)$, the invariance is proved. □

Proof of Theorem 3.3.5. The acyclicity is obvious from (3.1). Let $\mathcal{S}$ be a triangulation of M. Let us take a closed tubular neighborhood U of the 1-skeleton $\mathcal{S}^{(1)}$, and let V be the closure of $M \setminus U$. These U and V are handlebodies, and it is well known that $M = U \cup V$ gives a Heegaard splitting (see, for example, [OS04]). Let $(\mathcal{H}_2, \boldsymbol{\beta})$ be the corresponding Heegaard diagram.

We can regard U as the 0- and 1-handles of M and V as dual handles of U of 2- and 3-handles of M. Let K be a spine of V. When we take a maximal tree $\mathcal{T}$ of $\mathcal{S}^{(1)}$, we can choose the maximal tree $\mathcal{T}^*$ of K as the dual handle of $\mathcal{T}$.

Then, we can define the CW structure induced on the quotient space $M/(\mathcal{T} \cup \mathcal{T}^*)$ by passage $\mathcal{T} \cup \mathcal{T}^*$ as a single point. Note that the 0-cell and the 3-cell are single and that the cellular complex of $M/(\mathcal{T} \cup \mathcal{T}^*)$ is identified with $C_*(\mathcal{H}_2, \boldsymbol{\beta})$. Thus, Theorems 3.3.2 and 3.3.4 deduce

$$T_\varphi(M) = \tau_\varphi(M/(\mathcal{T} \cup \mathcal{T}^*)) \in \mathbb{F}^\times/\{\pm 1\}.$$

Therefore, if the invariance for the operation crushing any edge in the maximal tree is proved, then we have the desired equality $\tau_\varphi(M) = \tau_\varphi(M/(\mathcal{T} \cup \mathcal{T}^*)) = T_\varphi(M)$. Let X be the CW complex before applying this operation, and X' be that after applying the operation. We write C_* and C'_* for the cellular complex $C_*(\widetilde{X}; \mathbb{Z})$ and $C_*(\widetilde{X'}; \mathbb{Z})$, respectively. Assume that C'_* is represented as follows:

$$C'_* : 0 \longrightarrow C'_3 \xrightarrow{\partial'_3} C'_2 \xrightarrow{\partial'_2} C'_1 \xrightarrow{\partial'_1} C'_0 \longrightarrow 0.$$

By carefully observing the cellular complex with local coefficients, we can verify that C_* is represented by the following matrices:

$$C_* : 0 \longrightarrow C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0,$$

$$C_i = C'_i \oplus \mathbb{Z}[\pi_1(M)], \quad \partial_3 = \begin{pmatrix} \partial'_3 & * \\ O & \mathrm{id} \end{pmatrix}, \quad \partial_2 = \begin{pmatrix} \partial'_2 & * \\ * & 0 \end{pmatrix}, \quad \partial_1 = \begin{pmatrix} \partial'_1 & * \\ O & \mathrm{id} \end{pmatrix}.$$

That is, C_* is chain isomorphic to $C'_* \oplus E_*(\mathbb{F}, 0) \oplus E_*(\mathbb{F}, 2)$. By Remark 2.1.1(ii), we have

$$\begin{aligned} \tau_\varphi(M) &= \tau(\mathbb{F}^n \otimes_{\mathbb{Z}[\pi_1(M)]} C_*, \mathbf{c}) = \tau(\mathbb{F}^n \otimes_{\mathbb{Z}[\pi_1(M)]} C'_*, \mathbf{c}') \\ &= \tau(\mathbb{F}^n \otimes_{\mathbb{Z}[\pi_1(M)]} C_*(\mathcal{H}_2, \boldsymbol{\beta}), \mathbf{c}') = T_\varphi(M) \in \mathbb{F}^\times/\{\pm 1\}. \end{aligned}$$

Here, $\mathbf{c}$ and $\mathbf{c}'$ are bases of $C_*^{\varphi}(X)$ and $C_*^{\varphi}(X')$ as in Section 2.2, respectively. Thus, $\mathbf{c}$ and $\mathbf{c}'$ are related by a finite sequence of Remark 2.1.1. To summarize, we complete the proof. $\square$

(Additional remark) In the paper [Wak21] or the proof of Theorem 3.3.5, we discuss the torsions up to sings. However, if φ is a adjoint representation, we can define the normalized torsion, which does not depend on the choices of the finite cellular decomposition, as mentioned in Section 2.3; thus, following the proof, we can show the same equality $T_\varphi(M) = \tau_\varphi(M)$ in $\mathbb{C}$ without omitting sings.

Chapter 5

Examples of computation

In this chapter, we compute the torsions of some closed 3-manifolds with respect to $\mathrm{SL}_2(\mathbb{C})$-representations. Section 5.1 focuses on some Seifert 3-manifolds $M_{n,m,\ell}$ defined for three integers $n, m, \ell \geq 2$, and compute all the twisted Reidemeister torsions of $M_{n,m,\ell}$. Section 5.2 considers some 3-manifolds M obtained by some Dehn-surgeries on the figure-eight knot and the 5_2-knot, and computes all the adjoint Reidemeister torsions of M with respect to all irreducible $\mathrm{SL}_2(\mathbb{C})$-representations. Throughout this chapter, we assume that $G = \mathrm{SL}_2(\mathbb{C})$.

5.1 Some Seifert 3-manifolds

In this section, we focus on 3-dimensional manifolds discussed in [Sie86, p.127] and compute the twisted torsions. Let $n, m, \ell \geq 2$ be integers. Sieradski considers the group presentation $\langle g, h \mid s, r\rangle$, where $r = (gh)^n h^{-m}$, $s = (hg)^n g^{-\ell}$, and he constructs a taut identity $W = (1,r)(h,r)^{-1}(1,s)(g,s)^{-1}$. Consequently, according to Remark 3.2.3(ii), there is a certain 3-manifold $M = M_{n,m,\ell}$ such that the presentation of $\pi_1(M)$ obtained by the algorithm in Section 3.1 turns out to be

$$\langle g, h \mid (hg)^n g^{-\ell}, (gh)^n h^{-m}\rangle.$$

Moreover, we show the homeomorphism type of the manifold $M_{n,m,\ell}$:

Proposition 5.1.1. *The manifold $M_{n,m,\ell}$ is homeomorphic to the Seifert 3-manifold over the 2-sphere of the form*

$$M(0; (1,0); (n,-1), (m,-1), (\ell,1))).$$

Furthermore, if $-m\ell + n\ell + nm = \pm 1$, then $M_{n,m,\ell}$ is homeomorphic to the Brieskorn manifold $\Sigma(\ell, m, n)$, which is defined by

$$\Sigma(\ell, m, n) = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1^\ell + z_2^m + z_3^n = 0, |z_1|^2 + |x_2|^2 + |z_3|^2 = 1\}.$$

Proof. It is well known (see [Kit94, §3]) that the fundamental group of

$$M(0;(1,0);(\ell,-1),(m,-1),(n,1))$$

has the presentation

$$\langle x, y, z, h \mid h \text{ central}, x^{\ell}h^{-1} = y^{m}h^{-1} = z^{n}h = xyz = 1\rangle. \tag{5.1}$$

Here, let us eliminate the generators z and h by the relations $z^{-1} = xy$, $h^{-1} = z^n$. Namely, we can verify that the correspondence $x \mapsto g,\ y \mapsto h, z \mapsto (gh)^{-1}, h \mapsto g^{\ell}$ defines an isomorphism from (5.1) to $\pi_1(M_{n,m,\ell})$. The required homeomorphism immediately follows from a rigidity theorem for Seifert fiber spaces in [Sco83].

For the latter part, suppose $-m\ell + n\ell + nm = \pm 1$. Then, $\pi_1\big(\Sigma(\ell, m, n)\big)$ is isomorphic to (5.1); see [Kho07, §1]. Therefore, by the work of Scott again, $M_{n,m,\ell}$ is homeomorphic to $\Sigma(\ell, m, n)$. $\square$

For instance, the latter assumption in Proposition 5.1.1 establishes for $(n, m, \ell) = (k, 2k + 1, 2k - 1)$ where $k \geq 3$, and

$$(n, m, \ell) = (2, 7, 3), (3, 11, 4), (3, 13, 4), (3, 8, 5), (4, 19, 5), (5, 17, 7), \ \textit{etc.}$$

5.1.1 Results

First, we will show the following lemmas for classifying $\mathrm{SL}_2(\mathbb{C})$-representations. (The proofs of all the statements in this subsection will appear in the next subsection.) In what follows, the field $\mathbb{F}$ is assumed to be the complex field $\mathbb{C}$.

Lemma 5.1.2. *Let $\alpha \in \mathbb{C}^{\times} \setminus \{\pm 1\}$, and $y \neq 0$ or $z \neq 0$. Then, two matrices $A = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, B = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$ satisfy*

$$(AB)^n = B^m, (BA)^n = A^{\ell} \tag{5.2}$$

if and only if the following conditions hold:

(IA) $\alpha^{\ell} = \beta^m = \gamma^n \in \{\pm 1\}$,

(IB) $x = \dfrac{\alpha^{-1}(\beta + \beta^{-1}) - (\gamma + \gamma^{-1})}{\alpha^{-1} - \alpha}$, $w = \dfrac{(\gamma + \gamma^{-1}) - \alpha(\beta + \beta^{-1})}{\alpha^{-1} - \alpha}$.

Here, β, β^{-1} are the eigenvalues of B, and γ, γ^{-1} are the eigenvalues of AB.

Lemma 5.1.3. *Two matrices $A = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha^{-1} \end{pmatrix}, B = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$ satisfy* (5.2) *if and only if there exists $\beta \in \{\pm 1\}$ satisfying the following conditions:*

(IIA) $\alpha \in \{\pm 1\}$, $\alpha^{\ell} = \beta^m = (\alpha\beta)^n$,

(IIB) $m \neq n,\ \ \ell = mn(m-n)^{-1}$,

(IIC) $x = w = \beta,\ \ y = \alpha\beta n(m-n)^{-1}, z = 0$.

Proposition 5.1.4. *For any nontrivial homomorphism $\varphi : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$, there exist matrices A, B satisfying the conditions (IA) and (IB) in Lemma 5.1.2, or (IIA), (IIB) and (IIC) in Lemme 5.1.3, and the homomorphism $\varphi' : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ defined by $\varphi'(g) = A$ and $\varphi'(h) = B$ is conjugate to φ.*

Proof. It is straightforwardly proved by using Jordan decomposition of $\varphi'(g) = A$. □

For $i \in \{1, 2\}$, let $\varphi^i_\alpha : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ be a representation such that $\varphi^i_\alpha(g) = A$ and $\varphi^i_\alpha(h) = B$ satisfy the following conditions: when the eigenvalues of B (and AB, resp.) are β, β^{-1} (and γ, γ^{-1}, resp.),

(φ-I) $A = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$ if $i = 1$, and α, β, γ satisfy the conditions (IA), (IB) in Lemma 5.1.2.

(φ-II) $A = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha^{-1} \end{pmatrix}$ if $i = 2$, and α, β, γ satisfy the conditions (IIA), (IIB), (IIC) in Lemma 5.1.3.

Let $(\mathcal{H}_2, \boldsymbol{\beta})$ be a Heegaard diagram of M, and let X be the corresponding CW structure of M. Then, the torsion with respect to φ is computed as follows:

Theorem 5.1.5. *For $\alpha \in \mathbb{C}^* \setminus \{\pm 1\}$, let φ^1_α be a representation given by (φ-I). Suppose $y \neq 0$ or $z \neq 0$. Then, $\alpha^\ell = \beta^m = \gamma^n \in \{\pm 1\}$, and the following holds:*

(i) If $\alpha^\ell = \beta^m = \gamma^n = -1$, then $C^{\varphi^1_\alpha}_*(X)$ is acyclic, and the torsion is computed as

$$T_{\varphi^1_\alpha}(M) = -\frac{4\alpha\beta\gamma}{(\alpha-1)^2(\beta-1)^2(\gamma-1)^2} \in \mathbb{C}^\times.$$

(ii) If $\alpha^\ell = \beta^m = \gamma^n = 1$, then $C^{\varphi^1_\alpha}_*(X)$ is not acyclic.

Theorem 5.1.6. *For $\alpha \in \mathbb{C}$, let φ^2_α be a representation given by (φ-II). Then, $\alpha, \beta \in \{\pm 1\}$, and the followings hold:*

(iii) If $\alpha = -1$ or $\beta = -1$, then $C^{\varphi}_*(X)$ is acyclic and the torsion is computed as

$$T_\varphi(M) = \begin{cases} \ell^2/16 & \text{if } \alpha = 1, \beta = -1, \\ m^2/16 & \text{if } \alpha = -1, \beta = 1, \\ n^2/16 & \text{if } \alpha = \beta = -1. \end{cases}$$

(iv) If $\alpha = \beta = 1$, then $C_*^{\varphi}(X)$ is not acyclic.

Remark 5.1.7. More generally, Kitano [Kit94] has completely computed all the twisted torsions of any acyclic irreducible $\mathrm{SL}_2(\mathbb{C})$-representations of any Seifert manifolds. We can verify the result (i) is equal to that of Kitano. However, we point out that our procedure has advantages to compute the torsions of reducible $\mathrm{SL}_2(\mathbb{C})$-representations and to compute the torsions of other (e.g., adjoint) $\mathrm{SL}_2(\mathbb{C})$-representations.

5.1.2 Proofs of Theorems 5.1.5 and 5.1.6

We prepare the following lemma to prove Lemmas 5.1.2 and 5.1.3.

Lemma 5.1.8. *Take $r \in \mathbb{N}$ and $C \in \mathrm{GL}_2(\mathbb{C})$. Let $c, d \in \mathbb{C}$ be the eigenvalues of C, and E be the (2×2)-identity matrix. Then,*

$$C^r = \begin{cases} \dfrac{c^r - d^r}{c - d} C - \dfrac{c^r d - c d^r}{c - d} E, & \text{if } c \neq d, \\ r c^{r-1} C - (r-1) c^r E, & \text{if } c = d. \end{cases}$$

Proof. It immediately follows from the Cayley-Hamilton theorem and the induction on r. □

If $A, B \in \mathrm{SL}_2(\mathbb{C})$ satisfy (5.2), we obtain

$$A^{\ell} = (BA)^n = B(AB)^n B^{-1} = B B^m B^{-1} = B^m. \tag{5.3}$$

Proof of Lemma 5.1.2. We claim that the eigenvalues β, β^{-1} of B are distinct. In fact, if we assume $\beta = \beta^{-1}$, then B^m is not diagonalizable, which contradicts $B^m = A^{\ell} = \begin{pmatrix} a^{\ell} & 0 \\ 0 & a^{-\ell} \end{pmatrix}$. Thus, $\beta \neq \beta^{-1}$, i.e. $\beta \neq \pm 1$. According to Lemma 5.1.8, we have

$$A^{\ell} = B^m = \frac{\beta^m - \beta^{-m}}{\beta - \beta^{-1}} B - \frac{\beta^{m-1} - \beta^{1-m}}{\beta - \beta^{-1}} E.$$

From the $(1,2)$-th and $(2,1)$-th entries, we have $\beta^m = \beta^{-m}$, i.e. $\beta^{2m} = 1$. Then,

$$A^{\ell} = B^m = -\frac{\beta^{m-1} - \beta^{1-m}}{\beta - \beta^{-1}} E = -\frac{\beta^{m-1} - \beta^{1+m}}{\beta - \beta^{-1}} E = \beta^m E$$

yields $\alpha^{\ell} = \beta^m$. By observing the eigenvalues γ, γ^{-1} of AB, we can similarly show $\gamma \neq \pm 1$ and $\alpha^{\ell} = \gamma^n$. In summary, we obtain (IA).

Note that the sum of the eigenvalues of B (and AB) coincides with the sum of the diagonal components of B (and AB, respectively). Thus, we have

$$x + w = \beta + \beta^{-1}, \quad \alpha x + \alpha^{-1} w = \gamma + \gamma^{-1},$$

which leads to (IB). The "if" part is proven.

Conversely, the "only if" part can be checked by calculating A^ℓ, B^m, $(AB)^n$, $(BA)^n$ with Lemma 5.1.8. □

Proof of Lemma 5.1.3. Let the matrices $A, B \in \mathrm{SL}_2(\mathbb{C})$ satisfy (5.2). Then A is written as

$$A^\ell = \begin{pmatrix} \alpha^\ell & \ell\alpha^{\ell-1} \\ 0 & \alpha^\ell \end{pmatrix}, \tag{5.4}$$

and $\alpha^2 = xw - yz = 1$ follows from $\det A = \det B = 1$. For the eigenvalues β, β^{-1} of $B \in \mathrm{SL}_2(\mathbb{C})$, we have $\beta = \beta^{-1}$. Indeed, if $\beta \neq \beta^{-1}$, then B^m is diagonalizable, and A^ℓ is not diagonalizable, which means a contradiction.

Next, let us prove $z = 0$. By Lemma 5.1.8, $B^m = m\beta^{m-1}B - (m-1)\beta^m E$. If $z \neq 0$, then $A^\ell = B^m$ implies $\beta^{m-1} = 0$, which contradicts $\beta \neq 0$. Therefore, we have $z = 0$.

Since $\beta = \beta^{-1}$ and $z = 0$ as above, it follows that

$$B^m = \begin{pmatrix} \beta & y \\ 0 & \beta \end{pmatrix}^m = \begin{pmatrix} \beta^m & m\beta^{m-1}y \\ 0 & \beta^m \end{pmatrix},$$

$$(AB)^n = \begin{pmatrix} \alpha\beta & \alpha y + \beta \\ 0 & \alpha\beta \end{pmatrix}^n = \begin{pmatrix} (\alpha\beta)^n & n(\alpha\beta)^{n-1}(\alpha y + \beta) \\ 0 & (\alpha\beta)^n \end{pmatrix}.$$

By these equalities, (5.4), and $A^\ell = B^m = (AB)^n$, we obtain

$$\alpha^\ell = \beta^m = (\alpha\beta)^n, \quad \ell\alpha^{\ell-1} = m\beta^{m-1}y = n(\alpha\beta)^{n-1}(\alpha y + \beta).$$

Thus, we have $m \neq n$, $\ell = nm(m-n)^{-1}$. Furthermore, by $\alpha^2 = \beta^2 = 1$, we get $y = \alpha^{-1}\beta\ell m^{-1} = \alpha\beta n(m-n)^{-1}$, which means the "if" part.

The "only if" part can be shown by following the reverse process of the above calculation. □

Proofs of Theorems 5.1.5 and 5.1.6. We denote $\pi_1(M)$ by π_1. Since the identity is given by $W_M = (1, r)(h, r)^{-1}(1, s)(g, s)^{-1}$, the chain complex $C_*(\widetilde{X}; \mathbb{Z}) = C_*(\mathcal{H}_2, \boldsymbol{\beta})$ can be written as

$$C_*(\mathcal{H}_2, \boldsymbol{\beta}) : 0 \to \mathbb{Z}[\pi_1] \xrightarrow{\partial_3} \mathbb{Z}[\pi_1] \oplus \mathbb{Z}[\pi_1] \xrightarrow{\partial_2} \mathbb{Z}[\pi_1] \oplus \mathbb{Z}[\pi_1] \xrightarrow{\partial_1} \mathbb{Z}[\pi_1] \to 0,$$

$$\partial_1 = (1 - g, 1 - h), \quad \partial_2 = \begin{pmatrix} \frac{\partial s}{\partial g} & \frac{\partial s}{\partial h} \\ \frac{\partial r}{\partial g} & \frac{\partial r}{\partial h} \end{pmatrix}, \quad \partial_3 = \begin{pmatrix} \mu \circ \psi(\frac{\partial W_M}{\partial \rho_1}) \\ \mu \circ \psi(\frac{\partial W_M}{\partial \rho_2}) \end{pmatrix}.$$

Here,

$$
\begin{aligned}
\frac{\partial r}{\partial g} &= \frac{\partial((hg)^n g^{-\ell})}{\partial g} = h\sum_{i=0}^{n-2}(gh)^i - h(gh)^{n-1}\sum_{j=1}^{\ell-1} g^{-j}, \\
\frac{\partial s}{\partial h} &= \frac{\partial((gh)^n h^{-m})}{\partial h} = g\sum_{i=0}^{n-2}(hg)^i - g(hg)^{n-1}\sum_{j=1}^{m-1} h^{-j}, \\
\frac{\partial r}{\partial h} &= \frac{\partial((hg)^n g^{-\ell})}{\partial h} = \sum_{i=0}^{n-1}(gh)^i, \\
\frac{\partial s}{\partial g} &= \frac{\partial((gh)^n h^{-m})}{\partial g} = \sum_{i=0}^{n-1}(hg)^i, \\
\mu\circ\psi\left(\frac{\partial W_M}{\partial \rho_1}\right) &= rhr^{-1}h^{-1} - rhr^{-1}h^{-1}sgs^{-1} = 1-g, \\
\mu\circ\psi\left(\frac{\partial W_M}{\partial \rho_2}\right) &= 1 - rhr^{-1} = 1-h.
\end{aligned}
$$

Under the identification of $\mathbb{C}^2 \otimes \mathbb{Z}[\pi_1]$ with $\mathbb{C}^2$ through the isomorphism defined by

$$
\mathbb{C}^2 \otimes \mathbb{Z}[\pi_1] \longrightarrow \mathbb{C}^2; \qquad \begin{pmatrix} u \\ v \end{pmatrix} \otimes \sum_{\gamma\in\pi_1} a_\gamma \gamma \mapsto \sum_{\gamma\in\pi_1} a_\gamma \varphi(\gamma)^{-1}\begin{pmatrix} u \\ v \end{pmatrix},
$$

the chain complex $C_*^\varphi(X) = \mathbb{C}^2 \otimes_{\mathbb{Z}[\pi_1]} C_*(\widetilde{X};\mathbb{Z})$ is represented by the following matrices:

$$
C_*^\varphi(X) : 0 \longrightarrow \mathbb{C}^2 \xrightarrow{\partial_3^\varphi} \mathbb{C}^4 \xrightarrow{\partial_2^\varphi} \mathbb{C}^4 \xrightarrow{\partial_1^\varphi} \mathbb{C}^2 \longrightarrow 0,
$$

$$
\partial_1^\varphi = (E - A^{-1}, E - B^{-1}), \quad \partial_2^\varphi = \begin{pmatrix} S_g & S_h \\ R_g & R_h \end{pmatrix}, \quad \partial_3^\varphi = \begin{pmatrix} E - A^{-1} \\ E - B^{-1} \end{pmatrix}. \tag{5.5}
$$

Here,

$$
S_g = \sum_{i=0}^{n-2}(B^{-1}A^{-1})^i B^{-1} - \sum_{j=1}^{\ell-1} A^j (B^{-1}A^{-1})^{n-1} B^{-1},
$$

$$
R_h = \sum_{i=0}^{n-2}(A^{-1}B^{-1})^i A^{-1} - \sum_{j=1}^{m-1} B^j (A^{-1}B^{-1})^{n-1} A^{-1},
$$

$$S_h = \sum_{i=0}^{n-1} \left(B^{-1}A^{-1}\right)^i, \qquad R_g = \sum_{i=0}^{n-1} \left(A^{-1}B^{-1}\right)^i.$$

From now on, let $\{e_1^{(4)}, e_2^{(4)}, e_3^{(4)}, e_4^{(4)}\}$ and $\{e_1^{(2)}, e_2^{(2)}\}$ be the canonical bases of $\mathbb{C}^4$ and $\mathbb{C}^2$, respectively. Set $\mathbf{c}_0 = \mathbf{c}_3 = (e_1^{(2)}, e_2^{(2)})$ and $\mathbf{c}_1 = \mathbf{c}_2 = (e_1^{(4)}, e_2^{(4)}, e_3^{(4)}, e_4^{(4)})$.

Let us calculate the twisted torsions in the two cases (i) and (iii).

The case of (i) Assume $y \neq 0$. We first discuss ∂_1^φ. By (5.5), let $b_1^0 = \partial_1^\varphi(e_1^{(4)}) = \begin{pmatrix} 1-\alpha^{-1} \\ 0 \end{pmatrix}$, $b_2^0 = \partial_1^\varphi(e_2^{(4)}) = \begin{pmatrix} 0 \\ 1-\alpha \end{pmatrix}$. The set of these two is a basis of $B_0^\varphi(X) = \operatorname{Im}\partial_1^\varphi = \mathbb{C}^2$ since $\alpha \neq \pm 1$. Note that $H_0^\varphi(X) = \mathbb{C}^2/B_0^\varphi(X) = 0$. If we take $\mathbf{b}_0 = (b_1^0, b_2^0)$, then $\widetilde{\mathbf{b}}_0 = (e_1^{(4)}, e_2^{(4)})$, and by rank-nullity theorem, $\dim Z_1^\varphi(X) = 2$, where $Z_1^\varphi(X) = \operatorname{Ker}\partial_1^\varphi$.

Next, we examine ∂_2^φ. By Lemma 5.1.8,

$$A^j = \begin{pmatrix} \alpha^{-j} & 0 \\ 0 & \alpha^j \end{pmatrix}, \qquad B^j = \frac{\beta^j - \beta^{-j}}{\beta - \beta^{-1}} B - \frac{\beta^{j-1} - \beta^{1-j}}{\beta - \beta^{-1}} E,$$

$$(B^{-1}A^{-1})^j = \frac{\gamma^j - \gamma^{-j}}{\gamma - \gamma^{-1}} B^{-1}A^{-1} - \frac{\gamma^{j-1} - \gamma^{1-j}}{\gamma - \gamma^{-1}} E,$$

$$(A^{-1}B^{-1})^j = \frac{\gamma^j - \gamma^{-j}}{\gamma - \gamma^{-1}} A^{-1}B^{-1} - \frac{\gamma^{j-1} - \gamma^{1-j}}{\gamma - \gamma^{-1}} E.$$

Using the above equalities and Lemma 5.1.2, with the help of a computer program of Mathematica, we can obtain

$$S_g = \begin{pmatrix} \frac{2\alpha(\beta\gamma+\alpha^2\beta\gamma-\alpha(\beta+\gamma+\beta\gamma(-2+\beta+\gamma)))}{(\alpha^2-1)\beta(\gamma-1)^2} & \frac{2\gamma y}{(\gamma-1)^2} \\ \frac{2(\alpha\beta-\gamma)(\beta\gamma-\alpha)(\gamma\alpha-\beta)(\alpha\beta\gamma-1)}{(\alpha^2-1)\beta^2(\gamma-1)^2\gamma y} & \frac{-2\beta\gamma-2\alpha^2\beta\gamma+2\alpha(\beta+\gamma+\beta\gamma(-2+\beta+\gamma))}{\alpha(\alpha^2-1)\beta(\gamma+1)^2} \end{pmatrix},$$

$$R_g = \begin{pmatrix} \frac{2(\beta\gamma-\alpha(1+\beta^2)\gamma+\alpha^2\beta(1+(-1+\gamma)\gamma)}{(\alpha^2-1)\beta(\gamma-1)^2} & \frac{2\gamma y}{\alpha(\gamma-1)^2} \\ \frac{2(\alpha\beta-\gamma)(\beta\gamma-\alpha)(\gamma\alpha-\beta)(\alpha\beta\gamma-1)}{(\alpha^2-1)\beta^2(\gamma-1)^2\gamma y} & \frac{2\alpha\gamma+2\alpha\beta^2\gamma-2\beta(1+\gamma(-1+\alpha^2+\gamma))}{(\alpha^2-1)\beta(\gamma-1)^2} \end{pmatrix},$$

where $\partial_2^\varphi = \begin{pmatrix} S_g & S_h \\ R_g & R_h \end{pmatrix}$. We can show that the rank of ∂_2^φ is 2 and that $\mathbf{b}_1$ is a basis of $B_1^\varphi(X) = \operatorname{Im}\partial_2^\varphi$ when we take $b_1^1 = \partial_2^\varphi(e_1^{(4)})$, $b_2^1 = \partial_2^\varphi(e_2^{(4)})$, and $\mathbf{b}_1 = (b_1^1, b_2^1)$. Note that $\widetilde{\mathbf{b}}_1 = (e_1^{(4)}, e_2^{(4)})$ and that $\dim Z_1^\varphi(X) = \dim B_1^\varphi(X)$ leads to $H_1^\varphi(X) = Z_1^\varphi(X)/B_1^\varphi(X) = 0$.

Finally, let us discuss ∂_3^φ. If we take vectors of the forms

$$b_1^2 = \partial_2^\varphi(e_1^{(2)}) = \begin{pmatrix} b_1^0 \\ 1-w \\ 1+z \end{pmatrix}, \qquad b_2^2 = \partial_2^\varphi(e_2^{(2)}) = \begin{pmatrix} b_2^0 \\ 1+y \\ 1-x \end{pmatrix},$$

then $\mathbf{b}_2 = (b_1^2, b_2^2)$ provides a basis of $B_2^\varphi(X) = \operatorname{Im}\partial_3^\varphi$. Note that $\widetilde{\mathbf{b}}_2 = (e_1^{(2)}, e_2^{(2)})$, and $H_2^\varphi(X) = Z_2^\varphi(X)/B_2^\varphi(X) = 0$ follows from $\dim Z_2^\varphi(X) = \dim B_2^\varphi(X)$.

From this, we have

$$\begin{aligned}
T_\varphi(M) &= [\mathbf{b}_3, \widetilde{\mathbf{b}}_2/\mathbf{c}_3][\mathbf{b}_2, \widetilde{\mathbf{b}}_1/\mathbf{c}_2]^{-1}[\mathbf{b}_1, \widetilde{\mathbf{b}}_0/\mathbf{c}_0][\mathbf{b}_0, \widetilde{\mathbf{b}}_{-1}/\mathbf{c}_0]^{-1} \\
&= \det(\widetilde{\mathbf{b}}_2)(\det(\mathbf{b}_2, \widetilde{\mathbf{b}}_1))^{-1}\det(\mathbf{b}_1, \widetilde{\mathbf{b}}_0)(\det(\mathbf{b}_0))^{-1} \\
&= \det\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\det\begin{pmatrix}1-\alpha^{-1} & 0 & 1 & 0\\ 0 & 1-\alpha & 0 & 1\\ 1-w & 1+y & 0 & 0\\ 1+z & 1-x & 0 & 0\end{pmatrix}^{-1} \\
&\qquad \det\begin{pmatrix}(S_g)_{(1,1)} & (S_g)_{(1,2)} & 1 & 0\\ (S_g)_{(2,1)} & (S_g)_{(2,2)} & 0 & 1\\ (R_g)_{(1,1)} & (R_g)_{(1,2)} & 0 & 0\\ (R_g)_{(2,1)} & (R_g)_{(2,2)} & 0 & 0\end{pmatrix}\det\begin{pmatrix}1-\alpha^{-1} & 0\\ 0 & 1-\alpha\end{pmatrix}^{-1} \\
&= -\frac{4\alpha\beta\gamma}{(\alpha-1)^2(\beta-1)^2(\gamma-1)^2} \in \mathbb{C}^\times.
\end{aligned}$$

We use the relations of Lemma 5.1.2 for the last equation.

In the case where $z \neq 0$, the proof can be given in the same way.

The case of (iii) We assume $\alpha = 1$ and $\beta = -1$. Concerning the boundary map ∂_1^φ, it follows from (5.5) that

$$\partial_1^\varphi = \begin{pmatrix}1-\alpha & 1 & 1-\beta & y\\ 0 & 1-\alpha & 0 & 1-\beta\end{pmatrix} = \begin{pmatrix}0 & 1 & 2 & y\\ 0 & 0 & 0 & 2\end{pmatrix}.$$

By $B_0^\varphi(M) = \operatorname{Im}\partial_1^\varphi = \mathbb{C}^2$, we can take a basis $\mathbf{b}_0 = (e_1^{(2)}, \begin{pmatrix}y\\2\end{pmatrix})$ of B_0. Note that $\widetilde{\mathbf{b}}_0 = (e_2^{(4)}, e_4^{(4)})$ and $\dim Z_1^\varphi(X) = \dim(\operatorname{Ker}\partial_1^\varphi) = 2$ by rank-nullity theorem.

Next, to describe ∂_2^φ, we shall notice that

$$A^j = \begin{pmatrix}\alpha^j & -j\alpha^{j-1}\\ 0 & \alpha^j\end{pmatrix} = \begin{pmatrix}1 & -j\\ 0 & 1\end{pmatrix}, \quad B^j = \begin{pmatrix}\beta^j & jy\beta^{j-1}\\ 0 & \beta^j\end{pmatrix} = (-1)^j\begin{pmatrix}1 & -jy\\ 0 & 1\end{pmatrix},$$

$$(B^{-1}A^{-1})^j = (A^{-1}B^{-1})^j = \begin{pmatrix}\alpha\beta & -\alpha y - \beta\\ 0 & \alpha\beta\end{pmatrix}^j = (-1)^j\begin{pmatrix}1 & j(1-y)\\ 0 & 1\end{pmatrix}.$$

By these equalities and the relations in Lemma 5.1.3, it can be calculated with the help of a computer program in Mathematica that

$$S_g = \begin{pmatrix}\frac{mn}{n-m} & \frac{mn(m(n-2)+2n)}{2(m-n)^2}\\ 0 & \frac{mn}{n-m}\end{pmatrix},\ S_h = R_g = \begin{pmatrix}0 & \frac{mn}{2(m-n)}\\ 0 & 0\end{pmatrix},\ R_h = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}.$$

Therefore, we can show that the rank of ∂_2^φ is 2, and $\mathbf{b}_1$ is a basis of $B_1^\varphi(X) = \operatorname{Im}\partial_2^\varphi$ when we select

$$b_1^1 = \partial_2^\varphi(e_1^{(4)}) = \begin{pmatrix} \frac{mn}{n-m} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad b_2^1 = \partial_2^\varphi(e_2^{(4)}) = \begin{pmatrix} \frac{mn(m(n-2)+2n)}{2(m-n)^2} \\ \frac{mn}{n-m} \\ \frac{mn}{2(m-n)} \\ 0 \end{pmatrix},$$

and $\mathbf{b}_1 = (b_1^1, b_2^1)$. We can take $\widetilde{\mathbf{b}}_1 = (e_1^{(4)}, e_2^{(4)})$. Note that $\dim B_1^\varphi(X) = 2 = \dim Z_1^\varphi(X)$ yields $H_1^\varphi(X) = Z_1^\varphi(X)/B_1^\varphi(X) = 0$, and $\dim Z_2^\varphi(X) = \dim \partial_2^\varphi = 2$.

Likewise, regarding ∂_3^φ, it follows from (5.5) that

$$\partial_3^\varphi = \begin{pmatrix} 1-\alpha & 1 \\ 0 & 1-\alpha \\ 1-\beta & y \\ 0 & 1-\beta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 2 & y \\ 0 & 2 \end{pmatrix}.$$

Therefore, by choosing

$$b_1^2 = \partial_3^\varphi(e_1^{(2)}) = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \quad b_2^2 = \partial_3^\varphi(e_2^{(2)}) = \begin{pmatrix} 1 \\ 0 \\ y \\ 2 \end{pmatrix},$$

and setting $\mathbf{b}_2 = (b_1^2, b_2^2)$, we can regard $\mathbf{b}_2$ as a basis of $B_2^\varphi(X) = \operatorname{Im}\partial_3^\varphi$. We can take $\widetilde{\mathbf{b}}_2 = (e_1^{(2)}, e_2^{(2)})$. Note that $\dim B_2^\varphi(X) = 2 = \dim Z_2^\varphi(X)$ implies $H_2^\varphi(X) = Z_2^\varphi(X)/B_2^\varphi(X) = 0$, and $H_3^\varphi(X) = Z_3^\varphi(X) = 0$.

For $\mathbf{b}_0$, $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3(=\emptyset)$, $\widetilde{\mathbf{b}}_{-1}(=\emptyset)$, $\widetilde{\mathbf{b}}_0$, $\widetilde{\mathbf{b}}_1$ and $\widetilde{\mathbf{b}}_2$, we obtain

$$\begin{aligned}
T_\varphi(M) &= [\mathbf{b}_3, \widetilde{\mathbf{b}}_2/\mathbf{c}_3][\mathbf{b}_2, \widetilde{\mathbf{b}}_1/\mathbf{c}_2]^{-1}[\mathbf{b}_1, \widetilde{\mathbf{b}}_0/\mathbf{c}_0][\mathbf{b}_0, \widetilde{\mathbf{b}}_{-1}/\mathbf{c}_0]^{-1} \\
&= \det(\widetilde{\mathbf{b}}_2)(\det(\mathbf{b}_2, \widetilde{\mathbf{b}}_1))^{-1}\det(\mathbf{b}_1, \widetilde{\mathbf{b}}_0)(\det(\mathbf{b}_0))^{-1} \\
&= \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\det\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & \frac{n}{n-m} & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}^{-1} \\
&\qquad \det\begin{pmatrix} \frac{mn}{n-m} & \frac{mn(m(n-2)+2n)}{2(m-n)^2} & 0 & 0 \\ 0 & \frac{mn}{n-m} & 1 & 0 \\ 0 & \frac{mn}{2(m-n)} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\det\begin{pmatrix} 1 & y \\ 0 & 2 \end{pmatrix}^{-1} \\
&= \ell^2/16 \in \mathbb{C}^\times.
\end{aligned}$$

Here, we use the relations of Lemma 5.1.3 for the third equation.

In the case where $\alpha = -1, \beta = 1$ or $\alpha = \beta = -1$, it can be calculated similarly. Thus, we will omit writing the details.

The cases of (ii) and (iv) The non-acyclicity can be proved by computing the homology $H^{\varphi}_*(X)$ in the same way as in (i) and (iii). □

5.2 3-manifolds obtained by some Dehn-surgeries

For $p/q \in \mathbb{Q}$ and a knot K in S^3, let $S^3_{p/q}(K)$ be the closed 3-manifold obtained by (p/q)-Dehn surgery on K. In this section, we assume that M is one of $S^3_{p/1}(4_1)$ and $S^3_{1/q}(4_1)$ for some integers $p, q \neq 0$.

According to [Nos22a], we can give the group presentations of $\pi_1(M)$ as follows:

$$\begin{aligned}
\pi_1(S^3_{p/1}(4_1)) &\cong \langle x_1, x_2, \mathfrak{m} \mid \mathfrak{m}x_1x_2\mathfrak{m}^{-1}x_1^{-1}, \mathfrak{m}x_2x_1x_2\mathfrak{m}^{-1}x_2^{-1}, [x_1,x_2]\mathfrak{m}^p\rangle,\\
\pi_1(S^3_{1/q}(4_1)) &\cong \langle x_1, x_2, \mathfrak{m}, \mathfrak{m}' \mid \mathfrak{m}x_1x_2\mathfrak{m}^{-1}x_1^{-1}, \mathfrak{m}x_2x_1x_2\mathfrak{m}^{-1}x_2^{-1}, \\
&\qquad\qquad \mathfrak{m}[x_1,x_2]^q, \mathfrak{m}'[x_1,x_2]^{-1}\rangle.
\end{aligned} \tag{5.6}$$

Here, $[x, y]$ is $xyx^{-1}y^{-1}$. Let g be the number of generators of the group presentation above. Replace $\mathfrak{m}$ by x_3, $\mathfrak{m}'$ by x_4, and let r_i denote the i-th relator in (5.6). Under the identification of $\mathfrak{g} \otimes \mathbb{Z}[\pi_1(M)]$ with $\mathfrak{g} = \{\begin{pmatrix} u & v \\ w & -u \end{pmatrix} \mid u, v, w \in \mathbb{C}\}$ through the isomorphism defined by

$$\mathfrak{g} \otimes \mathbb{Z}[\pi_1(M)] \longrightarrow \mathfrak{g}; \qquad \nu \otimes \sum_{\gamma\in\pi_1} a_\gamma \gamma \mapsto \sum_{\gamma\in\pi_1} a_\gamma \varphi(\gamma)\nu\varphi(\gamma)^{-1},$$

the chain complex $C^{\varphi}_*(M : \mathfrak{g}) = \mathfrak{g} \otimes_{\mathbb{Z}[\pi_1(M)]} C_*(\widetilde{M}; \mathbb{Z})$ is represented by the following matrices:

$$C^{\varphi}_*(M; \mathfrak{g}) : 0 \to \mathfrak{g} \xrightarrow{\partial_3} \mathfrak{g}^g \xrightarrow{\partial_2} \mathfrak{g}^g \xrightarrow{\partial_1} \mathfrak{g} \to 0.$$

We now describe the differentials ∂_* in detail. Let F and P be the free groups $\langle x_1, \ldots, x_g \mid \,\rangle$ and $\langle \rho_1, \ldots, \rho_g \mid \,\rangle$, respectively. Let $\psi : P * F \to F$ be a homomorphism defined by (3.2), and μ be the natural surjection from F to $\pi_1(M)$. According to [Nos22a, §3.1], we can describe ∂_* by the words of the presentations (5.6) as follows: let $W_M \in P * F$ be

$$\rho_1{\cdot}x_1\rho_2x_1^{-1}{\cdot}(x_1x_2x_1^{-1})\rho_1^{-1}(x_1x_2x_1^{-1})^{-1}{\cdot}([x_1,x_2])\rho_2^{-1}([x_1,x_2])^{-1}{\cdot}\rho_4^{-1}{\cdot}\mathfrak{m}'\rho_3\mathfrak{m}'^{-1}{\cdot}\rho_4{\cdot}\rho_3^{-1},$$

if $M = S^3_{p/1}(4_1)$. Let $W_M \in P * F$ be

$$\rho_1{\cdot}x_1\rho_2x_1^{-1}{\cdot}(x_1x_2x_1^{-1})\rho_1^{-1}(x_1x_2x_1^{-1})^{-1}{\cdot}([x_1,x_2])\rho_2^{-1}([x_1,x_2])^{-1}{\cdot}\rho_4^{-1}{\cdot}\mathfrak{m}'\rho_3\mathfrak{m}'^{-1}{\cdot}\rho_4{\cdot}\rho_3^{-1},$$

if $M = S^3_{1/q}(4_1)$. Then, W_M is an identity satisfying Theorem 3.2.2, and each ∂_* can be written as the matrices

$$\partial_1 = (1 - x_j)_{j=1,\dots,g}\,, \quad \partial_2 = \left(\frac{\partial r_j}{\partial x_i}\right)_{i,j=1,\dots,g}, \quad \partial_3 = \mu \circ \psi \left(\frac{\partial W}{\partial \rho_i}\right)_{i=1,\dots,g}, \tag{5.7}$$

where $\frac{\partial *}{\partial *}$ is Fox derivative. Although each entry of the matrices is described in $\mathbb{Z}[\pi_1(M)]$, we regard the entry as an automorphism of $\mathfrak{g}$ via the adjoint action.

5.2.1 Results

To state Proposition 5.2.1 and Theorem 5.2.2 in this subsection, let us consider a domain D in $\mathbb{C}$ of the form

$$D \coloneqq \{a \in \mathbb{C} \mid |a| < 1\} \cup \{a \in \mathbb{C} \mid \operatorname{Im}(a) > 0, |a| = 1\} \cup \{-\sqrt{-1}\},$$

as in Figure 5.1, and define the polynomial $Q_M(x) \in \mathbb{Z}[x]$ by setting

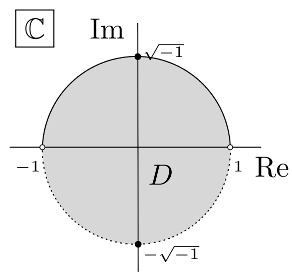

Figure 5.1: $D \subset \mathbb{C}$

$$Q_M(x) \coloneqq \begin{cases} 1 - x^{p-4} + x^{p-2} + 2x^p + x^{p+2} - x^{p+4} + x^{2p}, & \text{if } M = S^3_{p/1}(4_1), \\ 1 - x^{2q} - x^{4q-1} - 2x^{4q} - x^{4q+1} - x^{6q} + x^{8q}, & \text{if } M = S^3_{1/q}(4_1). \end{cases} \tag{5.8}$$

Let $Q_M^{-1}(0)$ denote the zero set of the polynomial Q_M.

Proposition 5.2.1 [Wak23, Proposition 3.1]**.** *Let $M = S^3_{p/1}(4_1)$ or $M = S^3_{1/q}(4_1)$ for some integers p, $q \neq 0$. If $p \neq 0$, then there is a bijection $\Phi_M : R^{\mathrm{irr}}_G(M) \to Q_M^{-1}(0) \cap D$. Here, for $[\varphi] \in R^{\mathrm{irr}}_G(M)$, we define*

$$\Phi_M([\varphi]) \coloneqq (\text{The eigenvalue of } \varphi(\mathfrak{m}) \text{ that lies in } D \setminus \{\pm\sqrt{-1}\}),$$

when the eigenvalues of $\varphi(\mathfrak{m})$ are not $\pm\sqrt{-1}$. If $\varepsilon\sqrt{-1} \in Q_M^{-1}(0)$ for some $\varepsilon \in \{\pm 1\}$, $\Phi_M^{-1}(\varepsilon\sqrt{-1})$ is a representation φ defined by

$$\varphi(\mathfrak{m}) = \begin{pmatrix} \varepsilon\sqrt{-1} & 0 \\ 0 & \sqrt{-1} \end{pmatrix}, \quad \varphi(x_1) = \begin{pmatrix} \frac{1}{4}(-1 + \varepsilon\sqrt{5}) & 1 \\ \frac{1}{8}(-5 - \varepsilon\sqrt{5}) & \frac{1}{4}(-1 + \varepsilon\sqrt{5}) \end{pmatrix}.$$

If $p = 0$ and $M = S^3_{p/1}(4_1)$, there is a bijection $\Phi_M : R^{\mathrm{irr}}_G(M) \to \{\pm\sqrt{-1}, \pm(1 - \sqrt{5})/2\}$.

Proof. Let $M = S^3_{p/1}(4_1)$ with $p \neq 0$. For an irreducible representation $\varphi : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$, take $x, y, z, w \in \mathbb{C}$ so that $\varphi(x_1) = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ and $xw - yz = 1$. We first claim that $\varphi(\mathfrak{m})$ is diagonalizable. In fact, if not so, we may suppose $\varphi(\mathfrak{m}) = \begin{pmatrix} \eta & b \\ 0 & \eta \end{pmatrix}$ for some $b \in \mathbb{C}^\times$ and $\eta \in \{\pm 1\}$. Since $\varphi(r_1) = E$, we have

$$\varphi(x_2) = \varphi(x_1)\varphi(\mathfrak{m})^{-1}\varphi(x_1)\varphi(\mathfrak{m}) = \begin{pmatrix} 1 - \eta bwz & -b\left(bwz + \eta w^2 - \eta\right) \\ \eta bz^2 & b^2z^2 + \eta bwz + 1 \end{pmatrix}. \tag{5.9}$$

It follows from (5.9) that

$$\begin{aligned} \varphi(r_3) &= \varphi(x_1)\varphi(x_2)\varphi(x_1)^{-1}\varphi(x_2)^{-1}\varphi(\mathfrak{m})^p \\ &= \eta^p \begin{pmatrix} \begin{array}{l} b^4z^4 + \eta b^3wz^3 \\ \quad - b^2z^2\left(x^2 + yz - 2\right) + \eta bz(w - x) + 1 \end{array} & * \\ -\eta b^3z^4 - b^2z^3(w + x) - \eta 2bz^2 & \begin{array}{l} -b^4pz^4 + \eta b^3z^3(-pw - px - w) \\ \quad - b^2z^2\left(2p + w^2 + yz\right) + \eta bz(x - w) + 1 \end{array} \end{pmatrix}. \end{aligned}$$

Then, the condition $\varphi(r_3) = E$ and $b \neq 0$ leads to $z = 0$. By substituting $z = 0$ into $\varphi(r_2)$, we obtain

$$E = \varphi(r_2) = \varphi(\mathfrak{m})\varphi(x_2)\varphi(x_1)\varphi(x_2)\varphi(\mathfrak{m})^{-1}\varphi(x_2)^{-1} = \begin{pmatrix} x & y - \eta b\left(w^3 - 2w + x\right) \\ 0 & w \end{pmatrix}.$$

Thus, $x = w = 1$ and $y = 0$; therefore, $\varphi(x_1)$ and $\varphi(x_2)$ are upper triangular matrices, which leads to a contradiction to the irreducibility.

By the above claim, we may suppose $\varphi(\mathfrak{m}) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ for some $a \in D$. Since we consider φ up to conjugacy, we may suppose $y = 1$. Thus, $z = xw - 1$. Since $\varphi(r_1) = \varphi(r_2) = \varphi(r_3) = E$, with the help of a computer program of Mathematica, we have

$$\begin{aligned} x &= \frac{1 + a^2 - a^4 + \eta(1 - 2a^2 - a^4 - 2a^6 + a^8)^{1/2}}{2(1 - a^2)}, \\ z &= -\frac{1 - 3a^2 + a^4 + \eta(1 - 2a^2 - a^4 - 2a^6 + a^8)^{1/2}}{2(a^2 - 1)^2}, \\ w &= \frac{-1 + a^2 + a^4 + \eta(1 - 2a^2 - a^4 - 2a^6 + a^8)^{1/2}}{2a^2(a^2 - 1)}, \end{aligned} \tag{5.10}$$

and $Q_M(a) = 0$ when $a \neq \pm\sqrt{-1}$. Here, we fix a branch of the $(1/2)$-th power on $\mathbb{C}^\times \setminus \mathbb{R}$, and define the signs $\eta \in \{\pm 1\}$ by setting

$$\eta = \begin{cases} +1, & \text{if } \ -1 + a^2 + 2a^4 + a^6 - a^8 + 2a^{p+4} = (a^4 - 1)(1 - 2a^2 - a^4 - 2a^6 + a^8)^{1/2}, \\ -1, & \text{if } \ -1 + a^2 + 2a^4 + a^6 - a^8 + 2a^{p+4} = -(a^4 - 1)(1 - 2a^2 - a^4 - 2a^6 + a^8)^{1/2}. \end{cases}$$

When $a = \pm\sqrt{-1}$, we have $x = (-1+\varepsilon\sqrt{5})/4$, $z = (-5-\varepsilon\sqrt{5})/8$, $w = (-1+\varepsilon\sqrt{5})/4$ for some $\varepsilon \in \{\pm 1\}$ by the condition $\varphi(r_1) = \varphi(r_2) = \varphi(r_3) = E$. In summary, the map Φ_M is well-defined and injective. Finally, we can easily show the surjectivity of Φ_M by following the reverse process of the above calculation.

In the remaining cases of $M = S^3_{0/1}(4_1)$ and $M = S^3_{1/q}(4_1)$, the proofs can be shown in similar ways. Thus, we omit the details. $\square$

Theorem 5.2.2 [Wak23, Theorem 3.2]**.** *Let $p \neq 0$ and $q \neq 0$. For $a \in Q_M^{-1}(0) \cap D$ as in Proposition 5.2.1, we denote the representative $\mathrm{SL}_2(\mathbb{C})$-representation of $\Phi_M^{-1}(a)$ by φ_a. Then, the adjoint Reidemeister torsion of M with respect to φ_a is computed as follows:*

$$\tau^{\mathrm{ad}}_{\varphi_a}(M) = -\frac{4-p+(-2+p)a^2+2pa^4+(2+p)a^6-(4+p)a^8+2pa^{4+p}}{2(a^2-1)^3(1+a^2)}, \quad \text{if } M = S^3_{p/1}(4_1),\ a \notin \{\pm\sqrt{-1}\}. \tag{5.11}$$

$$\tau^{\mathrm{ad}}_{\varphi_a}(M) = \tfrac{1}{8}(10 + ap\sqrt{-5}) \qquad \text{if } M = S^3_{p/1}(4_1),\ a \in \{\pm\sqrt{-1}\}. \tag{5.12}$$

$$\tau^{\mathrm{ad}}_{\varphi_a}(M) = -\frac{a^{6q}(-1+4q+(1-2q)a^{2q}+2(1+a)a^{4q}+(1+2q)a^{6q}-(1+4q)a^{8q})}{2(a^{4q}-1)^3(1-2a^{2q}-a^{4q}-2a^{6q}+a^{8q})}, \quad \text{if } M = S^3_{1/q}(4_1). \tag{5.13}$$

Proof. Under the identification of $\mathfrak{g} \cong \mathbb{C}^3$, we can concretely describe each ∂_i as the matrices according to (5.7) and the description of Φ_M in the proof of Proposition 5.2.1. Applying the τ-chain method in [Tur01, §2.1] to the chain complex $C^*_\varphi(M;\mathfrak{g})$, with the help of a computer program of Mathematica, we can directly obtain the resulting $\tau^{\mathrm{ad}}_{\varphi_a}(M)$. $\square$

Remark 5.2.3.

(i) While this paper deals with the adjoint torsion via adjoint action, the classical twisted Reidemeister torsion of $M = S^3_{p/q}(4_1)$ with respect to the $\mathrm{SL}_2(\mathbb{C})$-representation was computed in [Kit15a].

(ii) When $M = S^3_{p/1}(4_1)$, the torsion $\tau^{\mathrm{ad}}_{\varphi_a}(M)$ in the quotient set $\mathbb{C}/\{\pm 1\}$ was computed by [OT19]. The advantage of Theorem 5.2.2 is that the sign of the torsion is recovered; thus, we can compute the sum of $\tau^{\mathrm{ad}}_{\varphi_a}(M)^n$'s, as is seen later.

(iii) We can easily check that $\tau^{\mathrm{ad}}_{\varphi_{a^{-1}}}(M) = \tau^{\mathrm{ad}}_{\varphi_a}(M) \in \mathbb{C}^\times$ by using the relation $Q_M(a) = 0$ when $a \neq \pm\sqrt{-1}$, and that $Q_M(\pm\sqrt{-1}) = 0$ with $M = S^3_{p/1}(4_1)$ if and only if p is divisible by 4.

(iv) If $p = 0$, that is $M = S^3_{0/1}(4_1)$, then we can similarly compute $\tau^{\mathrm{ad}}_{\varphi_a}(M)$ as $5/4$, $5/4$, 5, and 5 with respect to $a = \sqrt{-1}$, $-\sqrt{-1}$, $(1-\sqrt{5})/2$, and $-(1-\sqrt{5})/2$, respectively.

5.2.2 Surgeries on the 5_2-knot

We compute the adjoint torsions of $M = S^3_{1/q}$ in the case where K is 5_2-knot. Since the outline of the discussion in this subsection is almost the same as that in the case of the figure-eight knot, we now roughly describe the discussion.

As in (5.6), the fundamental group $\pi_1(S^3_{1/q}(5_2))$ is known to be presented as

$$\pi_1(M) \cong \langle x_1, x_2, \mathfrak{m}, \mathfrak{m}' \mid \mathfrak{m}x_1^2x_2^{-1}\mathfrak{m}^{-1}x_1^{-2}, \mathfrak{m}x_2^{-1}\mathfrak{m}^{-1}x_1^{-1}x_2, \mathfrak{m}[x_1^2, x_2^{-1}]^q, \mathfrak{m}'[x_1^2, x_2^{-1}]^{-1}\rangle.$$

Recall the free groups F, P, and the homomorphism ψ in Section 3.2. Let $W \in P * F$ be

$$\rho_1 \cdot x_1^2\rho_2 x_1^{-2} \cdot (x_1^2x_2^{-1}x_1^{-1})\rho_1^{-1}(x_1^2x_2^{-1}x_1^{-1})^{-1} \cdot$$
$$(x_1^2x_2^{-1}x_1^{-2}x_2)\rho_2^{-1}(x_1^2x_2^{-1}x_1^{-2}x_2)^{-1} \cdot \cdot \rho_4^{-1} \cdot x_4\rho_3x_4^{-1} \cdot \rho_4 \cdot \rho_3^{-1}.$$

Then, each ∂_* can be written as in (5.7) according to [Nos22a, §3.1]. Let $Q_M(x)$ be the polynomial of the form

$$1-x-2x^{2q}-x^{4q-1}-2x^{4q}+x^{6q-1}+x^{8q}-2x^{10q-1}-x^{10q}-2x^{12q-1}-x^{14q-2}+x^{14q-1}.$$

The same statement in Proposition 5.2.1 holds for $M = S^3_{1/q}(5_2)$ and $q \neq 0$, namely, $R^{\mathrm{irr}}_G(M)$ is bijective to $Q_M^{-1}(0) \cap D$. For $a \in Q_M^{-1}(0) \cap D$, let us denote the representative $\mathrm{SL}_2(\mathbb{C})$-representation of $\Phi_M^{-1}(a)$ by φ_a as in Proposition 5.2.1. Then, the adjoint torsion $\tau^{\mathrm{ad}}_{\varphi_a}(M)$ can be computed as

$$\tau^{\mathrm{ad}}_{\varphi_a}(M) = -P(a)/(2a^2(a^2-1)^4)$$

with the help of a computer program of Mathematica. Here, $P(a) \in \mathbb{Z}[a]$ is a polynomial defined by setting

$$\begin{aligned}
&1 - 2q + a(28q+2) + a^2(3-42q) + a^3(36q-8) + a^4(2-20q)\\
&+ a^{2q}\left((4q-1)a^{-1} + 18q - 3 + (3-32q)a + (4-54q)a^2 - 2a^3 + (8q-1)a^4\right)\\
&+ a^{4q}\left((1-4q)a^{-1} - 10q + (-8q-3)a + (38q-4)a^2 + (5-34q)a^3 + (1-10q)a^4\right)\\
&+ a^{6q}\left((10q-1)a^{-1} + (18q+2)a + (7-56q)a + (74q-8)a^2 + 10qa^3\right)\\
&+ a^{8q}\left((14q-3)a^{-1} + 18q + (9-76q)a - 3a^2 + (16q-3)a^3\right)\\
&+ a^{10q}\left((24q-2)a^{-1} + (1-10q)a + (2-52q)a + (-18q-1)a^2\right)\\
&+ a^{12q}\left((4q-1)a^{-2} + 8qa^{-1} + 5 - 62q + (56q-6)a + (2-6q)a^2\right).
\end{aligned}$$

Chapter 6

The conjecture on the adjoint torsions

In this chapter, we discuss Conjecture 1.0.1 when $G = \mathrm{SL}_2(\mathbb{C})$. Section 6.1 focus on the case where $n = -1$, and $M = S^3_{p/1}(4_1)$ or $M = S^3_{1/q}(4_1)$. The main purpose of Section 6.1 is to prove Theorem 1.0.2. Section 6.2 deals with the case of $n > 0$.

6.1 The conjecture with $n = -1$

As preliminaries of the proof of Theorem 1.0.2, we prepare two lemmas in Section 6.1.1, and give the proof of the theorem in Section 6.1.2.

6.1.1 Two key lemmas

Let $Q_M(x) \in \mathbb{Z}[x]$ be the polynomial (5.8). We prepare two lemmas:

Lemma 6.1.1 [Wak23, Lemma 3.4]**.** *Define a polynomial $\kappa_p(x) \in \mathbb{Z}[x]$ by setting*

$$\kappa_p(x) = \begin{cases} (1+x)^2, & \text{if } \ p = 2m+1, \\ (1+x^2)^2, & \text{if } \ p = 4m, \\ 1, & \text{if } \ p = 4m+2, \end{cases}$$

for some $m \in \mathbb{Z}$. Then, $Q_M(x)$ with $M = S^3_{p/1}(4_1)$ is divisible by $\kappa_p(x)$, and the quotient $Q_M(x)/\kappa_p(x)$ has no repeated roots. Furthermore, $Q_M(x)$ with $M = S^3_{1/q}(4_1)$ is divisible by $(1+x)^2$, and the quotient $Q_M(x)/(1+x)^2$ also has no repeated roots.

Proof. The required statement with $|p| \leq 4$ and $|q| \leq 4$ can be directly shown, we may assume $|p| \geq 5$ and $|q| \geq 5$. We first focus on the case $M = S^3_{p/1}(4_1)$. By a computation of $\frac{\mathrm{d}^{\mathrm{n}}}{\mathrm{dx}^{\mathrm{n}}}(Q_M(x))\mid_{x=b}$ with $b = \pm 1, \pm\sqrt{-1}$, we can easily verify the multiplicity of $Q_M(x)$; thus, $Q_M(x)$ is divisible by $\kappa_p(x)$, and $Q_M(x)/\kappa_p(x)$ is not divisible by $x \pm 1$ and x^2+1.

Next, we suppose a repeated root $a \in \mathbb{C}$ of $Q_M(x)$ with $a \neq \pm 1, \pm\sqrt{-1}$. Then, $Q_M(a) = 0$ and $Q'_M(a) = 0$, which are equivalent to

$$1 - a^p(a^{-4} + a^{-2} + 2 + a^2 - a^4) + (a^p)^2 = 0, \tag{6.1}$$

$$(p-4)a^{-4} + (p-2)a^{-2} + 2p + (p+2)a^2 - (p+4)a^4 = -2pa^p. \tag{6.2}$$

Applying (6.2) to (6.1) to kill the term a^p, we equivalently have

$$(1+a)^2(1+a^2)^2\big(p^2-16+(16-2p^2)a^2-(36+p^2)a^4+(16-2p^2)a^6+(p^2-16)a^8\big) = 0.$$

Since $a^2 \neq \pm 1$, the last term as a quartic equation can be solved as

$$a^2 = \frac{p^2 - 8 + 2\eta p\sqrt{p^2-15} + \varepsilon\sqrt{(40-3p^2+2\eta p\sqrt{p^2-15})(p^2-24+2\eta p\sqrt{p^2-15})}}{2p^2-32},$$

for some $\varepsilon, \eta \in \{\pm 1\}$. In particular, for any $k \in \mathbb{N}$, we can obtain a formula of the k-th power a^k as a linear combination of $1, a, a^2, \ldots, a^7$. However, when $k = p+4$, the coefficients in the combination contradict (6.2). In summary, $Q_M(x)/\kappa_p(x)$ has no repeated roots as required.

On the other hand, if $M = S^3_{1/q}(4_1)$, we can easily show that $Q_M(x)$ is divisible not by $(1+x)^3$ but by $(1+x)^2$. Similarly, we suppose a repeated root a of $Q_M(x)$ with $a \neq \pm 1$. Then, $Q_M(a) = Q'_M(a) = 0$. We can easily see $Q_M(1/a) = Q'_M(1/a) = 0$ by reciprocity of Q_M. Thus, we obtain $(x^{-4q}Q_M)'(a) = (x^{-4q}Q_M)'(1/a) = 0$, which are equivalent to

$$2(1+a) = (2q+1)a^{2q} + (-2q+1)a^{-2q} - (4q+1)a^{4q} - (-4q+1)a^{-4q}, \tag{6.3}$$

$$2(1+a^{-1}) = (2q+1)a^{-2q} + (-2q+1)a^{2q} - (4q+1)a^{-4q} - (-4q+1)a^{4q}. \tag{6.4}$$

Since $a^{-4q}Q_M(a) = 0$ is equivalent to

$$2(1+a)2(1+a^{-1}) = 4(a^{4q} - a^{2q} - a^{-2q} + a^{-4q}), \tag{6.5}$$

the substitution of (6.3) and (6.4) into (6.5) gives the equation

$$(1-b)^2(1+b)^2(-1+2b+b^2+2b^3-b^4+16q^2-16bq^2+36b^2q^2-16b^3q^2+16b^4q^2) = 0, \tag{6.6}$$

where we replace a^{2q} by b. If $\omega^{2q} = \pm 1$ and $\omega \in \mathbb{C}$, we can easily check $Q_M(\omega) \neq 0$ by definition. Thus, a^q is a solution of the quartic equation in (6.6) and does not lie in $\mathbb{Q}$, for any $q \in \mathbb{Z}$. Let $F/\mathbb{Q}$ be the field extension by the quartic equation. By definition, F does not contain a and $2 + a + a^{-1}$, which contradicts (6.5) since $|q| > 4$. In summary, $Q_M(x)/(1+x)^2$ has no repeated roots as required. □

Next, we should mention a slight modification of Jacobi's residue theorem:

Lemma 6.1.2 [Wak23, Lemma 3.5]**.** *Fix $\eta \in \{0,1,2\}$ and $\varepsilon \in \{1,2\}$. Suppose a polynomial $k(x) \in \mathbb{Q}[x]$ such that $k(x)$ has no repeated roots and $k(0) \neq 0$. Take another polynomial $g(x) \in \mathbb{Q}[x]$ such that $\deg(g) \leq \deg(k) - \varepsilon\eta - 2$. Then, the following sum is zero:*

$$\sum_{a \in k^{-1}(0)} \frac{(1+a^{\varepsilon})^{\eta} g(a)}{\frac{\mathrm{d}}{\mathrm{d}x}((1+x^{\varepsilon})^{\eta} k(x))|_{x=a}} = 0. \tag{6.7}$$

Proof. If $\eta = 0$, the statement is Jacobi's residue theorem exactly (see, e.g., [TY, Section 6]). Thus, we may suppose $\eta = 2$. Note that the derivative of $(1+x^{\varepsilon})^{\eta} k(x)$ is $\eta\varepsilon(1+x^{\varepsilon})^{\eta-1}k(x) + (1+x^{\varepsilon})^{\eta} k'(x)$. Hence, the left hand side of (6.7) is computed as $\sum_{a \in k^{-1}(0)} g(a)/k'(a)$, which is equal to zero by the residue theorem. □

6.1.2 Proof of Theorem 1.0.2

We suppose $n = -1$ and give the proof of Theorem 1.0.2. Recall the fact that $M = S^3_{p/1}(4_1)$ and $M = S^3_{1/q}(4_1)$ are hyperbolic if and only if $|p| \geq 5$ and $|q| \geq 2$, respectively.

First, we focus on the case where $p \geq 5$ and $M = S^3_{p/1}(4_1)$, and p is not divisible by 4. From the definition of $Q_M(x)$ and Theorem 5.2.2, we can easily verify

$$\frac{1}{\tau^{\mathrm{ad}}_{\varphi_a}(M)} = \frac{2(1-a^2)^3(1+a^2)a^{p-5}}{Q'_M(a)} \qquad \text{for any } a \in \left(Q_M^{-1}(0) \cap D\right) \setminus \{\pm\sqrt{-1}\}. \tag{6.8}$$

If $p-2$ is divisible by 4, we replace $g(x)$ and $k(x)$ by $2(1-x^2)^3(1+x^2)x^{p-5}$ and $Q_M(x)$, respectively. Then, Lemma 6.1.2 with $\eta = 0$ deduces to the required conclusion as

$$0 = \sum_{a \in Q_M^{-1}(0)} \frac{g(a)}{Q'_M(a)} = \sum_{a \in Q_M^{-1}(0)} \frac{1}{\tau^{\mathrm{ad}}_{\varphi_a}(M)} = 2 \sum_{a \in Q_M^{-1}(0) \cap D} \frac{1}{\tau^{\mathrm{ad}}_{\varphi_a}(M)} = \sum_{\varphi \in R_G^{\mathrm{irr}}} \frac{2}{\tau^{\mathrm{ad}}_{\varphi}(M)}. \tag{6.9}$$

Here, the second, third, and fourth equalities immediately follow from (6.8), Remark 5.2.3 (iii), and Proposition 5.2.1, respectively. Meanwhile, when $p-1$ is divisible by 2, we replace $g(x)$ and $k(x)$ by $2(x-1)(x^4-1)x^{p-5}$ and $Q_M(x)/(1+x)^2$, respectively. Then, we can readily show similar equalities to (6.9).

We further discuss the case of $p/4 \in \mathbb{Z}$. By Lemma 6.1.1, $Q_M(x)/(1+x^2)$ lies in $\mathbb{Z}[x]$, and has no double roots. We let $g(x)$ and $k(x)$ be $2(1-x^2)^3x^{p-5}$ and

$Q_M(x)/(1+x^2)$, respectively. By Lemma 6.1.2 with $\eta=1$ and $\varepsilon=2$, we have

$$\begin{aligned}
0 &= \sum_{a\in k^{-1}(0)} \frac{g(a)}{k'(a)} = \frac{g(\sqrt{-1})}{k'(\sqrt{-1})} + \frac{g(-\sqrt{-1})}{k'(-\sqrt{-1})} + \sum_{a\in Q_M^{-1}(0)\cap D\setminus\{\pm\sqrt{-1}\}} \frac{2}{\tau^{\mathrm{ad}}_{\varphi_a}(M)} \\
&= \frac{32\sqrt{-1}}{20-p^2} + \sum_{a\in Q_M^{-1}(0)\cap D\setminus\{\pm\sqrt{-1}\}} \frac{2}{\tau^{\mathrm{ad}}_{\varphi_a}(M)} \\
&= \frac{2}{\tau^{\mathrm{ad}}_{\varphi_{\sqrt{-1}}}(M)} + \frac{2}{\tau^{\mathrm{ad}}_{\varphi_{-\sqrt{-1}}}(M)} + \sum_{a\in Q_M^{-1}(0)\cap D\setminus\{\pm\sqrt{-1}\}} \frac{2}{\tau^{\mathrm{ad}}_{\varphi_a}(M)} \\
&= \sum_{a\in Q_M^{-1}(0)\cap D} \frac{2}{\tau^{\mathrm{ad}}_{\varphi_a}(M)} = \sum_{\varphi\in R_G^{\mathrm{irr}}} \frac{2}{\tau^{\mathrm{ad}}_{\varphi}(M)},
\end{aligned}$$

which is the required vanishing identity. Here, the second, fourth, and sixth equalities follow from (6.8), Theorem 5.2.2, and Proposition 5.2.1, respectively.

Next, we focus on the case of $q\geq 2$ and $M=S^3_{1/q}(4_1)$. Similarly to (6.8), we can show

$$\frac{1}{\tau^{\mathrm{ad}}_{\varphi_a}(M)} = \frac{2(a^{4q}-1)^3(a^{4q}-(a^2+a+1)a^{2q-1}+1)}{\frac{\mathrm{d}}{\mathrm{d}x}(x^{4q+1}Q_M(x))|_{x=a}} \qquad \text{for any } a\in Q_M^{-1}(0)\cap D. \tag{6.10}$$

By a Euclidean Algorithm, we can choose a polynomial $h(x)\in\mathbb{Q}[x]$ such that

$$2(x^{4q}-1)^3(x^{4q}-(x^2+x+1)x^{2q-1}+1) \equiv x^{4q+1}h(x) \qquad (\text{modulo } Q_M(x)),$$

and $\deg h(x) < 8q-2$. Recall from Lemma 6.1.1 that $Q_M(x)$ is divisible by $(1+x)^2$; thus so is $h(x)$. In summary, we can define polynomials $g(x)$ and $k(x)$ to be $h(x)/(1+x)^2$ and $Q_M(x)/(1+x)^2$, respectively. Then, Lemma 6.1.2 with $\eta=2$ and $\varepsilon=1$ readily leads to the same equalities as (6.9).

The proof of the cases of $p\leq -5$ and $q\leq -2$ can be shown in the same manner; so we here do not carry out the detailed proof.

Finally, in the remaining cases of $|p|\leq 4$ for $M=S^3_{p/1}(4_1)$, we can obtain the following by a direct calculation:

$$\sum_{\varphi\in R_G^{\mathrm{irr}}(M)} \frac{1}{\tau^{\mathrm{ad}}_{\varphi}(M)} = \begin{cases} 2, & \text{if } p\in\{0,\pm1,\pm2,\pm3\},\\ 8, & \text{if } p\in\{\pm4\}.\end{cases}$$

For example, we now discuss the detail in the case $p=4$ for $M=S^3_{p/1}(4_1)$. The roots of $Q_M(x)=x^2+2x^4+x^6=0$ are $x=\pm\sqrt{-1}$. By Theorem 5.2.2, we have $\tau^{\mathrm{ad}}_{\varphi_{\sqrt{-1}}}(M)=(5-2\sqrt{5})/4$ and $\tau_{\varphi^{\mathrm{ad}}_{-\sqrt{-1}}}(M)=(5+2\sqrt{5})/4$, leading to $\sum_{\varphi\in R_G^{\mathrm{irr}}(M)}\tau^{\mathrm{ad}}_{\varphi}(M)^{-1}=8$. Similarly, the computations in the other cases run well.

Remark 6.1.3. When we replace the figure-eight knot with 5_2-knot, and letting $M = S^3_{1/q}(5_2)$, we can show that Theorem 1.0.2 is true for any integers $q \neq 0$. The proof can be shown in the same fashion as Section 6.1.2. However, the concrete substitutions of $g(x)$ and $k(x)$ into Lemma 6.1.2 are slightly complicated. For this reason, we do not go into detailed proof in this paper.

Incidentally, we give comments on the case $M = S^3_{p/1}(5_2)$ with $p \in \mathbb{Z}$. With the help of a computer program, we can similarly obtain the polynomial $Q_M(x)$ and determine the associated torsions $\tau^{\mathrm{ad}}_\varphi(M)$. However, the resulting computation of $\tau^{\mathrm{ad}}_\varphi(M)$ is more intricate; we do not describe the details. More generally, to show Conjecture 1.0.1 with $M = S^3_{p/q}(K)$ for other (twist) knots K, we might need other ideas. This is a subject for future analysis.

6.2 The conjecture with $n > 0$

We end this paper by discussing Conjecture 1.0.1 with $n > 0$. Hereafter, we assume that $R^{\mathrm{irr}}_G(M)$ is of finite order for $G = \mathrm{SL}_2(\mathbb{C})$ and a 3-manifold M as above. We abbreviate $\tau^{\mathrm{ad}}_\varphi(M)$ as τ_φ.

First, it is almost obvious that the sum (1.1) is a real number: precisely,

Proposition 6.2.1 [Wak23, Proposition 5.1]**.** *Let $n \in \mathbb{Z}$. The imaginary part of the sum $\sum_{\varphi \in R^{\mathrm{irr}}_G(M)} {\tau_\varphi}^n$ is zero.*

Proof. For a homomorphism $\varphi : \pi_1(M) \to G$, we denote by $\bar{\varphi}$ the conjugate representation. Then, $\tau_{\bar{\varphi}} = \overline{\tau_\varphi}$ by definition. Since we can select representatives $\varphi_1, \ldots, \varphi_m, \overline{\varphi_1}, \ldots, \overline{\varphi_m}, \eta_1, \ldots, \eta_n$ of $\mathrm{R}^{\mathrm{irr}}_G(M)$ such that $[\eta_i] = [\overline{\eta_i}] \in \mathrm{R}^{\mathrm{irr}}_G(M)$, the imaginary part is zero as required. $\square$

Furthermore, we will discuss the rationality of the sum (1.1), with $G = \mathrm{SL}_2(\mathbb{C})$. For a subfield $F \subset \mathbb{C}$, let $\mathrm{R}^{\mathrm{irr}}_{\mathrm{SL}_2(F)}(M)$ be the set of the conjugacy classes of all irreducible representations $\pi_1(M) \to \mathrm{SL}_2(F)$.

Proposition 6.2.2 [Wak23, Proposition 5.2]**.** *Let $F/\mathbb{Q}$ be a Galois extension with embedding $F \hookrightarrow \mathbb{C}$. Suppose that the inclusion $\mathrm{R}^{\mathrm{irr}}_{\mathrm{SL}_2(F)}(M) \subset \mathrm{R}^{\mathrm{irr}}_{\mathrm{SL}_2(\mathbb{C})}(M)$ is bijective as a finite set, and is closed under the Galois action of* $\mathrm{Gal}(F/\mathbb{Q})$*. Then, for any $n \in \mathbb{Z}$, the sum $\sum_{\varphi \in R^{\mathrm{irr}}_G(M)} \tau^n_\varphi$ is a rational number.*

Proof. By definition, $\tau_\varphi \in F^\times$, and the map $\tau_\bullet : \mathrm{R}^{\mathrm{irr}}_{\mathrm{SL}_2(F)}(M) \to F^\times$ is $\mathrm{Gal}(F/\mathbb{Q})$-equivariant. Thus, the sum lies in the invariant part $F^{\mathrm{Gal}(F/\mathbb{Q})}$. Hence, by $F^{\mathrm{Gal}(F/\mathbb{Q})} = \mathbb{Q}$, the sum (1.1) lies in $\mathbb{Q}$ as desired. $\square$

Corollary 6.2.3 [Wak23, Corollary 5.3]**.** *Suppose that p is even, and is relatively prime to q. Let K be a twist knot, and M be $S^3_{p/q}(K)$. Then, for any $n \in \mathbb{Z}$, the sum $\sum_{\varphi \in R^{\mathrm{irr}}_G(M)} \tau^n_\varphi$ is a rational number.*

Proof. As is shown in [Nos22b, Section 2], there is a Galois extension $F/\mathbb{Q}$ satisfying the condition in Proposition 6.2.2. $\square$

Meanwhile, the integrality of the sum (1.1) with $M = S^3_{p/q}(K)$ remains a future problem. When K is either 4_1- or 5_2-knot, we know the resulting computation of $\tau^{\mathrm{ad}}_\varphi(M)$ by Theorem 5.2.2. Accordingly, it is not so hard to check numerically the conjecture from the computation of $\tau^{\mathrm{ad}}_\varphi(M)$ for some small p, q.

However, we give the proof of the conjecture multiplied by 2^{2n+1} with $M = S^3_{2m/1}(4_1)$. Precisely,

Proposition 6.2.4 [Wak23, Proposition 5.4]**.** *As in Theorem 1.0.2, let $M = S^3_{2m/1}(4_1)$. If $n > 1$, then the 8-fold sum $2\sum_{\varphi \in R^{\mathrm{irr}}_G(M)} (8\tau_\varphi)^n$ is an integer.*

Proof. Since the proof with $|2m| \leq 4$ is a direct computation, we may suppose $|2m| > 4$. We can easily verify integral coefficient polynomials $h(x), k(x) \in \mathbb{Z}[x]$ such that

$$\frac{Q_M(x)}{(1-x^2)^3} = h(x) + \frac{m^2 - 2m + 3}{1 - x^2} + \frac{4m}{(1-x^2)^2} + \frac{4}{(1-x^2)^3},$$

$$\frac{Q_M(x)}{1+x^2} = k(x) + \frac{2(1 + (-1)^{m-1})}{1 + x^2}.$$

By Girard–Newton formula, the sums $\sum_{\alpha \in Q_M^{-1}(0) \cap D} h(\alpha)^n$ and $\sum_{\alpha \in Q_M^{-1}(0)} k(\alpha)^n$ are integers; thus, $\sum_{\alpha \in Q_M^{-1}(0) \cap D} 8^n (1 + \alpha^2)^{-n}$ and $\sum_{\alpha \in Q_M^{-1}(0) \cap D} 8^n (1 + \alpha^2)^{-3n+\varepsilon}$ are integers, where $\varepsilon \in \{1, 2, 3\}$. Recall from Theorem 5.2.2 the value of the torsion τ_{φ_a} for $a \in Q_M^{-1}(0) \cap D$; by the Euclidean Algorithm, we can show

$$2\tau_{\varphi_a} = \ell(a) + \frac{6 + 2m - 2m^2 - m^3}{1 - a^2} + \frac{-6 + 6m + 2m^2}{(1-a^2)^2} + \frac{-4m}{(1-a^2)^3} + \frac{m(-1 + (-1)^m)}{2(1 + a^2)} \tag{6.11}$$

for some $\ell(a) \in \mathbb{Z}[a]$. Thus, the sum $2\sum_{\varphi \in R^{\mathrm{irr}}_G(M)} (8\tau_\varphi)^n$ is equal to $\sum_{a \in Q_M^{-1}(0)} (8\tau_{\varphi_a})^n$ by Proposition 5.2.1, and is a sum of the above sums $\sum_{\alpha \in Q_M^{-1}(0) \cap D} 8^n (1 \pm \alpha^2)^{-3n+\varepsilon}$. Thus, it is an integer as required. $\square$

Similarly, we can show the same claim in the cases of $M = S^3_{(2m+1)/1}(4_1)$ and $M = S^3_{1/q}(4_1)$; however, it seems not easy to reduce the 2-torsion in the sum and to replace $(8\tau_\varphi)^n$ by $(2\tau_\varphi)^n$.

Appendix A

Construction of identities from a 3-manifold

We explain the identity referred to in Theorem 3.2.2 in detail. There is nothing new in this section since the discussion essentially follows from [Sie80, Sie86, Tro62].

As in Section 3.1, let us fix the group presentation $\langle x_1, \ldots, x_g \mid r_1, \ldots, r_g \rangle$ of $G = \pi_1(M)$, where M is a closed 3-manifold. Then, we can naturally consider the 2-dimensional CW complex X_G such that $\pi_1(X_G) \cong G$ with a single 0-cell, g 1-cells, and g 2-cells.

Next, we now construct $q_s : S^2 \to X_G$ from the identity s as follows. Let us assume that $s = \prod_{m=1}^n w_m \rho_{j_m}^{\epsilon_m} w_m^{-1} \in P * F$ and that ρ_{j_m}, w_m can be written in

$$\rho_{j_m} = x_{m,1}^{\epsilon_{m,1}} \cdots x_{m,\ell_m}^{\epsilon_{m,\ell_m}}, \qquad w_m = x_{m,1}^{\eta_{m,1}} \cdots x_{m,k_m}^{\eta_{m,k_m}}, \qquad (\epsilon_{i,j}, \eta_{i,j} \in \{\pm 1\}).$$

For each $w_m \rho_{j_m}^{\epsilon_m} w_m^{-1}$, take the labeled ℓ_m-gon D_{j_m}, and prepare the segment $I_m = [0, k_m]$ of length k_m. Divide $I_m = [0, k_m]$ into k_m intervals of length 1, and label the intervals $x_{m,1}^{\epsilon_{m,1}}, \ldots, x_{m,k_m}^{\epsilon_{m,k_m}}$ in order (and orient each of the intervals according to $\epsilon_{m,i}$). In addition, attach $\{k_m\} \in I_m$ to the first vertex of D_{j_m}, and take the one-point union of $I_1, \ldots, I_n$ joining at $\{0\}$. Suppose that $w'_m : I_m \to X_G^{(1)}$ is a cellular map which realizes the word w_m where $X_G^{(1)}$ is the 1-skeleton of X_G. By construction of X_G, note that the relator r_{j_m} means the attaching map $y_m : D_{j_m} \to X_G$. These maps define the cellular map

$$p_s : \cup_{m=1}^n \big(I_m \cup D_{j_m}\big) \longrightarrow X_G \tag{A.1}$$

(see Fig. A.1). Since $\cup_{m=1}^n \big(I_m \cup D_{j_m}\big)$ is homotopy equivalent to a disk D^2 (by taking a tubular neighborhood of I_j), it can be regarded as the continuous pair map $p_s : (D^2, S^1) \to (X_G, X_G^{(1)})$. Here, let us consider the long exact sequence of relative homotopy groups:

$$0 = \pi_2(X_G^{(1)}) \longrightarrow \pi_2(X_G) \xrightarrow{\text{proj}} \pi_2(X_G, X_G^{(1)}) \xrightarrow{\partial} \pi_1(X_G^{(1)}) \longrightarrow \pi_1(X_G).$$

By the definition of the identity, $\partial([p_s])$ is equal to zero. Therefore, by the exactness, p_s is uniquely lifted to $q_s : S^2 \to X_G$ up to homotopy, and this is the required q_s.

Let us consider the opposite discussion in a sense. For each m, prepare the labeled ℓ_m-gon disk D_{j_m} and the segment $I_m = [0, k_m]$ of length k_m which is divided into k_m intervals of length 1. To recover the cellular map (A.1), let us introduce some concepts.

(1) A self-bijection

$$\mathcal{I} : \cup_{m=1}^n \{(m,1), \ldots, (m, \ell_m)\} \to \cup_{m=1}^n \{(m,1), \ldots, (m, \ell_m)\}$$

is called a *syllable* if $x_{\mathcal{I}(i,j)} = x_{i,j} \in F$ and $\epsilon_{i,j} = -\epsilon_{\mathcal{I}(i,j)} \in \{\pm 1\}$.

(2) For a syllable $\mathcal{I}$, consider the following equivalence on the disjoint union $\sqcup_{i=1}^n D_{r_i}$: that is, we identify the intervals with labeling $x_{i,j}$ with the intervals with labeling $x_{\mathcal{I}(i,j)}$.

(3) A cellular map (A.1) is called *taut* if there is a syllable $\mathcal{I}$ such that the quotient space $\sqcup_{i=1}^n D_{r_i}/\sim$ of $\sqcup_{i=1}^n D_{r_i}$ under the above equivalence $\sim$ is homeomorphic to S^2, and if there are continuous maps

$$\lambda_m : [0, k_m] \to \sqcup_{i=1}^n \partial D_{r_i}/\sim \qquad , \qquad \kappa_m : [0, \ell_m] \to \partial D_{r_m}/\sim$$

satisfying the following condition (*).

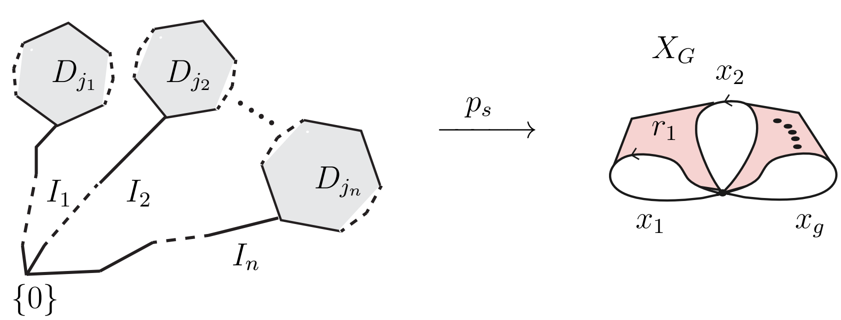

Figure A.1: The map p_s

(*) For each m, the image $\lambda_m([i-1,i])$ of the interval $[k_i-1,k_i]$ coincides with an interval with labeling $x_{m,i}$ compatible with the orientations, and $\kappa_m([j-1,j])$ coincides with the j-th side of D_{r_m} compatible with the orientations. Furthermore, $\lambda_m(k_m)=\lambda_m(0)=\kappa_m(\ell_m)$ is satisfied.

Conversely, for a given taut cellular map (A.1), it is easy to construct the identity s.

Now, we explain the construction of the identity W_M. By Morse theory, X_G is homotopy equivalent to the CW complex that is given by removing a single 3-cell e^3 from M. The cellular structure of $\partial(M\setminus e^3)$ is given by 1- and 2-handles of X_G, and it can be confirmed that the attaching map $\tau: S^2\to M\setminus e^3$ is realized as a taut cellular map. Consequently, the identity W_M associated with τ is defined.

Proof of Theorem 3.2.2. For an identity s, we can construct the map q_s as above. Regarding q_s as an attaching map of D^3, let Y be $X_G\cup_{q_s} D^3$. According to [Tro62, Section 2.4], it is known that the boundary map $\partial_3: C_3(\widetilde{Y})\to C_2(\widetilde{Y})$ of the cellular chain complex may be written as $\partial_3(a)=\sum_j a\mu\left([\psi\frac{\partial s}{\partial\rho_j}]\right)b_j$. Thus, if we substitute W_M for s, we obtain the required theorem. $\square$

In addition, by the construction of W_M, it is not difficult to prove the following proposition, which plays a key role in the proofs in Section 3.3.

Proposition A.0.1. *(1) If we reverse the orientation of β_i, then W_M is transposed to another identity W'_M which is obtained by replacing the factors $(r_i,w)^\epsilon$ of W_M with $(r_i^{-1},w)^{-\epsilon}$.*

(2) If we apply a handleslide which replaces the relator r_i with r_jr_i, then W_M is transposed to another identity W'_M which is obtained by replacing the factors $(r_i,w)^\epsilon$ of W_M with $(r_j,w)^{-\epsilon}(r_jr_i,w)^\epsilon$.

(3) If we apply a stabilization which adds the generator x_{g+1} and the relator r_{g+1}, then W_M is transposed to $W_M(r_{g+1},x_{g+1})(r_{g+1},1)^{-1}$.

References

[AC65] J. J. Andrews and M. L. Curtis. "Free groups and handlebodies". *Proc. Amer. Math. Soc.* 16 (1965), pp. 192–195.

[Bag21] N. Bagherifard. *Three-manifolds with boundary and the Andrews-Curtis transformations*. 2021. arXiv: `2109.13844 [math.GT]`.

[BGPZ20] F. Benini, D. Gang, and L. A. Pando Zayas. "Rotating black hole entropy from M5-branes". *J. High Energy Phys.* 3 (2020), pp. 057, 39.

[DHY09] J. Dubois, V. Huynh, and Y. Yamaguchi. "Non-abelian Reidemeister torsion for twist knots". *J. Knot Theory Ramifications* 18.3 (2009), pp. 303–341.

[Fox53] R. H. Fox. "Free differential calculus. I. Derivation in the free group ring". *Ann. of Math. (2)* 57 (1953), pp. 547–560.

[GKPZ20] D. Gang, N. Kim, and L. A. Pando Zayas. "Precision microstate counting for the entropy of wrapped M5-branes". *J. High Energy Phys.* 3 (2020), pp. 164, 42.

[GKY21] D. Gang, S. Kim, and S. Yoon. "Adjoint Reidemeister torsions from wrapped M5-branes". *Adv. Theor. Math. Phys.* 25.7 (2021), pp. 1819–1845.

[HAMS93] C. Hog-Angeloni, W. Metzler, and A. J. Sieradski, eds. *Two-dimensional homotopy and combinatorial group theory*. Vol. 197. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1993, pp. xii+412.

[Joh] D. Johnson. "A geometric form of Casson's invariant and its connection to Reidemeister torsion". unpublished lecture notes.

[Kho07] V. T. Khoi. "On the $\mathrm{SU}(2,1)$ representation space of the Brieskorn homology spheres". *J. Math. Sci. Univ. Tokyo* 14.4 (2007), pp. 499–510.

[KL99] P. Kirk and C. Livingston. "Twisted Alexander invariants, Reidemeister torsion, and Casson-Gordon invariants". *Topology* 38.3 (1999), pp. 635–661.

[Kit94] T. Kitano. "Reidemeister torsion of Seifert fibered spaces for $\mathrm{SL}(2;\mathbf{C})$-representations". *Tokyo J. Math.* 17.1 (1994), pp. 59–75.

[Kit15a] T. Kitano. *Reidemeister torsion of a 3-manifold obtained by a Dehn-surgery along the figure-eight knot.* arXiv:1506.00712. 2015.

[KN20] T. Kitano and Y. Nozaki. "Finiteness of the image of the Reidemeister torsion of a splice". *Ann. Math. Blaise Pascal* 27.1 (2020), pp. 19–36.

[Kit15b] T. Kitayama. "Normalization of twisted Alexander invariants". *Internat. J. Math.* 26.10 (2015), pp. 1550077, 21.

[Kod07] Y. Koda. "Spines, Heegaard splittings and the Reidemeister-Turaev torsion". *Tokyo J. Math.* 30.2 (2007), pp. 417–439.

[Lyn50] R. C. Lyndon. "Cohomology theory of groups with a single defining relation". *Ann. of Math. (2)* 52 (1950), pp. 650–665.

[MFP14] P. Menal-Ferrer and J. Porti. "Higher-dimensional Reidemeister torsion invariants for cusped hyperbolic 3-manifolds". *J. Topol.* 7.1 (2014), pp. 69–119.

[Mil66] J. Milnor. "Whitehead torsion". *Bull. Amer. Math. Soc.* 72 (1966), pp. 358–426.

[Mil62] J. Milnor. "A duality theorem for Reidemeister torsion". *Ann. of Math. (2)* 76 (1962), pp. 137–147.

[Nic03] L. I. Nicolaescu. *The Reidemeister torsion of 3-manifolds.* Vol. 30. De Gruyter Studies in Mathematics. Walter de Gruyter & Co., Berlin, 2003, pp. xiv+249.

[Nos22a] T. Nosaka. "Cellular chain complexes of universal covers of some 3-manifolds". *J. Math. Sci. Univ. Tokyo* 29.1 (2022), pp. 89–113.

[Nos22b] T. Nosaka. *Reciprocity of the Chern-Simons invariants of 3-manifolds.* arXiv:2212.07569. 2022.

[OT19] T. Ohtsuki and T. Takata. "On the quantum SU(2) invariant at $q = \exp(4\pi\sqrt{-1}/N)$ and the twisted Reidemeister torsion for some closed 3-manifolds". *Comm. Math. Phys.* 370.1 (2019), pp. 151–204.

[OS04] P. Ozsváth and Z. Szabó. "Holomorphic disks and topological invariants for closed three-manifolds". *Ann. of Math. (2)* 159.3 (2004), pp. 1027–1158.

[Por97] J. Porti. "Torsion de Reidemeister pour les variétés hyperboliques". *Mem. Amer. Math. Soc.* 128.612 (1997), pp. x+139.

[PY] J. Porti and S. Yoon. *The adjoint Reidemeister torsion for the connected sum of knots.* arXiv:2104.08150.

[Rei33] K. Reidemeister. "Zur dreidimensionalen Topologie". *Abh. Math. Sem. Univ. Hamburg* 9.1 (1933), pp. 189–194.

[Rei35] K. Reidemeister. "Homotopieringe und Linsenräume". *Abh. Math. Sem. Univ. Hamburg* 11.1 (1935), pp. 102–109.

[Sch02] M. Scharlemann. "Heegaard splittings of compact 3-manifolds". In: *Handbook of geometric topology.* North-Holland, Amsterdam, 2002, pp. 921–953.

[Sco83] P. Scott. "There are no fake Seifert fibre spaces with infinite π_1". *Ann. of Math. (2)* 117.1 (1983), pp. 35–70.

[Sie80] A. J. Sieradski. "Framed links for Peiffer identities". *Math. Z.* 175.2 (1980), pp. 125–137.

[Sie86] A. J. Sieradski. "Combinatorial squashings, 3-manifolds, and the third homology of groups". *Invent. Math.* 84.1 (1986), pp. 121–139.

[TY] A. T. Tran and Y. Yamaguchi. *Adjoint Reidemeister torsions of once-punctured torus bundles.* arXiv:2109.07058.

[Tro62] H. F. Trotter. "Homology of group systems with applications to knot theory". *Ann. of Math. (2)* 76 (1962), pp. 464–498.

[Tur01] V. Turaev. *Introduction to combinatorial torsions.* Lectures in Mathematics ETH Zürich. Notes taken by Felix Schlenk. Birkhäuser Verlag, Basel, 2001, pp. viii+123.

[Wak21] N. Wakijo. "Twisted Reidemeister torsions via Heegaard splittings". *Topology Appl.* 299 (2021), Paper No. 107731, 22.

[Wak23] N. Wakijo. "Adjoint Reidemeister torsions of some 3-manifolds obtained by Dehn surgeries". preprint, arXiv:2305.08402. 2023.

[Wit91] E. Witten. "On quantum gauge theories in two dimensions". *Comm. Math. Phys.* 141.1 (1991), pp. 153–209.

[Yam17] Y. Yamaguchi. "A surgery formula for the asymptotics of the higher-dimensional Reidemeister torsion and Seifert fibered spaces". *Indiana Univ. Math. J.* 66.2 (2017), pp. 463–493.

[Yoo22] S. Yoon. "Adjoint Reidemeister torsions of two-bridge knots". *Proc. Amer. Math. Soc.* 150.10 (2022), pp. 4543–4556.

[Yoo23] S. Yoon. private communication and unpublished notes. 2023.