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On successive minimal bases of Drinfeld modules and their applications to the ramification of torsion points

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Abstract

Let p be a prime number and q be some power of p. Let K be the localization of some function field at a prime v. We define successive minimal bases (SMBs) for the free $\mathbb{F}_q[t]/u^n$-module $\phi[u^n]$ of u^n-division points of a Drinfeld $\mathbb{F}_q[t]$-module ϕ over K, where u is a monic irreducible element of $\mathbb{F}_q[t]$ and n is a positive integer. These SMBs share similar properties to those of SMBs of the lattices associated to Drinfeld modules via the (Tate) uniformizations. Especially, the valuations of the elements of an SMB are independent of the choice of the SMBs, i.e., the valuations are invariants of $\phi[u^n]$. If v is infinite, then an exponential map e_ϕ and a lattice Λ are associated to the Drinfeld module ϕ via the uniformization. For an SMB $\{\omega_i\}_{i=1,\ldots,r}$ of Λ, we show that $\{e_\phi(\omega_i/u^n)\}_{i=1,\ldots,r}$ is an SMB of $\phi[u^n]$. Conversely, for an SMB $\{\lambda_i\}_{i=1,\ldots,r}$ of $\phi[u^n]$ and a large enough n, we show that $\{u^n \log_\phi(\lambda_i)\}_{i=1,\ldots,r}$ is an SMB of Λ, where $\log_\phi(\lambda_i)$ is the only preimage of λ_i under e_ϕ with minimal valuation. When v is finite and ϕ has stable and bad reduction, we show similar results.

On the practical side, we restrict ourselves to the case where ϕ is a rank r Drinfeld $\mathbb{F}_q[t]$-module over K such that $\phi_t(X) = tX + a_s X^{q^s} + a_r X^{q^r} \in K[X]$, where s is a positive integer $< r$. Assume that u is not divisible by v. We first calculate the valuations of elements of SMBs of $\phi[u^n]$ for all positive integer n. When $s = 1$ and $\deg(u) = 1$, under certain assumptions, we obtain the Herbrand ψ-funtion of $K(\phi[u])/K$ and the action of the wild ramification subgroup of the Galois group $G(K(\phi[u])/K)$ on an SMB of $\phi[u]$. Next, we assume $r = 2$ and allow u to have an arbitrary degree. Under certain assumptions, we obtain the Herbrand ψ-function of $K(\phi[u^n])/K$ and the action of the wild ramification subgroup of $G(K(\phi[u^n])/K)$ on an SMB of $\phi[u^n]$.

For a rank r Drinfeld $\mathbb{F}_q[t]$-module ϕ over a function field F such that $\phi_t(X) = tX + a_s X^{q^s} + a_r X^{q^r} \in F[X]$, we show a formula involving the J-height and the differential height of ϕ. Finally, we define and calculate the conductor of ϕ at each prime of F and show a function field analogue of Szpiro's conjecture for rank 2 Drinfeld $\mathbb{F}_q[t]$-modules over F under a certain limited situation.

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Contents

Introduction

1. Notation

Let us introduce the notation used throughout this paper. Let $A := \mathbb{F}_q[t]$ be the polynomial ring in t over the field $\mathbb{F}_q$ whose order is a power of a rational prime p. Let F be a global function field which is a finite extension of the fraction field of A. An infinite prime of F is a prime of F lying above the prime $(1/t)$ of $\mathbb{F}_q(t)$. A finite prime of F is a prime that is not infinite. Let K be the completion of F at a prime v. We also let v denote the valuation associated to K normalized so that $v(K^\times) = \mathbb{Z}$. Fix K^{sep} (resp. K^{alg}) a separable (resp. algebraic) closure of K. For a Galois extension L of K within K^{sep}/K, let $G(L/K)$ denote the Galois group. Let $\mathbb{C}_v$ denote the completion of K^{alg}. If v is an infinite prime, we also let $\mathbb{C}_\infty$ denote $\mathbb{C}_v$.

Let ϕ be a rank r Drinfeld A-module over K. For an element a in A, let $\phi[a]$ be the A/a-module of a-division points in K^{sep}. It is a A/a-free module of rank r. Fix a finite prime u of A, i.e., u is a monic irreducible polynomial in A and a positive integer n. The main research objects in this paper are successive minimal bases of $\phi[u^n]$ defined below. For $a \in A$ and $x \in \phi[u^n]$, write $a \cdot_\phi x := \phi_a(x)$ for the action of a on x. Let K_n denote the extension of K generated by elements in $\phi[u^n]$.

Let $|-|$ denote one of the following functions.

(F1) If v is an infinite prime, we have the absolute value $|-|$ on K which extends the absolute value $|-| = q^{\deg(-)}$ on $\mathbb{F}_q((\frac{1}{t}))$. This absolute value may be extended to $\mathbb{C}_\infty$.

(F2) Assume that v is a finite prime of F and ϕ has stable reduction over K. If the reduction of ϕ has rank r', following [**Gar02**, Section 1], define a function $|-|$ on K by

$$\text{for } x \in K,\ |x| = \begin{cases} (-v(x))^{1/r'} & v(x) < 0, \\ -v(x)^{1/r'} & v(x) \geq 0, \\ |0| = -\infty & x = 0. \end{cases}$$

We may extend this function to $\mathbb{C}_v$. This function is not an absolute value or a norm on K. However, the ultrametric inequality holds. We still call $|x|$ the norm of x.

The main definition is

Definition 1.1. Let $|-|$ denote the function in (F1) or (F2). We call a family of elements $\{\lambda_i\}_{i=1,\ldots,r}$ an *SMB* (*successive minimal basis*) of $\phi[u^n]$ if for each i, the elements $\lambda_1, \ldots, \lambda_i$ in $\phi[u^n]$ satisfy

(1) $\lambda_1, \ldots, \lambda_i$ are A/u^n-linearly independent;
(2) $|\lambda_i|$ is minimal among the values $|\lambda|$ of elements λ in $\phi[u^n]$ such that $\lambda_1, \ldots, \lambda_{i-1}, \lambda$ are A/u^n-linearly independent.

In the definition of SMBs of $\phi[u^n]$, we have imitated the definition of SMBs of the lattices Λ (defined below) (See [**Tag92**, Section 4] or [**Gek19A**, Section 3]). Note that (1) in the definition implies that $\{\lambda_1, \ldots, \lambda_r\}$ is an A/u^n-basis (or a generating set) of $\phi[u^n]$.

It turns out that an SMB of $\phi[u^n]$ has the following properties.

Proposition 1.2. *Let $\{\lambda_i\}_{i=1,\ldots,r}$ be an SMB of $\phi[u^n]$.*

(1) (Proposition 1.1.8) *The sequence $|\lambda_1| \leq |\lambda_2| \leq \cdots \leq |\lambda_r|$ associated to an SMB of $\phi[u^n]$ is an invariant of $\phi[u^n]$, i.e., for any SMB $\{\lambda'_i\}_{i=1,\ldots,r}$ of $\phi[u^n]$, we have $|\lambda'_i| = |\lambda_i|$ for all i.*
(2) (Proposition 1.2.13 and 1.3.16) *Assume that u is not divisible by the prime v, i.e., $v(u) \leq 0$. Then we have*
$$\left|\sum_i a_i \cdot_\phi \lambda_i\right| = \max_i\{|a_i \cdot_\phi \lambda_i|\}$$
for any $a_i \in A \mod u^n$.
(3) (Proposition 1.1.11) *There exists an SMB $\{\lambda'_i\}_{i=1,\ldots,r}$ of $\phi[u^{n+1}]$ such that $u \cdot_\phi \lambda'_i = \lambda_i$ for all i. The elements $u \cdot_\phi \lambda_i$ for $i = 1, \ldots, r$ form an SMB of $\phi[u^{n-1}]$.*

Here the properties (1) and (2) are similar to those of SMBs of lattices (See Proposition 1.1.6 and 1.1.5). We remark that (2) essentially follows from similar properties of SMBs of lattices (See Proposition 1.1.5 or [**Tag92**, Lemma 4.2]). We hope to know whether the condition "$v(u) \leq 0$" in (2) can be removed.

2. Relations between SMBs of $\phi[u^n]$ and those of lattices

If v is an infinite prime, let Λ denote the rank r A-lattice in $\mathbb{C}_\infty$ and e_ϕ the exponential function from $\mathbb{C}_\infty$ to $\mathbb{C}_\infty$ associated to ϕ via the uniformization. Here we consider Λ and the domain of e_ϕ as A-modules via the natural embedding $A \to \mathbb{C}_\infty$.

If v is a finite prime, we assume throughout this subsection that ϕ has stable reduction and the reduction of ϕ has rank $r' \leq r$. Let ψ denote the rank r' Drinfeld module over K having good reduction, Λ the rank $r - r'$ A-lattice in $\mathbb{C}_v$, and e_ϕ the exponential function from $\mathbb{C}_v$ to $\mathbb{C}_v$ associated to ϕ via the Tate uniformization (See [**Dri74**, Section 7] or

Section 1.1). Here we consider Λ and the domain of e_ϕ as A-modules via ψ, i.e., we have the action of a on ω to be $a \cdot_\psi \omega := \psi_a(\omega)$ for any $a \in A$ and any ω in Λ or $\mathbb{C}_v$.

If v is a finite prime, let $u^{-n}\Lambda$ denote the A-module consisting of all roots of $\psi_{u^n}(X)-\omega$ for all $\omega \in \Lambda$. For any infinite or finite prime w, by the uniformization or the Tate uniformization of ϕ, we have an isomorphism of A/u^n-modules

$$\mathcal{E}_\phi : u^{-n}\Lambda/\Lambda \to \phi[u^n]$$

induced by e_ϕ. Hence one may expect that there are relations between SMBs of $\phi[u^n]$ and those of Λ.

Let $|-|$ denote the absolute value in (F1) (resp. the function in (F2)) if w is an infinite prime (resp. a finite prime). Put $|u^n|_\infty = q^{\deg(u^n)}$.

Theorem 2.1. (1) *Let w be an infinite prime.*

- (Theorem 1.2.3) *Let $\{\omega_i\}_{i=1,\dots,r}$ be an SMB of Λ. Then the images $e_\phi(\omega_i/u^n)$ for $i=1,\dots,r$ form an SMB of $\phi[u^n]$.*
- (Corollary 1.2.12 (1)) *Let l be a positive integer and $\{\eta_i\}_{i=1,\dots,r}$ an SMB of $\phi[u^l]$. Let $\{\lambda_i\}_{i=1,\dots,r}$ be an SMB of $\phi[u^n]$. Assume that n satisfies $|u^n|_\infty > |\eta_r|/|\eta_1|$. Under this assumption, for each $i=1,\dots,r$, the element λ_i has only one preimage under e_ϕ, denoted $\log_\phi(\lambda_i)$, with absolute value $< |\omega|$ for any $\omega \in \Lambda \setminus \{0\}$. Then the family of elements $\{u^n \log_\phi(\lambda_i)\}_{i=1,\dots,r} \subset \mathbb{C}_\infty$ is an SMB of Λ.*

(2) *Let w be a finite prime.*

- (Theorem 1.3.7) *Let $\{\omega_i\}_{i=1,\dots,r'}$ (resp. $\{\omega_i^0\}_{i=r'+1,\dots,r}$) be an SMB of $\psi[u^n]$ (resp. Λ). Let ω_i be a root of $\psi_{u^n}(X)-\omega_i^0$ for $i=r'+1,\dots,r$. Then the images $e_\phi(\omega_i)$ for $i=1,\dots,r$ form an SMB of $\phi[u^n]$.*
- (Corollary 1.3.9 (1) and (2)) *Let l be a positive integer and $\{\eta_i\}_{i=1,\dots,r}$ an SMB of $\phi[u^l]$. Let $\{\lambda_i\}_{i=1,\dots,r}$ be an SMB of $\phi[u^n]$. Assume that n satisfies $|u^n|_\infty > |\eta_r|/|\eta_{r'+1}|$. Under this assumption, for each $i=1,\dots,r$, the element λ_i has only one preimage under e_ϕ, denoted $\log_\phi(\lambda_i)$, with absolute value $< |\omega|$ for any $\omega \in \Lambda \setminus \{0\}$. Then the family of elements $\{\log_\phi(\lambda_i)\}_{i=1,\dots,r'} \subset \mathbb{C}_v$ (resp. $\{u^n \cdot_\psi \log_\phi(\lambda_i)\}_{i=r'+1,\dots,r} \subset \mathbb{C}_v$) is an SMB of $\psi[u^n]$ (resp. of Λ).*

Let $K(\Lambda)$ (resp. $K(u^{-n}\Lambda)$ and K_n) denote the extension of K generated by all elements in Λ (resp. $u^{-n}\Lambda$ and $\phi[u^n]$). By Theorem 2.1, we are able to show

Proposition 2.2. *Let l be a positive integer and $\{\eta_i\}_{i=1,\dots,r}$ an SMB of $\phi[u^l]$. Let $\{\lambda_i\}_{i=1,\dots,r}$ be an SMB of $\phi[u^n]$.*

(1) (Corollary 1.2.12 (2)) *If w is an infinite prime and n is large enough so that $|u^n|_\infty > |\eta_r|/|\eta_1|$, then we have $K(\Lambda) = K_n$.*

(2) (Corollary 1.3.9 (3)) *If w is a finite prime and n is large enough so that $|u^n|_\infty > |\eta_r|/|\eta_{r'+1}|$, then we have $K(u^{-n}\Lambda) = K_n$.*

The claim (1) is an effective version of [**Mau19**, Proposition 2.1].

3. Application to certain Drinfeld modules

Let K be a local field which is the completion of some global function field at a prime v. For a positive integer $r \geq 2$, let ϕ be a Drinfeld A-module over K (ϕ not necessarily has stable reduction over K) such that

$$\phi_t(X) = tX + a_s X^{q^s} + a_r X^{q^r} \in K[X],$$

where s and r are two positive integers satisfying $s < r$. Put

$$\boldsymbol{j} = \frac{a_s^{(q^r-1)/(q-1)}}{a_r^{(q^s-1)/(q-1)}}.$$

Let u be a finite prime of A. As a preparation for the later results, we are to calculate the valuations of the elements of the SMBs of $\phi[u^n]$ for each n in Chapter 2. With this preparation, in Chapter 3, we are to study the ramification of $K(\phi[u^n])/K$. The results in Chapter 1 allow us to obtain Theorem 3.2 below.

3.1. The valuations of elements of SMBs of $\phi[u^n]$. In Chapter 2, we first calculate the valuations of the elements of $\phi[u^n]$ for the case where the prime u of A has degree 1 (See Section 1 of Chapter 2). When the prime u has arbitrary degree, the valuations of elements in the SMBs are completely worked out for the cases where the prime v is infinite prime or the prime v is finite satisfying $v \nmid u$ (See Proposition 2.2.1 and Proposition 2.2.6). For the case where $u \mid v$, the valuations are calculated under certain assumptions (See Proposition 2.2.11). Assume ϕ has stable reduction over K when the prime v is finite. For the lattice Λ (or the pair (ψ, Λ)) associated to ϕ via (Tate) uniformization, we also calculate the valuations of SMBs of Λ and $\psi[u^n]$.

3.2. Explicit actions of the wild ramification subgroup. In Chapter 3, we study the ramification of $K(\phi[u^n])/K$. We only know the result for certain limited cases. For a positive integer n, let $G(n)_1$ denote the wild ramification subgroup, i.e., the first lower ramification subgroup, of $\mathrm{Gal}(K(\phi[u^n])/K)$.

Assume $s = 1$ and $\deg(u) = 1$. For the extension $K(\phi[u])/K$, we work out its Herbrand ψ-function in Corollary 3.3.15 (3). We can describe the action of $G(1)_1$ on $\phi[u]$ as follows:

Theorem 3.1 (Theorem 3.3.16). *Let v be an infinite prime or finite prime. Let u be a finite prime of A with* $\deg(u) = 1$ *(we do not require $v \nmid u$). Assume $r \geq 3$, $s = 1$ such that $\phi_t(X) = tX + a_1 X^q + a_r X^{q^r}$. Let $\{\xi_{i,1}\}_{i=1,\ldots,r}$ be an SMB of $\phi[u^n]$. Assume $p \nmid v(\boldsymbol{j})$ and $v(\boldsymbol{j}) < \frac{v(u)q(q^{r-1}-1)}{q-1}$. Let V denote the A/u-module $A \cdot_\phi \xi_{1,1}$. Then the map*

$$G(1)_1 \to V^{r-1};\ \sigma \mapsto (\sigma(\xi_{2,1}) - \xi_{2,1}, \ldots, \sigma(\xi_{r,1}) - \xi_{r,1})$$

is an isomorphism.

Note that the case where $s \mid r$ is also included in this theorem. In fact, if $s \mid r$, up to a constant field extension of K, we may consider ϕ as a Drinfeld $\mathbb{F}_{q^s}[t]$-module. We hope to know either the Herbrand ψ-function or the action $G(n)_1$ on $\phi[u^n]$ for $n \geq 2$ when $r \geq 3$.

Assume moreover $r = 2$ from now on. For an infinite prime v, in Section 5.1, we study the action of $G(n)_1$ on an SMB of $\phi[u^n]$ for large enough n. For this, we study the case where $\deg(u) = 1$ in Section 4. For a finite prime $v \nmid u$, in Section 5.2, we study the action of $G(n)_1$ on an SMB of $\phi[u^n]$ for any positive integer n. In summary, we have

Theorem 3.2. *Assume $r = 2$ such that $\phi_t(X) = tX + a_1X^q + a_2X^{q^2}$. Let u be a finite prime of A with $\deg(u) = d$. Let $\{\lambda_i\}_{i=1,2}$ be an SMB of $\phi[u^n]$.*

(1) (Theorem 3.5.2) *Let v be an infinite prime. Assume $v(\boldsymbol{j}) < v(t)q$ and $p \nmid v(\boldsymbol{j})$. Let m be the integer such that $v(\boldsymbol{j}) \in (v(t)q^{m+1}, v(t)q^m)$. Put $d = \deg(u)$. Assume $n \geq m/d$.*
   - *Any element in $G(\Lambda)_1$ fixes λ_1;*
   - *For a positive integer i, let $A^{<i}$ denote the subgroup of A consists of elements with degree $< i$. Then the map*

$$G(\Lambda)_1 \to A^{<m} \cdot_\phi \lambda_1;\ \sigma \mapsto \sigma(\lambda_2) - \lambda_2$$

   is an isomorphism of groups.

(2) (Theorem 3.5.5) *Let v be a finite prime satisfying $v \nmid u$. Assume $v(\boldsymbol{j}) < 0$, and $p \nmid v(\boldsymbol{j})$.*
   - *Any element in $G(n)_1$ fixes λ_1;*
   - *There is an isomorphism of groups*

$$G(n)_1 \to A \cdot_\phi \lambda_1;\ \sigma \mapsto \sigma(\lambda_2) - \lambda_2.$$

For the case where the valuation $v(\boldsymbol{j})$ is large enough, we have

Proposition 3.3 (A special case of Lemma 3.6.1). *Let v be an infinite prime or a finite prime satisfying $v \nmid u$. Let ϕ be a rank r Drinfeld A-module over K such that $\phi_t(X) = tX + a_sX^{q^s} + a_rX^{q^r}$. For a degree 1 prime u' of A not divisible by v, if $v(\boldsymbol{j}) \geq \frac{v(u')q^s(q^{r-s}-1)}{q-1}$, the extension $K(\phi[u^n])/K$ is at worst tamely ramified such that $G(n)_1$ is a trivial group.*

Finally, we hope to know the ramification of $K(\phi[u^n])/K$ for an integer n and a finite prime u of A (See Remark 3.3.17).

4. Analogues of results for elliptic curves

Let F be a global function field. Let ϕ be a Drinfed A-module over F such that

$$\phi_t(X) = tX + a_sX^{q^s} + a_rX^{q^r} \in F[X].$$

where s and r positive integers satisfying $s < r$. Put

$$\boldsymbol{j} := \frac{a_s^{(q^r-1)/(q-1)}}{a_r^{(q^s-1)/(q-1)}}.$$

Drinfeld modules can be considered as analogues of elliptic curves over a number field. In this section, we introduce results that are analogues of those for elliptic curves over a number field.

4.1. A formula involving two heights. In Section 1 of Chatper 4, we apply the results in Chapter 2 to show a formula in Corollary 4.1.4 that can be regarded as a relation between the differential height and the J-height of ϕ (See (54) and (55) for the definition). The differential height and the J-height are respectively defined in [**Tag92**, Section 5] and [**BPR21**, Section 2]. They are analogues of certain heights of elliptic curves (See Remarks 4.1.1 and 4.1.2). Hence one may regard this formula as an analogue of Silverman's formula in [**Sil86**, Proposition 2.1]. We hope that there is a generalization of the formula in Corollary 4.1.4 for arbitrary Drinfeld A-modules (See Remark 4.1.6).

4.2. An analogue of Szpiro's conjecture. Section 4.2.1 is devoted to a review of the conductors and Szpiro's conjecture for elliptic curves over number fields. Then for each prime v of F and a certain rank 2 Drinfeld module ϕ over F_v, we introduce an analogue of these conductors for ϕ at v. Finally, we claim a numerical relation between the J-heights and these analogues. This relation can be regarded to be an analogue of Szpiro's conjecture.

4.2.1. *Review on conductors of elliptic curves.* Let E be an elliptic curve over a local number field K of residue characteristic $p > 0$. For a prime number $\ell \nmid p$, let $E[\ell]$ denote the vector space of the ℓ-division points of E. Let G_i (resp. G^y) denote the i-th lower (resp. y-th upper) ramification subgroup of the Galois group of the extension $K(E[\ell])/K$ generated by the ℓ-division points of E. Define the wild part of the conductor of E/K to be the quantity

$$\delta(E/K) := \int_0^{+\infty} \frac{\# G_i}{\# G_0} \mathrm{codim}_{\mathbb{F}_\ell}(E[\ell]^{G_i}) di = \int_0^{+\infty} \mathrm{codim}_{\mathbb{F}_\ell}(E[\ell]^{G^y}) dy,$$

where $E[\ell]^{G_i}$ is the subspace of elements of $E[\ell]$ fixed by G_i and $E[\ell]^{G^y}$ is similarly defined. Define the tame part of the conductor to be

$$\varepsilon(E/K) := \mathrm{codim}_{\mathbb{Q}_\ell}(V_\ell(E)^{I(K^{\mathrm{sep}}/K)}) = \begin{cases} 0, & E \text{ has good reduction;} \\ 1, & E \text{ has multiplicative reduction;} \\ 2, & E \text{ has additive reduction.} \end{cases}$$

Here $V_\ell(E) = \varprojlim E[\ell^n] \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ denotes the rational ℓ-adic Tate module and $I(K^{\mathrm{sep}}/K)$ is the inertia subgroup of the absolute Galois group of K. Put as in [**Sil94**, p. 380]

$$f(E/K) = \delta(E/K) + \varepsilon(E/K) \tag{1}$$

the conductor of E over K. The quantity $f(E/K)$ is an integer independent of the choice of ℓ.

Let E be an elliptic curve over a (global) number field F. The conductor of E/F is the ideal $\mathfrak{f}(E/F) = \prod_{\mathfrak{p} \text{ finite}} \mathfrak{p}^{f(E/F_\mathfrak{p})}$ defined by all conductors of $E/F_\mathfrak{p}$, where $F_\mathfrak{p}$ is the completion of F at $\mathfrak{p}$. Here the product extends over all finite primes $\mathfrak{p}$ of F. The conductor measures the extent to which an elliptic curve has bad reduction.

There is another invariant, called the minimal discriminant, which measures how bad the reduction is. The minimal discriminant $\mathcal{D}(E/F)$ of E/F is the product of the minimal discriminants of integral models of $E/F_\mathfrak{p}$ for all finite primes $\mathfrak{p}$ of F. Szpiro proposed a conjecture (see [**Szp90**, p. 10] or [**Sil94**, Chapter IV, 10.6]) concerning a relation between these two invariants. A stronger form of this conjecture was proposed by Lockhart-Rosen-Silverman in [**LRS93**, Remark 5].

Conjecture 4.1. (1) *Fix a number field F and a real positive number ε. Then there exists a constant $C(F,\varepsilon)$ such that: for any elliptic curve E over F, its minimal discriminant $\mathcal{D}(E/F)$ and its conductor $\mathfrak{f}(E/F)$ satisfy*

$$N_{F/\mathbb{Q}}(\mathcal{D}(E/F)) \le C(F,\varepsilon)(N_{F/\mathbb{Q}}(\mathfrak{f}(E/F)))^{6+\varepsilon}.$$

(2) (Stronger form) *Put $\mathfrak{f}^{\mathrm{tame}}(E/F) := \prod_{\mathfrak{p} \text{ finite}} \mathfrak{p}^{\varepsilon(E/F_\mathfrak{p})}$. Then there exists a constant $C(F,\varepsilon)$ such that for any elliptic curve E over F,*

$$N_{F/\mathbb{Q}}(\mathcal{D}(E/F)) \le C(F,\varepsilon)(N_{F/\mathbb{Q}}(\mathfrak{f}^{\mathrm{tame}}(E/F)))^{6+\varepsilon}.$$

The conductor $f(E/K)$ of E over a local number field K is estimated by

Theorem 4.2 (Lockhart-Rosen-Silverman [**LRS93**], Brumer-Kramer [**BK94**]). *Let $K/\mathbb{Q}_p$ be a local field with normalized valuation v_K, and let E/K be an elliptic curve. Then $f(E/K)$ has an upper bound*

$$f(E/K) \le 2 + 3v_K(3) + 6v_K(2).$$

See also [**Sil94**, Chapter IV, Theorem 10.4]. This estimate is very important for the study of Szpiro's conjecture since it implies that (1) and (2) in the above conjecture are equivalent.

For an abelian variety A over a local number field K, its conductor $f(A/K)$ is defined in [**LRS93**, (12), (13)] (initially defined by Serre-Tate in [**ST68**, p. 500]). The definition is similar to (1). Lockhart-Rosen-Silverman also proposed a "partial generalization of Szpiro's conjecture".

Conjecture 4.3 ([**LRS93**, (10)]). *Let A be an abelian variety of dimension d over a number field F. Put $\mathfrak{f}(A/F) := \prod_{\mathfrak{p} \text{ finite}} \mathfrak{p}^{f(A/F_{\mathfrak{p}})}$. Let $h_{\mathrm{FP}}(A/F)$ denote the Faltings-Parshin height of A/F. Then there are constants $C_1(F,d)$ and $C_2(F,d)$, depending only on F and d, such that*

$$h_{\mathrm{FP}}(A/F) \leq C_1(F,d) \cdot \log|N_{F/\mathbb{Q}}\mathfrak{f}(A/F)| + C_2(F,d).$$

Szpiro's conjecture for $\mathbb{Q}$ follows from the *abc* conjecture [**Szp90**, Section 2, Remarque]. Mochizuki [**Mo21**] announced the proof of the *abc* conjecture via the inter-universe Teichmüller theory.

4.2.2. *An analogue of Szpiro's conjecture.* Assume throughout this subsection that the rank of ϕ is $r = 2$. We first define an analogue of the conductor above. The estimate in Theorem 4.2 suggests that when working with Szpiro's conjecture and its variant for elliptic curves over a number field, one may ignore the contribution of wild ramification. On the contrary, for the extensions generated by division points of Drinfeld modules, the wild ramification can be made arbitrarily large. So it is worth investigating a relation between the height and the wild part of the conductor of a Drinfeld module.

Rather than the vector space of division points, we consider the G_v-module T_u, where G_v denotes the absolute Galois group of F_v. Since G_v is a profinite group, a definition similar to that of "$\delta(E/K)$" in (1) using lower ramification subgroups is not valid.

Using the notion of the upper ramification subgroups, we define for a rank 2 Drinfeld A-module ϕ over K the quantity

$$\mathfrak{f}_v(\phi) := \int_0^{+\infty} (2 - \mathrm{rank}_{A_u} T_u^{G_v^y}) dy$$

as an analogue of the wild part of the conductor $\delta(E/K)$ of an elliptic curve. Here G_v^y denotes the y-th upper ramification subgroup of G_v. Note that the prime v can be infinite or finite. In fact, the infinite part of a height (e.g. J-height) is not bounded. When we want to relate the height to the conductor (as in Theorem 4.5) in the function field case, we must define the conductors at infinite primes, unlike in the number field case.

Proposition 4.4 (Lemma-Definitions 4.2.1 and 4.2.2). *Let ϕ be a rank 2 Drinfeld A-module over F. Assume one of the following four cases happens*

(1) *v is infinite, $v(\boldsymbol{j}) < v(t)q$, and $p \nmid v(\boldsymbol{j})$.*
(2) *v is infinite and $v(\boldsymbol{j}) \geq v(t)q$;*
(3) *v is finite, $p \nmid v(\boldsymbol{j})$, and $v(\boldsymbol{j}) < 0$;*
(4) *v is finite, and $v(\boldsymbol{j}) \geq 0$.*

Then the integral $\mathfrak{f}_v(\phi)$ is independent of the choice of the finite prime u used in the definition of $\mathfrak{f}_v(\phi)$ and we have

$$\mathfrak{f}_v(\phi) = \begin{cases} \begin{cases} 0 & v(\boldsymbol{j}) \in [v(t)q, +\infty); \\ \frac{-v(\boldsymbol{j})+v(t)q}{q-1} & v(\boldsymbol{j}) \in (-\infty, v(t)q),\ p \nmid v(\boldsymbol{j}), \end{cases} & v \text{ is infinite}; \\ \begin{cases} 0 & v(\boldsymbol{j}) \in [0, +\infty); \\ \frac{-v(\boldsymbol{j})}{q-1} & v(\boldsymbol{j}) \in (-\infty, 0),\ p \nmid v(\boldsymbol{j}),\ \text{and } q \neq 2, \end{cases} & v \text{ is finite}. \end{cases}$$

The cases (1) and (3) follow from Lemma 3.3. The other cases follow from Theorem 3.2.

For a prime v of F and the completion F_v, let $\deg(v)$ denote the degree of the residue field of F_v over $\mathbb{F}_q$. Put $\mathfrak{f}(\phi) := \sum_v \deg(v) \cdot \mathfrak{f}_v(\phi)$ and we call it the *global conductor* of ϕ, where the sum extends over all primes v of F.

The conductors for certain Drinfeld modules are determined by the j-invariant. We obtain the following formula involving the global conductor of ϕ and the J-heights (See (54) for definition) of ϕ. This formula can be regarded as an analogue of Conjecture 4.3. Here the J-height is initially defined in Breuer-Pazuki-Razafinjatovo [**BPR21**] using the j-invariants and can be regarded as a replacement of the Faltings-Parshin height in Conjecture 4.3.

Theorem 4.5 (Theorem 4.2.6). *Let ϕ be a rank 2 Drinfeld A-module over F. For each prime of F, assume that ϕ satisfies one of the four conditions in Proposition 4.4. Let $h_J(\phi)$ denote the J-height of ϕ. Then we have the inequality*

$$h_J(\phi) \leq \mathfrak{f}(\phi) \cdot \frac{q-1}{[F : \mathbb{F}_q(t)]} + q.$$

4.3. Remark on the contents. The results Chapter 1 are contained in [**Hua23**, Sections 2, 3, and 4]. We slightly generalize the results in [**AH22**, Section 2 and Appendix] in Section 1 in Chapter 2. We slightly generalize certain results in [**Hua23**, Section 5.1 and Section 6.1] in Section 2 in Chapter 2. In Section 2 of Chapter 3, we slightly generalize [**AH22**, Proposition 3.2]. In Sections 1 and Section 6 of Chapter 3, we slightly generalize the [**AH22**, Lemmas 3.4, 3.5, 3.17, and 3.18]. In Section 3, we show a nontrivial generalization of [**AH22**, Lemmas 3.6] and derive Theorem 3.3.16 (the above Theorem 3.1). The results in Section 4 are straightly taken from [**AH22**, Section 3.2] and there is nothing new. The results in Section 5 are straightly taken from [**Hua23**, Sections 5.2 and 6.2]. The results in [**AH22**, Section 3.3] are covered by Corollary 3.5.5 (the above Theorem 3.2 (2)). The formula Corollary 4.1.4 in Section 1 of Chapter 4 has not appeared in the previous literature. The results in Section 2.2 is similar to those in Section 2.1, but have not appeared in the previous literature. The results in Sections 2.1

and 2.3 of Chapter 4 are taken from [**Hua23**, Sections 5.2, 6.2, and 6.3]. These result extends those in [**AH22**, Section 4].

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CHAPTER 1

On successive minimal bases

We consider Drinfeld A-modules over a localization K of a global function field. In Section 1, we first review the basics of the SMB of lattices. The rest of this section is devoted to the basics of SMBs of $\phi[u^n]$. In Section 2, we mainly show Theorem 1.2.3 (the infinite prime case of Theorem 2.1). For an element ω_i of an SMB of the lattice Λ associated to the Drinfeld module and an element a_i in A with a limited degree, we describe the absolute value of $e_\phi(a_i\omega_i)$ in Corollary 1.2.2 (1). This is the key result of this section and its proof is inspired by that of [**Gek19A**, Lemma 3.4]. Section 3 consists of finite prime analogues of the results in Section 2. The analogue of Corollary 1.2.2 (1) is Corollary 1.3.6 (1).

1. Basics of SMBs

For an infinite prime v (resp. a finite prime v), let $|-|$ denote the absolute value in (F1) (resp. the function in (F2)) defined at the beginning of Section 1 in the Introduction.

1.1. SMBs of lattices. In this subsection, we recall first the basics of SMBs of lattices and then the (Tate) uniformization of Drinfeld modules. Consider $\mathbb{C}_\infty$ as an A-module via the embedding $A \to \mathbb{C}_\infty$. If v is a finite prime, consider $\mathbb{C}_v$ as an A-module via a Drinfeld module ψ having good reduction of rank r'. The next lemma will be applied implicitly in this paper.

Lemma 1.1.1. (1) *If v is an infinite prime, we have $|a\omega| = |a| \cdot |\omega|$ for any $a \in A$ and $\omega \in \mathbb{C}_\infty$.*

(2) ([**Gar02**, Section 1]) *Let v be a finite prime. Then we have $|a \cdot_\psi \omega| = |a|_\infty \cdot |\omega|$, i.e., $v(a \cdot_\psi \omega) = |a|_\infty^{r'} \cdot v(\omega)$ for any $a \in A$ and any $\omega \in \mathbb{C}_v$ having valuation < 0, where $|a|_\infty = q^{\deg(a)}$.*

Proof. (1) is clear. We show (2). Put $g = r' \cdot \deg(a)$, $a_0 = a$, and $\sum_{i=0}^{g} a_i X^{q^i} = \psi_a(X)$. As the Drinfeld module ψ has good reduction, we have $v(a_i) \geq 0$ and $v(a_g) = 0$. Hence the assumption $v(\omega) < 0$ implies that the valuation $v(a_g\omega^{q^g})$ is the strictly smallest among $v(a_i\omega^{q^i})$ for all i. As $v(a_g) = 0$, we have $v(a_g\omega^{q^g}) = q^g v(\omega)$, i.e., $|a\omega| = |a|_\infty \cdot |\omega|$. □

Let L be an A-lattice of rank r in $\mathbb{C}_\infty$ or an A-lattice of rank r in $\mathbb{C}_v$ such that each nonzero element in the lattice has valuation < 0.

Definition 1.1.2 ([**Tag92**, Section 4] or [**Gek19A**, Section 3]). A family of elements $\{\omega_i\}_{i=1,\dots,r}$ in L is called an SMB of L if for each i, the elements $\omega_1, \dots, \omega_i$ satisfy

(1) $\omega_1, \dots, \omega_i$ are A-linearly independent;
(2) $|\omega_i|$ is minimal among the absolute values of elements ω in L such that $\omega_1, \dots, \omega_{i-1}, \omega$ are A-linearly independent.

Remark 1.1.3. The condition Definition 1.1 (1) implies that $\{\lambda_i\}_{i=1,\dots,r}$ is a basis (or a generating set) of $\phi[u^n]$. However, if $\{\omega_i\}_{i=1,\dots,r}$ is a family of elements in L that satisfies only the condition (1) in the above definition, then it is not necessarily a generating set.

Proposition 1.1.4. *Let $\{\omega_i\}_{i=1,\dots,r}$ be a family of elements in L.*

(1) *This family is an SMB if and only if for each i, the elements $\omega_1, \dots, \omega_i$ satisfy*
- *$\omega_1, \dots, \omega_i$ are A-linearly independent;*
- *we have $|\omega_i| = l_i$, where*

$$l_i = \min\left\{\rho \in \mathbb{R} \,\middle|\, \begin{array}{l} \text{the ball in } \mathbb{C}_\infty \text{ or } \mathbb{C}_v \text{ around } 0 \text{ of radius } \rho \text{ contains at least} \\ i \text{ elements in } L \text{ which are } A\text{-linearly independent} \end{array}\right\}.$$

(2) *The sequence $|\omega_1| \leq |\omega_2| \leq \cdots \leq |\omega_r|$ for an SMB $\{\omega_i\}_{i=1,\dots,r}$ is an invariant of L, i.e., for any SMB $\{\omega'_i\}_{i=1,\dots,r}$ of L, we have $|\omega_i| = |\omega'_i|$ for all i.*

Proof. As (2) immediately follows from (1), we only show (1). The first dot is the same as Definition 1.1.2 (1). For "$\Leftarrow$," we show that for A-linearly independent elements $\omega_1, \dots, \omega_{i-1}, \omega$ in L, we have $|\omega_i| \leq |\omega|$. Assume conversely $|\omega| < |\omega_i|$. We have $|\omega| < l_i$, which contradicts the definition of l_i. For "$\Rightarrow$," we fix any i and show $l_i = |\omega_i|$. Clearly, we have $l_i \leq |\omega_i|$ and $l_1 = \omega_1$. Then we do induction. Assume $l_j = |\omega_j|$ for $j < i$. If $l_i < |\omega_i|$, then there are A-linear independent elements $\mu_1, \dots, \mu_i \in \Lambda$ satisfying $|\mu_j| \leq l_i < |\omega_i|$ for $j = 1, \dots, i$. There exists k such that $\mu_k, \omega_1, \dots, \omega_{i-1}$ are A-linear independent. For otherwise, for each j there are some $a_j \in A$ such that $a_j\mu_j$ are A-linear combinations of all $\mu_1, \dots, \mu_i$, and hence the elements $\omega_1, \dots, \omega_{i-1}$ generate a rank i free A-module, which is absurd. As $|\mu_k| < |\omega_i|$, the elements $\mu_k, \omega_1, \dots, \omega_{i-1}$ being A-linear independence contradicts Definition 1.1.2 (2). □

Proposition 1.1.5. *In the proposition, we put $a\omega := a \cdot_\psi \omega$ for any $a \in A$ and $\omega \in L$ when the prime v is finite. Let $\{\omega_i\}_{i=1,\dots,r}$ be a family of elements in L so that $|\omega_1| \leq |\omega_2| \leq \cdots \leq |\omega_r|$. Then this family is an SMB of L if and only if*

(1) *$\omega_1, \dots, \omega_r$ form an A-basis of L;*
(2) *we have $|\sum_i a_i\omega_i| = \max_i\{|a_i\omega_i|\}$ for any $a_i \in A$.*

Proof. ([**Tag92**, Lemma 4.2]) We show $\Rightarrow$. Let k be the largest index so that $|a_k\omega_k| = \max_i\{|a_i\omega_i|\}$. Assume conversely $|\sum_i a_i\omega_i| < |a_k\omega_k|$. Then there is an index

$j \neq k$ so that $|a_j\omega_j| = |a_k\omega_k|$. Let J denote the set of all these j. As $|\omega_k| \geq |\omega_j|$ for each $j \in J$, there exist $b_j, c_j \in A$ such that $a_j = a_k b_j + c_j$ with $|b_j|_\infty > 1$ and $|c_j|_\infty < |a_k|_\infty$. If we admit the claim

$$\left|\sum_j b_j\omega_j + \omega_k\right| < |\omega_k|,$$

then $\omega_1, \ldots, \omega_{k-1}, \sum_j b_j\omega_j + \omega_k$ being A-linearly independent contradicts Definition 1.1.2 (2). As for the claim, by $|c_j|_\infty < |a_k|_\infty$, we have

$$\begin{aligned}\left|\sum_{j\in J} a_k b_j\omega_j + a_k\omega_k\right| &= \left|\sum_{j\in J} a_j\omega_j + a_k\omega_k\right| \\ &= \left|\sum_i a_i\omega_i\right| < |a_k\lambda_k|.\end{aligned}$$

The desired inequality follows.

As for $\Leftarrow$, fixing a positive integer $k < r$, we know that $|\sum_i a_i\omega_i| < |\omega_k|$ implies $a_i = 0$ for $i > k$ by the equation in (2). Hence we have $|\omega_k| = l_k$ and the proof follows from Proposition 1.1.4 (1). □

For the subfield K of $\mathbb{C}_v$, we say that L is $G(K^{\mathrm{sep}}/K)$-invariant if each element in the Galois group maps L into L. The following lemma concerns the extension generated by elements in the lattice with the minimal norm.

Lemma 1.1.6. *Let $\{\omega_i\}_{i=1,\ldots,r}$ be an SMB of L such that $\omega_1, \ldots, \omega_s$ satisfies $|\omega_1| = \cdots = |\omega_s| < |\omega_{s+1}|$. Assume that*

- *the extension M of K generated by ω_i for $i = 1, \ldots, s$ is separable;*
- *the lattice L is $G(K^{\mathrm{sep}}/K)$-invariant.*

Then the extension M/K is Galois and is at worst tamely ramified.

Proof. Let $\widehat{M}$ denote the Galois closure of M/K so that $\widehat{M}$ is exactly the compositum of ςM for all $\varsigma \in G(\widehat{M}/K)$. We have $\widehat{M} = M$. Indeed, if $\widehat{M}/M$ is nontrivial, there exists some element $\varsigma \in G(\widehat{M}/K)$ such that $\varsigma(\omega_j) \notin M$ for j to be one of $1, \ldots, s$. Note that M contains the A-module $\bigoplus_{i=1,\ldots,s} A\omega_i$ (here $A\omega_i := \{a \cdot_\psi \omega_i \mid a \in A\}$ if the prime v is finite). As elements in $A \setminus \bigoplus_{i=1,\ldots,s} A\omega_i$ have strictly smaller valuations than that of ω_i for $i = 1, \ldots, s$ and Galois actions preserve valuations, this implies that $\varsigma(\omega_j) \notin L$. If ς also denotes a preimage of ς under $G(K^{\mathrm{sep}}/K) \to G(\widehat{M}/K)$, then $\varsigma(\omega_j) \notin L$ contradicts that L is $G(K^{\mathrm{sep}}/K)$-invariant.

We show that M/K is tamely ramified. Assume the converse so that the wild ramification subgroup $G(M/K)_1$ is nontrivial. Let v_M denote the normalized valuation associated to M. For σ to be a nontrivial element in $G(M/K)_1$, we have for each i

$$1 \leq v_M(\sigma(\omega_i)\omega_i^{-1} - 1).$$

We also have $\sigma(\omega_j) - \omega_j \neq 0$ for j to be one of $1, \ldots, s$. Note that $v_M(\omega_j)$ is the largest among the valuations of all nonzero elements in L. As $\sigma(\omega_j) - \omega_j \in L$ (L is $G(K^{\mathrm{sep}}/K)$-invariant), we have

$$v_M(\sigma(\omega_j)\omega_j^{-1} - 1) = v_M(\sigma(\omega_j) - \omega_j) - v_M(\omega_j) \leq 0.$$

This gives a contradiction. ☐

Next, we briefly recall the uniformization and the Tate uniformization. If w is an infinite prime, then the uniformization (See [**Pap23**, Section 5.2] for more details) associates to the Drinfeld module ϕ a $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-invariant A-lattice Λ and an exponential function e_ϕ on $\mathbb{C}_\infty$ such that for each $a \in A$, the following diagram commutes, and its two rows are short exact sequences

$$\begin{array}{ccccc} \Lambda & \hookrightarrow & \mathbb{C}_\infty & \xrightarrow{e_\phi} & \mathbb{C}_\infty \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle \phi_a} \\ \Lambda & \hookrightarrow & \mathbb{C}_\infty & \xrightarrow{e_\phi} & \mathbb{C}_\infty . \end{array}$$

Here the exponential function is explicitly

$$e_\phi : \mathbb{C}_\infty \to \mathbb{C}_\infty;\ \omega \mapsto \omega \prod_{\mu \in \Lambda \setminus \{0\}} (1 - \omega/\mu)$$

and the coefficients of the polynomial $\phi_a(X)$ map to $\mathbb{C}_\infty$ via the embedding $K \hookrightarrow \mathbb{C}_\infty$. The commutativity of the right square in the diagram means $e_\phi(a\omega) = a \cdot_\phi e_\phi(\omega)$ for any $\omega \in \mathbb{C}_\infty$.

Remark 1.1.7 (SMBs and isomorphic Drinfeld modules). For any $b \in K^\times$, we have the Drinfeld module $b\phi b^{-1}$ isomorphic to ϕ. The uniformization associates to $b\phi b^{-1}$ the lattice $b\Lambda$. If the family $\{\omega_i\}_{i=1,\ldots,r}$ is an SMB of Λ, then $\{b\omega_i\}_{i=1,\ldots,r}$ is an SMB of $b\Lambda$.

If v is a finite prime of K, assume that ϕ has stable reduction and the reduction of ϕ has rank $r' < r$. According to [**Dri74**, Section 7] (See also [**Pap23**, Section 6.2]), there are the following data associated to ϕ :

(1) A rank r' Drinfeld A-module ψ over K has good reduction;
(2) A $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-invariant A-lattice Λ has rank $r - r'$ with the A action induced by ψ. Each element of Λ has valuation < 0.
(3) An analytic entire surjective homomorphism

$$e_\phi : \mathbb{C}_v \to \mathbb{C}_v;\ \omega \mapsto \omega \prod_{\mu \in \Lambda \setminus \{0\}} (1 - \omega/\mu)$$

such that for each $a \in A$, the following diagram commutes, and its two rows are short exact sequences

$$\begin{array}{ccccc} \Lambda & \hookrightarrow & \mathbb{C}_v & \xrightarrow{e_\phi} & \mathbb{C}_v \\ \downarrow{\scriptstyle \psi_a} & & \downarrow{\scriptstyle \psi_a} & & \downarrow{\scriptstyle \phi_a} \\ \Lambda & \hookrightarrow & \mathbb{C}_v & \xrightarrow{e_\phi} & \mathbb{C}_v. \end{array}$$

The commutativity of the right square means $e_\phi(a \cdot_\psi \omega) = a \cdot_\phi e_\phi(\omega)$ for any $\omega \in \mathbb{C}_v$. We call these data the Tate uniformization of ϕ.

1.2. SMBs of the module of u^n-division points. Let ϕ be a rank r Drinfeld A-module over K. For a positive integer n and a finite prime u of A, this subsection concerns with two basic properties of SMBs of $\phi[u^n]$.

Proposition 1.1.8. *Let $\{\lambda_i\}_{i=1,\dots,r}$ be a family of elements in $\phi[u^n]$.*

(1) *Then this family is an SMB if and only if for each i, the elements $\lambda_1, \dots, \lambda_i$ satisfy*
- *$\lambda_1, \dots, \lambda_i$ are A/u^n-linearly independent;*
- *we have $|\lambda_i| = l_i$, where*

$$l_i = \min \left\{ \rho \in \mathbb{R} \,\middle|\, \begin{array}{l} \text{the ball in } K^{\text{sep}} \text{ around } 0 \text{ of radius } \rho \text{ contains at least} \\ i \text{ elements in } \phi[u^n] \text{ which are } A/u^n\text{-linearly independent} \end{array} \right\}.$$

(2) *The sequence $|\lambda_1| \leq |\lambda_2| \leq \cdots \leq |\lambda_r|$ is an invariant of $\phi[u^n]$.*

Proof. (2) is straightly follows from (1). We then show (1). The "$\Leftarrow$" is straightforward. For "$\Rightarrow$," the first dot in (1) is the same as Definition 1.1 (1). Clearly, we have $l_i \leq |\lambda_i|$ for all i and $l_1 = |\lambda_1|$. Then we proceed by induction. We fix any i, assume $l_j = |\lambda_j|$ for $j < i$, and show $l_i = |\lambda_i|$. We assume $l_i < |\lambda_i|$ and find a contradiction. There exists elements $\eta_1, \dots, \eta_i \in \phi[u^n]$ such that $\eta_1, \dots, \eta_i$ are A/u^n-linearly independent and $|\eta_j| \leq l_i < |\lambda_i|$ for $j = 1, \dots, i$.

Put $\overline{\eta}_j := u^{n-1} \cdot_\phi \eta_j$ for $j \leq i$ and $\overline{\lambda}_j := u^{n-1} \cdot_\phi \lambda_j$ for $j < i$. We claim that there is some k such that $\overline{\eta}_k$ and $\overline{\lambda}_1, \dots, \overline{\lambda}_{i-1}$ are A/u-linearly independent. Assume the inverse. Then we have equations

$$b_l \cdot_\phi \overline{\eta}_l + \sum_{j=1}^{i-1} a_{l,j} \cdot_\phi \overline{\lambda}_j = 0$$

for all $l = 1, \dots, i$, where $a_{l,j} \in A \mod u$ and $b_l \in A \mod u$ with $b_l \not\equiv 0 \mod u$ for each l. Hence for each l, we obtain

$$\overline{\eta}_l = \sum_{j=1}^{i-1} a_{l,j}/b_l \cdot_\phi \overline{\lambda}_j,$$

where each $a_{l,j}/b_l \in A \mod u$ satisfies $b_l(a_{l,j}/b_l) \equiv a_{l,j} \mod u$. Hence $\overline{\lambda}_1, \dots, \overline{\lambda}_{i-1}$ generate an i-dimensional A/u-vector space, which is absurd.

Next, we claim that η_k and $\lambda_1, \ldots, \lambda_{i-1}$ are A/u^n-linearly independent. Assume the inverse and we have

$$c_k \cdot_\phi \eta_k + \sum_{j=1}^{i-1} a_j \cdot_\phi \lambda_j = 0, \tag{2}$$

where each $a_j \in A \mod u^n$ and $c_k \in A \mod u^n$ with $c_k \not\equiv 0 \mod u^n$. We may write $c_k = c'_k u^m$ with $m < n$ and $c'_k \in A$ not divisible by u. Then we have $u^m \mid a_j$ for all $j < i$, for otherwise, by (2), we have $\sum_{j=1}^{i-1} a_j u^{n-m} \cdot_\phi \lambda_j = 0$ with $a_j u^{n-m} \not\equiv 0 \mod u^n$ for some j. We may write $a_j = a'_j u^m$ for $a'_j \in A$. Hence we have by (2)

$$0 = c_k u^{n-1-m} \cdot_\phi \eta_k + \sum_{j=1}^{i-1} a_j u^{n-1-m} \cdot_\phi \lambda_j = c'_k \cdot_\phi \overline{\eta}_k + \sum_{j=1}^{i-1} a'_j \cdot_\phi \overline{\lambda}_j$$

with $c'_k \in A$ not divisible by u. This contradicts that $\overline{\eta}_k$ and $\overline{\lambda}_1, \ldots, \overline{\lambda}_{i-1}$ are A/u-linearly independent. We have obtained A/u^n-linearly independent elements $\lambda_1, \ldots, \lambda_{i-1}, \eta_k$ such that $|\eta_k| \leq l_i < |\lambda_i|$. This contradicts Definition 1.1.2 (2). □

In the remainder of this subsection, we construct an SMB of $\phi[u^n]$ for any positive integer n.

Lemma 1.1.9. *Let $\{\lambda_i\}_{i=1,\ldots,r}$ be an SMB of $\phi[u^n]$. For each i and $a \in A$ with $a \not\equiv 0$ mod u^n, the element λ_i has the largest valuation among the roots λ of $\phi_a(X) - a \cdot_\phi \lambda_i$ such that $\lambda \in \phi[u^n]$.*

Proof. Let λ be a root of $\phi_a(X) - a \cdot_\phi \lambda_i$ such that $\lambda \in \phi[u^n]$. Assume $v(\lambda) > v(\lambda_i)$. It suffices to show that $\lambda_1, \ldots, \lambda_{i-1}, \lambda$ are A/u^n-linearly independent because this implies that the inequality $v(\lambda) > v(\lambda_i)$ contradicts Definition 1.1 (2). Assume that there exists $b_j \in A \mod u^n$ with $b_i \not\equiv 0$ such that $b_i \cdot_\phi \lambda + \sum_{j<i} b_j \cdot_\phi \lambda_j = 0$. Let c be the minimal common multiple of a and b_i such that $c = b'_i b_i = a' a$ for some b'_i and $a' \in A$. Consider the equation $b'_i \cdot_\phi (b_i \cdot_\phi \lambda + \sum_{j<i} b_j \cdot_\phi \lambda_j) = 0$. Since $b'_i b_i \cdot_\phi \lambda = a' a \cdot_\phi \lambda = a' a \cdot_\phi \lambda_i = c \cdot_\phi \lambda_i$, we have

$$c \cdot_\phi \lambda_i + \sum_{j<i} b'_i b_j \cdot_\phi \lambda_j = 0. \tag{3}$$

We have $u^n \nmid c$, for otherwise one of a or b_i is divisible by u^n. Hence the nonzero coefficients in the equation (3) contradict that $\lambda_1, \ldots, \lambda_i$ are A/u^n-linearly independent. □

Corollary 1.1.10. *With the notation in the lemma, for each i and $a \in A$ being a power of u, the element λ_i has the largest valuation among the roots of $\phi_a(X) - a \cdot_\phi \lambda_i$.*

Proposition 1.1.11. *Let $\{\lambda_i\}_{i=1,\ldots,r}$ be an SMB of $\phi[u^n]$.*

(1) *For each i, put λ'_i to be a root of $\phi_u(X) - \lambda_i$ having the largest valuation. Then $\{\lambda'_i\}_{i=1,\dots,r}$ is an SMB of $\phi[u^{n+1}]$.*
(2) *The family of elements $\{u \cdot_\phi \lambda_i\}_{i=1,\dots,r}$ is an SMB of $\phi[u^{n-1}]$.*

Proof. (1) We show that $\lambda'_1, \dots, \lambda'_i$ are A-linear independent for any i. Assume conversely that there are $a_j \in A \mod u^{n+1}$ with $a_i \not\equiv 0$ such that $\sum_{j=1}^{i} a_j \cdot_\phi \lambda'_j = 0$. For $j = 1, \dots, i$, since $u \cdot_\phi \lambda'_j = \lambda_j$ and $\lambda_1, \dots, \lambda_i$ are A/u^n-linearly independent, we have $ua_j \equiv 0 \mod u^{n+1}$ and hence $u^n \mid a_j$. There are $b_j \in A$ with $b_i \not\equiv 0 \mod u$ such that $a_j = b_j u^n$ for all j. Hence

$$0 = \sum_{j=1}^{i} a_i \cdot_\phi \lambda'_i = \sum_{j=1}^{i} b_j u^{n-1} \cdot_\phi \lambda_i$$

with $b_i u^{n-1}$ not divisible by u^n, which is absurd.

As for Definition 1.1 (2), we show $v(\lambda'_i) \geq v(\lambda)$ for each $\lambda \in \phi[u^{n+1}]$ such that $\lambda'_1, \dots, \lambda'_{i-1}, \lambda$ are A/u^{n+1}-linearly independent. Notice $u \cdot_\phi \lambda \in \phi[u^n]$ and that the elements $\lambda_1, \cdots, \lambda_{i-1}, u \cdot_\phi \lambda$ are A/u^n-linearly independent. We have $v(\lambda_i) \geq v(u \cdot_\phi \lambda)$ as $\{\lambda_i\}_{i=1,\dots,r}$ is an SMB of $\phi[u^n]$. By Corollary 1.1.10, we know that $v(\lambda'_i)$ is the largest among the valuations of roots of $\phi_u(X) - \lambda_i$. By comparing the Newton polygons of $\phi_u(X) - \lambda_i$ and $\phi_u(X) - u \cdot_\phi \lambda$, this implies $v(\lambda'_i) \geq v(\lambda)$.

(2) It is straightforward to check Definition 1.1 (1). Let λ be an element of $\phi[u^{n-1}]$ such that $u \cdot_\phi \lambda_1, \dots, u \cdot_\phi \lambda_{i-1}, \lambda$ are A/u^{n-1}-linearly independent. To show $|u \cdot_\phi \lambda_i| \leq |\lambda|$, we assume conversely $v(u \cdot_\phi \lambda_i) < v(\lambda)$. By comparing the Newton polygon of $\phi_u(X) - u \cdot_\phi \lambda_i$ and $\phi_u(X) - \lambda$, there is a root λ' of $\phi_u(X) - \lambda$ such that $v(\lambda') > v(\lambda_i)$. We have $\lambda' \in \phi[u^n]$ as all roots of $\phi_u(X) - \lambda$ belong to $\phi[u^n]$. Similarly to the proof of (1), one shows that $\lambda_1, \cdots, \lambda_{i-1}, \lambda'$ are A/u^n-linearly independent. Hence the inequality $v(\lambda') > v(\lambda_i)$ contradicts Definition 1.1 (2). ☐

We can find an SMB of $\phi[u]$ in the following way. Put

$$\begin{aligned} &\lambda_{1,1} \coloneqq \text{an element in } \phi[u] \setminus \{0\} \text{ with the largest valuation and} \\ &\lambda_{i,1} \coloneqq \text{an element in } \phi[u] \setminus \bigoplus_{j<i} (A/u) \cdot_\phi \lambda_{j,1} \text{ with the largest valuation} \end{aligned} \tag{4}$$

for $i = 2, 3, \dots, r$. Since A/u is a field, the elements $\lambda_{i,1}$ for $i = 1, \cdots, r$ are A/u-linearly independent and form an SMB of $\phi[u]$. Applying the proposition, we have

Corollary 1.1.12. *Let $\{\lambda_{i,1}\}_{i=1,\dots,r}$ be an SMB of $\phi[u]$ defined above. Inductively, put $\lambda_{i,j}$ to be a root of $\phi_u(X) - \lambda_{i,j-1}$ having the largest valuation for each i and each integer $j \geq 2$. Then for each positive integer n, we have that $\{\lambda_{i,n}\}_{i=1,\dots,r}$ is an SMB of $\phi[u^n]$.*

Remark 1.1.13. For $b \in K$, we have the Drinfeld module $b\phi b^{-1}$ isomorphic to ϕ. If $\{\lambda_i\}_{i=1,\dots,r}$ is an SMB of $\phi[u^n]$, then the family $\{b\lambda_i\}_{i=1,\dots,r}$ is an SMB of $b\phi b^{-1}[u^n]$.

2. Relations between SMBs, the infinite prime case

Let w denote an infinite prime, $|-|$ the absolute value in (F1) and $\{\omega_i\}_{i=1,\dots,r}$ an SMB of Λ throughout this section. For a positive integer n and a finite prime u of A, we study the relations between SMBs of Λ and those of $\phi[u^n]$.

Lemma 1.2.1. *Let a be an element in A. For $\omega = \sum_j a_j\omega_j \in \Lambda$ with $a_j \in A$, let i be an index so that $|a_i\omega_i| = |\omega|$, i.e., $|a_i\omega_i| = \max_j\{|a_j\omega_j|\}$. Assume $\deg(a_i) < \deg(a)$. Then we have*

$$\left|e_\phi\left(\frac{\omega}{a}\right)\right| = \left|e_\phi\left(\frac{a_i\omega_i}{a}\right)\right|.$$

PROOF. We have

$$e_\phi\left(\frac{\omega}{a}\right) = \frac{\omega}{a}\prod_{\mu\in\Lambda\setminus\{0\}}\left(1-\frac{\omega}{a\mu}\right).$$

Its absolute value is

$$\left|\frac{\omega}{a}\right| \cdot \prod_{\substack{\mu\in\Lambda\setminus\{0\}\\ |a\mu|\le|\omega|}}\left|1-\frac{\omega}{a\mu}\right|.$$

For $\mu\in\Lambda$ satisfying $|a\mu|<|\omega|$, we have by the ultrametric inequality

$$\left|1-\frac{\omega}{a\mu}\right| = \left|\frac{\omega}{a\mu}\right| = \left|\frac{a_i\omega_i}{a\mu}\right| = \left|1-\frac{a_i\omega_i}{a\mu}\right|.$$

Next, for $\mu\in\Lambda$ satisfying $|a\mu| = |\omega| = |a_i\omega_i|$, we show

$$\left|1-\frac{\omega}{a\mu}\right| = \left|1-\frac{a_i\omega_i}{a\mu}\right| = 1.$$

It suffices to show

$$|\omega - a\mu| = |\omega| \text{ and } |a_i\omega_i - a\mu| = |a_i\omega_i|. \tag{5}$$

Since $|a_i|<|a|$, we have μ belonging to $\bigoplus_{j<i} A\omega_j$, for otherwise we have $|a\mu|\ge|a\omega_i|>|a_i\omega_i|$ by Proposition 1.1.5 (2). Applying Proposition 1.1.5 (2) to $|\omega-a\mu|$ and $|a_i\omega_i-a\mu|$, we obtain the desired equalities. □

Corollary 1.2.2. *Let a be an element in A.*

(1) *For any $i=1,\dots,r$ and any $a_i\in A$ satisfying $\deg(a_i)<\deg(a)$, we have*

$$\left|e_\phi\left(\frac{a_i\omega_i}{a}\right)\right| = \left|\frac{a_i\omega_i}{a}\right| \cdot \prod_{\substack{\mu\in\Lambda\setminus\{0\}\\ |a\mu|<|a_i\omega_i|}} |a_i\omega_i|/|a\mu|.$$

(2) *For any positive integers $i,j\le r$, let a_i and a_j be elements in A with degrees strictly smaller than that of a. Assume $|a_j\omega_j|\le|a_i\omega_i|$. Then*

$$\left|e_\phi\left(\frac{a_j\omega_j}{a}\right)\right| \le \left|e_\phi\left(\frac{a_i\omega_i}{a}\right)\right|.$$

(3) *With the notation in the lemma, we have*
$$\left|e_\phi\left(\frac{\omega}{a}\right)\right| = \max_j \left\{\left|a_j \cdot_\phi e_\phi\left(\frac{\omega_j}{a}\right)\right|\right\}.$$
(4) *For any positive integer $i \leq r$ and $b \in A$ satisfying* $\deg(b) < \deg(a)$, *we have*
$$|b| \cdot \left|e_\phi\left(\frac{\omega_i}{a}\right)\right| \leq \left|b \cdot_\phi e_\phi\left(\frac{\omega_i}{a}\right)\right|.$$

Proof. (1) has been shown in the proof of the lemma. As for (2), by the assumption, we have
$$\{\mu \in \Lambda \mid |a\mu| < |a_j\omega_j|\} \subset \{\mu \in \Lambda \mid |a\mu| < |a_i\omega_i|\}. \tag{6}$$
If μ satisfies $|a\mu| < |a_j\omega_j|$, we have $|a_j\omega_j|/|a\mu| \leq |a_i\omega_i|/|a\mu|$. Combining this inequality and (6), we have the desired inequality by (1). For (3), as $a \cdot_\phi e_\phi(\omega) = e_\phi(a\omega)$ for any $a \in A$ and any $\omega \in \mathbb{C}_\infty$, it remains to show
$$\left|e_\phi\left(\frac{\omega}{a}\right)\right| = \max_j \left\{\left|e_\phi\left(\frac{a_j\omega_j}{a}\right)\right|\right\}.$$
This equality follows from Lemma 1.2.1 and (2). As for (4), note $|\omega_i| < |b\omega_i|$. One can show (4) similarly to the proof of (2). □

Theorem 1.2.3. *For any finite prime u of A and any positive integer n, the family of elements $\{e_\phi(\omega_i/u^n)\}_{i=1,\dots,r}$ is an SMB of $\phi[u^n]$.*

Proof. Put $\lambda_i = e_\phi(\omega_i/u^n)$ for all i. Note that $\omega_1/u^n, \dots, \omega_r/u^n$ are A/u^n-linearly independent as elements in $u^{-n}\Lambda/\Lambda$. By the A/u^n-module isomorphism $\mathcal{E}_\phi : u^{-n}\Lambda/\Lambda \to \phi[u^n]$ induced by e_ϕ, we have that $\lambda_1, \dots, \lambda_r$ are A/u^n-linearly independent.

Fix a positive integer $i \leq r$. To check Definition 1.1 (2), we show that $|\lambda_i|$ is minimal among the absolute values of elements in $\phi[u^n] \setminus \bigoplus_{j<i}(A/u^n) \cdot_\phi \lambda_j$ (in $\phi[u^n] \setminus \{0\}$ if $i = 1$). Put $\lambda = \sum_j a_j \cdot_\phi \lambda_j$ with $a_j \in A \mod u^n$ such that there is $a_k \not\equiv 0$ for some $k \geq i$. We show $|\lambda_i| \leq |\lambda|$. Without loss of generality, we assume that $\deg(a_j) < \deg(u^n)$ for any j. Let l be an index so that $|a_l\omega_l| = |\sum_j a_j\omega_j|$. By Corollary 1.2.2 (3), we have
$$|\lambda| = |a_l \cdot_\phi \lambda_l|.$$
As $|a_k\omega_k| \leq |a_l\omega_l|$, Corollary 1.2.2 (2) implies
$$\left|e_\phi\left(\frac{a_k\omega_k}{u^n}\right)\right| \leq \left|e_\phi\left(\frac{a_l\omega_l}{u^n}\right)\right|,$$
hence $|a_k \cdot_\phi \lambda_k| \leq |a_l \cdot_\phi \lambda_l|$. As $|\omega_i| \leq |\omega_k|$, Corollary 1.2.2 (2) also implies $|\lambda_i| \leq |\lambda_k|$. By Corollary 1.2.2 (4), we have $|a_k| \cdot |\lambda_k| \leq |a_k \cdot_\phi \lambda_k|$. Combining the equality and inequalities, we have
$$|\lambda_i| \leq |\lambda_k| \leq |a_k| \cdot |\lambda_k| \leq |a_k \cdot_\phi \lambda_k| \leq |a_l \cdot_\phi \lambda_l| = |\lambda|.$$
□

Remark 1.2.4. We have shown in the above proof that $|\lambda_1|$ is minimal among the absolute values of nonzero elements in $\phi[u^n]$. Let $\{\lambda'_i\}_{i=1,\dots,r}$ be an SMB of $\phi[u^n]$. By Theorem 1.2.9 below, we can show that there exists an SMB $\{\omega'_i\}_{i=1,\dots,r}$ on Λ such that $e_\phi(\omega'_i/u^n) = \lambda'_i$ for all i. Hence λ'_1 has the minimal absolute value among elements in $\phi[u^n] \setminus \{0\}$.

Corollary 1.2.5. *Let $\{\lambda_i\}_{i=1,\dots,r}$ be an SMB of $\phi[u^n]$.*

(1) *If n is large enough so that $|u^n| \geq |\omega_r|/|\omega_1|$, then for $i = 1, \dots, r$, we have $|\lambda_i| \cdot |u^n| = |\omega_i|$.*
(2) *For any positive integer n, we have $|\lambda_r|/|\lambda_1| \geq |\omega_r|/|\omega_1|$.*
(3) *If n is large enough such that $|u^n| > |\omega_r|/|\omega_1|$, then we have $|\lambda_i| < |\omega_1|$ for $i = 1, \dots, r$.*

Proof. We show (1). Fix i to be one of $1, \dots, r$. Corollary 1.2.2 (1) implies

$$\left|e_\phi\left(\frac{\omega_i}{u^n}\right)\right| = \left|\frac{\omega_i}{u^n}\right| \cdot \prod_{\substack{\mu\in\Lambda\setminus\{0\}\\ |u^n\mu|<|\omega_i|}} |\omega_i|/|u^n\mu|. \tag{7}$$

For any $\mu \in \Lambda$, we have

$$|u^n\mu| \geq |u^n\omega_1| \geq |\omega_r| \geq |\omega_i|$$

by the hypothesis. Hence (7) implies

$$\left|e_\phi\left(\frac{\omega_i}{u^n}\right)\right| = \left|\frac{\omega_i}{u^n}\right|.$$

By Theorem 1.2.3, the family $\{e_\phi(\omega_i/u^n)\}_{i=1,\dots,r}$ is an SMB of $\phi[u^n]$. Hence we have

$$|\lambda_i| = \left|e_\phi\left(\frac{\omega_i}{u^n}\right)\right| \text{ for any } i \tag{8}$$

by Proposition 1.1.8 (2). (1) follows. Notice that (7) implies

$$\left|e_\phi\left(\frac{\omega_1}{u^n}\right)\right| = \left|\frac{\omega_1}{u^n}\right| \text{ and } \left|e_\phi\left(\frac{\omega_i}{u^n}\right)\right| \geq \left|\frac{\omega_i}{u^n}\right| \text{ for any } i.$$

(2) follows from (8). Since we know $|\lambda_r| = |\omega_r|/|u^n|$ by (1), we have

$$|\lambda_i| \leq |\lambda_r| = |\omega_r|/|u^n| < |\omega_r|/(|\omega_r|/|\omega_1|) = |\omega_1|$$

and (3) follows. □

Remark 1.2.6. By Corollary 1.2.5 (1) and (2), we have $|\lambda_i| \cdot |u^n| = |\omega_i|$ if n is large enough so that $|u^n| \geq |\lambda_r|/|\lambda_1|$.

Put $B := \{\omega \in \mathbb{C}_\infty \mid |\omega| < |\omega_1|\}$. Since $B \cap \Lambda = \varnothing$, the exponential function e_ϕ is injective on B. For any $\omega \in \mathbb{C}_\infty$, we have

$$|e_\phi(\omega)| = |\omega| \cdot \prod_{\substack{\mu\in\Lambda\setminus\{0\}\\ |\mu|\leq|\omega|}} \left|1 - \frac{\omega}{\mu}\right|. \tag{9}$$

Hence $|e_\phi(\omega)| = |\omega|$ for $\omega \in B$. This implies $e_\phi(B) \subset B$. Put $C := e_\phi(B)$. There is an inverse $\log_\phi : C \to B$ of e_ϕ defined by a power series with coefficients in K and $e_\phi : B \rightleftarrows C : \log_\phi$ are inverse to each other (See [**Pap23**, Section 5.1]).

Lemma 1.2.7. (1) *We have* $C \cap \phi[u^n] = B \cap \phi[u^n]$.
(2) *We have the following maps which are inverse to each other*

$$e_\phi : B \cap \mathcal{L} \rightleftarrows B \cap \phi[u^n] : \log_\phi,$$

where

$$\mathcal{L} := \left\{ \sum_i a_i(\omega_i/u^n) \,\middle|\, a_i \in A \text{ with } \deg(a_i) < \deg(u^n) \right\}$$

is a set of representatives of all elements in $u^{-n}\Lambda/\Lambda$.
(3) *For any* $\lambda \in B \cap \phi[u^n]$, *we have* $|\log_\phi(\lambda)| = |\lambda|$.

Proof. (1) We know $C \cap \phi[u^n] \subset B \cap \phi[u^n]$, which implies $\# B \cap \phi[u^n] \geq \# C \cap \phi[u^n]$. We show

$$\# C \cap \phi[u^n] \geq \# B \cap \mathcal{L} \geq \# B \cap \phi[u^n] \geq \# C \cap \phi[u^n].$$

As e_ϕ is injective on $\mathcal{L}$, we have $\# B \cap \mathcal{L} \leq \# C \cap \phi[u^n]$ and it remains to show $\# B \cap \mathcal{L} \geq \# B \cap \phi[u^n]$.

Put $B^c := \{\omega \in \mathbb{C}_\infty \mid |\omega| \geq |\omega_1|\}$, which is complementary to B in $\mathbb{C}_\infty$. Note that $\{\omega_i\}_{i=1,\dots,r}$ is an SMB. For any $\omega = \sum_j a_j(\omega_j/u^n) \in B^c \cap \mathcal{L}$, there is an index i so that $|\omega| = |a_i\omega_i|/|u^n|$. By Lemma 1.2.1, we have $|e_\phi(\omega)| = |e_\phi(a_i\omega_i/u^n)|$. By Corollary 1.2.2 (1), we have

$$|e_\phi(\omega)| = \left| e_\phi\left(\frac{a_i\omega_i}{u^n}\right)\right| \geq \left|\frac{a_i\omega_i}{u^n}\right| = |\omega| \geq |\omega_1|.$$

Since we have shown $|e_\phi(\omega)| \geq |\omega_1|$ for any $\omega \in B^c \cap \mathcal{L}$, we know $e_\phi(B^c \cap \mathcal{L}) \subset B^c \cap \phi[u^n]$. As e_ϕ is injective on $\mathcal{L}$, we have $\# B^c \cap \mathcal{L} \leq \# B^c \cap \phi[u^n]$. Notice that the cardinal of $\mathcal{L}$ and $\phi[u^n]$ are the same. We have $\# B \cap \mathcal{L} \geq \# B \cap \phi[u^n]$, as desired.

(2) The map $e_\phi : B \cap \mathcal{L} \to B \cap \phi[u^n]$ is injective. It is also surjective since $\# B \cap \mathcal{L} = \# B \cap \phi[u^n]$. Hence (2) follows.

(3) By (2), we have $\log_\phi(\lambda) \in B \cap \mathcal{L}$ and $e_\phi(\log_\phi(\lambda)) = \lambda$. Hence we have $|\log_\phi(\lambda)| = |\lambda|$ by (9). □

Remark 1.2.8. Do we have $e_\phi(B) = B$?

Let $\{\lambda_i\}_{i=1,\dots,r}$ denote an SMB of $\phi[u^n]$. Assume that the positive integer n is large enough so that $|u^n| > |\omega_r|/|\omega_1|$. By Corollary 1.2.5 (3) and Lemma 1.2.7 (1), for each i, we have $\lambda_i \in B \cap \phi[u^n] = C \cap \phi[u^n]$ and we put $\omega_i' := \log_\phi(\lambda_i)$.

Theorem 1.2.9. *The family* $\{u^n\omega_i'\}_{i=1,\dots,r}$ *is an SMB of* Λ.

We need a lemma in the proof.

Lemma 1.2.10. *Let $\{\eta_i\}_{i=1,\dots,r}$ be a family of elements in $u^{-n}\Lambda$. It is an SMB of $u^{-n}\Lambda$ if and only if $\{u^n\eta_i\}_{i=1,\dots,r}$ is an SMB of Λ.*

Proof of Lemma. For any $a_i \in A$, we have

$$\left|\sum_i a_i u^n \eta_i\right| = |u^n| \cdot \left|\sum_i a_i \eta_i\right|.$$

Then the lemma follows from Proposition 1.1.5. □

Proof of Theorem. By Lemma 1.2.10, it suffices to show that the family of elements $\{\omega'_i\}_{i=1,\dots,r}$ is an SMB of $u^{-n}\Lambda$. To check the first dot in Proposition 1.1.4 (1), we show that $\omega'_1, \dots, \omega'_r$ are A-linearly independent. Assume that there exist nonzero $a_i \in A$ such that $\sum_i a_i\omega'_i = 0$. We may assume $u^n \nmid a_i$ for some i, for otherwise we divide both sides of the equation $\sum_i a_i\omega'_i = 0$ by some power of u. Note that the map e_ϕ is A/u^n-linear. As some a_i satisfies $a_i \not\equiv 0 \mod u^n$ and $\lambda_1, \dots, \lambda_r$ are A/u^n-linearly independent, we have $e_\phi(\sum_i a_i\omega'_i) = \sum_i a_i \cdot_\phi \lambda_i \neq 0$. This is absurd.

Next, we check the second dot in Proposition 1.1.4 (1). Let $l_1 \le l_2 \le \cdots \le l_r$ be the invariant of $u^{-n}\Lambda$ as in Proposition 1.1.4 (2). Fix i to be a positive integer $\le r$. It suffices to show $l_i = |\omega'_i|$. We have $l_i \le |\omega'_i|$. Let us assume $l_i < |\omega'_i|$. As $\lambda_i \in B \cap \phi[u^n]$, we have $|\omega'_i| = |\lambda_i|$ by Lemma 1.2.7 (3). Hence $l_i < |\omega'_i| = |\lambda_i| < |\omega_1|$. By Proposition 1.1.4 (1), there is an SMB $\{\eta_j\}_{j=1,\dots,r}$ of $u^{-n}\Lambda$ such that $|\eta_i| = l_i < |\omega_1|$. As $|\eta_i| < |\omega_1|$, we know $|e_\phi(\eta_i)| = |\eta_i|$ from (9). We have

$$|e_\phi(\eta_i)| = |\eta_i| = l_i < |\omega'_i| = |\lambda_i|$$

and hence $|e_\phi(\eta_i)| < |\lambda_i|$. On the other hand, note that $\{u^n\eta_j\}_{j=1,\dots,r}$ is an SMB of Λ by Lemma 1.2.10. By Theorem 1.2.3, the elements $e_\phi(\eta_j)$ for $j = 1, \dots, r$ form an SMB of $\phi[u^n]$. By Proposition 1.1.8 (2), this contradicts $|e_\phi(\eta_i)| < |\lambda_i|$. □

Finally, we give two applications of Theorem 1.2.3 and 1.2.9.

Proposition 1.2.11. *If n is large enough so that $|u^n| > |\omega_r|/|\omega_1|$, then we have*

$$K(\Lambda) = K_n,$$

where $K(\Lambda)$ (resp. K_n) is the extension of K generated by all elements in Λ (resp. in $\phi[u^n]$).

Proof. (cf. the proof of [**Mau19**, Proposition 2.1]) Note that e_ϕ is given by a power series with coefficients in K. For any $x \in K^{\mathrm{sep}}$, we have $e_\phi(x) \in K(x)$ since the field $K(x)$ is complete. Since $\mathcal{E}_\phi : u^{-n}\Lambda/\Lambda \to \phi[u^n]$ is bijective, for any λ in $\phi[u^n]$, there exists $\omega \in u^{-n}\Lambda$ such that $e_\phi(\omega) = \lambda$. This implies $K(\lambda) \subset K(\omega)$ and $K_n \subset K(\Lambda)$.

Note that $\log_\phi$ is given by a power series with coefficients in K. For any $y \in c \cap K^{\mathrm{sep}}$, we similarly have $\log_\phi(y) \in K(y)$. Let $\{\lambda_i\}_{i=1,\dots,r}$ be an SMB of $\phi[u^n]$. As $|u^n| > |\omega_r|/|\omega_1|$, by

Theorem 1.2.9, the elements $u^n\omega_i'$ for $i = 1, \dots, r$ form an SMB of Λ, where $\omega_i' = \log_\phi(\lambda_i)$. Since $K(\omega_i') \subset K(\lambda_i)$ for each i, we have $K(\Lambda) \subset K_n$. □

Combining Corollary 1.2.5 (2), Theorem 1.2.9, and Proposition 1.2.11, we have

Corollary 1.2.12. *Let l be a positive integer and $\{\eta_i\}_{i=1,\dots,r}$ an SMB of $\phi[u^l]$. Let $\{\lambda_i\}_{i=1,\dots,r}$ be an SMB of $\phi[u^n]$. If n is large enough so that $|u^n| > |\eta_r|/|\eta_1|$, then we have*
(1) *the family $\{u^n \log_\phi(\lambda_i)\}_{i=1,\dots,r}$ is an SMB of Λ;*
(2) *$K(\Lambda) = K_n$.*

Proposition 1.2.13. *Let $\{\lambda_i\}_{i=1,\dots,r}$ be an SMB of $\phi[u^n]$. We have*

$$\left|\sum_i a_i \cdot_\phi \lambda_i\right| = \max_i\{|a_i \cdot_\phi \lambda_i|\}$$

for any $a_i \in A \mod u^n$.

Proof. Without loss of generality, we assume $\deg(a_i) < \deg(u^n)$ for all i. Assume first that n is large enough so that $|u^n| > |\lambda_r|/|\lambda_1|$ (Corollary 1.2.5 (2)). By Theorem 1.2.9, the elements $u^n\omega_i'$ for $i = 1, \dots, r$ form an SMB of Λ, where $\omega_i' = \log_\phi(\lambda_i)$. By Corollary 1.2.2 (3), we have

$$\left|e_\phi\left(\sum_i a_i\omega_i'\right)\right| = \max_i\{|a_i \cdot_\phi e_\phi(\omega_i')|\}.$$

As $e_\phi(\sum_i a_i\omega_i') = \sum_i a_i \cdot_\phi \lambda_i$, the claim follows.

For any n, let n' be an integer $\geq n$ so that $|u^{n'}| > |\lambda_r|/|\lambda_1|$. By Proposition 1.1.11 (1), there is an SMB $\{\lambda_i'\}_{i=1,\dots,r}$ of $\phi[u^{n'}]$ such that $u^{n'-n} \cdot_\phi \lambda_i' = \lambda_i$ for all i. Then the desired equation for $\{\lambda_i\}_{i=1,\dots,r}$ follows from that for $\{\lambda_i'\}_{i=1,\dots,r}$. □

3. Relations between SMBs, the finite prime case

Let v denote a finite prime. Throughout this section, unless otherwise specified, assume that ϕ has stable reduction and the reduction of ϕ has rank $r' < r$. Let ψ and Λ denote respectively the rank r' Drinfeld A-module associated to ϕ and A-lattice of the rank $r - r'$ associated to ϕ. Throughout this subsection, let $\{\omega_i^0\}_{i=r'+1,\dots,r}$ be an SMB of Λ. Let $|-|$ denote the function in (F2) and put $|a|_\infty := q^{\deg(a)}$ for any $a \in A$. For a positive integer n and a finite prime u of A, we study the relations between SMBs of $\psi[u^n]$, those of Λ, and those of $\phi[u^n]$.

Remark 1.3.1. For each Drinfeld A-module ϕ over K, there exists an element b in some extension L of K which is at worst tamely ramified such that the Drinfeld module $b\phi b^{-1}$ isomorphic to ϕ has stable reduction on L. For example, we can take L/K to be $K(\phi[u])/K$ or its certain subextension. For an SMB $\{\lambda_i\}_{i=1,\dots,r}$ of $\phi[u^n]$, the family

$\{b\lambda_i\}_{i=1,\dots,r}$ is an SMB of $b\phi b^{-1}[u^n]$. If $b\phi b^{-1}$ has bad reduction, for the Tate uniformization (ψ', Λ') associated to $b\phi b^{-1}$, we may apply the results in this section to $\{b\lambda_i\}_{i=1,\dots,r}$, the SMBs of $\psi'[u^n]$, and those of Λ'.

First, we are concerned with the valuations of the elements in the A-module $u^{-n}\Lambda$, i.e., the roots of $\psi_{u^n}(X) - \omega$ for all $\omega \in \Lambda$.

Lemma 1.3.2. *Let a be an element in A.*

(1) *Each root of $\psi_a(X)$ has valuation ≥ 0. Moreover, all nonzero roots of $\psi_a(X)$ have valuation $= 0$ if and only if $v(a) = 0$.*
(2) *For a nonzero element $\omega \in \Lambda$, each root of $\psi_a(X) - \omega$ has valuation < 0.*
(3) *An element $\omega \in a^{-1}\Lambda$ belongs to $\psi[a]$ if and only if it has valuation ≥ 0.*

Proof. Put $g := r' \cdot \deg(a)$, $a_0 := a$, $\sum_{i=0}^{g} a_i X^{q^i} := \psi_a(X)$, and $P_i = (q^i, v(a_i))$ for $i = 0, \dots, g$. As $v(a_i) \geq 0$ and $v(a_g) = 0$, the segments in the Newton polygon of $\psi_a(X)$ have slopes ≤ 0. If $v(a_0) = 0$, then the Newton polygon of $\psi_a(X)$ consists of exactly one segment P_0P_g which has slope 0 (We will always omit the segment in the Newton polygon with infinite slope). Hence each root of $\psi_a(X)$ has valuation $= 0$. If $v(a_0) > 0$, then the left-most segment in the Newton polygon of $\psi_a(X)$ has negative slope. Hence some root of $\psi_a(X)$ has valuation > 0.

As for (2), put $Q := (0, v(\omega))$. As $v(\omega) < 0$, $v(a_i) \geq 0$ for all i, and $v(a_g) = 0$, the Newton polygon of $\psi_a(X) - \omega$ consists of exactly one segment QP_g whose slope is $-v(\omega)/q^g > 0$. Hence (2) follows. From (1) and (2), we know (3). □

Fix a root ω_i of $\psi_{u^n}(X) - \omega_i^0$ for $i = r'+1, \dots, r$. The elements $\omega_{r'+1}, \dots, \omega_r$ are A-linearly independent. For all $a_i \in A$, we have

$$|u^n|_\infty \cdot \left| \sum_{i=r'+1}^{r} a_i \cdot_\psi \omega_i \right| = \left| \sum_{i=r'+1}^{r} a_i u^n \cdot_\psi \omega_i \right| = \left| \sum_{i=r'+1}^{r} a_i \cdot_\psi \omega_i^0 \right|.$$

Hence, by Proposition 1.1.5, we have

$$\left| \sum_{i=r'+1}^{r} a_i \cdot_\psi \omega_i \right| = \max_{i=r'+1,\dots,r} \{|a_i \cdot_\psi \omega_i|\} \tag{10}$$

for any $a_i \in A$.

In the remainder of this section, let $\{\omega_i\}_{i=1,\dots,r'}$ be an SMB of $\psi[u^n]$ and $\omega_{r'+1}, \dots, \omega_r$ be elements in $u^{-n}\Lambda$ defined as above. The family $\{\omega_i\}_{i=1,\dots,r}$ form an A/u^n-basis of $u^{-n}\Lambda/\Lambda$. Next, we study the relations between $\{\omega_i\}_{i=1,\dots,r}$ and SMBs of $\phi[u^n]$.

Lemma 1.3.3. (1) *For all $a_i \in A$, we have*

$$\Big|\sum_i a_i \cdot_\psi \omega_i\Big| = \begin{cases} |\sum_{i\le r'} a_i \cdot_\psi \omega_i| \le 0 & \text{all } a_i = 0 \text{ for } i > r'; \\ |\sum_{i>r'} a_i \cdot_\psi \omega_i| > 0 & \text{some } a_i \neq 0 \text{ for } i > r'. \end{cases}$$

(2) *Let a_i be elements in A for $i = 1, \dots, r$. Assume either $v(u) = 0$, or some a_i is nonzero for $i > r'$. Then we have*

$$\Big|\sum_i a_i \cdot_\psi \omega_i\Big| = \max_i\{|a_i \cdot_\psi \omega_i|\}.$$

Proof. (1) Since $\sum_{i\le r'} a_i \cdot_\psi \omega_i \in \psi[u^n]$, we have $|\sum_{i\le r'} a_i \cdot_\psi \omega_i| \le 0$ by Lemma 1.3.2 (3). Since $u^n \cdot_\psi \omega_i$ for all $i = r'+1, \dots, r$ are elements in Λ, we have $|u^n|_\infty \cdot |\omega_i| > 0$ and hence $|a_i|_\infty \cdot |\omega_i| > 0$ if a_i is nonzero. Hence, by (10) and the ultrametric inequality, we have $|\sum_i a_i \cdot_\psi \omega_i| = |\sum_{i>r'} a_i \cdot_\psi \omega_i| > 0$ if some a_i for $i > r'$ is nonzero. (1) follows.

(2) If some $a_i \neq 0$ for $i > r'$, the desired equality follows from (1) and (10). By Lemma 1.3.2 (1), the assumption $v(u) = 0$ implies that the elements in $\psi[u^n]$ have valuation 0. Hence $|\sum_{i\le r'} a_i \cdot_\psi \omega_i| = 0$ and $|a_i \cdot_\psi \omega_i| = 0$ for all $i \le r'$. The desired equality similarly follows. □

Recall for any $\omega \in \mathbb{C}_v$, we have

$$e_\phi(\omega) = \omega \prod_{\mu\in\Lambda\setminus\{0\}} \left(1 - \frac{\omega}{\mu}\right).$$

Its valuation is

$$v(e_\phi(\omega)) = v(\omega) + \sum_{\substack{\mu\in\Lambda\setminus\{0\} \\ v(\mu)\ge v(\omega)}} v\left(1 - \frac{\omega}{\mu}\right). \tag{11}$$

For certain $\omega = \sum_i a_i \cdot_\psi \omega_i \in u^{-n}\Lambda$, we are to calculate $|e_\phi(\omega)|$.

Lemma 1.3.4. *If $\omega = \sum_{i\le r'} a_i \cdot_\psi \omega_i$ with $a_i \in A \mod u^n$, we have*

$$|e_\phi(\omega)| = |\omega|.$$

Proof. By (11), it suffices to show $v(1 - \omega/\mu) = 0$ for each $\mu \in \Lambda$. Notice $v(\omega) \ge 0$ by Lemma 1.3.3 (1). Since $v(\mu) < 0$ for any $\mu \in \Lambda$, we have $v(1 - \omega/\mu) = 0$ by the ultrametric inequality. □

Lemma 1.3.5 (cf. Lemma 1.2.1). *For $\omega = \sum_j a_j \cdot_\psi \omega_j \in u^{-n}\Lambda$, assume some a_j for $j > r'$ is nonzero. Let i be an integer $> r'$ so that $|\omega| = |a_i \cdot_\psi \omega_i| = \max_j\{|a_j \cdot_\psi \omega_j|\}$ (By Lemma 1.3.3 (2), such an i exists). Assume $\deg(a_i) < \deg(u^n)$. Then we have*

$$|e_\phi(\omega)| = |e_\phi(a_i \cdot_\psi \omega_i)|.$$

PROOF. By (11), it suffices to show

$$v\left(1-\frac{\omega}{\mu}\right)=v\left(1-\frac{a_i\cdot_\psi \omega_i}{\mu}\right)$$

for each $\mu\in\Lambda$ with $v(\mu)\geq v(\omega)$. If $v(\mu)>v(\omega)$, then we have by the ultrametric inequality that

$$v\left(1-\frac{\omega}{\mu}\right)=v\left(\frac{\omega}{\mu}\right)=v\left(\frac{a_i\cdot_\psi \omega_i}{\mu}\right)=v\left(1-\frac{a_i\cdot_\psi \omega_i}{\mu}\right).$$

Next, we show

$$v\left(1-\frac{\omega}{\mu}\right)=v\left(1-\frac{a_i\cdot_\psi \omega_i}{\mu}\right)=0$$

if $v(\mu)=v(\omega)=v(a_i\cdot_\psi\omega_i)$. It suffices to show

$$v(\omega-\mu)=v(\omega)\text{ and }v(a_i\cdot_\psi\omega_i-\mu)=v(a_i\cdot_\psi\omega_i).$$

As $\deg(a_i)<\deg(u^n)$, we have

$$|\omega|=|a_i\cdot_\psi\omega_i|=|a_i|_\infty\cdot|\omega_i|<|u^n|_\infty\cdot|\omega_i|=|\omega_i^0|$$

and hence $|\mu|=|\omega|<|\omega_i^0|$. This implies $\mu\in\bigoplus_{j=r'+1}^{i-1}A\cdot_\psi\omega_j^0$, for otherwise we have $|\mu|\geq|\omega_i^0|$ by Proposition 1.1.5 (2). Applying Lemma 1.3.3 (2) to $|\omega-\mu|$ and $|a_i\cdot_\psi\omega_i-\mu|$, we obtain the desired equalities. □

Corollary 1.3.6 (cf. Corollary 1.2.2). (1) *With the notation in the lemma, we have*

$$v(e_\phi(\omega))=v(\omega)+\sum_{\substack{\mu\in\Lambda\setminus\{0\}\\ v(\mu)>v(\omega)}}\left(v(\omega)-v(\mu)\right).$$

Particularly, for any $i=1,\dots,r$ and any $a_i\in A\setminus\{0\}$ satisfying $\deg(a_i)<\deg(u^n)$, *we have*

$$v(e_\phi(a_i\cdot_\psi\omega_i))=v(a_i\cdot_\psi\omega_i)+\sum_{\substack{\mu\in\Lambda\setminus\{0\}\\ v(\mu)>v(a_i\cdot_\psi\omega_i)}}\big(v(a_i\cdot_\psi\omega_i)-v(\mu)\big).$$

(2) *For any positive integers $i,j\leq r$, let a_i and a_j be elements in A with degree strictly smaller than that of u^n. Assume $|a_j\cdot_\psi\omega_j|\leq|a_i\cdot_\psi\omega_i|$. Then*

$$|e_\phi(a_j\cdot_\psi\omega_j)|\leq|e_\phi(a_i\cdot_\psi\omega_i)|.$$

(3) *With the notation in the lemma, we have*

$$|e_\phi(\omega)|=\max_j\{|a_j\cdot_\phi e_\phi(\omega_j)|\}.$$

(4) *For any positive integer $i=r'+1,\dots,r$ and $b\in A$ satisfying* $\deg(b)<\deg(a)$, *we have*

$$|b|_\infty\cdot|e_\phi(\omega_i)|\leq|b\cdot_\phi e_\phi(\omega_i)|.$$

Proof. If $i \le r'$, then we have $v(e_\phi(a_i \cdot_\psi \omega_i)) = v(a_i \cdot_\psi \omega_i)$ by Lemma 1.3.4. The rest of (1) follows have been shown in the lemma. Similarly to the proof of Corollary 1.2.2 (2) (resp. (3)), the claim (2) (resp. (3)) follows from (1) (resp. the lemma and (2)).

We show (4). Note $b \cdot_\phi e_\phi(\omega_i) = e_\phi(b \cdot_\psi \omega_i)$. By (1), the desired inequality in (4) is equivalent to

$$\begin{aligned} &|b|_\infty^{r'} \cdot \Bigg(v(\omega_i) + \sum_{\substack{\mu\in\Lambda\setminus\{0\} \\ v(\mu)>v(\omega_i)}} \big(v(\omega_i) - v(\mu)\big)\Bigg) \\ &\ge v(b \cdot_\psi \omega_i) + \sum_{\substack{\mu\in\Lambda\setminus\{0\} \\ v(\mu)>v(b\cdot_\psi\omega_i)}} \big(v(b \cdot_\psi \omega_i) - v(\mu)\big). \end{aligned} \tag{12}$$

By Lemma 1.1.1 (2), we may write the left in this inequality to be

$$v(b \cdot_\psi \omega_i) + \sum_{\substack{\mu\in\Lambda\setminus\{0\} \\ v(\mu)>v(\omega_i)}} \big(v(b \cdot_\psi \omega_i) - v(b \cdot_\psi \mu)\big).$$

Then (12) follows from the inclusion

$$\{b \cdot_\psi \mu \in b \cdot_\psi \Lambda \mid v(b \cdot_\psi \mu) > v(b \cdot_\psi \omega_i)\} \subset \{\mu \in \Lambda \mid v(\mu) > v(b \cdot_\psi \omega_i)\}.$$

□

Theorem 1.3.7 (cf. Theorem 1.2.3). *For any finite prime u of A and any positive integer n, let $\{\omega_i\}_{i=1,\dots,r}$ be the elements in $u^{-n}\Lambda$ defined before Lemma 1.3.3. Then the family of elements $\{e_\phi(\omega_i)\}_{i=1,\dots,r}$ is an SMB of $\phi[u^n]$.*

Proof. Put $\lambda_i \coloneqq e_\phi(\omega_i)$ for all i. Since $\omega_1, \dots, \omega_r$ form an A/u^n-basis of $u^{-n}\Lambda/\Lambda$, their images under the A/u^n-module isomorphism $\mathcal{E}_\phi : u^{-n}\Lambda/\Lambda \to \phi[u^n]$ are A/u^n-linearly independent.

We check Definition 1.1 (2). Fix a positive integer $i \le r$. For $\lambda = \sum_j a_j \cdot_\phi \lambda_j$ with $a_j \in A \mod u^n$ such that $\lambda_1, \dots, \lambda_{i-1}, \lambda$ are A/u^n-linearly independent, we show $|\lambda_i| \le |\lambda|$. Without loss of generality, we assume $\deg(a_j) < \deg(u^n)$ for any j.

Assume first $i \le r'$. If $a_j = 0$ for all $j > r'$, the desired inequality follows from $\{\omega_j\}_{j=1,\dots,r'}$ being an SMB of $\psi[u^n]$ and Lemma 1.3.4. If $a_j \ne 0 \mod u^n$ for some $j > r'$, we can apply Corollary 1.3.6 (1), and we have $|\sum_j a_j \cdot_\psi \omega_j| \le |\sum_j a_j \cdot_\phi \lambda_j|$. We know $|\sum_j a_j \cdot_\psi \omega_j| \ge 0$ from Lemma 1.3.3 (1). By Lemma 1.3.3 (1) and 1.3.4, we have $|\lambda_i| = |\omega_i| < 0$. Hence

$$|\lambda_i| = |\omega_i| < 0 \le \Bigg|\sum_j a_j \cdot_\psi \omega_j\Bigg| = \Bigg|\sum_j a_j \cdot_\phi \lambda_j\Bigg|.$$

As for the case $i \geq r'+1$, note that there is $a_k \neq 0$ for some $k \geq i$ as $\lambda_1, \dots, \lambda_{i-1}, \lambda$ are A/u^n-linearly independent. Similarly to the proof of Theorem 1.2.3, one can apply Corollary 1.3.6 (2), (3), and (4) to show the inequality $|\lambda_i| \leq |\lambda|$. ☐

Corollary 1.3.8 (cf. Corollary 1.2.5). *Let $\{\lambda_i\}_{i=1,\dots,r}$ be an SMB of $\phi[u^n]$.*

(1) *If n is large enough so that $|u^n|_\infty \geq |\omega_r^0|/|\omega_{r'+1}^0|$, then for $i = 1, \dots, r$, we have $|\lambda_i| = |\omega_i|$.*
(2) *For any positive integer n, we have $|\lambda_r|/|\lambda_{r'+1}| \geq |\omega_r^0|/|\omega_{r'+1}^0|$.*
(3) *If n is large enough so that $|u^n|_\infty > |\omega_r^0|/|\omega_{r'+1}^0|$, then we have $|\lambda_i| < |\omega_{r'+1}^0|$ for $i = 1, \dots, r$.*

Proof. The equation $|\lambda_i| = |\omega_i|$ for $i = 1, \dots, r'$ follows from Lemma 1.3.4. Similarly to the proof of Corollary 1.2.5, one can apply Corollary 1.3.6 (1), Theorem 1.3.7, and Proposition 1.1.8 (2) to show the rest of the claims. ☐

Remark 1.3.9. By Corollary 1.3.8 (1) and (2), we have $|\lambda_i| \cdot |u^n| = |\omega_i|$ if n is large enough so that $|u^n| \geq |\lambda_r|/|\lambda_{r'+1}|$.

Put $B := \{\omega \in \mathbb{C}_v \mid |\omega| < |\omega_{r'+1}^0|\}$. Since $B \cap \Lambda = \varnothing$, the exponential function e_ϕ is injective on B. By (11), we have $|e_\phi(\omega)| = |\omega|$ for $\omega \in B$. This implies $e_\phi(B) \subset B$. Put $C := e_\phi(B)$. There is an inverse $\log_\phi : C \to B$ of e_ϕ defined by a power series with coefficients in K and $e_\phi : B \rightleftarrows C : \log_\phi$ are inverse to each other (Although Λ is an A-module via ψ, the claims in [**Pap23**, Lemma 5.1.5] can be applied due to the discreteness of Λ).

Lemma 1.3.10 (cf. Lemma 1.2.7). (1) *We have $C \cap \phi[u^n] = B \cap \phi[u^n]$.*
(2) *We have the following maps which are inverse to each other*

$$e_\phi : B \cap \mathcal{L} \rightleftarrows B \cap \phi[u^n] : \log_\phi,$$

where

$$\mathcal{L} := \left\{ \sum_i a_i \cdot_\psi \omega_i \,\middle|\, a_i \in A \text{ with } \deg(a_i) < \deg(u^n) \right\}$$

is a set of representatives of all elements in $u^{-n}\Lambda/\Lambda$.
(3) *For any $\lambda \in B \cap \phi[u^n]$, we have $|\log_\phi(\lambda)| = |\lambda|$.*

Proof. We show $\#B \cap \mathcal{L} \geq \#B \cap \phi[u^n]$. Then following the proof of Lemma 1.2.7, one can obtain the rest of the proof. Put $B^c := \{\omega \in \mathbb{C}_v \mid |\omega| \geq |\omega_{r'+1}^0|\}$, which is complementary to B in $\mathbb{C}_v$. For any $\omega = \sum_j a_j \cdot_\psi \omega_j \in B^c \cap \mathcal{L}$, there exists $a_j \neq 0$ for some $j > r'$, for otherwise we have $|\omega| < 0 < |\omega_{r'+1}^0|$ by Lemma 1.3.3 (1). By Corollary 1.3.6 (1), we have

$$|e_\phi(\omega)| \geq |\omega| \geq |\omega_{r'+1}^0|.$$

Hence $e_\phi(B^c \cap \mathcal{L}) \subset B^c \cap \phi[u^n]$. As e_ϕ is injective on $\mathcal{L}$, we have $\#B^c \cap \mathcal{L} \leq \#B^c \cap \phi[u^n]$. Notice that the cardinal of $\mathcal{L}$ and $\phi[u^n]$ are the same. We have $\#B \cap \mathcal{L} \geq \#B \cap \phi[u^n]$, as desired. □

Lemma 1.3.11. *Let $\{\lambda_i\}_{i=1,\dots,r}$ be an SMB of $\phi[u^n]$. We have $v(\lambda_i) \geq 0$ for $i \leq r'$ and $v(\lambda_i) < 0$ for $i > r'$.*

PROOF. For a positive integer j, let $\{\lambda_{i,j}\}_{i=1,\dots,r}$ be an SMB of $\phi[u^j]$ as in Corollary 1.1.12. By Proposition 1.1.8 (2), we have $v(\lambda_i) = v(\lambda_{i,n})$ for all i. It suffices to show $v(\lambda_{r',n}) \geq 0$ and $v(\lambda_{r'+1,n}) < 0$.

We first show $v(\lambda_{r',1}) \geq 0$ and $v(\lambda_{r'+1,1}) < 0$. Put $d \coloneqq \deg(u)$, $u_0 \coloneqq u$, $\sum_{i=0}^{rd} u_i X^{q^i} \coloneqq \phi_u(X)$, and $P_i \coloneqq (q^i, v(u_i))$ for $i = 0, \dots, rd$. As ϕ has stable reduction, we have $v(u_i) \geq 0$ for all i, $v(u_{r'd}) = 0$, and $v(u_i) > 0$ for all $i > r'd$. Hence the point $P_{r'd}$ is a vertex of the Newton polygon of $\phi_u(X)$. The segments on the left (resp. right) of $P_{r'd}$ have slopes ≤ 0 (resp. slopes > 0). Hence there are exactly $q^{r'd}$ roots with valuations ≥ 0. Here $0 \in \phi[u]$ is considered to have valuation > 0.

We show $v(\lambda_{r',1}) \geq 0$ and $v(\lambda_{r'+1,1}) < 0$ by induction. By (4), we have $v(\lambda_{1,1}) \geq 0$. Fix a positive integer $k \leq r'$ and assume $v(\lambda_{i,1}) \geq 0$ for $i < k$. Then the elements $\lambda_{i,1}$ for $i < k$ generates an A/u-vector subspace of $\phi[u]$ containing $q^{(k-1)d}$ many elements. Since ϕ has stable reduction, for any $a \in A$, all coefficients of $\phi_a(X)$ have valuation ≥ 0. By the ultrametric inequality, we have $v(a \cdot_\phi \lambda_{i,1})$ for any $a \in A \mod u$ and $i < k$. Hence all the elements in the vector subspace have valuations ≥ 0. Since $q^{(k-1)d} < q^{r'd}$, there are elements in $\phi[u] \setminus \bigoplus_{i<k}(A/u) \cdot_\phi \lambda_{i,1}$ having valuation ≥ 0. By (4), we have $v(\lambda_{k,1}) > 0$. For $k = r'+1$, we have the same inductive hypothesis as above. However, since $q^{(k-1)d} = q^{r'd}$, each element in $\phi[u] \setminus \bigoplus_{i<k}(A/u) \cdot_\phi \lambda_{i,1}$ has valuation < 0 and hence $v(\lambda_{r'+1,1}) < 0$.

Next, we show $v(\lambda_{r',n}) \geq 0$ (resp. $v(\lambda_{r'+1,n}) < 0$) by induction. Assume $v(\lambda_{r',j-1}) \geq 0$ (resp. $v(\lambda_{r'+1,j-1}) < 0$). By Corollary 1.1.12, the element $\lambda_{r',j}$ (resp. $\lambda_{r'+1,j}$) is a root of $\phi_u(X) - \lambda_{r',j-1}$ (resp. $\phi_u(X) - \lambda_{r'+1,j-1}$) having the largest valuation. By the induction hypothesis and the valuations of the coefficients of $\phi_u(X)$, the left-most segment in the Newton polygon of $\phi_u(X) - \lambda_{r',j-1}$ (resp. $\phi_u(X) - \lambda_{r'+1,j-1}$) has slope ≤ 0 (resp. > 0). Hence we have $v(\lambda_{r',j}) \geq 0$ and $v(\lambda_{r'+1,j}) < 0$. □

Remark 1.3.12. Assume $v \nmid u$. By the above proof, we have $v(\lambda_i) = 0$ and $v(\lambda_j) < 0$ for $i = 1, \dots, r'$ and $j = r'+1, \dots, r$. Similarly to Remark 1.2.4, the element λ_1 has the maximal valuation among elements in $\phi[u^n] \setminus \{0\}$. This may fail if $v \mid u$. Indeed, let ϕ be a rank 2 Drinfeld A-module over K so that $0 < v(\boldsymbol{j}) < v_0 q$. For a degree 1 finite prime u of A, let $\{\xi_{i,j}\}_{i=1,2}$ be an SMB of $\phi[u^j]$ for $j \geq 1$ obtained as in Corollary 1.1.12. By Proposition 2.1.7 ([**AH22**, Proposition A.3 (1)]), we have $v(\xi_{1,1}) > v(\xi_{1,m})$.

Let $\{\lambda_i\}_{i=1,\dots,r}$ denote an SMB of $\phi[u^n]$. Assume that the positive integer n is large enough so that $|u^n|_\infty > |\omega^0_r|/|\omega^0_{r'+1}|$. By Corollary 1.3.8 (3) and Lemma 1.3.10 (1), for each i, we have $\lambda_i \in B \cap \phi[u^n] = C \cap \phi[u^n]$ and we put $\omega'_i := \log_\phi(\lambda_i)$.

Theorem 1.3.13 (cf. Theorem 1.2.9). (1) *The family of elements* $\{\omega'_i\}_{i=1,\dots,r'}$ *is an SMB of* $\psi[u^n]$.

(2) *The family of elements* $\{u^n \cdot_\psi \omega'_i\}_{i=r'+1,\dots,r}$ *is an SMB of* Λ.

Proof. (1) To check Definition 1.1 (1), we show that the elements ω'_i for $i \le r'$ belong to $\psi[u^n]$ and are A/u^n-linearly independent. By Lemma 1.3.10 (3) and Lemma 1.3.11, we have $v(\omega'_i) = v(\lambda_i) \ge 0$ for $i \le r'$. By Lemma 1.3.2 (3), this implies that $\omega'_i \in \psi[u^n]$ for $i \le r'$. Note that $\mathcal{E}_\phi : u^{-n}\Lambda/\Lambda \to \phi[u^n]$ is an A/u^n-module isomorphism induced by e_ϕ and $e_\phi(\omega'_i) = \lambda_i$. If $\sum_{i\le r'} a_i \cdot_\psi \omega'_i = 0$ with $a_i \in A \mod u^n$, then we have $\sum_{i \le r'} a_i \cdot_\phi \lambda_i = 0$. This implies $a_i \equiv 0 \mod u^n$ and hence the desired linear independence.

As $\{\lambda_i\}_{i=1,\dots,r}$ is an SMB of $\phi[u^n]$, we can straightforwardly check Definition 1.1 (2) using Lemma 1.3.4.

(2) Similarly to (1), we can apply Lemma 1.3.11, 1.3.10 (3), 1.3.2 (3) to show $\omega'_i \notin \psi[u^n]$ such that $u^n \cdot_\psi \omega'_i$ for $i > r'$ belong to Λ. We check the two dots in Proposition 1.1.4 (1). Let us show that $\omega'_{r'+1}, \dots, \omega'_r$ are A-linearly independent first. If there exist $a_i \in A$ such that $\sum_{i>r'} a_i \cdot_\psi \omega'_i = 0$, we can show $a_i \equiv 0 \mod u^n$ for all i similarly to (1). Assume $a_i \ne 0$ for some i. Let m be the integer such that $u^m \mid a_i$ for all $i > r'$ and $u^{m+1} \nmid a_i$ for some i. Then there exist $b_i \in A$ such that $a_i = b_i u^m$ for all $i > r'$ and $b_i \not\equiv 0 \mod u$ for some i. Hence $\sum_{i>r'} b_i \cdot_\psi \omega'_i$ is a root of $\psi_{u^m}(X)$ and we denote this root by ω. On the other hand,

$$u^n \cdot_\psi \omega = \sum_{i>r'} b_i \cdot_\psi (u^n \cdot_\psi \omega'_i) \in \Lambda.$$

Since $\Lambda \cap \psi[u^m] = 0$, we have $u^n \cdot_\psi \omega = 0$ and hence $\omega \in \psi[u^n]$. By (1), there exist $b_i \in A \mod u^n$ for $i \le r'$ such that $\omega = \sum_{i \le r'} b_i \cdot_\psi \omega'_i$. This equality implies

$$0 = e_\phi\left(\sum_{i>r'} b_i \cdot_\psi \omega'_i - \sum_{i\le r'} b_i \cdot_\psi \omega'_i\right) = \sum_{i>r'} b_i \cdot_\phi \lambda_i - \sum_{i \le r'} b_i \cdot_\phi \lambda_i.$$

As some $b_i \not\equiv 0 \mod u^n$, this is absurd.

Finally, we check the second dots in Proposition 1.1.4 (1). Put $l_{r'+1} \le \cdots \le l_r$ to be invariant of Λ as in Proposition 1.1.4 (2). Fix i to be a positive integer satisfying $r' < i \le r$. It suffices to show $l_i = |u^n \cdot_\psi \omega'_i|$. We have $l_i \le |u^n \cdot_\psi \omega'_i|$. Let us assume $l_i < |u^n \cdot_\psi \omega'_i|$. Since $\lambda_i \in B \cap \phi[u^n]$, we have $|\omega'_i| = |\lambda_i|$ by Lemma 1.3.10 (3). Hence $l_i/|u^n|_\infty < |\omega'_i| = |\lambda_i| < |\omega^0_{r'+1}|$. By Proposition 1.1.4 (1), there is an SMB $\{\eta^0_j\}_{j=r'+1,\dots,r}$ of Λ such that $|\eta^0_i| = l_i$. Let η_j be a root of $\psi_{u^n}(X) - \eta^0_j$ for all j (cf. the definition of ω_j

before Lemma 1.3.3). As $|\eta_i| = l_i/|u^n|_\infty < |\omega^0_{r'+1}|$, we have $|e_\phi(\eta_i)| = |\eta_i|$ by (11). This implies

$$|e_\phi(\eta_i)| = |\eta_i| = l_i/|u^n|_\infty < |\omega'_i| = |\lambda_i|.$$

By Theorem 1.3.7, the elements $e_\phi(\omega'_j)$ for $j = 1, \dots, r'$ and $e_\phi(\eta_j)$ for $j = r' + 1, \dots, r$ form an SMB of $\phi[u^n]$. By Proposition 1.1.8 (2), this contradicts $|e_\phi(\eta_i)| < |\lambda_i|$. □

Proposition 1.3.14 (cf. Proposition 1.2.11). *If n is large enough such that $|u^n|_\infty \geq |\omega^0_r|/|\omega^0_{r'+1}|$, then we have*

$$K(u^{-n}\Lambda) = K_n,$$

where $K(u^{-n}\Lambda)$ (resp. K_n) is the extension of K generated by all elements in $u^{-n}\Lambda$ (resp. in $\phi[u^n]$).

Proof. Note that e_ϕ is given by a power series with coefficients in K and it induces an isomorphism $\mathcal{E}_\phi : u^{-n}\Lambda/\Lambda \to \phi[u^n]$. Similarly to the proof of Proposition 1.2.11, one can show $K_n \subset K(u^{-n}\Lambda)$.

Note that $\log_\phi$ is given by a power series with coefficients in K. For any $y \in C \cap K^{\mathrm{sep}}$, we have $\log_\phi(y) \in K(y)$. Let $\{\lambda_i\}_{i=1,\dots,r}$ be an SMB of $\phi[u^n]$. As $|u^n|_\infty > |\omega^0_r|/|\omega^0_{r'+1}|$, by Theorem 1.3.13, the families $\{\omega'_i\}_{i=1,\dots,r'}$ and $\{u^n \cdot_\psi \omega'_i\}_{i=r'+1,\dots,r}$ are respectively the SMB of $\psi[u^n]$ and Λ, where $\omega'_i = \log_\phi(\lambda_i)$. Since $K(\omega'_i) \subset K(\lambda_i)$ for each i, it suffices to show that ω'_i for all i form a generating set of $u^{-n}\Lambda$. For any $\omega \in u^{-n}\Lambda$, it is a root of $\psi_{u^n}(X) - u^n \cdot_\psi \omega$. Note $u^n \cdot_\psi \omega \in \Lambda$. Since $\{u^n \cdot_\psi \omega'_i\}_{i=r'+1,\dots,r}$ is an SMB of Λ, we have $u^n \cdot_\psi \omega = \sum_{i>r'} a_i \cdot_\psi (u^n \cdot_\psi \omega'_i)$ for some $a_i \in A$. Hence $\sum_{i>r'} a_i \cdot_\psi \omega'_i$ is also a root of $\psi_{u^n}(X) - u^n \cdot_\psi \omega$. Since $\{\omega'_i\}_{i=1,\cdots,r'}$ is an SMB of $\psi[u^n]$, we have $\sum_{i>r'} a_i \cdot_\psi \omega'_i - \omega = \sum_{i\leq r'} a_i \cdot_\psi \omega'_i$ for some $a_i \in A \mod u^n$, $i \leq r'$ and the claim follows. □

Combining Corollary 1.3.8 (2), Theorem 1.3.13, and Propostion 1.3.14, we have

Corollary 1.3.15 (cf. Corollary 1.2.12). *Let l be a positive integer and $\{\eta_i\}_{i=1,\dots,r}$ an SMB of $\phi[u^l]$. Let $\{\lambda_i\}_{i=1,\dots,r}$ be an SMB of $\phi[u^n]$. If n is large enough such that $|u^n|_\infty > |\eta_r|/|\eta_{r'+1}|$, then we have*

(1) *the family $\{\log_\phi(\lambda_i)\}_{i=1,\dots,r'}$ is an SMB of $\psi[u^n]$;*
(2) *the family $\{u^n \cdot_\psi \log_\phi(\lambda_i)\}_{i=r'+1,\dots,r}$ is an SMB of Λ;*
(3) *$K(u^{-n}\Lambda) = K_n$.*

Proposition 1.3.16 (cf. Proposition 1.2.13). *Let ϕ be a Drinfeld A-module over K (not necessarily have stable reduction). Assume $v(u) = 0$, i.e., u is not divisible by the prime w. Let $\{\lambda_i\}_{i=1,\dots,r}$ be an SMB of $\phi[u^n]$. Then we have*

$$\left|\sum_i a_i \cdot_\phi \lambda_i\right| = \max_i\{|a_i \cdot_\phi \lambda_i|\}$$

for any $a_i \in A \mod u^n$.

PROOF. We first show the claim for the case where ϕ has stable reduction such that the reduction has rank $< r$. Assume that n is large enough such that $|u^n|_\infty > |\omega^0_r|/|\omega^0_{r'+1}|$. By Theorem 1.3.13, the families $\{\omega'_i\}_{i=1,\dots,r'}$ and $\{u^n \cdot_\psi \omega'_i\}_{i=r'+1,\dots,r}$ are respectively an SMB of $\psi[u^n]$ and Λ, where $\omega'_i = \log_\phi(\lambda_i)$. Without loss of generality, we assume $\deg(a_i) < \deg(u^n)$. Assume that a_i is nonzero for some $i > r'$. By Corollary 1.3.6 (3), we have

$$\left| e_\phi\left(\sum_i a_i \cdot_\psi \omega'_i \right) \right| = \max_i \{ |a_i \cdot_\phi e_\phi(\omega'_i)| \}.$$

As $e_\phi(\sum_i a_i \cdot_\psi \omega'_i) = \sum_i a_i \cdot_\phi \lambda_i$, the claim follows. If $a_i = 0$ for all $i > r'$, then $\sum_{i \le r'} a_i \cdot_\psi \omega'_i$ belongs to $\psi[u^n]$. By Lemma 1.3.2 (1), we have $|\sum_{i\le r'} a_i \cdot_\psi \omega'_i| = 0$ and $|a_i \cdot_\psi \omega'_i| = 0$ for all $i \le r'$. The desired equality follows from Lemma 1.3.4. Similarly to the proof of Proposition 1.2.13, the case where n is arbitrary follows from the case where n is large enough.

If ϕ does not have stable reduction, there exists b in some extension of K that is at worst tamely ramified such that the Drinfeld module $b\phi b^{-1}$ has stable reduction. If $b\phi b^{-1}$ has good reduction, then each element in $b\phi b^{-1}[u^n]$ has valuation 0 (Lemma 1.3.2 (1)). In this case, the claim trivially follows. If the reduction of $b\phi b^{-1}$ has rank $r' < r$, then we have

$$\left| \sum_i a_i \cdot_{b\phi b^{-1}} b\lambda_i \right| = \max_i \{ |a_i \cdot_{b\phi b^{-1}} b\lambda_i| \}.$$

We can rewrite the equation to be

$$\left| \sum_i b(a_i \cdot_\phi \lambda_i) \right| = \max_i \{ |b(a_i \cdot_\phi \lambda_i)| \},$$

where $b(a_i \cdot_\phi \lambda_i)$ denotes the usual multiplication of b and $a_i \cdot_\phi \lambda_i$ in K^{sep}. This equation is equivalent to

$$v\left(\sum_i b(a_i \cdot_\phi \lambda_i) \right) = \min_i \{ v(b(a_i \cdot_\phi \lambda_i)) \}.$$

This equation is

$$v(b) + v\left(\sum_i a_i \cdot_\phi \lambda_i \right) = v(b) + \min_i \{ v(a_i \cdot_\phi \lambda_i) \}.$$

Hence the claim follows. □

We have assumed above that any Drinfeld module has stable reduction. For a Drinfeld A-module ϕ over K which does not have stable reduction, it turns out that ϕ is isomorphic to a Drinfeld module having stable reduction over an at worst tamely ramified subextension of $K(\phi[u])/K$ with $w \nmid u$.

Proposition 1.3.17. *Let ϕ be a rank r Drinfeld A-module over K (not necessarily have stable reduction). Let u be a finite prime of A with $v \nmid u$. Assume that ϕ is isomorphic to a Drinfeld module having stable reduction over some extension of K and the reduction has rank $r' < r$. Let $\{\lambda_i\}_{i=1,\dots,r}$ be an SMB of $\phi[u]$. Then $b\phi b^{-1}$ has stable reduction over $K(\lambda_1)$ for $b = \lambda_1^{-1}$ and the extension $K(\lambda_1)/K$ is at worst tamely ramified.*

Proof. For $\phi_t(X) = tX + \sum_{i=1}^{r} a_i X^{q^i} \in K[X]$. Put $v_i = v(a_i)$ for $i = 1, \dots, r$. Let M be a tamely ramified extension of K of degree $q^{r'} - 1$. Let b be an element in M with valuation $v(b) = \frac{v_{r'}}{q^{r'}-1}$. Then $b\phi b^{-1}$ has stable reduction on M. The family $\{b\lambda_i\}_{i=1,\dots,r}$ is an SMB of $b\phi b^{-1}[u]$.

Let ψ and Λ denote the Drinfeld A-module having good reduction and the lattice associated to ϕ via the Tate uniformization. The elements of $\psi[u]$ has valuation 0 by Lemma 1.3.2 (1). Then e_ϕ maps an SMB $\{\omega_i\}_{i=1,\dots,s}$ of $\psi[u]$ to the first r' elements of an SMB of $\phi[u]$ (Theorem 1.3.7) and $v(e_\phi(\omega_i)) = v(\omega_i) = 0$ for $i = 1, \dots, s$ (Lemma 1.3.4). We may take $b\lambda_1$ to be $e_\rho(\omega_1)$ [1]. Hence $v(b\lambda_1) = 0$ and $v(\lambda_1) = -\frac{v_{r'}}{q^{r'}-1}$. As $M(b^{-1}e_\rho(\omega_1)) \subset M(\omega_1)$ and $M(\omega_1)/M$ is unramified, the extension $K(\lambda_1)/K$ is at worst tamely ramified. The Drinfeld module $b\rho b^{-1}$ with $b \in K(\lambda_1)$ and $v(b) = -\frac{v_s}{q^s-1}$ is isomorphic to ρ and have stable reduction over $K(\lambda_1)$. □

[1] Indeed, by Corollary 1.1.12, we can find an SMB $\{\widehat{\lambda}_i\}_{i=1,\dots,r}$ of $\phi[u^n]$ for a large enough n so that $b\lambda_i = u^{n-1} \cdot_{b\phi b^{-1}} \widehat{\lambda}_i$ for $i = 1, \dots, r$ and we could apply Theorem 1.3.13 to $\{\widehat{\lambda}_i\}_{i=1,\dots,r}$. If we take $\{\omega_i\}_{i=1,\dots,r'}$ to be $\{u^{n-1} \cdot_\psi (\log_{b\phi b^{-1}}(\widehat{\lambda}_i))\}_{i=1,\dots,r'}$, then e_ϕ maps $\{\omega_i\}_{i=1,\dots,r'}$ to $\{b\lambda_i\}_{i=1,\dots,r'}$, as desired.

CHAPTER 2

Valuations of SMBs of $\phi[u^n]$

Let K be a local field which is the completion of some global function field at a prime v. For a positive integer $r \geq 2$, let ϕ be a Drinfeld A-module over K such that

$$\phi_t(X) = tX + a_s X^{q^s} + a_r X^{q^r} \in K[X],$$

where s and r are two positive integers satisfying $s < r$. Put

$$\boldsymbol{j} := \frac{a_s^{(q^r-1)/(q-1)}}{a_r^{(q^s-1)/(q-1)}}.$$

We call it the j-invariant of ϕ.

In Section 1, for a degree 1 finite prime u of A, we calculate the valuations of elements of $\phi[u^n]$. In Section 2, for a finite prime u of A with arbitrary degree, we apply the results in Chapter 1 to the Drinfeld module ϕ to obtain the results similar to those in Section 1. Explicitly, under certain conditions [1], we calculate the valuations of SMBs of $\phi[u^n]$. When v is an infinite prime or ϕ has potentially bad reduction, for the lattice Λ (or the pair (ψ, Λ)) associated to ϕ via the (Tate) uniformization, we also calculate the valuations of SMBs of Λ and $\psi[u^n]$.

1. Valuations of elements in $\phi[u^n]$ with $\deg(u) = 1$

Let u be a finite prime of A with degree 1 throughout this subsection. Let v also denote the valuation on K so that the uniformizer of K has valuation 1. Let v_0, v_s, and v_r denote respectively the valuations of the coefficients u, a_s, and a_r of $\phi_u(X)$. We have

$$\begin{cases} v(u) = v_0 < 0 & v \text{ is infinite;} \\ v(u) = v_0 = 0 & v \text{ is finite and } v \nmid u; \\ v(u) = v_0 > 0 & v \text{ is finite and } v \mid u. \end{cases} \tag{13}$$

For each $j \in \mathbb{Z}$, put

$$\alpha_j := \frac{v_0 q^{js}(q^{r-s}-1)}{q-1}.$$

[1] If the prime v is infinite, to obtain the valuations of SMBs of $\phi[u^n]$, we assume either that n is large enough or $v(\boldsymbol{j}) \neq \frac{v_0 q^{js}(q^{r-s}-1)}{q-1}$ for $j \in \mathbb{Z}_{\geq 1}$. See Proposition 2.2.1 and Corollary 2.2.3. If the prime v is finite, we require either $v(u) = 0$ or $v(a_r) > v(u) > 0$. See Proposition 2.2.6 and 2.2.11.

As in Corollary 1.1.12, let $\{\xi_{i,j}\}_{i=1,2}$ be an SMB of $\phi[u^j]$ for each positive integer j. In this section, we are to obtain the valuations of $\xi_{i,j}$ for all i, j so that we can obtain the valuations of all elements in $\phi[u^n]$.

1.1. Valuations of elements in $\phi[u]$. Put $P_0 := (1, v_0)$, $P_s := (q^s, v_s)$, and $P_r := (q^r, v_r)$. We define $\mu(P, Q)$ to be the slope of the segment PQ for P, $Q \in \mathbb{R}^2$. We have $\mu(P_0, P_s) = \frac{v_s - v_0}{q^s - 1}$, $\mu(P_0, P_r) = \frac{v_r - v_0}{q^r - 1}$, and

$$\mu(P_0, P_s) - \mu(P_0, P_r) = \frac{v(\boldsymbol{j})(q-1) - v_0 q^s(q^{r-s} - 1)}{(q^s - 1)(q^r - 1)}.$$

Note

$$\alpha_1 = \frac{v_0 q^s (q^{r-s} - 1)}{q - 1}.$$

We see that $\mu(P_0, P_s) < \mu(P_0, P_r)$ if and only if $v(\boldsymbol{j}) < \alpha_1$. If $v(\boldsymbol{j}) < \alpha_1$, then the Newton polygon of $\phi_u(X)$ is $P_0 P_s P_r$ having exactly two segments $P_0 P_s$ and $P_s P_r$ (We omit the segment with infinite slope). We have

$$\mu(P_0, P_s) = \frac{v_s - v_0}{q^s - 1} \text{ and } \mu(P_s, P_r) = \frac{v_r - v_s}{q^s(q^{r-s} - 1)}.$$

There are q^s roots (resp. $q^s(q^{r-s} - 1)$ roots) of $\phi[u]$ having valuations $\geq -\mu(P_0, P_s)$ (resp. equal to $-\mu(P_s, P_r)$). By (4), we have

$$v(\xi_{i,1}) = \begin{cases} -\mu(P_0, P_s) = -\frac{v_s - v_0}{q^s - 1} & i = 1, \ldots, s; \\ -\mu(P_s, P_r) = -\frac{v_r - v_s}{q^s(q^{r-s} - 1)} & i = s+1, \ldots, r. \end{cases} \tag{14}$$

In the next few subsections, we calculate the valuations of $\xi_{i,j}$ for all $i = 1, \ldots, r$ and $j \geq 1$ following Corollary 1.1.12. It turns out that the different valuations of u in (13) lead to different results.

1.2. Infinite prime cases. Now $v(u) = v_0 < 0$. We resume the notations in the previous subsection. We have the following lemma.

Lemma 2.1.1. *Let v be an infinite prime and n a positive integer. For each $i = 1, \ldots, r$, put $Q_{i,n} := (0, v(\xi_{i,n}))$. Assume $v(\boldsymbol{j}) < \alpha_1$. Let m be the integer such that $v(\boldsymbol{j}) \in (\alpha_{m+1}, \alpha_m]$.*

(1) *Fix i to be one of $1, \ldots, s$. For $n \geq 1$, we have*

$$v(\xi_{i,n}) = -\left(v_0(n-1) - \frac{v_0}{q^s - 1} + \frac{v_s}{q^s - 1}\right).$$

Then the Newton polygon of $\phi_u(X) - \xi_{i,n}$ is $Q_{i,n} P_0 P_s P_r$ having exactly three segments.

(2) *Fix i to be one of $s+1, \dots, r$. We have that*

$$v(\xi_{i,n}) = \begin{cases} -\left(\dfrac{v_s}{q^s-1} - \dfrac{v(\boldsymbol{j})(q-1)}{q^{ns}(q^s-1)(q^{r-s}-1)}\right) & 1 \le n \le m; \\ -\left(v_0(n-m) + \dfrac{v_s}{q^s-1} - \dfrac{v(\boldsymbol{j})(q-1)}{q^{ms}(q^s-1)(q^{r-s}-1)}\right) & n \ge m+1. \end{cases}$$

If $1 \le n \le m-1$, then the Newton polygon of $\phi_u(X) - \xi_{i,n}$ is $Q_{i,n}P_sP_r$ having exactly two segments. If $n \ge m$, then the Newton polygon of $\phi_u(X) - \xi_{i,n}$ is $Q_{i,n}P_0P_sP_r$ having exactly three segments.

Proof. We show (1). Put $\mu_k^n := \mu(Q_{i,n}, P_k)$ for $k = 0, s, r$. We first show that the Newton polygon of $\phi_u(X) - \xi_{i,1}$ is $Q_{i,1}P_0P_sP_r$ having exactly three segments. We have

$$\mu_0^1 = v_0 + \frac{v_s - v_0}{q^s - 1},\ \mu_s^1 = \frac{v_s + \frac{v_s - v_0}{q^s-1}}{q^s - 0},\ \mu_r^1 = \frac{v_r + \frac{v_s - v_0}{q^s - 1}}{q^r - 0}.$$

We calculate

$$\mu_0^1 - \mu_s^1 = \frac{v_0(q^s-1)}{q^s} < 0,$$
$$\mu_s^1 - \mu_r^1 = \frac{v(\boldsymbol{j})(q-1) - v_0(q^{r-s}-1)}{q^r(q^s-1)} < 0.$$

Hence P_0 is a vertex of the Newton polygon. By the argument in Section 1.1, the Newton polygon has the desired form. Assume that (1) for $n-1$ is valid. Then the valuation of $\xi_{i,n}$ is calculated by $-\mu_0^{n-1} = -v_0 + v(\xi_{i,n-1})$. We show that the Newton polygon of $\phi_u(X) - \xi_{i,n}$ is $Q_{i,n}P_0P_sP_r$ having exactly three segments. We have

$$\mu_0^n = v_0 - v(\xi_{i,n}),\ \mu_s^n = \frac{v_s - v(\xi_{i,n})}{q^s - 0},\ \mu_r^n = \frac{v_r - v(\xi_{i,n})}{q^r - 0}.$$

We calculate

$$\mu_0^n - \mu_s^n = \frac{v_0 n(q^s-1)}{q^s} < 0,$$
$$\mu_s^n - \mu_r^n = \frac{v(\boldsymbol{j})(q-1) + v_0(q^{r-s}-1)((n-1)(q^s-1)-1)}{q^r(q^s-1)} < 0.$$

Then P_0 is a vertex of the Newton polygon. By the argument in Section 1.1, the Newton polygon has the desired form.

We check (2). Now i is one of $1, \dots, s$. It is straight to check that the value $v(\xi_{i,1})$ coincides with the one in (14). Let us consider the Newton polygon of $\phi_u(X) - \xi_{i,n}$. Put $\mu_k^n = \mu(Q_{i,n}, P_k)$ for $k = 0, r, s$. (μ_k^n differs from the one in the proof of (1)). For $n = 1$, we have

$$\mu_0^1 = v_0 + \frac{v_r - v_s}{q^s(q^{r-s}-1)},\ \mu_s^1 = \frac{v_s + \frac{v_r - v_s}{q^s(q^{r-s}-1)}}{q^s - 0},\ \mu_r^1 = \frac{v_r + \frac{v_r - v_s}{q^s(q^{r-s}-1)}}{q^r - 0}.$$

We calculate

$$\mu_0^1 - \mu_s^1 = \frac{-v(\boldsymbol{j})(q-1) + v_0 q^{2s}(q^{r-s}-1)}{q^{2s}(q^{r-s}-1)},$$
$$\mu_s^1 - \mu_r^1 = \frac{v(\boldsymbol{j})(q-1)}{q^{s+r}} < 0.$$

If $v(\boldsymbol{j}) \in (\alpha_2, \alpha_1)$, we have $\mu_0^1 < \mu_s^1$ and hence P_0 is a vertex of the Newton polygon. In this case, the Newton polygon is $Q_{i,1}P_0P_sP_r$ having exactly three segments by the argument in Section 1.1. If $v(\boldsymbol{j}) \leq \alpha_2$, we have $\mu_0^1 \geq \mu_s^1$ and the Newton polygon is $Q_{i,1}P_sP_r$ having exactly two segments.

Assume that (2) for $n-1$ is valid. If $n \leq m-1$, the valuation of $\xi_{i,n}$ is

$$-\mu_s^{n-1} = -\frac{v_s - v(\xi_{i,n-1})}{q^s - 0} = -\left(\frac{v_s}{q^s-1} - \frac{v(\boldsymbol{j})(q-1)}{q^{(n+1)s}(q^s-1)(q^{r-s}-1)}\right).$$

Next, we determine the Newton polygon of $\phi_u(X) - \xi_{i,n}$. We have

$$\mu_0^n = v_0 - v(\xi_{i,n}),\ \mu_s^n = \frac{v_s - v(\xi_{i,n})}{q^s - 0},\ \mu_r^n = \frac{v_r - v(\xi_{i,n})}{q^r - 0}. \tag{15}$$

We calculate

$$\mu_0^n - \mu_s^n = \frac{-v(\boldsymbol{j})(q-1) + v_0 q^{(n+1)s}(q^{r-s}-1)}{q^{(n+1)s}(q^{r-s}-1)}, \tag{16}$$

$$\mu_s^n - \mu_r^n = \frac{v(\boldsymbol{j})(q-1)(q^{ns}-1)}{q^{ns+r}(q^s-1)} < 0. \tag{17}$$

Since $n \leq m-1$, we have $\mu_0^n \geq \mu_s^n$. This implies that $Q_{i,n}P_s$ is the first segment of the Newton polygon and the Newton polygon is $Q_{i,n}P_sP_r$ having exactly two segments.

When $n = m$, we have the same inductive hypothesis as above and $v(\xi_{i,m}) = -\mu_s^{m-1}$. However, we have $\mu_0^m < \mu_s^m$ by (16). Thus P_0 is a vertex of the Newton polygon of $\phi_u(X) - \xi_{i,m}$ by (17). By the argument in Section 1.1, the Newton polygon is $Q_{i,m}P_0P_sP_r$ having exactly three segments.

If $n \geq m+1$, then the valuation of $\xi_{i,n}$ is calculated by $-\mu_0^{n-1} = -v_0 + v(\xi_{i,n-1})$. We show that the Newton polygon of $\phi_u(X) - \xi_{i,n}$ is $Q_{i,n}P_0P_sP_r$ having exactly three segments. We have μ_0^n, μ_s^n, and μ_r^n as in (15). We calculate

$$\mu_0^n - \mu_s^n = \frac{-v(\boldsymbol{j}) + v_0(q^{r-s}-1)(n-m)q^{ms} + v_0\frac{q^{(m+1)s}(q^{r-s}-1)}{q-1}}{q^{(m+1)s}}\frac{q-1}{q^{r-s}-1} < 0,$$
$$\mu_s^n - \mu_r^n = \frac{v(\boldsymbol{j})(q-1)(q^{ms}-1) + v_0(n-m)(q^s-1)q^{ms}(q^{r-s}-1)}{q^{ms+r}(q^s-1)} < 0.$$

Then P_0 is a vertex of the Newton polygon. By the argument in Section 1.1, the Newton polygon has the desired form. □

For the rn elements $\xi_{i,j}$ for $i = 1, \ldots, r$ and $j = 1, \ldots, n$, the next proposition claims that they form a basis of the rn-dimensional $\mathbb{F}_q$-vector space $\phi[u^n]$ and can be arranged with respect to their valuations.

Proposition 2.1.2. *Let v be an infinite prime and n a positive integer. Assume $v(\boldsymbol{j}) < \alpha_1$. Let m be the integer such that $v(\boldsymbol{j}) \in (\alpha_{m+1}, \alpha_m]$. Then the elements $\xi_{i,j}$ for $i = 1 \ldots, r$ and $j = 1, \ldots, n$ form a basis of the $\mathbb{F}_p$-vector space $\phi[u^n]$. Fix k to be one of $1, \ldots, s$, and l to be one of $s+1, \ldots, r$. If $n \leq m$, then we have*

$$v(\xi_{k,n}) > v(\xi_{k,n-1}) > \cdots > v(\xi_{k,1}) \geq v(\xi_{l,n}) > v(\xi_{l,n-1}) > \cdots > v(\xi_{l,1}), \tag{18}$$

where the equality holds if and only if $n = m$ and $v(\boldsymbol{j}) = \alpha_m$. If $n \geq m+1$, then we have

$$\begin{aligned} & v(\xi_{k,n}) > v(\xi_{k,n-1}) > \cdots > v(\xi_{k,n-m+1}) \\ \geq\ & v(\xi_{l,n}) > v(\xi_{k,n-m}) \\ \geq\ & v(\xi_{l,n-1}) > v(\xi_{k,n-m-1}) \\ \geq\ & \cdots \\ \geq\ & v(\xi_{l,m+1}) > v(\xi_{k,1}) \\ \geq\ & v(\xi_{l,m}) > v(\xi_{l,m-1}) > \cdots > v(\xi_{l,1}), \end{aligned} \tag{19}$$

where each equality holds if and only if $v(\boldsymbol{j}) = \alpha_m$. For any $\xi = \sum_{i,j} a_{ij}\xi_{i,j} \in \phi[u^n]$ with $a_{ij} \in \mathbb{F}_q$, we have

$$v(\xi) = \min_{i,j}\{v(a_{ij}\xi_{i,j})\}.$$

Proof. The inequalities (18) and (19) follow from

(1) $v(\xi_{i,j+1}) > v(\xi_{i,j})$ for $i = 1, \ldots, r$ and $j \geq 1$;
(2) $v(\xi_{k,1}) \geq v(\xi_{l,j})$ for $k = 1, \ldots, s$, $l = s+1, \ldots, r$, and $j \leq m$, where the equality holds if and only if $j = m$ and $v(\boldsymbol{j}) = \alpha_m$;
(3) $v(\xi_{k,j+1}) \geq v(\xi_{l,j+m}) > v(\xi_{k,j})$ for $k = 1, \ldots, s$, $l = s+1, \ldots, r$, and $j = 1, \ldots, n-m$, where the equality holds if and only if $v(\boldsymbol{j}) = \alpha_m$.

These inequalities follow from Lemma 2.1.1, e.g., the left and the right inequalities of (3) are equivalent to $v(\boldsymbol{j}) \leq \alpha_m$ and $v(\boldsymbol{j}) > \alpha_{m+1}$, respectively.

As $\{\xi_{i,n}\}_{i=1,\ldots,r}$ is an SMB of $\phi[u^n]$, for any element ξ in $\phi[u^n]$, we have $\xi = \sum_{i=1}^r b_i \cdot_\phi \xi_{i,n}$ for some $b_i \in A \mod u^n$ and we may assume $\deg(b_i) < \deg(u^n)$. As $\deg(u) = 1$, for each $i = 1, \ldots, r$, there exist $a_{ij} \in \mathbb{F}_q$ satisfying $b_i = \sum_{j=0}^{n-1} a_{ij}u^j$ and hence

$$\xi = \sum_{\substack{i=1,\ldots,r \\ j=0,\ldots,n-1}} a_{ij}\xi_{i,n-j}.$$

Moreover, $\xi_{i,j}$ for all i,j are $\mathbb{F}_q$-linearly independent, for otherwise, $\xi_{i,n}$ for all i are A/u^n-linearly independent. By Proposition 1.2.13, we have

$$v(\xi) = \min_{i=1,\dots,r}\{v(b_i \cdot_\phi \xi_{i,n})\}.$$

By the inequality (1), we have $v(b_i \cdot_\phi \xi_{i,n}) = \min_j\{v(a_{ij}\xi_{i,n-j})\}$ and hence

$$v(\xi) = \min_{i,j}\{v(a_{ij}\xi_{i,n-j})\}.$$

□

Let us look at the case where $v(\boldsymbol{j}) \in [\alpha_1, +\infty)$. There are claims similar to Lemma 2.1.1 and Proposition 2.1.2.

Proposition 2.1.3. *Let v be an infinite prime and n a positive integer. Assume $v(\boldsymbol{j}) \in [\alpha_1, +\infty)$.*

(1) *Fix i to be one of $1, \dots, r$. We have*

$$v(\xi_{i,n}) = -\left(v_0(n-1) + \frac{v_r - v_0}{q^r - 1}\right).$$

Put $Q_n = (0, v(\xi_{i,n}))$. Then the Newton polygon of $\phi_u(X) - \xi_{i,n}$ is $Q_n P_0 P_r$ having exactly two segments.

(2) *The roots $\xi_{i,j}$ for $i = 1, \dots, r$ and $j = 1, \dots, n$ form a basis of the $\mathbb{F}_q$-vector space $\phi[u^n]$. For $n \geq 1$ and each i, we have*

$$v(\xi_{i,n}) > v(\xi_{i,n-1}) > \cdots > v(\xi_{i,1}).$$

For any $x = \sum_{i,j} a_{ij}\xi_{i,j} \in \phi[u^n]$ with $a_{ij} \in \mathbb{F}_q$, we have

$$v(x) = \min_{i,j}\{v(a_{ij}\xi_{i,j})\}.$$

Proof. Similarly to the proof of Lemma 2.1.1 (1), we can prove (1) by induction on n. We put $\mu_k^n := \mu(Q_n, P_k)$ for $k = 0, s, r$. The calculations below are enough for the proof of (1)

$$\mu_0^1 - \mu_s^1 = \frac{-v(\boldsymbol{j})(q-1) + v_0(q^{r+s} - 2q^s + 1)}{q^s(q^r - 1)} < 0,$$

$$\mu_0^1 - \mu_r^1 = \frac{v_0(q^r - 1)}{q^r} < 0.$$

For an integer $n > 1$, we have by the induction hypothesis

$$\mu_0^n - \mu_s^n = \frac{-v(\boldsymbol{j})(q-1) + v_0(q^{r+s} - 2q^s + 1) + v_0(q^s - 1)(q^r - 1)(n-1)}{q^s(q^r - 1)},$$

$$\mu_0^n - \mu_r^n = \frac{v_0 n(q^r - 1)}{q^r} < 0.$$

The proof of (2) is similar to that of Proposition 2.1.2. □

1.3. Finite prime cases with $v \nmid u$. We have the following claims similar to Lemma 2.1.1, Proposition 2.1.2, and Proposition 2.1.3. As in Section 1.1, put $P_0 = (1, v_0)$, $P_s = (q^s, v_s)$, and $P_r = (q^r, v_r)$.

Proposition 2.1.4. *Let v be a finite prime satisfying $v \mid u$ and n a positive integer.*

(1) *Fix k to be one of $1, \ldots, s$, and l to be one of $s+1, \ldots, r$. If $v(\boldsymbol{j}) < 0$, we have*

$$v(\xi_{k,n}) = -\frac{v_s}{q^s - 1},$$
$$v(\xi_{l,n}) = -\left(\frac{v_s}{q^s - 1} - \frac{v(j)(q-1)}{q^{ns}(q^s - 1)(q^{r-s} - 1)}\right).$$

Put $Q_{i,n} = (0, v(\xi_{i,n}))$ for $i = 1, \ldots, r$. Then for each i, the Newton polygon of $\phi_u(X) - \xi_{i,n}$ is $Q_{i,n}P_sP_r$ having exactly two segments.

Moreover, the elements $\xi_{i,j}$ for all $i = 1, \ldots, r$ and $j = 1, \ldots, n$ form a basis of the $\mathbb{F}_q$-vector space $\phi[u^n]$. Then we have

$$v(\xi_{k,n}) = v(\xi_{k,n-1}) = \cdots = v(\xi_{k,1}) > v(\xi_{l,n}) > v(\xi_{l,n-1}) > \cdots > v(\xi_{l,1}).$$

For any $\xi = \sum_{i,j} a_{ij}\xi_{i,j} \in \phi[u^n]$ with $a_{ij} \in \mathbb{F}_q$, we have

$$v(\xi) = \min_{i,j}\{v(a_{ij}\xi_{i,j})\}.$$

(2) *Fix i to be one of $1, \ldots, r$. If $v(\boldsymbol{j}) \geq 0$, we have*

$$v(\xi_{i,n}) = -\frac{v_r}{q^r - 1}.$$

Put $Q = (0, v(\xi_{i,n}))$. Then for each i, the Newton polygons of $\phi_u(X) - \xi_{i,n}$ are QP_2 having exactly one segment. The roots $\xi_{i,j}$ for all $i = 1, \ldots, r$ and $j = 1, \ldots, n$ form a basis of the $\mathbb{F}_q$-vector space of $\phi[u^n]$.

Proof. The claims for the valuations and the Newton polygons are proved by induction on n. For those in (1), put $\mu_i^{k,n} := \mu(Q_{k,n}, P_i)$ and $\mu_i^{l,n} := \mu(Q_{l,n}, P_i)$ for $i = 0, s, r$. The following calculations are enough for the proof (It turns out these calculations can be obtained by replacing the v_0 in the proof of Lemma 2.1.1 with 0). We have

$$\begin{cases} \mu_0^{k,1} - \mu_s^{k,1} = 0; \\ \mu_s^{k,1} - \mu_r^{k,1} = \dfrac{v(\boldsymbol{j})(q-1)}{q^r(q^s-1)} < 0, \end{cases} \quad \text{and} \quad \begin{cases} \mu_0^{l,1} - \mu_s^{l,1} = \dfrac{-v(\boldsymbol{j})(q-1)}{q^{2s}(q^{r-s}-1)} > 0; \\ \mu_s^{l,1} - \mu_r^{l,1} = \dfrac{v(\boldsymbol{j})(q-1)}{q^{s+r}} < 0. \end{cases}$$

For an integer $n > 1$, we have by the induction hypothesis

$$\begin{cases} \mu_0^{k,n} - \mu_s^{k,n} = 0; \\ \mu_s^{k,n} - \mu_r^{k,n} = \dfrac{v(\boldsymbol{j})(q-1)}{q^r(q^s-1)} < 0, \end{cases} \quad \text{and} \quad \begin{cases} \mu_0^{l,n} - \mu_s^{l,n} = \dfrac{-v(\boldsymbol{j})(q-1)}{q^{(n+1)s}(q^s-1)} > 0; \\ \mu_s^{l,n} - \mu_r^{l,n} = \dfrac{v(\boldsymbol{j})(q^{ns}-1)(q-1)}{q^{ns+r}(q^s-1)} < 0. \end{cases}$$

As for the claims of valuations and Newton polygons in (2), put $\mu_i = \mu(Q, P_i)$ for $i = 0, s, r$. We have

$$\mu_0 - \mu_r = 0,$$
$$\mu_s - \mu_r = \frac{v(j)(q-1)}{q^s(q^r-1)} < 0.$$

It remains to show all $\xi_{i,j}$ form an $\mathbb{F}_q$-basis of $\phi[u^n]$ and $v(\xi) = \min_{i,j}\{v(a_{ij}\xi_{i,j})\}$. The proof is similar to that of Proposition 2.1.2 but we apply Proposition 1.3.16 instead. ☐

1.4. Finite prime cases with $v \mid u$. For each integer $j \leq 1$, we have

$$\alpha_j := \frac{v_0 q^{js}(q^{r-s}-1)}{q-1}.$$

We have the following claims similar to Lemma 2.1.1.

Lemma 2.1.5. *Let v be a finite prime satisfying $v \mid u$ and n a positive integer. Put $P_0 = (1, v_0)$, $P_s = (q^s, v_s)$, $P_r = (q^r, v_r)$, and $Q_{i,n} = (0, v(\xi_{i,n}))$ for $i = 1, \ldots, r$.*

(1) *Assume $0 < v(\boldsymbol{j}) < \alpha_1$. Let m be the integer such that $v(\boldsymbol{j}) \in [\alpha_{-(m-1)}, \alpha_{-(m-2)})$.*

(i) *Fix i to be one of $1, \ldots, s$. We have*

$$v(\xi_{i,n}) = \begin{cases} \dfrac{v_0}{q^{(n-1)s}(q^s-1)} - \dfrac{v_s}{q^s-1} & 1 \leq n \leq m; \\ \dfrac{v_0}{q^{(m-1)s+(n-m)r}(q^s-1)} - \dfrac{v(\boldsymbol{j})(q-1)}{q^{(n-m)r}(q^r-1)(q^s-1)} - \dfrac{v_r}{q^r-1} & n \geq m+1. \end{cases}$$

If $n \leq m-1$, then the Newton polygon of $\phi_u(X) - \xi_{i,n}$ is $Q_{i,n}P_sP_r$ having exactly two segments. If $n \geq m$, then the Newton polygon of $\phi_u(X) - \xi_{i,n}$ is $Q_{i,n}P_r$ having exactly one segment.

(ii) *Fix i to be one of $s+1, \ldots, r$. For $n \geq 1$, we have*

$$v(\xi_{i,n}) = \frac{v(\boldsymbol{j})(q-1)}{q^{(n-1)r+s}(q^{r-s}-1)(q^r-1)} - \frac{v_r}{q^r-1}.$$

The Newton polygon of $\phi_u(X) - \xi_{i,n}$ is $Q_{i,n}P_r$ having exactly one segment.

(2) *Assume $v(\boldsymbol{j}) \leq 0$. Fix k to be one of $1, \ldots, s$, and l to be one of $s+1, \ldots, r$. We have*

$$v(\xi_{k,n}) = \frac{v_0}{q^{(n-1)s}(q^s-1)} - \frac{v_s}{q^s-1},$$
$$v(\xi_{l,n}) = \frac{v(\boldsymbol{j})(q-1)}{q^{ns}(q^s-1)(q^{r-s}-1)} - \frac{v_s}{q^s-1}.$$

If $v(\boldsymbol{j}) < 0$, then for each i, the Newton polygon of $\phi_u(X) - \xi_{i,n}$ is $Q_{i,n}P_sP_r$ having exactly two segments. If $v(\boldsymbol{j}) = 0$, then the Newton polygon of $\phi_u(X) - \xi_{k,n}$ (resp. $\phi_u(X) - \xi_{l,n}$) is $Q_{k,n}P_sP_r$ (resp. $Q_{l,n}P_r$) having exactly two segments (resp. one segment).

Proof. Similarly to the proof of Lemma 2.1.1 (1), we can prove the lemma by induction on n. For (i) of (1), we put $\mu_k^n := \mu(Q_{i,n}, P_k)$ for $k = 0, s, r$. The calculations below are enough for the proof of (i). We have

$$\mu_0^1 - \mu_s^1 = \frac{v_0(q^s - 1)}{q^s} > 0,$$
$$\mu_s^1 - \mu_r^1 = \frac{v(\boldsymbol{j})(q-1) - v_0(q^{r-s} - 1)}{q^r(q^s - 1)}.$$

For an integer $1 < n < m$, we have by the induction hypothesis

$$\mu_0^n - \mu_s^n = \frac{v_0(q^{ns} - 1)}{q^{ns}} > 0,$$
$$\mu_s^n - \mu_r^n = \frac{v(\boldsymbol{j})(q-1)}{q^r(q^s - 1)} - \frac{v_0(q^{r-s} - 1)}{q^{(n-1)s+r}(q^s - 1)}.$$

The inequality $\mu_s^n - \mu_r^n < 0$ holds if and only if $v(\boldsymbol{j}) < \alpha_{-(n-1)}$. This is the case if $n = 1, \ldots, m-1$ and we have $\geq$ holds if $n = m, m+1, \ldots$. For $n \geq m+1$, we have by the induction hypothesis

$$\mu_0^n - \mu_r^n = \frac{v(\boldsymbol{j})(q-1)}{(q^s - 1)q^{(n+1-m)r}} + v_0\left(1 - \frac{q^r - 1}{(q^s - 1)q^{(m-1)s+(n+1-m)r}}\right) > 0,$$

$$\mu_s^n - \mu_r^n = v(\boldsymbol{j})\left(\frac{q-1}{q^s(q^r - 1)} + \frac{(q-1)(q^{r-s} - 1)}{q^{(n+1-m)r}(q^r - 1)(q^s - 1)}\right) - \frac{v_0(q^{r-s} - 1)}{q^{(m-1)s+(n+1-m)r}(q^r - 1)(q^s - 1)}. \tag{20}$$

Put

$$f(\boldsymbol{j}) := v(\boldsymbol{j})\frac{q-1}{q^s(q^r - 1)} - \frac{v_0(q^{r-s} - 1)}{q^{(m-1)s+(n+1-m)r}(q^r - 1)(q^s - 1)}.$$

Notice (20) $> f(\boldsymbol{j})$. We have $f(\boldsymbol{j}) > 0$ if and only if

$$v(\boldsymbol{j}) > \frac{v_0(q^{r-s} - 1)}{q^{(m-1)s}(q-1)} \cdot \frac{1}{q^{(n-m)r+r-s}(q^s - 1)}.$$

As this inequality holds, we have (20) > 0.

As for (ii) of (1), now i is one of $s+1, \ldots, r$. We put $\mu_k^n := (Q_{i,n}, P_k)$ for $k = 0, s, r$. The calculations below are enough for its proof. We have

$$\mu_0^1 - \mu_r^1 = \frac{v_0(q^{r-s} - 1)q^{r+s} - v(\boldsymbol{j})(q-1)}{q^{r+s}(q^{r-s} - 1)} > 0,$$
$$\mu_s^1 - \mu_r^1 = \frac{v(\boldsymbol{j})(q-1)}{q^{r+s}} > 0.$$

For an integer $n > 1$, we have by the induction hypothesis

$$\mu_0^n - \mu_r^n = \frac{v_0(q^{r-s}-1)q^{nr+s} - v(\boldsymbol{j})(q-1)}{q^{rn+s}(q^{r-s}-1)} > 0,$$
$$\mu_s^n - \mu_r^n = \frac{v(\boldsymbol{j})(q-1)}{q^r-1}\left(\frac{1}{q^s} - \frac{1}{q^{nr+s}}\right) > 0.$$

Finally, we show the result in (2) by induction on n. Put $\mu_i^{k,n} := \mu(Q_{k,n}, P_i)$ and $\mu_i^{l,n} := \mu(Q_{l,n}, P_i)$ for $i = 0, s, r$. The calculations below are enough for the proof of (2). We have

$$\begin{cases} \mu_0^{k,1} - \mu_s^{k,1} = v_0\left(\dfrac{q^s-1}{q^s}\right) > 0; \\ \mu_s^{k,1} - \mu_r^{k,1} = \dfrac{v(\boldsymbol{j})(q-1) - v_0(q^{r-s}-1)}{q^r(q^s-1)} < 0, \end{cases} \quad \text{and}$$
$$\begin{cases} \mu_0^{l,1} - \mu_s^{l,1} = v_0 - \dfrac{v(\boldsymbol{j})(q-1)}{q^{2s}(q^{r-s}-1)} > 0; \\ \mu_s^{l,1} - \mu_r^{l,1} = \dfrac{v(\boldsymbol{j})(q-1)}{q^{s+r}} \leq 0. \end{cases}$$

Here $\mu_s^{l,1} - \mu_r^{l,1} = 0$ if and only if $v(\boldsymbol{j}) = 0$. For an integer $n > 1$, we have by the induction hypothesis

$$\begin{cases} \mu_0^{k,n} - \mu_s^{k,n} = v_0\left(\dfrac{q^{ns}-1}{q^{ns}}\right) > 0, \\ \mu_s^{k,n} - \mu_r^{k,n} = \dfrac{v(\boldsymbol{j})(q-1)q^{(n-1)s} - v_0(q^{r-s}-1)}{q^{(n-1)s+r}(q^s-1)} < 0, \end{cases} \quad \text{and}$$
$$\begin{cases} \mu_0^{l,n} - \mu_s^{l,n} = v_0 - \dfrac{v(j)(q-1)}{q^{(n+1)s}(q^{r-s}-1)} > 0; \\ \mu_s^{l,n} - \mu_r^{l,n} = \dfrac{v(\boldsymbol{j})(q-1)}{q^r(q^s-1)}\left(1 - \dfrac{1}{q^{ns}}\right) \leq 0. \end{cases}$$

Here $\mu_s^{l,n} - \mu_r^{l,n} = 0$ if and only if $v(\boldsymbol{j}) = 0$. □

The next result concerns the case $v(\boldsymbol{j}) \geq \alpha_1$.

Lemma 2.1.6. *Let v be a finite prime satisfying $v \mid u$ and n a positive integer. Assume $v(\boldsymbol{j}) \in [\alpha_1, +\infty)$. Fix i to be one of $1, \ldots, r$. We have*

$$v(\xi_{i,n}) = \frac{v_0}{q^{(n-1)r}(q^r-1)} - \frac{v_r}{q^r-1}.$$

Put $Q_n = (0, v(\xi_{i,n}))$. Then the Newton polygons of $\phi_u(X) - \xi_{i,n}$ are Q_nP_r having exactly one segment.

PROOF. Similarly to the proof of Lemma 2.1.1 (1), we can prove the lemma by induction on n. We put $\mu_k^n := (Q_{i,n}, P_k)$ for $k = 0, s, r$. The following calculations are enough for the proof. We have

$$\mu_0^1 - \mu_r^1 = v_0\left(\frac{q^r - 1}{q^r}\right) > 0,$$

$$\mu_s^1 - \mu_r^1 = \frac{v(\boldsymbol{j})(q-1)}{q^s(q^r-1)} - \frac{v_0(q^{r-s}-1)}{q^r(q^r-1)} > 0.$$

For an integer $n > 1$, we have

$$\mu_0^n - \mu_r^n = v_0\left(\frac{q^{nr} - 1}{q^{nr}}\right) > 0,$$

$$\mu_s^n - \mu_r^n = \frac{v(\boldsymbol{j})(q-1)}{q^s(q^r-1)} - \frac{v_0(q^{r-s}-1)}{q^{rn-s}(q^r-1)} > 0.$$

□

Similar to Proposition 2.1.2, we have the next proposition.

Proposition 2.1.7. *Let v be a finite prime satisfying $v \mid u$ and n a positive integer. Fix k to be one of $1, \dots, s$, and l to be one of $s+1, \dots, r$.*

(1) *Assume $0 < v(\boldsymbol{j}) < \alpha_1$. Let m be the integer such that $v(\boldsymbol{j}) \in [\alpha_{-(m-1)}, \alpha_{-(m-2)})$. Fix k to be one of $1, \dots, s$, and l to be one of $s+1, \dots, r$. If $n \leq m$, we have*

$$v(\xi_{k,1}) > v(\xi_{k,2}) > \cdots > v(\xi_{k,n}) > v(\xi_{l,1}) > v(\xi_{l,2}) > \cdots > v(\xi_{l,n}).$$

If $n \geq m+1$, we have

$$\begin{aligned} & v(\xi_{k,1}) > v(\xi_{k,2}) > \cdots > v(\xi_{k,m}) \\ > & v(\xi_{l,1}) \geq v(\xi_{k,m+1}) \\ > & v(\xi_{l,2}) \geq v(\xi_{k,m+2}) \\ > & \cdots \\ > & v(\xi_{l,n-m}) \geq v(\xi_{k,n}) \\ > & v(\xi_{l,n-m+1}) > v(\xi_{l,n-m+2}) > \cdots > v(\xi_{l,n}), \end{aligned}$$

where each equality holds if and only if $v(\boldsymbol{j}) = \alpha_{-(m-1)}$.

(2) *Assume $v(\boldsymbol{j}) \leq 0$. We have*

$$v(\xi_{k,1}) > v(\xi_{k,2}) > \cdots > v(\xi_{k,n}) > v(\xi_{l,n}) \geq v(\xi_{l,n-1}) \geq \cdots \geq v(\xi_{l,1}),$$

where each equality holds if and only if $v(\boldsymbol{j}) = 0$.

(3) *Assume $v(\boldsymbol{j}) \geq \alpha_1$. We have*

$$v(\xi_{k,1}) = v(\xi_{l,1}) > v(\xi_{k,2}) = v(\xi_{l,2}) > \cdots > v(\xi_{k,n}) = v(\xi_{l,n}).$$

(4) *The elements* $\xi_{i,j}$ *for* $i = 1, \dots, r$ *and* $j = 1, \dots, n$ *form a basis of the* $\mathbb{F}_q$*-vector space* $\phi[u^n]$*. For any* $\xi = \sum_{i,j} a_{ij}\xi_{i,j} \in \phi[u^n]$ *with* $a_{ij} \in \mathbb{F}_q$*, we have*

$$v(\xi) = \min_{i,j}\{v(a_{ij}\xi_{i,j})\}.$$

Proof. We show (1). The two inequalities follow from

- $v(\xi_{i,j}) > v(\xi_{i,j+1})$ for $i = 1, \dots, r$ and $j \geq 1$;
- $v(\xi_{k,n}) > v(\xi_{l,j})$ for $k = 1, \dots, s$, $l = s+1, \dots, r$, and $j \geq n-m+1$. Note $v(\xi_{k,n}) > v(\xi_{l,n-m+1})$ if and only if $v(\boldsymbol{j}) < \alpha_{-(m-2)}$;
- $v(\xi_{k,m+j-1}) > v(\xi_{l,j}) \geq v(\xi_{k,m+j})$ for $j = 1, \dots, n-m$, where the equality holds if and only if $v(\boldsymbol{j}) = \alpha_{-(m-1)}$.

These inequalities follow from Lemma 2.1.5, e.g., the left and the right inequalities in the third dot are equivalent to $v(\boldsymbol{j}) < \alpha_{-(m-2)}$ and $v(\boldsymbol{j}) \geq \alpha_{-(m-1)}$, respectively. The claims (2) and (3) follow similarly to (1).

Similarly to the proof of Proposition 2.1.2, we can show the first claim in (4). We show $v(\xi) = \min_{i,j}\{v(a_{ij}\xi_{i,j})\}$ for the case where $v(\boldsymbol{j}) \in (0, \alpha_1)$. By the ultrametric inequality, it suffices to show when $v(\boldsymbol{j}) = \alpha_{-(m-1)}$ that

$$v(a\xi_{l,j} + b\xi_{k,m+j}) = v(\xi_{l,j}) = v(\xi_{k,m+j}), \text{ for } a, b \in \mathbb{F}_q \text{ and } j = 1, \dots, n-m. \tag{21}$$

Notice $a\xi_{l,1} + b\xi_{k,m+1}$ is a root of $\phi_u(X) - b\xi_{k,m}$. By Lemma 2.1.5 (1), we know that the Newton polygon of $\phi_u(X) - b\xi_{k,m}$ has exactly one segment. This shows (21) for $j = 1$. We show (21) by induction on j. Assume (21) for positive integers $j-1$. Notice $a\xi_{l,j} + b\xi_{k,m+j}$ is a root of $\phi_u(X) - a\xi_{l,j-1} - b\xi_{k,m+j-1}$ and $v(a\xi_{l,j-1} + b\xi_{k,m+j-1}) = v(\xi_{k,m+j-1})$. The Newton polygon of $\phi_u(X) - a\xi_{l,j-1} - b\xi_{k,m+j-1}$ is the same as that of $\phi_u(X) - \xi_{k,m+j-1}$. By Lemma 2.1.5 (1), we know that the Newton polygon of $\phi_u(X) - a\xi_{l,j-1} - b\xi_{k,m+j-1}$ has exactly one segment. The desired equality hence follows.

The proofs of the cases where $v(\boldsymbol{j}) \leq 0$ and $v(\boldsymbol{j}) \geq \alpha_1$ similarly follow. □

2. Valuations of elements of SMBs of the lattice and of $\phi[u^n]$

If v is an infinite prime, then by the uniformization, there is a rank r A-lattice Λ associated to ϕ. If v is a finite prime and ϕ has stable reduction and the reduction is bad, then by the Tate uniformization, there is a rank s Drinfeld A-module ψ and rank $r-s$ A-lattice Λ associated to ϕ, where A acts on Λ via ψ. Let u be a finite prime of A with $\deg(u) = d$ for an arbitrary integer d. Our goal is to determine the valuations of elements of SMBs of Λ, $\psi[u^n]$, and $\phi[u^n]$ in terms of $v(t)$, $v(u)$, $v_s = v(a_s)$, and $v_r = v(a_r)$.

2.1. Infinite prime cases. Let v be an infinite prime. For any positive integer j, let $\{\xi_{i,j}\}_{i=1,\dots,r}$ denote an SMB of $\phi[t^j]$. In Lemma 2.1.1 and Proposition 2.1.3, we have determined the valuations $v(\xi_{i,j})$ for all i, j. As in Section 1, put for any positive integer $j \geq 1$

$$\alpha_j := \frac{v_0 q^{js}(q^{r-s}-1)}{q-1},$$

where $v_0 := v(t)$. If $v(\boldsymbol{j}) < \alpha_1$, let m be the integer satisfying $v(\boldsymbol{j}) \in (\alpha_{m+1}, \alpha_m]$. Now the condition $|t^n| \geq |\xi_{r,n}|/|\xi_{1,n}|$ in Remark 1.2.6 reads $-v_0 n \geq -v(\xi_{r,n}) + v(\xi_{1,n})$. For $n \geq m$, this inequality is equivalent to

$$-v_0 n \geq -v_0(m-1) + \frac{v_0}{q^s-1} - \frac{v(\boldsymbol{j})(q-1)}{q^{ms}(q^s-1)(q^{r-s}-1)}.$$

For any $n \geq m$, the inequality $|t^n| \geq |\xi_{r,n}|/|\xi_{1,n}|$ holds. If $v(\boldsymbol{j}) \geq \alpha_1$, for a fixed positive integer n, the valuations $v(\xi_{i,n})$ for $i = 1, \ldots, r$ are the same. Hence the condition $|t^n| \geq |\xi_{r,n}|/|\xi_{1,n}|$ is fulfilled for any positive integer n.

Proposition 2.2.1. *Put $v_0 := v(t)$. Let $\{\omega_i\}_{i=1,\ldots,r}$ be an SMB of Λ and $\{\lambda_i\}_{i=1,\ldots,r}$ an SMB of $\phi[u^n]$.*

(1) *If $v(\boldsymbol{j}) < \alpha_1$ and m is the integer such that $v(\boldsymbol{j}) \in (\alpha_{m+1}, \alpha_m]$, we have*

$$v(\omega_k) = v_0 + \frac{v_0}{q^s-1} - \frac{v_s}{q^s-1} \text{ for } k = 1, \ldots, s \text{ and}$$
$$v(\omega_l) = v_0 m + \frac{v(\boldsymbol{j})(q-1)}{q^{ms}(q^s-1)(q^{r-s}-1)} - \frac{v_s}{q^s-1} \text{ for } l = s+1, \ldots, r.$$

For $n \geq m/d$ and $i = 1, \ldots, r$, we have $|u^n| > |\omega_r|/|\omega_1|$ and $v(\lambda_i) = v(\xi_{i,nd}) = -v_0 nd + v(\omega_i)$.

(2) *Fix i to be one of $1, \ldots, r$. If $v(\boldsymbol{j}) \geq \alpha_1$, we have*

$$v(\omega_i) = v_0 + \frac{v_0}{q^r-1} - \frac{v_r}{q^r-1}.$$

For $n \geq 1$, we have $v(\lambda_i) = v(\xi_{i,nd})$.

We note that when $r = 2$, Chen-Lee has obtained the valuations $v(\omega_1)$ and $v(\omega_2)$ above in [**CL13**, Theorem 3.1 and Corollary 3.1]. One may also recover Gekeler's formula for "1-sparse Drinfeld A-modules" in [**Gek19B**, Proposition 3.2] (See also the rank 2 case in [**Pap23**, Proposition 5.5.8]).

Proof. The claims of $v(\omega_i)$ for $i = 1, \ldots, r$ follow from Remark 1.2.6, Corollary 1.2.5 (1), and the arguments before the proposition. Then the claims of $v(\lambda_k)$ and $v(\lambda_l)$ are proved by Corollary 1.2.5 (1). □

We also calculate the valuations of $a \cdot_\phi \lambda_i$ for any $a \in A$ and $i = 1, \ldots, r$.

Lemma 2.2.2. *Assume $v(\boldsymbol{j}) \in (\alpha_{m+1}, \alpha_m)$ for a positive integer m. Let n be an integer $\geq m/d$ and $\{\lambda_i\}_{i=1,\dots,r}$ an SMB of $\phi[u^n]$. Fix k to be one of $1, \dots, s$, and l to be one of $s+1, \dots, r$. Then we have*

$$v(t^j \cdot_\phi \lambda_k) = v(\xi_{k,nd-j}) \text{ and } v(t^j \cdot_\phi \lambda_l) = v(\xi_{l,nd-j}) \text{ for } 1 \leq j < nd.$$

Proof. We show the result for λ_l by induction. The proof of the result for λ_k is similar. By Proposition 2.2.1 (1), we have $v(\lambda_l) = v(\xi_{l,nd})$. To know $v(t \cdot_\phi \lambda_l) = v(t\lambda_l + a_s\lambda_l^{q^s} + a_r\lambda_l^{q^r})$, we calculate

$$v(t\lambda_l) - v(a_s\lambda_l^{q^s}) = \frac{v_0 q^{ms}(q^{r-s}-1)((q^s-1)(nd-m)+1) - v(\boldsymbol{j})(q-1)}{q^{ms}(q^{r-s}-1)},$$

$$v(a_s\lambda_l^{q^s}) - v(a_r\lambda_l^{q^r}) = v_0 q^s(q^{r-s}-1)(nd-m) + \frac{v(\boldsymbol{j})(q-1)(q^{(m-1)s}-1)}{q^{(m-1)s}(q^s-1)}.$$

Here $v(a_s\lambda_l^{q^s}) - v(a_r\lambda_l^{q^r}) \geq 0$ if and only if $nd = m = 1$. The case $nd = m = 1$ is not included in the claim and hence $v(a_s\lambda_l^{q^s}) - v(a_r\lambda_l^{q^r}) < 0$. We have

$$\begin{cases} v(t\lambda_l) - v(a_s\lambda_l^{q^s}) > 0 & nd = m; \\ v(t\lambda_l) - v(a_s\lambda_l^{q^s}) < 0 & nd > m. \end{cases}$$

Hence

$$v(t \cdot_\phi \lambda_l) = \begin{cases} v(a_s\lambda_l^{q^s}) = v(\xi_{l,m-1}) & nd = m; \\ v(t\lambda_l) = v(\xi_{l,nd-1}) & nd > m. \end{cases}$$

Assume that the result for $j-1$ is valid. Put $\lambda_l' := t^{j-1} \cdot_\phi \lambda_l$. If $j \leq nd - m$, to know $v(t \cdot_\phi \lambda_l')$, we calculate

$$v(t\lambda_l') - v(a_s\lambda_l'^{q^s}) = v_0((q^s-1)(nd-j+1-m)+1) - \frac{v(\boldsymbol{j})(q-1)}{q^{ms}(q^{r-s}-1)} < 0,$$

$$v(a_s\lambda_l'^{q^s}) - v(a_r\lambda_l'^{q^r}) = v_0(q^r - q^s)(nd-j+1-m) + \frac{v(\boldsymbol{j})(q-1)}{q^s-1}\left(1 - \frac{1}{q^{(m-1)s}}\right) < 0.$$

Hence we have $v(t \cdot_\phi \lambda_l') = v(t\lambda_l') = v(\xi_{2,nd-j})$. As for the case $j > nd - m$, to know $v(t \cdot_\phi \lambda_l')$, we calculate

$$v(t\lambda_l') - v(a_s\lambda_l'^{q^s}) = v_0 - \frac{v(\boldsymbol{j})(q-1)}{q^{(nd-j+1)s}(q^{r-s}-1)} > 0,$$

$$v(a_s\lambda_l'^{q^s}) - v(a_r\lambda_l'^{q^r}) = \frac{v(\boldsymbol{j})(q-1)}{q^s-1}\left(1 - \frac{1}{q^{(nd-j)s}}\right) < 0.$$

Hence $v(t \cdot_\phi \lambda_l') = v(a_s\lambda_l'^{q^s}) = v(\xi_{l,nd-j})$, and the result for λ_l follows. □

Corollary 2.2.3. *Assume $v(\boldsymbol{j}) \in (\alpha_{m+1}, \alpha_m)$ for a positive integer m. Fix k to be one of $1, \dots, s$, and l to be one of $s+1, \dots, r$. For $n \in \mathbb{Z}_{\geq 1}$, let $\{\lambda_i\}_{i=1,\dots,r}$ an SMB of $\phi[u^n]$.*

(1) *Assume* $n \geq m/d$. *For any* $a \in A$ *with* $\deg(a) < nd$, *we have*

$$v(a \cdot_\phi \lambda_k) = v(t^{\deg(a)} \cdot_\phi \lambda_k) = v(\xi_{k,nd-\deg(a)}), \tag{22}$$

$$v(a \cdot_\phi \lambda_l) = v(t^{\deg(a)} \cdot_\phi \lambda_l) = v(\xi_{l,nd-\deg(a)}). \tag{23}$$

(2) *Assume* $n \geq m/d$. *For* $\lambda \in \phi[u^n]$ *having valuation* $\geq v(\xi_{k,nd-m+1})$, *there exists some* $b_i \in A$ *for* $i = 1, \ldots, r$ *with* $\deg(b_i) < m$ *such that* $\sum_{i=1}^{s} b_i \cdot_\phi \lambda_i = \lambda$.

(3) *Let* n *be an arbitrary positive integer. We have*

$$v(\lambda_k) = v(\xi_{k,nd}) \text{ and } v(\lambda_l) = v(\xi_{l,nd}).$$

Proof. By Proposition 2.1.2, we have

$$v(\xi_{k,j}) > v(\xi_{l,nd}) \text{ for } j = nd, nd-1, \cdots, nd-m+1, \tag{24}$$

$$v(\xi_{i,j+1}) > v(\xi_{i,j}) \text{ for } i = 1, \ldots, r \text{ and positive integers } j < nd. \tag{25}$$

For (1), by (25) and the lemma, we have $v(t^{\deg(a)} \cdot_\phi \lambda_k) < v(t^j \cdot_\phi \lambda_k)$ for any positive integer $j < \deg(a)$. Hence the desired equality follows from the ultrametric inequality. The equation for λ_l follows in the same way.

We show (2). As $\{\lambda_i\}_{i=1,\ldots,r}$ is an SMB of $\phi[u^n]$, there exist $b_i \in A \mod u^n$ such that $\lambda = \sum_{i=1}^{r} b_i \cdot_\phi \lambda_i$. We may assume that all b_i have degree $< \deg(u^n) = nd$. To show $b_i = 0$ for $i > s$, assume conversely $b_{l'} \neq 0$ for some $l' > s$. By (23) and (25), we have

$$v(b_{l'} \cdot_\phi \lambda_{l'}) = v(t^{\deg(b_{l'})} \cdot_\phi \lambda_{l'}) = v(\xi_{l',nd-\deg(b_{l'})}) \leq v(\xi_{l',nd}) = v(\lambda_{l'}).$$

By Proposition 1.2.13, we have

$$v(\lambda) = \min_{i=1,\ldots,r} \{v(b_i \cdot_\phi \lambda_i)\}. \tag{26}$$

Hence $v(\lambda) \leq v(b_{l'} \cdot_\phi \lambda_{l'}) \leq v(\lambda_{l'}) = v(\lambda_l)$. On the other hand, by (24), we have $v(\lambda) \geq v(\xi_{k,nd-m+1}) > v(\xi_{l,nd}) = v(\lambda_l)$, which is a contradiction. By (22), for each $i \leq s$, we have

$$v(b_i \cdot_\phi \lambda_i) = v(t^{\deg(b_i)} \cdot_\phi \lambda_i) = v(\xi_{i,nd-\deg(b_i)}).$$

By (26), the element b_i satisfies $v(b_i \cdot_\phi \lambda_i) \geq v(\lambda) \geq v(\xi_{i,nd-m+1}) = v(\xi_{k,nd-m+1})$. Then (25) implies $\deg(b_i) < m$.

As for (3), if $nd < m$, we can find an SMB $\{\lambda'_i\}_{i=1,\ldots,r}$ of $\phi[u^{n+m}]$ by Proposition 1.1.11 satisfying $\lambda_i = u^m \cdot_\phi \lambda'_i$. Then (3) straightforwardly follows from (1). □

Remark 2.2.4. Assume $v(\boldsymbol{j}) \in (\alpha_{m+1}, \alpha_m)$. Let n be a positive integer. The elements $t^j \cdot_\phi \lambda_i$ for $i = 1, \ldots, r$ and $0 \leq j < nd$ form an $\mathbb{F}_q$-basis of $\phi[u^n]$ as a vector space. Moreover, by the equation $v(t^j \cdot_\phi \lambda_i) = v(\xi_{i,nd-j})$, the elements $t^j \cdot_\phi \lambda_i$ for all i and j

can be arranged with respect to their valuations as in Proposition 2.1.2. To see that the family $\{t^j \cdot_\phi \lambda_i\}_{i,j}$ is a basis, assume that we have

$$\sum_{\substack{i=1,\dots,r\\ j=0,\dots,nd-1}} a_{ij}(t^j \cdot_\phi \lambda_i) = 0,$$

where $a_{ij} \in \mathbb{F}_q$ and some a_{ij} are nonzero. Notice that the left of this equation equals to

$$\sum_{i=1,\dots,r} a_i \cdot_\phi \lambda_i = 0,$$

where $a_i = \sum_{j=0}^{nd-1} a_{ij}t^j$. As some $a_i \neq 0$, the equality implies λ_i are A/u^n-linear dependent, a contradiction.

Remark 2.2.5. Assume $v(\boldsymbol{j}) \in [\alpha_1, +\infty)$. Let n be a positive integer and $\{\lambda_i\}_{i=1,\dots,r}$ an SMB of $\phi[u^n]$. For $i = 1, \dots, r$, we claim that

$$v(t^j \cdot_\phi \lambda_i) = v(\xi_{i,nd-j}) \text{ for } j = 1, \dots, nd-1.$$

The proof of this claim is similar to that of Lemma 2.2.2. In fact, if the claim for $\lambda_i' := t^{j-1} \cdot_\phi \lambda_i$ is valid for $j = 1, \dots, nd-1$, then the following calculation is enough to show the result for $t^j \cdot_\phi \lambda_i$

$$\begin{aligned} v(t\lambda_i') - v(a_s\lambda_i'^{q^s}) &= v_0(q^s-1)(nd-j) + \frac{v_0q^s(q^{r-s}-1) - v(\boldsymbol{j})(q-1)}{q^r-1} < 0,\\ v(t\lambda_i') - v(a_r\lambda_i'^{q^r}) &= v_0(q^r-1)(nd-j). \end{aligned}$$

By Proposition 2.1.3, for $i = 1, \dots, r$, we also have for any $a \in A$ with $\deg(a) < nd$ that

$$v(a \cdot_\phi \lambda_i) = v(t^{\deg(a)} \cdot_\phi \lambda_i) = v(\xi_{i,nd-\deg(a)}).$$

2.2. Finite prime cases. Let v be a finite prime such that $v(u) \geq 0$. Assume that ϕ has stable reduction and the reduction is bad such that $v_s = 0$ and $v(\boldsymbol{j}) < 0$. Let $\{\xi_{i,j}\}_{i=1,\dots,r}$ denote an SMB of $\phi[t^j]$ throughout this subsection. In Proposition 2.1.4 and Lemma 2.1.5 (2), we have determined the valuations $v(\xi_{i,j})$ for $i = 1, \dots, r$ and all positive integers j. Since $v(\xi_{r,n}) = v(\xi_{s+1,n})$ for any positive integer n, the condition $|t^n| \geq |\xi_{r,n}|/|\xi_{s+1,n}|$ in Remark 1.3.9 always holds.

Proposition 2.2.6 (cf. Proposition 2.2.1). *Assume $v(u) = 0$. For a positive integer n, let $\{\omega_i\}_{i=1,\dots,s}$ be an SMB of $\psi[u^n]$, $\{\omega_i^0\}_{i=1,\dots,r-s}$ an SMB of Λ, and $\{\lambda_i\}_{i=1,\dots,r}$ an SMB of $\phi[u^n]$. Fix k to be one of $1, \dots, s$, and l to be one of $s+1, \dots, r$. Then for any positive integer n, we have*

$$v(\omega_k) = v(\lambda_k) = 0,\ v(\omega_{l-s}^0) = \frac{v(\boldsymbol{j})(q-1)}{(q^s-1)(q^{r-s}-1)},\ \text{and } v(\lambda_l) = \frac{v(\boldsymbol{j})(q-1)}{q^{nds}(q^s-1)(q^{r-s}-1)}.$$

PROOF. The results for ω^0_{l-s} and λ_l follow from the value $v(\xi_{l,n})$ and Corollary 1.3.8 (1). As for the equation $v(\omega_k) = v(\lambda_k) = 0$, as in the proof of Lemma 1.3.2, we can show that the Newton polygon of $\psi_{u^n}(X)$ is P_0P_{nds} having exactly one segment with slope 0, where $P_0 = (1, 0)$ and $P_{nds} = (q^{nds}, 0)$. As ω_k for $k = 1, \dots, s$ are nonzero roots of $\psi_{u^n}(X)$, hence $v(\omega_k) = 0$. The valuation $v(\lambda_k)$ for $k = 1, \dots, s$ follows from Lemma 1.3.4. □

Lemma 2.2.7. *Assume $v(u) = 0$. Let n be a positive integer and $\{\lambda_i\}_{i=1,\dots,r}$ an SMB of $\phi[u^n]$. Fix l to be one of $s+1, \dots, r$. Then we have*

$$v(t^j \cdot_\phi \lambda_l) = v(\xi_{l,nd-j}) \text{ for } 1 \le j < nd.$$

PROOF. Similar to Lemma 2.2.2, we apply Proposition 2.1.4 (1) to show the result. If the claim for $\lambda'_l := t^{j-1} \cdot_\phi \lambda_l$ is valid (One may take $j = 1$ and the base case for the induction is obtained), then the calculations below are enough to show the result for $t^j \cdot_\phi \lambda_l$

$$v(t\lambda'_l) - v(a_s\lambda'^{q^s}_l) = -\frac{v(\boldsymbol{j})(q-1)}{q^{(nd-j+1)s}(q^{r-s}-1)} > 0,$$
$$v(a_s\lambda'^{q^s}_l) - v(a_r\lambda'^{q^r}_l) = \frac{v(\boldsymbol{j})(q-1)}{q^s-1}\left(1 - \frac{1}{q^{(nd-j)s}}\right) < 0.$$

□

Corollary 2.2.8. *Assume $v(u) = 0$. Let n be a positive integer and $\{\lambda_i\}_{i=1,\dots,r}$ an SMB of $\phi[u^n]$. Fix k to be one of $1, \dots, s$ and l to be one of $s+1, \dots, r$. For any $a \in A$ with $\deg(a) < nd$, we have*

$$v(a \cdot_\phi \lambda_k) = 0,$$
$$v(a \cdot_\phi \lambda_l) = v(t^{\deg(a)} \cdot_\phi \lambda_l) = v(\xi_{l,nd-\deg(a)}).$$

PROOF. As the condition of Theorem 1.3.13 is fulfilled, the family $\{\omega_i\}_{i=1,\dots,s} := \{\log_\phi(\lambda_i)\}_{i=1,\dots,s}$ is an SMB of $\psi[u^n]$. We have that $a \cdot_\psi \omega_k$ is a root of $\psi_{u^n}(X)$ and hence $v(a \cdot_\psi \omega_k) = 0$ (In the proof of Proposition 2.2.6, we have seen that all roots of $\psi_{u^n}(X)$ have valuation 0). Lemma 1.3.4 implies $v(a \cdot_\phi \lambda_k) = v(a \cdot_\psi \omega_k) = 0$.

By Proposition 2.1.4, we have $v(\xi_{l,j+1}) > v(\xi_{l,j})$ for $1 \le j < nd$. Using the ultrametric inequality and Lemma 2.2.7, this implies the result for $a \cdot_\phi \lambda_l$. □

Let us assume $v(u) > 0$, i.e., $v \mid u$ then.

Lemma 2.2.9. *Let α denote the rank s Drinfeld A-module over K determined by $\alpha_t(X) = tX + a_sX^{q^s} \in K[X]$ so that $v(a_s) = 0$ and α has good reduction over K (We have $v(a_s) = 0$ as ϕ has bad reduction over K). Let $a_{u,i}$ denote the coefficients of $\alpha_u(X)$ such that $uX + \sum_{i=1}^d a_{u,i}X^{q^{is}} = \alpha_u(X)$. Then we have*

$$v(a_{u,d}) = 0 \text{ and } v(a_{u,i}) = v(u) \text{ for } i < d.$$

Proof. Put $Q := q^s$. Let K' denote the extension of K generated by all $Q-1$-th roots of the unity and some $b \in K^{\mathrm{sep}}$ with $b^{Q-1} = a_s$. We have $C = b\alpha b^{-1}$ as Drinfeld $\mathbb{F}_Q[t]$-modules over K', where C denotes the Carlitz $\mathbb{F}_Q[t]$-module. Put $u_0 := u$, $\sum_{i=0}^d u_i X^{Q^i} := C_u(X)$, and $P_i := (Q^i, v(u_i))$ for $i = 0, \dots, d$. By the explicit formula of u_i in [**Pap23**, Corollary 5.4.4] (initially given by Carlitz), we have $v(u_i) = v(u)$ for $i = 0, \dots, d-1$ and $v(u_d) = 0$. As

$$C_u(X) = (b\alpha b^{-1})_u(X) = uX + \sum_{i=1}^{s} a_{u,i} b^{-(Q^i-1)} X^{Q^i},$$

the result follows. □

Lemma 2.2.10. *Assume $v_r > v(u) > 0$. The leftmost segment in the Newton polygon of $\phi_u(X)$ is P_0P_{ds}, where $P_0 := (1, v(u))$ and $P_{ds} := (q^{ds}, 0)$. Here we omit the segment with the infinite slope.*

Proof. Admit the notations in Lemma 2.2.9. We have

$$\phi_u(X) = uX + \sum_{i=1}^{d} a_{u,i} q^{si} + \sum_{\substack{i=r+as+br \\ a,b\in\mathbb{Z}_{\geq 0} \\ i \leq dr}} b_i X^{q^i},$$

for some $b_i \in K$. For each b_i, we have $a_r \mid b_i$. This implies that $v(b_i) \geq v_r > v(u)$. By Lemma 2.2.9, we have $v(a_{u,i}) = v(u)$ for $i < d$ and $v(a_{u,d}) = 0$. Hence the leftmost segment in the Newton polygon of $\phi_u(X)$ has the desired form. □

Proposition 2.2.11. *Assume $v_r > v(u) > 0$. For a positive integer n, let $\{\omega_i\}_{i=1,\dots,s}$ be an SMB of $\psi[u^n]$, $\{\omega_i^0\}_{i=1,\dots,r-s}$ an SMB of Λ, and $\{\lambda_i\}_{i=1,\dots,r}$ an SMB of $\phi[u^n]$. Fix k to be one of $1, \dots, s$, and l to be one of $s+1, \dots, r$. Then for any positive integer n, we have $v(\omega_{l-s}^0)$ and $v(\lambda_l)$ as in Proposition 2.2.6 and*

$$v(\omega_k) = v(\lambda_k) = \frac{v(u)}{(q^{ds}-1)q^{(n-1)ds}}.$$

Proof. Following the proof of Proposition 2.2.6, we can obtain the claim for $v(\omega_{l-s}^0)$ and $v(\lambda_l)$.

Next, we show the claim for $v(\lambda_k)$ and then the claim for $v(\omega_k)$ follows from Lemma 1.3.4. For any positive integer j, let $\{\lambda_{i,j}\}_{i=1,\dots,r}$ be an SMB of $\phi[u^j]$ as in Corollary 1.1.12. By Lemma 2.2.10, there are $q^{ds}-1$ roots of $\phi_u(X)$ having valuation $\frac{v(u)}{q^{ds}-1}$. The other roots of $\phi_u(X)$ have valuations ∞ or < 0. As in Lemma 1.3.11, we can show $v(\lambda_{i,1}) = \frac{v(u)}{q^{ds}-1}$ for $i = 1, \dots, s$. Put $Q_j := (0, v(\lambda_{i,j}))$. By induction on j, we can show that the leftmost segment in the Newton polygon of $\phi_u(X) - \lambda_{i,j}$ is Q_jP_{ds} with $P_{ds} = (q^{ds}, 0)$ such that $v(\lambda_{i,j}) = \frac{v(u)}{(q^{ds}-1)q^{jds}}$. The result hence follows. □

Remark 2.2.12. If $s = 1$, then ψ is isomorphic to the Carlitz A-module so that the claim in the proposition can be proved without requiring the condition $v_r > v(u)$ (See [**Hua23**, Proposition 6.1]). If $s \mid r$, we may consider ϕ as a Drinfeld $\mathbb{F}_Q[t]$-module of rank r/s over K', where $Q \coloneqq q^s$ and K' is the extension of K generated by all $Q-1$-th roots of unity. Hence ψ is isomorphic to the Carlitz $\mathbb{F}_Q[t]$-module. Similarly, the condition $v_r > v(u)$ can be also dropped.

CHAPTER 3

On the extension generated by u^n-torsion points

Let K be a local field which is the completion of some (global) function field at a prime v. Let v also denote the normalized valuation such that $v(K^\times) = \mathbb{Z}$. In the applications of Krasner's lemma, we use the absolute value $|-|$ on K^{sep} given by $q^{-v(-)}$. Let ϕ be a rank r Drinfeld A-module over K such that

$$\phi_t(X) = tX + a_s X^{q^s} + a_r X^{q^r} \in K[X],$$

where s and r are two positive integers satisfying $s < r$. The j-invariant of ϕ is defined to be

$$\boldsymbol{j} := \frac{a_s^{(q^r-1)/(q-1)}}{a_r^{(q^s-1)/(q-1)}}.$$

For a finite prime u of A and a positive integer n, let $K_n = K(\phi[u^n])$ denote the extension of K generated by all u^n-torsion points of ϕ. Based on the results in Chapter 2, we are concerned with the ramification of K_n/K.

In Section 1, when u is an arbitrary degree 1 finite prime of K and $v(\boldsymbol{j}) < \alpha_1 = \frac{v_0 q(q^{r-1}-1)}{q-1}$, we apply Krasner's lemma to show: (1) $K(\phi[u])$ contains the splitting field of $a_s X^{q^s} + tX$; (2) if $v \nmid u$, then K_{n+1} is the compositum of extensions of K_n by roots of certain degree q^s polynomials. Section 2 is devoted to a review of the (Herbrand) ψ-function, where we also prove a result concerning the ψ-function of the extension generated by a certain polynomial with degree equal to a power of q. In Section 3, we study the extension K_1/K in the case $s = 1$. Under certain assumptions, we will obtain the ψ-function of K_1/K and the action of the wild ramification subgroup $G(K_1/K)_1$ of $G(K_1/K)$ on an SMB of $\phi[u]$ when the prime u is an arbitrary degree 1 finite prime of A.

Throughout Sections 4 and 5, For an infinite prime v, in Section 5.1, we obtain the ψ-function of K_n/K and the action of the wild ramification subgroup $G(K_n/K)_1$ of $G(K_n/K)$ on an SMB of $\phi[u^n]$ for large enough n (n is large enough so that $K(\phi[u^n]) = K(\Lambda)$ as in Corollary 1.2.12). For this, we first study the case where $\deg(u) = 1$ in Section 4. For a finite prime v, in Section 5.2, we study the action of $G(K_n/K)_1$ on an SMB of $\phi[u^n]$ for any positive integer n when $v \nmid u$.

we restrict ourselves to the case $r = 2$. If the prime v is infinite, in Section 4, under certain assumptions, when $\deg(u) = 1$. We extend these results to the case where $\deg(u) > 1$ in Section 5.1. In Section 5.2, we obtain similar results when v is a finite prime.

In Section 6, when u is a finite prime of A satisfying $v \nmid u$, we show that for a rank r Drinfeld A-module ρ such that the Newton polygon of $\rho_t(X)$ has exactly one segment, the extension $K(\rho[u^n])/K$ is at worst tamely ramified.

1. Applications of Krasner's lemma

Fix a finite prime u of A with degree 1. We first prepare a lemma concerning a subextension of the extension generated by u-torsion points of a Drinfeld A-module over K.

Assume $v \nmid u$. For any positive integer n, the extension K_{n+1}/K_n is generated by roots of polynomials $\phi_u(X) - \xi_{i,n}$ where $\{\xi_{i,n}\}_{i=1,\ldots,r}$ is an SMB of $\phi[u^n]$. By the calculations in Section 1.1 in Chapter 2, we know that the Newton polygon of $\phi_u(X) - \xi_{i,n}$ has two or more segments, and hence the polynomial is reducible. For each polynomial $\phi_u(X) - \xi_{i,n}$, we are to find a polynomial with a smaller degree and apply Krasner's lemma to show that its roots generate the same extension as the one generated by the roots of $\phi_u(X) - \xi_{i,n-1}$.

1.1. On subextension of the extension generated by u-torsion points. Let u be a finite prime of A with degree 1 (We admit the case where $v \mid u$ in this subsection). Let ρ be a rank r Drinfeld A-module such that

$$\rho_u(X) = uX + \sum_{k=1}^{r} a_k X^{q^k} \in K[X].$$

Put

$$v_0 \coloneqq v(u) \text{ and } v_k \coloneqq v(a_k) \text{ for each } k = 1, \ldots, r.$$

Let $\{\xi_{k,1}\}_{k=1,\ldots,r}$ be an SMB of $\rho[u]$ (following Corollary 1.1.12). There is a unique integer s such that for any positive integer $j < s$ and any integer $k > s$, we have

$$\frac{v_s - v_0}{q^s - 1} \leq \frac{v_j - v_0}{q^j - 1} \text{ and } \frac{v_s - v_0}{q^s - 1} < \frac{v_k - v_0}{q^k - 1}. \tag{27}$$

The segment determined by the points $(1, v_0)$ and (q^s, v_s) is the first segment of the Newton polygon of $\rho_u(X)$. By an argument similar to the one in Lemma 1.3.11, we know that $v(\xi_{k,1}) = -\frac{v_s - v_0}{q^s - 1}$ for $k = 1, \ldots, s$. We first study the extension of K generated by $\xi_{k,1}$ for $k \leq s$. Put

$$\psi(X) \coloneqq \prod_{\substack{k \leq s \\ a_k \in \mathbb{F}_q}} (X - a_k \cdot \xi_{k,1}) \in K[X].$$

Here we have $\psi(X) \in K[X]$ by [**Neu99**, Chapter II, Proposition 6.4]. Let K_ψ denote the extension of K generated by all roots of $\psi(X)$. It equals the extension of K generated by all $\xi_{k,1}$ for $k \leq s$.

Lemma 3.1.1. *Let v be an infinite prime or a finite prime. Let s be the integer defined above. Put $\eta(X) := uX + \sum_{k=1}^{s} a_k X^{q^k}$. Let K_η denote the extension of K generated by all roots of $\eta(X)$. Then $K_\psi = K_\eta$.*

Proof. Let $x, x_1, x_2, \dots, x_{q^s-2}$ be all nonzero roots of $\psi(X)$. Let x'_j for $j = 1, \dots, q^s-1$ denote all nonzero roots of $\eta(X)$. By (27), the Newton polygon of $\eta(X)$ has one segment (omit the one with infinite slope) and every nonzero root has valuation $-\frac{v_s - v_0}{q^s-1}$. If we can show that there exists some index i such that $|x - x'_i| < |x'_i - x'_j|$ for all $j \neq i$, then Krasner's lemma (See [**Neu99**, p.152]) implies that $K(x'_i) \subset K(x)$. Consider

$$\eta(x) = a_s \cdot x \prod_{j=1}^{q^s-1} (x - x'_j) = \sum_{k=1}^{s} a_k x^{q^k} + ux = -\sum_{k>s} a_k x^{q^k}.$$

Here the rightmost equality follows from $\rho_u(x) = 0$. Notice

$$v\left(a_s^{-1} \sum_{k>s} a_k x^{q^k-1}\right) \geq \min_{k>s} \left\{ v\left(a_s^{-1} a_k x^{q^k-1}\right) \right\}.$$

The inequality $\frac{v_k - v_0}{q^k-1} > \frac{v_s - v_0}{q^s-1}$ in (27) implies

$$v_k(q^s-1) > v_s(q^k-1) - v_0(q^k-1) + v_0(q^s-1).$$

For each $k \neq s$, we have

$$\begin{aligned} v(a_s^{-1} a_k x^{q^k-1}) &= -v_s + v_k - \frac{(v_s - v_0)(q^k-1)}{q^s-1} \\ &> -v_s + \frac{v_s(q^k-1) - v_0(q^k-1) + v_0(q^s-1)}{q^s-1} + \frac{v_0(q^k-1)}{q^s-1} - \frac{v_s(q^k-1)}{q^s-1} \\ &= -(v_s - v_0). \end{aligned}$$

This implies

$$\left| \prod_{k=1}^{q^s-1} (x - x'_i) \right| = \left| a_s^{-1} \sum_{k>s} a_k x^{q^k-1} \right| < q^{v_s - v_0}.$$

There must be some x'_i such that $|x - x'_i| < q^{\frac{v_s - v_0}{q^s-1}} = |x'_i - x'_j|$ for $j \neq i$, as desired.

There exists a root x'_k of $\eta(X)$ satisfying $|x'_k - x_j| < q^{\frac{v_s - v_0}{q^s-1}}$. As $|x_j - x| = q^{\frac{v_s - v_0}{q^s-1}}$, we have $|x_j - x'_i| = q^{\frac{v_s - v_0}{q^s-1}}$. This implies $x'_k \neq x'_i$. Hence for any root of $\eta(X)$, there exists a root of $\psi(X)$ so that the absolute value of the difference of these roots $< q^{\frac{v_s - v_0}{q^s-1}}$. Let x vary within the roots of $\psi(X)$. We have $K(x'_1, \dots, x'_{q^s-1}) \subset K(x, x_1, \dots, x_{q^s-2})$, i.e., $K_\eta \subset K_\psi$. Conversely, the conjugates of x are x_j for some j but $|x - x'_i| < |x - x_j| = q^{\frac{v_s - v_0}{q^s-1}}$. Apply Krasner's lemma again and $K_\psi \subset K_\eta$ follows. □

We apply the lemma to the Drinfeld module ϕ defined above. Let $\{\xi_{i,1}\}_{i=1,\ldots,r}$ be an SMB of $\phi[u]$ and put

$$\psi(X) := \prod_{\substack{k\leq s\\ a_k\in\mathbb{F}_q}} (X - a_k\cdot\xi_{k,1}) \in K[X].$$

Corollary 3.1.2. *Put*

$$\eta(X) := a_s X^{q^s} + uX. \tag{28}$$

Let v be an infinite or finite prime. Assume $v(\boldsymbol{j}) < \frac{v_0 q^s(q^{r-s}-1)}{q-1}$. Let K_ψ and K_η be respective the extension of K generated by the roots of $\psi(X)$ and those of $\eta(X)$. Then $K_\psi = K_\eta$.

We can choose an element $b\in K_\eta$ with $v(b) = \frac{v_s - v_0}{q^s-1}$ to modify $\phi_u(X)$, so that the coefficient of X^q is 1:

$$\Phi(X) = b_r X^{q^r} + X^{q^s} + b_0 X := b'(\phi_u(X/b)) \tag{29}$$

with $b' = b^{q^s}/a_s$. Then $v(b_0) = 0$, and

$$v(b_r) = \frac{-v(\boldsymbol{j})(q-1) + v_0 q^s(q^{r-s}-1)}{q^s-1} > 0. \tag{30}$$

1.2. Infinite primes. Let v be an infinite prime and u a finite prime of A with degree 1. As in Section 1.1 in Chapter 2, let $\{\xi_{i,j}\}_{i=1,\ldots,r}$ denote an SMB of $\phi[u^j]$ for each positive integer j obtained as in Corollary 1.1.12. Put

$$v_0 := v(t), v_s := v(a_s), v_r := v(a_r), \text{ and } \alpha_j := \frac{v_0 q^{js}(q^{r-s}-1)}{q-1} \text{ for } j\geq 1.$$

We will follow the notations for the Newton polygons in Chapter 2.

Lemma 3.1.3. *Assume $v(\boldsymbol{j}) < \alpha_1$. Let m be the integer satisfying $v(\boldsymbol{j}) \in (\alpha_{m+1}, \alpha_m]$. Fix k to be one of $1,\ldots,s$ and l to be one of $s+1,\ldots,r$. Then we have $\xi_{k,n+1}\in K_n$ for $n\geq 1$ and $\xi_{l,n+1}\in K_n$ for $n\geq m$, i.e.,*

$$K_n(\xi_{k,n+1}) = K_n \textit{ for } n\geq 1 \textit{ and } K_n(\xi_{l,n+1}) = K_n \textit{ for } n\geq m.$$

Proof. By Lemma 2.1.1 (1), for $n\geq 1$, the Newton polygon of $\phi_u(X) - \xi_{k,n}$ is $Q_{k,n}P_0P_sP_r$ having exactly three segments, where $Q_{k,n} = (0, v(\xi_{k,n}))$, $P_0 = (1, v_0)$, $P_s = (q^s, v_s)$, and $P_r = (q^r, v_r)$. Here, as in Chapter 2, by "the Newton polygon of $\phi_u(X)-\xi_{k,n}$ is $Q_{k,n}P_0P_sP_r$", we mean that the Newton polygon of $\phi_u(X)-\xi_{k,n}$ consists of three segments $Q_{k,n}P_0$, P_0P_s, and P_sP_r. Recall that we obtain $\{\xi_{i,n+1}\}_{i=1,\ldots,r}$ following Corollary 1.1.12. We see that $\xi_{k,n+1}$ is the only root of $\phi_u(X)-\xi_{k,n}$ whose valuation is $-\mu(Q_{k,n},P_0)$. Here, as in Chapter 2, $\mu(Q_{k,n},P_0)$ denotes the ratio of the segment $Q_{k,n}P_0$. Thus due to [**Neu99**, Chapter II, Proposition 6.4], we know $(X-\xi_{k,n+1})\in K_n[X]$ and thus $\xi_{k,n+1}\in K_n$. One can show $\xi_{l,n+1}\in K_n$ for $n\geq m$ in the same way. ☐

We now study the extension K_{n+1}/K_n for $1 \le n \le m-1$. It is the extension of K_n generated by all roots of $\phi_u(X)-\xi$ for all $\xi \in \phi[u^n]$. It is also the extension of K_n generated by an SMB $\{\xi_{i,n+1}\}_{i=1,\dots,r}$ of $\phi[u^{n+1}]$. By Lemma 3.1.3, the extension K_{n+1}/K_n is generated by $\xi_{l,n+1}$ for $l = s+1,\dots,r$. Hence it is the compositum of extensions $K_{l,n}/K_n$, where for each $l = s+1,\dots,r$, the extension $K_{l,n}/K_n$ is generated by the roots of $\phi_u(X)-\xi_{l,n}$. We are to show that for each $l = s+1,\dots,r$, the extension $K_{l,n}/K_n$ is generated by the roots of a degree q^s polynomials.

Assume $v(\boldsymbol{j}) < \alpha_1$. Let m be the integer satisfying $v(\boldsymbol{j}) \in (\alpha_{m+1}, \alpha_m]$. Fix l to be one of $s+1,\dots,r$. We can choose an element $b \in K_\eta$ (K_η is defined in Corollary 3.1.2) with $v(b) = \frac{v_s - v_0}{q^s-1}$ to modify $\phi_u(X)-\xi_{l,n}$, $\eta(X)$, and $\eta(X)-\xi_{l,n}$, so that the coefficient of X^q is 1:

$$\Phi_{l,n}(X) = b_r X^{q^r} + X^{q^s} + b_0 X + c_{l,n} \coloneqq b'(\phi_u(X/b) - \xi_{l,n}).$$

with $b' = b^{q^s}/a_s$. Then

$$v(c_{l,n}) = \frac{v(\boldsymbol{j})(q-1) - v_0 q^{s(n+1)}(q^{r-s}-1)}{q^{sn}(q^s-1)(q^{r-s}-1)}, \tag{31}$$

$v(b_0) = 0$, and

$$v(b_r) = \frac{-v(\boldsymbol{j})(q-1) + v_0 q^s (q^{r-s}-1)}{q^s-1} > 0.$$

We obtain modifications

$$\begin{aligned} \mathrm{H}(X) &\coloneqq X^{q^s} + b_0 X = b'(\eta(X/b)) \\ \mathrm{H}_{l,n}(X) &\coloneqq X^{q^s} + b_0 X + c_{l,n} = b'(\eta(X/b) - \xi_{l,n}) \end{aligned} \tag{32}$$

of $\eta(X)$ and $\eta(X)-\xi_{l,n}$.

In [**KL04**, Proposition 3], Kölle and Schmid applied Krasner's lemma to study unramified or tamely ramified extensions of a number field. Roughly speaking, their proposition claims that two polynomials yield the same field extension if their Newton polygons are the same. The following lemma is an analogue for certain wildly ramified extensions.

Lemma 3.1.4. *Assume $v(\boldsymbol{j}) \le \alpha_2$. Let $m \ge 2$ be the integer such that $v(\boldsymbol{j}) \in (\alpha_{m+1}, \alpha_m]$. If $1 \le n < m$, then any root ξ' of the polynomial $\eta(X) - \xi_{l,n}$ satisfies*

$$K_n(\xi') = K_{l,n},$$

where $K_{l,n}$ is the extension of K_n generated by all roots of $\phi_u(X) - \xi_{l,n}$.

Proof. By Proposition 2.1.2, the roots of $\phi_u(X)-\xi_{l,n}$ whose valuations equal $v(\xi_{l,n+1})$ are $\xi_{l,n+1} + \sum_{k=1}^{s} a_k \xi_{k,1}$ for all $a_k \in \mathbb{F}_q$. Due to [**Neu99**, Chapter II, Proposition 6.4], we have

$$\prod_{\substack{a_k\in\mathbb{F}_q\\ \text{for } k=1,\dots,s}} \left(X - \xi_{l,n+1} - \sum_{k=1}^{s} a_k \xi_{k,1} \right) \in K_n[X].$$

(Note that this polynomial and $\eta(X) - \xi_{l,n}$ have the same Newton polygon.) The field $K_{l,n}$ is the splitting field of this polynomial. Then we are reduced to showing $K_n(\xi') = K_n(\xi_{l,n+1})$. Put $x := b\xi_{l,n+1}$. Let x'_i for $i = 1, \dots, q^s$ denote the roots of $\mathrm{H}_{l,n}(X)$. The difference $b\xi' - x'_i$ belongs to $K_\eta \subset K_1$, as it is a root of $\mathrm{H}(X)$. Since $K_n(x'_i) = K_n(b\xi') = K_n(\xi')$, it suffices to show $K_n(x'_i) = K_n(x)$ for some i.

If there exists i such that $|x - x'_i| < |x'_i - x'_j|$ for all $j \neq i$, then we apply Krasner's lemma and obtain $K_n(x'_i) \subset K_n(x)$. We have $|x'_i - x'_j| = 1$ for all $i \neq j$ since $x'_i - x'_j$ is a nonzero root of $\mathrm{H}(X)$. It suffices to find a suitable root x'_i of $\mathrm{H}_{l,n}(X)$ such that $|x - x'_i| < 1$. To know $|x - x'_i|$, we consider the valuation of

$$\mathrm{H}_{l,n}(x) = \prod_{i=1}^{q^s} (x - x'_i) = x^{q^s} + b_0 x + c_{l,n} = -b_r x^{q^r},$$

where the rightmost equality comes from $\Phi_{l,n}(x) = 0$. By (30) and (31), we have

$$\begin{aligned} v(b_r x^{q^r}) &= v(b_r) + v(c_{l,n}) q^{r-s} \\ &= \frac{1}{q^s - 1} \left(-v(\boldsymbol{j})(q-1) \left(1 - \frac{q^{r-s}}{q^{sn}(q^{r-s} - 1)} \right) - v_0 q^r \right). \end{aligned}$$

As $v(\boldsymbol{j}) < 0$ and $v_0 < 0$, we have

$$v(b_r x^{q^r}) \geq \frac{1}{q^s - 1} \left(-v(\boldsymbol{j})(q-1) \left(\frac{q^r - q^s - q^{r-s}}{q^s(q^{r-s} - 1)} \right) - v_0 q^r \right) > 0, \tag{33}$$

which induces

$$\left| \prod_{i=1}^{q^s} (x - x'_i) \right| < 1.$$

Hence there exists some i to be one of $1, \dots, q^s$ such that $|x - x'_i| < 1$.

Conversely, all conjugates of $x = b\xi_{l,n+1}$ are of the form $b(\xi_{l,n+1} + \sum_{k=1}^{s} a_k \xi_{k,1})$ for $a_k \in \mathbb{F}_q$. If $a_k \neq a'_k$ for some k, then

$$\left| b\left(\xi_{l,n+1} + \sum_{k=1}^{s} a_k \xi_{k,1} \right) - b\left(\xi_{l,n+1} + \sum_{k=1}^{s} a'_k \xi_{k,1} \right) \right| = 1.$$

Since $|x - x'_i| < 1$, Krasner's lemma implies $K_n(\xi_{l,n+1}) \subset K_n(x'_i) = K_n(\xi')$. □

2. Basics of Herbrand ψ-functions

Throughout this section, let K be a complete discrete valuation field of characteristic p so that the residue field is a perfect field. Let us recall the definition of the (Herbrand) ψ-function $\psi_{L/K}$ for a finite Galois extension L/K of a complete valuation field of characteristic p. Let G^y denote the y-th upper ramification subgroup of the Galois group $G(L/K)$ of L/K. By the ψ-function of L/K, we mean the real-valued function on the interval $[0, +\infty)$ defined as

$$\psi_{L/K}(y) = \int_0^y \frac{\#G^0}{\#G^r} dr.$$

We extend $\psi_{L/K}$ to $[-1, +\infty)$ by letting $\psi_{L/K}(y) = y$ if $-1 \leq y \leq 0$. Then $\psi_{L/K}$ is a continuous and piecewise linear function on $[-1, +\infty)$. If $\psi_{L/K}$ is linear on some interval $[a, b] \subset [-1, \infty)$, then we have $G^b = G^y = G_{\psi_{L/K}(y)}$ for $y \in (a, b]$. By the (maximal) lower ramification break of L/K, we mean the real number $\psi_{L/K}(y)$, where $y \geq 0$ is the maximal real number such that $G^y \neq 1$. By the wild ramification subgroup of L/K, we mean the first lower ramification subgroup G_1, which is equal to the union of G^y for $y > 0$.

Lemma 3.2.1 (see e.g., [**FV02**, Chapter III, (3.3)]). *Let L/M and M/K be finite Galois extensions. Then*

$$\psi_{L/K} = \psi_{L/M} \circ \psi_{M/K}.$$

Assume that K contains $\mathbb{F}_q$, where q is a power of p. Let v_K denote the normalized valuation associated to K so that $v_K(K^\times) = \mathbb{Z}$. For a positive integer s, put

$$f(X) = X^{q^s} + \sum_{k=1}^{s-1} a_k X^{q^k} + aX \in K[X]$$

such that $\frac{v_K(a_k) - v_K(a)}{q^k - 1} \geq \frac{-v_K(a)}{q^s - 1}$ for $k = 1, \ldots, s-1$, i.e., the Newton polygon of $f(X)/X$ has exactly one segment. The extension generated by the roots of the polynomial $f(X) - c$ for certain $c \in K$ plays a key role in this chapter. To obtain its ψ-function, we will need the following fact. It is a slight generalization of the function field case of [**FV02**, Chapter III, Proposition 2.5] (cf. [**AH22**, Proposition 3.2]).

Proposition 3.2.2. *Let $f(X) - c$ be the polynomial above. Let F and L denote respectively the splitting field of $f(X)$ and that of $f(X) - c$. Put $v_c := v_K(c)$ and $v_a := v_K(a)$. Assume $p \nmid v_c$ and $\frac{-v_c}{q^s} < v_a - v_c$ so that the Newton polygon of $f(X) - c$ has exactly one segment and $R := \frac{v_a q^s}{q^s - 1} - v_c > 0$. Then*

(1) *The extension of F/K is at worst tamely ramified.*
(2) *We have a composition of field extensions*

$$K — F — L.$$

Moreover, the extension L/F is totally ramified of degree q^s and generated by one root x of $f(X)-c$. We have an isomorphism

$$g : G(L/F) \to V; \quad \sigma \mapsto \sigma(x) - x,$$

where $V \cong \mathbb{F}_q^s$ is the $\mathbb{F}_q$-vector space consisting of the roots of $f(X)$.

(3) *Let e denote the ramification index of F/K. The ψ-function of L/K is*

$$\psi_{L/K}(y) = \begin{cases} y, & -1 \leq y \leq 0; \\ ey, & 0 \leq y \leq R; \\ eq^s y - (q^s-1)eR, & R \leq y. \end{cases}$$

Proof. Let M be an extension of K with ramification index $q^s - 1$. We can take some $b \in M$ such that $v(b) = \frac{-v_a}{q^s-1}$. With $b' = b^{q^s}$, modify $f(X)$ to be

$$f_1(X) = X^{q^s} + \sum_{k=1}^{s-1} b_k X^{q^k} + b_0 X := b' f(X/b).$$

We have

$$v_K(b_0) = 0 \text{ and } v_K(b_k) = v_K(a_k) - \frac{v_a(q^s - q^k)}{q^s - 1} \geq 0 \text{ for } k = 1, \ldots, s-1.$$

Thus $f_1(X)$ is a monic polynomial whose reduction is separable. By Hensel's lemma [**Pap23**, Corollary 2.4.5], the extension of M generated by the roots of $f(X)$ is unramified. Hence the extension of K generated by the roots of $f(X)$ is at worst tamely ramified. This shows (1).

For (2), note that the difference of any two roots of $f(X)-c$ is a root of $f(X)$. The field F is contained in L and L is the extension of F generated by one root of $f(X)-c$. As the polynomial $f(X)$ is additive, it root form an $\mathbb{F}_q$-vector space of dimensional s, denoted V. Let x be a root of $f(X)-c$. For any $\sigma \in G(L/F)$, the difference $\sigma(x) - x$ is a root of $f(X)$ and hence we obtain a map $g : G(L/F) \to V$; $\sigma \mapsto \sigma(x) - x$. The element σ is determined by $\sigma(x)$ since x generates L/F. Hence the map g is injective. This implies that $\#G(L/F) \leq q^s$. As the Newton polygon of $f(X)-c$ has exactly one segment, we have $v_F(x) = ev_c/q^s$, where v_F denotes the normalized valuation associated to F and e denotes the ramification index of F/K. As $p \nmid e$, $p \nmid v_c$, we have $\#G(L/F) = q^s$. Therefore, the extension L/F is a totally ramified Galois extension of degree q^s. The map $G(L/F) \to V$ is surjective as the cardinal of $G(L/F)$ is equal to that of V. As each element $G(L/F)$ fixes each element of V, the map g is a morphism.

We show (3). Let π_L be a uniformizer of L. For a nontrivial element σ in $G(L/F)$, as $\sigma(x)/x$ is a unit of L (here x is a root of $f(X)-c$), we have

$$\sigma(x)/x = u_F \epsilon$$

for some $\epsilon \in 1 + (\pi_L)$ (the first higher unit group of L) and some u_F in the unit group of F. Notice

$$\sigma^2(x)/x = \sigma(xu_F\epsilon)/x = u_F\sigma(\epsilon)\sigma(x)/x = u_F^2\sigma(\epsilon)\epsilon,$$
$$\sigma^3(x)/x = \sigma(xu_F^2\sigma(\epsilon)\epsilon)/x = u_F^2\sigma^2(\epsilon)\sigma(\epsilon)\sigma(x)/x = u_F^3\sigma^2(\epsilon)\sigma(\epsilon)\epsilon \text{ and so on.}$$

As the Galois group of L/F is isomorphic to the $\mathbb{F}_q$-vector space of dimensional s, the Galois group element σ has order p. We have

$$1 = \sigma^p(x)/x = u_F^p \prod_{k=0}^{p-1} \sigma^k(\epsilon).$$

This implies $u_F^p \equiv 1 \mod (\pi_L)$. As p-th power map is injective on the residue field of L, we have $u_F \equiv 1 \mod (\pi_L)$. Hence $u_F \in 1 + (\pi_F)$, where π_F is a uniformizer of F. We know that $\sigma(x)/x \in 1 + (\pi_L)$. Hence there exists some u_L in the unit group of L and some positive integer b such that

$$\sigma(x)/x \equiv (1 + u_L\pi_L^b) \mod (\pi_L)^{b+1}. \tag{34}$$

From (2), we know $v_L(x) = ev_c$ and is prime to q^s (v_L denotes the normalized valuation associated to L). Hence there exist integers i, j satisfying $v_L(x^i\pi_F^j) = 1$. Here i is not divisible by p. The element $x^i\pi_F^j$ is a uniformizer of L. By [**Se79**, Chapter IV, Proposition 5], to know the ψ-function of L/F, we need to know $v_L(\sigma(x^i\pi_F^j)/x^i\pi_F^j - 1)$ for all nontrivial Galois group elements σ. By (34), we know

$$\sigma(x^i\pi_F^j)/x^i\pi_F^j \equiv (1 + u_L\pi_L^b)^i \equiv 1 + iu_L^i\pi_L^b \mod (\pi_L)^{b+1}.$$

On the other hand, as $v_L(\sigma(x) - x) = \frac{v_a e q^s}{q^s - 1}$ for any nontrivial σ, we know $b = v_L(\sigma(x) - x) - v_L(x) = eR$. The ψ-functions of F/K and L/F are respectively

$$\psi_{F/K}(y) = \begin{cases} y, & -1 \le y \le 0; \\ ey, & 0 \le y, \end{cases} \quad \text{and } \psi_{L/F}(y) = \begin{cases} y, & -1 \le y \le eR; \\ q^s y - (q^s - 1)eR, & eR \le y. \end{cases}$$

By Lemma 3.2.1, we obtain the ψ-function $\psi_{L/K}$ as the proposition describes. □

3. The extension K_1/K with $s \mid r$

Throughout this section, let ϕ be a Drinfeld A-module over K such that

$$\phi_t(X) = tX + a_s X^{q^s} + a_r X^{q^r} \in K[X],$$

where s and r are positive integers satisfying $s \mid r$. Let u be a finite prime of A with degree 1 and $v_0 = v(u)$ (We also consider the case where $v \mid u$). Put

$$\alpha_1 := \frac{v_0 q^s (q^{r-s} - 1)}{q - 1}.$$

In this section, we study the ramification of $K_1/K = K(\phi[u])/K$. Assume $v(\boldsymbol{j}) < \alpha_1$. We know that K_ψ/K generated by $\xi_{1,1}$ equals K_η/K generated by all roots of $\eta(X) = a_s X^{q^s} + tX$ in Corollary 3.1.2. Thereafter, to avoid confusion with the "ψ" in the Herbrand ψ-function, we use K_η instead of K_ψ. As K_η/K is at worst tamely ramified, it is enough for us to study the ramification of K_1/K_η.

Note that the field K_η contains (q^s-1)-th roots of unity and hence the field $\mathbb{F}_{q^s}$. As $s \mid r$, we may consider ϕ as a rank r/s Drinfeld $\mathbb{F}_{q^s}[t]$-module over K_η, and hence it suffices to obtain the results for the case where $s = 1$. In the rest of this section, assume $s = 1$ so that $\phi_t(X) = tX + a_1X^q + a_rX^{q^r}$, $\alpha_1 = v_0\frac{q(q^{r-1}-1)}{q-1}$, and K_η/K is the splitting field of $a_1X^q + tX$,

3.1. Decomposition of K_1/K_η. In this subsection, we are to study the subextensions of K_1/K_η. Let us prepare a lemma.

Lemma 3.3.1. *Let r be a positive integer ≥ 2. Put $S_r := \frac{q^{r-1}-1}{q-1}$, $Y_2(X) := X^q + X^{q-1} + \cdots + X$, and $Y_{j+1}(X) := Y_2(X)\big(Y_j(X)^q + 1\big)$ for all positive integers $j \geq 2$. Then we have $Y_r(X) = \sum_{i=1}^{qS_r} X^i$ and $Y_r(X) = \sum_{i=2}^{r} Y_2^{S_i}$.*

Proof. We show the equations by induction on r. Notice $S_2 = 1$. The second equation can be shown straightforwardly. As for the first equation, if $r = 3$, then

$$\begin{aligned} Y_3(X) &= (X^q + \cdots + X)\big((X^q + \cdots + X)^q + 1\big) \\ &= (X^q + \cdots + X)(X^{q^2} + X^{q(q-1)} + \cdots + X^q + 1) = \sum_{i=1}^{q^2+q} X^i. \end{aligned}$$

Assume that the first equation is valid for $r-1$. We have

$$\begin{aligned} \sum_{i=1}^{qS_r} X^i &= (X^q + \cdots + X)\big(X^{q(S_r-1)} + X^{q(S_r-2)} + \cdots + X^q + 1\big) \\ &= Y_2(X)\big((X^{S_r-1} + X^{S_r-2} + \cdots + X)^q + 1\big) \end{aligned}$$

As $S_r - 1 = qS_{r-1}$, we know from the assumption that

$$\sum_{i=1}^{qS_r} X^i = Y_2(X)\big((Y_{r-1})^q + 1\big) = Y_r(X).$$

□

Put $Z_r(X) := \sum_{i=2}^{r} X^{S_i}$ so that $Y_r(X) = Z_r(Y_2(X))$. We have $Z_2(X) = X$.

Lemma 3.3.2. *For a integer $r \geq 2$, let ϕ be a rank r Drinfeld A-module over K such that $\phi_t(X) = tX + a_1X^q + a_rX^{q^r} \in K[X]$. Assume $v(\boldsymbol{j}) < \alpha_1$. Let K_Z be the extension of*

K_η generated by all roots of

$$Z(X) \coloneqq Z_r(X) - \beta \text{ with } \beta = ua_r^{-1}\xi_{1,1}^{1-q^r}.$$

Let K_1' be the extension of K_Z generated by all roots of the degree q polynomials

$$\mathrm{H}_\gamma(X) = X^q - \gamma X - \gamma \text{ for all roots } \gamma^{-1} \text{ of } Z(X).$$

Then

- *K_1'/K_η is a subextension of K_1/K_η;*
- *K_1/K_1' is a compositum of Kummer extensions.*

Proof. Put $b = \xi_{1,1}^{-1}$ in the definition of $\Phi(X)$ in (29) so that any element in $\mathbb{F}_q$ is a root of $\Phi(X)$. We have that $X^q - X = \prod_{a\in\mathbb{F}_q}(X-a)$ divides $\Phi(X)$. Let $\Theta(X)$ be the polynomial such that $\Theta(X)(X^q - X) = \Phi(X)$. As $\Phi(1) = 0$, we have $-b_0 - b_r = 1$. Note $b_0/b_r = \beta$. We have

$$\Theta(X) = b_r\left(X^{q(q^{r-1}-1)} + X^{q(q^{r-1}-2)+1} + X^{q(q^{r-1}-3)+2} + \cdots + X^{q-1} - \beta\right),$$

whose roots generate K_1/K_η. We consider $\overline{\Theta}(X) \coloneqq (X^{qS_r} + X^{qS_r-1} + \cdots + X) - \beta$ so that $\overline{\Theta}(X^{q-1}) = \Theta(X)$. Let L denote the subextension of K_1/K_η generated by its roots. Then K_1/L is generated by the $(q-1)$-st roots of x for all $x \in L$ satisfying $\overline{\Theta}(x) = 0$, which implies that K_1/L is a compositum of Kummer extensions. If we can show $L = K_1'$, then two dots in the claim are proved.

Put $Y_2(X) = X^q + X^{q-1} + \cdots + X$. By Lemma 3.3.1, we have $\overline{\Theta}(X) = Y_r(X) - \beta = Z(Y_2(X))$. Hence L/K_Z is generated by all roots of the polynomials

$$X^q + \cdots + X - \gamma^{-1} \text{ for all roots } \gamma^{-1} \text{ of } Z(X).$$

We prove $L = K_1'$ by showing that the extension L/K_η is generated by all roots of $\mathrm{H}_\gamma(X)$ for all γ. Notice

$$\sum_{i=1}^{q} X^i = \frac{X(X^q-1)}{X-1} = \frac{X(X-1)^q}{X-1} = X(X-1)^{q-1}.$$

We have $Y_2(X+1) - \gamma^{-1} = X^q + X^{q-1} - \gamma^{-1}$. Then

$$\mathrm{H}_\gamma(X) = -\gamma X^q \cdot \left(Y_2\left(\frac{1}{X}+1\right) - \gamma^{-1}\right). \tag{35}$$

Thus $L = K_1'$, as desired. □

Notice $v(\beta) = -v(b_r) < 0$ (See (30) for $v(b_r)$). The Newton polygon of $Z(X)$ has exactly one segment determined by the points $(0, -v(b_r))$ and $(S_r, 0)$. Hence a root of

$Z(X)$ has valuation

$$c := -\frac{-v(\boldsymbol{j}) + \alpha_1}{S_r}. \tag{36}$$

If $v(\boldsymbol{j}) < \alpha_1$, then $c < 0$. Put $C := q^{-c}$. We have $C > 1$.

Remark 3.3.3. For a integer $r \geq 3$, let ϕ be a rank r Drinfeld $\mathbb{F}_q[t]$-module over K such that $\phi_t(X) = tX + a_1 X^q + a_r X^{q^r} \in K[X]$. We have shown that all roots of $X^q - \gamma X - \gamma$ for all roots γ^{-1} of $Y_r(X) - \beta$ generates K_1'/K_Z. The extension K_1'/K_Z contains a subextension K_Z^0/K_Z generated by all roots of polynomials $X^q - \gamma X$ with γ^{-1} varying within all roots of $Z(X)$. Let γ_1^{-1} and γ_2^{-1} be two different roots of $Z(X)$. Let K_{γ_1} be the extension of K_Z^0 generated by all roots of $\mathrm{H}_{\gamma_1}(X)$. If $p \nmid v(\boldsymbol{j})$, as $v(\gamma) = \frac{-v(\boldsymbol{j}) + v_0 q S_r}{S_r}$, we have $p \nmid v(\gamma_1) = -c$. By Proposition 3.2.2, the extension K_{γ_1}/K_Z^0 is a degree q wildly ramified extension. Hence if w is the normalized valuation associated to K_{γ_1} such that $w = Qv$ for some positive integer Q, then we have $q \mid Q$. As $w(\gamma_2) = -Qc$, this implies $p \mid w(\gamma_2)$. Hence one can not apply Proposition 3.2.2 to study the extension of K_{γ_1} generated by the roots of $\mathrm{H}_{\gamma_2}(X)$. Therefore, it may be hard to study K_1'/K_Z as an extension generated by the roots of $Y_r(X) - \beta$.

3.2. An alternative polynomial generating the same extension. We will continue to use the notation in Section 3.1. *Throughout this subsection*, for a integer $r \geq 3$, let ϕ be a rank r Drinfeld A-module over K such that

- $\phi_t(X) = tX + a_1 X^q + a_r X^{q^r} \in K[X]$;
- for the valuation $v(\boldsymbol{j})$ of the j-invariant of ϕ, we have $v(\boldsymbol{j}) < \alpha_1$.

Put $\widehat{Z}(X) := X^{S_r} - \beta$. In this subsection, we are to show that the roots of $\widehat{Z}(Y_2(X^{q-1}))$ also generate K_1/K_Z. The Newton polygons of $Z(X)$ and $\widehat{Z}(X)$ are the same. It turns out that we can obtain the ψ-function of K_1/K_Z by considering this extension as the one generated by the roots of $\widehat{Z}(Y_2(X^{q-1}))$ in Section 3.3.

Following [**KL04**, Proposition 3], we may show that the extension K_Z/K_η equals the extension generated by the roots of $\widehat{Z}(X)$.

Lemma 3.3.4. *Put $\widehat{Z}(X) = X^{S_r} - \beta$. Let $K_{\widehat{Z}}$ be the extension of K_η generated by the roots of $\widehat{Z}(X)$.*

(1) *The extension K_Z/K_η equals the extension of K_η generated by all roots of $\widehat{Z}(X)$.*
(2) *For each root x of $Z(X)$, there exists a unique root x' of $\widehat{Z}(X)$ such that $|x - x'| = C^{-q^{r-2}+1}$. For roots x_1 different from x and x_1' different from x', we have $|x - x_1'| = |x_1 - x'| = C$.*

Proof. We show (1). Let $x, x_1, \ldots, x_{S_r - 1}$ be all roots of $Z(X)$. Let x_j' for $j = 1, \ldots, S_r$ denote the roots of $\widehat{Z}(X)$. We first show $|x - x_j| = C$ for any j and $|x_i' - x_j'| = C$ for

$j \neq i$. We have $Z'(X) = \prod_{j=1}^{S_r-1}(X - x_j) = S_r X^{S_r-1} + S_{r-1}X^{S_{r-1}-1} + \cdots + 1$. Consider

$$Z'(x) = \prod_{j=1}^{S_r-1}(x - x_j) = S_r x^{S_r-1} + S_{r-1}x^{S_{r-1}-1} + \cdots + 1.$$

As $|x| = C > 1$, we have

$$\left|\prod_{j=1}^{S_r-1}(x - x_j)\right| = |x^{S_r-1}|.$$

By the ultrametric inequality, we have $|x - x_j| \leq |x|$. The above equation implies $|x - x_j| = |x| = C$ for any j. We have $\widehat{Z}'(X) = S_r X^{S_r-1}$ and hence

$$\prod_{j=1}^{S_r-1}(x'_i - x'_j) = \widehat{Z}'(x'_i) = S_r x_i'^{S_r-1}.$$

This implies

$$\left|\prod_{j=1}^{S_r-1}(x'_i - x'_j)\right| = |x'_i|^{S_r-1}.$$

As the Newton polygon of $\widehat{Z}(X)$ is the same as that of $Z(X)$, we have $|x'_i| = |x'_j| = C$. Similarly we have $|x'_i - x'_j| = |x'_i| = C$ for $j \neq i$.

If we can show for some index i that $|x - x'_i| < C = |x'_i - x'_j|$ for all $j \neq i$, then Krasner's lemma implies $K_\eta(x'_i) \subset K_\eta(x)$. To know $|x - x'_i|$, consider

$$Z(x) = \prod_{i=1}^{S_r}(x - x'_i) = x^{S_r} - \beta = x^{S_{r-1}} + x^{S_{r-2}} + \cdots + x^{S_2},$$

where the rightmost equation follows from $Z(x) = Z_r(X) - \beta = 0$. By (36), we have $v(x) = c$ and $|x| = C > 1$. Hence

$$\left|\prod_{j=1}^{S_r}(x - x'_j)\right| = |x^{S_{r-1}} + \cdots + x^{S_2}| = |x^{S_{r-1}}| = C^{S_{r-1}}. \tag{37}$$

There exists an index i such that $|x - x'_i| \leq C^{\frac{S_{r-1}}{S_r}} < C$, as desired.

There exists a root x'_k of $Z(X)$ satisfying $|x'_k - x_j| < C$. As $|x_j - x| = C$, we have $|x_j - x'_i| = C$. This implies $x'_k \neq x'_i$. Hence for any root of $\widehat{Z}(X)$, there exists a root of $Z(X)$ so that the absolute value of the difference of these roots $< C$. Let x vary within the roots of $Z(X)$. We have $K(x'_1, \cdots, x'_{S_r}) \subset K(x, x_1, \ldots, x_{S_r-1})$. Hence $K_Z \subset K_{\widehat{Z}}$. As $|x - x'_i| < C = |x - x_j|$ for all j, we can similarly show that $K_Z \subset K_{\widehat{Z}}$.

As for (2), one equation is already proved above. Similarly, as $|x'_i - x'_j| = C$ for $j \neq i$, we have $|x - x'_j| = C$. Hence x'_i is the only root of $\widehat{Z}(X)$ such that $|x - x'_i| < C$. By (37), we have $|x - x'_i| = C^{S_{r-1} - S_r + 1} = C^{-q^{r-2}+1}$. □

Lemma 3.3.4 (1) implies that the extension K_Z/K is at most tamely ramified. Alternatively, in the above proof, we have shown that $|x - x_j| = |x|$ for two different roots x and x_j of $Z(X)$. For an element $\sigma \in G(K_Z/K_\eta)$, we have that $\sigma(x) = x_j$ for some j if σ does not fix x. The equality $|x - x_j| = |x|$ in the proof implies that $v((\sigma(x) - x)/x) = 0$. Let x vary within the roots of $Z(X)$. We know that the wild ramification subgroup of the extension K_Z/K_η is trivial. Hence the extension K_Z/K_η is at worst tamely ramified.

The next corollary is concerned with Galois actions. Let V_Z and $V_{\widehat{Z}}$ denote respectively the set of roots of $Z(X)$ and those of $\widehat{Z}(X)$. By Lemma 3.3.4 (2), for each root x of $Z(X)$, we have found a unique root x' of $\widehat{Z}(X)$ satisfying $|x - x'| = C^{-q^{r-2}+1}$. This defines a map $f_Z : V_Z \to V_{\widehat{Z}}$. Note that the Galois group $G(K_Z/K_\eta)$ permutes both V_Z and $V_{\widehat{Z}}$.

Corollary 3.3.5. *The map f_Z is a bijection and is compatible with the $G(K_Z/K_\eta)$-action, i.e., if $x \mapsto x'$, then $\sigma(x)$ maps to $\sigma(x')$.*

Proof. Note that the polynomials $Z(X)$ and $\widehat{Z}(X)$ have the same degree. The bijectivity follows from Lemma 3.3.4 (2). As for the compatibility, if x maps to x', i.e., $|x - x'| = C^{-q^{r-2}+1}$, we have $|\sigma(x) - \sigma(x')| = |\sigma(x - x')| = C^{-q^{r-2}+1}$ as σ preserves the absolute value. Then $\sigma(x)$ maps to $\sigma(x')$ as $\sigma(x')$ is the only root of $Z(X)$ satisfying $|\sigma(x) - \sigma(x')| = C^{-q^{r-2}+1}$. □

Lemma 3.3.6. *Fix a root γ^{-1} of $Z(X)$ and put $\gamma'^{-1} := f_Z(\gamma^{-1})$ (f_Z in the above corollary).*

(1) *Let K_γ and $K_{\gamma'}$ be respectively the extension of K_Z generated by all roots of $\mathrm{H}_\gamma(X)$ and those of $\mathrm{H}_{\gamma'}(X)$. Then we have $K_\gamma = K_{\gamma'}$. Especially, the extension K_1'/K_Z equals the extension of K_Z generated by all roots of the polynomials $\mathrm{H}_{\gamma'}(X) = X^q - \gamma' X - \gamma'$ for γ' varying within roots of $\widehat{Z}(X)$.*
(2) *For each root x of $\mathrm{H}_\gamma(X)$, there exists a unique root x' of $\mathrm{H}_{\gamma'}(X)$ such that $|x - x'| = C^{-q^{r-2}}$. For each root x_1 different from x and each root x_1' different from x', we have $|x - x_1'| = |x_1 - x'| = C^{-\frac{1}{q-1}}$.*

As $|\gamma| = C^{-1} < 1$, the Newton polygons of $\mathrm{H}_\gamma(X)$ and $\mathrm{H}_{\gamma'}(X)$ are the same and both have exactly one segment determined by the points $(0, v(\gamma))$ and $(q, 0)$. Hence the roots of these two polynomials have absolute value $C^{-\frac{1}{q}}$.

Proof. We show (1). The root $\gamma'^{-1} = f(\gamma^{-1})$ of $Z(X)$ satisfies

$$|\gamma^{-1} - \gamma'^{-1}| = C^{-q^{r-2}+1}.$$

Note $|\gamma| = C^{-1}$. We have

$$|\gamma - \gamma'| = C^{-q^{r-2}-1}.$$

We are to show that the extension K_γ of K_Z generated by the roots of $\mathrm{H}_\gamma(X)$ equals the extension $K_{\gamma'}$ of K_Z generated by the roots of $\mathrm{H}_{\gamma'}(X)$. This is enough to claim (1).

Let $x, x_1, \ldots, x_{q-1}$ be all roots of $\mathrm{H}_\gamma(X)$. Let x'_i for $i = 1, \ldots, q$ denote the roots of $\mathrm{H}_{\gamma'}(X)$. Since the difference of two roots of $\mathrm{H}_\gamma(X)$ is a root of $X^q - \gamma X$, we have $|x - x_j| = |\gamma|^{\frac{1}{q-1}} = C^{-\frac{1}{q-1}}$. Similarly, we have $|x'_i - x'_j| = C^{-\frac{1}{q-1}}$ for $j \neq i$. If there exists an index i such that $|x - x'_i| < |x'_i - x'_j|$ for $j \neq i$, then Krasner's lemma implies $K_Z(x'_i) \subset K_Z(x)$. Consider

$$\mathrm{H}_{\gamma'}(x) = \prod_{i=1}^{q}(x - x'_i) = x^q - \gamma' x - \gamma' = (\gamma - \gamma')x + (\gamma - \gamma').$$

As $|x| = C^{-\frac{1}{q}} < 1$, we have

$$\left|\prod_{i=1}^{q}(x - x'_i)\right| = |\gamma - \gamma'| = C^{-q^{r-2}-1}. \tag{38}$$

Hence there exists an index i such that $|x - x'_i| \leq C^{-\frac{q^{r-2}+1}{q}} < C^{-\frac{1}{q-1}} = |x'_i - x'_j|$ for $j \neq i$. By Kranser's lemma, this shows $K_Z(x'_i) \subset K_Z(x)$.

There exists a root x'_k of $\mathrm{H}_\gamma(X)$ satisfying $|x'_k - x_j| < C^{-\frac{1}{q-1}}$. As $|x_j - x| = C^{-\frac{1}{q-1}}$, we have $|x_j - x'_i| = C^{-\frac{1}{q-1}}$. This implies $x'_k \neq x'_i$. Hence for any root of $\mathrm{H}_{\gamma'}(X)$, there exists a root of $\mathrm{H}_\gamma(X)$ so that the absolute value of the difference of these root $< C^{-\frac{1}{q-1}}$. We have $K(x'_1, \ldots, x'_q) \subset K(x, x_1, \ldots, x_{q-1})$. Hence $K_{\gamma'} \subset K_\gamma$. As $|x - x'_i| < C^{-\frac{1}{q-1}} = |x - x_j|$ for all j, we can similarly show that $K_\gamma \subset K_{\gamma'}$.

As for (2), one equation is already proved above. Similarly, as $|x'_i - x'_j| = C^{-\frac{1}{q}}$ for $j \neq i$, we have $|x - x'_j| = C^{-\frac{1}{q}}$. Hence x'_i is the only root of $\mathrm{H}_{\gamma'}(X)$ such that $|x - x'_i| < C^{-\frac{1}{q}}$. By (38), we have $|x - x'_i| = C^{-q^{r-2}}$. □

Put $\overline{\Theta}(X) = Z(Y_2(X))$ (as in the proof of Lemma 3.3.1) and $\widehat{\overline{\Theta}}(X) \coloneqq \widehat{Z}(Y_2(X))$.

Corollary 3.3.7. *Let δ be a root of $\overline{\Theta}(X)$.*

(1) *There exists a root δ' of $\widehat{\overline{\Theta}}(X)$ such that $|\delta - \delta'| = C^{-q^{r-2}+\frac{2}{q}}$.*
(2) *Let δ'_j for $j = 1, \ldots, q-1$ be roots of $Y_2(X) - Y_2(\delta')$ different from δ'. Then $|\delta - \delta'_j| = C^{\frac{q-2}{q(q-1)}}$. Let δ_j for $j = 1, \ldots, q-1$ be roots of $Y_2(X) - Y_2(\delta)$ different from δ. Then $|\delta' - \delta_j| = C^{\frac{q-2}{q(q-1)}}$.*
(3) *For any root δ'' of $\widehat{\overline{\Theta}}(X)$ such that $Y_2(\delta') \neq Y_2(\delta'')$, we have $|\delta - \delta''| = C^{\frac{1}{q}}$. For any root $\delta_{(1)}$ of $\overline{\Theta}(X)$ such that $Y_2(\delta_{(1)}) \neq Y_2(\delta)$, we have $|\delta' - \delta_{(1)}| = C^{\frac{1}{q}}$.*
(4) *Fix δ. The element δ' is the only root of $\widehat{\overline{\Theta}}(X)$ satisfying $|\delta - \delta'| = C^{-q^{r-2}+\frac{2}{q}}$. Fix δ'. The element δ is the only root of $\overline{\Theta}(X)$ satisfying $|\delta' - \delta| = C^{-q^{r-2}+\frac{2}{q}}$.*

PROOF. Put $x := (\delta - 1)^{-1}$. By (35), the element x is a root of $\mathrm{H}_\gamma(X)$, where γ^{-1} is a root of $Z(X)$ with $\gamma^{-1} = Y_2(\delta)$. As $|x| = |\gamma|^{\frac{1}{q}} = C^{-\frac{1}{q}} < 1$, we have

$$|\delta| = \left|\frac{1}{x} + 1\right| = C^{\frac{1}{q}}.$$

Hence $|\delta - 1| = |\delta|$.

For $\gamma'^{-1} = f_Z(\gamma^{-1})$, by Lemma 3.3.6, for the root x of $\mathrm{H}_\gamma(X)$, there exists a unique root x' of $\mathrm{H}_{\gamma'}(X)$ satisfying $|x - x'| = C^{-q^{r-2}}$. Put $\delta' := \frac{1}{x'} + 1$. Then

$$\left|(\delta - 1)^{-1} - (\delta' - 1)^{-1}\right| = |x - x'| = C^{-q^{r-2}}.$$

As $|x'| = |\gamma'|^{\frac{1}{q}} = C^{-\frac{1}{q}} < 1$, we have $|\delta' - 1| = |\delta'| = C^{\frac{1}{q}}$. Hence

$$|\delta - \delta'| = C^{-q^{r-2} + \frac{2}{q}}$$

and (1) follows.

We show the first claim of (2) and the second claim similarly follows. Let x'_j for $j = 1, \ldots, q-1$ denote the roots of $\mathrm{H}_{\gamma'}(X)$ different from x'. Put $\delta'_j := \frac{1}{x'_j} + 1$ for all j. These δ'_j are roots of $Y_2(X) - Y_2(\delta')$ different from δ'. We have $|\delta'_j - 1| = |\delta'_j| = C^{\frac{1}{q}}$. As $x' - x'_j$ is a nonzero root of $X^q - \gamma' X$, we have $|x' - x'_j| = C^{-\frac{1}{q-1}}$. Since

$$\left|(\delta' - 1)^{-1} - (\delta'_j - 1)^{-1}\right| = |x' - x'_j| = C^{-\frac{1}{q-1}},$$

we have

$$|\delta' - \delta'_j| = C^{\frac{q-2}{q(q-1)}}.$$

As $|\delta - \delta'| = C^{-q^{r-2} + \frac{2}{q}} < C^{\frac{q-2}{q(q-1)}}$, we have

$$|\delta - \delta'_j| = \max\{|\delta - \delta'|, |\delta' - \delta'_j|\} = |\delta' - \delta'_j| = C^{\frac{q-2}{q(q-1)}}.$$

and this shows (2).

We show the first claim of (3) and the second claim similarly follows. Let δ'' be as in the claim such that $Y_2(\delta'')$ is a root of $Z(X)$ different from $Y_2(\delta')$. In the proof of Lemma 3.3.4, we have shown $|Y_2(\delta') - Y_2(\delta'')| = C$. Let δ''_j for $j = 1, \ldots, q$ be the roots of $Y_2(X) - Y_2(\delta'')$ such that $\delta'' = \delta''_1$. We also have

$$Y_2(\delta') - Y_2(\delta'') = \prod_{j=1}^{q} (\delta' - \delta''_j).$$

Hence

$$\left|\prod_{j=1}^{q} (\delta' - \delta''_j)\right| = C.$$

For each j, as $|\delta'| = |\delta''_j| = C^{\frac{1}{q}}$, we have $|\delta' - \delta''_j| \leq |\delta'|$ and the above equation implies $|\delta' - \delta''_j| = |\delta'| = C^{\frac{1}{q}}$. As $|\delta - \delta'| = C^{-q^{r-2}+\frac{2}{q}} < C^{\frac{1}{q}}$, we have $|\delta - \delta''_j| = \max\{|\delta - \delta'|, |\delta' - \delta''_j|\} = C^{\frac{1}{q}}$.

(4) follows from (1), (2), and (3). □

Let $V_{\overline{\Theta}}$ and $V_{\widehat{\Theta}}$ denote respectively the set of roots of $\overline{\Theta}(X)$ and those of $\widehat{\Theta}(X)$. By Lemmas 3.3.1, 3.3.4, and 3.3.6, the extension K'_1/K_η is either generated by the elements in the set $V_{\overline{\Theta}}$, or the elements in the set $V_{\widehat{\Theta}}$. By Corollary 3.3.7, we have found for each root δ of $\overline{\Theta}(X)$ a unique root δ' of $\widehat{\Theta}(X)$ satisfying $|\delta - \delta'| = C^{-q^{r-2}+\frac{2}{q}}$. This defines a map $f_{\overline{\Theta}} : V_{\overline{\Theta}} \to V_{\widehat{\Theta}}$. Note that the Galois group $G(K'_1/K_\eta)$ permutes both $V_{\overline{\Theta}}$ and $V_{\widehat{\Theta}}$. Following the proof of Corollary 3.3.5, we can apply Corollary 3.3.7 (4) to show the following result.

Corollary 3.3.8. *The map $f_{\overline{\Theta}}$ is bijective and is compatible with the $G(K'_1/K_\eta)$-action.*

Lemma 3.3.9. *Fix δ to be a root of $\overline{\Theta}$. Put $\delta' = f_{\overline{\Theta}}(\delta)$ and $\gamma = Y_2(\delta)$. Let K_δ and $K_{\delta'}$ denote respectively the extensions of K_γ (See Lemma 3.3.6) generated by all roots of $X^{q-1} - \delta$ and those of $X^{q-1} - \delta'$.*

(1) *We have $K_\delta = K_{\delta'}$. Especially, the extension K_1/K'_1 equals the extension of K'_1 generated by all roots of $X^{q-1} - \delta'$ for all $\delta' \in V_{\widehat{\Theta}}$.*
(2) *For each root x of $X^{q-1} - \delta$, there exists a unique root x' of $X^{q-1} - \delta'$ such that $|x - x'| = C^{-q^{r-2}+\frac{1}{q-1}}$. For a root x_1 of $X^{q-1} - \delta$ different from x and a root x'_1 of $X^{q-1} - \delta'$ different from x', we have $|x - x'_1| = |x_1 - x'| = C^{\frac{1}{q(q-1)}}$.*

The Newton polygons of $X^{q-1} - \delta$ and $X^{q-1} - \delta'$ are the same and both have exactly one segment determined by the points $(0, v(\delta)) = (0, c/q)$ and $(q-1, 0)$.

Proof. If $K_\delta = K_{\delta'}$, we let δ vary within the roots of $V_{\widehat{\Theta}}$ and the second claim of (1) follows. We show $K_\delta = K_{\delta'}$.

Let $x, x_1, \ldots, x_{q-2}$ be all roots of $X^{q-1} - \delta$. Let x'_i for $i = 1, \ldots, q-1$ denote the roots of $X^{q-1} - \delta'$. We first show $|x - x_j| = C^{\frac{1}{q(q-1)}}$ and $|x'_i - x'_j| = C^{\frac{1}{q(q-1)}}$. Notice $(X^{q-1} - \delta)'|_{X=x} = \prod_{j=1}^{q-2}(x - x_j) = (q-1)x^{q-2}$. We have

$$\left|\prod_{j=1}^{q-2}(x - x_j)\right| = |x|^{q-2}.$$

As $|x| = |x_j| = C^{\frac{1}{q(q-1)}}$, this implies $|x - x_j| = |x| = C^{\frac{1}{q(q-1)}}$. Notice $(X^{q-1} - \delta')'|_{X=x'_i} = \prod_{j=1}^{q-2}(x'_i - x'_j) = (q-1)x_i'^{q-2}$. We have

$$\left|\prod_{j=1}^{q-2}(x'_i - x'_j)\right| = |x'_i|^{q-2}.$$

As $|x_i'| = |x_j'| = C^{\frac{1}{q(q-1)}}$, we have $|x_i' - x_j'| \leq |x_i'|$. This equation implies $|x_i' - x_j'| = |x_i'| = C^{\frac{1}{q(q-1)}}$.

If we can show for some i that $|x - x_i'| < C^{\frac{1}{q(q-1)}} = |x_i' - x_j'|$ for all $j \neq i$, then Kranser's lemma implies $K_\gamma(x_i') \subset K_\gamma(x)$. Consider

$$\prod_{i=1}^{q-1}(x - x_i') = x^{q-1} - \delta' = \delta - \delta'.$$

As $|\delta - \delta'| = C^{-q^{r-2}+\frac{2}{q}}$, we have

$$\left|\prod_{i=1}^{q-1}(x - x_i')\right| = C^{-q^{r-2}+\frac{2}{q}}. \tag{39}$$

There is some index i such that $|x - x_i'| \leq C^{-\frac{q^{r-2}}{q-1}+\frac{2}{q(q-1)}} < 1 < C^{\frac{1}{q(q-1)}}$ for any $j \neq i$, as desired. We have $K_{\delta'} = K_\gamma(x') \subset K_\gamma(x) = K_\delta$. As $|x - x_i'| < C^{\frac{1}{q(q-1)}} = |x - x_j|$ for any j, we have $K_\delta \subset K_{\delta'}$ by Kranser's lemma.

As for (2), for each $j = 1, \ldots, q-2$, we have

$$|x_j - x_i'| = \max\{|x_j - x|, |x - x_i'|\} = C^{\frac{1}{q(q-1)}},$$
$$|x - x_j'| = \max\{|x - x_i'|, |x_i' - x_j'|\} = C^{\frac{1}{q(q-1)}}.$$

Hence x_i' is the only root of $X^{q-1} - \delta'$ such that $|x - x_i'| < C^{\frac{1}{q(q-1)}}$. By (39), we have

$$|x - x_i'| = C^{-q^{r-2}+\frac{2}{q}-\frac{(q-2)}{q(q-1)}} = C^{-q^{r-2}+\frac{1}{q-1}}.$$

□

Put $\Theta(X) = \overline{\Theta}(X^{q-1})$ (as in the proof of Lemma 3.3.1) and $\widehat{\Theta} := \widehat{\overline{\Theta}}(X^{q-1})$.

Corollary 3.3.10. *Let x be a root of $\Theta(X)$ and x' be one root of $\widehat{\Theta}(X)$ satisfying $|x - x'| = C^{-q^{r-2}+\frac{1}{q-1}}$.*

(1) *For any root x'' of $\widehat{\Theta}(X)$ such that $|x^{q-1} - x''^{q-1}| = C^{\frac{q-2}{q(q-1)}}$ as in Corollary* 3.3.7 (2), *we have $|x - x''| = 1$ or $C^{\frac{1}{q(q-1)}}$. For any root $x_{(1)}$ of $\Theta(X)$ such that $|x_{(1)}^{q-1} - x'^{q-1}| = C^{\frac{q-2}{q(q-1)}}$, we have $|x_{(1)} - x'| = 1$ or $C^{\frac{1}{q(q-1)}}$.*

(2) *For any root x''' of $\widehat{\Theta}(X)$ such that $|x^{q-1} - x'''^{q-1}| = C^{\frac{1}{q}}$ as in Corollary* 3.3.7 (3), *we have $|x - x'''| = C^{\frac{1}{q(q-1)}}$. For any root $x_{(2)}$ of $\Theta(X)$ such that $|x_{(2)}^{q-1} - x'^{q-1}| = C^{\frac{1}{q}}$, we have $|x_{(2)} - x'| = C^{\frac{1}{q(q-1)}}$.*

(3) *The element x' is the only root of $\widehat{\Theta}(X)$ satisfying $|x - x'| = C^{-q^{r-2}+\frac{1}{q-1}}$. The element x is the only root of $\Theta(X)$ satisfying $|x' - x| = C^{-q^{r-2}+\frac{1}{q-1}}$.*

PROOF. We show the first claim of (1) and the second claim similarly follows. Let x''_j for $j = 1, \dots, q-1$ denote different roots of $X^{q-1} - x''^{q-1}$ so that $x''_1 = x''$. Consider

$$(X^{q-1} - x''^{q-1})|_{X=x} = \prod_{j=1}^{q-1}(x - x''_j) = x^{q-1} - x''^{q-1}.$$

We have

$$\left|\prod_{j=1}^{q-1}(x - x''_j)\right| = |x^{q-1} - x''^{q-1}| = C^{\frac{q-2}{q(q-1)}}. \tag{40}$$

There is an index i such that $|x - x''_i| \leq C^{\frac{q-2}{q(q-1)^2}}$. As $|x''_i - x''_j| = C^{\frac{1}{q(q-1)}}$ for $j \neq i$ (in the proof of Lemma 3.3.9), we have for $j \neq i$

$$|x - x''_j| = \max\{|x - x''_i|, |x''_i - x''_j|\} = C^{\frac{1}{q(q-1)}}.$$

By (40), we have $|x - x''_i| = 1$. If $x''_i = x''$, we have $|x - x''| = 1$, or else $|x - x''| = C^{\frac{1}{q(q-1)}}$. (1) follows.

We show the first claim of (2) and the second claim similarly follows. Let x'''_j for $j = 1, \dots, q-1$ denote different roots of $X^{q-1} - x'''^{q-1}$ so that $x'''_1 = x'''$. Consider

$$(X^{q-1} - x'''^{q-1})|_{X=x} = \prod_{j=1}^{q-1}(x - x'''_j) = x^{q-1} - x'''^{q-1}.$$

We have

$$\left|\prod_{j=1}^{q-1}(x - x'''_j)\right| = |x^{q-1} - x'''^{q-1}| = C^{\frac{1}{q}}. \tag{41}$$

Note that the absolute value of each root of $\Theta(X)$ and of $\widehat{\Theta}(X)$ is $C^{\frac{1}{q(q-1)}}$. We have $|x - x'''_j| \leq |x|$. Then (41) implies $|x - x'''_j| = C^{\frac{1}{q(q-1)}}$ for all j. (2) follows.

(3) follows from (1), (2), and Lemma 3.3.9 (2). □

Let V and $\widehat{V}$ denote respectively the set of roots of $\Theta(X)$ and that of $\widehat{\Theta}(X)$. By the proof of Lemma 3.3.2, the set V consists of elements $\xi_{1,1}^{-1}\xi$ for $\xi \in \phi[u]$ having valuation $v(\xi_{l,1})$, where $\{\xi_{i,1}\}_{i=1,\dots,r}$ is an SMB of $\phi[u]$ and l is one of $2, \dots, r$. By Lemma 3.3.9, the elements in this set $\widehat{V}$ also generate the extension K_1/K_η. We have found for each root x of $\Theta(X)$ a unique root x' of $\widehat{\Theta}(X)$ satisfying $|x - x'| = C^{-q^{r-2}+\frac{1}{q-1}}$. This defines a map $f : V \to \widehat{V}$. Note that the Galois group $G(K_1/K_\eta)$ permutes both V and $\widehat{V}$. Following the proof of Corollary 3.3.5, we can apply Corollary 3.3.10 (3) to show the following result.

Theorem 3.3.11. *The map $f : V \to \widehat{V}$ is bijective and compatible with the $G(K_1/K_\eta)$-action.*

3.3. The Herbrand ψ-function of K_1/K. We will continue to use the notations in previous subsections. *Throughout this subsection*, for a integer $r \geq 3$, let ϕ be a rank r Drinfeld A-module over K such that

- $\phi_t(X) = tX + a_1X^q + a_rX^{q^r} \in K[X]$;
- for the valuation $v(\boldsymbol{j})$ of the j-invariant of ϕ, we have $v(\boldsymbol{j}) < \alpha_1$.

In this subsection, we will work out the (Herbrand) ψ-function of K_1/K. In Lemma 3.3.2, we have the decomposition of K_1/K

$$K \text{ —— } K_\eta \text{ —— } K_Z \text{ —— } K_1' \text{ —— } K_1 \ .$$

The extensions K_Z/K and K_1/K_1' are tamely ramified and their ψ-functions are clear. We are to work out the ψ-function of the extension K_1'/K_Z.

Let ζ be a primitive S_r-th root of unity (Here $S_r = \frac{q^{r-1}-1}{q-1}$). Let ζ_1 be the preimage of ζ via the Frobenius map of $\mathbb{F}_{q^{r-1}}$ over $\mathbb{F}_q$ so that $\zeta_1^q = \zeta$. For each positive integer $j \geq 1$, let ζ_{j+1} denote the preimage of ζ_j via the Frobenius map so that $\zeta_{j+1}^q = \zeta_j$. For an integer $j \geq 1$, put $\xi_j \coloneqq \prod_{k=1}^{j} \zeta_k$.

Let us prepare a lemma.

Lemma 3.3.12. *For an integer j satisfying $0 \leq j \leq S_r - 1$ and an integer s satisfying $j \leq s \leq S_r - 1$, put*

$$\Delta_{s,j} \coloneqq \begin{cases} 1 & j = 0; \\ \sum_{i_1 \geq i_2 \geq \ldots \geq i_j \geq 0}^{\leq s-j} \zeta_1^{i_1} \cdots \zeta_j^{i_j} & j > 0, \end{cases}$$

such that $\Delta_{s,s} = 1$. For $j \geq 0$ and $s \geq j+1$, let $\delta_{s,j+1} = (\zeta^{s-j}\Delta_{s,j})^{-q}$ be the preimage of $\zeta^{s-j}\Delta_{s,j}$ via the Frobenius map. Then for $j \geq 0$ and $s \geq j+1$, we have

$$\Delta_{s,j} - \delta_{s,j+1} = (1 - \xi_{j+1})\,\Delta_{s,j+1}. \tag{42}$$

The elements $\Delta_{s,j}$ and $\delta_{s,j+1}$ for all s and j appearing in the lemma belong to $\mathbb{F}_{q^{r-1}}$.

Proof. For the case $j = 0$, the desired equation is

$$1 - \zeta_1^s = (1 - \zeta_1)(1 + \zeta_1 + \cdots + \zeta_1^{s-1}).$$

Assume that j is a positive integer then. The term $\delta_{s,j+1}$ equals

$$\zeta_1^{s-j} \left(\sum_{i_2 \geq \cdots \geq i_{j+1} \geq 0}^{\leq s-j} \zeta_2^{i_2} \cdots \zeta_{j+1}^{i_{j+1}} \right).$$

The left of (42) equals

$$\left(\sum_{i_1\geq\cdots\geq i_j\geq 0}^{\leq s-j}\zeta_1^{i_1}\cdots\zeta_j^{i_j}\right)-\left(\sum_{i_2\geq\cdots\geq i_{j+1}\geq 0}^{\leq s-j}\zeta_1^{s-j-i_{j+1}}\zeta_2^{i_2-i_{j+1}}\cdots\zeta_j^{i_j-i_{j+1}}\left(\prod_{k=1}^{j+1}\zeta_k\right)^{i_{j+1}}\right)$$

$$=\left(\sum_{i_1=0}^{s-j}\sum_{i_2\geq\cdots\geq i_j\geq 0}^{\leq i_1}\zeta_1^{i_1}\zeta_2^{i_2}\cdots\zeta_j^{i_j}\right)$$

$$-\left(\sum\zeta_1^{s-j-i_{j+1}}\left(\prod_{k=1}^{j+1}\zeta_k\right)^{i_{j+1}}\zeta_2^{i_2-i_{j+1}}\cdots\zeta_j^{i_j-i_{j+1}}\right),$$

where the last sum extends over indices $s-j-i_{j+1}, i_2-i_{j+1}, i_3-i_{j+1},\ldots,i_j-i_{j+1}$ satisfying

$$s-j\geq s-j-i_{j+1}\geq i_2-i_{j+1}\geq i_3-i_{j+1}\geq\cdots\geq i_j-i_{j+1}\geq 0.$$

By replacing the indices $s-j-i_{j+1}, i_2-i_{j+1},\ldots,i_j-i_{j+1}, i_{j+1}$ with $i_1,i_2,\ldots,i_j,s-j-i_1$, we have that the left of (42) equals (Recall $\xi_{j+1}=\prod_{k=1}^{j+1}\zeta_k$)

$$\left(\sum_{i_1=0}^{s-j}\zeta_1^{i_1}\left(\sum_{i_2\geq\cdots\geq i_j\geq 0}^{\leq i_1}\zeta_2^{i_2}\cdots\zeta_j^{i_j}\right)\right)-\left(\sum_{i_1=0}^{s-j}\zeta_1^{i_1}\xi_{j+1}^{s-j-i_1}\left(\sum_{i_2\geq\cdots\geq i_j\geq 0}^{\leq i_1}\zeta_2^{i_2}\cdots\zeta_j^{i_j}\right)\right)$$

$$=\sum_{i_1=0}^{s-j}\zeta_1^{i_1}\left(1-\xi_{j+1}^{s-j-i_1}\right)\left(\sum_{i_2\geq\cdots\geq i_j\geq 0}^{\leq i_1}\zeta_2^{i_2}\cdots\zeta_j^{i_j}\right)$$

$$=(1-\xi_{j+1})\left(\sum_{i_1=0}^{s-j-1}\zeta_1^{i_1}\left(\sum_{l=0}^{s-j-i_1-1}\xi_{j+1}^{l}\right)\left(\sum_{i_2\geq\cdots\geq i_j\geq 0}^{\leq i_1}\zeta_2^{i_2}\cdots\zeta_j^{i_j}\right)\right),\tag{43}$$

We note that i_1 in the leftmost sum in (43) does not take $s-j$ for if $i_1=s-j$, then $(1-\xi_{j+1}^{s-j-i_1})=0$. We have

$$(43)=(1-\xi_{j+1})\left(\sum_{i_1=0}^{s-j-1}\sum_{l=0}^{s-j-1-i_1}\zeta_1^{i_1+l}\zeta_{j+1}^{l}\left(\sum_{i_2\geq\cdots\geq i_j\geq 0}^{\leq i_1}\zeta_2^{i_2+l}\cdots\zeta_j^{i_j+l}\right)\right)$$

$$=(1-\xi_{j+1})\left(\sum\zeta_1^{i_1+l}\zeta_2^{i_2+l}\cdots\zeta_j^{i_j+l}\zeta_{j+1}^{l}\right),$$

where the last sum extends over indices $i_1+l, i_2+l,\ldots,i_j+l,l$ satisfying

$$s-j-1\geq i_1+l\geq i_2+l\geq\cdots\geq i_j+l\geq l\geq 0.$$

By replacing indices $i_1+l,\ldots,i_j+l,l$ with $i_1,\ldots,i_j,i_{j+1}$, we have

$$(43)=(1-\xi_{j+1})\left(\sum_{i_1\geq i_2\geq\cdots\geq i_j\geq i_{j+1}\geq 0}^{s-j-1}\zeta_1^{i_1}\zeta_2^{i_2}\cdots\zeta_j^{i_j}\zeta_{j+1}^{i_{j+1}}\right),$$

which is the right of (42). □

We continue to use the notations in the previous lemma. For an integer $j \geq 1$, let ξ_j be as above. Put $\xi_0 := 0$.

Proposition 3.3.13. *Let γ^{-1} be a root of $\widehat{Z}(X)$. Then the extension K_1'/K_Z equals K_Z^{r-1}/K_Z. Here we define K_Z^{r-1}/K_Z in the following manner.*

- *The extension K_Z^0/K_Z is generated by all roots of the polynomials $X^q - \zeta^i\gamma X$ for $i = 0, 1, \ldots, S_r - 1$, which is a compositum of Kummer extensions.*
- *Put $x_0 := 1$. For $0 \leq k \leq r-2$, put inductively x_{k+1} to be a root of*

$$\mathrm{H}^{(k)}(X) := X^q - \zeta^k\gamma X - \zeta^k(1-\xi_k)\gamma x_k \in K_Z^k[X].$$

For $0 \leq k \leq r-2$, the extension K_Z^{k+1}/K_Z^k is generated by x_{k+1}.

This proposition claims that K_1'/K_Z is the composition

$$K_Z \text{ —— } K_Z^0 \text{ —— } K_Z^1 \text{ —— } K_Z^2 \text{ —— } \cdots \text{ —— } K_Z^{r-2} \text{ —— } K_Z^{r-1} = K_1' .$$

Let $v_{Z,k}$ denote the normalized valuation corresponding to K_Z^k for $k = 0, \ldots, r-2$. As $p \nmid v(\gamma x_0) = \frac{-v(j)+\alpha_1}{S_r}$ and the Newton polygon of $\mathrm{H}^{(0)}(X) = \mathrm{H}_\gamma(X)$ has exactly one segment, we can apply Proposition 3.2.2 to the polynomial $\mathrm{H}^{(0)}(X) \in K_Z^0[X]$. It turns out that $p \nmid v_{Z,1}(\gamma x_1)$ and the Newton polygon of $\mathrm{H}^{(1)}(X)$ has exactly one segment. Hence we can apply Proposition 3.2.2 to $\mathrm{H}^{(1)}(X) \in K_Z^1[X]$. It turns out that $p \nmid v_{Z,2}(\gamma x_2)$ and the Newton polygon of $\mathrm{H}^{(2)}(X)$ has exactly one segment. Hence we can apply Proposition 3.2.2 again and so on. The ψ-function of K_1'/K_Z^0 eventually follows.

Proof. Fix a root γ^{-1} of $\widehat{Z}(X)$. The roots of $\widehat{Z}(X)$ are $\zeta^s\gamma^{-1}$ for $s = 0, 1, \ldots, S_r - 1$. By Lemma 3.3.6 (1), the extension K_1'/K_Z is generated by all roots of all polynomials $\mathrm{H}_{\zeta^s\gamma}(X) = X^q - \zeta^s\gamma X - \zeta^s\gamma$ for all s. Note that for any s, the difference of two roots of $\mathrm{H}_{\zeta^s\gamma}(X)$ is a root of $X^q - \zeta^s\gamma X$. The field K_Z^0 is contained in K_1'.

To show the proposition, put inductively $x_0' := 1$ and

$$\mathrm{H}^{(k)}_{\zeta^s\gamma}(X) := \begin{cases} \mathrm{H}_{\zeta^s\gamma}(X) & k = 0 \text{ and } 0 \leq s \leq S_r - 1; \\ \mathrm{H}_{\zeta^s\gamma}\left(X + \sum_{j=1}^k \delta_{s,j} x_j'\right) & k > 0 \text{ and } k \leq s \leq S_r - 1, \end{cases} \tag{44}$$

where x_j' is a root of $\mathrm{H}^{(j-1)}_{\zeta^{j-1}\gamma}(X)$ for each $j = 1, \ldots, k$. We claim

$$\mathrm{H}^{(k)}_{\zeta^s\gamma}(X) = X^q - \zeta^s\gamma X - \zeta^s(1-\xi_k)\Delta_{s,k}\gamma x_k \text{ for } k \leq s \leq S_r - 1. \tag{45}$$

Admit this claim. We show $K_1' = K_Z^{r-1}$. This claim implies that $\mathrm{H}^{(k)}_{\zeta^k\gamma}(X)$ equals $\mathrm{H}^{(k)}(X)$ for each $k = 0, \ldots, r-2$. Hence for each k, we have $x_{k+1}' = x_{k+1} \in K_Z^{k+1}$. By (44), we know that the extension K_Z^{k+1}/K_Z^k equals the extension of K_Z^k generated by the roots of $\mathrm{H}_{\zeta^k\gamma}(X)$. Hence the extension K_Z^{r-1} of K_Z^0 equals the one generated by the roots of polynomials $\mathrm{H}_{\zeta^k\gamma}(X)$ for $k = 0, \dots, r-2$. Hence the extension K_1'/K_Z^{r-1} is generated by all roots of polynomials $\mathrm{H}_{\zeta^s\gamma}(X)$ for $s = r-1, \dots, S_r - 1$. For each $s \geq r-1$, the extension of K_Z^{r-1} generated by the roots of the polynomial $\mathrm{H}_{\zeta^s\gamma}(X)$ equals the one generated by the roots of $\mathrm{H}_{\zeta^s\gamma}^{(r-1)}(X)$. As ζ_{r-1} is a S_r-th root of unity, we have

$$\xi_{r-1} = \prod_{i=1}^{r-1} \zeta_i = \prod_{i=0}^{r-2} \zeta_{r-1}^{q^i} = 1.$$

Hence

$$\begin{aligned} \mathrm{H}_{\zeta^s\gamma}^{(r-1)} &= X^q - \zeta^s\gamma X - \zeta^s \left(1 - \xi_{r-1}\right) \Delta_{s,r-1}\gamma x_{r-1} \\ &= X^q - \zeta^s\gamma X \end{aligned}$$

for each $s \geq r-1$. This implies that the extension of K_Z^{r-1} generated by the roots of polynomials $\mathrm{H}_{\zeta^s\gamma}(X)$ for $s \geq r-1$ is trivial. The equality $K_1' = K_Z^{r-1}$ follows.

It suffices to show the claim in (45). The case $k = 0$ is clear and hence $x_1' = x_1$. As for the case $k = 1$, notice $\zeta^s x_1^q - \zeta^s\gamma x_1 - \zeta^s\gamma = 0$. Then

$$\begin{aligned} \mathrm{H}_{\zeta^s\gamma}^{(1)}(X) &= \mathrm{H}_{\zeta^s\gamma}(X + \delta_{s,1}x_1) = \mathrm{H}_{\zeta^s\gamma}(X + \zeta_1^s x_1) \\ &= (X + \zeta_1^s x_1)^q - \zeta^s\gamma(X + \zeta_1^s x_1) - \zeta^s\gamma \\ &= X^q - \zeta^s\gamma X + \zeta^s x_1^q - \zeta^s\gamma - \zeta^s\zeta_1^s\gamma x_1 \\ &= X^q - \zeta^s\gamma X + \zeta^s(1 - \zeta_1^s)\gamma x_1 \\ &= X^q - \zeta^s\gamma X + \zeta^s(1 - \zeta_1)\left(\sum_{i\geq 0}^{\leq s-1} \zeta_1^i\right)\gamma x_1 \\ &= X^q - \zeta^s\gamma X + \zeta^s(1 - \xi_1)\Delta_{s,1}\gamma x_1. \end{aligned}$$

Hence the case $k = 1$ follows.

Assume the claim for $k-1$, i.e., we have for $k - 1 \leq s \leq S_r - 1$ that

$$\mathrm{H}_{\zeta^s\gamma}^{(k-1)}(X) = X^q - \zeta^s\gamma X - \zeta^s \left(1 - \xi_{k-1}\right) \Delta_{s,k-1}\gamma x_{k-1}.$$

This implies $x_k' = x_k$. By (44), we have

$$\mathrm{H}_{\zeta^s\gamma}^{(k)}(X) = \mathrm{H}_{\zeta^s\gamma}^{(k-1)}(X + \delta_{s,k}x_k) \text{ for } k \leq s \leq S_r - 1.$$

Hence

$$\begin{aligned} \mathrm{H}_{\zeta^s\gamma}^{(k)}(X) &= (X + \delta_{s,k}x_k)^q - \zeta^s\gamma(X + \delta_{s,k}x_k) - \zeta^s \left(1 - \xi_{k-1}\right) \Delta_{s,k-1}\gamma x_{k-1} \\ &= X^q - \zeta^s\gamma X + \delta_{s,k}^q x_k^q - \zeta^s \left(1 - \xi_{k-1}\right) \Delta_{s,k-1}\gamma x_{k-1} - \zeta^s\delta_{s,k}\gamma x_k. \end{aligned}$$

As x_k is a root of $\mathrm{H}^{(k-1)}_{\zeta^{k-1}\gamma}(X)$, we have

$$\begin{aligned} 0 &= \delta^q_{s,k}\mathrm{H}^{(k-1)}_{\zeta^{k-1}\gamma}(x_k) \\ &= \left(\zeta^{s-k+1}\Delta_{s,k-1}\right)\left(x_k^q - \zeta^{k-1}\gamma x_k - \zeta^{k-1}\left(1-\xi_{k-1}\right)\gamma x_{k-1}\right) \\ &= \delta^q_{s,k}x_k^q - \zeta^s\left(1-\xi_{k-1}\right)\Delta_{s,k-1}\gamma x_{k-1} - \zeta^s\Delta_{s,k-1}\gamma x_k. \end{aligned}$$

Hence

$$\mathrm{H}^{(k)}_{\zeta^s\gamma}(X) = X^q - \zeta^s\gamma X + \zeta^s(\Delta_{s,k-1} - \delta_{s,k})\gamma x_{k-1}.$$

For $s \geq k$, to show $\mathrm{H}^{(k)}_{\zeta^s\gamma}(X)$ equals the one in (45), it suffices to show

$$\Delta_{s,k-1} - \delta_{s,k} = \left(1-\xi_k\right)\Delta_{s,k}.$$

This has been proved in Lemma 3.3.12. □

Note that the Newton polygons of $Z(X)$ and $\widehat{Z}(X)$ are the same. By (36), for a root γ^{-1} of $\widehat{Z}(X)$, we have

$$v(\gamma) = \frac{-v(\boldsymbol{j}) + \alpha_1}{S_r}.$$

Lemma 3.3.14. *Resume the notations in Proposition* 3.3.13. *Fix* k *to be one of* $1, \ldots, r-1$. *The Newton polygon of* $\mathrm{H}^{(k-1)}(X)$ *has exactly one segment determined by the points* $(0, v(\gamma x_{k-1}))$, $(q, 0) \in \mathbb{R}^2$. *We have*

$$v(x_k) = v(\gamma)\left(\frac{1}{q} + \frac{1}{q^2} + \cdots + \frac{1}{q^k}\right).$$

Proof. We may show the claims by induction on k. As $v(\gamma) > 0$, the Newton polygon of $\mathrm{H}^{(0)}(X) = \mathrm{H}_\gamma(X)$ has exactly one segment, and hence $v(x_1) = \frac{v(\gamma)}{q}$. We assume that the claims are valid for $k-1$. Put $Q_{k-1} \coloneqq (0, v(\gamma x_{k-1}))$, $P_0 = (1, v(\gamma))$, and $P_1 = (q, 0)$. The slope of $Q_{k-1}P_0$ is

$$\frac{v(\gamma) - v(\gamma x_{k-1})}{1-0} = -v(x_{k-1}).$$

The slope of $Q_{k-1}P_1$ is

$$\frac{0 - v(\gamma x_{k-1})}{q-0} = -v(x_{k-1}) - \frac{v(\gamma)}{q^k}$$

and is smaller. Hence the Newton polygon of $\mathrm{H}^{(k-1)}(X)$ is $Q_{k-1}P_1$. Then

$$v(x_k) = v(\gamma)\left(\frac{1}{q} + \frac{1}{q^2} + \ldots + \frac{1}{q^k}\right).$$

The claim follows. □

By Lemma 3.3.9, the extension K_1/K_η is also the extension generated by the roots of $\widehat{\Theta}(X) = \widehat{Z}(Y_2(X^{q-1}))$. Hence we can apply Proposition 3.3.13 to obtain the ψ-function of the extension K_1/K_η (and hence the one of K_1/K) as follows:

Corollary 3.3.15. *For the valuation $v(\boldsymbol{j})$ of the j-invariant of ϕ, assume $p \nmid v(\boldsymbol{j})$. Put*

$$R \coloneqq \frac{v(\gamma)}{q-1} = \frac{-v(\boldsymbol{j}) + \alpha_1}{q^{r-1}-1}.$$

(1) *The extension K_1'/K_Z^0 is a totally ramified extension of degree q^{r-1}.*

(2) *The ψ-function of K_1'/K_Z^0 is*

$$\psi_{K_1'/K_Z^0}(y) = \begin{cases} y, & -1 \le y \le E_Z R; \\ q^{r-1}y - (q^{r-1}-1)E_Z R, & E_Z R \le y. \end{cases}$$

(3) *Put E_Z to be the ramification index of K_Z^0/K, e the ramification index of K_1/K_1', and $E \coloneqq eE_Z$. The ψ-function of K_1/K is*

$$\psi_{K_1/K}(y) = \begin{cases} y, & -1 \le y \le 0; \\ Ey, & 0 \le y \le R; \\ Eq^{r-1}y - (q^{r-1}-1)ER, & R \le y. \end{cases}$$

Proof. Let $v_{Z,k}$ denote the normalized valuation associated to K_Z^k for $k = 0, \dots, r-1$. We have $v_{Z,0} = E_Z v$. As K_Z^0/K is at worst tamely ramified, we know that E_Z is not divisible by p. As $p \mid \alpha_1$ and $p \nmid v(\boldsymbol{j})$, we have $p \nmid v_{Z,0}(\gamma)$. Note the Newton polygon $\mathrm{H}^{(0)}(X) = \mathrm{H}_\gamma(X)$ has exactly one segment. Apply Proposition 3.2.2 to the extension K_Z^1/K_Z^0 generated by a root of the polynomial $\mathrm{H}_\gamma(X)$. We know that the extension K_Z^1/K_Z^0 is a totally ramified Galois extension of degree q. Notice

$$\frac{v_{Z,0}(\gamma)q}{q-1} - v_{Z,0}(\gamma) = E_Z R.$$

We also know from Proposition 3.2.2 the ψ-function of K_Z^1/K_Z^0

$$\psi_{K_Z^1/K_Z^0}(y) = \begin{cases} y, & -1 \le y \le E_Z R; \\ qy - (q-1)E_Z R, & E_Z R \le y. \end{cases}$$

We claim that for each $k = 0, \dots, r-2$, the extension K_Z^{k+1}/K_Z^k is totally ramified Galois of degree q. We show this by induction and the base case has been proved above. The induction hypothesis implies that $v_{Z,k} = q^k v_{Z,0}$. By Lemma 3.3.14, we have

$$v_{Z,k}(\gamma x_k) = E_Z \cdot (1 + q + \cdots + q^k) \cdot v(\gamma),$$

which is not divisible by p. We also have that the Newton polygon of $\mathrm{H}^{(k)}(X)$ has exactly one segment. Apply Proposition 3.2.2. The extension K_Z^{k+1}/K_Z^k is totally ramified Galois of degree q, as desired. Hence (1) follows. Note

$$\frac{v_{Z,k}(\gamma)q}{q-1} - v_{Z,k}(\gamma x_k) = \frac{v_{Z,0}(\gamma)q^{k+1}}{q-1} - \frac{v_{Z,0}(\gamma)(q^{k+1}-1)}{q-1} = E_Z R.$$

We also know from Proposition 3.2.2 that the ψ-function of K_Z^{k+1}/K_Z^k is the same as that of K_Z^1/K_Z^0. Then (2) follows from Lemma 3.2.1. Notice K_1/K_1' is at worst tamely ramified. (3) follow. □

Resume the assumptions and notations in Corollary 3.3.15. As K_1/K_1' is tamely ramified (See Lemma 3.3.1), the natural projection $G(K_1/K_Z^0) \to G(K_1'/K_Z^0)$ induces an isomorphism $G(K_1/K_Z^0)_1 \cong G(K_1'/K_Z^0)_1$. As K_Z^0/K is tamely ramified, we have $G(K_1/K_Z^0)_1 = G(K_1/K)_1$. By the ψ-functions, we replace the indices as

$$G(K_1/K)_1 = G(K_1/K)_{ER} = G(K_1/K_Z^0)_{ER} \cong G(K_1'/K_Z^0)_{E_Z R}. \tag{46}$$

Theorem 3.3.16. *For a integer $r \geq 3$, let ϕ be a rank r Drinfeld A-module over K such that $\phi_t(X) = tX + a_1X^q + a_rX^{q^r} \in K[X]$, $v(\boldsymbol{j}) < \alpha_1$, and $p \nmid v(\boldsymbol{j})$. Let u be a degree one prime of A with degree 1 and $K_1 = K(\phi[u])$. Let $\{\xi_{i,1}\}_{i=1,\dots,r}$ be an SMB of $\phi[u]$. Let V be the 1-dimensional $\mathbb{F}_q$-vector space generated by $\xi_{1,1}$. Then each element in $G(K_1/K)_1$ fixes $\xi_{1,1}$ and there is an isomorphism*

$$g : G(K_1/K)_1 \to V^{r-1}; \quad \sigma \mapsto (\sigma(\xi_{2,1}) - \xi_{2,1}, \dots, \sigma(\xi_{r,1}) - \xi_{r,1}).$$

Proof. By (46), we have $G(K_1/K)_1 = G(K_1/K_Z^0)_{ER}$. Note $\xi_{1,1} \in K_\eta \subset K_Z^0$. Hence each element of $G(K_1/K)_1$ fixes $\xi_{1,1}$.

Let v_{K_1} denote the normalized valuation corresponding to K_1. By Corollary 3.3.15, we have $v_{K_1} = Eq^{r-1}v$. Let σ be an element in $G(K_1/K)_1 = G(K_1/K)_{ER}$. Fix i to be one of $2, \dots, r$. We have

$$v_{K_1}\left(\frac{\sigma(\xi_{i,1}) - \xi_{i,1}}{\xi_{i,1}}\right) \geq ER > 0.$$

By Proposition 2.1.3, each element in $\phi[u] \setminus V$ has valuation $v(\xi_{i,1})$. Hence if $\sigma(\xi_{i,1}) - \xi_{i,1} \notin V$, we have $v(\sigma(\xi_{i,1}) - \xi_{i,1}) = v(\xi_{i,1})$ and this implies $v_{K_1}\left(\frac{\sigma(\xi_{i,1}) - \xi_{i,1}}{\xi_{i,1}}\right) = 0$, which is a contradiction. Therefore $\sigma(\xi_{i,1}) - \xi_{i,1} \in V$.

For a nontrivial element $\sigma \in G(K_1/K)_1 = G(K_1/K_\eta)_1$ (Note that K_η/K is at most tamely ramified), as $\xi_{2,1}, \dots, \xi_{r,1}$ generates K_1/K_η, there exists some index $i = 2, \dots, r$ such that $\sigma(\xi_{i,1}) - \xi_{i,1} \neq 0$. This implies that g is injective. By Corollary 3.3.15 and (46), the cardinality of $G(K_1/K)_1$ is q^{r-1}. As the cardinal of V^{r-1} is q^{r-1}, this map is surjective. Let σ' be an element in $G(K_1/K)_1$. Put $a_i = (\sigma(\xi_{i,1}) - \xi_{i,1})/\xi_{1,1}$ and $a_i' = (\sigma'(\xi_{i,1}) - \xi_{i,1})/\xi_{1,1}$ for $i = 2, \dots, r$. As $\sigma, \sigma' \in G(K_1/K)_1 = G(K_1/K_\eta)_1$, we know that σ and σ' both fix $\xi_{1,1}$. Hence, for each i, we have

$$\sigma'\sigma(\xi_{i,1}) = \sigma'(\sigma(\xi_{i,1})) = \sigma'(\xi_{i,1} + a_i\xi_{1,1}) = \xi_{i,1} + (a_i + a_i')\xi_{1,1}.$$

This shows that this map is a morphism and hence an isomorphism. □

Remark 3.3.17. Let u be a finite prime of A with arbitrary degree. We hope to know the ψ-function of $K(\phi[u^n])/K$, the action of the wild ramification subgroup $G(K(\phi[u^n])/K)_1$ on $\phi[u^n]$, or the rank of $\phi[u^n]^{G^y}$ (cf. Corollary 4.2.5). Here G^y denotes the y-th upper ramification subgroup of the absolute Galois group of K.

Example 3.3.18. Let v be a prime of A and u a finite prime of A with $\deg(u) = 1$. For each integer i with $p \nmid i$, consider the Drinfeld $\mathbb{F}_q[t]$-module $\phi^{(i)}$ over $\mathbb{F}_q(t)_v$ characterized by

$$\phi_t^{(i)}(X) = tX + t^{i(q^{r-1}-1)(q-1)}X^q + (t+1)^{-\frac{q(q^{r-1}-1)}{q-1}}X^{q^r}.$$

It has potentially good reduction over $\mathbb{F}_q(t)_v$ if $v \neq t$. The j-invariant of $\phi^{(i)}$ is

$$\boldsymbol{j}^{(i)} := t^{i(q^r-1)(q^{r-1}-1)} \cdot (t+1)^{\frac{q(q^{r-1}-1)}{q-1}}.$$

(1) If the prime v is infinite, then $v(\boldsymbol{j}^{(i)}) < \alpha_1$ if and only if $i > 0$. The ramification break ER of K_1/K in Corollary 3.3.15 equals $Ei(q^r-1)$. Hence ER can be arbitrarily large.

(2) If $v \neq t$, then the extension K_1/K is at worst tamely ramified as ϕ has potentially good reduction over $\mathbb{F}_q(t)_v$. If $v = t$ and $i \geq 0$, then ϕ has good reduction. If $i < 0$, then ϕ has potentially stable reduction and the reduction over some extension of K is bad. The ramification break ER in Corollary 3.3.15 equals $-Ei(q^r-1)$. Hence ER can be arbitrarily large.

3.4. The action of the wild ramification subgroup of K_1'/K_Z on the generators. This subsection is a supplement to the previous section. There is no application of this subsection in this paper. By Theorem 3.3.16 and (46), we have that $G(K_1'/K_Z^0) \cong \mathbb{F}_q^{r-1}$. Our goal is to study the Galois action of $G(K_1'/K_Z^0)$ on the field extension generators $x_1, \ldots, x_{r-1}$ of the extension K_1'/K_Z^0. For this, we prepare a lemma. We continue to use the notations in Lemma 3.3.12 and Proposition 3.3.13.

Lemma 3.3.19. *Let $\zeta^{\frac{1}{q-1}}$ be a primitive $(q^{r-1}-1)$-st root of unity such that $(\zeta^{\frac{1}{q-1}})^{q-1} = \zeta$. Let $\gamma^{\frac{1}{q-1}}$ denote a root of $X^{q-1} - \gamma$. Define inductively $\mu_{k,l}$ for $k = 0, 1, \ldots, r-1$ and $l = 0, 1, \ldots, r-1$ as follows:*

(1) $\mu_{k,0} = 0$ *for all k and* $\mu_{0,l} = 0$ *for all l;*

(2) $\mu_{1,1} := \gamma^{\frac{1}{q-1}}$ *and* $\mu_{1,l} = 0$ *for* $l \geq 2$;

(3) $\mu_{k+1,l} := \mu_{k,l-1}\zeta^{\frac{1}{q-1}} + \xi_k\mu_{k,l}$ *for all* $k = 1, \ldots, r-2$ *and* $l \geq 1$.

Fix $a \in \mathbb{F}_q$, k to be one of $0, \ldots, r-2$, and l to be one of $0, \ldots, r-1$. Put

$$\mathrm{H}_{a,l}^{(k)}(X) := X^q - \zeta^k\gamma X - \zeta^k(1-\xi_k)\gamma a\mu_{k,l}.$$

Then $a\mu_{k+1,l}$ is a root of this polynomial.

We have $\mu_{k,1} = \gamma^{\frac{1}{q-1}} \prod_{i=1}^{k-1} \xi_i$ for $k = 2, \ldots, r-1$ and $\mu_{k,k} = \zeta^{\frac{k-1}{q-1}}\gamma^{\frac{1}{q-1}}$ for $k \geq 1$. We also have $\mu_{k,l} = 0$ for $l > k$.

PROOF. If $l > k+1$ or $l = 0$, then the lemma claims that 0 is a root of $X^q - \zeta^k\gamma X$. This is valid and we assume $1 \le l \le k+1$ then. We show these cases using induction on k and l. We first show the following claims as the base case.

(i) $a\mu_{k+1,1}$ is a root of $\mathrm{H}_{a,1}^{(k)}(X)$ for $k = 0, \dots, r-2$;
(ii) $a\mu_{k+1,k+1}$ is a root of $\mathrm{H}_{a,k+1}^{(k)}(X) = X^q - \zeta^k\gamma X$ for $k = 0, \dots, r-2$.

The proofs of (ii) and the case $k = 0, 1$ of (i) are straightforward. As for (i), we need to show for $k \ge 2$

$$\left(a\gamma^{\frac{1}{q-1}}\prod_{i=1}^{k}\xi_i\right)^q - \zeta^k\gamma\left(a\gamma^{\frac{1}{q-1}}\prod_{i=1}^{k}\xi_i\right) - \zeta^k(1-\xi_k)\gamma a\left(\gamma^{\frac{1}{q-1}}\prod_{i=1}^{k-1}\xi_i\right) = 0.$$

Note $a^q = a$, $\xi_1^q = \zeta$, and $\xi_i^q = \zeta\xi_{i-1}$ for $i \ge 2$. The left of this equation equals

$$\begin{aligned}
&\zeta^k\gamma^{\frac{q}{q-1}}a\left(\prod_{i=1}^{k-1}\xi_i\right) - \zeta^k\gamma^{\frac{q}{q-1}}a\left(\prod_{i=1}^{k}\xi_i\right) - \zeta^k\gamma^{\frac{q}{q-1}}(1-\xi_k)a\left(\prod_{i=1}^{k-1}\xi_i\right)\\
&= \zeta^k\gamma^{\frac{q}{q-1}}a\left((1-\xi_k)\left(\prod_{i=1}^{k-1}\xi_i\right) - (1-\xi_k)\left(\prod_{i=1}^{k-1}\xi_i\right)\right)\\
&= 0 \quad \text{(as desired)}.
\end{aligned}$$

Fix k and l to be integers satisfying $2 \le k \le r-2$ and $1 \le l \le k$. Assume $\mathrm{H}_{1,j}^{(i-1)}(\mu_{i,j}) = 0$ for integers i, j satisfying $i \le k$ and $j \le l$. We show

$$\mathrm{H}_{a,l}^{(k)}(a\mu_{k+1,l}) = (a\mu_{k+1,l})^q - \zeta^k\gamma(a\mu_{k+1,l}) - \zeta^k(1-\xi_k)\gamma a\mu_{k,l} = 0.$$

By the definition of $\mu_{k+1,l}$, we need to show

$$a^q(\mu_{k,l-1}\zeta^{\frac{1}{q-1}} + \xi_k\mu_{k,l})^q - \zeta^k\gamma a(\mu_{k,l-1}\zeta^{\frac{1}{q-1}} + \xi_k\mu_{k,l}) - \zeta^k(1-\xi_k)\gamma a\mu_{k,l} = 0.$$

As $a^q = a$, it suffices to show

$$(\mu_{k,l-1}\zeta^{\frac{1}{q-1}} + \xi_k\mu_{k,l})^q - \zeta^k\gamma(\mu_{k,l-1}\zeta^{\frac{1}{q-1}} + \xi_k\mu_{k,l}) - \zeta^k(1-\xi_k)\gamma\mu_{k,l} = 0.$$

Note $\xi_k^q = \zeta\xi_{k-1}$. The left of this desired equation equals

$$\begin{aligned}
&(\mu_{k,l-1}\zeta^{\frac{1}{q-1}} + \xi_k\mu_{k,l})^q - \zeta^k\zeta^{\frac{1}{q-1}}\gamma\mu_{k,l-1} - \zeta^k\gamma\mu_{k,l}\\
&= \zeta^{\frac{q}{q-1}}\left(\mu_{k,l-1}^q - \zeta^{k-1}\gamma\mu_{k,l-1}\right) + \zeta\xi_{k-1}\mu_{k,l}^q - \zeta^k\gamma\mu_{k,l}.
\end{aligned} \tag{47}$$

By $\mathrm{H}_{1,l-1}^{(k-1)}(\mu_{k,l-1}) = 0$, we have

$$\mu_{k,l-1}^q - \zeta^{k-1}\gamma\mu_{k,l-1} = \zeta^{k-1}(1-\xi_{k-1})\gamma\mu_{k-1,l-1}.$$

By $\mathrm{H}_{1,l}^{(k-1)}(\mu_{k,l}) = 0$, we have

$$\mu_{k,l}^q = \zeta^{k-1}\gamma\mu_{k,l} + \zeta^{k-1}(1-\xi_{k-1})\gamma\mu_{k-1,l}.$$

Hence

$$(47) = \zeta^{\frac{q}{q-1}}\zeta^{k-1}(1-\xi_{k-1})\gamma\mu_{k-1,l-1} + \zeta\xi_{k-1}\left(\zeta^{k-1}\gamma\mu_{k,l} + \zeta^{k-1}(1-\xi_{k-1})\gamma\mu_{k-1,l}\right) - \zeta^k\gamma\mu_{k,l}.$$

Note $\mu_{k,l} = \mu_{k-1,l-1}\zeta^{\frac{1}{q-1}} + \xi_{k-1}\mu_{k-1,l}$ for $k \geq 2$ and $l \geq 1$. We have

$$\begin{aligned}(47)/\zeta^k\gamma &= \zeta^{\frac{1}{q-1}}(1-\xi_{k-1})\mu_{k-1,l-1} + \xi_{k-1}\left(\mu_{k-1,l-1}\zeta^{\frac{1}{q-1}} + \xi_{k-1}\mu_{k-1,l}\right) \\ &\quad + \xi_{k-1}(1-\xi_{k-1})\mu_{k-1,l} - \left(\mu_{k-1,l-1}\zeta^{\frac{1}{q-1}} + \xi_{k-1}\mu_{k-1,l}\right) \\ &= 0 \text{ (as desired).}\end{aligned}$$

□

As the polynomial $X^q - \zeta^k\gamma X$ is additive, we have

Corollary 3.3.20. *Let $\underline{a} = (a_1, \ldots, a_{r-1})$ be an element of the $\mathbb{F}_q$-vector space $\mathbb{F}_q^{r-1}$. Put*

$$\mathrm{M}_{k,\underline{a}} := \sum_{l=1}^{r-1} a_l\mu_{k,l}$$

for $k = 0, \ldots, r-1$. Then for any $k = 0, \ldots, r-2$, we have that $\mathrm{M}_{k+1,\overline{a}}$ is a root of

$$\mathrm{H}_{\underline{a}}^{(k)}(X) := X^q - \zeta^k\gamma X - \zeta^k(1-\xi_k)\gamma\mathrm{M}_{k,\underline{a}}.$$

We are ready to state

Theorem 3.3.21. *For an integer k and an element $\underline{a} \in \mathbb{F}_q^{r-1}$, define $\mathrm{M}_{k,\underline{a}}$ as in Corollary 3.3.20. Let $\gamma^{\frac{1}{q-1}}$ denote a root of $X^{q-1} - \gamma$.*

(1) *Fix j to be one of $1, \ldots, r-1$. For an element $\sigma \in G(K_Z^j/K_Z^{j-1})$, let σ denote an extension of this element to $G(K_1'/K_Z^{j-1})$. Then there exists an element*

$$\underline{a} = (0, \ldots, 0, a_j, a_{j+1}, \ldots, a_{r-1}) \in \mathbb{F}_q^{r-1}$$

satisfying that for each $k = 1, \ldots, r-1$, we have

$$\sigma(x_k) - x_k = \mathrm{M}_{k,\underline{a}}.$$

(2) *The map*

$$G(K_1'/K_Z^0) \to \mathbb{F}_q^{r-1}; \sigma \mapsto \underline{a},$$

where $\underline{a}$ satisfies $\sigma(x_{r-1}) - x_{r-1} = \mathrm{M}_{r-1,\underline{a}}$, is an isomorphism.

Proof. We show (1) by induction on k. For $i = 1, \ldots, j-1$, as $x_i \in K_Z^{j-1}$, the action $\sigma(x_i)$ is trivial. By Proposition 3.2.2 (See the proof of Corollary 3.3.15), there exists some $a_j \in \mathbb{F}_q$ such that

$$\sigma(x_j) - x_j = a_j\zeta^{\frac{j-1}{q-1}}\gamma^{\frac{1}{q-1}} = a_j\mu_{j,j}.$$

Note $\mu_{j,l}=0$ for $l=j+1,\dots,r-1$. The equation in (1) is valid for $k=j$. Assume that the equation in (1) is valid for $1,\dots,k$ (some $k\geq j$) and some $\underline{a}\in\mathbb{F}_q^{r-1}$. Since x_{k+1} is a root of

$$\mathrm{H}^{(k)}(X)=X^q-\zeta^k\gamma X-\zeta^k(1-\xi_k)\gamma x_k,$$

the element $\sigma(x_{k+1})$ is a root of

$$\begin{aligned}\sigma\mathrm{H}^{(k)}(X)&\coloneqq X^q-\zeta^k\gamma X-\zeta^k(1-\xi_k)\gamma\sigma(x_k)\\&=X^q-\zeta^k\gamma X-\zeta^k(1-\xi_k)\gamma\left(x_k+\mathrm{M}_{k,\underline{a}}\right).\end{aligned}$$

By Corollary 3.3.20, $\mathrm{M}_{k+1,\underline{a}}$ is a root of $\mathrm{H}^{(k)}_{\underline{a}}(X)$. Hence the sum $x_{k+1}+\mathrm{M}_{k+1,\underline{a}}$ is a root of $\sigma\mathrm{H}^{(k)}$. On the other hand, the element $\sigma(x_{k+1})$ is a root of $\sigma\mathrm{H}^{(k)}$ and hence

$$\sigma(x_{k+1})=x_{k+1}+\mathrm{M}_{k+1,\underline{a}}+a\zeta^{\frac{k}{q-1}}\gamma^{\frac{1}{q-1}}=x_{k+1}+\mathrm{M}_{k+1,\underline{a}}+a\mu_{k+1,k+1}$$

for some $a\in\mathbb{F}_q$. We may replace $\underline{a}$ with $(0,\dots,0,a_j,a_{j+1},\dots,a_k,a_{k+1}+a,a_{k+2},\dots,a_{r-1})$ so that

$$\sigma(x_{k+1})-x_{k+1}=\mathrm{M}_{k+1,\underline{a}},$$

as desired.

As for (2), the map is injective as $x_1,\dots,x_{r-1}$ are generators of the extension K_1'/K_Z^0. Corollary 3.3.15 implies that the cardinal of $G(K_1'/K_Z^0)$ is q^{r-1} and hence the map is surjective. As $\widehat{Z}(X)=X^S-\beta$ for $S_r=\frac{q^{r-1}-1}{q-1}$, the field K_Z contains S_r-th roots of unity. As K_Z^0/K_Z generated by all roots of $X^q-\gamma X$ for γ^{-1} varying within the roots of $\widehat{Z}(X)$, the field K_Z^0 contains $q^{r-1}-1$-st roots of unity. Hence any element in $G(K_1'/K_Z^0)$ fixes $\mathrm{M}_{k,\underline{a}}\in K_Z^0$ for any k and any $\underline{a}\in\mathbb{F}_q^{r-1}$. Let σ' be an element of $G(K_1'/K_Z^0)$ such that $\sigma'(x_k)-x_k=\mathrm{M}_{k,\underline{a}'}$ for each k. We have

$$\begin{aligned}\sigma'\sigma(x_k)&=\sigma'(\sigma(x_k))=\sigma'(x_k+\mathrm{M}_{k,\underline{a}})\\&=x_k+\mathrm{M}_{k,\underline{a}}+\mathrm{M}_{k,\underline{a}'}=x_k+\mathrm{M}_{k,\underline{a}+\underline{a}'}.\end{aligned}$$

This shows that this map is a morphism and hence an isomorphism. □

4. The ψ-function of K_n/K with $r=2$ and $\deg(u)=1$ and v being infinite

Let v be an infinite prime and u a finite prime of A with degree 1. Throughout this section, let ϕ be a rank 2 Drinfeld A-module over K such that $\phi_t(X)=tX+a_1X^q+a_2X^{q^2}\in K[X]$. Put $v_0=v(t)$ and $v_1=v(a_1)$. Now, we have $\alpha_n=v_0q^n$ (Section 1.2 in Chapter 2). Assume $v(\boldsymbol{j})<\alpha_1=v_0q$. Let m be the positive integer such that $v(\boldsymbol{j})\in$

$(\alpha_{m+1}, \alpha_m]$, i.e., $v(\boldsymbol{j}) \in (v_0 q^{m+1}, v_0 q^m]$. By Lemma 2.1.1, we have

$$v(\xi_{1,n}) = -\left(v_0(n-1) - \frac{v_0}{q-1} + \frac{v_1}{q-1}\right) \text{ and}$$

(48)
$$v(\xi_{2,n}) = \begin{cases} -\left(\dfrac{v_1}{q-1} - \dfrac{v(\boldsymbol{j})}{q^n(q-1)}\right) & 1 \le n \le m; \\ -\left(v_0(n-m) + \dfrac{v_1}{q-1} - \dfrac{v(\boldsymbol{j})}{q^m(q-1)}\right) & n \ge m+1. \end{cases}$$

For each positive integer n, we are to work out the Herbrand ψ-function of the extensions K_n/K and the action of $G(K_n/K)$ on $\phi[u^n]$ when $v(\boldsymbol{j}) < v_0 q$.

By Lemma 3.3.2, we can decompose the extension K_1/K into

$$K \text{ —— } K_\eta \text{ —— } K_Z^0 \text{ —— } K_1' \text{ —— } K_1.$$

Here K_η/K is generated by the roots of $a_1 X^q + tX$. The extension K_Z^0/K_η is generated by the roots of $X^q - \gamma X$ with $\gamma = u^{-1} a_r \xi_{1,1}^{q^r-1}$ so that $v(\gamma) = -v(\boldsymbol{j}) + v_0 q$ (Note that $Z_2(X) = X$). The extension K_1'/K_η is generated by all roots of $\mathrm{H}_\gamma(X) = X^q - \gamma X - \gamma$. The extension K_1/K_1' is generated by all roots of $X^{q-1} - \delta$, where δ varies within the roots of $\mathrm{H}_\gamma(X)$.

Let E be the integer such that Eq is the ramification index of K_1/K. We then apply Lemma 3.2.1 and Proposition 3.2.2 to obtain the ψ-functions of K_n/K for all n. We first work out the ψ-functions of K_1/K and K_{n+1}/K_n as follows.

Lemma 3.4.1. *Assume $v(\boldsymbol{j}) < v_0 q$ and $p \nmid v(\boldsymbol{j})$. Let m be the integer satisfying $v(\boldsymbol{j}) \in (v_0 q^{m+1}, v_0 q^m)$.*

(1) *Let e and E_Z be respectively the ramification index of K_1/K_1' and of K_Z^0/K. Then we have $E = eE_Z$ and is not divisible by p. The ψ-function of K_1/K is*

$$\psi_{K_1/K}(y) = \begin{cases} y, & -1 \le y \le 0; \\ Ey, & 0 \le y \le R_1; \\ Eqy - (q-1)ER_1, & R_1 \le y, \end{cases}$$

where $R_1 := \frac{-v(\boldsymbol{j}) + v_0 q}{q-1}$.

(2) (i) *For $1 \le n \le m$, the ramification index of K_n/K is Eq^n;*

(ii) *For $1 \le n \le m-1$, we have*

$$\psi_{K_{n+1}/K_n}(y) = \begin{cases} y, & -1 \le y \le ER_{n+1}; \\ qy - (q-1)ER_{n+1}, & ER_{n+1} \le y, \end{cases}$$

where $R_{n+1} := \frac{-v(\boldsymbol{j}) + v_0 q^{n+1}}{q-1}$.

PROOF. We show (1). Due to Lemma 3.3.2, the extension K_1/K is decomposed into the tower

$$K — K_Z^0 — K_1' — K_1.$$

The extension K_Z^0/K is tamely ramified with the ramification index to be E_Z. Hence $p \nmid E_Z$ and we have

$$\psi_{K_Z^0/K}(y) = \begin{cases} y, & -1 \le y \le 0; \\ E_Z y, & 0 \le y. \end{cases}$$

The extension K_1/K_1' is a compositum of Kummer extensions and hence at worst tamely ramified. We have $p \nmid e$ and

$$\psi_{K_1/K_1'}(y) = \begin{cases} y, & -1 \le y \le 0; \\ ey, & 0 \le y. \end{cases}$$

Let v_Z denote the normalized valuation associated to K_Z^0 so that $v_Z = E_Z v$. We have $v_Z(\gamma) = E_Z(-v(\boldsymbol{j}) + v_0 q)$. As $v_Z(\gamma) > 0$ and $p \nmid v_Z(\gamma)$, we can apply Proposition 3.2.2 to $\mathrm{H}_\gamma(X) \in K_Z^0[X]$. Hence K_1'/K_Z^0 is a degree q totally ramified Galois extension. This implies $E = eE_Z$. The ψ-function of K_1'/K_Z^0 is

$$\psi_{K_1'/K_Z^0}(y) = \begin{cases} y, & -1 \le y \le E_Z R_1; \\ qy - (q-1)E_Z R_1, & E_Z R_1 \le y. \end{cases}$$

By Lemma 3.2.1, the ψ-function of K_1/K follows.

We show (i) of (2). The case $n = 1$ is known. Assume that (i) is valid for a positive integer $n \le m-1$. To show that K_{n+1}/K has ramification index Eq^{n+1}, it suffices to show that the ramification index of K_{n+1}/K_n is q. By Lemma 3.1.4, the extension K_{n+1}/K_n is generated by a root of $\mathrm{H}_{2,n}(X) = X^q + b_0 X + c_{2,n}$ in (32). By (31), we have

$$v(c_{2,n}) = \frac{v(\boldsymbol{j}) - v_0 q^{n+1}}{q^n(q-1)}.$$

Let v_{K_n} denote the valuation associated to K_n so that $v_{K_n} = Eq^n v$. As $p \nmid v(\boldsymbol{j})$, we know $p \nmid v_{K_n}(c_{2,n})$. Note that $v_{K_n}(c_{2,n}) < 0$ (as $n+1 \le m$) and $v_{K_n}(b_0) = 0$. We can apply Proposition 3.2.2 to $\mathrm{H}_{2,n}(X) \in K_n[X]$. Hence K_{n+1}/K_n is a degree q totally ramified Galois extension and this shows (i). We also know from Proposition 3.2.2 that

$$\psi_{K_{n+1}/K_n}(y) = \begin{cases} y, & -1 \le y \le ER_{n+1}; \\ qy - (q-1)ER_{n+1}, & ER_{n+1} \le y, \end{cases}$$

and (ii) of (2) follows. □

With the notation and assumptions in this lemma, we have the decomposition of K_n/K for $1 \le n \le m$,

$$K \overset{E_Z}{—} K_Z^0 \overset{q}{—} K_1' \overset{e}{—} K_1 \overset{q}{—} K_2 — \cdots — K_{n-1} \overset{q}{—} K_n \ ,$$

where each number indicates the ramification index of the corresponding extension. Let v_{K_n} denote the normalized valuation associated to K_n. We have $v_{K_n} = Eq^n v$ for $n = 1, \ldots, m$ and $v_{K_n} = v_{K_m}$ for $n \ge m$.

Due to Lemma 3.2.1, we can show by induction that

Lemma 3.4.2. *Assume $v(\boldsymbol{j}) < v_0 q$ and $p \nmid v(\boldsymbol{j})$. Let m be the integer satisfying $v(\boldsymbol{j}) \in (v_0 q^{m+1}, v_0 q^m)$. Put $R_n = \frac{-v(\boldsymbol{j})+v_0 q^n}{q-1}$ for any positive integer n as in Lemma 3.4.1. Then for $n \le m$, we have*

$$\psi_{K_n/K}(y) = \begin{cases} y, & -1 \le y \le 0; \\ Ey, & 0 \le y \le R_n; \\ E\big(q^j y - \sum_{i=0}^{j-1} q^i (q-1) R_{n-i}\big), & R_{n-j+1} \le y \le R_{n-j} \\ & \quad \text{for } j = 1, \ldots, n-1; \\ E\big(q^n y - \sum_{i=0}^{n-1} q^i (q-1) R_{n-i}\big), & R_1 \le y. \end{cases}$$

Assume the conditions in Lemma 3.4.2 in the rest of this subsection. We are to consider the wild ramification subgroup of $G(K_1/K)$ and how this group acts on the generators $\xi_{1,1}, \xi_{2,1}$ of K_1/K. By Lemma 3.4.1 (1) and Proposition 3.2.2, we know

$$G(K_1/K)_1 = G(K_1/K)_{ER_1} = G(K_1/K_Z^0)_{ER_1} \cong G(K_1'/K_Z^0)_{E_Z R_1} \cong \mathbb{F}_q. \tag{49}$$

For an element $\sigma \in G(K_1/K)_{ER_1}$, note that σ is characterized by $\sigma(\xi_{2,1})$ because the isomorphism (49) indicates that σ fixes $\xi_{1,1} \in K_Z^0$. Then we have

$$\begin{aligned} v_{K_1}(\sigma(\xi_{2,1}) - \xi_{2,1}) &= v_{K_1}(\sigma(\xi_{2,1})\xi_{2,1}^{-1} - 1) + v_{K_1}(\xi_{2,1}) \\ &\ge ER_1 + v_{K_1}(\xi_{2,1}) \\ &= E\left(\frac{v_0 q - v_1 q}{q-1}\right) = v_{K_1}(\xi_{1,1}). \end{aligned}$$

From $\sigma(\xi_{2,1}) - \xi_{2,1} \in \phi[u]$ and $v(\xi_{1,1}) > v(\xi_{2,1})$, we have $\sigma(\xi_{2,1}) = \xi_{2,1} + a \cdot \xi_{1,1}$ for some $a \in \mathbb{F}_q$. This defines a morphism

$$G(K_1/K)_{ER_1} \to \mathbb{F}_q; \quad \sigma \mapsto a,$$

which is injective. Comparing the cardinalities of the domain and codomain, we conclude that this map is an isomorphism.

For an integer $2 \le l \le m$, let us consider the wild ramification subgroup of $G(K_l/K_{l-1})$ and how this group acts on the generator $\xi_{2,l}$ of K_l/K_{l-1} (note that $\xi_{1,l} \in K_{l-1}$ by

Lemma 3.1.3). We know from Lemma 3.4.1 (ii) and Proposition 3.2.2 that $G(K_l/K_{l-1})_1 = G(K_l/K_{l-1})_{ER_l} \cong \mathbb{F}_q$. For an element $\sigma \in G(K_l/K_{l-1})_{ER_l}$, we have

$$\begin{aligned} v_{K_l}(\sigma(\xi_{2,l}) - \xi_{2,l}) &= v_{K_l}(\sigma(\xi_{2,l})\xi_{2,l}^{-1} - 1) + v_{K_l}(\xi_{2,l}) \\ &\geq ER_l + v_{K_l}(\xi_{2,l}) \\ &= -Eq^l \cdot \frac{v_1 - v_0}{q-1} = v_{K_l}(\xi_{1,1}), \end{aligned}$$

which shows $\sigma(\xi_{2,l}) = \xi_{2,l} + a \cdot \xi_{1,1}$ for some $a \in \mathbb{F}_q$. This defines an isomorphism

$$G(K_l/K_{l-1})_{ER_l} \to \mathbb{F}_q; \quad \sigma \mapsto a.$$

We now work out the action of the wild ramification subgroup of $G(K_n/K)$ on $\xi_{1,n}, \xi_{2,n} \in \phi[u^n]$ for infinite primes.

Theorem 3.4.3. *Let v be an infinite prime. Assume $v(\boldsymbol{j}) < v_0 q$ and $p \nmid v(\boldsymbol{j})$. Let m be the integer such that $v(\boldsymbol{j}) \in (v_0 q^{m+1}, v_0 q^m)$.*

(1) *For integers l and n satisfying $1 \leq l \leq m$ and $l \leq n$, put $R_n^l \coloneqq \psi_{K_n/K}(R_l)$. We set $K_0 \coloneqq K$. Then the natural projection $G(K_n/K_{l-1}) \to G(K_l/K_{l-1})$ induces an isomorphism $G(K_n/K_{l-1})_{R_n^l} \cong G(K_l/K_{l-1})_{R_l^l} \cong \mathbb{F}_q$. Let $\sigma_{l,a}$ for $a \in \mathbb{F}_q$ be the element in $G(K_l/K_{l-1})_{R_l^l}$ characterized by*

$$\sigma_{l,a}(\xi_{1,l}) = \xi_{1,l} \text{ and } \sigma_{l,a}(\xi_{2,l}) = \xi_{2,l} + a \cdot \xi_{1,1}.$$

Denote $\sigma_{l,a} \in G(K_n/K_{l-1})_{R_n^l}$ again its image under the isomorphism. Then $\sigma_{l,a}(\xi_{1,n}) = \xi_{1,n}$ and $\sigma_{l,u}(\xi_{2,n}) = \xi_{2,n} + a \cdot \xi_{1,n-l+1}$.

(2) *The wild ramification subgroup $G(K_m/K)_1$ of K_m/K is isomorphic to $\mathbb{F}_q^m$.*

Proof. (1) We first show the results for $l = 1$. The case $n = 1$ is known. Assume (1) for $n-1$. If $n \geq m+1$, then $K_n = K_{n-1}$ by Lemma 3.1.3, so the claim follows similarly as in the case $n \leq m$. Assume $n \leq m$. We have $G(K_n/K)_{R_n^1} \cap G(K_n/K_{n-1}) = G(K_n/K_{n-1})_{R_n^1}$ by [**Se79**, Chapter IV, Proposition 2]. As the ramification break of K_n/K_{n-1} is ER_n and $ER_n < R_n^1$, we have $G(K_n/K)_{R_n^1} \cap G(K_n/K_{n-1}) = 1$. Notice $G(K_n/K)^{R_1} = G(K_n/K)_{R_n^1}$. Hence $G(K_n/K)^{R_1} = G(K_n/K)^{R_1} G(K_n/K_{n-1})/G(K_n/K_{n-1})$. By [**Se79**, Chapter IV, Proposition 14], we have an isomorphism $G(K_n/K)^{R_1} \cong G(K_{n-1}/K)^{R_1}$. By the ψ-functions, this is the isomorphism $G(K_n/K)_{R_n^1} \cong G(K_{n-1}/K)_{R_{n-1}^1}$. The first claim follows.

As for the Galois action, by induction hypothesis, we know $\phi_u(\sigma_{1,a}(\xi_{1,n}) - \xi_{1,n}) = 0$ and thus $\sigma_{1,a}(\xi_{1,n}) - \xi_{1,n} \in \phi[u]$. Similarly, we have $\sigma_{1,a}(\xi_{2,n}) - \xi_{2,n} - a \cdot \xi_{1,n} \in \phi[u]$. So

$\sigma_{1,a}(\xi_{1,n}) - \xi_{1,n} = a' \cdot \xi_{1,1} + a'' \cdot \xi_{2,1}$ for a', $a'' \in \mathbb{F}_q$. If $a'' \neq 0$, we obtain

$$\begin{aligned} R_n^1 \leq v_{K_n}(\sigma_{1,a}(\xi_{1,n})\xi_{1,n}^{-1} - 1) &= v_{K_n}(\sigma_{1,a}(\xi_{1,n}) - \xi_{1,n}) - v_{K_n}(\xi_{1,n}) \\ &= v_{K_n}(\xi_{2,1}) - v_{K_n}(\xi_{1,n}) \\ &= \frac{E}{q-1}\left(v(\boldsymbol{j})q^{n-1} + v_0((n-1)q^{n+1} - nq^n)\right) < 0, \end{aligned}$$

which is a contradiction. Similarly, we can show $a' = 0$ and thus $\sigma_{1,a}(\xi_{1,n}) = \xi_{1,n}$. For $\sigma_{1,a}(\xi_{2,n})$, we have

$$\begin{aligned} & v_{K_n}(\sigma_{1,a}(\xi_{2,n}) - \xi_{2,n}) \\ &= v_{K_n}(\sigma_{1,a}(\xi_{2,n})\xi_{2,n}^{-1} - 1) + v_{K_n}(\xi_{2,n}) \\ &\geq R_n^1 + v_{K_n}(\xi_{2,n}) \\ &= \frac{E}{q-1}\left(-v(\boldsymbol{j}) - v_0((n-1)q^{n+1} - nq^n)\right) - Eq^n\left(\frac{v_1}{q-1} - \frac{v(\boldsymbol{j})}{q^n(q-1)}\right) \\ &= Eq^n\left(-v_0(n-1) - \frac{v_1 - v_0}{q-1}\right) = v_{K_n}(\xi_{1,n}). \end{aligned} \tag{50}$$

Since $v(\xi_{1,n}) > v(\xi_{1,1}) > v(\xi_{2,1})$ by Proposition 2.1.2, we have $\sigma_{1,u}(\xi_{2,n}) = \xi_{2,n} + a \cdot \xi_{2,n}$.

Then we show the case for all l. We again use induction. Similar to the proof in the case $l = 1$, we have the isomorphism $G(K_n/K_{l-1})_{R_n^l} \cong G(K_{n-1}/K_{l-1})_{R_{n-1}^l}$ by $R_n^l > ER_n$. We can show $\sigma_{l,a}(\xi_{1,n}) - \xi_{1,n}$, $\sigma_{l,a}(\xi_{2,n}) - \xi_{2,n} - a \cdot \xi_{1,n-l+1} \in \phi[u]$. Calculations similar to those in the case $l = 1$ show that they vanish.

(2) From Lemma 3.4.2, the wild ramification subgroup $G(K_m/K)_1$ is equal to $G(K_m/K)_{ER_m}$. By (1), it is generated by $\{\sigma_{l,a} \mid 1 \leq l \leq m,\ a \in \mathbb{F}_q\}$. For a basis $\{\xi_{1,m}, \ldots, \xi_{1,1}, \xi_{2,m}, \ldots, \xi_{2,1}\}$ of $\phi[u^m]$ with the order according to the valuations, we can identify each $\sigma_{l,a}$ as the representation matrix

$$\begin{pmatrix} I_m & 0 \\ a \cdot A_{m,l} & I_m \end{pmatrix}$$

with respect to this basis. Here I_m denotes the $m \times m$ identity matrix and $A_{m,l}$ is the $m \times m$ matrix defined by $(\delta_{i,j-l+1})_{ij}$ with the Kronecker delta δ. This gives a monomorphism $G(K_m/K)_{ER_m} \to \mathrm{GL}_{2m}(\mathbb{F}_q)$. Clearly, its image is isomorphic to the abelian group $\mathbb{F}_q^m$. □

5. On the extension generated by u^n-torsion points with arbitrary $\deg(u)$

Throughout this section, let ϕ be a rank 2 Drinfeld A-module over K such that $\phi_t(X) = tX + a_1X^q + a_2X^{q^2} \in K[X]$. Let u be a finite prime of A satisfying $v \nmid u$ with degree d, where d is an arbitrary positive integer. Fix a positive integer n. Let $\{\lambda_i\}_{i=1,2}$ be an SMB of $\phi[u^n]$. Let K_n denote the extension of K generated by the elements of $\phi[u^n]$. We are to work out the action of wild ramification subgroup $G(K_n/K)_1$ of $G(K_n/K)$ on $\{\lambda_i\}_{i=1,2}$ under certain assumptions.

Let us begin with a lemma.

Lemma 3.5.1 (cf. Lemma 1.1.6). *Assume $v \nmid u$. Let n be any positive integer. Let $\{\lambda_i\}_{i=1,2}$ be an SMB of $\phi[u^n]$. Let σ be an element of $G(K_n/K)_1$ which is not the unit. Then we have $\sigma(\lambda_1) = \lambda_1$ and $\sigma(\lambda_2) \neq \lambda_2$.*

Proof. Let v_{K_n} denote the normalized valuation associated to K_n. We have

$$1 \leq v_{K_n}(\sigma(\lambda_1)\lambda_1^{-1} - 1) = v_{K_n}(\sigma(\lambda_1) - \lambda_1) - v_{K_n}(\lambda_1).$$

By Remarks 1.2.4 and 1.3.12, the valuation $v_{K_n}(\lambda_1)$ is the largest among the valuations of all elements in $\phi[u^n] \setminus \{0\}$. As $\sigma(\lambda_1) - \lambda_1$ is an element in $\phi[u^n]$, we have

$$v_{K_n}(\sigma(\lambda_1) - \lambda_1) - v_{K_n}(\lambda_1) \leq 0 \text{ if } \sigma(\lambda_1) - \lambda_1 \neq 0.$$

Hence $\sigma(\lambda_1) - \lambda_1 = 0$.

As λ_1 and λ_2 are generators of $\phi[u^n]$ as an A/u^n-module, they generate the extension K_n/K. This implies that $\sigma(\lambda_2) - \lambda_2 \neq 0$. □

5.1. Infinite prime. Let v be an infinite prime and Λ the lattice associated to ϕ via the uniformization. Put $v_0 = v(t)$ and $v_1 = v(a_1)$. Assume $v(\boldsymbol{j}) < v_0 q$ and $p \nmid v(\boldsymbol{j})$. Let m be the positive integer such that $v(\boldsymbol{j}) \in (v_0 q^{m+1}, v_0 q^m)$. By Proposition 2.2.1 (now α_j^{inf} in Section 1.2 in Chapter 2 equals $v_0 q^j$), we know that if $n \geq m/d$, the condition "$|u^n| > |\omega_r|/|\omega_1|$" in Proposition 1.2.11 will be fulfilled and we have

$$K_n = K(\Lambda) = K(\phi[t^m]). \tag{51}$$

We are to work out the action of the ramification subgroup of $G(K(\Lambda)/K)$ on an SMB of $\phi[u^n]$ for $n \geq m/d$.

Theorem 3.5.2 (cf. Theorem 3.4.3). *Assume $v(\boldsymbol{j}) < v_0 q$ and $p \nmid v(\boldsymbol{j})$. Let m be the integer such that $v(\boldsymbol{j}) \in (v_0 q^{m+1}, v_0 q^m)$. Let n be an integer $\geq m/d$ and $\{\lambda_i\}_{i=1,2}$ an SMB of $\phi[u^n]$. Put $G(\Lambda) := G(K(\Lambda)/K)$. For a positive integer i, let $A^{<i}$ denote the subgroup of A consists of elements with degree $< i$.*

(1) *Any element in $G(\Lambda)_1$ fixes λ_1;*
(2) *Then the map*

$$g : G(\Lambda)_1 \to A^{<m} \cdot_\phi \lambda_1;\ \sigma \mapsto \sigma(\lambda_2) - \lambda_2$$

is an isomorphism.

(3) *Put $R_i := \frac{-v(\boldsymbol{j}) + v_0 q^i}{q-1}$ for $1 \leq i \leq m$. Let $G(\Lambda)^{R_i}$ denote the upper R_i-th ramification subgroup of $G(\Lambda)$. Then the restriction*

$$g : G(\Lambda)^{R_i} \to A^{<i} \cdot_\phi \lambda_1$$

is an isomorphism for $1 \leq i \leq m$.

PROOF. (1) has been shown in Lemma 3.5.1.

(2) By (51), the ψ-function of $K(\Lambda)/K$ is the one in Lemma 3.4.2, i.e., we have

$$\psi_{K(\Lambda)/K} = \psi_{K(\phi[t^m])/K}.$$

As in Theorem 3.4.3, put $R_m^i = \psi_{K(\Lambda)/K}(R_i)$ for $i = 1, \dots, m$ and we have

$$R_m^i = -v(\boldsymbol{j})E\frac{1}{q-1} - v_0Eq^m\left(m - i - \frac{1}{q-1}\right).$$

We show $\sigma(\lambda_2) - \lambda_2 \in A^{<m} \cdot_\phi \lambda_1$ for an element σ in $G(\Lambda)_1 = G(\Lambda)_{R_m^m}$ (the equality follows from Lemma 3.4.2). Clearly $\sigma(\lambda_2) - \lambda_2 \in \phi[u^n]$. By Corollary 2.2.3 (2), an element of $\phi[u^n]$ having valuation $\geq v(\xi_{1,nd-m+1})$ belongs to the $\mathbb{F}_q$-vector space $A^{<m} \cdot_\phi \lambda_1$ (see (48) for $v(\xi_{i,j})$). Hence it suffices to show $v(\sigma(\lambda_2) - \lambda_2) \geq v(\xi_{1,nd-m+1})$. By Proposition 2.2.1, we have $v(\lambda_i) = v(\xi_{i,nd})$. Let v_Λ denote the normalized valuation associated to $K(\Lambda)$. We have $v_\Lambda = Eq^m v$. Consider

$$\begin{aligned}
v_\Lambda(\sigma(\lambda_2) - \lambda_2) &= v_\Lambda(\sigma(\lambda_2)\lambda_2^{-1} - 1) + v_\Lambda(\lambda_2)\\
&\geq R_m^m + v_\Lambda(\lambda_2)\\
&= -v(\boldsymbol{j})E\frac{1}{q-1} - v_0Eq^m\left(-\frac{1}{q-1}\right)\\
&\quad - Eq^m\left(v_0(nd-m) + \frac{v_1}{q-1} - \frac{v(\boldsymbol{j})}{q^m(q-1)}\right)\\
&= -Eq^m\left(v_0(nd-m) + \frac{v_1 - v_0}{q-1}\right) = v_\Lambda(\xi_{1,nd-m+1}).
\end{aligned}$$

Hence we have a map

$$g : G(\Lambda)_1 \to A^{<m} \cdot_\phi \lambda_1; \quad \sigma \mapsto \sigma(\lambda_2) - \lambda_2.$$

Next, we show that g is an isomorphism. The map is injective since λ_1 and λ_2 generate $K(\Lambda)/K$ and $\sigma(\lambda_1) = \lambda_1$ for any $\sigma \in G(\Lambda)_1$. By Theorem 3.4.3, we know $G(\Lambda)_1 \cong \mathbb{F}_q^m$. As q^m is also the cardinal of $A^{<m} \cdot_\phi \lambda_1$, the map is bijective. It suffices to show that this map is a morphism. For any $\sigma \in G(\Lambda)_1$, we have that σ fixes λ_1 and $\sigma(\lambda_2) - \lambda_2 = b \cdot_\phi \lambda_1$ for some $b \in A$. Hence for any σ', $\sigma \in G(\Lambda)_1$, we have

$$\sigma'(\sigma(\lambda_2) - \lambda_2) = \sigma(\lambda_2) - \lambda_2.$$

This implies

$$\begin{aligned}
\sigma'(\sigma(\lambda_2)) - \lambda_2 &= \sigma'(\sigma(\lambda_2)) - \sigma'(\lambda_2) + \sigma'(\lambda_2) - \lambda_2\\
&= \sigma'(\sigma(\lambda_2) - \lambda_2) + \sigma'(\lambda_2) - \lambda_2\\
&= \sigma(\lambda_2) - \lambda_2 + \sigma'(\lambda_2) - \lambda_2,
\end{aligned}$$

which shows that the map is a morphism.

(3) Note $G(\Lambda)^{R_i} = G(\Lambda)_{R^i_m}$. We show that $g : G(\Lambda)_{R^i_m} \to A^{<i} \cdot_\phi \lambda_1$ is an isomorphism for each $1 \le i \le m$. By Corollary 2.2.3 and Proposition 2.1.2, the vector space $A^{<i} \cdot_\phi \lambda_1$ consists of elements of $\phi[u^n]$ having valuation $\ge v(\xi_{1,nd-i+1})$. For i to be one of $1, \dots, m$ and σ to be a nontrivial element in $G(\Lambda)_{R^i_m}$, we have

$$\begin{aligned} v_\Lambda(\sigma(\lambda_2) - \lambda_2) &= v_\Lambda(\sigma(\lambda_2)\lambda_2^{-1} - 1) + v_\Lambda(\lambda_2) \\ &\ge R^i_m + v_\Lambda(\lambda_2) \\ &= -v(\boldsymbol{j})E\frac{1}{q-1} - v_0 E q^m \left(m - i - \frac{1}{q-1}\right) \\ &\quad - Eq^m \left(v_0(nd-m) + \frac{v_1}{q-1} - \frac{v(\boldsymbol{j})}{(q-1)}\right) \\ &= -Eq^m \left(v_0(nd-i) + \frac{v_1 - v_0}{q-1}\right) = v_\Lambda(\xi_{1,nd-i+1}). \end{aligned}$$

This implies that $g(G(\Lambda)_{R^i_m}) \subset A^{<i} \cdot_\phi \lambda_1$. As the cardinal of $G(\Lambda)_{R^i_m}$ and $A^{<i} \cdot_\phi \lambda_1$ are both q^i, the map g induces an isomorphism

$$g : G(\Lambda)_{R^i_m} \to A^{<i} \cdot_\phi \lambda_1$$

for each i. □

5.2. Finite prime. Let v be a finite prime. Assume that ϕ has stable reduction such that $a_1 = 0$ and $v(\boldsymbol{j}) < 0$. Let ψ and Λ respectively denote the Drinfeld module having good reduction and the lattice associated to ϕ via the Tate uniformization. Let $\{\omega_1\}$ be an SMB of $\psi[u^n]$, $\{\omega_2^0\}$ an SMB of Λ, and ω_2 a root of $\psi_{u^n}(X) - \omega_2^0$. By Proposition 2.2.6, we have $v(\omega_1) = 0$ and $v(\omega_2^0) = \frac{v(\boldsymbol{j})}{q-1}$. We first study the action of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ on $\omega_1, \omega_2 \in u^{-n}\Lambda/\Lambda$. Using the $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-isomorphism $\mathcal{E}_\phi : u^{-n}\Lambda/\Lambda \to \phi[u^n]$, we then work out the action of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ on $\phi[u^n]$.

Let L be the extension of K generated by the elements in Λ and $L(\psi[u^n])$ the extension of L generated by the elements in $\psi[u^n]$. Let L_n denote $K(u^{-n}\Lambda)$ which is the extension of K generated by elements in $u^{-n}\Lambda$. As the condition "$|u^n|_\infty > |\omega_r^0|/|\omega_{r'+1}^0|$" in Section 3 in Chapter 1 is fulfilled for any positive integer n, by Proposition 1.3.14, we have $K_n = L_n$ for any positive integer n.

Lemma 3.5.3. *The extension L/K is at worst tamely ramified.*

Proof. We know that Λ is an A-lattice via ψ and is $G(K^{\mathrm{sep}}/K)$-invariant. As L/K is a subextension of K_1/K, we have that L/K is separable. Then the desired claim follows from Lemma 1.1.6. □

Theorem 3.5.4 (cf. Theorem 3.5.2). *Let ϕ be a rank 2 Drinfeld A-module over K having stable reduction such that $v(\boldsymbol{j}) < 0$. Assume $p \nmid v(\boldsymbol{j})$. Let n be any positive integer. Put $R := -v(\omega_2^0) = \frac{-v(\boldsymbol{j})}{q-1}$.*

(1) *There is an isomorphism*

$$G(L_n/L(\psi[u^n])) \to \psi[u^n];\ \sigma \mapsto \sigma(\omega_2) - \omega_2.$$

(2) *Let E be the ramification index of L/K. The (Herbrand) ψ-function of the extension L_n/K is*

$$\psi_{L_n/K}(y) = \begin{cases} y, & -1 \leq y \leq 0; \\ Ey, & 0 \leq y \leq R; \\ q^{nd}Ey - (q^{nd}-1)ER, & R \leq y. \end{cases}$$

Proof. Let v_L denote the normalized valuation associated to L. We have $v_L = Ev$. As the extension $L(\psi[u^n])/L$ is unramified, we may also denote by v_L the normalized valuation associated to $L(\psi[u^n])$. The field L_n is the splitting field of $\psi_{u^n}(X) - \omega_2^0$ over $L(\psi[u^n])$. Note that the difference between two roots of $\psi_{u^n}(X) - \omega_2^0$ belongs to $\psi[u^n]$. Hence the extension $L_n/L(\psi[u^n])$ is generated by ω_2. As E is not divisible by p (Lemma 3.5.3), we have $p \nmid ER = v_L(\omega_2^0)$. Applying Proposition 3.2.2 to $\psi_{u^n}(X) - \omega_2^0 \in L(\psi[u^n])[X]$, we know that the map $G(L_n/L(\psi[u^n])) \to \psi[u^n];\ \sigma \mapsto \sigma(\omega_2) - \omega_2$ is an isomorphism.

(2) By Lemma 3.5.3, we have the ψ-function of L/K to be

$$\psi_{L/K}(y) = \begin{cases} y, & -1 \leq y \leq 0; \\ Ey, & 0 \leq y. \end{cases}$$

The ψ-function of $L(\psi[u^n])/L$ is $\psi_{L(\psi[u^n])/L}(y) = y$. Applying Proposition 3.2.2 to $\psi_{u^n}(X) - \omega_2^0 \in L(\psi[u^n])$, we have

$$\psi_{L_n/L(\psi[u^n])}(y) = \begin{cases} y, & -1 \leq y \leq ER; \\ q^{nd}y - (q^{nd}-1)ER, & ER \leq y, \end{cases}$$

and the desired ψ-function follows from Lemma 3.2.1. □

In the reminder of this subsection, let ϕ be a rank 2 Drinfeld A-module over K, which does not necessarily have stable reduction over K. By Proposition 1.3.17, we have that ϕ is isomorphic to a Drinfeld module having stable reduction over $K(\lambda_{1,1})$, where $\{\lambda_{i,1}\}_{i=1,2}$ is an SMB of $\phi[u]$ and $K(\lambda_{1,1})/K$ is at worst tamely ramified. Let ψ and Λ denote respectively the Drinfeld module having good reduction and the lattice associated to the Drinfeld module having stable reduction via the Tate uniformization. Let L denote the extension of $K(\lambda_{1,1})$ generated by the elements in Λ. By Lemma 1.1.6, the extension L/K is at worst tamely ramified. For a positive integer n, we have $K_n = L_n$.

Corollary 3.5.5. *Let ϕ be a rank 2 Drinfeld A-module over K such that*

$$\phi_t(X) = tX + a_1X^q + a_2X^{q^2} \in K[X],$$

which does not necessarily have stable reduction over K. Assume $p \nmid v(\boldsymbol{j})$.

(1) *Let E be the ramification index of L/K. The ψ-function of the extension K_n/K is*

$$\psi_{K_n/K}(y) = \begin{cases} y, & -1 \leq y \leq 0; \\ Ey, & 0 \leq y \leq R; \\ q^{nd}Ey - (q^{nd} - 1)ER, & R \leq y. \end{cases}$$

(2) *Let $\{\lambda_i\}_{i=1,2}$ be an SMB of $\phi[u^n]$. Then each element in $G(n)_1$ fixes λ_1 and there is an isomorphism*

$$G(K_n/K)_1 \to A \cdot_\phi \lambda_1;\ \sigma \mapsto \sigma(\lambda_2) - \lambda_2.$$

Proof. Apply Theorem 3.5.4 (2) with K in the theorem being $K(\lambda_{1,1})$ and we obtain the ψ-function of $K_n/K(\lambda_{1,1})$. As $K(\lambda_{1,1})/K$ is at worst tamely ramified, its ψ-function is clear. Then (1) follows from Lemma 3.2.1.

We show (2). Note that $L(\psi[u^n])/K$ is at worst tamely ramified. By the ψ-function of L_n/K, we have the equation of the higher ramification subgroups

$$G(K_n/K)_1 = G(L_n/K)_1 = G(L_n/K)_{ER} = G(L_n/L(\psi[u^n])).$$

By Proposition 1.3.17, the Drinfeld module $b\phi b^{-1}$ for $b = \lambda_{1,1}^{-1}$ has stable reduction over $K(\lambda_{1,1})$. By Theorem 1.3.13, the element $\log_\phi(b\lambda_1)$ forms an SMB of $\psi[u^n]$ and the element $u^n \cdot_\psi \log_\phi(b\lambda_2)$ forms an SMB of Λ. Apply Theorem 3.5.4 (1) with $\omega_1 = \log_\phi(b\lambda_1)$ and $\omega_2 = \log_\phi(b\lambda_2)$. We have $\sigma(\log_\phi(b\lambda_1)) = \log_\phi(b\lambda_1)$ for any $\sigma \in G(K_n/K)_1$ and an isomorphism

$$G(K_n/K)_1 \to \psi[u^n];\ \sigma \mapsto \sigma(\log_\phi(b\lambda_2)) - \log_\phi(b\lambda_2).$$

Note $\psi[u^n] = A\cdot_\psi \omega_1$. The map $\mathcal{E}_{b\phi b^{-1}}|_{\psi[u^n]} : \psi[u^n] \to A\cdot_{b\phi b^{-1}} b\lambda_1$ induced by the exponential map e_ϕ is an isomorphism. Indeed, it is injective as $\psi[u^n] \cap \Lambda = \{0\}$. Since the sets $\psi[u^n]$ and $A \cdot_{b\phi b^{-1}} b\lambda_1$ both have cardinal q^{nd}, we have the surjectivity. By this isomorphism, we obtain $\sigma(b\lambda_1) = b\lambda_1$ and $\sigma(\log_\phi(b\lambda_2)) - \log_\phi(b\lambda_2)$ maps to $\sigma(b\lambda_2) - b\lambda_2$. The desired isomorphism is the composition

$$G(K_n/K)_1 \to \psi[u^n] \overset{\mathcal{E}_{b\phi b^{-1}}}{\longrightarrow} A \cdot_{b\phi b^{-1}} b\lambda_1 \overset{b^{-1}\cdot-}{\longrightarrow} A \cdot_\phi \lambda_1;\ \sigma \mapsto \sigma(\lambda_2) - \lambda_2.$$

□

6. Tamely ramified extensions

Let ρ be a rank r Drinfeld A-module over K such that (as in Section 1.1)

$$\rho_t(X) = tX + \sum_{k=1}^{r} a_k X^{q^k} \in K[X].$$

Put

$$v_0 := v(t) \text{ and } v_k := v(a_k) \text{ for each } k = 1, \dots, r.$$

Lemma 3.6.1. *Let v be an infinite prime or a finite prime, and u a finite prime of A satisfying $v \nmid u$. Assume for $k = 1, \dots, r-1$ that*

$$\frac{v_r - v_0}{q^r - 1} \leq \frac{v_k - v_0}{q^k - 1}.$$

(1) *The extension $K(\rho[u])/K$ is at worst tamely ramified. The ramification index of $K(\rho[u])/K$ divides $q^r - 1$.*

(2)
- *If v is an infinite prime, then we have $K(\rho[u^n]) = K(\rho[u])$ for any n.*
- *If v is a finite prime, then $K(\rho[u^n])/K(\rho[u])$ is unramified for any n.*

Let u be a finite prime of A with arbitrary degree and $v \nmid u$. Let us apply the lemma to the case $\rho = \phi$. If the prime v is infinite and $v(\boldsymbol{j}) \geq \alpha_1 = \frac{v_0 q^s (q^{r-s}-1)}{q-1}$, we have that $K(\phi[u])/K$ is at worst tamely ramified and $K(\phi[u^n]) = K(\phi[u])$. Moreover, if the prime v is finite satisfying $v \nmid u$ and $v(\boldsymbol{j}) \geq 0$, then we have that $K(\phi[u])/K$ is at worst tamely ramified and $K(\phi[u^n])/K(\phi[u])$ is unramified.

Proof. We show the result for the case where v is an infinite prime. Assume $\deg(u) = 1$. Let M be an extension of K with the ramification index being $e_{M/K} = q^r - 1$. We can take $b \in M$ such that $v(b) = \frac{v_r - v_0}{q^r - 1}$. With $b' = b^{q^r}/a_r$, modify $\rho_u(X)$ to be

$$\mathrm{P}(X) = X^{q^r} + \sum_{k=1}^{r-1} b_k X^{q^k} + b_0 X := b' \rho_u(X/b).$$

The valuations of $v(b_0)$ and $v(b_k)$ are respectively

$$0 \text{ and } \frac{1}{q^k - 1}\left(\frac{v_k - v_0}{q^k - 1} - \frac{v_r - v_0}{q^r - 1}\right) \text{ for } k = 1, \dots, r-1.$$

Thus $\mathrm{P}(X)$ is a monic polynomial whose reduction is separable. By Hensel lemma [**Pap23**, Corollary 2.4.5], we know the extension $M(\rho[u])$ of M generated by all roots of $\mathrm{P}(X)$ is unramified. As $K(\rho[u])/K$ is a subextension of $M(\rho[u])/K$, the ramification index of $K(\rho[u])/K$ divides the ramification index $e_{M(\rho[u])/K} = e_{M/K} = q^r - 1$. Under the present assumption, (1) follows.

As the Newton polygon of $\rho_u(X)$ has exactly one segment (still assume $\deg(u) = 1$), all elements of an SMB of $\rho[u]$ have the same valuation. By Corollary 1.2.12, we have

$K(\rho[u]) = K(\Lambda)$. Hence $K(\Lambda)/K$ is at worst tamely ramified with the ramification index dividing $q^r - 1$. For any element a of A with positive degree, we have $K(\rho[a]) = K(\Lambda)$. This shows (1) and the first dot of (2).

Now we deal with the case where the prime v is finite. Let M be a tamely ramified extension of K with ramification index $q^r - 1$. For $b \in M$ with $v(b) = \frac{v_r}{q^r-1}$, as $b\rho b^{-1}$ has good reduction, we have that $M(\rho[u])/M$ is unramified by the Néron-Ogg-Safarevich criterion (See [**Pap23**, Theorem 6.3.1]). Similarly to the infinite prime case, (1) follows. As $v \nmid u$, each element in $b\rho b^{-1}[u]$ has valuation 0 (cf. Lemma 1.3.2). As multiplying b establishes an isomorphism $\rho[u] \to b\rho b^{-1}[u]$, each element of $\rho[u]$ has valuation $\frac{-v_r}{q^r-1}$. Hence ρ is isomorphic to a Drinfeld module having good reduction over $K(\rho[u])$. Apply the Néron-Ogg-Safarevich criterion to the Drinfeld module $b\rho b^{-1}$ with $b \in K(\rho[u])$ with $v(b) = \frac{v_r}{q^r-1}$. The extension $K(\rho[u^n])/K(\rho[u])$ is unramified and the second dot of (2) follows. □

Let v be an infinite prime or a finite prime. Let u be a finite prime of A with degree 1. Assume for $k = 1, \ldots, r-1$ that

$$\frac{v_r - v_0}{q^r - 1} < \frac{v_k - v_0}{q^k - 1}.$$

We finish this section by determining the ψ-function of $K(\rho[u])/K$ under the above assumption. By Lemma 3.6.1 (1), the extension $K(\rho[u])/K$ is at worst tamely ramified. Hence determining the ψ-function of $K(\rho[u])/K$ is equivalent to determining the ramification index. By the proof of Lemma 3.6.1, if the prime v is infinite, then we have $K(\Lambda) = K(\rho[u]) = K(\rho[a])$, where Λ is the lattice associated to ρ and a is an arbitrary element in A having positive degree. Hence we also obtain the ψ-functions of $K(\Lambda)/K$ and $K(\rho[a])/K$ in this case.

It seems natural to ask whether there is an analogue of Lemma 3.1.1. Namely, does $K(\rho[u])$ contain the splitting field of some binomial whose terms come from $\rho_u(X)$? This is answered affirmatively in the following lemma.

Lemma 3.6.2. *Assume for $k = 1, \ldots, r-1$ that*

$$\frac{v_r - v_0}{q^r - 1} < \frac{v_k - v_0}{q^k - 1}. \tag{52}$$

Put $\eta(X) = a_r X^{q^r} + uX$. Let K_η denote the extension of K generated by all roots of $\eta(X)$. We claim $K(\rho[u]) = K_\eta$.

By this claim, if $v(\boldsymbol{j}) > \alpha_1 = \frac{q^s(q^{r-s}-1)}{q-1}$, the extension $K(\phi[u])/K$ is generated by the roots of $a_r X^{q^r} + uX$.

Proof. The proof of this claim is carried out by the strategy used in that of Lemma 3.1.1. We give an outline. Let x and x_j for $j = 1, \ldots, q^r - 2$ be all nonzero roots of $\rho_u(X)$. Let

x'_j for $j = 1, \ldots, q^r - 1$ denote all nonzero roots of $\eta(X)$. We have $v(x'_j) = -\frac{v_r - v_0}{q^r - 1}$ for all j. Since

$$\eta(x) = a_r \cdot x \prod_{j=1}^{q^r-1} (x - x'_j) = a_r x^{q^r} + ux = \sum_{k=1}^{r-1} a_k x^{q^k},$$

we have

$$\left| \prod_{j=1}^{q^r-1} (x - x'_j) \right| = \left| a_r^{-1} \sum_{k=1}^{r-1} a_k x^{q^k - 1} \right| \leq \max\{|a_r^{-1} a_k x^{q^k-1}|\}.$$

By (52), we have $v_k > v_0 + \frac{(v_r - v_0)(q^k - 1)}{q^r - 1}$ for each $k = 1, \ldots, r-1$. This implies for $k = 1, \ldots, r-1$

$$v(a_r^{-1} a_k x^{q^k}) = -v_r + v_k - \frac{(v_r - v_0)(q^k - 1)}{q^r - 1} > -(v_r - v_0).$$

Hence

$$\left| \prod_{j=1}^{q^r-1} (x - x'_j) \right| < q^{v_r - v_0}.$$

There exists some x'_i such that $|x - x'_i| < q^{\frac{v_r - v_0}{q^r - 1}}$. Put $x' := x'_i$ and rearrange the index so that x' and x'_j for $j = 1, \ldots, q^r - 2$ are different nonzero roots of $\eta(X)$. We have

$$|x - x'| < |x' - x'_j| = |x - x_j| = q^{\frac{v_r - v_0}{q^r - 1}} \text{ for all } j.$$

By Krasner's lemma, we have $K(x'_i) = K(x)$. Similarly, for j to be one of $1, \ldots, q^r - 2$, there exists x'' to be one of $x', x'_1, \ldots, x'_{q^r-2}$ such that $|x_j - x''| < q^{\frac{v_r - v_0}{q^r - 1}}$. As $|x_j - x'| = \max\{|x_j - x|, |x - x'|\} = q^{\frac{v_r - v_0}{q^r - 1}}$, we have $x'' \neq x'$. Let x vary within the nonzero roots of $\rho_u(X)$ and the equality $K_\eta = K(\rho[u])$ follows. $\square$

Proposition 3.6.3. *Assume for $k = 1, \ldots, r-1$ that*

$$\frac{v_r - v_0}{q^r - 1} < \frac{v_k - v_0}{q^k - 1}.$$

The ramifications index of $K(\rho[u])/K$ is $\frac{q^r - 1}{n}$ with $n = \gcd(v(u/a_r), q^r - 1)$.

Proof. Assume for the moment that K contains $\mathbb{F}_{q^r}$ so that the extension K_η/K is Kummer. By replacing ρ with some isomorphic Drinfeld A-module ρ' over K with the leading coefficient of $\rho'_u(X)$ having sufficiently negative valuation, we may assume that $v(u/a_r) > 0$. Note that n is unchanged under replacing ρ with ρ'. There exists some $\alpha' \in K$ with $v(\alpha') = n$ such that $K_\eta = K(\sqrt[q^r-1]{\alpha'})$ and the subextension $K(\sqrt[n]{\alpha})/K$ of K_η/K is unramified (see [**Bir67**, Section 2, Lemma 6]). Put $\alpha = \sqrt[n]{\alpha'}$. As $v(\alpha) = 1$, the extension $K(\sqrt[\frac{q^r-1}{n}]{\alpha})/K(\alpha)$ is totally ramified with degree $\frac{q^r - 1}{n}$. Therefore, the ramification index of K_1/K equals $\frac{q^r - 1}{n}$.

For general K, considering the compositum $K\mathbb{F}_{q^r}$ and the compositum $K(\rho[u])\mathbb{F}_{q^r}$ instead of K and $K(\rho[u])$ respectively, and using the fact that any residue field extension is unramified, we obtain the same result on the ramification index. $\square$

Example 3.6.4. [For results in Sections 4, 5, 6] Let v be a prime of A and u be a finite prime of A. Put $K_n := K(\phi[u^n])$. For the Drinfeld A-modules $\phi^{(i)}$ over $\mathbb{F}_q(t)_v$ for integers $p \nmid i$ in Example 3.3.18, we have

$$\phi_t^{(i)}(X) = tX + t^{i(q-1)^2} X^q + (t+1)^{-q} X^{q^2}.$$

The j-invariant of $\phi^{(i)}$ is

$$\boldsymbol{j}^{(i)} := t^{i(q^2-1)(q-1)} \cdot (t+1)^q.$$

(1) For v to be an infinite prime, if $i > 0$, then the ramification break R_1 of K_n/K (in Theorem 3.5.2) for any n equals $i(q^2-1)$. If $i \leq 0$, the extension K_n/K is at worst tamely ramified (Lemma 3.6.1).
(2) Let u be a finite prime of A with $u \neq v$. If $v = t$ and $i < 0$, then ϕ has potentially stable reduction and the reduction over some extension of K is bad. The ramification break R (in Theorem 3.5.4) of K_n/K equals $-i(q^2-1)$. If $v \neq t$ or $i > 0$, then ϕ has potential good reduction and the extension K_n/K is at worst tamely ramified (Lemma 3.6.1).

CHAPTER 4

Height functions, conductors, and Szpiro conjecture

Let F be a global function field. Let ϕ be a Drinfeld A-module over F such that

$$\phi_t(X) = tX + a_s X^{q^s} + a_r X^{q^r} \in F[X], \tag{53}$$

where s and r are two positive integers satisfying $s < r$. The j-invariant of ϕ is

$$\boldsymbol{j} := \frac{a_s^{(q^r-1)/(q-1)}}{a_r^{(q^s-1)/(q-1)}}.$$

The heights of ϕ measure the arithmetic complexity of ϕ. Applying Proposition 2.2.1 with the local field taking F_v and v being an infinite prime of F, we obtain a formula that can be regarded as a relation between the J-height and the differential height of ϕ in Section 1. In Section 2, using the results in Section 5 and 6 in Chapter 3 (and admitting the assumptions in these results), we define and calculate the conductor of ϕ at each prime v of F when $r = 2$. Finally, we show that there is a numerical relation between the J-heights and the conductors of certain rank 2 Drinfeld A-modules. The obtained numerical relation might be regarded as an analogue of Szpiro's conjecture for function fields.

1. Height functions

Let M_F (resp. M_F^{f} and M_F^{∞}) denote the set of all primes (resp. all finite primes and all infinite primes) of F. For each prime $v \in M_F^{\mathrm{f}}$, let $\deg(v)$ denote the degree of the residue field of F_v over $\mathbb{F}_q$. Put $F_0 := \mathrm{Frac}(A) = \mathbb{F}_q(t)$. Following [**BPR21**], put

$$h_J(\phi) := \frac{1}{[F:F_0]} \sum_{v \in M_F} \deg(v) \cdot \max\{-v(\boldsymbol{j}), 0\} \tag{54}$$

to be the J-height [1] of ϕ.

Remark 4.1.1. For a prime v of F, let $|-|_v$ denote the absolute value of F_v either satisfying $|u|_v = q^{-\deg(u)}$ for the finite prime u of A divisible by v (this is the case where

[1]In [**BPR21**], the J-height of ϕ is defined to be

$$\frac{d(q-1)}{(q^s-1)(q^r-1)} \cdot \frac{1}{[F:F_0]} \sum_{v \in M_F} \deg(v) \cdot \max\{-v(\boldsymbol{j}), 0\},$$

where d is the least common multiple of $q^i - 1$ for $i = 1, \ldots, r$.

the prime v is finite) or being the extension of $q^{\deg(-)}$ (this is the case where the prime v is infinite). Then we have

$$h_J(\phi) = \frac{1}{[F:F_0]} \sum_{v \in M_F} [F_v : F_{0,v}] \cdot \log \max\{|\boldsymbol{j}|_v, 0\},$$

where $F_{0,v}$ is the completion of F_0 at the prime of F_0 lying below v. On the other hand, let E be an elliptic curve over a number field N and j_E its j-invariant. For a prime w of N, let $|-|_w$ denote the absolute value of the local field N_w either satisfying $|p|_w = p^{-1}$ for p divisible by w or being the extension of the absolute value of $\mathbb{R}$. Let M_N denote the set of all primes of N. Following [**Sil94**, (10)], put

$$h(j_E) \coloneqq \frac{1}{[N:\mathbb{Q}]} \sum_{w \in M_N} [N_w : \mathbb{Q}_w] \log \max\{|j_E|_w, 0\},$$

where $\mathbb{Q}_w = \mathbb{Q}_p$ for p to be the prime number divisible by w if w is a finite prime or $\mathbb{Q}_w = \mathbb{R}$ if w is an infinite prime. One may consider the J-height of a Drinfeld A-module as an analogue of $h(j_E)$.

For each infinite prime v of F, let Λ_v be the A-lattice associated to ϕ as a Drinfeld module over F_v via the uniformization and $\{\omega_{v,i}\}_{i=1,\dots,r}$ an SMB of Λ_v. Following [**Tag92**, Section 5.3 and (5.9.1)], define the differential height of ϕ to be

$$(55)\quad \begin{aligned} h_d(\phi) \coloneqq \frac{1}{[F:F_0]} \Bigg(& \sum_{v \in M_F^{\mathrm{f}}} \deg(v) \frac{q-1}{(q^s-1)(q^r-1)} \Bigg(\max\{-v(\boldsymbol{j}), 0\} - v_r \frac{q^s-1}{q-1} \Bigg) \\ & + \sum_{v \in M_F^{\infty}} \deg(v) \frac{1}{r} \big(v(\omega_{v,1}) s + v(\omega_{v,r})(r-s) \big) \Bigg). \end{aligned}$$

Remark 4.1.2. Resume the above notations. Define the covolume of Λ_v to be

$$D(\Lambda_v) \coloneqq \prod_{i=1}^{r} |\omega_i|_v$$

We may write (by the product formula)

$$\begin{aligned} h_d(\phi) = \frac{1}{[F:F_0]} \Bigg(& \frac{q-1}{(q^s-1)(q^r-1)} \sum_{v \in M_F^{\mathrm{f}}} [F_v : F_{0,v}] \log \max\{|\boldsymbol{j}|_v, 0\} \\ & - \sum_{v \in M_F^{\infty}} [F_v : F_{0,v}] \Bigg(\log |a_r|_v^{1/(q^r-1)} + \log D(\Lambda_v)^{1/r} \Bigg) \Bigg). \end{aligned}$$

Assume that E/N has everywhere stable reduction. Let $\eta(z)$ denote the Dedekind eta function defined on the upper half plane of $\mathbb{C}$ and put $\Delta(z) = (2\pi)^{-12} \eta(z)^{24}$. By [**Sil86**, Proposition 1.1 and (10)], the Faltings-Parshin height of E/N equals to

$$h(E/N) = \frac{1}{[N:\mathbb{Q}]}\Bigg(\frac{1}{12}\sum_{w\in M_N^{\mathrm{f}}}[N_w:\mathbb{Q}_w]\log\max\{|\boldsymbol{j}|_w,0\} - \sum_{w\in M_K^{\infty}}[N_w:\mathbb{R}]\Big(\log|\Delta(\tau_w)|^{1/12}+\log\mathrm{Im}(\tau_w)^{1/2}\Big)\Bigg).$$

Notice that $\mathrm{Im}(\tau_w)$ is the volume of the parallelogram in $\mathbb{C}$ spanned by 1 and τ_w. One may consider the differential height of a Drinfeld A-module as an analogue of the Faltings height. The differential height has a similar behavior to that of the Faltings-Parshin height (See [**Tag92**, Section 5] and [**Wei18**, Theorem 5.3]).

Put $v_0 \coloneqq v(t)$. By Proposition 2.2.1 and the product formula, we can obtain the value of $h_d(\phi)$ in terms of v_0 and $v(\boldsymbol{j})$ for all primes v of F.

Lemma 4.1.3. *Let $M_F^{<\alpha_1}$ denote the set of infinite primes v of F such that $v(\boldsymbol{j}) < v_0 q$ and $M_F^{\geq v_0 q} \coloneqq M_F \setminus M_F^{<v_0 q}$. For each prime $v \in M_F^{<\alpha_1}$, let m_v denote the positive integer such that $v(\boldsymbol{j}) \in (\alpha_{m+1}, \alpha_m]$. Then we have*

$$\begin{aligned}[F:F_0]\cdot h_d(\phi) = &\sum_{v\in M_F^{\mathrm{f}}\cup M_F^{<\alpha_1}} \deg(v)\frac{q-1}{(q^s-1)(q^r-1)}\max\{-v(\boldsymbol{j}),0\} \\ &+ \sum_{v\in M_F^{<\alpha_1}}\deg(v)\frac{1}{r}\left(v_0\frac{q^s s}{q^s-1}+v_0 m_v(r-s)+\frac{v(\boldsymbol{j})(q-1)(r-s)}{q^{sm_v}(q^s-1)(q^{r-s}-1)}\right) \\ &+ \sum_{v\in M_F^{\geq\alpha_1}}\deg(v)\frac{v_0 q^r}{q^r-1}.\end{aligned}$$

Let $M_K^{[\alpha_1,0)}$ denote the set of infinite primes v of K such that $v(\boldsymbol{j}) \in [\alpha_1, 0)$. By (54), we have

$$[F:F_0]\frac{q-1}{(q^s-1)(q^r-1)}h_J(\phi) = \sum_{v\in M_F^{\mathrm{f}}\cup M_F^{<\alpha_1}\cup M_F^{[\alpha_1,0)}}\deg(v)\frac{q-1}{(q^s-1)(q^r-1)}\max\{-v(\boldsymbol{j}),0\}.$$

Hence we have the following corollary.

Corollary 4.1.4. *With the notations above, we have*

$$h_d(\phi) = \frac{q-1}{(q^s-1)(q^r-1)}h_J(\phi) + \frac{1}{[F:F_0]}\big(C_1(\phi)+C_2(\phi)+C_3(\phi)\big),$$

where

$$C_1(\phi) := \sum_{v \in M_K^{[\alpha_1, 0)}} \deg(v) \frac{q-1}{(q^s-1)(q^r-1)} v(\boldsymbol{j}),$$

$$C_2(\phi) := \sum_{v \in M_F^{<\alpha_1}} \deg(v) \frac{1}{r} \left(v_0 \frac{q^s s}{q^s - 1} + v_0 m_v (r-s) + \frac{v(\boldsymbol{j})(q-1)(r-s)}{q^{s m_v}(q^s-1)(q^{r-s}-1)} \right), \text{ and}$$

$$C_3(\phi) := \sum_{v \in M_F^{\geq \alpha_1}} \deg(v) v_0 \frac{q^r}{q^r - 1}.$$

Remark 4.1.5. For an elliptic curve E over a number field N, a relation between $h(j_E)$ and $h(E/N)$ has been claimed by Silverman (See [**Sil86**, Proposition 2.1]). The above corollary may be regarded as its analogue.

Remark 4.1.6. For arbitrary Drinfeld A-module over F, it is natural to consider the relation between its J-height and its differential height. As the above corollary heavily relies on Lemma 4.1.3 and hence on the calculations in Chapter 2, the assumption $\phi_t(X) = tX + a_s X^{q^s} + a_r X^{q^r}$ is essential. We guess that an explicit formula between the J-height and the differential height is intricate. On the other hand, similarly to Silverman's formula involving $h(j_E)$ and $h(E/N)$, finding an inequality involving these two heights might be a more feasible approach.

2. Conductors and Szpiro's conjecture

For a rank 2 Drinfeld A-module ϕ over F, let us first define and calculate the conductor of ϕ at each prime of F in Section 2.1 using the results in Section 5 in Chapter 3. Using these calculations, we then show the formula involving the conductors of ϕ and the J-height in Section 2.3. In Section 2.2, we apply the results in Section 3 of Chapter 3 to calculations similar to those in Section 2.1.

2.1. Definition of the conductors. For a prime v of F, put $K := F_v$ and ϕ a rank 2 Drinfeld A-module over K throughout this subsection. For a finite prime u of A, by the u-adic Tate module T_u of ϕ, we mean the rank 2 free A_u-module $\varprojlim_n \phi[u^n]$, where the projective limit is defined using the morphisms $\phi_u : \phi[u^{n+1}] \to \phi[u^n]$ for all integers $n \geq 1$.

Lemma-Definition 4.2.1. *Let v be an infinite prime. Put $v_0 = v(t)$. Assume that one of the following two cases happens*

(C1) $v(\boldsymbol{j}) < v_0 q$ *and* $p \nmid v(\boldsymbol{j})$;
(C2) $v(\boldsymbol{j}) \geq v_0 q$.

Write G^y for the y-th upper ramification subgroup of the Galois group $G(K^{\mathrm{sep}}/K)$. For any finite prime u of A, put

$$\mathfrak{f}_v(\phi) := \int_0^\infty \left(2 - \mathrm{rank}_{A_u} T_u^{G^y}\right) dy.$$

Then we have

(1) *the value $\mathfrak{f}_v(\phi)$ is independent of the choice of u;*

(2) $\mathfrak{f}_v(\phi) = \begin{cases} \frac{-v(\boldsymbol{j})+v_0 q}{q-1} & \text{if (C1) happens;} \\ 0 & \text{if (C2) happens.} \end{cases}$

Define the conductor *of ϕ at v to be the integral $\mathfrak{f}_v(\phi)$.*

Proof. We will show (2) for any finite prime u of A and (1) straightforwardly follows.

By Corollary 1.1.12, there is an SMB $\{\lambda_{i,n}\}_{i=1,2}$ of $\phi[u^n]$ for each integer $n \geq 1$ such that $u \cdot_\phi \lambda_{i,n+1} = \lambda_{i,n}$ for $i = 1, 2$. The tuples $(\lambda_{1,n})_{n\geq1}$ and $(\lambda_{2,n})_{n\geq1}$ form an A_u-basis of T_u.

Assume (C1) happens. By (51), i.e., $K(\phi[u^n]) = K(\Lambda) = K(\phi[t^m])$ for $n \geq m/d$, the action of G^y on $\phi[u^n]$ factors through $G(\Lambda)^y$ for any $y > 0$. Here $G(\Lambda)$ denotes $G(K(\Lambda)/K)$. Notice $G(\Lambda)_1 = \bigcup_{y>0} G(\Lambda)^y$. For any element $\sigma \in G(\Lambda)_1$, by Lemma 3.5.1, σ fixes $u^j \cdot_\phi \lambda_{1,n} = \lambda_{1,n-j}$ for any non-negative integer $j < n$. Hence σ fixes $(\lambda_{1,n})_{n\geq1}$. By Theorem 3.5.2 (2), if σ is not the unit and $n \geq m/d$, then it nontrivially acts on $\lambda_{2,n}$ and hence nontrivially acts on $(\lambda_{2,n})_{n\geq1}$. As $G(\Lambda)^{R_1} \neq \{e\}$ and $G(\Lambda)^y = \{e\}$ for $y > R_1$ (by Lemma 3.4.2), we have $\mathrm{rank}_{A_u} T_u^{G^y} = 1$ if $0 < y \leq R_1$ and $= 2$ if $y > R_1$. We have

$$\mathfrak{f}_v(\phi) = \int_0^{R_1} 1 dy = \frac{-v(\boldsymbol{j}) + v_0 q}{q-1}.$$

For the case (C2), we have $K(\Lambda) = K(\phi[u^n])$ for any $n \geq 1$ (by Proposition 2.2.1 (2), we can apply Proposition 1.2.13). The action of G^y on $\phi[u^n]$ for any $n \geq 1$ and any $y > 0$ factors through $G(\Lambda)^y$. By Lemma 3.6.1 (1), we have $G(\Lambda)^y = \{e\}$ if $y > 0$. The result for the case (C2) immediately follows. □

Lemma-Definition 4.2.2. *Let v be a finite prime. Assume that one of the following two cases happens*

(C1) $v(\boldsymbol{j}) < 0$ *and* $p \nmid v(\boldsymbol{j})$ *such that the reduction of ϕ over some extension of K has rank* 1;

(C2) $v(\boldsymbol{j}) \geq 0$ *such that ϕ has potentially good reduction.*

Write G^y for the y-th upper ramification subgroup of the Galois group $G(K^{\mathrm{sep}}/K)$. For any finite prime u of A not divisible by v, put

$$\mathfrak{f}_v(\phi) := \int_0^\infty \left(2 - \mathrm{rank}_{A_u} T_u^{G^y}\right) dy.$$

Then we have

(1) *the value* $\mathfrak{f}_v(\phi)$ *is independent of the choice of* u.

(2) $\mathfrak{f}_v(\phi) = \begin{cases} \frac{-v(\boldsymbol{j})}{q-1} & \text{(C1) } \textit{happens}; \\ 0 & \text{(C2) } \textit{happens}. \end{cases}$

Define the conductor *of* ϕ *at* v *to be the integral* $\mathfrak{f}_v(\phi)$.

Proof. We will show (2) for any finite prime u of A and (1) straightforwardly follows.

By Corollary 1.1.12, there is an SMB $\{\lambda_{i,n}\}_{i=1,2}$ of $\phi[u^n]$ for each integer $n \geq 1$ such that $u \cdot_\phi \lambda_{i,n+1} = \lambda_{i,n}$ for $i = 1, 2$. The tuples $(\lambda_{1,n})_{n\geq 1}$ and $(\lambda_{2,n})_{n\geq 1}$ form an A_u-basis of T_u. The action of G^y on $\phi[u^n]$ for any $n \geq 1$ and any $y > 0$ factors through $G(n)^y$. Here $G(n)$ denotes the Galois group of the extension $K(\phi[u^n])/K$.

Assume (C1) happens. By Corollary 3.5.5 (1), we have $G(n)^y = G(n)_1$ for any $0 < y \leq \frac{-v(\boldsymbol{j})}{q-1}$ and $= \{e\}$ for $y > \frac{-v(\boldsymbol{j})}{q-1}$. By Corollary 3.5.5 (2), for any $n \geq 1$ and $0 < y \leq \frac{-v(\boldsymbol{j})}{q-1}$, any nontrivial element in $G(n)^y$ fixes $\lambda_{1,n}$ and nontrivially acts on $\lambda_{2,n}$. Any element of $G(\infty)^y = \varprojlim_n G(n)^y$ for $0 < y \leq \frac{-v(\boldsymbol{j})}{q-1}$ fixes $\lambda_{1,n}$ for $n \geq 1$ and hence fixes $(\lambda_{1,n})_{n\geq 1}$. Any nontrivial element of $G(\infty)^y$ nontrivially acts on $\lambda_{2,n}$ for some $n \geq 1$ and hence nontrivially acts on $(\lambda_{2,n})_{n\geq 1}$. Note that G^y acts on T_u via $G(\infty)^y$. Hence $\mathrm{rank}_{A_u} T_u^{G^y} = 1$ if $0 < y \leq \frac{-v(\boldsymbol{j})}{q-1}$ and $= 2$ if $\frac{-v(\boldsymbol{j})}{q-1} < y$. We have

$$\mathfrak{f}_v(\phi) = \int_0^{\frac{-v(\boldsymbol{j})}{q-1}} 1 dy = \frac{-v(\boldsymbol{j})}{q-1}.$$

For the case (C2), by Lemma 3.6.1, we have $G(n)^y = \{e\}$ for any $y > 0$ and $n \geq 1$. Hence G^y fixes $\lambda_{i,n}$ for $i = 1, \ldots, r$ and any $n \geq 1$. The case (2) follows. $\square$

Corollary 4.2.3. *Let* v *be a prime of* F. *Let* ϕ *be a rank* 2 *Drinfeld* A*-module over* F_v. *Put* $v_0 = v(t)$. *Then we have*

$$\mathfrak{f}_v(\phi) = \begin{cases} \begin{cases} 0 & v(\boldsymbol{j}) \in [v_0 q, +\infty); \\ \frac{-v(\boldsymbol{j})+v_0 q}{q-1} & v(\boldsymbol{j}) \in (-\infty, v_0 q) \textit{ and } p \nmid v(\boldsymbol{j}), \end{cases} & v \textit{ is infinite}; \\ \begin{cases} 0 & v(\boldsymbol{j}) \in [0, +\infty); \\ \frac{-v(\boldsymbol{j})}{q-1} & v(\boldsymbol{j}) \in (-\infty, 0), p \nmid v(\boldsymbol{j}), \end{cases} & v \textit{ is finite}. \end{cases}$$

2.2. On the free submodule fixed by the higher ramification subgroups. Put $K = F_v$. Let ϕ be a rank r Drinfeld A-module over K such that

$$\phi_t(X) = tX + a_1 X^q + a_r X^{q^r} \in K[X].$$

Let G^y denote the y-th upper ramification subgroup of the Galois group $G(K^{\mathrm{sep}}/K)$. In this subsection, similar to Section 2.1, we apply the results in Section 3 in Chapter 3 to study $\phi[u]^{G^y}$ for $y > 0$.

Lemma 4.2.4. *Let u be a finite prime of A with degree 1. Put $v_0 := v(u)$. Assume $v(\boldsymbol{j}) < \alpha_1 = \frac{v_0 q(q^{r-1}-1)}{q-1}$ and $p \nmid v(\boldsymbol{j})$. Put $R := \frac{-v(\boldsymbol{j})+\alpha_1}{q^{r-1}-1}$. Write G^y for the y-th upper ramification subgroup of the Galois group $G(K^{\mathrm{sep}}/K)$. Then for any $0 < y \leq R$, we have*

$$\mathrm{rank}_{A/u}\phi[u]^{G^y} = 1.$$

Proof. Let $\{\xi_{i,1}\}_{i=1,\ldots,r}$ be an SMB of $\phi[u]$. For $\lambda \in \phi[u]$, we have $\lambda = \sum_{i=1}^r a_i \xi_{i,1}$ for some $a_i \in \mathbb{F}_q$. The group G^y acts on $\phi[u]$ via $G(K_1/K)^y$. By Corollary 3.3.15, we have $G(K_1/K)^y = G(K_1/K)_1$ for $0 < y \leq R$. We are to show $\sigma(\lambda) = \lambda$ for any $\sigma \in G(K_1/K)_1$ if and only if $a_2 = \cdots = a_r = 0$.

As any element in $G(K_1/K)_1$ fixes $\xi_{1,1}$ (Theorem 3.3.16), the "if" part follows. We show the "only if" part. By Theorem 3.3.16, the map

$$g : G(K_1/K)_1 \to V^{r-1};\ \sigma \mapsto (\sigma(\xi_{2,1}) - \xi_{2,1}, \ldots, \sigma(\xi_{r,1}) - \xi_{r,1}).$$

is an isomorphism, where V denotes the A/u-vector space generated by $\xi_{1,1}$. For each $i = 2, \ldots, r$, let σ_i denote the preimage of the vector in V^{r-1} whose $(i-1)$-st component is $\xi_{1,1}$ and other components are 0. Assume $a_i \neq 0$ for some $i \geq 2$. Then there exist some $b_i \in \mathbb{F}_q$ for $i = 2, \ldots, r$ such that $\sum_{i=2}^r a_i b_i \neq 0$. Then the element $\prod_{i=2}^r \sigma_i^{b_i}$ does not fix λ and the claim follows. □

Note $G(K_1/K)^y = \{e\}$ for $y > R$. Similarly to Lemma-Definitions 4.2.1 and 4.2.2, by the above lemma, we have

Corollary 4.2.5. *Resume the notations in the above lemma. We have*

$$\int_0^\infty \left(r - \mathrm{rank}_{A/u}\phi[u]^{G^y}\right) dy = R.$$

Especially, the value of this integral is independent of the choice of u.

2.3. An analogue of Szpiro's conjecture. For a global function field F, let ϕ be a rank 2 Drinfeld A-module over F. Let M_F denote the set of all primes of F. Let $\deg(v)$ denote the degree of the residue field of F_v over $\mathbb{F}_q$. We define the *global conductor* of the Drinfeld module ϕ to be

$$\mathfrak{f}(\phi) := \sum_{v \in M_F} \deg(v) \cdot \mathfrak{f}_v(\phi).$$

We have the following statement by Lemma-Definition 4.2.1 and 4.2.2.

Theorem 4.2.6. *Put $v_0 := v(t)$. Let ϕ be a rank 2 Drinfeld A-module over F such that for each prime v of F, its j-invariant $\boldsymbol{j}$ satisfies*

$$\begin{cases} \text{either } \big(v(\boldsymbol{j}) < v_0 q \text{ and } p \nmid v(\boldsymbol{j})\big), & \text{if } v \text{ is infinite;} \\ \text{or } v(\boldsymbol{j}) \geq v_0 q & \\ \text{either } \big(v(\boldsymbol{j}) < 0 \text{ and } p \nmid v(\boldsymbol{j})\big), & \text{if } v \text{ is finite.} \\ \text{or } v(\boldsymbol{j}) \geq 0 & \end{cases}$$

Then

$$h_J(\phi) \leq \mathfrak{f}(\phi) \cdot \frac{q-1}{[F : \mathbb{F}_q(t)]} + q. \tag{56}$$

Proof. We know from Corollary 4.2.3 that

$$\mathfrak{f}(\phi) = \left(\sum_{v \in M_F^f} \deg(v) \cdot \max\left\{ \frac{-v(\boldsymbol{j})}{q-1}, 0 \right\} \right) + \left(\sum_{v \in M_F^\infty} \deg(v) \cdot \max\left\{ \frac{-v(\boldsymbol{j}) + v_0 q}{q-1}, 0 \right\} \right).$$

Let f_v and e_v denote respectively the absolute residue degree and the absolute ramification index of v. We have

$$\begin{aligned} &\frac{q-1}{[F : \mathbb{F}_q(t)]} \mathfrak{f}(\phi) - h_J(\phi) \\ =& \frac{1}{[F : \mathbb{F}_q(t)]} \sum_{v \in M_F^\infty} \deg(v) \left(\max\{-v(\boldsymbol{j}) + v_0 q, 0\} - \max\{-v(\boldsymbol{j}), 0\} \right) \\ \geq& \frac{1}{[F : \mathbb{F}_q(t)]} \sum_{v \in M_F^\infty} \deg(v) \cdot v_0 q = \frac{1}{[F : \mathbb{F}_q(t)]} \sum_{v \in M_F^\infty} f_v \cdot e_v \cdot (-\deg(t) q) = -q, \end{aligned}$$

where we use the extension formula in the last equality. This shows the theorem. □

Although the conditions in the theorem seem strict, it is not hard to find infinitely many rank 2 Drinfeld modules fulfilling these conditions.

Example 4.2.7. Consider for each $i \geq 2$ with $p \nmid i$ the Drinfeld $\mathbb{F}_q[t]$-module $\phi^{(i)}$ over $\mathbb{F}_q(t)$ defined by $\phi_t^{(i)}(X) := tX + t^i X^q + X^{q^2}$. It has good reduction at all finite primes. The j-invariant of $\phi^{(i)}$ is $\boldsymbol{j}^{(i)} = t^{i(q+1)}$. The J-height of $\phi^{(i)}$ equals to $i(q+1)$. At the only infinite prime v, we have the conductor $\mathfrak{f}_v(\phi^{(i)}) = \frac{i(q+1)-q}{q-1}$. The global conductor of $\phi^{(i)}$ equals to $\frac{i(q+1)-q}{q-1}$. This family is an example that the conductor can be arbitrarily large. Note that the equality in (56) holds.

Example 4.2.8. For the Drinfeld $\mathbb{F}_q[t]$-module $\phi^{(i)}$ over $\mathbb{F}_q(t)$ for each integer i with $p \nmid i$ in Example 3.6.4, we have its j-invariant

$$\boldsymbol{j}^{(i)} := t^{i(q^2-1)(q-1)} \cdot (t+1)^q.$$

We have

(1) For each infinite prime v of A, we have

$$\mathfrak{f}_v(\phi^{(i)}) = \begin{cases} 0 & i \leq 0; \\ i(q^2-1) & i > 0. \end{cases}$$

For each finite prime v of A, we have

$$\mathfrak{f}_v(\phi^{(i)}) = \begin{cases} -i(q^2-1) & v = t \text{ and } i < 0; \\ 0 & v \neq t \text{ or } i \geq 0. \end{cases}$$

Hence

$$\mathfrak{f}(\phi^{(i)}) = \begin{cases} -i(q^2-1) & i < 0; \\ i(q^2-1) & i \geq 0. \end{cases}$$

(2) We have

$$h_J(\phi^{(i)}) = \begin{cases} -i(q^2-1)(q-1) & i < 0; \\ i(q^2-1)(q-1) + q & i \geq 0. \end{cases}$$

The conductors and the J-height of $\phi^{(i)}$ for all i can be arbitrarily large. Note the *strict inequality* in (56) holds if $i < 0$.

Bibliography

[AH22] T. Asayama, M. Huang, Ramification of Tate modules for rank 2 Drinfeld modules, arXiv:2204.13275 (2022).

[Bir67] B. J. Birch, Cyclotomic fields and Kummer extensions, in Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965), 85–93, Academic Press, London, 1967.

[BPR21] F. Breuer, F. Pazuki, and M. H. Razafinjatovo, Heights and isogenies of Drinfeld modules, Acta Arith. 197 (2021), no. 2, 111–128.

[BK94] A. Brumer and K. Kramer, The conductor of an abelian variety, Compos. Math. 92 (1994), no. 2, 227–248.

[CL13] I. Chen, Y. Lee, Newton polygons, successive minima, and different bounds for Drinfeld modules of rank 2, Proc. Amer. Math. Soc. 141 (2013), no. 1, 83–91

[Dri74] V. G., Drinfel'd, Elliptic modules, Mat. Sb. (N.S.) 94(136) (1974), 594–627, 656.

[FV02] I. B. Fesenko and S. Vostokov, Local fields and their extensions, second ed., Translations of Mathematical Monographs, vol. 121, American Mathematical Society, Providence, RI, 2002.

[Gar02] F. Gardeyn, Une borne pour l'action de l'inertie sauvage sur la torsion d'un module de Drinfeld, Arch. Math. (Basel) 79 (2002), no. 4, 241–251.

[Gek19A] E.-U. Gekeler, Towers of GL(r)-type of modular curves, J. Reine Angew. Math. 754 (2019), 87–141.

[Gek19B] E.-U. Gekeler, On the field generated by the periods of a Drinfeld module, Arch. Math. (Basel) 113 (2019), no. 6, 581–591.

[Hua23] M. Huang, A note on successive minimal bases of Drinfeld modules, arXiv:2304.02854 (2023).

[KL04] M. Kölle and P. Schmid, Computing Galois groups by means of Newton polygons, Acta Arith. 115 (2004), no. 1, 71–84.

[LRS93] P. Lockhart, M. Rosen, and J. H. Silverman, An upper bound for the conductor of an abelian variety, J. Algebraic Geom. 2 (1993), no. 4, 569–601.

[Mau19] A. Maurischat, On field extensions given by periods of Drinfeld modules, Arch. Math. (Basel) 113 (2019), no. 3, 247–254.

[Mo21] S. Mochizuki, Inter-universal Teichmüller theory IV: Log-volume computations and set-theoretic foundations, Publ. Res. Inst. Math. Sci. 57 (2021), no. 1–2, 627–723.

[Neu99] J. Neukirch, Algebraic number theory, Grundlehren der mathematischen Wissenschaften, vol. 322, Springer-Verlag, Berlin, 1999.

[Pap23] M. Papikian, Drinfeld modules, Graduate Texts in Mathematics, vol. 296, Springer Cham, 2023.

[Se79] J.-P. Serre, Local fields, Graduate Texts in Mathematics, vol. 67, Springer-Verlag, New York-Berlin, 1979.

[ST68] J.-P. Serre; J. Tate, Good reduction of abelian varieties, Ann. of Math. (2) 88 (1968), 492–517.

[Sil86] J. H. Silverman, Heights and elliptic curves, in Arithmetic geometry (Storrs, Conn., 1984), Springer, New York, 1986, 253–265.

[Sil94] J. H. Silverman, Advanced topics in the arithmetic of elliptic curves, Graduate Texts in Mathematics, vol. 151, Springer-Verlag, New York, 1994.

[Szp90] L. Szpiro, Discriminant et conducteur des courbes elliptiques, in Séminaire sur les Pinceaux de Courbes Elliptiques (Paris, 1988), Astérisque, no. 183 (1990), 7–18.

[Tag92] Y. Taguchi, Semi-simplicity of the Galois representations attached to Drinfel'd modules over fields of "infinite characteristics", J. Number Theory 44 (1993), no. 3, 292–314.

[Wei18] F.-T. Wei, On Kronecker terms over global function fields, Invent. Math. 220 (2020), no. 3, 847–907.