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Mordell–Weil groups over large algebraic extensions of finitely generated fields

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Abstract

The Mordell–Weil group defined over a finitely generated field over its prime field has been studied for many years. The most celebrated theorem in this area is the Mordell–Weil theorem, which states that the Mordell–Weil group of any abelian variety over a finitely generated field over its prime field is finitely generated. A natural question to ask is what happens to the Mordell–Weil group if one replaces the base field with an infinitely generated field.

Another question comes from anabelian geometry, which is a subject in algebraic geometry aiming to describe how to reconstruct an algebraic variety or related geometric objects from its algebraic fundamental group. It has been thought that the base field of an algebraic variety treated in anabelian geometry should be a finitely generated field over its prime field. However, recent studies have shown that there are much various fields suitable for the base field of anabelian geometry. Kummer-faithful fields seem to be suitable for developing anabelian geometry. A perfect field K is said to be Kummer-faithful if the Mordell–Weil group of every semi-abelian variety over every finite extension of K has no nonzero divisible element. What kind of fields are Kummer-faithful?

Motivated by these questions, we study the Mordell–Weil group defined over a large algebraic extension of a finitely generated field over its prime field. There are three themes in this thesis.

First, we introduce the notion of Drinfeld-Kummer-faithful fields by the triviality of the divisible part of Drinfeld modules and investigate its properties. This notion amounts to a function field analogue of that of Kummer-faithful fields. We present a sufficient condition for a Galois extension of a function field to be Drinfeld-Kummer-faithful in terms of ramification theory. We also give some examples of Drinfeld-Kummer-faithful fields, one of which is constructed inspired by Ozeki–Taguchi's examples of highly Kummer-faithful fields.

Second, we discuss the structure of the Mordell–Weil groups of semi-abelian varieties over finite extensions of $\overline{K}(\sigma)$ and $\overline{K}[\sigma]$ in the case where the field K has characteristic zero. Here, for a field K and an e-tuple σ of elements in the absolute Galois group G_K of K, we write $\overline{K}(\sigma)$ for the fixed field of σ in the fixed algebraic closure $\overline{K}$ of K and $\overline{K}[\sigma]$ for the maximal Galois extension of K in $\overline{K}(\sigma)$. We prove that, if $e \geq 2$, then the Mordell–Weil group of any semi-abelian variety over any finite extension of $\overline{K}[\sigma]$ reduced modulo torsion is free for almost all σ in the sense of the Haar measure on G_K^e. Combining with known results, we deduce that the Mordell–Weil group of any semi-abelian variety over such a field is the direct sum of a finite torsion subgroup and a free $\mathbb{Z}$-module of denumerable rank. We also show that $\overline{K}(\sigma)$ and $\overline{K}[\sigma]$ are Kummer-faithful for almost all σ in G_K^e, where the former is shown under the assumption $e \geq 2$. This is an improvement of the result by Ohtani.

Third, we examine the torsion submodule of Drinfeld modules over a finite extension of $\overline{K}(\sigma)$ for a finitely generated function field K over a finite field and an e-tuple σ of elements in G_K. We prove two finiteness results for the torsion submodule of Drinfeld modules over such a field; the first one is that the $\mathfrak{p}$-power torsion submodule of $\phi(M)$ is finite for almost all σ, any Drinfeld module ϕ over any finite extension M of $\overline{K}(\sigma)$, and any nonzero prime ideal $\mathfrak{p}$ of the ring of regular elements of the base function field; the second one is that the torsion submodule of $\phi(M)$ is finite for almost all σ and any Drinfeld module ϕ over any finite extension M of $\overline{K}(\sigma)$ if $e \geq 2$. These are extensions of the work by the author during his master program and give complete analogues for Drinfeld modules of generic characteristic of the finiteness theorems for abelian varieties over $\overline{K}(\sigma)$ conjectured by Geyer–Jarden and proved by Jacobson–Jarden. As an application of these results, we show the freeness result for Drinfeld modules over finite extensions of $\overline{K}[\sigma]$ reduced modulo torsion. This is a Drinfeld module analogue of the result in the second theme in this thesis.

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Chapter 1

Introduction

This thesis contains three main topics and all of these are motivated by the desire to understand the structure of the Mordell–Weil group over a large algebraic extension of a finitely generated field over its prime field. The structure of the Mordell–Weil group (in this thesis, the *Mordell–Weil group* means the abelian group of K-rational points of a semi-abelian variety or a Drinfeld module defined over some field K) over a finitely generated field over its prime field has been studied for a long time and many results are known. The following theorem is the most fundamental result in this area.

Mordell–Weil theorem (see [Lan83, Chapter 6, Theorem 1])**.** *Let K be a finitely generated field over its prime field and A an abelian variety over K. Then the group $A(K)$ of K-rational points of A is finitely generated.*

In 1901, Poincaré [Poi01] defined the rank of an elliptic curve, i.e., an abelian variety of dimension one, and implicitly assumed that it is finite. Mordell [Mor22] proved that the group of rational points of any elliptic curve over the field $\mathbb{Q}$ of rational numbers is indeed finitely generated, which amounted to the above theorem in the case where $K = \mathbb{Q}$ and $\dim A = 1$. Weil [Wei29] generalized the theorem to A of arbitrary dimension over any number field and Néron [Nér52] in 1952 proved the general case.

One of our motivation for this thesis comes from the question what happens to $A(K)$ if K is infinitely generated.

Drinfeld modules, introduced by Drinfeld [Dri74] under the name of elliptic modules, are function field analogues for elliptic curves over number fields (the precise definition is given in Section 2.1). Drinfeld used them to prove some special cases of the Langlands conjecture for function fields. There is an analogous result of the Mordell–Weil theorem for Drinfeld modules, which was proved by Poonen [Poo95] for function fields of transcendence degree one and by Wang [Wan01] for the general case. This states that, if K is a finitely

generated function field and ϕ is a Drinfeld module over K, then the group $\phi(K)$ of K-rational points of ϕ is the direct sum of a finite torsion submodule and a free module of rank $\aleph_0$ (for the precise statement, see Theorem 2.1.3). In contrast to the case of abelian varieties, the group $\phi(K)$ already has infinite rank even if K is finitely generated. Therefore, if we want to consider the structure of $\phi(K)$ in the case where K is infinitely generated, then what we are interested in are for example the size of the torsion submodule, whether or not this group is free modulo torsion, and whether or not it has non-trivial divisible points.

Another motivation for this thesis is to understand what kind of fields can be the base fields for developing anabelian geometry. Anabelian geometry is a subject which aims to describe how to reconstruct an algebraic variety or related geometric objects from its algebraic fundamental group. Anabelian geometry is proposed by Grothendieck in 1980s, and he considered that the base field of an algebraic variety treated in anabelian geometry should be a finitely generated field over its prime field. However, recent studies have shown that anabelian geometry is developed over much various fields, such as sub-p-adic fields. The class of Kummer-faithful fields, introduced by Mochizuki [Moc15], seems to be suitable for anabelian geometry. A perfect field K is said to be *Kummer-faithful* if the Mordell–Weil group of every semi-abelian variety over every finite extension of K has no nonzero divisible element. This property is thought to be an important one in developing anabelian geometry. Our interest lies in determining how large the class of Kummer-faithful fields is.

Ozeki and Taguchi [OT22] introduced the notion of highly Kummer-faithful fields and studied its properties in terms of ramification theory. The notion of highly Kummer-faithful fields is a specialization of that of Kummer-faithful fields in the sense that high Kummer-faithfulness implies Kummer-faithfulness for a Galois extension of a Kummer-faithful field of characteristic zero [OT22, Proposition 2.8]. They also provided some examples of highly Kummer-faithful fields. One of their examples is constructed by adjoining to a number field K the torsion points of all semi-abelian varieties A over K of bounded dimension, having order of bounded prime power. The precise statement is the following.

Theorem A (Ozeki–Taguchi [OT22, Theorem 3.3])**.** *Let K be a number field, g a positive integer, and $\boldsymbol{m} = (m_p)_p$ a family of non-negative integers, where p runs over all rational prime numbers. Let $K_{g,\boldsymbol{m}}$ be the extension of K generated by all coordinates of elements of $A[p^{m_p}]$ for all semi-abelian varieties A over K of dimension at most g and all rational prime numbers p. Then $K_{g,\boldsymbol{m}}$ is highly Kummer-faithful. In particular, it is Kummer-faithful.*

As the first main topic in this thesis, we investigate a function field analogue of the notion of Kummer-faithful fields in Chapter 3. We introduce Drinfeld-Kummer-faithful (DKF) fields by the triviality of the divisible parts of Drinfeld modules. A sufficient condition for a Galois extension of a function field to be DKF is provided in terms of ramification theory. Some examples of DKF fields are also given. One of these is constructed using torsion points of Drinfeld modules in a similar way as Theorem A, except that we add another parameter to limit the badness of reduction. Note that the maximal ramification break of the t-torsion points of a Drinfeld $\mathbb{F}_q[t]$-module ϕ over $\mathbb{F}_q(t)$ at a finite prime not dividing (t) may be arbitrarily large even if ϕ is of rank two. We will treat such Drinfeld modules in Section 3.2.

The second main topic in this thesis is the Mordell–Weil groups over finite extensions of $\overline{K}(\sigma)$ and $\overline{K}[\sigma]$ for a finitely generated field K over the field $\mathbb{Q}$ of rational numbers and $\sigma \in G_K^e$. To recall the definitions of $\overline{K}(\sigma)$ and of $\overline{K}[\sigma]$, and summarize their known properties, let us explain the notation used in this context. Fix an algebraic closure $\overline{K}$ of any field K and let K^{sep} be the separable closure of K in $\overline{K}$. Let G_K be the absolute Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ of a field K and e a positive number. For any $\sigma \in G_K^e$, set $\overline{K}(\sigma)$ to be the fixed field of σ in $\overline{K}$. We also set $\overline{K}[\sigma]$ to be the maximal Galois extension of K in $\overline{K}(\sigma)$. We equip the compact group G_K^e with the normalized Haar measure, which allows G_K^e to be regarded as a probability space. For a given probability space Ω, the term *almost all* $\omega \in \Omega$ is used in the sense of "all $\omega \in \Omega$ outside some measure zero set".

Studying the properties of $\overline{K}(\sigma)$ goes back to Ax [Ax67], and the structure of the Mordell–Weil groups of semi-abelian varieties over such fields has been investigated for more than half a century. The following is the summary of what has been found.

Theorem B. *Let K be a finitely generated field over its prime field and e a positive integer.*

(1) (Geyer–Jarden [GJ06, Theorem 2.4] (resp. Frey–Jarden [FreJ74, Theorem 9.1])) *Assume that K is an infinite field. Then for almost all $\sigma \in G_K^e$ and any abelian variety A of positive dimension over $\overline{K}[\sigma]$ (resp. $\overline{K}(\sigma)$), the group $A(\overline{K}[\sigma])$ (resp. $A(\overline{K}(\sigma))$) has rank $\aleph_0$.*

(2) (Jarden [Jar75, Theorems 8.1 and 8.2]) *Let ζ_n denote a primitive n-th root of unity in $\overline{K}$ for any positive integer n.*

   (2-i) *For almost all $\sigma \in G_K$ and any positive integer d, there exist infinitely many prime numbers l such that $[\overline{K}(\sigma)(\zeta_l) : \overline{K}(\sigma)] = d$. In particular, for almost all $\sigma \in G_K$ and any finite extension M*

of $\overline{K}(\sigma)$, there exist infinitely many roots of unity contained in M and a prime number l such that $\zeta_l \notin M$.

(2-ii) *Assume $e \geq 2$. Then for almost all $\sigma \in G_K^e$ and any positive integer d, there exist only finitely many positive integers n not divisible by the characteristic of K such that $[\overline{K}(\sigma)(\zeta_n) : \overline{K}(\sigma)] \leq d$. In particular, for almost all $\sigma \in G_K^e$ and any finite extension M of $\overline{K}(\sigma)$, there are only finitely many roots of unity contained in M.*

(3) *Consider the following statements on K:*

(a) *For almost all $\sigma \in G_K$ and any abelian variety A of positive dimension over $\overline{K}(\sigma)$, the group $A(\overline{K}(\sigma))_{\mathrm{tor}}$ is infinite. Moreover, there exist infinitely many prime numbers l such that $A(\overline{K}(\sigma))[l] \neq 0$.*

(b) *Assume $e \geq 2$. For almost all $\sigma \in G_K^e$ and any abelian variety A over $\overline{K}(\sigma)$, the group $A(\overline{K}(\sigma))_{\mathrm{tor}}$ is finite.*

(c) *For almost all $\sigma \in G_K^e$, any abelian variety A over $\overline{K}(\sigma)$, and any prime number l, the group $A(\overline{K}(\sigma))[l^\infty] = \bigcup_{i=1}^{+\infty} A(\overline{K}(\sigma))[l^i]$ is finite.*

Then these statements hold in the following situations:

(3-i) (Geyer–Jarden [GJ78, Theorem 1.1]) *Replace "any abelian variety" in each statement with "any elliptic curve". Then Statements* (a)–(c) *hold for any K.*

(3-ii) (Jacobson–Jarden [JJ84, Proposition 4.2]) *Statements* (a)–(c) *hold if K is a finite field.*

(3-iii) (Jacobson–Jarden [JJ01, Main Theorem (b), (a)] ((b) and (c)), Zywina [Zyw16, Theorem 1.1] ((a) for the number field case), Jarden–Petersen [JP19, Theorem C] ((a) for the general case)) *Statements* (a)–(c) *hold if K has characteristic zero.*

(3-iv) (Jacobson–Jarden [JJ01, Main Theorem (a)]) *Statement* (c) *holds for any K.*

(4) (Jarden–Petersen [JP22, Theorem 1.3 (ii)]) *Assume that K has characteristic zero and $e \geq 2$. Then for almost all $\sigma \in G_K^e$, any finite extension M of $\overline{K}(\sigma)$, and any abelian variety A over M, it holds that $\bigcap_{n \geq 1} n \cdot A(M) = 0$, where n runs over all positive integers.*

Geyer and Jarden [GJ78] conjectured that Statements (a)–(c) of (3) in this theorem holds for any finitely generated field K over its prime field. We note that the paper of Jacobson and Jarden [JJ84] involves a proof of

Statement (a) for K with positive characteristic, but it contains an error as indicated in [JJ85]. Statements (a) and (b) for K which is infinite and has positive characteristic remain open.

In Chapter 4, we describe more detailed structures of the Mordell–Weil groups of semi-abelian varieties over finite extensions of $\overline{K}(\sigma)$ and $\overline{K}[\sigma]$ in the case of characteristic zero. Section 4.1 is devoted to proving the freeness of the Mordell–Weil groups modulo torsion in the situation of Theorem B (1). Combining with Theorem B, (2-ii) and (3-iii), we deduce that the Mordell–Weil group of any semi-abelian variety over such a field is the direct sum of a finite torsion subgroup and a free $\mathbb{Z}$-module of denumerable rank. Section 4.2 concerns the Kummer-faithfulness of $\overline{K}(\sigma)$ and of $\overline{K}[\sigma]$. Since the Kummer-faithfulness means that the Mordell–Weil group does not contain groups like $\mathbb{Q}$ or $\mathbb{Q}/\mathbb{Z}$ as subgroups, our results are thought to describe more detailed structures of the Mordell–Weil groups of semi-abelian varieties over such fields.

The last main topic in this thesis is the finiteness on the torsion submodule of Drinfeld modules over $\overline{K}(\sigma)$ for a finitely generated field K of an algebraic function field F in one variable over a finite field and $\sigma \in G_K^e$. The author studied a Drinfeld module analogue of Theorem B (3) during his master program and obtained a partial answer. Let A be the ring of functions in F regular outside a fixed prime ∞ of F.

Theorem C (Asayama [Asa21, Theorem 1.4])**.** *Let K be a finitely generated field over F and e a positive integer. Then for almost all $\sigma \in G_K^e$ and any Drinfeld A-module ϕ over $\overline{K}(\sigma)$ with $\mathrm{End}_{\overline{K}}\phi = A$, the following statements hold:*

(a) *Assume $e = 1$. The A-module $\phi(\overline{K}(\sigma))_{\mathrm{tor}}$ is infinite. Moreover, there exist infinitely many nonzero prime ideals $\mathfrak{p}$ of A such that $\phi(\overline{K}(\sigma))[\mathfrak{p}] \neq 0$.*

(b) *Assume $e \geq 2$. The A-module $\phi(\overline{K}(\sigma))_{\mathrm{tor}}$ is finite.*

(c) *For any nonzero prime ideal $\mathfrak{p}$ of A, the A-module $\phi(\overline{K}(\sigma))[\mathfrak{p}^\infty]$ is finite.*

Note that it is not needed to mention Part (a) in the later discussion, but we refer it only for consistency. This theorem relies on the (adelic) open image theorems for Drinfeld modules proved by Pink and by Pink and Rütsche (see Theorem 2.1.5, (1-i) and (2-i), respectively). The assumption $\mathrm{End}_{\overline{K}}\phi = A$ comes from them. One expects that this assumption can be dropped from the theorem. In addition, one also expects that the theorem is still valid if $\overline{K}(\sigma)$ is replaced with its finite extension.

In Chapter 5, we establish the desired generalizations (Theorems 5.1.4 and 5.1.3) of Parts (b) and (c) of Theorem C. The DKF-ness result for $\overline{K}[\sigma]$ is obtained immediately from the generalization of Part (c) (Corollary 5.1.5). The proof is done by using the generalizations (Theorem 2.1.5, (1-ii) and (2-ii)) of the theorems for the image of Galois representations arising from Drinfeld modules described above. After establishing these results, we prove the freeness of Drinfeld modules modulo torsion over a finite extension of $\overline{K}[\sigma]$ for $\sigma \in G_K^e$ with $e \geq 2$, which is a Drinfeld module analogue of the result in Section 4.1. We show the Drinfeld module analogue of the proposition ([Moo09, Proposition 7], see Proposition 4.1.2) proved by Moon, which is the key proposition in Section 4.1.

We conclude the introduction by mentioning that Chapter 3 of this thesis is based on the papers [Asa23] and [AH23].

Chapter 2

Preliminaries

This chapter is dedicated to introducing the basic notions which we will use in the later chapters. We use the following notation in any situation throughout this thesis. For any field E, fix an algebraic closure $\overline{E}$ of E and let E^{sep} be the separable closure of E in $\overline{E}$. Denote by G_E the absolute Galois group $\mathrm{Gal}(E^{\mathrm{sep}}/E)$ of E. We extend each element in G_E to $\overline{E}$ in the unique way.

2.1 Drinfeld modules

Let F be an algebraic function field in one variable over the finite field $\mathbb{F}_q$ of q elements, where q is a power of a prime number p. We fix a prime ∞ of F. Denote by A the ring of functions in F which are regular outside ∞. Let $\mathfrak{M}_A$ be the set of nonzero prime ideals of A. An *A-field* is a field K which is equipped with a homomorphism of $\mathbb{F}_q$-algebras $\iota : A \to K$. Throughout this thesis, we consider only A-fields with ι injective. Such an A-field is said to have *generic characteristic*. For an A-field K with generic characteristic we can identify A with the image of ι. Thus it allows us to consider K as an extension of F.

Let $K\{\tau\}$ be a twisted polynomial ring over K generated by the q-th power Frobenius morphism τ, with the relation $\tau a = a^q \tau$ for all $a \in K$. We have the correspondence

$$\begin{array}{ccc} K\{\tau\} & \leftrightarrow & \left\{ \sum_{i=0}^{n} a_i X^{q^i} \;\middle|\; n \geq 0, a_i \in K \right\} \\ \sum_i a_i \tau^i & \leftrightarrow & \sum_i a_i X^{q^i}, \end{array}$$

where the right hand side forms a non-commutative ring with multiplication given by composition. Let $\partial : K\{\tau\} \to K$ be the morphism which takes each polynomial in $K\{\tau\}$ to its constant term.

Definition 2.1.1. A *Drinfeld A-module* over K is a ring homomorphism of $\mathbb{F}_q$-algebras $\phi : A \to K\{\tau\}$ such that $\partial \circ \phi = \iota$ and there is $a \in A$ with $\phi_a \neq \iota(a)\tau^0$.

Let L be a field extension of an A-field K. Each Drinfeld A-module ϕ over K defines a structure of an A-module on L by

$$ax = \phi_a(x) \quad \text{for } a \in A \text{ and } x \in L.$$

Here $f(x) = \sum_i a_i x^{q^i}$ for $f = \sum_i a_i \tau^i \in K\{\tau\}$ and $x \in L$. We use the notation $\phi(L)$ rather than L when we regard L as an A-module induced by ϕ in the above way.

For any $a \in A$, define the *a-torsion submodule* $\phi(L)[a]$ of $\phi(L)$ by

$$\phi(L)[a] = \{u \in \phi(L) \mid \phi_a(u) = 0\}.$$

Let $\mathfrak{a}$ be a nonzero ideal of A. Then the *$\mathfrak{a}$-torsion submodule* $\phi(L)[\mathfrak{a}]$ of $\phi(L)$ is defined by $\phi(L)[\mathfrak{a}] = \bigcap_{a \in \mathfrak{a}} \phi(L)[a]$. We usually write $\phi[a]$ for $\phi(\overline{K})[a]$ and $\phi[\mathfrak{a}]$ for $\phi(\overline{K})[\mathfrak{a}]$, respectively. We immediately find that if $\mathfrak{a} = (a)$ is a principal ideal, then $\phi(L)[\mathfrak{a}] = \phi(L)[a]$. The *$\mathfrak{a}$-power torsion submodule* $\phi(L)[\mathfrak{a}^\infty]$ is defined to be $\phi(L)[\mathfrak{a}^\infty] = \bigcup_{n \geq 1} \phi(L)[\mathfrak{a}^n]$, where n ranges over all positive integers. Finally, the *torsion submodule* $\phi(L)_{\mathrm{tor}}$ is defined by $\phi(L)_{\mathrm{tor}} = \bigcup_{0 \neq a \in A} \phi(L)[a] = \bigcup_{0 \neq \mathfrak{a} \subseteq A} \phi(L)[\mathfrak{a}]$. Namely, $\phi(L)_{\mathrm{tor}}$ consists of all $u \in \phi(L)$ such that there is a nonzero $a \in A$ with $\phi_a(u) = 0$.

We can show that, for every Drinfeld A-module ϕ over K, there exists a positive integer r satisfying $\deg_\tau \phi_a = r \deg a$ for each $a \in A$. This integer r is said to be the *rank* of ϕ [Pap23, Definition A.6 and Corollary A.14 (1)].

Theorem 2.1.2 (see [Pap23, Theorem A.12 (1) and Corollary A.15])**.** *Let ϕ be a Drinfeld A-module of rank r over K and n a positive integer. Then, for any nonzero prime ideal $\mathfrak{p}$ of A, we have*

$$\phi[\mathfrak{p}^n] \cong (A/\mathfrak{p}^n)^r.$$

More generally, for any nonzero ideal $\mathfrak{a}$ of A, we have

$$\phi[\mathfrak{a}] \cong (A/\mathfrak{a})^r.$$

The next result is a Drinfeld module analogue of the Mordell–Weil theorem. As usual, the *rank* of an A-module X means the dimension of the F-vector space $X \otimes_A F$.

Theorem 2.1.3 (Poonen [Poo95, Theorem 1] (the finite extension case), Wang [Wan01, Theorem 1] (the general case))**.** *Let K be a finitely generated extension of F and ϕ a Drinfeld A-module over K. Then the group $\phi(K)$ is the direct sum of a finite torsion submodule $\phi(K)_{\mathrm{tor}}$ and a free A-module of rank $\aleph_0$.*

We recall the notions of a morphism of Drinfeld modules and the ring of endomorphisms of a Drinfeld module.

Definition 2.1.4. Let ϕ and ψ be two Drinfeld A-modules over an A-field K and L a field extension of K. A *morphism* from ϕ to ψ over L is a polynomial $f \in L\{\tau\}$ satisfying $f\phi_a = \psi_a f$ for all $a \in A$. A morphism from ϕ to itself is called an *endomorphism* of ϕ. Denote by $\mathrm{End}_L\phi$ the set of all endomorphisms of ϕ over L.

Obviously, $\mathrm{End}_L\phi$ forms a subring of $L\{\tau\}$. For any $a \in A$, the polynomial ϕ_a is clearly an endomorphism of ϕ. Therefore we view $\mathrm{End}_L\phi$ as always containing A. However, not all Drinfeld A-modules have A as their endomorphism rings.

At the end of this section, we recall the theorems on Galois representations associated to Drinfeld modules. Let ϕ be a Drinfeld A-module over an A-field K. For any $\mathfrak{p} \in \mathfrak{M}_A$, the $\mathfrak{p}$-*adic Tate module* $T_\mathfrak{p}(\phi)$ of ϕ is defined by

$$T_\mathfrak{p}(\phi) = \varprojlim \phi[\mathfrak{p}^n].$$

From Theorem 2.1.2, if ϕ has rank r, then $T_\mathfrak{p}(\phi)$ is a free $A_\mathfrak{p}$-module of rank r, where $A_\mathfrak{p}$ is the completion of A at $\mathfrak{p}$. The $\mathfrak{p}$-adic Tate module $T_\mathfrak{p}(\phi)$ induces a continuous Galois representation

$$\rho_\mathfrak{p} : G_K \to \mathrm{GL}_{A_\mathfrak{p}}(T_\mathfrak{p}(\phi)) \cong \mathrm{GL}_r(A_\mathfrak{p}).$$

Since the action of the endomorphism ring $\mathrm{End}_K\phi$ commutes with the representation $\rho_\mathfrak{p}$, the image $\rho_\mathfrak{p}(G_K)$ of G_K under $\rho_\mathfrak{p}$ is contained in the centralizer $\mathrm{Cent}_{\mathrm{GL}_r(A_\mathfrak{p})}(\mathrm{End}_K\phi)$. Let $\mathbb{A}_F^{\mathrm{f}}$ be the ring of finite adeles of F. Then $\rho_\mathfrak{p}$ induces the adelic representation

$$\rho_{\mathrm{ad}} : G_K \to \prod_{\mathfrak{p} \in \mathfrak{M}_A} \mathrm{GL}_r(A_\mathfrak{p}) \subseteq \mathrm{GL}_r(\mathbb{A}_F^{\mathrm{f}}).$$

Theorem 2.1.5. *Let K be a finitely generated field of F and ϕ a Drinfeld A-module of rank r over K.*

(1) (Pink [Pin97, Theorems 0.1 and 0.2])

(1-i) *Suppose* $\mathrm{End}_{\overline{K}}\phi = A$. *Then, for any finite subset Λ in $\mathfrak{M}_A$, the image of the homomorphism*

$$G_K \to \prod_{\mathfrak{p} \in \Lambda} \mathrm{GL}_r(A_\mathfrak{p})$$

induced by $\rho_\mathfrak{p}$ for $\mathfrak{p} \in \Lambda$ is open.

(1-ii) *Suppose* $\mathrm{End}_{\overline{K}}\phi = \mathrm{End}_K\phi$. *Then, for any finite subset* Λ *in* $\mathfrak{M}_A$, *the image of the homomorphism*

$$G_K \to \prod_{\mathfrak{p}\in\Lambda} \mathrm{Cent}_{\mathrm{GL}_r(A_\mathfrak{p})}(\mathrm{End}_K\phi)$$

induced by $\rho_\mathfrak{p}$ *for* $\mathfrak{p} \in \Lambda$ *is open.*

(2) (Pink–Rütsche [PR09, Theorems 0.1 and 0.2])

(2-i) *Suppose* $\mathrm{End}_{\overline{K}}\phi = A$. *Then the image of the adelic representation*

$$\rho_{\mathrm{ad}} : G_K \to \mathrm{GL}_r(\mathbb{A}_F^{\mathrm{f}})$$

is open.

(2-ii) *Suppose* $\mathrm{End}_{\overline{K}}\phi = \mathrm{End}_K\phi$. *Then the image of the adelic representation*

$$\rho_{\mathrm{ad}} : G_K \to \prod_{\mathfrak{p}\in\mathfrak{M}_A} \mathrm{Cent}_{\mathrm{GL}_r(A_\mathfrak{p})}(\mathrm{End}_K\phi)$$

is open.

We remark that, since $\mathrm{End}_K\phi$ is finitely generated as an A-module [Pap23, Theorem A.16], there exists a finite extension K' of K such that $\mathrm{End}_{\overline{K}}\phi = \mathrm{End}_{K'}\phi$.

2.2 Ramification

Keep the notation of the previous section. We introduce the notion of the finiteness of the maximal breaks according to Ozeki–Taguchi [OT22, Definition 2.14]. For any algebraic field extension E of F, let $\mathfrak{M}_E^{\mathrm{f}}$ be the set of primes of E not lying above ∞ and we call the element of $\mathfrak{M}_E^{\mathrm{f}}$ a *finite prime*. Let $\mathfrak{M}_E^{\infty}$ be the set of primes of E lying above ∞ and we call the element of $\mathfrak{M}_E^{\infty}$ an *infinite prime*. Recall that $\mathfrak{M}_A$ is the set of nonzero prime ideals of A and we identify it with $\mathfrak{M}_F^{\mathrm{f}}$. Since, in contrast to the number field case, the influence of the ramification at infinite primes cannot be essentially ignored in the function field case, the following definition takes into account such circumstances.

Definition 2.2.1. (1) Let K be an algebraic extension of a local field E of positive characteristic. Let $\widetilde{K}$ be the Galois closure of the maximal separable subextension of K/E. The *maximal ramification break*

(*maximal break* for short) of K/E is defined to be $\inf\{u \in [-1, +\infty) \mid \mathrm{Gal}(\widetilde{K}/E)^u = 1\}$. Here, for $u \geq -1$, we write $\mathrm{Gal}(\widetilde{K}/E)^u$ for the u-th upper ramification group of $\mathrm{Gal}(\widetilde{K}/E)$ in the sense of Serre [Ser79, Chapter IV, Section 3]. The extension K/E is said to have *finite maximal break* if the maximal break of K/E is finite.

(2) Let K be a Galois extension of a function field F. For $\mathfrak{p} \in \mathfrak{M}_A$, the extension K/F is said to have *finite maximal break at* $\mathfrak{p}$ if the extension $K_{\mathfrak{P}}/F_{\mathfrak{p}}$ has finite maximal break, where $\mathfrak{P} \in \mathfrak{M}_K^{\mathrm{f}}$ is above $\mathfrak{p}$. Notice that this is defined independently of the choice of $\mathfrak{P}$.

(3) Let K be a Galois extension of a function field F. The extension K/F is said to have *finite maximal break outside* ∞ if K/F has finite maximal break at every $\mathfrak{p} \in \mathfrak{M}_A$.

The following proposition is used to construct Drinfeld $\mathbb{F}_q[t]$-modules over $\mathbb{F}_q(t)$ whose t-torsion points induce arbitrarily large maximal break in Chapter 3. We note that this is a slight generalization of the function field case of [FV02, Chapter III, Proposition 2.5].

Proposition 2.2.2. *Let K be a complete discrete valuation field of characteristic $p > 0$. Assume that K contains $\mathbb{F}_q$, where $q = p^\nu$ for some positive integer ν. Let v_K denote the normalized valuation. Let α be an element in K and λ a root of the polynomial $X^q - X - \alpha$. If $v_K(\alpha)$ is negative and not divisible by p, then the extension $K(\lambda)/K$ is totally ramified of degree q with Galois group isomorphic to $(\mathbb{Z}/p\mathbb{Z})^\nu$. Its maximal break is given by $-v_K(\alpha)$.*

Proof. See [AH23, Proposition 3.2]. □

2.3 Kummer-faithful fields

Kummer-faithful fields are defined by the triviality of the divisible parts of the Mordell–Weil groups of semi-abelian varieties as follows.

Definition 2.3.1 ([Moc15, Definition 1.5])**.** Let K be a perfect field. We say that K is *Kummer-faithful* if, for every finite extension L of K and every semi-abelian variety A over L, it holds that $\bigcap_{n \geq 1} n \cdot A(L) = 0$, where n runs over all positive integers.

The following facts immediately follow from the definition of Kummer-faithfulness.

(1) Any perfect subfield of a Kummer-faithful field is also Kummer-faithful.

(2) Let K' be a finite extension of a perfect field K. Then K' is Kummer-faithful if and only if K is Kummer-faithful.

It is also known that any sub-p-adic field is Kummer-faithful [Moc15, Remark 1.5.4]. Note that a *sub-p-adic field* for some rational prime number p means a field isomorphic to a subfield of a finitely generated extension of the field $\mathbb{Q}_p$ of p-adic numbers. In particular, any number field is Kummer-faithful. On the other hand, any algebraically closed field is obviously not Kummer-faithful.

In order to define Drinfeld-Kummer-faithful fields in Chapter 3, we generalize the notion of the divisible part to a module over a general commutative ring.

Definition 2.3.2. Let R be a commutative ring with identity. Let M be an R-module. An element x in M is called *divisible* if, for every nonzero $a \in R$, there exists y in M such that $x = ay$. For an ideal $\mathfrak{a}$ of R, we say that $x \in M$ is $\mathfrak{a}$-*divisible* if, for every positive integer n, there exist $a \in \mathfrak{a}^n$ and $y \in M$ such that $x = ay$. Denote by M_{div} (resp. $M_{\mathfrak{a}\text{-div}}$) the set of divisible (resp. $\mathfrak{a}$-divisible) elements in M, i.e.,

$$M_{\mathrm{div}} = \bigcap_{a \in R \setminus \{0\}} aM \quad \left(\text{resp. } M_{\mathfrak{a}\text{-div}} = \bigcap_{n \geq 1} \mathfrak{a}^n M \right).$$

For simplicity, if $\mathfrak{a} = (a)$ is a principal ideal, we say a-divisible for (a)-divisible and write $M_{a\text{-div}}$ for $M_{(a)\text{-div}}$.

Via this definition, we can rewrite the condition $\bigcap_{n \geq 1} n \cdot A(L) = 0$ in Definition 2.3.1 as $A(L)_{\mathrm{div}} = 0$ as a $\mathbb{Z}$-module.

Proposition 2.3.3. *A perfect field K is Kummer-faithful if and only if $\mathbb{G}_{\mathrm{m}}(L)_{\mathrm{div}} = 0$ for any finite extension L of K and $A(K)_{\mathrm{div}} = 0$ for any abelian variety A over K.*

Proof. See [OT22, Proposition 2.3]. □

We can extend (3-iii) in Theorem B to finite extensions of $\overline{K}(\sigma)$. For the convenience of applying this theorem in Chapter 4, we describe the assertion of this theorem again.

Theorem 2.3.4. *Let K be a finitely generated field over $\mathbb{Q}$ and e a positive integer.*

(1) *Assume $e \geq 2$. For almost all $\sigma \in G_K^e$, any finite extension M of $\overline{K}(\sigma)$, and any abelian variety A over M, the group $A(M)_{\mathrm{tor}}$ is finite.*

(2) *For almost all* $\sigma \in G_K^e$*, any finite extension* M *of* $\overline{K}(\sigma)$*, any abelian variety* A *over* M*, and any prime number* l*, the group* $A(M)[l^\infty]$ *is finite.*

Proof. Let A be an abelian variety over a finite extension M of $\overline{K}(\sigma)$. Using Weil restriction for abelian varieties [JP22, Lemma 6.1], we know that $B = \mathrm{Res}_{M/\overline{K}(\sigma)}(A)$ is an abelian variety over $\overline{K}(\sigma)$ and $A(M) \cong B(\overline{K}(\sigma))$. Hence the theorem follows from (3-iii) in Theorem B. □

2.4 The Haar measure of a profinite group

Let G be a profinite group and $\mathcal{B}$ its Borel algebra. A function $\mu : \mathcal{B} \to \mathbb{R}$ is a (*normalized*) *Haar measure* if μ is a probability measure and satisfies the following extra conditions:

(1) (translation invariance) If $B \in \mathcal{B}$ and $g \in G$, then $\mu(gB) = \mu(Bg) = \mu(B)$.

(2) (regularity) For $B \in \mathcal{B}$ and $\varepsilon > 0$, there exist an open set U and a closed set C in G such that $C \subseteq B \subseteq U$ and $\mu(U \setminus C) < \varepsilon$.

It is known that a Haar measure is uniquely defined on every profinite group G (see [FriJ23, Proposition 21.2.1]). We write this measure as μ_G, but other symbols may be used depending on convention made by chapters. We use the same notation for its completion. If G is a closed subgroup of another profinite group G' and $\alpha \in G'$, then we regard the function $\mu_{G\alpha}(X) = \mu_G(X\alpha^{-1})$ as a probability measure on $G\alpha$.

The probability measure μ has monotonicity; if B and B' are measurable sets with $B \subseteq B'$, then $\mu(B) \leq \mu(B')$. For a countable collection $\{B_i\}_{i=1}^{+\infty}$ of measurable sets, we have $\mu(\bigcup_{i=1}^{+\infty} B_i) \leq \sum_{i=1}^{+\infty} \mu(B_i)$ (countable subadditivity). In particular, the countable union of measure zero sets is again a measure zero set. If H is a closed subgroup of finite index in a profinite group G, then $\mu_G(H) = 1/(G : H)$.

If G_1 and G_2 are profinite groups, then the direct product $G_1 \times G_2$ is also a profinite group. Thus we can equip $G_1 \times G_2$ with the Haar measure $\mu_{G_1 \times G_2}$. On the other hand, we can also consider the *product measure* $\mu_{G_1} \times \mu_{G_2}$ of $G_1 \times G_2$, which satisfies $(\mu_{G_1} \times \mu_{G_2})(B_1 \times B_2) = \mu_{G_1}(B_1)\mu_{G_2}(B_2)$ for all measurable subsets $B_1 \subseteq G_1$ and $B_2 \subseteq G_2$. It can be proved that $\mu_{G_1 \times G_2}$ and $\mu_{G_1} \times \mu_{G_2}$ coincide (after completion) [FriJ23, Proposition 21.4.2]. Of course, this result generalizes to the direct product $G_1 \times \cdots \times G_e$ of a finite number of profinite groups $G_1, \ldots, G_e$, particularly to G^e, the direct product of e copies of a profinite group G.

Some of our theorems are proved using the following well-known lemma in probability theory.

Lemma 2.4.1 (The first Borel–Cantelli lemma, see [FriJ23, Lemma 21.3.5 (a)]). *Let $\{B_i\}_{i=1}^{+\infty}$ be a countable collection of measurable subsets of a profinite group G. Define*

$$B = \bigcap_{n=1}^{+\infty} \bigcup_{i=n}^{+\infty} B_i = \{g \in G \mid g \in B_i \text{ for infinitely many } i\text{'s}\}.$$

If $\sum_{i=1}^{+\infty} \mu_G(B_i) < +\infty$, then $\mu_G(B) = 0$.

Our results that hold for almost all elements in a profinite group are obtained, not only by using this lemma, but also by computing the measure of the exceptional set and showing that it is zero. Therefore these proofs are non-constructive and do not give an explicit element for which our statements hold.

Chapter 3

Kummer-faithfulness for function fields

We use the notation defined in Sections 2.1 and 2.2; F is a global function field over the finite field $\mathbb{F}_q$ of q elements, ∞ is a fixed prime of F, and A is the subring of F consisting of functions regular outside ∞; $\mathfrak{M}_E^{\mathrm{f}}$ and $\mathfrak{M}_E^{\infty}$ respectively are the sets of finite and infinite primes of any algebraic extension E of F, and $\mathfrak{M}_A$ is the set of nonzero prime ideals of A (we identify it with $\mathfrak{M}_F^{\mathrm{f}}$).

3.1 Drinfeld-Kummer-faithful fields

In this section, we define a notion of Drinfeld-Kummer-faithful fields and study its properties. Unlike that of Kummer-faithfulness, we do not assume that the field is perfect. Recall that we only consider Drinfeld modules of generic characteristic.

Definition 3.1.1. An extension K over F is called *Drinfeld-Kummer-faithful* (we abbreviate it as *DKF*) if, for every finite extension L of K and every Drinfeld A-module ϕ over L, it holds that $\phi(L)_{\mathrm{div}} = 0$.

As in the case of Kummer-faithfulness, it immediately follows from the definition that any intermediate field of a DKF field K/F is also DKF. Let K' be a finite extension of K/F. Then K' is DKF if and only if K is DKF.

If K is a finitely generated field over F, then, for any Drinfeld A-module ϕ over K, we have $\phi(K)_{\mathrm{div}} = 0$ from Theorem 2.1.3. Therefore K is DKF. On the other hand, the algebraic closure $\overline{F}$ and the separable closure F^{sep} of F are obviously not DKF. The DKF-ness represents that the field is small in the sense that Drinfeld modules have no nonzero divisible points.

Let ϕ be a Drinfeld A-module and $\mathfrak{p}$ a nonzero prime ideal. Recall that the $\mathfrak{p}$-*adic Tate module* of ϕ is defined by $T_{\mathfrak{p}}(\phi) = \varprojlim_n \phi[\mathfrak{p}^n]$. The following proposition is an analogue of [OT22, Proposition 2.4].

Proposition 3.1.2. *Let $K \subseteq L$ be two extensions in $\overline{F}$ of F. Let ϕ be a Drinfeld A-module over K.*

(1) *For any nonzero prime ideal $\mathfrak{p}$ of A, the following conditions are equivalent*:

  (i) $(\phi(L)[\mathfrak{p}^\infty])_{\mathfrak{p}\text{-div}} = 0$.

  (ii) $\phi(L)[\mathfrak{p}^\infty]$ *is finite.*

  (iii) $T_{\mathfrak{p}}(\phi)^{G_L} = 0$.

(2) *Consider the following conditions on ϕ*:

  (a) $\phi(L)_{\mathrm{div}} = 0$.

  (b) $(\phi(L)_{\mathrm{tor}})_{\mathrm{div}} = 0$.

  (c) $\phi(L)[\mathfrak{p}^\infty]$ *is finite for any nonzero prime ideal $\mathfrak{p}$ in A.*

  Then we have (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c). *If K is DKF and L/K is Galois, then we have* (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c).

Proof. (1) The definition of Tate modules implies the equivalence of (ii) and (iii). Since the class number of A is finite (see [FriJ23, Lemma 5.1.2]), there exists $a \in A$ such that the principal ideal (a) is a $\mathfrak{p}$-power. Then $T_{\mathfrak{p}}(\phi) = \varprojlim_n \phi[\mathfrak{p}^n] \cong \varprojlim_m \phi[a^m]$ and this isomorphism is compatible with the action of G_L. The natural isomorphisms $(\varprojlim_m \phi[a^m])^{G_L} \cong \mathrm{Hom}_A(A[1/a]/A, \phi(L)[a^\infty])$ and $(\phi(L)[a^\infty])_{a\text{-div}} = (\phi(L)[\mathfrak{p}^\infty])_{\mathfrak{p}\text{-div}}$ yield the equivalence of (i) and (ii).

(2) It is obvious that (a) implies (b). To see that (b) implies (c), we take $a_{\mathfrak{p}} \in A$ such that the principal ideal $(a_{\mathfrak{p}})$ is a $\mathfrak{p}$-power as in (1) for each $\mathfrak{p}$. Then we have the natural isomorphisms

$$\begin{aligned}
\prod_{\mathfrak{p}} T_{\mathfrak{p}}(\phi)^{G_L} &\cong \prod_{\mathfrak{p}} \mathrm{Hom}_A(A[1/a_{\mathfrak{p}}]/A, \phi(L)[a_{\mathfrak{p}}^\infty]) \\
&\cong \prod_{\mathfrak{p}} \mathrm{Hom}_A(A[1/a_{\mathfrak{p}}]/A, \phi(L)_{\mathrm{tor}}) \\
&\cong \mathrm{Hom}_A\left(\bigoplus_{\mathfrak{p}} A[1/a_{\mathfrak{p}}]/A, \phi(L)_{\mathrm{tor}}\right) \\
&\cong \mathrm{Hom}_A(F/A, \phi(L)_{\mathrm{tor}}),
\end{aligned}$$

where $\mathfrak{p}$ runs over the nonzero prime ideals of A. By (b), we have $T_{\mathfrak{p}}(\phi)^{G_L} = 0$ for all $\mathfrak{p}$. Thus (c) holds from (1).

To show that (c) implies (b), we prove the contraposition. Suppose that there exists a nonzero $x \in (\phi(L)_{\mathrm{tor}})_{\mathrm{div}}$. Let $\mathfrak{a}$ be the annihilator of x and $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ the prime ideals appearing in the prime decomposition of $\mathfrak{a}$. Set $S = A \setminus (\bigcup \mathfrak{p}_i)$ and $B = S^{-1}A$. Then B is a principal ideal domain since it is a Dedekind domain with only finitely many prime ideals (see [Neu99, Chapter I, Section 3, Exercise 4]). Hence there is $b \in B$ with $S^{-1}\mathfrak{a} = (b)$. In addition, x is a nonzero divisible point in $S^{-1}\phi(L)_{\mathrm{tor}}$ and (b) is the annihilator of x in $S^{-1}\phi(L)_{\mathrm{tor}}$. Let $\mathfrak{P} = (\pi)$ be a prime ideal of B dividing b. By renumbering the indices, we may assume $A \cap \mathfrak{P} = \mathfrak{p}_1$. Then it follows that $\pi^{-1}bx$ is a nonzero point in $(S^{-1}\phi(L))[\mathfrak{P}^\infty]_{\mathfrak{P}\text{-div}}$. Take $s \in S$ satisfying $s\pi^{-1}b \in A$ and let $y = s\pi^{-1}bx$. Then y is a nonzero point in $\phi(L)[\mathfrak{p}_1^\infty]_{\mathfrak{p}_1\text{-div}}$. By (1), $\phi(L)[\mathfrak{p}_1^\infty]$ is infinite and the negation of (c) holds.

Now we prove that (b) implies (a) if K is DKF and L/K is Galois. We use again the proof by contraposition. Suppose that there is a nonzero $x \in \phi(L)_{\mathrm{div}}$. For a nonzero $a \in A$, we set $X_a = \{y \in \phi(L) \mid x = ay\}$. Then $(X_a)_a$ forms a projective system. Since each X_a is a nonempty finite set by assumption, the projective limit $\varprojlim_a X_a$ is also not empty. Let $(y_a)_a \in \varprojlim_a X_a$ and K' a finite subextension of L/K with $x \in \phi(K')$. Then K' is DKF since K is so. Thus there exists $a_0 \in A$ such that $y_{a_0} \notin \phi(K')$. Let σ_0 be an element in $G_{K'}$ satisfying $\sigma_0 y_{a_0} \neq y_{a_0}$ and put $z_a = \sigma_0 y_a - y_a$ for each nonzero $a \in A$. Then $z_{a_0} \neq 0$ and $az_{aa_0} = z_{a_0}$ for each a. Since L/K is Galois, we have $z_{a_0} \in \phi(L)[a_0]$. This shows that z_{a_0} is a nonzero point in $(\phi(L)_{\mathrm{tor}})_{\mathrm{div}}$, which establishes the negation of (b). □

In the rest of this section, we assume $F = \mathbb{F}_q(t)$, $\infty = (1/t)$, and $A = \mathbb{F}_q[t]$.

Proposition 3.1.3. *Let K be a finite extension of F, ϕ a Drinfeld A-module over K, and $\mathfrak{p}$ a nonzero prime ideal of A. For a Galois extension L/K with finite maximal break outside ∞, we have $T_{\mathfrak{p}}(\phi)^{G_L} = 0$.*

Proof. We first remark that LK'/K' has finite maximal break outside ∞ for any finite separable extension K' of K. We start the proof by finding suitable K'.

Assume $T = T_{\mathfrak{p}}(\phi)^{G_L} \neq 0$. The Galois group $G = \mathrm{Gal}(L/K)$ acts on T and the representation arising from this is unramified at finite primes of K not lying above $\mathfrak{p}$ at which ϕ has good reduction [Tak82, Theorem 1]. Hence there exists a finite subset $S \subseteq \mathfrak{M}_K^{\mathrm{f}}$ such that $T_{\mathfrak{p}}(\phi)$ is unramified outside $S \cup \mathfrak{M}_K^\infty$. Let $\chi : G \to A_{\mathfrak{p}}^\times$ be the character induced from the action of G on $\det T \cong A_{\mathfrak{p}}$ and $K^\dagger$ the intermediate field of L/K corresponding to $\ker \chi$. Then $K^\dagger/K$ is abelian. For any $\mathfrak{l} \in S$, by [FV02, Chapter IV, Corollary of Theorem 6.2], the

reciprocity map $K_{\mathfrak{l}}^{\times} \to \mathrm{Gal}(K_{\mathfrak{l}}^{\mathrm{ab}}/K_{\mathfrak{l}})$ sends the u-th higher group $U_{K_{\mathfrak{l}}}^{(u)}$ of units of $K_{\mathfrak{l}}$ isomorphically onto $\mathrm{Gal}(K_{\mathfrak{l}}^{\mathrm{ab}}/K_{\mathfrak{l}})^u$ for any integer $u \geq 0$. Composing with the natural projection $\mathrm{Gal}(K_{\mathfrak{l}}^{\mathrm{ab}}/K_{\mathfrak{l}}) \to \mathrm{Gal}(K_{\mathfrak{L}}^{\dagger}/K_{\mathfrak{l}})$, where $\mathfrak{L}$ denotes a prime of $K^{\dagger}$ lying over $\mathfrak{l}$, we have the surjection $U_{K_{\mathfrak{l}}}^{(u)} \to \mathrm{Gal}(K_{\mathfrak{L}}^{\dagger}/K_{\mathfrak{l}})^u$. Then we obtain the surjection

$$U_{K_{\mathfrak{l}}}^{(0)}/U_{K_{\mathfrak{l}}}^{(u)} \to \mathrm{Gal}(K_{\mathfrak{L}}^{\dagger}/K_{\mathfrak{l}})^0/\mathrm{Gal}(K_{\mathfrak{L}}^{\dagger}/K_{\mathfrak{l}})^u$$

for $u \geq 0$. By assumption, $\mathrm{Gal}(K_{\mathfrak{L}}^{\dagger}/K_{\mathfrak{l}})^u$ is trivial for sufficiently large u. Since $U_{K_{\mathfrak{l}}}^{(0)}/U_{K_{\mathfrak{l}}}^{(u)}$ is finite, we know that $\mathrm{Gal}(K_{\mathfrak{L}}^{\dagger}/K_{\mathfrak{l}})^0$ is finite. Hence there exists a finite extension $K(\mathfrak{l})$ over K in $K^{\dagger}$ such that the restriction of χ to $\mathrm{Gal}(L/K(\mathfrak{l}))$ is unramified at all primes of $K(\mathfrak{l})$ lying over $\mathfrak{l}$.

For any $\mathfrak{l} \in \mathfrak{M}_K^{\infty}$, let Λ be the lattice in $K_{\mathfrak{l}}^{\mathrm{sep}}$ associated to ϕ. Then there exists a finite separable extension E of $K_{\mathfrak{l}}$ with $\Lambda \subseteq E$. The action of G_K on $T_{\mathfrak{p}}(\phi)$ induces the action of $G_{K_{\mathfrak{l}}}$ on $\Lambda \otimes_A A_{\mathfrak{p}}$, under which G_E acts trivially. We set $K(\mathfrak{l})$ to be a finite separable extension of K whose completion at some suitable prime lying over $\mathfrak{l}$ coincides with E. Then the representation $\mathrm{Gal}(L \cdot K(\mathfrak{l})/K(\mathfrak{l})) \to T$ is unramified at all primes of $K(\mathfrak{l})$ lying over $\mathfrak{l}$. We replace K with the compositum of $K(\mathfrak{l})$ for all $\mathfrak{l} \in S \cup \mathfrak{M}_K^{\infty}$ and may assume that χ is unramified at every prime of K.

Take $\mathfrak{l} \in \mathfrak{M}_K^{\mathrm{f}}$ not lying above S. Then the eigenvalue $\alpha_{\mathfrak{l}}$ of the action of the Frobenius element on $T_{\mathfrak{p}}(\phi)$ satisfies $|\alpha_{\mathfrak{l}}|_{\infty} = q_{\mathfrak{l}}^{1/r_{\phi}}$, where $q_{\mathfrak{l}}$ is the cardinality of the residue field at $\mathfrak{l}$ and r_{ϕ} is the rank of ϕ [Tak82, Proposition 3]. In particular, $|\alpha_{\mathfrak{l}}|_{\infty} > 1$. Thus the image $\chi(G)$ of G under the above χ is infinite. Let M be the extension of K satisfying $\chi(G) = \mathrm{Gal}(M/K)$. Then M is an infinite abelian extension over K unramified at every prime. Since the constant extension in $K(T_{\mathfrak{p}}(\phi))/K$ is finite [Gek19, Remark 4.2], there exists a finite constant extension K''/K in M such that M/K'' is geometric and infinite abelian extension unramified at every prime. However, this is impossible since the maximal geometric extension unramified at every prime must be finite (see [AT09, Chapter VIII, Section 3]). □

Theorem 3.1.4. *A Galois extension L/K with finite maximal break outside ∞ is DKF.*

Proof. Let L' be a finite Galois extension of L and ϕ a Drinfeld A-module over L'. There exists a finite subextension K' of L'/K such that L'/K' is Galois and that ϕ is defined over K'. From the assumption, L'/K' has finite maximal break outside ∞. By Proposition 3.1.3, for any nonzero prime ideal $\mathfrak{p}$ in A, we have $T_{\mathfrak{p}}(\phi)^{G_{L'}} = 0$. Hence $\phi(L')[\mathfrak{p}^{\infty}]$ is finite by Proposition 3.1.2 (1). Now K' is DKF as K is so. Then $\phi(L')_{\mathrm{div}} = 0$ by (2) of the same proposition. Therefore L is DKF. □

Corollary 3.1.5. *Let L be as in Theorem* 3.1.4. *Then $L\overline{\mathbb{F}_q}$ is DKF.*

Proof. We see that $L\overline{\mathbb{F}_q}/K$ has finite maximal break outside ∞ and apply Theorem 3.1.4. □

Corollary 3.1.6. *If a Galois extension L/K is tamely ramified, then L is DKF. In particular, the maximal tamely ramified extension of K is DKF.*

Proof. It immediately follows from Theorem 3.1.4. □

3.2 Drinfeld modules with torsion points yielding large maximal break

This section is devoted to observing the ramification arising from the torsion points of Drinfeld modules. All materials of this section are included in the paper by the author and Huang [AH23]. However, for the consistency of the present thesis, we reprove the existence of a family of Drinfeld modules of rank two having t-torsion points with arbitrarily large maximal break.

Assume $F = \mathbb{F}_q(t)$, $\infty = (1/t)$, $A = \mathbb{F}_q[t]$. Let K be a finite extension of F. Let $\mathfrak{p}$ be a finite prime of K not dividing the prime (t) and $v_\mathfrak{p}$ the valuation associated to $\mathfrak{p}$ normalized so that $v_\mathfrak{p}(K^\times) = \mathbb{Z}$. We uniquely extend $v_\mathfrak{p}$ to the separable closure $K_\mathfrak{p}^{\mathrm{sep}}$ of the completion $K_\mathfrak{p}$ of K at $\mathfrak{p}$.

Let ϕ be a Drinfeld A-module over $K_\mathfrak{p}$ of rank two defined by $\phi_t(X) = tX + a_1X^q + a_2X^{q^2}$, where $a_1, a_2 \in K_\mathfrak{p}$. Let $\boldsymbol{j} = a_1^{q+1}/a_2$ be the j-invariant of ϕ. Since we assume that $\mathfrak{p}$ does not divide (t), we have $v_\mathfrak{p}(t) = 0$. Put $s_1 = v_\mathfrak{p}(a_1)$ and $s_2 = v_\mathfrak{p}(a_2)$. Then $v_\mathfrak{p}(\boldsymbol{j}) = (q+1)s_1 - s_2$. To accomplish the goal of this section, we assume $v_\mathfrak{p}(\boldsymbol{j}) < 0$.

We determine the Newton polygon of $\phi_t(X)$ at $\mathfrak{p}$ under this assumption. We find that the slope of the segment connecting the two points $(1, 0)$ and (q, s_1) and the slope of the segment connecting the two points (q, s_1) and (q^2, s_2) are respectively $s_1/(q-1)$ and $(s_2 - s_1)/(q^2 - q)$. Since $v_\mathfrak{p}(\boldsymbol{j}) < 0$, we have

$$\frac{s_1}{q-1} < \frac{s_2 - s_1}{q^2 - q}.$$

Hence the Newton polygon of $\phi_t(X)$ at $\mathfrak{p}$ has exactly two segments. Let ξ_1 and ξ_2 be two roots of $\phi_t(X)$ with valuations

$$v_\mathfrak{p}(\xi_1) = -\frac{s_1}{q-1},\ v_\mathfrak{p}(\xi_2) = -\frac{s_2 - s_1}{q^2 - q},$$

respectively.

Let b be an element in $K_{\mathfrak{p}}(\xi_1)$ with valuation $s_1/(q-1)$, e.g., $b = 1/\xi_1$. Put $b' = b^q/a_1$ and $\Phi(X) = b'\phi_t(X/b)$. Let b_0 and b_2 be respectively the coefficient of X and X^{q^2} in $\Phi(X)$. Namely,

$$\Phi(X) = b_0X + X^q + b_2X^{q^2} = \frac{b^{q-1}t}{a_1}X + X^q + \frac{a_2}{b^{q^2-q}a_1}X^{q^2}.$$

Note that the coefficient of X^q in $\Phi(X)$ is 1. We have $v_{\mathfrak{p}}(b_0) = 0$ and $v_{\mathfrak{p}}(b_2) = -v_{\mathfrak{p}}(\boldsymbol{j}) > 0$.

Let $\eta(X) = tX + a_1X^q$. In the same manner as $\Phi(X)$, we define $\mathrm{H}(X) = b'\eta(X/b)$, i.e.,

$$\mathrm{H}(X) = b_0X + X^q.$$

Then the splitting field of $\mathrm{H}(X)$ over $K_{\mathfrak{p}}(\xi_1)$ coincides with that of $\eta(X)$ and any nonzero root of $\mathrm{H}(X)$ has valuation zero. Moreover, this field is a Kummer extension and any nonzero root of $\eta(X)$ can be a generator of this extension.

Lemma 3.2.1 ([AH23, Lemma 3.10 (1)])**.** *The field $K_{\mathfrak{p}}(\xi_1)$ coincides with the splitting field of $\eta(X) = tX + a_1X^q$ over $K_{\mathfrak{p}}$.*

Proof. Let $d = b\xi_1$. Then $v_{\mathfrak{p}}(d) = 0$ and $\Phi(d) = 0$. Let $x_1, \dots, x_{q-1}$ denote all nonzero roots of $\mathrm{H}(X)$. If $i \neq j$, then $x_i - x_j$ has valuation zero since it is again a nonzero root of $\mathrm{H}(X)$ and $v_{\mathfrak{p}}(b_0) = 0$. Hence, if we show that there exists i such that $v_{\mathfrak{p}}(x_i - d) > 0$, then Krasner's lemma [Pap23, Proposition 2.3.9] tells us that $x_i \in K_{\mathfrak{p}}(\xi_1)$, which is what we want to prove. Since $\Phi(d) = 0$, we have

$$\mathrm{H}(d) = \prod_{i=1}^{q-1}(d - x_i) = b_0d + d^q = -b_2d^{q^2}.$$

Hence

$$\sum_{i=1}^{q-1} v_{\mathfrak{p}}(x_i - d) = v_{\mathfrak{p}}\left(\prod_{i=1}^{q-1}(d - x_i)\right) = v_{\mathfrak{p}}(-b_2d^{q^2}) > 0.$$

This implies the existence of i such that $v_{\mathfrak{p}}(x_i - d) > 0$, as desired. □

Lemma 3.2.2 ([AH23, Lemma 3.10 (3)])**.** *Let $L_1 = K_{\mathfrak{p}}(\phi[t])$ and let M_1 be the splitting field of the degree q polynomial*

$$\widetilde{\mathrm{H}}(X) = X^q - \beta X - \beta \in K_{\mathfrak{p}}(\xi_1)[X]$$

with $\beta = \xi_1^{q^2-1}a_2/t$ over $K_{\mathfrak{p}}(\xi_1)$. Then the field M_1 is contained in L_1. The extension L_1/M_1 is a compositum of Kummer extensions.

Proof. In the proof of this lemma, we take $b = 1/\xi_1$. Then any element in $\mathbb{F}_q$ is a root of $\Phi(X)$ and the polynomial $X^q - X = \prod_{u\in\mathbb{F}_q}(X-u)$ divides $\Phi(X)$. Let $\Theta(X)$ be the polynomial such that $\Theta(X)(X^q - X) = \Phi(X)$. Noticing that $-b_0 - b_2 = 1$ since $\Phi(1) = 0$, and that $b_0/b_2 = 1/\beta$, we have

$$\Theta(X) = b_2\left(\sum_{i=1}^{q} X^{i(q-1)} - \frac{1}{\beta}\right),$$

whose roots generate L_1 over $K_{\mathfrak{p}}(\xi_1)$.

Consider the polynomial $\overline{\Theta}(X) = \sum_{i=1}^{q} X^i - 1/\beta$ and its splitting field M_1' over $K_{\mathfrak{p}}(\xi_1)$. Then any root of $\overline{\Theta}(X)$ is a $(q-1)$-st power of a root of $\Theta(X)$. This implies that the field M_1' is contained in L_1 and that the extension L_1/M_1' is a compositum of Kummer extensions. Hence it suffices to show that $M_1 = M_1'$. We have

$$\sum_{i=1}^{q} X^i = \frac{X(X^q-1)}{X-1} = \frac{X(X-1)^q}{X-1} = X(X-1)^{q-1}$$

and $\overline{\Theta}(X+1) = X^q + X^{q-1} - 1/\beta$. Then

$$\widetilde{\mathrm{H}}(X) = -\beta X^q \overline{\Theta}\left(\frac{1}{X}+1\right).$$

This shows that the splitting fields of $\widetilde{\mathrm{H}}(X)$ and of $\overline{\Theta}(X)$ are the same, which completes the proof. □

In the rest of this section, we assume $p \nmid v_{\mathfrak{p}}(\boldsymbol{j})$. The extension $K_{\mathfrak{p}}(\xi_1)/K_{\mathfrak{p}}$ is tamely ramified since it is generated by the $(q-1)$-st root of $-t/a_1$ by Lemma 3.2.1. The extension L_1/M_1 is also tamely ramified by Lemma 3.2.2. Let N_1 be the splitting field of the polynomial $X^q - \beta X$ over $K_{\mathfrak{p}}(\xi_1)$. It is tamely ramified and is a subextension of $M_1/K_{\mathfrak{p}}(\xi_1)$ since any root of $X^q - \beta X$ is the difference of some two roots of $\widetilde{\mathrm{H}}(X)$.

Let ω be a nonzero root of $X^q - \beta X$. We have $v_{\mathfrak{p}}(\omega) = -v_{\mathfrak{p}}(\boldsymbol{j})/(q-1)$. Consider the polynomial $\widehat{\mathrm{H}}(X) = \omega^{-q}\widetilde{\mathrm{H}}(\omega X) = X^q - X - \omega^{-1}$. Then any root of $\widehat{\mathrm{H}}(X)$ generates M_1 over N_1. We know that $-v_{N_1}(\omega^{-1}) = e_{N_1/K_{\mathfrak{p}}} v_{\mathfrak{p}}(\omega)$ is a positive integer not divisible by p, where $e_{N_1/K_{\mathfrak{p}}}$ denotes the ramification index of $N_1/K_{\mathfrak{p}}$. Applying Proposition 2.2.2 to $\widehat{\mathrm{H}}(X)$, the extension M_1/N_1 is a degree q totally ramified Galois extension with maximal break $v_{N_1}(\omega)$. As the extensions L_1/M_1 and $N_1/K_{\mathfrak{p}}$ are tamely ramified, the maximal break of $L_1/K_{\mathfrak{p}}$ is $v_{N_1}(\omega)/e_{N_1/K_{\mathfrak{p}}} = v_{\mathfrak{p}}(\omega) = -v_{\mathfrak{p}}(\boldsymbol{j})/(q-1)$.

This result implies that, if $v_{\mathfrak{p}}(\boldsymbol{j})$ is sufficiently negative and not divisible by p, then we obtain a Drinfeld module ϕ of rank two with the extension

$K_{\mathfrak{p}}(\phi[t])/K_{\mathfrak{p}}$ having arbitrarily large maximal break. For example, the case $\mathfrak{p} = (t+1)$ and the Drinfeld A-module ϕ over F defined by $\phi_t(X) = tX + X^q + (t+1)^i X^{q^2}$ for a positive integer i not divisible by p satisfies this assumption.

In conclusion, we have shown the following result.

Proposition 3.2.3. *Let K be a finite extension of F and $\mathfrak{p}$ a finite prime of K not dividing (t). Let C be a real number. Then there exists a Drinfeld A-module ϕ over K of rank two such that the maximal break of $K_{\mathfrak{p}}(\phi[t])/K_{\mathfrak{p}}$ is larger than C.*

3.3 Construction of Drinfeld-Kummer-faithful fields from Drinfeld modules

Recall that $\mathfrak{M}_E^{\mathrm{f}}$ denotes the set of primes of E not lying above ∞ for any finite field extension E over F. For $\mathfrak{l} \in \mathfrak{M}_E^{\mathrm{f}}$, denote by $v_{\mathfrak{l}}$ the valuation corresponding to $\mathfrak{l}$ normalized so that $v_{\mathfrak{l}}(E^{\times}) = \mathbb{Z}$. We uniquely extend $v_{\mathfrak{l}}$ to $E_{\mathfrak{l}}^{\mathrm{sep}}$. We use the notation v_L for the valuation corresponding to a finite extension L of $E_{\mathfrak{l}}$ normalized so that $v_L(L^{\times}) = \mathbb{Z}$. Denote by $\mathcal{O}_{\mathfrak{l}}$ the valuation ring corresponding to $E_{\mathfrak{l}}$. To simplify the notation, for a principal ideal $\mathfrak{a}$ of A, we will write again $\mathfrak{a}$ for its generator when the choice of the generator poses no problem.

The aim of this section is to give some examples of DKF fields which are infinitely generated over F. We will show that the field obtained by adjoining the torsion points, of bounded order, of Drinfeld modules of bounded rank with "not too bad" reduction, by referring to the strategy in Section 5 of the paper of Rosen [Ros03]. Because we are only interested in the finiteness of the maximal breaks, our estimate in this section will seem far from being optimal. For more accurate estimates, see [Tag92] or [CL13].

Proposition 3.3.1. *Let K be a finite extension of F. Let $\mathfrak{p} \in \mathfrak{M}_A$ and $\mathfrak{l} \in \mathfrak{M}_K^{\mathrm{f}}$. Let m, r, and N be non-negative integers. Assume that $\mathfrak{p}^m$ is a principal ideal. Let $a(\mathfrak{p}^m)$ be the highest coefficient of $\phi_{\mathfrak{p}^m}(X)$ for a Drinfeld A-module ϕ over $K_{\mathfrak{l}}$. Then there is a constant C depending only on $\mathfrak{p}$, $\mathfrak{l}$, m, r, and N such that, for any Drinfeld A-module ϕ over $\mathcal{O}_{\mathfrak{l}}$ of rank at most r with $v_{\mathfrak{l}}(a(\mathfrak{p}^m)) \leq N$, the maximal break of $K_{\mathfrak{l}}(\phi[\mathfrak{p}^m])/K_{\mathfrak{l}}$ is at most C.*

Proof. Looking at the Newton polygon of $\phi_{\mathfrak{p}^m}(X)$ at $\mathfrak{l}$, for a nonzero $x \in \phi[\mathfrak{p}^m]$, we have

$$v_{\mathfrak{l}}(x) \leq -\frac{0 - v_{\mathfrak{l}}(\mathfrak{p}^m)}{q-1} = \frac{v_{\mathfrak{l}}(\mathfrak{p}^m)}{q-1}$$

and

$$v_{\mathfrak{l}}(x) \geq -\frac{v_{\mathfrak{l}}(a(\mathfrak{p}^m)) - 0}{q^{r_\phi \deg(\mathfrak{p}^m)} - q^{r_\phi \deg(\mathfrak{p}^m)-1}} = \frac{-v_{\mathfrak{l}}(a(\mathfrak{p}^m))}{q^{r_\phi \deg(\mathfrak{p}^m)-1}(q-1)}.$$

Here r_ϕ denotes the rank of ϕ. Suppose that $G^u_{K_{\mathfrak{l}}} \neq 1$. For $\sigma \neq 1$ in $G^u_{K_{\mathfrak{l}}}$, there exists x in $\phi[\mathfrak{p}^m]$ such that $\sigma x \neq x$. Take $b \in K_{\mathfrak{l}}$ with $-v_{\mathfrak{l}}(x) \leq v_{\mathfrak{l}}(b) < -v_{\mathfrak{l}}(x) + 1$. Then $v_{\mathfrak{l}}(bx) \geq 0$ and

$$\begin{aligned} v_L(\sigma(bx) - bx) &= e_{L/K_{\mathfrak{l}}} v_{\mathfrak{l}}(\sigma(bx) - bx) \\ &= e_{L/K_{\mathfrak{l}}}(v_{\mathfrak{l}}(\sigma x - x) + v_{\mathfrak{l}}(b)) \\ &\leq e_{L/K_{\mathfrak{l}}} \left(\frac{v_{\mathfrak{l}}(\mathfrak{p}^m)}{q-1} + \left(-\frac{-v_{\mathfrak{l}}(a(\mathfrak{p}^m))}{q^{r_\phi \deg(\mathfrak{p}^m)-1}(q-1)} \right) + 1 \right). \end{aligned}$$

Here $L = K_{\mathfrak{l}}(\phi[\mathfrak{p}^m])$ and $e_{L/K_{\mathfrak{l}}}$ denotes the ramification index of $L/K_{\mathfrak{l}}$. Thus

$$u \leq e_{L/K_{\mathfrak{l}}} \left(\frac{v_{\mathfrak{l}}(\mathfrak{p}^m)}{q-1} + \frac{v_{\mathfrak{l}}(a(\mathfrak{p}^m))}{q^{r_\phi \deg(\mathfrak{p}^m)-1}(q-1)} + 1 \right) - 1.$$

Since $e_{L/K_{\mathfrak{l}}}$ is bounded in terms of $\mathfrak{p}$, m, and r (An obvious bound is $(q^{r \deg(\mathfrak{p}^m)})!$ because $L/K_{\mathfrak{l}}$ is the splitting field of the polynomial $\phi_{\mathfrak{p}^m}(X)$, whose degree is at most $q^{r \deg(\mathfrak{p}^m)}$), we obtain a desired upper bound of u. □

From now on, we assume $F = \mathbb{F}_q(t)$, $\infty = (1/t)$, and $A = \mathbb{F}_q[t]$. For a Drinfeld A-module ϕ over K with stable reduction, let (ψ, Γ) be the Tate uniformization of ϕ at $\mathfrak{l} \in \mathfrak{M}^{\mathrm{f}}_K$. Here ψ is a Drinfeld A-module over $\mathcal{O}_{\mathfrak{l}}$ with good reduction and Γ is a ψ-lattice, which means a finitely generated discrete projective A-submodule of $\psi(K^{\mathrm{sep}}_{\mathfrak{l}})$ stable under the action of $G_{K_{\mathfrak{l}}}$. Such a pair is called a *Tate datum*. The power series

$$e_\Gamma(X) = X \prod_{\gamma \in \Gamma \setminus \{0\}} \left(1 - \frac{X}{\gamma}\right)$$

defines an $\mathbb{F}_q$-linear entire function on $\psi(K^{\mathrm{sep}}_{\mathfrak{l}})$ and satisfies $e_\Gamma \psi_a = \phi_a e_\Gamma$ for all $a \in A$ (we can see e_Γ as an element in $\mathcal{O}_{\mathfrak{l}}\{\{\tau\}\}$, the non-commutative ring of formal power series in τ with coefficients in $\mathcal{O}_{\mathfrak{l}}$, whose multiplication is determined by $\tau x = x^q \tau$ for $x \in \mathcal{O}_{\mathfrak{l}}$). For positive integers r_ψ and r_Γ, there is a natural one-to-one correspondence (see [Dri74, Section 7])

$$\left\{ \begin{array}{l} \text{Drinfeld } A\text{-modules } \phi \text{ of rank } r_\psi + r_\Gamma \text{ over} \\ K_{\mathfrak{l}} \text{ with stable reduction of rank } r_\psi \end{array} \right\} \Big/ (K_{\mathfrak{l}}\text{-isom.})$$

$$\overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{l} \text{Tate data } (\psi, \Gamma), \text{ where } \psi \text{ is a Drinfeld} \\ A\text{-module over } \mathcal{O}_{\mathfrak{l}} \text{ with good reduction of} \\ \text{rank } r_\psi \text{ and } \Gamma \text{ is a } \psi\text{-lattice of rank } r_\Gamma \end{array} \right\} \Big/ (K_{\mathfrak{l}}\text{-isom.}).$$

Note that two Tate data (ψ, Γ) and (ψ', Γ') are $K_\mathfrak{l}$-isomorphic if there exists an isomorphism from ψ to ψ' over $K_\mathfrak{l}$ which induces an isomorphism from Γ to Γ'.

We introduce the notion of the size of a Tate datum (ψ, Γ). We notice that, as we have assumed $A = \mathbb{F}_q[t]$, any ψ-lattice Γ is a finitely generated free A-submodule in $\psi(K_\mathfrak{l}^{\mathrm{sep}})$. Since Γ is discrete and ψ has good reduction, we have $v_\mathfrak{l}(\gamma) < 0$ and $v_\mathfrak{l}(\psi_a(\gamma)) = q^{r_\psi \deg(a)} v_\mathfrak{l}(\gamma)$ for $\gamma \in \Gamma \setminus \{0\}$ and $a \in A \setminus \{0\}$. According to Gardeyn [Gar02, Section 1], we define

$$\|\gamma\|_\mathfrak{l} = (-v_{\mathfrak{l}_0}(\gamma))^{1/r_\psi}$$

for $\gamma \in \Gamma \setminus \{0\}$ and $\|0\|_\mathfrak{l} = 0$. Here $\mathfrak{l}_0 \in \mathfrak{M}_A$ lies under $\mathfrak{l}$. Then $\|-\|_\mathfrak{l}$ is a $G_{K_\mathfrak{l}}$-invariant norm relative to the absolute value $|-|_\infty = q^{\deg(-)}$ at ∞ and is unchanged under a finite extension of K. We recall the notion of successive minima of an A-lattice from the paper of Taguchi [Tag93, Section 4] in a general setting. Notice that an A-lattice means a finitely generated discrete free A-module with respect to some norm $\|-\|$.

Definition 3.3.2. Let Λ be an A-lattice with respect to a norm $\|-\|$. For a real number c, we set

$$B(c) = \{\lambda \in \Lambda \mid \|\lambda\| \leq c\}.$$

Let r be the rank of Λ. For $1 \leq i \leq r$, the *i-th successive minimum* c_i is the smallest real number c such that $B(c)$ contains at least i elements of Λ which are linearly independent over A. An A-basis $(\lambda_i)_{1 \leq i \leq r}$ of Λ is said to be a *successive minimum basis* if it satisfies $\|\lambda_i\| = c_i$ for all i.

Note that the set $B(c)$ is finite for any real number c. We easily verify that the successive minima and the successive minimum basis always exist. A useful property of successive minima is the next lemma.

Lemma 3.3.3 ([Tag93, Lemma 4.2])**.** *Let Λ be an A-lattice of rank r with respect to a norm $\|-\|$ and $(\lambda_i)_{1 \leq i \leq r}$ an A-basis of Λ such that $\|\lambda_1\| \leq \cdots \leq \|\lambda_r\|$. Then the following conditions are equivalent*:

(a) *$(\lambda_i)_{1 \leq i \leq r}$ is a successive minimum basis.*

(b) *$\|\sum a_i \lambda_i\| = \max\{\|a_i \lambda_i\|\}$ for all $(a_i)_{1 \leq i \leq r} \in A^r$.*

For an A-lattice Λ with successive minima $c_1, \ldots, c_r$, we define the *covolume* $D(\Lambda)$ of Λ as their product, i.e.,

$$D(\Lambda) = \prod_{i=1}^{r} c_i.$$

The next lemma is easy.

Lemma 3.3.4. *Let Λ be an A-lattice of rank r and $(c_i)_{1\leq i\leq r}$ its successive minima. If ε is a real positive number with $\varepsilon \leq c_1$, then we have $c_r \leq \varepsilon^{-(r-1)}D(\Lambda)$.*

Proof. Since $D(\Lambda) \geq c_1^{r-1}c_r$, we have $c_r \leq c_1^{-(r-1)}D(\Lambda) \leq \varepsilon^{-(r-1)}D(\Lambda)$. □

Now we turn to the situation of a Drinfeld A-module ϕ of rank at most r over K. From [Dri74, Proposition 7.1], there is a tamely ramified extension K' of K of ramification index dividing $d = \mathrm{lcm}\{q^i - 1 \mid 1 \leq i \leq r\}$ such that ϕ has stable reduction at $\mathfrak{l}' \in \mathfrak{M}_{K'}^{\mathrm{f}}$ lying above $\mathfrak{l}$. Let (ψ, Γ) be the Tate uniformization at $\mathfrak{l}'$. Recall that Γ is an A-lattice under the action of A via ψ and the norm $\|-\| = \|-\|_{\mathfrak{l}'}$. Put $D(\phi, \mathfrak{l}) = D(\Gamma)$. Notice that it is independent of the choices of an extension K', a prime $\mathfrak{l}'$, and a Tate datum (ψ, Γ). We also remark that, for a Galois extension L/K, the maximal break of LK'/K' is at most d times that of L/K. We write r_ϕ for the rank of ϕ, r_ψ for the rank of ψ, and r_Γ for the rank of Γ.

Proposition 3.3.5. *Let K be a finite extension of F. Let $\mathfrak{p} \in \mathfrak{M}_A$ and $\mathfrak{l} \in \mathfrak{M}_K^{\mathrm{f}}$. Let m, r, and N be non-negative integers. Then there is a constant C depending only on $\mathfrak{p}$, $\mathfrak{l}$, m, r, and N such that, for any Drinfeld A-module ϕ over $\mathcal{O}_\mathfrak{l}$ of rank at most r with $D(\phi, \mathfrak{l}) \leq N$, the maximal break of $K_\mathfrak{l}(\phi[\mathfrak{p}^m])/K_\mathfrak{l}$ is at most C.*

Proof. By Proposition 3.3.1, it suffices to give an upper bound of $v_\mathfrak{l}(a(\mathfrak{p}^m))$ in terms of $\mathfrak{p}$, $\mathfrak{l}$, m, r, and N. The above remark allows us to replace K with its tamely ramified extension of ramification index dividing d and to assume that ϕ has stable reduction at $\mathfrak{l}$ over K. Let (ψ, Γ) be a Tate uniformization of ϕ at $\mathfrak{l}$. If $r_\psi = r_\phi$, then ϕ has good reduction at $\mathfrak{l}$ and $v_\mathfrak{l}(a(\mathfrak{p}^m)) = 0$. We assume $r_\psi < r_\phi$ for the rest of the proof. Let $(\gamma_i)_{1\leq i\leq r_\Gamma}$ be a successive minimum basis of Γ. We have a natural isomorphism

$$\psi_{\mathfrak{p}^m}^{-1}(\Gamma)/\Gamma \xrightarrow{\sim} \phi[\mathfrak{p}^m]; \quad z + \Gamma \mapsto e_\Gamma(z).$$

Then

$$\phi_{\mathfrak{p}^m}(X) = \mathfrak{p}^m X \prod_{\xi \in \phi[\mathfrak{p}^m]\setminus\{0\}} \left(1 - \frac{X}{\xi}\right) = \mathfrak{p}^m X \prod_{z \in (\psi_{\mathfrak{p}^m}^{-1}(\Gamma)/\Gamma)\setminus\{0\}} \left(1 - \frac{X}{e_\Gamma(z)}\right).$$

Comparing the highest coefficient of both hand sides, we obtain

$$a(\mathfrak{p}^m) = \mathfrak{p}^m \Big/ \prod_z e_\Gamma(z),$$

where z runs over $(\psi_{\mathfrak{p}^m}^{-1}(\Gamma)/\Gamma) \setminus \{0\}$.

Let ξ_i be a solution of the equation $\psi_{\mathfrak{p}^m}(X) = \gamma_i$. We claim that the set

$$Z = \left\{ \sum_{i=1}^{r_\Gamma} \psi_{a_i}(\gamma_i) + \eta \;\middle|\; a_i \in A \text{ with } \deg(a_i) < \deg(\mathfrak{p}^m), \text{ and } \eta \in \psi[\mathfrak{p}^m] \right\}$$

is a system of representatives for $\psi_{\mathfrak{p}^m}^{-1}(\Gamma)/\Gamma$ with $0 \in Z$. Indeed, we can easily confirm that Z is contained in $\psi_{\mathfrak{p}^m}^{-1}(\Gamma)$, that its cardinality is equal to $q^{r_\phi \deg(\mathfrak{p}^m)} = \#\phi[\mathfrak{p}^m]$, and that no two elements in Z are congruent modulo Γ.

In order to obtain an upper bound of $v_\mathfrak{l}(a(\mathfrak{p}^m))$, we should find a lower bound of $v_\mathfrak{l}(e_\Gamma(z))$ for a nonzero $z \in Z$. We have

$$\begin{aligned} v_\mathfrak{l}(e_\Gamma(z)) &= v_\mathfrak{l}\left(z \prod_{\gamma \in \Gamma \setminus \{0\}} \left(1 - \frac{z}{\gamma}\right) \right) \\ &\geq v_\mathfrak{l}(z) + \sum_{\gamma \in \Gamma \setminus \{0\};\, v_\mathfrak{l}(\gamma) > v_\mathfrak{l}(z)} v_\mathfrak{l}\left(\frac{z}{\gamma}\right) \\ &\geq v_\mathfrak{l}(z) \cdot \#\{\gamma \in \Gamma \mid v_\mathfrak{l}(\gamma) > v_\mathfrak{l}(z)\}. \end{aligned}$$

The last inequality follows since $v_\mathfrak{l}(\gamma) < 0$ for a nonzero $\gamma \in \Gamma$. For a nonzero $z = \sum \psi_{a_i}(\xi_i) + \eta \in Z$, we evaluate

$$\begin{aligned} v_\mathfrak{l}(z) &\geq \min(\{v_\mathfrak{l}(\psi_{a_i}(\xi_i))\}_i \cup \{v_\mathfrak{l}(\eta)\}) \\ &= \min\{q^{r_\psi \deg(a_i)} v_\mathfrak{l}(\xi_i)\}_i \\ &\geq q^{r_\psi(\deg(\mathfrak{p}^m)-1)} \cdot q^{-r_\psi \deg(\mathfrak{p}^m)} \min\{v_\mathfrak{l}(\gamma_i)\}_i \\ &= -q^{-r_\psi} c_{r_\Gamma}^{r_\psi}. \end{aligned}$$

Then we find

$$\begin{aligned} &\#\{\gamma \in \Gamma \mid v_\mathfrak{l}(\gamma) > v_\mathfrak{l}(z)\} \\ &\leq \#\{\gamma \in \Gamma \mid v_\mathfrak{l}(\gamma) > -q^{-r_\psi} c_{r_\Gamma}^{r_\psi}\} \\ &= \#\left\{ (a_1, \ldots, a_{r_\Gamma}) \in A^{r_\Gamma} \;\middle|\; v_\mathfrak{l}\left(\sum_{i=1}^{r_\Gamma} \psi_{a_i}(\gamma_i) \right) > -q^{-r_\psi} c_{r_\Gamma}^{r_\psi} \right\} \\ &= \prod_{i=1}^{r_\Gamma} \#\{a_i \in A \mid v_\mathfrak{l}(\psi_{a_i}(\gamma_i)) > -q^{-r_\psi} c_{r_\Gamma}^{r_\psi}\} \\ &= \prod_{i=1}^{r_\Gamma} \#\left\{ a_i \in A \;\middle|\; \deg(a_i) < \log_q\left(\frac{c_{r_\Gamma}}{c_i}\right) - 1 \right\} \\ &< \prod_{i=1}^{r_\Gamma} \frac{c_{r_\Gamma}}{c_i}. \end{aligned}$$

On the other hand, the conjugates of γ_1 are contained in the set $\{\gamma \in \Gamma \mid \|\gamma\|_{\mathfrak{l}} = c_1\}$, whose cardinality is at most $q^{r_\Gamma} - 1$. Hence the degree of the splitting field L' of the minimal polynomial of γ_1 over $K_{\mathfrak{l}}$ is bounded and there exists a constant C' depending on r such that the ramification index $e_{L'/K_{\mathfrak{l}}}$ of $L'/K_{\mathfrak{l}}$ is at most C'. (An obvious bound is $C' = (q^r - 1)!$.) Since $v_{\mathfrak{l}}(\gamma_1) < 0$, we have $v_{\mathfrak{l}}(\gamma_1) \leq -1/C'$ so $c_1 = (-v_{\mathfrak{l}}(\gamma_1))^{1/r_\psi} \geq C'^{-1/r_\psi}$. Applying Lemma 3.3.4 with $\varepsilon = C'^{-1/r_\psi}$, we obtain

$$\begin{aligned}
v_{\mathfrak{l}}(a(\mathfrak{p}^m)) &= v_{\mathfrak{l}}(\mathfrak{p}^m) - \sum_{z \in Z \setminus \{0\}} v_{\mathfrak{l}}(e_\Gamma(z)) \\
&\leq v_{\mathfrak{l}}(\mathfrak{p}^m) - (q^{r_\phi \deg(\mathfrak{p}^m)} - 1)(-q^{-r_\psi} c_{r_\Gamma}^{r_\psi}) \prod_{i=1}^{r_\Gamma} \frac{c_{r_\Gamma}}{c_i} \\
&< v_{\mathfrak{l}}(\mathfrak{p}^m) + q^{r_\phi \deg(\mathfrak{p}^m) - 1} c_{r_\Gamma}^{r_\phi} \prod_{i=1}^{r_\Gamma - 1} c_i^{-1} \\
&\leq v_{\mathfrak{l}}(\mathfrak{p}^m) + q^{r_\phi \deg(\mathfrak{p}^m) - 1} (C'^{(r_\Gamma - 1)/r_\psi} D(\phi, \mathfrak{l}))^{r_\phi} C'^{r_\phi - 2} \\
&\leq v_{\mathfrak{l}}(\mathfrak{p}^m) + q^{r_\phi \deg(\mathfrak{p}^m) - 1} N^{r_\phi} C'^{(r_\phi + 1)(r_\phi - 2)},
\end{aligned}$$

which proves the proposition. □

For a positive integer r, a family of positive integers $\boldsymbol{N} = (N_{\mathfrak{l}})_{\mathfrak{l} \in \mathfrak{M}_K^{\mathrm{f}}}$ indexed by $\mathfrak{M}_K^{\mathrm{f}}$, and a family of non-negative integers $\boldsymbol{m} = (m_{\mathfrak{p}})_{\mathfrak{p} \in \mathfrak{M}_A}$ indexed by $\mathfrak{M}_A$, let $\Phi_K(r, \boldsymbol{N}, \boldsymbol{m})$ be the set of Drinfeld A-modules ϕ over K of rank at most r with $D(\phi, \mathfrak{l}) \leq N_{\mathfrak{l}}$ for all $\mathfrak{l} \in \mathfrak{M}_K^{\mathrm{f}}$. Define $K_{r,\boldsymbol{N},\boldsymbol{m}}$ to be the extension of K generated by all elements of $\phi[\mathfrak{p}^{m_{\mathfrak{p}}}]$ for all $\phi \in \Phi_K(r, \boldsymbol{N}, \boldsymbol{m})$ and all $\mathfrak{p} \in \mathfrak{M}_A$.

Theorem 3.3.6. *The field $K_{r,\boldsymbol{N},\boldsymbol{m}}$ is DKF.*

Proof. By Theorem 3.1.4, it is enough to show that $K_{r,\boldsymbol{N},\boldsymbol{m}}$ has finite maximal break outside ∞. We remark that Lemma 3.1 in [OT22] still holds in the context of function fields. Thus it suffices to prove that, for any $\mathfrak{l} \in \mathfrak{M}_K^{\mathrm{f}}$, there exists a constant $u(\mathfrak{l})$ such that the maximal break of $K_{\mathfrak{l}}(\phi[\mathfrak{p}^{m_{\mathfrak{p}}}])/K_{\mathfrak{l}}$ is at most $u(\mathfrak{l})$ for any $\phi \in \Phi_K(r, \boldsymbol{N}, \boldsymbol{m})$ and any $\mathfrak{p} \in \mathfrak{M}_A$.

Fix $\mathfrak{l} \in \mathfrak{M}_K^{\mathrm{f}}$, $\phi \in \Phi_K(r, \boldsymbol{N}, \boldsymbol{m})$, and $\mathfrak{p} \in \mathfrak{M}_A$. By the remark preceding Proposition 3.3.5, we may assume that ϕ has stable reduction at $\mathfrak{l}$ over K.

If $\mathfrak{p}$ lies beneath $\mathfrak{l}$, then there is a constant C_1 such that the maximal break of $K_{\mathfrak{l}}(\phi[\mathfrak{p}^{m_{\mathfrak{p}}}])/K_{\mathfrak{l}}$ is at most C_1 by Proposition 3.3.5. Suppose $\mathfrak{l} \nmid \mathfrak{p}$. Let (ψ, Γ) be the Tate uniformization of ϕ at $\mathfrak{l}$. Since ψ has good reduction, $K_{\mathfrak{l}}(\psi[\mathfrak{p}^{m_{\mathfrak{p}}}])/K_{\mathfrak{l}}$ is unramified [Tak82, Theorem 1]. Let u be a real number with $G_{K_{\mathfrak{l}}}^u \neq 1$ and $\sigma \neq 1$ belong to $G_{K_{\mathfrak{l}}}^u$. Then there is i with $\sigma\gamma_i \neq \gamma_i$. From

the definition of the first successive minimum c_1, we have $\|\sigma\gamma_i - \gamma_i\|_{\mathfrak{l}} \geq c_1$. Hence $v_{\mathfrak{l}}(\sigma\gamma_i - \gamma_i) \leq -c_1^{r_\psi}$. Take $b \in K_{\mathfrak{l}}$ with $-v_{\mathfrak{l}}(\gamma_i) \leq v_{\mathfrak{l}}(b) < -v_{\mathfrak{l}}(\gamma_i) + 1$. Then $v_{\mathfrak{l}}(b\gamma_i) \geq 0$ and

$$\begin{aligned} v_{K_{\mathfrak{l}}(\Gamma)}(\sigma(b\gamma_i) - b\gamma_i) &= e_{K_{\mathfrak{l}}(\Gamma)/K_{\mathfrak{l}}} v_{\mathfrak{l}}(\sigma(b\gamma_i) - b\gamma_i) \\ &= e_{K_{\mathfrak{l}}(\Gamma)/K_{\mathfrak{l}}}(v_{\mathfrak{l}}(\sigma\gamma_i - \gamma_i) + v_{\mathfrak{l}}(b)) \\ &< e_{K_{\mathfrak{l}}(\Gamma)/K_{\mathfrak{l}}}(-c_1^{r_\psi} - v_{\mathfrak{l}}(\gamma_i) + 1) \\ &\leq e_{K_{\mathfrak{l}}(\Gamma)/K_{\mathfrak{l}}}(-c_1^{r_\psi} + c_{r_\Gamma}^{r_\psi} + 1), \end{aligned}$$

where $e_{K_{\mathfrak{l}}(\Gamma)/K_{\mathfrak{l}}}$ is the ramification index of $K_{\mathfrak{l}}(\Gamma)/K_{\mathfrak{l}}$. Thus

$$u \leq e_{K_{\mathfrak{l}}(\Gamma)/K_{\mathfrak{l}}}(-c_1^{r_\psi} + c_{r_\Gamma}^{r_\psi} + 1) - 1.$$

By [Gar02, Proposition 4], on the different $\mathfrak{D}(K_{\mathfrak{l}}(\Gamma)/K_{\mathfrak{l}})$ of $K_{\mathfrak{l}}(\Gamma)/K_{\mathfrak{l}}$, we have

$$\operatorname{ord}_{\mathfrak{l}} \mathfrak{D}(K_{\mathfrak{l}}(\Gamma)/K_{\mathfrak{l}}) \leq 1 + 2\sum_{i=1}^{r_\Gamma} \left(q^{i-1} v_{\mathfrak{l}}\left(\frac{\gamma_1}{\gamma_i}\right) \prod_{j=1}^{i} \frac{c_i}{c_j} \right).$$

As in the proof of Proposition 3.3.5, there exists a constant C' depending only on r such that $c_1 \geq C'^{-1/r_\psi}$. Then by [Ser79, Chapter III, Section 6, Proposition 13], we evaluate

$$\begin{aligned} e_{K_{\mathfrak{l}}(\Gamma)/K_{\mathfrak{l}}} &\leq 1 + \operatorname{ord}_{\mathfrak{l}} \mathfrak{D}(K_{\mathfrak{l}}(\Gamma)/K_{\mathfrak{l}}) \\ &= 2 + 2\sum_{i=1}^{r_\Gamma} \left(q^{i-1}(-c_1^{r_\psi} + c_i^{r_\psi}) \prod_{j=1}^{i} \frac{c_i}{c_j} \right) \\ &\leq 2 + 2\sum_{i=1}^{r_\Gamma} q^{i-1} c_i^{r_\psi} \left(\frac{c_i}{c_1}\right)^{i-1} \\ &\leq 2 + 2c_{r_\Gamma}^{r_\psi} \sum_{i=1}^{r_\Gamma} \left(\frac{q c_{r_\Gamma}}{c_1}\right)^{i-1} \\ &< 2 + 2c_{r_\Gamma}^{r_\psi} \cdot 2 \left(\frac{q c_{r_\Gamma}}{c_1}\right)^{r_\Gamma - 1} \\ &= 2 + 4q^{r_\Gamma - 1} c_{r_\Gamma}^{r_\phi - 1} c_1^{-(r_\Gamma - 1)} \\ &\leq 2 + 4q^{r-2}(C'^{r-2} D(\phi, \mathfrak{l}))^{r-1} C'^{r-2} \\ &\leq 2 + 4q^{r-2} C'^{r(r-2)} N_{\mathfrak{l}}^{r-1}. \end{aligned}$$

Here we use $\sum_{i=1}^{r_\Gamma} \alpha^{i-1} < 2\alpha^{r_\Gamma - 1}$ for $\alpha \geq 2$, and apply Lemma 3.3.4. Hence

$$\begin{aligned} u &< (2 + 4q^{r-2} C'^{r(r-2)} N_{\mathfrak{l}}^{r-1})((C'^{r r_\Gamma - 1} N_{\mathfrak{l}})^{r_\psi} + 1) - 1 \\ &\leq (2 + 4q^{r-2} C'^{r(r-2)} N_{\mathfrak{l}}^{r-1})(C'^{(r-1)^2/4} N_{\mathfrak{l}}^{r} + 1) - 1. \end{aligned}$$

Set C_2 to be the rightmost hand side in the above inequality.

Now we put $u(\mathfrak{l}) = \max\{C_1, C_2\}$. We are going to show that this $u(\mathfrak{l})$ is the desired constant. The case $\mathfrak{l} \mid \mathfrak{p}$ is clear. Suppose that $\mathfrak{l} \nmid \mathfrak{p}$. We have the exact sequence

$$0 \to \psi[\mathfrak{p}^{m_\mathfrak{p}}] \xrightarrow{e_\Gamma} \phi[\mathfrak{p}^{m_\mathfrak{p}}] \to \Gamma/\mathfrak{p}^{m_\mathfrak{p}}\Gamma \to 0$$

on which $G_{K_\mathfrak{l}}$ acts compatibly. We view this sequence as that of $\mathbb{F}_q$-vector spaces. Then this sequence splits. If $u \geq u(\mathfrak{l})$, then $G^u_{K_\mathfrak{l}}$ acts trivially on $\psi[\mathfrak{p}^{m_\mathfrak{p}}]$ and $\Gamma/\mathfrak{p}^{m_\mathfrak{p}}\Gamma$. Therefore $G^u_{K_\mathfrak{l}}$ also acts trivially on $\phi[\mathfrak{p}^{m_\mathfrak{p}}]$ and the proof is completed. □

Remark 3.3.7. There exists a constant C depending only on K and r such that, for any Drinfeld A-module ϕ over K of rank r, the degree of the algebraic closure of $\mathbb{F}_q$ in $K(\phi(K^{\mathrm{sep}})_{\mathrm{tor}})$ is at most C [Gek19, Remark 4.2]. Therefore, the algebraic closure of $\mathbb{F}_q$ in $K_{r,\boldsymbol{N},\boldsymbol{m}}$ is finite. From Corollary 3.1.5, we see that $K_{r,\boldsymbol{N},\boldsymbol{m}}\overline{\mathbb{F}_q}$ is an infinite extension of $K_{r,\boldsymbol{N},\boldsymbol{m}}$ and still DKF.

Chapter 4

Mordell–Weil groups over large algebraic extensions of fields of characteristic zero

The goal of this chapter is to prove some properties on the Mordell–Weil groups over finite extensions of $\overline{K}(\sigma)$ and of $\overline{K}[\sigma]$ when the characteristic of K is zero. In this chapter, we simply write the Haar measure $\mu_{G_K^e}$ of G_K^e as μ.

4.1 Freeness of Mordell–Weil groups modulo torsion

The first result in this chapter is on the freeness of the Mordell–Weil groups over finite extensions of $\overline{K}[\sigma]$ modulo torsion. We note that, because these fields are countable, the rank of the Mordell–Weil groups over such a field has infinite rank if and only if it has rank $\aleph_0$.

Theorem 4.1.1. *Suppose that K is a finitely generated field over $\mathbb{Q}$ and $e \geq 2$. Then, for almost all $\sigma \in G_K^e$, the following statement holds: for any finite extension L of $\overline{K}[\sigma]$ and any semi-abelian variety A over L, the group $A(L)/A(L)_{\mathrm{tor}}$ is a free $\mathbb{Z}$-module of rank $\aleph_0$.*

Before proving this theorem, let us recall the proposition by Moon [Moo09], which plays a key role in our proof. It is notable that Moon seemingly proved this proposition only for the case where K is a number field and A is an abelian variety, but the same proof works in the more general setting.

Proposition 4.1.2 (Moon [Moo09, Proposition 7])**.** *Let K be a field of cardinality at most $\aleph_0$ and A a semi-abelian variety over K. Let L be a Galois extension of K such that $A(L)_{\mathrm{tor}}$ is finite. Then the group $A(L)/A(L)_{\mathrm{tor}}$ is a free $\mathbb{Z}$-module of rank at most $\aleph_0$.*

Proof of Theorem 4.1.1. We separate the proof of this theorem into two parts, one for the freeness of the group $A(L)/A(L)_{\mathrm{tor}}$ and the other for the infiniteness of the rank of the group $A(L)$. First, we show the former part.

Let $\sigma \in G_K^e$ satisfy the following: for any finite extension M of $\overline{K}(\sigma)$, only finitely many roots of unity belong to M and the group $B(M)_{\mathrm{tor}}$ is finite for any abelian variety B over M. By Theorems B (2-ii) and 2.3.4 (1), almost all $\sigma \in G_K^e$ satisfy this condition. Let L be a finite extension of $\overline{K}[\sigma]$ and $M = L \cdot \overline{K}(\sigma)$. Then M is a finite extension of $\overline{K}(\sigma)$. Let A be a semi-abelian variety over L. Then A is an extension of an abelian variety B by a torus T. By assumption, the groups $T(M)_{\mathrm{tor}}$ and $B(M)_{\mathrm{tor}}$ are finite, and so are $A(M)_{\mathrm{tor}}$ and $A(L)_{\mathrm{tor}}$. There exists a finite extension K' of K in L such that L/K' is Galois and A is defined over K'. Applying Proposition 4.1.2 to L/K' and A, we find that $A(L)/A(L)_{\mathrm{tor}}$ is a free $\mathbb{Z}$-module of rank at most $\aleph_0$.

It remains to show that almost all $\sigma \in G_K^e$ satisfy the following condition: for any finite extension L of $\overline{K}[\sigma]$ and any semi-abelian variety A over L, the group $A(L)$ has infinite rank. In fact, it turns out that we only need to prove this when A is an abelian variety of positive dimension. Theorem B (1) says that the following weaker claim than this statement holds for almost all $\sigma \in G_K^e$: for any abelian variety A of positive dimension over $\overline{K}[\sigma]$, the group $A(\overline{K}[\sigma])$ has rank $\aleph_0$. Let σ satisfy the statement in the above claim, L be a finite extension of $\overline{K}[\sigma]$, and A an abelian variety of positive dimension over L. Let $B = \mathrm{Res}_{L/\overline{K}[\sigma]}(A)$ be the Weil restriction of A with respect to $L/\overline{K}[\sigma]$. Then B is an abelian variety over $\overline{K}[\sigma]$ and we have $A(L) \cong B(\overline{K}[\sigma])$ by [JP22, Lemma 6.1]. The assumption on σ implies that $B(\overline{K}[\sigma])$ has rank $\aleph_0$ and $A(L)$ also does, which completes the proof. □

As described in the proof, for almost all $\sigma \in G_K^e$, any finite extension L of $\overline{K}[\sigma]$, and any semi-abelian variety A over L, the torsion group $A(L)_{\mathrm{tor}}$ is finite. Combining with the theorem, we obtain the structure of the group $A(L)$.

Corollary 4.1.3. *Let K and e be as in Theorem* 4.1.1*. Then, for almost all $\sigma \in G_K^e$, the following statement holds: for any finite extension L of $\overline{K}[\sigma]$ and any semi-abelian variety A over L, the group $A(L)$ is the direct sum of a finite torsion subgroup and a free $\mathbb{Z}$-module of rank $\aleph_0$.*

Proof. Let $\sigma \in G_K^e$ satisfy each statement in Theorems B (2-ii), 2.3.4 (1), and 4.1.1. We show that the statement in the corollary holds for σ. Let

L be a finite extension of $\overline{K}[\sigma]$ and A a semi-abelian variety over L. Then the group $A(L)/A(L)_{\mathrm{tor}}$ is a free $\mathbb{Z}$-module of rank $\aleph_0$ and in particular it is projective. Hence the identity map on $A(L)/A(L)_{\mathrm{tor}}$ can be lifted to a homomorphism $A(L)/A(L)_{\mathrm{tor}} \to A(L)$. This provides a section of the exact sequence

$$0 \to A(L)_{\mathrm{tor}} \to A(L) \to A(L)/A(L)_{\mathrm{tor}} \to 0$$

and we have $A(L) = A(L)_{\mathrm{tor}} \oplus A(L)/A(L)_{\mathrm{tor}}$. The corollary follows from this decomposition. □

Remark 4.1.4. If $e = 1$, then the proof of Theorem 4.1.1 is invalid. This is because, for almost all $\sigma \in G_K$ and any abelian variety A, the groups $\mathbb{G}_{\mathrm{m}}(\overline{K}(\sigma))_{\mathrm{tor}}$ and $A(\overline{K}(\sigma))_{\mathrm{tor}}$ are infinite (Theorem B, (2-i) and (3-iii)). It is not known whether Theorem 4.1.1 still holds in the case $e = 1$.

We also mention the following facts. Let K be a finitely generated field of $\mathbb{Q}$ and e a positive integer. Then, for almost all $\sigma \in G_K^e$, the field $\overline{K}(\sigma)$ is a Galois extension of no proper subfield of $\overline{K}(\sigma)$ and $\overline{K}(\sigma)/\overline{K}[\sigma]$ is an infinite extension. These facts follow from [BS09, Theorems 7.9 and 7.10].

4.2 Kummer-faithfulness for some large algebraic extensions

Ohtani showed that, if K is a number field and $e \geq 2$, then any finite extension of $\overline{K}[\sigma]$ is Kummer-faithful for almost all $\sigma \in G_K^e$ (see [Oht22, Corollary 1] and its corrigendum [Oht23, Corollary 1]). Our next result is on the Kummer-faithfulness for finite extensions of $\overline{K}(\sigma)$ and of $\overline{K}[\sigma]$, which is an extension of the result by Ohtani.

Theorem 4.2.1. *Suppose that K is a finitely generated field over $\mathbb{Q}$ and $e \geq 2$. Then, for almost all $\sigma \in G_K^e$, any finite extension of $\overline{K}(\sigma)$ is Kummer-faithful.*

Proof. Since Kummer-faithfulness is preserved under finite extensions, we only have to show that $\overline{K}(\sigma)$ is Kummer-faithful for almost all $\sigma \in G_K^e$. We know from Theorem B (4) that $A(\overline{K}(\sigma))_{\mathrm{div}} = 0$ for almost all $\sigma \in G_K^e$ and any abelian variety A over $\overline{K}(\sigma)$. By Proposition 2.3.3, it suffices to show $\mathbb{G}_{\mathrm{m}}(M)_{\mathrm{div}} = 0$ for almost all $\sigma \in G_K^e$ and any finite extension M of $\overline{K}(\sigma)$.

Let S be the set of $\sigma \in G_K^e$ such that $\mathbb{G}_{\mathrm{m}}(M)_{\mathrm{div}} \neq 0$ for some finite extension M of $\overline{K}(\sigma)$. For every $a \in \overline{K}^{\times} \setminus \{1\}$, let S_a be the set of $\sigma \in G_K^e$ such that $a \in \mathbb{G}_{\mathrm{m}}(M)_{\mathrm{div}}$ for some finite extension M of $\overline{K}(\sigma)$. Then we have

$S = \bigcup_{a \in \overline{K}^\times \setminus \{1\}} S_a$. Since $\overline{K}^\times \setminus \{1\}$ is countable, it suffices to show $\mu(S_a) = 0$ for each $a \in \overline{K}^\times \setminus \{1\}$.

If a is a root of unity, then $\overline{K}(\sigma)$ has infinitely many roots of unity for each $\sigma \in S_a$. Since the set of $\sigma \in G_K^e$ with $\overline{K}(\sigma)$ having infinitely many roots of unity has measure zero with respect to μ by Theorem B (2-ii), we have $\mu(S_a) = 0$.

Suppose that a is not a root of unity. For $n \geq 1$, put

$$T_a^{(n)} = \{\sigma \in G_K^e \mid \text{some } n\text{-th root of } a \text{ belongs to } \overline{K}(\sigma)\},$$

and let $\alpha_1^{(n)}, \ldots, \alpha_n^{(n)}$ be all solutions of $X^n = a$ in $\overline{K}$. Let $T_a^{(n,i)}$ be the set of $\sigma \in G_K^e$ with $\alpha_i^{(n)} \in \overline{K}(\sigma)$. Then we have $T_a^{(n)} = \bigcup_{i=1}^n T_a^{(n,i)}$ and $\mu(T_a^{(n,i)}) = 1/[K(\alpha_i^{(n)}) : K]^e$. The assumption on a tells us that there exists a prime number l_0 such that a is not an l-th power in $K(a)$ for all prime numbers $l \geq l_0$. If l is such a prime, then the polynomial $X^l - a$ is irreducible in $K(a)[X]$ [Lan02, Chapter VI, Theorem 9.1]. Hence we have

$$[K(\alpha_i^{(l)}) : K] = [K(\alpha_i^{(l)}) : K(a)][K(a) : K] \geq l$$

and

$$\mu(T_a^{(l)}) \leq \mu\left(\bigcup_{i=1}^{l} T_a^{(l,i)}\right) \leq \sum_{i=1}^{l} \mu(T_a^{(l,i)}) \leq \frac{l}{l^e} = \frac{1}{l^{e-1}}.$$

Since $e \geq 2$, we can take an infinite set Λ_a of prime numbers $l \geq l_0$ with $\sum_{l \in \Lambda_a} 1/l^{e-1} < +\infty$. The first Borel–Cantelli lemma (Lemma 2.4.1) implies that, for almost all $\sigma \in G_K^e$, there are only finitely many $l \in \Lambda_a$ with $\sigma \in T_a^{(l)}$. This shows that there are infinitely many $l \in \Lambda_a$ such that a is not an l-th power in $\overline{K}(\sigma)$ for such σ. By the same argument as above, we obtain

$$[\overline{K}(\sigma)(\alpha_i^{(l)}) : \overline{K}(\sigma)] \geq l$$

for infinitely many $l \in \Lambda_a$ and all $1 \leq i \leq l$. This implies that $a \notin \mathbb{G}_{\mathrm{m}}(M)_{\mathrm{div}}$ for almost all $\sigma \in G_K^e$ and all finite extensions M of $\overline{K}(\sigma)$, which yields $\mu(S_a) = 0$, as desired. □

Theorem 4.2.2. *Let K be a finitely generated field over $\mathbb{Q}$ and e a positive integer. Then, for almost all $\sigma \in G_K^e$, any finite extension of $\overline{K}[\sigma]$ is Kummer-faithful.*

Proof. If $e \geq 2$, then the proposition follows from Theorem 4.2.1 (it also follows from Theorem 4.1.1 and Theorem B (4)). Hence we may assume $e = 1$. Since Kummer-faithfulness is preserved under finite extensions, we only discuss whether $\overline{K}[\sigma]$ is Kummer-faithful. By Proposition 2.3.3, it suffices to show that the following two statements hold for almost all $\sigma \in G_K$:

(a) $\mathbb{G}_{\mathrm{m}}(L)_{\mathrm{div}} = 0$ for any finite extension L of $\overline{K}[\sigma]$.

(b) $A(\overline{K}[\sigma])_{\mathrm{div}} = 0$ for any abelian variety A over $\overline{K}[\sigma]$.

Since L is a Galois extension of some finite extension of K, the conditions $\mathbb{G}_{\mathrm{m}}(L)_{\mathrm{div}} = 0$ and $A(\overline{K}[\sigma])_{\mathrm{div}} = 0$ respectively can be replaced with $(\mathbb{G}_{\mathrm{m}}(L)_{\mathrm{tor}})_{\mathrm{div}} = 0$ and that $A(\overline{K}[\sigma])[l^{\infty}]$ is finite for any prime number l [OT22, Proposition 2.4 (2)]. Then Statement (a) holds for almost all $\sigma \in G_K$ from Theorem B (2-i). Statement (b) holds for almost all $\sigma \in G_K$ by Theorem 2.3.4 (2). □

Remark 4.2.3. If $e = 1$, then the proof of Theorem 4.2.1 is invalid because, in the estimation of $\mu(T_a^{(l)})$, the upper bound $1/l^{e-1}$ no longer goes to zero as $l \to +\infty$. Moreover, it is not known whether almost all $\sigma \in G_K$ have the property that $A(\overline{K}(\sigma))_{\mathrm{div}} = 0$ for any abelian variety A over $\overline{K}(\sigma)$ [JP22, Remark 5.5]. It remains open whether Theorem 4.2.1 holds even in the case $e = 1$.

Chapter 5

Finiteness of torsion of Drinfeld modules over large algebraic function fields

It is a natural question to ask whether there are analogous results for Drinfeld modules to the theorems in Chapter 4. If we follow the strategy of proving the original theorems to show this analogue, one finds that some of the results we need are not completely known. This chapter focuses on this problem and we prove some finiteness results on the torsion submodules of Drinfeld A-modules over a finite extension of $\overline{K}(\sigma)$. These results are generalizations of Parts (b) and (c) of our previous results (Theorem C). These also correspond to analogues for Drinfeld modules of generic characteristic of the finiteness theorems for abelian varieties over large algebraic extensions conjectured by Geyer–Jarden and proved by Jacobson–Jarden (see Theorem B, (3-iii) and (3-iv)). After that, we show the Drinfeld module analogue of Moon's result (Proposition 4.1.2) and deduce the analogue of Theorem 4.1.1. We follow the notation in Section 2.1; F is a global function field over the finite field $\mathbb{F}_q$ of q elements, ∞ is a fixed prime of F, A is the subring of F consisting of functions regular outside ∞, and $\mathfrak{M}_A$ is the set of nonzero prime ideals of A. Let K be a finitely generated field over F.

5.1 Torsion points of Drinfeld modules over large algebraic extensions

In this section, the letter μ without subscript denotes the Haar measure $\mu_{G_K^e}$ on G_K^e.

Let G be an arbitrary group and Z an abelian group on which G acts. If

G and Z have topologies, then we also assume that the action of G on Z is continuous. For given G, Z, and a positive integer N, set $\Sigma_N(G, Z)$ to be the set of $g \in G$ such that there exists nonzero $z \in Z$ such that $g^N z = z$. If G is a (closed) subgroup of $\mathrm{GL}_m(R)$ and $Z = R^m$ for some positive integer m and some commutative ring R with unity, and G operates on R^m in the usual manner, then $\Sigma_N(G, R^m)$ is the set of $g \in G$ which has some N-th root of unity as an eigenvalue. If Y is a (closed) subset (not necessarily a subgroup) of G, then we set $\Sigma_N(Y, Z) = Y \cap \Sigma_N(G, Z)$. In what follows, we identify each element in $R^\times$ with the corresponding scalar matrix in $\mathrm{GL}_m(R)$ and $R^\times$ with the subgroup of scalar matrices in $\mathrm{GL}_m(R)$.

We begin by showing the following lemma, which is a partial generalization of [Asa21, Lemma 5.1]. Its proof is done in a similar way as [JJ01, Lemma 3.1], which gives an upper bound on the cardinality of $\Sigma_1(G, \mathbb{F}_p)$ for a subgroup G of $\mathrm{GL}_m(\mathbb{F}_p)$.

Lemma 5.1.1. *Let m and N be positive integers. Then for any power q of a prime number p, any subgroup G in $\mathrm{GL}_m(\mathbb{F}_q)$, and any $\alpha \in \mathrm{GL}_m(\mathbb{F}_q)$, we have*

$$\frac{\#\Sigma_N(G\alpha, \mathbb{F}_q^m)}{\#(G\alpha)} \leq \frac{mN'}{q-1}(\mathbb{F}_q^\times : \mathbb{F}_q^\times \cap G).$$

Here we write $N = p^u N'$ with $u \geq 0$ and $\gcd(p, N') = 1$.

Proof. Put $H = \mathbb{F}_q^\times \cap G$ and write $\Sigma_N = \Sigma_N(G\alpha, \mathbb{F}_q^m)$ for short. Let U be the set of N-th roots of unity in $\overline{\mathbb{F}_q}$. Then $\#U = N'$. Consider the surjective map $f : H \times \Sigma_N \to H\Sigma_N$ defined by $f(\eta, X) = \eta X$. Take any $Y \in H\Sigma_N$. For any $\zeta \in U$, let Ξ_ζ be the set of $X \in \Sigma_N$ such that there is $\eta \in H$ with $\eta X = Y$. For any $X \in \Xi_\zeta$, the element $\eta \in H$ with $\eta X = Y$ is uniquely determined and $\eta\zeta$ is an eigenvalue of Y. Moreover, no two distinct elements in Ξ_ζ correspond to the same element in H. Since Y has at most m eigenvalues, we have $\#\Xi_\zeta \leq m$. Thus we obtain

$$\#f^{-1}(Y) \leq \#\left(\bigcup_{\zeta \in U} \Xi_\zeta\right) \leq mN'.$$

Hence $\#(H \times \Sigma_N) \leq mN'\#(H\Sigma_N) \leq mN'\#(G\alpha)$. Therefore we have

$$\frac{\#\Sigma_N}{\#(G\alpha)} \leq \frac{mN'}{\#H} = \frac{mN'}{q-1}(\mathbb{F}_q^\times : H),$$

as desired. □

The next lemma is a generalization of [Asa21, Lemma 5.3]. Note that the method of the proof is originally traced back to [JJ01, Lemma 3.2], which states that, if G is a closed subgroup of $\mathrm{GL}_m(\mathbb{Z}_p)$ having infinitely many scalar matrices, then $\Sigma_1(G, \mathbb{Z}_p^m)$ is a zero set in G.

Lemma 5.1.2. *Let m and N be positive integers and $\mathfrak{p} \in \mathfrak{M}_A$. Let G be a closed subgroup of $\mathrm{GL}_m(A_\mathfrak{p})$ and $\alpha \in \mathrm{GL}_m(A_\mathfrak{p})$. Suppose that G has infinitely many scalar matrices in $\mathrm{GL}_m(A_\mathfrak{p})$. Then $\Sigma_N(G\alpha, A_\mathfrak{p}^m)$ is a zero set in $G\alpha$.*

Proof. For any $x \in \overline{F_\mathfrak{p}}$, let $\Sigma(x)$ be the set of $g \in G\alpha$ of which x is an eigenvalue. Then we have

$$\Sigma_N(G\alpha, A_\mathfrak{p}^m) = \bigcup_{\zeta\colon N\text{-th root of unity}} \Sigma(\zeta).$$

It suffices to show that $\Sigma(\zeta)$ is a zero set in $G\alpha$ for any N-th root ζ of unity. Let H be the set of scalar matrices belonging to G. We claim that, if s is a positive integer and $\eta_1, \dots, \eta_s$ are distinct elements in H, then $\eta_1\Sigma(\zeta) \cap \cdots \cap \eta_s\Sigma(\zeta)$ is a zero set. The desired result is obtained as the special case of this claim where $s = 1$ and $\eta_1 = 1$.

Indeed, if $s \geq m + 1$ and $\eta_1, \dots, \eta_s$ are distinct elements in H, then

$$\eta_1\Sigma(\zeta) \cap \cdots \cap \eta_s\Sigma(\zeta) = \Sigma(\eta_1\zeta) \cap \cdots \cap \Sigma(\eta_s\zeta)$$

is the set of matrices which have $\eta_1\zeta, \dots, \eta_s\zeta$ as eigenvalues. Since any matrix in $G\alpha$ has at most m eigenvalues, this set is empty. In particular, it is a zero set.

Now assume that the claim holds for $s+1$. We prove the claim for s. Let $\eta_1, \dots, \eta_s$ be distinct elements in H and put $D = \eta_1\Sigma(\zeta) \cap \cdots \cap \eta_s\Sigma(\zeta)$. Since H is infinite, we can take a sequence $\{\xi_i\}_{i=1}^{+\infty}$ of H such that $\xi_j \neq \xi_i\eta_1/\eta_k$ for any $1 \leq i < j$ and any $1 \leq k \leq s$. If $1 \leq i < j$, then

$$\begin{aligned} \xi_i D \cap \xi_j D &\subseteq \xi_i(\eta_1\Sigma(\zeta)) \cap \xi_j(\eta_1\Sigma(\zeta) \cap \cdots \cap \eta_s\Sigma(\zeta)) \\ &= \xi_i\eta_1\Sigma(\zeta) \cap \xi_j\eta_1\Sigma(\zeta) \cap \cdots \cap \xi_j\eta_s\Sigma(\zeta) \end{aligned}$$

and $\xi_i\eta_1, \xi_j\eta_1, \dots, \xi_j\eta_s$ are distinct $s+1$ elements in H. The induction hypothesis implies $\mu_{G\alpha}(\xi_i\eta_1\Sigma(\zeta) \cap \xi_j\eta_1\Sigma(\zeta) \cap \cdots \cap \xi_j\eta_s\Sigma(\zeta)) = 0$ and $\mu_{G\alpha}(\xi_i D \cap \xi_j D) = 0$. Therefore, for each j, we obtain

$$\mu_{G\alpha}(D) = \frac{1}{j}\sum_{i=1}^{j} \mu_{G\alpha}(\xi_i D) = \frac{1}{j}\mu_{G\alpha}\left(\bigcup_{i=1}^{j} \xi_i D\right) \leq \frac{1}{j}.$$

Taking $j \to +\infty$, we have $\mu_{G\alpha}(D) = 0$. □

Now we are ready to prove the main theorem. We first show the statement for the $\mathfrak{p}$-power torsion submodule and then show the statement for the torsion submodule.

Theorem 5.1.3. *Let K be a finitely generated field over F and e a positive integer. Then for almost all $\sigma \in G_K^e$, the following statement holds: for any finite extension M of $\overline{K}(\sigma)$, any Drinfeld A-module ϕ over M, and any $\mathfrak{p} \in \mathfrak{M}_A$, the A-module $\phi(M)[\mathfrak{p}^\infty]$ is finite.*

Proof. Let X_e be the set of $\sigma \in G_K^e$ for which the statement in the theorem does not hold. Then it is easily seen that $X_e \subseteq X_1^e$. Hence, if the theorem holds for $e = 1$, then it establishes for arbitrary e. Therefore we may assume $e = 1$.

Let $\mathcal{Z}$ be the set of finite subsets in $\overline{K}$ stable under the action of G_K. Let $\sigma \in X_1$. Then there exist $Z \in \mathcal{Z}$, a Drinfeld A-module ϕ over $\overline{K}(\sigma)(Z)$, and $\mathfrak{p} \in \mathfrak{M}_A$ such that $\phi(\overline{K}(\sigma)(Z))[\mathfrak{p}^\infty]$ is infinite. Replacing Z with a larger one if necessary, we may assume that ϕ is defined over $K(Z)$ and satisfies $\mathrm{End}_{\overline{K}}\phi = \mathrm{End}_{K(Z)}\phi$. As a result, we obtain

$$X_1 = \bigcup_{Z \in \mathcal{Z}} \bigcup_{\substack{\phi/K(Z) \\ \mathrm{End}_{\overline{K}}\phi = \mathrm{End}_{K(Z)}\phi}} \bigcup_{\mathfrak{p} \in \mathfrak{M}_A} \{\sigma \in G_K \mid \phi(\overline{K}(\sigma)(Z))[\mathfrak{p}^\infty] \text{ is infinite}\}.$$

The right hand side is a denumerable union. Hence it is sufficient to prove that the set $\{\sigma \in G_K \mid \phi(\overline{K}(\sigma)(Z))[\mathfrak{p}^\infty] \text{ is infinite}\}$ is a zero set in G_K for each $Z \in \mathcal{Z}$, Drinfeld A-module ϕ over $K(Z)$ with $\mathrm{End}_{\overline{K}}\phi = \mathrm{End}_{K(Z)}\phi$, and $\mathfrak{p} \in \mathfrak{M}_A$.

In the rest of this proof, we fix Z, ϕ, and $\mathfrak{p}$ as above. Put $U = \{\sigma \in G_K \mid \phi(\overline{K}(\sigma)(Z))[\mathfrak{p}^\infty] \text{ is infinite}\}$. Let $K(Z)_{\mathrm{s}}$ be the maximal separable subextension of $K(Z)/K$. Let B be the Galois group of $K(Z)_{\mathrm{s}}/K$ and $N = \#B$. Extend each element in B to an element in G_K and we identify B as a representative system for $G_K/G_{K(Z)_{\mathrm{s}}}$. Let $U_\beta = U \cap G_{K(Z)_{\mathrm{s}}}\beta$ for $\beta \in B$. Then $U = \bigcup_{\beta \in B} U_\beta$. Let

$$\rho = \rho_{\mathfrak{p}} : G_{K(Z)_{\mathrm{s}}} \to \mathrm{GL}_{A_{\mathfrak{p}}}(T_{\mathfrak{p}}(\phi)) \cong \mathrm{GL}_r(A_{\mathfrak{p}})$$

be the Galois representation arising from the action of $G_{K(Z)_{\mathrm{s}}}$ on the $\mathfrak{p}$-adic Tate module $T_{\mathfrak{p}}(\phi)$ of ϕ. Here r denotes the rank of ϕ. For any $\beta \in B$, let ${}^\beta\phi$ be the Drinfeld A-module given by $({}^\beta\phi)_a = {}^\beta(\phi_a)$ for $a \in A$. We choose the isomorphism $\mathrm{GL}_{A_{\mathfrak{p}}}(T_{\mathfrak{p}}({}^\beta\phi)) \cong \mathrm{GL}_r(A_{\mathfrak{p}})$ so that the composition with $G_{K(Z)_{\mathrm{s}}} \to \mathrm{GL}_{A_{\mathfrak{p}}}(T_{\mathfrak{p}}({}^\beta\phi))$ corresponds to ρ. We define the group homomorphism

$$\theta : G_K \to \mathrm{GL}_{A_{\mathfrak{p}}}\left(\bigoplus_{\beta \in B} T_{\mathfrak{p}}({}^\beta\phi)\right) \cong \mathrm{GL}_{rN}(A_{\mathfrak{p}})$$

as follows. Let $\sigma \in G_K$ and $\gamma \in B$ satisfy $\sigma \in G_{K(Z)_\mathrm{s}}\gamma$. Then

$$\sigma \mapsto [(x_\beta)_{\beta\in B} \mapsto (\sigma x_{\gamma^{-1}\beta})_{\beta\in B}]$$

defines a representation of G_K on $\bigoplus_{\beta\in B} T_\mathfrak{p}({}^\beta\phi)$. The isomorphism

$$\mathrm{GL}_{A_\mathfrak{p}}\left(\bigoplus_{\beta\in B} T_\mathfrak{p}({}^\beta\phi)\right) \cong \mathrm{GL}_{rN}(A_\mathfrak{p})$$

is the one induced by the chosen isomorphisms $\mathrm{GL}_{A_\mathfrak{p}}(T_\mathfrak{p}({}^\beta\phi)) \cong \mathrm{GL}_r(A_\mathfrak{p})$ for all $\beta \in B$. The group $\mathrm{GL}_r(A_\mathfrak{p})$ acts diagonally on $\bigoplus_{\beta\in B} T_\mathfrak{p}({}^\beta\phi) \cong A_\mathfrak{p}^{rN}$ and this defines a group homomorphism

$$\delta : \mathrm{GL}_r(A_\mathfrak{p}) \to \mathrm{GL}_{A_\mathfrak{p}}\left(\bigoplus_{\beta\in B} T_\mathfrak{p}({}^\beta\phi)\right) \cong \mathrm{GL}_{rN}(A_\mathfrak{p}).$$

Since the action of the endomorphism ring $\mathrm{End}_{K(Z)}\phi$ commutes with the representation ρ, the image $\rho(G_{K(Z)_\mathrm{s}})$ of $G_{K(Z)_\mathrm{s}}$ under ρ is contained in the centralizer $C = C_\mathfrak{p} = \mathrm{Cent}_{\mathrm{GL}_r(A_\mathfrak{p})}(\mathrm{End}_{K(Z)}\phi)$. The theorem of Pink (Theorem 2.1.5 (1-ii)) asserts that $\rho(G_{K(Z)_\mathrm{s}})$ has finite index in C. Thus $\delta(\rho(G_{K(Z)_\mathrm{s}})) = \theta(G_{K(Z)_\mathrm{s}})$ is a finite index subgroup of $\delta(C)$. Hence

$$\mu_{\delta(C)\theta(\beta)}(\theta(G_{K(Z)_\mathrm{s}}\beta)) = \mu_{\delta(C)}(\theta(G_{K(Z)_\mathrm{s}})) > 0$$

for any $\beta \in G_K$.

Let $\beta \in B$ and $\sigma \in U_\beta$. Since $\phi(\overline{K}(\sigma)(Z))[\mathfrak{p}^\infty]$ is infinite, we have $T_\mathfrak{p}(\phi)^{G_{\overline{K}(\sigma)(Z)}} \neq 0$. Take nonzero $x \in T_\mathfrak{p}(\phi)^{G_{\overline{K}(\sigma)(Z)}}$. Since σ^N fixes $\overline{K}(\sigma)(Z)$, the image $\theta(\sigma^N)$ fixes a nonzero element $(x_\beta)_{\beta\in B} \in \bigoplus_{\beta\in B} T_\mathfrak{p}({}^\beta\phi)$ given by $x_1 = x$ and $x_\beta = 0$ for $\beta \neq 1$. Then we see that $\theta(\sigma) \in \Sigma_N(\theta(G_{K(Z)_\mathrm{s}}\beta), A_\mathfrak{p}^{rN})$. Hence $\theta(U_\beta) \subseteq \Sigma_N(\theta(G_{K(Z)_\mathrm{s}}\beta), A_\mathfrak{p}^{rN})$. As $\delta(C)$ contains all scalar matrices in $\mathrm{GL}_{rN}(A_\mathfrak{p})$, the above argument and Lemma 5.1.2 imply

$$\begin{aligned}
\mu_{\theta(G_{K(Z)_\mathrm{s}}\beta)}(\theta(U_\beta)) &\leq \mu_{\theta(G_{K(Z)_\mathrm{s}}\beta)}(\Sigma_N(\theta(G_{K(Z)_\mathrm{s}}\beta), A_\mathfrak{p}^{rN})) \\
&= \frac{\mu_{\delta(C)\theta(\beta)}(\Sigma_N(\theta(G_{K(Z)_\mathrm{s}}\beta), A_\mathfrak{p}^{rN}))}{\mu_{\delta(C)\theta(\beta)}(\theta(G_{K(Z)_\mathrm{s}}\beta))} \\
&\leq \frac{\mu_{\delta(C)\theta(\beta)}(\Sigma_N(\delta(C)\theta(\beta), A_\mathfrak{p}^{rN}))}{\mu_{\delta(C)\theta(\beta)}(\theta(G_{K(Z)_\mathrm{s}}\beta))} = 0.
\end{aligned}$$

Therefore $\mu_{G_{K(Z)_\mathrm{s}}\beta}(U_\beta) = 0$. Since $G_{K(Z)_\mathrm{s}}\beta$ has positive measure in G_K, we have $\mu(U_\beta) = 0$. This implies $\mu(U) = 0$, as desired. □

Theorem 5.1.4. *Let K be a finitely generated field over F and $e \geq 2$. Then for almost all $\sigma \in G_K^e$, the following statement holds: for any finite extension M of $\overline{K}(\sigma)$ and any Drinfeld A-module ϕ over M, the A-module $\phi(M)_{\mathrm{tor}}$ is finite.*

Proof. Let X be the set of $\sigma \in G_K^e$ for which the statement in the theorem does not hold. As in the proof of Theorem 5.1.3, we have

$$X = \bigcup_{Z \in \mathcal{Z}} \bigcup_{\substack{\phi/K(Z) \\ \mathrm{End}_{\overline{K}}\phi = \mathrm{End}_{K(Z)}\phi}} \{\sigma \in G_K^e \mid \phi(\overline{K}(\sigma)(Z))_{\mathrm{tor}} \text{ is infinite}\}$$

and it suffices to prove that the set $\{\sigma \in G_K^e \mid \phi(\overline{K}(\sigma)(Z))_{\mathrm{tor}} \text{ is infinite}\}$ is a zero set in G_K^e for each $Z \in \mathcal{Z}$ and Drinfeld A-module ϕ over $K(Z)$ with $\mathrm{End}_{\overline{K}}\phi = \mathrm{End}_{K(Z)}\phi$.

We fix Z and ϕ as above. Put $V = \{\sigma \in G_K^e \mid \phi(\overline{K}(\sigma)(Z))_{\mathrm{tor}} \text{ is infinite}\}$. It contains the set

$$W = \{\sigma \in G_K^e \mid \phi(\overline{K}(\sigma)(Z))[\mathfrak{p}] \neq 0 \text{ for infinitely many } \mathfrak{p} \in \mathfrak{M}_A\}.$$

Theorem 5.1.3 tells us that the set $V \setminus W$ is a zero set. Indeed, for each $\sigma \in V \setminus W$, there exists $\mathfrak{p} \in \mathfrak{M}_A$ such that $\phi(\overline{K}(\sigma)(Z))[\mathfrak{p}^\infty]$ is infinite and the statement in Theorem 5.1.3 does not hold for σ. Thus to prove that V is a zero set, it is sufficient to show that W is a zero set.

We continue to use some of the notation and the convention in the proof of the previous theorem. Let $K(Z)_{\mathrm{s}}$ be the maximal separable subextension of $K(Z)/K$, B the Galois group of $K(Z)_{\mathrm{s}}/K$, and $N = \#B$. We regard B as a representative system for $G_K/G_{K(Z)_{\mathrm{s}}}$. For any $\beta \in B$, let ${}^\beta\phi$ be the Drinfeld A-module given by $({}^\beta\phi)_a = {}^\beta(\phi_a)$ for $a \in A$.

For any $\mathfrak{p} \in \mathfrak{M}_A$, let $W_\mathfrak{p}$ be the set of $\sigma \in G_K^e$ with $\phi(\overline{K}(\sigma)(Z))[\mathfrak{p}] \neq 0$. Let

$$\overline{\rho_\mathfrak{p}} : G_{K(Z)_{\mathrm{s}}} \to \mathrm{GL}_{\mathbb{F}_\mathfrak{p}}(\phi[\mathfrak{p}]) \cong \mathrm{GL}_r(\mathbb{F}_\mathfrak{p})$$

be the Galois representation arising from the action of $G_{K(Z)_{\mathrm{s}}}$ on the $\mathfrak{p}$-torsion submodule $\phi[\mathfrak{p}]$ of ϕ. Here r denotes the rank of ϕ and $\mathbb{F}_\mathfrak{p} = A/\mathfrak{p}$. We notice that $\overline{\rho_\mathfrak{p}}$ is obtained as the composition of $\rho_\mathfrak{p} : G_{K(Z)_{\mathrm{s}}} \to \mathrm{GL}_r(A_\mathfrak{p})$ and the reduction modulo $\mathfrak{p}$, for which we write $\pi_\mathfrak{p} : \mathrm{GL}_r(A_\mathfrak{p}) \to \mathrm{GL}_r(\mathbb{F}_\mathfrak{p})$. For any $\beta \in B$, we choose the isomorphism $\mathrm{GL}_{\mathbb{F}_\mathfrak{p}}({}^\beta\phi[\mathfrak{p}]) \cong \mathrm{GL}_r(\mathbb{F}_\mathfrak{p})$ so that the composition with $G_{K(Z)_{\mathrm{s}}} \to \mathrm{GL}_{\mathbb{F}_\mathfrak{p}}({}^\beta\phi[\mathfrak{p}])$ corresponds to $\overline{\rho_\mathfrak{p}}$. We define the group homomorphism

$$\theta_\mathfrak{p} : G_K \to \mathrm{GL}_{\mathbb{F}_\mathfrak{p}}\left(\bigoplus_{\beta \in B} {}^\beta\phi[\mathfrak{p}]\right) \cong \mathrm{GL}_{rN}(\mathbb{F}_\mathfrak{p})$$

as follows. Let $\sigma \in G_K$ and $\gamma \in B$ satisfy $\sigma \in G_{K(Z)_{\mathrm{s}}}\gamma$. Then

$$\sigma \mapsto [(x_\beta)_{\beta\in B} \mapsto (\sigma x_{\gamma^{-1}\beta})_{\beta\in B}]$$

defines a representation of G_K on $\bigoplus_{\beta\in B} {}^{\beta}\phi[\mathfrak{p}]$. The isomorphism

$$\mathrm{GL}_{\mathbb{F}_\mathfrak{p}}\left(\bigoplus_{\beta\in B} {}^{\beta}\phi[\mathfrak{p}]\right) \cong \mathrm{GL}_{rN}(\mathbb{F}_\mathfrak{p})$$

is the one induced by the chosen isomorphisms $\mathrm{GL}_{\mathbb{F}_\mathfrak{p}}({}^{\beta}\phi[\mathfrak{p}]) \cong \mathrm{GL}_r(\mathbb{F}_\mathfrak{p})$ for all $\beta \in B$. The diagonal action of $\mathrm{GL}_r(\mathbb{F}_\mathfrak{p})$ on $\bigoplus_{\beta\in B} {}^{\beta}\phi[\mathfrak{p}] \cong \mathbb{F}_\mathfrak{p}^{rN}$ defines a group homomorphism

$$\delta_\mathfrak{p} : \mathrm{GL}_r(\mathbb{F}_\mathfrak{p}) \to \mathrm{GL}_{\mathbb{F}_\mathfrak{p}}\left(\bigoplus_{\beta\in B} {}^{\beta}\phi[\mathfrak{p}]\right) \cong \mathrm{GL}_{rN}(\mathbb{F}_\mathfrak{p}).$$

Since the action of the endomorphism ring $\mathrm{End}_{K(Z)}\phi$ commutes with the representation $\rho_\mathfrak{p}$, the image $\rho_\mathfrak{p}(G_{K(Z)_{\mathrm{s}}})$ of $G_{K(Z)_{\mathrm{s}}}$ under $\rho_\mathfrak{p}$ is contained in the centralizer $C_\mathfrak{p} = \mathrm{Cent}_{\mathrm{GL}_r(A_\mathfrak{p})}(\mathrm{End}_{K(Z)}\phi)$. The theorem of Pink–Rütsche (Theorem 2.1.5 (2-ii)) asserts that there exists a subset $\mathcal{P} \subseteq \mathfrak{M}_A$ such that all but finitely many $\mathfrak{p} \in \mathfrak{M}_A$ belong to $\mathcal{P}$ and $\rho_\mathfrak{p}(G_{K(Z)_{\mathrm{s}}}) = C_\mathfrak{p}$ for all $\mathfrak{p} \in \mathcal{P}$.

Let $\mathfrak{p} \in \mathcal{P}$. Then we have $\theta_\mathfrak{p}(G_{K(Z)_{\mathrm{s}}}) = \delta_\mathfrak{p} \circ \pi_\mathfrak{p} \circ \rho_\mathfrak{p}(G_{K(Z)_{\mathrm{s}}}) = \delta_\mathfrak{p} \circ \pi_\mathfrak{p}(C_\mathfrak{p})$. Let $\sigma = (\sigma_1, \ldots, \sigma_e) \in W_\mathfrak{p}$. Then there is nonzero $x \in \phi(\overline{K}(\sigma)(Z))[\mathfrak{p}]$. For $1 \le i \le e$, we see that $\theta_\mathfrak{p}(\sigma_i^N)$ fixes $(x_\beta)_{\beta\in B} \in \bigoplus_{\beta\in B} {}^{\beta}\phi[\mathfrak{p}]$ given by $x_1 = x$ and $x_\beta = 0$ for $\beta \ne 1$ since σ_i^N fixes $\overline{K}(\sigma)(Z)$. Thus we have $\theta_\mathfrak{p}(\sigma_i) \in \Sigma_N(\theta_\mathfrak{p}(G_K), \mathbb{F}_\mathfrak{p}^{rN})$ for each i. Hence we obtain $\theta_\mathfrak{p}^e(W_\mathfrak{p}) \subseteq \Sigma_N(\theta_\mathfrak{p}(G_K), \mathbb{F}_\mathfrak{p}^{rN})^e$ and

$$\mu(W_\mathfrak{p}) \le \frac{\#(\Sigma_N(\theta_\mathfrak{p}(G_K), \mathbb{F}_\mathfrak{p}^{rN})^e)}{\#\theta_\mathfrak{p}^e(G_K^e)} = \left(\frac{\#\Sigma_N(\theta_\mathfrak{p}(G_K), \mathbb{F}_\mathfrak{p}^{rN})}{\#\theta_\mathfrak{p}(G_K)}\right)^e.$$

We have

$$\begin{aligned}\Sigma_N(\theta_\mathfrak{p}(G_K), \mathbb{F}_\mathfrak{p}^{rN}) &= \bigcup_{\beta\in B} \Sigma_N(\theta_\mathfrak{p}(G_{K(Z)_{\mathrm{s}}}\beta), \mathbb{F}_\mathfrak{p}^{rN}) \\ &= \bigcup_{\beta\in B} \Sigma_N(\delta_\mathfrak{p} \circ \pi_\mathfrak{p}(C_\mathfrak{p})\theta_\mathfrak{p}(\beta), \mathbb{F}_\mathfrak{p}^{rN}).\end{aligned}$$

Since $\delta_\mathfrak{p} \circ \pi_\mathfrak{p}(C_\mathfrak{p})$ contains all scalar matrices in $\mathrm{GL}_{rN}(\mathbb{F}_\mathfrak{p})$, Lemma 5.1.1 gives

$$\frac{\#\Sigma_N(\delta_\mathfrak{p} \circ \pi_\mathfrak{p}(C_\mathfrak{p})\theta_\mathfrak{p}(\beta), \mathbb{F}_\mathfrak{p}^{rN})}{\#(\delta_\mathfrak{p} \circ \pi_\mathfrak{p}(C_\mathfrak{p})\theta_\mathfrak{p}(\beta))} \le \frac{rNN'}{\#\mathbb{F}_\mathfrak{p} - 1},$$

where $N = p^u N'$ with $u \geq 0$ and $\gcd(p, N') = 1$. Then we estimate

$$\begin{aligned}\frac{\#\Sigma_N(\theta_{\mathfrak{p}}(G_K), \mathbb{F}_{\mathfrak{p}}^{rN})}{\#\theta_{\mathfrak{p}}(G_K)} &= \frac{\#\left(\bigcup_{\beta\in B}\Sigma_N(\delta_{\mathfrak{p}}\circ\pi_{\mathfrak{p}}(C_{\mathfrak{p}})\theta_{\mathfrak{p}}(\beta), \mathbb{F}_{\mathfrak{p}}^{rN})\right)}{\#\theta_{\mathfrak{p}}(G_K)} \\ &\leq \sum_{\beta\in B}\frac{\#\Sigma_N(\delta_{\mathfrak{p}}\circ\pi_{\mathfrak{p}}(C_{\mathfrak{p}})\theta_{\mathfrak{p}}(\beta), \mathbb{F}_{\mathfrak{p}}^{rN})}{\#(\delta_{\mathfrak{p}}\circ\pi_{\mathfrak{p}}(C_{\mathfrak{p}}))} \\ &\leq \#B\cdot\frac{rNN'}{\#\mathbb{F}_{\mathfrak{p}}-1}.\end{aligned}$$

Hence we have

$$\mu(W_{\mathfrak{p}}) \leq \left(\#B\cdot\frac{rNN'}{\#\mathbb{F}_{\mathfrak{p}}-1}\right)^e.$$

for any $\mathfrak{p} \in \mathcal{P}$. Since $e \geq 2$, the sum $\sum_{\mathfrak{p}\in\mathfrak{M}_A}(\#\mathbb{F}_{\mathfrak{p}}-1)^{-e}$ converges. Thus $\sum_{\mathfrak{p}\in\mathfrak{M}_A}\mu(W_{\mathfrak{p}}) < +\infty$. The first Borel–Cantelli lemma (Lemma 2.4.1) implies that, for almost all $\sigma \in G_K^e$, there are only finitely many $\mathfrak{p} \in \mathfrak{M}_A$ such that $\sigma \in W_{\mathfrak{p}}$. This yields $\mu(W) = 0$, which completes the proof. □

We conclude this section by proving the following immediate corollary of Theorem 5.1.3.

Corollary 5.1.5. *Let K be a finitely generated field over F and e a positive integer. Then, for almost all $\sigma \in G_K^e$, any finite extension of $\overline{K}[\sigma]$ is DKF.*

Proof. Since DKF-ness is preserved under finite extensions, we only examine the DKF-ness of $\overline{K}[\sigma]$ itself. Let L be a finite extension of $\overline{K}[\sigma]$ and ϕ a Drinfeld A-module over L. Then we can take a finite subextension K' of L/K such that ϕ is defined over K' and that the extension L/K' is Galois. By the fact that K' is DKF and Proposition 3.1.2 (2), the condition $\phi(L)_{\mathrm{div}} = 0$ is equivalent to that $\phi(L)[\mathfrak{p}^\infty]$ is finite for any $\mathfrak{p} \in \mathfrak{M}_A$. Theorem 5.1.3 implies that the latter condition holds for almost all $\sigma \in G_K^e$, which proves the corollary. □

5.2 Freeness of Drinfeld modules modulo torsion

In this section, we investigate the structure of the Drinfeld modules over a finite extension of the maximal Galois extension $\overline{K}[\sigma]$ of K in $\overline{K}(\sigma)$, where $\sigma \in G_K^e$ with $e \geq 2$. We first prove the following key proposition.

Proposition 5.2.1. *Let K be an A-field which is finitely generated over F and ϕ a Drinfeld A-module over K. Let M be a Galois extension of K such that $\phi(M)_{\mathrm{tor}}$ is finite. Then the group $\phi(M)/\phi(M)_{\mathrm{tor}}$ is a free A-module of rank $\aleph_0$.*

The proof is carried out in parallel with that of Moon's result (Proposition 4.1.2). The only difficulty arises from the fact that A is not in general a principal ideal domain and finitely generated torsion-free A-modules are not always free. Note that the following Lemmas 5.2.2, 5.2.3, and 5.2.4 correspond to Lemmas 4, 5, and 6 in [Moo09], respectively.

Let X be an A-module. For an A-submodule Y of X, the *A-saturation* $Y^{\sim}$ of Y in X is defined by

$$Y^{\sim} = \{x \in X \mid ax \in Y \text{ for some } a \in A \setminus \{0\}\}.$$

It is an A-submodule of X containing Y. An A-submodule Y of X is said to be *A-saturated* if $Y^{\sim} = Y$. Note that Y is A-saturated if and only if the quotient group X/Y is a torsion-free A-module.

Lemma 5.2.2. *Let K be an A-field and ϕ a Drinfeld A-module over K. Let M be a Galois extension of K such that $\phi(M)_{\mathrm{tor}}$ is finite. We denote the annihilator of $\phi(M)_{\mathrm{tor}}$ by $\mathfrak{a}$. Let L be a finite extension of K contained in M. Then the A-saturation $\phi(L)^{\sim}$ of $\phi(L)$ in $\phi(M)$ is contained in $\mathfrak{a}^{-1}\phi(L) = \{x \in \phi(M) \mid \phi_b(x) \in \phi(L) \text{ for all } b \in \mathfrak{a}\}$ (actually this definition depends on M, but the reference to M is omitted from the notation for simplicity).*

Proof. Let x be an element of $\phi(L)^{\sim}$. Then there is a nonzero $a \in A$ with $\phi_a(x) \in \phi(L)$. For any $\tau \in \mathrm{Gal}(M/L)$, the element $\tau x - x$ is an a-torsion element in $\phi(M)$. Indeed, $\phi_a(\tau x - x) = \tau(\phi_a(x)) - \phi_a(x) = 0$. Then $\mathfrak{a}$ annihilates $\tau x - x$. Namely, $\tau(\phi_b(x)) = \phi_b(x)$ for all $\tau \in \mathrm{Gal}(M/L)$ and all $b \in \mathfrak{a}$. This implies $x \in \mathfrak{a}^{-1}\phi(L)$. □

Lemma 5.2.3. *Let X be a finitely generated A-module and Y an A-submodule of X. Suppose that Y is A-saturated. Then there exists an A-submodule Z of X such that $X = Y \oplus Z$.*

Proof. Since Y is A-saturated and X is finitely generated, the quotient A-module X/Y is a finitely generated torsion-free A-module. Thus it is projective since A is a Dedekind domain [DF04, Section 16.3, Corollary 23]. Then the exact sequence $0 \to Y \to X \to X/Y \to 0$ splits [DF04, Section 10.5, Proposition 30]. Hence X has an A-submodule isomorphic to X/Y and we obtain $X = Y \oplus X/Y$. □

Lemma 5.2.4. *Let X be a torsion-free A-module of rank $\aleph_0$. Let $(Y_i)_{i\geq 1}$ be an increasing sequence of finitely generated A-submodules Y_i of X such that $X = \bigcup_{i\geq 1} Y_i$. Suppose that there is a nonzero ideal $\mathfrak{a}$ of A such that $Y_i^{\sim} \subseteq \mathfrak{a}^{-1}Y_i$ for all $i \geq 1$. Then X is a free A-module.*

Proof. Since $Y_i^{\sim}$ contains Y_i, we have $X = \bigcup_{i\geq 1} Y_i^{\sim}$. As $Y_i^{\sim} \subseteq \mathfrak{a}^{-1}Y_i$ and X is torsion-free, $Y_i^{\sim}$ is finitely generated for all $i \geq 1$. By Lemma 5.2.3, $Y_i^{\sim} = Y_{i-1}^{\sim} \oplus Z_{i-1}$ for some A-submodule Z_{i-1} of $Y_i^{\sim}$. Then we have $X = Y_1^{\sim} \oplus \bigoplus_{i\geq 1} Z_i$. Since $Y_1^{\sim}$ and all Z_i are finitely generated torsion-free A-modules, they are isomorphic to the direct sums of ideals of A [DF04, Section 16.3, Theorem 22] and X is the direct sum of $\aleph_0$ ideals of A. By [Poo95, Lemma 12], we conclude that X is free. □

Proof of Proposition 5.2.1. If M is a finite extension of K, the proposition follows from Theorem 2.1.3. Assume that M is an infinite extension of K. Since M has cardinality $\aleph_0$, we can take an increasing sequence $(L_i)_{i\geq 1}$ of finite separable extensions L_i of K contained in M with $M = \bigcup_{i\geq 1} L_i$. Set $X = \phi(M)/\phi(M)_{\mathrm{tor}}$ and $Y_i = \phi(L_i)/\phi(L_i)_{\mathrm{tor}}$. Since $\phi(M)_{\mathrm{tor}}$ is finite, the annihilator $\mathfrak{a}$ of $\phi(M)_{\mathrm{tor}}$ is nonzero. By Lemma 5.2.2, we have $\phi(L_i)^{\sim} \subseteq \mathfrak{a}^{-1}\phi(L_i)$ in $\phi(M)$ for all $i \geq 1$. Then $Y_i^{\sim} \subseteq \mathfrak{a}^{-1}Y_i$ in X. Since $\phi(K)$ already has rank $\aleph_0$, we know that $\phi(M)$ has also rank $\aleph_0$ and so does X. Lemma 5.2.4 implies that X is free, as desired. □

We are now in a position to prove the structure theorem of Drinfeld modules over a finite extension of $\overline{K}[\sigma]$.

Theorem 5.2.5. *Let K be a finitely generated field of F and $e \geq 2$. Then, for almost all $\sigma \in G_K^e$, the following statement holds: for any finite extension L of $\overline{K}[\sigma]$ and any Drinfeld A-module ϕ over L, the group $\phi(L)/\phi(L)_{\mathrm{tor}}$ is a free A-module of rank $\aleph_0$.*

Proof. Fix $\sigma \in G_K^e$ satisfying the statement in Theorem 5.1.4. It suffices to show that the statement in the theorem holds for σ. Let L be a finite extension of $\overline{K}[\sigma]$ and set $M = L \cdot \overline{K}(\sigma)$. Then M is a finite extension of $\overline{K}(\sigma)$. By assumption, for any Drinfeld A-module ϕ over L, the A-module $\phi(M)_{\mathrm{tor}}$ is finite and so is $\phi(L)_{\mathrm{tor}}$. There exists a finite subextension K' of L/K such that ϕ is defined over K' and that the extension L/K' is Galois. Applying Proposition 5.2.1, we conclude that $\phi(L)/\phi(L)_{\mathrm{tor}}$ is a free A-module of rank $\aleph_0$. □

Corollary 5.2.6. *Let K and e be as in Theorem* 5.2.5. *Then, for almost all $\sigma \in G_K^e$, the following statement holds: for any finite extension L of $\overline{K}[\sigma]$ and any Drinfeld A-module ϕ over L, the group $\phi(L)$ is the direct sum of a finite torsion submodule and a free A-module of rank $\aleph_0$.*

Proof. As in the proof of Theorem 5.2.5, it is sufficient to prove that the statement in the corollary holds for any $\sigma \in G_K^e$ satisfying the statement in Theorem 5.1.4. Fix such $\sigma \in G_K^e$. Let L be a finite extension of $\overline{K}[\sigma]$ and ϕ a Drinfeld A-module over L. Since σ satisfies the statement in Theorem 5.2.5, the group $\phi(L)/\phi(L)_{\mathrm{tor}}$ is a free A-module of rank $\aleph_0$ and in particular it is projective. Therefore the exact sequence

$$0 \to \phi(L)_{\mathrm{tor}} \to \phi(L) \to \phi(L)/\phi(L)_{\mathrm{tor}} \to 0$$

splits [DF04, Section 10.5, Proposition 30], and we have

$$\phi(L) = \phi(L)_{\mathrm{tor}} \oplus \phi(L)/\phi(L)_{\mathrm{tor}}.$$

Since the statement in Theorem 5.1.4 holds for σ, the A-module $\phi(L)_{\mathrm{tor}}$ is finite and this is nothing but the desired decomposition. □

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