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Perfectoids, log-regularity, and commutative ring theory

(パーフェクトイド、対数的正則性と可換環論)

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Contents

Preface

In this thesis, we fix a prime p. In recent studies on commutative ring theory, one of the most remarkable results was solving the homological conjecture, including the direct summand conjecture due to Y. André [And18]. To solve the homological conjecture, André applied the theory of perfectoid geometry, which was introduced by P. Scholze in his thesis [Sch12]. In later studies on commutative ring theory in mixed characteristic, the perfectoid method has developed as an effective tool by L. Ma, K. Schwede, and other researchers (for example, [MS18], [MS21], and [CLM$^+$22]). In the series of these studies, many researchers recognize that perfectoid methods are the most powerful ones in mixed characteristic today. This thesis addressed the following two studies.

First, to obtain a class of examples of commutative rings in mixed characteristic, we study local log-regular rings. Local log-regular rings are defined by Kazuya Kato in [Kat94] to develop the theory of toric varieties without bases. He applied it to the toroidal embedding for arithmetic schemes. Hence we can formulate many ring-theoretic properties of local log-regular rings similar to those of semigroup rings. Indeed, Kato proved any local log-regular rings are Cohen–Macaulay and normal. In this thesis, we study other ring-theoretic properties of local log-regular rings, such as the structure of canonical modules (Theorem 1.9), the relationship of rational singularities (Proposition 1.6.3), and the finite generation of its divisor class group (Theorem 1.7.8).

Secondly, we improve the tilting to apply for Noetherian rings. In perfectoid theory, the *tilting operation* for a perfectoid ring is a central notion in this method, which makes a bridge between objects in mixed characteristic and objects in positive characteristic. However, perfectoid rings themselves are too big to fit into Noetherian ring theory. The typical construction of perfectoid rings from Noetherian rings is to take the direct limit of a deeply ramified tower such as

$$\mathbb{Z}_p \subseteq \mathbb{Z}_p[p^{1/p}] \subseteq \mathbb{Z}_p[p^{1/p^2}] \subseteq \cdots . \tag{1}$$

Moreover, in earlier work of K. Shimomoto [Shi11], the tilting for the tower such as (1) appears[1]. Hence, the question naturally arises whether we can axiomatize deeply ramified towers that admit a tower-theoretic analogue of the tilting operation appearing in [Shi11]. More precisely, we consider the following problem in this thesis.

Problem 1. *Can one axiomatize deeply ramified towers which satisfy the following properties?*

1. *The direct limits of towers are perfectoid rings,*

2. *they admit a tower-theoretic analogue of the tilting which is compatible with the tilting operation for perfectoid rings obtained by the direct limits, and*

[1] In the paper [Shi11], he treated more general tower consisting of formal power series rings over complete discrete valuation rings with perfect residue fields.

3. *the tower-theoretic tilting preserves Noetherian properties and their singularities.*

We provide an answer to the aforementioned problem by introducing perfectoid towers and their tilts (Definition 2.4.9 and Definition 2.4.18). By applying pefectoid towers and their tilts, we obtain the two cohomological comparison results between in mixed characteristic and in positive characteristic (Proposition 2.6.7 and Lemma 2.6.23).

Finally, we provide the outline of this thesis here. More detailed explanations are given at the beginning of each chapter as an introduction. In Chapter 1, we establish ring-theoretic properties of local log-regular rings mentioned above. We show the canonical module of a local log-regular ring is generated by the image of the interior of the associated monoid. Furthermore, using these results, we give a criterion of the Gorenstein property of local log-regular rings and show that local log-regular rings have rational singularities. Also, we show that the divisor class group of a local log-regular ring is finitely generated. The idea of this theorem is to prove that the divisor class group of a local log-regular ring is isomorphic to that of the associated monoid. This was already proved by Gabber and Ramero in [GR23]. They proved it by algebraic geometrical method. In contrast, our approach is purely ring-theoretical, allowing us to provide an alternative proof.

In Chapter 2, we introduce perfectoid towers and their tilts. Furthermore, we establish several basic properties. An important example of perfectoid towers is a tower consisting of local log-regular rings constructed by Gabber and Ramero. We can prove that these constructed towers are perfectoid towers and can compute their tilts explicitly. This implies that the tilts of their perfectoid towers are also consisting of local log-regular rings. As applications of perfectoid towers, we obtain the finiteness of étale cohomology groups and the vanishing of local cohomology modules. The result of the finiteness of étale cohomology modules is the reformulation of previous studies (for example [ČS19]) in the framework of perfectoid towers. The result of local cohomology modules such as Lemma 2.6.23 have never been observed in previous works. We hope that developing this method will lead to a new approach to research in the cohomology theory of commutative rings.

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Convention

- All rings are assumed to be commutative with unity. The unit of a ring is denoted by 1. Moreover, all ring homomorphisms are assumed to be unital.
- All monoids are assumed to be commutative with unity. The unit of a monoid is denoted by 0. Moreover, all monoid homomorphisms are assumed to be unital.
- We denote $\mathcal{Q}^*$ the set of units of $\mathcal{Q}$.
- We denote $\mathcal{Q}^{\mathrm{gp}}$ the set of elements $p - q$ where $p, q \in \mathcal{Q}$.
- Rings are not necessarily assumed to be Noetherian.

Chapter 1

Ring-theoretic properties of local log-regular rings

1.1 Introduction

This is based on papers [Ish22], [Ish24], and part of a joint paper [INS22] with Kei Nakazato and Kazuma Shimomoto. The aim of this chapter is to give ring-theoretic properties of local log-regular rings. We establish the two results of local log-regular rings, which are the structures of canonical modules and divisor class groups.

First, we introduce the result of canonical modules of local log-regular rings. Canonical modules and dualizing complexes are essential tools in studies on the commutative ring theory and algebraic geometry. For example, we need them to formulate the local duality theory. Moreover, it is related to a criterion of a Gorenstein property. In view of both of homological algebra and commutative ring theory, understanding the structure of canonical modules, such as their generators, is important in the view of commutative ring theory. The existence of a dualizing complex is described in [Kat94] and [GR23]. Their proof is not based on algebraic geometry (in particular, sheaf theory), but not on commutative ring theory. In order to know as commutative ring theory, we establish the proof of its existence and reveal its generators.

Main Theorem A (Theorem 1.5.1)**.** *Let $(R, \mathcal{Q}, \alpha)$ be a local log-regular ring, where $\mathcal{Q}$ is fine, sharp, and saturated (by Remark 1.2.25, we may assume that $\mathcal{Q} \subseteq \mathbb{N}^l$ for some $l > 0$). Let $x_1, \ldots, x_r$ be a sequence of elements of R such that $\overline{x_1}, \ldots, \overline{x_r}$ is a regular system of parameters for $R/I_{\mathcal{Q}}$ Then R admits a canonical module and its form is*

$$\langle (x_1 \cdots x_r)\alpha(a) \mid a \in \operatorname{relint}\mathcal{Q} \rangle, \tag{1.1}$$

where $\operatorname{relint}\mathcal{Q}$ *is the relative interior of* $\mathcal{Q}$*.*

We have one remark to Main Theorem A. Since affine semigroup rings are a homomorphic image of polynomial rings, they always admit canonical modules. In contrast, it is not known whether a local log-regular ring is a homomorphic image of Gorenstein rings or not by their definitions. Hence we remark that the existence of a canonical module is not obvious. Main Theorem A is based on the well-known result of canonical modules of affine semigroup rings whose form is $\langle x \in k[\mathcal{Q}] \mid x \in \operatorname{relint}\mathcal{Q} \rangle$.

We give two applications of Main Theorem A. The first application is to provide a criterion of the Gorenstein property of local log-regular rings (Corollary 1.5.5). We can provide it similarly to affine semigroup rings. Moreover, we can determine the generator of the associated monoid of a local log-regular ring whose dimension is two (Corollary 1.5.7). The second application is that local log-regular rings have pseudo-rational singularities. Pseudo-rational singularities are an important class in singularity theory defined by Lipman and Tessier. We show that any local log-regular ring has pseudo-rational singularities (Proposition 1.6.3).

Next, we introduce the result of divisor class groups of local log-regular rings. It is well-known that the divisor class groups of affine semigroup rings are generated by height one prime ideals of affine semigroups. To prove the finite generation of the divisor class groups of local log-regular rings and to investigate them, we establish the following theorem.

Main Theorem B (Theorem 1.7.8)**.** *Let* $(R, \mathcal{Q}, \alpha)$ *be a local log-regular ring. Then* $\mathrm{Cl}(\alpha) : \mathrm{Cl}(\mathcal{Q}) \to \mathrm{Cl}(R)$ *is isomorphism. In particular, the divisor class group* $\mathrm{Cl}(R)$ *is finitely generated.*

Main theorem B is reduced to the complete case. We should emphasize that the divisor class group of local log-regular rings is isomorphic to those of their completion though the divisor class groups of Noetherian domains are not isomorphic to their completion in general. We make this possible by considering a group homomorphism induced by a log structure.

Finally, we give the outline of this chapter. In §1.2, we provide several preliminaries on monoids for later sections. We also discuss monoids obtained by the adjoining p-th power root of elements of monoids, *Krull monoids*, and monoids appearing as lattice points of convex polyhedral cones. In §1.3, we introduce the definition of local log-regular rings and their basic properties. We also provide a typical example of non-complete local log-regular rings, which is called a *Jungian domain* (see Definition 1.3.11). In §1.4, we show that local log-regular rings are splinter. This is originally due to O. Gabber and L. Ramero, and we provide an elementary proof by using the direct summand theorem. In §1.5, we give an explicit description of canonical modules of local log-regular rings. We also provide a criterion of Gorenstein property of local log-regular rings. In particular, we give the structure theorem of Gorenstein log-regular rings consisting of two-dimensional monoids. In §1.6, we show that local log-regular rings have pseudo-rational singularities. In §1.7, we prove the Main Theorem B. We also discuss how to compute the divisor class group of local log-regular rings.

1.2 Preliminaries on monoids

1.2.1 Basics of monoids

In this subsection, we introduce basic terms on monoids. The references of the basic theory of monoids are [GR23], [GHK06], and [Ogu18]. Let us recall that the definition of monoids.

Definition 1.2.1. A monoid is a semigroup with a unity.

We use a symbol $\mathcal{Q}$ for almost all monoids, but we sometimes use $\mathcal{M}$ to be aware that given monoids are multiplicative. Also, unless otherwise noted, operations of monoids are usually denoted by "$+$" in this thesis.

Here we give examples of monoids.

Example 1.2.2. 1. *Let $\mathbb{N}^n$ be the n-th copy of the set of natural numbers $\mathbb{N}$. Then this is a monoid with its natural additive structure.*

2. *Let R be a commutative ring with the multiplication $\cdot$. Then $(R, \cdot)$ is a monoid. Moreover, assume that R is a domain. Set $R^\bullet := R \setminus \{0\}$. Then $(R^\bullet, \cdot)$ is also a monoid.*

Definition 1.2.3. Let $\mathcal{Q}$ be a monoid. Then $\mathcal{Q}$ is called finitely generated if there exists an elements $x_1, \ldots, x_n \in \mathcal{Q}$ such that the monoid homomorphism $\mathbb{N}^n \to \mathcal{Q}$; $\mathbf{e}_i \mapsto x_i$ is surjective where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are the canonical basis of $\mathbb{N}^n$. Here, the above sequence of elements $x_1, \ldots, x_n \in \mathcal{Q}$ is called a *generator of* $\mathcal{Q}$.

Let $\mathcal{Q}$ be a finitely generated monoid and let $x_1, \ldots, x_n$ be a generator of $\mathcal{Q}$. Then we denote $\mathcal{Q}$ by $\langle x_1, \ldots, x_n \rangle$. By an easy calculation, any element $a \in \mathcal{Q}$ can be denoted by $a = c_1x_1 + \cdots + c_nx_n$.

Definition 1.2.4 ($\mathcal{Q}$-module)**.** Let $\mathcal{Q}$ be a monoid.

1. A *$\mathcal{Q}$-module* is a set M equipped with a binary operation

$$\mathcal{Q} \times M \to M \ ; \ (q, x) \mapsto q + x$$

having the following properties:

(a) $0 + x = x$ for any $x \in M$;

(b) $(p + q) + x = p + (q + x)$ for any $p, q \in \mathcal{Q}$ and $x \in M$.

2. A *homomorphism of $\mathcal{Q}$-modules* is a (set-theoretic) map $f : M \to N$ between $\mathcal{Q}$-modules such that $f(q + x) = q + f(x)$ for any $q \in \mathcal{Q}$ and $x \in M$. We denote by $\mathcal{Q}$-**Mod** the category of $\mathcal{Q}$-modules and homomorphisms of $\mathcal{Q}$-modules.

We refer the reader to the definition of a monoid algebra $R[\mathcal{Q}]$ to [Ogu18]. We denote by e^q (resp. $e^{\mathcal{Q}}$) the image of an element q of $\mathcal{Q}$ (resp. the monoid $\mathcal{Q}$) in $R[\mathcal{Q}]$. For a monoid $\mathcal{Q}$, one obtains the functor

$$\mathcal{Q}\text{-}\mathbf{Mod} \to R[\mathcal{Q}]\text{-}\mathbf{Mod} \ ; \ M \mapsto R[M], \tag{1.2}$$

which is a left adjoint of the forgetful functor $R[\mathcal{Q}]$-**Mnd** $\to \mathcal{Q}$-**Mod**. Notice that (1.2) preserves coproducts (we use this property to prove Proposition 1.2.23).

Definition 1.2.5. Let $\mathcal{Q}$ be a monoid.

1. A subset I of $\mathcal{Q}$ is an *ideal* (or an *s-ideal*) if it is a $\mathcal{Q}$-module.

2. An ideal $\mathfrak{p}$ of $\mathcal{Q}$ is a *prime ideal* (or *prime s-ideal*) if $\mathfrak{p} \neq \mathcal{Q}$ and for elements $x, y \in \mathcal{Q}$ such that $x + y \in \mathfrak{p}$, $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$.

3. A subset F of $\mathcal{Q}$ is a *face* if it is a submonoid.

Example 1.2.6. *Let R be a domain and let $\mathfrak{a}$ be an ideal of R. Set $\mathfrak{a}^\bullet = \mathfrak{a} \setminus \{0\} \subseteq R^\bullet$. Then $\mathfrak{a}$ is prime in R if and only if $\mathfrak{a}^\bullet$ is prime s-ideal in $R^\bullet$.*

Proposition 1.2.7. *Let $\mathcal{Q}$ be a monoid.*

1. *There is a one-to-one correspondence*

$$\{\textit{The set of prime ideals}\} \longrightarrow \{\textit{The set of faces}\}$$

such that $\mathfrak{q} \mapsto R \setminus \mathfrak{q}$ *whose inverse is* $F \mapsto R \setminus \mathfrak{q}$.

2. *Let* $\mathcal{Q}^+$ *be the set of non-units of* $\mathcal{Q}$. *Then the empty set* $\emptyset$ *and* $\mathcal{Q}^+$ *are prime ideals.*

3. $\mathcal{Q}$ *is a group if and only if ideals of* $\mathcal{Q}$ *are only* $\emptyset$ *and* $\mathcal{Q}$.

Proof. (1): The fact that $R \setminus \mathfrak{p}$ is a face (resp. $R \setminus F$ is a prime ideal) is straightforward from the definition of prime ideals (resp. the definition of faces).

(2): This follows from (1).

(3): Suppose that $\mathcal{Q}$ is a group. Let I be a non-empty ideal of $\mathcal{Q}$. Since $\mathcal{Q}$ is a group, for any element $x \in I$, there exists an element $y \in \mathcal{Q}$ such that $x + y = 0$. Hence $0 \in I$ and this implies $I = \mathcal{Q}$. Conversely, assume that ideals of $\mathcal{Q}$ are only $\emptyset$ and $\mathcal{Q}$. Suppose that $\mathcal{Q}$ is not a group. Then we can choose a non-unit $x \in \mathcal{Q}$. Since an ideal $\langle x \rangle$ is not equal to $\mathcal{Q}$, it must be the empty set, but this is a contradiction. Hence $\mathcal{Q}$ is a group. □

Remark 1.2.8. By Corollary 1.2.7 (2), the empty set $\emptyset$ is the minimal prime, and the ideal $\mathcal{Q}^+$ is the maximal ideal of a monoid $\mathcal{Q}$.

Like a ring, one can define the dimension of a monoid.

Definition 1.2.9. Let $\mathcal{Q}$ be a monoid. Then the *Krull dimension of* $\mathcal{Q}$ is the supremum of the length of all chains of prime ideals. We denote it by $\dim(\mathcal{Q})$.

Definition 1.2.10. Let $\mathcal{Q}$ be a monoid.

1. $\mathcal{Q}$ is called *integral* (or *cancellative*) if for any elements $x, x', y \in \mathcal{Q}$, $x + y = x' + y$ implies $x = x'$.

2. $\mathcal{Q}$ is called *fine* if it is finitely generated and integral,

3. $\mathcal{Q}$ is called *sharp* (or *reduced*) if $\mathcal{Q}^* = 0$ where $\mathcal{Q}^*$ is the group of units of $\mathcal{Q}$.

4. $\mathcal{Q}$ is called *saturated* if it satisfies the following conditions:
   - $\mathcal{Q}$ is integral,
   - If $x \in \mathcal{Q}^{\mathrm{gp}}$ such that $nx \in \mathcal{Q}$ for some $n > 0$, then $x \in \mathcal{Q}$.

Definition 1.2.11. Let $\mathcal{Q}$ be a monoid. Then an equaivalent relation $\sim$ on $\mathcal{Q}$ is called *congruence* if $a \sim b$ implies $a + c \sim b + c$ for any $a, b, c \in \mathcal{Q}$.

Example 1.2.12 (Associated reduced monoids)**.** *Let* $\mathcal{Q}$ *be a monoid. Two elements* $a, b \in \mathcal{Q}$ *are called associates if there exists a unit* $u \in \mathcal{Q}^*$ *such that* $a = u + b$. *If* $a, b \in \mathcal{Q}$ *are associates, then we denote them by* $a \simeq b$. *The relation* $\simeq$ *is a congruence relation and the monoid* $\mathcal{Q}_{\mathrm{red}} := \mathcal{Q}/\simeq$ *is called the associated reduced monoid of* $\mathcal{Q}$. *By definition, we have* $[a] = a + \mathcal{Q}^*$ *where* $[a]$ *is an element of* $\mathcal{Q}_{\mathrm{red}}$. *This implies that if* $\mathcal{Q}$ *is sharp, we obtain* $\mathcal{Q} = \mathcal{Q}_{\mathrm{red}}$.

Next, we review the spectra of monoids. In the same as in spectra of commutative rings, the spectra of monoids are also topological spaces.

Definition 1.2.13 (Spectra of monoids)**.** For a monoid $\mathcal{Q}$, we denote by $\operatorname{Spec}(\mathcal{Q})$ the set of prime ideals of $\mathcal{Q}$. We call it the *spectrum of* $\mathcal{Q}$.

Lemma 1.2.14. *Let $\mathcal{Q}$ be a monoid and let I be an ideal of $\mathcal{Q}$. We denote by $V(I)$ the set of prime ideals of $\mathcal{Q}$ containing I. Then the following assertions hold.*

1. $V(\emptyset) = \operatorname{Spec}(\mathcal{Q})$ *and* $V(\mathcal{Q}) = \emptyset$.
2. *Let Λ be an index set and let $\{I_\lambda\}_{\lambda\in\Lambda}$ be a set of ideals of $\mathcal{Q}$. Then $V(\sum_{\lambda\in\Lambda} I_\lambda) = \bigcap_{\lambda\in\Lambda} V(I_\lambda)$.*
3. *Let I_1 and I_2 be ideals of $\mathcal{Q}$. Then $V(I_1 \cap I_2) = V(I_1) \cup V(I_2)$.*

Proof. Assertion (1) and assertion (2) are straightforward, hence we show assertion (3). An inclusion $V(I_1)\cup V(I_2) \subseteq V(I_1\cap I_2)$ obviously holds, hence we only prove the converse inclusion $V(I_1 \cap I_2) \subseteq V(I_1) \cup V(I_2)$. Pick $\mathfrak{p} \in V(I_1 \cap I_2)$. Suppose that $I_1 \not\subset \mathfrak{p}$ and $I_2 \not\subset \mathfrak{p}$. Then there exist elements $a \in I_1$ and $b \in I_2$ such that $a, b \notin \mathfrak{p}$. This implies that $a + b \in I_1 \cap I_2 \subseteq \mathfrak{p}$ and $a + b \notin \mathfrak{p}$. This is a contradiction. Hence $I_1 \subseteq \mathfrak{p}$ or $I_2 \subseteq \mathfrak{p}$, that is, $\mathfrak{p} \in V(I_1) \cup V(I_2)$, as desired. □

As a consequence, we can define a topology on $\operatorname{Spec}(\mathcal{Q})$ whose closed subsets are $V(I)$, which is called the *Zariski topology* on $\operatorname{Spec}(\mathcal{Q})$.

Proposition 1.2.15. *Let $\mathcal{Q}$ be a fine monoid. Then* $\operatorname{Spec}(\mathcal{Q})$ *is a finite set.*

Proof. This is [Ogu18, Chapter **I**, Propositoin 1.4.7 (1)]. □

Next, we review homomorphisms of monoids.

Definition 1.2.16. Let $\mathcal{Q}$ and $\mathcal{P}$ be monoids. Then a map $\varphi : \mathcal{Q} \to \mathcal{P}$ is a *homomorphism of monoids* if it is addition preserving and unit preserving.

Remark 1.2.17 ([Ogu18, Chapter **I**, P.2])**.** Unlike ring homomorphisms, even if a kernel of a homomorphism vanishes, the homomorphism is not necessarily injective. For example, let $\theta : \mathbb{N}^2 \to \mathbb{N}$ the homomorphism of monoids such that $(a, b) \mapsto a + b$. Then the kernel of θ is zero because $a + b = 0$ in $\mathbb{N}$ implies $a = b = 0$, but the both $(a, 0)$ and $(0, a)$, which is different in $\mathbb{N}^2$, maps to the same a.

We have a decomposition of the reduced part and the unit part for a monoid:

Lemma 1.2.18 ([GR23, Lemma 6.2.10])**.** *Let $\mathcal{Q}$ be a saturated monoid such that $\mathcal{Q}_{\mathrm{red}}$ is fine. Then there exists an isomorphism of monoids*

$$\mathcal{Q} \cong \mathcal{Q}_{\mathrm{red}} \times \mathcal{Q}^*.$$

As seen above, all monoids have a unique maximal ideal. Hence we define local homomorphisms of monoids.

Definition 1.2.19. Let $\mathcal{P}$ and $\mathcal{Q}$ be monoids. Let $\varphi : \mathcal{P} \to \mathcal{Q}$ be a homomorphism of monoids. Then φ is called *local* if $\varphi^{-1}(\mathcal{Q}^*) = \mathcal{P}^*$.

Lemma 1.2.20. *Let $\mathcal{P}$ and $\mathcal{Q}$ be monoids. Let $\varphi : \mathcal{P} \to \mathcal{Q}$ be a homomorphism of monoids. Then the following assertions are equivalent.*

1. *φ is local,*

2. *$\varphi(\mathcal{P}^+) \subseteq \mathcal{Q}^+$, and*

3. *$\varphi^{-1}(\mathcal{Q}^+) = \mathcal{P}^+$.*

Proof. The implication (2) $\Rightarrow$ (1) and the equivalence (2) $\Leftrightarrow$ (3) are straightforward. Hence we only show the implication (1) $\Rightarrow$ (2). Pick $x \in \mathcal{P}^+$. Suppose that $\varphi(x) \notin \mathcal{Q}^+$. Then we obtain $\varphi(x) \in \mathcal{Q}^*$. This implies that $x \in \varphi^{-1}(\mathcal{Q}^*) = \mathcal{P}^*$, and it is a contradiction. Therefore we obtain $\varphi(\mathcal{P}^+) \subseteq \mathcal{Q}^+$, as desired. □

Definition 1.2.21 (Exact homomorphisms)**.** Let $\mathcal{P}$ and $\mathcal{Q}$ be monoids.

1. A homomorphism of monoids $\varphi : \mathcal{P} \to \mathcal{Q}$ is said to be *exact* if the diagram of monoids:

$$\begin{array}{ccc} \mathcal{P}^{\mathrm{gp}} & \xrightarrow{\varphi^{\mathrm{gp}}} & \mathcal{Q}^{\mathrm{gp}} \\ \uparrow & & \uparrow \\ \mathcal{P} & \xrightarrow{\varphi} & \mathcal{Q} \end{array}$$

   is cartesian.

2. An *exact submonoid* of $\mathcal{Q}$ is a submonoid $\mathcal{Q}'$ of $\mathcal{Q}$ such that the inclusion map $\mathcal{Q}' \hookrightarrow \mathcal{Q}$ is exact (in other words, $(\mathcal{Q}')^{\mathrm{gp}} \cap \mathcal{Q} = \mathcal{Q}'$).

Here is a quite useful characterization of exact submonoids (Proposition 1.2.23). To see this, we recall a graded decomposition of a $\mathcal{Q}$-module attached to a submonoid. For a monoid $\mathcal{Q}$ and a submonoid $\mathcal{Q}' \subseteq \mathcal{Q}$, we denote by $\mathcal{Q} \to \mathcal{Q}/\mathcal{Q}'$ the cokernel of the inclusion map $\mathcal{Q}' \hookrightarrow \mathcal{Q}$.

Definition 1.2.22. Let $\mathcal{Q}$ be an integral monoid, and let $\mathcal{Q}' \subseteq \mathcal{Q}$ be a submonoid. Then for any $g \in \mathcal{Q}/\mathcal{Q}'$, we denote by $\mathcal{Q}_g$ a $\mathcal{Q}'$-module defined as follows.

- As a set, $\mathcal{Q}_g$ is the inverse image of $g \in \mathcal{Q}/\mathcal{Q}'$ under the cokernel $\mathcal{Q} \to \mathcal{Q}/\mathcal{Q}'$ of $\mathcal{Q}' \hookrightarrow \mathcal{Q}$.

- The operation $\mathcal{Q}' \times \mathcal{Q}_g \to \mathcal{Q}_g$ is defined by the rule: $(q, x) \mapsto q + x$ (where $q + x$ denotes the sum of q and x in $\mathcal{Q}$).

By definition, $\mathcal{Q} = \coprod_{g \in \mathcal{Q}/\mathcal{Q}'} \mathcal{Q}_g$ as sets. The right-hand side is viewed as the coproduct of $\mathcal{Q}'$-modules $\{\mathcal{Q}_g\}_{g \in \mathcal{Q}/\mathcal{Q}'}$, and hence a $\mathcal{Q}/\mathcal{Q}'$-graded decomposition of the $\mathcal{Q}'$-module $\mathcal{Q}$. Using this notion, one can refine a characterization of exact embeddings described in [Ogu18, Chapter **I**, Proposition 4.2.7].

Proposition 1.2.23 (cf. [Ogu18, Chapter **I**, Proposition 4.2.7])**.** *Let $\mathcal{Q}$ be an integral monoid, and let $\mathcal{Q}' \subseteq \mathcal{Q}$ be a submonoid. Let $\theta : \mathcal{Q}' \hookrightarrow \mathcal{Q}$ be the inclusion map, and let $\mathbb{Z}[\theta] : \mathbb{Z}[\mathcal{Q}'] \to \mathbb{Z}[\mathcal{Q}]$ be the induced ring map. Set $G := \mathcal{Q}/\mathcal{Q}'$. Then the following assertions hold.*

1. *The $\mathbb{Z}[\mathcal{Q}']$-module $\mathbb{Z}[\mathcal{Q}]$ admits a G-graded decomposition $\mathbb{Z}[\mathcal{Q}] = \bigoplus_{g \in G} \mathbb{Z}[\mathcal{Q}_g]$.*

2. *The following conditions are equivalent.*

   (a) The inclusion map $\theta : \mathcal{Q}' \hookrightarrow \mathcal{Q}$ is exact. In other words, $(\mathcal{Q}')^{\mathrm{gp}} \cap \mathcal{Q} = \mathcal{Q}'$.

   (b) $\mathcal{Q}_0 = \mathcal{Q}'$.

(c) $\mathbb{Z}[\theta]$ *splits as a* $\mathbb{Z}[\mathcal{Q}']$*-linear map.*

(d) $\mathbb{Z}[\theta]$ *is equal to the canonical embedding* $\mathbb{Z}[\mathcal{Q}_0] \hookrightarrow \bigoplus_{g\in G} \mathbb{Z}[\mathcal{Q}_g]$.

(e) $\mathbb{Z}[\theta]$ *is universally injective.*

Proof. (1): By applying the functor (1.2) (that admits a right adjoint) to the decomposition $\mathcal{Q} = \coprod_{g\in G} \mathcal{Q}_g$, we find that the assertion follows.

(2): Since $\mathcal{Q}_0 = (\mathcal{Q}')^{\mathrm{gp}} \cap \mathcal{Q}$ as sets by definition, the equivalence (a)⇔(b) follows. The assertion (a)⇔(c)⇔(e) is none other than [Ogu18, Chapter **I**, Proposition 4.2.7]. Moreover, (d) implies (c) obviously. Thus it suffices to show the implication (b)⇒(d). Assume that (b) is satisfied. Then one can decompose $\mathcal{Q}$ into the direct sum of $\mathcal{Q}'$-modules $\coprod_{g\in G} \mathcal{Q}_g$ with $\mathcal{Q}_0 = \mathcal{Q}'$. Hence the inclusion map $\mathcal{Q}' \hookrightarrow \mathcal{Q}$ is equal to the canonical embedding $\mathcal{Q}_0 \hookrightarrow \coprod_{g\in G} \mathcal{Q}_g$. Thus the induced homomorphism $\mathbb{Z}[\theta] : \mathbb{Z}[\mathcal{Q}_0] \hookrightarrow \mathbb{Z}[\coprod_{g\in G} \mathcal{Q}_g]$ satisfies (d), as desired. □

Remark 1.2.24. In the situation of Proposition 1.2.23, assume that the condition (d) is satisfied. Then the split surjection $\pi : \mathbb{Z}[\mathcal{Q}] \to \mathbb{Z}[\mathcal{Q}']$ has the property that $\pi(e^{\mathcal{Q}}) = e^{\mathcal{Q}'}$ by the construction of the G-graded decomposition $\mathbb{Z}[\mathcal{Q}] = \bigoplus_{g\in G} \mathbb{Z}[\mathcal{Q}_g]$. Moreover, $\pi(e^{\mathcal{Q}^+}) \subseteq e^{(\mathcal{Q}')^+}$ because $\mathcal{Q}^+ \cap \mathcal{Q}' \subseteq (\mathcal{Q}')^+$. We use this fact in our proof for Theorem 1.4.3.

Remark 1.2.25. If $\mathcal{Q}$ is fine, sharp, and saturated, then there is an exact injection $\mathcal{Q} \hookrightarrow \mathbb{N}^l$ for some $l \in \mathbb{N}$ (see [Ogu18, Chapter **I**, Proposition 1.3.5] and [Ogu18, Chapter **I**, Corollary 2.2.7]). Thus, in the following sections, we assume that a fine, sharp, and saturated monoid is a submodule of some $\mathbb{N}^r$.

Proposition 1.2.23 implies the following useful lemma.

Lemma 1.2.26. *Let* $\mathcal{Q}$ *be a fine, sharp, and saturated monoid. Let* A *be a ring. Then there is an embedding of monoids* $\mathcal{Q} \hookrightarrow \mathbb{N}^d$ *such that the induced map of monoid algebras*

$$A[\mathcal{Q}] \to A[\mathbb{N}^d]$$

splits as an $A[\mathcal{Q}]$*-linear map.*

Proof. Since $\mathcal{Q}$ is saturated, there exists an embedding $\mathcal{Q}$ into some $\mathbb{N}^d$ as an exact submonoid in view of Remark 1.2.25. Then by Proposition 1.2.23, the associated map of monoid algebras

$$\mathbb{Z}[\mathcal{Q}] \to \mathbb{Z}[\mathbb{N}^d] \tag{1.3}$$

splits as a $\mathbb{Z}[\mathcal{Q}]$-linear map. By tensoring (1.3) with A, we get the desired split map. □

Let $\mathcal{Q}$ be a monoid and let $\mathfrak{q} \subseteq \mathcal{Q}$ be a prime ideal. Then we define the *localization* $\mathcal{Q}_{\mathfrak{q}}$ *at* $\mathfrak{q}$ as the set of elements $a - b \in \mathcal{Q}^{\mathrm{gp}}$ such that $a, b \in \mathcal{Q}$ and $b \notin \mathfrak{q}$.

Lemma 1.2.27. *Let* $\mathcal{Q}$ *be a monoid.*

1. *If* $\mathcal{Q}$ *is integral, then any face of* $\mathcal{Q}$ *is an exact submonoid.*

2. *If* $\mathcal{Q}$ *is saturated, then every submonoid is saturated.*

3. *If* $\mathcal{Q}$ *is fine (resp. saturated) and* $\mathfrak{q} \in \mathcal{Q}$ *is a prime ideal, then so is* $\mathcal{Q}_{\mathfrak{q}}$.

Proof. (1): For a face $\mathcal{F}$, it suffices to show that $\mathcal{Q} \cap \mathcal{F}^{\mathrm{gp}} \subseteq \mathcal{Q}$. Let $a \in \mathcal{Q} \cap \mathcal{F}^{\mathrm{gp}}$ be an element. Then there exist elements $b, c \in \mathcal{F}$ such that $a = b - c \in \mathcal{F}$. Note that $a + c = b \in \mathcal{F}$. This implies that $a \in \mathcal{F}$ (and $c \in \mathcal{F}$) by the definition of faces, as desired.

(2): It follows from the definition of saturated and (1).

(3): The first assertion is straightforward, hence we prove the second assertion. Let $a - b \in (\mathcal{Q}_{\mathfrak{q}})^{\mathrm{gp}} = \mathcal{Q}^{\mathrm{gp}}$ be a element such that $n(a-b) \in \mathcal{Q}_{\mathfrak{q}}$ for some $n > 0$. Since $\mathcal{Q}_{\mathfrak{q}}$ and $\mathcal{F} := \mathcal{Q} \setminus \mathfrak{q}$ is saturated by the assumption and (2), $na \in \mathcal{Q}$ (resp. $nb \in \mathcal{F}$) implies $a \in \mathcal{Q}$ (resp. $b \in \mathcal{F}$). Thus $a - b \in \mathcal{Q}_{\mathfrak{q}}$, as desired. □

At the end of this subsection, we discuss monoid algebras and their completion. Let $\mathcal{Q}$ be a fine sharp monoid and let R be a commutative ring. Then we denote by $R[\![\mathcal{Q}]\!]$ the set of functions $\mathcal{Q} \to R$, viewed as an R-module using the usual point-wise structure and endowed with the product topology induced by the discrete topology on R, that is, we have the explicit description

$$R[\![\mathcal{Q}]\!] = \Big\{\sum_{q \in \mathcal{Q}} a_q e^q \mid a_q \in R\Big\}.$$

By using this description, the R-module $R[\![\mathcal{Q}]\!]$ admits the unique multiplication (see [Ogu18, Chapter **I**, Proposition 3.6.1 (2)]). Also, as a formal power series ring $R[\![x_1, \dots, x_n]\!]$ is the completion of a polynomial ring $R[x_1, \dots, x_n]$ with respect to an ideal $(x_1, \dots, x_n)$, a formal power series ring $R[\![\mathcal{Q}]\!]$ of $\mathcal{Q}$ can be view as the completion of $R[\mathcal{Q}]$ with respect to an ideal $R[\mathcal{Q}^+]$ (see [Ogu18, Chapter **I**, Proposition 3.6.1 (3)]).

Remark 1.2.28. It is easy to miss symbolically, but $R[\![\mathcal{Q}]\!]$ is not complete and separated with respect to a maximal ideal even if R is a field.

Proposition 1.2.29 ([Ogu18, Chapter **I**, Proposition 3.6.1 (4) and (5)])**.** *Keep the notation as above. Then the following assertions hold.*

1. *If $\mathcal{Q}^{\mathrm{gp}}$ is torsion free and R is also an integral domain, then $R[\![\mathcal{Q}]\!]$ is an integral domain.*

2. *If R is a local ring with maximal ideal $\mathfrak{m}$, then $R[\![\mathcal{Q}]\!]$ is a local ring, whose maximal ideal consists of elements of $R[\![\mathcal{Q}]\!]$ such that their constant term belongs $\mathfrak{m}$.*

1.2.2 Monoids adjoining with the p-th power roots

In this subsection, we construct and investigate monoids adjoining the p-th power roots of elements. It is important in the construction of perfectoid towers arising from local log-regular rings.

For an integral monoid $\mathcal{Q}$, we denote by $\mathcal{Q}_{\mathbb{Q}}$ the submonoid of $\mathcal{Q}^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ defined as

$$\mathcal{Q}_{\mathbb{Q}} := \{x \otimes r \in \mathcal{Q}^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q} \mid x \in \mathcal{Q},\ r \in \mathbb{Q}_{\geq 0}\}.$$

Using this, one can define the following monoid which plays a central role in Gabber-Ramero's construction of perfectoid towers consisting of local log-regular rings.

Definition 1.2.30. Let $\mathcal{Q}$ be an integral sharp monoid. Let c and i be non-negative integers with $c > 0$.

1. We denote by $\mathcal{Q}_c^{(i)}$ a submonoid of $\mathcal{Q}_{\mathbb{Q}}$ defined as

$$\mathcal{Q}_c^{(i)} := \{\gamma \in \mathcal{Q}_{\mathbb{Q}} \mid c^i\gamma \in \mathcal{Q}\}.$$

2. We denote by $\iota_c^{(i)} : \mathcal{Q}_c^{(i)} \hookrightarrow \mathcal{Q}_c^{(i+1)}$ the inclusion map, and by $\mathbb{Z}[\iota_c^{(i)}] : \mathbb{Z}[\mathcal{Q}_c^{(i)}] \to \mathbb{Z}[\mathcal{Q}_c^{(i+1)}]$ the induced ring map.

To prove several properties of $\mathcal{Q}_c^{(i)}$, the following one is important as a starting point.

Lemma 1.2.31. *Let $\mathcal{Q}$ be an integral sharp monoid. Then for every $c > 0$ and every $i \geq 0$, the following assertions hold.*

1. *$\mathcal{Q}_c^{(i)}$ is integral and sharp.*
2. *$\mathcal{Q}_c^{(i+1)} = (\mathcal{Q}_c^{(i)})_c^{(1)}$.*
3. *The c-times map on $\mathcal{Q}_{\mathbb{Q}}$ restricts to an isomorphism of monoids:*

$$f_c : \mathcal{Q}_c^{(i+1)} \xrightarrow{\cong} \mathcal{Q}_c^{(i)} \ ; \ \gamma \mapsto c\gamma.$$

Proof. (1): Since $\mathcal{Q}^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is an integral monoid, so is $\mathcal{Q}_c^{(i)}$. Let us show that $\mathcal{Q}_c^{(i)}$ is sharp. Pick $x, y \in \mathcal{Q}_c^{(i)}$ such that $x + y = 0$. Then $c^i x = 0$ because $\mathcal{Q}$ is sharp. Thus, since $\mathcal{Q}_c^{(i)}$ is a submonoid of the torsion-free group $\mathcal{Q}^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$, we have $x = 0$, as desired.

(2): Since any $g \in (\mathcal{Q}_c^{(i)})^{\mathrm{gp}}$ satisfies $c^i g \in \mathcal{Q}^{\mathrm{gp}}$, the inclusion map $\mathcal{Q}^{\mathrm{gp}} \hookrightarrow (\mathcal{Q}_c^{(i)})^{\mathrm{gp}}$ becomes an isomorphism $\varphi : \mathcal{Q}^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} (\mathcal{Q}_c^{(i)})^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ by extension of scalars along the flat ring map $\mathbb{Z} \to \mathbb{Q}$. The restriction $\widetilde{\varphi} : \mathcal{Q}_{\mathbb{Q}} \hookrightarrow (\mathcal{Q}_c^{(i)})_{\mathbb{Q}}$ of φ is also an isomorphism, and one can easily check that $\widetilde{\varphi}$ restricts to the desired canonical isomorphism $\mathcal{Q}_c^{(i+1)} \xrightarrow{\cong} (\mathcal{Q}_c^{(i)})_c^{(1)}$.

(3): It is easy to see that the c-times map on $\mathcal{Q}_{\mathbb{Q}}$ restricts to the homomorphism of monoids f_c. Since the abelian group $\mathcal{Q}_{\mathbb{Q}} = \mathcal{Q}^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is torsion-free, f_c is injective. Moreover, any element γ in $\mathcal{Q}_c^{(i)}$ is of the form $x \otimes r$ for some $x \in \mathcal{Q}^{\mathrm{gp}}$ and $r \in \mathbb{Q}$, which satisfy that $c(x \otimes \frac{r}{c}) = \gamma$ and $c^{i+1}(x \otimes \frac{r}{c}) \in \mathcal{Q}$. Hence f_c is also surjective, as desired. □

Let us inspect monoid-theoretic aspects of the inclusion $\iota_c^{(i)} : \mathcal{Q}_c^{(i)} \hookrightarrow \mathcal{Q}_c^{(i+1)}$. First, we observe that the assumption of fineness on $\mathcal{Q}$ induces several finiteness properties.

Lemma 1.2.32. *Let $\mathcal{Q}$ be a fine sharp monoid. Then for every $c > 0$ and every $i \geq 0$, the following assertions hold.*

1. *$\mathcal{Q}_c^{(i)}$ is fine and sharp.*
2. *The ring map $\mathbb{Z}[\iota_c^{(i)}] : \mathbb{Z}[\mathcal{Q}_c^{(i)}] \to \mathbb{Z}[\mathcal{Q}_c^{(i+1)}]$ is module-finite.*
3. *$\mathcal{Q}_c^{(i+1)}/\mathcal{Q}_c^{(i)} \cong (\mathcal{Q}_c^{(i+1)})^{\mathrm{gp}}/(\mathcal{Q}_c^{(i)})^{\mathrm{gp}}$ as monoids. Moreover, $\mathcal{Q}_c^{(i+1)}/\mathcal{Q}_c^{(i)}$ forms a finite abelian group.*
4. *For a prime $p > 0$, we have $|\mathcal{Q}_p^{(i+1)}/\mathcal{Q}_p^{(i)}| = p^s$ for some $s \geq 0$.*

Proof. In view of Lemma 1.2.31, it suffices to deal with the case when $i = 0$ only. Here notice that $\mathcal{Q}_c^{(0)} = \mathcal{Q}$.

(1): Since $\mathcal{Q}$ is fine, there exists a finite system of generators $\{x_1, \ldots, x_r\}$ of $\mathcal{Q}$. Hence $\mathcal{Q}_c^{(1)}$ also has a finite system of generators $\{x_j \otimes \frac{1}{c}\}_{j=1,\ldots,r}$. For $j = 1, \ldots, r$, we put $\frac{1}{c}x_j := x_j \otimes \frac{1}{c} \in \mathcal{Q}_c^{(1)}$.

(2): The $\mathbb{Z}[\mathcal{Q}]$-algebra $\mathbb{Z}[\mathcal{Q}_c^{(1)}]$ is generated by $\{e^{\frac{1}{c}x_1}, \dots, e^{\frac{1}{c}x_r}\}$, and each $e^{\frac{1}{c}x_j} \in \mathbb{Z}[\mathcal{Q}_c^{(1)}]$ is integral over $\mathbb{Z}[\mathcal{Q}]$. Hence $\mathbb{Z}[\iota_c^{(0)}]$ is module-finite, as desired.

(3): By [Ogu18, Chapter **I**, Proposition 1.3.3], $\mathcal{Q}_c^{(1)}/\mathcal{Q}$ is identified with the image of the composition

$$\mathcal{Q}_c^{(1)} \hookrightarrow (\mathcal{Q}_c^{(1)})^{\mathrm{gp}} \twoheadrightarrow (\mathcal{Q}_c^{(1)})^{\mathrm{gp}}/\mathcal{Q}^{\mathrm{gp}}. \tag{1.4}$$

Since $\mathcal{Q}_c^{(1)}$ is generated by $\frac{1}{c}x_1, \dots, \frac{1}{c}x_r$, $(\mathcal{Q}_c^{(1)})^{\mathrm{gp}}$ is generated by $\frac{1}{c}x_1, \dots, \frac{1}{c}x_r, -\frac{1}{c}x_1, \dots, -\frac{1}{c}x_r$ as a monoid. On the other hand, we have $-\frac{1}{c}x_j \equiv (c-1)\frac{1}{c}x_j \mod \mathcal{Q}^{\mathrm{gp}}$ for $j = 1, \dots, r$. Hence $(\mathcal{Q}_c^{(1)})^{\mathrm{gp}}/\mathcal{Q}^{\mathrm{gp}}$ is generated by $\{\frac{1}{c}x_j \mod \mathcal{Q}^{\mathrm{gp}}\}_{j=1,\dots,r}$ as a monoid. Therefore, the composite map (1.4) is surjective, and $(\mathcal{Q}_c^{(1)})^{\mathrm{gp}}/\mathcal{Q}^{\mathrm{gp}}$ is a finitely generated torsion abelian group. Thus, $\mathcal{Q}_c^{(1)}/\mathcal{Q}$ coincides with $(\mathcal{Q}_c^{(1)})^{\mathrm{gp}}/\mathcal{Q}^{\mathrm{gp}}$, which is a finite abelian group, as desired.

(4): Since there exists a surjective group homomorphism

$$f : \underbrace{\mathbb{Z}/p\mathbb{Z} \times \cdots \times \mathbb{Z}/p\mathbb{Z}}_{r} \twoheadrightarrow (\mathcal{Q}_p^{(1)})^{\mathrm{gp}}/\mathcal{Q}^{\mathrm{gp}} \; ; \; (\overline{n_1}, \dots, \overline{n_r}) \mapsto n_1\left(\frac{1}{p}x_1\right) + \cdots + n_r\left(\frac{1}{p}x_r\right) \mod \mathcal{Q}^{\mathrm{gp}} ,$$

we have $p^r = |(\mathcal{Q}_p^{(1)})^{\mathrm{gp}}/\mathcal{Q}^{\mathrm{gp}}||\operatorname{Ker}(f)|$. Hence $|(\mathcal{Q}_p^{(1)})^{\mathrm{gp}}/\mathcal{Q}^{\mathrm{gp}}| = p^s$ for some $s \geq 0$. Thus the assertion follows from (3). □

By assuming saturatedness, one finds the exactness of $\iota_c^{(i)} : \mathcal{Q}_c^{(i)} \hookrightarrow \mathcal{Q}_c^{(i+1)}$.

Lemma 1.2.33. *Let $\mathcal{Q}$ be an integral sharp saturated monoid. Then for every $c > 0$ and every $i \geq 0$, the following assertions hold.*

1. *$\mathcal{Q}_c^{(i)}$ is integral, sharp, and saturated.*

2. *$\iota_c^{(i)} : \mathcal{Q}_c^{(i)} \hookrightarrow \mathcal{Q}_c^{(i+1)}$ is exact (i.e. $\mathcal{Q}_c^{(i+1)} \cap (\mathcal{Q}_c^{(i)})^{\mathrm{gp}} = \mathcal{Q}_c^{(i)}$).*

Proof. (1): By Lemma 1.2.31, it suffices to show that $\mathcal{Q}_c^{(1)}$ is saturated. Pick an element x of $(\mathcal{Q}_c^{(1)})^{\mathrm{gp}}$ such that $nx \in \mathcal{Q}_c^{(1)}$. Then the element cx of $\mathcal{Q}^{\mathrm{gp}}$ satisfies $n(cx) = c(nx) \in \mathcal{Q}$. Hence $cx \in \mathcal{Q}$ because $\mathcal{Q}$ is saturated. This implies that $x \in \mathcal{Q}_c^{(1)}$, as desired.

(2): It suffices to show that $\mathcal{Q}_c^{(i+1)} \cap (\mathcal{Q}_c^{(i)})^{\mathrm{gp}} \subseteq \mathcal{Q}_c^{(i)}$. Pick an element $a \in \mathcal{Q}_c^{(i+1)} \cap (\mathcal{Q}_c^{(i)})^{\mathrm{gp}}$. Then $ca \in \mathcal{Q}_c^{(i)}$. Since $\mathcal{Q}_c^{(i)}$ is saturated by (1), it implies that $a \in \mathcal{Q}_c^{(i)}$, as desired. □

If further $\mathcal{Q}$ is fine, one can learn more about $\mathbb{Z}[\iota_c^{(i)}] : \mathbb{Z}[\mathcal{Q}_c^{(i)}] \to \mathbb{Z}[\mathcal{Q}_c^{(i+1)}]$ using the exactness of $\iota_c^{(i)}$ assured by Lemma 1.2.33 (2).

Lemma 1.2.34. *Let $\mathcal{Q}$ be a fine, sharp and saturated monoid. Let c and i be non-negative integers with $c > 0$. Set $G_i := \mathcal{Q}_c^{(i+1)}/\mathcal{Q}_c^{(i)}$ (which is a finite abelian group by Lemma 1.2.32 (3)) and $K_i := \operatorname{Frac}(\mathbb{Z}[\mathcal{Q}_c^{(i)}])$. Then the following assertions hold.*

1. *For any $g \in G_i$, we have an isomorphism of $\mathbb{Z}[\mathcal{Q}_c^{(i)}]$-modules $\mathbb{Z}[(\mathcal{Q}_c^{(i+1)})_g] \otimes_{\mathbb{Z}[\mathcal{Q}_c^{(i)}]} K_i \cong K_i$.*

2. *The base extension $K_i \to \mathbb{Z}[\mathcal{Q}_c^{(i+1)}] \otimes_{\mathbb{Z}[\mathcal{Q}_c^{(i)}]} K_i$ of $\mathbb{Z}[\iota_c^{(i)}]$ is isomorphic to the split injection*
$$K_i \hookrightarrow (K_i)^{\oplus |G_i|} \; ; \; a \mapsto (a, 0, \dots, 0)$$
as a K_i-linear map. In particular, $\dim_{K_i}\big(\mathbb{Z}[\mathcal{Q}_c^{(i+1)}] \otimes_{\mathbb{Z}[\mathcal{Q}_c^{(i)}]} K_i\big) = |\mathcal{Q}_c^{(i+1)}/\mathcal{Q}_c^{(i)}|$.

Proof. In view of Lemma 1.2.31 (2), Lemma 1.2.32 (1), and Lemma 1.2.33 (1), it suffices to show the assertions only for the case when $i = 0$.

(1): Let $y_g \in \mathcal{Q}_c^{(1)}$ be an element whose image in $\mathcal{Q}_c^{(1)}/\mathcal{Q}$ is equal to g. Then we obtain an injective homomorphism of $\mathcal{Q}$-modules

$$\iota_g : \mathcal{Q} \hookrightarrow (\mathcal{Q}_c^{(1)})_g \ ; \ x \mapsto x + y_g, \tag{1.5}$$

which induces an injective $\mathbb{Z}[\mathcal{Q}]$-linear map $\mathbb{Z}[\iota_g] : \mathbb{Z}[\mathcal{Q}] \hookrightarrow \mathbb{Z}[(\mathcal{Q}_c^{(1)})_g]$. Thus it suffices to show that $\mathrm{Coker}(\mathbb{Z}[\iota_g]) \otimes_{\mathbb{Z}[\mathcal{Q}]} K_0 = (0)$, i.e., $\mathrm{Coker}(\mathbb{Z}[\iota_g])$ is a torsion $\mathbb{Z}[\mathcal{Q}]$-module. On the other hand, we also have a homomorphism of $\mathcal{Q}$-modules

$$(\mathcal{Q}_c^{(1)})_g \to \mathcal{Q}^{\mathrm{gp}} \ ; \ y \mapsto y - y_g,$$

which induces an embedding of $\mathbb{Z}[\mathcal{Q}]$-modules $\mathrm{Coker}(\mathbb{Z}[\iota_g]) \hookrightarrow \mathbb{Z}[\mathcal{Q}^{\mathrm{gp}}]/\mathbb{Z}[\mathcal{Q}]$. Since $\mathbb{Z}[\mathcal{Q}^{\mathrm{gp}}]/\mathbb{Z}[\mathcal{Q}]$ is $\mathbb{Z}[\mathcal{Q}]$-torsion, the assertion follows.

(2): It immediately follows from the combination of Lemma 1.2.33 (2), Proposition 1.2.23 (2), and the assertion (1) of this lemma. □

1.2.3 Krull monoids and their divisor class group

In this subsection, we give an easy review of divisor class groups of Krull monoids. Krull monoids have a long history in factorization theory and they are related to many mathematical fields, such as algebraic number theory, analytic number theory, combinatorial theory, and commutative ring theory. First, we define fractional ideals of monoids.

Definition 1.2.35 (Fractional ideals of monoids)**.** Let $\mathcal{Q}$ be an integral monoid. Then a *fractional ideal* of $\mathcal{Q}$ is a $\mathcal{Q}$-submodule $I \subseteq \mathcal{Q}^{\mathrm{gp}}$ such that $I \neq \emptyset$ and $xI := \{x + a \mid a \in I\} \subset \mathcal{Q}$ for some $x \in \mathcal{Q}$.

Lemma 1.2.36. *Let $\mathcal{Q}$ be an integral monoid. Then the following hold.*

1. *If $I_1, \dots, I_n$ are fractional ideals of $\mathcal{Q}$, then $\bigcap_{i=1}^{n} I_i$ is fractional.*
2. *If I_1, I_2 are fractional ideals of $\mathcal{Q}$, then $I_1 I_2 := \{x + y \mid x \in I_1, y \in I_2\}$ is also fractional.*

Proof. (1): Since I_i is a fractional ideal, there exists an element $a_i \in \mathcal{Q}$ such that $a_i I_i \subseteq \mathcal{Q}$. Then $a_i J \subset a_i I_i \subset \mathcal{Q}$.

(2): Let $a_1, a_2 \in \mathcal{Q}$ such that $a_1 I_1 \subset \mathcal{Q}$ and $a_2 I_2 \subset \mathcal{Q}$. Thus $a_1 a_2 (I_1 I_2) \subset P$. □

We say that a fractional ideal I is *finitely generated* if it is finitely generated as a $\mathcal{Q}$-module. For any two fractional ideal I_1 and I_2, we define $(I_1 : I_2) := \{x \in \mathcal{Q}^{\mathrm{gp}} \mid x \cdot I_2 \subseteq I_1\}$.

Lemma 1.2.37. *Let $\mathcal{Q}$ be an integral monoid and let I_1 and I_2 be fractional ideals. Then $(I_1 : I_2)$ is also a fractional ideal.*

Proof. Let $a_1 \in \mathcal{Q}^{\mathrm{gp}}$ such that $a_1 I_1 \subseteq \mathcal{Q}$. Pick an element $a \in I_2$. For any $z \in (I_1 : I_2)$, $az \in I_1$. Thus $a_1 az \in a_1 I_1 \subset \mathcal{Q}$. This implies $(a_1 a) \cdot (I_1 : I_2) \subset \mathcal{Q}$, as desired. □

For a fractional ideal I of $\mathcal{Q}$, we set $I^{-1} := (\mathcal{Q} : I)$ and $I^* := (I^{-1})^{-1}$. We say that a fractional ideal I is *reflexive* if $I^* = I$ holds.

Lemma 1.2.38. *Let $\mathcal{Q}$ be an integral monoid and I and J be fractional ideals. Then the following hold.*

1. *If $I \subset J$, then $J^{-1} \subset I^{-1}$ and $I^* \subset J^*$ hold.*
2. *$I \subset I^*$ holds.*
3. *I^* is reflexive. Especially, I^* is the smallest reflexive fractional ideal containing I.*
4. *For any $a \in \mathcal{Q}^{\mathrm{gp}}$, $aI^{-1} = (a^{-1}I)^{-1}$ and $aI^* = (aI)^*$ hold.*
5. *$(IJ)^* = (I^*J^*)^*$ holds.*

Proof. (1): Let $a \in \mathcal{Q}^{\mathrm{gp}}$ such that $aJ \subset \mathcal{Q}$. Since $I \subseteq J$, we have $aI \subset aJ \subset \mathcal{Q}$, as desired. The latter assertion follows from the former assertion.

(2): Pick $a \in I$. For any $z \in (\mathcal{Q} : I)$, $zI \subset \mathcal{Q}$, in particular $za \in \mathcal{Q}$. Thus $a \in (\mathcal{Q} : (\mathcal{Q} : I))$.

(3): This is the same proof as in [SM64, Lemma 1.2 (1)].

(4): The inclusion $aI^{-1} \subseteq (a^{-1}I)^{-1}$ obviously holds. Conversely, pick an element $z \in (a^{-1}I)^{-1}$. Then we have $(a^{-1}z)I = z(a^{-1}I) \subseteq \mathcal{Q}$. This implies that $z \subseteq aI^{-1}$, as desired. Next, by the former equality, we obtain $aI^* = (a^{-1}I^{-1})^{-1} = (((a^{-1})^{-1}I)^{-1})^{-1} = (aI)^*$.

(5): Pick $a \in I^*$. Then $(aJ^*)^* = a(J^*)^* = aJ^*$. This implies that $(I^*J^*)^* = I^*J^*$. Pick $b \in J^*$. Then $I^*b = (Ib)^*$. This implies $I^*J^* = (IJ^*)^*$. Finally, pick $c \in I$. Then $(cJ^*)^* = (cJ)^{**} = (cJ)^*$. This implies that $(IJ^*)^* = (IJ)^*$. To summarize these, we obtain $(I^*J^*)^* = I^*J^* = IJ^* = (IJ)^*$, as desired. □

Definition 1.2.39. Let $\mathcal{Q}$ be an integral monoid. We denote by $\mathrm{Div}(\mathcal{Q})$ the set of all reflexive fractional ideals of $\mathcal{Q}$, We define a binary operation on $\mathrm{Div}(\mathcal{Q})$ by

$$I \bullet J := (IJ)^*.$$

Note that a monoid $\mathcal{Q}$ is a reflexive fractional ideal. Moreover, for a reflexive fractional ideal I, $\mathcal{Q} \bullet I = I \bullet \mathcal{Q} = I$. Hence $(\mathrm{Div}(\mathcal{Q}), \bullet)$ is a monoid. To discuss when $\mathrm{Div}(\mathcal{Q})$ becomes a group, we define the completely integrally closedness of a monoid.

Definition 1.2.40. Let $\mathcal{Q}$ be an integral monoid.

1. An element $x \in \mathcal{Q}^{\mathrm{gp}}$ is called *almost integral over* $\mathcal{Q}$ if there exists $c \in \mathcal{Q}$ such that $c + nx \in \mathcal{Q}$ for any $n \in \mathbb{Z}_{>0}$.
2. $\mathcal{Q}$ is called *completely integrally closed* if all almost integral elements over $\mathcal{Q}$ lie in $\mathcal{Q}$.

We note that the set of elements of $\mathcal{Q}^{\mathrm{gp}}$ which are almost integral over $\mathcal{Q}$ is a monoid. Indeed, for an almost integral element of $x, y \in \mathcal{Q}^{\mathrm{gp}}$, there exist elements $a, b \in \mathcal{Q}$ such that $a+nx, b+ny \in \mathcal{Q}$ for any $n \in \mathbb{Z}_{>0}$. Since we have $(a+b)+n(x+y) = (a+nx)+(b+ny) \in \mathcal{Q}$, $x + y$ is also almost integral over $\mathcal{Q}$.

Proposition 1.2.41. *Let $\mathcal{Q}$ be an integral monoid. Then the following assertions hold.*

1. *$(\mathrm{Div}(\mathcal{Q}), \bullet)$ is an abelian group if and only if $\mathcal{Q}$ is completely integrally closed.*
2. *If $\mathcal{Q}$ is fine and saturated, then $\mathcal{Q}$ is completely integrally closed.*

Proof. The assertion (1) is [GR23, Proposition 6.4.42 (i)] and the assertion (2) is [GR23, Proposition 6.4.42 (ii)]. □

Next, we define Krull monoids.

Definition 1.2.42 (Krull monoids)**.** Let $\mathcal{Q}$ be an integral monoid. Then $\mathcal{Q}$ is a *Krull monoid* if the following two condition hold:

1. The set of reflexive fractional ideals of $\mathcal{Q}$ contained in $\mathcal{Q}$ satisfies the ascending chain condition, that is, for any sequence $I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots$ of reflexive fractional ideals, there exists a number $n \geq 0$ such that $I_m = I_{m+1}$ for any $m \geq n$.
2. $\mathcal{Q}$ is completely integrally closed.

Lemma 1.2.43. *Let $\mathcal{Q}$ be an integral monoid. Then the following assertions hold.*

1. *$\mathcal{Q}$ is completely integrally closed if and only if $\mathcal{Q}_{\mathrm{red}}$ is completely integrally closed.*
2. *$\mathcal{Q}$ is a Krull monoid if and only if $\mathcal{Q}_{\mathrm{red}}$ is a Krull monoid.*

Proof. These are [GHK06, Corollary 2.3.6]. □

Krull monoids possess many properties that Krull rings have. In particular, the following Proposition 1.2.44 is important to compute divisor class groups.

Proposition 1.2.44. *Let $\mathcal{Q}$ be an integral monoid and let $D \subset \operatorname{Spec} \mathcal{Q}$ be the subset of all prime ideals of height one. Then $\mathcal{Q}$ is Krull if and only if there is an isomorphism $\mathbb{Z}^{\oplus D} \cong \operatorname{Div}(\mathcal{Q})$ as an abelian group.*

Proof. The proof is the same as in [SM64, Theorem 3.1]. □

Keep the notation as in Proposition 1.2.44. Let us denote $(n_{\mathfrak{p}})_{\mathfrak{p}\in D} \in \mathbb{Z}^{\oplus D}$ by $\sum_{\mathfrak{p}\in D} n_{\mathfrak{p}}\mathfrak{p}$. Also let us denote $\mathrm{div} : \mathbb{Z}^{\oplus D} \xrightarrow{\cong} \operatorname{Div}(\mathcal{Q})$.

Definition 1.2.45. Let $\mathcal{Q}$ be an integral monoid and let $a \in \mathcal{Q}^{\mathrm{gp}}$ be an element. Then we define a *principal fractional ideal* as $\{a + q \mid q \in \mathcal{Q}\}$. Moreover, we denote the set of principal fractional ideals by $\operatorname{Prin}(\mathcal{Q})$.

Lemma 1.2.46. *Let $\mathcal{Q}$ be an integral monoid and let I, J be fractional ideals of $\mathcal{Q}$. Here we define $I \sim J$ if there exists an element $a \in \mathcal{Q}^{\mathrm{gp}}$ such that $I = aJ$. Then $\sim$ is an equation relation.*

Proof. This is straightforward. □

Definition 1.2.47 (The divisor class groups of monoids)**.** Let $\mathcal{Q}$ be an integral monoid. Then we define *the divisor class group of $\mathcal{Q}$* as $\operatorname{Div}(\mathcal{Q})/\sim$ and denote this by $\operatorname{Cl}(\mathcal{Q})$.

For an integral monoid $\mathcal{Q}$, $\operatorname{Cl}(\mathcal{Q})$ is a monoid (its binary operation is induced by that of $\operatorname{Div}(\mathcal{Q})$). Moreover, if $\mathcal{Q}$ is completely saturated, then $\operatorname{Cl}(\mathcal{Q})$ is an abelian group.

Here assume that $\mathcal{Q}$ is a Krull monoid. Let $\mathfrak{p} \in \operatorname{Spec}(Q)$ be a height one prime ideal of $\mathcal{Q}$. If $\mathfrak{p}$ is a principal ideal, then $\mathrm{div}(\mathfrak{p})$ is contained in a principal fractional ideal of $\mathcal{Q}$ by Proposition 1.2.44. Hence we obtain $\mathrm{div}^{-1}(\operatorname{Prin}(Q)) = \{\sum_{\mathrm{ht}\,\mathfrak{p}=1} n_{\mathfrak{p}}\mathfrak{p} \in \mathbb{Z}^D \mid \mathfrak{p} \text{ is principal}\}$ and

$$\overline{\mathrm{div}} : \mathbb{Z}^D/\,\mathrm{div}^{-1}(\operatorname{Prin}(Q)) \xrightarrow{\cong} \operatorname{Cl}(\mathcal{Q}). \tag{1.6}$$

By this isomorphism, we obtain the following result.

Corollary 1.2.48. *Let $\mathcal{Q}$ be a Krull monoid. Then the following assertions are equivalent.*

1. $\operatorname{Cl}(\mathcal{Q}) = 0$.
2. *Any height one prime ideal of $\mathcal{Q}$ is principal.*

1.2.4 Monoids associated with convex polyhedral cones

Here we review on monoids associated with convex polyhedral cones appearing in toric theory. The terminologies are based on [Ful93] and [CLS11]. We refer the reader to [GR23, Section 6.3] for a more general theory of cones.

Let $M \cong \mathbb{Z}^n$ be a lattice with the $\mathbb{R}$-vector space $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ and let $N := \mathrm{Hom}(M, \mathbb{Z}) \cong \mathbb{Z}^n$ be the dual of M with $N_{\mathbb{R}}^* = N \otimes_{\mathbb{Z}} \mathbb{R}$.

A *convex polyhedral cone* is a set

$$\sigma := \mathrm{Cone}(S) = \{r_1 v_1 + \cdots + r_s v_s \in M_{\mathbb{R}} \mid r_i \in \mathbb{R}_{\geq 0}\}$$

where $S = \{v_1, \ldots, v_s\} \subset M_{\mathbb{R}}$.

For a convex polyhedral cone σ generated by $v_1, \ldots, v_r$, we denote by $-\sigma$ the convex polyhedral cone generated by $-v_1, \ldots, -v_s$. Then *the dimension of* σ is defined as the dimension of the $\mathbb{R}$-vector space $\mathbb{R} \cdot \sigma = \sigma + (-\sigma)$. Also, the dual of a set σ is

$$\sigma^\vee := \{u \in N_{\mathbb{R}} \mid \langle u, v \rangle \geq 0 \text{ for all } v \in \sigma\}.$$

Note that we have a duality $(\sigma^\vee)^\vee = \sigma$. Also, we have a duality for convex polyhedral cones, which is called *Farkas's theorem.*

Proposition 1.2.49 (Farkas's theorem)**.** *The dual of a convex polyhedral cone is a convex polyhedral cone.*

A convex polyhedral cone σ is *rational* if generators of σ are contained in M. Note that if σ is a rational convex polyhedral cone, so is $\sigma^\vee$. The following proposition called Gordon's theorem is essential to connect affine semigroup rings with toric varieties.

Proposition 1.2.50 (Gordon's lemma)**.** *If σ is a rational convex polyhedral cone, then $S_\sigma = \sigma^\vee \cap M$ is a finitely generated monoid.*

By Gordon's lemma, we obtain a monoid algebra $k[S_\sigma] = k[\sigma^\vee \cap M]$, which is called a *toric ring.* The class of toric rings reflects many combinatorial and geometric properties of convex polyhedral cones.

As one of the examples of toric rings, we introduce *Hibi rings.* For details, see [Hib87]. Let $P = \{p_1, \ldots, p_{d-1}\}$ be a finite partially ordered set equipped with a partial order $\preceq$. For two elements $p, q \in P$, we say that p *covers* q if $q \preceq p$ and there does not exist $p' \in P$ such that $q \preceq p' \preceq p$. We set $\hat{P} := P \cup \{p_0 := \hat{0}, p_d := \hat{1}\}$, where $\hat{0}$ (resp. $\hat{1}$) is the unique minimal (resp. maximal) element not belonging to P.

Next, let us consider the graph $\mathcal{H}(\hat{P})$ whose vertices are elements of $\widehat{P}$ and edges are $\{p_i, p_j\}$ where p_i covers p_j. Then the $d \times n$ matrix $H = (h_{p_i, e_j})$ is defined as follows:

$$h_{p_i e_j} = \begin{cases} 1 & (\text{if } p_i \text{ is the lower point of the edge } e_j) \\ -1 & (\text{if } p_i \text{ is the upper point of the edge } e_j) \\ 0 & (\text{othewise}). \end{cases}$$

We set the cone $\sigma_P := Cone(v_1, \ldots, v_n)$ where v_i is the i-th row vector of the matrix H. Then the dual $\sigma_P^\vee$ is $\{u \in N_{\mathbb{R}} \mid \langle u, v_i \rangle \geq 0 \text{ for any } i = 1, \ldots, n\}$ and the toric ring $k[\sigma_P^\vee \cap M]$ is called a *Hibi ring associated with* P.

Theorem 1.2.51. *Let P be a partially ordered set and let R be a Hibi ring associated with P. Then the divisor class group $\mathrm{Cl}(R)$ is isomorphic to $\mathbb{Z}^{d-n}$ where $d-1$ is the number of elements of P and n is the number of edges of $\mathcal{H}(\hat{P})$.*

In section 1.7, we discuss the divisor class groups of local log-regular rings. We also give an example of local log-regular rings whose monoids define Hibi rings.

At the final of this section, we provide important embeddings of monoids. This is obtained by using a discussion of convex polyhedral cones.

Lemma 1.2.52. *Let $\mathcal{Q}$ be a fine and sharp monoid.*

1. *The equality* $\dim(\mathcal{Q}) = \mathrm{rank}(\mathcal{Q}^{\mathrm{gp}})$ *holds.*
2. *Assume that $\mathcal{Q}^{\mathrm{gp}}$ is a torsion-free abelian group of rank r. Then there is an injective monoid homomorphism $\mathcal{Q} \hookrightarrow \mathbb{N}^r$.*

Proof. The assertion 1 is [GR23, Corollary 6.4.12 (i)] and the assertion 2 is [GR23, Corollary 6.4.12 (iv)]. □

1.3 Definition of local log-regular rings

Here, we introduce the definition of local log-regular rings. Local log-regular rings are defined as commutative rings equipped with log structures. First, we recall the definition of log rings.

Definition 1.3.1. Let R be a ring, let $\mathcal{Q}$ be a monoid, and let $\alpha : \mathcal{Q} \to R$ be a homomorphism of monoids. Here we regard R as a multiplicative monoid. Then we say that the triple $(R, \mathcal{Q}, \alpha)$ is a *log ring* (or a *prelog ring*).

Here we give examples of log rings.

Example 1.3.2.
1. *Let R be a ring, let R^* be the group of units of R, and let ι be the inclusion map $R^* \hookrightarrow R$. Then (R, R^*, ι) is a log ring, which is called the trivial log ring.*
2. *Let A be a ring, let $\mathcal{Q}$ be a monoid, and let $A[\mathcal{Q}]$ be the monoid ring. Then $(A[\mathcal{Q}], \mathcal{Q}, \iota_{\mathcal{Q}})$ is a log ring where $A[\mathcal{Q}]$ is the monoid algebra over A asssicated to $\mathcal{Q}$ and $\iota_{\mathcal{Q}}$ is the monoid homomorphism such that $q \mapsto \mathbf{e}^q$.*

The notion of locality of log rings is important to define local log-regular rings.

Definition 1.3.3. A log ring $(R, \mathcal{Q}, \alpha)$ is *local* if R is a local ring and the equality $\alpha^{-1}(R^*) = \mathcal{Q}^*$ holds.

Lemma 1.3.4. *Let $(R, \mathcal{Q}, \alpha)$ be a local log ring, and let S be a local ring with a local map $\phi : R \to S$. Then $(S, \mathcal{Q}, \phi \circ \alpha)$ is also a local log ring.*

Proof. This follows from the combination of equalities $\phi^{-1}(S^\times) = R^\times$ and $\alpha^{-1}(R^\times) = \mathcal{Q}$. □

We define the log-regularity of commutative rings.

Definition 1.3.5. Let $(R, \mathcal{Q}, \alpha)$ be a local log ring where R is Noetherian and $\overline{\mathcal{Q}}$ is fine and saturated. Let I_α be the ideal of R generated by the image of elements of $\mathcal{Q}^+$. Then we say that $(R, \mathcal{Q}, \alpha)$ is a *local log-regular ring* if it satisfies the following conditions:

1. R/I_α is a regular local ring.

2. The equality $\dim(R) = \dim(R/I_\alpha) + \dim(\mathcal{Q})$ holds.

We give typical examples of local log-regular rings.

Example 1.3.6. *1. Regular local rings are trivial examples of local log-regular rings. Namely, let R be a regular local ring, let $x_1, \ldots, x_d$ be a regular system of parameters of R, and let $\alpha : \mathbb{N}^d \to R$ be the monoid homomorphism such that $\alpha(\mathbf{e}_i) = x_i$. Then $(R, \mathbb{N}^d, \alpha)$ be a local log-regular ring.*

2. Complete monoid algebras over regular local rings are also local log-regular rings. Let R be a regular local ring and let $\mathcal{Q}$ be a fine, sharp, and saturated monoid. Then $(R[\![\mathcal{Q}]\!], \mathcal{Q}, \iota)$ be a local log-regular ring, where $\iota : \mathcal{Q} \to R[\![\mathcal{Q}]\!]$ is the monoid homomorphism such that $\iota(q) = e^q$.

Remark 1.3.7. We note that a monoid $\mathcal{Q}$ appearing in Definition 1.3.5 has a decomposition $\mathcal{Q} \cong \overline{\mathcal{Q}} \times \mathcal{Q}^*$ by Lemma 1.2.18. This implies that the natural projection $\pi : \mathcal{Q} \twoheadrightarrow \overline{\mathcal{Q}}$ splits as a monoid homomorphism. Thus α extends to the homomorphism $\overline{\alpha} : \overline{\mathcal{Q}} \to R$ along π. This implies that we obtain another log structure $(R, \overline{\mathcal{Q}}, \overline{\alpha})$, which becomes a local log-regular ring with a fine, sharp, and saturated monoid.

In order to know whether the class of local log-regular rings is nice or not, the definition is not enough. Indeed, the following structure theorem derives many properties.

Theorem 1.3.8 (Kato)**.** *Let $(R, \mathcal{Q}, \alpha)$ be a local log ring where R is Noetherian and $\mathcal{Q}$ is fine, sharp, and saturated. Let k be the residue field of R and let $\mathfrak{m}_R$ be the maximal ideal. Let r be the rank of $\mathcal{Q}^{\mathrm{gp}}$. Then the following assertions hold.*

1. Suppose that R is of equal characteristic. Then $(R, \mathcal{Q}, \alpha)$ is a local log-regular ring if there exists a following commutative diagram:

$$\begin{array}{ccc} \mathcal{Q} & \longrightarrow & k[\![\mathcal{Q} \oplus \mathbb{N}^r]\!] \\ \downarrow & & \downarrow \psi \\ R & \longrightarrow & \widehat{R}, \end{array} \tag{1.7}$$

where $\widehat{R}$ is the completion along the maximal ideal and ψ is an isomorphism.

2. Suppose that R is of mixed characteristic. Then $(R, \mathcal{Q}, \alpha)$ is a local log-regular ring if there exists a following commutative diagram:

$$\begin{array}{ccc} \mathcal{Q} & \longrightarrow & C(k)[\![\mathcal{Q} \oplus \mathbb{N}^r]\!] \\ \downarrow & & \downarrow \psi \\ R & \longrightarrow & \widehat{R}, \end{array} \tag{1.8}$$

where $\widehat{R}$ is the completion along the maximal ideal and ψ is a surjective ring map with $\mathrm{Ker}(\psi) = (\theta)$ for some element $\theta \in \mathfrak{m}_{\widehat{R}}$ whose constant term is p. Moreover, for any element $\theta' \in \mathrm{Ker}(\psi)$ whose constant term is p, $\mathrm{Ker}(\psi) = (\theta')$ holds.

Proof. The assertion 1 and the first part of the assertion 2 is [Ogu18, Chapter **III**, Theorem 1.11.2]. Thus let us prove the second part of the assertion 2. $\square$

The next lemma can be obtained by following the proof of [Ogu18, Chapter **III**, Theorem 1.11.2].

Lemma 1.3.9. *Let $(R, \mathcal{Q}, \alpha)$ be a local log-regular ring. Let $\mathbf{e}_1, \dots \mathbf{e}_r$ be the canonical bases on $\mathbb{N}^r$ and let $x_1, \dots, x_r$ be a sequence of elements of R such that $\overline{x_1}, \dots, \overline{x_r}$ is a regular system of parameters for R/I_α. Then a homomorphism ψ which appears in Theorem 1.3.8 sends $\mathbf{e}_i$ to $\widehat{x_i}$ where $\widehat{x_i}$ is the image of x_i in $\widehat{R}$.*

The above structure theorem provides many examples of local log-regular rings.

Example 1.3.10. *Let $\mathcal{Q}$ be the submonoid of $\mathbb{N}^3$ generated by*

$$x_1 = (1, 1, 0, 0), x_2 = (0, 0, 1, 1), x_3 = (1, 0, 1, 0), x_4 = (0, 1, 0, 1).$$

Set $R := \mathbb{Z}_p[\![\mathcal{Q}]\!]/(p - x_4)$. Let $\alpha : \mathcal{Q} \to R$ be the monoid homomorphism such that $q \mapsto e^q$. Then the triple $(R, \mathcal{Q}, \alpha)$ is a local log-regular ring.

Proof. The local property of $(R, \mathcal{Q}, \alpha)$ is obvious. By Theorem 1.3.8 (2), we know that $(R, \mathcal{Q}, \alpha)$ is a local log-regular ring. □

We provide another example, Jungian domains, which is defined by S. Abhyankar [Abh65] (see also [Kat94, §12]). Here we recall the definition of Jungian domains and give an induced log-structure.

Definition 1.3.11 ([Abh65, P23, Definition 2])**.** Let $(R, \mathfrak{m})$ be a Noetherian local domain. We say that $(R, \mathfrak{m})$ is a *Jungian domain* if it is a two-dimensional normal domain such that the following condition satisfies: There exist integers $m, n \in \mathbb{Z}$ with $0 \le m \le n$ and $\mathrm{GCD}(m, n) = 1$ and generators $x, y, z_1, \dots, z_{n-1}$ of $\mathfrak{m}$ such that $z_i^n = x^i y^{m_i}$ for any $i = 1, \dots, n-1$, where m_i is the unique integer such that $0 \le m_i \le n$ and $m_i = mi \pmod n$.

Lemma 1.3.12. *Let $(R, \mathfrak{m})$ be a Jungian domain, let $\mathcal{M}$ be the multiplicative submonoid $\langle x^{l_1} y^{l_2} z_1^{l_3} \cdots z_{n-1}^{l_{n+1}} \in R \mid l_1, \dots, l_{n+1} \ge 0 \rangle$, and let $\alpha : \mathcal{M} \hookrightarrow R$ be the inclusion map. Then $\mathcal{M}$ is fine, sharp, and saturated. Moreover, $(R, \mathcal{M}, \alpha)$ is a local log-regular ring.*

Proof. Since R is an normal domain and $\mathcal{M}$ is generated by $x, y, z_1, \dots, z_{i-1}$, $\mathcal{M}$ is obviously fine and saturated. Moreover, it follows from $I_\alpha = \mathfrak{m}$ that $\mathcal{M}$ is sharp and R/I_α is regular. Finally, we can easily check that any prime ideal of $\mathcal{M}$ forms $\mathfrak{p} \cap \mathcal{M}$ where $\mathfrak{p}$ is a prime ideal of R. Hence $\dim(\mathcal{M}) = \dim(R)$. □

S. Abhyankar explored how to construct Jungian local domains. For example, see [Abh65, Theorem 10] or [Abh65, Theorem 14].

Next, we list basic properties of local log-regular rings. The following theorem is an analogue of the Cohen–Macaulay property for affine normal semigroup rings proved by Hochster [Hoc72].

Theorem 1.3.13 (Kato)**.** *Every local log-regular ring is Cohen-Macaulay and normal.*

Definition 1.3.14. Let $(R, \mathcal{Q}, \alpha)$ be a log ring. Then R is *α-flat* if $\mathrm{Tor}_1^{\mathbb{Z}[\mathcal{Q}]}(\mathbb{Z}[\mathcal{Q}]/\mathbb{Z}[I], R) = 0$ for any ideal $I \subseteq \mathcal{Q}$.

Under the first condition in Definition 1.3.5, the second condition is equivalent to several conditions.

Proposition 1.3.15. *Keep the notation and the assumption as in Definition 1.3.5. Assume that R/I_α is regular. Then the following are equivalent:*

1. *$(R, \mathcal{Q}, \alpha)$ is a local log-regular ring.*
2. *For every prime ideal $\mathfrak{q}$ of $\mathcal{Q}$, the ideal $\mathfrak{q}R$ is generated by $\alpha(\mathfrak{q})$ is a prime ideal of R such that $\alpha^{-1}(\mathfrak{q}R) = \mathfrak{q}$ (then we say that α is very solid).*
3. *R is α-flat.*
4. $\mathrm{Tor}_1^{\mathbb{Z}[\mathcal{Q}]}(\mathbb{Z}[\mathcal{Q}]/\mathbb{Z}[\mathcal{Q}^+], R) = 0$.
5. $\mathrm{gr}_{\mathbb{Z}[\mathcal{Q}^+]}(\mathbb{Z}[\mathcal{Q}]) \otimes_{\mathbb{Z}} R/I_\alpha \cong \mathrm{gr}_{I_\alpha} R$ *is an isomorphism.*

Proof. The equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) are a combination of [Ogu18, Chapter **III**, Theorem 1.11.1] and [Ogu18, Chapter **III**, Proposition 1.11.5]. The equivalence (1) $\Leftrightarrow$ (3) is [Tho06, Proposition 52] □

Lemma 1.3.16. *Let $(R, \mathcal{Q}, \alpha)$ be a log ring, where $\mathcal{Q}$ is an integral monoid. Assume that α is injective. Then the image of α is contained in $R^\bullet = R\backslash\{0\}$.*

Proof. If $\mathcal{Q}$ is the zero monoid, the claim holds obviously. Thus we may assume that $\mathcal{Q}$ is a non-zero monoid. Suppose that there exists $x \in \mathcal{Q}$ such that $\alpha(x) = 0$. Then, for a non-zero element $y \in \mathcal{Q}$, we have the equality $\alpha(x + y) = \alpha(x)$. Since α is injective and $\mathcal{Q}$ is integral, we obtain $y = 0$. This is a contradiction. Thus $\mathrm{Im}\,\alpha \subseteq R^\bullet$ holds. □

In the situation of Lemma 1.3.16, we obtain the homomorphism of monoids $\alpha^\bullet : \mathcal{Q} \to R^\bullet$ which decomposes α.

Lemma 1.3.17. *Let $(R, \mathcal{Q}, \alpha)$ be a local log-regular ring. Assume that $\mathcal{Q}$ is fine, sharp, and saturated. Then $\alpha^\bullet$ is exact.*

Proof. Since $\mathcal{Q}$ is fine and saturated and $R^\bullet$ is integral, it suffices to show that $\mathrm{Spec}(\alpha^\bullet)$ is surjective by [Ogu18, Chapter **I**, Proposition 4.2.2]. For any $\mathfrak{q} \in \mathrm{Spec}(\mathcal{Q})$, $\mathfrak{q}R$ is prime of R and $\alpha^{-1}(\mathfrak{q}R) = \mathfrak{q}$ by Theorem 1.3.15 (2). Set $\mathfrak{q}^\bullet := \mathfrak{q}\backslash\{0\} \subseteq R^\bullet$. Since $\mathfrak{q}^\bullet$ is a prime ideal of $R^\bullet$, $\mathrm{Spec}(\alpha^\bullet)(\mathfrak{q}^\bullet) = \mathfrak{q}$ holds. Hence $\mathrm{Spec}(\alpha^\bullet)$ is surjective. □

1.4 Local log-regular ring are splinters

In this section, we show that any local log-regular ring is a splinter. The class of splinters is important in the recent studies on singularities theory. In [GR23], they prove that local log-regular rings are splinter by a logarithmic analogue of André's method to prove direct summand conjecture. In this section, we give an alternative and short proof of it by using the direct summand theorem. Let us recall the definition of a splinter.

Definition 1.4.1. A Noetherian ring A is a *splinter* if every finite ring map $f : A \to B$ such that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective admits an A-linear map $h : B \to A$ such that $h \circ f = \mathrm{id}_A$.

In general, it is not easy to see under what algebraic operations being a splinter is preserved. For instance, Datta and Tucker proved remarkable results ([DT23, Theorem B], [DT23, Theorem C], or [DT23, Example 3.2.1]). See also Murayama's work [Mur21] for the study of purity of ring extensions.

In order to prove the splinter property, we need a lemma on splitting a map under completion.

Lemma 1.4.2. *Let R be a ring and let $f : M \to N$ be an R-linear map. Consider a decreasing filtration of R-submodules $\{M_\lambda\}_{\lambda \in \Lambda}$ of M and a decreasing filtration of R-submodules $\{N_\lambda\}_{\lambda \in \Lambda}$ of N such that $f(M_\lambda) \subseteq N_\lambda$ for each $\lambda \in \Lambda$. Set*

$$\widehat{M} := \varprojlim_{\lambda \in \Lambda} M/M_\lambda \text{ and } \widehat{N} := \varprojlim_{\lambda \in \Lambda} N/N_\lambda,$$

respectively. Finally, assume that f is a split injection that admits an R-linear map $g : N \to M$ such that $g \circ f = \mathrm{id}_M$, $g(N_\lambda) \subseteq M_\lambda$ for each $\lambda \in \Lambda$. Then f extends to a split injection $\widehat{M} \to \widehat{N}$.

Proof. By assumption, there is an induced map

$$M/M_\lambda \xrightarrow{\overline{f}} N/N_\lambda \xrightarrow{\overline{g}} M/M_\lambda$$

which is an identity on M/M_λ. Taking inverse limits, we get an identity map $\widehat{M} \to \widehat{N} \to \widehat{M}$, which proves the lemma. □

The next result is originally due to Gabber and Ramero [GR23, Theorem 17.3.12],[1] and we provide an alternative and short proof, using the Direct Summand Theorem.

Theorem 1.4.3. *A local log-regular ring $(R, \mathcal{Q}, \alpha)$ is a splinter.*

Proof. First, we prove the theorem when R is complete. By Remark 1.3.7, we may assume that $\mathcal{Q}$ is fine, sharp, and saturated. By Theorem 1.3.8, we have

$$R \cong k[\![\mathcal{Q} \oplus \mathbb{N}^r]\!], \text{ or } R \cong C(k)[\![\mathcal{Q} \oplus \mathbb{N}^r]\!]/(p - f),$$

depending on whether R contains a field or not. Let us consider the mixed characteristic case. By Lemma 1.2.26, there is a split injection $C(k)[\mathcal{Q} \oplus \mathbb{N}^r] \to C(k)[\mathbb{N}^d]$ for some $d > 0$, which comes from an injection $\delta : \mathcal{Q} \oplus \mathbb{N}^r \to \mathbb{N}^d$ that realizes $\delta(\mathcal{Q} \oplus \mathbb{N}^r)$ as an exact submonoid of $\mathbb{N}^d$. After dividing out by the ideal $(p - f)$, we find that the map

$$C(k)[\![\mathcal{Q} \oplus \mathbb{N}^r]\!]/(p - f) \to C(k)[\![\mathbb{N}^d]\!]/(p - f)$$

splits as a $C(k)[\![\mathcal{Q} \oplus \mathbb{N}^r]\!]/(p - f)$-linear map by Remark 1.2.24 and Lemma 1.4.2. Hence, R becomes a direct summand of the complete regular local ring $A := C(k)[\![x_1, \ldots, x_d]\!]/(p - f)$. Pick a map $\alpha : A \to R$ that splits $R \to A$. Consider a module-finite extension $R \to S$ such that S is a domain. We want to show that this map splits. Now there is a commutative diagram:

$$\begin{array}{ccc} R^+ & \longrightarrow & A^+ \\ \uparrow & & \uparrow \\ S & \xrightarrow{\gamma} & B \\ \uparrow & & \uparrow \\ R & \longrightarrow & A \end{array}$$

where R^+ (resp. A^+) is the absolute integral closure of R (resp. A), and B is a subring of A^+ that is constructed as the chain of S and A, thus being finite over A. By the

[1] One notices that the treatment of logarithmic geometry in [GR23] is topos-theoretic, while [Kat94] considers mostly the Zariski sites.

Direct Summand Theorem [And18], there is a map $\beta : B \to A$ that splits $A \to B$. Therefore, the composite map $S \xrightarrow{\gamma} B \xrightarrow{\beta} A \xrightarrow{\alpha} R$ splits $R \to S$, as desired. The case where $R = k[\![\mathcal{Q} \oplus \mathbb{N}^r]\!]$ can be treated similarly.

Next, let us consider the general case. Let $R \to S$ be a module-finite extension with S being a domain, and let $\widehat{R}$ be as in Theorem 1.3.8. By applying the functor $(\ ) \otimes_R \widehat{R}$ to the exact sequence $0 \to R \to S \to S/R \to 0$, we get an exact sequence: $0 \to \widehat{R} \to S \otimes_R \widehat{R} \to S/R \otimes_R \widehat{R} \to 0$. We have proved that $\widehat{R}$ is a splinter, so the induced sequence

$$0 \to \mathrm{Hom}_{\widehat{R}}(S/R \otimes_R \widehat{R}, \widehat{R}) \to \mathrm{Hom}_{\widehat{R}}(S \otimes_R \widehat{R}, \widehat{R}) \to \mathrm{Hom}_{\widehat{R}}(\widehat{R}, \widehat{R}) \to 0$$

is exact. By the faithful flatness of $\widehat{R}$ over R, the above exact sequence induces the exact sequence:

$$0 \to \mathrm{Hom}_R(S/R, R) \to \mathrm{Hom}_R(S, R) \to \mathrm{Hom}_R(R, R) \to 0,$$

and we conclude. □

Next, we prove that F-finite local log-regular rings are strongly F-regular. Let us recall the definition of strong F-regularity.

Definition 1.4.4 (Strong F-regularity)**.** Let R be a Noetherian reduced $\mathbb{F}_p$-algebra that is F-finite. Let $F^e_* R$ be the same as R as its underlying abelian groups with its R-module structure via restrictions of scalars via the e-th iterated Frobenius endomorphism F^e_R on R. Then we say that R is *strongly F-regular*, if for any element $c \in R$ that is not in any minimal prime of R, there exists an $e > 0$ and a map $\phi \in \mathrm{Hom}_R(F^e_* R, R)$ such that $\phi(F^e_* c) = 1$.

It is well-known that strongly F-regular rings are splinter. Hence the following theorem gives an alternative proof of Theorem 1.4.3 in characteristic $p > 0$.

Lemma 1.4.5. *Let $(R, \mathcal{Q}, \alpha)$ be a local log-regular ring of characteristic $p > 0$ such that R is F-finite. Then R is strongly F-regular.*

Proof. The proof of this theorem is almost the same as Proposition 1.6.2. The completion of R with respect to its maximal ideal is isomorphic to the completion of $k[\mathcal{Q} \oplus \mathbb{N}^r]$, and $\mathcal{Q}$ is a fine, sharp and saturated monoid by Theorem 1.3.8 and [Ogu18, Chapter **I**, Proposition 3.4.1]. Then it follows from Lemma 1.2.26 that $\mathcal{Q} \oplus \mathbb{N}^r$ can be embedded into $\mathbb{N}^d$ for $d > 0$, and $k[\mathcal{Q} \oplus \mathbb{N}^r] \to k[\mathbb{N}^d] \cong k[x_1, \ldots, x_d]$ splits as a $k[\mathcal{Q} \oplus \mathbb{N}^r]$-linear map. Applying [HH89, Theorem 3.1], we see that $k[\mathcal{Q} \oplus \mathbb{N}^r]$ is strongly F-regular. After completion, the complete local ring $k[\![\mathcal{Q} \oplus \mathbb{N}^r]\!]$ is strongly F-regular in view of [Abe01, Theorem 3.6]. Then by faithful flatness of $R \to k[\![\mathcal{Q} \oplus \mathbb{N}^r]\!]$, [HH89, Theorem 3.1] applies to yield strong F-regularity of R. □

1.5 The canonical module of a local log-regular ring

In this section, we prove the structure theorem for the canonical module of a local log-regular ring and provide a Gorenstein criterion for local log-regular rings.

Theorem 1.5.1. *Let $(R, \mathcal{Q}, \alpha)$ be a local log-regular ring, where $\mathcal{Q}$ is fine, sharp, and saturated (by Remark 1.2.25, we may assume that $\mathcal{Q} \subseteq \mathbb{N}^l$ for some $l > 0$). Let $x_1, \ldots, x_r$*

be a sequence of elements of R such that $\overline{x_1}, \ldots, \overline{x_r}$ is a regular system of parameters for $R/I_{\mathcal{Q}}$ Then R admits a canonical module and its form is

$$\langle (x_1 \cdots x_r)\alpha(a) \mid a \in \mathrm{relint}\mathcal{Q}\rangle, \tag{1.9}$$

where relint$\mathcal{Q}$ *is the relative interior of* $\mathcal{Q}$.

Proof. First, assume that R is $\mathfrak{m}$-adically complete and separated. If R is of equal characteristic, then R is isomorphic to $k[\![\mathcal{Q} \oplus \mathbb{N}^r]\!]$ by Theorem 1.3.8. Let us check that

$$k[\mathrm{relint}\mathcal{Q} \oplus (\mathbf{e} + \mathbb{N}^r)] := \langle (q, \mathbf{e}) \mid q \in \mathrm{relint}\mathcal{Q}\rangle \subseteq k[\mathcal{Q} \oplus \mathbb{N}^r] \tag{1.10}$$

is a canonical module of $k[\mathcal{Q} \oplus \mathbb{N}^r]$, where $\mathbf{e} = (1, 1, \ldots, 1) \in \mathbb{N}^r$. Indeed, note that we have the ring isomorphism $k[\mathcal{Q}] \otimes_k k[\mathbb{N}^r] \cong k[\mathcal{Q} \oplus \mathbb{N}^r]$. Also note that the canonical module $\omega_{k[\mathcal{Q}]} = k[\mathrm{relint}\mathcal{Q}]$ and $\omega_{k[\mathbb{N}^r]} = k[\mathbf{e} + \mathbb{N}^r]$ by [BH98, Theorem 6.3.5 (b)]. This induces the following isomorphisms

$$\omega_{k[\mathcal{Q} \oplus \mathbb{N}^r]} \cong \omega_{k[\mathcal{Q}]} \otimes_k \omega_{k[\mathbb{N}^r]} = k[\mathrm{relint}\mathcal{Q}] \otimes_k k[\mathbf{e} + \mathbb{N}^r]. \tag{1.11}$$

If you trace (1.11) backwards, then it turns out that $\omega_{k[\mathcal{Q} \oplus \mathbb{N}^r]}$ is the form of (1.10). Since R is isomorphic to the completion of $k[\mathcal{Q} \oplus \mathbb{N}^r]$ along a maximal ideal $k[(\mathcal{Q} \oplus \mathbb{N}^r)^+]$, the image of (1.10) in R is the canonical module of R.

If R is of mixed characteristic, then R is isomorphic to $C(k)[\![\mathcal{Q} \oplus \mathbb{N}^r]\!]/(\theta)$ for some $\theta \in W(k)[\![\mathcal{Q} \oplus \mathbb{N}^r]\!]$. If $C(k)[\![\mathcal{Q} \oplus \mathbb{N}^r]\!]$ has a canonical module, then its image in R is the canonical module of R. Thus it suffices to show the case where $R = C(k)[\![\mathcal{Q} \oplus \mathbb{N}^r]\!]$.

Set $\omega_R := \langle p(q_i, \mathbf{e}) \mid q_i \in \mathrm{relint}\mathcal{Q}\rangle \subseteq C(k)[\![\mathcal{Q} \oplus \mathbb{N}^r]\!]$. Since $\omega_R/p\omega_R$ is a canonical module of $R/pR \cong k[\![\mathcal{Q} \oplus \mathbb{N}^r]\!]$ and p is a regular element on R and ω_R, ω_R is a maximal Cohen-Macaulay module of type 1. Finally, since R is a domain, ω_R is faithful. Thus ω_R is a canonical module of R.

Next, let us consider the general case. We define the ideal ω_R as (1.9). Then, by considering the diagrams (1.7) or (1.8), the image of ω_R in the $\mathfrak{m}$-adic completion of R is the canonical module. Thus, by [BH98, Theorem 3.3.14 (b)], ω_R is a canonical module of R. □

Remark 1.5.2. Set $\omega_R := \langle (x_1 \cdots x_r)\alpha(a) \mid a \in \mathrm{relint}\mathcal{Q}\rangle$ and $\omega'_R := \langle \alpha(a) \mid a \in \mathrm{relint}\mathcal{Q}\rangle$. Then we note that the homomorphism $\omega'_R \xrightarrow{\times x_1 \cdots x_r} \omega_R$ is isomorphism. Namely, the ideal of R generated by the image of the relative interior of the associated monoid is also the canonical module of R.

Remark 1.5.3. In Theorem 1.5.1, the case when $R = W(k)[\![\sigma^\vee \cap M]\!]$ follows from the following Marcus Robinson's result[2]: Let $A := W(k)[\sigma^\vee \cap M]$, where M is a lattice and σ is the strongly convex polyhedral cone. Let $X = \mathrm{Spec}(A)$. Then one can choose codimension one subschemes $D_1, \ldots, D_n$ of X such that $K_X = -\sum D_i$ is a canonical divisor on X. This result implies that the ideal $\omega_A := \bigcap \mathfrak{p}_i$ is a canonical module of A, where $\mathfrak{p}_i$ is the corresponding height one prime ideal to D_i. By taking the localization and the completion at the maximal ideal $W(k)[(\sigma^\vee \cap M)^+]$, we find out that ω_A is $W(k)[\![\sigma^\vee \cap M]\!]$ is the canonical module.

As an application of Theorem 1.5.1, let us provide a Gorenstein criterion of local log-regular rings. In order to prove it, we need the following proposition.

[2]For readers who are not familiar with algebraic geometry, see [ST12, Appendix B].

Proposition 1.5.4. *Let $(R, \mathcal{Q}, \alpha)$ be a local log-regular ring. Let $\underline{x} := x_1, \ldots, x_r$ be a sequence of elements of R such that $\overline{x_1}, \ldots, \overline{x_r}$ is a regular system of parameters for R/I_α. Set $R_i := R/(x_1, \ldots x_i)$ and $\alpha_i : \mathcal{Q} \to R \twoheadrightarrow R_i$. Then $\underline{x}$ is a regular sequence on R and $(R_i, \mathcal{Q}, \alpha_i)$ is also a local log-regular ring for any $1 \leq i \leq r$.*

Proof. Since a local homomorphism preserves the locality of the log structure (see [INS22, Lemma 2.16]), $(R_i, \mathcal{Q}, \alpha_i)$ is a local log ring. By the induction for i, it suffices to check that $i = 1$. Since R is a domain, x_1 is a regular element. Thus we obtain the isomorphism $R_1/I_{\alpha_1} \cong (R/I_\alpha)/x_1(R/I_\alpha)$. Since the image of x_1 is a regular element on R/I_α by the assumption and R/I_α is a regular local ring, R_1/I_{α_1} is regular. Moreover, the above isomorphism implies that the equality $\dim(R_1/I_{\alpha_1}) = \dim(R_1) - \dim(\mathcal{Q})$ holds. Thus $(R_1, \mathcal{Q}, \alpha_1)$ is a local log-regular ring. □

Corollary 1.5.5. *Keep the notation as in Theorem 1.5.1. The following assertions are equivalent:*

1. *R is Gorenstein.*
2. *For a fixed field k, $k[\mathcal{Q}]$ is Gorenstein.*
3. *There exists an element $c \in \operatorname{relint} Q$ such that $\operatorname{relint} \mathcal{Q} = c + \mathcal{Q}$.*

Proof. The equivalence of (2) and (3) is well-known (for example, see [BH98, Theorem 6.3.5 (a)]). Thus it suffices to show the equivalence of (1) and (3). Since the Gorenstein property of R is preserved under the completion and the quotient by a regular sequence, one can assume that α is injective by Theorem 1.3.8 and that $\dim R = \dim(\mathcal{Q})$ by Proposition 1.5.4. Hence $\omega_R = \langle \alpha(x) \mid x \in \operatorname{relint} \mathcal{Q} \rangle$. Now, assume that R is Gorenstein. There exists an element $c \in \operatorname{relint} \mathcal{Q}$ such that $\omega_R = (\alpha(c))$. This implies that for any $a \in \operatorname{relint} \mathcal{Q}$, there exists $x \in R$ such that $\alpha(a) = \alpha(c)x$. Since we have $\alpha(a) = \alpha^\bullet(a)$ and $\alpha(c) = \alpha^\bullet(c)$ by Lemma 1.3.16, we obtain

$$\alpha^\bullet(a) = \alpha^\bullet(c)x. \tag{1.12}$$

Hence $x = \alpha^\bullet(a - c) \in \operatorname{Im}\bigl((\alpha^\bullet)^{\mathrm{gp}}\bigr)$. Since $\alpha^\bullet$ is exact, we obtain $x \in \operatorname{Im} \alpha^\bullet$. Now, there exists $y \in \mathcal{Q}$ such that $x = \alpha^\bullet(y)$. By (1.12) and the injectivity of $\alpha^\bullet$, we obtain $a = c + y \in c + \mathcal{Q}$. Hence $\operatorname{relint} \mathcal{Q} \subseteq c + \mathcal{Q}$. Since $\operatorname{relint} \mathcal{Q}$ is an ideal of $\mathcal{Q}$, the converse inclusion holds. Therefore we obtain $\operatorname{relint} \mathcal{Q} = c + \mathcal{Q}$.

Conversely, assume that $\operatorname{relint} \mathcal{Q} = c + \mathcal{Q}$ for some $c \in \operatorname{relint} \mathcal{Q}$. Then we obtain the equalities $\omega_R = \alpha(c)\langle \alpha(x) \in R \mid x \in \mathcal{Q} \rangle = \alpha(c)R$. This implies that R is Gorenstein, as desired. □

If a Cohen-Macaulay local ring has a canonical module, it is a homomorphic image of a Gorenstein local ring. Namely, we obtain the following corollary (a toric ring is always a homomorphic image of a regular ring, but we don't know whether a local log-regular ring is a homomorphic image of a regular ring or not).

Corollary 1.5.6. *Let $(R, \mathcal{Q}, \alpha)$ be a local log-regular ring. Then R is a homomorphic image of a Gorenstein local ring.*

At the last of this section, we determine the form of Gorenstein local log-regular rings consisting of two-dimensional monoids by using Corollary 1.5.5.

Proposition 1.5.7. *Let $(R, \mathcal{Q}, \alpha)$ be a local log-regular ring where $\mathcal{Q}$ is fine, sharp, and saturated. Assume that $\mathcal{Q}$ is two-dimensional. Then R is Gorenstein if and only if $\mathcal{Q}$ is isomorphic to the submonoid of $\mathbb{N}^2$ generated by $(n+1, 0), (1, 1), (0, n+1)$ for some $n \geq 1$.*

Proof. By Corollary 1.5.5, one can reduce to the case of a toric ring $k[\mathcal{Q}]$ where k is an algebraically closed field, and in this case, we know that there exists $n \geq 1$ such that $k[\mathcal{Q}]$ is isomorphic to $k[\mathcal{P}]$ where $\mathcal{P}$ is the submonoid of $\mathbb{N}^2$ generated by $(n+1,0),(1,1),(0,n+1)$. By applying [Gub98, Theorem 2.1 (b)], we can show that $\mathcal{Q}$ is isomorphic to $\mathcal{P}$, as desired. Conversely, assume that $\mathcal{Q}$ is isomorphic to the submonoid generated by $(n+1,0),(1,1),(0,n+1) \in \mathbb{N}^2$. Then $k[\mathcal{Q}]$ is Gorenstein for an algebraically closed field k because this is an A_n-type singularity. Thus R is also Gorenstein by Corollary 1.5.5, as desired. □

From the above proposition, it follows that a complete Gorenstein local log-regular ring with an associated monoid that is two-dimensional has the following form.

Corollary 1.5.8. *Let $(R, \mathcal{Q}, \alpha)$ be a local log-regular ring where $\mathcal{Q}$ is a fine, sharp, and saturated monoid. Assume that the dimension of the monoid $\mathcal{Q}$ is two. Then the following assertions hold.*

1. *Suppose that R is of equal characteristic. Then R is Gorenstein if and only if $\widehat{R}$ is isomorphic to $k[\![s^{n+1}, st, t^{n+1}, x_1, \ldots, x_r]\!]$ for some $n \geq 1$.*

2. *Suppose that R is of mixed characteristic. Then R is Gorenstein if and only if $\widehat{R}$ is isomorphic to $C(k)[\![s^{n+1}, st, t^{n+1}, x_1, \ldots x_r]\!]/(\theta)$ for some $n \geq 1$ where $C(k)$ is the Cohen ring of the residue field k and θ is an element of $C(k)[\![s^{n+1}, st, t^{n+1}, x_1, \ldots x_r]\!]$ whose constant term is p.*

Proof. These follow from Proposition 1.5.7 and Theorem 1.3.8. □

We also give examples of non-Gorenstein local log-regular rings.

Example 1.5.9. *Let $\mathcal{P}$ be a monoid generated by $(1,0),(1,1),(1,2),(1,3)$.*

1. *Set $R := \mathbb{Z}_p[\![\mathcal{P}]\!]/(p-(1,0)) \cong \mathbb{Z}_p[\![s, st, st^2, st^3]\!]/(p-s)$ and set $\alpha : \mathcal{P} \to \mathbb{Z}_p[\![\mathcal{P}]\!] \to R$ Then $(R, \mathcal{Q}, \alpha)$ is a local log-regular ring. By Proposition 1.5.7, we know that R is not Gorenstein. Moreover, R is also isomorphic to $\mathbb{Z}_p[\![pt, pt^2, pt^3]\!]$. Since the relative interior of $\mathcal{Q}$ is generated by $(1,1)$ and $(1,2)$, its canonical module ω_R is isomorphic to the ideal of $\mathbb{Z}_p[\![pt, pt^2, pt^3]\!]$ generated by pt and pt^2.*

2. *Set $S := \mathbb{Z}_p[\![\mathcal{P}]\!]$. Then $(S, \mathcal{P}, \mathcal{P} \hookrightarrow S)$ is a local log-regular ring. The canonical module of S is isomorphic to the ideal of $\mathbb{Z}_p[\![s, st, st^2, st^3]\!]$ generated by pst and pst^2.*

1.6 Local log-regular rings are pseudo-rational singularities

In this section, we prove that local log-regular rings have pseudo-rational singularities. First of all, we introduce the definition of pseudo-rationality, defined by Lipman and Tessier.

Definition 1.6.1. Let $(R, \mathfrak{m})$ be a d-dimensional Noetherian local ring. Then we say that $(R, \mathfrak{m})$ is *pseudo-rational* if it is normal, Cohen–Macaualy, analytically unramified[3], and if for every projective birational map $\pi : W \to \operatorname{Spec}(R)$, the canonical map $H^d_{\mathfrak{m}}(R) \to H^d_E(W, \mathcal{O}_W)$ is injective, where $E := \pi^{-1}(\mathfrak{m})$ denotes the closed fibre.

[3]A Noetherian local ring $(R, \mathfrak{m})$ is *analytically unramified* if its completion $\widehat{R}$ is reduced.

The following theorem is usually called *Boutot's theorem.*

Theorem 1.6.2 ([Bou87], [HH90], [HM18])**.** *Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a pure map of local rings such that $(S, \mathfrak{n})$ is regular. Then R is pseudo-rational. In particular, direct summands of regular rings are pseudo-rational.*

Proposition 1.6.3. *Let $(R, \mathcal{Q}, \alpha)$ be a local log-regular ring with the maximal ideal $\mathfrak{m}_R$. Then $(R, \mathfrak{m}_R)$ has a pseudo-rational singularity.*

Proof. It suffices to show that X is pseudo-rational. To prove this, we show that R is a pure subring of a regular ring. Also, since a local log-regular ring has the canonical module by Theorem 1.5.1, by applying [Mur22, Proposition 4.20], we may assume that R is $\mathfrak{m}$-adically complete and separated. Namely, R is isomorphic to either $k[\![\mathcal{Q} \oplus \mathbb{N}^r]\!]$ or $C(k)[\![\mathcal{Q} \oplus \mathbb{N}^r]\!]/(\theta)$. Now, we prove that R is the direct summand of a regular local ring. Our approach is the same as in the proof of Lemma 1.4.5, so we give a sketch of the proof here. We refer the reader to it for the details. Since the same argument is made, we will show the case $R \cong C(k)[\![\mathcal{Q} \oplus \mathbb{N}^r]\!]/(\theta)$. An embedding $\mathcal{Q} \hookrightarrow \mathbb{N}^r$ given in Lemma 1.2.25 induces a split injection $C(k)[\mathcal{Q} \oplus \mathbb{N}^r] \hookrightarrow C(k)[\mathbb{N}^d]$ for some $d > 0$. This induces the split injection $C(k)[\![\mathcal{Q} \oplus \mathbb{N}^r]\!] \hookrightarrow C(k)[\![\mathbb{N}^d]\!]$. And after taking the quotient by some element $\theta \in A[\![\mathcal{Q} \oplus \mathbb{N}^r]\!]$, we also obtain the split injection $C(k)[\![\mathcal{Q} \oplus \mathbb{N}^r]\!]/(\theta) \hookrightarrow C(k)[\![\mathbb{N}^d]\!]/(\theta)$. Finally, applying Theorem 1.6.2, we obtain the desired claim. □

Remark 1.6.4. There is another way to prove equal characteristic cases. If R is F-finite complete local log-regular ring, then it is strongly F-regular ring. Since strong F-regularity implies F-rationality, R is F-rational. Hence we obtain R is pseudo-rational because an F-rational ring is pseudo-rational by [Smi97, Theorem 3.1]. Also, the equal characteristic 0 case is due to [Sch08, Main Theorem A] and the above discussion.

1.7 The divisor class group of a local log-regular ring

In this section, we show that the divisor class group of a local log-regular ring is isomorphic to that of the associated monoid. This theorem was proved in [GR23] by using the vanishing theorem of a sheaf cohomology. We provide an elementary proof of it by investigating the correspondence between height one prime ideals of monoids and height one prime ideals of local log-regular rings.

Lemma 1.7.1. *Let $(R, \mathcal{Q}, \alpha)$ be a local log-regular ring and let $\mathfrak{p}$ be a height one prime ideal of R. Then the following are equivalent.*

1. *There exists a height one prime ideal $\mathfrak{q}$ of $\mathcal{Q}$ such that $\mathfrak{p} = \mathfrak{q}R$,*

2. *The intersection of $\operatorname{Im} \alpha$ and $\alpha^{-1}(\mathfrak{p})$ is not empty.*

Proof. The implication (1) ⇒ (2) is obvious, hence let us consider the implication (2) ⇒ (1). Note that $\alpha^{-1}(\mathfrak{p})$ is a height one prime ideal by assertion (2). Since α is very solid and any element of $\mathcal{Q}$ does not map to 0 ([Ish22]), we obtain $\alpha^{-1}(\mathfrak{p})R = \mathfrak{p}$. Hence assertion (1) holds. □

Lemma 1.7.2. *Let $(R, \mathcal{Q}, \alpha)$ be a log ring and let I be an ideal of $\mathcal{Q}$. Then $\mathbb{Z}[I] \otimes_{\mathbb{Z}[\mathcal{Q}]} R$ is isomorphic to $\alpha(I)R$.*

Lemma 1.7.3. *Let $(R, \mathcal{Q}, \alpha)$ be a log ring and let I, J be ideals of $\mathcal{Q}$. Assume that R is α-flat. Then*

$$\alpha(I)R \cap \alpha(J)R = \alpha(I \cap J)R$$

holds.

Proof. Let us consider the following diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}[I \cap J] & \longrightarrow & \mathbb{Z}[I] & \longrightarrow & \mathbb{Z}[I]/\mathbb{Z}[I \cap J] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z}[J] & \longrightarrow & \mathbb{Z}[\mathcal{Q}] & \longrightarrow & \mathbb{Z}[\mathcal{Q}]/\mathbb{Z}[J] & \longrightarrow & 0. \end{array}$$

By the α-flatness of R, we obtain the following diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}[I \cap J] \otimes_{\mathbb{Z}[\mathcal{Q}]} R & \longrightarrow & \mathbb{Z}[I] \otimes_{\mathbb{Z}[\mathcal{Q}]} R & \longrightarrow & \mathbb{Z}[I]/\mathbb{Z}[I \cap J] \otimes_{\mathbb{Z}[\mathcal{Q}]} R & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z}[J] \otimes_{\mathbb{Z}[\mathcal{Q}]} R & \longrightarrow & \mathbb{Z}[\mathcal{Q}] \otimes_{\mathbb{Z}[\mathcal{Q}]} R & \longrightarrow & \mathbb{Z}[\mathcal{Q}]/\mathbb{Z}[J] \otimes_{\mathbb{Z}[\mathcal{Q}]} R & \longrightarrow & 0. \end{array}$$

This diagram is isomorphic to the following one:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \alpha(I \cap J)R & \longrightarrow & \alpha(I)R & \longrightarrow & \alpha(I)R/\alpha(I \cap J)R & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \alpha(J)R & \longrightarrow & R & \longrightarrow & R/\alpha(J)R & \longrightarrow & 0. \end{array}$$

Since the vertical arrows are injective, we obtain the following exact sequence by the snake lemma.

$$0 \to \alpha(J)R/\alpha(I \cap J)R \to R/\alpha(I)R \xrightarrow{p} R/\alpha(I)R + \alpha(J)R \to 0.$$

Thus since we obtain $\alpha(J)R/\alpha(I \cap J)R \cong \operatorname{Ker} p = \alpha(I)R + \alpha(J)R/\alpha(I)R \cong \alpha(J)R/\alpha(I)R \cap \alpha(J)R$, the equality $\alpha(I \cap J)R = \alpha(I)R \cap \alpha(J)R$ holds. □

Lemma 1.7.4. *Let $(R, \mathcal{Q}, \alpha)$ be a log ring where R is a domain and $\mathcal{Q}$ is integral. Let J, J' be a fractional ideal of $\mathcal{Q}$. Then the equality $(J \cap J')R = JR \cap J'R$ holds.*

Proof. Choose $x \in \mathcal{Q}$ such that $xJ, xJ' \subseteq \mathcal{Q}$. Then it suffices to show that $x(JR \cap J'R) = xJR \cap xJ'R$, but this follows from $x(J \cap J') = xJ \cap xJ'$. □

Lemma 1.7.5. *Let $(R, \mathcal{Q}, \alpha)$ be a local log-regular ring and let $I, J, J' \subseteq \mathcal{Q}^{\mathrm{gp}}$ be fractional ideals of $\mathcal{Q}$. Assume that I is finitely generated. Then the following assertions hold.*

1. *The equality $(J : I)R = (JR : IR)$ holds.*
2. *JR is equal to $J'R$ if and only if J is equal to J'.*
3. $\operatorname{Div}(\alpha) : \operatorname{Div}(\mathcal{Q}) \to \operatorname{Div}(R)$ *is well-defined and it is injective.*

Proof. We express $I = a_1\mathcal{Q} \cup \cdots \cup a_n\mathcal{Q}$ for some $a_1, \ldots, a_n \in \mathcal{Q}^{\mathrm{gp}}$. Thus we obtain $(J : I) = a_1^{-1}J \cap \cdots \cap a_n^{-1}J$ and $(JR : IR) = a_1^{-1}JR \cap \cdots \cap a_n^{-1}JR$. Here, by Lemma 1.7.4, the equality $a_1^{-1}JR \cap \cdots \cap a_n^{-1}JR = (a_1^{-1}J \cap \cdots \cap a_n^{-1}J)R$ holds. Hence the assertion (1) holds.

Next to prove the assertion (2), we may assume that $J \subseteq J'$ after replacing J' with $J' \cup J$. Assume that $JR = J'R$. Then, since the equality $\mathbb{Z}[xJ] \otimes_{\mathbb{Z}[\mathcal{Q}]} R = \mathbb{Z}[xJ'] \otimes_{\mathbb{Z}[\mathcal{Q}]} R$ holds, we obtain $\mathbb{Z}[xJ] = \mathbb{Z}[xJ']$ by faithfully α-flatness, and hence $xJ = xJ'$ holds. The converse implication is obvious, the assertion (2) holds.

Finally, the first assertion of (3) follows from (1), and the second follows from (2). □

Proposition 1.7.6. *Let $(R, \mathcal{Q}, \alpha)$ be a local log-regular ring. Then $\mathrm{Cl}(\alpha) : \mathrm{Cl}(\mathcal{Q}) \to \mathrm{Cl}(R)$ is well-defined and it is injective.*

Proof. It follows from [GR23, Proposition 6.4.55]. □

Lemma 1.7.7. *Let $(R, \mathcal{Q}, \alpha)$ be a complete local log-regular ring. Let S be the image of α. There exists an R-algebra T such that T is a regular local ring and $S^{-1}R \cong S^{-1}T$.*

Proof. By replacing the monoid $\mathcal{Q}$ with $\mathcal{Q} \oplus \mathbb{N}^{\dim(R/I_\alpha)}$, we may assume $\dim(R/I_\alpha) = 0$.

First, suppose that R is of equal characteristic. Then R is isomorphic to $k[\![\mathcal{Q}]\!]$ by Theorem 1.3.8 (1). Here, by Lemma 1.2.52 (2), the monoid homomorphism $\mathcal{Q} \hookrightarrow \mathbb{N}^r$ induces the injective ring homomorphism $k[\![\mathcal{Q}]\!] \hookrightarrow k[\![\mathbb{N}^r]\!]$. Moreover, since $S^{-1}k[\![\mathcal{Q}]\!]$ is isomorphic to $S'^{-1}k[\![\mathbb{N}^r]\!]$ where S is the multiplicatively closed subset of $k[\![\mathcal{Q}]\!]$ generated by an element of the relative interior of $\mathcal{Q}$ and S' is its image, $S^{-1}k[\![\mathcal{Q}]\!]$ is a unique factorization domain. Thus $k[\![\mathbb{N}^r]\!]$ is the desired regular local ring.

Next, suppose that R is of mixed characteristic. Then R is isomorphic to $V[\![\mathcal{Q}]\!]/(\theta)$ by Theorem 1.3.8 (2). By the same discussion of the equal characteristic case, we obtain the injection $V[\![\mathcal{Q}]\!]/(p-f)V[\![\mathcal{Q}]\!] \hookrightarrow V[\![\mathbb{N}^r]\!]/(p-f)V[\![\mathbb{N}^r]\!]$, and $S^{-1}(V[\![\mathcal{Q}]\!]/(p-f)V[\![\mathcal{Q}]\!])$ is isomorphic to $S'^{-1}(V[\![\mathbb{N}^r]\!]/(p-f)V[\![\mathbb{N}^r]\!])$. This also implies that $V[\![\mathbb{N}^r]\!]/(p-f)V[\![\mathbb{N}^r]\!]$ is the desired regular local ring. □

Theorem 1.7.8. *Let $(R, \mathcal{Q}, \alpha)$ be a local log-regular ring. Then $\mathrm{Cl}(\alpha) : \mathrm{Cl}(\mathcal{Q}) \to \mathrm{Cl}(R)$ is isomorphism. In particular, the divisor class group $\mathrm{Cl}(R)$ is finitely generated.*

Proof. Consider the composite map

$$\mathrm{Cl}(\mathcal{Q}) \to \mathrm{Cl}(R) \to \mathrm{Cl}(\widehat{R}).$$

Note that the former group homomorphism $\mathrm{Cl}(\mathcal{Q}) \to \mathrm{Cl}(R)$ is injective by Proposition 1.7.6 and it is well-known that the latter group homomorphism $\mathrm{Cl}(R) \to \mathrm{Cl}(\widehat{R})$ is injective. Since it suffices to show that $\mathrm{Cl}(\mathcal{Q}) \to \mathrm{Cl}(\widehat{R})$ is surjective, we may assume that R is complete. By Nagate's theorem, we obtain the following short exact sequence:

$$0 \to H \to \mathrm{Cl}(R) \to \mathrm{Cl}(S^{-1}R) \to 0,$$

where H is the subgroup of R generated by the isomorphic class of a height one prime ideal that meets S. Since $\mathrm{Cl}(S^{-1}R)$ is trivial by Lemma 1.7.7, we obtain $H = \mathrm{Cl}(R)$. Moreover, we have an isomorphism $\mathrm{Cl}(\alpha) : \mathrm{Cl}(\mathcal{Q}) \xrightarrow{\cong} \mathrm{Im}(\mathrm{Cl}(\alpha)) = H$ by Lemma 1.7.1. This implies that $\mathrm{Cl}(\alpha)$ is an isomorphism. Finally, since the set of height one primes of $\mathcal{Q}$ is finite, $\mathrm{Cl}(\mathcal{Q})$ is finitely generated. Thus so is $\mathrm{Cl}(R)$ by Lemma 1.2.15. □

By combining Theorem 1.7.8 with Chouinard's Theorem [Cho81], for a local log-regular ring $(R, \mathcal{Q}, \alpha)$, we obtain the following isomorphism

$$\mathrm{Cl}(R) \cong \mathrm{Cl}(\mathcal{Q}) \cong \mathrm{Cl}(k[\mathcal{Q}]). \tag{1.13}$$

By reducing the computation of the divisor class group of R to that of $k[\mathcal{Q}]$, we can obtain the many examples of finitely generated divisor class groups in of local log-regular rings in mixed characteristic.

Example 1.7.9. *Let $k[\sigma_P^\vee \cap M]$ be a Hibi ring associated with a partially ordered set P (see §1.2.4). We recall that the divisor class group of R is $\mathbb{Z}^{n-d}$ (Theorem 1.2.51). Thus we set $\mathcal{Q} := \sigma_P^\vee \cap M$ and $R := C(k)[\![\mathcal{Q}]\!]/(\theta)$ where $C(k)$ is the Cohen ring of k and θ is the element of $C(k)[\![\mathcal{Q}]\!]$ whose constant term is p. Then $(R, \mathcal{Q}, \alpha)$ is a local log-regular ring where α is the composition of $\mathcal{Q} \hookrightarrow C(k)[\![\mathcal{Q}]\!]$ and $C(k)[\![\mathcal{Q}]\!] \twoheadrightarrow R$. Moreover, by Theorem 1.7.8, we obtain $\mathrm{Cl}(R) \cong \mathbb{Z}^{n-d}$.*

Chapter 2

Perfectoid towers and their tilts

2.1 Introduction

This chapter is based on the two papers [INS22] and [AIS23]. In this chapter, we establish a tower-theoretic framework to deal with perfectoid objects by introducing the notion of *perfectoid towers* and *their tilts*. The main objects of this chapter are *towers of rings*. Hence we first introduce the definition of towers of rings.

Definition 2.1.1 (Towers of rings).

1. *A tower of rings* $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ is a direct system of rings of the form

$$R_0 \xrightarrow{t_0} R_1 \xrightarrow{t_1} R_2 \xrightarrow{t_2} \cdots \xrightarrow{t_{i-1}} R_i \xrightarrow{t_i} \cdots .$$

2. *A morphism of towers of rings* $f : (\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0}) \to (\{R'_i\}_{i\geq 0}, \{t'_i\}_{i\geq 0})$ is defined as a collection of ring maps $\{f_i : R_i \to R'_i\}_{i\geq 0}$ that is compatible with the transition maps.

We remark that for a tower of rings $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$, we often denote a direct limit $\varinjlim_{i\geq 0} R_i$ by R_∞. If $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ is isomorphic to $(\{S_i\}_{i\geq 0}, \{u_i\}_{i\geq 0})$, it is obvious that R_∞ is isomorphic to S_∞.

2.1.1 Perfect towers and purely inseparable towers

Perfectoid rings are a natural generalization of perfect rings and the tilting operation for perfectoid rings is none other than the inverse perfection for perfectoid rings. Thus, to consider a tower-theoretic analogue of perfectoid rings and its tilting operation, we first need to find a tower-theoretic analogue of perfect $\mathbb{F}_p$-algebras and the inverse perfection for towers.

When we construct a perfect ring from a reduced $\mathbb{F}_p$-algebra, we consider the following tower:

$$R \hookrightarrow R^{1/p} \hookrightarrow R^{1/p^2} \hookrightarrow \cdots . \tag{2.1}$$

A tower which is isomorphic to the tower (2.1) is called a *perfect tower* (Definition 2.2.2).

Next, let us define purely inseparable towers which are a class having an inverse perfection of towers. As the inverse perfection of a perfect $\mathbb{F}_p$-algebra coincides with itself, we hope that the inverse perfection for a tower also coincides with itself. The most naive idea to define the inverse perfection of towers is that we apply the inverse perfection of a ring for each layer, but, as the following example shows, we can not obtain a tower which has the desired property.

Example 2.1.2. *Let us consider the tower*

$$\mathbb{F}_p[\![t]\!] \hookrightarrow \mathbb{F}_p[\![t]\!][t^{1/p}] \hookrightarrow \mathbb{F}_p[\![t]\!][t^{1/p^2}] \hookrightarrow \cdots . \tag{2.2}$$

Then, for every $i \geq 0$, the inverse perfection of $\mathbb{F}_p[\![t]\!][t^{1/p^i}]$ is isomorphic to $\mathbb{F}_p$. Hence the obtained tower by applying the inverse perfection of a ring to each layer is

$$\mathbb{F}_p \xrightarrow{\cong} \mathbb{F}_p \xrightarrow{\cong} \cdots .$$

□

In the tower (2.2), we observe that the Frobenius endomorphism of $\mathbb{F}_p[\![t]\!][t^{1/p^{i+1}}]$ factors through $\mathbb{F}_p[\![t]\!][t^{1/p^i}]$. We denote by F_i the ring map $\mathbb{F}_p[\![t]\!][t^{1/p^{i+1}}] \to \mathbb{F}_p[\![t]\!][t^{1/p^i}]$. Since each F_i is an isomorphism, the inverse limit $\varprojlim\{\cdots \xrightarrow{F_{i+1}} \mathbb{F}_p[\![t]\!][t^{1/p^{i+1}}] \xrightarrow{F_i} \mathbb{F}_p[\![t]\!][t^{1/p^i}]\}$ is isomorphic to $\mathbb{F}_p[\![t]\!][t^{1/p^i}]$ for every $i \geq 0$. We can apply this observation for the deeply ramified tower

$$\mathbb{Z}_p \hookrightarrow \mathbb{Z}_p[p^{1/p}] \hookrightarrow \mathbb{Z}_p[p^{1/p^2}] \hookrightarrow . \tag{2.3}$$

Then the Frobenius endomorphism on $\mathbb{Z}_p[p^{1/p^{i+1}}]/(p)$ also factors through $\mathbb{Z}_p[p^{1/p^i}]/(p)$ and we denote by F_i the ring map $\mathbb{Z}_p[p^{1/p^{i+1}}]/(p) \to \mathbb{Z}_p[p^{1/p^i}]/(p)$. Also, the inverse limit $\varprojlim\{\cdots \to \mathbb{Z}_p[p^{1/p^{i+1}}]/(p) \xrightarrow{F_i} \mathbb{Z}_p[p^{1/p^i}]/(p)\}$ is isomorphic to $\mathbb{F}_p[\![t]\!][t^{1/p^i}]$ for every $i \geq 0$.

Based on these observations, we define *purely inseparable towers* which have an inverse perfection as above (see Definition 2.3.1).

2.1.2 Perfectoid towers

In the framework of towers, we introduce perfectoid towers as a specialized class of purely inseparable towers and define tilts to them as the inverse perfection of purely inseparable towers. The class of perfectoid towers gives a generalization of deeply ramified towers as (2.3) and perfect towers. Summarizing the above, these classes of towers form the following hierarchy:

$$\left\{\begin{array}{c}\text{Perfect towers}\\ \text{(Definition 2.2.2)}\end{array}\right\} \subseteq \left\{\begin{array}{c}\text{Perfectoid towers}\\ \text{(Definition 2.4.9)}\end{array}\right\} \subseteq \left\{\begin{array}{c}\text{Purely inseparable towers}\\ \text{(Definition 2.3.1 (1))}\end{array}\right\}$$

In the following of this subsection, let $R_0 \to R_1 \to \cdots$ be a perfectoid tower and let $R_0^{s.\flat} \to R_1^{s.\flat} \to \cdots$ is its inverse perfection (the inverse perfection of a perfectoid tower is called the *tilt*). Here, we list basic properties of perfectoid towers:

- There is an isomorphism of Koszul homology groups $H_j(R_i, f_0) \cong H_j(R_i^{s.\flat}, f_0^{s.\flat})$ for some specific elements f_0 and $f_0^{s.\flat}$ (see Remark 2.4.33). This leads to the results that the tilting operation for towers preserves some Noetherian properties (Proposition 2.4.35).

- The tilt of a perfectoid tower is also a perfectoid tower (Proposition 2.4.34). This is a tower-theoretic analogue of the result that the tilt of a perfectoid ring is also perfectoid.

- There is a compatibility with perfectoid ring, namely, the direct limit R_∞ of perfectoid towers $R_0 \to R_1 \to \cdots$ is a perfectoid ring in the sense of Bhatt-Morrow-Scholze (Corollary 2.4.42).

- The tilting operation also preserves information of toric singularities, that is, the tilt of a perfectoid tower consisting of log-regular rings in mixed characteristic also consists of log-regular rings in positive characteristic (Theorem 2.5.10).

Furthermore, these are answers to Question 1 in Preface.

2.1.3 Applications for cohomologies of commutative rings

Finally, let us give the two applications to cohomologies of rings. First of all, under a certain normality assumption, we obtain a comparison theorem on the finiteness of étale cohomology groups via tilting for towers (Proposition 2.6.7). This theorem relies on Česnavičius-Scholze's comparison theorem under tilting for schematic perfectoids [ČS19, Theorem 2.2.7]. We can not have the complete answer to what kind of classes of rings satisfy the normality condition in Proposition 2.6.7, but at least towers consisting of local log-regular rings constructed in Construction 2.5.5 satisfy its condition.

As an application of the comparison theorem, we obtain the finiteness of ℓ-torsion part of the divisor class group of a local log-regular ring for all prime $\ell \neq p$.

Main Theorem C (Theorem 2.6.13). *Let $(R, \mathcal{Q}, \alpha)$ be a local log-regular ring of mixed characteristic with perfect residue field k of characteristic $p > 0$, and denote by $\mathrm{Cl}(R)$ the divisor class group with its torsion subgroup $\mathrm{Cl}(R)_{\mathrm{tor}}$. Then the following assertions hold.*

1. *Assume that $R \cong W(k)[\![\mathcal{Q}]\!]$ for a fine, sharp, and saturated monoid $\mathcal{Q}$, where $W(k)$ is the ring of Witt vectors over k. Then $\mathrm{Cl}(R)_{\mathrm{tor}} \otimes \mathbb{Z}[\frac{1}{p}]$ is a finite group. In other words, the ℓ-primary subgroup of $\mathrm{Cl}(R)_{\mathrm{tor}}$ is finite for all primes $\ell \neq p$ and vanishes for almost all primes $\ell \neq p$.*

2. *Assume that $\widehat{R^{\mathrm{sh}}}[\frac{1}{p}]$ is locally factorial, where $\widehat{R^{\mathrm{sh}}}$ is the completion of the strict Henselization R^{sh}. Then $\mathrm{Cl}(R)_{\mathrm{tor}} \otimes \mathbb{Z}[\frac{1}{p}]$ is a finite group. In other words, the ℓ-primary subgroup of $\mathrm{Cl}(R)_{\mathrm{tor}}$ is finite for all primes $\ell \neq p$ and vanishes for almost all primes $\ell \neq p$.*

This theorem is recovered by Theorem 1.7.8, but the method of the proof is completely different. In characteristic $p > 0$, Polstra proved that the torsion part of the divisor class group of a strongly F-regular ring is surprisingly finite and we proved an F-finite local log-regular ring is strongly F-regular. Our approach is a reduction to Polstra's theorem by tilting. Let us explain our approach easily.

(1) After taking the strict Henselization and the completion, we reduce to the case where R is complete and its residue field is separably closed.

(2) By the assumption of the locally factoriality of $R[\frac{1}{p}]$, there is an open subset $U \subset \mathrm{Spec}(R)$ such that $\mathrm{Pic}(U) \cong \mathrm{Cl}(X)$ for $X \backslash V(pR) \subseteq U$ and $\mathrm{codim}_X(X \backslash U) \geq 2$ where $X := \mathrm{Spec}(R)$. Also, we obtain the isomorphisms

$$H^1(U_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}) \cong \mathrm{Pic}(U)[\ell^n] \cong \mathrm{Cl}(X)[\ell^n]. \tag{2.4}$$

(3) In characteristic $p > 0$ side, we have

$$H^1(U^{s.\flat}_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}) \cong \mathrm{Pic}(U^{s.\flat})[\ell^n] \hookrightarrow \mathrm{Cl}(R^{s.\flat})[\ell^n], \tag{2.5}$$

where $U^{s.\flat}$ is the open subset of $X^{s.\flat} := \mathrm{Spec}(R^{s.\flat})$ corresponding to U (see Definition 2.6.3).

(4) Applying Proposition 2.6.7 to the tower constructed in Construction 2.5.5, we obtain the inequality

$$|H^1(U_{\acute{e}t}, \mathbb{Z}/\ell^n\mathbb{Z})| \leq |H^1(U^{s.\flat}_{\acute{e}t}, \mathbb{Z}/\ell^n\mathbb{Z})|. \tag{2.6}$$

Hence by combining (2.4), (2.5), and (2.6), we obtain

$$|\operatorname{Cl}(X)[\ell^n]| = |H^1(U_{\acute{e}t}, \mathbb{Z}/\ell^n\mathbb{Z})| \leq |H^1(U^{s.\flat}_{\acute{e}t}, \mathbb{Z}/\ell^n\mathbb{Z})| \leq |\operatorname{Cl}(R^{s.\flat})[\ell^n]|.$$

(5) Finally, since $R^{s.\flat}$ is a local log-regular ring, particularly strongly F-regular, we can prove the finiteness of $\operatorname{Cl}(R^{s,\flat})[\ell^n]$ using Polstra's result.

Next, we apply perfectoid towers to the local cohomological dimension of regular local rings. In the history of the study on local cohomologies, finding the condition of the vanishing of the local cohomology modules has been considered important, which is called *Grothendieck's vanishing problem.* Let $(R, \mathfrak{m})$ be a Noetherian local ring. Then the *local cohomological dimension for a pair of an R-module M and an ideal I of R* is the supremum of the integer i such that $H^i_I(M) \neq 0$ and we denote it by $\operatorname{cd}(M, I)$. If $I = \mathfrak{m}$, then the Grothendieck vanishing theorem implies $\operatorname{cd}(R, \mathfrak{m}) = d$.[1] If $I \neq \mathfrak{m}$, then it is much more difficult to compute the cohomological dimension $\operatorname{cd}(M, I)$. The condition when $\operatorname{cd}(M, I) < d$ holds for any R-module M is given by Hartshorne, which is called Hartshorne-Lichtenbaum vanishing theorem. In particular, the theorem implies that for a complete Noetherian local domain R and an ideal I which is not the maximal ideal, $\operatorname{cd}(M, I) < d$ holds for any R-module M.

The condition when $\operatorname{cd}(M, I) < d - 1$ holds is called the *second vanishing theorem.* Let us state the second vanishing theorem for solved cases.

Theorem 2.1.3 (Peskine–Szpiro, Ogus, Hartshorne–Speiser, Huneke–Lyubeznik, Zhang)**.** *Let $(R, \mathfrak{m})$ be a d-dimensional regular local ring and I be an ideal of R. Assume that R is unramified (namely, R is of equal characteristic or R is of mixed characteristic $(0, p)$ and p is part of a regular system of parameters of $\mathfrak{m}$). Then the following assertions are equivalent.*

1. $\operatorname{cd}(M, I) < d - 1$ *for any R-module M.*

2. $\dim(R/I) \geq 2$ *and the puctured spectram of $\widehat{R^{\mathrm{sh}}}/I\widehat{R^{\mathrm{sh}}}$ is connected.*

Many researchers have studied this theorem, such as Hartshone, Peskine, Szpiro, Huneke, Lyubeznik, Zhang, and so on, but this theorem is not proved in the case of ramified regular local rings. In §2.6.3, we prove the similar type of the second vanishing theorem for a regular local ring in mixed characteristic. In our theorem, there is no need to distinguish between ramified and unramified.

Main Theorem D (Theorem 2.6.26)**.** *Let $(R, \mathfrak{m}, k)$ be a d-dimensional complete regular local ring of mixed characteristic with separably closed residue field k. Assume that $I \subset R$ is a proper ideal with* $\dim(R/\mathfrak{q}) \geq 3$ *and* $\ell_R(H^2_{\mathfrak{m}}(R/pR + \mathfrak{q})) < \infty$ *for all* $\mathfrak{q} \in \operatorname{Min}(R/I)$. *Then the following statements are equivalent.*

1. $H^{d-1}_I(R) = 0.$

[1] The Grothendieck vanishing theorem states the following: Let $(R, \mathfrak{m})$ be a d-dimensional Noetherian local ring, let M be a finitely generated R-module, and let t be the depth of M. Then $H^d_{\mathfrak{m}}(M)$ and $H^t_{\mathfrak{m}}(M)$ is non-zero. Also, if i is neither t nor d, then $H^i_{\mathfrak{m}}(M)$ is zero. This deduces that M is a maximal Cohen-Macaulay module if and only if $H^i_{\mathfrak{m}}(M) = 0$ for any $i \neq d$.

2. *The punctured spectrum* $\operatorname{Spec}^\circ(R/I)$ *is connected.*

The key to proving this theorem is that the comparison result of the cohomological dimensions between a complete regular local ring with a perfectoid tower constructed by the adjoining the p-th power roots of a regular system of parameters and the 0-th small tilt (Lemma 2.6.23). Therefore we can compute the cohomological dimension of a regular local ring of mixed characteristic by reducing to the case of positive characteristic.

This chapter is organized as follows. In §2.2, we define perfect towers and discuss their relationship to the classical direct perfection. We also discuss a relationship between perfect towers and lim Cohen–Macaulay sequences defined by Bhatt–Hochster–Ma. In §2.3, we define purely inseparable towers and their inverse perfection and discuss their basic properties. In §2.4, we define perfectoid towers and their tilts. We also discuss that they share some Noetherian properties between perfectoid towers with their tilts (Proposition 2.4.35). In §2.5, we give an example of perfectoid towers whose layers are local log-regular rings. This construction was given by Gabber and Ramero in [GR23] abstractly. However, in this thesis, we present it explicitly for the complete case. More specific examples are given in Example 2.5.11. In §2.6, we give applications of perfectoid towers to cohomologies of commutative rings, such as étale cohomology groups and local cohomology modules.

2.2 Perfect towers

In this section, we define perfect towers and study some of their basic properties. This type of towers naturally appears when one considers the perfect closure of a reduced $\mathbb{F}_p$-algebra. We also discuss the relationship with lim Cohen-Macaulay sequences.

2.2.1 Definition of perfect towers

Definition 2.2.1. Let R be a reduced $\mathbb{F}_p$-algebra.

1. For every $i \geq 0$, the ring R^{1/p^i} is defined as $\varinjlim\{R \underbrace{\xrightarrow{F_R} R \xrightarrow{F_R} \cdots \xrightarrow{F_R}}_{i \text{ Frobenius endomorphisms}} R\}$.

2. For every $i \geq 0$, the ring map $\iota_i : R^{1/p^i} \to R^{1/p^{i+1}}$ is the ring map induced by the commutative diagram

$$\begin{array}{ccccccccc} R & \xrightarrow{F_R} & R & \xrightarrow{F_R} & \cdots & \xrightarrow{F_R} & R & & \\ \downarrow{\scriptstyle \mathrm{id}_R} & & \downarrow{\scriptstyle \mathrm{id}_R} & & & & \downarrow{\scriptstyle \mathrm{id}_R} & \searrow{\scriptstyle F_R} & \\ R & \xrightarrow[F_R]{} & R & \xrightarrow[F_R]{} & \cdots & \xrightarrow[F_R]{} & R & \xrightarrow[F_R]{} & R. \end{array} \tag{2.7}$$

Then one can obtain the tower of rings $(\{R^{1/p^i}\}_{i\geq 0}, \{\iota_i\}_{i\geq 0})$.

Let us define perfect towers.

Definition 2.2.2. A *perfect* $\mathbb{F}_p$*-tower* (or simply a *perfect tower*) is a tower that is isomorphic to the $(\{R^{1/p^i}\}_{i\geq 0}, \{\iota_i\}_{i\geq 0})$ appearing in Definition 2.2.1.

Remark 2.2.3. R^{1/p^i} is isomorphic to the ring of p^i-th roots of elements of R. Indeed, let R_{1/p^j} be the ring of p^j-th roots of elements of R for every $j \geq 0$.[2] Then we have the isomorphism $F_j : R_{1/p^{j+1}} \to R_{1/p^j}$; $x \mapsto x^p$. Set $F_{0,j+1} := F_0 \circ \cdots \circ F_j$. Then we obtain the following commutative ladder:

$$\begin{array}{ccccccccc}
R & \longrightarrow & R_{1/p} & \longrightarrow & \cdots & \longrightarrow & R_{1/p^{i-1}} & \longrightarrow & R_{1/p^i} \\
\downarrow{\scriptstyle \mathrm{id}_R} & & \downarrow{\scriptstyle F_{0,1}} & & & & \downarrow{\scriptstyle F_{0,i-1}} & & \downarrow{\scriptstyle F_{0,i}} \\
R & \xrightarrow[F_R]{} & R & \xrightarrow[F_R]{} & \cdots & \xrightarrow[F_R]{} & R & \xrightarrow[F_R]{} & R,
\end{array}$$

where the top horizontal arrows are the natural inclusions. Since all the vertical arrows are isomorphisms, we obtain the isomorphism $R_{1/p^i} \cong R^{1/p^i}$.

The following lemma implies what perfectness is in tower theory.

Lemma 2.2.4. *Let $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ be a perfect tower. Let $F_i : R_{i+1} \to R_i$ be the ring map induced by the following commutative diagram:*

$$\begin{array}{ccccccccc}
R & \xrightarrow{F_R} & R & \xrightarrow{F_R} & \cdots & \xrightarrow{F_R} & R & \xrightarrow{F_R} & R \\
\downarrow{\scriptstyle F_R} & & \downarrow{\scriptstyle F_R} & & & & \downarrow{\scriptstyle F_R} & \swarrow{\scriptstyle \mathrm{id}_R} & \\
R & \xrightarrow[F_R]{} & R & \xrightarrow[F_R]{} & \cdots & \xrightarrow[F_R]{} & R. & &
\end{array} \tag{2.8}$$

Then, the following assertions hold.

1. *F_i is an isomorphism.*
2. *$F_i \circ \iota_i = F_{R^{1/p^i}}$.*
3. *$\iota_i \circ F_i = F_{R^{1/p^{i+1}}}$.*

Proof. (1): Since the rightest (diagonal) arrow in (2.8) is the identity map, F_i is an isomorphism.

(2): The composition of the diagrams (2.7) and (2.8) obviously induces the Frobenius endomorphism on R_i.

(3): This follows from the same reason as in (2). □

Corollary 2.2.5. *Let R be a reduced $\mathbb{F}_p$-algebra. Then the direct system $\{R \xrightarrow{F_R} R \xrightarrow{F_R} \cdots\}$ is a perfect tower. In particular, the direct limit of a perfect tower $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ is isomorphic to the perfection of R_0.*

Proof. Set $F_{0,i} := F_0 \circ \cdots \circ F_i : R \to R$ where $F_{0,0} := id_R$. Then we obtain the morphism of tower of rings $\{F_{0,i}\}_{i\geq 0} : \{R \xrightarrow{\iota_0} R^{1/p} \xrightarrow{\iota_1} \cdots\} \to \{R \xrightarrow{F_R} R \xrightarrow{F_R} \cdots\}$. By Lemma 2.2.4 (1), each $F_{0,i}$ is an isomorphism. Thus $\{F_{0,i}\}_{i\geq 0}$ is an isomorphism. □

[2] For more details of the ring of p-th roots of elements of a reduced ring, we refer to [MP21]

2.2.2 Perfect towers and lim Cohen–Macaulay sequences

Next, we discuss the relationship between perfect towers and lim Cohen–Macaulay sequences. In more detail, we refer the reader to [Hoc17] or [Ma23]. For functions $f, g : N \to (0, \infty)$ where $N \subseteq \mathbb{N}$ contains all $n \gg 0$, we denote $f(n) = o(g(n))$ if $\lim_{n\to\infty} f(n)/g(n) = 0$.

Definition 2.2.6. Let $(R, \mathfrak{m}, k)$ be a Noetherian local domain. If $\underline{x} = x_1, \ldots, x_d$ is a system of parameters, let $H_\bullet(\underline{x}; M)$ denote Koszul homology, and let $h_i(\underline{x}; M)$ denote its length (that is, $h_i(\underline{x}; M) = \ell_R(H_i(\underline{x}; M))$) when the length is finite. Then a sequence of nonzero finitely generated modules $\mathcal{M} := \{M_n\}_{n\geq 0}$ of dimension $\dim(R)$ over R is defined to be *lim Cohen–Macaulay* if for some (equivalently, every) system of parameters $\underline{x} = x_1, \ldots, x_d$, we have

$$h_i(\underline{x}; M_n) = o(\nu(M_n))$$

where $\nu(M) := \dim_k(k \otimes_R M)$ for a finitely generated R-module.

Remark 2.2.7. If R has a small Cohen–Macaulay module M, we may take the sequence to be the constant sequence $M, M, M, \ldots$.

The following theorem implies perfect towers are lim Cohen–Macaulay sequences.

Theorem 2.2.8. *Let $(R, \mathfrak{m}, k)$ be a complete Noetherian local domain in characteristic $p > 0$ with a perfect residue field. Then a perfect tower $\{R^{1/p^n}\}_{n\geq 0}$ is a lim Cohen–Macaulay sequence.*

Proof. Fix a system of parameters $\underline{x} := x_1, \ldots, x_d$. Then, by assumption, we have a module finite extension

$$A := k[\![x_1, \ldots, x_d]\!] \hookrightarrow R.$$

Consider the Koszul complex $H_i(\underline{x}; R^{1/p^n})$. Since A is regular, the complex $H_\bullet(\underline{x}; R)$ is a minimal free resolution as A-modules. Hence, by applying [Dut83, 1.5 Proposition], we obtain

$$\lim_{n\to\infty} \frac{h_i(\underline{x}; R^{1/p^n}))}{\nu(R^{1/p^n})} = \lim_{n\to\infty} \frac{\operatorname{rank}_A(R) \cdot \ell_A(H_i(\underline{x}; R^{1/p^n}))}{\nu(R^{1/p^n})} = \operatorname{rank}_A(R) \cdot \lim_{n\to\infty} \frac{\ell_A(\operatorname{Tor}_i^A(R^{1/p^n}, R))}{p^{pn}} = 0,$$

as desired. □

2.3 Purely inseparable towers and their quasi-inverse perfections

In this section, we define purely inseparable towers as a class of towers that have the inverse perfections called quasi-inverse perfection.

Definition 2.3.1 (Purely inseparable towers)**.** Let R be a ring, and let $I \subseteq R$ be an ideal. A tower $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ is called a *p-purely inseparable tower arising from* (R, I) (or simply a *purely inseparable tower*) if it satisfies the following axioms.

(a) $R_0 = R$ and $p \in I$.

(b) For any $i \geq 0$, the ring map $\overline{t_i} : R_i/IR_i \to R_{i+1}/IR_{i+1}$ induced by t_i is injective.

(c) For any $i \geq 0$, the image of the Frobenius endomorphism on R_{i+1}/IR_{i+1} is contained in the image of $\overline{t_i} : R_i/IR_i \to R_{i+1}/IR_{i+1}$.

By the axiom (b) and the axiom (c) in Definition 2.3.1, we obtain the following lemma.

Lemma 2.3.2. *Let R be a ring and let I be an ideal of R. Let $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ be a purely inseparable tower arising from (R, I). Then for any $i \geq 0$ there uniquely exists the ring map $F_i : R_{i+1}/IR_{i+1} \to R_i/IR_i$ such that the following diagram commutes:*

$$\begin{array}{ccc} R_{i+1}/IR_{i+1} & \xrightarrow{F_{R_{i+1}/IR_{i+1}}} & R_{i+1}/IR_{i+1} \\ & \searrow^{F_i} & \uparrow{\scriptstyle \overline{t_i}} \\ & & R_i/IR_i. \end{array} \tag{2.9}$$

Definition 2.3.3. Let $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ be a purely inseparable tower arising from (R, I). Then for every $i \geq 0$ the ring map F_i appearing in Lemma 2.3.2 is called the *i-th Frobenius projection of $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ associated to (R, I)*. If there is no confusion, we call it simply the *i-th Frobenius projection.*

Lemma 2.3.4. *Any perfect tower is a purely inseparable tower.*

Proof. We only check that $(\{R^{1/p^i}\}_{i\geq 0}, \{\iota_i\}_{i\geq 0})$ is a purely inseparable tower. Moreover, since the axiom (a) and the axiom (b) are obvious, we only prove that the axiom (c) holds. By Lemma 2.2.4 (1) and (3), we know that the axiom (c) holds and the ring map F_i defined in Lemma 2.2.4 is the i-th Frobenius projection of the perfect tower. □

The following lemma provides many examples of purely inseparable towers.

Lemma 2.3.5. *Let $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ be a tower of rings where R_0 contains p for any $i \geq 0$. Assume that the tower $(\{R_i/pR_i\}_{i\geq 0}, \{\overline{t_i}\}_{i\geq 0})$ is perfect. Then $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ is a purely inseparable tower arising from $(R_0, (p))$.*

Proof. A tower clearly satisfies the axiom (a). The assumption of the perfectness of $(\{R_i/pR_i\}_{i\geq 0}, \{\overline{t_i}\}_{i\geq 0})$ gives the following squares for any $i \geq 0$.

$$\begin{array}{ccc} R_i/pR_i & \xrightarrow{\overline{t_i}} & R_{i+1}/pR_{i+1} \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ R^{1/p^i} & \underset{F_i}{\overset{\iota_i}{\rightleftarrows}} & R^{1/p^{i+1}} \end{array}$$

where ι_i and F_i are the morphism appearing in Lemma 2.2.4. This induces the axiom (b) and the axiom (c). □

To explore properties of perfectoid towers, we often use the combination of the diagram (2.9) and the following diagram (2.10)

Lemma 2.3.6. *Let $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ be a purely inseparable tower arising from some pair (R, I). Then for every $i \geq 0$, the following assertions hold.*

1. *The kernel of F_i is equal to the kernel of the Frobenius endomorphism of R_{i+1}/IR_{i+1}.*

2. *The ring map* $\overline{t_i} : R_i/IR_i \to R_{i+1}/IR_{i+1}$ *is integral.*

3. *There is the following commutative diagram:*

$$\begin{array}{ccc} R_{i+1}/IR_{i+1} & & \\ \overline{t_i}\uparrow & \searrow^{F_i} & \\ R_i/IR_i & \xrightarrow[F_{R_i/IR_i}]{} & R_i/IR_i. \end{array} \tag{2.10}$$

Proof. Since $\overline{t_i}$ is injective, the diagram (2.9) yields the assertion (1). Also, the diagram (2.9) indicates any element $x \in R_{i+1}/IR_{i+1}$ is a solution of the equation $X^p - \overline{t_i}\big(F_i(x)\big)$. This implies that the assertion (2) holds. Finally, we prove the assertion (3). Note that we have the following equalities

$$\overline{t_i} \circ F_{R_i/IR_i} = F_{R_{i+1}/IR_{i+1}} \circ \overline{t_i} = \overline{t_i} \circ F_i \circ \overline{t_i}.$$

Since $\overline{t_i}$ is injective, we obtain the equality $F_i \circ \overline{t_i} = F_{R_i/IR_i}$, as desired. □

Next, we introduce the notion of the inverse perfection for purely inseparable towers.

Definition 2.3.7. Let $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ be a purely inseparable tower arising from some pair (R, I).

1. For any $j \geq 0$, we define the *j-th inverse quasi-perfection of* $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ *associated to* (R, I) as a limit:

$$(R_j)_I^{q.\text{frep}} := \varprojlim\{\cdots \to (R_{j+i+1}/IR_{j+i+1}) \xrightarrow{F_{j+i}} (R_{j+i}/IR_{j+i}) \to \cdots \xrightarrow{F_j} (R_j/IR_j\}.$$

2. For any $j \geq 0$, we define an injective ring map $(t_j)_I^{q.\text{frep}} : (R_j)_I^{q.\text{frep}} \hookrightarrow (R_{j+1})_I^{q.\text{frep}}$ such that $(a_i)_{i\geq 0} \mapsto (\overline{t_{j+i}}(a_i))_{i\geq 0}$. Now we say that the tower $(\{(R_i)_I^{q.\text{frep}}\}_{i\geq 0}, \{(t_i)_I^{q.\text{frep}}\}_{i\geq 0})$ is *the inverse perfection of* $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ associated to (R, I).

3. For any $j \geq 0$, we define a ring map $(F_j)_I^{q.\text{frep}} : (R_{j+1})_I^{q.\text{frep}} \to (R_j)_I^{q.\text{frep}}$ such that $(a_i)_{i\geq 0} \mapsto (F_{j+i}(a_i))_{i\geq 0}$.

4. For any $j \geq 0$ and for any $m \geq 0$, we denote by $\Phi_m^{(j)} : (R_j)_I^{q.\text{frep}} \to R_{j+m}/IR_{j+m}$ the m-th projection map

If no confusion arises, we denote $(R_j)_I^{q.\text{frep}}$ (resp. $(t_j)_I^{q.\text{frep}}$, resp. $(F_j)_I^{q.\text{frep}}$) by $R_j^{q.\text{frep}}$ (resp. $t_j^{q.\text{frep}}$, resp. $F_j^{q.\text{frep}}$) as an abbreviated form.

Example 2.3.8. *Let R be an $\mathbb{F}_p$-algebra. Then the tower $\{R \xrightarrow{\mathrm{id}_R} R \xrightarrow{\mathrm{id}_R} \cdots\}$ is purely inseparable tower arising from $(R, (0))$. Moreover, for every $j \geq 0$, the j-th Frobenius projection is the Frobenius endomorphism of R and the j-th inverse quasi-perfection $R_j^{q.\text{frep}} = \varprojlim\{\cdots \xrightarrow{F_R} R \xrightarrow{F_R} R\}$ is none other than the inverse perfection of R.*

In the situation of Definition 2.3.7, we obtain the following commutative diagram:

$$\begin{array}{ccc} (R_{j+1})_I^{q.\text{frep}} & \xrightarrow{F_{(R_{j+1})_I^{q.\text{frep}}}} & (R_{j+1})_I^{q.\text{frep}} \\ & \searrow_{(F_j)_I^{q.\text{frep}}} & \uparrow (t_j)_I^{q.\text{frep}} \\ & & (R_j)_I^{q.\text{frep}}. \end{array} \tag{2.11}$$

We immediately deduce the following lemma from the diagram (2.11).

Lemma 2.3.9. *Let $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ be a purely inseparable tower arising from some pair (R, I). Then its the inverse perfection $(\{(R_i)_I^{q.\mathrm{frep}}\}_{i\geq 0}, \{(t_i)_I^{q.\mathrm{frep}}\}_{i\geq 0})$ is also a purely inseparable tower arising from $((R_0)_I^{q.\mathrm{frep}}, (0))$.*

The next proposition is a list of basic properties of inverse quasi-perfection.

Proposition 2.3.10. *Let $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ be a purely inseparable tower arising from some pair (R, I). Then for any $i \geq 0$, the following assertions hold.*

1. *Let $J \subseteq (R_j)_I^{q.\mathrm{frep}}$ be a finitely generated ideal such that $J^k \subseteq \mathrm{Ker}(\Phi_0^j)$ for some $k > 0$. Then $(R_j)_I^{q.\mathrm{frep}}$ is J-adically complete and separated.*
2. *Let $x = (x_i)_{i\geq 0}$ be an element of $(R_j)_I^{q.\mathrm{frep}}$. Then x is a unit if and only if $x_0 \in R_j/IR_j$ is a unit.*
3. *The ring map $(F_j)_I^{q.\mathrm{frep}}$ is an isomorphism.*
4. *$(R_j)_I^{q.\mathrm{frep}}$ is reduced.*
5. *$(\{(R_i)_I^{q.\mathrm{frep}}\}_{i\geq 0}, \{(t_i)_I^{q.\mathrm{frep}}\}_{i\geq 0})$ is a perfect tower.*

Proof. Since $(\{(R_{j+i})^{q.\mathrm{frep}}\}_{i\geq 0}, \{(t_{j+i})_I^{q.\mathrm{frep}}\}_{i\geq 0})$ is also a purely inseparable tower, one can reduce to the case $j = 0$.

(1): By definition, $(R_0)_I^{q.\mathrm{frep}}$ is complete and separated with respect to the linear topology induced by the descending filtration

$$\mathrm{Ker}(\Phi_0^{(0)}) \supseteq \mathrm{Ker}(\Phi_1^{(0)}) \supseteq \mathrm{Ker}(\Phi_2^{(0)}) \supseteq \cdots .$$

Moreover, since $J^k \subseteq \mathrm{Ker}(\Phi_0^{(0)})$, we have $(J^k)^{[p^i]} \subseteq \mathrm{Ker}(\Phi_i^{(0)})$ for every $i \geq 0$ by the commutative diagram (2.9).[3] On the other hand, since J^k is finitely generated, $(J^k)^{p^i r} \subseteq (J^k)^{[p^i]}$ for some $r > 0$. Thus the assertion follows from [FGK11, Lemma 2.1.1].

(2): It is obvious that x_0 in a unit in R_0/IR_0 if x is a unit in $(R_j)_I^{q.\mathrm{frep}}$. Conversely, assume that x_0 is a unit in R_0/IR_0. Then the diagram (2.9) shows that x_i is a unit. Here the multiplication map $(R_0)_I^{q.\mathrm{frep}} \xrightarrow{\times x} (R_0)_I^{q.\mathrm{frep}}$ is an isomorphism because it is induced by the isomorphisms $R_i/IR_i \xrightarrow{\times x_i} R_i/IR_i$ for every $i \geq 0$. This implies that x is a unit, as desired.

(3): Let $s_0 : (R_0)_I^{q.\mathrm{frep}} \to (R_1)_I^{q.\mathrm{frep}}$ be the map such that $(a_i)_{i\geq 0} \mapsto (a_{i+1})_{i\geq 0}$. Then one can easily check that s_0 is the inverse map of $(F_0)_I^{q.\mathrm{frep}}$.

(4): This immediately follows from the combination of the injectivity of $(t_0)_I^{q.\mathrm{frep}}$, the diagram (2.11), and the assertion (3).

(5): Let us define $F_{0,i}^{q.\mathrm{frep}} : (R_i)_I^{q.\mathrm{frep}} \to (R_0)_I^{q.\mathrm{frep}}$ as the composite map $(F_0)_I^{q.\mathrm{frep}} \circ \cdots \circ (F_{i-1})_I^{q.\mathrm{frep}}$ where $F_{0,0}^{q.\mathrm{frep}} = \mathrm{id}_{(R_0)_I^{q.\mathrm{frep}}}$. Then by assertion (3), the map of towers $\{F_{0,i}^{q.\mathrm{frep}}\}_{i\geq 0} : (\{(R_i)_I^{q.\mathrm{frep}}\}_{i\geq 0}, \{(t_i)_I^{q.\mathrm{frep}}\}_{i\geq 0}) \to \left\{(R_0)_I^{q.\mathrm{frep}} \xrightarrow{F_{(R_0)_I^{q.\mathrm{frep}}}} (R_0)_I^{q.\mathrm{frep}} \xrightarrow{F_{(R_0)_I^{q.\mathrm{frep}}}} \cdots\right\}$ is isomorphism. Hence the assertion holds by Corollary 2.2.5. □

The operation of inverse quasi-perfection preserves the local property of rings and ring maps.

[3] The symbol $I^{[p^n]}$ for an ideal I in an $\mathbb{F}_p$-algebra A is the ideal generated by the elements x^{p^n} for $x \in I$.

Lemma 2.3.11. *Let $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ be a purely inseparable tower of local rings arising from some pair (R, I). Assume that $I \neq R_0$. Then for any $j \geq 0$, the following assertions hold.*

1. *The ring maps t_j, $\overline{t_j}$, and F_j are local.*
2. *$(R_j)^{q.\mathrm{frep}}_I$ is a local ring.*
3. *The ring map $t^{q.\mathrm{frep}}_j$ and $F^{q.\mathrm{frep}}_j$ is local.*

Proof. For the same reason as in Proposition 2.3.10, it suffices to show the assertions in the case when $j = 0$. (1): Since the diagrams (2.9) and (2.10) are commutative, $F_0 \circ \overline{t_0}$ and $\overline{t_0} \circ F_0$ are local. Hence $\overline{t_0}$ and F_0 are local. In particular, the composition $R_0 \twoheadrightarrow R_0/I \xrightarrow{\overline{t_0}} R_1/IR_1$ is local. Since this map factors through t_0, t_0 is also local, as desired.

(2): Let $\mathfrak{m}_0$ be the maximal ideal of R_0. Consider the ideal $(\mathfrak{m}_0)^{q.\mathrm{frep}}_I = \{(x_i)_{i\geq 0} \in (R_0)^{q.\mathrm{frep}}_I \mid x_0 \in \mathfrak{m}_0/IR_0\}$, where $\mathfrak{m}_0/IR_0$ is the maximal ideal of R_0/IR_0.

Then by Proposition 2.3.10 (2), $(\mathfrak{m}_0)^{q.\mathrm{frep}}_I$ is a unique maximal ideal of $(R_0)^{q.\mathrm{frep}}_I$. Hence the assertion follows.

(3): By the assertion (2) and Lemma 2.3.9, $(\{(R_i)^{q.\mathrm{frep}}_I\}_{i\geq 0}, \{(t_i)^{q.\mathrm{frep}}_I\}_{i\geq 0})$ is a purely inseparable tower of local rings. Hence by the assertion (1), $(t_0)^{q.\mathrm{frep}}_I$ and $F^{q.\mathrm{frep}}_j$ is local. □

At the final of this section, we prove the following property. This is well-known in positive characteristic, in which case $R_i \to R_{i+1}$ is a universal homeomorphism.

Lemma 2.3.12. *Let $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ be a purely inseparable tower arising from some pair (R, I). For every $i \geq 0$, assume that R_i is I-adically Henselian.*[4] *Then the ring map t_i induces an equivalence of categories:*

$$\mathbf{F.\acute{E}t}(R_i) \xrightarrow{\cong} \mathbf{F.\acute{E}t}(R_{i+1}),$$

where $\mathbf{F.\acute{E}t}(A)$ is the category of finite étale A-algebras for a ring A.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} R_i & \xrightarrow{\quad t_i \quad} & R_{i+1} \\ \downarrow{\scriptstyle \varphi_i} & & \downarrow{\scriptstyle \varphi_{i+1}} \\ R_i/IR_i & \xrightarrow{\quad \overline{t_i} \quad} & R_{i+1}/IR_{i+1} \\ \downarrow{\scriptstyle \phi_i} & & \downarrow{\scriptstyle \phi_{i+1}} \\ R_i/\sqrt{IR_i} & \xrightarrow{(\overline{t_i})_{\mathrm{red}}} & R_{i+1}/\sqrt{IR_{i+1}} \end{array} \tag{2.12}$$

where $\varphi_i, \varphi_{i+1}, \phi_i$ and ϕ_{i+1} are the natural quotient maps, and $(\overline{t_i})_{\mathrm{red}}$ is obtained from $\overline{t_i}$ by killing out the nilradical part. Recall that a universal homeomorphism induces an equivalence of respective categories of finite étale algebras in view of [Sta, Tag 0BQN]. By [Sta, Tag 054M], the maps ϕ_i and ϕ_{i+1} are universal homeomorphisms. By the axiom (b) in Definition 2.3.1, $IR_i = R_i \cap IR_{i+1}$. Hence $\sqrt{IR_i} = R_i \cap \sqrt{IR_{i+1}}$ and $(\overline{t_i})_{\mathrm{red}}$ is injective. Moreover by Lemma 2.3.6 (2), the image of $(\overline{t_i})_{\mathrm{red}}$ contains $\{x^p \mid x \in R_{i+1}/\sqrt{IR_{i+1}}\}$. So [Sta, Tag 0BRA] shows that $(\overline{t_i})_{\mathrm{red}}$ is a universal homeomorphism. Finally, as for φ_i and φ_{i+1}, these maps induce an equivalence of categories of finite étale algebras over respective rings by [Sta, Tag 09ZL]. By going around the diagram (2.12), we finish the proof. □

[4] This condition is realized if R_0 is I-adically Henselian and each $t_i : R_i \to R_{i+1}$ is integral.

2.4 Perfectoid towers and their tilts

In this section, we define perfectoid towers and their tilts, and investigate their properties. In §2.4.1, we first consider torsions of modules. This consideration is necessary to define perfectoid towers. After that, we define perfectoid towers. In §2.4.2, we define tilts of perfectoid towers. The main result of this subsection is to give an isomorphism between the Koszul homology of a perfectoid tower and that of its tilt. Also, we show that Noetherian properties of perfectoid towers are inherited by the tilting operation. In §2.4.3, we establish the theorem that the I_1-adic completion of the direct limit of a perfectoid tower is perfectoid where I_1 is a principal ideal defined in the axiom (f) of the definition of perfectoid towers.

2.4.1 Definition of perfectoid towers and their tilts

To define the perfectoid towers, we need some notation and some observation of torsions of modules over rings.

Definition 2.4.1. Let R be a ring, and let M be an R-module.

1. Let $x \in R$ be an element. Then we say that an element $m \in M$ is *x-torsion* if $x^n m = 0$ for some $n \in M$. We denote by $M_{x\text{-tor}}$ the R-submodule of M consisting of all x-torsion elements in M.

2. Let $I \subseteq R$ be an ideal. Then we say that an element $m \in M$ is *I-torsion* if m is x-torsion for every $x \in I$. We denote by $M_{I\text{-tor}}$ the R-submodule of M consisting of all I-torsion elements in M. Note that $M_{x\text{-tor}} = M_{(x)\text{-tor}}$.

3. For an element $x \in R$ (resp. an ideal $I \subseteq R$), we say that *M has bounded x-torsion* (resp. *bounded I-torsion*) if there exists some $l > 0$ such that $x^l M_{x\text{-tor}} = (0)$ (resp. $I^l M_{I\text{-tor}} = (0)$).

4. For an ideal $I \subseteq R$, we denote by $\varphi_{I,M} : M_{I\text{-tor}} \to M/IM$ the composition of natural R-linear maps $M_{I\text{-tor}} \hookrightarrow M \twoheadrightarrow M/IM$.

In the rest of this subsection, we assume that R is a ring, M is an R-module, $x \in R$ is an element, and $I \subseteq R$ is an ideal.

Lemma 2.4.2. *Let R be a ring, and let M be an R-module. Let $x \in R$ be an element. Then for every $n > 0$, we have*

$$M_{x\text{-tor}} \cap x^n M = x^n M_{x\text{-tor}}.$$

Proof. Pick an element $m \in M_{x\text{-tor}} \cap x^n M$. Then $m = x^n m_0$ for some $m_0 \in M$, and $x^l m = 0$ for some $l > 0$. Hence $x^{l+n} m_0 = 0$, which implies that $m_0 \in M_{x\text{-tor}}$ and thus $m \in x^n M_{x\text{-tor}}$. The containment $x^n M_{x\text{-tor}} \subseteq M_{x\text{-tor}} \cap x^n M$ is clear. □

Corollary 2.4.3. *Keep the notation as in Lemma 2.4.2, and suppose further that $xM_{x\text{-tor}} = (0)$. Then the map $\varphi_{(x),M} : M_{x\text{-tor}} \to M/xM$ (see Definition 2.4.1 (4)) is injective.*

Proof. It is clear from Lemma 2.4.2. □

Corollary 2.4.4. *Keep the notation as in Lemma 2.4.2, and suppose further that M has bounded x-torsion. Let $\widehat{M}$ be the x-adic completion of M, and let $\psi : M \to \widehat{M}$ be the natural map. Then the restriction $\psi_{\mathrm{tor}} : M_{x\text{-tor}} \to (\widehat{M})_{x\text{-tor}}$ of ψ is injective.*

Proof. By assumption, there exists some $l > 0$ such that $x^l M_{x\text{-tor}} = (0)$. On the other hand, $\mathrm{Ker}(\psi_{\mathrm{tor}}) = M_{x\text{-tor}} \cap \bigcap_{n=0}^{\infty} x^n M$ is contained in $M_{x\text{-tor}} \cap x^l M$, which is equal to $x^l M_{x\text{-tor}}$ by Lemma 2.4.2. Thus the assertion follows. □

Lemma 2.4.5. *Let R be a ring, and let M be an R-module. Let $x \in R$ be an element. Then for every $n > 0$, we have*

$$\mathrm{Ann}_{M/x^n M}(x) \subseteq \mathrm{Im}(\varphi_{(x^n),M}) + x^{n-1}(M/x^n M). \tag{2.13}$$

Proof. Pick an element $m \in M$ such that $xm \in x^n M$. Then $x(m - x^{n-1}m') = 0$ for some $m' \in M$. In particular, $m - x^{n-1}m' \in M_{x^n\text{-tor}}$. Hence $m \mod x^n M$ lies in the right-hand side of (2.13), as desired. □

In the case when $M = R$, we can regard $M_{I\text{-tor}}$ as a (possibly) non-unital subring of R. This point of view provides valuable insights. For example, "reducedness" for $R_{I\text{-tor}}$ induces a good property on the boundedness of torsions.

Lemma 2.4.6. *Let (R, I) be a pair such that $R_{I\text{-tor}}$ does not contain any non-zero nilpotent element of R. Then $IR_{I\text{-tor}} = (0)$.*

Proof. It suffices to show that $xR_{I\text{-tor}} = 0$ for every $x \in I$. Pick an element $a \in R_{I\text{-tor}}$. Then for a sufficiently large $n > 0$, $x^n a = 0$. Hence $(xa)^n = x^n a \cdot a^{n-1} = 0$. Thus we have $xa = 0$ by assumption, as desired. □

Corollary 2.4.7. *Let $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ be a purely inseparable tower arising from some pair (R, I). Then for every $i \geq 0$ and every ideal $J \subseteq (R_i)_I^{q.\mathrm{frep}}$, we have $J((R_i)_I^{q.\mathrm{frep}})_{J\text{-tor}} = (0)$.*

Proof. Since $(R_i)_I^{q.\mathrm{frep}}$ is reduced by Proposition 2.3.10 (4), the assertion follows from Lemma 2.4.6. □

Furthermore, we can treat $R_{I\text{-tor}}$ as a positive characteristic object (in the situation of our interest), even if R is not an $\mathbb{F}_p$-algebra.

Lemma-Definition 2.4.8. *Let (R, I) be a pair such that $p \in I$ and $IR_{I\text{-tor}} = (0)$. Then the multiplicative map:*

$$R_{I\text{-tor}} \to R_{I\text{-tor}};\ x \mapsto x^p \tag{2.14}$$

is also additive. We denote by $F_{R_{I\text{-tor}}}$ the map (2.14).

Proof. It immediately follows from the binomial theorem. □

Under the above preparation, we define perfectoid towers.

Definition 2.4.9 (Perfectoid towers)**.** Let R be a ring and let $I_0 \subseteq R$ be an ideal. Then we say that a tower $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ is *(p-)perfectoid towers arising from* (R, I_0) if it is a p-purely inseparable tower and satisfies the following additional axioms.

(d) For every $i \geq 0$, the i-th Frobenius projection $F_i : R_{i+1}/I_0 R_{i+1} \to R_i/I_0 R_i$ is surjective.

(e) For every $i \geq 0$, R_i is an I_0-adically Zariskian ring.

(f) I_0 is a principal ideal, and R_1 contains a principal ideal I_1 that satisfies the following axioms.

(f-1) $I_1^p = I_0R_1$.

(f-2) For every $i \geq 0$, $\mathrm{Ker}(F_i) = I_1(R_{i+1}/I_0R_{i+1})$.

(g) For every $i \geq 0$, $I_0(R_i)_{I_0\text{-tor}} = (0)$. Moreover, there exists a (unique) bijective map $(F_i)_{\mathrm{tor}} : (R_{i+1})_{I_0\text{-tor}} \to (R_i)_{I_0\text{-tor}}$ such that the diagram:

$$\begin{array}{ccc} (R_{i+1})_{I_0\text{-tor}} & \xrightarrow{\varphi_{I_0,R_{i+1}}} & R_{i+1}/I_0R_{i+1} \\ {\scriptstyle (F_i)_{\mathrm{tor}}}\downarrow & & \downarrow{\scriptstyle F_i} \\ (R_i)_{I_0\text{-tor}} & \xrightarrow[\varphi_{I_0,R_i}]{} & R_i/I_0R_i \end{array}$$

commutes.

Remark 2.4.10. If I_0 is generated by a regular sequence in R_∞, the axiom (g) is satisfied automatically. Consequently, the axiom (g) is satisfied if R_∞ is a domain. Also, if $I_0 = (0)$ (namely, R_i is $\mathbb{F}_p$-algebra for any $i \geq 0$), then the axiom (g) follows from the axioms (c) and (f).

We provide some examples of perfectoid towers.

Example 2.4.11. *1. (cf. [Shi11, Definition 4.4]) Let $(R, \mathfrak{m}, k)$ be a d-dimensional unramified regular local ring of mixed characteristic $p > 0$ whose residue field is perfect. Then R is isomorphic to the formal power series ring $W(k)[\![x_2, \ldots, x_d]\!]$ by Cohen's structure theorem. For every $i \geq 0$, set $R_i := R[p^{1/p^i}, x_2^{1/p^i}, \ldots, x_d^{1/p^i}] \in R^+$ where R^+ is the absolute integral closure of R and let $t_i : R_i \to R_{i+1}$ be the inclusion map. Then the tower $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ is a perfectoid tower arising from $(R, (p))$. Indeed, the Frobenius projection is given as the p-th power map.*

2. We note that t_i (resp. F_i) of a perfectoid tower is not necessarily the inclusion map (resp. the p-th power map). For instance, let R be a reduced $\mathbb{F}_p$-algebra. Set $R_i := R$, $t_i := F_R$, and $F_i := \mathrm{id}_R$ for every $i \geq 0$. Then $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ is a perfectoid tower arising from $(R, (0))$.

In the subsection 2.5, we provide the construction of perfectoid towers arising from *local log-regular rings*.

The following lemma implies that the class of perfectoid towers is a generalization of perfect towers.

Lemma 2.4.12. *Let $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ be a tower of $\mathbb{F}_p$-algebras. Then the following conditions are equivalent.*

1. *$(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ is a perfect $\mathbb{F}_p$-tower (cf. Definition 2.2.2).*

2. *$(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ is a p-perfectoid tower arising from $(R_0, (0))$.*

Proof. First, we verify the implication (1) $\Rightarrow$ (2). For this, we may assume that $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ is of the form $(\{R^{1/p^i}\}_{i\geq 0}, \{\iota_i\}_{i\geq 0})$ by definition. By Example 2.3.4, $(\{R^{1/p^i}\}_{i\geq 0}, \{\iota_i\}_{i\geq 0})$ is a purely inseparable tower arising from $(R, 0)$. The axioms (e) and (g) in Definition 2.4.9 are obvious. Moreover, the Frobenius projection F_i is an isomorphism for any $i \geq 0$ by Lemma 2.2.4 (1). Hence the axioms (d) and (f) are also satisfied, which yields the assertion.

Conversely, assume that $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ is a perfectoid tower arising from $(R_0, (0))$. By the combination of the axiom (d), the axiom (f-2), and Lemma 2.3.6, F_i is an isomorphism for any $i \geq 0$. Moreover, we have the following commutative ladder:

$$\begin{array}{ccccccccc} R_0 & \xrightarrow{t_0} & R_1 & \xrightarrow{t_1} & R_2 & \xrightarrow{t_2} & R_3 & \xrightarrow{t_3} & \cdots \\ \downarrow{\scriptstyle \mathrm{id}_{R_0}} & & \downarrow{\scriptstyle F_0} & & \downarrow{\scriptstyle F_0\circ F_1} & & \downarrow{\scriptstyle F_0\circ F_1\circ F_2} & & \\ R_0 & \xrightarrow[F_{R_0}]{} & R_0 & \xrightarrow[F_{R_0}]{} & R_0 & \xrightarrow[F_{R_0}]{} & R_0 & \xrightarrow[F_{R_0}]{} & \cdots . \end{array}$$

Hence $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ is isomorphic to $\{R_0 \xrightarrow{F_{R_0}} R_0 \xrightarrow{F_{R_0}} \cdots\}$, which is a perfect tower by Corollary 2.2.5. □

Let us verify the uniqueness of $I_1 \subseteq R_1$ appearing in the axiom (f). We carry out this in more general situations for later application.

Lemma 2.4.13. *Let $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ be a perfectoid tower arising from some pair (R_0, I_0). Fix an integer $i \geq 0$ and an ideal $J_i \subseteq R_i$ containing I_0R_i. Then if there exists an ideal J_{i+1} of R_{i+1} such that $J_{i+1}^p = J_iR_{i+1}$ and $F_i^{-1}(J_i(R_i/I_0R_i)) = J_{i+1}(R_{i+1}/I_0R_{i+1})$, the ideal J_{i+1} is unique.*

Proof. For every $r \in R_{i+1}$ such that $r^p \in J_iR_{i+1}$, the equality $J_iR_{i+1} = J_{i+1}$ implies $r \in J_{i+1}$. Hence for every ideal $J'_{i+1} \subseteq R_{i+1}$ with the same assumption as J_{i+1}, we have $J'_{i+1} = J_{j+1}$, as desired. □

Lemma 2.4.13 ensures the uniqueness of I_1.

Definition 2.4.14. We call the ideal I_1 appearing the axiom (f) of perfectoid towers the *first perfectoid pillar*.

The relationship between I_0 and I_1 can be observed also in higher layers.

Proposition 2.4.15. *Let $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ be a perfectoid tower arising from some pair (R_0, I_0) and let I_1 be the first perfectoid pillar. Set $\overline{R_i} := R_i/I_0R_i$ for every $i \geq 0$. Then the following assertions hold.*

1. *For a sequence of principal ideals $\{I_i \subseteq R_i\}_{i\geq 2}$, the following are equivalent.*
   - *(a) $F_i^{-1}(I_i\overline{R_i}) = I_{i+1}\overline{R_{i+1}}$ for every $i \geq 0$ for every $i \geq 0$.*
   - *(b) $F_i(I_{i+1}\overline{R_{i+1}}) = I_i\overline{R_i}$ for every $i \geq 0$.*
2. *Each one of the equivalent conditions in (1) implies that $I_{i+1}^p = I_iR_{i+1}$ for every $i \geq 0$.*
3. *There exists a unique sequence of principal ideals $\{I_i \subseteq R_i\}_{i\geq 0}$ that satisfies one of the equivalent condition in (1). Moreover, there exists a sequence of elements $\{\overline{f_i} \in \overline{R_i}\}_{i\geq 0}$ such that $I_i\overline{R_i} = (\overline{f_i})$ and $F_i(\overline{f_{i+1}}) = \overline{f_i}$ for every $i \geq 0$.*

Proof. (1) : Since the implication (a) ⇒ (b) follows from the axiom (d) in Definition 2.4.9, it suffices to show the converse. Assume that the condition (b) is satisfied. Then for every $i \geq 0$, the compatibility $\overline{t_i} \circ F_i = F_{\overline{R_{i+1}}}$ implies

$$I_{i+1}^p\overline{R_{i+1}} = I_i\overline{R_{i+1}} \tag{2.15}$$

because I_{i+1} is principal. In particular, $\mathrm{Ker}(F_i) = I_1\overline{R_{i+1}} \subseteq I_{i+1}\overline{R_{i+1}}$ (cf. the axiom (f-2)). On the other hand, by the surjectivity of F_i and the assumption again, we have $F_i(F_i^{-1}(I_i\overline{R_i})) = I_i\overline{R_i} = F_i(I_{i+1}\overline{R_{i+1}})$. Hence

$$F_i^{-1}(I_i\overline{R_i}) \subseteq I_{i+1}\overline{R_{i+1}} + \mathrm{Ker}(F_i) \subseteq I_{i+1}\overline{R_{i+1}} \subseteq F_i^{-1}(I_i\overline{R_i}),$$

which yields the assertion.

(2) : Let us deduce the assertion from (2.15) by induction. By definition, $I_1^p = I_0R_1$. We then fix some $i \geq 1$. Suppose that for every $1 \leq k \leq i$, $I_k^p = I_{k-1}R_k$. Then $I_0R_i = I_i^{p^i}$. In particular, R_i is I_i-adically Zariskian by the axiom (e). Moreover, by (2.15), we have the equalities of R_i-modules:

$$I_iR_{i+1} = I_{i+1}^p + I_0R_{i+1} = I_i^{p^i-1}(I_iR_{i+1}) + I_{i+1}^p.$$

Hence by the axiom (f) and Nakayama's lemma, we obtain $I_{i+1}^p = I_iR_{i+1}$ as desired.

(3): By the axiom of (dependent) choice, the existence follows from the axiom (d) in Definition 2.4.9. The uniqueness is due to Lemma 2.4.13. □

Definition 2.4.16. In the situation of Proposition (3), we call I_i the *i-th perfectoid pillar of* $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ *arising from* (R_0, I_0).

Lemma 2.4.17. *Let* $\{I_i\}_{i\geq 0}$ *denote the system of perfectoid pillars of* $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$, *and let* $\pi_i : R_i/I_0R_i \to R_i/I_iR_i$ $(i \geq 0)$ *be the natural projections. Then for every* $i \geq 0$, *there exists a unique isomorphism of rings:*

$$F_i' : R_{i+1}/I_{i+1}R_{i+1} \xrightarrow{\cong} R_i/I_iR_i$$

such that $\pi_i \circ F_i = F_i' \circ \pi_{i+1}$.

Proof. Since F_i and π_i are surjective, the assertion immediately follows from $\mathrm{Ker}(\pi_i \circ F_i) = F_i^{-1}(I_i(R_i/I_0R_i)) = I_{i+1}(R_{i+1}/I_0R_{i+1})$. □

2.4.2 Tilts of perfectoid towers

Here we establish tilting operation for perfectoid towers. For this, we first introduce the notion of *small tilt*, which originates in [Shi11].

Definition 2.4.18. Let $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ be a perfectoid tower arising from some pair (R, I_0).

1. For any $j \geq 0$, the j-th inverse quasi-perfection of $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ associated to (R, I_0) is called the *j-th small tilt* of $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ associated to (R, I_0) and is denoted by $(R_j)^{s.\flat}_{I_0}$ in distinction from $(R_j)^{q.\mathrm{frep}}_{I_0}$.

2. Let the notation be as in Lemma 2.4.17. Then we define $I_i^{s.\flat} := \mathrm{Ker}(\pi_i \circ \Phi_0^{(i)})$ for every $i \geq 0$.

Note that the ideal $I_i^{s.\flat} \subseteq R_i^{s.\flat}$ has the following property.

Lemma 2.4.19. *Let* $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ *be a perfectoid tower arising from some pair* (R_0, I_0). *Let* I_i *be the i-th perfectoid pillar. Then for every* $i \geq 0$ *and* $j \geq 0$, *we have* $\Phi_i^{(j)}(I_j^{s.\flat}) = I_{j+i}(R_{j+i}/I_0R_{j+i})$.

Proof. Since $\Phi_0^{(j)}$ is surjective, we have $\Phi_0^{(j)}(I_j^{s.\flat}) = I_j(R_j/I_0R_j)$. On the other hand, since $\Phi_0^{(j)} = F_j \circ \Phi_1^{(j)}$, we have

$$F_j^{-1}(\Phi_0^{(j)}(I_j^{s.\flat})) = \Phi_1^{(j)}(I_j^{s.\flat}) + \mathrm{Ker}(F_j) = \Phi_1^{(j)}(I_j^{s.\flat}).$$

Hence by the condition (a) in Proposition 2.4.15 (1), $\Phi_1^{(j)}(I_j^{s.\flat}) = I_{j+1}(R_{j+1}/I_0R_{j+1})$. By repeating this procedure recursively, we obtain the assertion. □

The next lemma provides some completeness of the small tilts attached to a perfectoid tower of characteristic $p > 0$.

Lemma 2.4.20. *Let $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ be a perfectoid tower arising from $(R,(0))$. Then, for any element $f \in R$ and any $j \geq 0$, the inverse limit $\varprojlim\{\cdots \xrightarrow{\overline{F_{j+1}}} R_{j+1}/fR_{j+1} \xrightarrow{\overline{F_j}} R_j/fR_j\}$ is isomorphic to the f-adic completion of R_j.*

Proof. It suffices to show the assertion when $j = 0$. Since $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ is a perfectoid tower arising from $(R,(0))$, each Frobenius projection $F_i : R_{i+1} \to R_i$ is an isomorphism. In particular, the 0-th projection map on $(R_0)^{s.\flat}_{(0)}$ is an isomorphism

$$(R_0)^{s.\flat}_{(0)} = \varprojlim\{\cdots \to R_1 \to R_0\} \xrightarrow{\cong} R_0. \tag{2.16}$$

Set the element $\mathbf{f} := (\ldots, (F_0 \circ F_1)^{-1}(f), F_0^{-1}(f), f) \in (R_0)^{s.\flat}_{(0)}$. Then for any $i \geq 0$, we obtain the following diagram

$$\begin{array}{ccccccc}
(R_0)^{s.\flat}_{(0)} & \xrightarrow{\times \mathbf{f}^{p^{i+1}}} & (R_0)^{s.\flat}_{(0)} & \xrightarrow{\phi_{i+1}} & R_{i+1}/fR_{i+1} & \longrightarrow & 0 \\
\downarrow{\scriptstyle \times \mathbf{f}^{p^i(p-1)}} & & \downarrow{\scriptstyle \mathrm{id}_{R_0^{s.\flat}}} & & \downarrow{\scriptstyle \overline{F_i}} & & \\
(R_0)^{s.\flat}_{(0)} & \xrightarrow{\times \mathbf{f}^{p^i}} & (R_0)^{s.\flat}_{(0)} & \xrightarrow{\phi_i} & R_i/fR_i & \longrightarrow & 0.
\end{array}$$

where ϕ_i is the composite map of the i-th projection $(R_0)^{s.\flat}_{(0)} \to R_i$ and the natural surjection $R_i \to R_i/fR_i$. Then taking the inverse limits for the above diagrams, we obtain the isomorphism

$$\varprojlim_{i\geq 0}(R_0)^{s.\flat}_{(0)}/\mathbf{f}^{p^i}(R_0)^{s.\flat}_{(0)} \xrightarrow{\cong} \varprojlim\{\cdots \to R_1/fR_1 \to R_0/fR_0\}. \tag{2.17}$$

On the other hand, (2.16) induces $\varprojlim_{i\geq 0}(R_0)^{s.\flat}_{(0)}/\mathbf{f}^{p^i}(R_0)^{s.\flat}_{(0)} \xrightarrow{\cong} \varprojlim_{i\geq 0} R_0/f^{p^i}R_0$. Hence the assertion follows. □

Example 2.4.21. *Let S be a perfect $\mathbb{F}_p$-algebra. Pick an arbitrary $f \in S$, and let $\widehat{S}$ denote the f-adic completion. Applying the argument of the above proof to the tower*

$$S \xrightarrow{\mathrm{id}_S} S \xrightarrow{\mathrm{id}_S} S \xrightarrow{\mathrm{id}_S} \cdots,$$

we obtain a canonical isomorphism $\widehat{S} \xrightarrow{\cong} \varprojlim_{\mathrm{Frob}} S/fS$.

Remark 2.4.22. In Lemma 2.4.20, the right-hand side of the isomorphism (2.17) can not be writen $(R_0)^{q.\mathrm{frep}}_{(f)}$ by the lack of the injectivity of $\overline{t_i}$. Thus if we add the assumption that $\overline{t_i} : R_i/f_0R_i \to R_{i+1}/f_0R_{i+1}$ is injective, then the tower $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ becomes a purely inseparable tower arising from $(R,(f_0))$, and the inverse limit $\varprojlim\{\cdots \xrightarrow{\overline{F_{j+1}}} R_{j+1}/f_0R_{j+1} \xrightarrow{\overline{F_j}} R_j/f_0R_j\}$ is the j-th inverse quasi-perfection $(R_j)^{q.\mathrm{frep}}_I$.

Now we define *tilts of perfectoid towers.*

Definition 2.4.23 (Tilts of perfectoid towers)**.** Let $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ be a perfectoid tower arising from some pair (R, I). Then the inverse perfection of $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ associated to (R, I) is called *the tilt of* $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ associated to (R, I), and is denoted by $(\{(R_i)_I^{s.\flat}\}_{i\geq 0}, \{(t_i)_I^{s.\flat}\}_{i\geq 0})$ in distinction from $(\{(R_i)_I^{q.\mathrm{frep}}\}_{i\geq 0}, \{(t_i)_I^{q.\mathrm{frep}}\}_{i\geq 0})$. Moreover, we set $(R_\infty)_I^{s.\flat} := \varinjlim_{i\geq 0}(R_i)_I^{s.\flat}$.

In [Shi11], he constructs a perfectoid tower arising from an unramified regular local ring and computes its tilt. We generalize these to those of a local log-regular ring. Before that, we discuss one of the most fundamental properties, which is a correspondence between I_0-torsions in R_j and $I_0^{s.\flat}$-torsions in $R_j^{s.\flat}$

Theorem 2.4.24. *Let $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ be a perfectoid tower arising from some pair (R, I_0), and let $\{I_i\}_{i\geq 0}$ be the system of perfectoid pillars. Let $(\{R_i^{s.\flat}\}_{i\geq 0}, \{t_i^{s.\flat}\}_{i\geq 0})$ denote the tilt associated to (R, I_0). Then the following assertions hold.*

1. *For every $j \geq 0$ and every element $f_j^{s.\flat} \in R_j^{s.\flat}$, the following conditions are equivalent.*
   - *(a) $f_j^{s.\flat}$ is a generator of $I_j^{s.\flat}$.*
   - *(b) For every $i \geq 0$, $\Phi_i^{(j)}(f_j^{s.\flat})$ is a generator of $I_{j+i}(R_{j+i}/I_0R_{j+i})$.*

   In particular, $I_j^{s.\flat}$ is a principal ideal, and $(I_{j+1}^{s.\flat})^p = I_j^{s.\flat}R_{j+1}^{s.\flat}$.
2. *We have isomorphisms of (possibly) non-unital rings $(R_j^{s.\flat})_{I_0^{s.\flat}\text{-tor}} \cong (R_j)_{I_0\text{-tor}}$ that are compatible with $\{t_j\}_{j\geq 0}$ and $\{t_j^{s.\flat}\}_{j\geq 0}$.*

To prove this theorem, we prepare several lemmas.

Lemma 2.4.25. *For every $i \geq 0$, let $(t_i)_{\mathrm{tor}} : (R_i)_{I_0\text{-tor}} \to (R_{i+1})_{I_0\text{-tor}}$ be the restriction of t_i. Then the following assertions hold.*

1. *$(t_i)_{\mathrm{tor}}$ is the unique map such that $\varphi_{I_0,R_{i+1}} \circ (t_i)_{\mathrm{tor}} = \overline{t_i} \circ \varphi_{I_0,R_i}$.*
2. *$(t_i)_{\mathrm{tor}} \circ (F_i)_{\mathrm{tor}} = (F_{i+1})_{\mathrm{tor}} \circ (t_{i+1})_{\mathrm{tor}} = F_{(R_{i+1})_{I_0\text{-tor}}}$.*

Proof. Since φ_{I_0,R_i} is injective by Corollary 2.4.3, the assertion (1) is clear from the construction. Hence we can regard $(t_i)_{\mathrm{tor}}$ and $(F_i)_{\mathrm{tor}}$ as the restrictions of $\overline{t_i}$ and F_i, respectively. Thus the assertion (2) follows from the compatibility $\overline{t_i} \circ F_i = F_{i+1} \circ \overline{t_{i+1}} = F_{\overline{R_{i+1}}}$ induced by Lemma 2.3.6 (3). □

The map $\varphi_{I_0,R_i} : (R_i)_{I_0\text{-tor}} \hookrightarrow R_i/I_0R_i$ restricts to $\mathrm{Ann}_{R_i}(I_i) \hookrightarrow \mathrm{Ann}_{\overline{R_i}}(\overline{I_i})$. On the other hand, $\mathrm{Ann}_{R_i}(I_i)$ turns out to be equal to $(R_i)_{I_0\text{-tor}}$ by the following lemma.

Lemma 2.4.26. *For every $i \geq 0$, $I_i(R_i)_{I_0\text{-tor}} = 0$. In particular, $\mathrm{Im}(\varphi_{I_0,R_i}) \subseteq \mathrm{Ann}_{\overline{R_i}}(\overline{I_i})$.*

Proof. By Lemma 2.4.25 (2) and the axiom (g) in Definition 2.4.9, we find that $F_{(R_i)_{I_0\text{-tor}}}$ is injective. In other words, $(R_i)_{I_0\text{-tor}}$ does not contain any non-zero nilpotent element. Moreover, $(R_i)_{I_0\text{-tor}} = (R_i)_{I_i\text{-tor}}$. Hence the assertion follows from Lemma 2.4.6. □

The following lemma is essential for proving Theorem 2.4.24.

Lemma 2.4.27. *For every $i \geq 0$, F_i restricts to a $\mathbb{Z}$-linear map $\mathrm{Ann}_{\overline{R_{i+1}}}(\overline{I_{i+1}}) \to \mathrm{Ann}_{\overline{R_i}}(\overline{I_i})$. Moreover, the resulting inverse system $\{\mathrm{Ann}_{\overline{R_i}}(\overline{I_i})\}_{i\geq 0}$ has the following properties.*

1. *For every $j \geq 0$, $\varprojlim^1_{i\geq 0} \mathrm{Ann}_{\overline{R_{j+i}}}(\overline{I_{j+i}}) = (0)$.*
2. *There are isomorphisms of $\mathbb{Z}$-linear maps $\varprojlim_{i\geq 0} \mathrm{Ann}_{\overline{R_{j+i}}}(\overline{I_{j+i}}) \cong (R_j)_{I_0\text{-tor}}$ $(j \geq 0)$ that are multiplicative, and compatible with $\{t_j^{s,\flat}\}_{j\geq 0}$ and $\{t_j\}_{j\geq 0}$.*

Proof. Since $F_i(\overline{I_{i+1}}) = \overline{I_i}$, F_i restricts to a $\mathbb{Z}$-linear map $(F_i)_{\mathrm{ann}} : \mathrm{Ann}_{\overline{R_{i+1}}}(\overline{I_{i+1}}) \to \mathrm{Ann}_{\overline{R_i}}(\overline{I_i})$. Let $\varphi_i : (R_i)_{I_0\text{-tor}} \hookrightarrow \mathrm{Ann}_{\overline{R_i}}(\overline{I_i})$ be the restriction of φ_{I_0,R_i}. By Lemma 2.4.5 and Lemma 2.4.26, we can write $\mathrm{Ann}_{\overline{R_i}}(\overline{I_i}) = \mathrm{Im}(\varphi_i) + \overline{I_i}^{p^i-1}$. Moreover, $\mathrm{Im}(\varphi_i) \cap \overline{I_i}^{p^i-1} = (0)$ by Lemma 2.4.2 and Lemma 2.4.26. Hence we have the following ladder with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (R_{i+1})_{I_0\text{-tor}} & \xrightarrow{\varphi_{i+1}} & \mathrm{Ann}_{\overline{R_{i+1}}}(\overline{I_{i+1}}) & \longrightarrow & \overline{I_{i+1}}^{p^{i+1}-1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle (F_i)_{\mathrm{tor}}} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (R_i)_{I_0\text{-tor}} & \xrightarrow{\varphi_i} & \mathrm{Ann}_{\overline{R_i}}(\overline{I_i}) & \longrightarrow & \overline{I_i}^{p^i-1} & \longrightarrow & 0 \end{array} \tag{2.18}$$

where the second and third vertical maps are the restrictions of F_i. Since $F_i(\overline{I_{i+1}}^{p^{i+1}-1}) = 0$, the both functors $\varprojlim_{i\geq 0}$ and $\varprojlim^1_{i\geq 0}$ assign (0) to the inverse system $\{\overline{I_{j+i}}^{p^{j+i}-1}\}_{i\geq 0}$. Moreover, since $(F_i)_{\mathrm{tor}}$ is bijective, $\varprojlim_{i\geq 0}(R_{j+i})_{I_0\text{-tor}} \cong (R_{j+i})_{I_0\text{-tor}}$ and $\varprojlim^1_{i\geq 0}(R_{j+i})_{I_0\text{-tor}} = (0)$. Hence we find that $\varprojlim^1_{i\geq 0} \mathrm{Ann}_{\overline{R_{j+i}}}(\overline{I_{j+i}}) = (0)$, which is the assertion (1). Furthermore, we obtain the isomorphisms of $\mathbb{Z}$-modules:

$$(R_j)_{I_0\text{-tor}} \xleftarrow{(\Phi_0^{(j)})_{\mathrm{tor}}} \varprojlim_{i\geq 0}(R_{j+i})_{I_0\text{-tor}} \xrightarrow{\varprojlim_{i\geq 0} \varphi_{j+i}} \varprojlim_{i\geq 0} \overline{R_{j+i}} \tag{2.19}$$

(where $(\Phi_0^{(j)})_{\mathrm{tor}}$ denotes the 0-th projection map), which are also multiplicative. Let us deduce (2) from it. Since we have $t_j^{s.\flat} = \varprojlim_{i\geq 0} \overline{t_{j+i}}$ by definition, the maps $\varprojlim_{i\geq 0} \varphi_{j+i}$ $(j \geq 0)$ are compatible with $\{\varprojlim_{i\geq 0}(t_{j+i})_{\mathrm{tor}}\}_{j\geq 0}$ (induced by Lemma 2.4.25 (2)) and $\{t_j^{s.\flat}\}_{j\geq 0}$ by Lemma 2.4.25 (1). On the other hand, the projections $(\Phi_0^{(j)})_{\mathrm{tor}}$ $(j \geq 0)$ are compatible with $\{\varprojlim_{i\geq 0}(t_{j+i})_{\mathrm{tor}}\}_{j\geq 0}$ and $\{(t_j)_{\mathrm{tor}}\}_{j\geq 0}$. Hence the assertion follows. □

Let us complete the proof of Theorem 2.4.24.

Proof of Theorem 2.4.24. (1): The implication (a) ⇒ (b) follows from Lemma 2.4.19. Let us show the converse (b)⇒(a). For every $i \geq 0$, put $\overline{f_{j+i}} := \Phi_i^{(j)}(f_j^{s.\flat})$, and let π_i and F'_i be as in Lemma 2.4.17. Then, by the assumption, we have the following commutative ladder with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\overline{f_{i+1}}) & \xrightarrow{\iota_{i+1}} & \overline{R_{i+1}} & \xrightarrow{\pi_{i+1}} & R_{i+1}/I_{i+1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle F_i} & & \downarrow{\scriptstyle F'_i} & & \\ 0 & \longrightarrow & (\overline{f_i}) & \xrightarrow{\iota_i} & \overline{R_i} & \xrightarrow{\pi_i} & R_i/I_i & \longrightarrow & 0 \end{array}$$

where ι_i is the inclusion map. Let us consider the exact sequence obtained by taking inverse limits for all columns of the above ladder. Then, since each F'_i is an isomorphism, the map $\varprojlim_{i\geq 0}\pi_{j+i}: R_j^{s.\flat}\to \varprojlim_{i\geq 0} R_{j+i}/I_{j+i}$ is isomorphic to $\pi_j\circ\Phi_0^{(j)}$. Thus we find that $I_j^{s.\flat}=\mathrm{Im}(\varprojlim_{i\geq 0}\iota_{j+i})$. Let us show that the ideal $\mathrm{Im}(\varprojlim_{i\geq 0}\iota_{j+i})\subseteq R_j^{s.\flat}$ is generated by $f_j^{s.\flat}$. For $i\geq 0$, let $\mu_i:\overline{R_i}\to(\overline{f_i})$ be the $\overline{R_i}$-linear map induced by multiplication by $\overline{f_i}$. Then we obtain the commutative ladder:

$$\begin{array}{ccccc} \overline{R_{i+1}} & \xrightarrow{\mu_{i+1}} & (\overline{f_{i+1}}) & \xrightarrow{\iota_{i+1}} & \overline{R_{i+1}} \\ \downarrow{\scriptstyle F_i} & & \downarrow & & \downarrow{\scriptstyle F_i} \\ \overline{R_i} & \xrightarrow{\mu_i} & (\overline{f_i}) & \xrightarrow{\iota_i} & \overline{R_i}. \end{array}$$

Then, since $\mathrm{Ker}\,\mu_i=\mathrm{Ann}_{\overline{R_i}}(\overline{I_i})$ for every $i\geq 0$, $\varprojlim_{i\geq 0}\mu_{j+i}$ is surjective by Lemma 2.4.27 (1). Hence we have $\mathrm{Im}(\varprojlim_{i\geq 0}\iota_{j+i})=\mathrm{Im}(\varprojlim_{i\geq 0}(\iota_{j+i}\circ\mu_{j+i}))$, where the right hand side is the ideal of $R_j^{s.\flat}$ generated by $f_j^{s.\flat}$. Thus we obtain the desired implication. Finally, note that by Proposition 2.4.15 (3), we can take a system of elements $\{f_j^{s.\flat}\in R_j^{s.\flat}\}_{j\geq 0}$ satisfying the condition (b) such that $(f_{j+1}^{s.\flat})^p=f_j^{s.\flat}$ $(j\geq 0)$.

(2): We have $I_0^{s.\flat}(R_j^{s.\flat})_{I_0^{s.\flat}\text{-tor}}=(0)$ by Corollary 2.4.7. Hence by the assertion (1),

$$(R_j^{s.\flat})_{I_0^{s.\flat}\text{-tor}}=\mathrm{Ann}_{R_j^{s.\flat}}(I_0^{s.\flat})=\mathrm{Ker}(\varprojlim_{i\geq 0}\mu_{j+i})=\varprojlim_{i\geq 0}\mathrm{Ann}_{\overline{R_{j+i}}}(\overline{I_{j+i}}).$$

Thus by Lemma 2.4.27 (2), we obtain an isomorphism $(R_j^{s.\flat})_{I_0^{s.\flat}\text{-tor}}\cong(R_j)_{I_0\text{-tor}}$ with the desired property. □

As applications, we give several basic properties of tilts.

Lemma 2.4.28. *For every $i\geq 0$, $R_i^{s.\flat}$ is $I_0^{s.\flat}$-adically complete and separated.*

Proof. By Theorem 2.4.24, the ideal $I_0^{s.\flat}R_i^{s.\flat}\subseteq R_i^{s.\flat}$ is principal. Hence one can apply Proposition 2.3.10 (1) to deduce the assertion. □

To discuss perfectoidness for the tilt $(\{R_i^{s.\flat}\}_{i\geq 0},\{t_i^{s.\flat}\}_{i\geq 0})$, we introduce the following maps.

Definition 2.4.29. For every $i\geq 0$, we define a ring map $(F_i)_{I_0}^{s.\flat}:(R_{i+1})_{I_0}^{s.\flat}/I_0^{s.\flat}(R_{i+1})_{I_0}^{s.\flat}\to(R_i)_{I_0}^{s.\flat}/I_0^{s.\flat}(R_i)_{I_0}^{s.\flat}$ by the rule:

$$(F_i)_{I_0}^{s.\flat}(\alpha_{i+1}\mod I_0^{s.\flat}(R_{i+1})_{I_0}^{s.\flat})=(F_i)_{I_0}^{q.\mathrm{frep}}(\alpha_{i+1})\mod I_0^{s.\flat}(R_i)_{I_0}^{s.\flat}$$

where $\alpha_{i+1}\in(R_{i+1})_{I_0}^{s.\flat}$.

Remark 2.4.30. Although the symbols $(\,\cdot\,)^{s.\flat}$ and $(\,\cdot\,)^{q.\mathrm{frep}}$ had been used interchangeably before Definition 2.4.29, $(F_i)_{I_0}^{s.\flat}$ differs from $(F_i)_{I_0}^{q.\mathrm{frep}}$ in general.

The following lemma is an immediate consequence of Theorem 2.4.24 (1), but quite useful.

Lemma 2.4.31. *For every $j\geq 0$, $\Phi_0^{(j)}$ induces an isomorphism*

$$\overline{\Phi_0^{(j)}}:R_j^{s.\flat}/I_0^{s.\flat}R_j^{s.\flat}\xrightarrow{\cong}R_j/I_0R_j;\quad a\mod I_0^{s.\flat}R_j^{s.\flat}\mapsto\Phi_0^{(j)}(a). \tag{2.20}$$

Moreover, $\{\overline{\Phi_0^{(i)}}\}_{i\geq 0}$ is compatible with $\{t_i\}_{i\geq 0}$ (resp. $\{F_{R_i^{s.\flat}/I_0^{s.\flat}R_i^{s.\flat}}\}_{i\geq 0}$, resp. $\{F_i^{s.\flat}\}_{i\geq 0}$) and $\{t_i^{s.\flat}\}_{i\geq 0}$ (resp. $\{F_{R_i/I_0R_i}\}_{i\geq 0}$, resp. $\{F_i\}_{i\geq 0}$).

Proof. By the axiom (d) in Definition 2.4.9, (2.20) is surjective. Let us check the injectivity. By Theorem 2.4.24 (1), $I_0^{s.\flat}$ is generated by an element $f_0^{s.\flat} \in R_0^{s.\flat}$ such that $\Phi_i^{(0)}(f_0^{s.\flat})$ is a generator of $I_i(R_i/I_0R_i)$ ($i \geq 0$). Note that $(\{R_{j+i}\}_{i\geq 0}, \{t_{j+i}\}_{i\geq 0})$ is a perfectoid tower arising from (R_j, I_0R_j). Moreover, $\{I_iR_{j+i}\}_{i\geq 0}$ is the system of perfectoid pillars associated to (R_j, I_0R_j) (cf. the condition (b) in Proposition 2.4.15 (1)). Put $J_0 := I_0R_j$. Then by Theorem 2.4.24 (1) again, we find that $J_0^{s.\flat} = f_0^{s.\flat}R_j^{s.\flat} = I_0^{s.\flat}R_j^{s.\flat}$. Since $J_0^{s.\flat} = \operatorname{Ker}\Phi_0^{(j)}$, we obtain the first assertion.

One can deduce that $\{\overline{\Phi_0^{(i)}}\}_{i\geq 0}$ is compatible with the Frobenius projections from the commutativity of (2.9), because the other compatibility assertions immediately follow from the construction. □

The following corollary is immediately obtained from the previous lemma.

Corollary 2.4.32. *Let $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ be a perfectoid tower arising from some pair $(R, (f_0))$. Fix a generator $f_0^{s.\flat}$ of $I_0^{s.\flat}$ (see Theorem 2.4.24). Then the following sequence of $R_0^{s.\flat}$-modules is exact for every $i \geq 0$:*

$$0 \longrightarrow (R_j^{s.\flat})_{I_0^{s.\flat}\text{-tor}} \longrightarrow R_i^{s.\flat} \xrightarrow{\times f_0^{s.\flat}} R_i^{s.\flat} \xrightarrow{\Phi_0^{(i)}} R_i/I_0R_i \longrightarrow 0. \tag{2.21}$$

Proof. The assertion follows from Lemma 2.4.31 because $\Phi_0^{(i)}$ is surjective and its kernel is $I_0^{s.\flat}R_i^{s.\flat}$ whose generator is the image of $f_0^{s.\flat}$. □

Remark 2.4.33. Theorem 2.4.24 (2) and Lemma 2.4.31 can be interpreted as a correspondence of homological invariants between R_i and $R_i^{s.\flat}$ by using Koszul homologies. Indeed, for any generator f_0 (resp. $f_0^{s.\flat}$) of I_0 (resp. $I_0^{s.\flat}$), the Koszul homology $H_q(f_0^{s.\flat}; R_i^{s.\flat})$ is isomorphic to $H_q(f_0; R_i)$ for any $q \geq 0$ as an abelian group.[5]

Now we can show the invariance of several properties of perfectoid towers under tilting. The first one is perfectoidness, which is most important in our framework.

Proposition 2.4.34. $(\{R_i^{s.\flat}\}_{i\geq 0}, \{t_i^{s.\flat}\}_{i\geq 0})$ *is a perfectoid tower arising from* $(R_0^{s.\flat}, I_0^{s.\flat})$*.*

Proof. Set $\overline{R_i} := R_i/I_0R_i$ for every $i \geq 0$. By Lemma 2.4.31, we find that $(\{R_i^{s.\flat}\}_{i\geq 0}, \{t_i^{s.\flat}\}_{i\geq 0})$ is a purely inseparable tower arising from $(R_0^{s.\flat}, I_0^{s.\flat})$ with Frobenius projections $\{F_i^{s.\flat}\}_{i\geq 0}$, and satisfies the the axiom (d) in Definition 2.4.9. Moreover, Lemma 2.4.31 also implies that

$$\operatorname{Ker}(F_i^{s.\flat}) = (\overline{\Phi_0^{(i+1)}})^{-1}(\operatorname{Ker}(F_i)) = (\overline{\Phi_0^{(i+1)}})^{-1}(I_1\overline{R_{i+1}}) = (\overline{\Phi_0^{(1)}})^{-1}(I_1\overline{R_1})R_{i+1}^{s.\flat} = I_1^{s.\flat}R_{i+1}^{s.\flat}.$$

Hence the axiom (f) follows from Theorem 2.4.24 (1). The axiom (e) holds by Lemma 2.4.28. Let us check that the axiom (g) holds. By Corollary 2.4.7, $I_0^{s.\flat}(R_i^{s.\flat})_{I_0^{s.\flat}\text{-tor}} = (0)$. Let $(t_i^{s.\flat})_{\text{tor}} : (R_i^{s.\flat})_{I_0^{s.\flat}\text{-tor}} \to (R_{i+1}^{s.\flat})_{I_0^{s.\flat}\text{-tor}}$ be the restriction of $t_i^{s.\flat}$. Then by Theorem 2.4.24 (2), there exists a bijection $(F_i^{s.\flat})_{\text{tor}} : (R_{i+1}^{s.\flat})_{I_0^{s.\flat}\text{-tor}} \to (R_i^{s.\flat})_{I_0^{s.\flat}\text{-tor}}$ such that $(t_i^{s.\flat})_{\text{tor}} \circ (F_i^{s.\flat})_{\text{tor}} = F_{(R_{i+1}^{s.\flat})_{I_0^{s.\flat}\text{-tor}}}$ (cf. Lemma 2.4.8). Thus we have

$$\overline{t_i^{s.\flat}} \circ \varphi_{I_0^{s.\flat}, R_i^{s.\flat}} \circ (F_i^{s.\flat})_{\text{tor}} = \varphi_{I_0^{s.\flat}, R_{i+1}^{s.\flat}} \circ (t_i^{s.\flat})_{\text{tor}} \circ (F_i^{s.\flat})_{\text{tor}} = \varphi_{I_0^{s.\flat}, R_{i+1}^{s.\flat}} \circ F_{(R_{i+1}^{s.\flat})_{I_0^{s.\flat}\text{-tor}}} = \overline{t_i^{s.\flat}} \circ F_i^{s.\flat} \circ \varphi_{I_0^{s.\flat}, R_{i+1}^{s.\flat}}.$$

Hence the injectivity of $\overline{t_i^{s.\flat}}$ yields the assertion. □

[5] Note that $(R_i)_{I_0\text{-tor}} = \operatorname{Ann}_{R_i}(I_0)$ by the axiom (g), and $(R_i^{s.\flat})_{I_0^{s.\flat}\text{-tor}} = \operatorname{Ann}_{R_i^{s.\flat}}(I_0^{s.\flat})$ by Corollary 2.4.7.

Next, we focus on finiteness properties. "Small" in the name of small tilts comes from the following fact.

Proposition 2.4.35. *For every $j \geq 0$, the following assertions hold.*

1. *If $t_j : R_j \to R_{j+1}$ is module-finite, then so is $t_j^{s.\flat} : R_j^{s.\flat} \to R_{j+1}^{s.\flat}$. Moreover, the converse holds true when R_j is I_0-adically complete and separated.*

2. *If R_j is a Noetherian ring, then so is $R_j^{s.\flat}$. Moreover, the converse holds true when R_j is I_0-adically complete and separated.*

3. *Assume that R_j is a Noetherian local ring, and a generator of $I_0 R_j$ is regular. Then the dimension of R_j is equal to that of $R_j^{s.\flat}$.*

Proof. (1): By Lemma 2.4.31, $\overline{t_j} : R_j/I_0R_j \to R_{j+1}/I_0R_{j+1}$ is module-finite if and only if $\overline{t_j^{s.\flat}} : R_j^{s.\flat}/I_0^{s.\flat}R_j^{s.\flat} \to R_{j+1}^{s.\flat}/I_0^{s.\flat}R_{j+1}^{s.\flat}$ is so. Thus by Lemma 2.4.28 and [Mat89, Theorem 8.4], the assertion follows.

(2): One can prove this assertion by applying Lemma 2.4.28, Lemma 2.4.31, and [Sta, Tag 05GH].

(3): By Theorem 2.4.24, $I_0^{s.\flat}R_j^{s.\flat}$ is also generated by a regular element. Thus we obtain the equalities $\dim(R_j) = \dim(R_j/I_0R_j) + 1$ and $\dim(R_j^{s.\flat}) = \dim(R_j^{s.\flat}/I_0^{s.\flat}R_j^{s.\flat}) + 1$. By combining these equalities with Lemma 2.4.31, we deduce the assertion. □

Proposition 2.4.35 (2) says that "Noetherianness" for a perfectoid tower is preserved under tilting.

Definition 2.4.36. We say that $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ is *Noetherian* if R_i is Noetherian for each $i \geq 0$.

Corollary 2.4.37. *If $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ is Noetherian, then so is the tilt $(\{R_i^{s.\flat}\}_{i\geq 0}, \{t_i^{s.\flat}\}_{i\geq 0})$. Moreover, the converse holds when R_i is I_0-adically complete and separated for each $i \geq 0$.*

Proof. It immediately follows from Proposition 2.4.35 (2). □

Finally, let us consider perfectoid towers of Henselian rings. Then we obtain the equivalence of categories of finite étale algebras over each layer.

Proposition 2.4.38. *Assume that R_i is I_0-adically Henselian for any $i \geq 0$. Then we obtain the following equivalences of categories:*

$$\mathbf{F.\acute{E}t}(R_i^{s.\flat}) \xrightarrow{\cong} \mathbf{F.\acute{E}t}(R_i).$$

Proof. This follows from Lemma 2.4.28, Lemma 2.4.31 and [Sta, Tag 09ZL]. □

2.4.3 Relation with perfectoid rings

For a ring R, we use the following notation. Set the inverse limit

$$R^\flat := \varprojlim\{\cdots \to R/pR \to R/pR \to \cdots \to R/pR\},$$

where each transition map is the Frobenius endomorphism on R/pR. It is called the *tilt* (or *tilting*) of R. Moreover, we denote by $W(R)$ the ring of p-typical Witt vectors over R.

If R is p-adically complete and separated, we denote by $\theta_R : W(R^\flat) \to R$ the ring map such that the diagram:

$$\begin{array}{ccc} W(R^\flat) & \xrightarrow{\theta_R} & R \\ \downarrow & & \downarrow \\ R^\flat & \longrightarrow & R/pR \end{array} \tag{2.22}$$

(where the vertical maps are induced by reduction modulo p and the bottom map is the first projection) commutes.

Recall the definition of perfectoid rings.

Definition 2.4.39. ([BMS18, Definition 3.5]) A ring S is *perfectoid* if the following conditions hold.

1. S is ϖ-adically complete and separated for some element $\varpi \in S$ such that ϖ^p divides p.

2. The Frobenius endomorphism on S/pS is surjective.

3. The kernel of $\theta_S : W(S^\flat) \to S$ is principal.

We have a connection between perfectoid towers and perfectoid rings. To see this, we use the following characterization of perfectoid rings.

Theorem 2.4.40 (cf. [GR23, Corollary 16.3.75])**.** *Let S be a ring. Then S is a perfectoid ring if and only if S contains an element ϖ with the following properties.*

1. *ϖ^p divides p, and S is ϖ-adically complete and separated.*

2. *The ring map $S/\varpi S \to S/\varpi^p S$ induced by the Frobenius endomorphism on $S/\varpi^p S$ is an isomorphism.*

3. *The multiplicative map*
$$S_{\varpi\text{-tor}} \to S_{\varpi\text{-tor}} \ ; \ s \mapsto s^p \tag{2.23}$$
is bijective.

Proof. ("if" part): It follows from [GR23, Corollary 16.3.75].

("only if" part): Let $\varpi \in S$ be as in Definition 2.4.39. Then, such ϖ clearly has the property (1) (in Theorem 2.4.40), and also has the property (2) by [BMS18, Lemma 3.10 (i)]. To show the remaining part, we set $\widetilde{S} := S/S_{\varpi\text{-tor}}$. By [ČS19, §2.1.3], the diagram of rings:

$$\begin{array}{ccc} S & \xrightarrow{\pi_2} & (S/\varpi S)_{\mathrm{red}} \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_4} \\ \widetilde{S} & \xrightarrow[\pi_3]{} & (\widetilde{S}/\varpi\widetilde{S})_{\mathrm{red}} \end{array}$$

(where π_i is the canonical projection map for $i = 1, 2, 3, 4$) is cartesian. Hence $S_{\varpi\text{-tor}}$ $(= \mathrm{Ker}(\pi_1))$ is isomorphic to $\mathrm{Ker}(\pi_4)$ as a (possibly) non-unital ring. Since $(S/\varpi S)_{\mathrm{red}}$ is a perfect $\mathbb{F}_p$-algebra, it admits the Frobenius endomorphism and the inverse Frobenius. Moreover, $\mathrm{Ker}(\pi_4)$ is closed under these operations because $(\widetilde{S}/\varpi\widetilde{S})_{\mathrm{red}}$ is reduced. Consequently, it follows that one has a bijection (2.23). Hence ϖ has the property (3), as desired. □

Remark 2.4.41. In view of the above proof, the "only if" part of Theorem 2.4.40 can be refined as follows. For a perfectoid ring S, an element $\varpi \in S$ such that $p \in \varpi^p S$ and S is ϖ-adically complete and separated satisfies the properties (2) and (3) in Theorem 2.4.40.

Corollary 2.4.42. *Let $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ be a perfectoid tower arising from some pair (R_0, I_0). Let $\widehat{R_\infty}$ denote the I_1-adic completion of R_∞. Then $\widehat{R_\infty}$ is a perfectoid ring.*

Proof. Since we have $\varinjlim_{i\geq 0} F_{R_i/I_0R_i} = (\varinjlim_{i\geq 0} \overline{t_i}) \circ (\varinjlim_{i\geq 0} F_i)$ and $\varinjlim_{i\geq 0} \overline{t_i}$ is a canonical isomorphism, the Frobenius endomorphism on $\widehat{R_\infty}$ can be identified with $\varinjlim_{i\geq 0} F_i$. Hence one can immediately deduce from the axioms in Definition 2.4.9 that any generator of $I_1\widehat{R_\infty}$ has the all properties assumed on ϖ in Theorem 2.4.40. □

In view of Theorem 2.4.40, one can regard perfectoid rings as a special class of perfectoid towers.

Example 2.4.43. *Let S be a perfectoid ring. Let $\varpi \in S$ be such that $p \in \varpi^p S$ and S is ϖ-adically complete and separated. Set $S_i = S$ and $t_i = \mathrm{id}_S$ for every $i \geq 0$, and $I_0 = \varpi^p S$. Then by Remark 2.4.41, the tower $(\{S_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ is a perfectoid tower arising from (S, I_0). In particular, $I_0 S_{I_0\text{-tor}} = (0)$, and $F_{S_{I_0\text{-tor}}}$ is bijective.*

Moreover, we can treat more general rings in a tower-theoretic way.

Example 2.4.44 (Zariskian preperfectoid rings)**.** *Let R be a ring that contains an element ϖ such that $p \in \varpi^p R$, R is ϖ-adically Zariskian, and R has bounded ϖ-torsion. Assume that the ϖ-adic completion $\widehat{R}$ is a perfectoid ring. Set $R_i = R$ and $t_i = \mathrm{id}_R$ for every $i \geq 0$, and $I_0 = \varpi^p R$. Then the tower $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ is a perfectoid tower arising from (R, I_0). Indeed, the axioms (a) and (e) are clear from the assumption. Moreover, since $\widehat{R}$ is perfectoid and $R/\varpi^p R \cong \widehat{R}/\varpi^p \widehat{R}$, the axioms (b), (c), (d) and (f) hold by Example 2.4.43. In view of Lemma 2.4.6, for proving that the axiom (g) holds, it suffices to show that the map:*

$$R_{I_0\text{-tor}} \to R_{I_0\text{-tor}};\ x \mapsto x^p \tag{2.24}$$

is bijective. By Corollary 2.4.4, the natural map $\psi_{\mathrm{tor}} : R_{I_0\text{-tor}} \to (\widehat{R})_{I_0\text{-tor}}$ is injective. Hence so is $F_{(\widehat{R})_{I_0\text{-tor}}} \circ \psi_{\mathrm{tor}}$, which factors through (2.24). Therefore, (2.24) is injective (in particular, $I_0 R_{I_0\text{-tor}} = (0)$). To check the surjectivity, we pick an element $x \in R_{I_0\text{-tor}}$. Then $\psi_{\mathrm{tor}}(x) = \eta^p$ for some $\eta \in (\widehat{R})_{I_0\text{-tor}}$. Let $y \in R$ be such that $y \equiv \eta \mod I_0^2\widehat{R}$. Then $y^p \equiv x \mod I_0^2$ and $I_0 y \subseteq I_0^2$. By Lemma 2.4.5, the second property implies that $y \equiv z \mod I_0$ for some $z \in R_{I_0\text{-tor}}$. Hence by the binomial theorem, we have $x \equiv y^p \equiv z^p \mod I_0^2$. On the other hand, $R_{I_0\text{-tor}} = R_{I_0^2\text{-tor}}$, and hence $\varphi_{I_0^2, R}$ is injective by Corollary 2.4.3. Thus we have $x = z^p$, as desired.

Recall that we have two types of tilting operation at present; one is defined for perfectoid rings, and the other is for perfectoid towers. The following result asserts that they are compatible.

Lemma 2.4.45. *Let $(\{R_i^{s.\flat}\}_{i\geq 0}, \{t_i^{s.\flat}\}_{i\geq 0})$ be the tilt of $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ associated to (R_0, I_0). Let $\widehat{R_\infty^{s.\flat}}$ be the $I_0^{s.\flat}$-adic completion of $R_\infty^{s.\flat}$. Let $I_0^\flat \subseteq R_\infty^\flat$ be the ideal that is the inverse image of $I_0 R_\infty \mod pR_\infty$ via the first projection. Then there exists a canonical isomorphism $\widehat{R_\infty^{s.\flat}} \xrightarrow{\cong} R_\infty^\flat$ that sends $I_0^{s.\flat}\widehat{R_\infty^{s.\flat}}$ onto $I_0^\flat$.*

Proof. Since $R^{s.\flat}_\infty$ is perfect, one can deduce the following isomorphism from Example 2.4.21:

$$\widehat{R^{s.\flat}_\infty} \xrightarrow{\cong} \varprojlim_{\text{Frob}} R^{s.\flat}_\infty / I^{s.\flat}_0 R^{s.\flat}_\infty \ ; \ (s_i \mod (I^{s.\flat}_0)^{p^i} R^{s.\flat}_\infty)_{i\geq 0} \mapsto (s_i^{1/p^i} \mod I^{s.\flat}_0 R^{s.\flat}_\infty)_{i\geq 0}.$$

On the other hand, (2.20) in Lemma 2.4.31 induces a canonical isomorphism

$$\varprojlim_{\text{Frob}} R^{s.\flat}_\infty / I^{s.\flat}_0 R^{s.\flat}_\infty \xrightarrow{\cong} \varprojlim_{\text{Frob}} R_\infty / I_0 R_\infty.$$

Moreover, by [BMS18, Lemma 3.2 (i)], we can identify $R^\flat_\infty$ with $\varprojlim_{\text{Frob}} R_\infty/I_0R_\infty$, and the ideal $I^\flat_0 \subseteq R^\flat_\infty$ corresponds to the kernel of the first projection map on $\varprojlim_{\text{Frob}} R_\infty/I_0R_\infty$. Thus the resulting composite map $\widehat{R^{s.\flat}_\infty} \xrightarrow{\cong} R^\flat_\infty$ has the desired property. □

2.5 Perfectoid towers arising from local log-regular rings

In this section, we provide the construction of perfectoid towers whose layers are local log-regular rings. In §2.5.1, we review the maximality of sequences of elements. We need its properties when we consider constructions of perfectoid towers by adjoining the p-th power roots of elements. In §2.5.2, we construct perfectoid towers arising from local log-regular rings adjoining the p-th power roots of elements of monoids. For this construction, we also need the consideration in §1.2.2.

2.5.1 Maximality of sequences of elements and differential modules

The content of this subsection is taken from Gabber-Ramero's treatise [GR23] whose purpose is to supply the corrected version of Grothendieck's original presentation in EGA. So we give only a sketch of the constructions of relevant modules and maps. The readers are encouraged to look into [GR23] for more details as well as proofs. We are motivated by the following specific problem.

Problem 2. *Let $(A, \mathfrak{m}_A)$ be a Noetherian regular local ring and fix a system of elements $f_1, \dots, f_n \in A$ and a system of integers $e_1, \dots, e_n$ with $e_i > 1$ for every $i = 1, \dots, n$. We set*

$$B := A[T_1, \dots, T_n]/(T_1^{e_1} - f_1, \dots, T_d^{e_n} - f_n).$$

Then find a sufficient condition that ensures that the localization B with respect to a maximal ideal $\mathfrak{n}$ with $\mathfrak{m}_A = A \cap \mathfrak{n}$ is regular.

From the construction, it is obvious that the induced ring map $A \to B$ is a flat finite injective extension. Let now $(A, \mathfrak{m}_A, k)$ be a Noetherian local ring with residue field $k_A := A/\mathfrak{m}_A$ of characteristic $p > 0$. Following the presentation in [GR23, (9.6.15)], we define a certain $k_A^{1/p}$-vector space $\boldsymbol{\Omega}_A$ together with a map $\boldsymbol{d}_A : A \to \boldsymbol{\Omega}_A$ as follows.

Case I: $(p \notin \mathfrak{m}_A^2)$

Let $W_2(k_A)$ denote the p-typical ring of length 2 Witt vectors over k_A. Then there is the ghost component map $\overline{\omega}_0 : W_2(k_A) \to k_A$, and set $V_1(k_A) := \text{Ker}(\overline{\omega}_0)$. More specifically, we have $W_2(k_A) = k_A \times k_A$ as sets with addition and multiplication given respectively by

$$(a,b)+(c,d) = \Big(a+c, b+d+\frac{a^p+c^p-(a+c)^p}{p}\Big) \text{ and } (a,b)(c,d) = (ac, a^pd+c^pb).$$

Using this structure, we see that $V_1(k_A) = 0 \times k_A$ as sets, which is an ideal of $W_2(k_A)$ and $V_1(k_A)^2 = 0$. This makes $V_1(k_A)$ equipped with the structure as a k_A-vector space by letting $x(0,a) := (x,0)(0,a)$ for $x \in k_A$. One can define the map of k_A-vector spaces

$$k_A^{1/p} \to V_1(k_A)\ ;\ a \mapsto (0, a^p), \tag{2.25}$$

which is a bijection. With this isomorphism, we may view $V_1(k_A)$ as a $k_A^{1/p}$-vector space. Next, we form the fiber product ring:

$$A_2 := A \times_{k_A} W_2(k_A).$$

It gives rise to a short exact sequence of A_2-modules

$$0 \to V_1(k_A) \to A_2 \to A \to 0, \tag{2.26}$$

where $A_2 \to A$ is the natural projection, and the A_2-module structure of $V_1(k_A)$ is via the restriction of rings $A_2 \to W_2(k_A)$. From (2.26), we obtain an exact sequence of A-modules:

$$V_1(k_A) \to \overline{\Omega}_A \to \Omega^1_{A/\mathbb{Z}} \to 0,$$

where we put $\overline{\Omega}_A = \Omega^1_{A_2/\mathbb{Z}} \otimes_{A_2} A$. After applying $(\ ) \otimes_A k_A$ to this sequence, we have another sequence of k_A-vector spaces:

$$0 \to V_1(k_A) \xrightarrow{j_A} \overline{\Omega}_A \otimes_A k_A \to \Omega^1_{A/\mathbb{Z}} \otimes_A k_A \to 0. \tag{2.27}$$

Then this is right exact. Moreover, (2.25) yields a unique k_A-linear map $\psi_A : V_1(k_A) \otimes_{k_A} k_A^{1/p} \to V_1(k_A)$. Define $\boldsymbol{\Omega}_A$ as the push-out of the diagram:

$$V_1(k_A) \xleftarrow{\psi_A} V_1(k_A) \otimes_{k_A} k_A^{1/p} \xrightarrow{j_A \otimes k_A^{1/p}} \overline{\Omega}_A \otimes_A k_A^{1/p}.$$

More concretely, we have

$$\boldsymbol{\Omega}_A = \frac{V_1(k_A) \oplus (\overline{\Omega}_A \otimes_A k_A^{1/p})}{T},$$

where $T = \big\{(\psi(x), -(j_A \otimes k_A^{1/p})(x)) \bigm| x \in V_1(k_A) \otimes_{k_A} k_A^{1/p}\big\}$. By the universality of push-outs, we get the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_1(k_A) \otimes_{k_A} k_A^{1/p} & \longrightarrow & \overline{\Omega}_A \otimes_A k_A^{1/p} & \longrightarrow & \Omega^1_{A/\mathbb{Z}} \otimes_A k_A^{1/p} & \longrightarrow & 0 \\ & & {\scriptstyle \psi_A}\downarrow & & {\scriptstyle \boldsymbol{\psi}_A}\downarrow & & \| & & \\ 0 & \longrightarrow & V_1(k_A) & \longrightarrow & \boldsymbol{\Omega}_A & \longrightarrow & \Omega^1_{A/\mathbb{Z}} \otimes_A k_A^{1/p} & \longrightarrow & 0 \end{array}$$

We define the map

$$\boldsymbol{d}_A : A \to \boldsymbol{\Omega}_A$$

as the composite mapping

$$A \xrightarrow{1\times \tau_{k_A}} A_2 = A \times_{k_A} W_2(k_A) \xrightarrow{d} \Omega^1_{A_2/\mathbb{Z}} \xrightarrow{\mathrm{id}\otimes 1} \overline{\Omega}_A = \Omega^1_{A_2/\mathbb{Z}} \otimes_A k_A^{1/p} \xrightarrow{\psi_A} \boldsymbol{\Omega}_A.$$

Here, $d : A_2 \to \Omega^1_{A_2/\mathbb{Z}}$ is the universal derivation, and $\tau_{k_A} : A \to k_A \to W_2(k_A)$, where the first map is the natural projection and the second one is the Teichmüller map.

Case II: $(p \in \mathfrak{m}_A^2)$

We just set $\boldsymbol{\Omega}_A := \Omega^1_{A/\mathbb{Z}} \otimes_A k_A^{1/p}$, and define $\boldsymbol{d}_A : A \to \boldsymbol{\Omega}_A$ as the map induced by the universal derivation $d_A : A \to \Omega^1_{A/\mathbb{Z}}$.

Combining both Case I and Case II together, we have a map $\boldsymbol{d}_A : A \to \boldsymbol{\Omega}_A$. Moreover, if $\phi : (A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ is a local ring map of local rings, it gives rise to the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{\boldsymbol{d}_A} & \boldsymbol{\Omega}_A \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\boldsymbol{\Omega}_\phi} \\ B & \xrightarrow{\boldsymbol{d}_B} & \boldsymbol{\Omega}_B \end{array}$$

With this in mind, one can consider the functor $A \mapsto \boldsymbol{\Omega}_A$ from the category of local rings $(A, \mathfrak{m}_A)$ of residual characteristic $p > 0$ to the category of the $k_A^{1/p}$-vector spaces $\boldsymbol{\Omega}_A$. Some distinguished features in the construction above are expressed by the following proposition.

Proposition 2.5.1 ([GR23, Proposition 9.6.20]). *Let $\phi : (A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ be a local ring map of Noetherian local rings such that the residual characteristic of A is $p > 0$. Then*

1. *Suppose that ϕ is formally smooth for the $\mathfrak{m}_A$-adic topology on A and the $\mathfrak{m}_B$-adic topology on B. Then the maps induced by ϕ and $\boldsymbol{\Omega}_\phi$ respectively*

$$(\mathfrak{m}_A/\mathfrak{m}_A^2) \otimes_{k_A} k_B \to \mathfrak{m}_B/\mathfrak{m}_B^2, \;\; \boldsymbol{\Omega}_A \otimes_{K_A^{1/p}} k_B^{1/p} \to \boldsymbol{\Omega}_B$$

are injective.

2. *Suppose that*

   (a) $\mathfrak{m}_A B = \mathfrak{m}_B$.

   (b) The residue filed extension $k_A \to k_B$ is separable algebraic.

   (c) ϕ is flat.

Then $\boldsymbol{\Omega}_\phi$ induces an isomorphism of $k_A^{1/p}$-vector spaces:

$$\boldsymbol{\Omega}_A \otimes_A B \cong \boldsymbol{\Omega}_B.$$

3. *If $B = A/\mathfrak{m}_A^2$ and $\phi : A \to B$ is the natural map, then $\boldsymbol{\Omega}_\phi$ is an isomorphism.*

4. *The functor $\boldsymbol{\Omega}_\bullet$ and the natural transformation $\boldsymbol{d}_\bullet$ commute with filtered colimits.*

We provide an answer to Problem 2 as follows.

Theorem 2.5.2 ([GR23, Corollary 9.6.34]). *Let $f_1, \dots, f_n$ be a sequence of elements in A, and let $e_1, \dots, e_n$ be a system of integers with $e_i > 1$ for every $i = 1, \dots, n$. Set*

$$C := A[T_1, \dots, T_n]/(T_1^{e_1} - f_1, \dots, T_n^{e^n} - f_n).$$

Fix a prime ideal $\mathfrak{n} \subseteq C$ such that $\mathfrak{n} \cap A = \mathfrak{m}_A$, and let $B := C_{\mathfrak{n}}$. Let $E \subseteq \boldsymbol{\Omega}_A$ be the $k_A^{1/p}$-vector space spanned by $\boldsymbol{d}_A f_1, \dots, \boldsymbol{d}_A f_n$. Then the following conditions are equivalent.

1. *A is a regular local ring, and $\dim_{k_A^{1/p}} E = n$.*

2. *B is a regular local ring.*

In particular, in the situation of the above theorem, B is a regular local ring if A is a regular local ring and $f_1, \dots, f_n$ is *maximal* in the sense of the following definition.

Definition 2.5.3. Let $(A, \mathfrak{m}_A, k_A)$ be a local ring with residual characteristic $p > 0$. Then we say that a sequence of elements $f_1, \dots, f_n$ in A is *maximal* if $\boldsymbol{d}_A f_1, \dots, \boldsymbol{d}_A f_n$ forms a basis of the $k_A^{1/p}$-vector space $\boldsymbol{\Omega}_A$.

In general, we have the following fact.

Lemma 2.5.4. *Let $(A, \mathfrak{m}_A, k_A)$ be a regular local ring of mixed characteristic and assume that $f_1, \dots, f_d$ is a regular system of parameters of A. Then the following hold:*

1. *$f_1, \dots, f_d$ satisfies the condition (1) of Theorem 2.5.2.*

2. *If the residue field k_A of A is perfect, then the sequence $f_1, \dots, f_d$ is maximal.*

Proof. (1): In the case that $p \notin \mathfrak{m}_A^2$, [GR23, Proposition 9.6.17] gives a short exact sequence:

$$0 \to \mathfrak{m}_A/\mathfrak{m}_A^2 \otimes_{k_A} k_A^{1/p} \to \boldsymbol{\Omega}_A \to \Omega^1_{k_A/\mathbb{Z}} \otimes_{k_A} k_A^{1/p} \to 0. \tag{2.28}$$

Then the images $\overline{f_1}, \dots, \overline{f_d}$ form a basis of the $k_A^{1/p}$-vector space $\mathfrak{m}_A/\mathfrak{m}_A^2 \otimes_{k_A} k_A^{1/p}$. The desired claim follows from the left exactness of (2.28).

In the case that $p \in \mathfrak{m}_A^2$, [GR23, Lemma 9.6.6] gives a short exact sequence

$$0 \to \mathfrak{m}_A/(\mathfrak{m}_A^2 + p\mathfrak{m}_A) \to \Omega_A \to \Omega^1_{k_A/\mathbb{Z}} \to 0. \tag{2.29}$$

and we can argue as in the case $p \notin \mathfrak{m}_A^2$.

(2): If k_A is perfect, then $\Omega^1_{k_A/\mathbb{Z}} = 0$. Therefore, (2.28) and (2.29) (in the latter case, one tensors it with $k_A^{1/p}$ over k_A) gives the desired conclusion. □

2.5.2 A construction of perfectoid towers arising from local log-regular rings

As an example of tilts of Noetherian perfectoid towers, we calculate them for certain towers of local log-regular rings. Firstly, we review a perfectoid tower constructed in [GR23].

Construction 2.5.5. Let $(R, \mathcal{Q}, \alpha)$ be a *complete* local log-regular ring with perfect residue field of characteristic $p > 0$. Assume that $\mathcal{Q}$ is fine, sharp, and saturated (see Remark 1.3.7). Set $A := R/I_\alpha$. Let $(f_1, \dots, f_r)$ be a sequence of elements of R whose image in A is *maximal* (see Definition 2.5.3). Since the residue field of R is perfect, r is the dimension of A (see §2.5.1). For every $i \geq 0$, we consider the ring

$$A_i := A[T_1, \dots, T_r]/(T_1^{p^i} - \overline{f_1}, \dots, T_r^{p^i} - \overline{f_r}),$$

where each $\overline{f_j}$ denotes the image of f_j in A ($j = 1, \dots, r$). Notice that A_i is regular by Theorem 2.5.2. Moreover, we set $\mathcal{Q}^{(i)} := \mathcal{Q}_p^{(i)}$ (see Definition 1.2.30). Furthermore, we define

$$R_i' := \mathbb{Z}[\mathcal{Q}^{(i)}] \otimes_{\mathbb{Z}[\mathcal{Q}]} R,\ R_i'' := R[T_1, \dots, T_r]/(T_1^{p^i} - f_1, \dots, T_r^{p^i} - f_r), \tag{2.30}$$

and

$$R_i := R_i' \otimes_R R_i''. \tag{2.31}$$

Let $t_i : R_i \to R_{i+1}$ be the ring map that is naturally induced by the inclusion map $\iota^{(i)} : \mathcal{Q}^{(i)} \hookrightarrow \mathcal{Q}^{(i+1)}$. Since R_{i+1}'' is a free R_i''-module, t_i is universally injective by Lemma 1.2.33 (2) and the condition (e) in Proposition 1.2.23 (2).

Proposition 2.5.6. *Keep the notation as in Construction 2.5.5. Let $\alpha_i : \mathcal{Q}^{(i)} \to R_i$ be the natural map. Then $(R_i, \mathcal{Q}^{(i)}, \alpha_i)$ is a local log-regular ring.*

Proof. We refer the reader to [GR23, 17.2.5]. □

By the construction, we obtain the tower of rings $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ (see Definition 2.1.1).

Proposition 2.5.7. *Keep the notation as in Construction 2.5.5. Then the tower $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ of local log-regular rings defined above is a perfectoid tower arising from $(R, (p))$.*

Proof. We verify (a)-(g) in Definition 2.3.1 and Definition 2.4.9. The axiom (a) is trivial. Since t_i is universally injective, the axiom (b) follows. The axioms (c) and (d) follow from [GR23, (17.2.10) and Lemma 17.2.11]. Since R is of residual characteristic p, the axiom (e) follows from the locality. Since R_i is a domain for any $i \geq 0$, the axiom (g) holds by Remark 2.4.10. Finally, let us check that the axiom (f) holds. In the case when $p = 0$, it follows from [GR23, Theorem 17.2.14 (i)]. Otherwise, there exists an element $\varpi \in R_1$ that satisfies $\varpi^p = pu$ for some unit $u \in R_1$ by [GR23, Theorem 17.2.14 (ii)]. Set $I_1 := (\varpi)$. Then the axiom (f-1) holds. Moreover, the axiom (f-2) follows from [GR23, Theorem 17.2.14 (iii)]. Thus the assertion follows. □

For calculating the tilt of the perfectoid tower constructed above, the following lemma is quite useful.

Lemma 2.5.8. *Keep the notation as in Proposition 2.5.6. Let k be the residue field of R. Then there exists a family of ring maps $\{\phi_i : C(k)[\![\mathcal{Q}^{(i)} \oplus (\mathbb{N}^r)^{(i)}]\!] \to R_i\}_{i\geq 0}$ which is compatible with the log structures of $\{(R_i, \mathcal{Q}^{(i)}, \alpha_i)\}_{i\geq 0}$ such that the following diagram commutes for every $i \geq 0$:*

$$\begin{array}{ccc} C(k)[\![\mathcal{Q}^{(i)} \oplus (\mathbb{N}^r)^{(i)}]\!] & \hookrightarrow & C(k)[\![\mathcal{Q}^{(i+1)} \oplus (\mathbb{N}^r)^{(i+1)}]\!] \\ \downarrow{\scriptstyle \phi_i} & & \downarrow{\scriptstyle \phi_{i+1}} \\ R_i & \overset{t_i}{\hookrightarrow} & R_{i+1} \end{array} \tag{2.32}$$

(where the top arrow is the natural inclusion). Moreover, there exists an element $\theta \in C(k)[\![\mathcal{Q} \oplus \mathbb{N}^r]\!]$ whose constant term is p such that the kernel of ϕ_i is generated by θ for every $i \geq 0$.

Proof. First, we remark on the following. Let k_i be the residue field of R_i. Then by Lemma 2.3.11 (1) and Lemma 2.3.6 (2), the transition maps induce a purely inseparable extension $k \hookrightarrow k_i$. Moreover, this extension is trivial because k is perfect. Therefore, we can identify k_i (resp. the Cohen ring of R_i) with k (resp. $C(k)$).

Next, let us show the existence of a family of ring maps $\{\phi_i\}_{i \geq 0}$ with the desired compatibility. Since $(R_i, \mathcal{Q}^{(i)}, \alpha_i)$ is a complete local log-regular ring, we can take a surjective ring map $\psi_i : C(k)[\![\mathcal{Q}^{(i)} \oplus \mathbb{N}^r]\!] \to R_i$ as in Theorem 1.3.8; its kernel is generated by an element θ_i whose constant term is p, and the diagram:

$$\begin{array}{ccc} \mathcal{Q}^{(i)} & \longrightarrow & C(k)[\![\mathcal{Q}^{(i)} \oplus \mathbb{N}^r]\!] \\ & \searrow^{\alpha_i} & \downarrow{\scriptstyle \psi_i} \\ & & R_i \end{array}$$

commutes. For $j = 1, \ldots, r$, let us denote by f_j^{1/p^i} the image of $T_j \in R[T_1, \ldots, T_r]$ in R_i (see (2.30) and (2.31)). Note that the sequence $f_1^{1/p^i}, \ldots, f_r^{1/p^i}$ in R_i becomes a regular system of parameters of R_i/I_{α_i} by the reduction modulo I_{α_i} (see [GR23, 17.2.3] and [GR23, 17.2.5]). Thus, for the set of the canonical basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_r\}$ of $\mathbb{N}^r$, we may assume $\psi_i(e^{\mathbf{e}_j}) = f_j^{1/p^i}$ by the construction of ψ_i (see the proof of [Ogu18, Chapter III, Theorem 1.11.2]). Hence we can choose $\{\psi_i\}_{i \geq 0}$ so that the diagram:

$$\begin{array}{ccc} C(k)[\![\mathcal{Q}^{(i)} \oplus \mathbb{N}^r]\!] & \hookrightarrow & C(k)[\![\mathcal{Q}^{(i+1)} \oplus \mathbb{N}^r]\!] \\ {\scriptstyle \psi_i}\downarrow & & \downarrow{\scriptstyle \psi_{i+1}} \\ R_i & \xrightarrow{t_i} & R_{i+1} \end{array} \tag{2.33}$$

commutes. Thus it suffices to define $\phi_i : C(k)[\![\mathcal{Q}^{(i)} \oplus (\mathbb{N}^r)^{(i)}]\!] \to R_i$ as the composite map of the isomorphism $C(k)[\![\mathcal{Q}^{(i)} \oplus (\mathbb{N}^r)^{(i)}]\!] \xrightarrow{\cong} C(k)[\![\mathcal{Q}^{(i)} \oplus \mathbb{N}^r]\!]$ obtained by Lemma 1.2.31 (3) and ψ_i.

Finally, note that the image of $\theta_0 \in \operatorname{Ker}(\psi_0)$ in $C(k)[\![\mathcal{Q}^{(i)} \oplus \mathbb{N}^r]\!]$ is contained in $\operatorname{Ker}(\psi_i)$, and its constant term is still p. Thus, by the latter assertion of Theorem 1.3.8 (2), $\operatorname{Ker}(\psi_i)$ is generated by θ_0. Hence by taking θ_0 as θ, we complete the proof. □

Let us consider the monoids $\mathcal{Q}^{(i)}$ for an integral sharp monoid $\mathcal{Q}$. Since there is the natural inclusion $\iota^{(i)} : \mathcal{Q}^{(i)} \hookrightarrow \mathcal{Q}^{(i+1)}$ for any $i \geq 0$, we obtain a direct system of monoids $(\{\mathcal{Q}^{(i)}\}_{i \geq 0}, \{\iota^{(i)}\}_{i \geq 0})$. Moreover, the p-times map on $\mathcal{Q}^{(i+1)}$ gives a factorization:

$$\begin{array}{ccc} \mathcal{Q}^{(i+1)} & \xrightarrow{\times p} & \mathcal{Q}^{(i+1)} \\ & \searrow^{\times p} & \uparrow{\scriptstyle \iota^{(i)}} \\ & & \mathcal{Q}^{(i)}. \end{array}$$

From this discussion, we define the small tilt of $\{\mathcal{Q}^{(i)}\}_{i \geq 0}$.

Definition 2.5.9. Let $\mathcal{Q}$ be an integral sharp monoid, and let $(\{\mathcal{Q}^{(i)}\}_{i\geq 0}, \{\iota^{(i)}\}_{i\geq 0})$ be as above. Then for an integer $j \geq 0$, we define the *j-th small tilt of* $(\{\mathcal{Q}^{(i)}\}_{i\geq 0}, \{\iota^{(i)}\}_{i\geq 0})$ as the inverse limit

$$\mathcal{Q}_j^{s.\flat} := \varprojlim\{\cdots \to \mathcal{Q}^{(j+1)} \to \mathcal{Q}^{(j)}\}, \tag{2.34}$$

where the transition map $\mathcal{Q}^{(i+1)} \to \mathcal{Q}^{(i)}$ is the p-times map of monoids.

Now we can derive important properties of the tilt of the perfectoid tower given in Construction 2.5.5.

Theorem 2.5.10. *Keep the notation as in Lemma 2.5.8. Then the following assertions hold.*

1. *The tower $(\{(R_i)_{(p)}^{s.\flat}\}_{i\geq 0}, \{(t_i)_{(p)}^{s.\flat}\}_{i\geq 0})$ is isomorphic to $(\{k[\![\mathcal{Q}^{(i)}\oplus(\mathbb{N}^r)^{(i)}]\!]\}_{i\geq 0}, \{u_i\}_{i\geq 0})$, where u_i is the ring map induced by the natural inclusion $\mathcal{Q}^{(i)} \oplus (\mathbb{N}^r)^{(i)} \hookrightarrow \mathcal{Q}^{(i+1)} \oplus (\mathbb{N}^r)^{(i+1)}$.*

2. *For every $j \geq 0$, there exists a homomorphism of monoids $\alpha_j^{s.\flat} : \mathcal{Q}_j^{s.\flat} \to (R_j)_{(p)}^{s.\flat}$ such that $((R_j)_{(p)}^{s.\flat}, \mathcal{Q}_j^{s.\flat}, \alpha_j^{s.\flat})$ is a local log-regular ring.*

3. *For every $j \geq 0$, $(t_j)_{(p)}^{s.\flat} : (R_j)_{(p)}^{s.\flat} \to (R_{j+1})_{(p)}^{s.\flat}$ is module-finite and $(R_j)_{(p)}^{s.\flat}$ is F-finite.*

Proof. (1): By Lemma 2.5.8, each R_i is isomorphic to $C(k)[\![\mathcal{Q}^{(i)}\oplus(\mathbb{N}^r)^{(i)}]\!]/(p-f)C(k)[\![\mathcal{Q}^{(i)}\oplus(\mathbb{N}^r)^{(i)}]\!]$ where f is an element of $C(k)[\![\mathcal{Q}\oplus\mathbb{N}^r]\!]$ which has no constant term. Set $S_i := k[\![\mathcal{Q}^{(i)} \oplus (\mathbb{N}^r)^{(i)}]\!]$ for any $i \geq 0$ and let $u_i : S_i \hookrightarrow S_{i+1}$ be the inclusion map induced by the natural inclusion $\mathcal{Q}^{(i)} \oplus (\mathbb{N}^r)^{(i)} \hookrightarrow \mathcal{Q}^{(i+1)} \oplus (\mathbb{N}^r)^{(i+1)}$. Then the tower $(\{S_i\}_{i\geq 0}, \{u_i\}_{i\geq 0})$ is a perfect tower. Indeed, each S_i is reduced by Theorem 1.3.13; moreover, by the perfectness of k and Lemma 1.2.31 (3), the Frobenius endomorphism on S_{i+1} factors through a surjection $G_i : S_{i+1} \to S_i$. In particular, $(\{S_i\}_{i\geq 0}, \{u_i\}_{i\geq 0})$ is a perfectoid tower arising from $(S_0, (0))$ and G_i is the i-th Frobenius projection (cf. Lemma 2.4.12).

Put $\overline{f} := f \mod pC(k)[\![\mathcal{Q}\oplus\mathbb{N}^r]\!] \in S_0$. Then each S_i is $\overline{f}$-adically complete and separated by [FGK11, Lemma 2.1.1]. Moreover, the commutative diagram (2.32) yields the commutative squares ($i \geq 0$):

$$\begin{array}{ccc} S_{i+1}/\overline{f}S_{i+1} & \xrightarrow{\cong} & R_{i+1}/pR_{i+1} \\ \downarrow{\scriptstyle \overline{G_i}} & & \downarrow{\scriptstyle F_i} \\ S_i/\overline{f}S_i & \xrightarrow{\cong} & R_i/pR_i \end{array}$$

that are compatible with $\{\overline{u_i} : S_i/\overline{f}S_i \to S_{i+1}/\overline{f}S_{i+1}\}_{i\geq 0}$ and $\{\overline{t_i}\}_{i\geq 0}$. Hence by Lemma 2.4.20, we obtain the isomorphisms

$$(R_j)_{(p)}^{s.\flat} \xleftarrow{\cong} \varprojlim\{\cdots \xrightarrow{\overline{G_{j+1}}} S_{j+1}/\overline{f}S_{j+1} \xrightarrow{\overline{G_j}} S_j/\overline{f}S_j\} \xrightarrow{\cong} S_j \qquad (j \geq 0) \tag{2.35}$$

that are compatible with the transition maps of the towers. Thus the assertion follows.

(2): Considering the inverse limit of the composite maps $\mathcal{Q}^{(j+i)} \xrightarrow{\alpha_{j+i}} R_{j+i} \twoheadrightarrow R_{j+i}/pR_{j+i}$ ($i \geq 0$), we obtain a homomorphism of monoids $\alpha_j^{s.\flat} : \mathcal{Q}_j^{s.\flat} \to (R_j)_{(p)}^{s.\flat}$. On the other hand, let $\overline{\alpha_j} : \mathcal{Q}^{(j)} \to S_j$ be the natural inclusion. Then, since S_j is canonically isomorphic to $k[\![\mathcal{Q}^{(j)}\oplus\mathbb{N}^r]\!]$, $(S_j, \mathcal{Q}^{(j)}, \overline{\alpha_j})$ is a local log-regular ring by Theorem 1.3.8 (1). Thus it suffices to show that $((R_j)^{s.\flat}_{(p)}, \mathcal{Q}^{s.\flat}_j, \alpha^{s.\flat}_j)$ is isomorphic to $(S_j, \mathcal{Q}^{(j)}, \overline{\alpha_j})$ as a log ring. Since the transition maps in (2.34) are isomorphisms by Lemma 1.2.31 (3), we obtain the isomorphisms of monoids

$$\mathcal{Q}^{s.\flat}_j \xleftarrow{\mathrm{id}_{\mathcal{Q}^{s.\flat}_j}} \mathcal{Q}^{s.\flat}_j \xrightarrow{\cong} \mathcal{Q}^{(j)} \qquad (j \geq 0). \tag{2.36}$$

Then one can connect (2.36) to (2.35) to construct a commutative diagram using $\alpha^{s.\flat}_j$ and $\overline{\alpha_j}$. Hence the assertion follows.

(3): By Lemma 1.2.32 (2), $t_j : R_j \to R_{j+1}$ is module-finite. Hence by Proposition 2.4.35 (1), $(t_j)^{s.\flat}_{(p)} : (R_j)^{s.\flat}_{(p)} \to (R_{j+1})^{s.\flat}_{(p)}$ is also module-finite. Finally let us show that $(R_j)^{s.\flat}_{(p)}$ is F-finite. By the assertion (2), $(R_j)^{s.\flat}_{(p)}$ is a complete Noetherian local ring, and the residue field is F-finite because it is perfect. Thus the assertion follows from [Mat89, Theorem 8.4]. □

Example 2.5.11. *1. A tower of regular local rings which is treated in [Česka19] and [ČS19] is a perfectoid tower in our sense. Let $(R, \mathfrak{m}, k)$ be a d-dimensional regular local ring whose residue field k is perfect, and let $x_1, \ldots, x_d$ be a regular sequence of parameters. Let $\mathbf{e}_1, \ldots, \mathbf{e}_d$ be the canonical basis of $\mathbb{N}^d$. Then $(R, \mathbb{N}^d, \alpha)$ is a local log-regular ring where $\alpha : \mathbb{N}^d \to R$ is a homomorphism of monoids which maps $\mathbf{e}_i$ to x_i. Furthermore, assume that R is $\mathfrak{m}$-adically complete. Then, by Cohen's structure theorem, R is isomorphic to*

$$W(k)[\![x_1, \ldots, x_d]\!]/(p - f)$$

where $f = x_1$ or $f \in (p, x_1, \ldots, x_d)^2$ (the former case is called unramified, and the latter is called ramified). Let us construct a perfectoid tower arising from $(R, (p))$ along Construction 2.5.5. Since k is perfect, $\mathbf{\Omega}_k$ is zero by the short exact sequences (2.28) and the definition of itself. This implies that the image of the empty subset of R in k forms a maximal sequence. Hence R''_i in Construction 2.5.5 is equal to R. Moreover, $(\mathbb{N}^d)^{(i)}$ is generated by $\frac{1}{p^i}\mathbf{e}_1, \ldots, \frac{1}{p^i}\mathbf{e}_d$. Thus, applying Construction 2.5.5, we obtain

$$R_i = \mathbb{Z}[(\mathbb{N}^d)^{(i)}] \otimes_{\mathbb{Z}[\mathbb{N}^d]} R \cong R[T_1, \ldots, T_d]/(T_1^{p^i} - x_1, \ldots, T_d^{p^i} - x_d) \cong W(k)[\![x_1^{1/p^i}, \ldots, x_d^{1/p^i}]\!]/(p - f).$$

Set the natural injection $t_i : R_i \to R_{i+1}$ for any $i \geq 0$. Then, by Proposition 2.5.7, $(\{R_i\}_{i \geq 0}, \{t_i\}_{i \geq 0})$ is a perfectoid tower arising from $(R, (p))$. By Theorem 2.5.10, its tilt $(\{(R_i)^{s.\flat}_{(p)}\}_{i \geq 0}, \{(t_i)^{s.\flat}_{(p)}\}_{i \geq 0})$ is isomorphic to the tower $k[\![\mathbb{N}^d]\!] \hookrightarrow k[\![(\mathbb{N}^d)^{(1)}]\!] \hookrightarrow k[\![(\mathbb{N}^d)^{(2)}]\!] \hookrightarrow \cdots$, which can be written as

$$k[\![x_1, \ldots, x_d]\!] \hookrightarrow k[\![x_1^{1/p}, \ldots, x_d^{1/p}]\!] \hookrightarrow k[\![x_1^{1/p^2}, \ldots, x_d^{1/p^2}]\!] \hookrightarrow \cdots.$$

2. Consider the surjection:

$$S := W(k)[\![x, y, z, w]\!]/(xy - zw) \twoheadrightarrow R := W(k)[\![x, y, z, w]\!]/(xy - zw, p - w).$$

where k is a perfect field. Let $\mathcal{Q} \subseteq \mathbb{N}^4$ be a saturated submonoid generated by

$$(1, 1, 0, 0), (0, 0, 1, 1), (1, 0, 0, 1), \text{ and } (0, 1, 1, 0).$$

Then S admits a homomorphism of monoids $\alpha_S : \mathcal{Q} \to S$ by letting $(1, 1, 0, 0) \mapsto x, (0, 0, 1, 1) \mapsto y, (1, 0, 0, 1) \mapsto z$ and $(0, 1, 1, 0) \mapsto w$. With this, $(S, \mathcal{Q}, \alpha_S)$ is a local *log-regular ring. The composite map $\alpha_R : \mathcal{Q} \to S \to R$ makes R into a local log ring. Indeed, we can write $R \cong W(k)[\![\mathcal{Q}]\!]/(p - e^{(0,1,1,0)})$, hence $(R, \mathcal{Q}, \alpha_R)$ is log-regular by Theorem 1.3.8.*

Next, note that $R/I_{\alpha_R} \cong k$. Then, for the same reason in (1), R''_i is equal to R. Moreover, $\mathcal{Q}^{(i)}$ is generated by

$$\Big(\frac{1}{p^i}, \frac{1}{p^i}, 0, 0\Big), \Big(0, 0, \frac{1}{p^i}, \frac{1}{p^i}\Big), \Big(\frac{1}{p^i}, 0, 0\frac{1}{p^i}\Big), \Big(0, \frac{1}{p^i}, \frac{1}{p^i}, 0\Big).$$

Thus, applying Construction 2.5.5, we obtain

$$\begin{aligned} R_i &= R[\![\mathcal{Q}^{(i)}]\!] \\ &\cong W(k)[\![\mathcal{Q}^{(i)}]\!]/(p - e^{(0,1,1,0)}) \\ &\cong W(k)[\![x^{1/p^i}, y^{1/p^i}, z^{1/p^i}, w^{1/p^i}]\!]/(x^{1/p^i}y^{1/p^i} - z^{1/p^i}w^{1/p^i}, p - w). \end{aligned}$$

Set a natural injection $t_i : R_i \to R_{i+1}$. Then, by Proposition 2.5.7, $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ is a perfectoid tower arising from $(R, (p))$. Hence

$$R_\infty = \varinjlim_{i\geq 0} R_i \cong \bigcup_{i\geq 0} W(k)[\![x^{1/p^i}, y^{1/p^i}, z^{1/p^i}, w^{1/p^i}]\!]/(x^{1/p^i}y^{1/p^i} - z^{1/p^i}w^{1/p^i}, p - w),$$

and its p-adic completion is perfectoid. Moreover, one can calculate the tilt $(\{R_i^{s.\flat}\}_{i\geq 0}, \{t_i^{s.\flat}\}_{i\geq 0})$ to be $k[\![\mathcal{Q}]\!] \hookrightarrow k[\![\mathcal{Q}^{(1)}]\!] \hookrightarrow k[\![\mathcal{Q}^{(2)}]\!] \hookrightarrow \cdots$ by Theorem 2.5.10, or, more explicitly,

$$k[\![x, y, z, w]\!]/(xy - zw) \hookrightarrow k[\![x^{1/p}, y^{1/p}, z^{1/p}, w^{1/p}]\!]/(x^{1/p}y^{1/p} - z^{1/p}w^{1/p}) \hookrightarrow \cdots.$$

Let us recall that Hansen and Kedlaya introduced a new class of topological rings that guarantees sheafiness on the associated adic spectra (see [HK23, Definition 7.1]).

Definition 2.5.12. Let A be a complete and separated Tate ring such that a prime $p \in A$ is topologically nilpotent. We say that A is *sousperfectoid*, if there exists a perfectoid ring B in the sense of Fontaine (see [HK23, Definition 2.13]) with a continuous A-linear map $f : A \to B$ that splits in the category of topological A-modules. That is, there is a continuous A-linear map $\sigma : B \to A$ such that $\sigma \circ f = \mathrm{id}_A$.

Let us show that a perfectoid tower consisting of split maps induces sousperfectoid rings. In view of Theorem 1.4.3, one can apply this result to the towers discussed above. See [NS22] for detailed studies on algebraic aspects of Tate rings.

Proposition 2.5.13. *Let $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ be a perfectoid tower arising from some pair $(R, (f_0))$. Assume that f_0 is regular, R is f_0-adically complete and separated, and t_i splits as an R_i-linear map for every $i \geq 0$. We equip $R[\frac{1}{f_0}]$ with the linear topology in such a way that $\{f_0^n R\}_{n\geq 1}$ defines a fundamental system of open neighborhoods at $0 \in R[\frac{1}{f_0}]$. Then $R[\frac{1}{f_0}]$ is a sousperfectoid Tate ring. In particular, it is stably uniform.*

In order to prove this, we need the following lemma.

Lemma 2.5.14. *Keep the notations and assumptions as in Proposition 2.5.13. Then the natural map $R_0 \to \varinjlim_{i\geq 0} R_i$ splits as an R_0-linear map.*

Proof. We use the fact that each $t_i : R_i \to R_{i+1}$ splits as an R_i-linear map by assumption. This implies that the short exact sequence of R-modules

$$0 \to R_0 \to R_i \to R_i/R \to 0$$

splits for any $i \geq 0$. It induces a commutative diagram of R-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_{R_0}(R_{i+1}/R_0, R_0) & \longrightarrow & \mathrm{Hom}_{R_0}(R_{i+1}, R_0) & \longrightarrow & \mathrm{Hom}_{R_0}(R_0, R_0) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_i} & & \downarrow{\scriptstyle \beta_i} & & \| & & \\ 0 & \longrightarrow & \mathrm{Hom}_{R_0}(R_i/R_0, R_0) & \longrightarrow & \mathrm{Hom}_{R_0}(R_i, R_0) & \longrightarrow & \mathrm{Hom}_{R_0}(R_0, R_0) & \longrightarrow & 0 \end{array}$$

where each horizontal sequence is split exact, and each vertical map forms an inverse system induced by $t_i : R_i \to R_{i+1}$. In particular, β_i is surjective and it thus follows from the snake lemma that α_i is surjective as well. By taking inverse limits, we obtain the short exact sequence:

$$0 \to \varprojlim_{i \geq 0} \mathrm{Hom}_{R_0}(R_i/R_0, R_0) \to \varprojlim_{i \geq 0} \mathrm{Hom}_{R_0}(R_i, R_0) \xrightarrow{h} \mathrm{Hom}_{R_0}(R_0, R_0) \to 0.$$

It follows from [Sch14, Lemma 4.1] that h is the canonical surjection $\mathrm{Hom}_{R_0}(R_\infty, R_0) \twoheadrightarrow \mathrm{Hom}_{R_0}(R_0, R_0)$. Then choosing an inverse image of $\mathrm{id}_{R_0} \in \mathrm{Hom}_{R_0}(R_0, R_0)$ gives a splitting of $R_0 \to R_\infty$. □

Proof of Proposition 2.5.13. We have constructed an infinite extension $R \to R_\infty$ such that if $\widehat{R_\infty}$ is the f_0-adic completion, then the associated Tate ring $\widehat{R_\infty}[\frac{1}{f_0}]$ is a perfectoid ring in the sense of Fontaine by Theorem 2.4.42 and [BMS18, Lemma 3.21].

By Lemma 1.4.2 and Lemma 2.5.14, it follows that the map $R[\frac{1}{f_0}] \to \widehat{R_\infty}[\frac{1}{f_0}]$ splits in the category of topological $R[\frac{1}{f_0}]$-modules (notice that R is f_0-adically complete and separated). Thus, $R[\frac{1}{f_0}]$ is a sousperfectoid Tate ring. The combination of [HK23, Corollary 8.10], [HK23, Proposition 11.3] and [HK23, Lemma 11.9] allows us to conclude that $R[\frac{1}{f_0}]$ is stably uniform. □

As a corollary, one can obtain the stable uniformity for complete local log-regular rings (see also Construction 2.5.5 and Theorem 1.4.3).

Corollary 2.5.15. *Let $(R, \mathcal{Q}, \alpha)$ be a complete local log-regular ring of mixed characteristic with perfect residue field. We equip $R[\frac{1}{p}]$ with the structure of a complete and separated Tate ring in such a way that $\{p^n R\}_{n \geq 1}$ defines a fundamental system of open neighborhoods at $0 \in R[\frac{1}{p}]$. Then $R[\frac{1}{p}]$ is stably uniform.*

2.6 Applications of perfectoid towers and their tilts

In this section, we establish several results on étale cohomology groups and local cohomologies of Noetherian rings, as applications of the theory of perfectoid towers developed in §2.4. In §2.6.1, for a ring that admits a certain type of a perfectoid tower, we prove that the finiteness of étale cohomology groups on the positive characteristic side carries over to the mixed characteristic side (Proposition 2.6.7). In §2.6.2, we apply this result to a problem on divisor class groups of log-regular rings. Finally, in §2.6.3, we provide a partial answer to the second vanishing theorem in mixed characteristic.

2.6.1 Application to étale cohomology groups

We prepare some notation. Let X be a scheme and let $X_{\text{ét}}$ denote the category of schemes that are étale over X, and for any étale X-scheme Y, we specify the covering $\{Y_i \to Y\}_{i\in I}$ so that Y_i is étale over Y and the family $\{Y_i\}_{i\in I}$ covers surjectively Y. For an abelian sheaf $\mathcal{F}$ on $X_{\text{ét}}$, we denote by $H^i(X_{\text{ét}}, \mathcal{F})$ the value of the i-th derived functor of $U \in X_{\text{ét}} \mapsto \Gamma(U, \mathcal{F})$. For the most part of applications, we consider *torsion* sheaves, such as $\mathbb{Z}/n\mathbb{Z}$ and μ_n for $n \in \mathbb{N}$. However, for the multiplicative group scheme $\mathbb{G}_m$, we often use the following isomorphism:

$$H^1(X_{\text{ét}}, \mathbb{G}_m) \cong \operatorname{Pic}(X).$$

For the basics on étale cohomology, we often use [Fu11] or [Mil80] as references.

Let A be a ring with an ideal J and let $U \subseteq \operatorname{Spec}(A)$ be an open subset. Then we define the J-adic completion of U to be the open subset $\widehat{U} \subseteq \operatorname{Spec}(\widehat{A})$, which is the inverse image of U via $\operatorname{Spec}(\widehat{A}) \to \operatorname{Spec}(A)$. We will use the following result for deriving results on the behavior of étale cohomology under the tilting operation as well as some interesting results on the divisor class groups of Noetherian normal domains (see Proposition 2.6.10 and Proposition 2.6.11).

Theorem 2.6.1 (Fujiwara-Gabber)**.** *Let (A, J) be a Henselian pair with $X := \operatorname{Spec}(A)$ and let $\widehat{A}$ be the J-adic completion of A. Then the following assertions hold.*

1. *For any abelian torsion sheaf $\mathscr{F}$ on $X_{\text{ét}}$, we have*

$$\mathbf{R}\Gamma(\operatorname{Spec}(A)_{\text{ét}}, \mathscr{F}) \simeq \mathbf{R}\Gamma(\operatorname{Spec}(A/J)_{\text{ét}}, \mathscr{F}|_{\operatorname{Spec}(A/J)}).$$

2. *Assume that J is finitely generated. Then for any abelian torsion sheaf $\mathscr{F}$ on $X_{\text{ét}}$ and any open subset $U \subseteq X$ such that $X \setminus V(J) \subseteq U$, we have*

$$\mathbf{R}\Gamma(U_{\text{ét}}, \mathscr{F}) \simeq \mathbf{R}\Gamma(\widehat{U}_{\text{ét}}, \mathscr{F}).$$

Proof. The first statement is known as *Affine analog of proper base change* in [Gab94], while the second one is known as *Formal base change theorem* which is [Fuj95, Theorem 7.1.1] in the Noetherian case, and [ILO14, XX, 4.4] in the non-Noetherian case. □

We will need the tilting invariance of (local) étale cohomology from [ČS19, Theorem 2.2.7]. To state the theorem and establish a variant of it, we give some notations.

Definition 2.6.2. Let (A, I) and (B, J) be pairs such that there exists an isomorphism of rings $\Phi : A/I \xrightarrow{\cong} B/J$. Then for any open subset $U \subseteq \operatorname{Spec}(B)$ containing $\operatorname{Spec}(B)\setminus V(J)$, we define an open subset $F_{A,\Phi}(U) \subseteq \operatorname{Spec}(A)$ as the complement of the closed subset $\operatorname{Spec}(\Phi)\big(\operatorname{Spec}(B) \setminus U\big) \subseteq \operatorname{Spec}(A)$.

One can define small tilts of Zariski-open subsets.

Definition 2.6.3. Let $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ be a perfectoid tower arising from some pair (R, I_0), and let $(\{R_i^{s.\flat}\}_{i\geq 0}, \{t_i^{s.\flat}\}_{i\geq 0})$ be the tilt associated to (R, I_0). Recall that we then have an isomorphism of rings $\overline{\Phi_0^{(i)}} : R_i^{s.\flat}/I_0^{s.\flat}R_i^{s.\flat} \xrightarrow{\cong} R_i/I_0R_i$ for every $i \geq 0$. For every $i \geq 0$ and every open subset $U \subseteq \operatorname{Spec}(R_i)$ containing $\operatorname{Spec}(R_i) \setminus V(I_0R_i)$, we define

$$U_{I_0}^{s.\flat} := F_{R_i^{s.\flat}, \overline{\Phi_0^{(i)}}}(U).$$

We also denote $U_{I_0}^{s.\flat}$ by $U^{s.\flat}$ as an abbreviated form.

Note that by the compatibility described in Lemma 2.4.31, the operation $U \rightsquigarrow U^{s.\flat}$ is compatible with the base extension along the transition maps of a perfectoid tower.

Let us give some examples of $U^{s.\flat}$.

Example 2.6.4 (Punctured spectra of regular local rings)**.** *Keep the notation as in Example 2.5.11 (1). In this situation, the isomorphism* $\overline{\Phi_0^{(0)}} : R_0^{s.\flat}/I_0^{s.\flat} \xrightarrow{\cong} R_0/I_0$ *in Definition 2.6.3 can be written as*

$$k[\![x_1, \ldots, x_d]\!]/(p^{s.\flat}) \xrightarrow{\cong} R/pR, \tag{2.37}$$

where $p^{s.\flat} \in k[\![x_1, \ldots, x_d]\!]$ *is some element. Set* $U := \mathrm{Spec}(R) \setminus V(\mathfrak{m})$*. Then, since the maximal ideal* $\overline{\mathfrak{m}} \subseteq R/pR$ *corresponds to the (unique) maximal ideal of* $k[\![x_1, \ldots, x_d]\!]/(p^{s.\flat})$*, we have*

$$U^{s.\flat} \cong \mathrm{Spec}(k[\![x_1, \ldots, x_d]\!]) \setminus V((x_1, \ldots, x_d)).$$

Example 2.6.5 (Tilting for preperfectoid rings)**.** *Keep the notation as in Example 2.4.44. Then by Lemma 2.4.45,* $\overline{\Phi_0^{(0)}} : R_0^{s.\flat}/I_0^{s.\flat} \xrightarrow{\cong} R_0/I_0$ *is identified with the isomorphism:*

$$\overline{\theta_{\widehat{R}}} : (\widehat{R})^\flat/I_0^\flat(\widehat{R})^\flat \xrightarrow{\cong} \widehat{R}/I_0\widehat{R} \tag{2.38}$$

which is induced by the bottom map in the diagram (2.22). In this case, we denote $F_{R^\flat, \overline{\Phi_0^{(0)}}}(U)$ *by* $U^\flat$ *in distinction from* $U^{s.\flat}$*.*

The comparison theorem we need, due to Česnavičius and Scholze [ČS19], is stated as follows.

Theorem 2.6.6 (Česnavičius-Scholze)**.** *Let A be a ϖ-adically Henselian ring with bounded ϖ-torsion for an element $\varpi \in A$ such that $p \in \varpi^p A$. Assume that the ϖ-adic completion of A is perfectoid. Let* $U \subseteq \mathrm{Spec}(A)$ *be a Zariski-open subset such that* $\mathrm{Spec}(A) \setminus V(\varpi A) \subseteq U$*, and let* $U^\flat \subseteq \mathrm{Spec}(A^\flat)$ *be its tilt (see Example 2.6.5).*

1. *For every torsion abelian group G, we have* $\mathbf{R}\Gamma(U_{\text{ét}}, G) \cong \mathbf{R}\Gamma(U^\flat_{\text{ét}}, G)$ *in a functorial manner with respect to A, U, and G.*
2. *Let Z be the complement of* $U \subseteq \mathrm{Spec}(A)$*. Then for a torsion abelian group G, we have* $\mathbf{R}\Gamma_Z(\mathrm{Spec}(A)_{\text{ét}}, G) \cong \mathbf{R}\Gamma_Z(\mathrm{Spec}(A^\flat)_{\text{ét}}, G)$*.*

Now we come to the main result on tilting étale cohomology groups. Recall that we have fixed a prime $p > 0$.

Proposition 2.6.7. *Let* $(\{R_j\}_{j \geq 0}, \{t_j\}_{j \geq 0})$ *be a perfectoid tower arising from some pair* (R, I_0)*. Suppose that R_j is I_0-adically Henselian for every $j \geq 0$. Let ℓ be a prime different from p. Suppose further that for every $j \geq 0$, $t_j : R_j \to R_{j+1}$ is a module-finite extension of Noetherian normal domains whose generic extension is of p-power degree.*[6] *Fix a Zariski-open subset* $U \subseteq \mathrm{Spec}(R)$ *such that* $\mathrm{Spec}(R) \setminus V(pR) \subseteq U$ *and the corresponding open subset* $U^{s.\flat} \subseteq \mathrm{Spec}(R^{s.\flat})$ *(cf. Definition 2.6.3). Then, for any fixed $i, n \geq 0$ such that* $|H^i(U^{s.\flat}_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z})| < \infty$*, one has*

$$|H^i(U_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z})| \leq |H^i(U^{s.\flat}_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z})|.$$

In particular, if $H^i(U^{s.\flat}_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}) = 0$*, then* $H^i(U_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}) = 0$*.*

[6] The existence of such towers is quite essential for applications to étale cohomology, because the extension degree of each $R_j \to R_{j+1}$ is controlled in such a way that the p-adic completion of its colimit is a perfectoid ring.

Proof. Since each R_j is a p-adically Henselian normal domain, so is $R_\infty = \varinjlim_{j\geq 0} R_j$. Moreover, every prime ℓ different from p is a unit in R_j and R_∞. Attached to the tower $(\{R_j\}_{i\geq 0}, \{t_j\}_{j\geq 0})$, we get a tower of finite (not necessarily flat) maps of normal schemes:

$$U = U_0 \leftarrow \cdots \leftarrow U_j \leftarrow U_{j+1} \leftarrow \cdots . \tag{2.39}$$

More precisely, let $h_j : \mathrm{Spec}(R_{j+1}) \to \mathrm{Spec}(R_j)$ be the associated scheme map. Then the open set U_{j+1} is defined as the inverse image $h_j^{-1}(U_j)$, thus defining the map $U_{j+1} \to U_j$ in the tower (2.39). Since h_j is a finite morphism of normal schemes, [Bha14, Lemma 3.4] applies to yield a well-defined trace map: $\mathrm{Tr} : h_{j*}h_j^*\mathbb{Z}/\ell^n\mathbb{Z} \to \mathbb{Z}/\ell^n\mathbb{Z}$ such that

$$\mathbb{Z}/\ell^n\mathbb{Z} \xrightarrow{h_j^*} h_{j*}h_j^*\mathbb{Z}/\ell^n\mathbb{Z} \xrightarrow{\mathrm{Tr}} \mathbb{Z}/\ell^n\mathbb{Z} \tag{2.40}$$

is multiplication by the generic degree of h_j (=p-power order). Then this is bijective, as the multiplication map by p on $\mathbb{Z}/\ell^n\mathbb{Z}$ is bijective. We have the natural map: $H^i(U_{j,\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}) \to H^i(U_{j+1,\text{ét}}, h_j^*\mathbb{Z}/\ell^n\mathbb{Z})$. Since h_j is affine, the Leray spectral sequence gives $H^i(U_{j+1,\text{ét}}, h_j^*\mathbb{Z}/\ell^n\mathbb{Z}) \cong H^i(U_{j,\text{ét}}, h_{j*}h_j^*\mathbb{Z}/\ell^n\mathbb{Z})$. Composing these maps, the composite map (2.40) induces

$$H^i(U_{j,\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}) \to H^i(U_{j+1,\text{ét}}, h_j^*\mathbb{Z}/\ell^n\mathbb{Z}) \xrightarrow{\cong} H^i(U_{j,\text{ét}}, h_{j*}h_j^*\mathbb{Z}/\ell^n\mathbb{Z}) \xrightarrow{\mathrm{Tr}} H^i(U_{j,\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z})$$

and the composition is bijective. Since $h_j^*\mathbb{Z}/\ell^n\mathbb{Z} \cong \mathbb{Z}/\ell^n\mathbb{Z}$, we get an injection

$$H^i(U_{j,\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}) \hookrightarrow H^i(U_{j+1,\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}). \tag{2.41}$$

Set $U_\infty = \varprojlim_j U_j$. Since each morphism $U_{j+1} \to U_j$ is affine, by using (2.41) and [Sta, Tag 09YQ], we have

$$H^i(U_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}) \hookrightarrow \varinjlim_j H^i(U_{j,\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}) \cong H^i(U_{\infty,\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}).$$

Thus, it suffices to show that $|H^i(U_{\infty,\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z})| \leq |H^i(U^{s.\flat}_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z})|$. Hence by tilting étale cohomology using Theorem 2.6.6, we are reduced to showing

$$|H^i(U^\flat_{\infty,\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z})| \leq |H^i(U^{s.\flat}_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z})|, \tag{2.42}$$

where $U^\flat_\infty$ is the open subset of $\mathrm{Spec}(R^\flat_\infty)$ that corresponds to $U_\infty \subseteq \mathrm{Spec}(R_\infty)$ in view of Example 2.6.5. On the other hand, considering the tilt of $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ associated to (R_0, I_0), we have a perfect $\mathbb{F}_p$-tower $(\{R_i^{s.\flat}\}_{i\geq 0}, \{t_i^{s.\flat}\}_{i\geq 0})$. Note that each $R_j^{s.\flat}$ is $I_0^{s.\flat}$-adically Henselian Noetherian ring[7] by Lemma 2.4.28 and Proposition 2.4.35 (2), and $t_j^{s.\flat}$ is module-finite by Proposition 2.4.35 (1). Considering the small tilts of the Zariski-open subsets appearing in (2.39) (see Definition 2.6.3), we get a tower of finite maps:

$$U^{s.\flat} = U_0^{s.\flat} \leftarrow \cdots \leftarrow U_j^{s.\flat} \leftarrow U_{j+1}^{s.\flat} \leftarrow \cdots .$$

So let $U^{s.\flat}_\infty$ be the inverse image of $U^{s.\flat}$ under $\mathrm{Spec}(R^{s.\flat}_\infty) \to \mathrm{Spec}(R^{s.\flat})$. Since $U^{s.\flat}_\infty \to U^{s.\flat}$ is a universal homeomorphism, the preservation of the small étale sites ([Sta, Tag 03SI]) gives an isomorphism:

$$H^i(U^{s.\flat}_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}) \cong H^i(U^{s.\flat}_{\infty,\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}). \tag{2.43}$$

Now the combination of Lemma 2.4.45 and Theorem 2.6.1 (2) together with the assumption finishes the proof of the theorem. □

[7] It is not obvious whether $R_j^{s.\flat}$ is normal. However, the normality was used only in the trace argument and we do not need it in the following argument.

Remark 2.6.8. One can formulate and prove the version of Proposition 2.6.7 for the étale cohomology with support in a closed subscheme of $\mathrm{Spec}(R)$, using Theorem 2.6.6. Then the resulting assertion gives a generalization of Česnavičius-Scholze's argument in [Čes19, Theorem 3.1.3] which is a key part of their proof for the absolute cohomological purity theorem. One of the advantages of Proposition 2.6.7 is that it can be used to answer some cohomological questions on possibly singular Noetherian schemes (e.g. log-regular schemes) in mixed characteristic.

2.6.2 Tilting the divisor class groups of local log-regular rings

We need a lemma of Grothendieck on the relationship between the divisor class group and the Picard group via direct limit. Its proof is found in [Gro67, Proposition (21.6.12)] or [Gro62, XI Proposition 3.7.1].

Lemma 2.6.9. *Let X be an integral Noetherian normal scheme, and let $\{U_i\}_{i\in I}$ be a family of open subsets of X. Consider the following conditions.*

1. *$\{U_i\}_{i\in I}$ forms a filter base. In particular, one can define a partial order on I so that it is a directed set and $\{U_i\}_{i\in I}$ together with the inclusion maps forms an inverse system.*
2. *Let $V_i := X \setminus U_i$. Then $\mathrm{codim}_X(V_i) \geq 2$.*
3. *For any $x \in \bigcap_{i\in I} U_i$, the local ring $\mathcal{O}_{X,x}$ is factorial.*

If $\{U_i\}_{i\in I}$ satisfies conditions (1) and (2), then the natural map $\mathrm{Pic}(U_i) \to \mathrm{Cl}(X)$ is injective for any $i \in I$. If $\{U_i\}_{i\in I}$ satisfies conditions (1), (2) and (3), then $\varinjlim_{i\in I} \mathrm{Pic}(U_i) \cong \mathrm{Cl}(X)$. In particular, if $U \subseteq X$ is any open subset that is locally factorial with $\mathrm{codim}_X(X\setminus U) \geq 2$, then $\mathrm{Pic}(U) \cong \mathrm{Cl}(X)$.

Next, we establish the following two results on the torsion part of the divisor class group of a (Noetherian) normal domain; these are a part of numerous applications of Theorem 2.6.1 of independent interest.

Proposition 2.6.10. *Let $(R, \mathfrak{m}, k)$ be a strictly Henselian Noetherian local normal $\mathbb{F}_p$-domain of dimension ≥ 2, let $X := \mathrm{Spec}(R)$ and fix an ideal $J \subseteq \mathfrak{m}$. Let $\{U_i\}_{i\in I}$ be any family of open subsets of X satisfying conditions (1), (2) and (3) as in the hypothesis of Lemma 2.6.9 and let U_i^{∞} be the $\mathbb{F}_p$-scheme which is the perfection of U_i.*

1. *For any prime $\ell \neq p$,*
$$\mathrm{Cl}(X)[\ell^n] \cong \varinjlim_{i\in I} H^1\big((U_i^{\infty})_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}\big).$$
2. *Let $\widehat{R^{1/p^{\infty}}}$ denote the J-adic completion of $R^{1/p^{\infty}}$. If moreover each U_i has the property that $X \setminus V(J) \subseteq U_i$, then for any prime $\ell \neq p$,*
$$\mathrm{Cl}(X)[\ell^n] \cong \varinjlim_{i\in I} H^1\big((\widehat{U_i^{\infty}})_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}\big),$$
where $\widehat{U_i^{\infty}}$ is inverse image of U_i^{∞} via the scheme map $\mathrm{Spec}(\widehat{R^{1/p^{\infty}}}) \to \mathrm{Spec}(R^{1/p^{\infty}})$.

Proof. Let us begin with a remark on the direct limit of étale cohomology groups. Note that for the transition morphism $g : U_i^\infty \to U_j^\infty$ which is affine, there is a functorial map: $H^1\big((U_j^\infty)_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}\big) \to H^1\big((U_i^\infty)_{\text{ét}}, g^*(\mathbb{Z}/\ell^n\mathbb{Z})\big) \cong H^1\big((U_i^\infty)_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}\big)$, which defines the direct system of cohomology groups.

(1): First we prove the following claim:

- There is an injection of abelian groups:

$$H^1(U_{et}, \mathbb{Z}/\ell^n\mathbb{Z}) \cong \mathrm{Pic}(U)[\ell^n] \subseteq \mathrm{Cl}(X)[\ell^n]$$

  for any $n \in \mathbb{N}$, where $U \subseteq X$ is an open subset whose complement is of codimension ≥ 2.

To prove this, consider the Kummer exact sequence

$$0 \to \mathbb{Z}/\ell^n\mathbb{Z} \cong \mu_{\ell^n} \to \mathbb{G}_m \xrightarrow{(\ )^{\ell^n}} \mathbb{G}_m \to 0,$$

where the identification of étale sheaves $\mu_{\ell^n} \cong \mathbb{Z}/\ell^n\mathbb{Z}$ follows from the fact that R is strict Henselian (one simply sends $1 \in \mathbb{Z}/\ell^n\mathbb{Z}$ to the primitive ℓ^n-th root of unity in R). Let $U \subseteq X$ be an open subset with its complement $V = X \setminus U$ having codimension ≥ 2. Then we have an exact sequence ([Mil80, Proposition 4.9; Chapter III]):

$$\Gamma(U_{\text{ét}}, \mathbb{G}_m) \xrightarrow{(\ )^{\ell^n}} \Gamma(U_{\text{ét}}, \mathbb{G}_m) \to H^1(U_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}) \to \mathrm{Pic}(U) \xrightarrow{(\ )^{\ell^n}} \mathrm{Pic}(U).$$

Since R is strict local and $\ell \neq p$, Hensel's lemma yields that $R^\times = (R^\times)^{\ell^n}$. Moreover, since $\mathrm{codim}_X(V) \geq 2$ and X is normal, we have $\Gamma(U_{\text{ét}}, \mathbb{G}_m) = R^\times$. Thus, $H^1(U_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}) \cong \mathrm{Pic}(U)[\ell^n]$. Note that $\mathrm{Pic}(U) \hookrightarrow \mathrm{Cl}(U)$ restricts to $\mathrm{Pic}(U)[\ell^n] \hookrightarrow \mathrm{Cl}(U)[\ell^n]$. Moreover, the natural homomorphism $\mathrm{Cl}(X) \to \mathrm{Cl}(U)$ is an isomorphism, thanks to $\mathrm{codim}_X(V) \geq 2$. Hence $H^1(U_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}) \cong \mathrm{Pic}(U)[\ell^n] \subseteq \mathrm{Cl}(X)[\ell^n]$, which proves the claim.

Since R is normal, the regular locus has a complement with codimension ≥ 2. Using this fact, we can apply Lemma 2.6.9 to get an isomorphism $\mathrm{Cl}(X)[\ell^n] \cong \varinjlim_{i\in I} H^1\big((U_i)_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}\big)$. By étale invariance of cohomology under taking perfection of $\mathbb{F}_p$-schemes ([Sta, Tag 03SI]), we get

$$\mathrm{Cl}(X)[\ell^n] \cong \varinjlim_{i\in I} H^1\big((U_i)_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}\big) \cong \varinjlim_{i\in I} H^1\big((U_i^\infty)_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}\big),$$

as desired.

(2): Since R is Henselian along $\mathfrak{m}$ and $J \subseteq \mathfrak{m}$, it is Henselian along J by [Sta, Tag 0DYD]. Moreover, the perfect closure of R still preserves Henselian property along J. Theorem 2.6.1 yields

$$H^1\big((U_i^\infty)_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}\big) \cong H^1\big((\widehat{U_i^\infty})_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}\big)$$

and the conclusion follows from (1). □

Proposition 2.6.11. *Let A be a Noetherian ring with a regular element $t \in A$ such that A is t-adically Henselian and $A \to A/tA$ is the natural surjection between locally factorial domains. Pick an integer $n > 0$ that is invertible on A. Then if* $\mathrm{Cl}(A)$ *has no torsion element of order n, the same holds for* $\mathrm{Cl}(A/tA)$*. If moreover A is a $\mathbb{Q}$-algebra and* $\mathrm{Cl}(A)$ *is torsion-free, then so is* $\mathrm{Cl}(A/tA)$.

Proof. The Kummer exact sequence $0 \to \mu_n \to \mathbb{G}_m \xrightarrow{(\ )^n} \mathbb{G}_m \to 0$ induces the following commutative diagram:

$$\begin{array}{ccccc} H^1(\mathrm{Spec}(A)_{\text{ét}}, \mu_n) & \xrightarrow{\delta_1} & \mathrm{Pic}(A) & \xrightarrow{(\ )^n} & \mathrm{Pic}(A) \\ {\scriptstyle\alpha}\downarrow & & \downarrow & & \downarrow \\ H^1(\mathrm{Spec}(A/tA)_{\text{ét}}, \mu_n) & \xrightarrow{\delta_2} & \mathrm{Pic}(A/tA) & \xrightarrow{(\ )^n} & \mathrm{Pic}(A/tA) \end{array}$$

By Theorem 2.6.1, the map α is an isomorphism. Then if $\mathrm{Pic}(A)$ has no torsion element of order n, δ_1 is the zero map. This implies that δ_2 is also the zero map and hence, $\mathrm{Pic}(A/tA)$ has no element of order n. Since both A and A/tA are locally factorial by assumption, we have $\mathrm{Cl}(A) \cong \mathrm{Pic}(A)$ and $\mathrm{Cl}(A/tA) \cong \mathrm{Pic}(A/tA)$. So the assertion follows. □

It is not necessarily true that δ_1 (resp. δ_2) is injective because we do not assume A to be strictly Henselian.

Lemma 2.6.12. *Let $(R, \mathcal{Q}, \alpha)$ be a log-regular ring. Then a strict Henselization $(R^{\mathrm{sh}}, \mathcal{Q}, \alpha^{\mathrm{sh}})$ is also a log-regular ring where $\alpha^{\mathrm{sh}} : \mathcal{Q} \to R \to R^{\mathrm{sh}}$ is the composition of homomorphisms.*

Proof. Since $R \to R^{\mathrm{sh}}$ is a local ring map, $(R^{\mathrm{sh}}, \mathcal{Q}, \alpha^{\mathrm{sh}})$ is a local log ring by Lemma 1.3.4. Note that we have the equality $I_{\alpha^{\mathrm{sh}}} = I_\alpha R^{\mathrm{sh}}$. Since we have the isomorphism $R^{\mathrm{sh}}/I_{\alpha^{\mathrm{sh}}} \cong (R/I_\alpha)^{\mathrm{sh}}$ by [Sta, Tag 05WS] and $(R/I_\alpha)^{\mathrm{sh}}$ is a regular local ring by [Sta, Tag 06LN], $R^{\mathrm{sh}}/I_{\alpha^{\mathrm{sh}}}$ is a regular local ring. Moreover, since the dimension of R is equal to the dimension of a strict henselization R^{sh}, we obtain the following equalities:

$$\dim(R^{\mathrm{sh}}) - \dim(R^{\mathrm{sh}}/I_{\alpha^{\mathrm{sh}}}) = \dim(R^{\mathrm{sh}}) - \dim((R/I_\alpha)^{\mathrm{sh}}) = \dim(R) - \dim(R/I_\alpha) = \dim(\mathcal{Q}).$$

So the local log ring $(R^{\mathrm{sh}}, \mathcal{Q}, \alpha^{\mathrm{sh}})$ is log-regular. □

Now we can prove the following result on the divisor class groups of local log-regular rings, as an application of the theory of perfectoid towers.

Theorem 2.6.13. *Let $(R, \mathcal{Q}, \alpha)$ be a local log-regular ring of mixed characteristic with perfect residue field k of characteristic $p > 0$, and denote by $\mathrm{Cl}(R)$ the divisor class group with its torsion subgroup $\mathrm{Cl}(R)_{\mathrm{tor}}$. Then the following assertions hold.*

1. *Assume that $R \cong W(k)[\![\mathcal{Q}]\!]$ for a fine, sharp, and saturated monoid $\mathcal{Q}$, where $W(k)$ is the ring of Witt vectors over k. Then $\mathrm{Cl}(R)_{\mathrm{tor}} \otimes \mathbb{Z}[\frac{1}{p}]$ is a finite group. In other words, the ℓ-primary subgroup of $\mathrm{Cl}(R)_{\mathrm{tor}}$ is finite for all primes $\ell \neq p$ and vanishes for almost all primes $\ell \neq p$.*

2. *Assume that $\widehat{R^{\mathrm{sh}}}[\frac{1}{p}]$ is locally factorial, where $\widehat{R^{\mathrm{sh}}}$ is the completion of the strict Henselization R^{sh}. Then $\mathrm{Cl}(R)_{\mathrm{tor}} \otimes \mathbb{Z}[\frac{1}{p}]$ is a finite group. In other words, the ℓ-primary subgroup of $\mathrm{Cl}(R)_{\mathrm{tor}}$ is finite for all primes $\ell \neq p$ and vanishes for almost all primes $\ell \neq p$.*

Proof. We note that we may assume that $\mathcal{Q}$ is fine, sharp, and saturated by Remark 1.3.7.

(1): Since $R \cong C(k)[\![\mathcal{Q}]\!]$, we have

$$R/pR \cong k[\![\mathcal{Q}]\!],$$

which is a local F-finite log-regular ring. There is an induced map $\mathrm{Cl}(R) \to \mathrm{Cl}(R/pR)$. By restriction, we have $\mathrm{Cl}(R)_{\mathrm{tor}} \to \mathrm{Cl}(R/pR)_{\mathrm{tor}}$. Then Lemma 1.4.5 together with Polstra's result [Pol22] says that $\mathrm{Cl}(R/pR)_{\mathrm{tor}}$ is finite. Let C_ℓ be the maximal ℓ-subgroup of $\mathrm{Cl}(R)_{\mathrm{tor}}$. Since $\ell \neq p$, we find that the map $\mathrm{Cl}(R)_{\mathrm{tor}} \to \mathrm{Cl}(R/pR)_{\mathrm{tor}}$ restricted to C_ℓ is injective in view of [GW94, Theorem 1.2]. In conclusion, C_ℓ is finite for all $\ell \neq p$, and C_ℓ vanishes for almost all $\ell \neq p$, as desired.

(2): The proof given below works for the first case under the assumption of local factoriality of $\widehat{R^{\mathrm{sh}}}[\frac{1}{p}]$. Since $R \to \widehat{R^{\mathrm{sh}}}$ is a local flat ring map, the induced map $\mathrm{Cl}(R) \to \mathrm{Cl}(\widehat{R^{\mathrm{sh}}})$ is injective by Mori's theorem (c.f. [For17, Corollary 6.5.2]). Thus, it suffices to prove the theorem for $\widehat{R^{\mathrm{sh}}}$. Moreover, $\widehat{R^{\mathrm{sh}}}$ is log-regular with respect to the induced log ring structure $\alpha : \mathcal{Q} \to R \to \widehat{R^{\mathrm{sh}}}$ by Lemma 2.6.12. So without loss of generality, we may assume that the residue field of R is separably closed (hence algebraically closed in our case).

Henceforth, we denote $\widehat{R^{\mathrm{sh}}}$ by R for brevity and fix a prime ℓ that is different from p. By Lemma 2.6.9 and the local factoriality of $R[\frac{1}{p}]$, we claim that there is an open subset $U \subseteq X := \mathrm{Spec}(R)$ such that the following holds:

- $\mathrm{Pic}(U) \cong \mathrm{Cl}(X)$, $X \setminus V(pR) \subseteq U$ and $\mathrm{codim}_X(X \setminus U) \geq 2$.

Indeed, note that X is a normal integral scheme by Kato's theorem (Theorem 1.3.13) and let U be the union of the regular locus of X and the open $\mathrm{Spec}(R[\frac{1}{p}]) \subseteq X$. Then by Serre's normality criterion, we see that $\mathrm{codim}_X(X \setminus U) \geq 2$. We fix such an open $U \subseteq X$ once and for all. Taking the cohomology sequence associated to the exact sequence

$$0 \to \mathbb{Z}/\ell^n\mathbb{Z} \to \mathbb{G}_m \xrightarrow{(\ )^{\ell^n}} \mathbb{G}_m \to 0$$

on the strict local scheme X and arguing as in the proof of Proposition 2.6.10, we have an isomorphism:

$$H^1(U_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}) \cong \mathrm{Pic}(U)[\ell^n] \cong \mathrm{Cl}(X)[\ell^n]. \tag{2.44}$$

On the other hand, there is a perfectoid tower of module-finite extensions of local log-regular rings arising from $(R, (p))$:

$$(R, \mathcal{Q}, \alpha) = (R_0, \mathcal{Q}^{(0)}, \alpha_0) \to \cdots \to (R_j, \mathcal{Q}^{(j)}, \alpha_j) \to (R_{j+1}, \mathcal{Q}^{(j+1)}, \alpha_{j+1}) \to \cdots. \tag{2.45}$$

Notice that each map is generically of p-power rank in view of Lemma 1.2.34 (2) and Lemma 1.2.32 (4). Moreover, the tilt of (2.45) (associated to $(R, (p))$) is given by

$$(R^{s.\flat}, \mathcal{Q}^{s.\flat}, \alpha^{s.\flat}) = ((R_0)^{s.\flat}_{(p)}, \mathcal{Q}^{s.\flat}_0, \alpha^{s.\flat}_0) \to \cdots \to ((R_j)^{s.\flat}_{(p)}, \mathcal{Q}^{s.\flat}_j, \alpha^{s.\flat}_j) \to ((R_{j+1})^{s.\flat}_{(p)}, \mathcal{Q}^{s.\flat}_{j+1}, \alpha^{s.\flat}_{j+1}) \to \cdots,$$

where $((R_j)^{s.\flat}_{(p)}, \mathcal{Q}^{s.\flat}_j, \alpha^{s.\flat}_j)$ is a complete local log-regular ring of characteristic $p > 0$ in view of Theorem 2.5.10. The local ring $R^{s.\flat}$ is strictly Henselian and the complement of $U^{s.\flat}(= U^{s.\flat}_{(p)})$ has codimension ≥ 2 in $\mathrm{Spec}(R^{s.\flat})$, and by repeating the proof of Proposition 2.6.10, we obtain an isomorphism

$$H^1(U^{s.\flat}_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}) \cong \mathrm{Pic}(U^{s.\flat})[\ell^n]. \tag{2.46}$$

By Lemma 2.6.9, the map

$$\mathrm{Pic}(U^{s.\flat})[\ell^n] \to \mathrm{Cl}(R^{s.\flat})[\ell^n] \tag{2.47}$$

is injective. Combining (2.44), (2.46), (2.47) and Proposition 2.6.7 together, it is now sufficient to check that

$$\mathrm{Cl}(R^{s.\flat})[\ell^n] \text{ is finite for all } \ell, \text{ and zero for almost all } \ell \neq p.$$

Since we know that $R^{s.\flat}$ is strongly F-regular by Theorem 2.5.10 and Lemma 1.4.5, the aforementioned result of Polstra finishes the proof. □

Remark 2.6.14. One can also deduce a special case of Polstra's result on the divisor class group of a strongly F-regular local $\mathbb{F}_p$-domain R, using étale cohomology and the main result of [CRST16]. Recall that for a connected (separated) Noetherian scheme X, any integer $n > 0$ and a finite abelian group G, there are isomorphisms:

$$H^1(X_{\text{ét}}, G) \cong \mathrm{Hom}_{\mathrm{cont}}(\pi_1^{\text{ét}}(X), G) \cong \mathrm{Hom}_{\mathrm{cont}}(\pi_1^{\mathrm{ab}}(X), G), \tag{2.48}$$

where $\pi_1^{\mathrm{ab}}(X)$ is the maximal abelian quotient of the étale fundamental group $\pi_1^{\text{ét}}(X)$. (2.48) is found in [Fu11, Proposition 5.7.20] via an interpretation of classifying G-torsors over X. Let us replace R with $\widehat{R^{\mathrm{sh}}}$ by [Abe01, Theorem 3.6]. Since R is normal, the non-singular locus $U \subseteq X := \mathrm{Spec}(R)$ is of codimension ≥ 2, and Lemma 2.6.9 gives an isomorphism: $\mathrm{Cl}(X) \cong \mathrm{Pic}(U)$. On the other hand, we know $|\pi_1^{\text{ét}}(U)| < \infty$ by [CRST16, Theorem 5.1]. For any prime $\ell \neq p$,

$$\mathrm{Cl}(X)[\ell^n] \cong H^1(U_{\text{ét}}, \mu_{\ell^n}) \cong \mathrm{Hom}_{\mathrm{cont}}(\pi_1^{\text{ét}}(U), \mu_{\ell^n}) \tag{2.49}$$

by (2.48). Then the finiteness of $\pi_1^{\text{ét}}(U)$ implies that (2.49) vanishes for almost all $\ell \neq p$, while the right-hand side of (2.49) is bounded for a fixed ℓ and varying n by Pontryagin duality for finite abelian groups (one notices that the sheaf μ_{ℓ^n} is constant because we are assuming that R is strictly Henselian). In conclusion, $\mathrm{Cl}(R)_{\mathrm{tor}} \otimes \mathbb{Z}[\frac{1}{p}]$ is finite. We should note that Polstra proved that $\mathrm{Cl}(R)_{\mathrm{tor}}$ is indeed finite and his proof is more elementary.

Example 2.6.15. *Here we compute the divisor class group of a log-regular local ring using a method different from that provided in Theorem 1.7.8. Let us consider the local log-regular ring $(R, \mathcal{Q}, \alpha_R)$ defined as in Example 2.5.11 (2). Assume that k is an algebraically closed field of characteristic $p > 0$. Then we have*

$$R[\frac{1}{p}] = W(k)[\![x, y, z]\!][\frac{1}{p}]/(xy - pz) = W(k)[\![x, y, z]\!][\frac{1}{p}]/((p^{-1}x)(p^{-1}y) - (p^{-1}z))$$

which is isomorphic to the regular ring

$$W(k)[\![s, t, u]\!][\frac{1}{p}]/(st - u) \cong W(k)[\![s, t]\!][\frac{1}{p}]$$

which is a UFD. Moreover, $p \in R$ is irreducible. Indeed, assume that $f, g \in W(k)[\![x, y, z]\!]$ are non-unit elements such that $p - fg \in (xy - pz)$. Then we have $p - fg = (xy - pz)h$ for some $h \in W(k)[\![x, y, z]\!]$ and so

$$p(1 + zh) = fg + xyh.$$

This gives $p = (1 + zh)^{-1}(fg + xyh)$, which is impossible. The n-th symbolic power $(p)^{(n)}$ never becomes principal for $n \geq 1$. Therefore, $\mathrm{Cl}(R) \cong \mathbb{Z}$ and $\mathrm{Cl}(R)_{\mathrm{tor}} = 0$.

2.6.3 Application to local cohomology modules

As the second application of perfectoid towers, we give an inequality of the cohomological dimension between in positive characteristic and in mixed characteristic.

First of all, we recall the definition of local cohomology modules. Let R be a Noetherian ring, let I be an ideal of R, and let M be an R-module. Then we define

$$\Gamma_I(M) := \{x \in M \mid I^n x = 0 \text{ for some } n > 0\}.$$

$\Gamma_I(-)$ is a left exact functor of the R-modules called the *I-torsion functor*. The i-th right derived functor of $\Gamma_I(-)$ is denoted by $H^i_I(-)$ and $H^i_I(M)$ is called *the i-th local cohomology of M with support in I*. Also, we often use characterization of local cohomology modules with Čech complex. Let $f_1, \ldots, f_r$ be a generator of I. Then we have the Čech complex

$$C^\bullet(I; M) : 0 \longrightarrow M \longrightarrow \prod_{1 \leq j \leq r} M_{f_j} \longrightarrow \cdots \longrightarrow M_{f_1 \cdots f_r} \longrightarrow 0. \tag{2.50}$$

Then its i-th cohomology is isomorphic to $H^i_I(M)$ as R-modules.

We deal with the long exact sequences of three different types. First for a short exact sequence $0 \to N \to M \to L \to 0$, we obtain the long exact sequence

$$\cdots \longrightarrow H^i_I(N) \longrightarrow H^i_I(M) \longrightarrow H^i_I(L) \longrightarrow H^{i+1}_I(N) \longrightarrow \cdots . \tag{2.51}$$

Take an element $f \in R$, an ideal $I \subset R$ and an R-module M. Then we obtain the long exact sequence

$$\cdots \longrightarrow H^i_{I+f}(M) \longrightarrow H^i_I(M) \longrightarrow H^i_I(M_f) \longrightarrow H^{i+1}_{I+f}(M) \longrightarrow \cdots \tag{2.52}$$

which is induced from the short exact sequence of complexes $0 \to C^\bullet(I + f; M) \to C^\bullet(I; M) \to C^\bullet(I; M_f) \to 0$.

Finally, take ideals $I, J \subset R$ and an R-module M. Then we obtain the long exact sequence, called the *Mayer-Vietoris long exact sequence*

$$\cdots \longrightarrow H^i_{I+J}(M) \longrightarrow H^i_I(M) \oplus H^i_J(M) \longrightarrow H^i_{I\cap J}(M) \longrightarrow H^{i+1}_{I+J}(M) \longrightarrow \cdots . \tag{2.53}$$

One of the most important invariants associated with local cohomology modules is the *local cohomological dimension* of I.

Definition 2.6.16. Let M be an R-module and let I be an ideal of R. Then the *local cohomological dimension* of I is defined by

$$\mathrm{cd}(I, M) := \sup\{i \in \mathbb{Z}_{\geq 0} \mid H^i_I(M) \neq 0\}.$$

The purpose of this subsection is to consider the following question which is proposed by A. Grothendieck.

Question 1 (Grothendieck)**.** *Find the condition that* $\mathrm{cd}(M, I) \geq i$ *for some* $i \in \mathbb{N}$.

The punctured spectrum $\mathrm{Spec}^\circ(R)$ of a local ring $(R, \mathfrak{m}, k)$ is the set of all primes $\mathfrak{p} \neq \mathfrak{m}$ with the topology induced by the Zariski topology on $\mathrm{Spec}\, R$. Let I be an ideal of R. The punctured spectrum $\mathrm{Spec}^\circ(R/I)$ is *connected* if the following property holds; For any ideals $\mathfrak{a}$ and $\mathfrak{b}$ of R such that $\sqrt{\mathfrak{a} \cap \mathfrak{b}} = \sqrt{I}$ and $\sqrt{\mathfrak{a} + \mathfrak{b}} = \mathfrak{m}$, we have $\sqrt{\mathfrak{a}}$ or $\sqrt{\mathfrak{b}}$ equals $\mathfrak{m}$. Or equivalently, $\sqrt{\mathfrak{a}}$ or $\sqrt{\mathfrak{b}}$ equals $\sqrt{I}$. Note that if R is a local domain, then it is easy to see that the punctured spectrum $\mathrm{Spec}^\circ(R)$ is connected.

First, our main result of this subsection is that local cohomology modules of a regular local ring in characteristic $p > 0$ are divisible under some finiteness condition. Let us recall the definition of divisible modules here.

Definition 2.6.17. Let R be a ring and let M be an R-module. Then M is *divisible* if for every regular element $r \in R$, and every element $m \in M$, there exists an element $m' \in M$ such that $m = rm'$.

Note that every injective module is divisible. Huneke and Sharp investigated the condition under which local cohomology modules of regular local rings of positive characteristic are injective (see [HS93]).

The following theorem is a key in this subsection. We emphasize the importance of the surjectivity of a map of local cohomology modules.

Theorem 2.6.18. *Let $(R, \mathfrak{m}, k)$ be a d-dimensional regular local ring of characteristic $p > 0$. Fix an ideal $I \subset \mathfrak{m}$. Suppose $\ell_R(H^{d-i}_{\mathfrak{m}}(R/I)) < \infty$ for some fixed $i \geq 0$. Then $H^i_I(R)$ is a divisible module. In particular, for any nonzero element $x \in R$, the multiplication map $H^i_I(R) \xrightarrow{\times x} H^i_I(R)$ is surjective.*

Proof. Since the R-module $H^{d-i}_{\mathfrak{m}}(R/I)$ is of finite length, it follows from [Lyu06, Corollary 3.3] and [Lyu06, Corollary 3.4] that $H^i_I(R)$ is an artinian R-module. Combining [HS93, 1.8 Lemma] with this fact and [HS93, 3.6 Corollary], $H^i_I(R)$ is an injective R-module. Thus, it is divisible. Since R is a domain, $H^i_I(R) \xrightarrow{\times x} H^i_I(R)$ is surjective for any nonzero element $x \in R$, as desired. □

Recall the definitions of two graphs, which are important to prove the main theorem.

Definition 2.6.19. Let $(R, \mathfrak{m}, k)$ be a local ring.

1. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_t$ be the set of minimal primes of R. Then the graph Θ_R has vertices labeled $1, \dots, t$ and there is an edge between two (distinct) vertices i and j, precisely when $\mathfrak{p}_i + \mathfrak{p}_j$ is not $\mathfrak{m}$-primary.

2. ([HH94, Definition 3.4]) Let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be the set of minimal primes of R such that $\dim(R/\mathfrak{p}_i) = \dim(R)$. Then the Hochster-Huneke graph of R, which is denoted by Γ_R, has vertices $1, \dots, r$ and there is an edge between two (distinct) vertices i and j, precisely if $\mathfrak{p}_i + \mathfrak{p}_j$ has height one.

Huneke and Lyubeznik pointed out the importance of the graph Θ_R when we investigate the connectedness of punctured spectra.

Lemma 2.6.20 (Huneke-Lyubeznik)**.** *Let $(R, \mathfrak{m}, k)$ be a local ring. Then* $\mathrm{Spec}^\circ(R)$ *is connected if and only if Θ_R is connected.*

The following lemma plays an important role in the proof of Lemma 2.6.25.

Lemma 2.6.21 ([HNBPW18, Proposition 3.5])**.** *Let $(R, \mathfrak{m}, k)$ be a complete local domain, and let $x \in \mathfrak{m}$ be a nonzero element. Then $\Gamma_{R/xR}$ is connected.*

Here we prepare some notation needed for later discussion.

Notation 2.6.22. Let $(R, \mathfrak{m}, k)$ be a complete regular local ring where k is perfect. Then by Example 2.5.11 (1), we can construct a perfectoid tower $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ arising from $(R, (p))$ such that

$$R_i \cong W(k)[\![x_1^{1/p^i}, \dots, x_d^{1/p^i}]\!]/(p - f)$$

and we know that i-th small tilt $R_i^{s.\flat}$ is isomorphic to $k[\![x_1^{1/p^i}, \dots, x_d^{1/p^i}]\!]$. Let I be an ideal of R containing p. Then, by Lemma 2.4.31 (that is, $R_0/pR_0 \cong R_0^{s.\flat}/p^{s.\flat}R_0^{s.\flat}$), one can define the ideal $I^{s.\flat}$ of $R_0^{s.\flat}$ as the inverse image of $I(R_0/pR_0)$ by $R_0^{s.\flat} \twoheadrightarrow R_0^{s.\flat}/p^{s.\flat}R_0^{s.\flat} \xrightarrow{\cong} R_0/pR_0$.

Lemma 2.6.23. *Keep the notation as in Notation 2.6.22. Let n be an integer. Assume that $\ell_R(H^{d-n+1}_{\mathfrak{m}}(R/I)) < +\infty$. then $H^n_{I^{s.\flat}}(R_0^{s.\flat}) = 0$ implies $H^n_I(R_0) = 0$.*

Proof. Since p is a non-zero divisor in R_0, we obtain the short exact sequence

$$0 \to R_0^{s.\flat} \xrightarrow{\times p^{s.\flat}} R_0^{s.\flat} \to R_0/pR_0 \to 0$$

by Corollary 2.4.32 (2). Hence we obtain the long exact sequence

$$\cdots \to H^{n-1}_{I^{s.\flat}}(R_0^{s.\flat}) \xrightarrow{\times p^{s.\flat}} H^{n-1}_{I^{s.\flat}}(R_0^{s.\flat}) \to H^{n-1}_I(R_0/pR_0) \to H^n_{I^{s.\flat}}(R_0^{s.\flat}) \xrightarrow{\times p^{s.\flat}} \cdots . \quad (2.54)$$

Since we have $\ell_{R_0^{s.\flat}}(H^{d-n+1}_{\mathfrak{m}^{s.\flat}}(R_0^{s.\flat}/I^{s.\flat})) = \ell_{R_0}(H^{d-i+1}_{\mathfrak{m}}(R/I)) < +\infty$ by [Sta, Tag 00IX] and $R_0^{s.\flat}$ is a regular local ring, the map $H^{n-1}_{I^{s.\flat}}(R_0^{s.\flat}) \xrightarrow{\times p^{s.\flat}} H^{n-1}_{I^{s.\flat}}(R_0^{s.\flat})$ is surjective by Theorem 2.6.18. Also, by assumption, we obtain $H^n_{I^{s.\flat}}(R_0^{s.\flat}) = 0$. Here, applying these to the long exact sequence (2.54), we obtain that $H^{n-1}_I(R_0/pR_0) \to H^n_{I^{s.\flat}}(R_0^{s.\flat}) = 0$ is injective. Hence $H^{n-1}_I(R_0/pR_0) = 0$ holds. Finally, consider the long exact sequence

$$\cdots \to H^{n-1}_I(R_0/pR_0) = 0 \to H^n_I(R_0) \xrightarrow{\times p} H^n_I(R_0) \to \cdots .$$

Then $H^n_I(R_0) \xrightarrow{\times p} H^n_I(R_0)$ is injective, but it is necessary that $H^n_I(R_0) = 0$ if this holds Assume that $H^n_I(R_0) \neq 0$. Since $H^n_I(R_0)$ is I-torsion and $p \in I$, every element of $H^n_I(R_0)$ is annihilated by p^j for some $j > 0$. This gives a contradiction to the injectivity of $H^n_I(R_0) \xrightarrow{\times p} H^n_I(R_0)$. Thus we obtain the vanishing $H^n_I(R_0) = 0$. □

Remark 2.6.24. To prove the converse statement of Lemma 2.6.23, we need to prove a mixed characteritstic analogue of Theorem 2.6.18.

We prove the following lemma which is similar to [HNBPW18, Lemma 3.7], but the proof is different.

Lemma 2.6.25. *Let $(R, \mathfrak{m}, k)$ be a d-dimensional complete regular local ring of mixed characteristic with separably closed residue field k. Let $\mathfrak{q}$ be a prime ideal of R such that $\dim(R/\mathfrak{q}) \geq 3$. Put $J := pR + \mathfrak{q}$. Assume $\ell_R(H^2_{\mathfrak{m}}(R/J)) < \infty$. Then $H^{d-1}_{\mathfrak{q}}(R) = 0$.*

Proof. Since R is a complete regular local ring of mixed characteristic, it is isomorphic to

$$C(k)[\![x_1, \ldots, x_d]\!]/(p - f)$$

where f is an element in $\mathfrak{m}^2_{C(k)[\![x_1,\ldots,x_d]\!]} \setminus \mathfrak{m}_{C(k)[\![x_1,\ldots,x_d]\!]}$ or $f = x_1$, and $C(k)$ is a complete discrete valuation ring such that $C(k)/pC(k) \cong k$. By considering a faithfully flat extension

$$C(k)[\![x_1, \ldots, x_d]\!]/(p - f) \to C(k_{\mathrm{perf}})[\![x_1, \ldots, x_d]\!]/(p - f)$$

where k_{perf} is the perfection of k, we may assume that the residue field of R is perfect.

First, consider the case $p \in \mathfrak{q}$. By Hartshorne-Lichtenbaum vanishing theorem and (SVT) for a regular local ring containing a field (see [HL90, Theorem 2.9]), we obtain $H^{d-1}_{\mathfrak{q}^{s.\flat}}(R^{s.\flat}) = H^d_{\mathfrak{q}^{s.\flat}}(R^{s.\flat}) = 0$, that is, $\mathrm{cd}(\mathfrak{q}^{s.\flat}, R^{s.\flat}) < d - 1$. Hence we obtain $\mathrm{cd}(\mathfrak{q}, R) < d - 1$ by Lemma 2.6.23.

Next, we consider the case $p \notin \mathfrak{q}$. For distinct minimal primes $\mathfrak{p}_1$ and $\mathfrak{p}_2$ of $\mathrm{Spec}(R/J)$, if $\mathrm{ht}_{R/J}(\mathfrak{p}_1 + \mathfrak{p}_2) = 1$, then $\mathfrak{p}_1 + \mathfrak{p}_2$ can not be primary to the maximal ideal of R/J, since we have the inequality $\dim(R/J) \geq 2$. This implies that $\Gamma_{R/J}$ is a subgraph of $\Theta_{R/J}$.

Moreover, the two graphs $\Gamma_{R/J}$ and $\Theta_{R/J}$ have the same vertices. Indeed, since $R/\mathfrak{q}$ is a complete local domain which is catenary, R/J is equidimensional.

Since $R/\mathfrak{q}$ is a complete local domain, $\Gamma_{R/J}$ connected by Lemma 2.6.21 by taking $x = p$. Since $\Gamma_{R/J}$ and $\Theta_{R/J}$ have the same vertices, $\Theta_{R/J}$ is also connected. This implies that $\operatorname{Spec}^\circ(R/J)$ is connected by Lemma 2.6.20. Remark that $R/J \cong S/\overline{J}$, where $\overline{J} \subset S$ is the inverse image of the ideal $J(R/pR)$ under $S \twoheadrightarrow R/pR$. and $\dim(R/J) \geq 2$. In view of (SVT) [HL90, Theorem 2.9], we observe that $H^{d-1}_{\overline{J}}(S) = 0$. Applying the above discussion replacing $\mathfrak{p}$ for $\overline{J}$, we obtain $H^{d-1}_J(R) = 0$.

Finally, recall from (4.2) the following long exact sequence

$$\cdots \to H^{d-1}_J(R) \to H^{d-1}_{\mathfrak{q}}(R) \to H^{d-1}_{\mathfrak{q}}(R_p) \to \cdots .$$

We already proved $H^{d-1}_J(R) = 0$. So it suffices to show that $H^{d-1}_{\mathfrak{q}}(R_p) = 0$. Since p is in the maximal ideal $\mathfrak{m}$, we obtain $\dim(R_p) = d - 1$ and $\dim(R_p/\mathfrak{q}R_p) \geq 2$. It then suffices to show that the localization of $H^{d-1}_{\mathfrak{q}}(R_p)$ at every prime $\mathfrak{p} \in \operatorname{Spec} R$ vanishes, where $p \notin \mathfrak{p}$, $\mathfrak{q} \subset \mathfrak{p}$ and $\dim(R_{\mathfrak{p}}) = d-1$. Notice that $(R_p)_{\mathfrak{p}} \cong R_{\mathfrak{p}}$ is a regular local ring of dimension $d-1$ and $\dim(R_{\mathfrak{p}}/\mathfrak{q}R_{\mathfrak{p}}) \geq 2$. We obtain $(H^{d-1}_{\mathfrak{q}}(R_p))_{\mathfrak{p}} \cong H^{d-1}_{\mathfrak{q}R_{\mathfrak{p}}}(R_{\mathfrak{p}}) = 0$ by the Hartshorne-Lichtenbaum vanishing theorem. Thus, $H^{d-1}_{\mathfrak{q}}(R_p) = 0$. □

Now let us prove the main theorem of this subsection. The proof is based on [HNBPW18, Theorem 3.8].

Theorem 2.6.26. *Let $(R, \mathfrak{m}, k)$ be a d-dimensional complete regular local ring of mixed characteristic with separably closed residue field k. Assume that $I \subset R$ is a proper ideal with $\dim(R/\mathfrak{q}) \geq 3$ and $\ell_R(H^2_{\mathfrak{m}}(R/pR+\mathfrak{q})) < \infty$ for all $\mathfrak{q} \in \operatorname{Min}(R/I)$. Then the following statements are equivalent.*

1. $H^{d-1}_I(R) = 0$.

2. *The punctured spectrum* $\operatorname{Spec}^\circ(R/I)$ *is connected in the Zariski topology.*

Proof. Suppose that $H^{d-1}_I(R) = 0$. If $\operatorname{Spec}^\circ(R/I)$ is not connected, there exist ideals J_1, J_2 of R such that $\sqrt{J_1 + J_2} = \mathfrak{m}$ and $\sqrt{J_1 \cap J_2} = \sqrt{I}$, but $\sqrt{J_1}$ and $\sqrt{J_2}$ are not equal to both $\mathfrak{m}$ and $\sqrt{I}$. By (2.53), we get

$$\cdots \to H^{d-1}_I(R) \to H^d_{\mathfrak{m}}(R) \to H^d_{J_1}(R) \oplus H^d_{J_2}(R) \to \cdots .$$

Then we have $H^{d-1}_I(R) = 0$ by assumption and $H^d_{J_1}(R) = H^d_{J_2}(R) = 0$ by the Hartshorne-Lichtenbaum vanishing theorem. This is a contradiction for $H^d_{\mathfrak{m}}(R) \neq 0$ by the Grothendieck's vanishing theorem.

Conversely, suppose that $\operatorname{Spec}^\circ(R/I)$ is connected. We proceed by induction on $t := |\operatorname{Min}(R/I)|$. The case that $t = 1$ was established in Lemma 2.6.25. Assume that the implication holds for all ideals $\mathfrak{a}$ of R with $t-1 \geq 0$ minimal primes, for which $\dim(R/\mathfrak{p}) \geq 3$ for some minimal primes $\mathfrak{p}$ of $\mathfrak{a}$.

Fix an ideal I with t minimal primes such that $\dim(R/\mathfrak{p}) \geq 3$ for any minimal prime $\mathfrak{p}$ of I, and for which $\operatorname{Spec}^\circ(R/I)$ is connected. Then $\Theta_{R/I}$ is also connected by Lemma 2.6.20. Thus, there is an ordering $\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_t$ of the minimal primes of I such that the induced subgraph of $\Theta_{R/I}$ on induces $\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_i$ is connected for all $1 \leq i \leq t$. This means that given $1 \leq i \leq t$, if $J_i = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_i$, then Θ_{R/J_i} is connected. So we deduce that $\operatorname{Spec}^\circ(R/J_i)$ is connected. Apply (2.53) associated to J_{t-1} and $\mathfrak{q}_t$ to get

$$\cdots \to H^{d-1}_{J_{t-1}}(R) \oplus H^{d-1}_{\mathfrak{q}_t}(R) \to H^{d-1}_I(R) \to H^d_{J_{t-1}+\mathfrak{q}_t}(R) \to 0.$$

By the inductive hypothesis, $H^{d-1}_{J_{t-1}}(R) = 0$. In addition, $H^{d-1}_{\mathfrak{q}_t}(R) = 0$ by Lemma 2.6.25. Moreover, since the punctured spectrum of R/I is connected, $\sqrt{J_{t-1} + \mathfrak{q}_t} \subsetneq \mathfrak{m}$, so that $H^d_{J_{t-1}+\mathfrak{q}_t}(R) = 0$ by Hartshorne-Lichtenbaum vanishing theorem, which proves $H^{d-1}_I(R) = 0$. □

Here, we construct some examples fitting into the setting of Theorem 2.6.26.

Example 2.6.27. *Let $(R, \mathfrak{m})$ be a local complete generalized Cohen-Macaulay integral domain in mixed characteristic with dimension at least four. By Cohen's structure theorem, there is a regular local ring A together with an ideal I such that $R = A/I$. Since R is an integral domain, the ideal must be prime. Put $J := p + I$. If p is contained in I, then $\ell_R(H^2_{\mathfrak{m}}(A/J))$ is obviously finite (more precisely is zero) since $I = J$ and R is generalized Cohen-Macaulay of dimension at least four. If p is not contained in I, then p is a non-zero element in A/I. We look at the short exact sequence $0 \to R \xrightarrow{\times p} R \to A/J \to 0$, and the induced long exact sequence:*

$$\cdots \longrightarrow H^2_{\mathfrak{m}}(R) \xrightarrow{g} H^2_{\mathfrak{m}}(A/J) \xrightarrow{f} H^3_{\mathfrak{m}}(R) \longrightarrow \cdots .$$

We define $K := \mathrm{Ker}(f)$ and $L := \mathrm{Im}(f)$. This fits in the following short exact sequence

$$0 \longrightarrow K \longrightarrow H^2_{\mathfrak{m}}(A/J) \longrightarrow L \longrightarrow 0. \tag{2.55}$$

Note that the value of the length function of R is equal to that of A. Since $L \subseteq H^3_{\mathfrak{m}}(R)$, we know that $\ell_A(L) < \infty$. Recall that $K = \mathrm{Ker}(f) = \mathrm{Im}(g)$ and that $H^2_{\mathfrak{m}}(A/I) \xrightarrow{g} \mathrm{Im}(g) \to 0$. We combine these along with $\ell_A(H^2_{\mathfrak{m}}(R)) < \infty$, and deduce that $\ell_A(K) < \infty$. By plugging these in (2.55), we observe that $\ell_A(H^2_{\mathfrak{m}}(A/J))$ is finite. In particular, we are in the situation of Lemma 2.6.25. So we have $H^{d-1}_I(A) = 0$.

Next, we present the more explicit examples as follows:

Example 2.6.28. *Let k and $W(k)$ be as before. Let $n \geq 2$ and $d_1, \ldots, d_n \geq 3$. Define:*

a) $A := W(k)[\![X_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq d_i]\!]$,

b) $I := \bigcap_{1 \leq i \leq n} (X_{1,1}, \ldots, X_{1,d_1-1}, X_{2,1}, \ldots, X_{2,d_2-1}, \ldots, X_{i-1,d_{i-1}-1}, X_{i+1,1}, \ldots, X_{n,d_n-1}) \subset A$

c) $f \in I$ *: a non-zero element.*

Let $R := A/(p - f)$. Then the following assertions hold:

i) R *is a ramified regular local ring of dimension* $d := \sum_{i=1}^n d_i$.

ii) $\dim(R/\mathfrak{q}) \geq 3$ *and* $\ell_R(H^2_{\mathfrak{m}}(R/(pR + \mathfrak{q}))) = 0 < \infty$ *for all* $\mathfrak{q} \in \mathrm{Min}(R/IR)$.

iii) $\mathrm{Spec}^\circ(R/IR)$ *is connected.*

iv) $H^{d-1}_I(R) = 0$.

Proof. i) Due to the relation $p - f = 0$, we know the maximal ideal of R is generated by the set $\{X_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq d_i\}$. Then as

$$d \geq \mu(\mathfrak{m}) \geq \dim(R) = \dim(A) - 1 = d,$$

we have $\mu(\mathfrak{m}) = \dim(R)$. In other words, R is regular and of dimension d. Since $p - f = 0$ and $f \in \mathfrak{m}^2$, R is ramified.

ii) Since f is part of monomials appearing in the generating set of I, we obtain an isomorphism $R/(pR + I) \cong R/IR$. Set

$$\mathfrak{q}_i := (X_{1,1}, \dots, X_{1,d_1-1}, X_{2,1}, \dots, X_{2,d_2-1}, \dots, X_{i-1,d_{i-1}-1}, X_{i+1,1}, \dots, X_{n,d_n-1}).$$

Then

$$\operatorname{Min}(R/IR) = \{\mathfrak{q}_i \mid 1 \leq i \leq n\}.$$

Now, we deduce from

$$R/pR + \mathfrak{q}_i \cong R/\mathfrak{q}_i \cong k[\![X_{1,d_1}, X_{2,d_2}, \dots, X_{i-1,d_{i-1}} X_{i,1}, \dots, X_{i,d_i}, X_{i+1,d_{i+1}}, \dots, X_{n,d_n}]\!]$$

that $H^2_{\mathfrak{m}}(R/pR+\mathfrak{q}_i) = 0$ for any $\mathfrak{q}_i \in \operatorname{Min}(R/IR)$, because $\dim(R/pR+\mathfrak{q}_i) = \dim(R/\mathfrak{q}_i) \geq 3$ and it is regular. In particular, it is of finite length.

iii) For any distinct two primes $\mathfrak{q}_i, \mathfrak{q}_j \in \operatorname{Min}(R/I)$, it is obvious that $\mathfrak{q}_i + \mathfrak{q}_j \neq \mathfrak{m}$. This implies that the graph $\Theta_{R/IR}$ is connected, hence $\operatorname{Spec}^\circ(R/IR)$ is connected.

iv) Apply part iii) along with Theorem 2.6.26 to deduce $H^{d-1}_I(R) = 0$. □

Example 2.6.29. *Let k and $W(k)$ be as before. Let $n \geq 2$ and $d_1, \dots, d_n \geq 3$. Define:*

a) $A := W(k)[\![X_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq d_i]\!]$,

b) $I := \bigcap_{1 \leq i \leq n} (X_{1,1}, \dots, X_{1,d_1}, X_{2,1}, \dots, X_{2,d_2}, \dots, X_{i-1,d_{i-1}}, X_{i+1,1}, \dots, X_{n,d_n}) \subset A$

c) $f \in I$ *: a non-zero element.*

Let $R := A/(p - f)$. Then the following assertions hold:

i) R is a ramified regular local ring and of dimension $d := \sum_{i=1}^n d_i$.

ii) $\dim(R/\mathfrak{q}) \geq 3$ *and* $\ell_R(H^2_{\mathfrak{m}}(R/(pR + \mathfrak{q}))) = 0 < \infty$ *for all* $\mathfrak{q} \in \operatorname{Min}(R/IR)$.

iii) $\operatorname{Spec}^\circ(R/IR)$ *is not connected.*

iv) $H^{d-1}_I(R) \neq 0$.

Proof. i) This follows from the similar reason as in part i) of Example 2.6.28.

ii) Since f is part of monomials appearing in the generating set of I, we obtain the isomorphisms $R/(pR + I) \cong R/IR$. Let

$$\mathfrak{q}_i := (X_{1,1}, \dots, X_{1,d_1}, X_{2,1}, \dots, X_{2,d_2}, \dots, X_{i-1,d_{i-1}}, X_{i+1,1}, \dots, X_{n,d_n}),$$

and recall that

$$\operatorname{Min}(R/IR) = \{\mathfrak{q}_i \mid 1 \leq i \leq n\}.$$

Then we deduce from

$$R/(pR + \mathfrak{q}_i) \cong R/\mathfrak{q}_i \cong k[\![X_{i,1}, \dots, X_{i,d_i}]\!]$$

that $H^2_{\mathfrak{m}}(R/pR+\mathfrak{q}_i)=0$ for any $\mathfrak{q}_i \in \mathrm{Min}(R/IR)$, because $\dim(R/pR+\mathfrak{q}_i)\geq 3$ and it is regular. In particular, it is of finite length. Moreover, we obtain $\dim(R/\mathfrak{q}_i)\geq 3$.

iii) For any distinct two primes $\mathfrak{q}_i,\mathfrak{q}_j \in \mathrm{Min}(R/IR)$, it is obvious that $\mathfrak{q}_i+\mathfrak{q}_j=\mathfrak{m}$. This implies that the graph $\Theta_{R/IR}$ is not connected, hence $\mathrm{Spec}^\circ(R/IR)$ is not connected.

iv) Apply part iii) along with Theorem 2.6.26 to deduce $H^{d-1}_I(R)\neq 0$. □

Remark 2.6.30. Adopt the notation of Example 2.6.29. Suppose that $n=2$, $d_1=d_2=4$ and $f=x_{1i}x_{2j}$ for some $1\leq i,j\leq 4$.

i) We claim that $H^{d-1}_I(R)$ is the injective envelop of k. In particular, $H^{d-1}_I(R)\neq 0$.

ii) Let R be an analytically unramified quasi-Gorenstein local ring of dimension d together with an ideal I such that $\mathrm{Spec}^\circ(R/I)$ is not connected. Then we claim that $\mathrm{Ann}_R(H^{d-1}_I(R))=0$. In particular, $H^{d-1}_I(R)\neq 0$.[8]

Proof. i): Recall from the relation $p-x_iy_j=0$ that $\mathfrak{q}_1+\mathfrak{q}_2=\mathfrak{m}$. Also as R is regular, we have $H^d_{\mathfrak{m}}(R)\cong E_R(k)$. By (2.53), we deduce the following exact sequence

$$0=H^7_{\mathfrak{q}_1}(R)\oplus H^7_{\mathfrak{q}_2}(R)\longrightarrow H^7_I(R)\longrightarrow H^8_{\mathfrak{m}}(R)\longrightarrow H^8_{\mathfrak{q}_1}(R)\oplus H^8_{\mathfrak{q}_2}(R)=0,$$

and the desired claim follows.

ii): There are two ideals J_1,J_2 of R such that $\sqrt{J_1+J_2}=\mathfrak{m}$ and $\sqrt{J_1\cap J_2}=\sqrt{I}$, but $\sqrt{J_1}$ and $\sqrt{J_2}$ are not equal to both $\mathfrak{m}$ and $\sqrt{I}$. We apply the Hartshorne-Lichtenbaum vanishing theorem along with (2.53) to get

$$\cdots\longrightarrow H^{d-1}_I(R)\longrightarrow H^d_{\mathfrak{m}}(R)\longrightarrow H^d_{J_1}(R)\oplus H^d_{J_2}(R)=0.$$

Recall that $H^d_{\mathfrak{m}}(R)\cong E_R(k)$. If $r\in \mathrm{Ann}(H^{d-1}_I(R))$, then r annihilates any homomorphic image of $H^{d-1}_I(R)$, e.g., $r\in \mathrm{Ann}(E_R(k))=0$. □

Example 2.6.31. *Let k and $W(k)$ be as before. Define $R := W(k)[[X_i \mid 1\leq i\leq 6]]$, $I := (X_1,X_2)\cap(X_3,X_4)\cap(X_5,X_6)$ and $A := R/I$. Then the following assertions hold:*

i) R is an unramified regular local ring and of dimension 7.

ii) $\mathrm{Spec}^\circ(A)$ is connected.

iii) $H^6_I(R)=0$.

Proof. i): This is easy.

ii): This follows from Hartshorne's criteria, by showing that $\mathrm{depth}(A)\neq 1$. Instead, we show it more directly. Indeed, since $\mathrm{Min}(A)$ is equal to $\{(x_1,x_2),(x_3,x_4),(x_5,x_6)\}$ and for example $(x_1,x_2)+(x_3,x_4)$ is not primary to the maximal ideal, Θ_A is connected. In view of Lemma 2.6.20, $\mathrm{Spec}^\circ(A)$ is connected.

iii): Apply ii) along with Zhang's result. □

[8] This part is valid without any use of quasi-Gorenstein assumption. Indeed, apply the Grothendieck'sW non-vanishing theorem along with the displayed exact sequence.

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