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Stability and Dimension in Feedback Systems:
A Differential Lyapunov Framework

by

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Abstract

This thesis is devoted to the development of a differential Lyapunov framework for stability analysis and dimension estimates. The main motivation of this work comes from a theoretical gap between studies in the systems and control community and those in the dynamical systems community. The thesis can be divided into three parts.

In the first part, we study the feedback system consisting of a linear time-invariant system and a memoryless nonlinearity. A differential framework is incorporated to reduce conservativeness of the classical results regarding the absolute stability problem. We suggested the use of a differential Lyapunov function of the Leonov form, which is constructed from a quadratic form and a Lyapunov-like function.

The second part focuses on the concept of Lyapunov dimension. We derive upper and exact estimates of Lyapunov dimension through a differential Lyapunov framework. In particular, our result provides a suitable characterization of the validity of the Eden conjecture on local estimation of Lyapunov dimension. As an example, the classical Lorenz attractor is investigated to obtain the exact Lyapunov dimension formula.

The spirit of the third part is in the combination of the notions of volume contracting systems and differentially dissipative systems. A general framework for the analysis of attractor dimension for interconnected systems is proposed. Based on Smith's method, we also present a computationally efficient way to estimate the attractor dimension of a large-scale system.

Contents

Mathematical Notations

$\mathbb{N}$	the set of all natural numbers (without 0)
$\mathbb{Z}$	the set of all integers
$\mathbb{R}$	the set of all real numbers
$\mathbb{C}$	the set of all complex numbers
$\mathbb{F}_+$	the set of all elements with nonnegative real parts of $\mathbb{F}$
$\mathbb{R}^n$	the n-dimensional Euclidean space
$\mathbb{R}^{n\times m}$	the set of all $n \times m$ real matrices
$\mathcal{S}_n$	the set of all $n \times n$ symmetric real matrices
$\mathcal{L}_2(I, \mathbb{R}^n)$	the Hilbert space of all square integrable functions from an interval $I \subset \mathbb{R}$ to $\mathbb{R}^n$
$\mathcal{L}_2^{\mathrm{loc}}(I, \mathbb{R}^n)$	the linear space of all locally square integrable functions from an interval $I \subset \mathbb{R}$ to $\mathbb{R}^n$
A^T	the transpose of a matrix A
A^H	the conjugate transpose of a matrix A
$\det A$	the determinant of a square matrix A
$\operatorname{tr} A$	the trace of a square matrix A
$\sigma_i(A)$	the i-th largest singular value of a square matrix A
$\eta_i(A)$	the i-th largest eigenvalue of the Hermitian part of a square matrix A

$\lambda_{\max}(A)$	the largest eigenvalue of a Hermitian matrix A
$\lambda_{\min}(A)$	the smallest eigenvalue of a Hermitian matrix A
$\kappa(A)$	the condition number of a Hermitian matrix A
$A \oplus B$	the direct sum of matrices A and B
$\dot{x}$	the time derivative of a function x
$\nabla f(x)$	the gradient of a scalar-valued function f at x
$\mathrm{D}f(x)$	the Jacobian matrix of a vector-valued function f at x
$\partial_x f(x, y)$	the partial derivative of a function f with respect to x
$\mathfrak{G}(f)$	the graph of a map f

Chapter 1

Introduction

Lyapunov's first and second methods, or the theories of Lyapunov exponents and Lyapunov functions, are powerful mathematical tools to study the asymptotic behaviors of dynamical systems. There are currently much broader interpretations of these tools than originally thought. Historically, Lyapunov exponents have been examined in the dynamical systems community [FK60, Ose68], whereas Lyapunov functions have been employed in the systems and control community [KB60a, KB60b]. For this reason, there is a large gap between theoretical approaches in these two research communities. However, only the different between dynamical system theory and control theory is whether some dynamical variables can be manipulated or not. Therefore, these two aspects should be integrated at a quite general level [CK00].

In this thesis, we are interested in *feedback systems*, which typically appear in the context of control theory [Wil71, DV75]. A key difference from the existing works is that we follow a differential approach, that is, the investigation of infinitesimal changes in the variables of interest. In this context, Lyapunov function techniques are utilized to estimate Lyapunov exponents (see [Lew80, Woj85, KB94]). We follow a similar approach to study interconnections of a number of subsystems, where input/output properties of each component play a crucial role in the analysis. This line of research was inspired by the recent papers [FS14, MVFS18, FS19], where the notions of differential Lyapunov and storage functions were introduced. The basic theory is known as *differential dissipativity theory* after the work of Jan Willems.

Our focus is mainly on the general asymptotic behaviors of dynamical systems. Although there are many concepts which characterize the possible asymptotic behaviors, we will investigate the notion of *Lyapunov dimension* [Kuz16]. It is widely known that

Lyapunov dimension is an upper bound of the Hausdorff and box dimensions of an invariant set. Another important property of Lyapunov dimension is that it is an invariant of a dynamical system with respect to diffeomorphisms. Moreover, some relations with Lyapunov exponents and topological entropy can be found in the literature. Some decades ago, Lyapunov's second method was extended to the analysis of Lyapunov dimension [LB92,PN00]. However, how to estimate the value of Lyapunov dimension from input/output properties of the subsystems which form components of a feedback system is still unclear. This problem is important to integrate the feedback theory pioneered by Norbert Wiener and the complexity theory pioneered by Andrey Kolmogorov. Of course these two theories are closely related to each other, but our aim is to combine the theoretical tools for studying feedback and complexity.

In what follows, some motivations to study dimension-like characteristics of feedback systems are explained. We then describe the theoretical background on the theories of Lyapunov exponents and Lyapunov functions. Finally, we summarize the contents and contributions of the thesis.

1.1 Motivations

We describe two motivations to study dimension-like characteristics in light of control systems. The first one comes from traditional control problems, whereas the second one comes from nontraditional control problems. In both cases, it is difficult to study fundamental issues within the usual framework of stability and stabilization.

1.1.1 Control with Limited Information

The primal objective of feedback control is to stabilize an unstable system. Hence, the Youla–Kučera parametrization of stabilizing controllers is one of the most useful results in control theory. On the other hand, modern cyber-physical systems consist of the interactions between physical systems and computer systems. As represented in Fig. 1.1, cyber-physical systems have an interface between analog and digital worlds, which approximates continuous quantities with discrete ones. Because the errors due to such approximations are different from traditional stochastic processes, the effects of A/D and D/A conversions bring new challenges in the design of feedback controllers.

One of the obstacles in the design of feedback controllers using information of finite

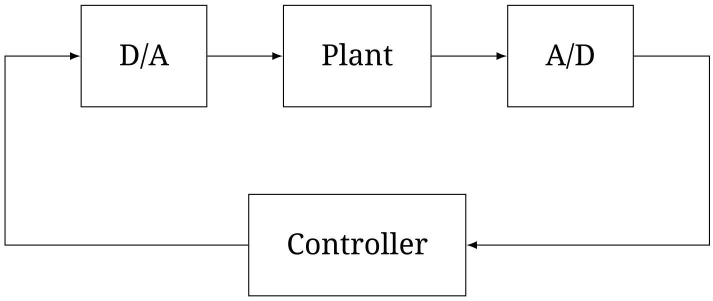

Figure 1.1: Quantized control system

precision is the difficulty of asymptotic stabilization [Del90]. As pointed out by Kalman in [Kal56], the effects of quantization prevent asymptotic stabilization and make control systems chaotic. Although several techniques for feedback stabilization with quantized measurements have been proposed [BL00, EM01], the fundamental problem described in [Del90] does not seem to be solved. In what sense can we stabilize an unstable system with quantized feedback? We need to answer this question from both qualitative and quantitative points of view. Nevertheless, there is still a lack of theoretical understanding for assessing chaotic behaviors of quantized control systems.

The aforementioned fact motivated us to develop a theoretical framework to investigate dimension-like characteristics of feedback systems. Although we do not address the discontinuity aspect of quantization, chaotic behaviors of feedback systems will be studied by extending absolute stability theory. It is worth mentioning that nonlinear effects of quantizers can be analyzed based on absolute stability theory. This point was investigated in [FX05], where logarithmic quantizers were employed to guarantee asymptotic stabilization. We will see that absolute stability theory provides a powerful framework for studying chaotic behaviors. To consider discontinuous nonlinearities, we need further developments with more advanced theories from differential inclusions such as those in [YLG04].

On the other hand, topological entropy has recently attracted attentions in the context of networked control systems as it provides information-theoretic perspectives in control over communication channels [Sav06]. Recently, novel concepts of topological feedback entropy and invariance entropy were introduced in [NEMM04] and [KC09], respectively. These concepts are defined based on invariant sets under some control functions and provide the minimum data rate required for feedback stabilization. It is also known that dimension-like characteristics are related to the concepts of control entropy (see

[Kaw11]). Possibly, such a relation comes from the fact that topological entropy can be interpreted as a special case of dimension-like characteristics [Pes97]. However, explicit relations between dimension and entropy in the context of feedback control remain an open question.

1.1.2 Reduction of Complex Dynamics

As Haken's slaving principle indicates, most variables involving a dynamical system are in general dominated by few variables. Such reduction of complex dynamics appears in many problems such as averaging methods, singular perturbation methods, phase reduction methods, and Chapman–Enskog theory. It is widely known that an infinite-dimensional system can possess an attractor whose Hausdorff dimension is finite [MP76]. Dimension-like characteristics including Hausdorff dimension are important when we embed such a complex set to an Euclidean space, where the degrees of freedom of the underlying dynamics are equal to the dimension of the space. The basic embedding method was developed by Mañé in [Mn81], where it was shown that if the box dimension of a set M is less than or equal to d, then almost all projections of rank $2d + 1$ are injective on M (see also [BAEFN93] for the modification of Mañé's original result). This method is called the Mañé projection theorem. Another approach is to use the Takens embedding theorem, by which the attractor is reconstructed from experimentally measured data [Tak81, SYC91]. For further topics on attractors and embeddings, we refer to [Rob11, Zel22].

Reduction of complex dynamics has also been studied in the theory of Navier–Stokes equations and turbulence [CFT85, FMRT01]. In this context, the concepts of determining modes and determining nodes were introduced in [FMTT83, FT84]. These concepts are useful to identify a specific choice of dominant variables differently from the embedding theorems mentioned above. The basic idea can be explained as follows: If we consider two different solutions and if dominant Fourier modes or dominant measurements have the same asymptotic behavior, then the other Fourier modes or unmeasured variables must converge to the same behavior. Thus, the asymptotic behaviors are uniquely determined by such dominant modes or points in the space. Relations between the determining properties and attractor dimensions are of special interest as the notion of determining dimension was defined in [Zel22].

In recent years, determining nodes were studied in the context of ordinary differ-

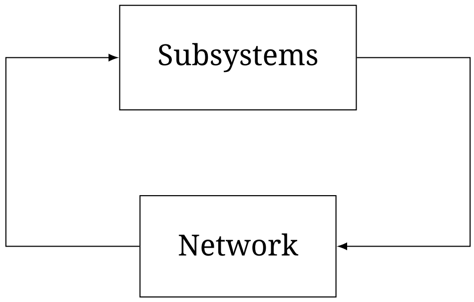

Figure 1.2: Networked system

ential equations [FMKS13, MFKS13], where it was shown that a certain graph-theoretic notion is sufficient to specify determining nodes. Interestingly, the results in those papers also provide a methodology to study controllability in the biological sense. Compared with controllability in the sense of Kalman [Kal60], controlling the internal state to an arbitrary target attractor is sufficient for cell fate reprogramming [MS11]. From this new perspective, controllability with respect to attractors have been studied for the purpose of control of biological systems [ZA15, ZYA17]. The motivation to combine dimension theory and interconnection theory actually comes from these works. Because an interconnected system can be considered as a feedback system as shown in Fig. 1.2, this problem can be addressed in the framework developed in this thesis.

1.2 Theoretical Background

In 1892, Lyapunov defined the concepts of stability and asymptotic stability and developed fundamental tools for stability analysis [Lya92]. These tools include Lyapunov exponents and Lyapunov functions. This section is dedicated to surveys of some selected topics on Lyapunov exponents and Lyapunov functions. Our emphasis is on stability of feedback systems and dimension/entropy concepts in dynamical systems. We also discuss a certain relation between the analyses via these two tools.

1.2.1 Theory of Lyapunov Exponents

In his original work, Lyapunov considered asymptotic stability of a solution to a nonlinear differential equation. He showed that if all Lyapunov exponents associated with the variational equation are negative and the so-called regularity condition is fulfilled, then the original solution is asymptotically stable. Roughly speaking, a Lyapunov exponent represents the exponential growth (decay) rate of a solution to the variational equation in a particular direction. If all Lyapunov exponents are negative, then any small perturbation in the initial state is expected to approach zero as time tends to infinity. However, because the decay rate is in general nonuniform with respect to the initial time, the property of asymptotic stability may be changed under a small nonlinear perturbation of the variational equation; hence, the regularity condition cannot be removed. The reversal of the stability property under a nonlinear perturbation is called the Perron effect after his discovery of a counterexample (see [Leo08]).

The stability analysis mentioned above is based on the sign of the largest Lyapunov exponent. By considering several Lyapunov exponents, we can characterize more complex behaviors of dynamical systems. In this regard, a certain splitting of the tangent bundle is one of the starting points. The regularity condition also plays a crucial role for an appropriate decomposition of a tangent space. A remarkable result is the multiplicative ergodic theorem by Oseledets [Ose68], which generalizes the Furstenberg–Kesten theorem involving the existence of the largest and smallest Lyapunov exponents [FK60]. The multiplicative ergodic theorem claims that the regularity condition is fulfilled almost everywhere with respect to an invariant measure. In particular, if the invariant measure is ergodic, then the Lyapunov exponents are constant almost everywhere. The work of Oseledets stimulated further developments of nonuniform hyperbolicity theory and its relation with ergodic theory (see [BP07]).

After the discovery of chaotic phenomena by Ueda and Lorenz in 1960s, chaos became an attractive topic for many researchers[1]. A key property of chaos is the sensitivity with respect to the initial condition. Because of this property, it is difficult to predict the long-term behaviors of a chaotic dynamical system. To understand chaotic phenomena, an appropriate characterization of complexity of chaotic attractors is an important task. The notions of Hausdorff dimension and box dimension are useful to measure the geometric complexity of a set. Nevertheless, these non-integral dimensions are hopelessly difficult to calculate. Motivated by this fact, Kaplan and Yorke introduced the concept of Lyapunov dimension to study chaotic attractors through Lyapunov exponents [KY79]. At the same time, a fundamental result for estimating the Hausdorff dimension of a compact invariant set was established by Douady and Oesterlé [DO80]. The Lyapunov dimension is currently defined in terms of finite-time Lyapunov exponents, and it is equivalent

[1]The term ”chaos” was coined by Li and Yorke in [LY75].

to the estimate by Douady and Oesterlé (see [Kuz16]). Roughly speaking, the notions of dimension for dynamical systems represent the *degrees of freedom* to capture the asymptotic behaviors. Dimension theory for dynamical systems is now an active research area of mathematics [Pes97, BLR05, Bar11, KR21].

On the other hand, the notion of entropy in dynamical systems was introduced by Kolmogorov [Kol58] and was improved by Sinai [Sin59]. The Kolmogorov–Sinai (KS) entropy is an analogue of the entropy concepts by Boltzmann in statistical thermodynamics and by Shannon in information theory[2]. Several years later, Adler, Konheim, and McAndrew defined the topological entropy for topological dynamics based on the idea of the KS entropy [AKM65]. The definition of topological entropy based on a certain metric considered by Bowen [Bow71] and Dinaburg [Din70] is often used in the literature. Roughly speaking, the notions of entropy for dynamical systems represent the *amount of information* which is required to understand the asymptotic behaviors. The KS entropy and the topological entropy are connected to each other by the so-called variational principle. In particular, the variational principle indicates that the topological entropy is equal to the supremum of the KS entropies over all invariant probability measures. Because a dynamical system may possess many invariant measures, the variational principle tells us which measure has a nice property. Relations between dynamical entropy and Lyapunov exponents have also been studied in the literature. We refer to the survey [Kat07] and the books [Wal82, Dow11] for more details.

Our special interest is in relations between dimension, entropy, and Lyapunov exponents (see [ER85] for a related review). This topic has been well studied in, e.g., [Led81, Man81, You82, Thi92, Fra98, Gel05]. On the other hand, these works are devoted to autonomous systems, and the connection to control theory has not mostly been clarified. Another interesting connection of dimension and entropy with control theory can be found in the context of adaptation and learning [Zam79]. The notions of dimension and entropy in this line of research are slightly different from the aforementioned ones but originate in Kolmogorov's work. Although robust control of uncertain systems has been deeply studied in the literature, there are many unsolved problems on dimension and entropy (see [Zam98b, Zam98a]).

[2]The term “entropy” was originally coined by Clausius in 1850.

1.2.2 Theory of Lyapunov Functions

The initial contributions to stability analysis of feedback systems with aim of engineering applications were made by Maxwell [Max67] and Vyshnegradskii [Vys76]. In particular, Vyshnegradskii studied global stability through local stability although the rigorous investigation remained unsolved. In 1944, Lur'e and Postnikov published a seminal paper on global stability of feedback systems [LP44]. The model adopted in that paper consists of a linear time-invariant system and a memoryless nonlinearity. Moreover, the use of a Lyapunov function of the special type "a quadratic form plus an integral of the nonlinearity" was suggested. Nowadays, such a function is called a Lyapunov function of the Lur'e–Postnikov form. The existence of a Lyapunov function of the Lur'e–Postnikov form and its relation with the absolute stability problem, which was formulated by Aizerman in [Aiz49], were central issues in stability theory from the 1940s to the 1950s.

Around 1960, the absolute stability problem was essentially solved by Popov based on the frequency-domain method [Pop61]. The so-called Popov criterion indicates that a frequency condition on the transfer function of the linear block is equivalent to the existence of a Lyapunov function of the Lur'e–Postnikov form. Stimulated by Popov's work, Yakubovich and Kalman derived their celebrated results on the equivalence between frequency-domain inequalities and linear matrix inequalities [Yak62, Kal63]. These results are nowadays known as the Kalman–Yakubovich–Popov (KYP) lemma. On the other hand, to address time-varying nonlinearities, several authors derived the so-called circle criterion [Bon63, Yak63, Ros63], where the investigated class of Lyapunov functions is restricted to quadratic forms compared with Popov's work. The circle criterion is also called the Rosenwasser–Yakubovich–Bongiorno criterion.

The method of Lyapunov functions for global stability was further studied by many researchers. For example, the Barbashin–Krasovskii theorem [BK52] and the LaSalle invariance principle [LaS60] are applicable in general situations. The concepts of radially unbounded functions and invariant sets play important roles in these results. However, there is no general method to construct a particular Lyapunov function for global stability. Another important tool is the partial stability theorem, which unifies classical stability and uniform stability results (see [HC08, Chapter 4]). Partial stability is of special importance in this thesis. Also, the converse Lyapunov theorem is useful to estimate a gap between necessity and sufficiency for global stability (see [Kel15] for a recent survey).

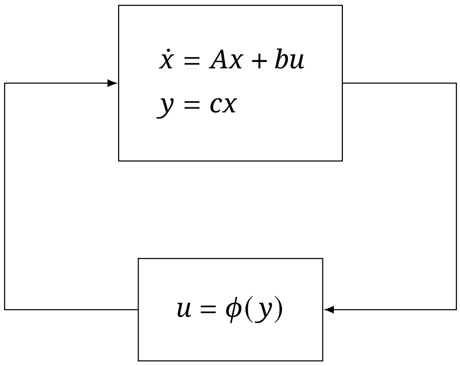

Figure 1.3: Lur'e system

In the early 1960s, Zames and Sandberg studied nonlinear systems from a functional analysis perspective, where the concept of extended spaces plays a fundamental role [Zam60, San64]. The so-called small-gain and passivity theorems are useful results for studying interconnections of open systems or operators (see [vdS17]). More recently, integral quadratic constraints (IQCs) [MR97, Sei15] and Zames–Falb multipliers [ZF68, CTH16] were introduced by extending the Popov and circle criteria. Another method for stability analysis of feedback systems is to utilize graph separations [Saf80, Tee96]. The aforementioned results mostly concentrate on input/output systems. Relations between Lyapunov stability and input/output theory were investigated by Willems [Wil72a, Wil72b], Hill and Moylan [HM80], and others. In particular, Lyapunov functions techniques were extended to those of storage functions and supply rates. The comprehensive theory of dissipative systems by means of Willems plays a central role in this thesis.

1.2.3 Linear Analysis of Nonlinear Systems

In 1877, Vyshnegradskii studied a three-dimensional model of the Watt governor and derived a sufficient condition for local stability based on its linear approximation [Vys77]. Besides, Vyshnegradskii conjectured that the condition is also sufficient for global stability. It was the first result that explores the equivalence between local stability and global stability. Vyshnegradskii's conjecture was positively solved by Andronov and Mayer in 1945 (see the historical survey [KLY+20]). Here, we review some important problems involving local analysis of nonlinear systems.

In general, local stability does not imply global stability. The so-called Aizerman and Kalman conjectures claim that global stability can be guaranteed by some local properties. Consider a feedback system in Fig. 1.3, which was essentially studied by Lur'e

[Lur63]. The closed-loop dynamics are described by

$$\dot{x} = Ax + b\phi(cx),$$

where $A \in \mathbb{R}^{n\times n}$, $b, c^\top \in \mathbb{R}^n$, and $\phi\colon \mathbb{R} \to \mathbb{R}$ is such that $\phi(0) = 0$. This feedback system is said to be *absolutely stable* if it is globally asymptotically stable for every nonlinearity ϕ which belongs to a given class. There are two important classes of nonlinearities. On one hand, Aizerman considered the class of nonlinearities ϕ satisfying the following sector condition [Aiz49]:

$$k_1 \le \frac{\phi(\sigma)}{\sigma} \le k_2, \quad \forall \sigma \neq 0, \tag{1.1}$$

where $k_1, k_2 > 0$ (see Fig. 1.4). Aizerman conjectured that the feedback system is absolutely stable with respect to the nonlinearity class given in (1.1) if $A + kbc$ is Hurwitz for all $k \in [k_1, k_2]$. On the other hand, Kalman considered the class of nonlinearities ϕ such that the following condition is satisfied [Kal57]:

$$k_1 \le \phi'(\sigma) \le k_2, \quad \forall \sigma \in \mathbb{R}, \tag{1.2}$$

where $k_1, k_2 > 0$. Kalman conjectured that the feedback system is absolutely stable with respect to the nonlinearity class given in (1.2) if $A + kbc$ is Hurwitz for all $k \in [k_1, k_2]$. The Aizerman conjecture is false even if $n \le 2$, while the Kalman conjecture is true if $n \le 3$. When $n \ge 4$, there is a counterexample of the Kalman conjecture. Identifying the class of feedback systems which satisfies the Aizerman or Kalman conjecture is now an active research topic (see [LKB10]).

The so-called Markus–Yamabe conjecture corresponds to the case where feedback is not present. Consider a system of nonlinear differential equations

$$\dot{x} = f(x),$$

where $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a smooth function. The Markus–Yamabe conjecture states that if the Jacobian matrix $\mathrm{D}f(x)$ is Hurwitz uniformly in $x \in \mathbb{R}^n$, then the above nonlinear system is globally asymptotically stable. This conjecture is not true in general, and it is important to justify local analysis of nonlinear systems. Recently, since the landmark paper [LS98], contraction analysis of nonlinear systems has attracted much attention in the systems and control community (see the surveys [AS14, TCS21, GHK23]). There, it was shown that every two solutions converge to each other if the symmetric part of the Jacobian matrix is uniformly negative definite.

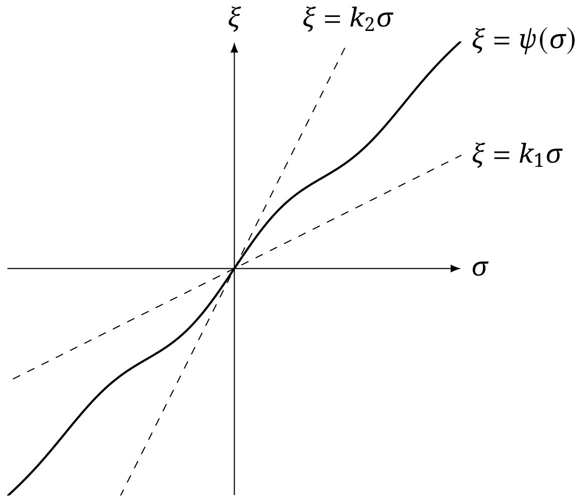

Figure 1.4: Sector condition

This idea was further studied in [FS14], where a framework of Lyapunov functions for contraction analysis was developed. The essential idea is to investigate partial stability of the following prolonged system:

$$\begin{aligned} \dot{x} &= f(x), \\ \delta\dot{x} &= \mathrm{D}f(x)\delta x. \end{aligned}$$

The analysis of partial stability with respect to δx is thus converted to the analysis of (the family of) linear time-varying systems. Another approach of linearization without approximation is to use the Koopman operator [MM16]. Compared with the differential framework, the Koopman operator converts a finite-dimensional nonlinear system to an infinite-dimensional linear system. An advantage of the differential approach is that we can employ Lyapunov function techniques. More precisely, a differential Lyapunov function can be employed to estimate a Lyapunov exponent (in terms of singular values) [KR21, Sect. 2.6]. The basic idea in the proof is to use the cone invariance via a quadratic form and the Fischer–Courant theorem.

Moreover, the linearization approach is useful to study not only stationary solutions but also more general attractors. The Poincaré–Bendixson theorem is the most fundamental result for the analysis of convergence to simple attractors. Although the original result is restricted to dynamical systems on planes, multidimensional generalizations were developed in [LM93, LM95b, Smi87]. On the other hand, the Andronov–Vitt theorem provides sufficient conditions for orbital stability (Poincaré stability) in terms of the two largest Lyapunov exponents [Leo06]. Beyond stationary and periodic orbits, it is not easy to characterize chaotic trajectories as there is no pertinent definition of chaos. For example, Lyapunov stability and orbital stability are inadequate for the study of chaos, and the notion of Zhukovskii stability is more appropriate [Leo08].

1.3 Contributions of the Thesis

As mentioned above, the theories of Lyapunov exponents and Lyapunov functions provide different techniques for studying the asymptotic behaviors of dynamical systems. From a historical perspective, Lyapunov exponents have been studied to characterize weak properties such as hyperbolicity and ergodicity. On the other hand, Lyapunov functions have been studied to consider strong properties such as stability and stabilization. In this thesis, we employ differential Lyapunov function techniques to investigate Lyapunov dimension, which is more general than stability. Stability analysis is a special case of dimension estimates in the sense that global stability can be interpreted as the convergence to a 0-dimensional attractor.

Now, we summarize the key contributions of this thesis more explicitly. The results in this thesis are related to some classical and modern works on dynamical systems. We explain the contributions from three aspects.

Absolute Stability Theory

In Chapter 3, we investigate a feedback system consisting of a linear time-invariant system and a memoryless nonlinearity. Such a feedback system is called a Lur'e system [Lur63]. The first contribution of the thesis is to provide new sufficient conditions for global stability of a Lur'e system. Although absolute stability of Lur'e systems has a long history in control theory [AG64, NT73, YLG04], our approach is different from such traditional studies. In particular, we focus on the derivative of the nonlinearity. Note that under the smoothness assumption, we can integrate the derivative of the nonlinearity. It is also shown that conservativeness of the sufficient conditions for global stability can be dramatically reduced by the proposed framework. Our method is useful to study the gap between necessity and sufficiency for global stability of a Lur'e system.

More specifically, we suggest the use of what we call a differential Lyapunov function of the Leonov form [Leo91b]. A differential Lyapunov function of the Leonov form consists of two separated functions: a quadratic form and the exponential of a Lyapunov-like function. The use of such a Lyapunov-like function provides novel flexibility. Although there is no general method to obtain a Lyapunov-like function, the converse Lyapunov theorem guarantees the existence of a radially unbounded function whose total derivative is negative except for the equilibrium. Moreover, it is allowed that the linear time-invariant system and the memoryless nonlinearity have different input/output properties. This is different from the classical approach such as IQCs [MR97, Sei15], which require the linear block and the nonlinear block to have the same properties.

Dimension Theory

In Chapter 4, we study a Lur'e system with unstable dynamics. To estimate the Lyapunov dimension of a Lur'e system, we generalize the method developed in Chapter 3. Our framework can be considered as an analogue of Leonov's method in dimension theory [LB92, Kuz16]. Although the frequency-domain method for dimension estimates has been studied [LPS96], the main contribution is to characterize the validity of the Eden conjecture on local estimation of Lyapunov dimension [Ede89a, Ede89b, Ede90]. A key point of our result is that a suitable characterization of the class of nonlinearities for which local and global Lyapunov dimensions coincide can be obtained.

Furthermore, our framework is related to recent generalizations of contraction analysis [WKM22, WPMS22, OOM23]. Compared with these works, we consider Riemannian metrics rather than vector norms. In particular, the metric adopted in this thesis is constructed from a Lyapunov-like function. This makes a connection with the classical Lyapunov stability theory.

Dissipativity Theory

The objective of Chapter 5 is to formalize a generalized framework of dimension theory for feedback systems. A key concept is the differential dissipativity theory, which was recently introduced in [FS13, van13]. Differently from the results in [MVFS18, FS19], where differential dissipativity was generalized with nondegenerate metric tensors, our aim is to consider volume contracting dynamics. In particular, we define the concept of volume dissipative systems, which combines the volume contraction properties originating in the classical Liouville theorem and the dissipation inequalities originating in Willems' celebrated results [Wil72a, Wil72b].

In addition, we generalize Smith's method for dimension estimates [Smi86], which is based on the dissipation inequality with reversed direction. The developed framework is useful in the sense that large-scale systems can be analyzed in a scalable way. Our result is closely related to entropy estimates for interconnected systems in [Lib21, TZ22]. An energy-like interpretation of dimension and entropy would be important to understand complexity of open systems.

1.4 Outline of the Thesis

The rest of this thesis is organized as follows.

In Chapter 2, we highlight central ideas of a differential Lyapunov framework. First, the notions of nonincremental and incremental stability are recalled. We then describe contraction analysis via a differential Lyapunov function. Some operator-theoretic perspectives are reviewed, and a characterization of differentially dissipative systems is introduced.

In Chapter 3, we investigate the feedback system consisting of a linear time-invariant system and a memoryless nonlinearity. We show that a differential framework is an effective tool to derive less conservative global stability conditions. In this chapter, we propose a new class of differential Lyapunov functions which we call a Lyapunov function of the Leonov form.

In Chapter 4, we study Lyapunov dimension of the feedback system considered in Chapter 3. We start with introducing the definition of Lyapunov dimension and its properties. The Eden conjecture on local estimation of Lyapunov dimension is also reviewed. We show a certain characterization for the validity of the Eden conjecture. The derivation of the exact Lyapunov dimension formula of a Lorenz system is also presented.

In Chapter 5, we study a more general setting. An interconnection of two or more than two nonlinear systems is investigated. We introduce the notions of volume contracting systems and volume dissipative systems. These concepts are useful to analyze dimension-like characteristics of interconnected systems. Also, a computationally efficient method for dimension estimates is proposed.

Finally, in Chapter 6, we summarize the thesis and describe future directions. Appendix A provides some mathematical background.

Chapter 2

A Differential Lyapunov Framework

In this chapter, two notions of stability and dissipativity are reviewed: one is a nonincremental concept and the other is an incremental concept. We also introduce the main tools to study incremental notions of stability and dissipativity in this thesis. A particular emphasis is on the differential Lyapunov framework developed by Forni and Sepulchre [FS14]. The underlying idea was originally brought to the control community by Lohmiller and Slotine [LS98].

Notations: Throughout this chapter, the following notations are employed [Kha02].

Definition 2.1. A continuous function $\alpha\colon \mathbb{R}_+ \to \mathbb{R}_+$ is of class-$\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. If, in addition, $\lim_{t\to\infty} \alpha(t) = \infty$, then it is of class-$\mathcal{K}_\infty$.

Definition 2.2. A continuous function $\sigma\colon \mathbb{R}_+ \to \mathbb{R}_+$ is of class-$\mathcal{L}$ if it is strictly decreasing and $\lim_{t\to\infty} \beta(t) = 0$.

Definition 2.3. A continuous function $\beta\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of class-$\mathcal{KL}$ if the function $t \mapsto \beta(t, s)$ is of class-$\mathcal{K}$ and the function $s \mapsto \beta(t, s)$ is of class-$\mathcal{L}$.

In what follows, we omit the domains and codomains of these functions as they are clear from the context.

2.1 Stability of Dynamical Systems

Consider an ordinary differential equation

$$\dot{x} = f(x), \tag{2.1}$$

where $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a C^1-function. Assume that f is forward complete, i.e., for any $x_0 \in \mathbb{R}^n$, there is a unique solution $t \mapsto \phi(t, x_0)$ to (2.1) such that $\phi(0, x_0) = x_0$ defined on $\mathbb{R}_+$. The differential equation (2.1) thus defines the dynamical system $\{\varphi^t\}_{t\in\mathbb{R}_+}$ by setting $\varphi^t(x) := \phi(t, x)$ for $t \geq 0$ and $x \in \mathbb{R}^n$. In what follows, we study global stability of the dynamical system $\{\varphi^t\}_{t\in\mathbb{R}_+}$ generated by (2.1).

2.1.1 On Notions of Global Stability

The classical definition of global stability is as follows:

Definition 2.4. An equilibrium point $p \in \mathbb{R}^n$ of the dynamical system $\{\varphi^t\}_{t\in\mathbb{R}_+}$ is said to be

(i) *globally asymptotically stable* if there exists a class-$\mathcal{KL}$ function β such that

$$\|\varphi^t(x) - p\| \leq \beta(\|x - p\|, t)$$

for all $t \geq 0$ and all $x \in \mathbb{R}^n$;

(ii) *globally exponentially stable* if there exist $K \geq 1$ and $\lambda > 0$ such that

$$\|\varphi^t(x) - p\| \leq K\mathrm{e}^{-\lambda t}\|x - p\|$$

for all $t \geq 0$ and all $x \in \mathbb{R}^n$.

Recently, the notion of incremental stability has attracted much attention in the systems and control community [Ang02, PPvdWN04, Ang09, RvdWM13].

Definition 2.5. The dynamical system $\{\varphi^t\}_{t\in\mathbb{R}_+}$ is said to be

(i) *incrementally asymptotically stable* if there exists a class-$\mathcal{KL}$ function β such that

$$\|\varphi^t(x) - \varphi^t(y)\| \leq \beta(\|x - y\|, t)$$

for all $t \geq 0$ and all $x, y \in \mathbb{R}^n$;

(ii) *incrementally exponentially stable* if there exist $K \geq 1$ and $\lambda > 0$ such that

$$\|\varphi^t(x) - \varphi^t(y)\| \leq K\mathrm{e}^{-\lambda t}\|x - y\|$$

for all $t \geq 0$ and all $x, y \in \mathbb{R}^n$.

Incremental stability possesses some useful properties especially when the system is nonautonomous (see [RdBS10]). For an overview of recent developments, we refer to [AS14, TCS21, GHK23, Bul23].

2.1.2 Lyapunov's First Method

The following result is closely related to Lyapunov's first method [Leo98, Leo08].

Theorem 2.1. Suppose that there exists a continuous function $\alpha\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\sup_{x\in\mathbb{R}^n} \|D\varphi^t(x)\| \le \alpha(t), \quad \forall t \ge 0.$$

Then, the following statements are true:

(i) If α is bounded and $\lim_{t\to\infty} \alpha(t) = 0$, then the dynamical system $\{\varphi^t\}_{t\in\mathbb{R}_+}$ is incrementally asymptotically stable.

(ii) If there exist $K \ge 1$ and $\lambda > 0$ such that $\alpha(t) \le K\mathrm{e}^{-\lambda t}$ for all $t \ge 0$, then the dynamical system $\{\varphi^t\}_{t\in\mathbb{R}_+}$ is incrementally exponentially stable.

For the sake of completeness, we provide the proof of the above theorem. The underlying idea is to measure the distance between two trajectories by the integral of an infinitesimal displacement (see [LS98, Son10, SPB14]).

Proof. First, we fix any two points $x, y \in \mathbb{R}^n$. Let $\gamma\colon [0,1] \to \mathbb{R}^n$ be the line segment defined by

$$\gamma(\sigma) := \sigma x + (1-\sigma)y, \quad \sigma \in [0,1].$$

By the fundamental theorem of calculus, we have

$$\begin{aligned}
\varphi^t(x) - \varphi^t(y) &= (\varphi^t \circ \gamma)(1) - (\varphi^t \circ \gamma)(0) \\
&= \int_0^1 (\varphi^t \circ \gamma)'(\sigma)\,\mathrm{d}\sigma \\
&= \int_0^1 D\varphi^t(\gamma(\sigma))\,\mathrm{d}\sigma\,(x-y).
\end{aligned}$$

It thus follows that

$$\begin{aligned}
\|\varphi^t(x) - \varphi^t(y)\| &\le \int_0^1 \|D\varphi^t(\gamma(\sigma))\|\,\mathrm{d}\sigma\,\|x-y\| \\
&\le \alpha(t)\|x-y\|.
\end{aligned}$$

If α is bounded and $\alpha(t) \to 0$ as $t \to \infty$, then there exists a class-$\mathcal{KL}$ function β such that

$$\|\varphi^t(x) - \varphi^t(y)\| \le \beta(\|x-y\|, t), \quad \forall t \ge 0.$$

If there exist $K \ge 1$ and $\lambda > 0$ such that $\alpha(t) \le K\mathrm{e}^{-\lambda t}$ for all $t \ge 0$, then

$$\|\varphi^t(x) - \varphi^t(y)\| \le K\mathrm{e}^{-\lambda t}\|x-y\|, \quad \forall t \ge 0.$$

The proof is complete. □

2.1.3 Lyapunov's Second Method

The following result can be found in [Yos66, Ang02].

Theorem 2.2. The following statements are true:

(i) If there exist a C^1-function $V\colon \mathbb{R}^{2n} \to \mathbb{R}$, class-$\mathcal{K}_\infty$ functions α, β, and a class-$\mathcal{K}$ function γ such that for all $x, y \in \mathbb{R}^n$,

$$\alpha(\|x-y\|) \leq V(x,y) \leq \beta(\|x-y\|),$$
$$\partial_x V(x,y) f(x) + \partial_y V(x,y) f(y) \leq -\gamma(\|x-y\|),$$

then the dynamical system $\{\varphi^t\}_{t\in\mathbb{R}_+}$ is incrementally asymptotically stable.

(ii) If there exist a C^1-function $V\colon \mathbb{R}^{2n} \to \mathbb{R}$ and positive constants $a, b, c > 0$ such that for all $x, y \in \mathbb{R}^n$,

$$a\|x-y\|^2 \leq V(x,y) \leq b\|x-y\|^2,$$
$$\partial_x V(x,y) f(x) + \partial_y V(x,y) f(y) \leq -c\|x-y\|^2,$$

then the dynamical system $\{\varphi^t\}_{t\in\mathbb{R}_+}$ is incrementally exponentially stable.

The basic idea behind the above theorem is to consider stability of the invariant subspace $\{(x,y) : x = y\}$ of the following product system:

$$\begin{cases} \dot{x} = f(x), \\ \dot{y} = f(y). \end{cases}$$

The proof is given below.

Proof. We only prove the case (i); the case (ii) can be proven in the same way. First, we note that

$$\partial_x V(x,y) f(x) + \partial_y V(x,y) f(y) \leq -\gamma \circ \beta(V(x,y)).$$

By the comparison lemma, there exists a class-$\mathcal{KL}$ function ρ such that

$$V(\varphi^t(x), \varphi^t(y)) \leq \rho(V(x,y), t), \quad \forall t \geq 0.$$

Hence,

$$\begin{aligned} \|\varphi^t(x) - \varphi^t(y)\| &\leq \alpha^{-1}(V(\varphi^t(x), \varphi^t(y))) \\ &\leq \alpha^{-1} \circ \rho(V(x,y), t) \\ &\leq \alpha^{-1} \circ \rho(\beta(\|x-y\|), t). \end{aligned}$$

The proof is complete. $\square$

2.2 A Differential Framework for Stability Analysis

The bulk of this thesis studies the differential of a flow. We continue to consider the dynamical system $\{\varphi^t\}_{t\in\mathbb{R}_+}$ generated by (2.1). Here, we review the differential framework developed in [FS14], which is a central tool in this thesis.

2.2.1 Differential of a Flow

For $t \in \mathbb{R}_+$, the differential of φ^t at $x \in \mathbb{R}^n$ is the linear map

$$\mathrm{D}\varphi^t(x)\colon \mathbb{R}^n \to \mathbb{R}^n\colon \delta x \mapsto \mathrm{D}\varphi^t(x)\delta x.$$

It can be observed that the following cocycle property is fulfilled:

$$\mathrm{D}\varphi^{t+s}(x) = \mathrm{D}\varphi^t(\varphi^s(x))\mathrm{D}\varphi^s(x), \quad \forall t, s \geq 0, \quad \forall x \in \mathbb{R}^n.$$

Define a transformation $\pi^t\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ by

$$\pi^t(x, \delta x) := (\varphi^t(x), \mathrm{D}\varphi^t(x)\delta x), \quad (x, \delta x) \in \mathbb{R}^{2n}.$$

Then, the one-parameter family $\{\pi^t\}_{t\in\mathbb{R}_+}$ of transformations satisfies the axioms of a dynamical system, i.e.,

$$\begin{aligned}
\pi^0(x, \delta x) &= (x, \delta x), \quad \forall (x, \delta x) \in \mathbb{R}^{2n};\\
\pi^{t+s}(x, \delta x) &= \pi^t(\pi^s(x, \delta x)), \quad \forall t, s \geq 0, \quad \forall (x, \delta x) \in \mathbb{R}^{2n}.
\end{aligned}$$

Therefore, the differential operation of a flow on $\mathbb{R}^n$ defines a new dynamical system on $\mathbb{R}^{2n}$. In this thesis, we call the flow $\{\pi^t\}_{t\in\mathbb{R}_+}$ constructed above the *differential flow*.

Remark 2.1. In the context of nonautonomous dynamical systems, a similar procedure is employed to construct a linear skew-product flow from a process (see, e.g., [Hal88]). An important point in the current problem is that the differential flow is partially linear, that is, the map $\delta x \mapsto \pi^t(x, \delta x)$ is linear for any fixed x.

Now, we are interested in the differential equation which generates the differential flow $\{\pi^t\}_{t\in\mathbb{R}_+}$. Since $\{\varphi^t\}_{t\in\mathbb{R}_+}$ is generated by (2.1), the associated variational equation is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{D}\varphi^t(x) = \mathrm{D}f(\varphi^t(x))\mathrm{D}\varphi^t(x).$$

Thus, the matrix $\mathrm{D}\varphi^t(x)$ is the Cauchy matrix of the nonautonomous linear differential equation[1]

$$\delta\dot{x} = \mathrm{D}f(\varphi^t(x))\delta x.$$

We conclude that the differential flow $\{\pi^t\}_{t\in\mathbb{R}_+}$ is generated by the composite differential equation

$$\begin{bmatrix} \dot{x} \\ \delta\dot{x} \end{bmatrix} = \begin{bmatrix} f(x) \\ \mathrm{D}f(x)\delta x \end{bmatrix}. \tag{2.2}$$

The differential equation (2.2) is called the *prolonged system* with respect to the original nonlinear system (2.1) [CvdS87].

2.2.2 Differential Lyapunov Functions

Recently, a Lyapunov function method for the prolonged system (2.2) was developed in [FS14]. In that paper, a general Finsler metric was considered to measure the distance between any two points. In this thesis, however, we restrict our attention to dynamical systems on Euclidean spaces and metric tensors defined by state-dependent positive-definite matrices.

The following theorem is an analogue of Theorem 2.2.

Theorem 2.3. The following statements are true:

(i) If there exist a C^1-function $V\colon \mathbb{R}^{2n} \to \mathbb{R}$, class-$\mathcal{K}_\infty$ functions α, β, and a class-$\mathcal{K}$ function γ such that for all $(x, \delta x) \in \mathbb{R}^{2n}$,

$$\alpha(\|\delta x\|) \leq V(x, \delta x) \leq \beta(\|\delta x\|),$$
$$\partial_x V(x, \delta x) f(x) + \partial_{\delta x} V(x, \delta x)\mathrm{D}f(x)\delta x \leq -\gamma(\|\delta x\|),$$

then the dynamical system $\{\varphi^t\}_{\in\mathbb{R}_+}$ is incrementally asymptotically stable.

(ii) If there exist a C^1-function $V\colon \mathbb{R}^{2n} \to \mathbb{R}$ and positive constants $a, b, c > 0$ such that for all $(x, \delta x) \in \mathbb{R}^{2n}$,

$$a\|\delta x\|^2 \leq V(x, \delta x) \leq b\|\delta x\|^2$$
$$\partial_x V(x, \delta x) f(x) + \partial_{\delta x} V(x, \delta x)\mathrm{D}f(x)\delta x \leq -c\|\delta x\|^2,$$

then the dynamical system $\{\varphi^t\}_{\in\mathbb{R}_+}$ is incrementally exponentially stable.

[1] By the notation $\delta\dot{x}$, we mean $(\mathrm{d}/\mathrm{d}t)\delta x$.

Note that similar conditions appear in the context of partial stability [HC08]. The proof of this result is based on the combination of Lyapunov's first and second methods.

Proof. For the case (i), there exists a class-$\mathcal{KL}$ function ρ such that

$$V(\varphi^t(x), \mathrm{D}\varphi^t(x)\delta x) \leq \rho(V(x, \delta x), t), \quad \forall t \geq 0.$$

Thus, we obtain

$$\|\mathrm{D}\varphi^t(x)\delta x\| \leq \alpha^{-1} \circ \rho(\beta(\|\delta x\|), t).$$

It follows that

$$\begin{aligned}\|\mathrm{D}\varphi^t(x)\| &= \sup_{\|\delta x\|=1} \|\mathrm{D}\varphi^t(x)\delta x\| \\ &\leq \sup_{\|\delta x\|=1} \alpha^{-1} \circ \rho(\beta(\|\delta x\|), t) \\ &= \alpha^{-1} \circ \rho(\beta(1), t).\end{aligned}$$

This means that there exists a continuous function σ such that

$$\sup_{x \in \mathbb{R}^n} \|\mathrm{D}\varphi^t(x)\| \leq \sigma(t), \quad \forall t \geq 0.$$

It is clear that σ is bounded and $\sigma(t) \to 0$ as $t \to \infty$. The rest of the proof is the same as that of Theorem 2.1. The proof of the case (ii) is almost the same. □

2.2.3 Reduction to Matrix Differential Inequalities

An important class of differential Lyapunov functions is that of quadratic forms (with respect to δx). Consider a function $V\colon \mathbb{R}^{2n} \to \mathbb{R}$ of the form

$$V(x, \delta x) = \delta x^{\mathsf{T}} P(x) \delta x, \tag{2.3}$$

where $P\colon \mathbb{R}^n \to \mathcal{S}_n$ is a C^1-function. Then, the condition (ii) in Theorem 2.3 is equivalent to the existence of positive constants $a, b, c > 0$ such that for all $x \in \mathbb{R}^n$,

$$\begin{gathered}\lambda_{\min}(P(x)) \geq a, \quad \lambda_{\max}(P(x)) \leq b, \\ \dot{P}(x) + \mathrm{D}f(x)^{\mathsf{T}} P(x) + P(x)\mathrm{D}f(x) \preceq -cI,\end{gathered}$$

where

$$\dot{P}(x) = \begin{bmatrix} \nabla P_{11}(x)^{\mathsf{T}} f(x) & \cdots & \nabla P_{1n}(x)^{\mathsf{T}} f(x) \\ \vdots & \ddots & \vdots \\ \nabla P_{n1}(x)^{\mathsf{T}} f(x) & \cdots & \nabla P_{nn}(x)^{\mathsf{T}} f(x) \end{bmatrix}.$$

Because $P(x)$ is positive definite for all $x \in \mathbb{R}^n$, the function V in (2.3) can be interpreted as a Riemannian metric. If the system is linear, then the above matrix differential inequality reduces to the standard Lyapunov inequality.

2.3 Operator-Theoretic Properties

Now, we study dynamical systems from an operator-theoretic perspective [DV75]. Consider a causal operator $\mathcal{G}\colon \mathcal{U}_\mathrm{e} \to \mathcal{Y}_\mathrm{e}$, where

$$\mathcal{U}_\mathrm{e} := \mathcal{L}_2^{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^m) \quad \text{and} \quad \mathcal{Y}_\mathrm{e} := \mathcal{L}_2^{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^p).$$

Recall that these spaces are the extensions of

$$\mathcal{U} := \mathcal{L}_2(\mathbb{R}_+, \mathbb{R}^m) \quad \text{and} \quad \mathcal{Y} := \mathcal{L}_2(\mathbb{R}_+, \mathbb{R}^p),$$

respectively.

2.3.1 Nonincremental Dissipativity

The following definition of dissipativity is due to Hill and Moylan [HM80, Moy14][2]:

Definition 2.6. Let $\Phi \in \mathcal{S}_{p+m}$ be a symmetric matrix. The causal operator $\mathcal{G}\colon \mathcal{U}_e \to \mathcal{Y}_e$ is said to be

(i) Φ-*dissipative* if

$$\int_0^T \begin{bmatrix} y(t) \\ u(t) \end{bmatrix}^\top \Phi \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} \mathrm{d}t \geq 0$$

for all $T \geq 0$ and all $(u, y) \in \mathfrak{G}(\mathcal{G})$;

(ii) Φ-*ultimately dissipative* if

$$\int_0^\infty \begin{bmatrix} y(t) \\ u(t) \end{bmatrix}^\top \Phi \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} \mathrm{d}t \geq 0$$

for all $(u, y) \in \mathfrak{G}(\mathcal{G}|_{\mathcal{U}})$.

There are some important classes of dissipative operators (see, e.g., [vdS17]).

[2] Note that dissipative operators in this thesis are different from those in operator theory.

- Let
$$\Phi = \begin{bmatrix} -I & 0 \\ 0 & \gamma^2 I \end{bmatrix}, \quad \gamma > 0.$$
Then, the $\mathcal{L}_2$ gain of a Φ-dissipative operator $\mathcal{G}$ is no more than γ. If $\mathcal{G}$ is Φ-ultimately dissipative, then $\mathcal{G}|_{\mathcal{U}}$ is a bounded operator of a Hilbert space.

- Let
$$\Phi = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$
Then, a Φ-dissipative operator $\mathcal{G}$ is passive. If $\mathcal{G}$ is Φ-ultimately dissipative, then $\mathcal{G}|_{\mathcal{U}}$ is a positive operator of a Hilbert space.

The above properties are closely related to the classical concepts of gain and phase of a transfer function (see [KA01] for a historical review). Some graphical interpretations of nonlinear operators were provided in [Saf80, Tee96].

2.3.2 Incremental Dissipativity

Recently, incremental notions of dissipativity has attracted a lot of attention [FMNC96b, FMNC96a, SS07, PM08, Cha22, CFS23]. The study of incremental analysis can date back to the seminal works by Zames [Zam66a, Zam66b] and Willems [Wil71]. Following Definition 2.6, we introduce the notions below.

Definition 2.7. Let $\Phi \in \mathcal{S}_{p+m}$ be a symmetric matrix. The causal operator $\mathcal{G}\colon \mathcal{U}_e \to \mathcal{Y}_e$ is said to be

(i) Φ-*incrementally dissipative* if
$$\int_0^T \begin{bmatrix} y(t) - z(t) \\ u(t) - v(t) \end{bmatrix}^\top \Phi \begin{bmatrix} y(t) - z(t) \\ u(t) - v(t) \end{bmatrix} \mathrm{d}t \geq 0$$
for all $T \geq 0$ and all $(u, y), (v, z) \in \mathfrak{G}(\mathcal{G})$;

(ii) Φ-*incrementally ultimately dissipative* if
$$\int_0^\infty \begin{bmatrix} y(t) - z(t) \\ u(t) - v(t) \end{bmatrix}^\top \Phi \begin{bmatrix} y(t) - z(t) \\ u(t) - v(t) \end{bmatrix} \mathrm{d}t \geq 0$$
for all $(u, y), (v, z) \in \mathfrak{G}(\mathcal{G}|_{\mathcal{U}})$.

As mentioned in the previous subsection, there are some important cases.

- Let
$$\Phi = \begin{bmatrix} -I & 0 \\ 0 & \gamma^2 I \end{bmatrix}, \quad \gamma > 0.$$
Then, the incremental $\mathcal{L}_2$ gain of a Φ-incrementally dissipative operator $\mathcal{G}$ is no more than γ. If $\mathcal{G}$ is Φ-incrementally ultimately dissipative, then $\mathcal{G}|_{\mathcal{U}}$ is a Lipschitz operator.

- Let
$$\Phi = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$
Then, a Φ-dissipative operator $\mathcal{G}$ is called incrementally passive. If $\mathcal{G}$ is Φ-incrementally ultimately dissipative, then $\mathcal{G}|_{\mathcal{U}}$ is a monotone operator.

In recent years, graphical analysis of incremental dissipativity was studied based on the concept of scaled relative graphs [CFS23]. We remark that dissipativity is a central concept in control theory, while ultimate dissipativity typically appears in the context of convex optimization [RY22].

2.4 Differentially Dissipative Systems

In this section, we describe the concept of differential dissipativity, which was introduced in [FS13, van13] based on the differential Lyapunov framework of [FS14].

2.4.1 Differential of a Behavior

Consider the nonlinear system
$$\begin{cases} \dot{x} = f(x,u), \\ y = h(x,u), \end{cases} \tag{2.4}$$
where $f\colon \mathbb{R}^{n+m} \to \mathbb{R}^n$ and $h\colon \mathbb{R}^{n+m} \to \mathbb{R}^p$ are C^1-functions. We suppose that $\mathcal{L}_2^{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^m)$ and $\mathcal{L}_2^{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^p)$ are the admissible classes of inputs and outputs, respectively. Also, we

assume that every solution $x \in \mathcal{L}_2^{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n)$ to the above differential equation is well defined for any input $u \in \mathcal{L}_2^{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^m)$. Thus, the behavior under study is given by

$$\mathfrak{B} := \{(x, u, y) \in \mathcal{L}_2^{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^{n+m+p}) : \text{(2.4) is satisfied}\}.$$

The differential representation of the above dynamical system can be introduced as follows: For a given state, input, and output trajectories $(x, u, y) \in \mathfrak{B}$, their variations, which are denoted by $(\delta x, \delta u, \delta y)$, satisfy the equations

$$\begin{aligned}
\delta \dot{x}(t) &= \partial_x f(x(t), u(t))\delta x(t) + \partial_u f(x(t), u(t))\delta u(t),\\
\delta y(t) &= \partial_x h(x(t), u(t))\delta x(t) + \partial_u h(x(t), u(t))\delta u(t).
\end{aligned}$$

It is assumed that $(\delta x, \delta u, \delta y)$ belongs to the same signal space as (x, u, y), i.e.,

$$(\delta x, \delta u, \delta y) \in \mathcal{L}_2^{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^{n+m+p}).$$

We combine the above equations with the original ones to obtain the following differential representation

$$\begin{cases}
\begin{bmatrix} \dot{x} \\ \delta \dot{x} \end{bmatrix} = \begin{bmatrix} f(x, u) \\ \partial_x f(x, u)\delta x + \partial_u f(x, u)\delta u \end{bmatrix}, \\
\begin{bmatrix} y \\ \delta y \end{bmatrix} = \begin{bmatrix} h(x, u) \\ \partial_x h(x, u)\delta x + \partial_u h(x, u)\delta u \end{bmatrix}.
\end{cases} \tag{2.5}$$

This system is called a *prolonged system* [CvdS87]. Then, we define

$$\mathrm{d}\mathfrak{B} := \{(x, \delta x, u, \delta u, y, \delta y) \in \mathcal{L}_2^{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^{2n+2m+2p}) : \text{(2.5) is satisfied}\}.$$

We call the set $\mathrm{d}\mathfrak{B}$ the *differential behavior*.

2.4.2 Differential Dissipation Inequalities

In 1972, Willems introduced fundamental characterizations of dissipation inequalities for nonlinear systems [Wil72a] and for linear systems [Wil72b]. Now, we describe the definition of differentially dissipative systems in the same way as Willems' formulation.

Let $s\colon \mathcal{L}_2^{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^{2m+2p}) \to \mathbb{R}$ be a measurable function. The following definition is based on [FS13, van13].

Definition 2.8. The dynamical system $\mathfrak{B}$ is said to be *differentially dissipative* with respect to s if there exists a nonnegative continuous function $S\colon \mathbb{R}^{2n} \to \mathbb{R}_+$ such that

$$S(x(T), \delta x(T)) \leq S(x(0), \delta x(0)) + \int_0^T s(u(\tau), \delta u(\tau), y(\tau), \delta y(\tau))\, \mathrm{d}\tau$$

for all $T \geq 0$ and all $(x, \delta x, u, \delta u, y, \delta y) \in \mathrm{d}\mathfrak{B}$.

The following notion is useful in stability analysis (see [HC08]):

Definition 2.9. The dynamical system $\mathfrak{B}$ is said to be *differentially exponentially dissipative* with respect to s if there exists a continuous function $S\colon \mathbb{R}^{2n} \to \mathbb{R}$ and constants $a, b, c > 0$ such that the following conditions hold:

(i) The inequalities

$$a\|\delta x\|^2 \leq S(x, \delta x) \leq b\|\delta x\|^2$$

hold for all $(x, \delta x) \in \mathbb{R}^{2n}$.

(ii) The inequality

$$\begin{aligned} S(x(T), \delta x(T)) \leq{} & \mathrm{e}^{-ct} S(x(0), \delta x(0)) \\ & + \int_0^T \mathrm{e}^{-c(t-\tau)} s(u(\tau), \delta u(\tau), y(\tau), \delta y(\tau))\, \mathrm{d}\tau \end{aligned}$$

holds for all $T \geq 0$ and all $(x, \delta x, u, \delta u, y, \delta y) \in \mathrm{d}\mathfrak{B}$.

The function S is called a *differential storage function*, and the function s is called a *differential supply rate*. If the dynamical system $\mathfrak{B}$ is differentially dissipative in the sense of Definition 2.8 with the differential supply rate

$$s(u, \delta u, y, \delta y) = \begin{bmatrix} \delta y \\ \delta u \end{bmatrix}^\mathsf{T} \Phi \begin{bmatrix} \delta y \\ \delta u \end{bmatrix},$$

then

$$S(x(T), \delta x(T)) \leq S(x(0), \delta x(0)) + \int_0^T \begin{bmatrix} \delta y(t) \\ \delta u(t) \end{bmatrix}^\mathsf{T} \Phi \begin{bmatrix} \delta y(t) \\ \delta u(t) \end{bmatrix} \mathrm{d}t.$$

Because S is a nonnegative function, this implies

$$\int_0^T \begin{bmatrix} \delta y(t) \\ \delta u(t) \end{bmatrix}^\mathsf{T} \Phi \begin{bmatrix} \delta y(t) \\ \delta u(t) \end{bmatrix} \mathrm{d}t \geq -S(x(0), \delta x(0)).$$

If $S(x(0), \delta x(0)) = 0$, then the associated input/output relation satisfies

$$\int_0^T \begin{bmatrix} \delta y(t) \\ \delta u(t) \end{bmatrix}^\top \Phi \begin{bmatrix} \delta y(t) \\ \delta u(t) \end{bmatrix} \mathrm{d}t \geq 0, \quad \forall T \geq 0.$$

If the upper-left block of Φ is positive semidefinite, then the above differential dissipation inequality can be integrated to obtain the incremental dissipation inequality [VKHT23]. Thus, differentially dissipative dynamical systems and incrementally dissipative operators are related to each other.

2.4.3 Interconnections

An interconnection of differentially dissipative systems was recently studied in [MS14, FS19]. Here, we follow the approach in [AMP16]. Suppose that there are N differentially dissipative dynamical systems. Each system is described by

$$\begin{cases} \dot{x}_i = f_i(x_i, u_i), \\ y_i = h_i(x_i, u_i), \end{cases}$$

where $f_i\colon \mathbb{R}^{n_i+m_i} \to \mathbb{R}^{n_i}$ and $h_i\colon \mathbb{R}^{n_i+m_i} \to \mathbb{R}^{p_i}$ are as before for $i \in \{1, \ldots, N\}$. Let $x := \mathrm{col}(x_1, \ldots, x_N)$, $u := \mathrm{col}(u_1, \ldots, u_N)$, and $y := \mathrm{col}(y_1, \ldots, y_N)$. Now, we have a single open system described by (2.4) with

$$f(x,u) = \begin{bmatrix} f_1(x_1, u_1) \\ \vdots \\ f_N(x_N, u_N) \end{bmatrix}, \quad h(x,u) = \begin{bmatrix} h_1(x_1, u_1) \\ \vdots \\ h_N(x_N, u_N) \end{bmatrix}.$$

The associated prolonged system is also given by (2.5).

Let $M \in \mathbb{R}^{(m_1+\cdots+m_N)\times(p_1+\cdots+p_N)}$. Then, we consider an interconnection written by the following equation:

$$\begin{bmatrix} u \\ e \end{bmatrix} = \begin{bmatrix} M & I \\ I & 0 \end{bmatrix} \begin{bmatrix} y \\ d \end{bmatrix},$$

where d and e are the external input and the external output, respectively. Because this is a linear equation, the differential is just a copy of the above equation, that is,

$$\begin{bmatrix} u \\ \delta u \\ e \\ \delta e \end{bmatrix} = \begin{bmatrix} M \oplus M & I \\ I & 0 \end{bmatrix} \begin{bmatrix} y \\ \delta y \\ d \\ \delta d \end{bmatrix}. \tag{2.6}$$

For $i \in \{1, \dots, N\}$, let S_i and s_i denote a differential storage function and a differential supply rate of the i-th subsystem, respectively. Then, we define

$$S(x, \delta x) := \sum_{i=1}^{N} \alpha_i S_i(x_i, \delta x_i),$$

where $\alpha_1, \dots, \alpha_N > 0$ are positive constants. For the interconnected system, it can be confirmed that the inequality

$$S(x(T), \delta x(T)) \leq S(x(0), \delta x(0)) + \int_0^T \sum_{i=1}^{N} \alpha_i s_i(u_i(t), \delta u_i(t), y_i(t), \delta y_i(t))\, \mathrm{d}t$$

holds for all $T \geq 0$. Given an external differential supply rate s_e, we can show that the interconnected system is differentially dissipative with the differential storage function S and the differential supply rate s_e if

$$\sum_{i=1}^{N} \alpha_i s_i(u_i, \delta u_i, y_i, \delta y_i) \leq s_\mathrm{e}(d, \delta d, e, \delta e)$$

for all $(y, \delta y, d, \delta d, u, \delta u, e, \delta e)$ satisfying (2.6). In [FS19], the differential versions of the classical small-gain and passivity theorems were investigated.

Chapter 3

Global Stability of Lur'e Systems

In this chapter, we develop a new framework for the analysis of global stability of Lur'e systems. A Lur'e system consists of a linear time-invariant system and a memoryless nonlinearity [Lur63]. We propose a differential Lyapunov function of the Leonov form to study global stability of such a feedback system. It is shown that the gap between necessity and sufficiency for global stability can be dramatically reduced when the nonlinearity is smooth. This chapter partly contains the results in [KI24].

3.1 Introduction

The absolute stability problem, which was started by Lur'e and Postnikov in 1944, is very important in the development of modern control theory [AG64, NT73, YLG04]. This problem involves global stability of feedback systems, and the conjectures by Aizerman and Kalman were well studied in 1940s and 1950s. These conjectures suggest the equivalence between linear stability and nonlinear stability. The problem was essentially solved by Popov around 1960 through the frequency-domain method. In recent years, absolute stability has been reconsidered with their connection with hidden attractors [LK13]. Moreover, the Aizerman and Kalman conjectures have remained attractive topics [LKB10, DD18, NMIG19]. These facts motivate us to study the exact global stability boundary in a thorough manner.

There are numerous results on Lur'e systems in the literature (see [BLME20, Chapter 3] for a list of references). In particular, the IQC framework proposed by Megretski and Rantzer [MR97] is an effective approach to characterize stability of feedback systems. This framework is based on the frequency-domain method and extends the Popov and circle criteria. An interpretation of the IQC framework with Lyapunov's method was considered in [Sei15], where dissipation inequalities and suitable factorizations of multipliers were utilized. More recently, global stability of Lur'e systems was considered with a differential framework in [MVFS18, OOM23]. In these papers, the authors studied not only global stability but also dominated splittings and volume contraction.

The contributions of this chapter are summarized as follows: Based on the differential Lyapunov and dissipativity frameworks [FS14, MVFS18, FS19], we provide sufficient conditions for global stability of a Lur'e system. In our method, there is a certain gap between a storage function for the linear time-invariant system and a Lyapunov function for the closed-loop system. The underlying idea in this chapter originates in Leonov's work on dimension theory [Leo91b], and we suggest the use of a special differential Lyapunov function which we call a differential Lyapunov function of the Leonov form. It is also shown that the existence of a differential Lyapunov function of the Leonov form is necessary and sufficient for global stability of dissipative systems in the sense of Levinson. Compared with the previous works mentioned above, we do not consider indefinite metrics, but it is not difficult to generalize our results to more general situations. Furthermore, our characterization of a class of nonlinearities is a generalization of Kalman's sector condition. This is different from the the IQC framework, which generalizes Aizerman's sector condition in absolute stability theory.

Notations: Let $\varphi\colon X \to Y$ be a differentiable map, where X and Y are smooth manifolds. Recall that the graph of φ is defined by the set

$$\mathfrak{G}(\varphi) := \{(x, y) \in X \times Y : y = \varphi(x)\}.$$

The *differential graph* of φ is defined by the set

$$\mathrm{d}\mathfrak{G}(\varphi) := \{(x, \delta x, y, \delta y) \in \mathrm{T}X \times \mathrm{T}Y : y = \varphi(x),\ \delta y = (\mathrm{d}_x\varphi)\delta x\},$$

where $\mathrm{T}X$ and $\mathrm{T}Y$ are the tangent bundles of X and Y, respectively.

3.2 The Absolute Stability Problem

In this section, we briefly review a classical solution to the absolute stability problem in a general setting.

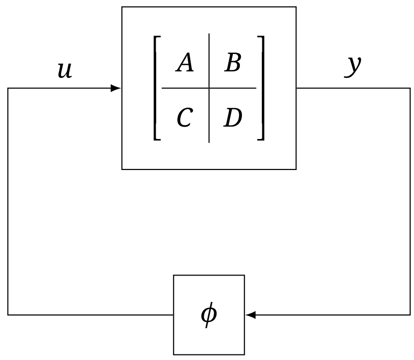

Figure 3.1: Lur'e system

3.2.1 Basic Setting

Consider the linear time-invariant system described by

$$\begin{cases} \dot{x} = Ax + Bu, \\ y = Cx + Du, \end{cases} \tag{3.1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and $D \in \mathbb{R}^{p\times m}$ are given matrices. Also, consider the memoryless nonlinearity described by

$$u = \phi(y), \tag{3.2}$$

where $\phi\colon \mathbb{R}^p \to \mathbb{R}^m$ is a C^1-map such that $\phi(0) = 0$. Note that the origin $x = 0$ is an equilibrium of the closed-loop system. The feedback system consisting of (3.1) and (3.2) is called a Lur'e system (see Fig. 3.1). We notice that all differential equations which possess an equilibrium can be converted to the Lur'e form by suitable coordinate changes.

To formulate the absolute stability problem, we impose the following constraint on the memoryless nonlinearity: Let $\Phi \in \mathcal{S}_{p+m}$ be a symmetric matrix. We then assume that the nonlinear map ϕ satisfies the quadratic inequality

$$\begin{bmatrix} y \\ u \end{bmatrix}^{\mathsf{T}} \Phi \begin{bmatrix} y \\ u \end{bmatrix} \le 0, \quad \forall (y,u) \in \mathfrak{G}(\phi). \tag{3.3}$$

The absolute stability problem can be stated as follows: Find a condition on (A,B,C,D) for which the closed-loop system is globally asymptotically stable for every nonlinear map ϕ satisfying (3.3).

Remark 3.1. In the literature, the following sector condition is often considered:

$$(u - K_1 y)^{\mathsf{T}}(K_2 y - u) \ge 0, \quad \forall (y,u) \in \mathfrak{G}(\phi),$$

where $K_1, K_2 \in \mathbb{R}^{m\times p}$ are given matrices. The classical problem formulated by Aizerman corresponds to the case where $m = p = 1$. Notice that the left-hand side of the above inequality can be rewritten as

$$\begin{aligned}(u - K_1 y)^\top (K_2 y - u) &= -\frac{1}{2} y^\top (K_1^\top K_2 + K_2^\top K_1) y + y^\top (K_1 + K_2)^\top u - u^\top u \\ &= -\frac{1}{2}\begin{bmatrix} y \\ u \end{bmatrix}^\top \begin{bmatrix} K_1^\top K_2 + K_2^\top K_1 & -(K_1 + K_2)^\top \\ -(K_1 + K_2) & 2I \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix}.\end{aligned}$$

Hence, this sector condition is a special case of (3.3).

Remark 3.2. The closed-loop system must satisfy the equality $y = Cx + D\phi(y)$. For well-posedness, it is required that there is a unique y for each x. In addition, since we are interested in stability of solutions, the closed-loop system should be forward complete. However, we do not address details of these issues (see, e.g., [BLME20, Chapter 3]).

3.2.2 Quadratic Lyapunov Functions

Following [YLG04], we now derive a sufficient condition for absolute stability based on a quadratic Lyapunov function candidate. As a Lyapunov function candidate, we consider the one in the following form:

$$V(x) = x^\top P x,$$

where $P \in \mathcal{S}_n$ is a positive-definite matrix. According to the Barbashin–Krasovskii theorem, for the closed-loop system to be globally asymptotically stable, it is sufficient that the inequality

$$\dot{V}(x) = 2x^\top P(Ax + Bu) < 0 \tag{3.4}$$

holds for all $(x, u) \in \mathbb{R}^{n+m}$ such that $x \neq 0$ and $u = \phi(Cx)$. Since $\phi(0) = 0$, this is equivalent to that the inequality (3.4) is valid for all $(x, u) \in \mathbb{R}^{n+m} \setminus \{0\}$ such that $u = \phi(Cx)$.

We apply the constraint (3.3) to the above argument. The constraint (3.3) is equivalent to that the inequality

$$\begin{bmatrix} x \\ u \end{bmatrix}^\top \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^\top \Phi \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \leq 0$$

holds whenever $u = \phi(Cx)$. Thus, we only need to examine the following relation:

$$\begin{bmatrix} x \\ u \end{bmatrix}^\top \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^\top \Phi \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \leq 0 \implies \begin{bmatrix} x \\ u \end{bmatrix}^\top \begin{bmatrix} A^\top P + PA & PB \\ B^\top P & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} < 0.$$

By the $\mathcal{S}$-procedure and the losslessness theorem [Yak71], the above relation is true if and only if there exists a nonnegative number $\tau \geq 0$ such that

$$\begin{bmatrix} A^\mathsf{T}P + PA & PB \\ B^\mathsf{T}P & 0 \end{bmatrix} - \tau \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^\mathsf{T} \Phi \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \prec 0. \tag{3.5}$$

Since the inequality is strict, τ must be positive. Hence, absolute stability is guaranteed if there exists a pair (τ, P) with $\tau > 0$ and $P \succ 0$ such that (3.5) holds. By multiplying both sides with τ^{-1} and by replacing P with $\tau^{-1}P$, we can set $\tau = 1$ without loss of generality.

3.2.3 Frequency Conditions

Here, we convert the condition (3.5) to an equivalent frequency condition [YLG04]. We first assume that Φ is partitioned as

$$\Phi = \begin{bmatrix} Q & L \\ L^\mathsf{T} & R \end{bmatrix},$$

where $Q \in \mathcal{S}_n$, $R \in \mathcal{S}_m$, and $L \in \mathbb{R}^{n\times m}$. Note that the question under consideration is the existence of a positive-definite matrix $P \in \mathcal{S}_n$ such that

$$\begin{bmatrix} A^\mathsf{T}P + PA & PB \\ B^\mathsf{T}P & 0 \end{bmatrix} - \begin{bmatrix} \widehat{Q} & \widehat{L} \\ \widehat{L}^\mathsf{T} & \widehat{R} \end{bmatrix} \prec 0, \tag{3.6}$$

where

$$\begin{aligned} \widehat{Q} &:= C^\mathsf{T}QC, \\ \widehat{L} &:= C^\mathsf{T}QD + C^\mathsf{T}L, \\ \widehat{R} &:= D^\mathsf{T}QD + D^\mathsf{T}L + L^\mathsf{T}D + R. \end{aligned}$$

By the Schur complement, (3.6) is satisfied if and only if

$$\widehat{R} \succ 0, \quad A^\mathsf{T}P + PA - \widehat{Q} + (PB - \widehat{L})\widehat{R}^{-1}(PB - \widehat{L})^\mathsf{T} \prec 0.$$

We define the transfer matrix $G(s) := C(sI - A)^{-1}B + D$ and assume that A has no eigenvalue on the imaginary axis. Then, the LMI (3.6) is feasible for a symmetric matrix $P \in \mathcal{S}_n$ if and only if the following frequency condition is fulfilled:

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^\mathsf{H} \begin{bmatrix} Q & L \\ L^\mathsf{T} & R \end{bmatrix} \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \succ 0, \quad \forall \omega \in [-\infty, \infty].$$

Under some additional assumptions, P is positive definite.

In general, the converse Lyapunov theorem does not guarantee the existence of a quadratic Lyapunov function. Because only quadratic Lyapunov functions are considered, the aforementioned approach to the absolute stability problem can be conservative. Another approach is to consider a Lyapunov function of the Lur'e–Postnikov form, the existence of which is equivalent to the Popov criterion. However, it is hopelessly hard to consider general classes of Lyapunov function candidates. In what follows, we utilize the differential framework in Chapter 2 to provide a new approach to the investigation of global stability.

3.3 Main Results

In this section, we provide new sufficient conditions for global stability based on the differential framework. We then show that necessary conditions can be obtained in a similar form.

3.3.1 Problem Formulation

Consider the linear time-invariant system (3.1) and the memoryless nonlinearity (3.2). For simplicity, we assume that $D = 0$. It is not difficult to remove this assumption. In our setting, the associated prolonged system can be written as

$$\begin{cases} \begin{bmatrix} \dot{x} \\ \delta\dot{x} \end{bmatrix} = (A \oplus A) \begin{bmatrix} x \\ \delta x \end{bmatrix} + (B \oplus B) \begin{bmatrix} u \\ \delta u \end{bmatrix}, \\ \begin{bmatrix} y \\ \delta y \end{bmatrix} = (C \oplus C) \begin{bmatrix} x \\ \delta x \end{bmatrix}. \end{cases} \tag{3.7}$$

Also, the corresponding memoryless nonlinearity is given by

$$\begin{bmatrix} u \\ \delta u \end{bmatrix} = \begin{bmatrix} \phi(y) \\ \mathrm{D}\phi(y)\delta y \end{bmatrix}. \tag{3.8}$$

In what follows, we provide sufficient conditions as well as necessary conditions.

3.3.2 Sufficiency for Global Stability

The theorem below specifies a class of nonlinearities ϕ with which the closed-loop system is globally asymptotically stable.

Theorem 3.1. Let $\Phi \in \mathcal{S}_{n+m}$ be a symmetric matrix. Suppose that the following conditions hold:

(i) There exist a positive-definite matrix $P \in \mathcal{S}_n$ and a negative constant $\mu < 0$ such that

$$\begin{bmatrix} A^\top P + PA - 2\mu P & PB \\ B^\top P & 0 \end{bmatrix} - \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}^\top \Phi \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} \preceq 0. \tag{3.9}$$

(ii) There exists a bounded C^1-function $v\colon \mathbb{R}^n \to \mathbb{R}$ such that

$$\begin{bmatrix} \delta x \\ \delta u \end{bmatrix}^\top \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}^\top \Phi \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \delta x \\ \delta u \end{bmatrix} + \dot{v}(x)\delta x^\top P \delta x \le 0 \tag{3.10}$$

for all $(x, \delta x, u, \delta u) \in \mathrm{d}\mathfrak{G}(\phi \circ C)$.

Then, the Lur'e system $\dot{x} = Ax + B\phi(Cx)$ is globally asymptotically stable.

Proof. From the condition (3.9), the inequality

$$\begin{bmatrix} \delta x \\ \delta u \end{bmatrix}^\top \begin{bmatrix} A^\top P + PA - 2\mu P & PB \\ B^\top P & 0 \end{bmatrix} \begin{bmatrix} \delta x \\ \delta u \end{bmatrix} - \begin{bmatrix} \delta x \\ \delta u \end{bmatrix}^\top \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}^\top \Phi \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \delta x \\ \delta u \end{bmatrix} \le 0$$

holds for all $(\delta x, \delta u) \in \mathbb{R}^{n+m}$. Because $\delta u = \mathrm{D}\phi(Cx)C\delta x$, we can observe from the condition (3.10) that

$$\begin{bmatrix} \delta x \\ \delta u \end{bmatrix}^\top \begin{bmatrix} A^\top P + PA - 2\mu P & PB \\ B^\top P & 0 \end{bmatrix} \begin{bmatrix} \delta x \\ \delta u \end{bmatrix} + \dot{v}(x)\delta x^\top P \delta x \le 0.$$

Hence, the closed-loop system satisfies

$$2\delta x^\top P J(x)\delta x + \dot{v}(x)\delta x^\top P \delta x \le 2\mu \delta x^\top P \delta x,$$

where $J(x) := A + B\mathrm{D}\phi(Cx)C$. This can be converted to the following LMI:

$$J(x)^\top P + PJ(x) + \dot{v}(x)P \preceq 2\mu P. \tag{3.11}$$

Let us now introduce the function $p\colon \mathbb{R}^n \to \mathbb{R}$ defined by $v(x) = \ln p(x)^2$. Then, we define the positive-definite matrix

$$\widetilde{P}(x) := p(x)^2 P.$$

It can be seen that $\dot{\widetilde{P}}(x) = 2p(x)\dot{p}(x)P = \dot{v}(x)p(x)^2 P$. Multiplying both sides of (3.11) by $p(x)^2$, we obtain

$$J(x)^\top \widetilde{P}(x) + \widetilde{P}(x)J(x) + \dot{\widetilde{P}}(x) \preceq 2\mu \widetilde{P}(x).$$

From the results in Section 2.2, we conclude that

$$\|\mathrm{D}\varphi^t(x)\| \leq \left[\frac{\lambda_{\max}(\widetilde{P}(x))}{\lambda_{\min}(\widetilde{P}(\varphi^t(x)))}\right]^{1/2} \mathrm{e}^{\mu t}$$

for all $t \geq 0$ and all $x \in \mathbb{R}^n$. Because of the boundedness of ν, there exist constants $p_{\min}, p_{\max} > 0$ such that

$$p_{\min} \leq p(x) \leq p_{\max}$$

for all $x \in \mathbb{R}^n$. It follows that

$$\left[\frac{p(x)^2 \lambda_{\max}(P)}{p(\varphi^t(x))^2 \lambda_{\min}(P)}\right]^{1/2} = \sqrt{\kappa(P)} \frac{p(x)}{p(\varphi^t(x))} \leq \sqrt{\kappa(P)} \left(\frac{p_{\max}}{p_{\min}}\right)$$

for all $x \in \mathbb{R}^n$. Therefore, from Theorem 2.1, the system is incrementally exponentially stable. The proof is complete. □

Remark that the condition (i) in Theorem 3.1 means that the linear time-invariant system is differentially exponentially dissipative with the differential storage function

$$S(x, \delta x) = \delta x^{\mathsf{T}} P \delta x$$

and the differential supply rate

$$s(u, \delta u, y, \delta y) = \begin{bmatrix} \delta y \\ \delta u \end{bmatrix}^{\mathsf{T}} \Phi \begin{bmatrix} \delta y \\ \delta u \end{bmatrix}.$$

On the other hand, stability analysis is based on a differential Lyapunov function of the form

$$V(x, \delta x) = p(x)^2 \delta x^{\mathsf{T}} P \delta x, \tag{3.12}$$

where p is given by the formula $\nu(x) = \ln p(x)^2$. We call a differential Lyapunov function of the form (3.12) a *differential Lyapunov function of the Leonov form* as the idea originates in Leonov's work on dimension theory [Leo91b]. Note that Leonov considered the function ν as a Lyapunov-like function. The relation between Leonov's interpretation and Riemannian metrics was pointed out in [PN00]. By employing this class of differential Lyapunov functions, we can admit a gap between a storage function of the linear time-invariant system and a Lyapunov function of the closed-loop system.

Compared with the related work in [MVFS18], the introduction of an arbitrary function ν can reduce conservativeness of the global stability condition. Because a constant function v satisfies $\dot{v}(x) = 0$ for all $x \in \mathbb{R}^n$, we can assume without loss of generality that $\dot{v}$ is a nonpositive function. Notice that the term $\dot{v}(x)$, which appears in the condition (3.10), can be made arbitrarily small as long as it is strictly negative. This is possible because multiplication of $v(x)$ by some large positive number does not affect the feasibility of the inequality. This is the main feature of Theorem 3.1, and we can characterize less conservative global stability conditions.

Remark 3.3. The matrix P appears in both two conditions (3.9) and (3.10). It is, however, noted that the those conditions can in fact be checked independently as follows: The inequality (3.10) can be replaced by the inequality

$$\begin{bmatrix} \delta x \\ \delta u \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}^{\mathsf{T}} \Phi \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \delta x \\ \delta u \end{bmatrix} + \dot{v}(x)\|\delta x\|^2 \leq 0.$$

Whenever $\dot{v}(x) \leq 0$, we can find another function $\widetilde{v}(x)$ such that $\dot{\widetilde{v}}(x)P \leq \dot{v}(x)I$. Then, we can employ the new function $\widetilde{v}$ instead of v in the above inequality to ensure the condition in the theorem.

Remark 3.4. The condition (3.10) can be divided into two parts. The first part is to find a function v such that $\dot{v}$ is negative semidefinite. Since v need not be positive definite, this requirement is similar to that of the LaSalle invariance principle. However, as in the classical case, there is no general method to find such a function. The second part is to check the inequality (3.10). Note that the inequality (3.10) depends on both the nonlinearity ϕ and its derivative $\mathrm{D}\phi$ because

$$\dot{v}(x) = \nabla v(x)^{\mathsf{T}}(Ax + B\phi(Cx)).$$

Thus, our condition differs from the classical sector conditions formulated by Aizerman and Kalman. However, if v is a constant function, then the inequality (3.10) is the same as Kalman's formulation.

3.3.3 Necessity for Global Stability

So far, we have provided sufficient conditions for global stability. Here, we study the necessity part. Suppose that the Lur'e system is globally asymptotically stable. Then, by the converse Lyapunov theorem, there exists a positive-definite and radially unbounded function $v\colon \mathbb{R}^n \to \mathbb{R}$ such that

$$\dot{v}(x) = \nabla v(x)^{\mathsf{T}}(Ax + B\phi(Cx)) < 0 \tag{3.13}$$

for all $x \in \mathbb{R}^n \setminus \{0\}$. This leads us to obtain the following result.

Theorem 3.2. Let $\Phi \in \mathcal{S}_{p+m}$ be a symmetric matrix. Suppose that $\dot{x} = Ax + B\phi(Cx)$ is globally asymptotically stable. If the condition (i) in Theorem 3.1 holds and if $\mathrm{D}\phi$ is bounded, then there exists a C^1-function $v\colon \mathbb{R}^n \to \mathbb{R}$ such that

$$\begin{bmatrix} \delta x \\ \delta u \end{bmatrix}^\top \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}^\top \Phi \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \delta x \\ \delta u \end{bmatrix} + \dot{v}(x)\|\delta x\|^2 \leq 0$$

for all $(x, \delta x, u, \delta u) \in \mathrm{d}\mathfrak{G}(\phi \circ C)$.

Proof. The proof immediately follows from the converse Lyapunov theorem. We have a function v satisfying (3.13). Multiplying the function v with some large constant implies that $\dot{v}(x)$ can be made arbitrarily small whenever it is negative. Because of the boundedness of $\mathrm{D}\phi$, we obtain the desired result. □

We should notice that the converge Lyapunov theorem guarantees the existence of a radially unbounded function v. This indicates that the function V defined in (3.12) may not be a differential Lyapunov function in the sense that there do not exist constants $\alpha, \beta > 0$ such that

$$\alpha\|\delta x\|^2 \leq V(x, \delta x) \leq \beta\|\delta x\|^2$$

for all $(x, \delta x) \in \mathbb{R}^{2n}$. This means that the necessary conditions in Theorem 3.2 and the sufficient conditions in Theorem 3.1 do not necessarily coincide even though $\mathrm{D}\phi$ is bounded. However, both conditions are consistent in that the function V converges to zero along any solution to the prolonged system described by (3.7) and (3.8). Probably, the reason for this gap is due to a trajectory which is unbounded in the Euclidean coordinates but bounded in a certain Riemannian distance.

3.3.4 Relation with Absolute Stability

The results obtained so far focused on relaxing the quadratic constraint on the memoryless nonlinearity. Here, we pay attention to a condition of absolute stability with respect to a given class of nonlinearities. To do so, we start by assuming that

$$\begin{bmatrix} \delta y \\ \delta u \end{bmatrix}^\top \Phi \begin{bmatrix} \delta y \\ \delta u \end{bmatrix} \leq 0, \quad \forall (y, \delta y, u, \delta u) \in \mathrm{d}\mathfrak{G}(\phi), \tag{3.14}$$

where $\Phi \in \mathcal{S}_{n+m}$ is a symmetric matrix. This is a generalization of the sector condition formulated by Kalman. The objective of this subsection is to find a condition for which the closed-loop system is globally asymptotically stable for every nonlinear map ϕ satisfying the above constraint. The following theorem can be proved in the same way as Theorem 3.1.

Theorem 3.3. The Lur'e system $\dot{x} = Ax + B\phi(Cx)$ is globally asymptotically stable for every nonlinear map ϕ satisfying the condition (3.14) if there exist a positive-definite matrix $P \in \mathcal{S}_n$, a negative number $\mu < 0$, and a bounded C^1-function $v\colon \mathbb{R}^n \to \mathbb{R}$ such that

$$\begin{bmatrix} A^\top P + PA - 2\mu P + \dot{v}(x)P & PB \\ B^\top P & 0 \end{bmatrix} - \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}^\top \Phi \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} \preceq 0$$

for all $x \in \mathbb{R}^n$.

If v is a constant function, then we can recover the classical approach discussed in Section 3.2. The above result allows additional flexibility in the class of Lyapunov functions by introducing a function v. The problem on the search of a class of functions v for every nonlinear map ϕ satisfying (3.14) is still open.

3.4 Conclusion

In this chapter, we studied global stability of Lur'e systems. There is no doubt that the absolute stability problem is one of the most important problems in control theory. However, many insightful questions have remained open. The results in this chapter contributed to reduce a gap between necessity and sufficiency for global stability when the memoryless nonlinearity is smooth. We finally provide some open questions.

- Is there some relation between the differential Lyapunov framework developed in this chapter and the dissipativity-based approach in the IQC framework [Sei15, SV18]?
- If the positive definiteness of P is removed as in [MVFS18, FS19], is it possible to completely characterize hyperbolicity-like properties and structural stability with a differential Lyapunov function of the Leonov form?

Chapter 4

Lyapunov Dimension of Lur'e Systems

In this chapter, we study Lyapunov dimension of Lur'e systems. The previous chapter was devoted to the equivalence between local stability and global stability. The subject in the current chapter is the equivalence between local Lyapunov dimension and global Lyapunov dimension, which is known as the Eden conjecture. Our framework is based on Leonov's method and provides a rich insight in the connection between dissipativity theory and dimension theory. Also, we demonstrate that our framework can be applied to the classical Lorenz system to obtain the exact Lyapunov dimension as in [Leo02]. This chapter partly contains the results in [KI24].

4.1 Introduction

In the seminal paper [KY79], Kaplan and Yorke introduced the concept of Lyapunov dimension to study strange attractors (see also [FKYY83]). The so-called Kaplan–Yorke conjecture states that the box dimension of an attractor can be calculated by some formula of Lyapunov exponents. At the same time, Douady and Oesterlé provided a key contribution in [DO80], where an upper estimate of the Hausdorff dimension of a negatively invariant set was derived in terms of singular values. A similar result was independently obtained by Il'yashenko [Il'92]. Later, Hunt showed that the Douady–Oesterlé theorem is also valid for box dimension [Hun96]. There are several other related works by Constantin et al. [CF85], Smith [Smi86], Eden [Ede89b], Leonov [Leo91b], Pogromsky and Nijmeijer [PN00], Kuznetsov [Kuz16], and Wu et al. [WPMS22]. We refer to the survey [BG11] and books [Tem97, BLR05] for other results.

The aforementioned results involve upper estimates of Lyapunov dimension. Thus, the obtained estimates may be conservative. In contrast, Eden conjectured that the Lyapunov dimension can be computed locally at one of the equilibria contained in the attractor [Ede89a, Ede89b, Ede90, EFT91]. In his papers, Eden introduced the concept of local Lyapunov dimension and tried to obtain the exact Lyapunov dimension of a Lorenz system. Roughly, the Eden conjecture states that the global Lyapunov dimension is dominated by the local Lyapunov dimension at an equilibrium. A crucial point is that if this is the case, the local Lyapunov dimension coincides with the global Lyapunov dimension, and thus, the exact Lyapunov dimension can be obtained. The Eden conjecture was positively solved for the Lorenz system by Leonov [Leo02, LKKK16]. Also, exact Lyapunov dimensions have been found for some well-known systems (see [KR21, Chapter 6]).

The contributions of this chapter can be stated as follows. We first derive an upper estimate of the Lyapunov dimension of a Lur'e system. This result can be seen as a natural generalization of the global stability result in the previous chapter. Then, we provide a characterization of the validity of the Eden conjecture. Our method is essentially based on the general framework in [PN00], but we restrict our attention to feedback systems of the Lur'e form. In this regard, the most relevant work is [OOM23], where volume contraction of Lur'e systems was explored. Compared with that work, our goal is to obtain the exact Lyapunov dimension formula through the differential Lyapunov framework. Another related work is frequency-domain estimation of Lyapunov dimension [LB92, LPS96]. These previous results relied on reversed dissipation inequalities, whereas our approach is based on conventional dissipation inequalities. With our method, a class of Lur'e systems in which the Eden conjecture is valid can be characterized based on Leonov's method. Finally, we demonstrate the effectiveness of our framework with a Lorenz system.

Notations: Let $A \in \mathbb{R}^{n\times n}$. The eigenvalues of $(A^\top A)^{1/2}$ and $(A^\top + A)/2$ are denoted by

$$\sigma_1(A) \geq \cdots \geq \sigma_n(A) \quad \text{and} \quad \eta_1(A) \geq \cdots \geq \eta_n(A),$$

respectively. For a real number $d \in [0, n]$, we define

$$\omega_d(A) := \begin{cases} 1 & \text{if } d = 0, \\ \sigma_1(A) \cdots \sigma_j(A) \sigma_{j+1}(A)^s & \text{otherwise,} \end{cases}$$

where $j \in \{0, 1, \ldots, n-1\}$ and $s \in (0, 1]$ are chosen such that $d = j + s$. Similarly, we define

$$\zeta_d(A) := \begin{cases} 0 & \text{if } d = 0, \\ \eta_1(A) + \cdots + \eta_j(A) + s\eta_{j+1}(A) & \text{otherwise,} \end{cases}$$

where $j \in \{0, 1, \ldots, n-1\}$ and $s \in (0, 1]$ are chosen such that $d = j+s$. In addition, we define

$$\omega^d(A) := 1/\omega_d(A^{-1}) \quad \text{and} \quad \zeta^d(A) := -1 \cdot \zeta_d(-A),$$

where A has been assumed to be invertible.

4.2 Reviews on Lyapunov Dimension

First, we review some basic facts on Lyapunov dimension. When introducing several terminologies, we mostly follow the recent book [KR21]. Throughout this section, let $\{\varphi^t\}_{t \in \mathbb{R}_+}$ denote a smooth dynamical system on $\mathbb{R}^n$.

4.2.1 The Douady–Oesterlé Theorem

The following definition is inspired by the work of Douady and Oesterlé.

$$\dim_{\mathrm{L}}^{\mathrm{DO}}(\varphi^t, x) := \sup\{d \in [0, n] : \omega_d(\mathrm{D}\varphi^t(x)) \geq 1\}.$$

In their seminal paper [DO80], the above notion was used to estimate the Hausdorff dimension of a compact negatively invariant set. The definition of Hausdorff dimension is given in Appendix A.1.3. The Douady–Oesterlé theorem is stated as follows:

Theorem 4.1 (Douady–Oesterlé [DO80]). Let $K \subset \mathbb{R}^n$ be a compact set such that

$$K \subset \varphi^t(K)$$

for a fixed $t > 0$. Then, we have

$$\dim_{\mathrm{H}}(K) \leq \sup_{x \in K} \dim_{\mathrm{L}}^{\mathrm{DO}}(\varphi^t, x).$$

If K is also positively invariant, then the same estimate is valid for box dimension [Hun96] (see Appendix A.1.3 for the definition).

Theorem 4.2 (Hunt [Hun96]). Let $K \subset \mathbb{R}^n$ be a compact set such that

$$\varphi^t(K) = K$$

for a fixed $t > 0$. Then, we have

$$\dim_{\mathrm{B}}(K) \leq \sup_{x \in K} \dim_{\mathrm{L}}^{\mathrm{DO}}(\varphi^t, x).$$

Remark 4.1. Some remarks are given below [KR21].

(i) The function $d \mapsto \omega_d(\mathrm{D}\varphi^t(x))$ is continuous except for the points where d takes an integer. Note that it is still left-continuous at those points. Thus, we have the following equality:

$$\sup\{d \in [0,n] : \omega_d(\mathrm{D}\varphi^t(x)) \geq 1\} = \max\{d \in [0,n] : \omega_d(\mathrm{D}\varphi^t(x)) \geq 1\}.$$

(ii) The function $x \mapsto \max\{d \in [0,n] : \det_d \mathrm{D}\varphi^t(x) \geq 1\}$ is continuous on $\mathbb{R}^n$ except for the points x where $\sigma_1(\mathrm{D}\varphi^t(x)) = 1$. It is still upper semicontinuous at those points. From this fact, we have

$$\sup_{x \in K} \dim_{\mathrm{L}}^{\mathrm{DO}}(\varphi^t, x) = \max_{x \in K} \dim_{\mathrm{L}}^{\mathrm{DO}}(\varphi^t, x).$$

(iii) An alternative characterization is as follows:

$$\max_{x \in K} \dim_{\mathrm{L}}^{\mathrm{DO}}(\varphi^t, x) = \max\left\{d \in [0,n] : \max_{x \in K} \omega_d(\mathrm{D}\varphi^t(x)) \geq 1\right\}.$$

(iv) If K is a flow-invariant set, then

$$\max_{x \in K} \omega_d(\mathrm{D}\varphi^{t+s}(x)) \leq \left(\max_{x \in K} \omega_d(\mathrm{D}\varphi^t(x))\right)\left(\max_{x \in K} \omega_d(\mathrm{D}\varphi^s(x))\right).$$

Hence, for each $t \geq 0$, there exists a number $s = s(t) > 0$ such that

$$\max_{x \in K} \dim_{\mathrm{L}}^{\mathrm{DO}}(\varphi^{t+s}, x) \leq \max_{x \in K} \dim_{\mathrm{L}}^{\mathrm{DO}}(\varphi^t, x).$$

Since K is an invariant set, we have

$$\dim_{\mathrm{H}}(K) \leq \dim_{\mathrm{B}}(K) \leq \inf_{t>0} \max_{x \in K} \dim_{\mathrm{L}}^{\mathrm{DO}}(\varphi^t, x)$$

(v) If K is a flow-invariant set, then

$$\inf_{t>0} \max_{x \in K} \dim_{\mathrm{L}}^{\mathrm{DO}}(\varphi^t, x) = \liminf_{t \to \infty} \max_{x \in K} \dim_{\mathrm{L}}^{\mathrm{DO}}(\varphi^t, x).$$

4.2.2 The Kaplan–Yorke Formula

For a fixed $t > 0$, the i-th finite-time Lyapunov exponent at x is defined by

$$\lambda_i(t,x) := \frac{1}{t} \ln \sigma_i(\mathrm{D}\varphi^t(x)), \quad i \in \{1,\dots,n\}.$$

Let $j(t,x)$ be the integer defined by $j(t,x) := 0$ if $\lambda_1(t,x) < 0$ and

$$j(t,x) := \max \left\{ k \in \{1,\dots,n-1\} : \sum_{i=1}^{k} \lambda_i(t,x) \geq 0 \right\}$$

otherwise. Then, we introduce the quantity

$$\dim_{\mathrm{L}}^{\mathrm{KY}}(\{\lambda_i(t,x)\}_{i=1}^n) := \begin{cases} j(t,x) + \dfrac{\lambda_1(t,x) + \cdots + \lambda_{j(t,x)}(t,x)}{|\lambda_{j(t,x)+1}(t,x)|} & \text{if } \sum_{i}^{n} \lambda_i(t,x) < 0, \\ n & \text{otherwise.} \end{cases}$$

We remark that the Lyapunov dimension defined with finite-time Lyapunov exponents was considered by Kuznetsov and coworkers [KAL16, Kuz16] and is different from the original notion by Kaplan and Yorke [KY79]. A slightly different definition based on global Lyapunov exponents was introduced in [CF85, CFT85]. Relations between various definitions of Lyapunov dimension can be found in [KR21, Chapter 6].

Remark 4.2. The concepts of Lyapunov exponents, or characteristic exponents, are important in the theory of dynamical systems. For instance, the sum of positive Lyapunov exponents is related to the notion of Kolmogorov–Sinai entropy (also known as the metric entropy). Two important results are the Margulis–Ruelle inequality on upper estimates and the Pesin formula on exact estimates (see, e.g., [Dow11, Chapter 10]). Relations between dimension and entropy notions are of great interest.

4.2.3 Definition of Lyapunov Dimension

We now introduce the Lyapunov dimension of a dynamical system. The definition below is due to [Ede90], where it was called the Douady–Oesterlé dimension.

Definition 4.1. Let $K \subset \mathbb{R}^n$ be a compact invariant set. The *Lyapunov dimension* of $\{\varphi^t\}_{t\in\mathbb{R}_+}$ with respect to K is the number

$$\dim_{\mathrm{L}}(\{\varphi^t\}_{t\in\mathbb{R}_+}, K) := \inf \left\{ d \in [0,n] : \lim_{t\to\infty} \frac{1}{t} \ln \left(\sup_{x\in K} \omega_d(\mathrm{D}\varphi^t(x)) \right) < 0 \right\}.$$

Note that the limit in the above definition exists because of the invariance of K. We show two important lemmas below. The latter one is called the Kaplan–Yorke formula.

Lemma 4.1. Let $K \subset \mathbb{R}^n$ be a compact invariant set. Then, we have

$$\dim_{\mathrm{L}}(\{\varphi^t\}_{t\in\mathbb{R}_+}, K) = \liminf_{t\to\infty} \sup_{x\in K} \dim_{\mathrm{L}}^{\mathrm{DO}}(\varphi^t, x).$$

Lemma 4.2. Let $K \subset \mathbb{R}^n$ be a compact invariant set. Then, we have

$$\dim_{\mathrm{L}}(\{\varphi^t\}_{t\in\mathbb{R}_+}, K) = \liminf_{t\to\infty} \sup_{x\in K} \dim_{\mathrm{L}}^{\mathrm{KY}}(\{\lambda_i(t,x)\}_{i=1}^n).$$

4.2.4 The Eden Conjecture

A local concept of Lyapunov dimension was introduced by Eden [Ede90].

Definition 4.2. The *local Lyapunov dimension* of $\{\varphi^t\}_{t\in\mathbb{R}_+}$ at $x \in \mathbb{R}^n$ is the number

$$\dim_{\mathrm{L}}^{\mathrm{loc}}(\{\varphi^t\}_{t\in\mathbb{R}_+}, x) := \inf\left\{d \in [0,n] : \limsup_{t\to\infty} \frac{1}{t} \ln \omega_d(\mathrm{D}\varphi^t(x)) < 0\right\}.$$

Notice that the above definition is different from the (global) Lyapunov dimension in Definition 4.1 in that the local Lyapunov dimension can be defined even if $\{x\}$ is not an invariant set. The relation between local and global Lyapunov dimensions are stated in the following lemma [Ede90, Corollaly 5.6].

Lemma 4.3. Let $K \subset \mathbb{R}^n$ be a compact invariant set. Then, the following statements are true:

(i) The inequality

$$\dim_{\mathrm{L}}^{\mathrm{loc}}(\{\varphi^t\}_{t\in\mathbb{R}_+}, x) \leq \dim_{\mathrm{L}}(\{\varphi^t\}_{t\in\mathbb{R}_+}, K)$$

holds for all $x \in K$.

(ii) There exists a point $x^* \in K$ such that

$$\dim_{\mathrm{L}}^{\mathrm{loc}}(\{\varphi^t\}_{t\in\mathbb{R}_+}, x^*) = \dim_{\mathrm{L}}(\{\varphi^t\}_{t\in\mathbb{R}_+}, K).$$

The second statement is of special importance since $\dim_{\mathrm{L}}^{\mathrm{loc}}(\{\varphi^t\}_{t\in\mathbb{R}_+}, x^*)$ may be calculated exactly. However, it is difficult to localize such a critical point x^*, and that point may not be an equilibrium. In a subsequent section, we consider the *Eden conjecture*, which can be stated as follows: Let $K \subset \mathbb{R}^n$ be a compact invariant set containing at least one equilibrium. Then, there exists an equilibrium $x_{\mathrm{e}} \in K$ such that

$$\dim_{\mathrm{L}}^{\mathrm{loc}}(\{\varphi^t\}_{t\in\mathbb{R}_+}, x_{\mathrm{e}}) = \dim_{\mathrm{L}}(\{\varphi^t\}_{t\in\mathbb{R}_+}, K).$$

In general, there is no known class of dynamical systems for which the Eden conjecture is valid. We will provide a certain characterization on the validity of this conjecture.

4.3 Preliminaries

Before moving on to the main results, we offer some preliminaries.

4.3.1 Generalized Ważewski Inequalities

Consider a linear differential equation

$$\dot{x} = A(t)x, \tag{4.1}$$

where $A\colon \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ is a C^0-function. We recall that a fundamental matrix of (4.1), denoted by $X\colon \mathbb{R}_+ \to \mathbb{R}^{n\times n}$, is a square matrix whose columns are linearly independent solutions to (4.1). If $X(0) = I$, then it is called the Cauchy matrix.

The Ważewski inequalities provide lower and upper estimates of the solutions to (4.1). More precisely, if $x\colon \mathbb{R}_+ \to \mathbb{R}^n$ is a solution to (4.1), then

$$\|x(0)\| \exp \int_0^t \eta_n(A(\tau))\,\mathrm{d}\tau \le \|x(t)\| \le \|x(0)\| \exp \int_0^t \eta_1(A(\tau))\,\mathrm{d}\tau$$

for all $t \ge 0$. This result is also known as the Wintner inequalities and is a special case of the more general Coppel inequalities [Cop65].

The Ważewski inequalities were generalized by Smith.

Lemma 4.4 (Smith [Smi86]). Let $X\colon \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ be the Cauchy matrix of (4.1), and let $k \in \{1,\dots,n\}$ be an integer. Then, we have

$$\omega_k(X(t)) \le \exp \int_0^t \zeta_k(A(\tau))\,\mathrm{d}\tau \quad \text{and} \quad \omega^k(X(t)) \ge \exp \int_0^t \zeta^k(A(\tau))\,\mathrm{d}\tau$$

for all $t \ge 0$.

In what follows, we only consider the first inequality since the second inequality can be derived from the first one by reversing the time direction. The above result can be generalized further. For a fixed real number $d \in [0,n]$, let $j \in \{0,1,\dots,n-1\}$ and $s \in (0,1]$ be chosen such that $d = j + s$. Then, we have

$$\omega_d(X(t)) = \omega_j(X(t))^{1-s}\omega_{j+1}(X(t))^s,$$
$$\zeta_d(X(t)) = (1-s)\zeta_j(X(t)) + s\zeta_{j+1}(X(t)).$$

Therefore,

$$\begin{aligned}\omega_d(X(t)) &\le \exp \int_0^t [(1-s)\zeta_k(A(\tau)) + s\zeta_{j+1}(A(\tau))\,\mathrm{d}\tau] \\ &= \exp \int_0^t \zeta_d(A(\tau))\,\mathrm{d}\tau\end{aligned}$$

for all $t \geq 0$.

Notice that there exists a nonsingular matrix $C \in \mathbb{R}^{n\times n}$ such that $X_1(t) = X_2(t)C$ for any two fundamental matrices X_1, X_2. If X is a fundamental matrix but is not necessarily the Cauchy matrix, then we can show that

$$\omega_d(X(t)) \leq \omega_d(X(0)) \exp \int_0^t \zeta_d(A(\tau))\,\mathrm{d}\tau, \quad \forall t \geq 0.$$

When $d = n$, the above result is consistent with the well-known Liouville formula:

$$|\det X(t)| = |\det X(0)| \exp \int_0^t \operatorname{tr} A(\tau)\,\mathrm{d}\tau, \quad \forall t \geq 0.$$

4.3.2 Lyapunov-Based Analysis

For the linear system (4.1), we consider the matrix inequalities

$$P(t) \succ 0, \quad \dot{P}(t) + A(t)^\top P(t) + P(t)A(t) - Q(t) \preceq 0, \tag{4.2}$$

where $P\colon \mathbb{R}_+ \to \mathcal{S}_n$ is a C^1-function and $Q\colon \mathbb{R}_+ \to \mathcal{S}_n$ is a C^0-function. We note that the latter matrix inequality is satisfied if and only if for every solution $x\colon \mathbb{R}_+ \to \mathbb{R}^n$ to (4.1), the inequality

$$\begin{bmatrix} \dot{x}(t) \\ x(t) \end{bmatrix}^\top \begin{bmatrix} 0 & P(t) \\ P(t) & \dot{P}(t) - Q(t) \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ x(t) \end{bmatrix} \leq 0$$

holds for all $t \geq 0$.

The following result is fundamental in the subsequent developments.

Lemma 4.5. Let $X\colon \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ be the Cauchy matrix of (4.1), and let $d \in [0, n]$ be a real number. Suppose that there exist a C^1-function $P\colon \mathbb{R}_+ \to \mathcal{S}_n$ and a C^0-function $Q\colon \mathbb{R}_+ \to \mathcal{S}_n$ such that the matrix inequalities in (4.2) hold for all $t \geq 0$. Then, we have

$$\omega_d(X(t)) \leq \left[\frac{\omega_d(P(0))}{\omega^d(P(t))}\right]^{1/2} \exp \int_0^t (1/2)\zeta_d(Q(\tau)P(\tau)^{-1})\,\mathrm{d}\tau$$

for all $t \geq 0$.

Proof. Let $T(t) := P(t)^{1/2}$. Note that $T(t) \succ 0$ and $\dot{P}(t) = \dot{T}(t)T(t) + T(t)\dot{T}(t)$. By multiplying the matrix inequality (4.2) with $T(t)^{-1}$ from both sides, we obtain

$$\widetilde{A}(t)^\top + \widetilde{A}(t) \preceq T(t)^{-1}Q(t)T(t)^{-1},$$

where $\widetilde{A}(t) := [T(t)A(t) + \dot{T}(t)]T(t)^{-1}$. This implies that

$$\zeta_d(\widetilde{A}(t)) \leq \frac{1}{2}\zeta_d(T(t)^{-1}Q(t)T(t)^{-1})$$

for all $d \in [0, n]$. Note that the eigenvalues of $T(t)^{-1}Q(t)T(t)^{-1}$ are the solutions of the algebraic equation

$$\det(Q(t) - \lambda(t)P(t)) = 0.$$

Thus,

$$\zeta_d(T(t)^{-1}Q(t)T(t)^{-1}) = \zeta_d(Q(t)P(t)^{-1}) = \zeta_d(P(t)^{-1}Q(t)).$$

Defining $Y(t) := T(t)X(t)$, we have

$$\begin{aligned}\dot{Y}(t) &= T(t)\dot{X}(t) + \dot{T}(t)X(t)\\ &= [T(t)A(t) + \dot{T}(t)]T(t)^{-1}Y(t)\\ &= \widetilde{A}(t)Y(t).\end{aligned}$$

Since $Y(0) = T(0)$, the generalized Ważewski inequality indicates that

$$\begin{aligned}\omega_d(Y(t)) &\leq \omega_d(Y(0)) \exp\int_0^t \zeta_d(\widetilde{A}(s))\,\mathrm{d}s\\ &\leq \omega_d(T(0)) \exp\int_0^t (1/2)\zeta_d(Q(s)P(s)^{-1})\,\mathrm{d}s.\end{aligned}$$

By Horn's inequality, it follows that

$$\omega_d(X(t)) \leq \omega_d(T(t)^{-1})\omega_d(Y(t)) = \frac{\omega_d(Y(t))}{\omega^d(T(t))}.$$

Thus, we obtain

$$\omega_d(X(t)) \leq \frac{\omega_d(T(0))}{\omega^d(T(t))} \exp\int_0^t (1/2)\zeta_d(Q(s)P(s)^{-1})\,\mathrm{d}s.$$

Since $\sigma_i(T(t)) = \sigma_i(P(t))^{1/2}$ for all $i \in \{1, \ldots, n\}$, we complete the proof. □

4.3.3 Estimation of Lyapunov Dimension

We now provide an upper estimate of Lyapunov dimension. Let $\{\varphi^t\}_{t\in\mathbb{R}_+}$ be a dynamical system generated by the differential equation

$$\dot{x} = f(x),$$

where $f: \mathbb{R}^n \to \mathbb{R}^n$ is a C^1-function. The associated prolonged system can be written as

$$\begin{bmatrix} \dot{x} \\ \delta\dot{x} \end{bmatrix} = \begin{bmatrix} f(x) \\ \mathrm{D}f(x)\delta x \end{bmatrix}.$$

Then, we obtain the following result.

Theorem 4.3. Let $K \subset \mathbb{R}^n$ be a compact invariant set. Suppose that there exist a C^1-function $P: K \to \mathcal{S}_n$ and a C^0-function $Q: K \to \mathcal{S}_n$ such that $P(x) \succ 0$ and

$$\begin{bmatrix} \delta\dot{x} \\ \delta x \end{bmatrix}^\top \begin{bmatrix} 0 & P(x) \\ P(x) & \dot{P}(x) - Q(x) \end{bmatrix} \begin{bmatrix} \delta\dot{x} \\ \delta x \end{bmatrix} \leq 0 \tag{4.3}$$

for all $x \in K$. If there exists a real number $d \in (0, n]$ such that

$$\sup_{x \in K} \zeta_d(Q(x)P(x)^{-1}) < 0,$$

then $\dim_{\mathrm{L}}(\{\varphi^t\}_{t\in\mathbb{R}_+}, K) < d$.

Proof. Note that when (4.3) holds with equality, the result is the same as that of [PN00]. From Lemma 4.5, we have

$$\omega_d(\mathrm{D}\varphi^t(x)) \leq \left[\frac{\omega_d(P(0))}{\omega^d(P(t))}\right]^{1/2} \exp \int_0^t (1/2)\zeta_d(Q(\tau)P(\tau)^{-1})\,\mathrm{d}\tau$$

for all $t \geq 0$ and all $x \in K$. Since K is compact, there exists $T > 0$ sufficiently large such that

$$\sup_{x \in K} \omega_d(\mathrm{D}\varphi^T(x)) < 1.$$

Then, the Douady–Oesterlé theorem (Theorem 4.1) yields the desired result. We complete the proof. □

4.4 Main Results

We are now ready to state our main results in this chapter. The Lur'e system under consideration is the same as the previous chapter. That is, we consider the Lur'e system described by (3.1) and (3.2) with $D = 0$.

4.4.1 Upper Estimates

We first derive an upper estimate of Lyapunov dimension. The theorem below generalizes stability analysis to dimension estimates.

Theorem 4.4. Let $K \subset \mathbb{R}^n$ be a compact invariant set of the Lur'e system $\dot{x} = Ax + B\phi(Cx)$. Let $\Phi \in \mathcal{S}_{n+m}$ be a symmetric matrix. Suppose that the following conditions hold:

(i) There exist a positive-definite matrix $P \in \mathcal{S}_n$ and a number $\mu \in \mathbb{R}$ such that

$$\begin{bmatrix} A^\top P + PA - 2\mu P & PB \\ B^\top P & 0 \end{bmatrix} - \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}^\top \Phi \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} \preceq 0. \tag{4.4}$$

(ii) Let $\lambda_1(x) \geq \cdots \geq \lambda_n(x)$ denote the solutions to the algebraic equation

$$\det\left(C^\top \begin{bmatrix} I \\ \mathrm{D}\phi(Cx) \end{bmatrix}^\top \Phi \begin{bmatrix} I \\ \mathrm{D}\phi(Cx) \end{bmatrix} C - \lambda(x) P \right) = 0.$$

There exist a real number $d \in (0, n]$ and a C^1-function $v\colon K \to \mathbb{R}$ such that for all $x \in K$,

$$\frac{\lambda_1(x) + \cdots + \lambda_{\lfloor d \rfloor}(x) + (d - \lfloor d \rfloor)\lambda_{\lfloor d \rfloor + 1}(x)}{d} + \dot{v}(x) + 2\mu < 0. \tag{4.5}$$

Then, $\dim_{\mathrm{L}}(\{\varphi^t\}_{t\in\mathbb{R}_+}, K) < d$.

Proof. From (4.4), we have

$$\begin{bmatrix} \delta x \\ \delta u \end{bmatrix}^\top \begin{bmatrix} A^\top P + PA - 2\mu P & PB \\ B^\top P & 0 \end{bmatrix} \begin{bmatrix} \delta x \\ \delta u \end{bmatrix} - \begin{bmatrix} \delta x \\ \delta u \end{bmatrix}^\top \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}^\top \Phi \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \delta x \\ \delta u \end{bmatrix} \leq 0$$

for all $(\delta x, \delta u) \in \mathbb{R}^{n+m}$. The relation $\delta u = \mathrm{D}\phi(Cx)C\delta x$ implies

$$2\delta x^\top P J(x) \delta x \leq \delta x^\top \left(C^\top \begin{bmatrix} I \\ \mathrm{D}\phi(Cx) \end{bmatrix}^\top \Phi \begin{bmatrix} I \\ \mathrm{D}\phi(Cx) \end{bmatrix} C + 2\mu P \right) \delta x,$$

where $J(x) := A + B\mathrm{D}\phi(Cx)C$. This is equivalent to the matrix inequality

$$J(x)^\top P + PJ(x) \preceq C^\top \begin{bmatrix} I \\ \mathrm{D}\phi(Cx) \end{bmatrix}^\top \Phi \begin{bmatrix} I \\ \mathrm{D}\phi(Cx) \end{bmatrix} C + 2\mu P.$$

By adding $\dot{v}(x)P$ to both sides, we obtain

$$J(x)^\top P + PJ(x) + \dot{v}(x)P \preceq C^\top \begin{bmatrix} I \\ \mathrm{D}\phi(Cx) \end{bmatrix}^\top \Phi \begin{bmatrix} I \\ \mathrm{D}\phi(Cx) \end{bmatrix} C + 2\mu P + \dot{v}(x)P.$$

Let us introduce the function $p\colon \mathbb{R}^n \to \mathbb{R}$ defined by $v(x) = \ln p(x)^2$. Then, we define the positive-definite matrix

$$\widetilde{P}(x) := p(x)^2 P.$$

Since $\dot{\widetilde{P}}(x) = \dot{v}(x)p(x)^2 P$, by multiplying both sides by $p(x)^2$ in the above inequality, we obtain

$$\begin{aligned} & J(x)^\mathsf{T}\widetilde{P}(x) + \widetilde{P}(x)J(x) + \dot{\widetilde{P}}(x) \\ & \quad \le p(x)^2 \left(C^\mathsf{T} \begin{bmatrix} I \\ \mathrm{D}\phi(Cx) \end{bmatrix}^\mathsf{T} \Phi \begin{bmatrix} I \\ \mathrm{D}\phi(Cx) \end{bmatrix} C + \dot{v}(x)P + 2\mu P \right). \end{aligned}$$

We define

$$Q(x) := p(x)^2 \left(C^\mathsf{T} \begin{bmatrix} I \\ \mathrm{D}\phi(Cx) \end{bmatrix}^\mathsf{T} \Phi \begin{bmatrix} I \\ \mathrm{D}\phi(Cx) \end{bmatrix} C + \dot{v}(x)P + 2\mu P \right).$$

Now, we note that

$$\zeta_d(Q(x)\widetilde{P}(x)^{-1}) = \lambda_1(x) + \cdots + \lambda_{\lfloor d \rfloor}(x) + (d - \lfloor d \rfloor)\lambda_{\lfloor d \rfloor + 1}(x) + d(\dot{v}(x) + 2\mu).$$

From Theorem 4.3, we conclude that $\dim_\mathrm{L}(\{\varphi^t\}_{t\in\mathbb{R}_+}, K) < d$. □

In the above theorem, we do not require that μ is negative. Thus, by setting μ such that $A - \mu I$ is Hurwitz, then the condition (i) can be considered as the standard dissipation inequality for linear systems.

4.4.2 Exact Formula

Remark that we can assume that $\mathrm{D}\phi(0) = 0$ without loss of generality. In this case, the local Lyapunov dimension at the origin can be determined by the matrix A. For example, the number

$$d_0 := \inf \left\{ d \in (0, n] : \zeta_d(P^{1/2}AP^{-1/2}) < 0,\ P \succ 0 \right\}$$

is an upper bound of the local Lyapunov dimension at the origin. The exact Lyapunov dimension can be obtained when A has a special structure (see [Leo02]). The following theorem characterizes a class of nonlinearities ϕ with which the global Lyapunov dimension is dominated by the local one, or equivalently, the Eden conjecture is valid.

Theorem 4.5. Let $K \subset \mathbb{R}^n$ be a compact invariant set of the Lur'e system $\dot{x} = Ax + B\phi(Cx)$. Let $d_0 \in [0, n]$ be an upper bound of the local Lyapunov dimension at the origin. Suppose that the following conditions hold:

(i) There exist a positive-definite matrix $P \in \mathcal{S}_n$ and a constant $\mu \in \mathbb{R}$ such that

$$\begin{bmatrix} A^\top P + PA - 2\mu P & PB \\ B^\top P & 0 \end{bmatrix} - \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}^\top \Phi \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} \preceq 0. \tag{4.6}$$

(ii) Let $\lambda_1(x) \geq \cdots \geq \lambda_n(x)$ denote the solutions to the algebraic equation

$$\det\left(C^\top \begin{bmatrix} I \\ \mathrm{D}\phi(Cx) \end{bmatrix}^\top \Phi \begin{bmatrix} I \\ \mathrm{D}\phi(Cx) \end{bmatrix} C - \lambda(x) P \right) = 0.$$

For every $d \in (d_0, n]$, there exists a C^1-function $v\colon K \to \mathbb{R}$ such that

$$\frac{\lambda_1(x) + \cdots + \lambda_{\lfloor d \rfloor}(x) + (d - \lfloor d \rfloor)\lambda_{\lfloor d \rfloor + 1}(x)}{d} + \dot{v}(x) + 2\mu < 0 \tag{4.7}$$

for all $x \in K$.

Then, $\dim_{\mathrm{L}}(\{\varphi^t\}_{t \in \mathbb{R}_+}, K) \leq d_0$. Moreover, if d_0 is the exact local Lyapunov dimension at the origin, then $\dim_{\mathrm{L}}(\{\varphi^t\}_{t \in \mathbb{R}_+}, K) = d_0$.

Proof. Similarly to the proof of Theorem 4.3, we obtain $\dim_{\mathrm{L}}(\{\varphi^t\}_{t \in \mathbb{R}_+}, K) \leq d_0$. By definition, we have

$$\dim_{\mathrm{L}}^{\mathrm{loc}}(\{\varphi^t\}_{t \in \mathbb{R}_+, K}, 0) \leq \dim_{\mathrm{L}}(\{\varphi^t\}_{t \in \mathbb{R}_+}, K).$$

Hence, if $\dim_{\mathrm{L}}^{\mathrm{loc}}(\{\varphi^t\}_{t \in \mathbb{R}_+}, K, 0) = d_0$, then $\dim_{\mathrm{L}}(\{\varphi^t\}_{t \in \mathbb{R}_+}, K) = d_0$. □

The above theorem provides the exact Lyapunov dimension formula in some cases. This point is further investigated through Lorenz systems in the next section. Because all differential equations which possess an equilibrium can be converted to the Lur'e form, our result is quite general. If the system has no equilibrium, then the linearized system becomes time-varying. In such a case, the current framework cannot be applied.

4.4.3 Convergence Theorem

Multidimensional generalizations of the Bendixson negative criterion was considered by Smith [Smi86] and Leonov [Leo91a] based on similar frameworks. In particular, the following result is a natural consequence of Theorem 4.4 with $d = 2$.

Theorem 4.6. Suppose that the conditions in Theorem 4.4 hold with $d = 2$. Then, every bounded solution of the Lur'e system $\dot{x} = Ax + B\phi(Cx)$ converges to an equilibrium point.

The conditions in Theorem 4.6 are weaker than those in Theorem 3.1 on global stability. However, there may exist multiple equilibria in the current case. In particular, Theorem 4.6 is useful to characterize the absence of closed orbits.

4.5 Application to Lorenz Systems

The exact Lyapunov dimension formula of a Lorenz system was first derived in [Leo02]. We verify that the same formula can be obtained with our framework. Fortunately, we can choose the matrix Φ and the function v to reproduce the result of the meritorious paper [Leo02]. Consider the Lorenz system

$$\begin{aligned}\dot{x} &= -\sigma x + \sigma y,\\ \dot{y} &= rx - y - xz,\\ \dot{z} &= -bz + xy,\end{aligned}$$

where r, b, and σ are positive parameters. As in [Leo02], we assume that

(i) $\sigma + 1 \geq b \geq 2$ and

(ii) $r\sigma^2(4-b) + 2\sigma(b-1)(2\sigma - 3b) > b(b-1)^2$.

The Lorenz system can be written in the form (3.1) with

$$A = \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = I, \quad D = 0.$$

The memoryless nonlinearity in (3.2) and its derivative are given by

$$\phi(x,y,z) = \begin{bmatrix} 0 \\ xz \\ xy \end{bmatrix}, \quad \mathrm{D}\phi(x,y,z) = \begin{bmatrix} 0 & 0 & 0 \\ z & 0 & x \\ y & x & 0 \end{bmatrix}.$$

To satisfy the conditions in Theorem 4.5, we set $\mu = 0$ and

$$P = \begin{bmatrix} a^{-2} + \sigma^{-2}(b-1)^2 & -\sigma^{-1}(b-1) & 0 \\ -\sigma^{-1}(b-1) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $a := \sigma/\sqrt{r\sigma + (b-1)(\sigma - b)}$. This matrix appeared in [Leo02]. Furthermore, we set

$$\Phi = \begin{bmatrix} A^\top P + PA & PB \\ B^\top P & 0 \end{bmatrix}.$$

In this case, the linear time-invariant system is conservative in the sense that the storage function $S(x) = x^\top P x$ satisfies

$$\dot{S}(x(t)) = \begin{bmatrix} y(t) \\ u(t) \end{bmatrix}^\top \Phi \begin{bmatrix} y(t) \\ u(t) \end{bmatrix}$$

for all $t \geq 0$. On the other hand, the nonlinear map ϕ satisfies

$$\begin{aligned} \begin{bmatrix} I \\ \mathrm{D}\phi(x,y,z) \end{bmatrix}^\top \Phi \begin{bmatrix} I \\ \mathrm{D}\phi(x,y,z) \end{bmatrix} &= (A + B\mathrm{D}\phi(x,y,z))^\top P + P(A + B\mathrm{D}\phi(x,y,z)) \\ &= J(x,y,z)^\top P + PJ(x,y,z), \end{aligned}$$

where $J(x,y,z)$ is the Jacobian matrix associated with the closed-loop system.

As a result, the condition (ii) in Theorem 4.5 is equivalent to the condition in [Leo02]. First, we note that the local Lyapunov dimension at the origin is given by

$$\dim_{\mathrm{L}}^{\mathrm{loc}}(\{\varphi^t\}_{t\in\mathbb{R}_+}, 0) = 3 - \frac{2(\sigma + b + 1)}{\sigma + 1 + \sqrt{(\sigma-1)^2 + 4r\sigma}}.$$

Let

$$s_0 := \frac{\sqrt{4r\sigma + (\sigma-1)^2} - 2b - \sigma - 1}{\sqrt{4r\sigma + (\sigma-1)^2} + \sigma + 1}.$$

From [Leo02], we can find a C^1-function v such that for all $s \in (s_0, 1]$,

$$\lambda_1(x,y,z) + \lambda_2(x,y,z) + s\lambda_3(x,y,z) + (2+s)\dot{v}(x,y,z) < 0,$$

where $\lambda_1(x,y,z) \leq \lambda_2(x,y,z) \leq \lambda_3(x,y,z)$ are the solutions of

$$\det(J(x,y,z)^\top P + PJ(x,y,z) - \lambda(x,y,z)P) = 0.$$

Note that v in general depends on s. Therefore, we conclude that

$$\dim_{\mathrm{L}}(\{\varphi^t\}_{t\in\mathbb{R}_+}, K) \leq 3 - \frac{2(\sigma + b + 1)}{\sigma + 1 + \sqrt{(\sigma-1)^2 + 4r\sigma}}.$$

Because the global Lyapunov dimension must be larger than or equal to the local Lyapunov dimension, we obtain

$$\dim_{\mathrm{L}}(\{\varphi^t\}_{t\in\mathbb{R}_+}, K) = 3 - \frac{2(\sigma + b + 1)}{\sigma + 1 + \sqrt{(\sigma-1)^2 + 4r\sigma}}.$$

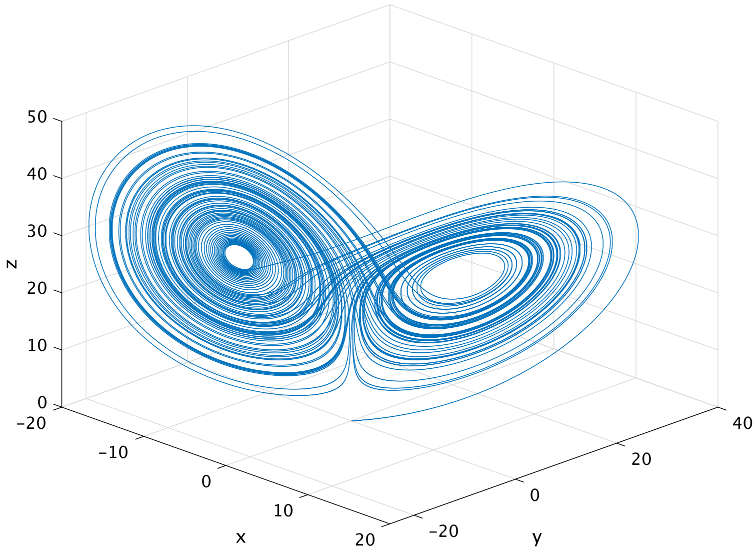

Figure 4.1: Lorenz attractor

We illustrate the Lorenz attractor with the parameters $r = 28$, $b = 8/3$, and $\sigma = 10$ in Fig. 4.1. In general, the exact calculation of the Hausdorff dimension of such a strange attractor is hopeless, and the best known estimate is the exact Lyapunov dimension. In the above case, we have $\dim_{\mathrm{L}}(\{\varphi^t\}_{t\in\mathbb{R}_+}, K) = 2.401$. The Hausdorff dimension of the Lorenz attractor is known to be about 2.062, which is a numerically derived value. It may be difficult to reduce this gap. Furthermore, lower estimates of the Hausdorff dimension are generally harder to obtain than upper estimates (see [LM95a]).

4.6 Conclusion

In this chapter, we revisited the Eden conjecture on local estimation of Lyapunov dimension in a general setting. Furthermore, our results in this chapter suggested a unified treatment of stability and dimension of feedback systems. There are several directions of this study.

- Estimation of topological entropy can be conducted in a similar manner [BL98, PM11]. Since entropy-like concepts have recently attracted attention in the control community [Kaw13], it would be interesting to combine some results on dimension and entropy.

- We have not considered specific structures of attractors such as hyperbolicity. The combination of our results with those in [MVFS18, FS19] is expected to characterize many useful properties of asymptotic behaviors.
- The theory of hidden attractors [LK13] is one of the recent innovative results. Relation with validity of the Eden conjecture and the existence of hidden attractors is an important topic.

Chapter 5

Interconnection of Volume Dissipative Systems

In this chapter, we introduce the concept of volume dissipative systems, which combines the notions of volume contracting systems and differentially dissipative systems. With this concept, we derive an upper bound of the Hausdorff dimension of the attractor of an interconnected system in a scalable way. This chapter partly contains the results in [KI23a, KI23b].

5.1 Introduction

A general framework for stability analysis of interconnected systems was developed by Willems [Wil72a, Wil72b]. Also, instability results for interconnected systems were summarized in [HM83]. On the other hand, quantitative characterizations of unstable dynamics such as dimension and entropy have seldom been studied in the literature. Motivated by the intuitive fact that an interconnection of dynamical systems increases dynamic complexity, we consider attractor dimension of an interconnected system.

Interconnected systems with unstable dynamics were studied in [FS19], where convergence to low-dimensional attractors was characterized by certain Lyapunov and dissipation inequalities. A similar result was obtained in [MS14], where limit cycles were investigated through transverse contraction. Applications of these results have also attracted attentions in recent years [MVFS22, DCS22]. In particular, some relations between unstable dynamics and interconnection structures are important in understanding of biological phenomena [BG21]. Furthermore, some entropy concepts for interconnected systems were considered with applications to networked control systems [Lib21, TZ22]. It is thus meaningful to generalize stability frameworks to instability cases.

The contributions of this chapter are summarized as follows. First, we describe an idea to develop a general framework for dimension analysis of interconnected systems. The notion of volume dissipative systems is introduced by combining volume contraction and differential dissipativity. Then, we consider to estimate attractor dimension of an interconnected system through a scalable way as in the classical stability results [AMP16]. Also, we provide a computationally efficient method by extending the result by Smith. Finally, a numerical example is studied to demonstrate our result.

5.2 From Volume Contraction to Volume Dissipation

In this section, we study a generalization of the framework of differentially dissipative systems.

5.2.1 Volume Contracting Systems

Let $\{\varphi^t\}_{t\in\mathbb{R}_+}$ be a smooth dynamical system on $\mathbb{R}^n$. Here, we introduce the concept of *volume contracting systems*. The following notion comes from the work by Il'yashenko [Il'83, Il'92].

Definition 5.1. Let $d \in (0, n]$ be a real number. The dynamical system is *d-volume contracting* if

$$\dim_{\mathrm{L}}(\{\varphi^t\}_{t\in\mathbb{R}_+}, \mathbb{R}^n) < d.$$

We refer to [BSDM23] for a modern perspective with compound matrices. In particular, any compact attractor of a d-volume contracting system has the Hausdorff dimension less than d.

5.2.2 Volume Dissipative Systems

Consider the nonlinear system described by

$$\begin{cases} \dot{x} = f(x, u), \\ y = h(x, u), \end{cases}$$

where $f\colon \mathbb{R}^{n+m} \to \mathbb{R}^n$ and $h\colon \mathbb{R}^{n+m} \to \mathbb{R}^p$ are C^1-functions. Recall that the associated prolonged system is represented as

$$\begin{cases} \begin{bmatrix} \dot{x} \\ \delta\dot{x} \end{bmatrix} = \begin{bmatrix} f(x,u) \\ \partial_x f(x,u)\delta x + \partial_u f(x,u)\delta u \end{bmatrix}, \\ \begin{bmatrix} y \\ \delta y \end{bmatrix} = \begin{bmatrix} h(x,u) \\ \partial_x h(x,u)\delta x + \partial_u h(x,u)\delta u \end{bmatrix}. \end{cases}$$

Let $\mathrm{d}\mathfrak{B}$ be the differential behavior given in Section 2.4.1.

Let $\Phi\colon \mathbb{R}^m \times \mathbb{R}^p \to \mathcal{S}_n$ be a measurable function. Assume that there exist a C^1-function $P\colon \mathbb{R}^n \to \mathcal{S}_n$ and a C^0-function $Q\colon \mathbb{R}_+ \to \mathcal{S}_n$ such that

(i) $P(x)$ is positive definite for all $x \in \mathbb{R}^n$ and

(ii) the inequality

$$\begin{bmatrix} \delta\dot{x}(t) \\ \delta x(t) \end{bmatrix}^\top \begin{bmatrix} 0 & P(x(t)) \\ P(x(t)) & \dot{P}(x(t)) - Q(x(t)) \end{bmatrix} \begin{bmatrix} \delta\dot{x}(t) \\ \delta x(t) \end{bmatrix} \le \begin{bmatrix} \delta y(t) \\ \delta u(t) \end{bmatrix}^\top \Phi(u(t), y(t)) \begin{bmatrix} \delta y(t) \\ \delta u(t) \end{bmatrix}$$

holds for all $t \ge 0$ and all $(x, \delta x, u, \delta u, y, \delta y) \in \mathrm{d}\mathfrak{B}$.

If there exists $d \in [0, n]$ such that

$$\sup_{x \in \mathbb{R}^n} \zeta_d(Q(x)P(x)^{-1}) < 0,$$

then we say that the system is *d-volume dissipative*. This is because the system with zero input is d-volume contracting.

We would like to discuss two possible extensions of the concept mentioned above. While our notion is based on internal dynamics, a related concept for systems with inputs was proposed in the context of networked control [NEMM04, KC09]. On the other hand, the notion of compound systems is appropriate for volume contraction analysis [MW21, WKM22, WPMS22]. However, there is a technical difficulty in the definition of compound systems with inhomogeneous types [MS10]. These problems are left for future studies.

5.2.3 Interconnections

Here, we describe an interconnection of N volume dissipative systems. For $i \in \{1, \ldots, N\}$, consider the nonlinear system

$$\begin{cases} \dot{x}_i = f_i(x_i, u_i), \\ y_i = h_i(x_i, u_i), \end{cases}$$

where $f_i \colon \mathbb{R}^{n_i+m_i} \to \mathbb{R}^{n_i}$ and $h_i \colon \mathbb{R}^{n_i \times m_i} \to \mathbb{R}^{p_i}$ are C^1 functions for each $i \in \{1, \ldots, N\}$. We denote by $\mathrm{d}\mathfrak{B}_i$ the differential behavior of the i-th subsystem.

Let $\Phi_i \colon \mathbb{R}^{p_i+m_i} \to \mathcal{S}_{p_i+m_i}$ be a measurable function. It is assumed that there exist a C^1-function $P_i \colon \mathbb{R}^{n_i} \to \mathcal{S}_{n_i}$ and a C^0-function $Q_i \colon \mathbb{R}^{n_i} \to \mathcal{S}_{n_i}$ such that

(i) $P_i(x_i)$ is positive definite for all $x_i \in \mathbb{R}^{n_i}$ and

(ii) the inequality

$$\begin{bmatrix} \delta\dot{x}_i(t) \\ \delta x_i(t) \end{bmatrix}^\top \begin{bmatrix} 0 & P_i(x_i(t)) \\ P_i(x_i(t)) & \dot{P}_i(x_i(t)) - Q_i(x_i(t)) \end{bmatrix} \begin{bmatrix} \delta\dot{x}_i(t) \\ \delta x_i(t) \end{bmatrix} \le \begin{bmatrix} \delta y_i(t) \\ \delta u_i(t) \end{bmatrix}^\top \Phi_i(u_i(t), y_i(t)) \begin{bmatrix} \delta y_i(t) \\ \delta u_i(t) \end{bmatrix}$$

holds for all $t \ge 0$ and all $(x_i, \delta x_i, u_i, \delta u_i, y_i, \delta y_i) \in \mathrm{d}\mathfrak{B}_i$.

Let $x := \mathrm{col}(x_1, \ldots, x_N)$, $u := \mathrm{col}(u_1, \ldots, u_N)$, and $y := \mathrm{col}(y_1, \ldots, y_N)$. Then, the composite system is given by

$$f(x,u) := \begin{bmatrix} f_1(x_1, u_1) \\ \vdots \\ f_N(x_N, u_N) \end{bmatrix}, \quad h(x,u) := \begin{bmatrix} h_1(x_1, u_1) \\ \vdots \\ h_N(x_N, u_N) \end{bmatrix}.$$

An interconnection can be interpreted as the linear output feedback represented by

$$u = My,$$

where $M \in \mathbb{R}^{(m_1+\cdots+m_N)\times(p_1+\cdots+p_N)}$. Suppose that

$$\begin{bmatrix} I \\ M \end{bmatrix}^\top \Phi_1(u_1, y_1) \oplus \cdots \oplus \Phi_N(u_N, y_N) \begin{bmatrix} I \\ M \end{bmatrix} \le 0$$

for all $(u_i, y_i) \in \mathbb{R}^{m_i+p_i}$ and all $i \in \{1,\dots,N\}$. Under this assumption, it is not difficult to see that the closed-loop system preserves the dissipativity property [AMP16]. Indeed, we sum up all dissipation inequalities to obtain

$$\begin{aligned}
&\begin{bmatrix} \delta\dot{x}(t) \\ \delta x(t) \end{bmatrix}^\top \begin{bmatrix} 0 & P(x(t)) \\ P(x(t)) & \dot{P}(x(t)) - Q(x(t)) \end{bmatrix} \begin{bmatrix} \delta\dot{x}(t) \\ \delta x(t) \end{bmatrix} \\
&\quad \le \delta y(t)^\top \begin{bmatrix} I \\ M \end{bmatrix}^\top \Phi_1(u_1, y_1) \oplus \cdots \oplus \Phi_N(u_N, y_N) \begin{bmatrix} I \\ M \end{bmatrix} \delta y(t),
\end{aligned}$$

where $P(x) := P_1(x_1) \oplus \cdots \oplus P_N(x_N)$ and $Q(x) := Q_1(x_1) \oplus \cdots \oplus Q_N(x_N)$. Thus,

$$\begin{bmatrix} \delta\dot{x}(t) \\ \delta x(t) \end{bmatrix}^\top \begin{bmatrix} 0 & P(x(t)) \\ P(x(t)) & \dot{P}(x(t)) - Q(x(t)) \end{bmatrix} \begin{bmatrix} \delta\dot{x}(t) \\ \delta x(t) \end{bmatrix} \le 0, \quad \forall t \ge 0,$$

As a result, the interconnected system is d-volume contracting if

$$\sup_{x \in \mathbb{R}^n} \zeta_d(Q(x)P(x)^{-1}) < 0.$$

If the i-th subsystem is d_i-volume dissipative, then the closed-loop system is at least $(d_1 + \cdots + d_N)$-volume dissipative. This can be confirmed as follows: If

$$\zeta_{d_i}(Q_i(x_i)P_i(x_i)^{-1}) < 0, \quad \forall i \in \{1,\dots,N\},$$

then

$$\zeta_{d_1+\cdots+d_N}(Q(x)P(x)^{-1}) \le \sum_{i=1}^{N} \zeta_{d_i}(Q_i(x_i)P_i(x_i)^{-1}) < 0.$$

The above result is analogues to those in [FS19] and [TZ22]. In particular, our result and that in [TZ22] provide upper bounds of dimension and entropy of an interconnected system, respectively. On the other hand, the result in [FS19] indicates a dominated splitting for an interconnected system.

5.3 On Smith's Method

In [Smi86], Smith proposed a method for estimation of the Hausdorff dimension with a low computational cost. This section aims to generalize Smith's method to interconnected systems.

5.3.1 Trace Inequalities

Consider a linear differential equation

$$\dot{x} = A(t)x, \tag{5.1}$$

where $A\colon \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ is a C^0-function. The following lemma generalizes Corollary 1.2 in [Smi86], where P is restricted to a constant matrix.

Lemma 5.1. Let $X\colon \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ be the Cauchy matrix of (5.1), and let $k \in \{1,\dots,k\}$ be an integer. Suppose that there exist a C^1-function $P\colon \mathbb{R}_+ \to \mathcal{S}_n$ and a C^0-function $\theta\colon \mathbb{R}_+ \to \mathbb{R}$ such that

(i) $P(t)$ is positive definite for all $t \geq 0$ and

(ii) the matrix inequality

$$\dot{P}(t) + A(t)^\top P(t) + P(t)A(t) + 2\theta(t)P(t) \geq 0$$

holds for all $t \geq 0$.

Then, we have

$$\omega_k(X(t)) \leq \frac{\omega_{n-k}(P(t))}{\omega^{n-k}(P(0))} \exp \int_0^t [\operatorname{tr} A(\tau) + (n-k)\theta(\tau)]\, \mathrm{d}\tau$$

for all $t \geq 0$.

Proof. Letting $T(t) := P(t)^{1/2}$, we have

$$\widetilde{A}(t)^\top + \widetilde{A}(t) + 2\theta(t)I \geq 0,$$

where $\widetilde{A}(t) := [T(t)A(t) + \dot{T}(t)]T(t)^{-1}$. This implies that

$$\eta_i(\widetilde{A}(t)) + \theta(t) \geq 0, \quad \forall i \in \{1,\dots,n\}.$$

Thus, we have

$$\begin{aligned}\zeta_k(\widetilde{A}(t)) &= \operatorname{tr} \widetilde{A}(t) - \sum_{i=k+1}^{n} \eta_i(\widetilde{A}(t)) \\ &\leq \operatorname{tr} \widetilde{A}(t) + (n-k)\theta(t).\end{aligned}$$

We define $Y(t) := Q(t)X(t)$. It satisfies the matrix differential equation

$$\dot{Y}(t) = \widetilde{A}(t)Y(t).$$

The generalized Ważewski inequalities (Lemma 4.4) indicate that

$$\begin{aligned}\omega_k(Y(t)) &\le \omega_k(Y(0)) \exp \int_0^t \zeta_k(\widetilde{A}(\tau))\,\mathrm{d}\tau \\ &\le \omega_k(Q(0)) \exp \int_0^t [\operatorname{tr} \widetilde{A}(\tau) + (n-k)\theta(\tau)]\,\mathrm{d}\tau.\end{aligned}$$

By Horn's inequality, it follows that

$$\omega_k(X(t)) \le \omega_k(Q(t)^{-1})\omega_k(Y(t)) = \frac{1}{\omega^k(Q(t))}\omega_k(Y(t)).$$

Thus, we have

$$\omega_k(X(t)) \le \frac{\omega_k(Q(0))}{\omega^k(Q(t))} \exp \int_0^t [\operatorname{tr} \widetilde{A}(s) + (n-k)\theta(s)]\,\mathrm{d}s.$$

Note that $\operatorname{tr} \widetilde{A}(t) = \operatorname{tr} A(t) + \operatorname{tr} \dot{Q}(t)Q(t)^{-1}$. The integral of $\operatorname{tr} \dot{Q}(t)Q(t)^{-1}$ can be calculated as follows. By Jacobi's formula, we have

$$\operatorname{tr} \dot{Q}(t)Q(t)^{-1} = \frac{(\mathrm{d}/\mathrm{d}t) \det Q(t)}{\det Q(t)}.$$

By integrating both sides over time, we can derive

$$\begin{aligned}\int_0^t \operatorname{tr} \dot{Q}(s)Q(s)^{-1}\,\mathrm{d}s &= \int_0^t \frac{(\mathrm{d}/\mathrm{d}s) \det Q(s)}{\det Q(s)}\,\mathrm{d}s \\ &= \log\det Q(t) - \log\det Q(0).\end{aligned}$$

Finally, we obtain

$$\begin{aligned}\omega_k(X(t)) &\le \frac{\omega_k(Q(0))}{\omega^k(Q(t))} \frac{\det Q(t)}{\det Q(0)} \exp \int_0^t [\operatorname{tr} A(s) + (n-k)\theta(s)]\,\mathrm{d}s \\ &= \frac{\omega_{n-k}(Q(t))}{\omega^{n-k}(Q(0))} \exp \int_0^t [\operatorname{tr} A(s) + (n-k)\theta(s)]\,\mathrm{d}s.\end{aligned}$$

Since $\sigma_i(Q(t)) = \sigma_i(P(t))^{1/2}$ for all $i \in \{1, \dots, n\}$, we complete the proof. □

5.3.2 Dimension Estimates

Consider a nonlinear differential equation

$$\dot{x} = f(x), \tag{5.2}$$

where $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a C^1-function. We assume that this system is forward complete. Now, we generalize Corollary 2.2 in [Smi86] based on state-dependent metric tensors.

Note that more general periodic time-varying systems were considered in [Smi86]; however, our focus in on the extension to the use of time-varying matrix P as in the previous sections. In the theorem below, we use the following notation:

$$\dot{P}(x) := \begin{bmatrix} \nabla P_{11}(x)^\mathsf{T} f(x) & \cdots & \nabla P_{1n}(x)^\mathsf{T} f(x) \\ \vdots & \ddots & \vdots \\ \nabla P_{n1}(x)^\mathsf{T} f(x) & \cdots & \nabla P_{nn}(x)^\mathsf{T} f(x) \end{bmatrix}.$$

Theorem 5.1. Let $K \subset \mathbb{R}^n$ be a compact invariant set of (5.2). Suppose that there exist a matrix-valued function $P\colon K \to \mathcal{S}_n$ and a function $\lambda\colon K \to \mathbb{R}$ such that

(i) $P(x)$ is positive definite for all $x \in K$ and

(ii) the matrix inequality

$$\dot{P}(x) + \mathrm{D}f(x)^\mathsf{T} P(x) + P(x)\mathrm{D}f(x) + 2\theta(x)P(x) \geq 0. \tag{5.3}$$

holds for all $x \in K$.

If there exists $d \in (0, n]$ such that

$$\sup_{x\in K}[\operatorname{tr} \mathrm{D}f(x) + (n-d)\theta(x)] < 0, \tag{5.4}$$

then $\dim_\mathrm{H}(K) < d$.

Proof. By applying Lemma 5.1, we obtain

$$\omega_k(\mathrm{D}\varphi^t(x)) \leq \frac{\omega_{n-k}(P(t))}{\omega^{n-k}(P(0))} \exp\int_0^t [\operatorname{tr} \mathrm{D}f(\varphi^\tau(x)) + (n-k)\theta(\varphi^\tau(x))]\,\mathrm{d}\tau$$

for all $t \geq 0$. If $d = j + s$ with $j \in \{0, \dots, n-1\}$ and $s \in (0, 1]$, then

$$\omega_d(\mathrm{D}\varphi^t(x)) = \omega_j(\mathrm{D}\varphi^t(x))^{1-s}\omega_{j+1}(\mathrm{D}\varphi^t(x))^s.$$

It follows that

$$\omega_d(\mathrm{D}\varphi^t(x)) \leq \frac{\omega_{n-d}(P(\varphi^t(x)))}{\omega^{n-d}(P(x))} \exp\int_0^t [\operatorname{tr} \mathrm{D}f(\varphi^\tau(x)) + (n-d)\theta(\varphi^\tau(x))]\,\mathrm{d}\tau$$

for all $t \geq 0$. Because K is compact, $\omega_{n-d}(P(\varphi^t(x)))/\omega^{n-d}(P(x))$ is bounded over $t \geq 0$ and $x \in K$. This implies that there exists $t \geq 0$ sufficiently large such that

$$\sup_{x\in K} \omega_d(\mathrm{D}\varphi^t(x)) < 0.$$

By the Douady–Oesterlé theorem, we conclude that $\dim_\mathrm{H}(K) < d$. □

5.3.3 Interconnected Systems

We then investigate an interconnection of N systems. For each $i \in \{1, \dots, N\}$, the i-th subsystem is described by

$$\begin{cases} \dot{x}_i = f_i(x_i) + B_i u_i, \\ y_i = C_i x_i, \end{cases} \tag{5.5}$$

where $f_i \colon \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$, $B_i \in \mathbb{R}^{n_i \times m_i}$, and $C_i \in \mathbb{R}^{p_i \times n_i}$. These systems can be written in the following composite form:

$$\begin{cases} \dot{x} = f(x) + Bu, \\ y = Cx, \end{cases}$$

where

$$f(x) := \begin{bmatrix} f_1(x_1) \\ \vdots \\ f_N(x_N) \end{bmatrix}, \quad B := B_1 \oplus \cdots \oplus B_N, \quad C := C_1 \oplus \cdots \oplus C_N.$$

It is assumed that f is a C^1-function. Let us now consider the interconnection represented by $u = My$, where $M \in \mathbb{R}^{(m_1 + \cdots + m_N) \times (p_1 + \cdots + p_N)}$ represents the interconnection structure.

We explain the basic assumptions. For $i \in \{1, \dots, N\}$, let $\Phi_i \in \mathcal{S}_n$ be a symmetric matrix. Then, assume that there exist a C^1-function $P_i \colon \mathbb{R}^{n_i} \to \mathcal{S}_{n_i}$ and a C^0-function $\theta_i \colon \mathbb{R}^{n_i} \to \mathbb{R}$ such that

(i) $P_i(x_i)$ is positive definite for all $x_i \in \mathbb{R}^{n_i}$ and

(ii) the inequality

$$\begin{aligned} &\begin{bmatrix} \delta\dot{x}_i(t) \\ \delta x_i(t) \end{bmatrix}^\top \begin{bmatrix} 0 & P_i(x_i(t)) \\ P_i(x_i(t)) & \dot{P}_i(x_i(t)) + 2\theta_i(x_i(t))P_i(x_i(t)) \end{bmatrix} \begin{bmatrix} \delta\dot{x}_i(t) \\ \delta x_i(t) \end{bmatrix} \\ &\qquad \geq \begin{bmatrix} \delta y_i(t) \\ \delta u_i(t) \end{bmatrix}^\top \Phi_i(t) \begin{bmatrix} \delta y_i(t) \\ \delta u_i(t) \end{bmatrix} \end{aligned} \tag{5.6}$$

holds for all $t \geq 0$ and all $(x_i, \delta x_i, u_i, \delta u_i, y_i, \delta y_i) \in \mathrm{d}\mathfrak{B}_i$.

Moreover, we assume that the diagonal blocks of M are all zeros, that is, M has the form

$$M = \begin{bmatrix} 0 & * & \cdots & * \\ * & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ * & \cdots & * & 0 \end{bmatrix}. \tag{5.7}$$

This assumption means that the interconnection does not include any self loop.

In the above setting, the closed-loop system can be written as

$$\dot{x} = f(x) + BMCx.$$

We notice that $\operatorname{tr} BMC = 0$ because of the structure of M in (5.7). This point is of special importance as the trace of the Jacobian is independent of the interconnection. Now, we state the main result of this chapter.

Theorem 5.2. Let $K \subset \mathbb{R}^{n_1+\cdots+n_N}$ be a compact invariant set of the interconnected system. Assume that the matrix M satisfies

$$\begin{bmatrix} I \\ M \end{bmatrix}^\top (\Phi_1 \oplus \cdots \oplus \Phi_N) \begin{bmatrix} I \\ M \end{bmatrix} \geq 0.$$

Also, assume that for each $i \in \{1, \ldots, N\}$, there exists $d_i \in (0, n_i]$ such that

$$\sup_{x_i \in \mathbb{R}^{n_i}} [\operatorname{tr} \mathrm{D} f_i(x_i) + (n_i - d_i)\theta_i(x_i)] < 0.$$

Then, we have $\dim_{\mathrm{H}}(K) < d$ with d satisfying

$$\sup_{x \in K} \left[\zeta_{n-d}(\theta_1(x_1)I_{n_1} \oplus \cdots \oplus \theta_{n_N}(x_N)I_{n_N}) - \sum_{i=1}^{N} (n_i - d_i)\theta_i(x_i) \right] \leq 0. \tag{5.8}$$

Proof. For the sake of convenience, let

$$P(x) := P_1(x_1) \oplus \cdots \oplus P_N(x_N) \quad \text{and} \quad \Theta(x) := \theta_1(x_1)I_{n_1} \oplus \cdots \oplus \theta_{n_N}(x_N)I_{n_N}.$$

By summing up all inequalities (5.6), we obtain

$$\begin{bmatrix} \delta\dot{x} \\ \delta x \end{bmatrix}^\top \begin{bmatrix} 0 & P(x) \\ P(x) & \dot{P}(x) + 2\Theta(x)P(x) \end{bmatrix} \begin{bmatrix} \delta\dot{x} \\ \delta x \end{bmatrix} \geq \delta x^\top C^\top \begin{bmatrix} I \\ M \end{bmatrix}^\top \Theta(x) \begin{bmatrix} I \\ M \end{bmatrix} C\delta x.$$

Note that we have used the fact that $P(x)$ and $\Theta(x)$ commute. Since the right-hand side is nonnegative, we have

$$\begin{bmatrix} \delta\dot{x} \\ \delta x \end{bmatrix}^\top \begin{bmatrix} 0 & P(x) \\ P(x) & \dot{P}(x) + 2\Lambda(x)P(x) \end{bmatrix} \begin{bmatrix} \delta\dot{x} \\ \delta x \end{bmatrix} \geq 0.$$

Thus, we have

$$\dot{P}(x) + J(x)^\top P(x) + P(x)J(x) + 2\Theta(x)P(x) \geq 0,$$

where $J(x) := \mathrm{D}f(x) + BMC$. It follows that

$$\omega_d(\mathrm{D}\varphi^t(x)) \le \frac{\omega_{n-d}(P(\varphi^t(x)))}{\omega^{n-d}(P(x))} \exp \int_0^t [\operatorname{tr} J(\varphi^\tau(x)) + \zeta_{n-d}(\Theta(\varphi^\tau(x)))] \,\mathrm{d}\tau$$

Because $\operatorname{tr} J(x) = \operatorname{tr} \mathrm{D}f(x)$, we obtain

$$\begin{aligned}\operatorname{tr} J(x) + \zeta_{n-d}(\Theta(x)) &= \sum_{i=1}^N \operatorname{tr} \mathrm{D}f_i(x_i) + \zeta_{n-d}(\Theta(x)) \\ &\le -\sum_{i=1}^N (n_i - d_i)\theta_i(x_i) + \zeta_{n-d}(\Theta(x)).\end{aligned}$$

By hypothesis, it follows that

$$\sup_{x\in K}[\operatorname{tr} J(x) + \omega_{n-d}(\Theta(x))] < 0.$$

Then, we can observe that $\sup_{x\in K} \omega_d(\mathrm{D}\varphi^t(x)) < 1$ for some $t \ge 0$ sufficiently large. The Douady–Oesterlé theorem (Theorem 4.1) insures that $\dim_{\mathrm{H}}(K) < d$. The proof is complete.

□

In the above theorem, the numbers d_i are independent of the interconnection. Thus, the condition (5.8) is relatively easy to verify. The main advantage of the above theorem is scalability of dimension estimates.

5.4 Application to Coupled Lorenz Systems

Consider the following coupled Lorenz system:

$$\begin{cases} \dot{x}_1 = \sigma(y_1 - x_1) - x_2, \\ \dot{y}_1 = \rho x_1 + x_1 z_1 - y_1, \\ \dot{z}_1 = x_1 y_1 - \beta z_1, \end{cases} \qquad \begin{cases} \dot{x}_2 = \sigma(y_2 - x_2) + x_1, \\ \dot{y}_2 = \rho x_2 + x_2 z_2 - y_2, \\ \dot{z}_2 = x_2 y_2 - \beta z_2, \end{cases}$$

where $(\sigma, \rho, \beta) = (10, 28, 8/3)$. Since it is difficult to satisfy the inequality (5.6) on the whole state space, we verify the conditions on the following ranges: $x_i \in [-20, 30]$, $y_i \in [-30, 30]$, $z_i \in [-10, 50]$. These ranges over approximate the closed-loop global attractor. We consider $P_i = I$ and the supply rates with $(Q_i, S_i, R_i) = (0, 1, 0)$ for both subsystems. Then, we can choose $\theta_i = 35$ to satisfy the inequality (5.6). Also, d_i in Theorem 5.2 are given by $d_1 = d_2 = 2.61$ on the region mentioned above. These values are conservative since the actual Hausdorff dimension of the Lorenz attractor is about 2.06, but it may be made less conservative by using more appropriate storage functions and supply rates. Because the interconnection matrix is skew symmetric, the condition in Theorem 5.2 immediately holds. Therefore, the attractor dimension of the closed-loop system can be estimated as $d_1 + d_2 = 5.22$.

5.5 Conclusion

In this chapter, we considered a general framework on dimension theory for interconnected systems. It was shown that attractor dimensions of large-scale interconnected systems can be estimated in a scalable way. There is the possibility to generalize the concept of volume dissipation. In the current framework, energy inequalities and volume contractions are independent properties. It may be possible to combine these two properties to develop a comprehensive theory for energy dissipative and volume contracting systems. Another open problem is its relation with entropy concepts as studied in the systems and control community [BL98, NEMM04, KC09, PM11, Lib21, TZ22]. In this regard, it is worth mentioning that some entropy notions can be interpreted in dimension-like characteristics [Pes97].

Chapter 6

Conclusion

6.1 Summary

A primal interest of this study was the combination of feedback theory and dimension theory. In what follows, we summarize the achievements of each chapter.

In Chapter 3, we provided new sufficient conditions and necessary conditions for global stability of a Lur'e system. Our framework is based on the method of differential Lyapunov functions. Particularly, we suggested to use a differential Lyapunov function of the Leonov form, which consists of a usual quadratic function and a novel Lyapunov-like function. It was shown that such an additional Lyapunov-like function reduces conservativeness of our stability conditions.

In Chapter 4, we gave an upper bound of Lyapunov dimension for a Lur'e system. The result is an extension of that in the previous chapter to systems with unstable dynamics. Moreover, a key point in this chapter is the investigation of Eden's conjecture on the equivalence between local and global Lyapunov dimensions. A suitable characterization of the class of nonlinearities for which Eden's conjecture is valid was provided. The method developed in this chapter is useful to derive the exact Lyapunov dimension formula.

In Chapter 5, we considered a general idea to integrate some theoretical tools in stability theory, dissipativitiy theory, and dimension theory. We formally defined the concept of volume dissipative systems, which combines the volume contraction properties and the dissipation inequalities. Moreover, we provided a generalization of Smith's method for dimension estimates. The generalized method is able to treat nonconstant metrics and feedback interconnections. Such a generalization is important when we need to control complex systems in nontraditional fields, where control objective is not necessarily stabilization.

6.2 Outlook

As mentioned in the introduction, it is important to study dimension-like characteristics and their relations with quantized control systems and with determining nodes of complex networks. Moreover, there are several directions of this research. Our particular interest is in the theory of dimension and entropy and that of Lyapunov exponents and Lyapunov functions. We conclude this thesis by indicating overviews of some ideas.

Adaptation and Learning

This thesis is based on the concept of dimension. In general, dimension-like characteristics include Hausdorff dimension, box dimension, metric entropy, topological entropy, and so on [Pes97]. From this perspective, it is expected to develop a comprehensive framework to study dimension and entropy concepts for dynamical systems.

Furthermore, dimension and entropy can be defined for not only a dynamical system itself but also a set of dynamical systems. This fact leads us to study complexity concepts for adaptation and learning in uncertain environments as pointed out in [Zam79]. Although machine learning has been a recent major trend, it would be important to consider the questions posed by Zames [Zam98b]:

> What should the terms "adaptive" and "learning" mean in the context of control? Is it possible to tell whether or not a black box is adaptive without knowledge of its internal structure? In design, is it possible to determine beforehand whether it is necessary for a controller to adapt and learn in order to meet performance specifications, or is adaptation a matter of choice?

In particular, it is interesting to analyze dimension and entropy for uncertain systems which are modeled as a subset of some space.

Theory of Oscillations

From a personal perspective, Lyapunov methods are fundamental tools in not only stability theory but also oscillation theory. There are essentially two types of oscillations that we would like to understand. One is self-excited, and the other is hidden [LK13].

The former type is essentially based on instability of an equilibrium and boundedness of the trajectories. This type of oscillations is important to understand the mechanism of oscillations in biological systems [SDF19]. In the generation of self-excited oscillations, some balance between positive energy and negative energy [MVFS22] or between positive feedback and negative feedback [DCS22] plays a crucial role. Thus, indefinite energy functions and some specific feedback structures are expected to be important.

The later type is relatively new, and it is different from the self-excited type because the equilibrium and the limit cycle both remain stable. Relations of hidden oscillations with stability theory of feedback systems have now been an attractive topic (see [Kuz20] for an overview). In the recent paper [KMKK20], the classical Lorenz model was studied in details. However, oscillation phenomena are far from complete understanding even in 3-dimensional dynamical systems.

Appendix A

Mathematical Background

A.1 Dynamical Systems and Attractors

In this appendix, we present some mathematical concepts of dynamical systems and attractors. We mostly follow the book [BLR05]. Other good references are [Tem97, Rob01].

A.1.1 Flows

First, we recall the basic definition of dynamical systems.

Definition A.1. A (continuous-time) *dynamical system* is the triplet

$$\Sigma = (T, X, \Phi)$$

consisting of the following sets and map:

- A set $T \in \{\mathbb{R}, \mathbb{R}_+\}$, called the *time set*;
- A nonempty set X, called the *phase space*;
- A map $\Phi \colon T \times X \to X$, called the *evolution map*, such that
  - (i) $\Phi(0, x) = x$ for all $x \in X$;
  - (ii) $\Phi(t + s, x) = \Phi(t, \Phi(s, x))$ for all $t, s \in T$ and all $x \in X$.

Usually, X is required to be at least a metric space or a topological space, and Φ is required to be a continuous map. The two properties in the items (i) and (ii) represent the deterministic rule of the time evolution. Thus, there is no stochastic effect in the dynamics.

For $t \in T$, we define a transformation $\varphi^t : X \to X$ by

$$\varphi^t(x) := \Phi(t, x), \quad x \in X.$$

The one-parameter (semi)group $\{\varphi^t\}_{t\in T}$ is simply referred to as a dynamical system or flow. In what follows, we suppose that X is a metric space and that $\{\varphi^t\}_{t\in\mathbb{R}_+}$ is a smooth dynamical system on X. We define some important concepts.

Definition A.2. A nonempty set $M \subset X$ is called

(i) an *invariant set* if $\varphi^t(M) = M$ for all $t \geq 0$;

(ii) a *positively invariant set* if $\varphi^t(M) \subset M$ for all $t \geq 0$;

(iii) a *negatively invariant set* if $M \subset \varphi^t(M)$ for all $t \geq 0$.

Now, we define the point-to-set distance by

$$\operatorname{dist}(x, M) := \sup_{y\in M} d(x, y), \quad x \in M, \quad M \subset X.$$

Definition A.3. An invariant set $M \subset X$ is said to be

(i) *stable* if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\varphi^t(\mathcal{N}_\delta(M)) \subset \mathcal{N}_\varepsilon(M), \quad \forall t \geq 0;$$

(ii) *asymptotically stable* if it is stable and there exists $\delta > 0$ such that

$$\lim_{t\to\infty} \operatorname{dist}(\varphi^t(x), M) = 0, \quad \forall x \in \mathcal{N}_\delta(M);$$

(iii) *uniformly asymptotically stable* if it is stable and there exists $\delta > 0$ such that

$$\lim_{t\to\infty} \operatorname{dist}(\varphi^t(\mathcal{N}_\delta(M)), M) = 0.$$

A.1.2 Attractors

There are several definitions of attractors. The most important concept in the current context is the maximal global attractor, which contains all information on the asymptotic behaviors of a dynamical system.

Definition A.4. An invariant set $M \subset X$ is said to

(i) *attract* a point $p \in X$ if

$$\lim_{t\to\infty} \mathrm{dist}(\varphi^t(x), M) = 0;$$

(ii) *uniformly attract* a set $S \subset X$ if

$$\lim_{t\to\infty} \sup_{x\in S} \mathrm{dist}(\varphi^t(x), M) = 0.$$

Definition A.5. A compact invariant set $\mathcal{A} \subset X$ is called

(i) an *attractor* if it attracts every point in a neighborhood of X;

(ii) a *global attractor* if it attracts every point in X.

(iii) a *global $\mathcal{B}$-attractor* if it uniformly attracts every bounded set in X.

(iv) a *maximal global attractor* (resp., *maximal global $\mathcal{B}$-attractor*) if it is the maximal set among all global attractors (resp., global $\mathcal{B}$-attractors).

(v) a *minimal global attractor* (resp., *minimal global $\mathcal{B}$-attractor*) if it is the minimal set among all global attractors (resp., global $\mathcal{B}$-attractors).

Sometimes, the invariance of $\mathcal{A}$ can be replaced by the positively invariance. Also, the compactness of $\mathcal{A}$ is not assumed in some literature. The difference between the maximal and minimal global attractors can be explained as follows: Suppose that there is a stable limit cycle and an unstable equilibrium on a plane. Then, the maximal global attractor is the disk enclosed by the limit cycle, while the minimal global attractor is the union of the limit cycle and the equilibrium.

The following concept is known as dissipativity in the sense of Levinson.

Definition A.6. The dynamical system $\{\varphi^t\}_{t\in\mathbb{R}_+}$ is said to be

(i) *$\mathcal{L}$-dissipative* if there exists a bounded set $B \subset X$ which attracts every point in X.

(ii) *uniformly $\mathcal{L}$-dissipative* if there exists a bounded set $B \subset X$ which attracts every bounded set in X.

Thus, every $\mathcal{L}$-dissipative dynamical system possesses a compact global attractor.

A.1.3 Hausdorff and Box Dimensions

We here introduce the definitions of Hausdorff dimension and box dimension. The basic idea was first introduced by Carathéodory [Car14]. The notion of Hausdorff dimension was then defined by Hausdorff himself [Hau18]. According to [Fal14], it seems difficult to trace the origin of the notion of box dimension. For more details, we refer to [BLR05, Rob11, Fal14].

Let X be a metric space, and let $K \subset X$ be a compact set. We denote by $\mathcal{B}_r(x)$ the (open) ball in X centered at x with radius r. Given $d \geq 0$ and $\varepsilon > 0$, we set

$$\mu(K,d,\varepsilon) := \inf\left\{\sum_i (r_i)^d : K \subset \bigcup_i \mathcal{B}_{r_i}(x_i),\ x_i \in K,\ r_i \leq \varepsilon\right\},$$

where the infimum is taken over all finite covers by balls of K. If K and d are fixed, then $\mu(K,d,\varepsilon)$ is nonincreasing as ε is increasing. We then define the *Hausdorff d-measure* of K by

$$\mu_{\mathrm{H}}(K,d) := \lim_{\varepsilon\to 0^+} \mu(K,d,\varepsilon) = \sup_{\varepsilon>0} \mu(K,d,\varepsilon).$$

The Hausdorff dimension of K is defined as follows:

Definition A.7. The *Hausdorff dimension* of K is defined by

$$\dim_{\mathrm{H}}(K) := \inf\{d \geq 0 : \mu_{\mathrm{H}}(K,d) = 0\}.$$

Let $n_\varepsilon(K)$ denote the minimal number of balls of radius ε which are necessary to cover K. The box dimension of K is defined as follows:

Definition A.8. The *(upper-)box dimension* of K is defined by

$$\dim_{\mathrm{B}}(K) := \limsup_{\varepsilon\to 0^+} \frac{\log n_\varepsilon(K)}{\log(1/\varepsilon)}.$$

It is known that the following relation holds:

$$\dim_{\mathrm{H}}(K) \leq \dim_{\mathrm{B}}(K).$$

A.2 Lyapunov and Riccati Inequalities

Here, we review some useful facts from linear systems theory.

A.2.1 Lyapunov Lemma

Let $A \in \mathbb{R}^{n\times n}$. The spectral abscissa of A is defined by

$$\alpha(A) := \max\{\mathrm{Re}(\lambda) : \lambda \text{ is an eigenvalue of } A\}.$$

Definition A.9. The matrix A is called a *Hurwitz matrix* if $\alpha(A) < 0$.

The following lemma describes the solvability of the Lyapunov inequality.

Lemma A.1. The following statements are equivalent:

(i) A is a Hurwitz matrix.

(ii) There exists a positive-definite matrix $P \in \mathcal{S}_n$ such that

$$A^\top P + PA \prec 0. \tag{A.1}$$

Let $Q \in \mathcal{S}_n$ be a positive-definite matrix. If A is Hurwitz, then the matrix

$$P := \int_0^\infty \mathrm{e}^{tA^\top} Q \mathrm{e}^{tA}\,\mathrm{d}t$$

is positive definite and satisfies the Lyapunov equation

$$A^\top P + PA = -Q.$$

This is a solution to the Lyapunov inequality (A.1).

Consider now a linear time-varying system

$$\dot{x} = A(t)x, \tag{A.2}$$

where $A\colon \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ is a C^0-function. Let $X\colon \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ be a fundamental matrix of (A.2).

Definition A.10. The linear system (A.2) is said to be

(i) *nonuniformly exponentially stable* if there exist $K \geq 1$ and $\lambda > 0$ such that

$$\|X(t)\| \leq K\mathrm{e}^{-\lambda t}$$

for all $t \geq 0$;

(ii) *uniformly exponentially stable* if there exist $K \geq 1$ and $\lambda > 0$ such that

$$\|X(t)X(s)^{-1}\| \leq K\mathrm{e}^{-\lambda(t-s)}$$

for all $t \geq s$ and all $s \geq 0$.

The following lemma is useful to investigate stability of (A.2).

Lemma A.2. Suppose that there exist a C^1-function $P\colon \mathbb{R}_+ \to \mathcal{S}_n$ and a C^0-function $\mu\colon \mathbb{R}_+ \to \mathbb{R}$ such that

(i) $P(t)$ is positive definite for all $t \geq 0$ and

(ii) the matrix inequality

$$\dot{P}(t) + A(t)^\top P(t) + P(t)A(t) - 2\mu(t)P(t) \preceq 0$$

holds for all $t \geq 0$.

Then, we have

$$\frac{\|X(t)\|}{\|X(s)\|} \leq \left[\frac{\lambda_{\max}(P(s))}{\lambda_{\min}(P(t))}\right]^{1/2} \exp \int_0^t \mu(\tau)\,\mathrm{d}\tau$$

for all $t \geq s$ and all $s \geq 0$.

We define a function $V\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ by

$$V(t,x) = x^\top P(t)x, \quad t \geq 0, \quad x \in \mathbb{R}^n.$$

Under the hypothesis in the above lemma, for every solution $x\colon \mathbb{R}_+ \to \mathbb{R}^n$ to (A.2), we have

$$\dot{V}(t,x(t)) \leq 2\mu(t)V(t,x(t)), \quad \forall t \geq 0.$$

Because this is a scalar differential inequality, the comparison lemma states that

$$V(t,x(t)) \leq V(s,x(s)) \exp \int_s^t 2\mu(\tau)\,\mathrm{d}\tau$$

for all $t \geq s$ and all $s \geq 0$.

A.2.2 Kalman–Yakubovich–Popov Lemma

The Kalman–Yakubovich–Popov lemma states a certain equivalence between linear matrix inequalities and frequency-domain inequalities. This relation was first proved by Yakubovich for strict inequalities [Yak62] and was then investigated by Kalman for non-strict inequalities [Kal63]. In [Kal63], the solvability of so-called Lur'e equations was also considered. These initial contributions were devoted to the single-input/single-output case. A generalization to the multiple-input/multiple-output case was established by

Popov [Pop64]. Special cases of the Kalman–Yakubovich–Popov lemma are known as the bounded real lemma and the positive real lemma [AV73].

Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ be given. We first recall the definitions of controllability and observability.

Definition A.11. The pair (A, B) is said to be *controllable* if

$$\operatorname{rank}\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = n.$$

Let $M \in \mathcal{S}_{n+m}$ be a symmetric matrix partitioned as

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^{\mathsf{T}} & M_{22} \end{bmatrix}.$$

Two versions of the Kalman–Yakubovich–Popov lemma are stated below. The first one involves strict inequalities, and the second one involves nonstrict inequalities.

Lemma A.3. Suppose that A has no eigenvalue on the imaginary axis. Then, the following statements are equivalent:

(i) There exists a symmetric matrix $P \in \mathcal{S}_n$ such that

$$\begin{bmatrix} A^{\mathsf{T}}P + PA & PB \\ B^{\mathsf{T}}P & 0 \end{bmatrix} + M \prec 0.$$

(ii) For all $\omega \in [-\infty, \infty]$,

$$\begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^{\mathsf{H}} M \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} \prec 0.$$

Note that P in the above lemma is positive definite if A is Hurwitz and M_{11} is positive semidefinite.

Lemma A.4. Suppose that (A, B) is controllable. Then, the following statements are equivalent:

(i) There exists a symmetric matrix $P \in \mathcal{S}_n$ such that

$$\begin{bmatrix} A^{\mathsf{T}}P + PA & PB \\ B^{\mathsf{T}}P & 0 \end{bmatrix} + \begin{bmatrix} Q & L \\ L^{\mathsf{T}} & R \end{bmatrix} \preceq 0.$$

(ii) For all $\omega \in (-\infty, \infty)$ such that $\det(j\omega I - A) \neq 0$,

$$\begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^{\mathsf{H}} \begin{bmatrix} Q & L \\ L^{\mathsf{T}} & R \end{bmatrix} \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} \preceq 0.$$

Note that P in the above lemma is positive definite if A is Hurwitz, M_{11} is positive semidefinite, and (A^{T}, M_{12}) is controllable.

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