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Automorphism groups of K3 surfaces and their group invariants

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Abstract

This thesis consists of two parts. In the first part of this thesis, we study the constructions of certain K3 surfaces in particular singular K3 surfaces with discriminant 3, 4 and 7. We give a construction of the singular K3 surface with discriminant 3 as a double covering over the projective plane. Moreover, we can generalize this construction and obtain 3-dimensional moduli space of K3 surfaces and give the characterization of the K3 surfaces that appear in this moduli space.

In the second part of this thesis, we focus on the natural action of automorphism groups of K3 surfaces on hyperbolic spaces, which come from the action on the cohomology of degree 2. Then we can consider the symplectic automorphism group of a K3 surface as a geometrically finite group. By applying the result of Tomohiro Fukaya, we show that the virtual cohomological dimension of the automorphism group of a K3 surface is determined by the covering dimension of the blown-up boundary associated with the ample cone of the K3 surface.

Contents

Chapter 1

Introduction

1.1 Singular K3 surfaces

A compact complex surface X is called a *K3 surface* if the canonical bundle is trivial and the irregularity is zero. By Torelli theorem for K3 surfaces, lattice theory can be applied in classification of K3 surfaces.

For example, Shioda and Inose ([30, Theorem 5]) showed that any singular K3 surface (i.e., a K3 surface with Picard number 20) admits infinitely many automorphisms. In spite of their significance, there are only eleven such K3 surfaces such that the automorphism groups are actually calculated as shown in the following table:

No.	$[a,b,c]$	
1	[2,1,2]	Vinberg [36]
2	[2,0,2]	Vinberg [36]
3	[2,1,4]	Ujikawa [35]
4	[2,0,4]	Shimada [27]
5	[2,1,6]	Shimada [27]
6	[2,0,6]	Shimada [27]
7	[4,2,4]	Keum and Kondo [14]
8	[2,1,8]	Shimada [27]
9	[4,1,4]	Shimada [27]
10	[2,0,8]	Shimada [27]
11	[4,0,4]	Keum and Kondo [14]

Here, each of the K3 surfaces X is specified by a triple $[a, b, c]$, which indicates the Gram matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ of the transcendental lattice T_X. Note that X in the cases treated by Keum and Kondo [14] (No. 7 and 11) is a Kummer surface.

In the first part of this thesis, we focus on the singular K3 surfaces with discriminant 3, 4 and 7, and aim at finding common properties among them, which should give us clue to understand the geometry of these surfaces more in depth. From now on we denote by X_d these three K3 surfaces with discriminant d ($d = 3, 4, 7$), which correspond to No. 1, 2 and 3 in the table above, respectively.

Naruki [22] gave some constructions of the singular K3 surface of discriminant 7 (No. 3 in the above table). One of the constructions is given by the double covering of $\mathbb{P}^2$ branched over a specific plane sextic, which is induced from the quotient by the involution associated to an elliptic fibration. We will discuss this construction in §2, where we will see how one can find the explicit equation of the sextic, as Naruki did in Section 5 of [22]. In Section 3.2, we will see, moreover, that this method can be extended to the cases of discriminant 3 and 4 to obtain the double covering constructions and the explicit equations of the corresponding sextics. Interestingly, as we will show in Section 3.2, these sextics have several properties in common, especially in their singularity types. For example, they have a singular point of type D_4, and the pencil of the lines passing through this point gives rise to the elliptic fibration on the K3 surfaces. From these common properties, one can read a general recipe to construct all of these K3 surfaces, and obtain a three-dimensional period domain $\mathcal{B}$.

To state our main theorem precisely, let

$$
\begin{aligned}
\mathcal{B} &:= \{\omega \in \mathbb{P}(T \otimes \mathbb{C}) |\ \langle \omega, \omega \rangle = 0, \langle \omega, \overline{\omega} \rangle > 0\}, \\
\mathcal{H}_\delta &:= \{\omega \in \mathcal{B} |\ \langle \omega, \delta \rangle = 0\} \text{ for } \delta \in T, \\
\mathcal{H} &:= \bigcup_{\delta \in T} \mathcal{H}_\delta,
\end{aligned}
$$

where $\langle\ ,\ \rangle$ is the inner product of T. Here, T is a lattice defined in §3.

A *marked K3 surface* is a pair of a K3 surface X and an isometry α_X from $H^2(X, \mathbb{Z})$ to the unimodular even lattice $U^{\oplus 3} \oplus E_8^{\oplus 2}$ with signature (3,19), where U is the hyperbolic lattice defined in Section 2.1.

Theorem 1 ([34, Theorem 7.1])**.** Let X be a K3 surface and let ω_X be a nowhere vanishing holomorphic 2-form of X. Then the following conditions are equivalent.

1. There exists a sextic curve C such that X is isomorphic to the minimal resolution of the double cover of $\mathbb{P}^2$ branched along the sextic curve C, that the Mordell Weil group associated to C is trivial, and that C satisfies the following three conditions (in Section 3.3, we define the Jacobian elliptic fibration associated to C) :

    (i) C has a D_4 (resp. A_3) singularity at a point q (resp. three distinct points $p_i, i = 1, 2, 3$), and is smooth elsewhere.

(ii) $\{p_1, p_2, p_3, q\}$ lies in general position.

(iii) The intersection multiplicity of C and the line $p_i q$ at p_i (resp. q) is 2 (resp. 4).

2. There exists an elliptic fibration $f : X \to \mathbb{P}^1$ such that f has three singular fibres of types I_6, and others are of types I_1 or II, and the associated Mordell-Weil group $MW(f)$ is trivial.

3. There exists a marking α_X for X such that $\alpha_X(\omega_X) \in \mathcal{B}\backslash\mathcal{H}$.

This result is already published in the paper of the author [34]. Here, by the definition of $\mathcal{B}$, three singular K3 surfaces X_d ($d = 3, 4, 7$) correspond to points in the period domain $\mathcal{B}$, and by the result of Nikulin ([24, Theorem 0.2.2]), a general member of $\mathcal{B}$ admits an infinite automorphism group.

Recently, Sienkiewicz [32] has discussed the construction of X_4 as the double covering of the del-Pezzo quintic, and its automorphism group. This construction is essentially the same as the construction in [34].

1.2 Automorphism groups of K3 surfaces

In the latter part of this thesis, we discuss the classification problem of automorphism groups of K3 surfaces, which is located at the intersection of several research fields, such as lattice theory, reflection group theory, and finite sporadic simple group theory.

There are two approaches to automorphisms of K3 surfaces. The first one is the classification under some finiteness conditions, such as the classification of finite subgroups, and that of possible finite order of automorphisms. For example, a well-known and important result in this approach is Mukai's result ([20, Theorem (0.3)]), which says that symplectic finite subgroups of the automorphism groups of K3 surfaces are isomorphic to subgroups of the Mathieu group M_{23}, one of the sporadic simple groups.

The other approach is the application of Borcherds' lattice theory (often called Borcherds' method) to compute a generating set of infinite automorphism groups. The problem of finding a generating set of the automorphism groups of Kummer surfaces was first proposed by Klein. This problem took more than 100 years to be solved by Kondo [16] using Borcherds' method. Although this method is very useful and, in fact, the automorphism groups of some specific K3 surfaces have been calculated (the singular K3 surface is shown in the table in the previous section), this method is not applicable to the general case.

However, Shigeru Mukai [21] recently proposed at the occasion of the Algebra Symposium in Tokyo in 2018 the following conjecture on discrete group theory and automorphism groups, from which one may hopefully find a general framework for dealing with infinite automorphism groups of elliptic K3 surfaces.

Conjecture 1 (S. Mukai [21]). Let X be an elliptic K3, Enriques or Coble surface, and let Aut(X) be the automorphism group of X. Then, the virtual cohomological dimension vcd(Aut(X)) of Aut(X) is equal to the maximum of the Mordell-Weil rank induced by all elliptic surface structures on X.

Remark 1.2.1. The virtual cohomological dimension of Aut(X) is easily seen to be larger than the rank of the Mordell-Weil group of any elliptic fibration. Hence the main point of Mukai's conjecture lies in finding the upper-bound of vcd(Aut(X)) as stated therein.

This conjecture could be a turning point from the conventional study of automorphism groups of elliptic K3, Enriques and Coble surfaces by computing their generators to that by computing their cohomological invariants. Our main result is a formula that determines the virtual cohomological dimension of the automorphism group of a K3 surface by the covering dimension of the blown-up boundary associated with the ample cone of the K3 surface.

Theorem 2. If $\mathrm{Aut}_s(X)$ is non-elementary, then

$$\mathrm{vcd}(\mathrm{Aut}(X)) = \dim \partial A_X^{bl} + 1,$$

where ∂A_X^{bl} is the blown-up boundary associated with the ample cone of X (see Chapter 6 for the detailed discussion.)

To show this result, we focus on the natural action of automorphism groups on hyperbolic spaces, which come from the action on the cohomology of degree 2. Then we can consider the symplectic automorphism group of a K3 surface as a discrete subgroup of Möbius transformations of a hyperbolic space. Moreover, we can show the following theorem:

Theorem 3. Let X be a K3 surface, and let $\mathrm{Aut}_s(X)$ be the symplectic automorphism group of X. Then $\mathrm{Aut}_s(X)$ is geometrically finite.

In [5], Bestvina and Mess showed that the cohomological dimension of a torsion free hyperbolic group is determined by the covering dimension of its Gromov boundary. Recently, Tomohiro Fukaya [12] generalized the formula for relatively hyperbolic groups satisfying a certain condition. Geometrically finite groups are typical examples of relative hyperbolic groups. We can immediately obtain Theorem 1 by applying Fukaya's result [12, Corollary B].

On the other hand, the paper of Fukaya is still unpublished, and his theorem and his proof are stated using geometric group theory. For readability to algebraic geometers, this paper also contains the result and the proof of Fukaya, which are simplified and modified to fit in with our situation, in Section 6.2, using the geometry of ample cones of K3 surfaces and hyperbolic geometry.

Remark 1.2.2. After the author finished the first draft of this paper, Tomohiro Fukaya informed the author that Theorem 2 has been independently obtained and proved by Kohei Kikuta. He showed geometrical finiteness for the automorphism groups of irreducible symplectic varieties over a field of characteristic zero, and his proof is different in some points from our proof given in this paper. For more details, the reader is advised to refer to his forthcoming paper.

An application of Theorem 1 is the case that the ample cones of K3 surfaces are sphere packings.

Theorem 4. *Let X be an elliptic K3 surface. If ∂A_X is a connected sphere packing, and every elliptic divisor of X corresponds to the tangent point of some two spheres associated with (-2)-roots, then*

$$\mathrm{vcd}(\mathrm{Aut}(X)) = \max\{\mathrm{rk}\,\mathrm{MW}(f)\} = \rho(X) - 3,$$

where f runs all elliptic fibrations of X.

Baragar [2] constructed K3 surfaces whose ample cones are sphere packings. Then we can compute the virtual cohomological dimension of their automorphism groups. Note that these K3 surfaces provide examples that affirm Conjecture 1, and, to the best of our knowledge, they are the first non-trivial examples in the sense that the verification of the conjecture does not rely on explicit calculation of the group structure of the automorphism groups.

1.3 The contents of this thesis

In Chapter 2, we recall some basic definitions and facts of K3 surfaces and some related topics. By the Torelli type theorem for K3 surfaces, the theory of K3 surfaces can be interpreted as the theory of lattices. In Section 2.1, we recall the definition and some basic facts of lattices, and in Section 2.2, we study K3 surfaces and the Torelli type theorem for K3 surfaces. In Section 2.3, we recall the definition of elliptic surfaces, and in Section 2.4, we review the result of Shoda-Inose. In Section 2.5, we study the construction of singular K3 surfaces with discriminant 3, 4, and 7.

In Chapter 3, we study K3 surfaces, which are given by the double plane model of $\mathbb{P}^2$ branched over sextic curves with D_4 singularities. In particular, in Section

3.2, we give the double plane model of singular K3 surfaces with discriminant 3, 4 and 7. In Section 3.3, we prove Theorem 1.

In Chapter 4, we recall some basics of hyperbolic geomery. In hyperbolic geometry, it is useful to use some models of hyperbolic spaces in different situations. Therefore, we intruduce the hyperboloid model, the conformal ball model, and the upper half-space model of hyperbolic spaces. To prove Theorem 2, the notion of geometrically finite subgroups of Möbius transformations plays an important role. So we recall the definition of gemetrically finite subgroups and some basic properties.

In Chapter 5, we recall some basic facts of the cohomological dimension of groups, and we review the result of Bestvina and Mess [5].

In Chapter 6, we introduce the natural action of automorphism groups of K3 surfaces on hyperbolic spaces, and we show that the symplectic automorphism group of a K3 surface is a geometrically finite subgroup. By the result of Fukaya, we prove Theorem 2.

Chapter 2

K3 surfaces

2.1 Lattices

In this section, we recall some definitions and facts about lattice theory, which are contained in [17] and [23].

Definition 2.1.1. Let L be a free abelian group of rank r, and let $\langle\ ,\ \rangle : L \times L \to \mathbb{Z}$ be a symmetric bilinear form. A symmetric bilinear form $\langle\ ,\ \rangle$ is said to be *non-degenerate* if $\langle x, y\rangle = 0$ for any $y \in L$ implies $x = 0$. We call a pair $(L, \langle\ ,\ \rangle)$ of L and a non-degenerate symmetric bilinear form a *lattice* of rank r.

Unless there is any risk of confusion, we denote the lattice $(L, \langle\ ,\ \rangle)$ by L. The *orthogonal group* of L is the group consisting of isometries of L, which is denoted by $O(L)$. The *dual lattice* $\mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$ is denoted by L^*. Since $\langle\ ,\ \rangle$ is non-degenerate, a map

$$L \to L^* : \ x \mapsto \langle x, \cdot\rangle$$

is injective.

Let $\{v_1, \ldots, v_r\}$ be a basis of a lattice L of rank r. For each $x \in L$, write $x = \sum_{i=1}^{r} x_i v_i$ and also define $a_{ij} = \langle v_i, v_j\rangle$. Then, $f(x) = \langle x, x\rangle = \sum_{i,j} a_{i,j} x_i x_j$ is a quadratic form. By Sylvester's theorem, we can write

$$f(x) = t_1^2 + \cdots t_p^2 - t_{p+1}^2 - \cdots - t_{p+q}^2, \quad r = p + q,$$

where t_i are variables in $\mathbb{R}$. A pair (p, q) is called the signature of lattice L. A symmetric matrix $A = (a_{i,j})$ is called a *Gram matrix* with respect to a basis $\{v_1, \ldots, v_r\}$. We denote by $d(L)$ the absolute value of the determinant of the Gram matrix A. This is well-defined.

Definition 2.1.2. Let L, p, and q be as above. A lattice L is said to be *positive* (resp. *negative*) definite if $p = r$ (resp. $q = r$). A lattice L is said to be *unimodular* if $d(L) = 1$.

Remark 2.1.3. For a lattice L and a basis of L, we defined the quadratic form $f(x)$ and the Gram matrix A. Conversely, a quadratic form or a Gram matrix determines the structure of a lattice L.

Example 2.1.4. We denote the lattice of rank 1 given by the quadratic form $f(x) = \pm x^2$ by $I_\pm$. Then $I_+^{\oplus p} \oplus I_-^{\oplus q}$ is an odd unimodular lattice of signature (p, q). The Gram matrix of $I_+^{\oplus p} \oplus I_-^{\oplus q}$ is given by as follows:

$$\begin{pmatrix} I_p & \\ & -I_q \end{pmatrix},$$

where I_p (resp. I_q) is the identity matrix of rank p (resp. q).

Example 2.1.5. The *hyperbolic lattice U* is the lattice of rank 2 with the Gram matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $(L, \langle\ ,\ \rangle)$ be a lattice. A *sublattice* of L is a subgroup S of L such that $(S, \langle\ ,\ \rangle|_S)$ is a lattice. A sublattice S of L is said to be *primitive* if the quotient group L/S has no torsion element. The orthogonal complement of S is denoted by $S^\perp$.

Definition 2.1.6. An element δ of L is called a (-2)-*root* if $\delta^2 = -2$. A lattice is called a *root lattice* if it is a negative-definite even lattice and generated by (-2)-roots.

Example 2.1.7. We define the sublattice A_m of $I_-^{\oplus m+1}$ by

$$A_m = \{(x_1, \dots, x_{m+1}) \in \mathbb{Z}^{m+1} : \sum_{i=1}^{m+1} x_i = 0\}.$$

With the standard basis $\{e_1 = (1, 0, \dots, 0), e_2 = (0, 1, 0, \dots, 0), \dots, e_{m+1} = (0, \dots, 1)\}$ of $\mathbb{Z}^{m+1}$, it is easy to show that A_m is generated by $r_i = e_i - e_{i+1}$ $(i = 1, \dots, m)$ and A_m is of rank m. Since $r_i^2 = -2$, A_m is a root lattice.

Root lattices can be easy to understand by means of Dynkin diagrams. The correspondence is as follows. Let r_i $(i = 1, \dots, n)$ be a basis of a root lattice. Then each r_i corresponds to a vertex of a Dynkin diagram, and two vertices corresponding to r_i and r_j is connected by $\langle r_i, r_j\rangle$-edges. A root lattice is said to be *irreducible* if the corresponding Dynkin diagram is a connected graph. It is well-known that any root lattice is isomorphic to a sum of irreducible root lattices of types A_l, D_m

and E_n ($l \geq 1,\ m \geq 4,\ n = 6, 7, 8$). The lattice in Example 2.1.7 is the irreducible root lattice of type A_m. The list of Dynkin diagrams is in the last page of this section.

It is easy to show the following lemma.

Lemma 2.1.8. Let L and S be lattices. Assume that L has a sublattice of finite index isomorphic to S. Then

$$d(S) = d(L) \times |L/S|^2.$$

Definition 2.1.9. A lattice L is called an *even lattice* if $\langle x, x\rangle$ is even for any element x of L. For an even lattice L, a quotient group L^*/L is denoted by A_L.

Definition 2.1.10. The *discriminant quadratic form* of L is defined to be

$$q_L : A_L \to \mathbb{Q}/2\mathbb{Z}\ ,\ q_L(x + L) := \langle x, x\rangle\ mod\ 2\mathbb{Z}.$$

Let $O(q_L)$ be the group consisting of automorphisms of A_L that preserves q_L.

In lattice theory, the following lemma is well-known.

Lemma 2.1.11 ([23, Theorem 1.14.4])**.** Let T be an even lattice with the signature (t_+, t_-), and let q_T be the discriminant quadratic form of T, and let L be an even unimodular lattice of signature (s_+, s_-). Assume that

(i) $t_+ < s_+$,

(ii) $t_- < s_-$,

(iii) $l(A_T) + 2 \leq rank(L) - rank(T)$, where $l(A_T)$ is the length of A_T.

Then there exists a unique primitive embedding of T into L.

Let L be a unimodular even lattice, and let S be a primitive sublattice of L. Let T be the orthogonal complement of S. The lattice T is also a primitive sublattice of L. Let H be a quotient group $L/(S \oplus T)$. We denote by p_S (resp. p_T) the projection map from $A_S \oplus A_T$ to A_S (resp. A_T). Then p_S and p_T satisfy the following Lemma.

Lemma 2.1.12 ([17, Lemma 1.31])**.** $p_S|_H : H \to A_S$ and $p_T|_H : H \to A_T$ are bijective maps.

Proof. It is enough to show the injectivity of p_S and p_T because $|A_S| \cdot |A_T| = |H|^2$. Let $(x \text{ mod } S,\ y \text{ mod } T) \in H$ be an element of the kernel of $p_s|H$, where $x \in S^*$, $y \in T^*$. Since $x + y \in L$ and $x \in S \subset L$, we have $y \in L \cap T^*$. Since T is primitive, we have $y \in T$. Hence, $(x \text{ mod } S,\ y \text{ mod } T) = 0$. In the same way, $p_T|H$ is injective. □

Hence the map

$$r_{S,T} = p_T \circ (p_S|H)^{-1} : A_S \to A_T$$

is an isomorphism of groups.

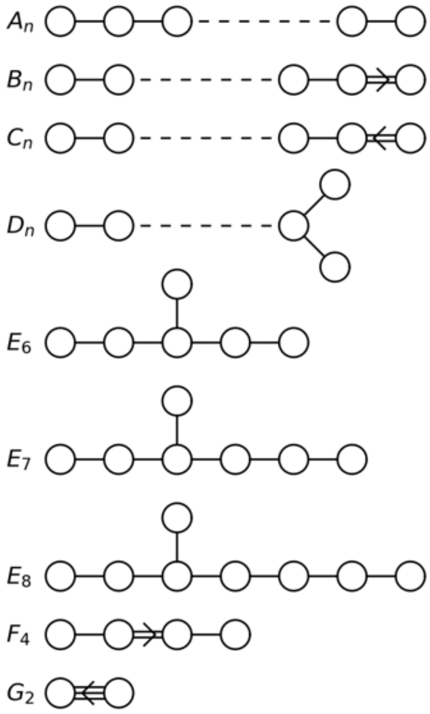

Figure 2.1

2.2 K3 surfaces and the period map of K3 surfaces

2.2.1 K3 surfaces

Definition 2.2.1. Let X be a complex surface. Suppose X is compact and connected. If the canonical bundle K_X is trivial and the irregularity $q(X)$ is equal to 0, X is called a *K3 surface*.

Example 2.2.2. Let Y_4 be a nonsingular quartic surface in $\mathbb{P}^4$. By the adjunction formula, the canonical line bundle K_{Y_4} is trivial. By Lefschetz's theorem, the irregularity of Y_4 is 0. Then Y_4 is a K3 surface. In the same way, $Y_6 = Z_2 \cap Z_3$ and

$Y_8 = Q_1 \cap Q_2 \cap Q_3$ are K3 surfaces, where Z_i is a hypersurface of degree i in $\mathbb{P}^4$ and Q_i is a hypersurface of degree2 in $\mathbb{P}^5$.

Theorem 2.2.3 ([17, Theorem 4.5])**.** For a K3 surface X, the second cohomology group $H^2(X, \mathbb{Z})$ with the cup product is a unimodular even lattice with the signature $(3, 19)$. Then the Gram matrix of $H^2(X, \mathbb{Z})$ is isomorphic to $U^{\oplus 3} \oplus E_8^{\oplus 2}$.

Definition 2.2.4. Let ω_X be a nowhere vanishing holomorphic 2-form of X. The *Néron-Severi lattice* $NS(X)$ is defined by

$$NS(X) := \{x \in H^2(X, \mathbb{Z}) \ : \ \langle x, \omega_X \rangle = 0\} = H^2(X, \mathbb{Z}) \cap H^{1,1}(X, \mathbb{R}).$$

The Néron-Severi lattice $NS(X)$ is a lattice by the cup product. The Néron-Severi lattice is also denoted by S_X. The *transcendental lattice* T_X of X is defined by $NS(X)^{\perp}$.

Definition 2.2.5. The rank of $NS(X)$ is called the *Picard number* of X. The Picard number of X is denoted by $\rho(X)$.

By Riemann-Roch theorem and Serre duality, we have the following lemma.

Lemma 2.2.6 ([17, Lemma 4.16])**.** Let X be a K3 surface, and let δ be an element of S_X with $\delta^2 \geq -2$. Then δ or $-\delta$ is represented by an effective divisor.

Proof. Let L be a line bundle on X which represents $\delta \in NS(X)$. By Riemann-Roch theorem and Serre duality, we have

$$\begin{aligned}
&\dim H^0(X, \mathcal{O}(L)) + \dim H^2(X, \mathcal{O}(L)) \\
&= \dim H^0(X, \mathcal{O}(L)) + \dim H^0(X, \mathcal{O}(-L)) \\
&\geq 2 + \delta^2/2 \\
&\geq 1.
\end{aligned} \tag{2.2.1}$$

Hence we have $\dim H^0(X, \mathcal{O}(L)) > 0$ or $\dim H^0(X, \mathcal{O}(-L)) > 0$. Then δ or $-\delta$ is represented by an effective divisor. $\square$

Definition 2.2.7. Let X be a K3 surface. We define

$$\Delta(X) := \{\delta \in S_X : \delta \text{ is a } (-2)\text{-root } \}$$

and

$$\Delta^+(X) := \{\delta \in \Delta(X) : \delta \text{ is effective}\}.$$

For $\delta \in NS(X)$, the *reflection* s_δ in $H^{1,1}(X, \mathbb{R})$ is defined by

$$s_\delta(x) = x + \langle x, \delta \rangle \delta \ \ for \ \ x \in L.$$

By an easy calculation, the reflection s_δ preserves the bilinear form of L, and satisfies $s_\delta^2 = 1$. Hence s_δ is an element of $O(L)$. The *reflection group* $W(X)$ is the subgroup of $O(H^{1,1}(X,\mathbb{R}))$ generated by the set of reflections $\{s_\delta : \delta \in \Delta(X)\}$. The cone $P(X) = \{x \in H^{1,1}(X,\mathbb{R}) : \langle x, x\rangle > 0\}$ has two connected components and the cone containing the Kähler class is denoted by $P^+(X)$. The cone $P^+(X)$ is called the *positive cone*.

Definition 2.2.8. The group $W(X)$ acts on the space $P^+(X)$ in a canonical way. The *Kähler cone* of X is the one of the fundamental domains given by

$$D(X) = \{x \in P^+(X) : \langle x, \delta\rangle > 0 \text{ for any } \delta \in \Delta(X)^+\}$$

containing Kähler class of X.

For a K3 surface X, the following theorem is well-known.

Definition 2.2.9. Let X be an algebraic K3 surface. The *ample cone* $A(X)$ is defined by

$$A(X) = \{x \in NS(X) \otimes \mathbb{R} \cap P^+(X) : \text{for any } \delta \in \Delta(X)^+, \langle x, \delta\rangle > 0\}.$$

Theorem 2.2.10 (The Torelli type theorem for K3 surfaces [7] and [25])**.** Let X and X' be K3 surfaces, and let ω_X (resp. $\omega_{X'}$) be a nowhere vanishing holomorphic 2-form on X (resp. X'). Assume that the isomorphism $\phi : H^2(X,\mathbb{Z}) \to H^2(X',\mathbb{Z})$ satisfies the following conditions:

1. $\phi(\omega_X) \in \mathbb{C}\omega_{X'}$,
2. $\phi(D(X)) = D(X')$.

Then, there exists uniquely an isomorphism $\psi : X' \to X$ such that $\psi^* = \phi$.

2.2.2 The period map of K3 surfaces

Definition 2.2.11. Let L be a unimodular even lattice with the signature $(3, 19)$. The *period domain* Ω is defined by

$$\Omega = \{\omega \in L \otimes \mathbb{C} : \langle \omega, \omega\rangle = 0,\ \langle \omega, \overline{\omega}\rangle > 0\}.$$

Let X be a K3 surface, and let $\alpha_X : H^2(X,\mathbb{Z}) \to L$ be an isometry of lattices. The pair (X, α_X) is called a *marked K3 surface*.

Let $\mathcal{M}$ be the set of equivelence classes of marked K3 surfaces. The period map of marked K3 surfaces is defined to be:

$$\lambda : \mathcal{M} \to \Omega,\ (X, \alpha_X) \to \alpha(\omega_X).$$

It is known that this map is surjective ([18], [19]).

2.3 Elliptic fibrations

Definition 2.3.1. An *elliptic surface* is a morphism $\pi : X \to C$ from a surface X to a curve C with connected fibres such that any fibre except over finitely many points of C is an elliptic curves. If a fibre of an elliptic fibration is not smooth, this fibre is called a *singular fibre*.

Remark 2.3.2. If the number of components of a singular fibre is larger than 2, any irreducible component of the singular fibre is a rational curve. In this case, The dual graph associated to the singular fibre is defined as follows in a similar way of root lattices.

Let F be a singular fibre of an elliptic surface, and let C_i $(i = 1, \cdots, m)$ be irreducible components of F. Each irreducible component of a singular fibre corresponds to a vertex of the dual graph, and and two vertices corresponding to C_i and C_j is connected by $\langle C_i, C_j \rangle$-edges, where $\langle C_i, C_j \rangle$ is the intersection number of C_i and C_j. The dual graph obtained by this way is called the *extended Dynkin diagram* of F. The list of extended Dynkin diagrams is in the next page.

Let $f : X \to \mathbb{P}^1$ be an elliptic K3 surface with a global section g. We regard each smooth fibre as an elliptic curve with the zero element being the intersection with g. In particular, we have a rational selfmap of X given by inversion on each fiber. By the minimality of K3 surfaces, this rational map extends to an automorphism of X. This automorphism is called the *inversion involution*.

Let $F_\nu = \sum_{i=0}^{m_\nu} C_\nu^i$ $(0 \leq \nu \leq k)$ be singular fibres of f, where C_ν^i is an irreducible component of F_ν. We assume that a global section g intersects the rational curve C_ν^0. By Kodaira's classification of singular fibres ([15, Theorem 9.1]), the lattice generated by C_ν^i $(1 \leq i \leq m_\nu)$ is isomorphic to an irreducible root lattice L_ν. We denote by F a divisor defined by a general fiber of f, and by L a divisor $F + g$. Since $F^2 = L^2 = 0$ and $F \cdot L = 1$, the lattice generated by F and L is isomorphic to the hyperbolic lattice U. Since $C_\nu^i \cdot F = C_\nu^i \cdot L = C_\lambda^i \cdot C_\mu^j = 0$ $(\lambda \neq \mu, i \geq 1)$, these lattices are orthogonal to each other. Hence S_X has a sublattice isomorphic to $U \oplus (\oplus_{\nu=1}^k L_\nu)$. We call this sublattice the *trivial sublattice associated with f*. The trivial sublattice associated with f is denoted by L_f. This means that L_f is determined by the types of singular fibres of f.

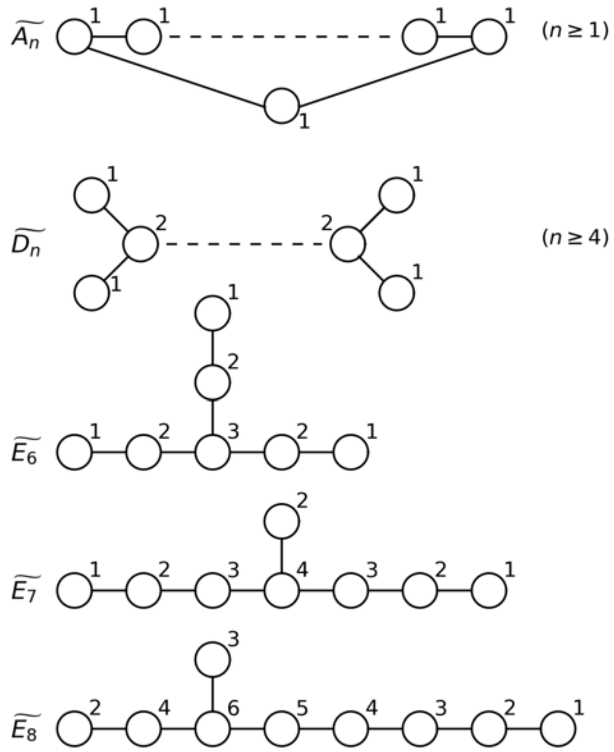

Figure 2.2

2.4 The result of Shioda and Inose

Let $f : X \to \mathbb{P}^1$ be an elliptic K3 surface with a section, and let L_f be the trivial sublattice. The sections of f generate a group. The group is called the *Mordell-Weil group* $MW(f)$. The Mordell-Weil group $MW(f)$ is a finitely generated abelian group, and satisfies the following isomorphism ([29, Theorem 1.3]):

$$MW(f) \cong NS(X)/L_f. \tag{2.4.1}$$

We denote by $r(f)$ the rank of $MW(f)$. When $r(f) = 0$, let $n(f)$ denote the order of $MW(f)$.

We will use the following theorem and lemmata by Shioda and Inose [30].

Lemma 2.4.1 ([30, Lemma 1.1])**.** Assume that an effective divisor D on a K3 surface X has the same type as a simple singular fibre of an elliptic surface in the sense of Kodaira [15] §6. Then there is a unique elliptic pencil $f : X \to \mathbb{P}^1$ of which D is a singular fibre. Moreover, any irreducible curve C on X with $(CD) = 1$ defines a (holomorphic) section of f.

Lemma 2.4.2 ([30, Lemma 1.2])**.** If a divisor D on an elliptic surface has its support contained in a (simple) singular fibre, then the self-intersection number D^2 is non-positive, and $D^2 = 0$ if and only if D is a multiple of a singular fibre.

Lemma 2.4.3 ([30, Lemma 1.3])**.** Let $f : X \to \mathbb{P}^1$ denote an elliptic K3 surface, and $F_v = f^{-1}(t_v)$ $(1 \leq v \leq k)$ be the singular fibres of f. We denote by m_v and $m_v^{(1)}$ respectively the number of irreducible components of F_v, and the number of components of F_v of multiplicity 1. Assume that f has a section. Then

1. The Picard number $\rho(X)$ of X is given by the following formula:
$$\rho(X) = r(f) + 2 + \sum_{v=1}^{k} (m_v - 1). \tag{2.4.2}$$

2. When $r(f) = 0$, we have
$$|\det T_X| = |\det S_X| = (\prod_{v=1}^{k} m_v^{(1)})/n(f)^2. \tag{2.4.3}$$

Theorem 2.4.4 ([30, Theorem 4])**.** There exists a bijective map from the set $\mathcal{S}$ of equivalence classes of singular K3 surfaces to the set $\mathcal{T}$ of isomorphism classes of positive definite oriented even lattices of rank 2.

Remark 2.4.5. Let $f : X \to \mathbb{P}^1$ be an elliptic fibration of a K3 surface X. If f has no section, we put $j : J \to \mathbb{P}^1$ be the jacobian fibration of f. The sections of j generate a group $MW(f)$, which is also callded the Mordell-Weil group of the elliptic fibration $f : X \to \mathbb{P}^1$. We also denote by $r(f)$ the rank of $MW(f)$, and $\rho(X)$ is given by the same equation as in the equation 2.4.2 (see [11, Formula (4.3.2)]).

The map is defined to be $X \mapsto T_X$. By Torelli theorem for K3 surfaces, it is easy to show the injectivity of this map. For any singular abelian surface A, Shioda and Inose explicitly constructed the singular K3 surface X which is a double covering of a Kummer surface Km(A) and satisfies $T_X = T_A$. Then the surjectivity follows from the similar result on singular abelian surfaces ([31, Theorem 3.1]). Moreover they gave another construction for singular K3 surfaces with discriminant 3 and 4. We will explain this construction in the next section.

2.5 The construction of X_3, X_4 and X_7

Let $\mathcal{T}$ be the set of equivalence classes of positive definite even lattices of rank 2. There is a classification table of $\mathcal{T}$ in terms of discriminant in Chapter 15 of [10]:

Table 15.1. **Reduced positive definite binary forms.**

d	Forms
1	$1^{0}1$
2	$1^{0}2$
3	$1^{0}3,\ 2^{1}2$
4	$1^{0}4,\ 2^{0}2$
5	$1^{0}5,\ 2^{1}3$
6	$1^{0}6,\ 2^{0}3$
7	$1^{0}7,\ 2^{1}4$
8	$1^{0}8,\ 2^{0}4,\ 3^{1}3$
9	$1^{0}9,\ 3^{0}3,\ 2^{1}5$
10	$1^{0}10,\ 2^{0}5$
11	$1^{0}11,\ 2^{1}6,\ 3^{\pm1}4$
12	$1^{0}12,\ 2^{0}6,\ 3^{0}4,\ 4^{2}4$
13	$1^{0}13,\ 2^{1}7$
14	$1^{0}14,\ 2^{0}7,\ 3^{\pm1}5$
15	$1^{0}15,\ 3^{0}5,\ 2^{1}8,\ 4^{1}4$
16	$1^{0}16,\ 2^{0}8,\ 4^{0}4,\ 4^{2}5$
17	$1^{0}17,\ 2^{1}9,\ 3^{\pm1}6$
18	$1^{0}18,\ 2^{0}9,\ 3^{0}6$
19	$1^{0}19,\ 2^{1}10,\ 4^{\pm1}5$
20	$1^{0}20,\ 2^{0}10,\ 4^{0}5,\ 3^{\pm1}7,\ 4^{2}6$
21	$1^{0}21,\ 3^{0}7,\ 2^{1}11,\ 5^{2}5$
22	$1^{0}22,\ 2^{0}11$
23	$1^{0}23,\ 2^{1}12,\ 3^{\pm1}8,\ 4^{\pm1}6$
24	$1^{0}24,\ 2^{0}12,\ 3^{0}8,\ 4^{0}6,\ 5^{1}5,\ 4^{2}7$
25	$1^{0}25,\ 5^{0}5,\ 2^{1}13$
26	$1^{0}26,\ 2^{0}13,\ 3^{\pm1}9,\ 5^{\pm2}6$
27	$1^{0}27,\ 3^{0}9,\ 2^{1}14,\ 4^{\pm1}7,\ 6^{3}6$
28	$1^{0}28,\ 2^{0}14,\ 4^{0}7,\ 4^{2}8$
29	$1^{0}29,\ 2^{1}15,\ 3^{\pm1}10,\ 5^{\pm1}6$
30	$1^{0}30,\ 2^{0}15,\ 3^{0}10,\ 5^{0}6$
31	$1^{0}31,\ 2^{1}16,\ 4^{\pm1}8,\ 5^{\pm2}7$
32	$1^{0}32,\ 2^{0}16,\ 4^{0}8,\ 3^{\pm1}11,\ 4^{2}9,\ 6^{2}6$
33	$1^{0}33,\ 3^{0}11,\ 2^{1}17,\ 6^{3}7$
34	$1^{0}34,\ 2^{0}17,\ 5^{\pm1}7$
35	$1^{0}35,\ 5^{0}7,\ 2^{1}18,\ 3^{\pm1}12,\ 4^{\pm1}9,\ 6^{1}6$
36	$1^{0}36,\ 2^{0}18,\ 3^{0}12,\ 4^{0}9,\ 6^{0}6,\ 4^{2}10,\ 5^{\pm2}8$
37	$1^{0}37,\ 2^{1}19$
38	$1^{0}38,\ 2^{0}19,\ 3^{\pm1}13,\ 6^{\pm2}7$
39	$1^{0}39,\ 3^{0}13,\ 2^{1}20,\ 4^{\pm1}10,\ 5^{\pm1}8,\ 6^{3}8$
40	$1^{0}40,\ 2^{0}20,\ 4^{0}10,\ 5^{0}8,\ 4^{2}11,\ 7^{3}7$
41	$1^{0}41,\ 2^{1}21,\ 3^{\pm1}14,\ 6^{\pm1}7,\ 5^{\pm2}9$
42	$1^{0}42,\ 2^{0}21,\ 3^{0}14,\ 6^{0}7$
43	$1^{0}43,\ 2^{1}22,\ 4^{\pm1}11$
44	$1^{0}44,\ 2^{0}22,\ 4^{0}11,\ 3^{\pm1}15,\ 5^{\pm1}9,\ 4^{2}12,\ 6^{\pm2}8$
45	$1^{0}45,\ 3^{0}15,\ 5^{0}9,\ 2^{1}23,\ 7^{2}7,\ 6^{3}9$
46	$1^{0}46,\ 2^{0}23,\ 5^{\pm2}10$
47	$1^{0}47,\ 2^{1}24,\ 3^{\pm1}16,\ 4^{\pm1}12,\ 6^{\pm1}8,\ 7^{\pm3}8$
48	$1^{0}48,\ 2^{0}24,\ 3^{0}16,\ 4^{0}12,\ 6^{0}8,\ 7^{1}7,\ 4^{2}13,\ 8^{4}8$
49	$1^{0}49,\ 7^{0}7,\ 2^{1}25,\ 5^{\pm1}10$
50	$1^{0}50,\ 2^{0}25,\ 5^{0}10,\ 3^{\pm1}17,\ 6^{\pm2}9$

Figure 2.3

According to this table, there exist unique elements of $\mathcal{T}$ with discriminant 3, 4 and 7, respectively; this means that the singular K3 surfaces with discriminant d are unique up to isomorphisms for $d = 3, 4, 7$, so we denote by X_d these K3 surfaces, respectively.

2.5.1 The construction of X_3 and X_4

This construction is given by Shioda and Inose ([30, Lemmas 5.1 and 5.2]).

1. Let E_3 be the elliptic curve $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\omega)$, where $\omega = \exp(2\pi\sqrt{-1}/3)$, and let $A = E_3 \times E_3$. The automorphism σ of A is defined to be:
$$\sigma : A \to A,\ \sigma(x, y) = (\omega x, \omega^2 y).$$
Then the minimal resolution Y_3 of the quotient A/σ is a singular K3 surface isomorphic to X_3.

2. Let E_4 be an elliptic curve $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}i)$, where $i = \sqrt{-1}$, and let $A = E_4 \times E_4$. The automorphism σ of A is defined to be:
$$\sigma : A \to A,\ \sigma(x, y) = (ix, -iy).$$
Then the minimal resolution Y_4 of the quotient A/σ is a singular K3 surface isomorphic to X_4.

Shioda and Inose constructed an elliptic fibration on Y_k ($k = 3, 4$) for which the trivial lattice is of rank 20. Hence Y_k is a singular K3 surface. By the formula 2.4.3, they calculated the discriminant of Y_k, respectively.

2.5.2 The construction of X_7 (the result of Naruki)

Here we give the result of Naruki [22]. We denote $\exp(2\pi\sqrt{-1}/7)$ by ζ. Let $\mathfrak{p}$ be a principal ideal of the ring of integers of the cyclotomic field $\mathbb{Q}(\zeta)$ generated by $1-\zeta$. Naruki defined a Hermitian form H_7:
$$H_7(z) = z_1\overline{z_1} + z_2\overline{z_2} + (\zeta + \overline{\zeta})z_3\overline{z_3} \quad \text{for } z = (z_1, z_2, z_3) \in \mathbb{C}^3.$$
Let B_7 be the set
$$B_7 := \{(z_1 : z_2 : z_3) \in \mathbb{P}^2 \mid H_7(z) < 0\}.$$
Since the signature of H_7 is (2,1), B_7 is isomorphic to the complex 2-ball. Let SU_7 be the group consisting of $(3, 3)$-matrices with determinant 1 that is unitary with respect to H_7. The group SU_7 acts on B_7. Let Γ_7 be the subgroup of SU_7 consisting of elements whose entries are integers of $\mathbb{Q}(\zeta)$, and let Γ_7' (resp. Γ_7'') be a subgroup of Γ_7 consisting of elements which are congruent to the identity modulo $\mathfrak{p}$ (resp. $\mathfrak{p}^2$). Then Naruki's main result is as follows:

Theorem 2.5.1 ([22, Theorem 1])**.** The quotient surface $B_7/\Gamma_7^{'}$ is a singular K3 surface isomorphic to X_7. The branch locus of $B_7/\Gamma_7^{''} \to B_7/\Gamma_7^{'}$ consists of twenty-eight rational curves, which are transitively permutated by the $\Gamma_7^{'}/\Gamma_7^{''}$.

Moreover, Naruki obtained another construction of $B_7/\Gamma_7^{'}$ as follows. Let $S(7)$ be the elliptic modular surface of level 7, where the definition of an elliptic modular surface is in Section 4 of [28]. This surface has an elliptic fibration $F : S(7) \to X(7)$, where $X(7)$ is the Klein quartic curve. $X(7)$ is given by the following quartic equation in homogenous coordinates $[x : y : z]$ on $\mathbb{P}^2$:

$$x^3y + y^3z + z^3z = 0.$$

Naruki showed that $B_7/\Gamma_7^{'}$ is a quotient of $S(7)$, and the fibration F induces an elliptic fibration $f : B_7/\Gamma_7^{'} \to \mathbb{P}^1$, which has three singular fibres of types I_7 and others are of type I_1:

$$\begin{array}{ccc} S(7) & \xrightarrow{F} & X(7) \\ \downarrow & & \downarrow \\ B_7/\Gamma_7^{'} & \xrightarrow{f} & \mathbb{P}^1 \end{array}$$

Here the map from $X(7) \to \mathbb{P}^1$ is branching over 0, 1 and ∞ and $\mathbb{P}^1$ can be regarded as the quotient of $X(7)$ by the action of $\mathrm{PSL}_2(7)$. Let ι be the inversion involution associated to $f : B_7/\Gamma_7^{'} \to \mathbb{P}^1$. Blowing down (-1)-curves of the quotient surface of $B_7/\Gamma_7^{'}$ by ι, Naruki obtained a construction of X_7 as double covering branched over the projective plane. The branch locus of this is given by the equation as

$$(x_0^2x_1 + x_1^2x_2 + x_2^2x_0 - 3x_0x_1x_2)^2 - 4x_0x_1x_2(x_0 - x_1)(x_1 - x_2)(x_2 - x_0) = 0,$$

which has a D_4 singularity at (1,1,1) and A_4 singularities at (1,0,0), (0,1,0) and (0,0,1).

Chapter 3

The double plane model of singular K3 surfaces with discriminant 3, 4 and 7

3.1 Sextic curves with D_4 singularities and elliptic fibrations

Let C be a sextic curve on $\mathbb{P}^2$. We assume its singular points are of ADE-type. Moreover we assume C has a singularity, denoted by p, of type D_4. We denote by X_C the minimal resolution of the double covering branched over C. Then X_C is a K3 surface. We denote by $\widetilde{\mathbb{P}^2}$ the surface given by blowing up $\mathbb{P}^2$ at p, E the exceptional divisor on $\widetilde{\mathbb{P}^2}$ and $\widetilde{C}$ the inverse image of C. A canonical projection ϕ from $\widetilde{\mathbb{P}^2}$ to E induces the projective bundle on E ($\cong \mathbb{P}^1$) as shown in Figures 3.1 and 3.2.

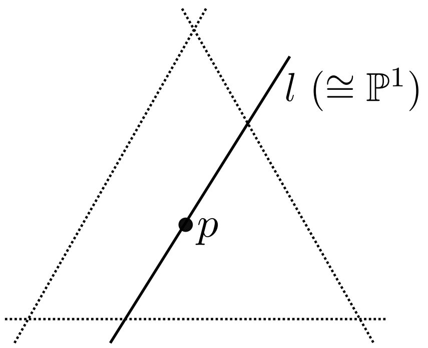

Figure 3.1

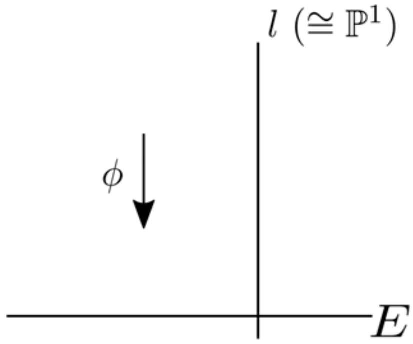

Figure 3.2

Since the general lines passing through p intersect C three times outside the triple point p, the inverse image of these lines are elliptic curves on X_C. We denote by g the rational curve on X_C corresponding to E. Since these elliptic curves intersect g, this family of elliptic curves defines an elliptic fibration of X_C associated to C. We denote this elliptic fibration by $\Phi : X_C \to \mathbb{P}^1$. Then g is a global section of Φ. Since p is a singularity of C and $\widetilde{\mathbb{P}^2}$ is the surface given by blowing up $\mathbb{P}^2$ at p, there exists a natural map from X_C to $\widetilde{\mathbb{P}^2}$. Then we have a diagram as follows:

$$\begin{tikzcd} X_C \arrow[r, "2:1"] \arrow[d, "\Phi"'] & \widetilde{\mathbb{P}^2} \arrow[d, "\phi"', "\text{projection}"] \\ \mathbb{P}^1 \arrow[r, "\cong"] & E \end{tikzcd}$$

Let l_0 be a line on $\mathbb{P}^2$ passing through p and let $\widetilde{l_0}$ be the strict transform of l_0 under the blowing-up $\widetilde{\mathbb{P}^2} \to \mathbb{P}^2$. We assume that $\widetilde{l_0}$ intersects a singular point of $\widetilde{C}$. Since the number of components of $\pi^{-1}(l_0)$ is larger than 1, $\pi^{-1}(l_0)$ is a singular fibre of Φ.

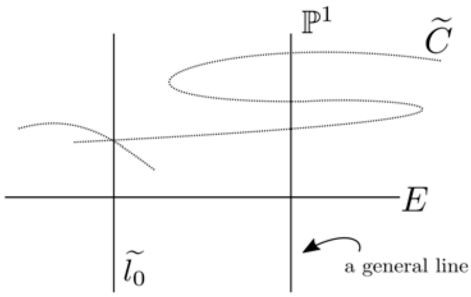

Figure 3.3: C and lines passing through p

3.2 Singular K3 surfaces with discriminant 3, 4, and 7

We denote by X_d the singular K3 surface of discriminant d and by C_d a branch locus if it can be constructed by a double covering $f : X_d \to \mathbb{P}^2$. As in 2.5.2, for the case $d = 7$, C_7 is given by the equation

$$(x_0^2x_1 + x_1^2x_2 + x_2^2x_0 - 3x_0x_1x_2)^2 - 4x_0x_1x_2(x_0 - x_1)(x_1 - x_2)(x_2 - x_0) = 0.$$

Since C_7 has a D_4 singularity at $(1, 1, 1)$, there exists an elliptic fibration $f' : X \to \mathbb{P}^1$ associated to C. The branch locus of $f : B_7/\Gamma_7' \to \mathbb{P}^1$ is equal to C_7, and the fibration f' is equivalent to the fibration $f : B_7/\Gamma_7' \to \mathbb{P}^1$, which has three singular fibres of type I_7 and the others are of type I_1 (see [22], pp.28–30). Then the trivial sublattice L_f is isomorphic to $U \oplus A_6^{\oplus 3}$, and the Néron-Severi lattice $NS(X_7)$ has a sublattice of finite index isomorphic to L_f. In the case of $d = 3$ and 4, we have similarly the following results.

Proposition 3.2.1 ([34, Propposition 6.1])**.** The surface X_d $(d = 3, 4)$ is isomorphic to the double cover of $\mathbb{P}^2$ branched along the plane sextic C_d, where

(i) C_3 is given by the equation

$$(x_0 - x_1)(x_1 - x_2)(x_2 - x_0)\{(x_0 + x_1 + x_2)^3 + (x_0 - x_1)(x_1 - x_2)(x_2 - x_0)\} = 0,$$

which has a D_4 singularity at $(1, 1, 1)$ and A_5 singularities at the intersection points of the lines and the cubic component of C_3. In particular, $NS(X_3)$ has a sublattice of finite index isomorphic to $U \oplus E_6^{\oplus 3}$.

(ii) C_4 is given by the equation

$$x_0x_1x_2(x_0 - x_1)(x_1 - x_2)(x_2 - x_0) = 0,$$

which has four D_4 singularities and three A_1 singularities at intersections of six lines. In particular, $NS(X_4)$ has a sublattice of finite index isomorphic to $U \oplus D_6^{\oplus 3}$.

Proof. By an easy calculation, it is easy to show these sextic curves have singularities as above and are smooth elsewhere. We denote by X_d' $(d = 3, 4)$ the K3 surface which are double covers of $\mathbb{P}^2$ branched along these sextic curves. Since C_d have a D_4 singularity at $(1, 1, 1)$, they have an elliptic fibration associated to C_d, denoted by $\Phi_d' : X_d' \to \mathbb{P}^1$. We define $\widetilde{\mathbb{P}}^2$, E, $\widetilde{C_d}$ and $\pi : X_d' \to \mathbb{P}^2$ in the same way as 3.1. We have a diagram as follows:

$$\begin{array}{ccc} X_d' & \xrightarrow{2:1} & \widetilde{\mathbb{P}}^2 \\ \Phi_d' \big\downarrow & & \phi \big\downarrow \text{projection} \\ \mathbb{P}^1 & \xrightarrow{\cong} & E \end{array}$$

The lines l_{ij} given by $x_i - x_j = 0$ $(i \neq j)$ are components of the branch locus. We denote by $\widetilde{l_{ij}}$ the strict transform of l_{ij} under the blowing-up. These curves pass through singular points of $\widetilde{C_d}$. Hence $\pi^{-1}(l_{ij})$ is a singular fibre of Φ_d'.

In the case of $d = 3$, there are two singular points p_{ij} and q_{ij} of $\widetilde{C_3}$ on $\widetilde{l_{ij}}$ as given in Figure 3.4.

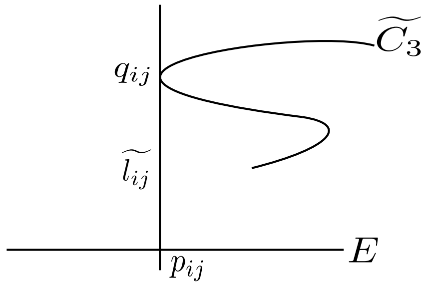

Figure 3.4

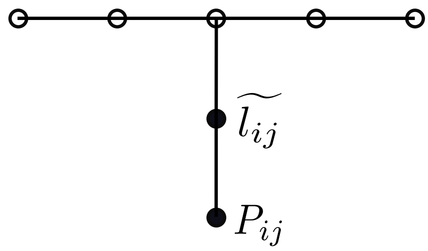

Figure 3.5

Here, p_{ij} (resp. q_{ij}) is an intersection of $\widetilde{l_{ij}}$ and E (resp. the elliptic component of $\widetilde{C_3}$). Since $\widetilde{C_3}$ has an A_1 (resp. A_5) singularity at p_{ij} (resp. q_{ij}), the dual graph of components of $\pi^{-1}(l_{ij})$ is given by Figure 3.5. Here, white circles are obtained by resolution at q_{ij} and P_{ij} is obtained by resolution at p_{ij}. Hence we conclude that the types of $\pi^{-1}(l_{ij})$ $(i \neq j)$ are IV^*. Since X_3' has a global section and three singular fibres of type IV^*, the trivial sublattice is isomorphic to $U \oplus E_6^{\oplus 3}$. In particular, X_3' is a singular K3 surface and $r(\Phi_3') = 0$. Now, X_3' has three global sections. One of them is denoted by g corresponding to E, and others are obtained by the inverse image of the equation: $x_0 + x_1 + x_2 = 0$. From the formula (4,2), we have $|\det T_{X_3'}| = \frac{27}{n(\Phi_3')^2} \leq 3$. Since $\det T_{X_3'}$ is integer, we deduce that

$$|\det T_{X_3'}| = \frac{27}{n(\Phi_3')^2} = 3.$$

By Figure 2.3, the binary quadratic form of discriminant 3 is unique. By Torelli theorem 2.2.10, we conclude that X_3' is isomorphic to X_3.

In the case of $d = 4$, there are three singular points of $\widetilde{C_4}$ on $\widetilde{l_{ij}}$, denoted by p_{ij}, q_{ij} and r_{ij}, as given in Figure 3.6.

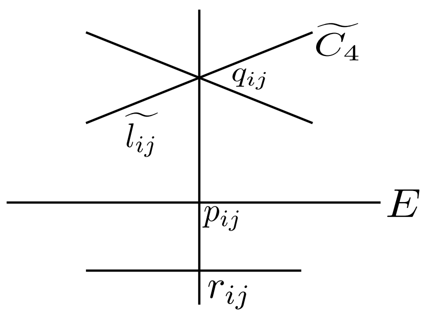

Figure 3.6

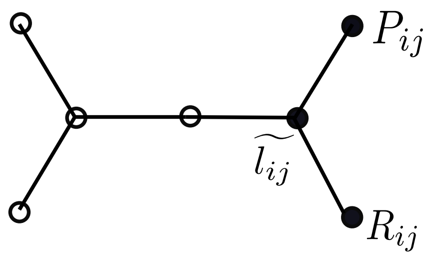

Figure 3.7

$\widetilde{C_4}$ has A_1 singularities at p_{ij} and r_{ij}, and a D_4 singularity at q_{ij}. Hence the dual graph of components of $\pi^{-1}(l_{ij})$ is given by Figure 3.7. Here, white circles in

Figure 3.7 are obtained by resolution at q_{ij} and P_{ij} (resp. R_{ij}) is obtained by resolution at p_{ij} (resp. q_{ij}). Hence we conclude that the types of $\pi^{-1}(l_{ij})$ with $(i \neq j)$ are of type I_2^*. Since the trivial sublattice is isomorphic to $U \oplus D_6^{\oplus 3}$, X_4' is a singular K3 surface. Now, X_4' has four global sections. One of them is denoted by g corresponding to E, and others are obtained by the inverse image of the equation: $x_i = 0$ $(i = 1, 2, 3)$. From the formula 2.4.3, we have $|\det T_{X_4'}| = \frac{64}{n(\Phi_4')^2} \leq 4$. Since $\det T_{X_4'}$ is an integer and $T_{X_4'}$ is a even lattice, we deduce that

$$|\det T_{X_4'}| = \frac{64}{n(\Phi_4')^2} = 4.$$

By Figure 2.3, the binary quadratic form of discriminant 4 is unique. Then we conclude X_4' is isomorphic to X_4. □

Remark 3.2.2. For X_4, an alternative proof can be given as follows. By the classification of non-symplectic involutions by Nikulin ([24, Theorem 4.2.2]), X_4 is the unique K3 surface with involution fixing 10 disjoint smooth rational curves. Since the fixed locus of the covering involution satisfies the requirements as above, we have $X_4' \cong X_4$.

3.3 Generalization of the construction

In this section, we generalize the construction of 3.2 in terms of singularity of branch locus.

Let C be a sextic curve and p_i $(i = 1, 2, 3)$ and q be points in $\mathbb{P}^2$. We denote by l_i the line passing through p_i and q. Now, we consider the following conditions:

(i) $\{p_i, q \mid i = 1, 2, 3\}$ lies in general position,

(ii) C has a D_4 (resp. A_3) singularity at q (resp. p_i), and is smooth elsewhere,

(iii) The intersection multiplicity of C and l_i at p_i (resp. q) is 2 (resp. 4).

Example 3.3.1. Let D_μ $(\mu \neq 0, -4)$ be a sextic curve given by the equation

$$(x_0^2 x_1 + x_1^2 x_2 + x_2^2 x_0 - 3x_0 x_1 x_2)^2 + \mu x_0 x_1 x_2 (x_0 - x_1)(x_1 - x_2)(x_2 - x_0) = 0.$$

Let p_1, p_2, p_3 be distinct vertices of the triangle $x_0 x_1 x_2 = 0$, and let q be $(1, 1, 1)$. Then D_μ, p_i and q satisfy the conditions as above.

Remark 3.3.2. For the case of $\mu = -4$ (resp. $\mu = \infty$), $D_{-4} = C_7$ (resp. $D_\infty = C_4$), where C_d is a branch locus of X_d $(d = 4, 7)$.

Lemma 3.3.3 ([34, Lamma 7.1])**.** Let X be a K3 surface. The following conditions are equivalent.

(1) X is isomorphic to X_C and $MW(\Phi_C)$ is trivial, where C satisfies the conditions (i), (ii) and (iii) as above.

(2) There exists an elliptic fibration $f : X \to \mathbb{P}^1$ with a global section g such that f has three singular fibres of types I_6, and the others are of types I_1 or II, and $MW(f)$ is trivial.

Proof. (1)⇒(2): Since C has a D_4 singularity at q, there exists an elliptic fibration $f : X_C \to \mathbb{P}^1$. We define $\widetilde{\mathbb{P}^2}$, E, $\widetilde{C}$ and $\pi : X_C \to \mathbb{P}^2$ in the same way as 3.1. We denote by l_i the line passing through p_i and q, and $\widetilde{l_i}$ the strict transform of l_i under the blowing-up $\widetilde{\mathbb{P}^2} \to \mathbb{P}^2$. Let $\widetilde{p_i}$ (resp. $\widetilde{q_i}$) be the inverse image of p_i (resp. the intersection of $\widetilde{l_i}$ and E). By the conditions (ii) and (iii), $\widetilde{C}$ has an A_1 singularity at $\widetilde{q_i}$ and an A_3 singularity at $\widetilde{p_i}$. Hence $\pi^{-1}(l_i)$ is a singular fibre of type I_6.

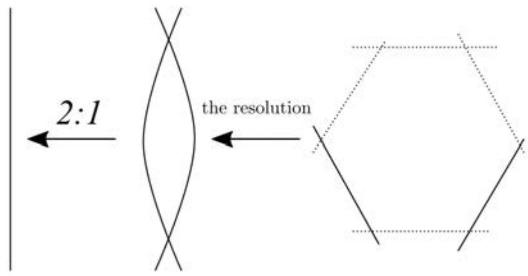

Figure 3.8

Let l be a line passing through q with $l \neq l_i$, and $\widetilde{l}$ the strict transform of l under the blowing-up $\widetilde{\mathbb{P}^2} \to \mathbb{P}^1$. Since $\widetilde{l}$ does not meet $\widetilde{C}$ at singular points of $\widetilde{C}$, the number of component of $\pi^{-1}(l)$ is 1. Hence, if it is a singular fibre, it is of types I_1 or II.

(2)⇒(1): The same argument as in Section 5 of Naruki [22] can be applied. Let $\iota : X \to X$ be the inversion involution. X/ι has the canonical projection $p : X/\iota \to \mathbb{P}^1$. Here, $\iota : X \to X$ fixes the zero section g and permutes components of singular fibres. On the surface X/ι, we have three configurations of the following figures coming from fibres of type I_6:

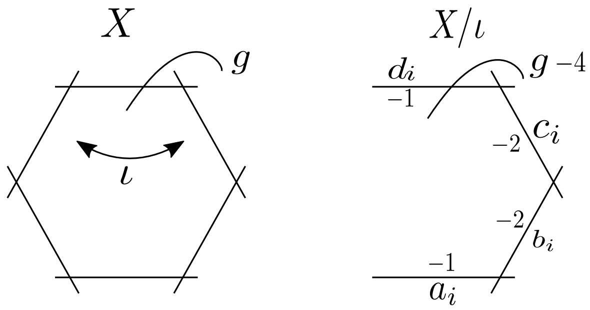

Figure 3.9

Here, a_i, b_i, c_i and d_i are the curves, and $-1, -2, -4$ are the self-intersection numbers. We blow down a_i and d_i ($i = 1, 2, 3$), and then the image of b_i is exceptional curve. Blowing down the image of b_i, we obtain a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. Since the self-intersection number of the image of g is -1, it is a tautological bundle. Hence if we blow down the image of g, we obtain the projective plane $\mathbb{P}^2$ and the map from X to $\mathbb{P}^2$, denoted by $\pi : X \to \mathbb{P}^2$. Now we define p_i and q as follows:

$$p_i = \pi(a_i) = \pi(b_i), \ \ q = \pi(g).$$

By the configuration of three singular fibres, the branch locus C is a curve on $\mathbb{P}^2$ which has a D_4 singularity at q and an A_3 singularity at p_i. The image of fibres of π are lines passing through q. Let l be a general line passing through q. Since an elliptic curve $\pi^{-1}(l)$ intersects the fixed locus of ι exactly three times outside a global section g, the line l intersects the branch locus C exactly three times outside triple point q. By Bezout theorem, C is a sextic curve.

If $\{p_i | i = 1, 2, 3\}$ does not lie in general position, there exists the line l_0 passing through p_i ($i = 1, 2, 3$). Since p_i is a A_3 singular point of C, the intersection multiplicity m_i of l_0 and C is larger than 2. By Bezout theorem, $m_i = 2$ and l_0 does not intersect C outside p_i. Then $\pi^{-1}(l_0)$ is the union of two rational curves and X has three global sections. This contradicts $n(f) = 1$. Therefore, $\{p_i | i = 1, 2, 3\}$ lies in general position. Hence X is isomorphic to the minimal resolution of the double covering branched over C which satisfies the condition (i), (ii) and (iii). □

Corollary 3.3.4. If X satisfies the conditions (1) and (2) in Lemma 3.3.3, we have $NS(X) \cong U \oplus A_5^{\oplus 3}$.

Proof. This corollary is an immediate consequence of (2) of Lemma 3.3.3 and the equation (2.4.1) in Section 2.4. □

Remark 3.3.5. By Lemma 2.1.11, there exists a unique primitive embedding $U \oplus A_5^{\oplus 3} \hookrightarrow L$ up to $O(L)$.

Let L be a unimodular even lattice with the signature $(3, 19)$. There exists a primitive sublattice T of L with the signature $(2, 3)$ such that $q_T = -q_{(U\oplus A_5^{\oplus 3})}$. We define a period domain $\mathcal{B}$, $\mathcal{H}_\delta$ and $\mathcal{H}$ in the same way as Section 1. Let (X, α_X) be a marked K3 surface and let ω_X be a nowhere vanishing 2-form of X. If there exists an element $\delta \in T$ such that $\langle \delta, \alpha_X(\omega_X)\rangle = 0$, then $\alpha_X^{-1}(\delta)$ is an element of S_X. Then $\alpha_X(T_X)$ is a proper subset of T. To exclude this case, we defined $\mathcal{H}_\delta$ and $\mathcal{H}$.

Remark 3.3.6. Let ω_{X_d} $(d = 3, 4, 7)$ be a nowhere vanishing holomorphic 2-form of X_d and let α_d be a marking of X_d Then $\omega_d = \alpha_d(\omega_{X_d}) \in \mathcal{H} \subset \mathcal{B}$. Indeed, it is enough to show that there exists a primitive embedding

$$T_{X_d} \hookrightarrow T \ (d = 3, 4, 7).$$

In the case of $d = 4, 7$, this is an immediate consequence of Example 1. In the case of $d = 3$, we can find a primitive sublattice of E_6 isomorphic to A_5. Then we have a natural embedding

$$U \oplus A_5^{\oplus 3} \hookrightarrow NS(X_3) \hookrightarrow L \tag{3.3.1}$$

The fact that this is a primitive embedding can be confirmed by direct calculation. If we take the orthogonal complements of these lattices in L, we have the desired primitive embeddings $T_{X_d} \hookrightarrow T$ $(d = 3, 4)$. Moreover, by computing the orthogonal complement of $U \oplus A_5^{\oplus 3}$ in the natural embedding map 3.3.1, we can also calculate the transcendental lattice T concretely. T is given by the following lattice:

$$T \cong \begin{pmatrix} -42 & 0 & -12 & 0 & -18 \\ 0 & -42 & -18 & 0 & -18 \\ -12 & -18 & -12 & 0 & -18 \\ 0 & 0 & 0 & 4 & 1 \\ -18 & -18 & -18 & 1 & -46 \end{pmatrix}.$$

Theorem 3.3.7 ([34, Theorem 7.1])**.** Let X be a $K3$ surface and ω_X be a nowhere vanishing holomorphic 2-form of X. Then the following conditions are equivalent.

(i) The condition (1) of Lemma 3.3.3,

(ii) The condition (2) of Lemma 3.3.3,

(ii) There exists a marking α_X for X such that $\alpha_X(\omega_X) \in \mathcal{B}\backslash\mathcal{H}$.

Proof. (i)⇔(ii): This is Lemma 3.3.3.

(ii)⇒(iii): This is an immediate consequence of Corollary 3.3.4 and Remark 3.3.5.

(iii)⇒(ii): Let $P^+(X)$ be the positive cone (i.e. the component of $\{x \in H^{1,1}(X,\mathbb{R}) : \langle x, x\rangle \geq 0\}$ that contains a Kähler class of X), let $W(X)$ be the subgroup of $O(H^{1,1}(X,\mathbb{R}))$ generated by reflections s_δ with $\delta^2 = -2$, and let $D(X)$ be the closure of the Káhler cone $D(X)$. By $\alpha_X(\omega_X) \in \mathcal{B}\backslash\mathcal{H}$, we have $NS(X) \cong U \oplus A_5^{\oplus 3}$ and $T_X \cong T$. Now, we denote by E and F the basis of the hyperbolic lattice U and by Θ_i^k ($i = 1, ..., 5, k = 1, 2, 3$) the basis of k-th component of $A_5^{\oplus 3}$, and define Θ_0^k as $E - \sum_{i=1}^5 \Theta_i^k$. There exists $\sigma \in W(X)$ such that $E' = \sigma(E) \in \overline{D}(X)$ and the linear system $|E'|$ contains an elliptic curve, where $\overline{D}(X)$ is the closure of $D(X)$. By Lemma 2.2.6, $\sigma(\Theta_i^k)$ is represented by an effective divisor D_i^k. By Bertini's theorem, we obtain an elliptic fibration $\phi_{|E'|} : X \to \mathbb{P}^1$. It is obvious that $F_k := \sum_{i=1}^5 D_i^k \in |E'|$ is a singular fibre. Since the number of the irreducible components of F_k is larger than six, the type of F_k is either I_n or I_n^* ($n \geq 6$). Let m_k^1 be the number of irreducible component of F_k. We deduce from the formula 2.4.2 that $m_1 + m_2 + m_3 \leq 18$. Therefore, $m_k = 6$ and $r(\phi_{|E'|}) = 0$. By the formula 2.4.3, we deduced that the condition (ii) holds. □

Theorem 3.3.8 ([34, Theorem 7.2])**.** Let X be a $K3$ surface which satisfies the conditions of Theorem 3.3.7. Then its automorphism group $\mathrm{Aut}(X)$ is of infinite order.

This theorem follows from Nikulin's classification ([24, Theorem 0.2.2]). Here we include an alternative proof by an argument similar to Shioda and Inose [30, Theorem 5].

Proof. For an elliptic fibration f with $r(f) \geq 1$, by the definition, there are infinitely many sections of f. Since a K3 surface is minimal, the Mordell-Weil group of f can be seen as a subgroup of the autmorphism group of X. Hence it is enough to show that there exists an elliptic fibration $f : X \to \mathbb{P}^1$ with $r(f) \geq 1$. By Proposition 3.3.7, X has an elliptic fibration Φ which satysfies the condition (ii) of Lemma 3.3.3. The elliptic fibration Φ has a global section g and the three singular fibers of type I_6. The components of fibers of type I_6 are denoted by Θ_l^k ($k = 1, 2, 3, l = 0, 1, ..., 5$). They satisfy the following conditions:

$$\langle \Theta_l^k, \Theta_{l+1}^k \rangle = 1,$$
$$\langle \Theta_0^k, g \rangle = 1.$$

Now, we define an effective divisor E and E_k as follows:

$$E := 3g + 2(\Theta_0^1 + \Theta_0^2 + \Theta_0^3) + (\Theta_1^1 + \Theta_1^2 + \Theta_1^3),$$
$$E_k := E - (\Theta_3^k + \Theta_4^k).$$

By Lemma 2.4.2, X has an elliptic fibration $f : X \to \mathbb{P}^1$ with a singular fibre E. The type of E is IV^*. By Lemma 2.4.2, Θ_2^k are global sections of f. We consider the three divisors

$$\Theta_3^k + \Theta_4^k \ (k = 1, 2, 3).$$

These divisors do not intersect the fibre E of f. Hence, the image $f(\Theta_3^k + \Theta_4^k)$ is a point t_k of $\mathbb{P}$, and (-2)-curves Θ_3^k and Θ_4^k are the components of a singular fibre $f^{-1}(t_k)$.

Now, let S be a lattice generated by the following divisors:

$$\Theta_l^k \ (k = 1, 2, 3\ , l = 0, 1, 3, 4) \ \textit{and}\ \Theta_2^1.$$

Let the divisors Θ^k be $\Theta_1^k + 2\Theta_2^k + \Theta_3^k - \Theta_5^k$. By an easy calculation, it is shown that a lattice $S^\perp$ which is the orthogonal complement of S in $NS(X)$ is generated by these four divisors

$$A = 2E + \Theta^1,$$

$$g, \Theta^2, \text{ and } \Theta^3.$$

Moreover, the lattice $S^\perp$ is a lattice with Gram matrix

$$\begin{pmatrix} -6 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -6 \end{pmatrix}.$$

The elliptic fibration f has the singular fibres E and $f^{-1}(t_k)$. Other singular fibres are of type I_1 or II because the lattice $S^\perp$ does not have (-2)-roots. Let L be the direct sum of the root lattices corresponding to the singular fibers of $f^{-1}(t_1)$, $f^{-1}(t_2)$, and $f^{-1}(t_3)$. L is a sublattice of L_Φ. Here, if $rk(NS(X)) = rk(L_\Phi)$, then $rk(L)$ is 9. From $d(L_\Phi) = d(NS(X)) \cdot [NS(X) : L_\Phi]^2$, $d(L)$ is larger than 72. In the case that all t_k's are the same, $L = A_9, D_9$ or E_9. This contradits that $d(L) \geq 72$. In the case that two of the t_k's are the same, by virtue of $d(A_m) = m + 1, d(D_m) = 4, d(E_m) \leq 3$ and $rk(L) = 9$, it can be seen that there is no lattice L that satisfies $d(L) \geq 72$. In the case that t_k's are mutually distinct, in the same way, L can be $A_2^{\oplus 3}$, $A_2 \oplus A_3 \oplus A_4$, $A_2 \oplus A_3 \oplus D_4$, $A_2 \oplus A_2 \oplus A_5$ or $A_2 \oplus A_2 \oplus D_5$. This contradits that $d(L) \geq 72$. □

Chapter 4

Hyperbolic geometry

In this chapter, we recall some basic facts of n-dimensional hyperbolic spaces and discrete groups of isometries of n-dimensional hyperbolic spaces, which are contained in [26].

Notation

Let (X, d) be a metric space, and let r be a positive real number. Throughout this section, we denote by

$B(a, r)$ the open ball of radius r centered at a point a of X,

$C(a, r)$ the closed ball of radius r centered at a point a of X,

$S(a, r)$ the sphere of radius r centerd at a point a of X,

$N(S, r)$ the r-neighborhood of S in X,

$I(X)$ the group of isometries of X together with multiplication defined by composition.

We denote by $\mathbb{R}^n$ the n-dimensional real vector space and by S^n the n-dimensional unit sphere. A vector in $\mathbb{R}^n$ is denoted by an ordered n-tuple $x = (x_1, \ldots, x_n)$. A vector e_i in $\mathbb{R}^n$ is the vector whose i-th entry is 1 and other entries are 0.

4.1 The Euclidean n-spaces

In this section, we summarize some basic notation of the Euclidean n-spaces.

Definition 4.1.1. Let x and y be vectors in $\mathbb{R}^n$. The *Euclidean inner product* of x and y are defined by

$$x \cdot y = x_1 y_1 + \cdots + x_n y_n.$$

The *Euclidean norm* of x is defined to be the real number

$$|x| = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}.$$

Definition 4.1.2. The *Euclidean distance* between vectors x and y in $\mathbb{R}^n$ is defined to be

$$d_E(x, y) = |x - y|.$$

The *Euclidean n-space* E^n is the metric space consisting of $\mathbb{R}^n$ together with the Euclidean metric d_E.

4.2 The hyperboloid model of hyperbolic n-spaces

In the hyperbolic geometry, it is useful to use some models of hyperbolic spaces for different situations. Therefore, we begin with recalling the *hyperboloid model* of hyperbolic spaces. In Section 4.4 and 4.5 we recall the *conformal ball model*, and the *upper half-plane model* of hyperbolic spaces.

Definition 4.2.1. Let n be a non-negative integer. Let $x = (x_1, \dots, x_n, x_{n+1})$ and $y = (y_1, \dots, y_n, y_{n+1})$ be vectors in the $(n+1)$-dimensional real vector space $\mathbb{R}^{n+1}$. The *Lorentzian inner product* of x and y is defined by

$$x \circ y = x_1 y_1 + \cdots + x_n y_n - x_{n+1} y_{n+1}.$$

The *Lorentzian (n+1)-space* is the vector space $\mathbb{R}^{n+1}$ together with the Lorentzian inner product. A real $(n+1) \times (n+1)$ matrix A is said to be *Lorentzian* if for any $x, y \in \mathbb{R}^{n+1}$, $x \circ y = Ax \circ Ay$.

Definition 4.2.2. The *hyperboloid model* H^n of hyperbolic n-spaces is defined to be the set $\{x \in \mathbb{R}^{n+1} : x \circ x = -1,\ x_1 > 0\}$. The *hyperbolic distance* $d_H(x, y)$ between x and y is defined by the equation

$$\cosh d_H(x, y) = -x \circ y.$$

By [26, Theorem 3.2.2], the hyperbolic distance d_H is a metric on H^n. We denote the group of isometries by $I(H^n)$.

Definition 4.2.3. Let H^n be the hyperboloid model of n-dimensional hyperbolic spaces. The set of all Lorentz $(n+1) \times (n+1)$ matrices with matrix multiplication is called the *Lorentz group* of $(n+1) \times (n+1)$ matrices, and is denoted by $O(n, 1)$. The *positive Lorentz group* $O^+(n, 1)$ is defined by

$$O^+(n, 1) = \{A \in O(n, 1) : AH^n = H^n\},$$

and acts isometrically on H^n.

By [26, Theorem 3.2.3], there is an isomorphism from the positive Lorentzian group $O^+(n, 1)$ to the group of hyperbolic isometries $I(H^n)$.

Definition 4.2.4. A *hyperbolic m-plane* of H^n is the nonempty intersection of H^n with an $(n+1)$-dimensional vector subspace of $\mathbb{R}^{n+1}$.

4.3 Möbius transformations

To define the conformal ball model and the upper half-space model of hyperbolic n-spaces, we use the hyperboloid model and *Möbius transformation*. In this section, we recall the definition and some basic facts of Möbius transformation.

Definition 4.3.1. Let a be a unit vector in E^n, and let t be a positive real number. Consider the hyperplane of E^n defined by

$$P(a,t) = \{x \in E^n : a \cdot x = t\}.$$

The *reflection* ρ of E^n in the plane $P(a,t)$ is defined by

$$\rho(x) = x + 2(t - a \cdot x)a.$$

Let a be a point in E^n, and let r be a positive real number. The *sphere* of E^n of radius r centered at a is defined by

$$S(a,r) = \{x \in E^n : |x - a| = r\}.$$

The *reflection* (or *inversion*) σ of E^n in the sphere $S(a,r)$ is defined by

$$\sigma(x) = a + \left(\frac{r}{|x-a|}\right)^2 (x - a).$$

Definition 4.3.2. We identify E^n with $E^n \times \{0\}$. The *stereographic projection* from E^n to $S^n \backslash \{e_{n+1}\}$ is defined by

$$\pi(x) = \left(\frac{2x_1}{1+|x|^2}, \dots, \frac{2x_n}{1+|x|^2}, \frac{|x|^2 - 1}{|x|^2 + 1}\right).$$

The image $\pi(x)$ of x is the intersection of $S^n \backslash \{e_{n+1}\}$ with the line connecting x to e_{n+1}.

Let $\hat{E}^n$ be the one-point compactification of E^n. Then $\hat{E}^n = E^n \cup \{\infty\}$, where ∞ is a point that is not in E^n. Now we extend π to a bijection $\hat{\pi} : \hat{E}^n \to S^n$ defined by $\hat{\pi}(\infty) = e_{n+1}$. The metric d on $\hat{E}^n$ is defined by

$$d(x,y) = |\pi(x) - \pi(y)|,$$

and is called the *chordal metric*.

Definition 4.3.3. Let a be a unit vector in E^n, and let t be a positive real number. The *extended hyperplane* of $\hat{E}^n$ is defined by $\hat{P}(a,t) = P(a,t) \cup \{\infty\}$. Let ρ be the reflection of the hyperplane $P(a,t)$. Extend ρ to a bijection $\hat{\rho} : \hat{E}^n \to \hat{E}^n$ by setting $\hat{\rho}(\infty) = \infty$. In a similar way, the inversion σ of an Euclidean sphere $S(a,r)$ in E^n is extended to $\hat{\sigma} : \hat{E}^n \to \hat{E}^n$ by setting $\hat{\sigma}(\infty) = \infty$. The extended map $\hat{\rho}$ (resp. $\hat{\sigma}$) is called the reflection of $\hat{P}(a,t)$ (resp. $\hat{S}(a,r)$) , which is a homeomorphism of $\hat{E}^n$.

Definition 4.3.4. A *sphere* of $\hat{E}^n$ is either an extended hyperplane or an Euclidean sphere. A *Möbius transformation* of $\hat{E}^n$ is a finite composition of reflection of a sphere in $\hat{E}^n$.

Theorem 4.3.5 ([26, Theorem 4.3.4])**.** Let ϕ be a Möbius transformation. The image of a sphere in $\hat{E}^n$ by ϕ is also a sphere in $\hat{E}^n$.

4.4 The conformal ball model

In this section, we recall the conformal ball model of hyperbolic space. Identify $\mathbb{R}^n$ with $\mathbb{R}^n \times \{0\}$ in $\mathbb{R}^{n+1}$. Let B^n be the open unit ball of $\mathbb{R}^n$, and let $|x|$ be the Euclidean norm of a vector x in $\mathbb{R}^n$: i.e., $|x| = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$.

The *stereographic projection* ζ from B^n to the hyperbolic space H^n is defined by the formula

$$\zeta(x) = \Big(\frac{2x_1}{1-|x|^2}, \dots, \frac{2x_n}{1-|x|^2}, \frac{1+|x|^2}{1-|x|^2}\Big).$$

This map ζ is bijective.

Definition 4.4.1. A metric d_B on B^n is defined by the formula $d_B(x, y) = d_H(\zeta(x), \zeta(y))$. The metric space (B^n, d_B) is called the conformal ball model of hyperbolic n-spaces.

Definition 4.4.2. A *Möbius transformation of* B^n is a Möbius transformation ϕ of $\hat{E}^n$ that leaves B^n invariant: ,i.e., ϕ satisfies $\phi(B^n) = B^n$. The group of Möbius transformation of B^n is denoted by $M(B^n)$.

Theorem 4.4.3 ([26, Theorem 4.5.2.])**.** Every Möbius transformation of B^n restricts to an isometry of the conformal ball model B^n. Conversely, every isometry of B^n extends to a unique Möbius transformation of B^n. In particular, restriction induces an isomorphism from $M(B^n)$ to $I(B^n)$.

Definition 4.4.4. The metric D_B on $M(B^n)$ is defined by

$$D_B(\phi, \psi) = sup_{x \in \partial B^n} |\phi(x) - \psi(x)|.$$

Let Γ be a subgroup of $M(B^n)$. Γ is *discrete* if all the point of Γ is open with respect to the metric D_B.

The *boundary of* B^n in $\mathbb{R}^n$ is the unit sphere $\{x \in \mathbb{R}^n : |x| = 1\}$, and is denoted by ∂B^n. A *hyperbolic m-plane* of B^n is the image of a hyperbolic m-plane of H^n by ζ^{-1}. By [26, Theorem 4.5.3], a hyperbolic m-plane of B^n is the intersection of B^n with either an m-dimensional vector subspace of $\mathbb{R}^n$ or an m-dimensional sphere of $\mathbb{R}^n$ orthogonal to ∂B^n.

Definition 4.4.5. Let c be a point in ∂B^n. Let S be an $(n-1)$-dimensional inscribed sphere of B^n tangent at c, and let T be the one of connected component of $\mathbb{R}^n \backslash S$ which is bounded. We call the intersection of B^n with S (resp. T) the *horosphere* of B^n (resp. *horoball* of B^n).

In Figure 4.1, the small circle is a horosphere in the 2-dimensional conformal ball modle B^2.

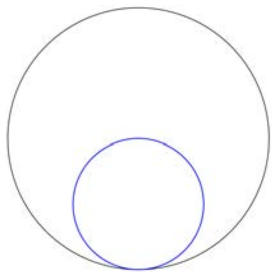

Figure 4.1: A horosphere of B^2

4.5 The upper half-space model

Definition 4.5.1. Let $x = (x_1, \ldots, x_n)$ be a vector in $\mathbb{R}^n$, and let e_n be the vector in $\mathbb{R}^n$ whose entries are all zero except for the n-th, which is one. Let ρ (resp. σ) be the reflection of $\hat{E}^n$ in $P(e_n, 0)$ (resp. $S(e_n, \sqrt{2})$). The maps ρ and σ are given by the formula

$$\rho(x_1, \ldots, x_{n-1}, x_n) = (x_1, \ldots, x_{n-1}, -x_n),$$
$$\sigma(x) = e_n + \frac{2(x - e_n)}{|x - e_n|^2} \text{ for } x \in \mathbb{R}^n \backslash \{e_n\}, \text{ respectively.}$$

The n-dimensional *upper half-space* U^n is defined to be

$$U^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}.$$

We define $\eta = \sigma\rho$. Then the map η is a map from the upper half-space U^n to the open unit ball B^n, and is called the *standard transformation* from U^n to B^n. The *Poincaré metric* on U^n is defined by the formula

$$d_U(x, y) = d_B(\eta(x), \eta(y)).$$

The pair (U^n, d_U) is a metric space, and is called the *upper half-model* of hyperbolic n-spaces. By the definition, this is isomorphic to the conformal ball model B^n of hyperbolic n-spaces. The group consisting of isometries of U^n is denoted by $I(U^n)$.

Definition 4.5.2. A *Möbius transformation of* U^n is a Möbius transformation ϕ that leaves U^n invariant: i.e. ϕ satisfies $\phi(U^n) = U^n$. The group of Möbius transformation of U^n is denoted by $M(U^n)$.

Theorem 4.5.3 ([26, Theorem 4.6.2.])**.** Every Möbius transformation of the upper half-space model U^n restricts to an isometry of U^n. Conversely, every isometry of U^n extends to a Möbius transformation of U^n. In particular, restriction induces an isomorphism from $M(U^n)$ to $I(U^n)$.

Example 4.5.4. In the case $n = 2$, identify $\mathbb{R}^2$ with $\mathbb{C}$ under the map

$$(x, y) \mapsto z = x + \boldsymbol{i}y.$$

Let ϕ be a *linear fractional transformation*

$$\phi(z) = \frac{az + b}{cz + d}$$

with a, b, c, d real numbers and $ad - bc = 1$. It is well-known that ϕ acts isometrically on the upper half-space $U^2 = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. The standard transformation η can be expressed as $\eta(z) = \boldsymbol{i} f(z)$, where the map $f(z)$ is the *Cayley transformation* defined by $f(z) = \dfrac{z - \boldsymbol{i}}{z + \boldsymbol{i}}$.

Definition 4.5.5. Identify E^{n-1} with $E^{n-1} \times \{0\}$ in E^n. Let ϕ be a Möbius transformation of $\hat{E}^{n-1}$, and let $\overline{a} = (a, 0)$. The extension $\overline{\phi}$ of ϕ is defined as follows:

1. If ϕ is the reflection of $\hat{E}^{n-1}$ in $P(a, t)$, then $\overline{\phi}$ is the reflection of $\hat{E}^n$ in $P(\overline{a}, t)$.
2. If ϕ is the reflection of $\hat{E}^{n-1}$ in $S(a, t)$, then $\overline{\phi}$ is the reflection of $\hat{E}^n$ in $S(\overline{a}, t)$.
3. In general case, ϕ is the composition $\phi = \sigma_1 \cdots \sigma_m$ of reflections. Then, we define the map $\overline{\phi}$ by the composition $\overline{\phi} = \overline{\sigma_1} \cdots \overline{\sigma_m}$.

We call the extension of ϕ the *Poincaré extension*

The Poincaré extension is well-defined. Moreover, the following holds:

Theorem 4.5.6. A Möbius transformation ϕ of $\hat{E}^n$ leaves U^n invariant if and only if ϕ is the Poincaré extension of a Möbius transformation of $\hat{E}^{n-1}$. In particular, the Poincaré extension induces the isomorphism from $M(\hat{E}^{n-1})$ to $M(U^n)$.

Definition 4.5.7. Let η be the standard transformation from U^n to B^n. A subset P is a *hyperbolic m-plane* (resp. *horosphere*, *horoball*) of U^n if the image $\eta(P)$ of P by η is a hyperbolic m-plane (resp. horosphere, horoball) in B^n.

Theorem 4.5.8 ([26, Theorems 4.6.3. and 4.6.5.] **).**

1. A subset P of U^n is a hyperbolic m-plane of U^n if and only if P is the intersection of U^n with either an m-plane of E^n orthogonal to E^{n-1} or an m-sphere of E^n orthogonal to E^{n-1}.

2. A subset Σ of U^n is a horopshere of U^n based at a point b of $\hat{E}^{n-1}$ if and only if Σ is either a Euclidean hyperplane in U^n parallel to E^{n-1} if $b = \infty$, or the intersection with U^n of a Euclidean sphere in $\overline{U}^n$ tangent to E^{n-1} at b if $b \neq \infty$.

4.6 Types of transformations and elementary groups

Let B^n be the conformal ball model of n-dimensional hyperbolic spaces. We write $\overline{B^n}$ for the closed unit ball $B^n \cup \partial B^n$.

Definition 4.6.1. Let ϕ be an element of $M(B^n)$. Then ϕ maps the closed unit ball $\overline{B^n}$ to itself. By the Brouwer fixed point theorem, ϕ has a fixed point in $\overline{B^n}$.

1. ϕ is *elliptic* if ϕ has a fixed point in B^n,

2. ϕ is *parabolic* if ϕ has no fixed point in B^n and has a unique fixed point in ∂B^n,

3. ϕ is *hyperbolic* if ϕ has no fixed point in B^n and fixes two points in ∂B^n.

Theorem 4.6.2. A Möbius transformation ϕ of $M(B^n)$ is elliptic if and only if ϕ is conjugate in $M(B^n)$ to an orthogonal transformation of E^n.

Example 4.6.3. Let ϕ be the rotation of B^2 by the angle θ around the origin of the form $\phi(z) = e^{i\theta}z$ for any $z \in \mathbb{C}$. Then ϕ is a Möbius transformationof elliptic type.

Let ϕ be an isometry of U^n, and let η be the standard transformation from U^n to B^n. Then ϕ is said to be *elliptic* (resp. *parabolic*, *hyperbolic*) if $\eta\phi\eta^{-1}$ is elliptic (resp. parabolic, hyperbolic).

Theorem 4.6.4 ([26, Theorems 4.7.3, and 4.7.5] **).** Let ϕ be a Möbius transformation of U^n. Then

1. ϕ is parabolic if and only if ϕ is conjugate in $M(U^n)$ to the Poincaré extension of an isometry ψ of E^{n-1} of the form $\psi(x) = a + Ax$ where $a \neq 0$ and A is an orthogonal transformation of E^{n-1} such that $Aa = a$.

2. ϕ is hyperbolic if and only if ϕ is conjugate in $M(U^n)$ to the Poincaré extension of a similarity ψ of E^{n-1} of the form $\psi(x) = kAx$, where $k > 1$ and A is an orthogonal transformation of E^{n-1}.

Example 4.6.5. Let f and g be isometries of U^2 defined by $f(z) = z + 1$ and $g(z) = 2z$. Then f (resp. g) is parabolic (resp. hyperbolic).

Definition 4.6.6. A subgroup G of $M(B^n)$ is *elementary* if G has a finite orbit in the closed ball B^n. Every elementary group is classified into three types as follows:

1. The group G is of *elliptic type* if G has a finite orbit in B^n.

2. The group G is of *parabolic type* if G fixes a point of ∂B^n and has no other finite orbit in $\overline{B}^n$

3. The group G is of *hyperbolic type* if G is neither of elliptic type nor parabolic type.

Theorem 4.6.7 ([26, Theorem 5.5.2.])**.** Let Γ be a subgroup of $M(B^n)$. Then Γ is an elementary discrete subgroup of elliptic type if and only if Γ is conjugate in $M(B^n)$ to a finite subgroup of the orthogonal group $O(n)$.

Theorem 4.6.8 ([26, Theorems 5.5.5, and 5.5.8.])**.** Let Γ be a subgroup of $M(U^n)$.

1. The group Γ is an elementary discrete subgroup of parabolic type if and only if Γ conjugate in $M(U^n)$ to an infinite discrete subgroup of $I(E^{n-1})$.

2. The group Γ is an elementary discrete subgroup of hyperbolic type if and only if Γ contains an infinite cyclic subgroup of finite index that is generated by a hyperbolic transformation.

4.7 Convex polyhedra and fundamental domains

Definition 4.7.1. Let B^n be the conformal ball model of n-dimensional hyperbolic spaces. Let x and y be points in $\overline{B^n}$. Then there exists the open circular arc C that is the intersection of B^n with the hyperbolic 1-plane of B^n that contains x and y. A subset $[x, y]$ of C is defined by

1. If x and y are in ∂B^n, then $[x, y]$ is the open circular arc C ending at x and y, and is called the *geodesic line* connecting x to y,

2. If x and y are in B^n, then $[x, y]$ is the closed circular arc ending at x and y, and is called the *geodesic segment* connecting x to y,

3. If x is in ∂B^n and y is in B^n, then $[x, y]$ is the circular arc ending at x and y, and is called the *geodesic ray* connecting x to y.

Definition 4.7.2. A subset C of B^n is *convex* if for any x and y, the geodesic segment $[x, y]$ is contained in C. Let S be a subset of B^n. The *convex hull* of S is the intersection of all convex subsets containing S. A subset P of B^n is a *polytope* in B^n if P is a convex hull of finitely many points of B^n.

Example 4.7.3. Let S be a $30° - 45°$ hyperbolic right triangle. A hyperbolic regular hexagon T is obtained by applying reflections one after another, keeping the vertex at the $30°$ angle fixed, as shown in Figure 4.2. S and T are convex polyhedra in B^2.

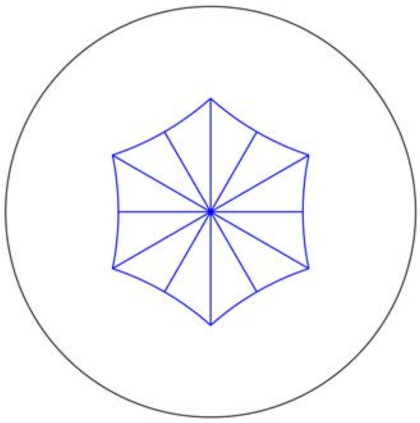

Figure 4.2: A hyperbolic regular hexagon

Definition 4.7.4. A subset C of $\overline{B^n}$ is *hyperbolic convex* if any two distinct points of C can be joined by the geodesic segment, the geodesic ray, or the geodesic line connecting x to y. Let K be a subset of $\overline{B^n}$. The *hyperbolic convex hull* $C(K)$ is the intersection of all hyperbolic convex subsets containing K. A subset P of B^n is a *generalized polytope* if P is the hyperbolic convex hull of finitely many points of $\overline{B}^n$. A point in $P \cap \partial B^n$ is called an *ideal vertex* of P.

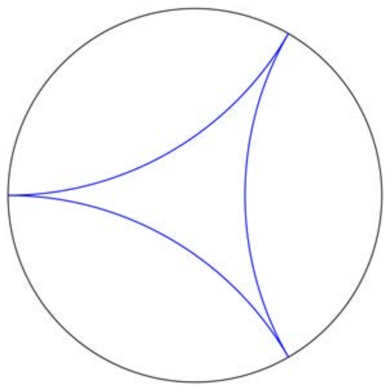

Figure 4.3: An ideal triangle in B^2

Example 4.7.5. The curvilinear triangle in Figure 4.3 is a generalized triangle.

Definition 4.7.6. A *side* is a convex subset C of B^n is a nonempty maximal convex subset of ∂C. A *convex polyhedron* P of B^n is a nonempty closed convex subset of B^n such that the collection of its sides is locally finite in B^n: i.e., for each $x \in B^n$, there is an $\epsilon > 0$ such that the open ball $B(x, \epsilon)$ of radius ϵ centered at x meets only finitely many sides of P.

Definition 4.7.7. Let P be a convex polyhedron in B^n, and let u be a point of $\overline{P} \cap \partial B^n$, where $\overline{P}$ is the closure of P in E^n. A side S of P is *incident* with u if u is in the closure of S in E^n.

Definition 4.7.8. Let X be a metric space. A subset D of X is a *fundamental domain* for a group Γ of isometries of X if

1. the set D is open and connected in X,
2. the elements of $\{gD : g \in \Gamma\}$ are mutually disjoint,
3. $X = \bigcup_{g \in \Gamma} g\overline{D}$,

   where $\overline{D}$ is the closure of D in X.

A fundamental domain D is *locally finite* if the set $\{g\overline{D} : g \in \Gamma\}$ is locally finite: i.e., for each $x \in X$, there is an $\epsilon > 0$ such that the open ball $B(x, \epsilon)$ of radius ϵ centered at x meets only finitely many members of $\{g\overline{D} : g \in \Gamma\}$.

Definition 4.7.9. Let X be a topological space, and let G be a group that acts on X. The action of G on X is *discontinuous* if for any compact subset K, the set $K \cap gK$ is nonempty for only finitely many g in G.

Let Γ be a discontinuous group of isometries of a hyperbolic n-space H^n. By [26, Theorems 6.6.10 and 6.6.12], there is a point a such that the stabilizer group Γ_a of a is trivial. For each $g \neq 1$ in Γ, we set $H_g^+(a) = \{x \in H^n : d_H(x, a) < d_H(x, ga)\}$.

Definition 4.7.10. The *Dirichlet domain* $D_\Gamma(a)$ for Γ, with center a, is either H^n if Γ is trivial or

$$D_\Gamma(a) = \bigcap_{g \in \Gamma \setminus \{1\}} H_g^+(a).$$

Let $D_\Gamma(a)$ be the Dirichlet domain for Γ with center a as above. By [26, Theorem 6.6.13], $D_\Gamma(a)$ is a locally finite fundamental domain for Γ.

Example 4.7.11. Let Γ be the group of all linear fractional transformations $\phi(z) = \dfrac{az+b}{cz+d}$ with a, b, c, d integers and $ad - bc = 1$. Then Γ is a discrete subgroup of $I(U^2)$, which is isomorphic to $\mathrm{PSL}(2, \mathbb{Z})$. Let T be the Dirichlet domain for Γ with center $a = t\boldsymbol{i}$ for any $t > 1$. Let $f(z) = z + 1$ and $g(z) = -\dfrac{1}{z}$. Then T is given by $T = H_f^+(a) \cap H_{f^{-1}}^+(a) \cap H_g^+(a)$ as the following figure 4.4.

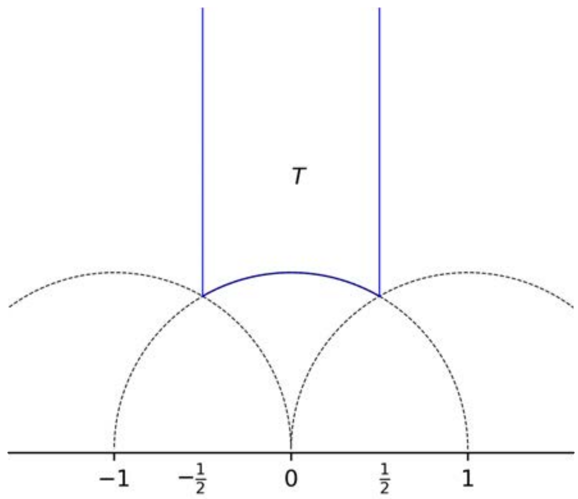

Figure 4.4: A Dirichlet domain T of $\mathrm{PSL}(2, \mathbb{Z})$

4.8 Limit sets

Definition 4.8.1. A point a of ∂B^n is a *limit point* of a subgroup G of $M(B^n)$ if there is a point $x \in B^n$ and a sequence $\{g_i\}$ of elements of G such that the sequence $\{g_i x\}_{i\in\mathbb{N}}$ converges to a. The set of limit points of G is called the *limit set* of G, and is denoted by $\Lambda(G)$. The complement of $\Lambda(G)$ in ∂B^n is called the *ordinary set* of G, and is denoted by $O(G)$.

Definition 4.8.2. A point a of ∂B^n is a *conical limit point* of a subgroup G of $M(B^n)$ if there is a point $x \in B^n$, a sequence $\{g_i\}$ of elements of G, a 1-plane R and a real number $r > 0$ such that the closure of R in $\overline{B^n}$ contains a, $g_i x$ is in the r-neightborhood $N(R, r)$ of R and the sequence $\{g_i x\}_{i\in\mathbb{N}}$ converges to a, where $N(R, r)$ is defined to be

$$N(R, r) = \{x \in B^n : d_B(x, y) < r \text{ for some } y \in R\}.$$

Example 4.8.3. Let g be the same isometry in Example 4.6.5, and let R be the y-axis. As the sequence $\{g^{-i}(\boldsymbol{i})\}_{i\in\mathbb{N}}$ converges to 0 and $g^{-i}(\boldsymbol{i}) \in R$, the limit point 0 is a conical limit point of a cyclic subgroup generated by g.

In general, it is known that a fixed point of hyperbolic transformation h of a subgroup G of $M(B^n)$ is a conical limit point of G.

Definition 4.8.4. A point a of ∂B^n is a *bounded parabolic limit point* of a discrete group Γ of $M(B^n)$ if a is a fixed point of a parabolic element of Γ and the orbit space $(\Lambda(\Gamma)\backslash\{a\})/\Gamma$ is compact.

Example 4.8.5. In Example 4.7.11, the set of all bounded parabolic points is the set of extended rational numbers $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$, and the set of all conical limit points is $\mathbb{R}\backslash\mathbb{Q}$.

4.9 Geometrically finite discrete groups

Definition 4.9.1. A convex polyhedron P in B^n is geometrically finite if for any $x \in \overline{P} \cap \partial B^{n-1}$, there exists an open neighborhood N of x in E^n such that N meets only the sides of P whose closure in E^n contains x.

Definition 4.9.2. A discrete group Γ of $M(B^n)$ is geometrically finite if Γ has a geometrically finite convex fundamental polyhedron P and P is exact: i.e., for each side S of P, there is an element γ of Γ such that $S = P \cap \gamma P$.

Definition 4.9.3. Let Γ be a discrete subgroup of $M(B^n)$, and let a be a point of ∂B^n fixed by some parabolic element of Γ. Let Γ_a be the stabilizer group of a. A horoball $B(a)$ based at a is said to be a *horocusped region* for Γ based at a if for each $g \in \Gamma\backslash\Gamma_a$, we have

$$B(a) \cap gB(a) = \emptyset.$$

If a horocusped region $B(a)$ based at a is not maximal with respect to the inclusion relation, $B(a)$ is said to be *proper*.

Definition 4.9.4. Let $C(\Gamma)$ be the hyperbolic convex hull of the limit set $\Lambda(\Gamma)$ of Γ. The *convex core* of $M = B^n/\Gamma$ is the set

$$C(M) = (C(\Gamma) \cap B^n)/\Gamma.$$

Theorem 4.9.5 ([26, Theorem 5.3.5])**.** Let Γ be a subgroup of $I(H^n)$. Γ is discrete if and only if the action of Γ on H^n is properly discontinuous.

From the proof of [26, Theorem 12.4.4], we have following lemma.

Lemma 4.9.6. Let Γ be a geometrically finite subgroup of $M(U^n)$, and let P be a geometrically finite exact convex fundamental polyhedron for Γ. Then, for every cusped limit point x, there exist a cusped region $U(x)$ based at x and elements $g_1, \cdots, g_k$ in Γ such that $U(x) \subset \bigcup_{f\in\Gamma_x}(\bigcup_{i=1}^{i=k} g_i\overline{P})$, and $U(x)$ meets just the sides of polyhedrons fg_iP incident with x.

Theorem 4.9.7 ([26, Theorem 12.4.5.])**.** Let Γ be a discrete subgroup of $M(B^n)$, let $M = B^n/\Gamma$, and let $C(M)$ be the convex core of M. The following are equivalent:

1. The group Γ is geometrically finite,
2. There is a (possibly empty) finite union V of proper horocusps of M, with disjoint closures, such that $C(M)\backslash V$ is compact,
3. Every limit point of Γ is either conical or bounded parabolic,
4. Every exact convex fundamental polyhedron for Γ is geometrically finite.

Theorem 4.9.8 ([26, Theorem 12.4.8])**.** Let Γ be a discrete subgroup of $M(B^n)$, and let H be a subgroup of finite index in Γ. Then H is geometrically finite if and only if Γ is geometrically finite.

Theorem 4.9.9 ([26, Theorem 12.4.9])**.** If Γ is a discrete subgroup of $M(B^n)$ such that B^n/Γ has finite volume, then Γ is geometrically finite.

Theorem 4.9.10 ([26, Theorem 12.4.10])**.** Let Γ be a discrete subgroup of $M(B^n)$ with an integer $n > 1$. Then the following are equivalent:

1. The orbit space B^n/Γ has finite volume.

2. The group Γ is geometrically finite and $\Lambda(\Gamma) = \partial B^n$.

3. There exists an exact convex fundamental polyhedron P for Γ such that P is a generalized polytope.

Example 4.9.11. Let Γ and T be the same as those in Example 4.7.11. As the volume of T is $\frac{\pi}{3}$, Γ is geometrically finite. In this case, the set of bounded parabolic limit points is the rationals $\mathbb{Q}$ and the set of conical limit points is $\mathbb{R}\backslash\mathbb{Q}$.

Theorem 4.9.12 ([26, Theorem 12.6.5])**.** Let Γ be a subgroup of $M(B^n)$, and let C be the set of cusped limit points of Γ. For each $c \in C$, there exists a cusped region $U(c)$ based at c for Γ such that

1. $\{\overline{U}(c) : c \in C\}$ is mutually disjoint,

2. $\gamma U(c) = U(\gamma c)$ for any $\gamma \in \Gamma$ and $c \in C$,

3. $\{\overline{U}(c)\backslash\{c\}\}$ is a locally finite family of closed subsets of $B^n \cup O(\Gamma)$.

4.10 Arithmetic hyperbolic groups

Let f be a quadratic form in $(n + 1)$-variables with real symmetric coefficient matrix $A = (a_{ij})$. Then we have

$$f(x) = x^t A x = \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} a_{ij} x_i x_j.$$

The *orthogonal group* of f over a subring R of $\mathbb{R}$ is defined to be

$$\begin{aligned} O(f, R) &= \{T \in \mathrm{GL}(n+1, R) : f(Tx) = f(x) \text{ for all } x \in \mathbb{R}^{n+1}\} \\ &= \{T \in \mathrm{GL}(n+1, R) : T^t A T = A\}. \end{aligned}$$

Let f_n be the *Lorentzian quadratic form* in $(n + 1)$-variables defined by

$$f_n(x) = -x_1^2 + x_2^2 + \cdots + x_{n+1}.$$

Then we have $O(n, 1) = O(f_n, \mathbb{R})$. If the signature of f is of type $(n, 1)$, there exists a matrix $M \in GL(n + 1, \mathbb{R})$ such that $f(Mx) = f_n(x)$ for all $x \in \mathbb{R}$. Then M maps the set $\{x \in \mathbb{R}^{n+1} : f_n(x) < 0\}$ onto the set $\{x \in \mathbb{R}^{n+1} : f(x) < 0\}$.

Let $O^+(f, R)$ be the subgroup of $O(f, R)$ leaves the component of $\{x \in \mathbb{R}^{n+1} : f(x) < 0\}$. Then there is an isomorphism from $O^+(n, 1)$ to $O^+(f, \mathbb{R})$.

Definition 4.10.1. Let K be a totally real number field, and let f be a quadratic form over K in $(n+1)$-variables, with an integer $n > 0$ and symmetric coefficient matrix $A = (a_{ij})$. The quadratic form f is said to be admissible if f has signature $(n, 1)$ and for each nonidentity field embedding $\sigma : K \to \mathbb{R}$, the quadratic form f^σ over $\sigma(K)$, with coefficient matrix $A^\sigma = (\sigma(a_{ij}))$, is positive definite.

Example 4.10.2. Any quadratic from f over $\mathbb{Q}$ of signature $(n, 1)$ is admissible.

Definition 4.10.3. Let K be a totally real number field, and let R_K be the ring of integers of K. A subgroup Γ of $O^+(n, 1)$ is an *arithmetic group of isometries of* H^n defined over K if there is a quadratic form f over K in $(n+1)$-variables, and there is $M \in GL(n+1, \mathbb{R})$ such that $f(Mx) = f_n(x)$ for all $x \in \mathbb{R}^{n+1}$, and the subgroup $M\Gamma M^{-1}$ and $O^+(f, R_K)$ of $O^+(f, \mathbb{R})$ are commensurable. If the quadratic form f is admissible, Γ is said to be *of simplest type*

Theorem 4.10.4 ([26, Theorem 12.8.4])**.** If Γ is an arithmetic group of isometries of $\mathbb{H}^n$, with an integer $n > 1$, of the simplest type defined over a totally real number field K, then $\Gamma\backslash\mathbb{H}^n$ has finite volume.

Example 4.10.5. Let $(L, \langle , \rangle)$ be a lattice of signature $(1, n)$, and let $O^+(L)$ be the subgroup of $O(L)$ leaves the component of the set $\{x \in L \otimes \mathbb{R} : \langle x, x\rangle < 0\}$. Then $O^+(L)$ is an arithmetic group of the simplest type defined over $\mathbb{Q}$. By Theorem 4.10.4 and Theorem 4.9.10, $O^+(L)$ is a geometrically finite discrete subgroup of $I(H^n)$.

Chapter 5

Cohomology of groups

5.1 Cohomological dimension of groups

In this chapter, we recall some definitions and basic facts about cohomology of groups from [9]. Group cohomology is a tool used to study groups using cohomology theory. For a given group Γ and a Γ-module M, we consider the abelian group $H^*(\Gamma, M)$. The passage $M \mapsto H^*(\Gamma, M)$ can be regarded as a derived functor and also it can be interpreted geometrically using *Eilenberg-MacLane spaces*.

Definition 5.1.1. Let Γ be a group and let $\mathbb{Z}\Gamma$ be the free $\mathbb{Z}$-module generated by the elements of Γ. Thus the elements of $\mathbb{Z}\Gamma$ can be uniquely expressed in the form

$$\sum_{\gamma\in\Gamma} a_\gamma \gamma,$$

where $a_\gamma \in \mathbb{Z}$ and $a_\gamma = 0$ for all but finitely many γ. The multiplication in Γ uniquely extends to $\mathbb{Z}\Gamma$; this makes $\mathbb{Z}\Gamma$ into a ring. $\mathbb{Z}\Gamma$ is called the *integral group ring of* Γ. A left (resp. right) $\mathbb{Z}\Gamma$-module is called a *left (resp. right)* Γ*-module.*

Let R be a ring. The tensor product $M \otimes_R N$ is defined for a right R-module M and a left R-module N. If R is a group ring $\mathbb{Z}\Gamma$, any left Γ-module M can be regarded as a right Γ-module by setting $m\gamma = \gamma^{-1}m$ ($m \in M$, $\gamma \in \Gamma$). Then, the tensor product $M \otimes_{\mathbb{Z}\Gamma} N$ can be defined for left Γ-modules M and N. The tensor product $M \otimes_{\mathbb{Z}\Gamma} N$ is also denoted by $M \otimes_\Gamma N$, and the set of homomorphisms from M to N is denoted by $\mathrm{Hom}_\Gamma(M, N)$.

Definition 5.1.2. Let F be a left projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$, where we regard $\mathbb{Z}$ as the left Γ-module with trivial Γ-action. Let M be a left Γ-module. We define the *homology* and *cohomology* of Γ with coefficients in M by

$$\begin{aligned} H_i(\Gamma, M) &:= H_i(F \otimes_\Gamma M), \\ H^i(\Gamma . M) &:= H^i(\mathrm{Hom}_\Gamma(F, M)), \text{ respectively.} \end{aligned}$$

By the fundamental property of projective resolutions (see Brown [9, Proposition 8.6]), we can also define the homology and cohomology of Γ with coefficients in M as follows:

$$\begin{aligned} H_i(\Gamma, M) &:= H_i(F \otimes_\Gamma P) = H_i(\mathbb{Z}, P), \\ H^i(\Gamma.M) &:= H^i(\mathrm{Hom}_\Gamma(F, Q)) = H_i(\mathrm{Hom}_\Gamma(\mathbb{Z}, Q)), \end{aligned} \tag{5.1.1}$$

where P (resp. Q) is a left projective (resp. injective) resolution of M. As noted above, the homology (resp. cohomology) of a group Γ can be regarded as a derived functor. To see this, we define the invariants (resp. co-invariants) of left Γ-modules.

Definition 5.1.3. For a left Γ-module M, we define

$$\begin{aligned} M^\Gamma &:= \{m \in M : \gamma m = m \ \text{ for } \ \forall\gamma \ \text{ in } \ \Gamma\}, \\ M_\Gamma &:= M/\{\gamma m - m : \gamma \in \Gamma, m \in M\} = M/IM, \end{aligned}$$

where I is the bisided ideal of $\mathbb{Z}\Gamma$ generated by $\gamma - 1$ ($\gamma \in \Gamma$), said to be the *augmentation ideal*. M^Γ (resp. M_Γ) is called the *invariant part* (resp. *coinvariant*) of M.

M_Γ is the maximum quotient module of M among those quotients with the induced Γ-action being trivial, and M^Γ is the maximum submodule of M whose Γ-action is trivial.

Proposition 5.1.4 ([9, Chapter II, eq. (2,1)])**.** For any left Γ-module M,

$$\begin{aligned} M_\Gamma &\simeq \mathbb{Z} \otimes_\Gamma M, \\ M^\Gamma &\simeq \mathrm{Hom}_\Gamma(\mathbb{Z}, M). \end{aligned}$$

Proof. We define the two maps defined to be

$$M_\Gamma \to \mathbb{Z} \otimes_\Gamma M : m \ \ mod \ \ IM \mapsto 1 \otimes m,$$

$$\mathbb{Z} \otimes_\Gamma M \to M_\Gamma : n \otimes m \mapsto n \cdot (m \ \ mod \ \ IM).$$

These two maps are inverse to each other. Then, we have $M_\Gamma \simeq \mathbb{Z} \otimes_\Gamma M$. By the definition of Γ-module $\mathbb{Z}$ and M_Γ, it is easy to see that the map $Hom_\Gamma(\mathbb{Z}, M) \to M^\Gamma : f \mapsto f(1)$ is an isomorphism. □

By Proposition 5.1.1, the functor from the category of left Γ-modules $Mod(\Gamma)$ to the category of abelian groups Ab defined by $M \mapsto M_\Gamma$ (resp. $M \mapsto M^\Gamma$) is an additive right exact functor (resp. an additive left exact functor). By equation

(5.1.1), we can deduce that the homology (resp. cohomology) of Γ with coefficients in M is the left derived functor of $M \mapsto M_\Gamma$ (resp. the right derived functor of $M \mapsto M^\Gamma$).

Next, we recall the topological interpretation of homology (resp. cohomology) of groups using Eilenberg-MacLane spaces.

Definition 5.1.5. A CW-complex Y is said to be an Eilenberg-MacLane complex of type $(\Gamma, 1)$ if Y satisfies the following conditions:

1. Y is connected.

2. $\pi_1 Y = \Gamma$.

3. The universal cover X of Y is contractible.

An Eilenberg-MacLane complex of type $(\Gamma, 1)$ is denoted by $K(\Gamma, 1)$.

Definition 5.1.6. Let X be a CW-comlex. If X admits a left Γ-action which permutes cells, X is said to be a Γ-*complex*.

If X is a Γ-complex, then the action of Γ on X induces an action of Γ on the cellular chain complex $C_*(X)$, which thereby becomes a chain complex of left Γ-modules. The *canonical augmentation* $\epsilon : C_0(X) \to \mathbb{Z}$, defined by $\epsilon(v) = 1$ for every 0-cell v of X, is a left Γ-module map. X is said to be *free* if the action of Γ on X is free for the set of cells of X. In this case, each chain module $C_n(X)$ is a $\mathbb{Z}\Gamma$-free module.

Proposition 5.1.7 ([9, Chapter I, Proposition (4.1)])**.** Let X be a contractible free Γ-complex. Then, the augmented chain complex

$$C_* : \cdots \to C_n(X) \to \cdots \to C_1(X) \to C_0(X) \xrightarrow{\epsilon} \mathbb{Z}$$

is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$.

Proof. Take the homology of $C_*(X)$. Since X is contractible, $H_i(X) = 0$ for $i \geq 1$ and $H_0(X) = \mathbb{Z}$. Hence, $C_*(X)$ is exact. □

Proposition 5.1.8 ([9, Chapter I, Proposition (4.2)])**.** If Y is a $K(\Gamma, 1)$-complex, then the augmented cellular complex of the universal cover X of Y is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$.

Proof. Let X be the universal cover of Y and let $p : X \to Y$ be the covering map. By the theory of regular coverings, X admits Γ-complex structure and the action of Γ on fibres $p^{-1}\sigma$ is transitive and free. Then each chain module $C_*(X)$ is a free left $\mathbb{Z}\Gamma$-module. By Proposition 5.1.7, $C_*(X)$ is exact. □

Let X and Y be as above. Y is isomorphic to X/Γ. This isomorphism induces the isomorphism from the chain complex $C_*(Y)$ to the complex $C_*(X)_\Gamma$ that is the invariants of $C_*(X)$. Then $H_i(\Gamma, \mathbb{Z}) = H_i(C_*(X) \otimes_\Gamma \mathbb{Z}) \simeq H_i(C_*(X)_\Gamma) \simeq H_i(C_*(Y))$. We have the following corollary:

Corollary 5.1.9. If Y is a $K(\Gamma, 1)$-complex, then $H_*(\Gamma, \mathbb{Z}) \simeq H_*(Y, \mathbb{Z})$.

Example 5.1.10. Let Γ be $\mathbb{Z}$ with generator t. Then we have a resolution

$$0 \to \mathbb{Z}\Gamma \xrightarrow{\times(t-1)} \mathbb{Z}\Gamma \to 0.$$

Hence $H_*(\Gamma, M)$ is the homology of the following complex:

$$\cdots \to 0 \to M \xrightarrow{\times(t-1)} M$$

and $H^*(\Gamma, M)$ is the cohomology of the following complex:

$$M \xrightarrow{\times(t-1)} M \to 0 \to \cdots .$$

Hence we have $H_0(\Gamma, M) = H^1(\Gamma, M) = M_\Gamma$ and $H_1(\Gamma, M) = H^0(\Gamma, M) = M^\Gamma$.

Example 5.1.11. Let Γ be the free abelian group $\mathbb{Z}^n$ of rank n and let Y be the CW-complex $S^1 \times S^1 \times \cdots \times S^1$ (n factors) , where S^1 is the circle. Then Y is a $K(\Gamma, 1)$-complex. Hence $H_i(\Gamma, M) = H_i(Y, M)$ and $H^i(\Gamma, M) = H^i(Y, M)$.

Remark 5.1.12. We have given the topological interpretation of the homology of groups with coefficients $\mathbb{Z}$. In general, for any left Γ-module M,

$$H_*(\Gamma, M) = H_*(K(\Gamma, 1), \mathcal{M}),$$

$$H^*(\Gamma, M) = H^*(K(\Gamma, 1), \mathcal{M}),$$

where $\mathcal{M}$ is the locally constant sheaf on Y induced by M. By Hurewicz's theorem, any $K(\Gamma, 1)$-complex are homotopy equivalent to each other. Then, it is important for computation of H_* or H^* to find an appropriate $K(\Gamma, 1)$-complex or find an appropriate free resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$.

For example, it is reasonable to try to choose $K(\Gamma, 1)$ with dimension as small as possible. This leads to the notion of the cohomological dimension of groups.

Definition 5.1.13. Let R be a ring and M be a left R-module. Let n is a non-negative integer. Then $\operatorname{projdim}_R M \leq n$ if there is a left projective resolution of length n:

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0.$$

Lemma 5.1.14 ([9, Chapter VIII, Lemma (2.1)])**.** The following conditions are equivalent:

1. $\operatorname{projdim}_R M \leq n$.
2. $\operatorname{Ext}^i_R(M, -) = 0$ for any $i > n$.
3. $\operatorname{Ext}^{n+1}_R(M, -) = 0$.
4. For any left R-module exact sequence $0 \to K \to P_{n-1} \to \cdots \to P_0 \to 0$ with P_i projective, K is projective.

Proof. It is obvious that (iv) $\Rightarrow$ (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). Then we will prove (iii) $\Rightarrow$ (iv). For a given partial projective resolution as in (iv), consider an extended projective resolution:

$$\begin{tikzcd}
\cdots \arrow[r, "f_{n+2}"] & P_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, two heads] & P_n \arrow[r, "f_n"] \arrow[d, two heads] & P_{n-1} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & M \arrow[r] & 0. \\
& L \arrow[ur, hook] & K \arrow[ur, hook] & & & & &
\end{tikzcd}$$

For any left R-module N, an $(n+1)$-cocycle in $\operatorname{Hom}_R(P, N)$ is a map $\phi : P_{n+1} \to N$ whose composition with $P_{n+1} \to P_n$ is zero. By $L \simeq \operatorname{Im} f_{n+1} \simeq P_{n+1}/\ker f_n = P_{n+1}/\operatorname{Im} f_{n+2}$, ϕ induces $\overline{\phi} : L \to N$. By the condition (iii), any map on L can be extended to P_n. In particular, the identity map on L extends to P_n. Hence $P_n \simeq L \oplus K$ and K are projective. $\square$

Definition 5.1.15. We define the *cohomological dimension* $\operatorname{cd}\Gamma$ as following:

$$\begin{aligned}
\operatorname{cd}\Gamma &= \operatorname{projdim}_{\mathbb{Z}\Gamma} \mathbb{Z} \\
&= \inf\{n : \mathbb{Z} \text{ admits a projective resolution of lengh } n\} \\
&= \inf\{n : H^n(\Gamma, -) = 0 \text{ for any } i > n\} \\
&= \sup\{n : H^n(\Gamma, -) \neq 0 \text{ for some } \Gamma\text{-module } M\}
\end{aligned}$$

Recall that the functor $\operatorname{Ext}^i_R(M, -) = H^i(\operatorname{Hom}_R(M, -))$. Then $\operatorname{Ext}^i_{\mathbb{Z}\Gamma}(M, -) = H^i(\Gamma, -)$. By the previous Lemma, the above definition is well-defined.

Example 5.1.16. Let Γ be the fundamental group of a closed surface Y whose genus is larger than 2. Since Y is a $K(\Gamma, 1)$-complex, then $\operatorname{cd}\Gamma \leq 2$. And $H^2(\Gamma, \mathbb{Z}/2\mathbb{Z}) \simeq H^2(Y, \mathbb{Z}/2\mathbb{Z}) \neq 0$. Hence $\operatorname{cd}\Gamma = 2$.

Example 5.1.17. Let Γ and Y be the same as those in Example 5.1.11. Since Y is a $K(\Gamma, 1)$-complex and $H^n(Y, \mathbb{Z}) = \mathbb{Z}$, $\operatorname{cd}\Gamma = n$.

Proposition 5.1.18 ([9, Chapter VIII, Proposition (2.4)])**.** Let Γ and $\Gamma^{'}$ be groups with $\Gamma^{'} \subset \Gamma$. Then $\operatorname{cd}\Gamma^{'} \leq \operatorname{cd}\Gamma$.

Proof. A projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$ can also be regarded as a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma^{'}$. □

Theorem 5.1.19 ([9, Chapter VIII, Proposition (8.1)])**.** Let Γ be a group, and let Y be an n-dimensional $K(\Gamma, 1)$-manifold (possibly with boundary). Then $\operatorname{cd}\Gamma \leq n$ and the equality holds if and only if Y is closed (i.e., compact and without boundary).

The following theorem claims well-definedness of the *virtual cohomological dimension*.

Theorem 5.1.20 ([9, Chapter VIII, Theorem (3.1)])**.** If Γ is a torsion-free group and $\Gamma^{'}$ is a subgroup of finite index, then $\operatorname{cd}\Gamma = \operatorname{cd}\Gamma^{'}$.

There are many important groups with torsion that have torsion-free subgroups of finite index (for example, $SL_n(\mathbb{Z})$). It is known that the cohomological dimension of groups with torsion is infinite. The following notion gives a useful framework in such situation: In general, for a property P in terms of group, a group is called *virtually* P if there is a subgroup of finite index that satisfies the property P.

Definition 5.1.21. Let Γ be a virtually torsion-free group. By Theorem 5.1.20, any torsion-free subgroup of finite index has the same cohomological dimension. Then the dimension is called the *virtual cohomological dimension*, and is denoted by $\operatorname{vcd}\Gamma$.

5.2 Bestvina-Mess formula

In [5], Bestvina and Mess proved the following formula for any virtually torsion free hyperbolic group G:

$$\operatorname{vcd} G = \dim \partial G + 1,$$

where ∂G is the Gromov boundary of G and $\dim \partial G$ is the *covering dimension* of ∂G. In [5], this formula, called the Bestvina-Mess formula, is generalized for groups which admits a *$\mathcal{Z}$-structure.* Recently, Fukaya [12] proved the Bestvina-mess formula for relatively hyperbolic groups.

Definition 5.2.1. Let A be a subset of a topological space X. If there exists a map $r : X \to A$ such that $r|A = id$, the subset A is called a *retract* of X. A *neighborhood retract* of X is a closed set in X that is a retract of some neighborhood in X.

Definition 5.2.2. A metrizable space X is called an *abusolute neighborhood retract (ANR)* (resp. an *abusolute retract (AR)*) if X is a neighborhood retract (resp. a retract) of an arbitrary metrizable space that contains X as a closed subspace.

It is well-known that an ANR X is an AR if and only if X is contractible.

Example 5.2.3.
1. Every topological manifold is an ANR.
2. $[0,1]^n$ is an AR.

Definition 5.2.4. A subset A of a topological space X is said to be *homotopy dense* in X if there exists a homotopy $H : X \times [0,1] :\to X$ such that for any $t > 0$, $H_t(X) \subset A$.

Definition 5.2.5. Let X be an ANR. A closed subset Z of X is called a $\mathcal{Z}$-set if $X \backslash Z$ is homotopy dense in X.

Definition 5.2.6. Let $\overline{X}$ be a compact AR, and let Z be a closed subset of $\overline{X}$. A pair $(\overline{X}, Z)$ is a $\mathcal{Z}$-structure for a group G if

1. Z is a $\mathcal{Z}$-set of $\overline{X}$.
2. There exists a proper metric d on $X = \overline{X} \backslash Z$ such that

   2-1. G geometrically acts on (X, d).

   2-2. For any compact subset K of X and any open covering $\mathcal{U}$ of $\overline{X}$, there exists $U_g \in \mathcal{U}$ such that $gK \subset U_g$ for all $g \in G$ exept finitely many elements of G.

Theorem 5.2.7 ([6, Theorem 1.7])**.** Let $(\overline{X}, Z)$ be a $\mathcal{Z}$-structure on a virtually torsion free group G. Then $\operatorname{vcd} G = \dim Z + 1$, where $\dim Z$ is the covering dimension of Z.

Example 5.2.8. Let $\mathbb{B}^n$ be the n-dimensional conformal model of hyperbolic spaces. and let Γ be a cocompact discrete subgroup of $M(B^n)$. Then the pair $(\overline{\mathbb{H}^n}, S^{n-1})$ is a $\mathcal{Z}$-structure of a group Γ, and so $\operatorname{vcd}\Gamma = \dim S^{n-1} + 1 = n$.

Chapter 6

Automorphism groups of K3 surfaces and virtual cohomological dimension

In this chapter, we deal with only algebraic K3 surfaces defined over $\mathbb{C}$. We consider the natural actions of the automorphism groups of K3 surfaces on hyperbolic spaces. First, we recall the constructions of hyperbolic spaces induced by hyperbolic lattices, and discuss the actions of the automorphism groups of K3 surfaces on certain hyperbolic spaces, and moreover some properties of automorphism groups. Secondly, we discuss some invariants of automorphism groups, including the virtual cohomological dimension.

6.1 Geometrical finiteness of $\mathrm{Aut}_s(X)$

Let X be an algebraic K3 surface defined over $\mathbb{C}$, and let f be the quadratic form defined by $f(x) = \langle x, x\rangle$ for $x \in NS(X) \otimes \mathbb{R}$, where $\langle\ ,\ \rangle$ is the inner product on $NS(X)$ induced by the cup product on $H^2(X, \mathbb{Z})$. f has the signature $(1, n)$, where $n = \rho(X) - 1$. Then there exists $M \in \mathrm{GL}(n+1, \mathbb{R})$ such that $f_n(Mx) = f(x)$, where $f_n(x) = x_0^2 - (x_1^2 + x_2^2 + \cdots + x_n^2)$. Therefore, we can identify the component of the set

$$\{x \in NS(X) \otimes \mathbb{R} : \langle x, x\rangle = 1\}$$

containing ample classes of X with the hyperboloid model of hyperbolic n-space. We denote the hyperboloid model of hyperbolic n-space by $\mathbb{H}^n_X$. The hyperbolic distance function d on $\mathbb{H}^n_X$ satisfies $\cosh d(x, y) = \langle x, y\rangle$.

Definition 6.1.1. We call $\mathbb{H}^n_X$ the *hyperbolic space associated to X*. Let $\overline{P^+}(X)$ be the closure of the ample cone $P^+(X)$ in $NS(X) \otimes \mathbb{R}$. We define $\mathbb{P}(\overline{P^+}(X))$ by

$$\mathbb{P}(\overline{P^+}(X)) = (\overline{P^+}(X)\backslash\{0\})/\mathbb{R}_{>0}.$$

For a point x in $\overline{P^+}(X)\backslash\{0\}$, we denote by $[x]$ the class in $\mathbb{P}(\overline{P^+}(X))$ corresponding to x. The inclusion from $\mathbb{H}^n_X$ to $\overline{P^+}(X)$ induces a continuous injective map $i : \mathbb{H}^n_X \to \mathbb{P}(\overline{P^+}(X))$. Since i is a homeomorphism from $\mathbb{H}^n_X$ to $i(\mathbb{H}^n_X)$, we identify $\mathbb{H}^n_X$ with $i(\mathbb{H}^n_X)$.

The *boundary of* $\mathbb{H}^n_X$ *at infinity* is the closure of $i(\mathbb{H}^n_X)$ in $\mathbb{P}(\overline{P^+}(X))$, which is equals to the set $\{x \in \overline{P^+}(X)\backslash\{0\} : \langle x, x\rangle = 0\}/\mathbb{R}_{>0}$, and is denoted by $\partial\mathbb{H}^n_X$. We set $\overline{\mathbb{H}}^n_X = \mathbb{H}^n_X \cup \partial\mathbb{H}^n_X$.

Definition 6.1.2. Let $A(X)$ be the ample cone. We define

$$\mathrm{Aut}(A(X)) := \{\phi \in O(NS(X)) : \phi A(X) = A(X)\} \text{ and } A_X := A(X) \cap \mathbb{H}^n_X.$$

The closure of A_X in $\overline{\mathbb{H}^n_X}$ is denoted by $\overline{A_X}$. We denote $\overline{A_X}\backslash A_X$ by ∂A_X. ∂A_X is called *the boundary associated with the ample cone of* X.

By the unimodulality of $H^2(X, \mathbb{Z})$, the discriminant groups of $NS(X)$ and T_X are isomorphic:

$$A_{NS(X)} = NS(X)^*/NS(X) \simeq T_X^*/T_X = A_{T_X}.$$

Each isometry ϕ of $NS(X)$ or T_X gives rise to an isometry of the group $A_{NS(X)} \simeq A_{T(X)}$, denoted by ϕ^*.

Since the automorphism group $\mathrm{Aut}(X)$ of X naturally acts on the ample cone of X, there is a natural homomorphism $\rho : \mathrm{Aut}(X) \to \mathrm{Aut}(A(X))$. By Torelli theorem, the image of ρ is equal to the normal subgroup $\mathrm{Ker}\{O(S_X) \to O(q_{S_X})\} \cap \mathrm{Aut}(A(X))$ of $\mathrm{Aut}(A(X))$, where $O(q_{S_X})$ is the same as in Definition 2.1.10. As for this homomorphism, the following theorem is well-known.

Theorem 6.1.3 ([17, Theorem 8.1])**.** The kernel and the cokernel of ρ are finite groups.

Theorem 6.1.4 ([33, Proposition 2.2])**.** Let X be an algebraic K3 surface over $\mathbb{C}$. Then $\mathrm{Aut}(X)$ is finitely generated.

An elementary algebraic proof of the following lemma can be found in [1].

Lemma 6.1.5 (Selberg's lemma, [1])**.** Let K be a field of characteristic zero. Any finitely generated subgroup of $\mathrm{GL}(n, K)$ is virtually torsion-free. In other words, such a group has a torsion free subgroup of finite index.

Definition 6.1.6. Let X be a K3 surface, and let ω_X be a nowhere vanishing holomorphic 2-form of X. A automorphism ϕ of X is said to be *symplectic* if ϕ acts trivially on ω_x. The *symplectic automorphism group* of X is the group of symplectic automorphisms of X, and is denoted by $\mathrm{Aut}_s(X)$.

The following lemma is well-known.

Lemma 6.1.7 ([14, Lemma 8.11])**.** The symplectic automorphism of X acts trivially on the transcendental lattice T_X.

Let ϕ be the symplectic automorphism of X. Then, by Lemma 6.1.7, ϕ acts trivially on $A_{NS(X)} \simeq A_{T_X}$. By the Torelli theorem, we can identify $\mathrm{Aut}_s(X)$ with the subgroup $\{\phi \in O^+(NS(X)) : \phi A(X) = A(X),\ \phi^*|_{A_{NS(X)}} = \mathrm{id}_{A_{NS(X)}}\}$. Since $A_{NS(X)}$ is a finite abelian group, $\mathrm{Aut}_s(X)$ is a subgroup of finite index in $\mathrm{Aut}(A(X))$. Since any element of $O^+(NS(X))$ is an isometry of the hyperbolic space $\mathbb{H}^n_X$ associated X, the symplectic automorphism group $\mathrm{Aut}_s(X)$ of X is an n-dimensional discrete subgroup of isometries of $\mathbb{H}^n_X$.

Proposition 6.1.8. Let X be a K3 surface, and let $a \in A_X$. Then the Dirichlet domain $D(a)$ of $\mathbb{H}^n_X$, with center a, for the reflection group $W(X)$ is equal to $A_X = A(X) \cap \mathbb{H}^n_X$.

Proof. Recall that $H^+_w(a) = \{x \in \mathbb{H}^n : d(x, a) < d(x, wa)\}$ for $w \in W_X$. If w is the reflection s_δ for some (-2)-root δ,

$$\begin{aligned} & d(x,a) < d(x, s_\delta a) \\ \Leftrightarrow\ & \cosh d(x,a) < \cosh d(x, s_\delta a) \\ \Leftrightarrow\ & \langle x, a\rangle < \langle x, s_\delta a\rangle \\ \Leftrightarrow\ & 0 < \langle a, \delta\rangle\langle x, \delta\rangle \\ \Leftrightarrow\ & 0 < \langle x, \delta\rangle. \end{aligned}$$

Then we have that $H^+_{s_\delta}(a) = \{x \in \mathbb{H}^n_X : \langle x, \delta\rangle > 0\}$. Since $A(X)$ is a fundamental domain for $W(X)$ on the positive cone $P^+(X)$ of X whose sides are bounded by (-2)-roots, we have $A_X = \cap_{\delta \in \Delta^+(X)} H^+_{s_\delta}$.

By the definition of $D(a)$, we have $D(a) \subset A_X$. Since $D(a)$ and A_X are fundamental domains for $W(X)$, we have $D(a) = A_X$. □

$O^+(NS(X))$ is a geometrically finite subgroup of the first kind. By Theorem 4.9.10, we have the following:

Lemma 6.1.9. Let X be a K3 surface. Then a fundamental domain of $\mathbb{H}^n_X$ for $O^+(NS(X))$ is a generalized polytope. In particular, the action of $O^+(NS(X))$ on $\mathbb{H}^n_X$ is finite covolume.

Lemma 6.1.10. Let $a \in A_X$, and let P be the Dirichlet domain, centered at a, of $\mathbb{H}^n_X$ for $O^+(NS(X))$. Then, $\mathrm{Aut}(A(X)) \cdot \overline{P} = \overline{A_X}^H$, where $\overline{P}$ (resp. $\overline{A_X}^H$) are the closure of P (resp. A_X) in $\mathbb{H}^n_X$.

Proof. By Lemma 6.1.8 and the definition of P, we have

$$P = \{x \in \mathbb{H}^n_X : d(x, a) < d(x, ga) \text{ for any } g \in O^+(NS(X))\},$$
$$A_X = \{x \in \mathbb{H}^n_X : d(x, a) < d(x, wa) \text{ for any } w \in W(X)\}.$$

Then we have $P \subset A_X$ and so $\overline{P} \subset \overline{A_X}^H$. Hence we obtain $\mathrm{Aut}(A(X)) \cdot \overline{P} \subset \overline{A_X}^H$.

Conversely, because the tessellation $\{g\overline{P} : g \in O^+(NS(X))\}$ of $\mathbb{H}^n_X$ is locally finite, the union $\mathrm{Aut}(A(X)) \cdot \overline{P}$ is a closed subset. Then it is enough to show that $A_X \subset \mathrm{Aut}(A(X)) \cdot \overline{P}$. For any $x \in A_X$, there exist $w \in W(X)$ and $\gamma \in \mathrm{Aut}(A(X))$ such that $x \in w\gamma\overline{P}$. Then we have $x \in w\gamma\overline{P} \cap A_X \subset w\overline{A_X}^H \cap A_X$. By [26, Theorem 6.6.4], we have $w = 1$. In particular, we have $x \in \gamma\overline{P}$. Hence we have $A_X \subset \mathrm{Aut}(A(X)) \cdot \overline{P}$. □

Definition 6.1.11. Let P be the same as in Lemma 6.1.10. We define sets of bounded parabolic points of $O^+(NS(X))$ as follows:

- I_P = the set of ideal vertices of P,
- $V = Aut(A(X)) \cdot I_P$,
- $V_{NS} = O^+(NS(X)) \cdot I_P$,
- $C_X = \{c \in V : \mathrm{vcd}\, W(X)_c < n - 1\}$.

Definition 6.1.12. Let X be a K3 surface. Since $\mathrm{Aut}_s(X)$ is of finite index in $\mathrm{Aut}(A(X))$, the limit set of $\mathrm{Aut}_s(X)$ is equal to that of $\mathrm{Aut}(A(X))$. We denote this limit set by Λ_X. Λ_X is called the *limit set of the K3 surface X*. We denote the hyperbolic convex hull (we defined in Definition 4.7.4) of Λ_X by $C(X)$, and put $B(X) = C(X) \cap \mathbb{H}^n_X$.

Lemma 6.1.13. For any $c \in V \backslash C_X$, there exists a horoball $U(c)$ based at c such that $B(X) \cap U(c) = \emptyset$. Moreover, the point c is isolated in the boundary ∂A_X associated with the ample cone.

Proof. If $\mathrm{Aut}(A(X))$ is elementary, this lemma is obvious.

Suppose that $\mathrm{Aut}(A(X))$ is non-elementary. Let $\phi : \mathbb{H}^n_X \to U^n$ be an isometry from $\mathbb{H}^n_X$ to the upper half-space model U^n such that $\phi(c) = \infty$. The isometry ϕ induces the natural homeomorphisms from $\partial\mathbb{H}^n_X$ to ∂U^n. We also denote this homeomorphism by ϕ. Let b be a point of U^n corresponding to a point a in A_X.

We can consider the stabilizer group $W(X)_\infty$ as the Poincaré extension of a subgroup of $I(E^{n-1})$. Since $\mathrm{vcd}\, W(X)_c = n-1$, $W(X)_\infty$ is a crystallographic group of E^{n-1}. Then the Dirichlet domain K of E^{n-1} for $W(X)_\infty$, centered at b, is compact. Proposition 6.1.8 implies that $\Lambda_X \subset \partial A_X$. Then we have $\phi(\Lambda_X) \subset K \cup \{\infty\}$. Since $\mathrm{Aut}(A(X))$ is non-elementary, the limit set Λ_X is perfect: i.e., every point of Λ_X is not isolated. Hence the infinity ∞ is not in $\phi(\Lambda_X)$. By the compactaness of K, there exists a horoball B_∞ based at ∞ such that $C(K) \cap B_\infty = \emptyset$, where $C(K)$ is the hyperbolic convex hull of K. In particular, $B(X) \cap B_\infty = \emptyset$. The inverse image $\phi^{-1}(B_\infty)$ of B_∞ by ϕ is the desired horoball. □

Lemma 6.1.14. Let e be an elliptic divisor of X, let f_e be an elliptic fibration associated to e, and let $\mathrm{Aut}(X)_e$ be the stabilizer group of e. Then

$$\mathrm{vcdAut(X)_e} = \mathrm{vcd}\,\mathrm{MW}(f_e), \text{ and } \mathrm{vcd}\, W(X)_e = \sum_{v\in\mathbb{P}^1} (m_v - 1),$$

where $\mathrm{MW}(f_e)$ is the Mordell-Weil group of f_e, and m_v be the number of connected components of the fiber $f^{-1}(v)$ for $v \in \mathbb{P}^1$.

Proof. Let v be a point in $\mathbb{P}^1$, and let m_v be the number of connected components of the fiber $f^{-1}(v)$. By Shioda-Tate formula,

$$\rho(X) - 2 = \mathrm{vcd}\, MW(f_e) + \sum_{v\in\mathbb{P}^1} (m_v - 1).$$

If $m_v \neq 1$, we denote by W_v the Weyl group generated by the connected components of $f^{-1}(v)$, which is of type $\tilde{A}$, $\tilde{D}$, or $\tilde{E}$. If $m_v = 1$, we denote by W_v the trivial group. W_v fixes the elliptic divisor e, and for any two different $u, v \in \mathbb{P}^1$, W_u and W_v are commute. Then $\oplus_{v\in\mathbb{P}^1} W_v$ is a subgroup of $W(X)_e$. Here, for each v with $m_v \neq 1$, the Weyl group W_v acts on the (m_v-1)-dimensional Euclidean space, and admits an (m_v-1)-dimensional simplex Δ_v as a fundamental domain for the action of W_v. Hence $\oplus W_v$ acts on the $(\sum_{v\in\mathbb{P}^1}(m_v-1))$-dimensional Euclidean space, and has the fundamental domain $\prod_v \Delta_v$. Then we have

$$\sum_{v\in\mathbb{P}^1}(m_v - 1) = \mathrm{vcd}(\oplus W_v) \leq \mathrm{vcd}\, W(X)_e.$$

Since the Mordell-Weil group of f_e is a subgroup of $\mathrm{Aut}(X)_e$, then $\mathrm{rk}\,\mathrm{MW}\, f_e \leq \mathrm{vcd}\,\mathrm{Aut}(X)_e$, where $\mathrm{rk}\,\mathrm{MW}\, f_e$ is the rank of Mordell-Weil group of f_e.

Since both $W(X)_e$ and $\mathrm{Aut}(X)_e$ fix the elliptic divisor e, which is a cusped limit points of $O^+(NS(X))$, these two groups are elementary of either elliptic or parabolic type. Therefore, there exists an integer l (resp. m) such that $W(X)_e$

(resp. $\mathrm{Aut}(X)_e$) has a subgroup that is isomoprhic to $\mathbb{Z}^l$ (resp. $\mathbb{Z}^m$) of finite index in $W(X)_e$ (resp. $\mathrm{Aut}(X)_e$). Hence we have

$$\sum_{v\in\mathbb{P}^1}(m_v - 1) \leq l, \tag{6.1.1}$$

$$\mathrm{rk}\,\mathrm{MW}\, f_e \leq m, \tag{6.1.2}$$

so that $\rho(X) - 2 \leq l + m$. On the other hand, $O^+(NS(X))_e$ has a subgroup that is isomorphic to $\mathbb{Z}^{l+m}$. Then $l + m = \rho(X) - 2$. The inequality 6.2.1 and 6.2.2 imply that

$$l = \sum_{v\in\mathbb{P}^1}(m_v - 1),$$

$$m = \mathrm{rk}\,\mathrm{MW}\, f_e.$$

These are namely the desired equalities. □

By Lemma 6.1.14, we have $C_X = \{c \in V : \mathrm{vcd}\,\mathrm{Aut}(X)_c > 1\} = \{c \in V : \mathrm{vcd}\,\mathrm{Aut}(A(X))_c > 1\}$. Then each element of C_X is fixed by some parabolic transformation of $\mathrm{Aut}(A(X))_c$.

As noted in the introduction, the following theorem is also found by Kohei Kikuta at about the same time (see the introduction for details).

Theorem 6.1.15. Let X be an elliptic K3 surface, and let $\mathrm{Aut}_s(X)$ be the symplectic automorphism group of X. Then $\mathrm{Aut}_s(X)$ is geometrically finite.

Proof. By Theorem 6.1.3, the symplectic automorphism group $\mathrm{Aut}_s(X)$ is a subgroup of finite index in $\mathrm{Aut}(A(X))$. Then it is enough to show that $\mathrm{Aut}(A(X))$ is geometrically finite. Recall the notation of P, V, V_{NS}, and C_X in Definition 6.1.11

By Theorem 4.9.12, there exist proper horoballs at $c \in V_{NS}$ such that

1. The elements of $\{\overline{U(c)} : c \in V_{NS}\}$ are mutually disjoint,
2. $gU(c) = U(gc)$ for any element $g \in O^+(NS(X))$,
3. $\{\overline{U(c)}\backslash c\}$ is locally finite in $\mathbb{H}^n_X$.

By Lemma 6.1.13, shrinking the horoballs $U(c)$ sufficiently small, we obtain that

$$B(X)\backslash \bigcup_{c\in C_X} U(c) = B(X)\backslash \bigcup_{c\in V} U(c).$$

By applying Lemma 6.1.10,

$$\begin{aligned}
B(X)\backslash \bigcup_{c\in C_X} U(c) &\subset \overline{A_X}^H \backslash \bigcup_{c\in V} U(c)\\
&= \Big(\bigcup_{g\in \mathrm{Aut}(A(X))} g\overline{P}\Big)\backslash \bigcup_{c\in V} U(c)\\
&= \bigcup_{g\in \mathrm{Aut}(A(X))} \Big(g\overline{P}\backslash \bigcup_{c\in V} U(c)\Big)\\
&= \bigcup_{g\in \mathrm{Aut}(A(X))} g\Big(\overline{P}\backslash \bigcup_{c\in V} U(c)\Big).
\end{aligned}$$

Hence we have

$$B(X)\backslash \bigcup_{c\in C_X} U(c) \subset \bigcup_{g\in \mathrm{Aut}(A(X))} g\Big(\overline{P}\backslash \bigcup_{c\in V} U(c)\Big). \tag{6.1.3}$$

Let $\pi : \mathbb{H}^n_X \to \mathbb{H}^n_X/\mathrm{Aut}(A(X))$ be the natural quotient map. We define $V_c = U(c)/\mathrm{Aut}(A(X))$. By the equality 6.1.3, we obtain

$$(B(X)/\,\mathrm{Aut}(A(X)))\backslash \bigcup_{c\in I_P} V_c \subset \pi\Big(\overline{P}\backslash \bigcup_{c\in V} U(c)\Big).$$

Then $(B(X)/\,\mathrm{Aut}(A(X)))\backslash \bigcup_{c\in I_P} V_c$ is compact and $\{V_c\}_{c\in I_P}$ is a set of finitely many horocusps as disjoint closures. By Theorem 4.9.7, $\mathrm{Aut}(A(X))$ is geometrically finite. □

Proposition 6.1.16. Let p be a cusped limit point of $\mathrm{Aut}(A(X))$. Then p is a horopoint of $\overline{A_X}$, i.e., there exists a horoball U based at p such that U meets just the sides of $\overline{A_X}$ incident with p.

Proof. We now pass to the upper half-space model U^n and conjugate $\mathrm{Aut}(A(X))$ so that $p = \infty$. Let P be a Dirichlet polyhedron P, centerd at $a \in A_X$, for $O^+(NS(X))$. By Lemma 4.9.6, there exist a horoball Σ based at ∞ and elements $g_1, \cdots, g_k$ in $O^+(NS(X))$ such that

$$\Sigma \subset \bigcup_{f\in \mathrm{Aut}(A(X))_\infty} \Big(\bigcup_{i=1}^{k} g_i P\Big)$$

and Σ meets just the vertical sides of fg_iP.

Let S be a side of A_X with $S \cap \Sigma \neq \emptyset$. There exists an element $x \in S$ and $r > 0$ such that $B(x,r) \cap S$ is homeomorphic to B^{n-1}. Since P is locally finite, shrinking r, there exists $g_1, \cdots, g_k \in O^+(NS(X))$ such that $x \in g_i\overline{P}$ and $B(x,r) \subset \bigcup_{i=1}^{k} g_i\overline{P}$. In particular, we have

$$B^{n-1} \cong B(x,r) \cap S \subset \bigcup \{g_iT : g_i \in \mathrm{Aut}(A(X)), T \text{ is a side of } \overline{P}, \text{and } x \in gT\}.$$

Hence there exists a side T of $g\overline{P}$ such that $T^\circ \cap S \neq \emptyset$. By [26, Theorem 6.3.8], we have $T \subset S$. By $x \in T \cap \Sigma$, then T is a vertical side of $g\overline{P}$. Therefore, we have $\infty \in T \subset S$. $\square$

There is a natural one-to-one correspondence from the set of maximal parabolic subgroups of $\mathrm{Aut}_s(X)$ to the rational ray of $\partial A(X)$. By multiplying a positive integer, we have a primitive element in $NS(X)$. From this argument and Lemma 6.1.14, we have the following proposition:

Proposition 6.1.17. Let P be a maximal parabolic subgroup of $\mathrm{Aut}_s(X)$. Then there exists an element $w \in W(X)$ such that the fixed point of wPw^{-1} is an elliptic divisor e. Moreover, the rank of P is equal to the rank of the Mordell-Weil group of the elliptic fibration f_e associated with e.

6.2 The blown-up boundaries associated with ample cones of K3 surfaces

The notion of the blown-up coronae for relatively hyperbolic groups (a generalization of geometrically finite groups) is defined in the paper of Fukaya-Oguni [13], and Fukaya [12] established Bestvina-Mess type formula for relatively hyperbolic groups. We can immediately obtain Theorem 1 by applying Fukaya's result [12, Corollary B].

For readability to algebraic geometers, in this section, we give a simplified version of Fukaya's result, using the geometry of ample cones of K3 surfaces and hyperbolic geometry. Note that the proof method and some definitions in this section are not itself new.

Definition 6.2.1. Let U be an open subset of $\mathbb{R}^n$, and let p be a point in U. We denote the $(n-1)$-dimensional unit sphere in $\mathbb{R}^n$ by S^{n-1}. We put

$$\mathrm{Bl}_p(U) = \{(x,t) \in U \times S^{n-1} : |x-p|t = (x-p)\},$$

where $|\cdot|$ is the Eulidean norm. We define a function $\pi_p : \mathrm{Bl}_p(U) \to U$ by setting $\pi_p(x,t) = x$. We call the function π_p the *blowing-up of U at a point p (to S^{n-1}).* For a subset A of U, the closure of $\pi_p^{-1}(A\backslash\{p\})$ in $\mathbb{R}^n$ is called the *strict transform of A at a point p*, and is denoted by $\mathrm{St}_p(A)$.

Definition 6.2.2. Let $\mathcal{P} = \{p_1, \ldots, p_m\}$ be a subset of U. For each $p \in \mathcal{P}$, we take an open neighborhood U_p of p such that $U_p \cap \mathcal{P} = \{p\}$. Let $\pi_{p_i} : \mathrm{Bl}_{p_i}(U_{p_i}) \to U_{p_i}$ be the blowing-up of U_{p_i} at a point p_i. We define $\mathrm{Bl}_{\mathcal{P}}(U)$ by

$$\mathrm{Bl}_{\mathcal{P}}(U) = (U\backslash\mathcal{P}) \amalg (\coprod_{p\in\mathcal{P}} \mathrm{Bl}_p(U_p))/\sim,$$

where the equivalence relation $\sim$ is generated by $x \sim \pi_{p_i}(x)$ for all $x \in \coprod_{p \in \mathcal{P}} \pi_p^{-1}(U_p \backslash \{p\})$.

We denote by $[x]$ the class in $\mathrm{Bl}_{\mathcal{P}}(U)$ corresponding to x. We define the function $\pi_{\mathcal{P}} : \mathrm{Bl}_{\mathcal{P}}(U) \to U$ by setting

$$\pi_{\mathcal{P}}([x]) = \begin{cases} x & (x \in U \backslash \mathcal{P}) \\ \pi_p(x) & (x \in \mathrm{Bl}_p(U_p), p \in \mathcal{P}) \end{cases}$$

We call the function $\pi_{\mathcal{P}}$ the *blowing-up of U at a subset $\mathcal{P}$*. For a subset A of U, the closure of $\pi_{\mathcal{P}}^{-1}(A \backslash \mathcal{P})$ in $\mathbb{R}^n$ is called the *strict transform of A at a subset $\mathcal{P}$*, and is denoted by $\mathrm{St}_{\mathcal{P}}(A)$.

Definition 6.2.3. Let Γ be a geometrically finite subgroup of $M(B^n)$, and let C be the set of cusped limit points of Γ. For finite subsets $\mathcal{P}$ and Q of C with $\mathcal{P} \subset Q$, there exists the narural projection $\pi_{\mathcal{P},Q} : \mathrm{Bl}_Q(\mathbb{R}^n) \to \mathrm{Bl}_{\mathcal{P}}(\mathbb{R}^n)$ such that $\pi_{\mathcal{P}} \circ \pi_{\mathcal{P},Q} = \pi_Q$.

We denote the projective limit of $(\mathrm{Bl}_{\mathcal{P}}(\mathbb{R}^n), \pi_{\mathcal{P},Q})$ by $\pi_C = \varprojlim \pi_{\mathcal{P}} : \mathrm{Bl}_C(\mathbb{R}^n) \to \mathbb{R}^n$. We call the function π_C the *blowing-up of $\mathbb{R}^n$ at C*. For a subset A of $\mathbb{R}^n$, the closure of $\pi_C^{-1}(A \backslash \mathcal{P})$ in $\mathbb{R}^n$ is called the *strict transform of A at C*, and is denoted by $\mathrm{St}_C(A)$. The strict transform of $\overline{B^n}$ (resp. ∂B^n) is denoted by $\overline{B}^{bl}$ (resp. ∂B^{bl}).

Lemma 6.2.4. The action of a gemetrically finite subgroup Γ of $M(B^n)$ on $\mathbb{R}^n$ (resp. B^n, ∂B^n) is extended to an action on $\mathrm{Bl}_C(\mathbb{R}^n)$ (resp. $\overline{B}^{bl}$, ∂B^{bl}).

Proof. Since Γ preserves the cusped limit point of Γ, the action of Γ on $\mathbb{R}^n$ can be extend to that on $\mathrm{Bl}_C(\mathbb{R}^n)$. As Γ preserves $\overline{B^n}$ and ∂B^n, Γ acts on $\overline{B}^{bl}$ (resp. ∂B^{bl}) □

Definition 6.2.5. Let Γ be a geometrically finite subgroup of $M(B^n)$. A point a of ∂B^{bl} is a *limit point of Γ in ∂B^{bl}* if there is a point $x \in B^n$ and there exists a sequence $\{g_i\}_{i \in \mathbb{N}}$ of Γ such that $a = \lim g_i x$. The set of limit points of Γ is called *the blown-up limit set of Γ*, and is denoted by $\Lambda^{bl}(\Gamma)$.

Lemma 6.2.6. Let $\eta : U^n \to B^n$ be the standard transformation from U^n to B^n. For a non-zero vector $v \in \partial U = E^{n-1}$ and $a \in \overline{U^n}$, we define a half line $l : [0, \infty) \to \overline{U^n}$ by $l(t) = tv + a$, and put $L(t) = (\eta \circ l)(t)$. Then

$$\mathrm{St}_{e_n}(L(t)) \cap \pi^{-1}(e_n) = \left\{(e_n, \frac{v}{|v|})\right\} \subset \mathrm{Bl}_{e_n}(\mathbb{R}^n).$$

In particular, the intersection of $\mathrm{St}_{e_n}(L(t))$ with $\pi^{-1}(e_n)$ is determined by the direction of v.

Proof. By direct calculation, we obtain

$$L(t) - e_n = \frac{2(tv + \rho(a) - e_n)}{|tv + \rho(a) - e_n|^2}.$$

Then

$$\lim_{t\to\infty} \frac{L(t) - e_n}{|L(t) - e_n|} = \lim_{t\to\infty} \frac{tv + \rho(a) - e_n}{|tv + \rho(a) - e_n|} = \frac{v}{|v|}.$$

Hence, $\mathrm{St}_{e_n}(L) = L \cup \{(e_n, \frac{v}{|v|})\}$. □

Corollary 6.2.7. Let V be a vector subspace of $\mathbb{R}^n$, and let a be a point of $\overline{U^n}$. Let A be an open r-neighborhood of $V + a$ in U^n for a positive number r. Then,

$$\mathrm{St}_{e_n}(\eta(A)) \cap \pi_{e_n}^{-1}(e_n) = \left\{(e_n, \frac{v}{|v|}) \in \mathrm{Bl}_{e_n}(\mathbb{R}^n) : v \in V\backslash\{0\}\right\} \cong S^{r-1},$$

where r is the dimension of the vector space V.

Lemma 6.2.8. Let Γ be a geometrically finite subgroup of $M(B^n)$. For every cusped limit point p of Γ, we have

$$\Lambda^{bl}(\Gamma) \cap \pi_C^{-1}(p) = \Lambda^{bl}(\Gamma_p) \cong S^{r-1},$$

where r is the rank of the stabilizer group Γ_p, i.e., the rank of maximal abelian subgroup of Γ_p.

Proof. By conjugating Γ in $M(B^n)$, we can suppose $p = e_n$. Let $\eta : U^n \to B^n$ be the standard transformation. The group Γ acts on U^n by setting $\gamma(x) = (\eta^{-1}\gamma\eta)(x)$ for any $\gamma \in \Gamma$.

First, we show that $\Lambda^{bl}(\Gamma_p) \cong S^{r-1}$. There is a Γ_∞-inveriant r-plane Q of E^{n-1} such that Q/Γ_∞ is compact. For any $r > 0$, the r-neighborhood $N(Q, r)$ of Q in U^n is Γ_∞-invariant. The r-plane Q has a description of the form $Q = V + a$, where V is the r-dimensional vector subspace of E^{n-1} and a is a vector in Q. By Corollary 6.2.7, we have

$$\Lambda^{bl}(\Gamma_p) \subset \{(e_n, \frac{v}{|v|}) \in \mathrm{Bl}_{e_n}(\mathbb{R}^n : v \in V\backslash\{0\}\}.$$

Conversely, for $v \in V\backslash\{0\}$ and any integer $n > 0$, there is an element γ_n of Γ_∞ such that $\gamma_n a \in N(nv + a, R)$, where R is the Euclidean diameter of a fundamental domain of Q for Γ_∞. By Lemma 6.2.6,

$$\lim_{n\to\infty} \gamma_n x = \frac{v}{|v|} \in \Lambda^{bl}(\Gamma_p).$$

Hence, we have

$$\Lambda^{bl}(\Gamma_p) = \{(e_n, \frac{v}{|v|}) \in \mathrm{Bl}_{e_n}(\mathbb{R}^n) : v \in V\backslash\{0\}\} \cong S^{r-1}.$$

Secondly, we show that $\Lambda^{bl}(\Gamma) \cap \pi_C^{-1}(p) = \Lambda^{bl}(\Gamma_p)$. Let a be a point in $\Lambda^{bl}(\Gamma) \cap \pi_C^{-1}(p)$. There exist a point $x \in B^n$ and a sequence $\{g_i\}$ of Γ such that $a = \lim_{n\to\infty} g_i x$. If the sequence $\{g_i\}$ has infinitely many elements of Γ_p, by the definition of $\Lambda^{bl}(\Gamma_p)$, we have $a \in \Lambda^{bl}(\Gamma_p)$. Otherwise, there exists a proper cusped region $U(Q, r)$ such that $g_i x \in N(Q, r)$ for infinitely many $i \in \mathbb{N}$. By Corollary 6.2.7, a is in $\Lambda^{bl}(\Gamma_p)$. Hence we have $\Lambda^{bl}(\Gamma) \cap \pi_C^{-1}(p) \subset \Lambda^{bl}(\Gamma_p)$. The converse inclusion is obvious. □

Lemma 6.2.9. For any $x \in B^n$,

$$\Lambda^{bl}(\Gamma) = \overline{\Gamma x} \cap \partial B^{bl}.$$

In particular, the blown-up limit set of Γ is a Γ-inveriant closed subset of B^{bl}.

Proof. By the definition of $\Lambda^{bl}(\Gamma)$, it is obvious that $\Lambda^{bl}(\Gamma) \supset \overline{\Gamma x} \cap \partial B^{bl}$.

Conversely, for any $a \in \Lambda^{bl}(\Gamma)$, we put $\overline{a} = \pi_C(a) \in \Lambda(\Gamma)$. If a does not lie in the set C of cusped limit points of Γ, we can show that a is a limit point of Γ in ∂B^{bl} in the same way as a limit point of Γ in ∂B^{n-1}. If $a \in C$, applying Lemma 6.2.8 to the cusped limit point a, we have

$$a \in \Lambda^{bl}(\Gamma) \cap \pi_C^{-1}(\overline{a}) = \Lambda^{bl}(\Gamma_{\overline{a}}) = \overline{\Gamma_{\overline{a}} x} \cap \partial B^{bl} \subset \overline{\Gamma x} \cap \partial B^{bl}.$$

Therefore $\Lambda^{bl}(\Gamma) \subset \overline{\Gamma x} \cap \partial B^{bl}$. □

Lemma 6.2.10. Let Γ be a non-elementary geometrically finite subgroup of $M(B^n)$. Then,

$$\Lambda^{bl}(\Gamma) = \mathrm{St}_C(\Lambda(\Gamma)).$$

Proof. By Lemma 6.2.9, we have $\Lambda^{bl}(\Gamma) \supset \overline{\pi_C^{-1}(\Lambda(\Gamma)\backslash C)} = \mathrm{St}_C(\Lambda(\Gamma))$.

Conversely, it is enough to show that $\mathrm{St}_C(\Lambda(\Gamma)) \cap \pi_C^{-1}(p) \subset \Lambda^{bl}(\Gamma_p)$ for each $p \in C$. By conjugating Γ in $M(B^n)$, we can suppose $p = e_n$. As the similar way in Lemma 6.2.8, we pass the upper half-space model U^n, and Γ acts on U^n. Let $U(Q, r)$ be a proper cusped region based at ∞. Then $\Lambda(\Gamma) \subset N(Q, r) \cup \{\infty\}$. There exist a vector subspace V of E^{n-1} and a vector a of Q such that $Q = V + a$. By Corollary 6.2.7 and Lemma 6.2.8 we have

$$\mathrm{St}_C(\Lambda(\Gamma)) \cap \pi_C^{-1}(p) \subset \{(e_n, \frac{v}{|v|}) : v \in V\backslash\{0\}\} = \Lambda^{bl}(\Gamma_p).$$

□

Definition 6.2.11. Let X be a K3 surface, and let V (resp. C_X) be the same as in Definition 6.1.11. We denote the strict transform of ∂A_X (resp. $\overline{A_X}$) at C_X by ∂A_X^{bl} (resp. $\overline{A_X}^{bl}$).

The following result is given by Baragar [3]. By using the geometrically finiteness of $O^+(NS(X))$, we can give another proof of the result.

Theorem 6.2.12 ([3, Theorem 3.3])**.** For a K3 surface X, the set $\partial A_X \backslash \Lambda_X$ is equal to $V \backslash C_X = \{c \in V : \mathrm{Aut}(A(X))_c \text{ is finite}\}$.

Proof. Let x be a point of A_X. For any $y \in \partial A_X$, we denote the hyperbolic line connecting x to y by $l_{x,y}$.

Let P be the Dirichlet domain of $O^+(NS(X))$ centered at x. Then there exists a horoball $U(c)$ based at c such that

1. The elements of $\{\overline{U(c)} : c \in V_{NS}\}$ are mutually disjoint,
2. $gU(c) = U(gc)$ for any element $g \in O^+(NS(X))$,
3. $\{\overline{U(c)} \backslash c\}$ is locally finite in $\mathbb{H}^n_X$,

where the set V_{NS} is the same as in Definition 6.1.11. We put $K = P \backslash \bigcup_{c \in C} U(c)$. Then, by Lemma 6.1.10, $\overline{A_X} \backslash \bigcup_{c \in C} U(c) = \mathrm{Aut}(A(X)) \cdot K$.

If $y \notin C$, since cusped regions $\{U(c) : c \in V_{NS}\}$ are mutually disjoint, there exists a sequence $\{a_n\}_{n \in \mathbb{N}}$ and $\gamma_n \in \mathrm{Aut}(A(X))$ such that a_n converges to infinity as $n \to \infty$ and $l_{x,y}(a_n) \in \gamma_n K$. Since the Euclidean diameter of $\gamma_n K$ goes to zero as n goes to infinity, we have $y = \lim_{n\to\infty} l_{x,y}(a_n) = \lim_{n\to\infty} \gamma_n x \in \Lambda_X$.

If $y \in C$ and the rank of the stabilizer group $\mathrm{Aut}(A(X))_y$ is 0, by Lemma 6.1.13, the point y is isolated in ∂A_X. In particular, y is not in the limit set Λ_X.

Otherwise, y is in C_X, where $C_X = \{c \in C : \mathrm{vcd}\, W(X)_c < n - 1\}$. Then y is a bounded parabolic point. In particular, y is in the limit set Λ_X. □

Definition 6.2.13. Let X be a K3 surface. In Definition 6.1.12, we defined the limit set Λ_X of X. We denote the blown-up limit set of $\mathrm{Aut}_s(X)$ by Λ_X^{bl}.

Corollary 6.2.14. If the symplectic automorphism group $\mathrm{Aut}_s(X)$ of a K3 surface X is non-elementary, then $\partial A_X^{bl} \backslash \Lambda_X^{bl} = V \backslash C_X$.

Proof. By Theorem 6.2.12, we have $\partial A_X \backslash V = \Lambda(\Gamma) \backslash C_X$. Then, by Lemma 6.2.10, we have the desired equality. □

Let C be the set of cusped limit points of $\mathrm{Aut}(A(X))$. By Lemma 4.9.12 and Lemma 6.1.16, there exist cusped regions $U(c)$ based at $c \in C$ such that

1. The elements of $\{U(c) : c \in C\}$ are mutually disjoint,
2. $gU(c) = U(gc)$ for any $g \in \mathrm{Aut}(A(X))$ and $c \in C$,
3. $\{\overline{U(c)} \backslash \{c\}\}$ is locally finite in $B^n \cup O(\mathrm{Aut}(A(X)))$,

4. $U(c)$ meets just the sides of $\overline{A_X}$ incident with c.

Since a cusped region $U(c)$ is homeomorphic to its strict transformation on $\overline{B}^{bl}$, we identify $U(c)$ with its strict transformation. To apply the Bestvina-Mess formula (Theorem 5.2.7) to automorphism groups of K3 surfaces, we construct a homotopy on $\overline{A_X}^{bl} \setminus \cup_{c\in C} U(c)$.

Lemma 6.2.15. Let a be a point of A_X, and let $K(a,t) = C(a,t) \cap \overline{A_X} \setminus \cup_{c\in C} U(c)$, where $C(a,t)$ is the closed t-ball centered at a. For every $x \in \overline{A_X} \setminus (\cup_{c\in C} U(c) \cup C)$, let S_x be the smallest (horo)sphere (based) centered at x that meets $K(a,t)$. Then $K(a,t) \cap S_x$ is a singleton.

Proof. Suppose that there exists y_1, y_2 in $K(a,t) \cap S_x$ with $y_1 \neq y_2$. Then the hyperbolic line $[y_1, y_2]$ connecting y_1 to y_2 is contained in $C(a,t) \cap \overline{A_X}$. For any $z \in (y_1, y_2)$, we have $d(x,z) < d(x,y_1) = d(x,y_2)$. Then we have $z \in \cup_{c\in C} U(c)$. Since cusped regions $U(c)$ are mutually disjoint, there exists $c \in C$ such that $(y_1, y_2) \subset U(c)$ and $y_1, y_2 \in K_t \cap \partial U(c)$.

We now pass to the upper half-space model U^n and conjugate $\mathrm{Aut}(A(X))$ so that $c = \infty$. We denote by Σ (resp. w_1, w_2) the horoball (resp. the point) on U^n corresponding to $\partial U(c)$ (resp. y_1, y_2). By Lemma 6.1.16, the Euclidean segment l connecting y_1 to y_2 is contained in $K(a,t) \cap \Sigma$. This contradicts the minimality of S_x. □

Definition 6.2.16. Let $K(a,t)$ be the same as in Lemma 6.2.15. We put

$$U = \cup_{c\in C} U(c) \cup C.$$

For every $x \in \overline{A_X} \setminus U$ and $t \geq 0$, we define the function

$$\rho_a : [0,\infty) \times \overline{A_X} \setminus U \to \overline{A_X} \setminus U$$

as follows: if $x \in K(a,t)$, we define $\rho_a(x,t) = x$, otherwise, we define $\rho_a(x,t)$ by the unique point of $K(a,t) \cap S_x$.

Proposition 6.2.17. Let ρ_a be the same as above. Then ρ_a is continuous.

Proof. If ρ_a is not continuous, there exists a sequence $\{(x_n, t_n)\}$ such that (x_n, t_n) converges to (x,t) and $\rho_a(x_n, t_n)$ converges to $y \neq \rho_a(x,t)$.

If x is in B^n, passing to a subsequence of $\{(x_n, t_n)\}$, we can assume that each x_n is in B^n. By the definition of $\rho_a(x_n, t_n)$ (resp. $\rho_a(x,t)$), there exists r_n (resp. r) such that $S(x_n, r_n) \cap K(a, t_n) = \{\rho_a(x_n, t_n)\}$ (resp. $S(x,r) \cap K(a,t) = \{\rho_a(x,t)\}$). Passing to a subsequence of (x_n, t_n), we can suppose that r_n converges to r_0. Since

$d((x_n, t_n), \rho_a(x_n, t_n)) = r_n$, we have $d(x, y) = r_0$. By the definition of $\rho_a(x, t)$, we obtain $r_0 > r$ and

$$\begin{aligned} r_n - d((x_n, t_n), \rho_a(x, t)) &\geq r_n - d((x_n, t_n), (x, t)) - d((x, t), \rho_a(x, t)) \\ &= (r_n - r_0) + (r_0 - r) - d((x_n, t_n), (x, t)). \end{aligned}$$

Hence, for a sufficiently large n, we have $r_n > d((x_n, t_n), \rho_a(x, t))$. This contradicts the definition of r_n.

If x is in ∂B^n, passing to the upper half-space model U^n, there exists a (horo)sphere S_k of U^n (based) centered at x_k such that $S_k \cap K(a, t_k) = \{\rho_a(t_k, x_k)\}$. We denote the Euclidean center (resp. radius) of S_k by $p_k = (p_{k,1}, \ldots, p_{k,n})$ (resp. r_k). Then we have that $|p_k - \rho_a(t_k, x_k)| = r_k$, $|x_k - p_k| \leq r_k$, and $r_k \leq p_{k,n}$, where $|\ |$ is the Euclidean norm. Passing to a subsequence of $\{(x_n, t_n)\}$, we can assume that p_k (resp. r_k) converges to some point $p = (p_1, \ldots, p_n)$ (resp. $r \geq 0$). As k goes to infinity, we have

$$|p - y| = r, \ \ |x - p| \leq r, \ \text{ and } r \leq p_n.$$

Since x is in ∂B^n, we have that $p = (x_1, \ldots, x_{n-1}, r)$ and $|x - p| = r$. Since $|p - y| = r$, this contradicts the uniqueness of $\rho_a(x, t)$. □

Proposition 6.2.18. ρ_a extends to $\overline{\rho_a} : [0, \infty) \times \overline{A_X}^{bl} \setminus \cup_{c \in C} U(c) \to \overline{A_X}^{bl} \setminus \cup_{c \in C} U(c)$.

Proof. Let x be a point in $\overline{A_X}^{bl} \setminus \cup_{c \in C} U(c)$ with $\pi_C(x) = c$. We now pass to the upper half-space model U^n so that $a = e_n$ and $c = \infty$ by an isometry $\phi : B^n \to U^n$. Let Σ be the horosphere $\{x \in U^n : x_n = r\}$ based at ∞ corresponding to $U(c)$.

If $C(e_n, t) \cap \overline{A_X} \cap U(c) = \emptyset$, we define $\overline{\rho_a}(x, t)$ by the intersection $S(e_n, t)$ with $\mathbb{R}_{>0} e_n$. If $C(e_n, t) \cap \overline{A_X} \cap U(c) \neq \emptyset$, there exists the half-line l_x on Σ starting at re_n such that $\mathrm{St}_C(\phi^{-1} \circ l) \cap \pi_C^{-1}(c) = \{x\}$. For any $t \geq 0$, there exists a sufficiently large $R_t > 0$ such that $\rho_a(s, x)$ is in Σ for any $s \in (0, t]$ and any x with $|x| > R_t$. Then, if we define $\overline{\rho_a}(t, x)$ by the intersection of $K(a, t)$ with l_x, this is the desired extension of ρ_a. □

Theorem 6.2.19. If $\mathrm{Aut}_s(X)$ is non-elementary, then $(\overline{A_X}^{bl} \setminus \cup_{c \in C} U(c), \partial A_X^{bl})$ is a $\mathcal{Z}$-structure for $\mathrm{Aut}(X)$. In particular,

$$\mathrm{vcd}(\mathrm{Aut}(X)) = \dim \partial A_X^{bl} + 1.$$

Proof. We define the homotopy $H : [0, 1] \times \overline{A_X}^{bl} \setminus \cup_{c \in C} U(c) \to \overline{A_X}^{bl} \setminus \cup_{c \in C} U(c)$ setting

$$H(t, x) = \begin{cases} (t, x) & (t = 1) \\ \rho_a(\log \frac{t+1}{t-1}, x) & (t \neq 1) \end{cases}$$

By Proposition 6.2.18 and the definition of ρ_a, H is continuous. Hence $\overline{A}_X^{bl} \setminus \cup_{c \in C} U(c)$ is an AR, and ∂A_X^{bl} is a $\mathcal{Z}$-set of $\overline{A}_X^{bl} \setminus \cup_{c \in C} U(c)$. Then $(\overline{A}_X^{bl} \setminus \cup_{c \in C} U(c), \partial A_X^{bl})$ is a $\mathcal{Z}$-structure for $\mathrm{Aut}_s(X)$ (resp. $\mathrm{Aut}(X)$). By Theorem 5.2.7, we have the desired equality. □

Corollary 6.2.20. If Λ_X is homeomorphic to a Cantor set and X has an elliptic fibration whose rank is larger than 0, then

$$\mathrm{vcd}(\mathrm{Aut}(X)) = \max \mathrm{MW}(f : X \to \mathbb{P}^1),$$

where f runs all the elliptic fibrations of X.

Proof. For any $c \in C$, we have $\Lambda_X^{bl} \cap \pi_C^{-1}(c) \cong S^{r-1}$, where r is the rank of $\mathrm{Aut}(A(X))_c$. By Proposition 6.1.17, r is equal to the rank of the Mordell weil group corresponding to c. If $c \notin C$, since Λ_X is totally disconnected, $\{c\}$ itself is an open neighborhood. Then the topological dimension of Λ_X^{bl} is equal to $\max \mathrm{MW}(f : X \to \mathbb{P}^1)$. □

Example 6.2.21. Let X be the K3 surface whose Néron-Severi lattice $NS(X)$ is given by

$$NS(X) = \begin{pmatrix} 2 & 4 & 1 \\ 4 & 2 & 0 \\ 1 & 0 & -2 \end{pmatrix}.$$

Baragar [4] studied the automorphism group of this K3 surface and the growth of orbits of curves. He showed that Λ_X is a Cantor set and $\mathrm{Aut}(A(X))$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$.

Although the automorphism group of X is completely determined in [4] as above and then this is not a new example of computing the virtual cohomological dimensions, we can also apply Corollary 6.2.20 to this K3 surface X.

6.3 The ample cones of K3 surfaces and sphere packings

Let X be a K3 surface, and let $\phi : \mathbb{H}_X^n \to B^n$ be an isometry from $\mathbb{H}_X^n$ to the conformal ball model B^n. The isometry ϕ induces the natural homeomorphism from $\partial\mathbb{H}_X^n$ to ∂B^n. For a (-2)-root δ and an ample class a of X, we set

$$H_\delta = \{x \in \mathbb{H}_X^n : d(x, a) = d(x, s_\delta a)\},$$
$$H_\delta^+ = \{x \in \mathbb{H}_X^n : d(x, a) < d(x, s_\delta a)\}.$$

The closure of H_δ (resp. H_δ^+) in $\overline{\mathbb{H}_X^n}$ is denoted by $\overline{H_\delta}$ (resp. $\overline{H_\delta^+}$). We set

$$\mathcal{S}_X := \{\delta \in \Delta^+(X) : H_\delta \cap \overline{A_X}^H \text{ is a side of } \overline{A_X}^H\},$$

where $\overline{A_X}^H$ is the closure of A_X in $\mathbb{H}_X^n$.

Definition 6.3.1. Let δ be an element of $\mathcal{S}_X$. We define the *sphere associated with* δ by $S_\delta = \phi(\overline{H_\delta}) \cap \partial B^n$, which is isomorphic to the $(n-2)$-sphere. We define $B_\delta = \phi(\overline{H_\delta^+} \backslash \overline{H_\delta}) \cap \partial B^n$. We call B_δ the *open ball associated with* δ.

By Proposition 6.1.8, we have $A_X = \cap_{\delta \in \mathcal{S}_X} H_\delta^+$ and $\partial A_X = \cap_{\delta \in \mathcal{S}_X} \overline{B_\delta}$, where $\overline{B_\delta}$ is the closure of B_δ in ∂B^n.

Definition 6.3.2. The boundary ∂A_X associated with the ample cone of X is a *sphere packing* if for any two different δ_1, δ_2 in $\mathcal{S}_X$, their associated spheres S_{δ_1} and S_{δ_2} are either disjoint or tangent to each other.

∂A_X is called a *connected sphere packing* if ∂A_X is a sphere packing, and for any two different δ, δ' in $\mathcal{S}_X$, there exist (-2)-roots $\delta_1, \dots, \delta_n$ in $\mathcal{S}_X$ such that $\delta = \delta_1$, $\delta' = \delta_n$, and two spheres S_{δ_i} and $S_{\delta_{i+1}}$ are tangent for $i = 1, \dots, n-1$.

Theorem 6.3.3. Let X be an elliptic K3 surface. If ∂A_X is a connected sphere packing, and every elliptic divisor of X corresponds to the tangent point of some two spheres associated with elements of $\mathcal{S}_X$, then

$$\mathrm{vcd}(\mathrm{Aut}(X)) = \max\{\mathrm{rk}\,\mathrm{MW}(f)\} = \rho(X) - 3,$$

where f runs all elliptic fibrations of X.

Proof. As ∂A_X is connected, we can sort the set of the open balls associated with elements of $\mathcal{S}_X$ to $\{B_i\}_{i \in \mathbb{N}}$ such that for any $i \in \mathbb{N}$, the intersection of B_i with $\cup_{j=1}^{i-1} B_j$ is not empty. Let $\mathcal{P}_m$ be the set $\{p : p \in B_i \cap B_j$ for some two distinct $i, j \leq m\}$. We define the following sets:

- $U_m = \bigcap_{i=1}^m B_i$,
- $F_m = \bigcap_{i=1}^m \overline{B_i}$,
- $F_m^{bl} = \mathrm{St}_{\mathcal{P}_m}(F_m)$ = the strict transform of F_m at $\mathcal{P}_m$.

For any $l < m$, let $\pi_{\mathcal{P}_l, \mathcal{P}_m}$ be the same as in Definition 6.2.3. The map $\pi_{\mathcal{P}_l, \mathcal{P}_m}$ restricts to a map from F_m^{bl} to F_l^{bl}. These maps form a projective system. Since ∂A_X is a subset of F_m for any $m \in \mathbb{N}$, there exists the natural map $\phi_m : \partial A_X^{bl} \to F_m^{bl}$. Then we have the natural map $\phi : \partial A_X^{bl} \to \varprojlim F_m^{bl}$. The map ϕ is a homeomorphism. To show this, we consider the following:

$$\psi_m : \varprojlim F_i^{bl} = \varprojlim \mathrm{St}_{\mathcal{P}_i}(F_i) \to \cap_{i \in \mathbb{N}} \mathrm{St}_{\mathcal{P}_m}(F_i) = \mathrm{St}_{\mathcal{P}_m}(\partial A_X).$$

Since $\partial A_X^{bl} = \varprojlim \mathrm{St}_{\mathcal{P}_m}(\partial A_X)$, the maps ψ_m induce the natural map $\psi : \varprojlim F_m^{bl} \to \partial A_X^{bl}$, which is the inverse of ϕ.

For each $m \in \mathbb{N}$, F_m^{bl} is a topological manifold with its interior U_m, and then F_m^{bl} and U_m are homotopy equivalent. Hence we have

$$H^*(\partial A_X^{bl}, \mathbb{Z}) \cong \lim_{m\to\infty} H^*(F_m^{bl}, \mathbb{Z}) \cong \lim_{m\to\infty} H^*(U_m, \mathbb{Z}). \tag{6.3.1}$$

For each $m \in \mathbb{N}$ and a sufficiently small $\epsilon > 0$, we define V_{m+1} by the ϵ-neighborhood of $\partial B^n \backslash B_{m+1}$ in U_m. Then we can see as follows:

- V_{m+1} is contractible,
- $U_{m+1} \cup V_{m+1} = U_m$,
- $U_{m+1} \cap V_{m+1}$ is homotopy equivalent to the $(n-2)$-sphere with finite k punctures,

where k is the number of the set $\{p : p$ is a tangent point of B_i and B_{m+1} for some $i \leq m\}$. By Mayer-Vietoris exact sequence and the $(n-2)$-sphere with k punctures is homotopy equivalent to the wedge sum of $(k-1)$-copies of S^{n-3}, we have the following exact sequence:

$$\cdots \to H^*(U_m) \to H^*(U_{m+1}) \oplus H^*(V_{m+1}) \to H^*(\vee_{i=1}^{k-1} S^{n-3}) \to H^{*+1}(U_m) \to \cdots .$$

Therefore, we can inductively conclude that

$$H^*(U_m) = 0 \quad (* \geq n-2).$$

In particular, by Theorem 6.2.19 and the equation 6.3.1, we have

$$\begin{aligned} \operatorname{vcd} \operatorname{Aut}(X) &= \max\{m : H_c^m(A_X, \mathbb{Z}) \neq 0\} \\ &= \max\{m : H^m(\partial A_X^{bl}, \mathbb{Z}) \neq 0\} + 1 \\ &\leq n-2 = \rho(X) - 3, \end{aligned}$$

where H_c^* is the cohomology with compact support. Lemma 6.1.14 implies that there exists an elliptic fibration f such that $\operatorname{rk} \operatorname{MW}(f) = \rho(X) - 3$. □

Example 6.3.4. The Apollonian circle packing is generated as follows: we begin with four mutually tangent circles. There is four curvlinear triangles. We can put the inscribed circle for each curvlinear triangle, so that the curvlinear triangle separated into new curvlinear triangles. Inductively, we put inscribed circles for each curvelinear triangle, and the closure of the union of the inscribed circles is the Apollonian circle packing, which is shown in the following figure.

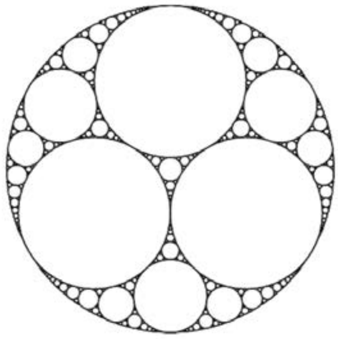

Figure 6.1: The Apollonian circle packing

In [2], Baragar constructed a K3 surface Y_2 with ample cone the Apollonian circle packing, whose Néron-Severi lattice is given by

$$\begin{pmatrix} -2 & 2 & 2 & 4 \\ 2 & -2 & 2 & 4 \\ 2 & 2 & -2 & 0 \\ 4 & 4 & 0 & 0 \end{pmatrix}.$$

By Theorem 6.3.3, we have $\operatorname{vcd}\operatorname{Aut}(Y_2) = 1$.

Example 6.3.5. The Apollonian sphere packing is the three dimensional analog of the Apollonian circle packing. We begin with five mutually tangent spheres, and in the space between any four of them, we put the inscribed sphere. As the same manner in the previous example, we can inductively put a sphere so that this sphere touch other four spheres. The closure of the union of the inscribed spheres is the Apollonian sphere packing. Baragar [2] also constructed a K3 surface Y_3 with ample cone the Apollonian sphere packing, whose Néron-Severi lattice is given by

$$\begin{pmatrix} -2 & 2 & 2 & 2 & 4 \\ 2 & -2 & 2 & 2 & 4 \\ 2 & 2 & -2 & 2 & 4 \\ 2 & 2 & 2 & -2 & 0 \\ 4 & 4 & 4 & 0 & 0 \end{pmatrix}.$$

By Theorem 6.3.3, we have $\operatorname{vcd}\operatorname{Aut}(Y_3) = 2$.

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