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Completely Positive and Copositive Cones over Symmetric Cones: Approximations and Facial Structure

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ABSTRACT

This thesis studies approximations and the facial structure of completely positive cones and their duals, copositive cones, over symmetric cones from the optimization aspect.

In the part on the approximations, we first provide approximation hierarchies for copositive cones over general and specific symmetric cones. These hierarchies are described by positive semidefinite constraints. Additionally, we compare them with existing ones. Through numerical experiments, we demonstrate that by combining the proposed approximation hierarchies with the existing ones, we can evaluate the optimal values of copositive programming problems more accurately and efficiently.

We second examine various extensions of doubly nonnegative cones. Inner-approximation hierarchies for generalized (in the sense that the cones underlying completely positive and copositive cones are beyond nonnegative orthants) copositive cones are utilized to obtain two generalized doubly nonnegative cones from the doubly nonnegative cones. We conduct theoretical and numerical comparisons to assess the relaxation strengths of the generalized doubly nonnegative cones over a direct product of a nonnegative orthant and second-order or positive semidefinite cones. These comparisons also include an analysis of an existing generalized doubly nonnegative cone. Through numerical experiments, we demonstrate that the three generalized doubly nonnegative cones provide significantly tighter bounds for generalized completely positive programming problems than positive semidefinite cones.

In the part on the facial structure, we first investigate two values decided by the facial structure of closed convex cones: the length of the longest chain of faces and the distance to polyhedrality. We prove that the two lengths of completely positive cones over nonnegative orthants, doubly nonnegative cones, and their duals are the worst cases possible.

We second show that copositive cones over symmetric cones are never facially exposed when the underlying symmetric cone has dimension at least 2. We do so by explicitly exhibiting a non-exposed extreme ray. Our result extends the known fact that copositive cones over nonnegative orthants are not facially exposed in general.

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Sets and set operators

$\mathbb{N}$ Set of nonnegative integers

$\mathbb{R}x$ Line generated by x, i.e., $\{\alpha x \mid \alpha \in \mathbb{R}\}$

$\mathbb{R}$ Space of real numbers

$\mathbb{R}_+x$ Half-line originated from the origin and generated by x, i.e., $\{\alpha x \mid \alpha \geq 0\}$

$\operatorname{cl} S$ Closure of S

$\operatorname{conv} S$ Convex hull of S

$\dim K$ Dimension of the smallest subspace that contains a cone K

ℓ_K Length of the longest chain of faces of K

$\ell_{\text{poly}}(K)$ Distance to polyhedrality of K

$|S|$ Number of elements in S

$\mathfrak{S}_n$ Symmetric group of order n

$\operatorname{int} S$ Interior of S

$\operatorname{span} S$ Smallest subspace that contains S

K^* Dual cone of a cone K

$S^\perp$ Space of elements orthogonal to every element in S

Vectors and matrices

$(\boldsymbol{v}_1; \dots; \boldsymbol{v}_k)$ Vector obtained by vertically concatenating vectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_k$

$\mathbb{L}^n$ n-dimensional second-order cone

$\mathbb{R}^I$ Space of $|I|$-dimensional real vectors with elements indexed by I

$\mathbb{R}^n$ Space of n-dimensional real vectors

$\mathbb{R}^n_+$ Cone of entrywise nonnegative vectors in $\mathbb{R}^n$ (nonnegative orthant)

$\mathbb{S}^I$ Space of real $|I| \times |I|$ symmetric matrices with rows and columns indexed by I

$\mathbb{S}^n$ Space of real $n \times n$ symmetric matrices

$\mathbb{S}^n_+$ Cone of positive semidefinite matrices in $\mathbb{S}^n$

$\mathbb{S}^I_+$ Cone of positive semidefinite matrices in $\mathbb{S}^I$

$\mathbf{0}_n, \mathbf{0}$ Zero vector (of dimension n)

$\mathbf{1}_n, \mathbf{1}$ Vector with all elements 1 (of dimension n)

$\boldsymbol{A}, \boldsymbol{B}$ Real matrices (boldface uppercase letters)

$\boldsymbol{a}, \boldsymbol{b}$ Real column vectors (boldface lowercase letters)

$\boldsymbol{a} \odot \boldsymbol{b}$ Entrywise product of $\boldsymbol{a}$ and $\boldsymbol{b}$

$\boldsymbol{A}^\top$ Transpose of $\boldsymbol{A}$

$\boldsymbol{e}_i$ Vector with the ith element being 1 and all other elements 0

$\boldsymbol{E}_n$ $n \times n$ matrix with all elements 1

$\boldsymbol{I}_n, \boldsymbol{I}$ $(n \times n)$ identity matrix

$\boldsymbol{O}_n, \boldsymbol{O}$ $(n \times n)$ zero matrix

$\mathcal{N}^n$ Cone of $n \times n$ symmetric entrywise nonnegative matrices

$\mathcal{COP}^n$ Cone of $n \times n$ copositive matrices

$\mathcal{CP}^n$ Cone of $n \times n$ completely positive matrices

$\Delta^n_=$ Set of $\boldsymbol{x} \in \mathbb{R}^{n+1}_+$ the sum of elements of which is 1

$\mathrm{diag}(\boldsymbol{A})$ Vector of the diagonal elements of $\boldsymbol{A}$

$\mathcal{DNN}^n$ Cone of $n \times n$ symmetric matrices that are entrywise nonnegative and positive semidefinite (doubly nonnegative cone)

$\|\boldsymbol{a}\|_2$ Norm of a vector $\boldsymbol{a}$ defined as $\|\boldsymbol{a}\|_2 := \sqrt{\boldsymbol{a}^\top \boldsymbol{a}}$

$\mathrm{rank}(\boldsymbol{A})$ Rank of $\boldsymbol{A}$

$\mathcal{SPN}^n$ Cone of sums of a real $n \times n$ symmetric positive semidefinite matrix and a symmetric entrywise nonnegative matrix (SPN cone)

$\mathrm{svec}(\boldsymbol{A})$ Vectorization of a symmetric matrix $\boldsymbol{A}$

a_i ith element of a vector $\boldsymbol{a}$

$A_{i,j}, A_{ij}$ (i,j)th element of a matrix $\boldsymbol{A}$

S^n Set of $\boldsymbol{x} \in \mathbb{R}^{n+1}$ satisfying $\|\boldsymbol{x}\|_2 = 1$

Polynomials, tensors, and functions

$[\boldsymbol{\alpha}]$ Tuple $(\underbrace{1,\dots,1}_{\alpha_1 \text{ factors}},\dots,\underbrace{n,\dots,n}_{\alpha_n \text{ factors}})$

$\mathbb{I}^n_{=m}$ Set of $\boldsymbol{\alpha} \in \mathbb{N}^n$ such that $\sum_{i=1}^n \alpha_i = m$

$\mathbb{I}^n_{\leq m}$ Set of $\boldsymbol{\alpha} \in \mathbb{N}^n$ such that $\sum_{i=1}^n \alpha_i \leq m$

$\mathbb{N}^m_n$ Set of $(i_1,\dots,i_m) \in \mathbb{N}^m$ such that $1 \leq i_k \leq n$ for all $k = 1,\dots,m$

$\boldsymbol{\alpha}!$ Factorial $\prod_{i=1}^n \alpha_i!$

$\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{B}}$ Real tensors (boldface calligraphic letters)

$\boldsymbol{x}^{\boldsymbol{\alpha}}$ Power $\prod_{i=1}^n x_i^{\alpha_i}$

$\mathcal{S}(\mathbb{V})$ Space of self-adjoint linear transformations on $\mathbb{V}$

$\mathcal{S}^{n,m}$ Space of real symmetric tensors of order m and dimension n

$\mathcal{S}^{n,m}(\mathbb{V})$ Space of symmetric tensors of order m over an n-dimensional real inner product space $\mathbb{V}$

$\mathcal{COP}(\mathbb{K})$ Cone of symmetric tensors of order 2 that are copositive over $\mathbb{K}$

$\mathcal{COP}^{n,m}$ Cone of real symmetric tensors of order m and dimension n that are copositive over $\mathbb{R}^n_+$

$\mathcal{COP}^{n,m}(\mathbb{K})$ Cone of symmetric tensors of order m and dimension n that are copositive over $\mathbb{K}$

$\mathcal{CP}(\mathbb{K})$ Cone of symmetric tensors of order 2 that are completely positive over $\mathbb{K}$

$\mathcal{CP}^{n,m}$ Cone of real symmetric tensors of order m and dimension n that are completely positive over $\mathbb{R}^n_+$

$\mathcal{CP}^{n,m}(\mathbb{K})$ Cone of symmetric tensors of order m and dimension n that are completely positive over $\mathbb{K}$

$\langle \mathcal{A}, \mathcal{B} \rangle$ Inner product of tensors $\mathcal{A}$ and $\mathcal{B}$

$|\boldsymbol{\alpha}|$ Sum $\sum_{i=1}^n \alpha_i$

$\|\mathcal{A}\|_{\mathrm{F}}$ Norm of a tensor $\mathcal{A}$

$\mathscr{S}\mathcal{A}$ Symmetrization of a tensor $\mathcal{A}$

$\Sigma^{n,2m}$ Cone of sums of squares of elements in $H^{n,m}$

$\Sigma^{n,2m}(\mathbb{V})$ Cone of sums of squares of elements in $H^{n,m}(\mathbb{V})$

$\mathrm{SOS}^{n,2m}(\mathbb{V})$ Cone $\mathrm{conv}\{\mathscr{S}(\mathcal{A} \otimes \mathcal{A}) \mid \mathcal{A} \in \mathcal{S}^{n,m}(\mathbb{V})\}$

$H^{n,m}$ Space of homogeneous polynomials in n variables of degree m with real coefficients

$H^{n,m}(\mathbb{V})$ Space of homogeneous polynomial functions of degree m on an n-dimensional real inner product space $\mathbb{V}$

T_n Value $\frac{n(n+1)}{2}$

$x^{\otimes m}$ m-fold tensor product of x with itself

Part I

Prologue

Chapter 1

Introduction

Completely positive programming is a class of optimization problems that minimize a linear objective function subject to the intersection of an affine space and a *completely positive cone*. In particular, we focus on completely positive cones *over symmetric cones*. With completely positive cones over symmetric cones, we can represent many realistic and hard-to-solve problems. This means that completely positive cones also have complicated structures. In this thesis, we study approximations and the facial structure of completely positive cones and their duals, *copositive cones*, over symmetric cones from the optimization perspective.

In Section 1.1, we mention the background of this thesis. In Section 1.2, we state the main results of this thesis. In Section 1.3, we provide the organization of subsequent chapters.

1.1 Background

Many real-world problems can be formulated as optimization problems with combinatorial or nonconvex structures. As an example of combinatorial structure, binary constraints can encode decision-making such as "do or not" and "yes or no." As that of nonconvex structure, rank constraints can represent low dimensionality of data [16] and are also utilized in the problems of estimating unknown locations of sensors [83] and finding the nearest low-rank correlation matrix [46], for example. Such problems with combinatorial or nonconvex structures are hard to solve, and in fact, include many NP-hard problems such as the maximum stable set problem and the max-cut problem [60].

We can reformulate such NP-hard optimization problems as conic linear programming problems with a completely positive cone constraint, namely completely positive programming problems. A conic linear programming problem in the standard form is written as

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \langle c, x\rangle \\ \text{subject to} \quad & \langle a_i, x\rangle = b_i \ (i = 1,\dots,m), \\ & x \in K, \end{aligned} \tag{1.1}$$

where c and $a_1,\dots,a_m$ belong to a finite-dimensional real vector space V equipped with an inner product $\langle \cdot, \cdot \rangle$, $b_1, \dots, b_m$ are real numbers, and K is a closed convex cone in V. Conic linear programming has several favorable structural properties. First, it is a subclass of convex programming and so any locally optimal solution is globally optimal [11, Section 4.2.2]. Second, we can use conic duality theorems developed in the conic optimization field [5, Section 2]. The dual problem of (1.1) is

$$\begin{aligned} \underset{\boldsymbol{y}, s}{\text{maximize}} \quad & \sum_{i=1}^{m} b_i y_i \\ \text{subject to} \quad & \sum_{i=1}^{m} y_i a_i + s = c, \\ & s \in K^*, \end{aligned} \tag{1.2}$$

where K^* is the dual cone of K defined in Section 2.1. As stated in the following theorem, weak duality holds between (1.1) and (1.2), and under an assumption, strong duality also holds.

Theorem 1.1 ([5, Theorem 2.4.1])**.** *Let v_{p} and v_{d} be the optimal values of* (1.1) *and* (1.2)*, respectively. Then, $v_{\mathrm{p}} \geq v_{\mathrm{d}}$ holds. Furthermore, we assume that* (1.1) *and* (1.2) *satisfy Slater's condition, i.e., there exist a feasible solution x_0 of* (1.1) *with x_0 in the interior of K and a feasible solution $(\boldsymbol{y}_0, s_0)$ of* (1.2) *with s_0 in the interior of K^*. Then, both* (1.1) *and* (1.2) *have optimal solutions and $v_{\mathrm{p}} = v_{\mathrm{d}}$ holds.*

Completely positive programming is a subclass of conic linear programming (1.1) in which K is a cone of completely positive tensors. Its dual (1.2) is *copositive programming*, where K^* is a cone of copositive tensors.

Complete positivity and its dual, copositivity, over the nonnegative orthant $\mathbb{R}^n_+$ (the cone of vectors with only nonnegative elements) are the most fundamental. A real symmetric tensor $\boldsymbol{\mathcal{A}} = (\mathcal{A}_{i_1 \cdots i_m})_{i_1, \dots, i_m = 1}^{n}$ of order m and dimension n is completely positive over the nonnegative orthant $\mathbb{R}^n_+$ if there exist $k \in \mathbb{N}$ and $\boldsymbol{a}_1, \dots, \boldsymbol{a}_k \in \mathbb{R}^n_+$ such that

$$\boldsymbol{\mathcal{A}} = \sum_{i=1}^{k} \underbrace{\boldsymbol{a}_i \otimes \cdots \otimes \boldsymbol{a}_i}_{m \text{ factors}}$$

and copositive over the nonnegative orthant $\mathbb{R}^n_+$ if the homogeneous polynomial function

$$\sum_{i_1, \dots, i_m = 1}^{n} \mathcal{A}_{i_1 \cdots i_m} x_{i_1} \cdots x_{i_m}$$

is nonnegative for all $\boldsymbol{x} \in \mathbb{R}^n_+$. Note that tensors include matrices as a special case. The aforementioned complete positivity and copositivity have been deeply studied since the 1950s in the field of applied mathematics [6, 52, 53, 79, 100, 111]. From the aspect of optimization, researchers have found since the 2000s that by utilizing completely positive cones over nonnegative orthants, many NP-hard optimization problems can be equivalently reformulated as completely positive programming problems. The problems include standard quadratic programming problems [8], the maximum stable set problem [23], a certain graph partitioning problem [97], the vertex coloring problem [20, 29, 51], the quadratic assignment problem [98], polynomial optimization problems [94], and so on. Among them, Burer's study [13] is especially seminal, which displayed a completely positive reformulation of nonconvex quadratic programming problems with nonnegative variables that may be binary.

By analogy with the complete positivity and copositivity over nonnegative orthants, we obtain complete positivity and copositivity over closed cones, respectively. We are especially interested in those over *symmetric cones*. The class of symmetric cones includes nonnegative orthants, second-order cones, positive semidefinite cones, and their direct product. Therefore, complete positivity and copositivity over symmetric cones are generalizations of those over nonnegative orthants.

Complete positivity and copositivity over symmetric cones beyond nonnegative orthants also naturally arise especially in the convex conic reformulation of nonconvex conic optimization problems. Burer [14] extended his previous result [13] and suggested the potential importance of further studying complete positivity over a direct product of a nonnegative orthant, second-order cones, and positive semidefinite cones. Utilizing the result of [14], Burer and Dong [15] reformulated quadratically constrained quadratic programming problems with bounded feasible sets as completely positive programming problems. The resultant problems involve a constraint forcing a matrix variable to belong to a completely positive cone over a direct product of a nonnegative orthant and second-order or positive semidefinite cones. Bai, Mitchell, and Pang [4, Theorem 5] provided a completely positive reformulation of rank-constrained semidefinite programming problems by using a completely positive cone over a direct product of a nonnegative orthant and three positive semidefinite cones. Additionally, the findings of Kim, Kojima, and Toh [61, Corollary 4.7] suggest a convex conic reformulation of polynomial optimization problems with a symmetric cone constraint. This reformulation involves completely positive *tensor*, beyond *matrix*, cones over symmetric cones. Besides convex conic reformulation, Gowda [43] discussed weighted linear complementarity problems over symmetric cones, in which the copositivity over the symmetric cones was utilized to guarantee the existence of a solution.

As we can see from the various results of completely positive reformulation, if we can solve completely positive programming problems, we can also solve a variety of NP-hard problems in a unified manner. At the same time, as completely positive programming can represent such formidable problems equivalently, it is also hard to solve directly. Indeed, the membership problems for completely positive and copositive cones even over nonnegative orthants are computationally difficult [26, 80]. Although Gaddum [40, Theorem 3.2] suggested an implementable recursive procedure to detect copositivity over nonnegative orthants, it necessitates solving a series of linear programming problems, the quantity of which grows exponentially with the order of a matrix (see also [57, Section 3.1]). Orlitzky [87, Theorem 4] extended this method to symmetric cones.

In this thesis, to remedy and understand the intricate structure of completely positive and copositive cones over symmetric cones, we study their approximations and facial structure.

1.1.1 Approximations

To deal with complete positivity or copositivity efficiently in a computer, tractable approximations for completely positive and copositive cones are often used in practice. Two such examples are *doubly nonnegative cones* and their duals, the cones of sums of a positive semidefinite matrix and a nonnegative matrix, also known as *SPN cones*. The doubly nonnegative cone, the set of positive semidefinite matrices with only nonnegative elements, includes the completely positive cone over the nonnegative orthant and is included in the positive semidefinite cone, thereby providing a tighter approximation for the completely positive cone over the nonnegative orthant than the positive semidefinite cone [98, 122]. Since we can describe the doubly nonnegative cone using a positive semidefinite constraint, we may apply efficient algorithms to conic linear programming problems with the doubly nonnegative cone constraint. Burer and Dong [15, Section 4] generalized the doubly nonnegative cone to provide a tighter approximation for completely positive cones over closed convex cones than the positive semidefinite cone.

Furthermore, to approximate completely positive and copositive cones in arbitrary accuracy, many types of *approximation hierarchies* have been proposed [1, 12, 23, 27, 28, 42, 48, 50, 58, 65, 89, 90, 93, 113, 121, 124, 125]. An approximation hierarchy, e.g., $\{\mathcal{K}_l\}_{l\in\mathbb{N}}$, gradually approaches the completely positive or copositive cone from the inside or outside as the approximation order l, which determines the approximation accuracy, increases and, in a sense, agrees with the cone in the limit. By defining each $\mathcal{K}_l$ as a set represented by nonnegativity, second-order cone, or positive semidefinite constraints, we can tentatively handle the complete positivity or copositivity on a computer.

Most of the aforementioned works provided approximation hierarchies for completely positive and copositive cones over nonnegative orthants, and only a few studies [27, 48, 65, 125] have considered those for completely positive and copositive cones over cones beyond nonnegative orthants. Zuluaga, Vera, and Peña [125] provided an inner-approximation hierarchy described by sum-of-squares constraints, which reduce to positive semidefinite constraints, for copositive cones over cones that are pointed (including no lines), semialgebraic, and convex. The term "semialgebraic" means that the set is defined by finitely many nonnegativity constraints of homogeneous polynomials. The class of pointed semialgebraic cones includes nonnegative orthants, second-order cones, and positive semidefinite cones or their direct product, which are also symmetric cones. The positive semidefiniteness of a matrix of order n is characterized by the nonnegativity of all the $2^n - 1$ principal minors. That is, although positive semidefinite cones are semialgebraic, the semialgebraic representation requires an exponential number of nonnegativity constraints in its order. In such a case, their approximation hierarchy is no longer tractable even if positive semidefinite constraints can describe it. Lasserre [65] provided an outer-approximation hierarchy described by a positive semidefinite constraint for copositive cones over general closed convex cones. The hierarchy is implementable only for the case in which the moments of a finite Borel measure dependent on the underlying closed convex cone are obtainable. We will describe these two approximation hierarchies mathematically in Section 2.5. The study by Dickinson and Povh [27] is a variant of that of Lasserre [65]. They considered the special case in which the underlying closed convex cone is included in a nonnegative orthant to provide a tighter approximation than Lasserre [65].

1.1.2 Facial structure

Faces of convex sets are basic concepts of convex analysis [106, Section 18]. In addition, studying the facial structure is also important in the context of linear algebra and optimization for several reasons.

The *length* ℓ_K *of the longest chain of faces* and the *distance* $\ell_{\text{poly}}(K)$ *to polyhedrality* of closed convex cones K are decided by the facial structure of K and related to various quantities. The first example is the *Carathéodory number* [47], which can be viewed as the maximum *cp-rank* [111, Chapter 4] of completely positive matrices for completely positive cones over nonnegative orthants. Ito and Lourenço [59, Theorem 4] showed that $\ell_K - 1$ is an upper bound for the Carathéodory number of a pointed closed convex cone K. The second example is the *singularity degree*. The singularity degree of a conic linear feasibility problem over K, a problem of finding a feasible solution of (1.1), is the minimum number of facial reduction steps needed to satisfy Slater's condition. This quantity is known to be related to error bounds for the feasibility problem [69, 72, 115]. Waki and Muramatsu [119, Corollary 3.1] showed that the singularity degree of a conic linear feasibility problem over a closed convex cone K is bounded by ℓ_K (see also [92, Theorem 28.5.3], [70, Theorem 1], and [73, page 2315]). Lourenço, Muramatsu, and Tsuchiya [73, Theorem 10] improved this bound and revealed that the singularity degree is bounded by $\ell_{\text{poly}}(K) + 1$.

The *facial exposedness* of closed convex cones also plays a crucial part in optimization. First, it arises when ensuring the invariance of strict complementarity for conic linear programming under duality [19, Theorem 1]. Second, facial exposedness is a necessary condition for a number of useful stronger exposure properties such as *niceness* [91, Theorem 3] (or *facially dual completeness* [107]), *tangential exposedness* [107, Proposition 2.2], *amenability* [72, Proposition 13], and *projectional exposedness* [10], [118, Corollary 4.4]. See also [74] for a discussion on some of those properties. It is also a necessary condition for certain algebraic properties such as *spectrahedrality* [102, Corollary 1] and *hyperbolicity* [103, Theorem 23]. See also [74, Corollary 3.5] and [75, Theorem 1.1] for the connection between these algebraic properties and amenability. In this way, proving that a cone is *not facially exposed* provides an easy way to certify that it is neither spectrahedral nor hyperbolic.

The facial structure of completely positive and copositive cones over nonnegative orthants, although not completely understood, has been a subject of several papers. Findings up to around the year 2021 are covered in the book [111] and references therein. See [56, 62, 63, 77] for more recent results. However, to the best of our knowledge, there is no systematic study on the facial structure of copositive cones over symmetric cones other than the nonnegative orthants.

1.2 Main Results

Approximation Hierarchies for Copositive Cones over Symmetric Cones (Chapter 3)

In Chapter 3, we aim to provide approximation hierarchies for copositive cones over symmetric cones and compare them with existing ones. First, we provide an inner-approximation hierarchy described by a sum-of-squares constraint for copositive cones over general symmetric cones. It is a generalization of the approximation hierarchy proposed by Parrilo [90] for copositive cone over nonnegative orthants. Second, we characterize copositive cones over symmetric cones using copositive cones over nonnegative orthants. By replacing the copositive cones over nonnegative orthants appearing in this characterization with the approximation hierarchies given by de Klerk and Pasechnik [23] and Yıldırım [121], we obtain inner- and outer-approximation hierarchies described by positive semidefinite but not by sum-of-squares constraints for the cone of copositive *matrices* over a direct product of a nonnegative orthant and a single second-order cone, respectively. We demonstrate that by combining the proposed approximation hierarchies with existing hierarchies, especially that provided by Zuluaga, Vera, and Peña [125], we can evaluate the optimal values of copositive programming problems more accurately and efficiently.

Generalizations of Doubly Nonnegative Cones and Their Comparison (Chapter 4)

In Chapter 4, we examine various extensions of doubly nonnegative cones. To provide tighter relaxation for generalized (in the sense that underlying cones are beyond nonnegative orthants) completely positive cones than positive semidefinite cones, the inner-approximation hierarchies presented in Chapter 3 for generalized copositive cones are exploited to obtain two generalized doubly nonnegative cones from the doubly nonnegative cones. We conduct theoretical and numerical comparisons to assess the relaxation strengths of the two generalized doubly nonnegative cones over a direct product of a nonnegative orthant and second-order or positive semidefinite cones. These comparisons also include an analysis of the existing generalized doubly nonnegative cone proposed by Burer and Dong [15]. The findings from solving several generalized doubly nonnegative programming relaxation problems for generalized completely positive programming problems demonstrate that the three generalized doubly nonnegative cones provide significantly tighter bounds for generalized completely positive programming than positive semidefinite cones.

Length of the Longest Chain of Faces and Distance to Polyhedrality of the Completely Positive and Copositive Cones (Chapter 5)

In Chapter 5, we consider a wide class of closed convex cones $\mathcal{K}$ in the space of real $n \times n$ symmetric matrices and establish the existence of a chain of faces of $\mathcal{K}$, the length of which is maximized at $\frac{n(n+1)}{2} + 1$. Examples of such cones include but are not limited to, the completely positive and copositive cones over the nonnegative orthant

$\mathbb{R}^n_+$. Using this chain, we prove that the distance to polyhedrality of any closed convex cone $\mathcal{K}$ that is sandwiched between the completely positive cone over $\mathbb{R}^n_+$ and the doubly nonnegative cone of order $n \geq 2$, as well as its dual, is at least $\frac{n(n+1)}{2} - 2$, which is also the worst-case scenario. To the best of our knowledge, these lengths, except for the length of the longest chain of faces of the doubly nonnegative cone [73, Proposition 21], have not been computed so far (see [59, Table 1]).

The main results of this chapter are disappointing for the Carathéodory number and the singularity degree. As mentioned in Section 1.1.2, by utilizing the length of the longest chain of faces and the distance to polyhedrality, we can provide their upper bounds. However, the results of this chapter imply that the upper bounds equal trivial ones.

Non-facial Exposedness of Copositive Cones over Symmetric Cones (Chapter 6)

In Chapter 6, we show that copositive cones over symmetric cones are never facially exposed when the underlying cone has a dimension of at least 2. We do so by explicitly exhibiting a non-exposed extreme ray. This result extends the known fact that the copositive cone over the nonnegative orthant $\mathbb{R}^n_+$ of order $n \geq 2$ is not facially exposed [24, Theorem 4.4].

1.3 Organization

The remainder of this thesis is organized as follows. In the rest of this part, i.e., Chapter 2, we introduce the notation and some concepts used in this thesis. Almost all of the statements in Chapter 2 have already been established or are folklore, but (i) and (ii) of Proposition 2.11 originally proved by us [84, Proposition 2.10] have not been known to the best of my knowledge.

Parts II and III contain the main results of this thesis. Because the two parts are independent of each other, readers may begin reading either of them first.

In Part II, we focus on approximations for completely positive and copositive cones over symmetric cones. In Chapter 3, we first provide approximation hierarchies for copositive cones. The contents of this chapter are based on our paper [84] published in the *Journal of Global Optimization*. In Chapter 4, utilizing approximation hierarchies, we then generalize doubly nonnegative cones. The contents of this chapter are based on our paper [85] published in the *Journal of the Operations Research Society of Japan*. Readers had better read this part in order because the discussion in Chapter 4 builds on the results of Chapter 3.

In Part III, we discuss the facial structure of completely positive and copositive cones over symmetric cones. This part consists of two chapters and they are independent of each other. In Chapter 5, we compute the length of the longest chain of faces and the distance to polyhedrality. The contents of this chapter are based on the author's paper [81] published in the *Linear Algebra and its Applications*. In Chapter 6, we show the non-facial exposedness of copositive cones over symmetric cones. The contents of this chapter are based on our preprint [82].

In Part IV, we finish with some concluding remarks.

Chapter 2

Preliminaries

2.1 Convex Sets, Cones, and Faces

Throughout this section, let V be an n-dimensional real vector space.

Let S be a nonempty subset of V. We call S *convex* if $\alpha x + (1-\alpha)y \in S$ for all $x, y \in S$ and $\alpha \in [0,1]$. We use $\operatorname{conv} S$ to denote the convex hull of S, namely the smallest convex set containing S. we also write $\operatorname{span} S$ for the smallest subspace containing S, $\operatorname{cl} S$ for the closure of S, $\operatorname{int} S$ for the interior of S, and $S^\perp$ for the space of $x \in V$ such that a specified inner product, which is determined from the context, between x and y is 0 for all $y \in S$. For $x \in V$, we define

$$\begin{aligned} \mathbb{R}x &:= \{\alpha x \mid \alpha \in \mathbb{R}\}, \\ \mathbb{R}_+ x &:= \{\alpha x \mid \alpha \geq 0\}, \end{aligned}$$

where $\mathbb{R}$ denotes the space of real numbers.

A nonempty subset K of V is called a *cone* if $\alpha x \in K$ for all $\alpha \geq 0$ and $x \in K$.[1] For a cone K, its *dual cone* denoted by K^* is the set of $x \in V$ such that a specified inner product between x and y is nonnegative for all $y \in K$. A cone K is said to be *self-dual* if $K^* = K$ and *pointed* if $K \cap (-K) = \{0\}$. We define the *dimension* of K, denoted by $\dim K$, as that of the space $\operatorname{span} K$.

The following theorem characterizes the convex hull of a cone.

Theorem 2.1 ([36, Theorem 7]). *Let K be a cone of dimension d. Then, every element x in $\operatorname{conv} K$ can be written as the sum of at most d elements in K. Conversely, a sum of elements in K belongs to $\operatorname{conv} K$.*

The following properties of a cone and its dual cone are established.

Theorem 2.2 ([11, Section 2.6.1]). *Let K, K_1, and K_2 be cones in V.*

(i) K^* is a closed convex cone.

(ii) If K is a pointed closed convex cone, K^* has a nonempty interior.

[1] Some literature, [106] for example, does not require cones to contain the origin. However, in this thesis, we follow Boyd and Vandenberghe [11] and require cones to contain the origin.

(iii) *If K is convex, $(K^*)^* = \mathrm{cl}\, K$ holds. If K is also closed, $(K^*)^* = K$ holds.*

(iv) *If $K_1 \subseteq K_2$, $K_2^* \subseteq K_1^*$.*

Let K be a closed convex cone in V. A nonempty convex subcone F of K is called a *face* of K if for any $a, b \in K$, they belong to F whenever $a + b \in F$. For a nonzero $x \in K$, if $\mathbb{R}_+ x$ is a face of K, we say that x generates the *extreme ray* $\mathbb{R}_+ x$. We use $\mathrm{Ext}\, K$ to denote the set of elements generating extreme rays of K.

A face F of K is said to be *exposed* if there exists $h \in K^*$ such that $F = K \cap \{h\}^\perp$. If all the faces of K are exposed, then we call K *facially exposed*.

A *chain of faces* of K with length l is a sequence

$$F_l \subsetneq \cdots \subsetneq F_1 \tag{2.1}$$

such that every F_i is a face of K. We use ℓ_K to denote the *length of the longest chain of faces* of K. In addition, we define the *distance $\ell_{\mathrm{poly}}(K)$ to polyhedrality* of K as the length *minus one* of the longest chain (2.1) of faces of K such that F_l is polyhedral and every F_i with $i < l$ is not polyhedral.

The following two lemmas provide the upper bounds for the length of the longest chain of faces and the distance to polyhedrality of a closed convex cone, respectively.

Lemma 2.3. *For a closed convex cone K in an n-dimensional real vector space, $\ell_K \leq n + 1$ holds.*

Proof. Consider any chain (2.1) of faces of K with length l. The dimensions of the faces $F_1, \ldots, F_l$ are integers between 0 and n. In addition, by [106, Corollary 18.1.3], they must satisfy $\dim F_l < \cdots < \dim F_1$, which implies that $l \leq n + 1$. Since the chain (2.1) we choose is arbitrary, we obtain the desired result. □

Lemma 2.4. *Let K be a closed convex cone in an n-dimensional real vector space.*

(i) *If $n \leq 2$, then $\ell_{\mathrm{poly}}(K) = 0$.*

(ii) *If $n \geq 3$, then $\ell_{\mathrm{poly}}(K) \leq n - 2$.*

Proof. First, note that every closed convex cone with dimension at most 2 is polyhedral (see, for example, [111, Exercise 1.65]). If $n \leq 2$, then the dimension of K is at most 2, and thus (i) holds. In what follows, we prove (ii). To obtain a contradiction, we assume that $l := \ell_{\mathrm{poly}}(K) \geq n - 1$. It follows from $n \geq 3$ that $l \geq n - 1 \geq 2$. Then, there exists a chain $F_{l+1} \subsetneq \cdots \subsetneq F_1$ of faces of K such that F_l is not polyhedral. For the same reason as the proof of Lemma 2.3, we have $\dim F_l \leq n - l + 1 \leq 2$. This implies that F_l is polyhedral, which is a contradiction. □

2.2 Tensor Background

In this section, we introduce fundamental notation and known facts concerning tensors, encompassing vectors, matrices, and homogeneous polynomials.

2.2.1 Vectors

For a finite set I, the number of elements in I is denoted by $|I|$. We use $\mathbb{R}^I$ to denote the space of $|I|$-dimensional real vectors with elements indexed by I. All real vectors in $\mathbb{R}^I$ that appear in this thesis are column vectors and written in boldface lowercase letters such as $\boldsymbol{a}$. The ith element of a vector $\boldsymbol{a}$ is denoted by a_i. We write $\mathbb{R}^{\{1,\dots,n\}}$ as $\mathbb{R}^n$ for short. The space $\mathbb{R}^I$ is endowed with the standard inner product and $\|\cdot\|_2$ denotes the induced norm.

We use $\boldsymbol{e}_i$ to denote the vector with the ith element being 1 and all other elements 0, whose size is determined from the context. In addition, we use $\mathbf{0}$ and $\mathbf{1}$ to denote the zero vector and vector with all elements 1, respectively. We sometimes use a subscript, such as $\mathbf{0}_n$ and $\mathbf{1}_n$, to specify the size. For a collection of vectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_k$, we use $(\boldsymbol{v}_1; \dots; \boldsymbol{v}_k)$ to denote the vector obtained by vertically concatenating them. The entrywise product of two vectors is denoted by $\odot$.

We use S^n and $\Delta^n_=$ to denote the n-dimensional unit sphere and standard simplex in $\mathbb{R}^{n+1}$, i.e.,

$$\begin{aligned} S^n &= \{\boldsymbol{x} \in \mathbb{R}^{n+1} \mid \|\boldsymbol{x}\|_2 = 1\}, \\ \Delta^n_= &= \{\boldsymbol{x} \in \mathbb{R}^{n+1} \mid \boldsymbol{x}^\top \mathbf{1} = 1 \text{ and } x_i \geq 0 \text{ for all } i = 1, \dots, n+1\}, \end{aligned}$$

respectively.

2.2.2 Matrices

Every real matrix is denoted by boldface uppercase letters such as $\boldsymbol{A}$. The (i,j)th element of $\boldsymbol{A}$ is expressed as $A_{i,j}$. If there is no danger of misunderstanding, $A_{i,j}$ and A_{ij} are used interchangeably. For index sets $\mathcal{I}, \mathcal{J}$, $\boldsymbol{A}_{\mathcal{I},\mathcal{J}}$ and $\boldsymbol{A}_{\mathcal{I}\mathcal{J}}$ denote the submatrix obtained by extracting the rows of $\boldsymbol{A}$ indexed by $\mathcal{I}$ and the columns indexed by $\mathcal{J}$.

The space of real $n \times m$ matrices is denoted by $\mathbb{R}^{n \times m}$. For a finite set I, we use $\mathbb{S}^I$ and $\mathbb{S}^I_+$ to denote the space of real $|I| \times |I|$ symmetric matrices with rows and columns indexed by I and the cone of positive semidefinite matrices in $\mathbb{S}^I$, respectively. We write $\mathbb{S}^{\{1,\dots,n\}}$ and $\mathbb{S}^{\{1,\dots,n\}}_+$ as $\mathbb{S}^n$ and $\mathbb{S}^n_+$ for short, respectively. The space $\mathbb{S}^I$ is endowed with the trace inner product defined by $\langle \boldsymbol{A}, \boldsymbol{B} \rangle := \sum_{i,j \in I} A_{ij} B_{ij}$ for $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{S}^I$. For $n \in \mathbb{N}$, we let $T_n := \frac{n(n+1)}{2}$, which is the dimension of $\mathbb{S}^n$.

We use $\boldsymbol{O}$, $\boldsymbol{I}$, $\boldsymbol{E}$ to denote the zero matrix, identity matrix, and matrix with all elements 1, respectively. As in vectors, we sometimes use a subscript, such as $\boldsymbol{O}_n$ and $\boldsymbol{I}_n$, to specify the order. We use $\mathrm{diag}(\boldsymbol{A})$ to denote the vector of the diagonal elements of a matrix $\boldsymbol{A}$. The rank of a matrix $\boldsymbol{A}$ is defined by $\mathrm{rank}(\boldsymbol{A})$.

2.2.3 Homogeneous polynomials

Let $H^{n,m}$ be the space of homogeneous polynomials in n variables of degree m with real coefficients. In this thesis, we use the following multi-index notation:

$$\begin{aligned} |\boldsymbol{\alpha}| &:= \sum_{i=1}^{n} \alpha_i, \\ \boldsymbol{x}^{\boldsymbol{\alpha}} &:= \prod_{i=1}^{n} x_i^{\alpha_i}, \\ \boldsymbol{\alpha}! &:= \prod_{i=1}^{n} \alpha_i! \,. \end{aligned}$$

Let $\mathbb{N}$ denote the set of nonnegative integers. Defining $\mathbb{I}^n_{=m} := \{\boldsymbol{\alpha} \in \mathbb{N}^n \mid |\boldsymbol{\alpha}| = m\}$ and $\boldsymbol{\sigma}_m(\boldsymbol{x})$ as the vector of monomials of degree m, we can express each homogeneous polynomial $\theta \in H^{n,m}$ as

$$\theta(\boldsymbol{x}) = \sum_{\boldsymbol{\alpha} \in \mathbb{I}^n_{=m}} \theta_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}} = \boldsymbol{\theta}^\top \boldsymbol{\sigma}_m(\boldsymbol{x})$$

for a vector $\boldsymbol{\theta} = (\theta_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha} \in \mathbb{I}^n_{=m}} \in \mathbb{R}^{\mathbb{I}^n_{=m}}$. Subsequently, we define $\Sigma^{n,2m} := \operatorname{conv}\{\theta^2 \mid \theta \in H^{n,m}\}$ as the cone of sums of squares of elements in $H^{n,m}$.

We can verify whether a homogeneous polynomial in $H^{n,2m}$ belongs to the set $\Sigma^{n,2m}$ by solving a semidefinite programming problem. By the definition of $\Sigma^{n,2m}$ and Theorem 2.1, a polymonomial $\theta \in H^{n,2m}$ belongs to $\Sigma^{n,2m}$ if and only if there exist $\theta_1, \dots, \theta_k \in H^{n,m}$ such that

$$\theta(\boldsymbol{x}) = \sum_{i=1}^{k} \theta_i(\boldsymbol{x})^2 = \boldsymbol{\sigma}_m(\boldsymbol{x})^\top \left(\sum_{i=1}^{k} \boldsymbol{\theta}_i \boldsymbol{\theta}_i^\top \right) \boldsymbol{\sigma}_m(\boldsymbol{x}),$$

where $\boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_k \in \mathbb{R}^{\mathbb{I}^n_{=m}}$ are the vectors corresponding to the polynomials $\theta_1, \dots, \theta_k$. The matrix $\sum_{i=1}^{k} \boldsymbol{\theta}_i \boldsymbol{\theta}_i^\top$ is clearly positive semidefinite. Conversely, suppose that a polynomial $\theta \in H^{n,2m}$ is represented as $\theta(\boldsymbol{x}) = \boldsymbol{\sigma}_m(\boldsymbol{x})^\top \boldsymbol{\Theta} \boldsymbol{\sigma}_m(\boldsymbol{x})$ for some $\boldsymbol{\Theta} \in \mathbb{S}_+^{\mathbb{I}^n_{=m}}$. Then, using the Cholesky decomposition of $\boldsymbol{\Theta}$, we have $\theta \in \Sigma^{n,2m}$. Therefore, the constraint that a homogeneous polynomial is sum-of-squares reduces to a positive semidefinite constraint.

2.2.4 Tensors

Let $(\mathbb{V}, \bullet)$ be an n-dimensional real inner product space. Note that $\mathbb{V}$ can be identified with the dual space, the set of linear mappings from $\mathbb{V}$ to $\mathbb{R}$, by the natural isomorphism $x \mapsto (x \bullet \cdot)$. We use

$$\mathbb{V}^{\otimes m} := \underbrace{\mathbb{V} \otimes \cdots \otimes \mathbb{V}}_{m \text{ factors}}$$

to denote the space of tensors of order m over $\mathbb{V}$. Let $\{v_1, \dots, v_n\}$ be a basis for $\mathbb{V}$. For notational convenience, we define

$$\mathbb{N}_n^m := \{(i_1, \dots, i_m) \in \mathbb{N}^m \mid 1 \le i_k \le n \text{ for all } k = 1, \dots, m\}$$

and $\tilde{v}_{i_1 \cdots i_m} := v_{i_1} \otimes \cdots \otimes v_{i_m}$ for each $(i_1, \dots, i_m) \in \mathbb{N}_n^m$. Then, $\{\tilde{v}_{i_1 \cdots i_m} \mid (i_1, \dots, i_m) \in \mathbb{N}_n^m\}$ forms a basis for $\mathbb{V}^{\otimes m}$. That is, each $\mathcal{A} \in \mathbb{V}^{\otimes m}$ can be written in the following form:

$$\mathcal{A} = \sum_{i_1, \dots, i_m = 1}^{n} \mathcal{A}_{i_1 \cdots i_m} \tilde{v}_{i_1 \cdots i_m} \tag{2.2}$$

with coefficients $\mathcal{A}_{i_1 \cdots i_m} \in \mathbb{R}$. Let $\langle \cdot, \cdot \rangle$ be the inner product on $\mathbb{V}^{\otimes m}$ induced by that on $\mathbb{V}$. That is, it satisfies

$$\langle x_1 \otimes \cdots \otimes x_m, y_1 \otimes \cdots \otimes y_m \rangle = \prod_{i=1}^{m} x_i \bullet y_i \tag{2.3}$$

for $x_1 \otimes \cdots \otimes x_m$, $y_1 \otimes \cdots \otimes y_m \in \mathbb{V}^{\otimes m}$. We write $\|\cdot\|_{\mathrm{F}}$ for the norm on $\mathbb{V}^{\otimes m}$ induced by the inner product $\langle \cdot, \cdot \rangle$.

Let $\mathfrak{S}_m$ be the symmetric group of order m. For $\sigma \in \mathfrak{S}_m$, the linear transformation π_σ on $\mathbb{V}^{\otimes m}$ is defined by $\pi_\sigma(\tilde{v}_{i_1 \cdots i_m}) := \tilde{v}_{i_{\sigma(1)} \cdots i_{\sigma(m)}}$. The definition of π_σ does not depend on the choice of the basis for $\mathbb{V}$. Then,

$$\mathcal{S}^{n,m}(\mathbb{V}) := \{\mathcal{A} \in \mathbb{V}^{\otimes m} \mid \pi_\sigma \mathcal{A} = \mathcal{A} \text{ for all } \sigma \in \mathfrak{S}_m\}$$

denotes the space of symmetric tensors of order m over $\mathbb{V}$, which is a subspace of $\mathbb{V}^{\otimes m}$. For $x \in \mathbb{V}$, let

$$x^{\otimes m} := \underbrace{x \otimes \cdots \otimes x}_{m \text{ factors}} \in \mathcal{S}^{n,m}(\mathbb{V}).$$

Let $\mathscr{S}$ be the linear transformation on $\mathbb{V}^{\otimes m}$ defined as

$$\mathscr{S} := \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} \pi_\sigma.$$

Then, $\mathscr{S}\mathcal{A} \in \mathcal{S}^{n,m}(\mathbb{V})$ for each $\mathcal{A} \in \mathbb{V}^{\otimes m}$. The following lemma states that the inner product of two tensors is invariant under the operation of $\mathscr{S}$.

Lemma 2.5. *For $\mathcal{A} \in \mathbb{V}^{\otimes m}$ and $\mathcal{B} \in \mathcal{S}^{n,m}(\mathbb{V})$, it follows that $\langle \mathscr{S}\mathcal{A}, \mathcal{B} \rangle = \langle \mathcal{A}, \mathcal{B} \rangle$.*

Proof. Using an orthonormal basis $\{v_1, \dots, v_n\}$ for $\mathbb{V}$, we write $\mathcal{A}$ and $\mathcal{B}$ in the form

(2.2). It then follows from the symmetry of $\mathcal{B}$ that

$$\begin{aligned}
\langle \mathscr{S}\mathcal{A}, \mathcal{B}\rangle &= \left\langle \frac{1}{m!}\sum_{\sigma\in\mathfrak{S}_m} \pi_\sigma \mathcal{A}, \mathcal{B}\right\rangle \\
&= \frac{1}{m!}\sum_{\sigma\in\mathfrak{S}_m}\sum_{i_1,\dots,i_m=1}^{n} \mathcal{A}_{i_1\cdots i_m}\mathcal{B}_{i_{\sigma(1)}\cdots i_{\sigma(m)}} \\
&= \frac{1}{m!}\sum_{\sigma\in\mathfrak{S}_m}\sum_{i_1,\dots,i_m=1}^{n} \mathcal{A}_{i_1\cdots i_m}\mathcal{B}_{i_1\cdots i_m} \\
&= \frac{1}{m!}\sum_{\sigma\in\mathfrak{S}_m} \langle \mathcal{A}, \mathcal{B}\rangle \\
&= \langle \mathcal{A}, \mathcal{B}\rangle.
\end{aligned}$$

□

We consider the case of $\mathbb{V} = \mathbb{R}^n$ with the canonical basis $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$ and write $\mathcal{S}^{n,m}(\mathbb{R}^n)$ as $\mathcal{S}^{n,m}$. Every tensor in $\mathcal{S}^{n,m}$ is denoted by boldface calligraphy letters such as $\boldsymbol{\mathcal{A}}$. Each element in $\mathcal{S}^{n,m}$ can be considered a multi-dimensional array. Thus, we write $\mathcal{A}_{i_1\cdots i_m}$ for the $(i_1, \dots, i_m)$th element of $\boldsymbol{\mathcal{A}} \in \mathcal{S}^{n,m}$.

We next consider the case of $m = 2$. For $a, b \in \mathbb{V}$, the tensor product $a \otimes b$ corresponds to the linear transformation on $\mathbb{V}$ such that $(a \otimes b)(x) = (b \bullet x)a$, for all $x \in \mathbb{V}$. Under this correspondence, we can identify $\mathcal{S}^{n,2}(\mathbb{V})$ with the space of self-adjoint linear transformations on $\mathbb{V}$, which is denoted by $\mathcal{S}(\mathbb{V})$. See Section 6.2 for additional notation and properties of linear mappings. In addition, the symmetric tensor space $\mathcal{S}^{n,2}$ equals the symmetric matrix space $\mathbb{S}^n$. Note that the inner product that the space $\mathcal{S}^{n,2}(\mathbb{V})$ equips with is consistent with that on the space $\mathbb{S}^n$.

2.3 Symmetric Cones and Euclidean Jordan Algebras

In this section, we introduce symmetric cones. We also present Euclidean Jordan algebras, which are closely related to symmetric cones as mentioned later. The book written by Faraut and Korányi [34] is a standard textbook on this field, but the papers written by Faybusovich [35] and Sturm [116] might be more readable, especially for optimizers. In writing this section, we also refer to these two references.

A closed cone $\mathbb{K}$ in a finite-dimensional real inner product space $\mathbb{E}$ is called *symmetric* if it satisfies the following two conditions:[2]

(i) (Self-duality) $\mathbb{K}^* = \mathbb{K}$.

(ii) (Homogeneity) For all x and y belonging to the interior of $\mathbb{K}$, there exists a bijective linear transformation $\mathcal{G}$ on $\mathbb{E}$ such that $\mathcal{G}(\mathbb{K}) = \mathbb{K}$ and $\mathcal{G}(x) = y$.

[2] In the book written by Faraut and Korányi [34], symmetric cones are *open*. In the context of optimization, however, symmetric cones are often considered *closed* [87, 116] and we also follow the latter.

By self-duality, symmetric cones are pointed convex cones with a nonempty interior.

A *Jordan algebra* is a finite-dimensional real vector space $\mathbb{E}$ equipped with a bilinear product $\circ\colon \mathbb{E} \times \mathbb{E} \to \mathbb{E}$ satisfying the following two conditions for all $x, y \in \mathbb{E}$:

(J1) (Commutativity) $x \circ y = y \circ x$,

(J2) (Jordan identity) $x \circ ((x \circ x) \circ y) = (x \circ x) \circ (x \circ y)$.

We assume in this thesis that every Jordan algebra $(\mathbb{E}, \circ)$ has an identity element, denoted by e, concerning the product $\circ$. A Jordan algebra $(\mathbb{E}, \circ)$ is *Euclidean* if there exists an inner product $\bullet\colon \mathbb{E} \times \mathbb{E} \to \mathbb{R}$ satisfying

(J3) (Associativity) $(x \circ y) \bullet z = x \bullet (y \circ z)$

for all $x, y, z \in \mathbb{E}$. In this thesis, we fix such an associative inner product $\bullet$ and write a Euclidean Jordan algebra as a triple $(\mathbb{E}, \circ, \bullet)$. However, we may merely write $\mathbb{E}$ for $(\mathbb{E}, \circ, \bullet)$ if it is clear from the context that $\mathbb{E}$ has the Euclidean Jordan algebraic structure.

Throughout this section, let $\mathbb{E}$ be a Euclidean Jordan algebra. For convenience, let $x^2 := x \circ x$ for each $x \in \mathbb{E}$. It is known that the cone of squares in $\mathbb{E}$ defined as $\mathbb{E}_+ := \{x^2 \mid x \in \mathbb{E}\}$ is a symmetric cone with e in its interior [34, Theorem III.2.1]. Conversely, for a given symmetric cone $\mathbb{K}$ in a finite-dimensional real inner product space $(\hat{\mathbb{E}}, \hat{\bullet})$, we can define a bilinear product $\hat{\circ}$ on $\hat{\mathbb{E}}$, so that $(\hat{\mathbb{E}}, \hat{\circ}, \hat{\bullet})$ is a Euclidean Jordan algebra and $\mathbb{K}$ agrees with the cone $\hat{\mathbb{E}}_+$ [34, Theorem III.3.1].

An element $c \in \mathbb{E}$ is termed an *idempotent* if $c^2 = c$. Note that any idempotent $c \in \mathbb{E}$ belongs to the symmetric cone $\mathbb{E}_+$. An idempotent c is said to be *primitive* if it is nonzero and cannot be represented as the sum of two nonzero idempotents. Two idempotents c and d are termed *orthogonal* if $c \circ d = 0$. The system $c_1, \dots, c_k$ is called a *complete system of orthogonal idempotents* if each c_i is an idempotent, $c_i \circ c_j = 0$ if $i \neq j$, and $\sum_{i=1}^k c_i = e$. In addition, if each c_i is also primitive, the system is called a *Jordan frame*. Each Jordan frame is known to consist of exactly r elements, where r is the *rank* of the Euclidean Jordan algebra $\mathbb{E}$ and the rank depends only on the algebra [34, Section III.1]. In this thesis, for the convenience of the proofs, we also consider an ordered Jordan frame and let $\mathfrak{F}(\mathbb{E})$ be the set of ordered Jordan frames, i.e.,

$$\mathfrak{F}(\mathbb{E}) = \{(c_1, \dots, c_r) \mid \text{The system } c_1, \dots, c_r \text{ is a Jordan frame}\}.$$

Note that $\mathfrak{F}(\mathbb{E})$ is a compact subset of $\mathbb{E}^r$ [34, Exercise IV.5]. Each element of $\mathbb{E}$ can be decomposed into a linear combination of a Jordan frame [34, Theorem III.1.2]. In particular, for each $x \in \mathbb{E}$, there exist $x_1, \dots, x_r \in \mathbb{R}$ and $(c_1, \dots, c_r) \in \mathfrak{F}(\mathbb{E})$ such that

$$x = \sum_{i=1}^r x_i c_i. \tag{2.4}$$

Note that when each $x \in \mathbb{E}_+$ is decomposed into the form (2.4), all coefficients x_i are nonnegative. Conversely, for any nonnegative scalars $x_1, \dots, x_r$ and $(c_1, \dots, c_r) \in \mathfrak{F}(\mathbb{E})$, it follows that $\sum_{i=1}^r x_i c_i \in \mathbb{E}_+$.

Suppose that the rank of $\mathbb{E}$ is r. For an idempotent $c \in \mathbb{E}$ and $\lambda = 0, \frac{1}{2}, 1$, we define

$$\mathbb{E}(c, \lambda) \coloneqq \{x \in \mathbb{E} \mid c \circ x = \lambda x\}.$$

The subspaces $\mathbb{E}(c, 0)$ and $\mathbb{E}(c, 1)$ are Euclidean Jordan subalgebras of $\mathbb{E}$ [34, Proposition IV.1.1]. Using the subspaces $\mathbb{E}(c, \lambda)$, the space $\mathbb{E}$ decomposes into the following orthogonal direct sum [34, page 62]:

$$\mathbb{E} = \mathbb{E}(c, 0) \oplus \mathbb{E}(c, \tfrac{1}{2}) \oplus \mathbb{E}(c, 1). \tag{2.5}$$

We refer to (2.5) as the *Peirce decomposition of* $\mathbb{E}$ *with respect to the idempotent* c. Furthermore, we can decompose $\mathbb{E}$ more finely. We fix a Jordan frame $c_1, \ldots, c_r$ of $\mathbb{E}$ and consider the following subspaces of $\mathbb{E}$:

$$\begin{aligned} \mathbb{E}_{ii} &\coloneqq \mathbb{E}(c_i, 1) = \mathbb{R}c_i && (i = 1, \ldots, n), \\ \mathbb{E}_{ij} &\coloneqq \mathbb{E}(c_i, \tfrac{1}{2}) \cap \mathbb{E}(c_j, \tfrac{1}{2}) && (i, j = 1, \ldots, n,\ i \neq j). \end{aligned} \tag{2.6}$$

Then, we can decompose $\mathbb{E}$ into the following orthogonal direct sum of the above subspaces [34, Theorem IV.2.1.i]:

$$\mathbb{E} = \bigoplus_{1 \leq i \leq j \leq r} \mathbb{E}_{ij}. \tag{2.7}$$

We call (2.7) the *Peirce decomposition of* $\mathbb{E}$ *with respect to the Jordan frame* $c_1, \ldots, c_r$. For each $i, j = 1, \ldots, n$ with $i \neq j$ and every $x \in \mathbb{E}_{ij}$, we have

$$x^2 = \underbrace{c_i \circ x^2}_{\in \mathbb{E}_{ii}} + \underbrace{c_j \circ x^2}_{\in \mathbb{E}_{jj}}, \tag{2.8}$$

see [34, Proposition IV.1.1] and its proof.

The following three examples represent typical Euclidean Jordan algebras that appear frequently throughout this thesis, especially in Part II.

Example 2.6 (Nonnegative orthant). Let $\mathbb{E}$ be an n-dimensional Euclidean space $\mathbb{R}^n$. If we define

$$\begin{aligned} \boldsymbol{x} \circ \boldsymbol{y} &\coloneqq \boldsymbol{x} \odot \boldsymbol{y}, \\ \boldsymbol{x} \bullet \boldsymbol{y} &\coloneqq \boldsymbol{x}^\top \boldsymbol{y} \end{aligned}$$

for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{E}$, then $(\mathbb{E}, \circ, \bullet)$ is a Euclidean Jordan algebra, and the associated symmetric cone $\mathbb{E}_+$ is the nonnegative orthant $\mathbb{R}^n_+ \coloneqq \{\boldsymbol{x} \in \mathbb{R}^n \mid x_i \geq 0 \text{ for all } i = 1, \ldots, n\}$. The identity element of the Euclidean Jordan algebra is $\boldsymbol{1}_n$ and the set $\mathfrak{F}(\mathbb{E})$ of ordered Jordan frames is $\{(\boldsymbol{e}_{\sigma(1)}, \ldots, \boldsymbol{e}_{\sigma(n)}) \mid \sigma \in \mathfrak{S}_n\}$.

Example 2.7 (Second-order cone). Let $\mathbb{E}$ be an n-dimensional Euclidean space $\mathbb{R}^n$. For each $\boldsymbol{x} \in \mathbb{E}$, we represent $\boldsymbol{x}$ as $(x_1; \boldsymbol{x}_{2:n})$, where $\boldsymbol{x}_{2:n}$ is the $(n-1)$-dimensional vector comprising the second through the nth element of $\boldsymbol{x}$. If we define

$$\begin{aligned} \boldsymbol{x} \circ \boldsymbol{y} &\coloneqq \begin{pmatrix} \boldsymbol{x}^\top \boldsymbol{y} \\ x_1 \boldsymbol{y}_{2:n} + y_1 \boldsymbol{x}_{2:n} \end{pmatrix}, \\ \boldsymbol{x} \bullet \boldsymbol{y} &\coloneqq \boldsymbol{x}^\top \boldsymbol{y} \end{aligned}$$

for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{E}$, then $(\mathbb{E}, \circ, \bullet)$ is a Euclidean Jordan algebra, and the associated symmetric cone $\mathbb{E}_+$ is the second-order cone $\mathbb{L}^n$ defined as

$$\mathbb{L}^n := \begin{cases} \mathbb{R}_+ & (n = 1), \\ \{\boldsymbol{x} \in \mathbb{R}^n \mid x_1 \geq \|\boldsymbol{x}_{2:n}\|_2\} & (n \geq 2). \end{cases}$$

The identity element of the Euclidean Jordan algebra is $(1; \mathbf{0}_{n-1})$ and the set $\mathfrak{F}(\mathbb{E})$ of ordered Jordan frames is

$$\left\{ \left(\frac{1}{2} \begin{pmatrix} 1 \\ \boldsymbol{v} \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 \\ -\boldsymbol{v} \end{pmatrix} \right) \,\middle|\, \boldsymbol{v} \in S^{n-2} \right\}.$$

Example 2.8 (Positive semidefinite cone). Consider the space $\mathbb{S}^n$ of real $n \times n$ symmetric matrices. If we define

$$\begin{aligned} \boldsymbol{X} \lozenge \boldsymbol{Y} &:= \frac{\boldsymbol{X}\boldsymbol{Y} + \boldsymbol{Y}\boldsymbol{X}}{2}, \\ \boldsymbol{X} \blacklozenge \boldsymbol{Y} &:= \langle \boldsymbol{X}, \boldsymbol{Y} \rangle \end{aligned}$$

for $\boldsymbol{X}, \boldsymbol{Y} \in \mathbb{S}^n$, then $(\mathbb{S}^n, \lozenge, \blacklozenge)$ is a Euclidean Jordan algebra, and the associated symmetric cone is the positive semidefinite cone $\mathbb{S}^n_+$. The identity element of the Euclidean Jordan algebra is $\boldsymbol{I}_n$.

We focus on the vectorized positive semidefinite cone in this thesis. We define the linear mapping svec: $\mathbb{S}^n \to \mathbb{R}^{T_n}$ as

$$\operatorname{svec}(\boldsymbol{X}) := (X_{1,1}, \sqrt{2}X_{1,2}, X_{2,2}, \dots, \sqrt{2}X_{1,n}, \dots, \sqrt{2}X_{n-1,n}, X_{n,n})^\top$$

for each $\boldsymbol{X} \in \mathbb{S}^n$. Then, the mapping $\operatorname{svec}(\cdot)$ is an isometry between $\mathbb{S}^n$ and $\mathbb{R}^{T_n}$; i.e., $\langle \boldsymbol{X}, \boldsymbol{Y} \rangle = \operatorname{svec}(\boldsymbol{X})^\top \operatorname{svec}(\boldsymbol{Y})$ holds for all $\boldsymbol{X}, \boldsymbol{Y} \in \mathbb{S}^n$. Let $\operatorname{smat}(\cdot)$ denote the inverse mapping of $\operatorname{svec}(\cdot)$. If we define

$$\begin{aligned} \mathbb{E} &:= \mathbb{R}^{T_n} \\ \boldsymbol{x} \circ \boldsymbol{y} &:= \operatorname{svec}(\operatorname{smat}(\boldsymbol{x}) \lozenge \operatorname{smat}(\boldsymbol{y})), \\ \boldsymbol{x} \bullet \boldsymbol{y} &:= \boldsymbol{x}^\top \boldsymbol{y} \end{aligned}$$

for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{E}$, then $(\mathbb{E}, \circ, \bullet)$ is a Euclidean Jordan algebra, and the associated symmetric cone $\mathbb{E}_+$ is $\operatorname{svec}(\mathbb{S}^n_+)$. We also refer to the vectorized positive semidefinite cone $\operatorname{svec}(\mathbb{S}^n_+)$ as the positive semidefinite cone hereafter.

Consider the case in which a Euclidean Jordan algebra $(\mathbb{E}, \circ, \bullet)$ can be written as the direct product (sum) of two Euclidean Jordan algebras $(\mathbb{E}_i, \circ_i, \bullet_i)$ with rank r_i and identity element e_i $(i = 1, 2)$. Note that the following discussion can be directly extended to the case of finitely many Euclidean Jordan algebras. The product $\circ$ and associative inner product $\bullet$ are defined as

$$\begin{aligned} (x_1, x_2) \circ (y_1, y_2) &:= (x_1 \circ_1 y_1, x_2 \circ_2 y_2), \\ (x_1, x_2) \bullet (y_1, y_2) &:= x_1 \bullet_1 y_1 + x_2 \bullet_2 y_2 \end{aligned}$$

for $(x_1, x_2), (y_1, y_2) \in \mathbb{E} = \mathbb{E}_1 \times \mathbb{E}_2$, the rank of $\mathbb{E}$ is $r_1 + r_2$, and the identity element e of $\mathbb{E}$ is (e_1, e_2). The cone of squares in $\mathbb{E}$ is the direct product $[\mathbb{E}_1]_+ \times [\mathbb{E}_2]_+$, whence we see that a direct product of multiple symmetric cones is also a symmetric cone. In the following, we derive the set of ordered Jordan frames of $\mathbb{E}$. This result will be exploited in Section 3.3 to obtain approximation hierarchies described by finitely many semidefinite constraints.

Lemma 2.9. *For a primitive idempotent $f = (f_1, f_2) \in \mathbb{E}$, exactly one of the following two statements holds:*

(a) f_1 is a primitive idempotent of $\mathbb{E}_1$ and $f_2 = 0$.

(b) $f_1 = 0$ and f_2 is a primitive idempotent of $\mathbb{E}_2$.

Proof. We assume that $f_1 \neq 0$. If $f_2 \neq 0$, then f can be decomposed into the sum of two nonzero elements $(f_1, 0)$ and $(0, f_2)$. The two elements are actually idempotents of $\mathbb{E}$, which contradicts f being primitive. Thus, we have $f_2 = 0$. Because $f = (f_1, 0)$ is a primitive idempotent, f_1 is also a primitive idempotent. This case falls under the case (a).

Next, we assume that $f_1 = 0$. In this case, as in the above discussion, we note that f_2 is a primitive idempotent. This case falls under the case (b). $\square$

Proposition 2.10 (The set of ordered Jordan frames of $\mathbb{E}$)**.** *An element $(f_1, \ldots, f_{r_1+r_2}) \in \mathbb{E}^{r_1+r_2}$ belongs to $\mathfrak{F}(\mathbb{E})$ if and only if there exists a partition (I_1, I_2) of $\{1, \ldots, r_1 + r_2\}$ such that the following two conditions hold:*

(i) $f_i = (f_{1i}, 0) \in \mathbb{E}$ for all $i \in I_1$ and $(f_{1i})_{i \in I_1} \in \mathfrak{F}(\mathbb{E}_1)$.

(ii) $f_i = (0, f_{2i}) \in \mathbb{E}$ for all $i \in I_2$ and $(f_{2i})_{i \in I_2} \in \mathfrak{F}(\mathbb{E}_2)$.

Proof. We first prove the "if" part. Because $(f_1, \ldots, f_{r_1+r_2})$ satisfying the two conditions is evidently a complete system of orthogonal idempotents, we only prove that each f_i is primitive. To prove this, it suffices to show that $(f, 0) \in \mathbb{E}$ is primitive if $f \in \mathbb{E}_1$ is primitive. Assume that $(f, 0)$ can be written as the sum of two idempotents (f_1, f_2) and (g_1, g_2), i.e.,

$$(f, 0) = (f_1, f_2) + (g_1, g_2). \tag{2.9}$$

First, (2.9) implies that $f_2 = -g_2$. In addition, we note that f_1, g_1, f_2, and g_2 are idempotents because (f_1, f_2) and (g_1, g_2) are idempotents. Therefore, $f_2 = g_2 = 0$. Second, (2.9) implies again that $f = f_1 + g_1$. As f_1 and g_1 are idempotents and f is primitive, either f_1 or g_1 must be 0. We can assume that $f_1 = 0$ without loss of generality. We then obtain $(f_1, f_2) = 0$, which implies that $(f, 0)$ is primitive.

We now prove the "only if" part. Let $(f_1, \ldots, f_{r_1+r_2}) \in \mathfrak{F}(\mathbb{E})$ and set $f_i = (f_{1i}, f_{2i})$ for each i. Then, each $f_i = (f_{1i}, f_{2i})$ falls into exactly one of the two cases in Lemma 2.9; thus, we define

$$\begin{aligned} I_1 &:= \{i \in \{1, \ldots, r_1 + r_2\} \mid f_{1i} \text{ is a primitive idempotent and } f_{2i} = 0\}, \\ I_2 &:= \{i \in \{1, \ldots, r_1 + r_2\} \mid f_{1i} = 0 \text{ and } f_{2i} \text{ is a primitive idempotent}\}. \end{aligned}$$

Evidently, (I_1, I_2) is a partition of $\{1, \ldots, r_1 + r_2\}$. In the following, we show that $(f_{1i})_{i\in I_1} \in \mathfrak{F}(\mathbb{E}_1)$ and $(f_{2i})_{i\in I_2} \in \mathfrak{F}(\mathbb{E}_2)$. From the assumption on $(f_1, \ldots, f_{r_1+r_2})$, it follows that

$$e = (e_1, e_2) = \sum_{i=1}^{r_1+r_2} f_i = \left(\sum_{i\in I_1} f_{1i}, \sum_{i\in I_2} f_{2i} \right),$$

from which we obtain $\sum_{i\in I_1} f_{1i} = e_1$ and $\sum_{i\in I_2} f_{2i} = e_2$. In addition, it follows that $0 = f_i \circ f_j = (f_{1i} \circ_1 f_{1j}, f_{2i} \circ_2 f_{2j})$ for any $i \neq j$. In particular, we have $f_{1i} \circ_1 f_{1j} = 0$ for all $i \neq j \in I_1$ and $f_{2i} \circ_2 f_{2j} = 0$ for all $i \neq j \in I_2$. □

2.4 Completely Positive and Copositive Cones

Let $\mathbb{K}$ be a closed cone in an n-dimensional real inner product space $(\mathbb{V}, \bullet)$. We refer to a symmetric tensor $\mathcal{A} \in \mathcal{S}^{n,m}(\mathbb{V})$ as *completely positive over* $\mathbb{K}$ if there exist a positive integer k and $x_1, \ldots, x_k \in \mathbb{K}$ such that

$$\mathcal{A} = \sum_{i=1}^{k} x_i^{\otimes m} \tag{2.10}$$

and *copositive over* $\mathbb{K}$ if $\langle \mathcal{A}, x^{\otimes m} \rangle \geq 0$ for all $x \in \mathbb{K}$, respectively. The *completely positive* (resp. *copositive*) *cone over* $\mathbb{K}$ is defined as the set of symmetric tensors that are completely positive (resp. copositive) over $\mathbb{K}$. Specifically, they are denoted by

$$\begin{aligned} \mathcal{CP}^{n,m}(\mathbb{K}) &:= \operatorname{conv}\{x^{\otimes m} \mid x \in \mathbb{K}\}, \\ \mathcal{COP}^{n,m}(\mathbb{K}) &:= \{\mathcal{A} \in \mathcal{S}^{n,m}(\mathbb{V}) \mid \langle \mathcal{A}, x^{\otimes m} \rangle \geq 0 \text{ for all } x \in \mathbb{K}\}, \end{aligned}$$

respectively. According to Theorem 2.1, every element in $\mathcal{CP}^{n,m}(\mathbb{K})$ can be represented in the form of (2.10), where k does not surpass the dimension of $\mathcal{S}^{n,m}(\mathbb{V})$. We call the cone $\mathbb{K}$ appearing in $\mathcal{CP}^{n,m}(\mathbb{K})$ and $\mathcal{COP}^{n,m}(\mathbb{K})$ the *underlying cone*.[3]

In the case in which $\mathbb{V}$ is the space $\mathbb{R}^n$ with the canonical basis and $\mathbb{K} = \mathbb{R}^n_+$, we write the completely positive cone $\mathcal{CP}^{n,m}(\mathbb{R}^n_+)$ (resp. copositive cone $\mathcal{COP}^{n,m}(\mathbb{R}^n_+)$) in the space $\mathcal{S}^{n,m}$ as $\mathcal{CP}^{n,m}$ (resp. $\mathcal{COP}^{n,m}$) for brevity. In the case of $m = 2$, we write $\mathcal{CP}^{n,2}(\mathbb{K})$ (resp. $\mathcal{COP}^{n,2}(\mathbb{K})$) as $\mathcal{CP}(\mathbb{K})$ (resp. $\mathcal{COP}(\mathbb{K})$) for short. Under the identification between $\mathcal{S}^{n,2}(\mathbb{V})$ and $\mathcal{S}(\mathbb{V})$, $\mathcal{COP}(\mathbb{K})$ can be equivalently written as

$$\mathcal{COP}(\mathbb{K}) = \{\mathcal{A} \in \mathcal{S}(\mathbb{V}) \mid x \bullet \mathcal{A}(x) \geq 0 \text{ for all } x \in \mathbb{K}\}.$$

In particular, when $\mathbb{V}$ is the space $\mathbb{R}^n$ with the canonical basis, we have

$$\begin{aligned} \mathcal{CP}(\mathbb{K}) &= \operatorname{conv}\{\boldsymbol{x}\boldsymbol{x}^\top \mid \boldsymbol{x} \in \mathbb{K}\}, \\ \mathcal{COP}(\mathbb{K}) &= \{\boldsymbol{A} \in \mathbb{S}^n \mid \boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} \geq 0 \text{ for all } \boldsymbol{x} \in \mathbb{K}\}. \end{aligned}$$

As with tensors, we write $\mathcal{CP}(\mathbb{R}^n_+)$ (resp. $\mathcal{COP}(\mathbb{R}^n_+)$), which is $\mathcal{CP}^{n,2}$ (resp. $\mathcal{COP}^{n,2}$), as $\mathcal{CP}^n$ (resp. $\mathcal{COP}^n$).

[3]Some researchers [9, 39] refer to it as the ground cone.

We may refer to $\mathcal{CP}^{n,m}(\mathbb{K})$ (resp. $\mathcal{COP}^{n,m}(\mathbb{K})$) as the *generalized* completely positive (resp. copositive) cones over $\mathbb{K}$ when we emphasize that the underlying cone $\mathbb{K}$ is a closed cone beyond the nonnegative orthant, and we may omit the term "over $\mathbb{K}$" unless we need to specify the underlying cone.[4] We may also refer to $\mathcal{CP}^{n,m}$ (resp. $\mathcal{COP}^{n,m}$) as the *standard* completely positive (resp. copositive) cones when we specifically stress that the underlying cone is the nonnegative orthant.

We now discuss the duality between $\mathcal{CP}^{n,m}(\mathbb{K})$ and $\mathcal{COP}^{n,m}(\mathbb{K})$. The following proposition extends the known fact that $\mathcal{CP}^{n,m}$ and $\mathcal{COP}^{n,m}$ are mutually dual [94, 101].

Proposition 2.11. *Let $\mathbb{K}$ be a closed cone in $\mathbb{V}$.*

(i) $\mathcal{COP}^{n,m}(\mathbb{K}) = \mathcal{CP}^{n,m}(\mathbb{K})^*$. In particular, $\mathcal{COP}^{n,m}(\mathbb{K})$ is a closed convex cone.

(ii) If $\mathbb{K}$ is also pointed and convex, then $\mathcal{CP}^{n,m}(\mathbb{K})$ is a closed convex cone. In particular, $\mathcal{COP}^{n,m}(\mathbb{K})$ and $\mathcal{CP}^{n,m}(\mathbb{K})$ are dual to each other.

(iii) $\mathcal{CP}(\mathbb{K})$ and $\mathcal{COP}(\mathbb{K})$ are dual to each other.

Proof. Throughout this proof, we let $d \coloneqq \dim \mathcal{S}^{n,m}(\mathbb{V})$. We first prove (i). Let $\mathcal{A} \in \mathcal{COP}^{n,m}(\mathbb{K})$. Then, for any $x_1, \dots, x_d \in \mathbb{K}$, we have

$$\left\langle \mathcal{A}, \sum_{i=1}^{d} x_i^{\otimes m} \right\rangle = \sum_{i=1}^{d} \langle \mathcal{A}, x_i^{\otimes m} \rangle. \tag{2.11}$$

As $\langle \mathcal{A}, x_i^{\otimes m} \rangle$ is nonnegative for all $i = 1, \dots, d$, (2.11) is also nonnegative, which implies that $\mathcal{A} \in \mathcal{CP}^{n,m}(\mathbb{K})^*$. Conversely, let $\mathcal{A} \in \mathcal{CP}^{n,m}(\mathbb{K})^*$. For any $x \in \mathbb{K}$, because $x^{\otimes m} \in \mathcal{CP}^{n,m}(\mathbb{K})$, $\langle \mathcal{A}, x^{\otimes m} \rangle \geq 0$ follows from the definition of dual cones. Therefore, we obtain $\mathcal{A} \in \mathcal{COP}^{n,m}(\mathbb{K})$. By (i) of Theorem 2.2, $\mathcal{COP}^{n,m}(\mathbb{K})$ is a closed convex cone.

We now prove (ii). The convexity of $\mathcal{CP}^{n,m}(\mathbb{K})$ follows from its definition. In addition, $\mathcal{CP}^{n,m}(\mathbb{K})$ is a cone because $\mathbb{K}$ is a cone. To prove the closedness, let $\{\mathcal{A}_k\}_k \subseteq \mathcal{CP}^{n,m}(\mathbb{K})$ and suppose it converges to some $\mathcal{A}_\infty \in \mathcal{S}^{n,m}(\mathbb{V})$. For every k, there exist $x_{k1}, \dots, x_{kd} \in \mathbb{K}$ such that $\mathcal{A}_k = \sum_{i=1}^{d} x_{ki}^{\otimes m}$. By (ii) of Theorem 2.2, we see that $\operatorname{int} \mathbb{K}^*$ is nonempty, so we take $a \in \operatorname{int} \mathbb{K}^*$ arbitrarily. Then, we obtain

$$\langle \mathcal{A}_k, a^{\otimes m} \rangle = \sum_{i=1}^{d} \langle x_{ki}^{\otimes m}, a^{\otimes m} \rangle = \sum_{i=1}^{d} (x_{ki} \bullet a)^m \to \langle \mathcal{A}_\infty, a^{\otimes m} \rangle \ (k \to \infty).$$

Given that $x_{ki} \in \mathbb{K}$, $x_{ki} \bullet a \geq 0$ for any k and i. Therefore, $\{x_{ki} \bullet a\}_k$ is bounded for each i.

Then, $\{x_{ki}\}_k$ is bounded for each i. To observe this, let $\{k(l) \mid l \in \mathbb{N}\}$ denote the set of indices k such that $x_{ki} \neq 0$. Showing the boundedness of $\{x_{k(l)i}\}_l$ is sufficient.

[4] The terminology used to denote complete positivity/copositivity and completely positive/copositive cones in a general sense basically follows from [4, 15]. However, other terms are also employed to refer to them: $\mathbb{K}$-completely positive/copositive [65], set-completely positive/copositive [3, 68], completely positive/copositive with respect to $\mathbb{K}$ [55, 120]. Eichfelder and Povh [32] also use the terms "$\mathbb{K}$-semidefinite" and "set-semidefinite" to denote copositivity over $\mathbb{K}$.

Let $B \coloneqq \mathbb{K} \cap \{y \in \mathbb{V} \mid y \bullet a = 1\}$. Note that B is compact. Then, for each l, there exist $\alpha_{li} > 0$ and $y_{li} \in B$ such that $x_{k(l)i} = \alpha_{li} y_{li}$. Given that $x_{k(l)i} \bullet a = \alpha_{li}$, the sequence $\{\alpha_{li}\}_l$ is bounded. Combining it with the boundedness of $\{y_{li}\}_l \subseteq B$ leads to the boundedness of $\{x_{k(l)i}\}_l = \{\alpha_{li} y_{li}\}_l$.

Thus, by taking a subsequence, if necessary, we assume that $\{x_{ki}\}_k$ converges to some $x_{\infty i}$ for each i. The closedness of $\mathbb{K}$ implies that $x_{\infty i} \in \mathbb{K}$. Therefore,

$$\mathcal{A}_\infty = \lim_{k\to\infty} \mathcal{A}_k = \lim_{k\to\infty} \sum_{i=1}^{d} x_{ki}^{\otimes m} = \sum_{i=1}^{d} x_{\infty i}^{\otimes m} \in \mathcal{CP}^{n,m}(\mathbb{K}),$$

which means that $\mathcal{CP}^{n,m}(\mathbb{K})$ is closed.

Now that $\mathcal{CP}^{n,m}(\mathbb{K})$ is a closed convex cone, by taking the dual of (i) of Proposition 2.11 and using (iii) of Theorem 2.2, we obtain $\mathcal{COP}^{n,m}(\mathbb{K})^* = \mathcal{CP}^{n,m}(\mathbb{K})$.

See [117, Section 2] for the proof of (iii). □

Note that when $m = 2$, neither pointedness nor convexity of $\mathbb{K}$ is necessary to guarantee the duality between $\mathcal{CP}(\mathbb{K})$ and $\mathcal{COP}(\mathbb{K})$. Furthermore, in this thesis, we focus mainly on the case in which $\mathbb{K}$ is a symmetric cone, which is pointed, closed, and convex. In this case, by (ii) of Proposition 2.11, $\mathcal{CP}^{n,m}(\mathbb{K})$ and $\mathcal{COP}^{n,m}(\mathbb{K})$ are dual to each other. Although complete positivity and copositivity over general sets that are neither necessarily closed nor cones are also considered [25, 30, 31, 33, 117], we have to modify the definition of completely positive cones to guarantee the duality to copositive cones.

2.4.1 Standard completely positive and copositive matrix cones

In this subsection, we focus on the standard completely positive matrix cone $\mathcal{CP}^n$ and copositive cone $\mathcal{COP}^n$. We introduce some cones approximating them from the inside or outside and describe their properties.

The *nonnegative cone* $\mathcal{N}^n$ is defined by the set of $n \times n$ symmetric matrices with only nonnegative elements. The *doubly nonnegative cone* $\mathbb{S}^n_+ \cap \mathcal{N}^n$ is denoted by $\mathcal{DNN}^n$ and the *SPN cone* $\mathbb{S}^n_+ + \mathcal{N}^n$ by $\mathcal{SPN}^n$. Let

$$\mathcal{DD}^n \coloneqq \left\{ \boldsymbol{A} \in \mathbb{S}^n \;\middle|\; A_{ii} \geq \sum_{j\neq i} |A_{ij}| \text{ for all } i = 1, \dots, n \right\}$$

denote the cone of $n \times n$ symmetric diagonally dominant matrices with nonnegative diagonal elements and let $\mathcal{SDD}^n$ be the cone of matrices $\boldsymbol{DAD}$ such that $\boldsymbol{A} \in \mathcal{DD}^n$ and $\boldsymbol{D} \in \mathbb{S}^n$ is a diagonal matrix with positive diagonal elements. In addition, we define $\mathcal{DD}^n_+ \coloneqq \mathcal{DD}^n \cap \mathcal{N}^n$ and $\mathcal{SDD}^n_+ \coloneqq \mathcal{SDD}^n \cap \mathcal{N}^n$.

The nonnegative cone is self-dual. In addition, $\mathcal{DNN}^n$ and $\mathcal{SPN}^n$ are mutually dual [111, Theorem 1.167]. In particular, by (i) of Theorem 2.2, $\mathcal{SPN}^n$ is a closed convex cone. The dual of $\mathcal{DD}^n_+$ is

$$\begin{aligned}(\mathcal{DD}^n_+)^* &= (\mathcal{DD}^n)^* + (\mathcal{N}^n)^* \\ &= \{\boldsymbol{A} \in \mathbb{S}^n \mid A_{ii} + A_{jj} \pm 2A_{ij} \geq 0 \text{ for all } i, j = 1, \dots, n\} + \mathcal{N}^n, \end{aligned} \tag{2.12}$$

where we use Corollaries 16.4.2 and 9.1.3 of [106] to derive the first equation and use [95, Table 1] to derive the second equation. Similarly, the dual of $\mathcal{SDD}_+^n$ is

$$\begin{aligned}
&(\mathcal{SDD}_+^n)^* \\
&\quad= (\mathcal{SDD}^n)^* + (\mathcal{N}^n)^* \\
&\quad= \left\{ \boldsymbol{A} \in \mathbb{S}^n \;\middle|\; \begin{array}{l} A_{ii}A_{jj} \geq A_{ij}^2 \text{ for all } i, j = 1, \dots, n \text{ with } i \neq j, \\ A_{ii} \geq 0 \text{ for all } i = 1, \dots, n \end{array} \right\} + \mathcal{N}^n,
\end{aligned} \tag{2.13}$$

where we use [42, Proposition 1.ii] to derive the first equation and use [95, Table 1] to derive the second equation.

Equation (2.13) leads to the following lemma. Its proof is straightforward and thus omitted.

Lemma 2.12. *Let $\boldsymbol{A} \in (\mathcal{SDD}_+^n)^*$. Then, the following statements hold:*

(i) $A_{ii} \geq 0$ for all $i = 1, \dots, n$.

(ii) If $A_{ii} = 0$, then $A_{ij} \geq 0$ for all $j = 1, \dots, n$.

The following inclusions hold:

$$\mathcal{DD}_+^n \subseteq \mathcal{SDD}_+^n \overset{\text{(a)}}{\subseteq} \mathcal{CP}^n \overset{\text{(b)}}{\subseteq} \mathcal{DNN}^n \subseteq \mathcal{N}^n, \tag{2.14}$$

where the inclusion (a) follows from [42, Proposition 1.iii] and the others follow from their definitions. In addition, $\mathcal{DNN}^n \subseteq \mathbb{S}_+^n$ also holds by definition. Taking their dual and using (iv) of Theorem 2.2, we also obtain the following inclusions:

$$\mathcal{N}^n \subseteq \mathcal{SPN}^n \overset{\text{(c)}}{\subseteq} \mathcal{COP}^n \subseteq (\mathcal{SDD}_+^n)^* \subseteq (\mathcal{DD}_+^n)^*. \tag{2.15}$$

The inclusions (b) and (c) hold with equality if and only if $n \leq 4$; see Sections 2.9 and 2.10 of [111]. We state this fact explicitly because it will be exploited to compute the distance to polyhedrality in Chapter 5.

Lemma 2.13. *The equations $\mathcal{CP}^n = \mathcal{DNN}^n$ and $\mathcal{SPN}^n = \mathcal{COP}^n$ hold if and only if $n \leq 4$.*

2.4.2 Completely positive reformulation for nonconvex programming problems

One of the motivations for introducing completely positive cones is that they can reformulate nonconvex programming problems as convex ones. As stated in Section 1.1, there are a lot of studies that deal with the complete positive reformulation of nonconvex programming problems. In this subsection, among them, we present the completely positive representation of conic quadratically constrained quadratic programming problems failing Slater's condition proved by [4, Theorem 4].

We consider the following conic quadratically constrained quadratic programming problem:

$$
\begin{aligned}
\underset{\boldsymbol{x}}{\text{minimize}} \quad & \frac{1}{2}\boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{c}^\top \boldsymbol{x} \\
\text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \\
& q(\boldsymbol{x}) := \frac{1}{2}\boldsymbol{x}^\top \boldsymbol{P}\boldsymbol{x} + \boldsymbol{q}^\top \boldsymbol{x} + h \leq 0, \\
& \boldsymbol{x} \in K,
\end{aligned} \tag{2.16}
$$

where $\boldsymbol{Q}, \boldsymbol{P} \in \mathbb{S}^n$, $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{c}, \boldsymbol{q} \in \mathbb{R}^n$, $\boldsymbol{b} \in \mathbb{R}^m$, $h \in \mathbb{R}$, and K is a closed convex cone in $\mathbb{R}^n$. In addition, $q(\boldsymbol{x})$ is assume to be nonnegative for all $\boldsymbol{x} \in K$ satisfying $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, which ensures that (2.16) fails Slater's condition. Although this form and condition seem to be restrictive, general conic quadratically constrained quadratic programming problems with finitely many inequality constraints can be recast as (2.16) with a convex objective function [4, Corollary 1]. It is easy to see that (2.16) is equivalent to the following problem with a rank-1 constraint:

$$
\begin{aligned}
\underset{\boldsymbol{x},\boldsymbol{X}}{\text{minimize}} \quad & \frac{1}{2}\langle \boldsymbol{Q}, \boldsymbol{X}\rangle + \boldsymbol{c}^\top \boldsymbol{x} \\
\text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \\
& \operatorname{diag}(\boldsymbol{A}\boldsymbol{X}\boldsymbol{A}^\top) = \boldsymbol{b} \odot \boldsymbol{b}, \\
& \frac{1}{2}\langle \boldsymbol{P}, \boldsymbol{X}\rangle + \boldsymbol{q}^\top \boldsymbol{x} + h = 0, \\
& \begin{pmatrix} 1 & \boldsymbol{x}^\top \\ \boldsymbol{x} & \boldsymbol{X} \end{pmatrix} \in \mathcal{CP}(\mathbb{R}_+ \times K), \\
& \operatorname{rank}\begin{pmatrix} 1 & \boldsymbol{x}^\top \\ \boldsymbol{x} & \boldsymbol{X} \end{pmatrix} = 1.
\end{aligned} \tag{2.17}
$$

By dropping the rank constraint in (2.17), we have the following completely positive programming relaxation problem for (2.16):

$$
\begin{aligned}
\underset{\boldsymbol{x},\boldsymbol{X}}{\text{minimize}} \quad & \frac{1}{2}\langle \boldsymbol{Q}, \boldsymbol{X}\rangle + \boldsymbol{c}^\top \boldsymbol{x} \\
\text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \\
& \operatorname{diag}(\boldsymbol{A}\boldsymbol{X}\boldsymbol{A}^\top) = \boldsymbol{b} \odot \boldsymbol{b}, \\
& \frac{1}{2}\langle \boldsymbol{P}, \boldsymbol{X}\rangle + \boldsymbol{q}^\top \boldsymbol{x} + h = 0, \\
& \begin{pmatrix} 1 & \boldsymbol{x}^\top \\ \boldsymbol{x} & \boldsymbol{X} \end{pmatrix} \in \mathcal{CP}(\mathbb{R}_+ \times K).
\end{aligned} \tag{2.18}
$$

In general, (2.16) and (2.18) are not equivalent and the optimal value of (2.18) is smaller than that of (2.16). However, under an assumption on the coefficient matrix $\boldsymbol{Q}$ in the objective function, we can guarantee the equivalence between them.

Theorem 2.14 ([4, Theorem 4]). *Let L be the set of $\boldsymbol{d} \in K$ satisfying $\boldsymbol{A}\boldsymbol{d} = \boldsymbol{0}$ and $\boldsymbol{d}^\top \boldsymbol{P}\boldsymbol{d} = 0$, and suppose that $\boldsymbol{Q} \in \mathcal{COP}(L)$.*[5] *Then, the conic quadratically constrained quadratic programming problem (2.16) and its completely positive programming relaxation problem (2.18) is equivalent in the following sense:*

(i) Their optimal values coincide, including the case of $\pm\infty$.

(ii) There exists an optimal solution of (2.16) if and only if there exists an optimal solution of (2.18).

(iii) If $(\boldsymbol{x}^*, \boldsymbol{X}^*)$ is an optimal solution of (2.18), then $\boldsymbol{x}^*$ belongs to the convex hull of the set of optimal solutions of (2.16).

An aspect deserving attention is that if K is a symmetric cone in $\mathbb{R}^n$, then the underlying cone $\mathbb{R}_+ \times K$ is also a symmetric cone in $\mathbb{R}^{n+1}$. This observation spurs us to study completely positive and copositive cones over symmetric cones.

In what follows, as conducted in [4], we apply this theorem to specific nonconvex programming problems. We first consider the following nonconvex conic quadratic programming problem with binary and continuous variables:

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & \frac{1}{2}\boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \\ & x_i \in \{0,1\} \ (i \in B), \\ & \boldsymbol{x} \in K, \end{array} \tag{2.19}$$

where $\boldsymbol{Q} \in \mathbb{S}^n$ is not necessarily positive semidefinite, $\boldsymbol{c} \in \mathbb{R}^n$, $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{b} \in \mathbb{R}^m$, $B \subseteq \{1, \dots, n\}$, and K is a closed convex cone in $\mathbb{R}^n$. To derive a completely positive reformulation of (2.19), we impose the following essential assumption on it.

Assumption 2.15. For any $\boldsymbol{x} \in K$ satisfying $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, x_i lies in the interval $[0, 1]$ for all $i \in B$.

Each binary constraint $x_i \in \{0, 1\}$ is equivalent to $-x_i^2 + x_i = 0$. By Assumption 2.15, we can express the $|B|$ binary constraints in (2.19) as a single constraint

$$q(\boldsymbol{x}) \coloneqq \sum_{i\in B}(-x_i^2 + x_i) \le 0,$$

and the function $q(\boldsymbol{x})$ is nonnegative for all $\boldsymbol{x} \in K$ satisfying $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$. Therefore, we can express (2.19) in the form of (2.16). Considering the case where the assumption on the coefficient matrix $\boldsymbol{Q}$ in Theorem 2.14 does not hold,[6] we have the following result, which was originally proved in [14].

[5]For example, this condition holds when (2.16) has a linear objective function, i.e., $\boldsymbol{Q} = \boldsymbol{O}$.

[6]If the coefficient matrix $\boldsymbol{Q}$ does not satisfy the assumption in Theorem 2.14, both of (2.19) and its completely positive programming relaxation problem are unbounded.

Corollary 2.16. *Under Assumption 2.15, the optimal value of (2.19) is equal to that of the following completely positive programming problem:*

$$\begin{array}{ll} \underset{\boldsymbol{x},\boldsymbol{X}}{\text{minimize}} & \frac{1}{2}\langle \boldsymbol{Q}, \boldsymbol{X}\rangle + \boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \\ & \operatorname{diag}(\boldsymbol{A}\boldsymbol{X}\boldsymbol{A}^\top) = \boldsymbol{b} \odot \boldsymbol{b}, \\ & -X_{ii} + x_i = 0 \ (i \in B), \\ & \begin{pmatrix} 1 & \boldsymbol{x}^\top \\ \boldsymbol{x} & \boldsymbol{X} \end{pmatrix} \in \mathcal{CP}(\mathbb{R}_+ \times K). \end{array}$$

We next consider the following semidefinite programming problem with a rank inequality constraint:

$$\begin{array}{ll} \underset{\boldsymbol{X}}{\text{minimize}} & \langle \boldsymbol{C}, \boldsymbol{X}\rangle \\ \text{subject to} & \langle \boldsymbol{A}_i, \boldsymbol{X}\rangle = b_i \ (i = 1, \dots, m), \\ & \operatorname{rank}(\boldsymbol{X}) \le p, \\ & \boldsymbol{X} \in \mathbb{S}^n_+, \end{array} \tag{2.20}$$

where $\boldsymbol{C}, \boldsymbol{A}_1, \dots, \boldsymbol{A}_m \in \mathbb{S}^n$, $b_1, \dots, b_m \in \mathbb{R}$, and p is an integer satisfying $1 \le p \le n-1$. To apply Theorem 2.14, we utilize the equivalence between (2.20) and a semidefinite programming problem with a complementarity constraint [67, Theorem 1.1]. Specifically, (2.20) is equivalent to the problem

$$\begin{array}{ll} \underset{\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}}{\text{minimize}} & \langle \boldsymbol{C}, \boldsymbol{X}\rangle \\ \text{subject to} & \langle \boldsymbol{A}_i, \boldsymbol{X}\rangle = b_i \ (i = 1, \dots, m), \\ & \boldsymbol{Y} = \boldsymbol{I} - \boldsymbol{Z}, \\ & \displaystyle\sum_{i=1}^n Z_{ii} = p, \\ & q(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}) := \langle \boldsymbol{X}, \boldsymbol{Y}\rangle \le 0, \\ & \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in \mathbb{S}^n_+ \end{array} \tag{2.21}$$

in the sense that the feasible set of (2.20) agrees with the projection of the feasible set of (2.21) onto the $\boldsymbol{X}$ component. Since (2.21) has a linear objective function and the function $q(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z})$ is nonnegative under the condition that $\boldsymbol{X}, \boldsymbol{Y} \in \mathbb{S}^n_+$, Theorem 2.14 implies that (2.21), namely (2.20) always has an equivalent completely positive reformulation. Specifically, let $\hat{\boldsymbol{a}}_i$ be the T_n-dimensional vector defined as

$$\hat{\boldsymbol{a}}_i := \begin{cases} \boldsymbol{e}_i & (\text{if there exists } j = 1, \dots, n \text{ such that } i = T_j), \\ \frac{\sqrt{2}}{2}\boldsymbol{e}_i & (\text{otherwise}) \end{cases}$$

and $\hat{\boldsymbol{A}} \in \mathbb{R}^{(m+1+T_n)\times 3T_n}$ be defined as

$$\hat{\boldsymbol{A}} \coloneqq \begin{pmatrix} \mathrm{svec}(\boldsymbol{A}_1)^\top & \boldsymbol{0}_{T_n}^\top & \boldsymbol{0}_{T_n}^\top \\ \vdots & \vdots & \vdots \\ \mathrm{svec}(\boldsymbol{A}_m)^\top & \boldsymbol{0}_{T_n}^\top & \boldsymbol{0}_{T_n}^\top \\ \boldsymbol{0}_{T_n}^\top & \mathrm{svec}(\boldsymbol{I}_n)^\top & \boldsymbol{0}_{T_n}^\top \\ \boldsymbol{0}_{T_n}^\top & \hat{\boldsymbol{a}}_1^\top & \hat{\boldsymbol{a}}_1^\top \\ \vdots & \vdots & \vdots \\ \boldsymbol{0}_{T_n}^\top & \hat{\boldsymbol{a}}_{T_n}^\top & \hat{\boldsymbol{a}}_{T_n}^\top \end{pmatrix}.$$

In addition, let $\hat{\boldsymbol{c}} \in \mathbb{R}^{3T_n}$, $\hat{\boldsymbol{b}} \in \mathbb{R}^{m+1+T_n}$, and $\hat{\boldsymbol{Q}} \in \mathbb{S}^{3T_n}$ be defined as

$$\begin{aligned} \hat{\boldsymbol{c}} &\coloneqq (\mathrm{svec}(\boldsymbol{C}); \boldsymbol{0}_{2T_n}), \\ \hat{\boldsymbol{b}} &\coloneqq (b_1; \dots; b_m; p; \mathrm{svec}(\boldsymbol{I}_n)), \\ \hat{\boldsymbol{Q}} &\coloneqq \begin{pmatrix} & & \frac{1}{2}\boldsymbol{I}_{T_n} \\ & \boldsymbol{O}_{T_n} & \\ \frac{1}{2}\boldsymbol{I}_{T_n} & & \end{pmatrix}, \end{aligned}$$

respectively. Then, the completely positive reformulation of (2.20) is described as follows.

Corollary 2.17. *The optimal value of* (2.20) *is equal to that of the following completely positive programming problem:*

$$\begin{aligned} \underset{\hat{\boldsymbol{x}}, \hat{\boldsymbol{X}}}{\text{minimize}} \quad & \hat{\boldsymbol{c}}^\top \hat{\boldsymbol{x}} \\ \text{subject to} \quad & \hat{\boldsymbol{A}}\hat{\boldsymbol{x}} = \hat{\boldsymbol{b}}, \\ & \mathrm{diag}(\hat{\boldsymbol{A}}\hat{\boldsymbol{X}}\hat{\boldsymbol{A}}^\top) = \hat{\boldsymbol{b}} \odot \hat{\boldsymbol{b}}, \\ & \langle \hat{\boldsymbol{Q}}, \hat{\boldsymbol{X}} \rangle = 0, \\ & \begin{pmatrix} 1 & \hat{\boldsymbol{x}}^\top \\ \hat{\boldsymbol{x}} & \hat{\boldsymbol{X}} \end{pmatrix} \in \mathcal{CP}(\mathbb{R}_+ \times \mathrm{svec}(\mathbb{S}_+^n)^3). \end{aligned}$$

2.5 Approximation Hierarchies for Copositive Cones

In this section, we introduce several existing inner- and outer-approximation hierarchies for copositive cones. We first define the terms "inner/outer- approximation hierarchies."

Definition 2.18 (Inner-approximation hierarchy)**.** We call a sequence $\{\mathcal{I}_l\}_{l\in\mathbb{N}}$ an *inner-approximation hierarchy* for $\mathcal{K}$ if the following three conditions hold:

(i) (Non-decreasingness) $\mathcal{I}_l \subseteq \mathcal{I}_{l+1}$ for all $l \in \mathbb{N}$,

(ii) (Inclusion) $\mathcal{I}_l \subseteq \mathcal{K}$ for all $l \in \mathbb{N}$,

(iii) (Convergence) $\operatorname{int} \mathcal{K} \subseteq \bigcup_{l \in \mathbb{N}} \mathcal{I}_l$.

Definition 2.19 (Outer-approximation hierarchy)**.** We call a sequence $\{\mathcal{O}_l\}_{l \in \mathbb{N}}$ an *outer-approximation hierarchy* for $\mathcal{K}$ if the following two conditions hold:

(i) (Non-increasingness) $\mathcal{O}_{l+1} \subseteq \mathcal{O}_l$ for all $l \in \mathbb{N}$,

(ii) (Convergence) $\bigcap_{l \in \mathbb{N}} \mathcal{O}_l = \mathcal{K}$.

Note that (ii) of Definition 2.19 implies that an outer-approximation hierarchy $\{\mathcal{O}_l\}_l$ for $\mathcal{K}$ satisfies the inclusion $\mathcal{K} \subseteq \mathcal{O}_l$ for all $l \in \mathbb{N}$.

2.5.1 Inner-approximation hierarchies for copositive cones

First, we introduce two inner-approximation hierarchies for the standard copositive cone $\mathcal{COP}^n$. Since there exists a one-to-one correspondence between the elements in $\mathbb{R}^n_+$ and the elements $\boldsymbol{x} \odot \boldsymbol{x}$ with $\boldsymbol{x} \in \mathbb{R}^n$, a symmetric matrix $\boldsymbol{A} \in \mathbb{S}^n$ belongs to $\mathcal{COP}^n$ if and only if the quartic homogeneous polynomial

$$(\boldsymbol{x} \odot \boldsymbol{x})^\top \boldsymbol{A} (\boldsymbol{x} \odot \boldsymbol{x}) = \sum_{i,j=1}^n A_{ij} x_i^2 x_j^2 \tag{2.22}$$

is nonnegative for all $\boldsymbol{x} \in \mathbb{R}^n$. However, the nonnegativity of quartic homogeneous polynomials is difficult to verify computationally; see, for example, [2] and references therein. Alternatively, as a sufficient and computationally tractable condition for the nonnegativity of (2.22), Parrilo [90] (see also [89]) considers the condition that the polynomial

$$p_l(\boldsymbol{x}; \boldsymbol{A}) \coloneqq (\boldsymbol{x}^\top \boldsymbol{x})^l (\boldsymbol{x} \odot \boldsymbol{x})^\top \boldsymbol{A} (\boldsymbol{x} \odot \boldsymbol{x}) \in H^{n,2(l+2)}$$

is sum-of-squares for each fixed $l \in \mathbb{N}$. We define $\mathcal{I}^n_{\mathrm{Par},l}$ as the set of $\boldsymbol{A} \in \mathbb{S}^n$ satisfying this condition; i.e.,

$$\mathcal{I}^n_{\mathrm{Par},l} \coloneqq \{\boldsymbol{A} \in \mathbb{S}^n \mid p_l(\boldsymbol{x}; \boldsymbol{A}) \in \Sigma^{n,2(l+2)}\}. \tag{2.23}$$

de Klerk and Pasechnik [23] consider a stronger condition than the polynomial $p_l(\boldsymbol{x}; \boldsymbol{A})$ being sum-of-squares. Since only squared variables such as $x_1^{2l} x_2^2 x_3^2$ appear in each term of $p_l(\boldsymbol{x}; \boldsymbol{A})$, if all the coefficients of $p_l(\boldsymbol{x}; \boldsymbol{A})$ are nonnegative, it is sum-of-squares. Based on this idea, we define $\mathcal{I}^n_{\mathrm{dP},l}$ as

$$\begin{aligned} \mathcal{I}^n_{\mathrm{dP},l} &\coloneqq \{\boldsymbol{A} \in \mathbb{S}^n \mid p_l(\boldsymbol{x}; \boldsymbol{A}) \text{ has only nonnegative coefficients}\} \\ &= \left\{ \boldsymbol{A} \in \mathbb{S}^n \;\middle|\; \left(\sum_{i=1}^n x_i \right)^l \boldsymbol{x}^\top \boldsymbol{A} \boldsymbol{x} \text{ has only nonnegative coefficients} \right\}. \end{aligned}$$

Both of the sequences $\{\mathcal{I}^n_{\mathrm{Par},l}\}_l$ and $\{\mathcal{I}^n_{\mathrm{dP},l}\}_l$ are indeed inner-approximation hierarchies for $\mathcal{COP}^n$. It is easy to see that they are non-decreasing. Because the multiplier $(\boldsymbol{x}^\top \boldsymbol{x})^l$ does not affect the sign of $p_l(\boldsymbol{x}; \boldsymbol{A})$, $\mathcal{COP}^n$ includes $\mathcal{I}^n_{\mathrm{Par},l}$. Since $\mathcal{I}^n_{\mathrm{Par},l}$ includes

$\mathcal{I}_{\mathrm{dP},l}^n$, $\mathcal{COP}^n$ also includes $\mathcal{I}_{\mathrm{dP},l}^n$. Therefore, if the sequence $\{\mathcal{I}_{\mathrm{dP},l}^n\}_l$ converges to $\mathcal{COP}^n$ in the sense of Definition 2.18, the sequence $\{\mathcal{I}_{\mathrm{Par},l}^n\}_l$ also converges to it. The convergence of the sequence $\{\mathcal{I}_{\mathrm{dP},l}^n\}_l$ results from *Pólya's Positivstellensatz*, which is stated as follows.

Theorem 2.20 ([54, Theorem 56]). *Let f be a homogeneous polynomial and suppose that $f(\boldsymbol{x})$ is positive for all $\boldsymbol{x} \in \mathbb{R}_+^n \setminus \{\boldsymbol{0}\}$. Then, there exists $l \in \mathbb{N}$ such that the polynomial $\left(\sum_{i=1}^n x_i\right)^l f(\boldsymbol{x})$ has only positive coefficients.*

In general, *Positivstellensatz* is a theorem providing algebraic descriptions of positive polynomials. In this thesis, other types of Positivstellensatz will be used to prove the convergence of other approximation hierarchies for copositive cones.

We now prove the convergence. Suppose that $\boldsymbol{A} \in \operatorname{int} \mathcal{COP}^n$. Then, $\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x}$ is positive for all $\boldsymbol{x} \in \mathbb{R}_+^n \setminus \{\boldsymbol{0}\}$ [111, Theorem 2.25]. Therefore, it follows from Theorem 2.20 that $\boldsymbol{A} \in \mathcal{I}_{\mathrm{dP},l}^n$ for some $l \in \mathbb{N}$. As stated in the following theorem, we can provide an explicit bound on l that depends on $\boldsymbol{A} \in \operatorname{int} \mathcal{COP}^n$. In Section 3.3.1, we use this result to provide an inner-approximation hierarchy for copositive cones over symmetric cones.

Theorem 2.21 ([23, Corollary 3.5]; also see [99, Theorem 1]). *Let $\boldsymbol{A} \in \operatorname{int} \mathcal{COP}^n$, and set*

$$
\begin{aligned}
L &:= \max_{1 \le i,j \le n} |A_{ij}|, \\
\lambda &:= \min_{\boldsymbol{x} \in \Delta_=^{n-1}} \boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} > 0.
\end{aligned}
$$

If $l \in \mathbb{N}$ satisfies $l > \frac{L}{\lambda} - 2$, then $\boldsymbol{A} \in \mathcal{I}_{\mathrm{dP},l}^n$.

The dual cone of the zeroth level of Parrilo's hierarchy, i.e., $(\mathcal{I}_{\mathrm{Par},0}^n)^*$, is known to be the doubly nonnegative cone $\mathcal{DNN}^n$ [90, Section 5.3]. Utilizing this fact, we provide some generalized doubly nonnegative cones in Chapter 4.

We second introduce the inner-approximation hierarchy proposed by Zuluaga, Vera, and Peña [125] for copositive cones over pointed semialgebraic convex cones. Specifically, we suppose that the pointed semialgebraic convex cone $\mathbb{K}$ is expressed as

$$
\{\boldsymbol{x} \in \mathbb{R}^n \mid \phi_i(\boldsymbol{x}) \ge 0 \ (i = 1, \dots, q)\}, \tag{2.24}
$$

where $\phi_i \in H^{n,m_i}$ for $i = 1, \dots, q$. Then, there exists some $\boldsymbol{a} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}$ such that the following holds (see [125, Section 6]):

$$
\mathbb{K} \subseteq \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{a}^\top \boldsymbol{x} \ge 0\}, \tag{2.25}
$$

$$
\mathbb{K} \cap \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{a}^\top \boldsymbol{x} = 0\} = \{\boldsymbol{0}\}. \tag{2.26}
$$

In fact, the set of $\boldsymbol{a}$ satisfying (2.25) and (2.26) is exactly $\operatorname{int} \mathbb{K}^*$, which is nonempty because $\mathbb{K}$ is a pointed closed convex cone; see (ii) of Theorem 2.2. From (2.25), the geometric property of $\mathbb{K}$ does not change even if we add the nonnegativity constraint $\boldsymbol{a}^\top \boldsymbol{x} \ge 0$ with $\boldsymbol{a} \in \operatorname{int} \mathbb{K}^*$ into (2.24). Thus, we may assume that there is some i such that $\phi_i(\boldsymbol{x}) = \boldsymbol{a}^\top \boldsymbol{x}$. Such a redundant constraint is necessary to construct the inner-approximation hierarchy (see [125, Assumption 1]). Note that a different choice of $\boldsymbol{a}$ may yield a different hierarchy even if the geometric property of $\mathbb{K}$ does not change. Namely, the hierarchy depends on the algebraic description of $\mathbb{K}$. Therefore, we may use the notation $(\mathbb{K}; \boldsymbol{a})$ to emphasize the choice of $\boldsymbol{a}$. Under the assumption on $\mathbb{K}$, the inner-approximation hierarchy presented by Zuluaga, Vera, and Peña for $\mathcal{COP}(\mathbb{K})$ is described as follows.

Theorem 2.22 ([125, Proposition 17])**.** *Let*

$$E^{n,m}(\mathbb{K}) \coloneqq \operatorname{conv}\left\{ \psi^2 \prod_{j=1}^{k} \phi_{i_j} \;\middle|\; \begin{array}{l} k \in \mathbb{N},\ m - \sum_{j=1}^{k} m_{i_j} \in \mathbb{N} \text{ is even}, \\ \psi \in H^{n,(m-\sum_{j=1}^{k} m_{i_j})/2}, \\ i_j \in \{1,\dots,q\}\ (j = 1,\dots,k) \end{array} \right\}. \tag{2.27}$$

Equivalently, $E^{n,m}(\mathbb{K})$ is the set of sums of m-dimensional homogeneous polynomials expressed in the form $\psi^2 \prod_{j=1}^{k} \phi_{i_j}$. Here, each ϕ_{i_j} is selected from the polynomials $\phi_1,\dots,\phi_q$, and ψ is a homogeneous polynomial. The degree of ψ is adjusted so that the entire polynomial $\psi^2 \prod_{j=1}^{k} \phi_{i_j}$ attains a degree of m. In addition, let

$$\mathcal{I}_{\mathrm{ZVP},l}(\mathbb{K}) \coloneqq \{\boldsymbol{A} \in \mathbb{S}^n \mid (\boldsymbol{a}^\top \boldsymbol{x})^l \boldsymbol{x}^\top \boldsymbol{A} \boldsymbol{x} \in E^{n,l+2}(\mathbb{K})\} \tag{2.28}$$

for each $l \in \mathbb{N}$. Then, the sequence $\{\mathcal{I}_{\mathrm{ZVP},l}(\mathbb{K})\}_l$ is an inner-approximation hierarchy for $\mathcal{COP}(\mathbb{K})$.

To prove the convergence of the sequence $\{\mathcal{I}_{\mathrm{ZVP},l}(\mathbb{K})\}_l$, they utilize *Schmüdgen's Positivstellensatz* [110, Corollary 3], which provides a sum-of-squares representation of a (not necessarily homogeneous) polynomial that is positive over a compact semialgebraic set. In the following, we call the sequence $\{\mathcal{I}_{\mathrm{ZVP},l}(\mathbb{K})\}_l$ the Zuluaga–Vera–Peña (ZVP)-type inner-approximation hierarchy.

As pointed out in [125, Remark 2], the ZVP-type inner-approximation hierarchy is a generalization of Parrilo's inner-approximation hierarchy presented in (2.23). When $\mathbb{K} = \mathbb{R}^n_+ = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{e}_i^\top \boldsymbol{x} \geq 0\ (i = 1,\dots,n)\}$ and $\boldsymbol{a}$ is given as $\mathbf{1}_n \in \operatorname{int} \mathbb{K}^*$, the hierarchy $\{\mathcal{I}_{\mathrm{ZVP},l}(\mathbb{R}^n_+; \mathbf{1}_n)\}_l$ agrees with Parrilo's hierarchy.

2.5.2 Outer-approximation hierarchies for copositive cones

We first introduce the outer-approximation hierarchy provided by Yıldırım [121] for the standard copositive cone $\mathcal{COP}^n$. A symmetric matrix $\boldsymbol{A} \in \mathbb{S}^n$ belongs to $\mathcal{COP}^n$ if and only if $\boldsymbol{x}^\top \boldsymbol{A} \boldsymbol{x}$ is nonnegative for all $\boldsymbol{x} \in \Delta^{n-1}_{=}$. To provide a necessary and computationally tractable condition for the copositivity, we consider the condition that $\boldsymbol{x}^\top \boldsymbol{A} \boldsymbol{x}$ is nonnegative for all $\boldsymbol{x}$ belonging to a finite subset of $\Delta^{n-1}_{=}$, which we can describe as a linear inequality system. In particular, as a finite subset of $\Delta^{n-1}_{=}$, the set δ^{n-1}_l defined as

$$\delta^{n-1}_l \coloneqq \bigcup_{k=0}^{l} \{\boldsymbol{x} \in \Delta^{n-1}_{=} \mid (k+2)\boldsymbol{x} \in \mathbb{N}^n\} \tag{2.29}$$

for each $l \in \mathbb{N}$ is exploited in [121]. If we define $\mathcal{O}^n_{\text{Yıl},l}$ as

$$\mathcal{O}^n_{\text{Yıl},l} \coloneqq \bigcap_{\boldsymbol{x} \in \delta^{n-1}_l} \{\boldsymbol{A} \in \mathbb{S}^n \mid \boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} \geq 0\}, \tag{2.30}$$

then the sequence $\{\mathcal{O}^n_{\text{Yıl},l}\}_l$ is an outer-approximation hierarchy for $\mathcal{COP}^n$ because the sequence $\{\delta^{n-1}_l\}_l$ is non-decreasing and its union is dense in $\Delta^{n-1}_{=}$. Because $|\delta^{n-1}_l|$ is bounded by $n^2(\frac{n^{l+1}-1}{n-1})$ [121, Equation (10)], the number of the nonnegativity constraints defining $\{\mathcal{O}^n_{\text{Yıl},l}\}_l$ is polynomial in n.

Second, we introduce the outer-approximation hierarchy given by Lasserre [65] for copositive cones over closed convex cones. For simplicity, we assume the underlying closed convex cone $\mathbb{K} \subseteq \mathbb{R}^n$ is pointed. Since the symmetric cones we focus on in this thesis are pointed, this assumption is not restrictive. See [65, Section 2.4] for the case of general closed convex cones. Under the assumption that $\mathbb{K}$ is pointed, as mentioned in Section 2.5.1, there exists $\boldsymbol{a} \in \operatorname{int} \mathbb{K}^*$. For such $\boldsymbol{a}$, let $\Delta_{\boldsymbol{a}}(\mathbb{K}) \coloneqq \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{a}^\top \boldsymbol{x} \leq 1\}$, which is compact. Note that a symmetric matrix $\boldsymbol{A} \in \mathbb{S}^n$ belongs to $\mathcal{COP}(\mathbb{K})$ if and only if the polynomial function $f_{\boldsymbol{A}}(\boldsymbol{x}) \coloneqq \boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x}$ is nonnegative over the compact set $\Delta_{\boldsymbol{a}}(\mathbb{K})$. Exploiting the characterization of the nonnegativity of a polynomial function over a compact set shown in [64, Theorem 3.2], we can construct an outer-approximation hierarchy for $\mathcal{COP}(\mathbb{K})$, which is described as follows.

Theorem 2.23 ([65, Section 2.4]). *Let ν be an arbitrary finite Borel measure whose support is $\Delta_{\boldsymbol{a}}(\mathbb{K})$ and let $\boldsymbol{y} = (y_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha} \in \mathbb{N}^n}$ be the sequence of moments of ν defined as*

$$y_{\boldsymbol{\alpha}} \coloneqq \int_{\mathbb{R}^n} \boldsymbol{x}^{\boldsymbol{\alpha}} d\nu$$

for each $\boldsymbol{\alpha} \in \mathbb{N}^n$. For each $l \in \mathbb{N}$, we define

$$\mathcal{O}_{\text{Las},l}(\mathbb{K}) \coloneqq \left\{ \boldsymbol{A} \in \mathbb{S}^n \;\middle|\; \boldsymbol{M}_l(f_{\boldsymbol{A}}\boldsymbol{y}) \in \mathbb{S}^{\mathbb{I}^n_{\leq l}}_+ \right\}, \tag{2.31}$$

where $\mathbb{I}^n_{\leq l} \coloneqq \{\boldsymbol{\alpha} \in \mathbb{N}^n \mid |\boldsymbol{\alpha}| \leq l\}$ and $\boldsymbol{M}_l(f_{\boldsymbol{A}}\boldsymbol{y})$ is the symmetric matrix with the $(\boldsymbol{\alpha}, \boldsymbol{\beta})$th element $\sum_{i,j=1}^n A_{ij} y_{\boldsymbol{\alpha}+\boldsymbol{\beta}+\boldsymbol{e}_i+\boldsymbol{e}_j}$ for each $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{I}^n_{\leq l}$. Then, the sequence $\{\mathcal{O}_{\text{Las},l}(\mathbb{K})\}_l$ is an outer-approximation hierarchy for $\mathcal{COP}(\mathbb{K})$.

In the following, we call the sequence $\{\mathcal{O}_{\text{Las},l}(\mathbb{K})\}_l$ the Lasserre-type outer-approximation hierarchy. To implement this hierarchy, we need to calculate the moments $\boldsymbol{y}$ involving integration. The following case of $\mathbb{K}$ being a direct product of a nonnegative orthant and a second-order cone is an example in which the moments can be theoretically obtained.

Part II

Approximations

Chapter 3

Approximation Hierarchies for Copositive Cones over Symmetric Cones

3.1 Introduction

In this chapter, we focus on approximation hierarchies for copositive cones over symmetric cones.

The first part of this chapter is the proposal of approximation hierarchies for copositive cones over general and specific symmetric cones. In Section 3.2, we provide an inner-approximation hierarchy described by a sum-of-squares constraint for copositive cones over general symmetric cones and discuss its dual. In Section 3.3, we provide inner- and outer-approximation hierarchies described by positive semidefinite but not by sum-of-squares constraints for copositive matrix cones over a direct product of a nonnegative orthant and a single second-order cone. We also discuss their concise expressions.

The second part of this chapter is the theoretical and numerical comparisons of the proposed approximation hierarchies with the ZVP- and Lasserre-type approximation hierarchies introduced in Section 2.5. In Section 3.4, we compare the number and size of positive semidefinite constraints defining the proposed and existing approximation hierarchies. In Section 3.5, we compare the approximation hierarchies numerically by solving optimization problems obtained by approximating copositive cones. Moreover, we also investigate the effect of the concise expressions mentioned in Section 3.3. Combining these approximation hierarchies, we can evaluate the optimal values of copositive programming problems more accurately and efficiently.

Finally, Section 3.6 provides concluding remarks.

3.2 Sum-of-squares-based Inner-approximation Hierarchy

Let $(\mathbb{E}, \circ, \bullet)$ be a Euclidean Jordan algebra of dimension n. In this section, we aim to provide an inner-approximation hierarchy described by a sum-of-squares constraint for the copositive cone $\mathcal{COP}^{n,m}(\mathbb{E}_+)$. In Section 3.2.1, we first provide an inner-approximation hierarchy for the cones of homogeneous polynomials that are nonnegative over symmetric cones in $\mathbb{R}^n$. Leveraging the outcomes from Section 3.2.1, we then construct the desired approximation hierarchy in Section 3.2.2. Finally, in Section 3.2.3, we delve into its dual. Subsequently, we fix an orthonormal basis $\{v_1, \dots, v_n\}$ for the given Euclidean Jordan algebra $(\mathbb{E}, \circ, \bullet)$ and let $\phi\colon \mathbb{E} \to \mathbb{R}^n$ be the associated isometry.

3.2.1 Approximation hierarchy for homogeneous polynomials

If we define $\boldsymbol{x} \diamondsuit \boldsymbol{y} := \phi(\phi^{-1}(\boldsymbol{x}) \circ \phi^{-1}(\boldsymbol{y}))$ and $\boldsymbol{x} \blacklozenge \boldsymbol{y} := \boldsymbol{x}^\top \boldsymbol{y}$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, then $(\mathbb{R}^n, \diamondsuit, \blacklozenge)$ is also a Euclidean Jordan algebra. Hereafter, to emphasize that $\mathbb{R}^n$ is a Euclidean Jordan algebra and $(\mathbb{R}^n, \diamondsuit, \blacklozenge)$ depends on the choice of ϕ, we write the Euclidean Jordan algebra $\mathbb{R}^n$ as $\phi(\mathbb{E})$. We note that the symmetric cone $\phi(\mathbb{E})_+$ associated with the Euclidean Jordan algebra $(\phi(\mathbb{E}), \diamondsuit, \blacklozenge)$ satisfies

$$\phi(\mathbb{E})_+ = \{\boldsymbol{x} \diamondsuit \boldsymbol{x} \mid \boldsymbol{x} \in \phi(\mathbb{E})\} = \{\phi(x) \diamondsuit \phi(x) \mid x \in \mathbb{E}\} = \phi(\mathbb{E}_+). \tag{3.1}$$

Here, we derive an inner-approximation hierarchy for the cone of homogeneous polynomials that are nonnegative over the symmetric cone $\phi(\mathbb{E}_+)$. Let

$$\widetilde{\mathcal{COP}}^{n,m}(\phi(\mathbb{E}_+)) := \{\theta \in H^{n,m} \mid \theta(\boldsymbol{x}) \geq 0 \text{ for all } \boldsymbol{x} \in \phi(\mathbb{E}_+)\}$$

be the cone of homogeneous polynomials in n variables of degree m that are nonnegative over the symmetric cone $\phi(\mathbb{E}_+)$. For each $l \in \mathbb{N}$, we define

$$\tilde{\mathcal{I}}^{n,m}_{\mathrm{NN},l}(\phi(\mathbb{E}_+)) := \{\theta \in H^{n,m} \mid (\boldsymbol{x}^\top \boldsymbol{x})^l \theta(\boldsymbol{x} \diamondsuit \boldsymbol{x}) \in \Sigma^{n,2(l+m)}\}.$$

Proposition 3.1. *Each $\tilde{\mathcal{I}}^{n,m}_{\mathrm{NN},l}(\phi(\mathbb{E}_+))$ is a closed convex cone, and the sequence $\{\tilde{\mathcal{I}}^{n,m}_{\mathrm{NN},l}(\phi(\mathbb{E}_+))\}_l$ is an inner-approximation hierarchy for $\widetilde{\mathcal{COP}}^{n,m}(\phi(\mathbb{E}_+))$.*

To prove Proposition 3.1, we utilize *Reznick's Positivstellensatz*, which is described as follows.

Theorem 3.2 ([105, Theorem 3.12])**.** *Let $\theta \in H^{n,2m}$. If $\theta(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}$, then there exists $l_0 \in \mathbb{N}$ such that $(\boldsymbol{x}^\top \boldsymbol{x})^{l_0} \theta(\boldsymbol{x}) \in \Sigma^{n,2(l_0+m)}$.*

Proof of Proposition 3.1. Note that $\theta(\boldsymbol{x} \diamondsuit \boldsymbol{x}) \in H^{n,2m}$ for each $\theta \in H^{n,m}$ because the product $\diamondsuit$ is bilinear. Thus, the set $\tilde{\mathcal{I}}^{n,m}_{\mathrm{NN},l}(\phi(\mathbb{E}_+))$ is well-defined for each $l \in \mathbb{N}$. $\tilde{\mathcal{I}}^{n,m}_{\mathrm{NN},l}(\phi(\mathbb{E}_+))$ can be shown to be a closed convex cone from the counterpart properties of $\Sigma^{n,2(l+m)}$ [104, Proposition 3.6]. In the following, we prove that $\tilde{\mathcal{I}}^{n,m}_{\mathrm{NN},l}(\phi(\mathbb{E}_+))$ is an inner-approximation hierarchy for $\widetilde{\mathcal{COP}}^{n,m}(\phi(\mathbb{E}_+))$.

To prove the non-decreasingness, let $\theta \in \tilde{\mathcal{I}}^{n,m}_{\mathrm{NN},l}(\phi(\mathbb{E}_+))$. Then, there exist $\theta_1, \dots, \theta_N \in H^{n,l+m}$ such that

$$(\boldsymbol{x}^\top \boldsymbol{x})^l \theta(\boldsymbol{x} \diamondsuit \boldsymbol{x}) = \sum_{i=1}^N \theta_i^2(\boldsymbol{x}). \tag{3.2}$$

Using this, we obtain

$$(\boldsymbol{x}^\top \boldsymbol{x})^{l+1} \theta(\boldsymbol{x} \diamondsuit \boldsymbol{x}) = \sum_{i=1}^N \sum_{j=1}^n (x_j \theta_i(\boldsymbol{x}))^2 \in \Sigma^{n,2(l+m+1)},$$

which means that $\theta \in \tilde{\mathcal{I}}^{n,m}_{\mathrm{NN},l+1}(\phi(\mathbb{E}_+))$.

To prove the inclusion $\tilde{\mathcal{I}}^{n,m}_{\mathrm{NN},l}(\phi(\mathbb{E}_+)) \subseteq \widetilde{\mathcal{COP}}^{n,m}(\phi(\mathbb{E}_+))$, we assume that there exists $\theta \in \tilde{\mathcal{I}}^{n,m}_{\mathrm{NN},l}(\phi(\mathbb{E}_+))$ such that $\theta \notin \widetilde{\mathcal{COP}}^{n,m}(\phi(\mathbb{E}_+))$. Then, from (3.1), there exists $\tilde{\boldsymbol{x}} \in \phi(\mathbb{E})$ such that $\theta(\tilde{\boldsymbol{x}} \lozenge \tilde{\boldsymbol{x}}) < 0$. As $\tilde{\boldsymbol{x}} \neq \mathbf{0}$ and $\tilde{\boldsymbol{x}}^\top \tilde{\boldsymbol{x}} > 0$, we have $(\tilde{\boldsymbol{x}}^\top \tilde{\boldsymbol{x}})^l \theta(\tilde{\boldsymbol{x}} \lozenge \tilde{\boldsymbol{x}}) < 0$. However, (3.2) implies that $(\tilde{\boldsymbol{x}}^\top \tilde{\boldsymbol{x}})^l \theta(\tilde{\boldsymbol{x}} \lozenge \tilde{\boldsymbol{x}})$ must take a nonnegative value, which is a contradiction. Therefore, we obtain $\theta \in \widetilde{\mathcal{COP}}^{n,m}(\phi(\mathbb{E}_+))$.

To prove the convergence, let $\theta \in \operatorname{int} \widetilde{\mathcal{COP}}^{n,m}(\phi(\mathbb{E}_+))$. Then, as $\phi(\mathbb{E}_+)$ is a closed cone, it follows that $\theta(\boldsymbol{y}) > 0$ for all $\boldsymbol{y} \in \phi(\mathbb{E}_+) \setminus \{\mathbf{0}\}$, i.e., $\theta(\boldsymbol{x} \lozenge \boldsymbol{x}) > 0$ for all $\boldsymbol{x} \in \phi(\mathbb{E}) \setminus \{\mathbf{0}\} = \mathbb{R}^n \setminus \{\mathbf{0}\}$ (see [125, Observation 1], for example). Thus, by Theorem 3.2, there exists $l_0 \in \mathbb{N}$ such that $(\boldsymbol{x}^\top \boldsymbol{x})^{l_0} \theta(\boldsymbol{x} \lozenge \boldsymbol{x}) \in \Sigma^{n,2(l_0+m)}$, which means that $\theta \in \tilde{\mathcal{I}}^{n,m}_{\mathrm{NN},l_0}(\phi(\mathbb{E}_+)) \subseteq \bigcup_{l\in\mathbb{N}} \tilde{\mathcal{I}}^{n,m}_{\mathrm{NN},l}(\phi(\mathbb{E}_+))$. $\square$

Note that the set $\tilde{\mathcal{I}}^{n,m}_{\mathrm{NN},l}(\phi(\mathbb{E}_+))$ is defined by a sum-of-squares constraint. This constraint can be written as a positive semidefinite constraint of size $|\mathbb{I}^n_{=l+m}| = \binom{n+l+m-1}{n-1}$, which is polynomial in n and m for each fixed $l \in \mathbb{N}$.

3.2.2 Approximation hierarchy for symmetric tensors

In this subsection, we translate the result of Proposition 3.1 to the case of symmetric tensors. To realize this, we start by generalizing the notion of polynomials.

A homogeneous polynomial function of degree m on $\mathbb{E}$ is the mapping

$$\mathbb{E} \to \mathbb{R};\ x \mapsto \langle \mathcal{A}, x^{\otimes m} \rangle$$

for some $\mathcal{A} \in \mathcal{S}^{n,m}(\mathbb{E})$.[1] $H^{n,m}(\mathbb{E})$ denotes the space of homogeneous polynomial functions of degree m on $\mathbb{E}$, i.e.,

$$H^{n,m}(\mathbb{E}) = \{\langle \mathcal{A}, x^{\otimes m} \rangle \mid \mathcal{A} \in \mathcal{S}^{n,m}(\mathbb{E})\}.$$

As $H^{n,m} = \{\langle \boldsymbol{\mathcal{A}}, \boldsymbol{x}^{\otimes m} \rangle \mid \boldsymbol{\mathcal{A}} \in \mathcal{S}^{n,m}\}$, $H^{n,m}(\mathbb{R}^n)$ agrees with $H^{n,m}$.

As the definition of $\Sigma^{n,2m}$, $\Sigma^{n,2m}(\mathbb{E})$ denotes the cone of sums of squares of homogeneous polynomial functions of degree m on $\mathbb{E}$. To represent the set $\Sigma^{n,2m}(\mathbb{E})$ more explicitly, we prove the following lemma.

Lemma 3.3. *$\langle \mathcal{A}, x^{\otimes m} \rangle^2 = \langle \mathscr{S}(\mathcal{A} \otimes \mathcal{A}), x^{\otimes 2m} \rangle$ for any $\mathcal{A} \in \mathcal{S}^{n,m}(\mathbb{E})$.*

Proof. Using the basis $\{v_1, \dots, v_n\}$, we write $\mathcal{A} \in \mathcal{S}^{n,m}(\mathbb{E})$ in the form (2.2). Given that

$$\langle \mathcal{A}, x^{\otimes m} \rangle = \sum_{i_1,\dots,i_m=1}^{n} \mathcal{A}_{i_1\cdots i_m} \prod_{k=1}^{m} x \bullet v_{i_k},$$

we have

$$\langle \mathcal{A}, x^{\otimes m} \rangle^2 = \sum_{i_1,\dots,i_{2m}=1}^{n} \mathcal{A}_{i_1\cdots i_m} \mathcal{A}_{i_{m+1}\cdots i_{2m}} \prod_{k=1}^{2m} x \bullet v_{i_k}. \tag{3.3}$$

[1] In our original paper [84], we define homogeneous polynomial functions on a general inner product space rather than a Euclidean Jordan algebra. They can be defined by analogy with those on a Euclidean Jordan algebra.

Moreover, as $\mathcal{A}\otimes\mathcal{A} = \sum_{i_1,\dots,i_{2m}=1}^{n} \mathcal{A}_{i_1\cdots i_m}\mathcal{A}_{i_{m+1}\cdots i_{2m}}\tilde{v}_{i_1\cdots i_{2m}}$, $\langle \mathcal{A}\otimes\mathcal{A}, x^{\otimes 2m}\rangle$ agrees with (3.3). Therefore, by applying Lemma 2.5, we obtain the desired result. □

Using Lemma 3.3, we can express $\Sigma^{n,2m}(\mathbb{E})$ as

$$\Sigma^{n,2m}(\mathbb{E}) = \mathrm{conv}\{\langle \mathscr{S}(\mathcal{A}\otimes\mathcal{A}), x^{\otimes 2m}\rangle \mid \mathcal{A}\in\mathcal{S}^{n,m}(\mathbb{E})\}. \tag{3.4}$$

Through the isomorphism

$$H^{n,2m}(\mathbb{E}) \to \mathcal{S}^{n,2m}(\mathbb{E});\ \langle \mathcal{A}, x^{\otimes 2m}\rangle \mapsto \mathcal{A},$$

the set $\Sigma^{n,2m}(\mathbb{E})$ is mapped onto the set

$$\mathrm{SOS}^{n,2m}(\mathbb{E}) \coloneqq \mathrm{conv}\{\mathscr{S}(\mathcal{A}\otimes\mathcal{A}) \mid \mathcal{A}\in\mathcal{S}^{n,m}(\mathbb{E})\}.$$

Here, we provide an inner-approximation hierarchy for $\mathcal{COP}^{n,m}(\mathbb{E}_+)$. Given that $(x\bullet x)^l\langle \mathcal{A}, (x\circ x)^{\otimes m}\rangle \in H^{n,2(l+m)}(\mathbb{E})$, for each $l\in\mathbb{N}$ and each $\mathcal{A}\in\mathcal{S}^{n,m}(\mathbb{E})$, there exists a unique $\mathcal{A}^{(l)}\in\mathcal{S}^{n,2(l+m)}(\mathbb{E})$ such that

$$(x\bullet x)^l\langle \mathcal{A}, (x\circ x)^{\otimes m}\rangle = \langle \mathcal{A}^{(l)}, x^{\otimes 2(l+m)}\rangle.$$

Using the symmetric tensor $\mathcal{A}^{(l)}$, we define

$$\mathcal{I}_{\mathrm{NN},l}^{n,m}(\mathbb{E}_+) \coloneqq \{\mathcal{A}\in\mathcal{S}^{n,m}(\mathbb{E}) \mid \mathcal{A}^{(l)}\in\mathrm{SOS}^{n,2(l+m)}(\mathbb{E})\}.$$

Theorem 3.4. *Each $\mathcal{I}_{\mathrm{NN},l}^{n,m}(\mathbb{E}_+)$ is a closed convex cone, and the sequence $\{\mathcal{I}_{\mathrm{NN},l}^{n,m}(\mathbb{E}_+)\}_l$ is an inner-approximation hierarchy for $\mathcal{COP}^{n,m}(\mathbb{E}_+)$.*

Proof. The mapping

$$\psi\colon \mathcal{S}^{n,m}(\mathbb{E}) \to H^{n,m};\ \mathcal{A}\mapsto \langle \mathcal{A}, \phi^{-1}(\boldsymbol{x})^{\otimes m}\rangle$$

is a linear isomorphism. By Proposition 3.1 and the linear isomorphism of ψ, it suffices to show that $\psi(\mathcal{COP}^{n,m}(\mathbb{E}_+)) = \widetilde{\mathcal{COP}}^{n,m}(\phi(\mathbb{E}_+))$ and $\psi(\mathcal{I}_{\mathrm{NN},l}^{n,m}(\mathbb{E}_+)) = \tilde{\mathcal{I}}_{\mathrm{NN},l}^{n,m}(\phi(\mathbb{E}_+))$ to prove the theorem.

We prove the first equation. Let $\theta = \psi(\mathcal{A}) \in \psi(\mathcal{COP}^{n,m}(\mathbb{E}_+))$ and $\mathcal{A}\in\mathcal{COP}^{n,m}(\mathbb{E}_+)$. Then, for any $x\in\mathbb{E}$, we have

$$\begin{aligned}\theta(\phi(x)\lozenge\phi(x)) &= \langle \mathcal{A}, \phi^{-1}(\phi(x)\lozenge\phi(x))^{\otimes m}\rangle\\ &= \langle \mathcal{A}, (x\circ x)^{\otimes m}\rangle\\ &\ge 0,\end{aligned}$$

using $\phi^{-1}(\phi(x)\lozenge\phi(x)) = x\circ x$ and $\mathcal{A}\in\mathcal{COP}^{n,m}(\mathbb{E}_+)$. Therefore, we have $\theta\in\widetilde{\mathcal{COP}}^{n,m}(\phi(\mathbb{E}_+))$. Conversely, let $\theta\in\widetilde{\mathcal{COP}}^{n,m}(\phi(\mathbb{E}_+))$ and $\mathcal{A}\coloneqq\psi^{-1}(\theta)\in\mathcal{S}^{n,m}(\mathbb{E})$. Then, for any $x\in\mathbb{E}$, in the same manner as the discussion above, we obtain

$$\langle \mathcal{A}, (x\circ x)^{\otimes m}\rangle = \theta(\phi(x)\lozenge\phi(x)) \ge 0,$$

using $\phi(x) \diamondsuit \phi(x) \in \phi(\mathbb{E}_+)$. Therefore, $\mathcal{A} \in \mathcal{COP}^{n,m}(\mathbb{E}_+)$, and thus, $\theta = \psi(\mathcal{A}) \in \psi(\mathcal{COP}^{n,m}(\mathbb{E}_+))$.

We prove the second equation. Let $\theta = \psi(\mathcal{A}) \in \psi(\mathcal{I}^{n,m}_{\mathrm{NN},l}(\mathbb{E}_+))$ and $\mathcal{A} \in \mathcal{I}^{n,m}_{\mathrm{NN},l}(\mathbb{E}_+)$. As $\mathcal{A} \in \mathcal{I}^{n,m}_{\mathrm{NN},l}(\mathbb{E}_+)$, there exist $\mathcal{A}_1, \ldots, \mathcal{A}_N \in \mathcal{S}^{n,l+m}(\mathbb{E})$ such that

$$(x \bullet x)^l \langle \mathcal{A}, (x \circ x)^{\otimes m} \rangle = \sum_{i=1}^{N} \langle \mathcal{A}_i, x^{\otimes (l+m)} \rangle^2.$$

As $\langle \mathcal{A}_i, \phi^{-1}(\boldsymbol{x})^{\otimes (l+m)} \rangle \in H^{n,l+m}$ for each i, it follows that

$$\begin{aligned}(\boldsymbol{x}^\top \boldsymbol{x})^l \theta(\boldsymbol{x} \diamondsuit \boldsymbol{x}) &= (\phi^{-1}(\boldsymbol{x}) \bullet \phi^{-1}(\boldsymbol{x}))^l \langle \mathcal{A}, \phi^{-1}(\boldsymbol{x} \diamondsuit \boldsymbol{x})^{\otimes m} \rangle \\ &= (\phi^{-1}(\boldsymbol{x}) \bullet \phi^{-1}(\boldsymbol{x}))^l \langle \mathcal{A}, (\phi^{-1}(\boldsymbol{x}) \circ \phi^{-1}(\boldsymbol{x}))^{\otimes m} \rangle \\ &= \sum_{i=1}^{N} \langle \mathcal{A}_i, \phi^{-1}(\boldsymbol{x})^{\otimes (l+m)} \rangle^2 \in \Sigma^{n,2(l+m)}.\end{aligned}$$

Therefore, we have $\theta \in \tilde{\mathcal{I}}^{n,m}_{\mathrm{NN},l}(\phi(\mathbb{E}_+))$. Conversely, let $\theta \in \tilde{\mathcal{I}}^{n,m}_{\mathrm{NN},l}(\phi(\mathbb{E}_+))$. Then, there exist $\mathcal{A}_1, \ldots, \mathcal{A}_N \in \mathcal{S}^{n,l+m}(\mathbb{E})$ such that

$$(\boldsymbol{x}^\top \boldsymbol{x})^l \theta(\boldsymbol{x} \diamondsuit \boldsymbol{x}) = \sum_{i=1}^{N} \langle \mathcal{A}_i, \phi^{-1}(\boldsymbol{x})^{\otimes (l+m)} \rangle^2.$$

Let $\mathcal{A} := \psi^{-1}(\theta) \in \mathcal{S}^{n,m}(\mathbb{E})$. Then, we have

$$\begin{aligned}(x \bullet x)^l \langle \mathcal{A}, (x \circ x)^{\otimes m} \rangle &= (\phi(x)^\top \phi(x))^l \theta(\phi(x) \diamondsuit \phi(x)) \\ &= \sum_{i=1}^{N} \langle \mathcal{A}_i, x^{\otimes (l+m)} \rangle^2 \in \Sigma^{n,2(l+m)}(\mathbb{E}).\end{aligned}$$

□

Note that $\mathcal{I}^{n,m}_{\mathrm{NN},l}(\mathbb{E}_+)$ does not depend on the choice of the orthonormal basis $\{v_1, \ldots, v_n\}$. We call the sequence $\{\mathcal{I}^{n,m}_{\mathrm{NN},l}(\mathbb{E}_+)\}_l$ the Nishijima–Nakata (NN)-type inner-approximation hierarchy.

In the case where the symmetric cone $\mathbb{E}_+$ is the nonnegative orthant $\mathbb{R}^n_+$, this hierarchy is essentially identical to the sum-of-squares-based one provided by Iqbal and Ahmed [58, Equation (19)]. They are generalizations of Parrilo's hierarchy (2.23) to tensors.

Remark 3.5. The NN-type inner-approximation hierarchy can be extended to the *sum-of-squares cones* proposed by Papp and Alizadeh [88]. Let $\mathbb{A}$ and $\mathbb{B}$ be real inner product spaces of dimensions k and n, respectively, and let $\diamond\colon \mathbb{A} \times \mathbb{A} \to \mathbb{B}$ be a bilinear mapping. The sum-of-squares cone is then defined as $\Sigma_\diamond := \mathrm{conv}\{x_i \diamond x_i \mid x_i \in \mathbb{A}\}$. Note that each element in $\Sigma_\diamond$ can be written as the sum of at most n elements $x_1 \diamond x_1, \ldots, x_n \diamond x_n$ such that $x_1, \ldots, x_n \in \mathbb{A}$, by Theorem 2.1. If $(\mathbb{A}, \mathbb{B}, \diamond)$ is formally real, or equivalently, if $\Sigma_\diamond$ is a pointed closed convex cone with a nonempty interior [88, Theorem 3.3], then

$$\mathcal{I}_l^{n,m}(\Sigma_\diamond) := \left\{ \mathcal{A} \in \mathcal{S}^{n,m}(\mathbb{B}) \;\middle|\; \begin{array}{l} \left(\sum_{i=1}^n x_i \bullet_{\mathbb{A}} x_i\right)^l \left\langle \mathcal{A}, \left(\sum_{i=1}^n x_i \diamond x_i\right)^{\otimes m} \right\rangle \\ \qquad \in \Sigma^{kn,2(l+m)}(\mathbb{A}^n) \end{array} \right\}$$

is a closed convex cone for each $l \in \mathbb{N}$ and $\{\mathcal{I}_l^{n,m}(\Sigma_\diamond)\}_l$ is an inner-approximation hierarchy for $\mathcal{COP}^{n,m}(\Sigma_\diamond)$, where $\bullet_{\mathbb{A}}$ denotes the inner product on $\mathbb{A}$ and $\Sigma^{kn,2(l+m)}(\mathbb{A}^n)$ is defined in the same manner as (3.4).

For the subsequent discussions, we may consider the matrix case, i.e., the case where $m = 2$ and $\mathbb{E} = \mathbb{R}^n$. In this case, the lth level of the NN-type inner-approximation hierarchy for $\mathcal{COP}(\mathbb{E}_+)$ is described as

$$\mathcal{I}_{\mathrm{NN},l}(\mathbb{E}_+) := \{\boldsymbol{A} \in \mathbb{S}^n \mid (\boldsymbol{x}^\top \boldsymbol{x})^l (\boldsymbol{x} \circ \boldsymbol{x})^\top \boldsymbol{A} (\boldsymbol{x} \circ \boldsymbol{x}) \in \Sigma^{n,2(l+2)}\}. \tag{3.5}$$

3.2.3 Dual of $\mathcal{I}_{\mathrm{NN},l}^{n,m}(\mathbb{E}_+)$

Next, we discuss the dual cone of $\mathcal{I}_{\mathrm{NN},l}^{n,m}(\mathbb{E}_+)$. By considering its dual, we can provide an outer-approximation hierarchy for the completely positive cone $\mathcal{CP}^{n,m}(\mathbb{E}_+)$. Although the closure operator is generally required to describe the dual cone, we succeeded in removing it for the case in which the symmetric cone $\mathbb{E}_+$ is the nonnegative orthant $\mathbb{R}_+^n$. This implies that the outer-approximation hierarchy for $\mathcal{CP}^{n,m}$ has an explicit expression described by a positive semidefinite constraint and without the closure, as shown later.

Let $\mathcal{C}^{(l)}\colon \mathcal{S}^{n,2(l+m)}(\mathbb{E}) \to \mathcal{S}^{n,m}(\mathbb{E})$ be the adjoint of the linear mapping $\mathcal{A} \mapsto \mathcal{A}^{(l)}$, so that

$$\langle \mathcal{A}, \mathcal{C}^{(l)}(\mathcal{X})\rangle = \langle \mathcal{A}^{(l)}, \mathcal{X}\rangle \text{ for all } \mathcal{A} \in \mathcal{S}^{n,m}(\mathbb{E}) \text{ and } \mathcal{X} \in \mathcal{S}^{n,2(l+m)}(\mathbb{E}). \tag{3.6}$$

Using the linear mapping $\mathcal{C}^{(l)}$, we define $\mathcal{C}_l^{n,m}(\mathbb{E}_+) := \{\mathcal{C}^{(l)}(\mathcal{X}) \mid \mathcal{X} \in \mathrm{SOS}^{n,2(l+m)}(\mathbb{E})^*\}$.

Proposition 3.6. *It follows that $\mathcal{I}_{\mathrm{NN},l}^{n,m}(\mathbb{E}_+) = \mathcal{C}_l^{n,m}(\mathbb{E}_+)^*$, and thus, $\mathcal{I}_{\mathrm{NN},l}^{n,m}(\mathbb{E}_+)^* = \operatorname{cl} \mathcal{C}_l^{n,m}(\mathbb{E}_+)$.*

Proof. Let $\mathcal{A} \in \mathcal{I}_{\mathrm{NN},l}^{n,m}(\mathbb{E}_+)$. Then, $\mathcal{A}^{(l)} \in \mathrm{SOS}^{n,2(l+m)}(\mathbb{E})$ by definition. For any $\mathcal{X} \in \mathrm{SOS}^{n,2(l+m)}(\mathbb{E})^*$, it follows from (3.6) that $\langle \mathcal{A}, \mathcal{C}^{(l)}(\mathcal{X})\rangle = \langle \mathcal{A}^{(l)}, \mathcal{X}\rangle \geq 0$, which means that $\mathcal{A} \in \mathcal{C}_l^{n,m}(\mathbb{E}_+)^*$.

Conversely, let $\mathcal{A} \in \mathcal{C}_l^{n,m}(\mathbb{E}_+)^*$. Then, for any $\mathcal{X} \in \mathrm{SOS}^{n,2(l+m)}(\mathbb{E})^*$, because $\mathcal{C}^{(l)}(\mathcal{X}) \in \mathcal{C}_l^{n,m}(\mathbb{E}_+)$, it follows that $\langle \mathcal{A}^{(l)}, \mathcal{X}\rangle = \langle \mathcal{A}, \mathcal{C}^{(l)}(\mathcal{X})\rangle \geq 0$. In addition, the convex cone $\mathrm{SOS}^{n,2(l+m)}(\mathbb{E})$ is closed [18, Lemma 2.2]. Therefore, by (iii) of Theorem 2.2, $\mathcal{A}^{(l)} \in (\mathrm{SOS}^{n,2(l+m)}(\mathbb{E})^*)^* = \mathrm{SOS}^{n,2(l+m)}(\mathbb{E})$, which means that $\mathcal{A} \in \mathcal{I}_{\mathrm{NN},l}^{n,m}(\mathbb{E}_+)$.

Given that $\mathcal{C}_l^{n,m}(\mathbb{E}_+)$ is a convex cone, by taking the dual and using (iii) of Theorem 2.2, we have $\mathcal{I}_{\mathrm{NN},l}^{n,m}(\mathbb{E}_+)^* = \operatorname{cl} \mathcal{C}_l^{n,m}(\mathbb{E}_+)$. ◻

We could not prove that $\mathcal{C}_l^{n,m}(\mathbb{E}_+)$ itself is closed, and the closure operator is required to describe the dual of $\mathcal{I}_{\mathrm{NN},l}^{n,m}(\mathbb{E}_+)$. Therefore, whether $\mathcal{I}_{\mathrm{NN},l}^{n,m}(\mathbb{E}_+)$ and $\mathcal{C}_l^{n,m}(\mathbb{E}_+)$

itself are dual to each other is an open problem for a general symmetric cone $\mathbb{E}_+$. The difficulty originates from a lack of understanding of the dual cone of $\mathrm{SOS}^{n,2(l+m)}(\mathbb{E})$.

As a special case, we consider the symmetric cone $\mathbb{E}_+$ to be the nonnegative orthant $\mathbb{R}^n_+$. In this case, we demonstrated in Proposition 3.10 that the dual of $\mathrm{SOS}^{n,2m}(\mathbb{R}^n)$ (written as $\mathrm{SOS}^{n,2m}$ hereafter) in the space $\mathcal{S}^{n,2m}$ can be explicitly described by a positive semidefinite constraint. This answers a question posed by Chen, Li, and Qi [18, Section 2.1] of whether the membership problem of the dual of $\mathrm{SOS}^{n,2m}(\mathbb{R}^n)$ can be solved in polynomial time or not. Furthermore, the closedness of $\mathcal{C}^{n,m}_l(\mathbb{R}^n_+)$ (written as $\mathcal{C}^{n,m}_l$ hereafter) is shown by exploiting this result.

For $\boldsymbol{\alpha} \in \mathbb{I}^n_{=m}$, let

$$[\boldsymbol{\alpha}] := (\underbrace{1,\dots,1}_{\alpha_1 \text{ factors}},\dots,\underbrace{n,\dots,n}_{\alpha_n \text{ factors}}).$$

Moreover, for $\boldsymbol{\mathcal{X}} \in \mathcal{S}^{n,2m}$, let $\boldsymbol{M}^{n,m}(\boldsymbol{\mathcal{X}})$ be a matrix in $\mathbb{S}^{\mathbb{I}^n_{=m}}$ with the $(\boldsymbol{\alpha},\boldsymbol{\beta})$th element $\mathcal{X}_{[\boldsymbol{\alpha}+\boldsymbol{\beta}]}$ for $\boldsymbol{\alpha},\boldsymbol{\beta} \in \mathbb{I}^n_{=m}$. Then, we define

$$\mathcal{M}^{n,2m} := \{\boldsymbol{\mathcal{X}} \in \mathcal{S}^{n,2m} \mid \boldsymbol{M}^{n,m}(\boldsymbol{\mathcal{X}}) \in \mathbb{S}^{\mathbb{I}^n_{=m}}_+\}.$$

We aim to demonstrate that the dual of $\mathrm{SOS}^{n,2m}$ agrees with $\mathcal{M}^{n,2m}$.

The following lemma provides an orthonormal basis for the symmetric tensor space $\mathcal{S}^{n,m}$ and the representation of the elements of $\mathcal{S}^{n,m}$ with the orthonormal basis.

Lemma 3.7. *We now define $\boldsymbol{\mathcal{F}}_{\boldsymbol{\alpha}} := \sqrt{m!/\boldsymbol{\alpha}!}\mathscr{S}\tilde{\boldsymbol{e}}_{[\boldsymbol{\alpha}]}$ for each $\boldsymbol{\alpha} \in \mathbb{I}^n_{=m}$. Then, $\{\boldsymbol{\mathcal{F}}_{\boldsymbol{\alpha}} \mid \boldsymbol{\alpha} \in \mathbb{I}^n_{=m}\}$ is an orthonormal basis for $\mathcal{S}^{n,m}$. In addition, using the orthonormal basis, we can represent each $\boldsymbol{\mathcal{A}} \in \mathcal{S}^{n,m}$ as $\boldsymbol{\mathcal{A}} = \sum_{\boldsymbol{\alpha}\in\mathbb{I}^n_{=m}} \sqrt{m!/\boldsymbol{\alpha}!}\mathcal{A}_{[\boldsymbol{\alpha}]}\boldsymbol{\mathcal{F}}_{\boldsymbol{\alpha}}$.*

Proof. Note that $\dim \mathcal{S}^{n,m} = |\mathbb{I}^n_{=m}|$. By [108, page 248], $\{\boldsymbol{\mathcal{F}}_{\boldsymbol{\alpha}} \mid \boldsymbol{\alpha} \in \mathbb{I}^n_{=m}\}$ is a basis for $\mathcal{S}^{n,m}$. We show its orthonormality. Let $\boldsymbol{\alpha},\boldsymbol{\beta} \in \mathbb{I}^n_{=m}$. We first consider the case of $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$. Let $(i_1,\dots,i_m) := [\boldsymbol{\alpha}]$ and $(j_1,\dots,j_m) := [\boldsymbol{\beta}]$. Then, as $(i_1,\dots,i_m) \neq (j_1,\dots,j_m)$, there exists $k_0 \in \{1,\dots,m\}$ such that $i_{k_0} \neq j_{k_0}$, which implies that the number of values i_{k_0} in the tuple $(i_1,\dots,i_m)$ is not equal to that in the tuple $(j_1,\dots,j_m)$. Using Lemma 2.5, we then have

$$\begin{aligned}
\langle \boldsymbol{\mathcal{F}}_{\boldsymbol{\alpha}}, \boldsymbol{\mathcal{F}}_{\boldsymbol{\beta}} \rangle &= \left\langle \sqrt{\frac{m!}{\boldsymbol{\alpha}!}}\mathscr{S}\tilde{\boldsymbol{e}}_{i_1\cdots i_m}, \sqrt{\frac{m!}{\boldsymbol{\beta}!}}\mathscr{S}\tilde{\boldsymbol{e}}_{j_1\cdots j_m} \right\rangle \\
&= \sqrt{\frac{m!}{\boldsymbol{\alpha}!}}\sqrt{\frac{m!}{\boldsymbol{\beta}!}}\langle \mathscr{S}\tilde{\boldsymbol{e}}_{i_1\cdots i_m}, \tilde{\boldsymbol{e}}_{j_1\cdots j_m} \rangle \\
&= \sqrt{\frac{m!}{\boldsymbol{\alpha}!}}\sqrt{\frac{m!}{\boldsymbol{\beta}!}}\left\langle \frac{1}{m!}\sum_{\sigma\in\mathfrak{S}_m} \tilde{\boldsymbol{e}}_{i_{\sigma(1)}\cdots i_{\sigma(m)}}, \tilde{\boldsymbol{e}}_{j_1\cdots j_m} \right\rangle \\
&= \sqrt{\frac{m!}{\boldsymbol{\alpha}!}}\sqrt{\frac{m!}{\boldsymbol{\beta}!}}\frac{1}{m!}\sum_{\sigma\in\mathfrak{S}_m}\prod_{k=1}^m \boldsymbol{e}_{i_{\sigma(k)}}^\top \boldsymbol{e}_{j_k} \\
&= 0.
\end{aligned}$$

Second, we consider the case of $\boldsymbol{\alpha} = \boldsymbol{\beta}$. Let $(i_1, \ldots, i_m) \coloneqq [\boldsymbol{\alpha}]$. Then,

$$\begin{aligned}
\|\boldsymbol{\mathcal{F}}_{\boldsymbol{\alpha}}\|_{\mathrm{F}}^2 &= \frac{m!}{\boldsymbol{\alpha}!}\langle \mathscr{S}\tilde{\boldsymbol{e}}_{i_1\cdots i_m}, \tilde{\boldsymbol{e}}_{i_1\cdots i_m}\rangle \\
&= \frac{m!}{\boldsymbol{\alpha}!}\frac{1}{m!}\sum_{\sigma\in\mathfrak{S}_m}\prod_{k=1}^{m}\boldsymbol{e}_{i_k}^{\top}\boldsymbol{e}_{i_{\sigma(k)}} \\
&= 1.
\end{aligned}$$

Therefore, $\{\boldsymbol{\mathcal{F}}_{\boldsymbol{\alpha}} \mid \boldsymbol{\alpha} \in \mathbb{I}^n_{=m}\}$ is an orthonormal basis for $\mathcal{S}^{n,m}$.

In addition, for $\boldsymbol{\alpha} \in \mathbb{I}^n_{=m}$, let

$$\mathbb{N}^m_n[\boldsymbol{\alpha}] \coloneqq \{(i_1, \ldots i_m) \in \mathbb{N}^m_n \mid (i_{\sigma(1)}, \ldots, i_{\sigma(m)}) = [\boldsymbol{\alpha}] \text{ for some } \sigma \in \mathfrak{S}_m\}.$$

From the symmetry of $\boldsymbol{\mathcal{A}}$, it then follows that

$$\begin{aligned}
\boldsymbol{\mathcal{A}} &= \sum_{i_1,\ldots,i_m=1}^{n} \mathcal{A}_{i_1\cdots i_m}\tilde{\boldsymbol{e}}_{i_1\cdots i_m} \\
&= \sum_{\boldsymbol{\alpha}\in\mathbb{I}^n_{=m}}\sum_{(i_1,\ldots,i_m)\in\mathbb{N}^m_n[\boldsymbol{\alpha}]} \mathcal{A}_{i_1\cdots i_m}\tilde{\boldsymbol{e}}_{i_1\cdots i_m} \\
&= \sum_{\boldsymbol{\alpha}\in\mathbb{I}^n_{=m}} \mathcal{A}_{[\boldsymbol{\alpha}]}\left(\sum_{(i_1,\ldots,i_m)\in\mathbb{N}^m_n[\boldsymbol{\alpha}]}\tilde{\boldsymbol{e}}_{i_1\cdots i_m}\right) \\
&= \sum_{\boldsymbol{\alpha}\in\mathbb{I}^n_{=m}} \mathcal{A}_{[\boldsymbol{\alpha}]}\left(\frac{1}{\boldsymbol{\alpha}!}\sum_{\sigma\in\mathfrak{S}_m}\pi_\sigma\tilde{\boldsymbol{e}}_{[\boldsymbol{\alpha}]}\right) \\
&= \sum_{\boldsymbol{\alpha}\in\mathbb{I}^n_{=m}} \sqrt{\frac{m!}{\boldsymbol{\alpha}!}}\mathcal{A}_{[\boldsymbol{\alpha}]}\left(\sqrt{\frac{m!}{\boldsymbol{\alpha}!}}\mathscr{S}\tilde{\boldsymbol{e}}_{[\boldsymbol{\alpha}]}\right) \\
&= \sum_{\boldsymbol{\alpha}\in\mathbb{I}^n_{=m}} \sqrt{\frac{m!}{\boldsymbol{\alpha}!}}\mathcal{A}_{[\boldsymbol{\alpha}]}\boldsymbol{\mathcal{F}}_{\boldsymbol{\alpha}}.
\end{aligned}$$

□

For convenience, we write the coefficients of $\boldsymbol{\mathcal{A}} \in \mathcal{S}^{n,m}$ with respect to the orthonormal basis $\{\boldsymbol{\mathcal{F}}_{\boldsymbol{\alpha}} \mid \boldsymbol{\alpha} \in \mathbb{I}^n_{=m}\}$ taken in Lemma 3.7 as $(\mathcal{A}^{\mathcal{F}}_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathbb{I}^n_{=m}}$. That is, each $\boldsymbol{\mathcal{A}} \in \mathcal{S}^{n,m}$ is written as

$$\boldsymbol{\mathcal{A}} = \sum_{\boldsymbol{\alpha}\in\mathbb{I}^n_{=m}} \mathcal{A}^{\mathcal{F}}_{\boldsymbol{\alpha}}\boldsymbol{\mathcal{F}}_{\boldsymbol{\alpha}}. \tag{3.7}$$

Because $\mathcal{S}^{n,m}$ and $H^{n,m}$ are linearly isomorphic, for each $\boldsymbol{\mathcal{A}} \in \mathcal{S}^{n,m}$, there exists $\theta \in H^{n,m}$ such that $\langle \boldsymbol{\mathcal{A}}, \boldsymbol{x}^{\otimes m}\rangle = \theta(\boldsymbol{x})$. The following lemma links the coefficients $(\mathcal{A}^{\mathcal{F}}_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathbb{I}^n_{=m}}$ with the coefficients of θ.

Lemma 3.8. *Suppose that $\boldsymbol{\mathcal{A}} \in \mathcal{S}^{n,m}$ and $(\theta_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathbb{I}^n_{=m}} \in \mathbb{R}^{\mathbb{I}^n_{=m}}$ satisfy*

$$\langle \boldsymbol{\mathcal{A}}, \boldsymbol{x}^{\otimes m}\rangle = \sum_{\boldsymbol{\alpha}\in\mathbb{I}^n_{=m}} \theta_{\boldsymbol{\alpha}}\boldsymbol{x}^{\boldsymbol{\alpha}}. \tag{3.8}$$

Then, $\mathcal{A}^{\mathcal{F}}_{\boldsymbol{\alpha}} = \sqrt{\boldsymbol{\alpha}!\,/m!}\theta_{\boldsymbol{\alpha}}$ *for all* $\boldsymbol{\alpha} \in \mathbb{I}^n_{=m}$.

Proof. Let $\boldsymbol{\mathcal{A}}$ be in the form (3.7). It then follows from Lemma 2.5 that

$$\begin{aligned}
\langle \boldsymbol{\mathcal{A}}, \boldsymbol{x}^{\otimes m} \rangle &= \sum_{\boldsymbol{\alpha} \in \mathbb{I}^n_{=m}} \left\langle \mathcal{A}^{\mathcal{F}}_{\boldsymbol{\alpha}} \sqrt{\frac{m!}{\boldsymbol{\alpha}!}} \mathscr{S} \tilde{\boldsymbol{e}}_{[\boldsymbol{\alpha}]}, \boldsymbol{x}^{\otimes m} \right\rangle \\
&= \sum_{\boldsymbol{\alpha} \in \mathbb{I}^n_{=m}} \mathcal{A}^{\mathcal{F}}_{\boldsymbol{\alpha}} \sqrt{\frac{m!}{\boldsymbol{\alpha}!}} \langle \boldsymbol{e}_1^{\otimes \alpha_1} \otimes \cdots \otimes \boldsymbol{e}_n^{\otimes \alpha_n}, \boldsymbol{x}^{\otimes m} \rangle \\
&= \sum_{\boldsymbol{\alpha} \in \mathbb{I}^n_{=m}} \mathcal{A}^{\mathcal{F}}_{\boldsymbol{\alpha}} \sqrt{\frac{m!}{\boldsymbol{\alpha}!}} \boldsymbol{x}^{\boldsymbol{\alpha}}.
\end{aligned}$$

As this equals $\sum_{\boldsymbol{\alpha} \in \mathbb{I}^n_{=m}} \theta_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}}$ and by comparing the coefficients, we obtain the desired result. $\square$

Lemma 3.9. *For* $\boldsymbol{\mathcal{A}} \in \mathcal{S}^{n,m}$*, let* $\boldsymbol{\theta}$ *be the element of* $\mathbb{R}^{\mathbb{I}^n_{=m}}$ *satisfying* (3.8)*. Then,* $\langle \mathscr{S}(\boldsymbol{\mathcal{A}} \otimes \boldsymbol{\mathcal{A}}), \boldsymbol{\mathcal{X}} \rangle = \boldsymbol{\theta}^\top \boldsymbol{M}^{n,m}(\boldsymbol{\mathcal{X}}) \boldsymbol{\theta}$ *for all* $\boldsymbol{\mathcal{X}} \in \mathcal{S}^{n,2m}$.

Proof. From Lemma 3.3 and (3.8), we have

$$\langle \mathscr{S}(\boldsymbol{\mathcal{A}} \otimes \boldsymbol{\mathcal{A}}), \boldsymbol{x}^{\otimes 2m} \rangle = \langle \boldsymbol{\mathcal{A}}, \boldsymbol{x}^{\otimes m} \rangle^2 = \sum_{\boldsymbol{\gamma} \in \mathbb{I}^n_{=2m}} \left(\sum_{\substack{\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{I}^n_{=m} \\ \boldsymbol{\gamma} = \boldsymbol{\alpha} + \boldsymbol{\beta}}} \theta_{\boldsymbol{\alpha}} \theta_{\boldsymbol{\beta}} \right) \boldsymbol{x}^{\boldsymbol{\gamma}}.$$

Therefore, from Lemma 3.8, when we represent $\mathscr{S}(\boldsymbol{\mathcal{A}} \otimes \boldsymbol{\mathcal{A}})$ with the orthonormal basis $\{\boldsymbol{\mathcal{F}}_{\boldsymbol{\gamma}} \mid \boldsymbol{\gamma} \in \mathbb{I}^n_{=2m}\}$ for $\mathcal{S}^{n,2m}$, the coefficient $\mathscr{S}(\boldsymbol{\mathcal{A}} \otimes \boldsymbol{\mathcal{A}})^{\mathcal{F}}_{\boldsymbol{\gamma}}$ can be written as

$$\mathscr{S}(\boldsymbol{\mathcal{A}} \otimes \boldsymbol{\mathcal{A}})^{\mathcal{F}}_{\boldsymbol{\gamma}} = \sqrt{\frac{\boldsymbol{\gamma}!}{(2m)!}} \sum_{\substack{\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{I}^n_{=m} \\ \boldsymbol{\gamma} = \boldsymbol{\alpha} + \boldsymbol{\beta}}} \theta_{\boldsymbol{\alpha}} \theta_{\boldsymbol{\beta}}$$

for each $\boldsymbol{\gamma} \in \mathbb{I}^n_{=2m}$. In addition, from Lemma 3.7, each $\boldsymbol{\mathcal{X}} \in \mathcal{S}^{n,2m}$ can be written as

$$\boldsymbol{\mathcal{X}} = \sum_{\boldsymbol{\gamma} \in \mathbb{I}^n_{=2m}} \sqrt{\frac{(2m)!}{\boldsymbol{\gamma}!}} \mathcal{X}_{[\boldsymbol{\gamma}]} \boldsymbol{\mathcal{F}}_{\boldsymbol{\gamma}}.$$

Then, we have

$$\begin{aligned}
\langle \mathscr{S}(\boldsymbol{\mathcal{A}} \otimes \boldsymbol{\mathcal{A}}), \boldsymbol{\mathcal{X}} \rangle &= \sum_{\boldsymbol{\gamma} \in \mathbb{I}^n_{=2m}} \left(\sqrt{\frac{\boldsymbol{\gamma}!}{(2m)!}} \sum_{\substack{\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{I}^n_{=m} \\ \boldsymbol{\gamma} = \boldsymbol{\alpha} + \boldsymbol{\beta}}} \theta_{\boldsymbol{\alpha}} \theta_{\boldsymbol{\beta}} \right) \left(\sqrt{\frac{(2m)!}{\boldsymbol{\gamma}!}} \mathcal{X}_{[\boldsymbol{\gamma}]} \right) \\
&= \sum_{\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{I}^n_{=m}} \mathcal{X}_{[\boldsymbol{\alpha} + \boldsymbol{\beta}]} \theta_{\boldsymbol{\alpha}} \theta_{\boldsymbol{\beta}} \\
&= \boldsymbol{\theta}^\top \boldsymbol{M}^{n,m}(\boldsymbol{\mathcal{X}}) \boldsymbol{\theta}.
\end{aligned}$$

$\square$

Proposition 3.10. *The dual of* $\mathrm{SOS}^{n,2m}$ *is* $\mathcal{M}^{n,2m}$. *In particular,* $\mathcal{M}^{n,2m}$ *is closed.*

Proof. Let $\boldsymbol{\mathcal{X}} \in \mathcal{M}^{n,2m}$. For each $\boldsymbol{\mathcal{A}} \in \mathrm{SOS}^{n,2m}$, there exist $\boldsymbol{\mathcal{A}}^{(1)}, \dots, \boldsymbol{\mathcal{A}}^{(k)} \in \mathcal{S}^{n,m}$ such that $\boldsymbol{\mathcal{A}} = \sum_{i=1}^{k} \mathscr{S}(\boldsymbol{\mathcal{A}}^{(i)} \otimes \boldsymbol{\mathcal{A}}^{(i)})$. For each $\boldsymbol{\mathcal{A}}^{(i)}$, let $\boldsymbol{\theta}^{(i)} = (\theta^{(i)}_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha} \in \mathbb{I}^n_{=m}} \in \mathbb{R}^{\mathbb{I}^n_{=m}}$ be such that it satisfies (3.8). Then, it follows from Lemma 3.9 and $\boldsymbol{M}^{n,m}(\boldsymbol{\mathcal{X}}) \in \mathbb{S}^{\mathbb{I}^n_{=m}}_{+}$ that

$$\langle \boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{X}} \rangle = \sum_{i=1}^{k} \langle \mathscr{S}(\boldsymbol{\mathcal{A}}^{(i)} \otimes \boldsymbol{\mathcal{A}}^{(i)}), \boldsymbol{\mathcal{X}} \rangle = \sum_{i=1}^{k} (\boldsymbol{\theta}^{(i)})^\top \boldsymbol{M}^{n,m}(\boldsymbol{\mathcal{X}}) \boldsymbol{\theta}^{(i)} \geq 0,$$

which means that $\boldsymbol{\mathcal{X}} \in (\mathrm{SOS}^{n,2m})^*$.

Conversely, suppose that $\boldsymbol{\mathcal{X}} \in (\mathrm{SOS}^{n,2m})^*$. We take $\boldsymbol{\theta} = (\theta_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha} \in \mathbb{I}^n_{=m}} \in \mathbb{R}^{\mathbb{I}^n_{=m}}$ arbitrarily and let $\boldsymbol{\mathcal{A}}$ be the element of $\mathcal{S}^{n,m}$ satisfying (3.8). Then, it follows from Lemma 3.9 and $\mathscr{S}(\boldsymbol{\mathcal{A}} \otimes \boldsymbol{\mathcal{A}}) \in \mathrm{SOS}^{n,2m}$ that $\boldsymbol{\theta}^\top \boldsymbol{M}^{n,m}(\boldsymbol{\mathcal{X}}) \boldsymbol{\theta} = \langle \mathscr{S}(\boldsymbol{\mathcal{A}} \otimes \boldsymbol{\mathcal{A}}), \boldsymbol{\mathcal{X}} \rangle \geq 0$. Therefore, $\boldsymbol{\mathcal{X}} \in \mathcal{M}^{n,2m}$.

Finally, the closedness of $\mathcal{M}^{n,2m}$ is a consequence of (i) of Theorem 2.2. ☐

Using Proposition 3.10, we show the closedness of $\mathcal{C}^{n,m}_l$. When the Euclidean Jordan algebra $(\mathbb{E}, \circ, \bullet)$ is that given by Example 2.6, the $(i_1, \dots, i_m)$th element of $\boldsymbol{\mathcal{C}}^{(l)}(\boldsymbol{\mathcal{X}}) \in \mathcal{S}^{n,m}$ is

$$\mathcal{C}^{(l)}(\boldsymbol{\mathcal{X}})_{i_1 \cdots i_m} = \sum_{\boldsymbol{\alpha} \in \mathbb{I}^n_{=l}} \frac{l!}{\boldsymbol{\alpha}!} \mathcal{X}_{[2\boldsymbol{\alpha} + 2\sum_{t=1}^{m} \boldsymbol{e}_{i_t}]}. \tag{3.9}$$

See [58, Equation (29)] for this derivation.

Lemma 3.11. *Suppose that* $\boldsymbol{\mathcal{X}} \in \mathcal{M}^{n,2m}$, *i.e.,* $\boldsymbol{M}^{n,m}(\boldsymbol{\mathcal{X}}) \in \mathbb{S}^{\mathbb{I}^n_{=m}}_{+}$. *Using* $\boldsymbol{\mathcal{X}}$, *we define* $\boldsymbol{\mathcal{X}}' \in \mathcal{S}^{n,2m}$ *as*

$$\mathcal{X}'_{[\boldsymbol{\gamma}]} := \begin{cases} \mathcal{X}_{[\boldsymbol{\gamma}]} & (\text{if all of the elements of } \boldsymbol{\gamma} \text{ are even}), \\ 0 & (\text{if some of the elements of } \boldsymbol{\gamma} \text{ are odd}) \end{cases}$$

for each $\boldsymbol{\gamma} \in \mathbb{I}^n_{=2m}$. *It then follows that* $\boldsymbol{M}^{n,m}(\boldsymbol{\mathcal{X}}') \in \mathbb{S}^{\mathbb{I}^n_{=m}}_{+}$.

Proof. The set $\mathbb{I}^n_{=m}$ is partitioned as two disjoint sets $\mathbb{I}^n_{=m,\mathrm{even}}$ and $\mathbb{I}^n_{=m,\mathrm{odd}}$, where

$$\begin{aligned} \mathbb{I}^n_{=m,\mathrm{even}} &:= \{\boldsymbol{\alpha} \in \mathbb{I}^n_{=m} \mid \text{All of the elements of } \boldsymbol{\alpha} \text{ are even}\}, \\ \mathbb{I}^n_{=m,\mathrm{odd}} &:= \{\boldsymbol{\alpha} \in \mathbb{I}^n_{=m} \mid \text{Some of the elements of } \boldsymbol{\alpha} \text{ are odd}\}. \end{aligned}$$

In addition, the elements of $\{0,1\}^n \setminus \{\boldsymbol{0}\}$ are ordered as $\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_{2^n-1}$ (e.g., $\boldsymbol{\delta}_1 = (1, 0, \dots, 0)^\top$). Then, let

$$\mathbb{I}^n_{=m,\mathrm{odd},i} := \{\boldsymbol{\alpha} \in \mathbb{I}^n_{=m,\mathrm{odd}} \mid \text{All of the elements of } \boldsymbol{\alpha} - \boldsymbol{\delta}_i \text{ are even}\}$$

for $i = 1, \dots, 2^n - 1$. Note that the parity between $\boldsymbol{\alpha}$ and $\boldsymbol{\delta}_i$ agrees for every $\boldsymbol{\alpha} \in \mathbb{I}^n_{=m,\mathrm{odd},i}$ and that the sets $\mathbb{I}^n_{=m,\mathrm{odd},i}$ $(i = 1, \dots, 2^n-1)$ are disjoint. Then, for $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{I}^n_{=m}$, all elements of $\boldsymbol{\alpha} + \boldsymbol{\beta}$ are even if and only if $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ belong to the same set of $\mathbb{I}^n_{=m,\mathrm{even}}, \mathbb{I}^n_{=m,\mathrm{odd},1}, \dots, \mathbb{I}^n_{=m,\mathrm{odd},2^n-1}$. Therefore, from the definition of $\boldsymbol{\mathcal{X}}'$, we note that

$$\boldsymbol{M}^{n,m}(\boldsymbol{\mathcal{X}}')_{IJ} = \begin{cases} \boldsymbol{M}^{n,m}(\boldsymbol{\mathcal{X}})_{IJ} & (I = J), \\ \boldsymbol{O} & (\text{otherwise}) \end{cases}$$

for $I, J = \mathbb{I}^n_{=m,\mathrm{even}}, \mathbb{I}^n_{=m,\mathrm{odd},1}, \ldots, \mathbb{I}^n_{=m,\mathrm{odd},2^n-1}$. Because $\boldsymbol{M}^{n,m}(\boldsymbol{\mathcal{X}})$ is positive semidefinite, $\boldsymbol{M}^{n,m}(\boldsymbol{\mathcal{X}}')$ is as well. □

Proposition 3.12. *$\mathcal{C}^{n,m}_l$ is a closed convex cone, and thus, $\mathcal{I}^{n,m}_{\mathrm{NN},l}(\mathbb{R}^n_+)$ and $\mathcal{C}^{n,m}_l$ are dual to each other.*

Proof. Showing the closedness of $\mathcal{C}^{n,m}_l$ is sufficient. Let $\{\boldsymbol{\mathcal{A}}_k\}_k \subseteq \mathcal{C}^{n,m}_l$ and suppose that the sequence converges to some $\boldsymbol{\mathcal{A}}_\infty \in \boldsymbol{\mathcal{S}}^{n,m}$. For each k, there exists $\boldsymbol{\mathcal{X}}_k \in \mathcal{M}^{n,2(l+m)}$ such that $\boldsymbol{\mathcal{A}}_k = \boldsymbol{\mathcal{C}}^{(l)}(\boldsymbol{\mathcal{X}}_k)$. We note from (3.9) that $\boldsymbol{\mathcal{C}}^{(l)}(\boldsymbol{\mathcal{X}}_k)$ is independent of $(\mathcal{X}_k)_{[\boldsymbol{\gamma}]}$ such that some elements of $\boldsymbol{\gamma} \in \mathbb{I}^n_{=2(l+m)}$ are odd. Thus, by Lemma 3.11, we can assume that such $(\mathcal{X}_k)_{[\boldsymbol{\gamma}]}$ is 0 without loss of generality. As $\boldsymbol{\mathcal{C}}^{(l)}(\boldsymbol{\mathcal{X}}_k) \to \boldsymbol{\mathcal{A}}_\infty$ $(k \to \infty)$, we observe that

$$\|\boldsymbol{\mathcal{C}}^{(l)}(\boldsymbol{\mathcal{X}}_k)\|_{\mathrm{F}} = \sqrt{\sum_{i_1,\ldots,i_m=1}^{n} \left(\sum_{\boldsymbol{\alpha} \in \mathbb{I}^n_{=l}} \frac{l!}{\boldsymbol{\alpha}!} (\mathcal{X}_k)_{[2\boldsymbol{\alpha}+2\sum_{t=1}^m \boldsymbol{e}_{i_t}]} \right)^2} \to \|\boldsymbol{\mathcal{A}}_\infty\|_{\mathrm{F}}.$$

Note that each $(\mathcal{X}_k)_{[2\boldsymbol{\alpha}+2\sum_{t=1}^m \boldsymbol{e}_{i_t}]}$ is nonnegative because it is a diagonal element of the positive semidefinite matrix $\boldsymbol{M}^{n,l+m}(\boldsymbol{\mathcal{X}}_k)$. Therefore, $\{(\mathcal{X}_k)_{[2\boldsymbol{\alpha}+2\sum_{t=1}^m \boldsymbol{e}_{i_t}]}\}_k$ is bounded for each $(i_1, \ldots, i_m) \in \mathbb{N}^m_n$ and $\boldsymbol{\alpha} \in \mathbb{I}^n_{=l}$, and $\{\boldsymbol{\mathcal{X}}_k\}_k$ is as well. Thus, by taking a subsequence if necessary, we assume that the sequence $\{\boldsymbol{\mathcal{X}}_k\}_k \subseteq \mathcal{M}^{n,2(l+m)}$ converges to some $\boldsymbol{\mathcal{X}}_\infty$. Since $\mathcal{M}^{n,2(l+m)}$ is closed by Proposition 3.10, we have $\boldsymbol{\mathcal{X}}_\infty \in \mathcal{M}^{n,2(l+m)}$. Therefore,

$$\boldsymbol{\mathcal{A}}_\infty = \lim_{k\to\infty} \boldsymbol{\mathcal{C}}^{(l)}(\boldsymbol{\mathcal{X}}_k) = \boldsymbol{\mathcal{C}}^{(l)}(\boldsymbol{\mathcal{X}}_\infty) \in \mathcal{C}^{n,m}_l,$$

which implies that $\mathcal{C}^{n,m}_l$ is closed. □

3.3 Approximation Hierarchies Exploiting Those for the Standard Copositive Cone

In this section, we provide other approximation hierarchies for copositive cones over symmetric cones by exploiting those for the standard copositive cone. Let $(\mathbb{E}, \circ, \bullet)$ be a Euclidean Jordan algebra of dimension n and rank r. In addition, for $\mathcal{A} \in \mathcal{S}^{n,m}(\mathbb{E})$ and an ordered Jordan frame $(c_1, \ldots, c_r) \in \mathfrak{F}(\mathbb{E})$, let $\mathcal{A}(c_1, \ldots, c_r) \in \mathcal{S}^{r,m}$ be the tensor with the $(i_1, \ldots, i_m)$th element $\langle \mathcal{A}, c_{i_1} \otimes \cdots \otimes c_{i_m} \rangle$. The tensor is guaranteed to be symmetric by the symmetry of $\mathcal{A}$.

The following lemma is key to providing the desired approximation hierarchies.

Lemma 3.13.

$$\mathcal{COP}^{n,m}(\mathbb{E}_+) = \bigcap_{(c_1,\ldots,c_r) \in \mathfrak{F}(\mathbb{E})} \{\mathcal{A} \in \mathcal{S}^{n,m}(\mathbb{E}) \mid \mathcal{A}(c_1, \ldots, c_r) \in \mathcal{COP}^{r,m}\}. \tag{3.10}$$

Proof. Let $\mathcal{A} \in \mathcal{COP}^{n,m}(\mathbb{E}_+)$. For any $(c_1, \dots, c_r) \in \mathfrak{F}(\mathbb{E})$ and $\boldsymbol{x} \in \mathbb{R}^r_+$, we have

$$\begin{aligned}\langle \mathcal{A}(c_1, \dots, c_r), \boldsymbol{x}^{\otimes m} \rangle &= \sum_{i_1,\dots,i_m=1}^{r} \langle \mathcal{A}, c_{i_1} \otimes \cdots \otimes c_{i_m} \rangle x_{i_1} \cdots x_{i_m} \\ &= \left\langle \mathcal{A}, \left(\sum_{i=1}^{r} x_i c_i \right)^{\otimes m} \right\rangle,\end{aligned}$$

which is nonnegative given that $\sum_{i=1}^{r} x_i c_i \in \mathbb{E}_+$.

Conversely, suppose that $\mathcal{A}$ belongs to the right-hand side set of (3.10). For any $x \in \mathbb{E}_+$, there exist $(x_1, \dots, x_r)^\top \in \mathbb{R}^r_+$ and $(c_1, \dots, c_r) \in \mathfrak{F}(\mathbb{E})$ such that $x = \sum_{i=1}^{r} x_i c_i$. As $\mathcal{A}(c_1, \dots, c_r) \in \mathcal{COP}^{r,m}$, we can show $\langle \mathcal{A}, x^{\otimes m} \rangle \geq 0$, i.e., $\mathcal{A} \in \mathcal{COP}^{n,m}(\mathbb{E}_+)$ in the same manner as the above discussion. $\square$

The characterization of copositive cones over symmetric cones is somewhat redundant because the Jordan frames we considered are ordered, and thus, the same set appears multiple times on the right-hand side of (3.10). To solve this problem, we provide a more concise characterization.

Definition 3.14. For each $\sigma \in \mathfrak{S}_n$, let $\mathscr{P}_\sigma$ be the linear transformation on $\mathcal{S}^{n,m}$ defined by $(\mathscr{P}_\sigma \boldsymbol{\mathcal{A}})_{i_1 \cdots i_m} := \mathcal{A}_{\sigma(i_1) \cdots \sigma(i_m)}$ for each $\boldsymbol{\mathcal{A}} \in \mathcal{S}^{n,m}$ and $(i_1, \dots, i_m) \in \mathbb{N}^m_n$. A set $\mathcal{K} \subseteq \mathcal{S}^{n,m}$ is said to be *permutation-invariant* if $\mathscr{P}_\sigma \mathcal{K} = \mathcal{K}$ for all $\sigma \in \mathfrak{S}_n$.

Note that because $\mathscr{P}_\sigma$ is invertible and $\mathscr{P}_\sigma^{-1} = \mathscr{P}_{\sigma^{-1}}$ for all $\sigma \in \mathfrak{S}_n$, Definition 3.14 can be equivalently stated such that $\mathscr{P}_\sigma \boldsymbol{\mathcal{A}} \in \mathcal{K}$ holds for all $\boldsymbol{\mathcal{A}} \in \mathcal{K}$ and $\sigma \in \mathfrak{S}_n$.

Lemma 3.15. *$\mathcal{COP}^{n,m}$ is permutation-invariant.*

Proof. Let $\boldsymbol{\mathcal{A}} \in \mathcal{COP}^{n,m}$. Then, for any $\sigma \in \mathfrak{S}_n$ and $\boldsymbol{x} \in \mathbb{R}^n_+$, we have

$$\begin{aligned}\langle \mathscr{P}_\sigma \boldsymbol{\mathcal{A}}, \boldsymbol{x}^{\otimes m} \rangle &= \sum_{i_1,\dots,i_m=1}^{n} \mathcal{A}_{\sigma(i_1) \cdots \sigma(i_m)} x_{i_1} \cdots x_{i_m} \\ &= \sum_{i_1,\dots,i_m=1}^{n} \mathcal{A}_{i_1 \cdots i_m} x_{\sigma^{-1}(i_1)} \cdots x_{\sigma^{-1}(i_m)} \\ &= \langle \boldsymbol{\mathcal{A}}, (x_{\sigma^{-1}(1)}; \dots; x_{\sigma^{-1}(n)})^{\otimes m} \rangle \\ &\geq 0,\end{aligned}$$

for which the last inequality follows from the fact that $(x_{\sigma^{-1}(1)}; \dots; x_{\sigma^{-1}(n)}) \in \mathbb{R}^n_+$ if $\boldsymbol{x} \in \mathbb{R}^n_+$. $\square$

Lemma 3.16. *Let $\mathcal{K} \subseteq \mathcal{S}^{r,m}$ be a permutation-invariant set and let $\mathcal{A} \in \mathcal{S}^{n,m}(\mathbb{E})$. If $\mathcal{A}(c_1, \dots, c_r) \in \mathcal{K}$ for a given $(c_1, \dots, c_r) \in \mathfrak{F}(\mathbb{E})$, then $\mathcal{A}(c_{\sigma(1)}, \dots, c_{\sigma(r)}) \in \mathcal{K}$ for all $\sigma \in \mathfrak{S}_r$.*

Proof. Because $\mathcal{K}$ is permutation-invariant, we have $\mathscr{P}_\sigma\mathcal{A}(c_1,\dots,c_r) \in \mathcal{K}$. Then, the $(i_1,\dots,i_m)$th element of $\mathscr{P}_\sigma\mathcal{A}(c_1,\dots,c_r)$ is

$$\mathcal{A}(c_1,\dots,c_r)_{\sigma(i_1)\cdots\sigma(i_m)} = \langle \mathcal{A}, c_{\sigma(i_1)} \otimes \cdots \otimes c_{\sigma(i_m)} \rangle,$$

which equals the $(i_1,\dots,i_m)$th element of $\mathcal{A}(c_{\sigma(1)},\dots,c_{\sigma(r)})$. Therefore, we have

$$\mathcal{A}(c_{\sigma(1)},\dots,c_{\sigma(r)}) = \mathscr{P}_\sigma\mathcal{A}(c_1,\dots,c_m) \in \mathcal{K}.$$

□

The symmetric group $\mathfrak{S}_r$ acts on the set $\mathfrak{F}(\mathbb{E})$ by $\sigma \cdot (c_1,\dots,c_r) = (c_{\sigma(1)},\dots,c_{\sigma(r)})$. Let $\mathfrak{F}_{\mathrm{c}}(\mathbb{E})$ be a complete set of representatives of $\mathfrak{S}_r$-orbits in $\mathfrak{F}(\mathbb{E})$. Then, using Lemmas 3.15 and 3.16, we obtain the following, more concise, characterization of $\mathcal{COP}^{n,m}(\mathbb{E}_+)$, compared with that using Lemma 3.13.

Theorem 3.17. *Let $\mathcal{K} \subseteq \mathcal{S}^{r,m}$ be a permutation-invariant set and consider $\mathfrak{F}_{\mathrm{c}}(\mathbb{E}) \subseteq \mathfrak{F} \subseteq \mathfrak{F}(\mathbb{E})$. Then, the set*

$$\bigcap_{(c_1,\dots,c_r)\in\mathfrak{F}} \{\mathcal{A} \in \mathcal{S}^{n,m}(\mathbb{E}) \mid \mathcal{A}(c_1,\dots,c_r) \in \mathcal{K}\}$$

is the same regardless of the choice of $\mathfrak{F}$. In particular, the claim holds when $\mathcal{K} = \mathcal{COP}^{r,m}$, and it follows that

$$\mathcal{COP}^{n,m}(\mathbb{E}_+) = \bigcap_{(c_1,\dots,c_r)\in\mathfrak{F}} \{\mathcal{A} \in \mathcal{S}^{n,m}(\mathbb{E}) \mid \mathcal{A}(c_1,\dots,c_r) \in \mathcal{COP}^{r,m}\}. \tag{3.11}$$

The idea for providing inner- and outer-approximation hierarchies for $\mathcal{COP}^{n,m}(\mathbb{E}_+)$ is to approximate $\mathcal{COP}^{r,m}$ on the right-hand side set in (3.11) from the inside and outside.

3.3.1 Inner-approximation hierarchy

Throughout this subsection, we only consider the case of $m = 2$. Generally, even if $\{\mathcal{I}_l^r\}_l$ is an inner-approximation hierarchy for $\mathcal{COP}^r$, the sequence obtained by replacing $\mathcal{COP}^r$ on the right-hand side set in (3.11) with $\mathcal{I}_l^r$ is not guaranteed to converge to $\mathcal{COP}(\mathbb{E}_+)$. However, if we use the de Klerk and Pasechnik's inner-approximation hierarchy $\{\mathcal{I}_{\mathrm{dP},l}^r\}_l$ introduced in Section 2.5.1, the induced sequence is indeed an inner-approximation hierarchy for $\mathcal{COP}(\mathbb{E}_+)$.

Proposition 3.18. *We fix a set $\mathfrak{F}$ with $\mathfrak{F}_{\mathrm{c}}(\mathbb{E}) \subseteq \mathfrak{F} \subseteq \mathfrak{F}(\mathbb{E})$. Let*

$$\mathcal{I}_{\mathrm{dP},l}(\mathbb{E}_+) := \bigcap_{(c_1,\dots,c_r)\in\mathfrak{F}} \{\mathcal{A} \in \mathcal{S}(\mathbb{E}) \mid \mathcal{A}(c_1,\dots,c_r) \in \mathcal{I}_{\mathrm{dP},l}^r\}. \tag{3.12}$$

Then, each $\mathcal{I}_{\mathrm{dP},l}(\mathbb{E}_+)$ is a closed convex cone and $\{\mathcal{I}_{\mathrm{dP},l}(\mathbb{E}_+)\}_l$ is an inner-approximation hierarchy for $\mathcal{COP}(\mathbb{E}_+)$.

Proof. Given that each $\mathcal{I}^r_{\mathrm{dP},l}$ is a closed convex cone and that the mapping $\mathcal{A} \mapsto \langle \mathcal{A}, c_i \otimes c_j \rangle$ is linear for each $(c_1, \dots, c_r) \in \mathfrak{F}$ and $i, j = 1, \dots, r$, $\mathcal{I}_{\mathrm{dP},l}(\mathbb{E}_+)$ is a closed convex cone. In the following, we prove that $\{\mathcal{I}_{\mathrm{dP},l}(\mathbb{E}_+)\}_l$ is an inner-approximation hierarchy for $\mathcal{COP}(\mathbb{E}_+)$. As the sequence $\{\mathcal{I}^r_{\mathrm{dP},l}\}_l$ satisfies $\mathcal{I}^r_{\mathrm{dP},l} \subseteq \mathcal{I}^r_{\mathrm{dP},l+1}$ and $\mathcal{I}^r_{\mathrm{dP},l} \subseteq \mathcal{COP}^r$ for all $l \in \mathbb{N}$, it follows from Theorem 3.17 that the sequence $\{\mathcal{I}_{\mathrm{dP},l}(\mathbb{E}_+)\}_l$ satisfies $\mathcal{I}_{\mathrm{dP},l}(\mathbb{E}_+) \subseteq \mathcal{I}_{\mathrm{dP},l+1}(\mathbb{E}_+)$ and $\mathcal{I}_{\mathrm{dP},l}(\mathbb{E}_+) \subseteq \mathcal{COP}(\mathbb{E}_+)$ for all $l \in \mathbb{N}$. Next, let $\mathcal{A} \in \operatorname{int} \mathcal{COP}(\mathbb{E}_+)$. Using $\mathcal{A}$, we define

$$
\begin{aligned}
L(\mathcal{A}; c_1, \dots, c_r) &:= \max_{1 \le i,j \le r} |\mathcal{A}(c_1, \dots, c_r)_{ij}| = \max_{1 \le i,j \le r} |\langle \mathcal{A}, c_i \otimes c_j \rangle|, \\
\lambda(\mathcal{A}; c_1, \dots, c_r) &:= \min_{\boldsymbol{x} \in \Delta^{r-1}_{=}} \boldsymbol{x}^\top \mathcal{A}(c_1, \dots, c_r) \boldsymbol{x}
\end{aligned}
$$

for each $(c_1, \dots, c_r) \in \mathfrak{F}(\mathbb{E})$, and also define

$$
\begin{aligned}
L(\mathcal{A}; \tilde{\mathfrak{F}}) &:= \sup_{(c_1, \dots, c_r) \in \tilde{\mathfrak{F}}} L(\mathcal{A}; c_1, \dots, c_r), \\
\lambda(\mathcal{A}; \tilde{\mathfrak{F}}) &:= \inf_{(c_1, \dots, c_r) \in \tilde{\mathfrak{F}}} \lambda(\mathcal{A}; c_1, \dots, c_r)
\end{aligned}
$$

for $\tilde{\mathfrak{F}} \in \{\mathfrak{F}, \mathfrak{F}(\mathbb{E})\}$. Given that $\mathcal{A} \in \operatorname{int} \mathcal{COP}(\mathbb{E}_+)$ and that the sets $\Delta^{r-1}_{=}$ and $\mathfrak{F}(\mathbb{E})$ are compact, we have $L(\mathcal{A}; \mathfrak{F}(\mathbb{E})) < +\infty$ and $\lambda(\mathcal{A}; \mathfrak{F}(\mathbb{E})) > 0$. Because $L(\mathcal{A}; \mathfrak{F}) \le L(\mathcal{A}; \mathfrak{F}(\mathbb{E}))$ and $\lambda(\mathcal{A}; \mathfrak{F}) \ge \lambda(\mathcal{A}; \mathfrak{F}(\mathbb{E}))$, we obtain $L(\mathcal{A}; \mathfrak{F}) < +\infty$ and $\lambda(\mathcal{A}; \mathfrak{F}) > 0$. Now, let l_0 be the smallest integer greater than or equal to $\frac{L(\mathcal{A}; \mathfrak{F})}{\lambda(\mathcal{A}; \mathfrak{F})}$ and fix $(c_1, \dots, c_r) \in \mathfrak{F}$ arbitrarily. Then, because $\mathcal{A}(c_1, \dots, c_r) \in \operatorname{int} \mathcal{COP}^r$ and

$$
l_0 > \frac{L(\mathcal{A}; \mathfrak{F})}{\lambda(\mathcal{A}; \mathfrak{F})} - 2 \ge \frac{L(\mathcal{A}; c_1, \dots, c_r)}{\lambda(\mathcal{A}; c_1, \dots, c_r)} - 2,
$$

Theorem 2.21 implies that $\mathcal{A}(c_1, \dots, c_r) \in \mathcal{I}^r_{\mathrm{dP},l_0}$. Because l_0 is independent of the choice of $(c_1, \dots, c_r) \in \mathfrak{F}$, we obtain $\mathcal{A} \in \mathcal{I}_{\mathrm{dP},l_0}(\mathbb{E}_+) \subseteq \bigcup_{l \in \mathbb{N}} \mathcal{I}_{\mathrm{dP},l}(\mathbb{E}_+)$. □

Remark 3.19. As it can be seen from the proof of Proposition 3.18, if a non-decreasing sequence $\{\mathcal{I}^r_l\}_l$ satisfies $\mathcal{I}^r_{\mathrm{dP},l} \subseteq \mathcal{I}^r_l \subseteq \mathcal{COP}^r$ for all $l \in \mathbb{N}$, the sequence obtained by replacing $\mathcal{COP}^r$ on the right-hand side set in (3.11) with $\mathcal{I}^r_l$ is also an inner-approximation hierarchy for $\mathcal{COP}(\mathbb{E}_+)$. In addition, Proposition 3.18 can be extended to the case of general m by using the polyhedral inner-approximation hierarchy provided by Iqbal and Ahmed [58, Equation (20)], which is a generalization of that provided by de Klerk and Pasechnik [23], for $\mathcal{COP}^{r,m}$.

Note that $\mathcal{I}_{\mathrm{dP},l}(\mathbb{E}_+)$ is defined as the intersection of the *infinitely* many sets in general even if m, the order of tensors, is limited to 2. This means that the inner-approximation hierarchy induces a semi-infinite conic constraint. Initially, the tractability of the approximation hierarchy seems to be the same as the copositive cone because $\mathcal{COP}(\mathbb{E}_+)$ can also be described by a semi-infinite constraint. However, when the symmetric cone $\mathbb{E}_+$ is the direct product of a nonnegative orthant and a *single* second-order cone, each $\mathcal{I}_{\mathrm{dP},l}(\mathbb{E}_+)$ can be represented by *finitely* many positive semidefinite constraints.

3.3.1.1 Full expression of $\mathcal{I}_{\mathrm{dP},l}(\mathbb{E}_+)$

Let $(\mathbb{E}_1, \circ_1, \bullet_1)$ and $(\mathbb{E}_2, \circ_2, \bullet_2)$ be the Euclidean Jordan algebras associated with the nonnegative orthant $\mathbb{R}^{n_1}_+$ and second-order cone $\mathbb{L}^{n_2}$ shown in Examples 2.6 and 2.7, respectively. Then, $\mathbb{E} := \mathbb{E}_1 \times \mathbb{E}_2 = \mathbb{R}^{n_1+n_2}$ is the Euclidean Jordan algebra with the induced symmetric cone $\mathbb{R}^{n_1}_+ \times \mathbb{L}^{n_2}$ and rank $r = n_1 + 2$. We set $n := n_1 + n_2$ and reindex $(1, \dots, r)$ as $(11, \dots, 1n_1, 21, 22)$, i.e., $1i := i$ for $i = 1, \dots, n_1$ and $2i := n_1 + i$ for $i = 1, 2$. Section 3.3.2 uses this notation. In addition,

$$\mathfrak{F} = \left\{ \left(\begin{pmatrix} \boldsymbol{e}_{11} \\ \hline \boldsymbol{0}_{n_2} \end{pmatrix}, \dots, \begin{pmatrix} \boldsymbol{e}_{1n_1} \\ \hline \boldsymbol{0}_{n_2} \end{pmatrix}, \frac{1}{2} \begin{pmatrix} \boldsymbol{0}_{n_1} \\ \hline 1 \\ \boldsymbol{v} \end{pmatrix}, \frac{1}{2} \begin{pmatrix} \boldsymbol{0}_{n_1} \\ \hline 1 \\ -\boldsymbol{v} \end{pmatrix} \right) \;\middle|\; \boldsymbol{v} \in S^{n_2-2} \right\} \tag{3.13}$$

is a set satisfying $\mathfrak{F}_{\mathrm{c}}(\mathbb{E}) \subseteq \mathfrak{F} \subseteq \mathfrak{F}(\mathbb{E})$ (see Example 2.6, Example 2.7, and Proposition 2.10).

Under the identification between $\mathcal{S}^{n,2}$ and $\mathbb{S}^n$, the inner-approximation hierarchy (3.12) with the set (3.13) is

$$\mathcal{I}_{\mathrm{dP},l}(\mathbb{E}_+) = \bigcap_{\boldsymbol{v} \in S^{n_2-2}} \{\boldsymbol{A} \in \mathbb{S}^n \mid f_l(\boldsymbol{x}; \boldsymbol{A}, \boldsymbol{v}) \text{ has only nonnegative coefficients}\},$$

where

$$f(\boldsymbol{x}; \boldsymbol{A}, \boldsymbol{v}) := \begin{pmatrix} 2\sum_{i=1}^{n_1} x_{1i}\boldsymbol{e}_{1i} \\ x_{21} + x_{22} \\ (x_{21} - x_{22})\boldsymbol{v} \end{pmatrix}^\top \boldsymbol{A} \begin{pmatrix} 2\sum_{i=1}^{n_1} x_{1i}\boldsymbol{e}_{1i} \\ x_{21} + x_{22} \\ (x_{21} - x_{22})\boldsymbol{v} \end{pmatrix}, \tag{3.14}$$

$$f_l(\boldsymbol{x}; \boldsymbol{A}, \boldsymbol{v}) := \left(\sum_{i=1}^{n} x_i\right)^l f(\boldsymbol{x}; \boldsymbol{A}, \boldsymbol{v}).$$

Note that $f(\boldsymbol{x}; \boldsymbol{A}, \boldsymbol{v})$ is doubled for the convenience of the following calculation, but the set $\mathcal{I}_{\mathrm{dP},l}(\mathbb{E}_+)$ is unchanged. Let $\boldsymbol{A} \in \mathbb{S}^n$ be partitioned as follows:

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{A}^{(11)} & \boldsymbol{A}^{(121)} & (\boldsymbol{A}^{(122)})^\top \\ (\boldsymbol{A}^{(121)})^\top & A^{(2121)} & (\boldsymbol{A}^{(2122)})^\top \\ \boldsymbol{A}^{(122)} & \boldsymbol{A}^{(2122)} & \boldsymbol{A}^{(2222)} \end{pmatrix}$$

with $\boldsymbol{A}^{(11)} \in \mathbb{S}^{n_1}$, $\boldsymbol{A}^{(121)} \in \mathbb{R}^{n_1}$, $\boldsymbol{A}^{(122)} \in \mathbb{R}^{(n_2-1)\times n_1}$, $A^{(2121)} \in \mathbb{R}$, $\boldsymbol{A}^{(2122)} \in \mathbb{R}^{n_2-1}$, and $\boldsymbol{A}^{(2222)} \in \mathbb{S}^{n_2-1}$. In addition, for such $\boldsymbol{A} \in \mathbb{S}^n$ and

$$\boldsymbol{\alpha} = (\underbrace{\alpha_{11}; \dots; \alpha_{1n_1}}_{:=\boldsymbol{\alpha}_1}; \alpha_{21}; \alpha_{22}) \in \mathbb{N}^r,$$

we let

$$\begin{aligned} M^{(11)}(\boldsymbol{A}, \boldsymbol{\alpha}) &:= 4\{\boldsymbol{\alpha}_1^\top \boldsymbol{A}^{(11)}\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_1^\top \operatorname{diag}(\boldsymbol{A}^{(11)})\} + 4(\alpha_{21} + \alpha_{22})\boldsymbol{\alpha}_1^\top \boldsymbol{A}^{(121)} \\ &\quad + (\alpha_{21} + \alpha_{22})(\alpha_{21} + \alpha_{22} - 1)A^{(2121)}, \\ \boldsymbol{M}^{(21)}(\boldsymbol{A}, \boldsymbol{\alpha}) &:= 2(\alpha_{21} - \alpha_{22})\boldsymbol{A}^{(122)}\boldsymbol{\alpha}_1 + (\alpha_{21} - \alpha_{22})(\alpha_{21} + \alpha_{22} - 1)\boldsymbol{A}^{(2122)}, \\ \boldsymbol{M}^{(22)}(\boldsymbol{A}, \boldsymbol{\alpha}) &:= \{(\alpha_{21} - \alpha_{22})^2 - (\alpha_{21} + \alpha_{22})\}\boldsymbol{A}^{(2222)} \end{aligned}$$

and define

$$\boldsymbol{M}(\boldsymbol{A},\boldsymbol{\alpha}) := \begin{pmatrix} M^{(11)}(\boldsymbol{A},\boldsymbol{\alpha}) & \boldsymbol{M}^{(21)}(\boldsymbol{A},\boldsymbol{\alpha})^\top \\ \boldsymbol{M}^{(21)}(\boldsymbol{A},\boldsymbol{\alpha}) & \boldsymbol{M}^{(22)}(\boldsymbol{A},\boldsymbol{\alpha}) \end{pmatrix} \in \mathbb{S}^{n_2}.$$

Then, after some calculations, we have

$$f_l(\boldsymbol{x};\boldsymbol{A},\boldsymbol{v}) = \sum_{\boldsymbol{\alpha}\in\mathbb{I}^r_{=l+2}} \frac{l!}{\boldsymbol{\alpha}!} \begin{pmatrix} 1 \\ \boldsymbol{v} \end{pmatrix}^\top \boldsymbol{M}(\boldsymbol{A},\boldsymbol{\alpha}) \begin{pmatrix} 1 \\ \boldsymbol{v} \end{pmatrix} \boldsymbol{x}^{\boldsymbol{\alpha}}.$$

Therefore,

$$\begin{aligned} \mathcal{I}_{\mathrm{dP},l}(\mathbb{E}_+) &= \bigcap_{\boldsymbol{\alpha}\in\mathbb{I}^r_{=l+2}} \bigcap_{\boldsymbol{v}\in S^{n_2-2}} \left\{ \boldsymbol{A}\in\mathbb{S}^n \;\middle|\; \begin{pmatrix} 1 \\ \boldsymbol{v} \end{pmatrix}^\top \boldsymbol{M}(\boldsymbol{A},\boldsymbol{\alpha}) \begin{pmatrix} 1 \\ \boldsymbol{v} \end{pmatrix} \geq 0 \right\} \\ &= \bigcap_{\boldsymbol{\alpha}\in\mathbb{I}^r_{=l+2}} \left\{ \boldsymbol{A}\in\mathbb{S}^n \;\middle|\; \begin{array}{l} \text{There exists } t_{\boldsymbol{\alpha}}\in\mathbb{R} \text{ such that} \\ \boldsymbol{M}(\boldsymbol{A},\boldsymbol{\alpha}) - t_{\boldsymbol{\alpha}} \begin{pmatrix} 1 & \\ & -\boldsymbol{I}_{n_2-1} \end{pmatrix} \in \mathbb{S}^{n_2}_+ \end{array} \right\}, \end{aligned} \tag{3.15}$$

where the last equation follows from the S-lemma with equality [96, Proposition 3.1]. In summary, $\mathcal{I}_{\mathrm{dP},l}(\mathbb{E}_+)$ can be described by $|\mathbb{I}^r_{=l+2}|$ positive semidefinite constraints, which is bounded by r^{l+2} and whose size is n_2. We call the sequence $\{\mathcal{I}_{\mathrm{dP},l}(\mathbb{E}_+)\}_l$ the de Klerk–Pasechnik (dP)-type inner-approximation hierarchy.

3.3.1.2 Concise expression of $\mathcal{I}_{\mathrm{dP},l}(\mathbb{E}_+)$

In this subsubsection, we explain that the expression (3.15) can be made more concise. First, we show that the number of constraints in (3.15) can be halved. For $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1;\alpha_{21};\alpha_{22}) \in \mathbb{I}^r_{=l+2}$, let $\tilde{\boldsymbol{\alpha}} := (\boldsymbol{\alpha}_1;\alpha_{22};\alpha_{21}) \in \mathbb{I}^r_{=l+2}$. As $M^{(11)}(\boldsymbol{A},\tilde{\boldsymbol{\alpha}}) = M^{(11)}(\boldsymbol{A},\boldsymbol{\alpha})$, $\boldsymbol{M}^{(21)}(\boldsymbol{A},\tilde{\boldsymbol{\alpha}}) = -\boldsymbol{M}^{(21)}(\boldsymbol{A},\boldsymbol{\alpha})$, and $\boldsymbol{M}^{(22)}(\boldsymbol{A},\tilde{\boldsymbol{\alpha}}) = \boldsymbol{M}^{(22)}(\boldsymbol{A},\boldsymbol{\alpha})$ for each $\boldsymbol{A}\in\mathbb{S}^n$, we have

$$\begin{aligned} &\begin{pmatrix} 1 \\ \boldsymbol{v} \end{pmatrix}^\top \boldsymbol{M}(\boldsymbol{A},\tilde{\boldsymbol{\alpha}}) \begin{pmatrix} 1 \\ \boldsymbol{v} \end{pmatrix} \geq 0 \text{ for all } \boldsymbol{v}\in S^{n_2-2} \\ &\iff \begin{pmatrix} 1 \\ \boldsymbol{v} \end{pmatrix}^\top \begin{pmatrix} M^{(11)}(\boldsymbol{A},\boldsymbol{\alpha}) & -\boldsymbol{M}^{(21)}(\boldsymbol{A},\boldsymbol{\alpha})^\top \\ -\boldsymbol{M}^{(21)}(\boldsymbol{A},\boldsymbol{\alpha}) & \boldsymbol{M}^{(22)}(\boldsymbol{A},\boldsymbol{\alpha}) \end{pmatrix} \begin{pmatrix} 1 \\ \boldsymbol{v} \end{pmatrix} \geq 0 \text{ for all } \boldsymbol{v}\in S^{n_2-2} \\ &\iff \begin{pmatrix} 1 \\ -\boldsymbol{v} \end{pmatrix}^\top \begin{pmatrix} M^{(11)}(\boldsymbol{A},\boldsymbol{\alpha}) & \boldsymbol{M}^{(21)}(\boldsymbol{A},\boldsymbol{\alpha})^\top \\ \boldsymbol{M}^{(21)}(\boldsymbol{A},\boldsymbol{\alpha}) & \boldsymbol{M}^{(22)}(\boldsymbol{A},\boldsymbol{\alpha}) \end{pmatrix} \begin{pmatrix} 1 \\ -\boldsymbol{v} \end{pmatrix} \geq 0 \text{ for all } \boldsymbol{v}\in S^{n_2-2}. \end{aligned} \tag{3.16}$$

When $\boldsymbol{v}$ takes every element of S^{n_2-2}, $-\boldsymbol{v}$ also takes every element of S^{n_2-2}. Thus, (3.16) is equivalent to

$$\begin{pmatrix} 1 \\ \boldsymbol{v} \end{pmatrix}^\top \boldsymbol{M}(\boldsymbol{A},\boldsymbol{\alpha}) \begin{pmatrix} 1 \\ \boldsymbol{v} \end{pmatrix} \geq 0 \text{ for all } \boldsymbol{v}\in S^{n_2-2}.$$

That is, taking the intersection with respect to $\boldsymbol{\alpha}\in\mathbb{I}^r_{=l+2}$ with $\alpha_{21}\leq\alpha_{22}$ in (3.15) is sufficient. This can be explained by the fact that the ordering of a Jordan frame can be ignored because $\mathcal{I}^r_{\mathrm{dP},l}$ is permutation-invariant; thus, we can apply Theorem 3.17.

Next, we show that some positive semidefinite constraints in (3.15) can be written as second-order cone or nonnegativity constraints. If $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1; \alpha_{21}; \alpha_{22}) \in \mathbb{I}^r_{=l+2}$ satisfies

$$\alpha_{21} = \frac{1}{2}k(k-1),\ \alpha_{22} = \frac{1}{2}k(k+1)$$

for some $k \in \mathbb{N}$, then $\boldsymbol{M}^{(22)}(\boldsymbol{A}, \boldsymbol{\alpha}) = \boldsymbol{O}$. In this case,

$$\begin{aligned}
&\begin{pmatrix} 1 \\ \boldsymbol{v} \end{pmatrix}^\top \boldsymbol{M}(\boldsymbol{A}, \boldsymbol{\alpha}) \begin{pmatrix} 1 \\ \boldsymbol{v} \end{pmatrix} \geq 0 \text{ for all } \boldsymbol{v} \in S^{n_2-2} \\
&\iff M^{(11)}(\boldsymbol{A}, \boldsymbol{\alpha}) + 2\boldsymbol{M}^{(21)}(\boldsymbol{A}, \boldsymbol{\alpha})^\top \boldsymbol{v} \geq 0 \text{ for all } \boldsymbol{v} \in S^{n_2-2} \\
&\iff M^{(11)}(\boldsymbol{A}, \boldsymbol{\alpha}) - 2\|\boldsymbol{M}^{(21)}(\boldsymbol{A}, \boldsymbol{\alpha})\|_2 \geq 0 \\
&\iff \begin{pmatrix} M^{(11)}(\boldsymbol{A}, \boldsymbol{\alpha}) \\ 2\boldsymbol{M}^{(21)}(\boldsymbol{A}, \boldsymbol{\alpha}) \end{pmatrix} \in \mathbb{L}^{n_2}.
\end{aligned}$$

In particular, when $k = 0$, i.e., $\alpha_{21} = \alpha_{22} = 0$, as $\boldsymbol{M}^{(21)}(\boldsymbol{A}, \boldsymbol{\alpha})$ is also zero, the above second-order cone constraint reduces to the nonnegativity constraint $M^{(11)}(\boldsymbol{A}, \boldsymbol{\alpha}) \geq 0$.

Finally, we show that the size of some positive semidefinite constraints can be reduced by 1. If $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1; \alpha_{21}; \alpha_{22}) \in \mathbb{I}^r_{=l+2}$ satisfies $\alpha_{21} = \alpha_{22} \neq 0$, we have $\boldsymbol{M}^{(21)}(\boldsymbol{A}, \boldsymbol{\alpha}) = \boldsymbol{0}$. Then,

$$\begin{aligned}
&\begin{pmatrix} 1 \\ \boldsymbol{v} \end{pmatrix}^\top \boldsymbol{M}(\boldsymbol{A}, \boldsymbol{\alpha}) \begin{pmatrix} 1 \\ \boldsymbol{v} \end{pmatrix} \geq 0 \text{ for all } \boldsymbol{v} \in S^{n_2-2} \\
&\iff M^{(11)}(\boldsymbol{A}, \boldsymbol{\alpha}) + \boldsymbol{v}^\top \boldsymbol{M}^{(22)}(\boldsymbol{A}, \boldsymbol{\alpha}) \boldsymbol{v} \geq 0 \text{ for all } \boldsymbol{v} \in S^{n_2-2} \\
&\iff M^{(11)}(\boldsymbol{A}, \boldsymbol{\alpha}) + \lambda_{\min}(\boldsymbol{M}^{(22)}(\boldsymbol{A}, \boldsymbol{\alpha})) \geq 0 \\
&\iff M^{(11)}(\boldsymbol{A}, \boldsymbol{\alpha}) \boldsymbol{I}_{n_2-1} + \boldsymbol{M}^{(22)}(\boldsymbol{A}, \boldsymbol{\alpha}) \in \mathbb{S}^{n_2-1}_+,
\end{aligned}$$

where $\lambda_{\min}(\cdot)$ denotes the minimum eigenvalue for an input matrix.

3.3.2 Outer-approximation hierarchy

Unlike the case of inner-approximation hierarchies, an outer-approximation hierarchy for $\mathcal{COP}^{r,m}$ always induces that for $\mathcal{COP}^{n,m}(\mathbb{E}_+)$.

Proposition 3.20. *Let $\{\mathcal{O}^{r,m}_l\}_l$ be an outer-approximation hierarchy for $\mathcal{COP}^{r,m}$ such that each $\mathcal{O}^{r,m}_l$ is a closed convex cone. We fix a set $\mathfrak{F}$ with $\mathfrak{F}_{\mathrm{c}}(\mathbb{E}) \subseteq \mathfrak{F} \subseteq \mathfrak{F}(\mathbb{E})$. Let*

$$\mathcal{O}^{n,m}_l(\mathbb{E}_+) := \bigcap_{(c_1,\dots,c_r)\in\mathfrak{F}} \{\mathcal{A} \in \mathcal{S}^{n,m}(\mathbb{E}) \mid \mathcal{A}(c_1, \dots, c_r) \in \mathcal{O}^{r,m}_l\}. \tag{3.17}$$

Then, each $\mathcal{O}^{n,m}_l(\mathbb{E}_+)$ is a closed convex cone and $\{\mathcal{O}^{n,m}_l(\mathbb{E}_+)\}_l$ is an outer-approximation hierarchy for $\mathcal{COP}^{n,m}(\mathbb{E}_+)$.

Proof. Because $\mathcal{O}^{n,m}_l(\mathbb{E}_+)$ can easily be shown to be a closed convex cone and $\mathcal{O}^{n,m}_{l+1}(\mathbb{E}_+) \subseteq \mathcal{O}^{n,m}_l(\mathbb{E}_+)$ for all $l \in \mathbb{N}$ in the same manner as Proposition 3.18, we only prove $\mathcal{COP}^{n,m}(\mathbb{E}_+) = \bigcap_{l\in\mathbb{N}} \mathcal{O}_l^{n,m}(\mathbb{E}_+)$. The "$\subseteq$" part follows from Theorem 3.17. To prove the "$\supseteq$" part, let $\mathcal{A} \in \bigcap_{l\in\mathbb{N}} \mathcal{O}_l^{n,m}(\mathbb{E}_+)$. Then, $\mathcal{A}(c_1,\dots,c_r) \in \mathcal{O}_l^{r,m}$ for all $l \in \mathbb{N}$ and $(c_1,\dots,c_r) \in \mathfrak{F}$. By $\mathcal{COP}^{r,m} = \bigcap_{l\in\mathbb{N}} \mathcal{O}_l^{r,m}$, we have $\mathcal{A}(c_1,\dots,c_r) \in \mathcal{COP}^{r,m}$ for all $(c_1,\dots,c_r) \in \mathfrak{F}$. Thus, from Theorem 3.17, we obtain $\mathcal{A} \in \mathcal{COP}^{n,m}(\mathbb{E}_+)$. $\square$

Remark 3.21. The set $\bigcup_{l\in\mathbb{N}} \mathcal{I}_{\mathrm{dP},l}(\mathbb{E}_+)$ involves the intersection with respect to Jordan frames and the union with respect to approximation orders. Therefore, to establish the convergence $\operatorname{int}\mathcal{COP}(\mathbb{E}_+) \subseteq \bigcup_{l\in\mathbb{N}} \mathcal{I}_{\mathrm{dP},l}(\mathbb{E}_+)$ in the proof of Proposition 3.18, it is necessary to take l_0 such that it is independent of Jordan frames.

On the other hand, the set $\bigcap_{l\in\mathbb{N}} \mathcal{O}_l^{n,m}(\mathbb{E}_+)$ involves the intersection with respect to approximation orders as well as Jordan frames. Thus, as seen in the proof of Proposition 3.20, the sequence of approximation orders and Jordan frames is interchangeable in showing the convergence $\mathcal{COP}^{n,m}(\mathbb{E}_+) = \bigcap_{l\in\mathbb{N}} \mathcal{O}_l^{n,m}(\mathbb{E}_+)$, which makes the proof easier than the inner-approximation hierarchy.

As in Proposition 3.18, the outer-approximation hierarchy induces a semi-infinite conic constraint. However, as with the inner-approximation hierarchy, when the symmetric cone $\mathbb{E}_+$ is the direct product of a nonnegative orthant and second-order cone and $m = 2$, if we choose an appropriate polyhedral outer-approximation hierarchy as $\{\mathcal{O}_l^{r,2}\}_l$ (written as $\{\mathcal{O}_l^{r}\}_l$ hereafter), each $\mathcal{O}_l^{n,2}(\mathbb{E}_+)$ (written as $\mathcal{O}_l(\mathbb{E}_+)$ hereafter) can be represented by finitely many positive semidefinite constraints.

3.3.2.1 Full expression of $\mathcal{O}_l(\mathbb{E}_+)$

Let $\mathbb{E}$ be the same Euclidean Jordan algebra with the induced symmetric cone $\mathbb{R}_+^{n_1} \times \mathbb{L}^{n_2}$ as defined in Section 3.3.1.1. The polyhedral outer-approximation hierarchy $\{\mathcal{O}_l^r\}_l$ we use is based on a discretization of the standard simplex and written as

$$\mathcal{O}_l^r = \bigcap_{\boldsymbol{x}\in\delta_l^{r-1}} \{\boldsymbol{A} \in \mathbb{S}^n \mid \boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} \geq 0\}, \tag{3.18}$$

where δ_l^{r-1} is a finite subset of $\Delta_=^{r-1}$ for each $l \in \mathbb{N}$. This type of outer-approximation hierarchy includes, for example, the hierarchy $\{\mathcal{O}_{\mathrm{Yl},l}^r\}_l$ introduced in Section 2.5.2. The outer-approximation hierarchy (3.17) induced by the set (3.13) and polyhedral outer-approximation hierarchy (3.18) is

$$\mathcal{O}_l(\mathbb{E}_+) = \bigcap_{\boldsymbol{v}\in S^{n_2-2}} \bigcap_{\boldsymbol{x}\in\delta_l^{r-1}} \{\boldsymbol{A} \in \mathbb{S}^n \mid f(\boldsymbol{x};\boldsymbol{A},\boldsymbol{v}) \geq 0\},$$

where $f(\boldsymbol{x};\boldsymbol{A},\boldsymbol{v})$ is defined as (3.14). Let

$$\boldsymbol{N}(\boldsymbol{x},\boldsymbol{A}) := \sum_{\boldsymbol{\alpha}\in\mathbb{I}_{=2}^r} \frac{1}{\boldsymbol{\alpha}!}\boldsymbol{M}(\boldsymbol{A},\boldsymbol{\alpha})\boldsymbol{x}^{\boldsymbol{\alpha}}.$$

Then,

$$\begin{aligned}\mathcal{O}_l(\mathbb{E}_+) &= \bigcap_{\boldsymbol{x}\in\delta_l^{r-1}} \bigcap_{\boldsymbol{v}\in S^{n_2-2}} \left\{ \boldsymbol{A}\in\mathbb{S}^n \;\middle|\; \begin{pmatrix}1\\ \boldsymbol{v}\end{pmatrix}^\top \boldsymbol{N}(\boldsymbol{x},\boldsymbol{A}) \begin{pmatrix}1\\ \boldsymbol{v}\end{pmatrix} \ge 0 \right\} \\ &= \bigcap_{\boldsymbol{x}\in\delta_l^{r-1}} \left\{ \boldsymbol{A}\in\mathbb{S}^n \;\middle|\; \begin{array}{l}\text{There exists } t_{\boldsymbol{x}}\in\mathbb{R} \text{ such that} \\ \boldsymbol{N}(\boldsymbol{x},\boldsymbol{A}) - t_{\boldsymbol{x}}\begin{pmatrix}1 & \\ & -\boldsymbol{I}_{n_2-1}\end{pmatrix} \in \mathbb{S}_+^{n_2}\end{array} \right\}. \end{aligned} \tag{3.19}$$

In summary, $\mathcal{O}_l(\mathbb{E}_+)$ can be described by the $|\delta_l^{r-1}|$ positive semidefinite constraints of size n_2. In particular, if we use the outer-approximation hierarchy $\{\mathcal{O}_l^r\}_l$ shown in (2.30), then δ_l^{r-1} is given by (2.29) and $|\delta_l^{r-1}|$ is polynomial in l for every fixed $l\in\mathbb{N}$. We call the sequence $\{\mathcal{O}_l(\mathbb{E}_+)\}_l$ obtained by exploiting the hierarchy (2.30) the Yıldırım-type outer-approximation hierarchy.

3.3.2.2 Concise expression of $\mathcal{O}_l(\mathbb{E}_+)$

As with the inner-approximation hierarchy, we can make the expression (3.19) more concise. Let $\boldsymbol{N}(\boldsymbol{x},\boldsymbol{A})$ be partitioned as follows:

$$\boldsymbol{N}(\boldsymbol{x},\boldsymbol{A}) = \begin{pmatrix} N^{(11)}(\boldsymbol{x},\boldsymbol{A}) & \boldsymbol{N}^{(21)}(\boldsymbol{x},\boldsymbol{A})^\top \\ \boldsymbol{N}^{(21)}(\boldsymbol{x},\boldsymbol{A}) & \boldsymbol{N}^{(22)}(\boldsymbol{x},\boldsymbol{A}) \end{pmatrix},$$

where $N^{(11)}(\boldsymbol{x},\boldsymbol{A})\in\mathbb{R}$, $\boldsymbol{N}^{(21)}(\boldsymbol{x},\boldsymbol{A})\in\mathbb{R}^{n_2-1}$, and $\boldsymbol{N}^{(22)}(\boldsymbol{x},\boldsymbol{A})\in\mathbb{S}^{n_2-1}$ and $\boldsymbol{x} = (\boldsymbol{x}_1; x_{21}; x_{22})\in\mathbb{R}^r$. Then, we have

$$\begin{aligned} N^{(11)}(\boldsymbol{x},\boldsymbol{A}) &= 4\boldsymbol{x}_1^\top \boldsymbol{A}^{(11)}\boldsymbol{x}_1 + 4(x_{21}+x_{22})\boldsymbol{x}_1^\top \boldsymbol{A}^{(121)} + (x_{21}+x_{22})^2 A^{(2121)}, \\ \boldsymbol{N}^{(21)}(\boldsymbol{x},\boldsymbol{A}) &= 2(x_{21}-x_{22})\boldsymbol{A}^{(122)}\boldsymbol{x}_1 + (x_{21}^2 - x_{22}^2)\boldsymbol{A}^{(2122)}, \\ \boldsymbol{N}^{(22)}(\boldsymbol{x},\boldsymbol{A}) &= (x_{21}-x_{22})^2\boldsymbol{A}^{(2222)}. \end{aligned}$$

First, the number of constraints in (3.19) may be reduced. Suppose that δ_l^{r-1} is permutation-invariant in the sense of Definition 3.14.[2] For example, (2.29) is permutation-invariant. For $\boldsymbol{x} = (\boldsymbol{x}_1; x_{21}; x_{22}) \in \delta_l^{r-1}$, let $\tilde{\boldsymbol{x}} := (\boldsymbol{x}_1; x_{22}; x_{21})$. Then $\tilde{\boldsymbol{x}}\in\delta_l^{r-1}$ because δ_l^{r-1} is permutation-invariant. Given that $N^{(11)}(\tilde{\boldsymbol{x}},\boldsymbol{A}) = N^{(11)}(\boldsymbol{x},\boldsymbol{A})$, $\boldsymbol{N}^{(21)}(\tilde{\boldsymbol{x}},\boldsymbol{A}) = -\boldsymbol{N}^{(21)}(\boldsymbol{x},\boldsymbol{A})$, and $\boldsymbol{N}^{(22)}(\tilde{\boldsymbol{x}},\boldsymbol{A}) = \boldsymbol{N}^{(22)}(\boldsymbol{x},\boldsymbol{A})$ for each $\boldsymbol{A}\in\mathbb{S}^n$, taking the intersection with respect to $\boldsymbol{x}\in\delta_l^{r-1}$ with $x_{21}\le x_{22}$ in (3.19) is sufficient for the same reason as for Section 3.3.1.2.

Second, some positive semidefinite constraints in (3.19) can be written as nonnegativity constraints. If $\boldsymbol{x} = (\boldsymbol{x}_1; x_{21}; x_{22}) \in \delta_l^{r-1}$ satisfies $x_{21} = x_{22}$, then we have $\boldsymbol{N}^{(21)}(\boldsymbol{x},\boldsymbol{A}) = \boldsymbol{0}$ and $\boldsymbol{N}^{(22)}(\boldsymbol{x},\boldsymbol{A}) = \boldsymbol{O}$. Therefore, the constraint

$$\begin{pmatrix}1\\ \boldsymbol{v}\end{pmatrix}^\top \boldsymbol{N}(\boldsymbol{x},\boldsymbol{A}) \begin{pmatrix}1\\ \boldsymbol{v}\end{pmatrix} \ge 0 \text{ for all } \boldsymbol{v}\in S^{n_2-2}$$

reduces to $N^{(11)}(\boldsymbol{x},\boldsymbol{A}) \ge 0$.

[2] Note that $\mathcal{S}^{r,1} = \mathbb{R}^r$.

3.4 Comparison with Other Approximation Hierarchies

The previous sections provide approximation hierarchies applicable to the copositive cone over the cone $\mathbb{K} = \mathbb{R}^{n_1}_+ \times \mathbb{L}^{n_2}$. For such $\mathbb{K}$, the ZVP- and Lasserre-type approximation hierarchies introduced in Section 2.5 are also applicable to $\mathcal{COP}(\mathbb{K})$. In this section, we analytically compare the number and size of positive semidefinite constraints defining these approximation hierarchies for the copositive cone $\mathcal{COP}(\mathbb{R}^{n_1}_+ \times \mathbb{L}^{n_2})$. Throughout this section, we let $\mathbb{K} = \mathbb{R}^{n_1}_+ \times \mathbb{L}^{n_2}$. In addition, we set $n := n_1 + n_2$ as in Section 3.3 and reindex $(1, \dots, n)$ as $(11, \dots, 1n_1, 21, \dots, 2n_2)$, i.e., $1i := i$ for $i = 1, \dots, n_1$ and $2i := n_1 + i$ for $i = 1, \dots, n_2$. We use this notation for numerical experiments in Section 3.5 as well.

First, the cone $\mathbb{K}$ can be represented as a semialgebraic set

$$\left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \phi^{(1)}_i(\boldsymbol{x}) := x_{1i} \geq 0 \ (i = 1, \dots, n_1), \\ \phi^{(1)}_{n_1+1}(\boldsymbol{x}) := x_{21} \geq 0, \\ \phi^{(1)}_{n_1+2}(\boldsymbol{x}) := \boldsymbol{e}^\top \boldsymbol{x} \geq 0, \\ \phi^{(2)}(\boldsymbol{x}) := x_{21}^2 - \sum_{i=2}^{n_2} x_{2i}^2 \geq 0 \end{array} \right\},$$

where $\boldsymbol{e} := (\boldsymbol{1}_{n_1+1}; \boldsymbol{0}_{n_2-1}) \in \operatorname{int} \mathbb{K}$ and the inequality $\boldsymbol{e}^\top \boldsymbol{x} \geq 0$ is redundant but necessary to deriving the hierarchy. Thus, the ZVP-type inner-approximation hierarchy $\{\mathcal{I}_{\mathrm{ZVP},l}(\mathbb{K})\}_l$ defined as (2.28) is applicable to $\mathcal{COP}(\mathbb{K})$.

Since $\mathcal{I}_{\mathrm{ZVP},l}(\mathbb{K}) = \{\boldsymbol{A} \in \mathbb{S}^n \mid (\boldsymbol{e}^\top \boldsymbol{x})^l \boldsymbol{x}^\top \boldsymbol{A} \boldsymbol{x} \in E^{n,l+2}(\mathbb{K})\}$, the size and number of the positive semidefinite constraints defining $\mathcal{I}_{\mathrm{ZVP},l}(\mathbb{K})$ precisely match those of the positive semidefinite constraints defining the set $E^{n,l+2}(\mathbb{K})$, defined as (2.27). To compute them, we furnish a recursive representation of $E^{n,m}(\mathbb{K})$ in the following lemma. Recall that

$$E^{n,m}(\mathbb{K}) = \operatorname{conv} \left\{ \psi^2 \prod_{j=1}^k \phi_{i_j} \;\middle|\; \begin{array}{l} k \in \mathbb{N},\ m - \sum_{j=1}^k \deg(\phi_{i_j}) \in \mathbb{N} \text{ is even}, \\ \psi \in H^{n,(m - \sum_{j=1}^k \deg(\phi_{i_j}))/2}, \\ \phi_{i_j} \in \{\phi^{(1)}_1, \dots, \phi^{(1)}_{n_1+2}, \phi^{(2)}\} \ (j = 1, \dots, k) \end{array} \right\},$$

where $\deg(\phi_{i_j})$ denotes the degree of ϕ_{i_j}.

Lemma 3.22. *If $m = 2s$ for some $s \in \mathbb{N}$, then*

$$\begin{aligned} E^{n,m}(\mathbb{K}) &= \operatorname{conv} \left(\Sigma^{n,2s} \cup \left\{ \psi \phi^{(1)}_i \;\middle|\; \begin{array}{l} i = 1, \dots, n_1 + 2, \\ \psi \in E^{n,2s-1}(\mathbb{K}) \end{array} \right\} \cup \{\psi \phi^{(2)} \mid \psi \in E^{n,2s-2}(\mathbb{K})\} \right) \\ &= \left\{ \psi^{(0)} + \sum_{i=1}^{n_1+2} \psi^{(1)}_i \phi^{(1)}_i + \psi^{(2)} \phi^{(2)} \;\middle|\; \begin{array}{l} \psi^{(0)} \in \Sigma^{n,2s}, \\ \psi^{(1)}_i \in E^{n,2s-1}(\mathbb{K}), \\ \psi^{(2)} \in E^{n,2s-2}(\mathbb{K}) \end{array} \right\}. \end{aligned}$$

If $m = 2s + 1$ for some $s \in \mathbb{N}$, then

$$\begin{aligned} E^{n,m}(\mathbb{K}) &= \operatorname{conv} \left(\left\{ \psi \phi^{(1)}_i \;\middle|\; \begin{array}{l} i = 1, \dots, n_1 + 2, \\ \psi \in E^{n,2s}(\mathbb{K}) \end{array} \right\} \cup \{\psi \phi^{(2)} \mid \psi \in E^{n,2s-1}(\mathbb{K})\} \right) \\ &= \left\{ \sum_{i=1}^{n_1+2} \psi^{(1)}_i \phi^{(1)}_i + \psi^{(2)} \phi^{(2)} \;\middle|\; \begin{array}{l} \psi^{(1)}_i \in E^{n,2s}(\mathbb{K}), \\ \psi^{(2)} \in E^{n,2s-1}(\mathbb{K}) \end{array} \right\}. \end{aligned}$$

Using Lemma 3.22, we can calculate the size and number of the positive semidefinite constraints defining $E^{n,m}(\mathbb{K})$.

Proposition 3.23. *Let*

$$a_m := \frac{1}{\sqrt{r^2+4}} \left\{ \left(\frac{r+\sqrt{r^2+4}}{2} \right)^{m+1} - \left(\frac{r-\sqrt{r^2+4}}{2} \right)^{m+1} \right\}$$

for each $m \in \mathbb{N}$. Note that a_m is the mth term of the recurrence $a_{m+2} = ra_{m+1} + a_m$ with initial conditions $a_0 = 1$ and $a_1 = r$ and is of order $O(n_1^m)$. If $m = 2s$ for some $s \in \mathbb{N}$, then $E^{n,m}(\mathbb{K})$ is described by a_{2i} positive semidefinite constraints of size $|\mathbb{I}^n_{=s-i}|$ $(i = 0, \ldots, s)$. If $m = 2s+1$ for some $s \in \mathbb{N}$, then $E^{n,m}(\mathbb{K})$ is described by a_{2i+1} positive semidefinite constraints of size $|\mathbb{I}^n_{=s-i}|$ $(i = 0, \ldots, s)$.

Second, we focus on the Lasserre-type outer-approximation hierarchy (2.31). Let ν be the finite Borel measure uniformly supported on $\Delta_{\boldsymbol{e}}(\mathbb{K})$, i.e., $\nu(B) := \int_B 1_{\Delta_{\boldsymbol{e}}(\mathbb{K})} d\boldsymbol{x}$ for each B in the Borel σ-algebra of $\mathbb{R}^n$, where $1_{\Delta_{\boldsymbol{e}}(\mathbb{K})}$ is the indicator function of $\Delta_{\boldsymbol{e}}(\mathbb{K})$, and the notation $d\boldsymbol{x}$ represents the Lebesgue measure. By using the measure ν, as shown in Theorem 2.23, we can construct the Lasserre-type outer-approximation hierarchy $\{\mathcal{O}_{\mathrm{Las},l}(\mathbb{K})\}_l$ for the copositive cone $\mathcal{COP}(\mathbb{K})$. The following lemma shows the calculation of the moments $\boldsymbol{y} = (y_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha} \in \mathbb{N}^n}$ of the measure ν.

Lemma 3.24. *The moments $\boldsymbol{y} = (y_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha} \in \mathbb{N}^n}$ of the measure ν satisfy*

$$y_{\boldsymbol{\alpha}} = \begin{cases} \frac{2\boldsymbol{\alpha}_1! \left(\sum_{i=1}^{n_2} \alpha_{2i} + n_2 - 1\right)! \prod_{i=2}^{n_2} \Gamma(\beta_{2i})}{\left(\sum_{i=2}^{n_2} \alpha_{2i} + n_2 - 1\right)(n+|\boldsymbol{\alpha}|)! \Gamma\left(\sum_{i=2}^{n_2} \beta_{2i}\right)} & \text{(if all of } \alpha_{22}, \ldots, \alpha_{2n_2} \text{ are even)}, \\ 0 & \text{(if some of } \alpha_{22}, \ldots, \alpha_{2n_2} \text{ are odd)} \end{cases} \tag{3.20}$$

for each $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1; \alpha_{21}; \ldots; \alpha_{2n_2}) \in \mathbb{N}^n$, where $\Gamma(\cdot)$ denotes the gamma function and $\beta_{2i} := \frac{\alpha_{2i}+1}{2}$ for $i = 2, \ldots, n_2$.

Proof. Note that we can represent the set $\Delta_{\boldsymbol{e}}(\mathbb{K})$ as

$$\left\{ \boldsymbol{x} = (\boldsymbol{x}_1; x_{21}; \ldots; x_{2n_2}) \in \mathbb{R}^n \;\middle|\; \begin{array}{l} (\boldsymbol{x}_1; x_{21}) \in \Delta_{\leq}^{n_1+1}, \\ (x_{22}, \ldots, x_{2n_2})^\top \in B^{n_2-1}(x_{21}) \end{array} \right\},$$

where

$$\begin{aligned} \Delta_{\leq}^{n_1+1} &:= \{\boldsymbol{z} \in \mathbb{R}^{n_1+1}_+ \mid \boldsymbol{z}^\top \mathbf{1} \leq 1\}, \\ B^{n_2-1}(x_{21}) &:= \{\boldsymbol{z} \in \mathbb{R}^{n_2-1} \mid \|\boldsymbol{z}\|_2 \leq x_{21}\}. \end{aligned}$$

Then, for $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1; \alpha_{21}; \ldots; \alpha_{2n_2}) \in \mathbb{N}^n$, it follows that

$$\begin{aligned} y_{\boldsymbol{\alpha}} &= \int_{\mathbb{R}^n} \boldsymbol{x}^{\boldsymbol{\alpha}} d\nu \\ &= \int_{\Delta_{\boldsymbol{e}}(\mathbb{K})} \boldsymbol{x}^{\boldsymbol{\alpha}} d\boldsymbol{x} \\ &= \int_{\Delta_{\leq}^{n_1+1}} \boldsymbol{x}_1^{\boldsymbol{\alpha}_1} x_{21}^{\alpha_{21}} \left(\int_{B^{n_2-1}(x_{21})} x_{22}^{\alpha_{22}} \cdots x_{2n_2}^{\alpha_{2n_2}} dx_{22} \cdots dx_{2n_2} \right) d\boldsymbol{x}_1 dx_{21}. \end{aligned}$$

Table 3.1: Comparison of the number and size of positive semidefinite constraints defining the lth level of each approximation hierarchy for $\mathcal{COP}(\mathbb{K})$. $\lfloor \cdot \rfloor$ denotes the greatest integer less than or equal to an input value.

Type	Direction	Number	Size
dP (proposed)	Inner	$\lvert \mathbb{I}^{r}_{=l+2} \rvert$	n_2
ZVP	Inner	Includes at most r positive semidefinite constraints of size $\lvert \mathbb{I}^{n}_{=\lfloor \frac{l}{2} \rfloor + 1} \rvert$ (see Proposition 3.23 for details)	
NN (proposed)	Inner	1	$\lvert \mathbb{I}^{n}_{=l+2} \rvert$
Yıldırım (proposed)	Outer	$\lvert \delta_l^{r-1} \rvert$ (see (2.29))	n_2
Lasserre	Outer	1	$\lvert \mathbb{I}^{n}_{\leq l} \rvert$

From [37, page 447], we note that

$$\begin{aligned}
&\int_{B^{n_2-1}(x_{21})} x_{22}^{\alpha_{22}} \cdots x_{2n_2}^{\alpha_{2n_2}} dx_{22} \cdots dx_{2n_2} \\
&\quad = \begin{cases} \frac{2\prod_{i=2}^{n_2} \Gamma(\beta_{2i}) x_{21}^{\sum_{i=2}^{n_2} \alpha_{2i} + n_2 - 1}}{\left(\sum_{i=2}^{n_2} \alpha_{2i} + n_2 - 1\right) \Gamma(\sum_{i=2}^{n_2} \beta_{2i})} & \text{(if all of } \alpha_{22}, \dots, \alpha_{2n_2} \text{ are even),} \\ 0 & \text{(if some of } \alpha_{22}, \dots, \alpha_{2n_2} \text{ are odd).} \end{cases}
\end{aligned} \tag{3.21}$$

As (3.21) implies that $y_{\boldsymbol{\alpha}} = 0$ where some of $\alpha_{22}, \dots, \alpha_{2n_2}$ are odd, we consider the case in which all $\alpha_{22}, \dots, \alpha_{2n_2}$ are even. In this case, we have

$$\begin{aligned}
y_{\boldsymbol{\alpha}} &= \frac{2\prod_{i=2}^{n_2} \Gamma(\beta_{2i})}{\left(\sum_{i=2}^{n_2} \alpha_{2i} + n_2 - 1\right) \Gamma(\sum_{i=2}^{n_2} \beta_{2i})} \int_{\Delta_{\leq}^{n_1+1}} \boldsymbol{x}_1^{\boldsymbol{\alpha}_1} x_{21}^{\sum_{i=1}^{n_2} \alpha_{2i} + n_2 - 1} d\boldsymbol{x}_1 dx_{21} \\
&= \frac{2\boldsymbol{\alpha}_1! \left(\sum_{i=1}^{n_2} \alpha_{2i} + n_2 - 1\right)! \prod_{i=2}^{n_2} \Gamma(\beta_{2i})}{\left(\sum_{i=2}^{n_2} \alpha_{2i} + n_2 - 1\right) (n + |\boldsymbol{\alpha}|)! \, \Gamma(\sum_{i=2}^{n_2} \beta_{2i})}.
\end{aligned}$$

See [114, Section 7.8], for example, to obtain the last equation. □

The above discussion indicates that the three (dP-, ZVP-, and NN-type) inner-approximation hierarchies and two (Yıldırım- and Lasserre-type) outer-approximation hierarchies for $\mathcal{COP}(\mathbb{K})$ are basically described by positive semidefinite constraints. Table 3.1 summarizes the approximation hierarchies, from which we can observe the characteristics of each approximation hierarchy. In particular, the dP- and Yıldırım-type approximation hierarchies have features that differ from those of the ZVP-, NN-, and Lasserre-type approximation hierarchies.

First, the number of positive semidefinite constraints defining the dP- and Yıldırım-type approximation hierarchies is exponential in l but depends only on n_1 and not on n_2 because $r = n_1 + 2$. In addition, their size is linear in n_2. Thus, they would not be affected much by the increase in n_2, and n_1 determines to extent to which approximation order l can be computationally increased. Conversely, the other approximation hierarchies include positive semidefinite constraints whose maximum size is exponential in l and dependent on $n = n_1 + n_2$. Thus, they would be considerably affected by the increase in n_2 as well as in n_1. The numerical experiment conducted in Section 3.5 demonstrates this theoretical comparison.

Second, the dP- and Yıldırım-type approximation hierarchies are defined by multiple but small positive semidefinite constraints. This means that by approximating $\mathcal{COP}(\mathbb{K})$ appearing in copositive programming problems with these hierarchies, we obtain semidefinite programming problems with a block diagonal matrix structure. To solve such problems, we can conduct some of the operations in the primal-dual interior-point methods independently for each block [38], thereby reducing the computational and spatial complexity.

3.5 Numerical Experiments

In this section, we solve the following copositive programming problem:

$$\begin{array}{ll} \underset{y,\boldsymbol{S}}{\text{maximize}} & y \\ \text{subject to} & y\boldsymbol{E}_n + \boldsymbol{S} = \boldsymbol{C}, \\ & \boldsymbol{S} \in \mathcal{COP}(\mathbb{K}), \end{array} \tag{3.22}$$

where $\boldsymbol{C} \in \mathbb{S}^n$ is a positive definite matrix and $\mathbb{K} = \mathbb{R}_+^{n_1} \times \mathbb{L}^{n_2}$. Note that the dual problem of (3.22) is

$$\begin{array}{ll} \underset{\boldsymbol{X}}{\text{minimize}} & \langle \boldsymbol{C}, \boldsymbol{X} \rangle \\ \text{subject to} & \langle \boldsymbol{E}_n, \boldsymbol{X} \rangle = 1, \\ & \boldsymbol{X} \in \mathcal{CP}(\mathbb{K}). \end{array} \tag{3.23}$$

Both (3.22) and its dual problem (3.23) satisfy Slater's condition. Thus, (3.22) is ideal in a sense.

Lemma 3.25. *Both problems (3.22) and (3.23) satisfy Slater's condition if $\boldsymbol{C}$ is a symmetric positive definite matrix.*

Proof. Let $y_0 := 0$ and $\boldsymbol{S}_0 := \boldsymbol{C}$. Then, $(y_0, \boldsymbol{S}_0)$ is a feasible solution of (3.22) and $\boldsymbol{S}_0$ lies in the interior of $\mathcal{COP}(\mathbb{K})$ since $\mathbb{S}_+^n \subseteq \mathcal{COP}(\mathbb{K})$. Next, let

$$\begin{aligned} \boldsymbol{X}_0' &:= \begin{pmatrix} \boldsymbol{E}_{n_1} + \boldsymbol{I}_{n_1} & \boldsymbol{1}_{n_1} & \\ \boldsymbol{1}_{n_1}^\top & 2n_2 + 1 & \\ & & 2\boldsymbol{I}_{n_2-1} \end{pmatrix}, \\ \boldsymbol{X}_0 &:= \frac{1}{n_1^2 + 3n_1 + 4n_2 - 1} \boldsymbol{X}_0' \end{aligned}$$

and we prove that $\boldsymbol{X}_0$ is a feasible solution of problem (3.23) and it lies in the interior of $\mathcal{CP}(\mathbb{K})$. Let

$$\begin{aligned}
\boldsymbol{u}_i^{(1)} &:= \boldsymbol{e}_i \in \mathbb{K} && (i = 11, \ldots, 1n_1),\\
\boldsymbol{u}_i^{(2)} &:= \boldsymbol{e}_{21} + \boldsymbol{e}_i \in \mathbb{K} && (i = 22, \ldots, 2n_2),\\
\boldsymbol{u}_i^{(3)} &:= \boldsymbol{e}_{21} - \boldsymbol{e}_i \in \mathbb{K} && (i = 22, \ldots, 2n_2),\\
\boldsymbol{u}^{(4)} &:= (\mathbf{1}_{n_1+1}; \mathbf{0}_{n_2-1}) \in \operatorname{int} \mathbb{K}.
\end{aligned}$$

Then, they span $\mathbb{R}^n$ as each $\boldsymbol{x} \in \mathbb{R}^n$ can be written as

$$\boldsymbol{x} = \sum_{i=11}^{1n_1} x_i \boldsymbol{u}_i^{(1)} + \frac{x_{21}}{2}(\boldsymbol{u}_{22}^{(2)} + \boldsymbol{u}_{22}^{(3)}) + \sum_{i=22}^{2n_2} \frac{x_i}{2}(\boldsymbol{u}_i^{(2)} - \boldsymbol{u}_i^{(3)}).$$

Therefore, it follows from [45, Theorem 3.3] that

$$\boldsymbol{X}_0' = \sum_{i=11}^{1n_1} \boldsymbol{u}_i^{(1)}(\boldsymbol{u}_i^{(1)})^\top + \sum_{i=22}^{2n_2} \boldsymbol{u}_i^{(2)}(\boldsymbol{u}_i^{(2)})^\top + \sum_{i=22}^{2n_2} \boldsymbol{u}_i^{(3)}(\boldsymbol{u}_i^{(3)})^\top + \boldsymbol{u}^{(4)}(\boldsymbol{u}^{(4)})^\top \in \operatorname{int} \mathcal{CP}(\mathbb{K}).$$

Given that $\langle \boldsymbol{E}_n, \boldsymbol{X}_0' \rangle = n_1^2 + 3n_1 + 4n_2 - 1 > 0$, we obtain $\langle \boldsymbol{E}_n, \boldsymbol{X}_0 \rangle = 1$ and $\boldsymbol{X}_0 \in \operatorname{int} \mathcal{CP}(\mathbb{K})$. $\square$

All experiments in this section were conducted on a computer with an Intel Core i5-8279U 2.40 GHz CPU and 16 GB of memory. The modeling language YALMIP [71] (version 20210331), the MOSEK solver [78] (version 9.3.3), and MATLAB (R2022a), were used to solve optimization problems.

Based on Lemma 3.25, a coefficient matrix $\boldsymbol{C}$ in problem (3.22) was randomly generated such that it was symmetric positive definite. We measured three types of time when solving the optimization problems:

- preparetime: Time taken before calling YALMIP commands `optimize` or `solvesos`.
- yalmiptime: Time between calling the above commands and beginning to solve an optimization problem in MOSEK.
- solvertime: Time required to solve an optimization problem in MOSEK.

We defined the total time as the sum of the three types of time, and the calculation was considered invalid when the total time exceeded 7200 s.

3.5.1 Comparison of approximation hierarchies

Table 3.1 lists five approximation hierarchies for $\mathcal{COP}(\mathbb{K})$ we have introduced. Here, we solve optimization problems obtained by replacing the copositive cone $\mathcal{COP}(\mathbb{K})$ in (3.22) with the approximation hierarchies. For convenience, we hereafter call such problems dP-type approximation problems (of order l), for example. The YALMIP command `optimize` was used when solving dP-, Yıldırım-, and Lasserre-type approximation problems, and `solvesos` was used when solving ZVP- and NN-type approximation

Table 3.2: Optimal values (optv) of the optimization problems obtained by replacing the copositive cone $\mathcal{COP}(\mathbb{R}^{20}_{+} \times \mathbb{L}^{5})$ in (3.22) with each approximation hierarchy, as well as the solver time (solt) and total time (tott) required to solve them. All values are rounded to the second decimal place. The asterisk $*$ indicates that the total time exceeded 7200 s. Because more than 7200 s were required to solve the dP-type approximation problem of order 6, only results up to order $l \leq 5$ are shown.

	dP			ZVP			NN			Yıldırım			Lasserre		
l	optv	solt	tott	optv	solt	tott	optv	solt	tott	optv	solt	tott	optv	solt	tott
0	$-\infty$	0.00	0.72	1.52	0.04	4.20	1.52	364.56	373.31	6.09	0.01	0.48	$+\infty$	0.00	1.28
1	$-\infty$	0.02	3.24	1.52	4.09	234.49	$*$	$*$	$*$	5.14	0.06	3.53	$+\infty$	0.00	26.11
2	-0.22	0.16	18.87	$*$	$*$	$*$	$*$	$*$	$*$	4.42	0.36	23.99	$+\infty$	0.11	4285.07
3	0.08	1.85	106.15	$*$	$*$	$*$	$*$	$*$	$*$	3.90	2.09	151.11	$*$	$*$	$*$
4	0.18	32.60	693.39	$*$	$*$	$*$	$*$	$*$	$*$	3.49	12.62	1200.10	$*$	$*$	$*$
5	0.26	513.54	6357.99	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$

Table 3.3: Optimal values (optv) of the optimization problems obtained by replacing the copositive cone $\mathcal{COP}(\mathbb{R}^{5}_{+} \times \mathbb{L}^{20})$ in (3.22) with each approximation hierarchy, as well as the solver time (solt) and total time (tott) required to solve them. All values are rounded to the second decimal place. The asterisk $*$ indicates that the total time exceeded 7200 s. Because more than 7200 s were required to solve the dP-type approximation problem of order 17, only results up to order $l \leq 16$ are shown.

	dP			ZVP			NN			Yıldırım			Lasserre		
l	optv	solt	tott	optv	solt	tott	optv	solt	tott	optv	solt	tott	optv	solt	tott
0	$-\infty$	0.00	0.23	1.79	0.03	1.17	1.79	270.90	280.22	2.25	0.01	0.18	20.21	0.00	0.17
1	$-\infty$	0.02	0.26	1.79	0.30	4.88	$*$	$*$	$*$	2.25	0.03	0.36	$+\infty$	0.00	19.52
2	$-\infty$	0.05	0.62	1.79	569.21	751.10	$*$	$*$	$*$	2.18	0.09	0.84	$+\infty$	0.13	2979.10
3	$-\infty$	0.15	1.32	$*$	$*$	$*$	$*$	$*$	$*$	2.10	0.23	2.06	$*$	$*$	$*$
4	-0.11	0.37	2.77	$*$	$*$	$*$	$*$	$*$	$*$	2.02	0.52	4.42	$*$	$*$	$*$
5	0.40	0.65	4.95	$*$	$*$	$*$	$*$	$*$	$*$	1.95	1.05	10.19	$*$	$*$	$*$
6	0.91	1.30	9.33	$*$	$*$	$*$	$*$	$*$	$*$	1.87	2.07	22.01	$*$	$*$	$*$
7	1.19	2.12	17.04	$*$	$*$	$*$	$*$	$*$	$*$	1.81	3.94	49.55	$*$	$*$	$*$
8	1.30	3.89	32.91	$*$	$*$	$*$	$*$	$*$	$*$	1.79	7.33	110.76	$*$	$*$	$*$
9	1.37	6.23	62.80	$*$	$*$	$*$	$*$	$*$	$*$	1.79	13.33	261.56	$*$	$*$	$*$
10	1.42	9.46	118.91	$*$	$*$	$*$	$*$	$*$	$*$	1.79	21.98	603.44	$*$	$*$	$*$
11	1.45	15.59	230.01	$*$	$*$	$*$	$*$	$*$	$*$	1.79	40.07	1491.77	$*$	$*$	$*$
12	1.47	22.72	440.28	$*$	$*$	$*$	$*$	$*$	$*$	1.79	63.83	3266.76	$*$	$*$	$*$
13	1.48	38.53	889.31	$*$	$*$	$*$	$*$	$*$	$*$	1.79	99.29	7158.94	$*$	$*$	$*$
14	1.49	49.62	1642.49	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
15	1.50	69.87	2987.17	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
16	1.52	101.21	5274.74	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$

Table 3.4: Optimal values (optv) of the optimization problems obtained by replacing the copositive cone $\mathcal{COP}(\mathbb{R}^5_+ \times \mathbb{L}^{25})$ in (3.22) with each approximation hierarchy, as well as the solver time (solt) and total time (tott) required to solve them. All values are rounded to the second decimal place. The asterisk $*$ indicates that the total time exceeded 7200 s. Because more than 7200 s were required to solve the dP-type approximation problem of order 16, only results up to order $l \leq 15$ are shown.

	dP			ZVP			NN			Yıldırım			Lasserre		
l	optv	solt	tott	optv	solt	tott	optv	solt	tott	optv	solt	tott	optv	solt	tott
0	$-\infty$	0.02	1.14	1.83	0.06	1.35	1.83	3864.82	3904.27	2.23	0.01	0.18	25.29	0.00	0.19
1	$-\infty$	0.02	0.47	1.83	0.47	5.59	$*$	$*$	$*$	2.23	0.05	0.38	$+\infty$	0.00	38.01
2	$-\infty$	0.09	0.76	1.83	6126.23	6587.04	$*$	$*$	$*$	2.18	0.12	0.89	$*$	$*$	$*$
3	$-\infty$	0.17	1.55	$*$	$*$	$*$	$*$	$*$	$*$	2.13	0.38	2.30	$*$	$*$	$*$
4	$-\infty$	0.45	3.85	$*$	$*$	$*$	$*$	$*$	$*$	2.06	0.68	5.00	$*$	$*$	$*$
5	0.25	1.24	6.70	$*$	$*$	$*$	$*$	$*$	$*$	1.98	1.65	12.36	$*$	$*$	$*$
6	0.95	2.56	13.70	$*$	$*$	$*$	$*$	$*$	$*$	1.92	3.18	28.62	$*$	$*$	$*$
7	1.22	4.02	25.25	$*$	$*$	$*$	$*$	$*$	$*$	1.87	5.71	69.53	$*$	$*$	$*$
8	1.33	5.73	48.05	$*$	$*$	$*$	$*$	$*$	$*$	1.84	12.62	173.91	$*$	$*$	$*$
9	1.41	11.50	102.73	$*$	$*$	$*$	$*$	$*$	$*$	1.83	22.40	431.75	$*$	$*$	$*$
10	1.46	15.19	199.37	$*$	$*$	$*$	$*$	$*$	$*$	1.83	38.45	1076.99	$*$	$*$	$*$
11	1.49	23.34	400.18	$*$	$*$	$*$	$*$	$*$	$*$	1.83	64.26	2689.87	$*$	$*$	$*$
12	1.52	39.72	780.18	$*$	$*$	$*$	$*$	$*$	$*$	1.83	94.82	6229.90	$*$	$*$	$*$
13	1.53	57.08	1571.46	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
14	1.54	81.10	2922.42	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
15	1.55	113.98	5315.38	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$

problems. For each approximation hierarchy, we continuously increased the approximation order l until the total time exceeded 7200 s. When solving dP- and Yıldırım-type approximation problems, the concise expressions mentioned in Sections 3.3.1.2 and 3.3.2.2 were adopted. (Section 3.5.2 investigates their numerical effect.) The pair (n_1, n_2) was set to $(20, 5)$, $(5, 20)$, and $(5, 25)$.

Tables 3.2, 3.3, and 3.4 show the results of $(n_1, n_2) = (20, 5)$, $(5, 20)$, and $(5, 25)$, respectively.[3] These tables report the solver and total times because some of the approximation hierarchies spent most of the total time before beginning to solve optimization problems in MOSEK. Although we used YALMIP for convenience, the total time would be substantially reduced if we implemented these approximation hierarchies directly.

As shown in Tables 3.3 and 3.4, the optimal values of the ZVP- and NN-type inner-approximation problems agree with those of the Yıldırım-type outer-approximation problems, which implies that the ZVP-, NN-, and Yıldırım-type approximation hierarchies almost approach the optimal value of the original copositive programming problem (3.22) for $(n_1, n_2) = (5, 20)$ and $(5, 25)$.

Although many Lasserre-type outer-approximation problems were considered unbounded, this would result from the moments (3.20) taking infinitesimal values. Indeed, the MOSEK solver provided a warning about treating nearly zero elements, and we determined that $y_{\boldsymbol{\alpha}}$ is approximately 4.06×10^{-18} for $(n_1, n_2) = (5, 25)$ and $\boldsymbol{\alpha} = \boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3 + \boldsymbol{e}_4 \in \mathbb{R}^{30}$, for instance. Hence, the Lasserre-type outer-approximation hierarchy is numerically unstable, whereas the Yıldırım-type outer-approximation hierarchy is numerically stable.[4]

The results of this numerical experiment support the theoretical comparison mentioned in Section 3.4. As shown in Tables 3.3 and 3.4, the dP- and Yıldırım-type approximation hierarchies are not affected much by the increase in n_2. The solver and total times required to solve the dP- and Yıldırım-type approximation problems with $(n_1, n_2) = (5, 25)$ are less than twice as long as those with $(n_1, n_2) = (5, 20)$ regardless of l, except for $l = 0$. Conversely, the others are considerably affected by the increase in n_2. For example, the solver and total times required to solve the NN-type inner-approximation problem with $(n_1, n_2) = (5, 25)$ of order 0 are more than ten times as long as those with $(n_1, n_2) = (5, 20)$. In addition, although the increase for $l = 0, 1$ is mild, those required to solve the ZVP-type inner-approximation problem with $(n_1, n_2) = (5, 25)$ of order 2 are also approximately ten times as long as those with $(n_1, n_2) = (5, 20)$.[5]

As shown in Tables 3.2 and 3.3, the dP- and Yıldırım-type approximation hierarchies are considerably affected by the increase in n_1. The solver and total times required

[3] We found a mistake in the code utilized in the original article [84] to solve Lasserre-type outer-approximation problems. Here, we fixed it and solved the problems once more. The tables presented herein display the corrected results.

[4] We observed the phenomenon that all Lasserre-type outer-approximation problems were considered unbounded in our original article [84] too.

[5] This is because, as shown in Table 3.1, the maximum size of the positive semidefinite constraints defining the ZVP-type inner-approximation hierarchy increases from $|\mathbb{I}^n_{=1}| = n$ to $|\mathbb{I}^n_{=2}| = \frac{n(n+1)}{2}$ when l increases from 1 to 2.

to solve the dP- and Yıldırım-type approximation problems with $(n_1, n_2) = (20, 5)$ are longer than those with $(n_1, n_2) = (5, 20)$ for each l, and the difference rapidly increases with l. Because $n = n_1 + n_2$ is the same for the two pairs and because n_2 in the pair $(n_1, n_2) = (20, 5)$ is smaller than that in the pair $(n_1, n_2) = (5, 20)$, we can conclude that the increase in the required time results from the increase in n_1. Conversely, if n_1 is small, we can increase approximation order l and, in this case, the Yıldırım-type outer-approximation hierarchy may approach nearly optimal values of copositive programming problems.

Finally, the ZVP- and NN-type inner-approximation hierarchies provided much tighter bounds than that of the dP-type in all cases, and, as mentioned above, the two hierarchies are guaranteed to approach nearly optimal values even in the case of $l = 0$ for $(n_1, n_2) = (5, 20)$ and $(5, 25)$. Moreover, the time required to solve the ZVP-type inner-approximation problems of order 0 is much shorter than that for the NN-type ones. Therefore, from a practical perspective, using the zeroth level of the ZVP-type inner-approximation hierarchy may be preferable if aiming to obtain a reasonable lower bound of the optimal value of the original problem (3.22).

3.5.2 Effect of concise expressions of dP- and Yıldırım-type approximation hierarchies

In this subsection, we investigate the numerical effect of the concise expressions of the dP- and Yıldırım-type approximation hierarchies mentioned in Sections 3.3.1.2 and 3.3.2.2. The dP- and Yıldırım-type approximation hierarchies with the full expressions are provided by (3.15) and (3.19), respectively. Except for the differences in the expressions of these approximation hierarchies, the experimental settings were the same as those in Section 3.5.1.

Table 3.5 provides the results of $(n_1, n_2) = (5, 25)$. The information of the optimal values is omitted because those of the dP- and Yıldırım-type approximation problems with the full expressions are theoretically (and numerically) the same as those with the concise expressions. The effect of the concise expressions is significant, and the solver and total times required to solve the dP- and Yıldırım-type approximation problems with the concise expressions are shorter than those with the full expressions, except for the total time of solving the dP-type inner-approximation problem of order 0. Note that the effect was also confirmed for $(n_1, n_2) = (20, 5)$.

3.6 Concluding Remarks

In this chapter, we initially introduced the inner-approximation hierarchies defined by a sum-of-squares constraint for copositive cones over general symmetric cones. Subsequently, we presented both inner- and outer-approximation hierarchies defined by positive semidefinite constraints but not by sum-of-squares constraints for copositive matrix cones over a direct product of a nonnegative orthant and a single second-order cone.

Unfortunately, the infinity of a set of Jordan frames is guaranteed to be solved

Table 3.5: Solver time (solt) and total time (tott) required to solve the optimization problems obtained by replacing the copositive cone $\mathcal{COP}(\mathbb{R}^5_+ \times \mathbb{L}^{25})$ in (3.22) with the dP- and Yıldırım-type approximation hierarchies with the full expressions. The "solt ratio" (resp. "tott ratio") column lists values obtained by dividing "solt" (resp. "tott") in this table (the case of not making the expressions more concise) by that in Table 3.4 (the case of making the expressions more concise). The asterisk $*$ indicates that the total time exceeded 7200 s, and the symbol $+\infty$ in the columns of "solt ratio" and "tott ratio" indicates that the total time in this table exceeded 7200 s, whereas that in Table 3.4 did not. The ratios that are more than 1 are in bold. All values are rounded to the second decimal place.

	dP				Yıldırım			
l	solt	solt ratio	tott	tott ratio	solt	solt ratio	tott	tott ratio
0	0.03	**1.23**	0.71	0.62	0.04	**2.97**	0.27	**1.48**
1	0.08	**3.63**	0.77	**1.64**	0.11	**2.34**	0.78	**2.06**
2	0.26	**2.95**	1.82	**2.40**	0.30	**2.58**	2.03	**2.27**
3	0.57	**3.34**	4.47	**2.88**	0.98	**2.60**	5.65	**2.46**
4	1.17	**2.57**	7.45	**1.93**	1.96	**2.86**	13.37	**2.67**
5	2.34	**1.89**	15.04	**2.25**	4.44	**2.70**	36.01	**2.91**
6	5.17	**2.02**	33.12	**2.42**	8.57	**2.70**	92.24	**3.22**
7	7.81	**1.94**	65.96	**2.61**	15.44	**2.71**	258.37	**3.72**
8	12.25	**2.14**	143.70	**2.99**	32.80	**2.60**	700.01	**4.03**
9	19.66	**1.71**	304.94	**2.97**	57.39	**2.56**	1893.38	**4.39**
10	34.48	**2.27**	690.06	**3.46**	95.71	**2.49**	4710.17	**4.37**
11	50.02	**2.14**	1465.11	**3.66**	$*$	$+\infty$	$*$	$+\infty$
12	71.78	**1.81**	2943.20	**3.77**	$*$	$+\infty$	$*$	$+\infty$
13	108.49	**1.90**	5723.38	**3.64**	$*$	$*$	$*$	$*$
14	$*$	$+\infty$	$*$	$+\infty$	$*$	$*$	$*$	$*$
15	$*$	$+\infty$	$*$	$+\infty$	$*$	$*$	$*$	$*$

in Sections 3.3.1 and 3.3.2 only when the symmetric cone is the direct product of a nonnegative orthant and a single second-order cone and $m = 2$. However, as shown by the following proposition, we obtain an outer-approximation hierarchy implementable on a computer for copositive cones over general symmetric cones by replacing a set $\mathfrak{F}$ of Jordan frames appearing in (3.17) with its finite subset $\mathfrak{F}_l$ such that the union $\bigcup_{l\in\mathbb{N}} \mathfrak{F}_l$ is dense in $\mathfrak{F}$. The proof is omitted here for the sake of brevity.

Proposition 3.26. *Let $\{\mathcal{O}_l^{r,m}\}_l$ be an outer-approximation hierarchy for $\mathcal{COP}^{r,m}$ such that each $\mathcal{O}_l^{r,m}$ is a closed convex cone. We fix a set $\mathfrak{F}$ with $\mathfrak{F}_{\mathrm{c}}(\mathbb{E}) \subseteq \mathfrak{F} \subseteq \mathfrak{F}(\mathbb{E})$ and let $\{\mathfrak{F}_l\}_l$ be a non-decreasing sequence of finite subsets of $\mathfrak{F}$ such that $\bigcup_{l\in\mathbb{N}} \mathfrak{F}_l$ is dense in $\mathfrak{F}$. Let*

$$\hat{\mathcal{O}}_l^{n,m}(\mathbb{E}_+) := \bigcap_{(c_1,\dots,c_r)\in\mathfrak{F}_l} \{\mathcal{A} \in \mathcal{S}^{n,m}(\mathbb{E}) \mid \mathcal{A}(c_1,\dots,c_r) \in \mathcal{O}_l^{r,m}\}.$$

Then, each $\hat{\mathcal{O}}_l^{n,m}(\mathbb{E}_+)$ is a closed convex cone and $\{\hat{\mathcal{O}}_l^{n,m}(\mathbb{E}_+)\}_l$ is an outer-approximation hierarchy for $\mathcal{COP}^{n,m}(\mathbb{E}_+)$.

Through the numerical experiments conducted in Section 3.5.1, a question arises regarding the inclusion between the ZVP- and NN-type inner-approximation hierarchies. From Tables 3.2, 3.3, and 3.4, we observe that the optimal values of ZVP-type approximation problems of order 0 and those of NN-type approximation problems of order 0 were identical. This finding prompts us to investigate whether the zeroth levels of the ZVP- and NN-type inner-approximation hierarchies coincide. We will address this inquiry in the subsequent chapter.

Chapter 4

Generalizations of Doubly Nonnegative Cones and Their Comparison

4.1 Introduction

Inspired by the question arising in the previous chapter, we are curious about the inclusion between ZVP- and NN-type inner-approximation hierarchies, especially their zeroth levels. As justified in Section 4.2, we can regard the dual cones of the zeroth levels of the ZVP- and NN-type hierarchies as generalizations of a doubly nonnegative cone. For that reason, we call them ZVP- and NN-type generalized doubly nonnegative cones, respectively.

In this chapter, we provide theoretical and numerical comparisons of the strength of relaxation by the above two generalized doubly nonnegative cones as well as that proposed by Burer and Dong [15], i.e., the Burer–Dong (BD)-type generalized doubly nonnegative cone. To compare these cones, we focus on the two specific cases in which (i) the underlying cone $\mathbb{K}$ is a direct product of a nonnegative orthant and second-order cones, and (ii) $\mathbb{K}$ is a direct product of a nonnegative orthant and positive semidefinite cones. The first case permits $\mathbb{K}$ to encompass *multiple* second-order cones, contrasting with Section 3.5, where $\mathbb{K}$ involved only a *single* second-order cone. Moreover, the completely positive reformulations of quadratically constrained quadratic programming problems and rank-constrained semidefinite programming problems, as discussed in Section 1.1, suggest that these two scenarios naturally emerge within the realm of optimization. Specifically, we investigate the inclusion relationship between the three generalized doubly nonnegative cones because inclusion is a crucial factor in the strength of relaxation. The results of this chapter are summarized as follows.

- When $\mathbb{K}$ is a direct product of a nonnegative orthant and second-order cones, no theoretical inclusion relationship is obtained between ZVP- and BD-type generalized doubly nonnegative cones (see Examples 4.16 and 4.17), and numerically, the ZVP-type generalized doubly nonnegative cone provides better relaxation than BD-type cone (Section 4.4.1).

- When $\mathbb{K}$ is a direct product of a nonnegative orthant and positive semidefinite cones, the ZVP-type generalized doubly nonnegative cone provides worse relaxation than the BD-type cone numerically (Section 4.4.2).

- In both cases, theoretically, the NN-type generalized doubly nonnegative cone is strictly included in the ZVP-type cone (Theorems 4.15 and 4.23). Solutions to the generalized doubly nonnegative programming relaxation of generalized completely positive programming problems show that the three generalized doubly nonnegative cones provide much tighter relaxation for generalized completely positive programming than the positive semidefinite cone (Section 4.4).

The rest of this chapter is organized as follows. In Section 4.2, we introduce two new generalized doubly nonnegative cones and describe the BD-type cone. In Section 4.3, we discuss the theoretical properties of the three generalized doubly nonnegative cones. In Section 4.4, we describe the results of numerical experiments conducted to investigate the effectiveness of relaxation by these cones.

4.2 Generalized Doubly Nonnegative Cones

For a class of closed cones $\mathbb{K} \subseteq \mathbb{R}^n$, we consider a generalized doubly nonnegative cone $\mathcal{DNN}(\mathbb{K})$ over $\mathbb{K}$. The following requirements are natural. First, to regard $\mathcal{DNN}(\mathbb{K})$ as a generalization of the doubly nonnegative cone, we require $\mathcal{DNN}(\mathbb{R}^n_+) = \mathcal{DNN}^n$. Second, we require that $\mathcal{DNN}(\mathbb{K})$ be a cone. Third, we require $\mathcal{CP}(\mathbb{K}) \subseteq \mathcal{DNN}(\mathbb{K}) \subseteq \mathbb{S}^n_+$. The inclusion $\mathcal{CP}(\mathbb{K}) \subseteq \mathbb{S}^n_+$ holds by the definition of $\mathcal{CP}(\mathbb{K})$. To provide tighter relaxation than semidefinite programming for generalized completely positive programming as the doubly nonnegative cone does, it is necessary to interpose a generalized doubly nonnegative cone between the generalized completely positive cone and the positive semidefinite cone.

Of note, ZVP- and NN-type inner-approximation hierarchies are generalizations of Parrilo's inner-approximation hierarchy (2.23) for the standard copositive cone, and the dual cone of the zeroth level of Parrilo's hierarchy is known to be the doubly nonnegative cone [90, Section 5.3]. In this section, focusing on this fact, we define the dual cone of the zeroth level of each of the two inner-approximation hierarchies for generalized copositive cones as a generalized doubly nonnegative cone. In addition, we introduce the generalized doubly nonnegative cone proposed by Burer and Dong [15].

4.2.1 ZVP-type generalized doubly nonnegative cone

Zuluaga, Vera, and Peña [125] presented the inner-approximation hierarchy (2.28) for copositive cones over pointed semialgebraic convex cones. More generally, they introduced inner-approximation hierarchies for the cone of copositive *homogeneous polynomials*, or *tensors* over given cones, as we also discussed in Chapter 3. However, in this chapter, we restrict our scope to *matrices* in accordance with Burer and Dong [15]. Using this hierarchy, we define ZVP-type generalized doubly nonnegative cones as follows.

Definition 4.1. Let $\mathbb{K}$ be a pointed semialgebraic convex cone in $\mathbb{R}^n$. Then, we define $\mathcal{I}_{\mathrm{ZVP},0}(\mathbb{K})^*$ as the ZVP-type generalized doubly nonnegative cone over $\mathbb{K}$, written as $\mathcal{DNN}_{\mathrm{ZVP}}(\mathbb{K})$.

Before checking whether the ZVP-type generalized doubly nonnegative cone satisfies the requirements mentioned at the beginning of Section 4.2, we provide an explicit expression of $\mathcal{DNN}_{\mathrm{ZVP}}(\mathbb{K})$, the dual cone of $\mathcal{I}_{\mathrm{ZVP},0}(\mathbb{K})$. Considering that $\mathcal{I}_{\mathrm{ZVP},0}(\mathbb{K}) = \{\boldsymbol{A} \in \mathbb{S}^n \mid \boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} \in E^{n,2}(\mathbb{K})\}$, it becomes imperative to explicitly characterize the set $E^{n,2}(\mathbb{K})$. As in (2.24), suppose that $\mathbb{K}$ is described by the nonnegativity of homogeneous polynomials $\phi_1, \dots, \phi_q$. In the representation of a polynomial in $E^{n,2}(\mathbb{K})$ as the sum of 2-dimensional homogeneous polynomials in the form

$$\psi^2 \prod_{j=1}^{k} \phi_{i_j} \tag{4.1}$$

(see Theorem 2.22), each ϕ_{i_j} is constrained to have a degree at most 2. Otherwise, the degree of the polynomial (4.1) surpasses 2. This implies that, among the polynomials $\phi_1, \dots, \phi_q$, those with a degree exceeding 2 are disregarded in constructing $E^{n,2}(\mathbb{K})$. Thus, assuming that the degrees of homogeneous polynomials $\phi_1, \dots, \phi_q$ are at most 2, we now express $\mathbb{K}$ as

$$\{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x}^\top \boldsymbol{Q}_i \boldsymbol{x} \geq 0 \ (i = 1, \dots, q_2),\ \boldsymbol{a}_i^\top \boldsymbol{x} \geq 0 \ (i = 1, \dots, q_1)\}, \tag{4.2}$$

where $\boldsymbol{Q}_i \in \mathbb{S}^n$ for $i = 1, \dots, q_2$ and $\boldsymbol{a}_i \in \mathbb{R}^n$ for $i = 1, \dots, q_1$. Under this assumption, each polynomial belonging to $E^{n,2}(\mathbb{K})$ can be expressed as

$$\sum_t (\boldsymbol{\psi}_t^\top \boldsymbol{x})^2 + \sum_t \psi_{2(t)}^2 (\boldsymbol{x}^\top \boldsymbol{Q}_{i_{2(t)}} \boldsymbol{x}) + \sum_t \psi_{1(t)}^2 (\boldsymbol{a}_{i_{1(t)}}^\top \boldsymbol{x})(\boldsymbol{a}_{j_{1(t)}}^\top \boldsymbol{x}), \tag{4.3}$$

where $\boldsymbol{\psi}_t \in \mathbb{R}^n$, $\psi_{2(t)} \in \mathbb{R}$, $i_{2(t)} \in \{1, \dots, q_2\}$, $\psi_{1(t)} \in \mathbb{R}$, and $i_{1(t)}, j_{1(t)} \in \{1, \dots, q_1\}$. The first summand in (4.3) corresponds to the case of k appearing in (4.1) equal to 0; in the second, $k = 1$; and in the third, $k = 2$.

Lemma 4.2. *Suppose that $\mathbb{K}$ is expressed as (4.2). Then,*

$$\mathcal{I}_{\mathrm{ZVP},0}(\mathbb{K}) = \mathbb{S}_+^n + \sum_{i=1}^{q_2} \mathbb{R}_+ \boldsymbol{Q}_i + \sum_{1 \leq i < j \leq q_1} \mathbb{R}_+ \boldsymbol{A}^{ij},$$

where $\boldsymbol{A}^{ij} := \frac{\boldsymbol{a}_i \boldsymbol{a}_j^\top + \boldsymbol{a}_j \boldsymbol{a}_i^\top}{2}$.

Proof. Let $\boldsymbol{A} \in \mathcal{I}_{\mathrm{ZVP},0}(\mathbb{K})$. Then, because $\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} \in E^{n,2}(\mathbb{K})$, it can be expressed as (4.3). This implies that

$$\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}^\top \left(\sum_t \boldsymbol{\psi}_t \boldsymbol{\psi}_t^\top + \sum_{t \in \mathcal{T}_=} \psi_{1(t)}^2 \boldsymbol{a}_{i_{1(t)}} \boldsymbol{a}_{i_{1(t)}}^\top + \sum_t \psi_{2(t)}^2 \boldsymbol{Q}_{i_{2(t)}} + \sum_{t \in \mathcal{T}_{\neq}} \psi_{1(t)}^2 \boldsymbol{A}^{i_{1(t)} j_{1(t)}} \right) \boldsymbol{x}, \tag{4.4}$$

where $\mathcal{T}_= := \{t \mid i_{1(t)} = j_{1(t)}\}$ and $\mathcal{T}_{\neq} := \{t \mid i_{1(t)} \neq j_{1(t)}\}$. Since (4.4) holds for all $\boldsymbol{x}$, it follows that

$$\boldsymbol{A} = \sum_t \boldsymbol{\psi}_t \boldsymbol{\psi}_t^\top + \sum_{t \in \mathcal{T}_=} \psi_{1(t)}^2 \boldsymbol{a}_{i_{1(t)}} \boldsymbol{a}_{i_{1(t)}}^\top + \sum_t \psi_{2(t)}^2 \boldsymbol{Q}_{i_{2(t)}} + \sum_{t \in \mathcal{T}_{\neq}} \psi_{1(t)}^2 \boldsymbol{A}^{i_{1(t)} j_{1(t)}},$$

which belongs to $\mathbb{S}_+^n + \sum_{i=1}^{q_2} \mathbb{R}_+ \boldsymbol{Q}_i + \sum_{1 \le i < j \le q_1} \mathbb{R}_+ \boldsymbol{A}^{ij}$.

Conversely, let $\boldsymbol{A} \in \mathbb{S}_+^n + \sum_{i=1}^{q_2} \mathbb{R}_+ \boldsymbol{Q}_i + \sum_{1 \le i < j \le q_1} \mathbb{R}_+ \boldsymbol{A}^{ij}$. The matrix $\boldsymbol{A}$ can then be written as $\boldsymbol{P} + \sum_{i=1}^{q_2} \psi_i \boldsymbol{Q}_i + \sum_{1 \le i < j \le q_1} \psi_{ij} \boldsymbol{A}^{ij}$, where $\boldsymbol{P} \in \mathbb{S}_+^n$, $\psi_i \ge 0$ for $i = 1, \dots, q_2$, and $\psi_{ij} \ge 0$ for $1 \le i < j \le q_1$. Because $\boldsymbol{P} \in \mathbb{S}_+^n$, there exist $\boldsymbol{p}_1, \dots, \boldsymbol{p}_n \in \mathbb{R}^n$ such that $\boldsymbol{P} = \sum_{i=1}^n \boldsymbol{p}_i \boldsymbol{p}_i^\top$. Therefore,

$$\boldsymbol{x}^\top \boldsymbol{A} \boldsymbol{x} = \sum_{i=1}^n (\boldsymbol{p}_i^\top \boldsymbol{x})^2 + \sum_{i=1}^{q_2} (\sqrt{\psi_i})^2 \boldsymbol{x}^\top \boldsymbol{Q}_i \boldsymbol{x} + \sum_{1 \le i < j \le q_1} (\sqrt{\psi_{ij}})^2 (\boldsymbol{a}_i^\top \boldsymbol{x})(\boldsymbol{a}_j^\top \boldsymbol{x}),$$

which belongs to $\mathcal{I}_{\mathrm{ZVP},0}(\mathbb{K})$. □

Proposition 4.3. *Suppose that $\mathbb{K}$ is expressed as (4.2). Then,*

$$\mathcal{DNN}_{\mathrm{ZVP}}(\mathbb{K}) = \mathbb{S}_+^n \cap \bigcap_{i=1}^{q_2} \{\boldsymbol{X} \in \mathbb{S}^n \mid \langle \boldsymbol{Q}_i, \boldsymbol{X} \rangle \ge 0\} \cap \bigcap_{1 \le i < j \le q_1} \{\boldsymbol{X} \in \mathbb{S}^n \mid \langle \boldsymbol{A}^{ij}, \boldsymbol{X} \rangle \ge 0\}.$$

Proof. Note that for $\boldsymbol{S} \in \mathbb{S}^n$, we have $(\mathbb{R}_+ \boldsymbol{S})^* = \{\boldsymbol{X} \in \mathbb{S}^n \mid \langle \boldsymbol{S}, \boldsymbol{X} \rangle \ge 0\}$. In addition, for any closed convex cones $\mathcal{K}_1, \dots, \mathcal{K}_k$, $(\sum_{i=1}^k \mathcal{K}_i)^* = \bigcap_{i=1}^k \mathcal{K}_i^*$ holds [106, Corollary 16.4.2]. Using these facts, we obtain the desired result by taking the dual in Lemma 4.2. □

It is reasonable to call $\mathcal{DNN}_{\mathrm{ZVP}}(\mathbb{K})$ a generalized doubly nonnegative cone. First, as mentioned in Section 2.5.1, the hierarchy $\{\mathcal{I}_{\mathrm{ZVP},l}(\mathbb{R}_+^n; \boldsymbol{1}_n)\}_l$ agrees with Parrilo's hierarchy. This implies that $\mathcal{DNN}_{\mathrm{ZVP}}(\mathbb{R}_+^n; \boldsymbol{1}_n) = \mathcal{DNN}^n$. Second, $\mathcal{DNN}_{\mathrm{ZVP}}(\mathbb{K})$ is a cone, as it is obtained by dualizing $\mathcal{I}_{\mathrm{ZVP},0}(\mathbb{K})$. Third, it follows from Theorem 2.22 that $\mathcal{CP}(\mathbb{K}) \subseteq \mathcal{DNN}_{\mathrm{ZVP}}(\mathbb{K})$ and from Proposition 4.3 that $\mathcal{DNN}_{\mathrm{ZVP}}(\mathbb{K}) \subseteq \mathbb{S}_+^n$.

Note that unlike $\mathbb{S}_+^n + \mathcal{N}^n$, which is closed, $\mathcal{I}_{\mathrm{ZVP},0}(\mathbb{K})$ is not closed in general. Here, we show an example of $\mathbb{K}$ such that $\mathcal{I}_{\mathrm{ZVP},0}(\mathbb{K})$ is not closed.

Example 4.4. Let $\boldsymbol{a} := (1, 0)^\top$ and $\boldsymbol{Q} := \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$. Then,

$$\mathbb{K} := \{\boldsymbol{x} \in \mathbb{R}^2 \mid \boldsymbol{x}^\top \boldsymbol{Q} \boldsymbol{x} = -x_2^2 \ge 0,\ \boldsymbol{a}^\top \boldsymbol{x} = x_1 \ge 0\}$$

is a pointed semialgebraic convex cone (the nonnegative x_1-axis) and the vector $\boldsymbol{a}$ satisfies $\boldsymbol{a} \in \operatorname{int} \mathbb{K}^*$. It follows from Lemma 4.2 that $\mathcal{I}_{\mathrm{ZVP},0}(\mathbb{K}) = \mathbb{S}_+^2 + \mathbb{R}_+ \boldsymbol{Q}$. For each k, we define $\boldsymbol{P}_k := \begin{pmatrix} \frac{1}{k} & 1 \\ 1 & k \end{pmatrix} \in \mathbb{S}_+^2$ and $\boldsymbol{X}_k := \boldsymbol{P}_k + k\boldsymbol{Q} \in \mathcal{I}_{\mathrm{ZVP},0}(\mathbb{K})$. Then, we have $\boldsymbol{X}_k = \begin{pmatrix} \frac{1}{k} & 1 \\ 1 & 0 \end{pmatrix} \to \boldsymbol{X}^* := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

To see that $\boldsymbol{X}^* \notin \mathcal{I}_{\mathrm{ZVP},0}(\mathbb{K})$, we assume that there exist $\boldsymbol{P}^* \in \mathbb{S}^2_+$ and $t^* \geq 0$ such that $\boldsymbol{X}^* = \boldsymbol{P}^* + t^*\boldsymbol{Q}$. This equation implies that $P^*_{11} = 0$ and $P^*_{12} = 1$. However, by $P^*_{11} = 0$ and the positive semidefiniteness of $\boldsymbol{P}^*$, P^*_{12} must be 0, which is a contradiction. Therefore, $\mathcal{I}_{\mathrm{ZVP},0}(\mathbb{K})$ is not closed.

This example implies that the dual cone of $\mathcal{DNN}_{\mathrm{ZVP}}(\mathbb{K})$ is not $\mathcal{I}_{\mathrm{ZVP},0}(\mathbb{K})$ itself but, in general, its closure. However, at least when $\mathbb{K}$ is a direct product of a nonnegative orthant and second-order or positive semidefinite cones, $\mathcal{I}_{\mathrm{ZVP},0}(\mathbb{K})$ is closed (see Sections 4.3.1 and 4.3.2).

4.2.2 NN-type generalized doubly nonnegative cone

In Chapter 3, we presented the inner-approximation hierarchy (3.5) for copositive matrix cones over symmetric cones. Using this hierarchy, we define NN-type generalized doubly nonnegative cones.

Definition 4.5. Let $(\mathbb{E}, \circ, \bullet)$ be a Euclidean Jordan algebra where $\mathbb{E}$ is the space $\mathbb{R}^n$. We then define $\mathcal{I}_{\mathrm{NN},0}(\mathbb{E}_+)^*$ as the NN-type generalized doubly nonnegative cone over $\mathbb{E}_+$, written as $\mathcal{DNN}_{\mathrm{NN}}(\mathbb{E}_+)$.

The set $\mathcal{DNN}_{\mathrm{NN}}(\mathbb{E}_+)$ may be considered a generalized doubly nonnegative cone. When the Euclidean Jordan algebra is taken as in Example 2.6, the associated symmetric cone $\mathbb{E}_+$ is the nonnegative orthant $\mathbb{R}^n_+$ and the hierarchy $\{\mathcal{I}_{\mathrm{NN},l}(\mathbb{R}^n_+)\}_l$ agrees with Parrilo's hierarchy (2.23). This implies that $\mathcal{DNN}_{\mathrm{NN}}(\mathbb{R}^n_+) = \mathcal{DNN}^n$. Moreover, $\mathcal{DNN}_{\mathrm{NN}}(\mathbb{E}_+)$ is a cone that includes $\mathcal{CP}(\mathbb{E}_+)$ for the same reason as does the ZVP-type generalized doubly nonnegative cone. Finally, for any $\boldsymbol{A} \in \mathbb{S}^n_+$, there exists $\boldsymbol{U} \in \mathbb{S}^n$ such that $\boldsymbol{A}$ can be decomposed into $\boldsymbol{U}^2$. For each $\boldsymbol{x} \in \mathbb{E}$, because each element of $\boldsymbol{x} \circ \boldsymbol{x}$ is a quadratic homogeneous polynomial, we can write $\boldsymbol{x} \circ \boldsymbol{x}$ as $(\phi_1(\boldsymbol{x}), \dots, \phi_n(\boldsymbol{x}))^\top$, where $\phi_i(\boldsymbol{x}) \in H^{n,2}$ for $i = 1, \dots, n$. Then, we have

$$(\boldsymbol{x} \circ \boldsymbol{x})^\top \boldsymbol{A} (\boldsymbol{x} \circ \boldsymbol{x}) = \|\boldsymbol{U}(\boldsymbol{x} \circ \boldsymbol{x})\|_2^2 = \sum_{i=1}^n \left(\sum_{j=1}^n U_{ij}\phi_j(\boldsymbol{x}) \right)^2 \in \Sigma^{n,4}.$$

Therefore, $\mathbb{S}^n_+ \subseteq \mathcal{I}_{\mathrm{NN},0}(\mathbb{E}_+)$, and so $\mathcal{DNN}_{\mathrm{NN}}(\mathbb{E}_+) \subseteq \mathbb{S}^n_+$ hold.

We should note, however, that a fully explicit expression of $\mathcal{DNN}_{\mathrm{NN}}(\mathbb{E}_+)$ has been unknown in the case of general symmetric cones. As discussed in Section 3.2.3, $\mathcal{DNN}_{\mathrm{NN}}(\mathbb{E}_+)$ is written as the *closure* of a set defined by a positive semidefinite constraint. We demonstrate that the closure operator can be removed in the case wherein $\mathbb{E}_+$ is a direct product of a nonnegative orthant and second-order or positive semidefinite cones. Namely, in such cases, $\mathcal{DNN}_{\mathrm{NN}}(\mathbb{E}_+)$ is fully captured by a positive semidefinite constraint. Note in addition that because $\mathcal{I}_{\mathrm{NN},0}(\mathbb{E}_+)$ is always closed as mentioned in Theorem 3.4, the dual of $\mathcal{DNN}_{\mathrm{NN}}(\mathbb{E}_+)$ is precisely $\mathcal{I}_{\mathrm{NN},0}(\mathbb{E}_+)$.

4.2.3 BD-type generalized doubly nonnegative cone

Above, we propose two generalized doubly nonnegative cones based on the inner-approximation hierarchies for generalized copositive cones. Independently, Burer and

Dong [15] proposed another generalized doubly nonnegative cone over closed convex cones without exploiting inner-approximation hierarchies.

Definition 4.6 ([15, Section 4])**.** For a closed convex cone $\mathbb{K}$ in $\mathbb{R}^n$, the BD-type generalized doubly nonnegative cone $\mathcal{DNN}_{\mathrm{BD}}(\mathbb{K})$ over $\mathbb{K}$ is defined by $\mathbb{S}^n_+ \cap \mathcal{N}(\mathbb{K})$, where $\mathcal{N}(\mathbb{K}) := \{\boldsymbol{X} \in \mathbb{S}^n \mid \boldsymbol{X}\boldsymbol{s} \in \mathbb{K} \text{ for all } \boldsymbol{s} \in \operatorname{Ext}\mathbb{K}^*\}$.

The set $\mathcal{DNN}_{\mathrm{BD}}(\mathbb{K})$ can also be considered a generalized doubly nonnegative cone based on the requirements listed at the beginning of Section 4.2. In fact, by the definition of $\mathcal{DNN}_{\mathrm{BD}}(\mathbb{K})$, it comprises a cone included in the positive semidefinite cone. In addition, as discussed in [15, Section 4], $\mathcal{DNN}_{\mathrm{BD}}(\mathbb{R}^n_+) = \mathcal{DNN}^n$ holds and $\mathcal{DNN}_{\mathrm{BD}}(\mathbb{K})$ includes $\mathcal{CP}(\mathbb{K})$.

4.3 Analysis on Three Generalized Doubly Nonnegative Cones

In the previous section, we provided two generalized doubly nonnegative cones $\mathcal{DNN}_{\mathrm{ZVP}}(\mathbb{K})$ and $\mathcal{DNN}_{\mathrm{NN}}(\mathbb{K})$, and introduced another, $\mathcal{DNN}_{\mathrm{BD}}(\mathbb{K})$, proposed by Burer and Dong [15]. Note that for a general closed cone $\mathbb{K}$, some generalized doubly nonnegative cones may not be defined, in which case we cannot compare the three generalized doubly nonnegative cones. That is, $\mathcal{DNN}_{\mathrm{ZVP}}(\mathbb{K})$ is defined for pointed semialgebraic convex cones, whereas $\mathcal{DNN}_{\mathrm{NN}}(\mathbb{K})$ is defined for symmetric cones. Therefore, in this section, we consider two special cases to define all three cones. In particular, we consider the case in which $\mathbb{K}$ is a direct product of a nonnegative orthant and second-order cones. In addition, as a preliminary investigation, we also consider the case in which $\mathbb{K}$ is a direct product of a nonnegative orthant and positive semidefinite cones. Recall that the sequence $\{\mathcal{I}_{\mathrm{ZVP},l}(\mathbb{R}^n_+; \mathbf{1}_n)\}_l$ agrees with Parrilo's hierarchy and the vector $\mathbf{1}_n$ is the identity element of the Euclidean Jordan algebra associated with $\mathbb{R}^n_+$, as shown in Example 2.6. Accordingly, we select the vector $\boldsymbol{a} \in \operatorname{int}\mathbb{K}^*$ in the ZVP-type generalized doubly nonnegative cone $\mathcal{DNN}_{\mathrm{ZVP}}(\mathbb{K}; \boldsymbol{a})$ as the identity element of a Euclidean Jordan algebra associated with $\mathbb{K}$, and no longer specify it. The selection of $\boldsymbol{a}$ is also consistent with Chapter 3.

4.3.1 When $\mathbb{K}$ is a direct product of a nonnegative orthant and second-order cones

In this subsection, we consider the case in which $\mathbb{K}$ is a direct product of a nonnegative orthant and second-order cones, i.e., $\mathbb{K} = \mathbb{R}^{n_1}_+ \times \prod_{h=2}^N \mathbb{L}^{n_h}$, and let $n := \sum_{h=1}^N n_h$. For convenience, we reindex $(1,\dots,n)$ as $(11,\dots,1n_1,21,\dots,2n_2,\dots,N1,\dots,Nn_N)$; i.e., let $hi := \sum_{k=1}^{h-1} n_k + i$ for $h = 1,\dots,N$ and $i = 1,\dots,n_h$. Moreover, we set $\mathcal{I}_h := \{h1,\dots,hn_h\}$ for $h = 1,\dots,N$, $\mathcal{I}_h^- := \mathcal{I}_h \setminus \{h1\}$ for $h = 2,\dots,N$, and $\mathcal{I}_{\geq 0} := \mathcal{I}_1 \cup \bigcup_{h=2}^N \{h1\}$. Let $(\mathbb{E}_1, \circ_1, \bullet_1)$ be the Euclidean Jordan algebra associated with the nonnegative orthant $\mathbb{R}^{n_1}_+$ shown in Example 2.6, and $(\mathbb{E}_h, \circ_h, \bullet_h)$ be the Euclidean Jordan algebra associated with the second-order cone $\mathbb{L}^{n_h}$ shown in Example 2.7 for $h = 2,\dots,N$. Taking the direct product of the N Euclidean Jordan algebras, we obtain the Euclidean Jordan algebra $(\mathbb{E}, \circ, \bullet)$ with $\mathbb{E}_+ = \mathbb{K}$. For each $\boldsymbol{x} \in \mathbb{E}$, the square $\boldsymbol{x} \circ \boldsymbol{x}$ is calculated as

$$\boldsymbol{x} \circ \boldsymbol{x} = \left((x_I^2)_{I \in \mathcal{I}_1}, \sum_{I \in \mathcal{I}_2} x_I^2, (2x_{21}x_I)_{I \in \mathcal{I}_2^-}, \dots, \sum_{I \in \mathcal{I}_N} x_I^2, (2x_{N1}x_I)_{I \in \mathcal{I}_N^-} \right)^\top .$$

The identity element $\boldsymbol{e}$ of the Euclidean Jordan algebra $(\mathbb{E}, \circ, \bullet)$ is the vector with the Ith ($I \in \mathcal{I}_{\geq 0}$) elements 1 and all other elements 0.

4.3.1.1 Explicit expression of generalized doubly nonnegative cones

First, we provide a simpler explicit expression of the ZVP-type generalized doubly nonnegative cone $\mathcal{DNN}_{\mathrm{ZVP}}(\mathbb{K})$. Note that the cone $\mathbb{K}$ has the following semialgebraic representation:

$$\left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \boldsymbol{e}_I^\top \boldsymbol{x} = x_I \geq 0 \ (I \in \mathcal{I}_{\geq 0}), \\ \boldsymbol{e}^\top \boldsymbol{x} = \sum_{I \in \mathcal{I}_{\geq 0}} x_I \geq 0, \\ \boldsymbol{x}^\top \boldsymbol{J}_h \boldsymbol{x} = x_{h1}^2 - \sum_{I \in \mathcal{I}_h^-} x_I^2 \geq 0 \ (h = 2, \dots, N) \end{array} \right\}, \tag{4.5}$$

where $\boldsymbol{J}_h$ is the $n \times n$ matrix such that the $(h1, h1)$th element is 1, the $(h2, h2), \dots, (hn_h, hn_h)$th elements are -1, and all other elements are 0 for $h = 2, \dots, N$. As mentioned in Section 2.5.1, remember that the redundant constraint $\boldsymbol{e}^\top \boldsymbol{x} \geq 0$ is necessary to construct the ZVP-type hierarchy. Consequently, Lemma 4.2 and Proposition 4.3 yield explicit expressions of $\mathcal{I}_{\mathrm{ZVP},0}(\mathbb{K})$ and $\mathcal{DNN}_{\mathrm{ZVP}}(\mathbb{K})$, respectively. However, in this case, they can be expressed in a simpler way.

Lemma 4.7.

$$\mathcal{I}_{\mathrm{ZVP},0}(\mathbb{K}) = \mathbb{S}_+^n + \sum_{h=2}^N \mathbb{R}_+ \boldsymbol{J}_h + \left\{ \boldsymbol{N} \in \mathbb{S}^n \;\middle|\; N_{I,J} \begin{cases} \geq 0 & (\text{if } I, J \in \mathcal{I}_{\geq 0} \text{ and } I \neq J) \\ = 0 & (\text{otherwise}) \end{cases} \right\}. \tag{4.6}$$

Throughout this subsection, for simplicity, we express the last set in the right-hand side of (4.6) as $\mathcal{N}_{\geq 0}$.

Proof. Let $\boldsymbol{E}^{IJ} := \frac{\boldsymbol{e}_I \boldsymbol{e}_J^\top + \boldsymbol{e}_J \boldsymbol{e}_I^\top}{2}$ and $\boldsymbol{E}_I := \frac{\boldsymbol{e}_I \boldsymbol{e}^\top + \boldsymbol{e}\boldsymbol{e}_I^\top}{2}$. By Lemma 4.2, we have

$$\mathcal{I}_{\mathrm{ZVP},0}(\mathbb{K}) = \mathbb{S}_+^n + \sum_{h=2}^N \mathbb{R}_+ \boldsymbol{J}_h + \sum_{\substack{I<J \\ I,J \in \mathcal{I}_{\geq 0}}} \mathbb{R}_+ \boldsymbol{E}^{IJ} + \sum_{I \in \mathcal{I}_{\geq 0}} \mathbb{R}_+ \boldsymbol{E}_I.$$

Therefore, it is sufficient to show that

$$\mathbb{S}_+^n + \sum_{\substack{I<J \\ I,J \in \mathcal{I}_{\geq 0}}} \mathbb{R}_+ \boldsymbol{E}^{IJ} + \sum_{I \in \mathcal{I}_{\geq 0}} \mathbb{R}_+ \boldsymbol{E}_I = \mathbb{S}_+^n + \mathcal{N}_{\geq 0}.$$

For each $I, J \in \mathcal{I}_{\geq 0}$ with $I < J$, since $\boldsymbol{E}^{IJ}$ is the matrix with the (I, J)th and (J, I)th elements equal to $\frac{1}{2}$ and other elements 0, we have $\boldsymbol{E}^{IJ} \in \mathcal{N}_{\geq 0}$. In addition, for each $I \in \mathcal{I}_{\geq 0}$, we have

$$\boldsymbol{E}_I = \boldsymbol{e}_I \boldsymbol{e}_I^\top + \sum_{\substack{J \in \mathcal{I}_{\geq 0} \\ J \neq I}} \boldsymbol{E}^{IJ} \in \mathbb{S}^n_+ + \mathcal{N}_{\geq 0}.$$

These imply the "$\subseteq$" component. Conversely, let $\boldsymbol{N} \in \mathcal{N}_{\geq 0}$. Then,

$$\boldsymbol{N} = \sum_{\substack{I<J \\ I,J \in \mathcal{I}_{\geq 0}}} 2N_{I,J} \boldsymbol{E}^{IJ} \in \sum_{\substack{I<J \\ I,J \in \mathcal{I}_{\geq 0}}} \mathbb{R}_+ \boldsymbol{E}^{IJ},$$

which implies the "$\supseteq$" component, thereby completing the proof. $\square$

Taking the dual in Lemma 4.7, we have a simpler explicit expression of $\mathcal{DNN}_{\mathrm{ZVP}}(\mathbb{K})$.

Proposition 4.8. *For $\mathbb{K} = \mathbb{R}^{n_1}_+ \times \prod_{h=2}^N \mathbb{L}^{n_h}$, it follows that*

$$\begin{aligned}
&\mathcal{DNN}_{\mathrm{ZVP}}(\mathbb{K}) \\
&\quad = \mathbb{S}^n_+ \cap \bigcap_{h=2}^N \{\boldsymbol{X} \in \mathbb{S}^n \mid \langle \boldsymbol{J}_h, \boldsymbol{X} \rangle \geq 0\} \cap \{\boldsymbol{N} \in \mathbb{S}^n \mid N_{I,J} \geq 0 \ (I, J \in \mathcal{I}_{\geq 0},\ I \neq J)\}.
\end{aligned}$$

Next, we provide an explicit expression of the NN-type generalized doubly nonnegative cone. For $\boldsymbol{y} = (y_{\boldsymbol{\delta}})_{\boldsymbol{\delta} \in \mathbb{I}^n_{=4}} \in \mathbb{R}^{\mathbb{I}^n_{=4}}$, let $\boldsymbol{C}^{(0)}(\boldsymbol{y}) \in \mathbb{S}^n$ be the matrix with the (I, J)th element:[1]

$$\begin{cases}
y_{2(\boldsymbol{e}_I + \boldsymbol{e}_J)} & (\text{if } I, J \in \mathcal{I}_1), & \text{(4.7a)} \\
\displaystyle\sum_{K \in \mathcal{I}_h} y_{2(\boldsymbol{e}_I + \boldsymbol{e}_K)} & (\text{if } I \in \mathcal{I}_1 \text{ and } J = h1 \text{ for some } h = 2, \dots, N), & \text{(4.7b)} \\
2y_{2\boldsymbol{e}_I + \boldsymbol{e}_{h1} + \boldsymbol{e}_J} & (\text{if } I \in \mathcal{I}_1 \text{ and } J \in \mathcal{I}_h^- \text{ for some } h = 2, \dots, N), & \\
\displaystyle\sum_{K \in \mathcal{I}_g} \sum_{L \in \mathcal{I}_h} y_{2(\boldsymbol{e}_K + \boldsymbol{e}_L)} & (\text{if } I = g1 \text{ and } J = h1 \text{ for some } g, h = 2, \dots, N), & \text{(4.7c)} \\
\displaystyle\sum_{K \in \mathcal{I}_g} 2y_{2\boldsymbol{e}_K + \boldsymbol{e}_{h1} + \boldsymbol{e}_J} & (\text{if } I = g1 \text{ and } J \in \mathcal{I}_h^- \text{ for some } g, h = 2, \dots, N), & \\
4y_{\boldsymbol{e}_{g1} + \boldsymbol{e}_I + \boldsymbol{e}_{h1} + \boldsymbol{e}_J} & (\text{if } I \in \mathcal{I}_g^- \text{ and } J \in \mathcal{I}_h^- \text{ for some } g, h = 2, \dots, N). &
\end{cases}$$

In addition, let $\boldsymbol{M}(\boldsymbol{y}) \in \mathbb{S}^{\mathbb{I}^n_{=2}}$ be the matrix with the $(\boldsymbol{\alpha}, \boldsymbol{\beta})$th element $y_{\boldsymbol{\alpha}+\boldsymbol{\beta}}$ for $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{I}^n_{=2}$, let $\mathcal{M} := \{\boldsymbol{y} \in \mathbb{R}^{\mathbb{I}^n_{=4}} \mid \boldsymbol{M}(\boldsymbol{y}) \in \mathbb{S}^{\mathbb{I}^n_{=2}}_+\}$, and we define $\mathcal{C}_0 := \{\boldsymbol{C}^{(0)}(\boldsymbol{y}) \mid \boldsymbol{y} \in \mathcal{M}\}$. Then, we can show that $\mathcal{DNN}_{\mathrm{NN}}(\mathbb{K}) = \mathrm{cl}\,\mathcal{C}_0$ by the same argument as in Section 3.2.3.[2] Since $\boldsymbol{C}^{(0)}(\boldsymbol{y})$ is linear with respect to $\boldsymbol{y}$ and $\mathcal{M}$ is a convex cone, it follows that $\mathcal{C}_0$ is a convex cone. We now demonstrate that $\mathcal{C}_0$ is closed, and the closure operator can be removed.

[1]Although we define only the upper triangular blocks of $\boldsymbol{C}^{(0)}(\boldsymbol{y})$ in (4.7), the lower triangular blocks are further defined so that $\boldsymbol{C}^{(0)}(\boldsymbol{y})$ is a symmetric matrix.

[2]Note that the matrix $\boldsymbol{M}(\boldsymbol{y})$ defined here corresponds to $\boldsymbol{M}^{n,2}(\boldsymbol{\mathcal{X}})$ defined in Section 3.2.3, the set $\mathcal{M}$ to $\mathcal{M}^{n,4}$, the matrix $\boldsymbol{C}^{(0)}(\boldsymbol{y})$ to a tensor $\boldsymbol{\mathcal{C}}^{(0)}(\boldsymbol{\mathcal{X}})$, and the set $\mathcal{C}_0$ to $\mathcal{C}_0^{n,2}(\mathbb{K})$.

Proposition 4.9. *The set $\mathcal{C}_0$ is closed.*

Proof. Let $\{\boldsymbol{y}^{(k)}\}_k \subseteq \mathcal{M}$ and suppose that $\boldsymbol{C}^{(0)}(\boldsymbol{y}^{(k)})$ converges to some $\boldsymbol{C}_*^{(0)}$ in the limit $k \to \infty$. It follows from the definition of $\mathcal{M}$ that $\boldsymbol{M}(\boldsymbol{y}^{(k)}) \in \mathbb{S}_+^{\mathbb{I}_{=2}^n}$, and in particular, the diagonal elements of $\boldsymbol{M}(\boldsymbol{y}^{(k)})$ are nonnegative. Hence, we see that

$$y_{2\boldsymbol{\gamma}}^{(k)} \geq 0 \quad \text{for all } \boldsymbol{\gamma} \in \mathbb{I}_{=2}^n. \tag{4.8}$$

Since $\boldsymbol{C}^{(0)}(\boldsymbol{y}^{(k)}) \to \boldsymbol{C}_*^{(0)}$, each element of $\boldsymbol{C}^{(0)}(\boldsymbol{y}^{(k)})$ is bounded. Specifically,

(i) $\{y_{2(\boldsymbol{e}_I+\boldsymbol{e}_J)}^{(k)}\}_k$ is bounded for all $I, J \in \mathcal{I}_1$ (see (4.7a)).

(ii) For each $h = 2, \dots, N$, $\{\sum_{J \in \mathcal{I}_h} y_{2(\boldsymbol{e}_I+\boldsymbol{e}_J)}^{(k)}\}$ is bounded for all $I \in \mathcal{I}_1$ (see (4.7b)). Therefore, we see from (4.8) that $\{y_{2(\boldsymbol{e}_I+\boldsymbol{e}_J)}^{(k)}\}_k$ is bounded for all $I \in \mathcal{I}_1$ and $J \in \mathcal{I}_h$.

(iii) For each $g, h = 2, \dots, N$, $\{\sum_{I \in \mathcal{I}_g} \sum_{J \in \mathcal{I}_h} y_{2(\boldsymbol{e}_I+\boldsymbol{e}_J)}^{(k)}\}_k$ is bounded (see (4.7c)). Therefore, $\{y_{2(\boldsymbol{e}_I+\boldsymbol{e}_J)}^{(k)}\}_k$ is bounded for all $I \in \mathcal{I}_g$ and $J \in \mathcal{I}_h$.

Because $\mathbb{I}_{=2}^n = \{\boldsymbol{e}_I + \boldsymbol{e}_J \mid 1 \leq I \leq J \leq n\}$, it follows from (i) to (iii) that $\{y_{2\boldsymbol{\gamma}}^{(k)}\}_k$ is bounded for all $\boldsymbol{\gamma} \in \mathbb{I}_{=2}^n$, which implies the boundedness of the diagonal elements of $\boldsymbol{M}(\boldsymbol{y}^{(k)})$. Combining the boundedness with the positive semidefiniteness of $\boldsymbol{M}(\boldsymbol{y}^{(k)})$ yields the boundedness of each element in $\boldsymbol{M}(\boldsymbol{y}^{(k)})$. Therefore, $\{\boldsymbol{y}^{(k)}\}_k$ is also bounded, as $y_{\boldsymbol{\delta}}^{(k)}$ is an element of $\boldsymbol{M}(\boldsymbol{y}^{(k)})$ for each $\boldsymbol{\delta} \in \mathbb{I}_{=4}^n$. Therefore, there exists a convergent subsequence $\{\boldsymbol{y}^{(k_r)}\}_r$ with a limit of $\boldsymbol{y}^*$. Then, because $\mathcal{M}$ is closed and $\boldsymbol{y}^{(k_r)} \in \mathcal{M}$, we have $\boldsymbol{y}^* \in \mathcal{M}$. Since $\boldsymbol{C}^{(0)}(\boldsymbol{y})$ is continuous with respect to $\boldsymbol{y}$, we obtain $\boldsymbol{C}^{(0)}(\boldsymbol{y}^{(k_r)}) \to \boldsymbol{C}^{(0)}(\boldsymbol{y}^*) = \boldsymbol{C}_*^{(0)}$. Thus, $\mathcal{C}_0$ is closed. ◻

Corollary 4.10. *For $\mathbb{K} = \mathbb{R}_+^{n_1} \times \prod_{h=2}^N \mathbb{L}^{n_h}$, it follows that $\mathcal{DNN}_{\mathrm{NN}}(\mathbb{K}) = \mathcal{C}_0$.*

Proof. It is clear from Proposition 4.9. ◻

Remark 4.11. For $\mathbb{K} = \mathbb{R}_+^{n_1} \times \prod_{h=2}^N \mathbb{L}^{n_h}$, in the same manner as Proposition 3.6 and Corollary 4.10, we can derive an explicit semidefinite representation of the dual cone of $\mathcal{I}_{\mathrm{NN},l}^{n,m}(\mathbb{K})$ for general cases of m and l.

4.3.1.2 Inclusion relationship between generalized doubly nonnegative cones

The following subsubsection discusses the inclusion relationship between ZVP-, NN-, and BD-type generalized doubly nonnegative cones.

To begin with, we show that $\mathcal{DNN}_{\mathrm{NN}}(\mathbb{K})$ is strictly included in $\mathcal{DNN}_{\mathrm{ZVP}}(\mathbb{K})$. The proof requires the closedness of $\mathcal{I}_{\mathrm{ZVP},0}(\mathbb{K})$, which does not hold in the general case. To prove closedness, we prepare an additional lemma that claims a sufficient condition that the Minkowski sum of pointed closed convex cones is closed. Because it is a slight modification of [106, Corollary 9.1.3], the proof is omitted.

Lemma 4.12. *Let $\mathcal{K}_1, \dots, \mathcal{K}_m$ be pointed closed convex cones in $\mathbb{S}^n$. Then, $\sum_{i=1}^m \mathcal{K}_i$ is closed if the following condition holds: for any $\boldsymbol{X}_i \in \mathcal{K}_i$ $(i = 1, \dots, m)$, if $\sum_{i=1}^m \boldsymbol{X}_i = \boldsymbol{O}$, then $\boldsymbol{X}_i = \boldsymbol{O}$ for all $i = 1, \dots, m$.*

Lemma 4.13. *For $\mathbb{K} = \mathbb{R}_+^{n_1} \times \prod_{h=2}^N \mathbb{L}^{n_h}$, the convex cone $\mathcal{I}_{\mathrm{ZVP},0}(\mathbb{K})$ is closed.*

Proof. Let $\boldsymbol{P} \in \mathbb{S}_+^n$, $t_2, \dots, t_N \geq 0$, and $\boldsymbol{N} \in \mathcal{N}_{\geq 0}$, and suppose that

$$\boldsymbol{A} \coloneqq \boldsymbol{P} + \sum_{h=2}^N t_h \boldsymbol{J}_h + \boldsymbol{N} = \boldsymbol{O}.$$

For every $h = 2, \dots, N$, because $A_{h1,h1} = P_{h1,h1} + t_h = 0$ and each term is nonnegative, we have $P_{h1,h1} = t_h = 0$. Given that $t_h = 0$ for all $h = 2, \dots, N$, for each $I \in \mathcal{I}_1 \cup \bigcup_{h=2}^N \mathcal{I}_h^-$, we have $A_{I,I} = P_{I,I} = 0$. Therefore, the diagonal elements of $\boldsymbol{P}$ are all 0 and $\boldsymbol{P} = \boldsymbol{O}$, as $\boldsymbol{P} \in \mathbb{S}_+^n$. Finally, we obtain $\boldsymbol{A} = \boldsymbol{N} = \boldsymbol{O}$. Since $\mathbb{S}_+^n$, $\mathbb{R}_+ \boldsymbol{J}_h$ $(h = 2, \dots, N)$, and $\mathcal{N}_{\geq 0}$ are pointed closed convex cones, $\mathcal{I}_{\mathrm{ZVP},0}(\mathbb{K})$ is therefore closed by Lemma 4.12. □

Lemma 4.14. *For $\mathbb{K} = \mathbb{R}_+^{n_1} \times \prod_{h=2}^N \mathbb{L}^{n_h}$, $\mathcal{I}_{\mathrm{ZVP},0}(\mathbb{K}) \subseteq \mathcal{I}_{\mathrm{NN},0}(\mathbb{K})$ holds. Moreover, the inclusion holds strictly if $n_1 \geq 1$ and $n_2 \geq 2$.*

Proof. Let

$$\boldsymbol{A} = \boldsymbol{P} + \sum_{h=2}^N t_h \boldsymbol{J}_h + \boldsymbol{N} \in \mathcal{I}_{\mathrm{ZVP},0}(\mathbb{K}), \tag{4.9}$$

where $\boldsymbol{P} \in \mathbb{S}_+^n$, $t_2, \dots, t_N \geq 0$, and $\boldsymbol{N} \in \mathcal{N}_{\geq 0}$. First, let $\boldsymbol{U}$ be a real $n \times n$ symmetric matrix such that $\boldsymbol{P} = \boldsymbol{U}^2$. Then, we see that

$$(\boldsymbol{x} \circ \boldsymbol{x})^\top \boldsymbol{P} (\boldsymbol{x} \circ \boldsymbol{x}) = \|\boldsymbol{U}(\boldsymbol{x} \circ \boldsymbol{x})\|_2^2 \in \Sigma^{n,4}.$$

Second, for each $h = 2, \dots, N$,

$$(\boldsymbol{x} \circ \boldsymbol{x})^\top (t_h \boldsymbol{J}_h)(\boldsymbol{x} \circ \boldsymbol{x}) = t_h \left(x_{h1}^2 - \sum_{I \in \mathcal{I}_h^-} x_I^2 \right)^2 \in \Sigma^{n,4}.$$

Finally,

$$\begin{aligned}(\boldsymbol{x} \circ \boldsymbol{x})^\top \boldsymbol{N} (\boldsymbol{x} \circ \boldsymbol{x}) = &\sum_{I,J \in \mathcal{I}_1} N_{I,J} (x_I x_J)^2 + 2 \sum_{h=2}^N \sum_{I \in \mathcal{I}_1} \sum_{J \in \mathcal{I}_h} N_{I,h1} (x_I x_J)^2 \\ &+ \sum_{g,h=2}^N \sum_{I \in \mathcal{I}_g} \sum_{J \in \mathcal{I}_h} N_{g1,h1} (x_I x_J)^2 \in \Sigma^{n,4}.\end{aligned}$$

Therefore, $(\boldsymbol{x} \circ \boldsymbol{x})^\top \boldsymbol{A} (\boldsymbol{x} \circ \boldsymbol{x}) \in \Sigma^{n,4}$, which means that $\boldsymbol{A} \in \mathcal{I}_{\mathrm{NN},0}(\mathbb{K})$.

The following example shows that the inclusion holds strictly. Suppose that $n_1 \geq 1$ and $n_2 \geq 2$. Let $\boldsymbol{A} \in \mathbb{S}^n$ be a matrix with the $(11,21)$th, $(11,22)$th, $(21,11)$th, and $(22,11)$th elements 1 and all other elements 0. Since

$$\begin{aligned}(\boldsymbol{x} \circ \boldsymbol{x})^\top \boldsymbol{A}(\boldsymbol{x} \circ \boldsymbol{x}) &= 2x_{11}^2 \sum_{I \in \mathcal{I}_2} x_I^2 + 4x_{11}^2 x_{21} x_{22} \\ &= 2x_{11}^2 \left\{ (x_{21} + x_{22})^2 + \sum_{I \in \mathcal{I}_2^- \setminus \{22\}} x_I^2 \right\} \in \Sigma^{n,4},\end{aligned}$$

it follows that $\boldsymbol{A} \in \mathcal{I}_{\mathrm{NN},0}(\mathbb{K})$.

On the other hand, we assume that $\boldsymbol{A} \in \mathcal{I}_{\mathrm{ZVP},0}(\mathbb{K})$ and express $\boldsymbol{A}$ as (4.9). Then, we have $0 = A_{11,11} = P_{11,11}$. Combining $P_{11,11} = 0$ with $\boldsymbol{P} \in \mathbb{S}_+^n$ yields $P_{11,22} = 0$. Therefore, $A_{11,22}$ must be 0, which contradicts the definition of $\boldsymbol{A}$. Hence, $\boldsymbol{A} \notin \mathcal{I}_{\mathrm{ZVP},0}(\mathbb{K})$. □

Theorem 4.15. *For $\mathbb{K} = \mathbb{R}_+^{n_1} \times \prod_{h=2}^N \mathbb{L}^{n_h}$, $\mathcal{DNN}_{\mathrm{NN}}(\mathbb{K}) \subseteq \mathcal{DNN}_{\mathrm{ZVP}}(\mathbb{K})$ holds. Moreover, the inclusion holds strictly if $n_1 \geq 1$ and $n_2 \geq 2$.*

Proof. By taking the dual in Lemma 4.14 and using (iv) of Theorem 2.2, we have $\mathcal{DNN}_{\mathrm{NN}}(\mathbb{K}) \subseteq \mathcal{DNN}_{\mathrm{ZVP}}(\mathbb{K})$. We next show by contradiction that the strict inclusion $\mathcal{DNN}_{\mathrm{NN}}(\mathbb{K}) \subsetneq \mathcal{DNN}_{\mathrm{ZVP}}(\mathbb{K})$ holds under the assumption of $n_1 \geq 1$ and $n_2 \geq 2$. Assume that $\mathcal{DNN}_{\mathrm{NN}}(\mathbb{K}) = \mathcal{DNN}_{\mathrm{ZVP}}(\mathbb{K})$ holds. By taking its dual, we have $\mathcal{DNN}_{\mathrm{ZVP}}(\mathbb{K})^* = \mathcal{DNN}_{\mathrm{NN}}(\mathbb{K})^*$. Remember that the double dual of a closed convex cone agrees with itself; see (iii) of Theorem 2.2. Since $\mathcal{I}_{\mathrm{ZVP},0}(\mathbb{K})$ and $\mathcal{I}_{\mathrm{NN},0}(\mathbb{K})$ are both closed convex cones (see Lemma 4.13 and Theorem 3.4, respectively), it follows from the definition of the ZVP- and NN-type generalized doubly nonnegative cones that $\mathcal{I}_{\mathrm{ZVP},0}(\mathbb{K}) = \mathcal{I}_{\mathrm{NN},0}(\mathbb{K})$. This contradicts the assumption and Lemma 4.14. □

We will demonstrate that, in general, there is no inclusion relationship between $\mathcal{DNN}_{\mathrm{ZVP}}(\mathbb{K})$ and $\mathcal{DNN}_{\mathrm{BD}}(\mathbb{K})$. Before presenting such examples, we provide a more concise expression of $\mathcal{N}(\mathbb{K})$ appearing in $\mathcal{DNN}_{\mathrm{BD}}(\mathbb{K})$ for the case where $\mathbb{K}$ is a direct product of a nonnegative orthant and second-order cones. It can be easily seen that the set $\mathfrak{J}$, defined as

$$\mathfrak{J} := \bigcup_{I \in \mathcal{I}_1} \{(\boldsymbol{e}_I; \mathbf{0}_{n_2}; \ldots; \mathbf{0}_{n_N})\} \cup \bigcup_{h=2}^N \left\{ \left(\mathbf{0}_{n_1}; \mathbf{0}_{n_2}; \ldots; \frac{1}{2}; \frac{\boldsymbol{v}}{2}; \ldots; \mathbf{0}_{n_N} \right) \;\middle|\; \boldsymbol{v} \in S^{n_h - 2} \right\},$$

satisfies $\mathrm{Ext}\,\mathbb{K} = \{\alpha \boldsymbol{s} \mid \alpha > 0, \boldsymbol{s} \in \mathfrak{J}\}$. Since the vector $\boldsymbol{X}\boldsymbol{s}$ is linear with respect to $\boldsymbol{s}$ and $\mathbb{K}$ is self-dual, we can equivalently represent $\mathcal{N}(\mathbb{K})$ as $\{\boldsymbol{X} \in \mathbb{S}^n \mid \boldsymbol{X}\boldsymbol{s} \in \mathbb{K} \text{ for all } \boldsymbol{s} \in \mathfrak{J}\}$.

Example 4.16 (a matrix that is in $\mathcal{DNN}_{\mathrm{ZVP}}(\mathbb{K})$ but not in $\mathcal{DNN}_{\mathrm{BD}}(\mathbb{K})$). Suppose that

$n_1 \geq 1$ and $n_2 \geq 2$. Let

$$\boldsymbol{A} = \begin{pmatrix} \boxed{\begin{matrix} n_2 - 1 & & & \boldsymbol{1}_{n_2-1}^\top \\ & \boldsymbol{O}_{n_1-1} & & \\ & & n_2 - 1 & \\ \boldsymbol{1}_{n_2-1} & & & \boldsymbol{I}_{n_2-1} \end{matrix}} & \\ & \boldsymbol{O} \end{pmatrix}.$$

We use Proposition 4.8 to show that $\boldsymbol{A} \in \mathcal{DNN}_{\mathrm{ZVP}}(\mathbb{K})$. We can easily check that

$$\boldsymbol{A} \in \bigcap_{h=2}^{N} \{\boldsymbol{X} \in \mathbb{S}^n \mid \langle \boldsymbol{J}_h, \boldsymbol{X} \rangle \geq 0\} \cap \{\boldsymbol{N} \in \mathbb{S}^n \mid N_{I,J} \geq 0 \ (I, J \in \mathcal{I}_{\geq 0},\ I \neq J)\}.$$

Therefore, it is sufficient to show that $\boldsymbol{A} \in \mathbb{S}^n_+$. $\boldsymbol{A} \in \mathbb{S}^n_+$ if and only if the block diagonal matrix

$$\begin{pmatrix} n_2 - 1 & & & \\ & n_2 - 1 & \boldsymbol{1}_{n_2-1}^\top & \\ & \boldsymbol{1}_{n_2-1} & \boldsymbol{I}_{n_2-1} & \\ & & & \boldsymbol{O} \end{pmatrix},$$

which is obtained by rearranging some rows and columns of $\boldsymbol{A}$, is positive semidefinite. This is true because $n_2 - 1 \geq 0$ and $(n_2 - 1) - \boldsymbol{1}_{n_2-1}^\top \boldsymbol{I}_{n_2-1} \boldsymbol{1}_{n_2-1} = 0$, where we apply the Schur complement lemma [5, Lemma 4.2.1]. Therefore, we have $\boldsymbol{A} \in \mathcal{DNN}_{\mathrm{ZVP}}(\mathbb{K})$.

Next, we let

$$\boldsymbol{s} = \left(\boldsymbol{0}_{n_1}; \frac{1}{2}; -\frac{\boldsymbol{1}_{n_2-1}}{2\sqrt{n_2 - 1}}; \boldsymbol{0}\right) \in \mathfrak{J}.$$

Then, since the first component of an element in $\mathbb{K}$ must be nonnegative and $(\boldsymbol{A}\boldsymbol{s})_1 = -\frac{\sqrt{n_2-1}}{2} < 0$, we see that $\boldsymbol{A}\boldsymbol{s} \notin \mathbb{K}$. Thus, $\boldsymbol{A} \notin \mathcal{DNN}_{\mathrm{BD}}(\mathbb{K})$.

Example 4.17 (a matrix that is in $\mathcal{DNN}_{\mathrm{BD}}(\mathbb{K})$ but not in $\mathcal{DNN}_{\mathrm{ZVP}}(\mathbb{K})$). Suppose that $n_2 \geq 3$. Let

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{O}_{n_1} & & & \\ & 1 & & \\ & & \frac{\boldsymbol{I}_{n_2-1}}{\sqrt{n_2-1}} & \\ & & & \boldsymbol{O} \end{pmatrix}.$$

On the one hand, $\boldsymbol{A} \in \mathcal{DNN}_{\mathrm{BD}}(\mathbb{K})$. Indeed, let $\boldsymbol{s} \in \mathfrak{J}$. If $\boldsymbol{s} = (\boldsymbol{0}_{n_1}; \frac{1}{2}; \frac{\boldsymbol{v}}{2}; \boldsymbol{0})$ for some $\boldsymbol{v} \in S^{n_2-2}$, then

$$\boldsymbol{A}\boldsymbol{s} = \left(\boldsymbol{0}_{n_1}; \frac{1}{2}; \frac{\boldsymbol{v}}{2\sqrt{n_2 - 1}}; \boldsymbol{0}\right).$$

Since

$$\left(\frac{1}{2}\right)^2 - \left\|\frac{\boldsymbol{v}}{2\sqrt{n_2 - 1}}\right\|_2^2 = \frac{1}{4}\left(1 - \frac{1}{n_2 - 1}\right) \geq 0,$$

$(\frac{1}{2}; \frac{\boldsymbol{v}}{2\sqrt{n_2-1}}) \in \mathbb{L}^{n_2}$, and therefore we get $\boldsymbol{A}\boldsymbol{s} \in \mathbb{K}$. Otherwise, $\boldsymbol{A}\boldsymbol{s} = \boldsymbol{0} \in \mathbb{K}$.

On the other hand, $\boldsymbol{A} \notin \mathcal{DNN}_{\mathrm{ZVP}}(\mathbb{K})$ because $\langle \boldsymbol{J}_2, \boldsymbol{A} \rangle = 1 - \sqrt{n_2 - 1} < 0$.

We consider the special case of $\mathbb{K} = \mathbb{L}^n$. As a consequence of the dual of the S-lemma [96, Theorem 2.2] (see also [117, Theorem 1]), it follows that

$$\mathcal{CP}(\mathbb{L}^n) = \mathbb{S}^n_+ \cap \left\{ \boldsymbol{X} \in \mathbb{S}^n \;\middle|\; X_{1,1} - \sum_{i=2}^{n} X_{i,i} \geq 0 \right\}. \tag{4.10}$$

Therefore, $\mathcal{CP}(\mathbb{L}^n)$ itself is semidefinite representable. Surprisingly, from Proposition 4.8 and Theorem 4.15, we see that $\mathcal{DNN}_{\mathrm{ZVP}}(\mathbb{L}^n)$ and $\mathcal{DNN}_{\mathrm{NN}}(\mathbb{L}^n)$ agree with $\mathcal{CP}(\mathbb{L}^n)$. However, Example 4.17 implies that $\mathcal{DNN}_{\mathrm{BD}}(\mathbb{L}^n)$ includes $\mathcal{CP}(\mathbb{L}^n)$ strictly if $n \geq 3$. Including the case of $n \leq 2$, we derive the necessary and sufficient condition that $\mathcal{CP}(\mathbb{L}^n)$ agrees with each generalized doubly nonnegative cone over $\mathbb{L}^n$.

Proposition 4.18. *The following inclusions*

$$\mathcal{CP}(\mathbb{L}^n) = \mathcal{DNN}_{\mathrm{NN}}(\mathbb{L}^n) = \mathcal{DNN}_{\mathrm{ZVP}}(\mathbb{L}^n) \subseteq \mathcal{DNN}_{\mathrm{BD}}(\mathbb{L}^n)$$

hold. Furthermore, the above inclusion of "$\subseteq$" holds strictly if and only if $n \geq 3$.

Proof. It is sufficient to show that $\mathcal{DNN}_{\mathrm{BD}}(\mathbb{L}^n) \subseteq \mathcal{CP}(\mathbb{L}^n)$ in the case of $n \leq 2$. The equality holds when $n = 1$ because $\mathcal{CP}(\mathbb{L}^1) = \mathbb{S}^1_+$ and $\mathcal{DNN}_{\mathrm{BD}}(\mathbb{L}^1)$ is sandwiched between $\mathcal{CP}(\mathbb{L}^1)$ and $\mathbb{S}^1_+$.

Let us consider the case of $n = 2$. For $\boldsymbol{X} \in \mathcal{DNN}_{\mathrm{BD}}(\mathbb{L}^2)$, $\boldsymbol{X}$ is positive semidefinite by the definition of $\mathcal{DNN}_{\mathrm{BD}}(\mathbb{L}^2)$. In addition, since the vector $(1,1)^\top$ generates an extreme ray of $\mathbb{L}^2$, it follows that $\boldsymbol{X}(1,1)^\top = (X_{1,1} + X_{1,2}, X_{1,2} + X_{2,2})^\top \in \mathbb{L}^2$, i.e., $X_{1,1} + X_{1,2} \geq |X_{1,2} + X_{2,2}|$. This implies that $X_{1,1} - X_{2,2}$ is greater than or equal to 0. Thus, we have $\boldsymbol{X} \in \mathcal{CP}(\mathbb{L}^2)$. $\square$

Finally, we discuss the inclusion relationship between $\mathcal{DNN}_{\mathrm{NN}}(\mathbb{K})$ and $\mathcal{DNN}_{\mathrm{BD}}(\mathbb{K})$. Because $\mathcal{DNN}_{\mathrm{NN}}(\mathbb{K}) \subseteq \mathcal{DNN}_{\mathrm{ZVP}}(\mathbb{K})$ (Theorem 4.15), the matrix presented in Example 4.17 is also in $\mathcal{DNN}_{\mathrm{BD}}(\mathbb{K})$, but not in $\mathcal{DNN}_{\mathrm{NN}}(\mathbb{K})$. Therefore, we are interested in whether there exists a matrix that is in $\mathcal{DNN}_{\mathrm{NN}}(\mathbb{K})$ but not in $\mathcal{DNN}_{\mathrm{BD}}(\mathbb{K})$. If no such matrix exists, $\mathcal{DNN}_{\mathrm{NN}}(\mathbb{K})$ is included in $\mathcal{DNN}_{\mathrm{BD}}(\mathbb{K})$. Although we were not able to determine this theoretically, the results of our numerical experiment imply that $\mathcal{DNN}_{\mathrm{NN}}(\mathbb{K})$ is included in $\mathcal{DNN}_{\mathrm{BD}}(\mathbb{K})$. For details, see Section 4.4.1.

4.3.2 When $\mathbb{K}$ is a direct product of a nonnegative orthant and positive semidefinite cones

In this subsection, we consider the case where $\mathbb{K}$ is a direct product of a nonnegative orthant and positive semidefinite cones; i.e., $\mathbb{K} = \mathbb{R}^{n_1}_+ \times \prod_{h=2}^{N} \mathrm{svec}(\mathbb{S}^{n_h}_+)$, and let $n := n_1 + \sum_{h=2}^{N} T_{n_h}$. For convenience, we reindex $(1, \ldots, n)$ as

$$(11, \ldots, 1n_1, 211, 212, 222, \ldots, 21n_2, \ldots, 2n_2n_2, \ldots, N11, \ldots, Nn_Nn_N),$$

i.e., $1i := i$ for $i = 1, \ldots, n_1$, and $hij := n_1 + \sum_{k=2}^{h-1} T_{n_k} + \frac{j(j-1)}{2} + i$ for $h = 2, \ldots, N$ and $1 \leq i \leq j \leq n_h$. For $h = 2, \ldots, N$ and $1 \leq j < i \leq n_h$, let $hij := hji$. Moreover, we set $\tilde{\mathcal{I}}_h := \{h11, h12, h22, \ldots, hn_hn_h\}$ for $h = 2, \ldots, N$ and

$$\tilde{\mathcal{I}}_{\geq 0} := \{11, \ldots, 1n_1, 211, 222, 233, \ldots, 2n_2n_2, \ldots, N11, \ldots, Nn_Nn_N\}.$$

Let $(\mathbb{E}_1, \circ_1, \bullet_1)$ be the Euclidean Jordan algebra associated with the nonnegative orthant $\mathbb{R}^{n_1}_+$ shown in Example 2.6, and $(\mathbb{E}_h, \circ_h, \bullet_h)$ be the Euclidean Jordan algebra associated with the positive semidefinite cone $\mathrm{svec}(\mathbb{S}^{n_h}_+)$ shown in Example 2.8 for $h = 2, \dots, N$. Taking the direct product of the N Euclidean Jordan algebras, we obtain the Euclidean Jordan algebra $(\mathbb{E}, \circ, \bullet)$ with $\mathbb{E}_+ = \mathbb{K}$. For each $\boldsymbol{x} \in \mathbb{E}$, the Ith element of the square $\boldsymbol{x} \circ \boldsymbol{x}$ is calculated as

$$\begin{cases} x_I^2 & (\text{if } I = 11, \dots, 1n_1), \\ x_I^2 + \dfrac{1}{2} \displaystyle\sum_{k:k\neq i} x_{hki}^2 & (\text{if } I = hii \text{ for some } h = 2, \dots, N \text{ and } i = 1, \dots, n_h), \\ x_{hii}x_I + x_I x_{hjj} & \\ \quad + \dfrac{1}{\sqrt{2}} \displaystyle\sum_{k:k\neq i,j} x_{hik}x_{hjk} & (\text{if } I = hij \text{ for some } h = 2, \dots, N \text{ and } 1 \leq i < j \leq n_h). \end{cases}$$

The identity element $\boldsymbol{e}$ of the Euclidean Jordan algebra $(\mathbb{E}, \circ, \bullet)$ is the vector with the Ith $(I \in \tilde{\mathcal{I}}_{\geq 0})$ elements 1 and all other elements 0.

As mentioned in Example 2.8, the positive semidefinite cone $\mathrm{svec}(\mathbb{S}^m_+)$ is a symmetric cone. In addition, $\mathrm{svec}(\mathbb{S}^m_+)$ is semialgebraic, as

$$\mathrm{svec}(\mathbb{S}^m_+) = \left\{ \boldsymbol{x} \in \mathbb{R}^{T_m} \;\middle|\; \begin{array}{l} \det \mathrm{smat}(\boldsymbol{x})_{\mathcal{I},\mathcal{I}} \geq 0 \text{ for all } \mathcal{I} \subseteq \{1, \dots, m\}, \\ \boldsymbol{e}^\top \boldsymbol{x} = \sum_{I \in \tilde{\mathcal{I}}_{\geq 0}} x_I \geq 0 \end{array} \right\}, \tag{4.11}$$

where $\det(\cdot)$ represents the determinant of an input matrix. Therefore, we can define all three generalized doubly nonnegative cones properly in this case.

Here, we investigate the inclusion relationship between the ZVP- and NN-type generalized doubly nonnegative cones. As in Section 4.3.1, we provide a simpler explicit expression of $\mathcal{I}_{\mathrm{ZVP},0}(\mathbb{K})$ and $\mathcal{DNN}_{\mathrm{ZVP}}(\mathbb{K})$.

Lemma 4.19. *Let $\boldsymbol{J}_h^{ij}$ be the $n \times n$ matrix with the (hii, hjj)th and (hjj, hii)th elements 1, the (hij, hij)th element -1, and all other elements 0. Then,*

$$\begin{aligned} &\mathcal{I}_{\mathrm{ZVP},0}(\mathbb{K}) \\ &= \mathbb{S}^n_+ + \sum_{h=2}^{N} \sum_{1\leq i<j\leq n_h} \mathbb{R}_+ \boldsymbol{J}_h^{ij} + \left\{ \boldsymbol{N} \in \mathbb{S}^n \;\middle|\; N_{I,J} \begin{cases} \geq 0 & (\textit{if } I, J \in \tilde{\mathcal{I}}_{\geq 0} \textit{ and } I \neq J) \\ = 0 & (\textit{otherwise}) \end{cases} \right\}. \end{aligned} \tag{4.12}$$

Note that the matrix $\boldsymbol{J}_h^{ij}$ arises from the nonnegativity of the determinant of a 2×2 principal submatrix, which is a necessary condition for positive semidefiniteness. For simplicity, we write the last set in the right-hand side of (4.12) as $\tilde{\mathcal{N}}_{\geq 0}$. Because Lemma 4.19 can be proven in the same way as Lemma 4.7, the proof is omitted. We can also show that $\mathcal{I}_{\mathrm{ZVP},0}(\mathbb{K})$ is closed using Lemma 4.12. Moreover, taking the dual in Lemma 4.19, we obtain a simpler explicit expression of $\mathcal{DNN}_{\mathrm{ZVP}}(\mathbb{K})$.

Proposition 4.20. *For $\mathbb{K} = \mathbb{R}^{n_1}_+ \times \prod_{h=2}^N \mathrm{svec}(\mathbb{S}^{n_h}_+)$, it follows that*

$$\mathcal{DNN}_{\mathrm{ZVP}}(\mathbb{K}) = \mathbb{S}^n_+ \cap \bigcap_{h=2}^N \bigcap_{1\le i<j\le n_h} \{\boldsymbol{X} \in \mathbb{S}^n \mid \langle \boldsymbol{J}^{ij}_h, \boldsymbol{X}\rangle \ge 0\}$$
$$\cap \{\boldsymbol{N} \in \mathbb{S}^n \mid N_{I,J} \ge 0 \ (I, J \in \tilde{\mathcal{I}}_{\ge 0},\ I \ne J)\}.$$

Remark 4.21. As for $\mathcal{DNN}_{\mathrm{NN}}(\mathbb{K})$, similarly to Corollary 4.10, we can obtain its explicit expression without the closure operator. In addition, this can be extended to the case of tensors.

We show that the NN-type generalized doubly nonnegative cone is strictly included in the ZVP-type cone as in the case that $\mathbb{K}$ involves second-order cones.

Lemma 4.22. *For $\mathbb{K} = \mathbb{R}^{n_1}_+ \times \prod_{h=2}^N \mathrm{svec}(\mathbb{S}^{n_h}_+)$, $\mathcal{I}_{\mathrm{ZVP},0}(\mathbb{K}) \subseteq \mathcal{I}_{\mathrm{NN},0}(\mathbb{K})$ holds. Moreover, the inclusion holds strictly if $n_1 \ge 1$ and $n_2 \ge 2$.*

Proof. Let

$$\boldsymbol{A} = \boldsymbol{P} + \sum_{h=2}^N \sum_{1\le i<j\le n_h} t^{ij}_h \boldsymbol{J}^{ij}_h + \boldsymbol{N}, \tag{4.13}$$

where $\boldsymbol{P} \in \mathbb{S}^n_+$, $t^{ij}_h \ge 0$, and $\boldsymbol{N} \in \tilde{\mathcal{N}}_{\ge 0}$. First, as the proof of Lemma 4.14, $(\boldsymbol{x}\circ\boldsymbol{x})^\top \boldsymbol{P}(\boldsymbol{x}\circ\boldsymbol{x}) \in \Sigma^{n,4}$. Second, for $h = 2, \dots, N$ and $1 \le i < j \le n_h$, because

$$\begin{aligned}
&2(\boldsymbol{x}\circ\boldsymbol{x})^\top \boldsymbol{J}^{ij}_h(\boldsymbol{x}\circ\boldsymbol{x})\\
&\quad= 4\left(x_{hii}^2 + \frac{1}{2}\sum_{k:k\ne i} x_{hki}^2\right)\left(x_{hjj}^2 + \frac{1}{2}\sum_{k:k\ne j} x_{hkj}^2\right)\\
&\qquad - 2\left(x_{hii}x_{hij} + x_{hij}x_{hjj} + \frac{1}{\sqrt{2}}\sum_{k:k\ne i,j} x_{hik}x_{hjk}\right)^2\\
&\quad= (x_{hij}^2 - 2x_{hii}x_{hjj})^2 + \sum_{k:k\ne i,j}(x_{hij}x_{hik} - \sqrt{2}x_{hii}x_{hjk})^2\\
&\qquad + \sum_{k:k\ne i,j}(x_{hij}x_{hjk} - \sqrt{2}x_{hjj}x_{hik})^2 + \sum_{\substack{k<l\\ k,l\ne i,j}}(x_{hik}x_{hjl} - x_{hil}x_{hjk})^2,
\end{aligned}$$

we have $(\boldsymbol{x}\circ\boldsymbol{x})^\top \boldsymbol{J}^{ij}_h(\boldsymbol{x}\circ\boldsymbol{x}) \in \Sigma^{n,4}$. Finally,

$$\begin{aligned}
&(\boldsymbol{x}\circ\boldsymbol{x})^\top \boldsymbol{N}(\boldsymbol{x}\circ\boldsymbol{x})\\
&= \sum_{i,j=1}^{n_1} N_{1i,1j}x_{1i}^2x_{1j}^2 + 2\sum_{h=2}^N\sum_{i=1}^{n_1}\sum_{j=1}^{n_h} N_{1i,hjj}x_{1i}^2\left(x_{hjj}^2 + \frac{1}{2}\sum_{k:k\ne j}x_{hkj}^2\right)\\
&\quad + \sum_{g,h=2}^N\sum_{i=1}^{n_g}\sum_{j=1}^{n_h} N_{gii,hjj}\left(x_{gii}^2 + \frac{1}{2}\sum_{k:k\ne i}x_{gki}^2\right)\left(x_{hjj}^2 + \frac{1}{2}\sum_{k:k\ne j}x_{hkj}^2\right).
\end{aligned} \tag{4.14}$$

As all variables that appear in (4.14) are squared and each element of $\boldsymbol{N}$ is nonnegative, we see that $(\boldsymbol{x}\circ\boldsymbol{x})^\top \boldsymbol{N}(\boldsymbol{x}\circ\boldsymbol{x}) \in \Sigma^{n,4}$. Therefore, $(\boldsymbol{x}\circ\boldsymbol{x})^\top \boldsymbol{A}(\boldsymbol{x}\circ\boldsymbol{x}) \in \Sigma^{n,4}$, which implies that $\boldsymbol{A} \in \mathcal{I}_{\mathrm{NN},0}(\mathbb{K})$.

The following example shows that the inclusion holds strictly. Suppose that $n_1 \geq 1$ and $n_2 \geq 2$. Let

$$\epsilon := \min\left\{\frac{2}{n_2-1}, \frac{2}{2+\sqrt{2}(n_2-2)}\right\} > 0 \tag{4.15}$$

and $\boldsymbol{A} \in \mathbb{S}^n$ be the matrix whose (I,J)th element is

$$\begin{cases} 1 & (\text{if } I = 11 \text{ and } J \in \{2ii \mid i = 1,\dots,n_2\}), \\ \epsilon & (\text{if } I = 11 \text{ and } J \in \tilde{\mathcal{I}}_2 \setminus \{2ii \mid i = 1,\dots,n_2\}), \\ 0 & (\text{otherwise}). \end{cases}$$

Then, some calculations yield the following:

$$\begin{aligned} (\boldsymbol{x}\circ\boldsymbol{x})^\top \boldsymbol{A}(\boldsymbol{x}\circ\boldsymbol{x}) = x_{11}^2 \Bigg[& \epsilon \sum_{i<j} \{(x_{2ii}+x_{2ij})^2 + (x_{2ij}+x_{2jj})^2\} + \frac{\epsilon}{\sqrt{2}} \sum_{i<j} \sum_{k:k\neq i,j} (x_{2ik}+x_{2jk})^2 \\ & + \{2-(n_2-1)\epsilon\} \sum_{i=1}^{n_2} x_{2ii}^2 + [2 - \{2+\sqrt{2}(n_2-2)\}\epsilon] \sum_{i<j} x_{2ij}^2 \Bigg], \end{aligned}$$

which is sum-of-squares by (4.15). Consequently, on the one hand, we have $\boldsymbol{A} \in \mathcal{I}_{\mathrm{NN},0}(\mathbb{K})$.

On the other hand, assume that $\boldsymbol{A}$ can be expressed as (4.13). Then, we have $0 = A_{11,11} = P_{11,11}$. Combining $P_{11,11} = 0$ with $\boldsymbol{P} \in \mathbb{S}_+^n$ yields $P_{11,2ij} = 0$ for all $i < j$. Therefore, for a pair (i,j) with $i<j$, $A_{11,2ij}$ must be 0. However, by the definition of $\boldsymbol{A}$, $A_{11,2ij} = \epsilon > 0$, which is a contradiction. $\square$

Taking the dual in Lemma 4.22, we obtain the following desired result.

Theorem 4.23. *For $\mathbb{K} = \mathbb{R}_+^{n_1} \times \prod_{h=2}^N \mathrm{svec}(\mathbb{S}_+^{n_h})$, $\mathcal{DNN}_{\mathrm{NN}}(\mathbb{K}) \subseteq \mathcal{DNN}_{\mathrm{ZVP}}(\mathbb{K})$ holds. Moreover, the inclusion holds strictly if $n_1 \geq 1$ and $n_2 \geq 2$.*

The theoretical inclusion relationship between ZVP- and BD-type generalized doubly nonnegative cones and between NN- and BD-type generalized doubly nonnegative cones remains unknown. See also Section 4.4.2 for the numerical comparison of the strength of relaxation by the three generalized doubly nonnegative cones.

4.4 Numerical Experiments

We conducted a series of numerical experiments to investigate the strength of relaxation by the three generalized doubly nonnegative cones. We consider in Section 4.4.1, the case where $\mathbb{K}$ is a direct product of a nonnegative orthant and second-order cones, and in Section 4.4.2, the case where $\mathbb{K}$ is a direct product of a nonnegative orthant and positive semidefinite cones. We conducted all of the experiments with MATLAB

(R2021b in Section 4.4.1 and R2022a in Section 4.4.2) on a computer with an Intel Core i5-8279U 2.40 GHz CPU and 16 GB of memory. The versions of the YALMIP modeling language [71] and the MOSEK solver [78] we used in the following two experiments are 20210331 and 9.3.3, respectively.

4.4.1 Mixed 0–1 second-order cone programming

We considered the following mixed 0–1 second-order cone programming problem with a linear objective function:

$$\begin{aligned} &\underset{\boldsymbol{x}}{\text{minimize}} && \boldsymbol{c}^\top \boldsymbol{x} \\ &\text{subject to} && 0 \leq x_1 \leq 2, \\ &&& 0 \leq x_i \leq 1 \ (i = 2, \dots, n), \\ &&& x_i \in \{0,1\} \ (i \in B \subseteq \{2, \dots, n\}), \\ &&& \boldsymbol{x} \in \mathbb{L}^n. \end{aligned} \tag{4.16}$$

As a consequence of Corollary 2.16, the second constraint of problem (4.16) ensures that we can equivalently transform problem (4.16) into a generalized completely positive programming problem. Specifically, by introducing $2n$ nonnegative slack variables, we can convert problem (4.16) into the primal standard form of generalized completely positive programming

$$\begin{aligned} &\underset{\boldsymbol{X}}{\text{minimize}} && \langle \boldsymbol{C}, \boldsymbol{X} \rangle \\ &\text{subject to} && \langle \boldsymbol{A}_i, \boldsymbol{X} \rangle = b_i \ (i = 1, \dots, m), \\ &&& \boldsymbol{X} \in \mathcal{CP}(\mathbb{K}) \end{aligned} \tag{4.17}$$

with $m = 4n + |B| + 1$ and $\mathbb{K} = \mathbb{R}_+^{2n+1} \times \mathbb{L}^n$ for some symmetric matrices $\boldsymbol{A}_1, \dots, \boldsymbol{A}_m, \boldsymbol{C}$ and scalars $b_1, \dots, b_m$. In this section, we solve the problem obtained by relaxing the generalized completely positive programming problem with each generalized doubly nonnegative cone and compare the results.

Reformulating problem (4.16) as a generalized completely positive programming problem would be impractical because problem (4.16) can be solved directly and quickly with an existing mixed-integer second-order cone programming solver. However, we decided to solve a mixed 0–1 second-order cone programming problem with a linear objective function rather than a nonlinear or nonconvex objective because we were able to obtain the exact optimal value of the problem with the solver and calculate the difference between the optimal value of the original problem and that of its relaxation problem with each generalized doubly nonnegative cone.

We created instances of problem (4.16) as follows. The number n of variables was changed between 5, 10, 30, and 50. The set B of indices to determine binary variables was generated by selecting $0.4n$ elements from $\{2, \dots, n\}$ randomly. All elements of $\boldsymbol{c}$ were independent and identically distributed, and each followed the standard normal distribution. For each n, we generated five instances varying the randomness of B and $\boldsymbol{c}$. For each instance, the optimal value of the relaxation problem with

Table 4.1: Optimal values of problem (4.16) and its associated generalized doubly nonnegative programming and semidefinite programming relaxation problems.

		Optimal value				
n	No.	MISOCP	NN	ZVP	BD	SDP
5	1	−4.01	−4.01	−4.01	−4.01	−30.56
	2	−3.18	−3.18	−3.18	−3.18	−4.93
	3	−0.28	−0.28	−0.31	−0.28	−16.86
	4	0.00	0.00	0.00	0.00	−106.50
	5	−1.47	−1.47	−1.47	−1.47	−163.54
10	1	−4.73	OOM	−4.74	−4.77	−42.56
	2	−5.43	OOM	−5.43	−5.43	−140.18
	3	−1.99	OOM	−1.99	−1.99	−13.20
	4	0.00	OOM	−0.86	0.00	−186.66
	5	−4.23	OOM	−4.23	−4.23	−53.90
30	1	−9.17	OOM	−9.17	−9.62	−328.08
	2	−9.42	OOM	−9.42	−9.95	−340.17
	3	−5.91	OOM	−5.91	−6.00	−120.07
	4	−3.05	OOM	−3.07	−3.25	−444.75
	5	−9.58	OOM	−9.58	−10.14	−362.77
50	1	−9.34	OOM	−9.34	−10.93	−489.81
	2	−9.70	OOM	−9.71	−10.25	−324.13
	3	−6.68	OOM	−6.69	−7.16	−293.58
	4	−3.30	OOM	−3.30	−3.78	−560.80
	5	−12.07	OOM	−12.10	−13.29	−750.69

$\mathcal{DNN}_{\mathrm{NN}}(\mathbb{K})$ (written as NN in Table 4.1), $\mathcal{DNN}_{\mathrm{ZVP}}(\mathbb{K})$ (ZVP), and $\mathcal{DNN}_{\mathrm{BD}}(\mathbb{K})$ (BD) was measured. For reference, we also solved the problem (4.16) itself (MISOCP), as well as the relaxation problem with the positive semidefinite cone (SDP). To improve numerical stability, $0.005\boldsymbol{I}$ was added to the coefficient matrix $\boldsymbol{C}$ in the standard form (4.17) when we solved the semidefinite programming relaxation problems. We solved a BD-type generalized doubly nonnegative programming problem as a semi-infinite conic programming problem by adopting an algorithm based on the explicit exchange method [86]. See Appendix A for the algorithm. We used the YALMIP modeling language and the MOSEK solver to solve the optimization problems.

Table 4.1 lists the optimal values of the problem (4.16) and its generalized doubly nonnegative programming and semidefinite programming relaxation problems, where "OOM" means that we could not solve the problem because of insufficient memory.

It may be observed from Table 4.1 that the optimal values of the generalized doubly nonnegative programming relaxation problems were much better than those of the semidefinite programming relaxation problems and were close to those of the original problems. This implies that generalized doubly nonnegative programming relaxation for generalized completely positive programming provides much tighter bounds than semidefinite programming relaxation. In particular, the NN- and ZVP-type generalized doubly nonnegative programming relaxations provided nearly optimal values for the original problems. These results are consistent with those of Section 3.5.1, in which

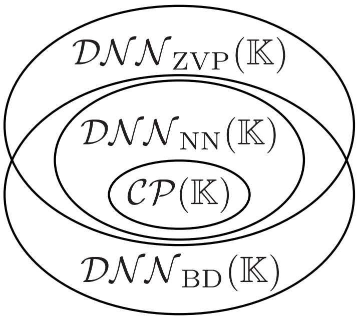

Figure 4.1: Inclusion relationship between ZVP-, NN-, and BD-type generalized doubly nonnegative cones (including Conjecture 4.24).

we approximated the generalized copositive cone with the dual cone of the NN- and ZVP-type generalized doubly nonnegative cones, i.e., $\mathcal{I}_{\mathrm{NN},0}(\mathbb{K})$ and $\mathcal{I}_{\mathrm{ZVP},0}(\mathbb{K})$.

ZVP-type generalized doubly nonnegative programming relaxation exhibited better numerical properties than BD-type generalized doubly nonnegative programming relaxation. As illustrated in Examples 4.16 and 4.17, the inclusion relationship between ZVP- and BD-type generalized doubly nonnegative cones does not hold. However, the optimal values of the ZVP-type generalized doubly nonnegative programming relaxation problems are better than those of the BD-type problems in most cases, especially when $n \geq 30$.

Given the theoretical inclusion relationship between NN- and ZVP-type generalized doubly nonnegative cones shown in Theorem 4.15 and the superior numerical performance of the ZVP-type generalized doubly nonnegative cone compared to the BD-type cone, it is reasonable to conjecture that the NN-type cone provides numerically tighter relaxation than the BD-type. It may be observed from Table 4.1 that the optimal values of the NN-type generalized doubly nonnegative programming relaxation problems were always not worse than those of the BD-type problems. Of note, the theoretical inclusion relationship between NN- and BD-type generalized doubly nonnegative cones remains unknown. We therefore presume that the NN-type generalized doubly nonnegative cone is included in the BD-type generalized doubly nonnegative cone.

Conjecture 4.24. *If $\mathbb{K}$ is a direct product of a nonnegative orthant and second-order cones, $\mathcal{DNN}_{\mathrm{NN}}(\mathbb{K}) \subseteq \mathcal{DNN}_{\mathrm{BD}}(\mathbb{K})$ holds.*

Note that this conjecture is true where $\mathbb{K}$ is a second-order cone by Proposition 4.18. Thus far, we have demonstrated the inclusion relationship between the three generalized doubly nonnegative cones in Theorem 4.15, Examples 4.16, 4.17, and Conjecture 4.24. These results are illustrated in Figure 4.1.

We have observed that the NN-type generalized doubly nonnegative cone provided the tightest relaxation of the three generalized doubly nonnegative cones from both theoretical and numerical perspectives. However, we could not compute the NN-type generalized doubly nonnegative programming relaxation problems in the case of $n \geq 10$ due to insufficient memory.[3] We suspect that one of the reasons is that the NN-type

[3] We tried to solve the NN-type generalized doubly nonnegative programming relaxation problems

generalized doubly nonnegative cone is described by the positive semidefiniteness of the matrix $\boldsymbol{M}(\boldsymbol{y})$. The NN-type generalized doubly nonnegative programming relaxation problem of problem (4.17) includes a positive semidefinite constraint of size $O(n^2)$, whereas the ZVP-type generalized doubly nonnegative programming relaxation problem includes a positive semidefinite constraint of only size $O(n)$.

4.4.2 Max-cut problem

Let $G = (V, E)$ be an undirected graph with nonnegative weights $(w_{ij})_{ij\in E}$, where $V \coloneqq \{1, \dots, n\}$ is a set of vertices, and $E \subseteq \{\{i, j\} \mid 1 \le i < j \le n\}$ is a set of edges. We write ij for $\{i, j\} \in E$. The max-cut problem, a well-known NP-hard problem [60], aims to find a subset $S \subseteq V$ that maximizes the sum of weights on edges connecting S and $V \setminus S$. For notational convenience, for $1 \le i \le j \le n$ with $ij \notin E$, we define $w_{ij} \coloneqq 0$. Then, the max-cut problem can be formulated as the following binary quadratic programming problem:

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{maximize}} \quad & \frac{1}{4}\sum_{i,j=1}^{n} w_{ij}(1 - x_i x_j) \\ \text{subject to} \quad & x_i^2 = 1 \ (i = 1, \dots, n). \end{aligned} \tag{4.18}$$

Introducing a matrix variable $\hat{\boldsymbol{X}} \coloneqq \boldsymbol{x}\boldsymbol{x}^\top$, up to a constant term and the sign of the optimal value, (4.18) is equivalent to a semidefinite programming problem with the rank constraint $\operatorname{rank}(\hat{\boldsymbol{X}}) \le 1$. Furthermore, by Corollary 2.17, we can equivalently transform the rank-constrained semidefinite programming problem into a generalized completely positive programming problem in the form of (4.17) with $m = n^2 + 3n + 4$ and $\mathbb{K} = \mathbb{R}_+ \times \operatorname{svec}(\mathbb{S}_+^n)^3$. As in Section 4.4.1, to solve the problem obtained, we relax the generalized completely positive programming problem with each generalized doubly nonnegative cone and compare the results.

Instances of problem (4.18) were created as follows. The number n of vertices was changed between 3 and 5.[4] For each $1 \le i < j \le n$, we independently generated an edge ij with probability $\frac{1}{2}$. All elements of $(w_{ij})_{ij\in E}$ were independent and identically distributed, and each followed the uniform distribution on the interval $(0, 1)$. For each n, we generated five instances by varying the randomness of E and $(w_{ij})_{ij\in E}$. For each instance, the optimal value of the relaxation problem with $\mathcal{DNN}_{\mathrm{NN}}(\mathbb{K})$ (written as NN in Table 4.2), $\mathcal{DNN}_{\mathrm{ZVP}}(\mathbb{K})$ (ZVP), and $\mathcal{DNN}_{\mathrm{BD}}(\mathbb{K})$ (BD) was measured. For reference, we also solved problem (4.18) (MC) and the associated relaxation problems with the positive semidefinite cone $\mathbb{S}_+^{3T_n+1}$ (SDP). Note that the semidefinite programming relaxation problem addressed here is distinct from the one utilized in presenting a 0.878-approximation algorithm demonstrated by Goemans and Williamson [41] for

on another computer with 64 GB of memory, but the same issue persisted.

[4]Comparing the three generalized doubly nonnegative cones for larger n generated numerical issues. For $n = 10$, the computational time for solving BD-type generalized doubly nonnegative programming relaxation problems exceeded 14400 s, and due to insufficient memory, we were unable to solve NN-type generalized doubly nonnegative programming relaxation problems.

Table 4.2: Optimal values of problem (4.18) and its associated generalized doubly nonnegative programming and semidefinite programming relaxation problems.

		Optimal value				
n	No.	MC	NN	ZVP	BD	SDP
3	1	0.81	0.81	0.81	0.81	13.58
	2	0.33	0.33	0.33	0.33	2.89
	3	1.00	1.00	1.00	1.00	15.14
	4	1.19	1.19	1.19	1.19	25.41
	5	0.87	0.87	0.87	0.87	19.38
5	1	2.10	OOM	2.10	2.10	36.55
	2	1.81	OOM	2.24	1.81	26.67
	3	3.04	OOM	3.32	3.04	45.36
	4	4.05	OOM	4.30	4.05	91.34
	5	1.94	OOM	2.13	1.94	35.26

the max-cut problem. To improve numerical stability, $0.005\boldsymbol{I}$ was added to the coefficient matrix $\boldsymbol{C}$ in the standard form (4.17) when solving semidefinite programming relaxation problems. As in Section 4.4.1, the BD-type generalized doubly nonnegative programming problem was solved by the algorithm shown in Appendix A. To solve NN-, ZVP-, and BD-type generalized doubly nonnegative programming and semidefinite programming relaxation problems, we used the YALMIP modeling language and the MOSEK solver. To solve (4.18), we used the Gurobi solver [49] (version 10.0.3).

Table 4.2 lists the optimal values of problem (4.18) and the accompanying generalized doubly nonnegative programming and semidefinite programming relaxation problems, where "OOM" means that we could not solve the problem because of insufficient memory.

Generalized doubly nonnegative programming relaxation for generalized completely positive programming provided significantly tighter bounds than semidefinite programming relaxation, a finding that aligns with the results presented in Section 4.4.1. However, unlike Section 4.4.1, the ZVP-type generalized doubly nonnegative cone provided worse relaxation than the BD-type cone. This might be because the semialgebraic representation (4.11) of $\mathrm{svec}(\mathbb{S}^n_+)$ involves polynomials of degree exceeding 2 when $n \geq 3$; whereas the semialgebraic representation (4.5) of a direct product of a nonnegative orthant and second-order cones only involves polynomials of degree at most 2. As mentioned in Section 4.2.1, polynomials of degree exceeding 2 appearing in a semialgebraic representation of $\mathbb{K}$ do not contribute to the construction of $\mathcal{DNN}_{\mathrm{ZVP}}(\mathbb{K})$. Therefore, we can assume that the more the polynomials of degree exceeding 2, the looser the ZVP-type generalized doubly nonnegative programming relaxation.

Part III

Facial Structure

Chapter 5

Length of the Longest Chain of Faces and Distance to Polyhedrality of the Completely Positive and Copositive Cones

5.1 Introduction

In this chapter, we focus on the length $\ell_{\mathcal{K}}$ of the longest chain of faces and the distance $\ell_{\text{poly}}(\mathcal{K})$ to polyhedrality for several matrix cones $\mathcal{K}$ related to the standard completely positive and copositive cones. We prove that for any closed convex cone $\mathcal{K}$ with $\mathcal{CP}^n \subseteq \mathcal{K} \subseteq \mathcal{DNN}^n$, $\ell_{\mathcal{K}}$ equals $T_n + 1$ and if $n \geq 2$, $\ell_{\text{poly}}(\mathcal{K})$ equals $T_n - 2$, which are the worst cases possible for these lengths (Corollary 5.3). The same holds for the dual cone, i.e., any closed convex cone $\mathcal{K}$ with $\mathcal{SPN}^n \subseteq \mathcal{K} \subseteq \mathcal{COP}^n$ (Corollary 5.8).

The cones $\mathcal{K}$ for which the above results hold are not limited to the completely positive, doubly nonnegative, SPN, and copositive cones: for example, the closure of the *completely positive semidefinite cone* [17, 66] and Parrilo's inner-approximation hierarchy (2.23). Moreover, the result of length $\ell_{\mathcal{K}}$ holds for a wider class of $\mathcal{K}$, specifically for any closed convex cone sandwiched between the nonnegative diagonally dominant cone $\mathcal{DD}_+^n$ and the nonnegative cone $\mathcal{N}^n$ (Proposition 5.2), or between the nonnegative cone $\mathcal{N}^n$ and the dual $(\mathcal{SDD}_+^n)^*$ of the nonnegative scaled diagonally dominant cone (Proposition 5.7).

We construct a chain of faces by specifying the pattern of zeros of the elements of the matrices belonging to each face to prove our main results. The idea is inspired by the work of Lourenço, Muramatsu, and Tsuchiya [73], but we use a different order in which the elements are restricted to zero. To compute the distance to polyhedrality, it is necessary to construct a chain of faces such that as few polyhedral cones as possible appear (see Remark 5.4).

The rest of this chapter consists of two sections. In Section 5.2, we present a result concerning the completely positive cone, while in Section 5.3, we demonstrate a result about the copositive cone.

Throughout this chapter, we use the following notation. Let $\mathcal{I}$ be a subset of $\{(i,j) \mid i,j = 1,\ldots,n\}$. Then, we define $\mathcal{K}[\mathcal{I}] \coloneqq \{\boldsymbol{A} \in \mathcal{K} \mid A_{ij} = 0 \text{ for all } (i,j) \in \mathcal{I}\}$. We also define $\boldsymbol{E}[\mathcal{I}] \in \mathbb{S}^n$ as the matrix with the (i,j)th and (j,i)th elements 0 for each $(i,j) \in \mathcal{I}$ and 1 for all other elements, which appears only in Section 5.3. Recall that the value $\frac{n(n+1)}{2}$ is denoted by T_n.

5.2 The Completely Positive Side

For $i = 1, \ldots, n$ and $j = i, \ldots, n$, let

$$\mathcal{I}_{i,j} := \bigcup_{k=1}^{i-1} \bigcup_{l=k}^{n} \{(k,l)\} \cup \bigcup_{l=j}^{n} \{(i,l)\}.$$

Here, $\mathcal{I}_{ij}$ and $\mathcal{I}_{i,j}$ are used interchangeably. For convenience, we let $\mathcal{I}_{00} := \emptyset$ and $\mathcal{I}_{i,n+1} := \mathcal{I}_{i-1,i-1}$ for $i = 1, \ldots, n$. Under this notation, for $i = 1, \ldots, n$ and $j = i, \ldots, n$, we have

$$\mathcal{I}_{ij} = \mathcal{I}_{i,j+1} \cup \{(i,j)\}. \tag{5.1}$$

In addition, we define $\boldsymbol{E}_{ij} := (\boldsymbol{e}_i + \boldsymbol{e}_j)(\boldsymbol{e}_i + \boldsymbol{e}_j)^\top$ for $i, j = 1, \ldots, n$.

Lemma 5.1. *Let $\mathcal{K} \subseteq \mathbb{S}^n$ be a closed convex cone with $\mathcal{K} \subseteq \mathcal{N}^n$. For $i = 1, \ldots, n$ and $j = i, \ldots, n$, the set $\mathcal{K}[\mathcal{I}_{ij}]$ is a face of $\mathcal{K}$.*

Proof. It can be observed that $\mathcal{K}[\mathcal{I}_{ij}]$ is a convex subcone in $\mathcal{K}$. Let $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{K}$. Note that all the elements of $\boldsymbol{A}$ and $\boldsymbol{B}$ are nonnegative because $\mathcal{K} \subseteq \mathcal{N}^n$. Now, we assume that $\boldsymbol{A} + \boldsymbol{B} \in \mathcal{K}[\mathcal{I}_{ij}]$. Then, for each $(k,l) \in \mathcal{I}_{ij}$, we have $A_{kl} + B_{kl} = 0$. By the nonnegativity of $\boldsymbol{A}$ and $\boldsymbol{B}$, A_{kl} and B_{kl} must be zero. Therefore, we obtain $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{K}[\mathcal{I}_{ij}]$. ☐

Proposition 5.2. *For any closed convex cone $\mathcal{K} \subseteq \mathbb{S}^n$ with $\mathcal{DD}_+^n \subseteq \mathcal{K} \subseteq \mathcal{N}^n$, the following inclusions hold:*

$$\begin{aligned} &\underbrace{\mathcal{K}[\mathcal{I}_{nn}]}_{1 \text{ face}} \subsetneq \underbrace{\mathcal{K}[\mathcal{I}_{n-1,n-1}] \subsetneq \mathcal{K}[\mathcal{I}_{n-1,n}]}_{2 \text{ faces}} \subsetneq \mathcal{K}[\mathcal{I}_{n-2,n-2}] \subsetneq \cdots \\ &\qquad \subsetneq \mathcal{K}[\mathcal{I}_{2n}] \subsetneq \underbrace{\mathcal{K}[\mathcal{I}_{11}] \subsetneq \cdots \subsetneq \mathcal{K}[\mathcal{I}_{1,n-1}] \subsetneq \mathcal{K}[\mathcal{I}_{1n}]}_{n \text{ faces}} \subsetneq \mathcal{K}[\mathcal{I}_{00}] = \mathcal{K}. \end{aligned} \tag{5.2}$$

In particular, we have $\ell_{\mathcal{K}} = T_n + 1$.

Figure 5.1 illustrates the chain (5.2) with $n = 3$.

Proof. For each $i = 1, \ldots, n$ and $j = i, \ldots, n$, (5.1) implies that $\mathcal{K}[\mathcal{I}_{ij}] \subseteq \mathcal{K}[\mathcal{I}_{i,j+1}]$. In what follows, we prove $\boldsymbol{E}_{ij} \in \mathcal{K}[\mathcal{I}_{i,j+1}] \setminus \mathcal{K}[\mathcal{I}_{ij}]$ to show the strictness of this inclusion. Refer to Figure 5.2 for a better understanding of the following proof. We have $\boldsymbol{E}_{ij} \in \mathcal{DD}_+^n \subseteq \mathcal{K}$ and only the (i,i)th, (j,j)th, (i,j)th, and (j,i)th elements of $\boldsymbol{E}_{ij}$ are nonzero. Therefore, for $\mathcal{I} \in \{\mathcal{I}_{i,j+1}, \mathcal{I}_{ij}\}$, the matrix $\boldsymbol{E}_{ij}$ belongs to $\mathcal{K}[\mathcal{I}]$ if and only if all the tuples (i,i), (j,j), and (i,j) do not belong to $\mathcal{I}$. On the one hand, by definition, $\boldsymbol{E}_{ij} \in \mathcal{K}[\mathcal{I}_{i,j+1}]$. On the other hand, as $(i,j) \in \mathcal{I}_{ij}$, we have $\boldsymbol{E}_{ij} \notin \mathcal{K}[\mathcal{I}_{ij}]$.

By $\mathcal{K} \subseteq \mathcal{N}^n$ and Lemma 5.1, each set in the sequence (5.2) is a face of $\mathcal{K}$. Because this chain comprises $(T_n + 1)$ faces of $\mathcal{K}$, we have $\ell_{\mathcal{K}} \geq T_n + 1$. Combining this with Lemma 2.3 yields $\ell_{\mathcal{K}} = T_n + 1$. ☐

$$
\underset{\mathcal{K}[\mathcal{I}_{00}]}{\begin{array}{|c|c|c|}\hline * & * & * \\ \hline * & * & * \\ \hline * & * & * \\ \hline \end{array}}
\to
\underset{\mathcal{K}[\mathcal{I}_{13}]}{\begin{array}{|c|c|c|}\hline * & * & 0 \\ \hline * & * & * \\ \hline 0 & * & * \\ \hline \end{array}}
\to
\underset{\mathcal{K}[\mathcal{I}_{12}]}{\begin{array}{|c|c|c|}\hline * & 0 & 0 \\ \hline 0 & * & * \\ \hline 0 & * & * \\ \hline \end{array}}
\to
\underset{\mathcal{K}[\mathcal{I}_{11}]}{\begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 0 & * & * \\ \hline 0 & * & * \\ \hline \end{array}}
$$

$$
\to
\underset{\mathcal{K}[\mathcal{I}_{23}]}{\begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 0 & * & 0 \\ \hline 0 & 0 & * \\ \hline \end{array}}
\to
\underset{\mathcal{K}[\mathcal{I}_{22}]}{\begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & * \\ \hline \end{array}}
\to
\underset{\mathcal{K}[\mathcal{I}_{33}]}{\begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array}}
$$

Figure 5.1: The chain (5.2) of faces of $\mathcal{K}$. For each matrix belonging to each face $\mathcal{K}[\mathcal{I}_{ij}]$, the elements corresponding to the symbol 0 must be zero, and those corresponding to the symbol $*$ can be nonzero.

$$
\underset{\mathcal{K}[\mathcal{I}_{i,j+1}]}{\begin{array}{c|c|c|c|c|c|c|}
 & 1 & & i & & j & j+1 \\ \hline
1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
i & 0 & 0 & * & * & * & 0 \\ \hline
 & 0 & 0 & * & * & * & * \\ \hline
j & 0 & 0 & * & * & * & * \\ \hline
j+1 & 0 & 0 & 0 & * & * & * \\ \hline
\end{array}}
\qquad
\underset{\mathcal{K}[\mathcal{I}_{ij}]}{\begin{array}{c|c|c|c|c|c|c|}
 & 1 & & i & & j & j+1 \\ \hline
1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
i & 0 & 0 & * & * & 0 & 0 \\ \hline
 & 0 & 0 & * & * & * & * \\ \hline
j & 0 & 0 & 0 & * & * & * \\ \hline
j+1 & 0 & 0 & 0 & * & * & * \\ \hline
\end{array}}
$$

Figure 5.2: Illustration of $\boldsymbol{E}_{ij} \in \mathcal{K}[\mathcal{I}_{i,j+1}] \setminus \mathcal{K}[\mathcal{I}_{ij}]$ in the proof of Proposition 5.2. For each matrix belonging to a face $\mathcal{K}[\mathcal{I}_{i,j+1}]$ or $\mathcal{K}[\mathcal{I}_{ij}]$, the elements corresponding to the symbol 0 must be zero, and those corresponding to the symbol $*$ can be nonzero. In addition, the grids where nonzero elements of the matrix $\boldsymbol{E}_{ij}$ exist are presented in gray. On the one hand, because all the "0" grids in $\mathcal{K}[\mathcal{I}_{i,j+1}]$ are not gray, we have $\boldsymbol{E}_{ij} \in \mathcal{K}[\mathcal{I}_{i,j+1}]$. On the other hand, as some of the "0" grids in $\mathcal{K}[\mathcal{I}_{ij}]$ are gray, we have $\boldsymbol{E}_{ij} \notin \mathcal{K}[\mathcal{I}_{ij}]$.

Corollary 5.3. *For any closed convex cone $\mathcal{K} \subseteq \mathbb{S}^n$ with $\mathcal{CP}^n \subseteq \mathcal{K} \subseteq \mathcal{DNN}^n$, we have $\ell_{\mathcal{K}} = T_n + 1$ and*

$$
\ell_{\text{poly}}(\mathcal{K}) = \begin{cases} 0 & (n = 1), \\ T_n - 2 & (n \geq 2). \end{cases} \tag{5.3}
$$

Proof. It follows from (2.14) and Proposition 5.2 that $\ell_{\mathcal{K}} = T_n + 1$ with the longest chain (5.2). In what follows, we prove (5.3). Because the case of $n = 1$ follows from (i) of Lemma 2.4, we only consider the case where $n \geq 2$. The inclusion $\mathcal{CP}^n \subseteq \mathcal{K} \subseteq \mathcal{DNN}^n$

implies that

$$\mathcal{CP}^n[\mathcal{I}_{n-2,n-2}] \subseteq \mathcal{K}[\mathcal{I}_{n-2,n-2}] \subseteq \mathcal{DNN}^n[\mathcal{I}_{n-2,n-2}]. \tag{5.4}$$

From Lemma 2.13, both of the first and third sets of (5.4) are equal to

$$\left\{ \begin{pmatrix} \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{A} \end{pmatrix} \in \mathbb{S}^n \;\middle|\; \boldsymbol{A} \in \mathcal{CP}^2 \right\}. \tag{5.5}$$

Therefore, the face $\mathcal{K}[\mathcal{I}_{n-2,n-2}]$ equals (5.5), which is not polyhedral since $\mathcal{CP}^2$ has infinitely many extreme rays [111, Remark 3.26]. Given the fact that each face of a polyhedral cone is also polyhedral [106, page 172], all the $(T_n - 2)$ faces $\mathcal{K}[\mathcal{I}_{n-2,n-2}] \subsetneq \cdots \subsetneq \mathcal{K}$ in the chain (5.2) are not polyhedral. Thus, we have $\ell_{\text{poly}}(\mathcal{K}) \geq T_n - 2$. Combining this with (ii) of Lemma 2.4 yields the desired result. $\square$

Remark 5.4. Lourenço, Muramatsu, and Tsuchiya [73] have provided a chain of faces of $\mathcal{DNN}^n$ with length $T_n + 1$. The chain is constructed by restricting first the $\frac{n(n-1)}{2}$ nondiagonal elements and then the n diagonal elements to zero. However, because of the existence of a polyhedral face of dimension n, the chain does not lead to $\ell_{\text{poly}}(\mathcal{DNN}^n) = T_n - 2$ when $n \geq 3$.

5.3 The Copositive Side

For $i = 1, \ldots, n$ and $j = i, \ldots, n$, let

$$\mathcal{J}_{i,j} := \bigcup_{k=1}^{i-1} \bigcup_{l=k}^{n} \{(k,l)\} \cup \bigcup_{l=i}^{j} \{(i,l)\}.$$

Here, $\mathcal{J}_{ij}$ and $\mathcal{J}_{i,j}$ are used interchangeably. For convenience, we let $\mathcal{J}_{0n} := \emptyset$ and $\mathcal{J}_{i,i-1} := \mathcal{J}_{i-1,n}$ for $i = 1, \ldots, n$. Under this notation, for $i = 1, \ldots, n$ and $j = i, \ldots, n$, we have

$$\mathcal{J}_{ij} = \mathcal{J}_{i,j-1} \cup \{(i,j)\}. \tag{5.6}$$

Lemma 5.5. *Let $\mathcal{K} \subseteq \mathbb{S}^n$ be a closed convex cone with $\mathcal{K} \subseteq (\mathcal{SDD}_+^n)^*$. For $i = 1, \ldots, n$ and $j = i, \ldots, n$, the set $\mathcal{K}[\mathcal{J}_{ij}]$ is a face of $\mathcal{K}$.*

Proof. It can be observed that $\mathcal{K}[\mathcal{J}_{ij}]$ is a convex subcone in $\mathcal{K}$. Let $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{K}$. Note that $\boldsymbol{A}$ and $\boldsymbol{B}$ also belong to $(\mathcal{SDD}_+^n)^*$. Now, we assume that $\boldsymbol{A} + \boldsymbol{B} \in \mathcal{K}[\mathcal{J}_{ij}]$. Then, for each $k = 1, \ldots, i$, as $(k,k) \in \mathcal{J}_{ij}$, we observe that $A_{kk} + B_{kk} = 0$. Because A_{kk} and B_{kk} are nonnegative (see (i) of Lemma 2.12), they must be zero. This implies that for each $(k,l) \in \mathcal{J}_{ij}$, A_{kl} and B_{kl} are nonnegative (see (ii) of Lemma 2.12). Combining this with $A_{kl} + B_{kl} = 0$, we observe that A_{kl} and B_{kl} are also zero. Thus, we obtain $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{K}[\mathcal{J}_{ij}]$. $\square$

Remark 5.6. Lemma 5.5 does not hold for $\mathcal{K} = (\mathcal{DD}_+^n)^*$. For example, let

$$\boldsymbol{A} := \begin{pmatrix} 0 & 1 & \\ 1 & 2 & \\ & & \boldsymbol{O} \end{pmatrix}, \; \boldsymbol{B} := \begin{pmatrix} 0 & -1 & \\ -1 & 2 & \\ & & \boldsymbol{O} \end{pmatrix}.$$

$$
\underset{\mathcal{K}[\mathcal{J}_{03}]}{\begin{array}{|c|c|c|}\hline * & * & * \\ \hline * & * & * \\ \hline * & * & * \\ \hline \end{array}} \to
\underset{\mathcal{K}[\mathcal{J}_{11}]}{\begin{array}{|c|c|c|}\hline 0 & * & * \\ \hline * & * & * \\ \hline * & * & * \\ \hline \end{array}} \to
\underset{\mathcal{K}[\mathcal{J}_{12}]}{\begin{array}{|c|c|c|}\hline 0 & 0 & * \\ \hline 0 & * & * \\ \hline * & * & * \\ \hline \end{array}} \to
\underset{\mathcal{K}[\mathcal{J}_{13}]}{\begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 0 & * & * \\ \hline 0 & * & * \\ \hline \end{array}}
$$

$$
\to \underset{\mathcal{K}[\mathcal{J}_{22}]}{\begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 0 & 0 & * \\ \hline 0 & * & * \\ \hline \end{array}} \to
\underset{\mathcal{K}[\mathcal{J}_{23}]}{\begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & * \\ \hline \end{array}} \to
\underset{\mathcal{K}[\mathcal{J}_{33}]}{\begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array}}
$$

Figure 5.3: The chain (5.7) of faces of $\mathcal{K}$. For each matrix belonging to each face $\mathcal{K}[\mathcal{J}_{ij}]$, the elements corresponding to the symbol 0 must be zero, and those corresponding to the symbol $*$ can be nonzero.

Then, it can be observed from (2.12) that $\boldsymbol{A}, \boldsymbol{B} \in (\mathcal{DD}_+^n)^*$. On the one hand, as

$$
\boldsymbol{A} + \boldsymbol{B} = \begin{pmatrix} 0 & 0 & \\ 0 & 4 & \\ & & \boldsymbol{O} \end{pmatrix} \in (\mathcal{DD}_+^n)^*
$$

and $A_{11} + B_{11} = A_{12} + B_{12} = 0$, it follows that $\boldsymbol{A} + \boldsymbol{B} \in (\mathcal{DD}_+^n)^*[\mathcal{J}_{12}]$. On the other hand, $\boldsymbol{A}, \boldsymbol{B} \notin (\mathcal{DD}_+^n)^*[\mathcal{J}_{12}]$ since $A_{12}, B_{12} \neq 0$. Thus, $(\mathcal{DD}_+^n)^*[\mathcal{J}_{12}]$ is not a face of $(\mathcal{DD}_+^n)^*$.

Proposition 5.7. *For any closed convex cone $\mathcal{K} \subseteq \mathbb{S}^n$ with $\mathcal{N}^n \subseteq \mathcal{K} \subseteq (\mathcal{SDD}_+^n)^*$, the following inclusions hold:*

$$
\begin{aligned}
&\underbrace{\mathcal{K}[\mathcal{J}_{nn}]}_{1\ \textit{face}} \subsetneq \underbrace{\mathcal{K}[\mathcal{J}_{n-1,n}] \subsetneq \mathcal{K}[\mathcal{J}_{n-1,n-1}]}_{2\ \textit{faces}} \subsetneq \mathcal{K}[\mathcal{J}_{n-2,n}] \subsetneq \cdots \\
&\qquad \subsetneq \mathcal{K}[\mathcal{J}_{22}] \subsetneq \underbrace{\mathcal{K}[\mathcal{J}_{1n}] \subsetneq \cdots \subsetneq \mathcal{K}[\mathcal{J}_{12}] \subsetneq \mathcal{K}[\mathcal{J}_{11}]}_{n\ \textit{faces}} \subsetneq \mathcal{K}[\mathcal{J}_{0n}] = \mathcal{K}.
\end{aligned} \tag{5.7}
$$

In particular, we have $\ell_{\mathcal{K}} = T_n + 1$.

Figure 5.3 illustrates the chain (5.7) with $n = 3$.

Proof. For each $i = 1, \dots, n$ and $j = i, \dots, n$, (5.6) implies that $\mathcal{K}[\mathcal{J}_{ij}] \subseteq \mathcal{K}[\mathcal{J}_{i,j-1}]$. In what follows, we prove $\boldsymbol{E}[\mathcal{J}_{i,j-1}] \in \mathcal{K}[\mathcal{J}_{i,j-1}] \setminus \mathcal{K}[\mathcal{J}_{ij}]$ to show the strictness of this inclusion. Refer to Figure 5.4 for a better understanding of the following proof. Because $\boldsymbol{E}[\mathcal{J}_{i,j-1}] \in \mathcal{N}^n \subseteq \mathcal{K}$, it suffices to check whether or not the (k,l)th element of the matrix $\boldsymbol{E}[\mathcal{J}_{i,j-1}]$ is zero for each (k,l) belonging to $\mathcal{J} \in \{\mathcal{J}_{i,j-1}, \mathcal{J}_{ij}\}$. On the one hand, as the (k,l)th element of $\boldsymbol{E}[\mathcal{J}_{i,j-1}]$ is zero for all $(k,l) \in \mathcal{J}_{i,j-1}$, we have $\boldsymbol{E}[\mathcal{J}_{i,j-1}] \in \mathcal{K}[\mathcal{J}_{i,j-1}]$. On the other hand, as $(i,j) \in \mathcal{J}_{ij}$ and the (i,j)th element of $\boldsymbol{E}[\mathcal{J}_{i,j-1}]$ is one, we have $\boldsymbol{E}[\mathcal{J}_{i,j-1}] \notin \mathcal{K}[\mathcal{J}_{ij}]$.

By $\mathcal{K} \subseteq (\mathcal{SDD}_+^n)^*$ and Lemma 5.5, each set in the sequence (5.7) is a face of $\mathcal{K}$. Because this chain is composed of $(T_n + 1)$ faces of $\mathcal{K}$, we have $\ell_{\mathcal{K}} = T_n + 1$. $\square$

$$
\begin{array}{c|cccccc|}
 & 1 & & i & j-1 & j & \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 & 0 \\
i & 0 & 0 & 0 & 0 & * & * \\
j-1 & 0 & 0 & 0 & * & * & * \\
j & 0 & 0 & * & * & * & * \\
 & 0 & 0 & * & * & * & * \\
\hline
\end{array}
\qquad
\begin{array}{c|cccccc|}
 & 1 & & i & j-1 & j & \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 & 0 \\
i & 0 & 0 & 0 & 0 & 0 & * \\
j-1 & 0 & 0 & 0 & * & * & * \\
j & 0 & 0 & 0 & * & * & * \\
 & 0 & 0 & * & * & * & * \\
\hline
\end{array}
$$

$\mathcal{K}[\mathcal{J}_{i,j-1}]$ $\qquad$ $\mathcal{K}[\mathcal{J}_{ij}]$

Figure 5.4: Illustration of $\boldsymbol{E}[\mathcal{J}_{i,j-1}] \in \mathcal{K}[\mathcal{J}_{i,j-1}] \setminus \mathcal{K}[\mathcal{J}_{ij}]$ in the proof of Proposition 5.7. For each matrix belonging to a face $\mathcal{K}[\mathcal{J}_{i,j-1}]$ or $\mathcal{K}[\mathcal{J}_{ij}]$, the elements corresponding to the symbol 0 must be zero, and those corresponding to the symbol $*$ can be nonzero. In addition, the grids where nonzero elements of the matrix $\boldsymbol{E}[\mathcal{J}_{i,j-1}]$ exist are presented in gray. On the one hand, as all the "0" grids in $\mathcal{K}[\mathcal{J}_{i,j-1}]$ are not gray, we have $\boldsymbol{E}[\mathcal{J}_{i,j-1}] \in \mathcal{K}[\mathcal{J}_{i,j-1}]$. On the other hand, as some of the "0" grids in $\mathcal{K}[\mathcal{J}_{ij}]$ are gray, we have $\boldsymbol{E}[\mathcal{J}_{i,j-1}] \notin \mathcal{K}[\mathcal{J}_{ij}]$.

Corollary 5.8. *For any closed convex cone $\mathcal{K} \subseteq \mathbb{S}^n$ with $\mathcal{SPN}^n \subseteq \mathcal{K} \subseteq \mathcal{COP}^n$, we have $\ell_{\mathcal{K}} = T_n + 1$ and*

$$
\ell_{\text{poly}}(\mathcal{K}) = \begin{cases} 0 & (n = 1), \\ T_n - 2 & (n \geq 2). \end{cases} \tag{5.8}
$$

Proof. It follows from (2.15) and Proposition 5.7 that $\ell_{\mathcal{K}} = T_n + 1$ with the longest chain (5.7). In what follows, we prove (5.8). Since the case $n = 1$ follows from (i) of Lemma 2.4, we only consider the case of $n \geq 2$. The inclusion $\mathcal{SPN}^n \subseteq \mathcal{K} \subseteq \mathcal{COP}^n$ implies that

$$
\mathcal{SPN}^n[\mathcal{J}_{n-2,n}] \subseteq \mathcal{K}[\mathcal{J}_{n-2,n}] \subseteq \mathcal{COP}^n[\mathcal{J}_{n-2,n}]. \tag{5.9}
$$

From Lemma 2.13, both the first and third sets of (5.9) are equal to

$$
\left\{ \begin{pmatrix} \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{A} \end{pmatrix} \in \mathbb{S}^n \;\middle|\; \boldsymbol{A} \in \mathcal{COP}^2 \right\}. \tag{5.10}
$$

Therefore, the face $\mathcal{K}[\mathcal{J}_{n-2,n}]$ equals (5.10), which is not polyhedral as $\mathcal{COP}^2$ has infinitely many extreme rays [111, Theorem 2.29]. Therefore, we obtain $\ell_{\text{poly}}(\mathcal{K}) = T_n - 2$. $\square$

Remark 5.9. The order in which the elements of a matrix are restricted to zero is essential for obtaining the longest chain of faces. Let us consider what would happen if we used $\mathcal{I}_{ij}$, defined in Section 5.2, instead of $\mathcal{J}_{ij}$ in the discussion of this section. Then, for a closed convex cone $\mathcal{K}$ with $\mathcal{SPN}^n \subseteq \mathcal{K} \subseteq \mathcal{COP}^n$, the set $\mathcal{K}[\mathcal{I}_{ij}]$ is not necessarily a face of $\mathcal{K}$. For example, consider the set $\mathcal{K}[\mathcal{I}_{1n}]$. Let

$$\boldsymbol{A} \coloneqq \begin{pmatrix} 1 & & -1 \\ & \boldsymbol{O} & \\ -1 & & 1 \end{pmatrix}, \ \boldsymbol{B} \coloneqq \begin{pmatrix} 1 & & 1 \\ & \boldsymbol{O} & \\ 1 & & 1 \end{pmatrix} \in \mathbb{S}_+^n \subseteq \mathcal{K}.$$

On the one hand, since

$$\boldsymbol{A} + \boldsymbol{B} = \begin{pmatrix} 2 & & 0 \\ & \boldsymbol{O} & \\ 0 & & 2 \end{pmatrix} \in \mathcal{K}$$

and $A_{1n} + B_{1n} = 0$, it follows that $\boldsymbol{A} + \boldsymbol{B} \in \mathcal{K}[\mathcal{I}_{1n}]$. On the other hand, $\boldsymbol{A}, \boldsymbol{B} \notin \mathcal{K}[\mathcal{I}_{1n}]$ since $A_{1n}, B_{1n} \neq 0$. Thus, $\mathcal{K}[\mathcal{I}_{1n}]$ is not a face of $\mathcal{K}$.

Conversely, for a closed convex cone $\mathcal{K}$ with $\mathcal{CP}^n \subseteq \mathcal{K} \subseteq \mathcal{DNN}^n$ and the set $\mathcal{J}_{ij}$, $\mathcal{K}[\mathcal{J}_{ij}]$ is indeed a face of $\mathcal{K}$. However, for each $i = 1, \dots, n$, all the inclusions in $\mathcal{K}[\mathcal{J}_{in}] \subseteq \cdots \subseteq \mathcal{K}[\mathcal{J}_{ii}]$ hold with equality. This is because for any $\boldsymbol{A} \in \mathcal{DNN}^n$, $A_{ii} = 0$ implies that $A_{ij} = 0$ for all $j = 1, \dots, n$. Therefore, if we use $\mathcal{J}_{ij}$ instead of $\mathcal{I}_{ij}$ in the discussion of Section 5.2, we cannot obtain a chain of faces of $\mathcal{K}$ with length $T_n + 1$.

Chapter 6

Non-facial Exposedness of Copositive Cones over Symmetric Cones

6.1 Introduction

In this chapter, we investigate the facial structure, and in particular, the facial exposedness of copositive cones over symmetric cones. Specifically, under the assumption that the symmetric cone $\mathbb{K}$ has a dimension greater than or equal to 2, for any c generating an extreme ray of $\mathbb{K}$, the rank-1 tensor $c^{\otimes 2}$ generates a non-exposed extreme ray of the copositive cone over $\mathbb{K}$. In particular, the copositive cone over such a symmetric cone is never facially exposed. Our result extends the known fact that the standard copositive cone $\mathcal{COP}^n$ of order $n \geq 2$ has non-exposed extreme rays generated by the matrix $\boldsymbol{e}_i\boldsymbol{e}_i^\top$ for each $i = 1, \ldots, n$ [24, Theorem 4.4]. We remark, however, that our proof will not be a straightforward extension of its proof.

The rest of this chapter is organized as follows. In Section 6.2, we supplement the notation and some basic properties of linear mappings. In Section 6.3, we provide a non-exposed extreme ray of copositive cones over symmetric cones.

6.2 Preliminaries on Linear Mappings

For a linear mapping f from a finite-dimensional real inner product space to another one, we denote by f^* its adjoint. For two functions f and g, we write $f \circ g$ for the composition of f and g. Note that the product of Jordan algebras is also denoted by the symbol $\circ$. Throughout this section, let $(\mathbb{V}, \bullet)$ be a finite-dimensional real inner product space with an orthogonal direct sum decomposition

$$\mathbb{V} = \bigoplus_{l=1}^{k} \mathbb{V}_l. \tag{6.1}$$

For each $i = 1, \ldots, k$, let $\mathcal{P}_{\mathbb{V}_i}\colon \mathbb{V} \to \mathbb{V}_i$ be the orthogonal projection onto $\mathbb{V}_i$, i.e.,:

$$\mathcal{P}_{\mathbb{V}_i}\colon \bigoplus_{l=1}^{k} \mathbb{V}_l \to \mathbb{V}_i; \sum_{l=1}^{k} x_l \mapsto x_i.$$

The adjoint $\mathcal{P}^*_{\mathbb{V}_l}\colon \mathbb{V}_l \to \mathbb{V}$ is the inclusion mapping. For $\mathcal{A} \in \mathcal{S}(\mathbb{V})$ and for each $i, j = 1, \dots, k$, we define $\mathcal{A}_{i,j} := \mathcal{P}_{\mathbb{V}_i} \circ \mathcal{A}|_{\mathbb{V}_j}$, which is a linear mapping from $\mathbb{V}_j$ to $\mathbb{V}_i$.

The following two lemmas justify the notation $\mathcal{A}_{i,j}$ for a self-adjoint linear transformation $\mathcal{A}$.

Lemma 6.1. *Let $\mathcal{A} \in \mathcal{S}(\mathbb{V})$. For each $i, j = 1, \dots, k$, $\mathcal{A}_{i,j}$ agrees with the adjoint $(\mathcal{A}_{j,i})^*$.*

Proof. Let $x_i \in \mathbb{V}_i$ and $x_j \in \mathbb{V}_j$ be arbitrary. To prove this lemma, it is sufficient to show that both $x_i \bullet \mathcal{A}_{i,j}(x_j)$ and $x_i \bullet (\mathcal{A}_{j,i})^*(x_j)$ agree with $x_i \bullet \mathcal{A}(x_j)$.

First, we have

$$x_i \bullet \mathcal{A}_{i,j}(x_j) = x_i \bullet \mathcal{P}_{\mathbb{V}_i}(\mathcal{A}(x_j)) = \mathcal{P}^*_{\mathbb{V}_i}(x_i) \bullet \mathcal{A}(x_j) = x_i \bullet \mathcal{A}(x_j). \tag{6.2}$$

Second, we have

$$x_i \bullet (\mathcal{A}_{j,i})^*(x_j) = x_j \bullet \mathcal{A}_{j,i}(x_i) = x_j \bullet \mathcal{A}(x_i) = x_i \bullet \mathcal{A}(x_j),$$

where we use the symmetry of the inner product and the definition of the adjoint $(\mathcal{A}_{j,i})^*$ to derive the first equality, the second equality follows for the same reason as in (6.2), and the third equality holds because of the self-adjointness of $\mathcal{A}$ and the symmetry of the inner product. This completes the proof. $\square$

Lemma 6.2. *Let $\mathcal{A} \in \mathcal{S}(\mathbb{V})$. For each $x \in \mathbb{V}$, we decompose x into $\sum_{l=1}^k x_l$ corresponding to the decomposition (6.1). Then, it follows that*

$$x \bullet \mathcal{A}(x) = \sum_{i=1}^k x_i \bullet \mathcal{A}_{i,i}(x_i) + 2 \sum_{1 \le i < j \le k} x_i \bullet \mathcal{A}_{i,j}(x_j).$$

Proof. Since $\mathcal{A}(x) = \sum_{j=1}^k \mathcal{A}|_{\mathbb{V}_j}(x_j)$, we have

$$\begin{aligned}
x \bullet \mathcal{A}(x) &= \sum_{i,j=1}^k x_i \bullet \mathcal{A}|_{\mathbb{V}_j}(x_j) \\
&= \sum_{i,j=1}^k \mathcal{P}^*_{\mathbb{V}_i}(x_i) \bullet \mathcal{A}|_{\mathbb{V}_j}(x_j) \\
&= \sum_{i,j=1}^k x_i \bullet (\mathcal{P}_{\mathbb{V}_i} \circ \mathcal{A}|_{\mathbb{V}_j})(x_j) \\
&= \sum_{i,j=1}^k x_i \bullet \mathcal{A}_{i,j}(x_j) \\
&= \sum_{i=1}^k x_i \bullet \mathcal{A}_{i,i}(x_i) + \sum_{1 \le i < j \le k} x_i \bullet \mathcal{A}_{i,j}(x_j) + \sum_{1 \le j < i \le k} x_i \bullet \mathcal{A}_{i,j}(x_j).
\end{aligned} \tag{6.3}$$

The third term in (6.3) is equal to

$$\sum_{1\leq j<i\leq k} x_j \bullet (\mathcal{A}_{i,j})^*(x_i) = \sum_{1\leq j<i\leq k} x_j \bullet \mathcal{A}_{j,i}(x_i) = \sum_{1\leq i<j\leq k} x_i \bullet \mathcal{A}_{i,j}(x_j), \tag{6.4}$$

where the first equality follows from Lemma 6.1. Since (6.4) agrees with the second term in (6.3), we obtain the desired result. $\square$

Lemma 6.2 implies that for each $\mathcal{A} \in \mathcal{S}(\mathbb{V})$ and for each $i, j = 1, \ldots, k$, $\mathcal{A}_{i,j}$ can be regarded as the "(i,j)th element" of $\mathcal{A}$ when an orthogonal direct sum decomposition (6.1) is fixed. Therefore, we may write a self-adjoint linear transformation $\mathcal{A} \in \mathcal{S}(\mathbb{V})$ in the following matrix-like form:

$$\begin{pmatrix} \mathcal{A}_{1,1} & \mathcal{A}_{1,2} & \cdots & \mathcal{A}_{1,k} \\ & \mathcal{A}_{2,2} & \cdots & \vdots \\ & & \ddots & \vdots \\ & & & \mathcal{A}_{k,k} \end{pmatrix},$$

where the strictly lower triangular portion of $\mathcal{A}$ can be omitted because of its self-adjointness. For a subset $\mathcal{S}$ in $\mathcal{S}(\mathbb{V}_k)$, we define

$$\{0\} \oplus \mathcal{S} \coloneqq \{\mathcal{P}_{\mathbb{V}_k}^* \circ \mathcal{A} \circ \mathcal{P}_{\mathbb{V}_k} \mid \mathcal{A} \in \mathcal{S}\} \subseteq \mathcal{S}(\mathbb{V}). \tag{6.5}$$

Using the matrix-like notation, we can write the set (6.5) as

$$\left\{ \begin{pmatrix} 0 & \cdots & 0 & 0 \\ & \ddots & \vdots & \vdots \\ & & 0 & 0 \\ & & & \mathcal{A} \end{pmatrix} \middle| \, \mathcal{A} \in \mathcal{S} \right\}.$$

6.3 Main Result

Lemma 6.3. *Let $(\mathbb{E}, \circ, \bullet)$ be a Euclidean Jordan algebra. For an idempotent $c \in \mathbb{E}$, consider the Peirce decomposition (2.5) with respect to c. Then, $\{0\} \oplus \mathcal{COP}(\mathbb{E}(c,1)_+)$ is a face of $\mathcal{COP}(\mathbb{E}_+)$.*

Proof. Since $\mathcal{COP}(\mathbb{E}(c,1)_+)$ is a convex cone, so is $\{0\} \oplus \mathcal{COP}(\mathbb{E}(c,1)_+)$. To prove the inclusion, let $\mathcal{A} \in \{0\} \oplus \mathcal{COP}(\mathbb{E}(c,1)_+)$. Then, there exists $\mathcal{G} \in \mathcal{COP}(\mathbb{E}(c,1)_+)$ such that $\mathcal{A} = \mathcal{P}_{\mathbb{E}(c,1)}^* \circ \mathcal{G} \circ \mathcal{P}_{\mathbb{E}(c,1)}$. For any $x \in \mathbb{E}_+$, we decompose it into $x = x_0 + x_1$, where $x_0 \in \mathbb{E}(c,1)^\perp$ and $x_1 \in \mathbb{E}(c,1)$. It follows from [72, Proposition 32.i] that $x_1 \in \mathbb{E}(c,1)_+$. Combining it with $\mathcal{G} \in \mathcal{COP}(\mathbb{E}(c,1)_+)$ yields

$$\begin{aligned} x \bullet \mathcal{A}(x) &= x \bullet (\mathcal{P}_{\mathbb{E}(c,1)}^* \circ \mathcal{G} \circ \mathcal{P}_{\mathbb{E}(c,1)})(x) \\ &= x_1 \bullet \mathcal{G}(x_1) \\ &\geq 0, \end{aligned}$$

which implies that $\mathcal{A} \in \mathcal{COP}(\mathbb{E}_+)$. Therefore, $\{0\} \oplus \mathcal{COP}(\mathbb{E}(c,1)_+)$ is a convex subcone of $\mathcal{COP}(\mathbb{E}_+)$.

Let r be the rank of the Euclidean Jordan algebra $\mathbb{E}$. We decompose the idempotent c into the sum of orthogonal primitive idempotents $c_p, \dots, c_r$. We can find $c_1, \dots, c_{p-1}$ such that $c_1, \dots, c_r$ is a Jordan frame of $\mathbb{E}$.[1] Consider the Peirce decomposition (2.7) with respect to the Jordan frame. We note that $c_p, \dots, c_r$ is a Jordan frame of the algebra $\mathbb{E}(c,1)$, so we have

$$\mathbb{E}(c,1) = \bigoplus_{p \le i \le j \le r} \mathbb{E}_{ij}, \tag{6.6}$$

which follows from (2.7) applied to $\mathbb{E}(c,1)$. Let $\preceq$ be the lexicographical order, whence the elements in the set $\{(i,j) \mid 1 \le i \le j \le r\}$ are ordered as $(1,1) \preceq (1,2) \preceq \cdots \preceq (1,r) \preceq (2,2) \preceq \cdots \preceq (r,r)$. For simplicity, when there is no danger of confusion, we write ij for (i,j). In accordance with the orthogonal decomposition (2.7), for $\mathcal{A} \in \mathcal{S}(\mathbb{E})$, we use the following matrix-like notation:

$$\mathcal{A} = (\mathcal{A}_{ij,kl})_{\substack{1 \le i \le j \le r \\ 1 \le k \le l \le r \\ ij \preceq kl}}.$$

Recall that $\mathcal{A}_{ij,kl} = \mathcal{P}_{\mathbb{E}_{ij}} \circ \mathcal{A}|_{\mathbb{E}_{kl}}$ is the (ij,kl)th element of $\mathcal{A}$.

For $\mathcal{A}, \mathcal{B} \in \mathcal{COP}(\mathbb{E}_+)$, we suppose that $\mathcal{A} + \mathcal{B} \in \{0\} \oplus \mathcal{COP}(\mathbb{E}(c,1)_+)$. Then, there exists $\mathcal{G} \in \mathcal{COP}(\mathbb{E}(c,1)_+)$ such that

$$\mathcal{A} + \mathcal{B} = \mathcal{P}^*_{\mathbb{E}(c,1)} \circ \mathcal{G} \circ \mathcal{P}_{\mathbb{E}(c,1)}. \tag{6.7}$$

We see that $\mathcal{P}_{\mathbb{E}(c,1)} \circ \mathcal{A}|_{\mathbb{E}(c,1)}$ belongs to $\mathcal{COP}(\mathbb{E}(c,1)_+)$. Indeed, let $x \in \mathbb{E}(c,1)_+$ be arbitrary. Then, we have

$$0 \overset{\text{(a)}}{\le} x \bullet \mathcal{A}(x) \overset{\text{(b)}}{=} \mathcal{P}^*_{\mathbb{E}(c,1)}(x) \bullet \mathcal{A}|_{\mathbb{E}(c,1)}(x) = x \bullet (\mathcal{P}_{\mathbb{E}(c,1)} \circ \mathcal{A}|_{\mathbb{E}(c,1)})(x),$$

where (a) holds because $x \in \mathbb{E}(c,1)_+ \subseteq \mathbb{E}_+$ and $\mathcal{A} \in \mathcal{COP}(\mathbb{E}_+)$ and (b) follows from $x \in \mathbb{E}(c,1)$. In addition, $\mathcal{P}_{\mathbb{E}(c,1)} \circ \mathcal{A}|_{\mathbb{E}(c,1)}$ is the "principal submatrix" of $\mathcal{A}$ obtained by extracting the $\frac{(r-p+1)(r-p+2)}{2}$ rows and columns indexed by $(p,p), (p,p+1) \dots, (r,r)$. The same is true for $\mathcal{B}$.

Thus, to prove $\mathcal{A}, \mathcal{B} \in \{0\} \oplus \mathcal{COP}(\mathbb{E}(c,1)_+)$, it is sufficient to show that

$$\mathcal{A}_{ij,kl} = \mathcal{B}_{ij,kl} = 0$$

for all (i,j,k,l) satisfying $1 \le i \le p-1$, $1 \le i \le j \le r$, $1 \le k \le l \le r$, and $ij \preceq kl$. See Figure 6.1 for an illustration of this proof. We also recall that $\mathcal{A}_{ij,kl}$ is a linear mapping between $\mathbb{E}_{kl}$ and $\mathbb{E}_{ij}$, so we have the following characterization:

$$\mathcal{A}_{ij,ij} = 0 \Longleftrightarrow x_{ij} \bullet \mathcal{A}_{ij,ij}(x_{ij}) = 0 \text{ for all } x_{ij} \in \mathbb{E}_{ij}, \tag{6.8}$$

$$\mathcal{A}_{ij,kl} = 0 \Longleftrightarrow x_{ij} \bullet \mathcal{A}_{ij,kl}(x_{kl}) = 0 \text{ for all } x_{ij} \in \mathbb{E}_{ij} \text{ and } x_{kl} \in \mathbb{E}_{kl}. \tag{6.9}$$

[1] Its proof is almost the same as [76, Lemma 23]. To be more precise, since $\mathbb{E}(c,0)$ is a Euclidean Jordan subalgebra of rank $p-1$, there exist a Jordan frame $c_1, \dots, c_{p-1}$ of $\mathbb{E}(c,0)$ and real numbers $\lambda_1, \dots, \lambda_{p-1}$ such that $0 = \sum_{i=1}^{p-1} \lambda_i c_i$; see (2.4). Then, $c_1, \dots, c_r$ is a Jordan frame of $\mathbb{E}$.

	11	12	13	14	22	23	24	33	34	44
11	(1)	(2)	(2)	(2)	(3)	(6)	(6)	(3)	(6)	(3)
12		(4)	(7)	(7)	(5)	(10)	(10)	(8)	(11)	(8)
13			(4)	(7)	(8)	(9)	(11)	(5)	(10)	(8)
14				(4)	(8)	(11)	(9)	(8)	(9)	(5)
22					(1)	(2)	(2)	(3)	(6)	(3)
23						(4)	(7)	(5)	(10)	(8)
24							(4)	(8)	(9)	(5)
33								*	*	*
34									*	*
44										*

Figure 6.1: Illustration of the proof of Lemma 6.3 in the case of $r = 4$ and $p = 3$. The cells marked (i) correspond to Case i for each $i = 1, \ldots, 11$. The 3×3 lower-right block marked the symbol $*$ corresponds to $\mathcal{P}_{\mathbb{E}(c,1)} \circ \mathcal{A}|_{\mathbb{E}(c,1)}$ and $\mathcal{P}_{\mathbb{E}(c,1)} \circ \mathcal{B}|_{\mathbb{E}(c,1)}$.

Moving on, a quadruple (i, j, k, l) satisfying the following three conditions

$$i \leq j, \; k \leq l, \; ij \preceq kl \tag{6.10}$$

must fall into exactly one of the following eleven cases:

Case 1: $i = j = k = l$;

Case 2: $i = j = k \neq l$;

Case 3: $i = j \neq k = l$;

Case 4: $i = k \neq j = l$;

Case 5: $i \neq j = k = l$;

Case 6: $i = j$ and i, k, l are different from each other;

Case 7: $i = k$ and i, j, l are different from each other;

Case 8: $k = l$ and i, j, l are different from each other;

Case 9: $j = l$ and i, k, l are different from each other;

Case 10: $j = k$ and i, j, l are different from each other;

Case 11: i, j, k, l are different from each other.

See Appendix B for the reasoning behind this case division. In the following discussion, for each $x \in \mathbb{E}$, we write $x \bullet \mathcal{A}(x)$ and $x \bullet \mathcal{B}(x)$ as $\alpha(x)$ and $\beta(x)$, respectively. Similarly, we define

$$\gamma(x) := x \bullet (\mathcal{P}^*_{\mathbb{E}(c,1)} \circ \mathcal{G} \circ \mathcal{P}_{\mathbb{E}(c,1)})(x). \tag{6.11}$$

Note that by (6.7), we have

$$\alpha(x) + \beta(x) = \gamma(x) \tag{6.12}$$

for all $x \in \mathbb{E}$. Also note that $\alpha(x)$ and $\beta(x)$ are nonnegative for any $x \in \mathbb{E}_+$ since $\mathcal{A}, \mathcal{B} \in \mathcal{COP}(\mathbb{E}_+)$.

$\boxed{\text{Case 1}}$ On the one hand, since $c_i \in \mathbb{E}_+$, we have $\alpha(c_i), \beta(c_i) \geq 0$. On the other hand, it follows from $c_i \notin \mathbb{E}(c,1)$ and (6.11) that $\gamma(c_i) = 0$. Therefore, by (6.12), both $\alpha(c_i) = c_i \bullet \mathcal{A}_{ii,ii}(c_i)$ and $\beta(c_i) = c_i \bullet \mathcal{B}_{ii,ii}(c_i)$ must be 0. Since $\mathbb{E}_{ii} = \mathbb{R}c_i$ (see (2.6)), we have $x_{ii} \bullet \mathcal{A}_{ii,ii}(x_{ii}) = 0$ for all $x_{ii} \in \mathbb{E}_{ii}$. In view of (6.8) and since an analogous argument holds for $\mathcal{B}$, we obtain $\mathcal{A}_{ii,ii} = \mathcal{B}_{ii,ii} = 0$.

$\boxed{\text{Case 2}}$ For any $x_{il} \in \mathbb{E}_{il}$ and $\epsilon > 0$, let

$$x(\epsilon) := (c_i + \epsilon x_{il})^2 = \underbrace{c_i + \epsilon^2 c_i \circ x_{il}^2}_{\in \mathbb{E}_{ii}} + \underbrace{\epsilon x_{il}}_{\in \mathbb{E}_{il}} + \underbrace{\epsilon^2 c_l \circ x_{il}^2}_{\in \mathbb{E}_{ll}} \in \mathbb{E}_+,$$

where the last equality holds by (2.6) and (2.8). Then, on the one hand, using Lemma 6.2 and considering the orthogonality among $\mathbb{E}_{ii}$, $\mathbb{E}_{il}$, and $\mathbb{E}_{ll}$, we have

$$\begin{aligned}
0 &\leq \alpha(x(\epsilon)) \\
&= x(\epsilon) \bullet \begin{array}{c} \begin{array}{ccc} ii & il & ll \end{array} \\ \begin{pmatrix} (1) & \mathcal{A}_{ii,il} & \mathcal{A}_{ii,ll} \\ & \mathcal{A}_{il,il} & \mathcal{A}_{il,ll} \\ & & \mathcal{A}_{ll,ll} \end{pmatrix} \end{array} (x(\epsilon)) & (6.13) \\
&= 2\epsilon c_i \bullet \mathcal{A}_{ii,il}(x_{il}) + \epsilon^2 \{x_{il} \bullet \mathcal{A}_{il,il}(x_{il}) + 2c_i \bullet \mathcal{A}_{ii,ll}(c_l \circ x_{il}^2)\} \\
&\quad + 2\epsilon^3 \{(c_i \circ x_{il}^2) \bullet \mathcal{A}_{ii,il}(x_{il}) + x_{il} \bullet \mathcal{A}_{il,ll}(c_l \circ x_{il}^2)\} + O(\epsilon^4), & (6.14)
\end{aligned}$$

where (1) appearing in (6.13) indicates that the (ii,ii)th element $\mathcal{A}_{ii,ii}$ of $\mathcal{A}$ is 0 by the proof of Case 1. By replacing $\mathcal{A}$ in (6.14) with $\mathcal{B}$, we can also calculate $\beta(x(\epsilon))$, which is greater that or equal to 0. On the other hand, recalling (2.5), (6.6), and the fact that $i \leq p-1$ holds by assumption, we have

$$\mathcal{P}_{\mathbb{E}(c,1)}(x(\epsilon)) = \begin{cases} \epsilon^2 c_l \circ x_{il}^2 & (\text{if } l \text{ satisfies } p \leq l \leq r), \\ 0 & (\text{otherwise}), \end{cases}$$

which implies that $\gamma(x(\epsilon)) = O(\epsilon^4)$. Therefore, setting $x = x(\epsilon)$ in (6.12), dividing by 2ϵ, and letting $\epsilon \downarrow 0$, we obtain

$$c_i \bullet \mathcal{A}_{ii,il}(x_{il}) + c_i \bullet \mathcal{B}_{ii,il}(x_{il}) = 0.$$

Since $\alpha(x(\epsilon))$ is nonnegative for all $\epsilon > 0$,

$$c_i \bullet \mathcal{A}_{ii,il}(x_{il}) = \lim_{\epsilon \downarrow 0} \frac{\alpha(x(\epsilon))}{2\epsilon}$$

is also nonnegative. Similarly, $c_i \bullet \mathcal{B}_{ii,il}(x_{il})$ is nonnegative. Thus, we have

$$c_i \bullet \mathcal{A}_{ii,il}(x_{il}) = c_i \bullet \mathcal{B}_{ii,il}(x_{il}) = 0.$$

Since $\mathbb{E}_{ii} = \mathbb{R}c_i$ and $x_{il} \in \mathbb{E}_{il}$ is arbitrary, we obtain $\mathcal{A}_{ii,il} = \mathcal{B}_{ii,il} = 0$ from (6.9).

$\boxed{\text{Case 3}}$ For any $\epsilon > 0$, let $x(\epsilon) := c_i + \epsilon c_l \in \mathbb{E}_+$. Then, on the one hand, from Case 1, we have $\mathcal{A}_{ii,ii} = 0$ so

$$0 \leq \alpha(x(\epsilon)) = 2\epsilon c_i \bullet \mathcal{A}_{ii,ll}(c_l) + \epsilon^2 c_l \bullet \mathcal{A}_{ll,ll}(c_l).$$

On the other hand, we have

$$\mathcal{P}_{\mathbb{E}(c,1)}(x(\epsilon)) = \begin{cases} \epsilon c_l & (\text{if } l \text{ satisfies } p \leq l \leq r), \\ 0 & (\text{otherwise}), \end{cases}$$

which implies that $\gamma(x(\epsilon)) = O(\epsilon^2)$. Therefore, setting $x = x(\epsilon)$ in (6.12), dividing by 2ϵ, and letting $\epsilon \downarrow 0$, we obtain

$$c_i \bullet \mathcal{A}_{ii,ll}(c_l) = c_i \bullet \mathcal{B}_{ii,ll}(c_l) = 0.$$

Recalling that $\mathbb{E}_{ii} = \mathbb{R}c_i$ and $\mathbb{E}_{ll} = \mathbb{R}c_l$ hold, we obtain $\mathcal{A}_{ii,ll} = \mathcal{B}_{ii,ll} = 0$, again by (6.9).

$\boxed{\text{Case 4}}$ For any $x_{il} \in \mathbb{E}_{il}$ and $\epsilon > 0$, let $x(\epsilon)$ be the same as in Case 2. Then, on the one hand, it follows from Cases 1, 2, 3, and a computation analogous to (6.14) that

$$0 \leq \alpha(x(\epsilon)) = \epsilon^2 x_{il} \bullet \mathcal{A}_{il,il}(x_{il}) + O(\epsilon^3).$$

On the other hand, we have $\gamma(x(\epsilon)) = O(\epsilon^4)$. Therefore, setting $x = x(\epsilon)$ in (6.12), dividing by ϵ^2, and letting $\epsilon \downarrow 0$, we obtain

$$x_{il} \bullet \mathcal{A}_{il,il}(x_{il}) = x_{il} \bullet \mathcal{B}_{il,il}(x_{il}) = 0.$$

Since $x_{il} \in \mathbb{E}_{il}$ is arbitrary, we obtain $\mathcal{A}_{il,il} = \mathcal{B}_{il,il} = 0$, again by (6.8).

$\boxed{\text{Case 5}}$ For any $x_{il} \in \mathbb{E}_{il}$ and $\epsilon > 0$, let

$$x(\epsilon) := (c_i + \epsilon^2 x_{il})^2 + \epsilon^3 c_l = \underbrace{c_i + \epsilon^4 c_i \circ x_{il}^2}_{\in \mathbb{E}_{ii}} + \underbrace{\epsilon^2 x_{il}}_{\in \mathbb{E}_{il}} + \underbrace{\epsilon^3 c_l + \epsilon^4 c_l \circ x_{il}^2}_{\in \mathbb{E}_{ll}} \in \mathbb{E}_+.$$

Then, on the one hand, we have

$$0 \leq \alpha(x(\epsilon)) = x(\epsilon) \bullet \begin{array}{c} \begin{array}{ccc} ii & il & ll \end{array} \\ \begin{pmatrix} (1) & (2) & (3) \\ & (4) & \mathcal{A}_{il,ll} \\ & & \mathcal{A}_{ll,ll} \end{pmatrix} \end{array} (x(\epsilon)) = 2\epsilon^5 x_{il} \bullet \mathcal{A}_{il,ll}(c_l) + O(\epsilon^6).$$

On the other hand, we have $\gamma(x(\epsilon)) = O(\epsilon^6)$. Therefore, setting $x = x(\epsilon)$ in (6.12), dividing by $2\epsilon^5$, and letting $\epsilon \downarrow 0$, we obtain

$$x_{il} \bullet \mathcal{A}_{il,ll}(c_l) = x_{il} \bullet \mathcal{B}_{il,ll}(c_l) = 0.$$

Since $x_{il} \in \mathbb{E}_{il}$ is arbitrary, we obtain $\mathcal{A}_{il,ll} = \mathcal{B}_{il,ll} = 0$.

$\boxed{\text{Case 6}}$ For any $x_{kl} \in \mathbb{E}_{kl}$ and $\epsilon > 0$, let

$$x(\epsilon) := c_i + \epsilon(c_k + x_{kl})^2 = \underbrace{c_i}_{\in \mathbb{E}_{ii}} + \underbrace{\epsilon c_k + \epsilon c_k \circ x_{kl}^2}_{\in \mathbb{E}_{kk}} + \underbrace{\epsilon x_{kl}}_{\in \mathbb{E}_{kl}} + \underbrace{\epsilon c_l \circ x_{kl}^2}_{\in \mathbb{E}_{ll}} \in \mathbb{E}_+.$$

Then, on the one hand, we have

$$\begin{aligned} 0 &\le \alpha(x(\epsilon)) \\ &= x(\epsilon) \bullet \begin{array}{c} \begin{array}{cccc} ii & kk & kl & ll \end{array} \\ \begin{pmatrix} (1) & (3) & \mathcal{A}_{ii,kl} & (3) \\ & \mathcal{A}_{kk,kk} & \mathcal{A}_{kk,kl} & \mathcal{A}_{kk,ll} \\ & & \mathcal{A}_{kl,kl} & \mathcal{A}_{kl,ll} \\ & & & \mathcal{A}_{ll,ll} \end{pmatrix} \end{array} (x(\epsilon)) \\ &= 2\epsilon c_i \bullet \mathcal{A}_{ii,kl}(x_{kl}) + O(\epsilon^2). \end{aligned}$$

On the other hand, we have $\gamma(x(\epsilon)) = O(\epsilon^2)$. Therefore, setting $x = x(\epsilon)$ in (6.12), dividing by 2ϵ, and letting $\epsilon \downarrow 0$, we obtain

$$c_i \bullet \mathcal{A}_{ii,kl}(x_{kl}) = c_i \bullet \mathcal{B}_{ii,kl}(x_{kl}) = 0.$$

Since $x_{kl} \in \mathbb{E}_{kl}$ is arbitrary, we obtain $\mathcal{A}_{ii,kl} = \mathcal{B}_{ii,kl} = 0$.

$\boxed{\text{Case 7}}$ For any $x_{ij} \in \mathbb{E}_{ij}$, $x_{il} \in \mathbb{E}_{il}$, and $\epsilon > 0$, let

$$\begin{aligned} x(\epsilon) &:= (c_i + \epsilon x_{ij})^2 + (c_i + \epsilon x_{il})^2 \\ &= \underbrace{2c_i + \epsilon^2 c_i \circ x_{ij}^2 + \epsilon^2 c_i \circ x_{il}^2}_{\in \mathbb{E}_{ii}} + \underbrace{\epsilon x_{ij}}_{\in \mathbb{E}_{ij}} + \underbrace{\epsilon x_{il}}_{\in \mathbb{E}_{il}} + \underbrace{\epsilon^2 c_j \circ x_{ij}^2}_{\in \mathbb{E}_{jj}} + \underbrace{\epsilon^2 c_l \circ x_{il}^2}_{\in \mathbb{E}_{ll}} \in \mathbb{E}_+. \end{aligned}$$

Then, on the one hand, we have

$$\begin{aligned} 0 &\le \alpha(x(\epsilon)) \\ &= x(\epsilon) \bullet \begin{array}{c} \begin{array}{ccccc} ii & ij & il & jj & ll \end{array} \\ \begin{pmatrix} (1) & (2) & (2) & (3) & (3) \\ & (4) & \mathcal{A}_{ij,il} & (5) & \mathcal{A}_{ij,ll} \\ & & (4) & \mathcal{A}_{il,jj} & (5) \\ & & & \mathcal{A}_{jj,jj} & \mathcal{A}_{jj,ll} \\ & & & & \mathcal{A}_{ll,ll} \end{pmatrix} \end{array} (x(\epsilon)) \\ &= 2\epsilon^2 x_{ij} \bullet \mathcal{A}_{ij,il}(x_{il}) + O(\epsilon^3). \end{aligned}$$

On the other hand, we have $\gamma(x(\epsilon)) = O(\epsilon^4)$. Therefore, setting $x = x(\epsilon)$ in (6.12), dividing by $2\epsilon^2$, and letting $\epsilon \downarrow 0$, we obtain

$$x_{ij} \bullet \mathcal{A}_{ij,il}(x_{il}) = x_{ij} \bullet \mathcal{B}_{ij,il}(x_{il}) = 0.$$

Since $x_{ij} \in \mathbb{E}_{ij}$ and $x_{il} \in \mathbb{E}_{il}$ are arbitrary, we obtain $\mathcal{A}_{ij,il} = \mathcal{B}_{ij,il} = 0$.

$\boxed{\text{Case 8}}$ For any $x_{ij} \in \mathbb{E}_{ij}$ and $\epsilon > 0$, let

$$x(\epsilon) := (c_i + \epsilon x_{ij})^2 + \epsilon^2 c_l = \underbrace{c_i + \epsilon^2 c_i \circ x_{ij}^2}_{\in \mathbb{E}_{ii}} + \underbrace{\epsilon x_{ij}}_{\in \mathbb{E}_{ij}} + \underbrace{\epsilon^2 c_j \circ x_{ij}^2}_{\in \mathbb{E}_{jj}} + \underbrace{\epsilon^2 c_l}_{\in \mathbb{E}_{ll}} \in \mathbb{E}_+.$$

Then, on the one hand, we have

$$\begin{aligned} 0 &\le \alpha(x(\epsilon)) \\ &= x(\epsilon) \bullet \begin{array}{c} \begin{array}{cccc} ii & ij & jj & ll \end{array} \\ \begin{pmatrix} (1) & (2) & (3) & (3) \\ & (4) & (5) & \mathcal{A}_{ij,ll} \\ & & \mathcal{A}_{jj,jj} & \mathcal{A}_{jj,ll} \\ & & & \mathcal{A}_{ll,ll} \end{pmatrix} \end{array} (x(\epsilon)) \qquad (6.15) \\ &= 2\epsilon^3 x_{ij} \bullet \mathcal{A}_{ij,ll}(c_l) + O(\epsilon^4). \end{aligned}$$

The matrix-like notation in (6.15) is for the case $j < l$, but the calculation is also valid if $l < j$. On the other hand, we have $\gamma(x(\epsilon)) = O(\epsilon^4)$. Therefore, setting $x = x(\epsilon)$ in (6.12), dividing by $2\epsilon^3$, and letting $\epsilon \downarrow 0$, we obtain

$$x_{ij} \bullet \mathcal{A}_{ij,ll}(c_l) = x_{ij} \bullet \mathcal{B}_{ij,ll}(c_l) = 0.$$

Since $x_{ij} \in \mathbb{E}_{ij}$ is arbitrary, we obtain $\mathcal{A}_{ij,ll} = \mathcal{B}_{ij,ll} = 0$.

$\boxed{\text{Case 9}}$ For any $x_{il} \in \mathbb{E}_{il}$, $x_{kl} \in \mathbb{E}_{kl}$, and $\epsilon > 0$, let

$$\begin{aligned} x(\epsilon) &:= (c_i + \epsilon x_{il})^2 + \epsilon^2 (c_k + x_{kl})^2 \\ &= \underbrace{c_i + \epsilon^2 c_i \circ x_{il}^2}_{\in \mathbb{E}_{ii}} + \underbrace{\epsilon x_{il}}_{\in \mathbb{E}_{il}} + \underbrace{\epsilon^2 c_k + \epsilon^2 c_k \circ x_{kl}^2}_{\in \mathbb{E}_{kk}} + \underbrace{\epsilon^2 x_{kl}}_{\in \mathbb{E}_{kl}} + \underbrace{\epsilon^2 c_l \circ x_{kl}^2 + \epsilon^2 c_l \circ x_{il}^2}_{\in \mathbb{E}_{ll}} \in \mathbb{E}_+. \end{aligned}$$

Then, on the one hand, we have

$$\begin{aligned} 0 &\le \alpha(x(\epsilon)) \\ &= x(\epsilon) \bullet \begin{array}{c} \begin{array}{ccccc} ii & il & kk & kl & ll \end{array} \\ \begin{pmatrix} (1) & (2) & (3) & (6) & (3) \\ & (4) & (8) & \mathcal{A}_{il,kl} & (5) \\ & & \mathcal{A}_{kk,kk} & \mathcal{A}_{kk,kl} & \mathcal{A}_{kk,ll} \\ & & & \mathcal{A}_{kl,kl} & \mathcal{A}_{kl,ll} \\ & & & & \mathcal{A}_{ll,ll} \end{pmatrix} \end{array} (x(\epsilon)) \\ &= 2\epsilon^3 x_{il} \bullet \mathcal{A}_{il,kl}(x_{kl}) + O(\epsilon^4). \end{aligned}$$

On the other hand, we have $\gamma(x(\epsilon)) = O(\epsilon^4)$. Therefore, setting $x = x(\epsilon)$ in (6.12), dividing by $2\epsilon^3$, and letting $\epsilon \downarrow 0$, we obtain

$$x_{il} \bullet \mathcal{A}_{il,kl}(x_{kl}) = x_{il} \bullet \mathcal{B}_{il,kl}(x_{kl}) = 0.$$

Since $x_{il} \in \mathbb{E}_{il}$ and $x_{kl} \in \mathbb{E}_{kl}$ are arbitrary, we obtain $\mathcal{A}_{il,kl} = \mathcal{B}_{il,kl} = 0$.

$\boxed{\text{Case 10}}$ For any $x_{ij} \in \mathbb{E}_{ij}$, $x_{jl} \in \mathbb{E}_{jl}$, and $\epsilon > 0$, let

$$\begin{aligned} x(\epsilon) &:= (c_i + \epsilon x_{ij})^2 + \epsilon^2 (c_j + x_{jl})^2 \\ &= \underbrace{c_i + \epsilon^2 c_i \circ x_{ij}^2}_{\in \mathbb{E}_{ii}} + \underbrace{\epsilon x_{ij}}_{\in \mathbb{E}_{ij}} + \underbrace{\epsilon^2 c_j + \epsilon^2 c_j \circ x_{ij}^2 + \epsilon^2 c_j \circ x_{jl}^2}_{\in \mathbb{E}_{jj}} + \underbrace{\epsilon^2 x_{jl}}_{\in \mathbb{E}_{jl}} + \underbrace{\epsilon^2 c_l \circ x_{jl}^2}_{\in \mathbb{E}_{ll}} \in \mathbb{E}_+. \end{aligned}$$

Then, on the one hand, we have

$$\begin{aligned} 0 &\leq \alpha(x(\epsilon)) \\ &= x(\epsilon) \bullet \begin{array}{c} \begin{array}{ccccc} ii & ij & jj & jl & ll \end{array} \\ \begin{pmatrix} (1) & (2) & (3) & (6) & (3) \\ & (4) & (5) & \mathcal{A}_{ij,jl} & (8) \\ & & \mathcal{A}_{jj,jj} & \mathcal{A}_{jj,jl} & \mathcal{A}_{jj,ll} \\ & & & \mathcal{A}_{jl,jl} & \mathcal{A}_{jl,ll} \\ & & & & \mathcal{A}_{ll,ll} \end{pmatrix} \end{array} (x(\epsilon)) \\ &= 2\epsilon^3 x_{ij} \bullet \mathcal{A}_{ij,jl}(x_{jl}) + O(\epsilon^4). \end{aligned}$$

On the other hand, we have $\gamma(x(\epsilon)) = O(\epsilon^4)$. Therefore, setting $x = x(\epsilon)$ in (6.12), dividing by $2\epsilon^3$, and letting $\epsilon \downarrow 0$, we obtain

$$x_{ij} \bullet \mathcal{A}_{ij,jl}(x_{jl}) = x_{ij} \bullet \mathcal{B}_{ij,jl}(x_{jl}) = 0.$$

Since $x_{ij} \in \mathbb{E}_{ij}$ and $x_{jl} \in \mathbb{E}_{jl}$ are arbitrary, we obtain $\mathcal{A}_{ij,jl} = \mathcal{B}_{ij,jl} = 0$.

$\boxed{\text{Case 11}}$ For any $x_{ij} \in \mathbb{E}_{ij}$, $x_{kl} \in \mathbb{E}_{kl}$, and $\epsilon > 0$, let

$$\begin{aligned} x(\epsilon) &:= (c_i + \epsilon x_{ij})^2 + \epsilon^2 (c_k + x_{kl})^2 \\ &= \underbrace{c_i + \epsilon^2 c_i \circ x_{ij}^2}_{\in \mathbb{E}_{ii}} + \underbrace{\epsilon x_{ij}}_{\in \mathbb{E}_{ij}} + \underbrace{\epsilon^2 c_j \circ x_{ij}^2}_{\mathbb{E}_{jj}} + \underbrace{\epsilon^2 c_k + \epsilon^2 c_k \circ x_{kl}^2}_{\in \mathbb{E}_{kk}} + \underbrace{\epsilon^2 x_{kl}}_{\in \mathbb{E}_{kl}} + \underbrace{\epsilon^2 c_l \circ x_{kl}^2}_{\in \mathbb{E}_{ll}} \in \mathbb{E}_+. \end{aligned}$$

Then, on the one hand, we have

$$\begin{aligned} 0 &\leq \alpha(x(\epsilon)) \\ &= x(\epsilon) \bullet \begin{array}{c} \begin{array}{cccccc} ii & ij & jj & kk & kl & ll \end{array} \\ \begin{pmatrix} (1) & (2) & (3) & (3) & (6) & (3) \\ & (4) & (5) & (8) & \mathcal{A}_{ij,kl} & (8) \\ & & \mathcal{A}_{jj,jj} & \mathcal{A}_{jj,kk} & \mathcal{A}_{jj,kl} & \mathcal{A}_{jj,ll} \\ & & & \mathcal{A}_{kk,kk} & \mathcal{A}_{kk,kl} & \mathcal{A}_{kk,ll} \\ & & & & \mathcal{A}_{kl,kl} & \mathcal{A}_{kl,ll} \\ & & & & & \mathcal{A}_{ll,ll} \end{pmatrix} \end{array} (x(\epsilon)) \\ &= 2\epsilon^3 x_{ij} \bullet \mathcal{A}_{ij,kl}(x_{kl}) + O(\epsilon^4). \end{aligned}$$

On the other hand, we have $\gamma(x(\epsilon)) = O(\epsilon^4)$. Therefore, setting $x = x(\epsilon)$ in (6.12), dividing by $2\epsilon^3$, and letting $\epsilon \downarrow 0$, we obtain

$$x_{ij} \bullet \mathcal{A}_{ij,kl}(x_{kl}) = x_{ij} \bullet \mathcal{B}_{ij,kl}(x_{kl}) = 0.$$

Since $x_{ij} \in \mathbb{E}_{ij}$ and $x_{kl} \in \mathbb{E}_{kl}$ are arbitrary, we obtain $\mathcal{A}_{ij,kl} = \mathcal{B}_{ij,kl} = 0$. □

Corollary 6.4. *Let $\mathbb{K}$ be a symmetric cone in a finite-dimensional real inner product space $\mathbb{E}$, and $\mathbb{F}$ be a face of $\mathbb{K}$. Also, let $\mathbb{V}_1 := (\operatorname{span}\mathbb{F})^\perp$ and $\mathbb{V}_2 := \operatorname{span}\mathbb{F}$, so that $\mathbb{E} = \mathbb{V}_1 \oplus \mathbb{V}_2$. Then, following the definition in (6.5), $\{0\} \oplus \mathcal{COP}(\mathbb{F})$ is a face of $\mathcal{COP}(\mathbb{K})$.*

Proof. Let $\circ$ be a bilinear product on $\mathbb{E}$ such that $(\mathbb{E}, \circ, \bullet)$ is a Euclidean Jordan algebra and $\mathbb{K} = \mathbb{E}_+$. For a face $\mathbb{F}$ of $\mathbb{K}$, there exists an idempotent c in the Euclidean Jordan algebra $\mathbb{E}$ such that $\mathbb{F} = \mathbb{E}(c,1)_+$ and $\operatorname{span}\mathbb{F} = \mathbb{E}(c,1)$ [44, Theorem 3.1]. Therefore, the claim follows from Lemma 6.3. □

Corollary 6.4 implies that the facial structure of the copositive cone over a symmetric cone is never simpler than that of the underlying symmetric cone. For a symmetric cone $\mathbb{K}$, consider faces $\mathbb{F}_1, \mathbb{F}_2$ of $\mathbb{K}$ such that $\mathbb{F}_2 \subseteq \mathbb{F}_1$. It follows from Corollary 6.4 that $\{0\} \oplus \mathcal{COP}(\mathbb{F}_1)$, which is isomorphic to $\mathcal{COP}(\mathbb{F}_1)$, is a face of $\mathcal{COP}(\mathbb{K})$. In addition, since $\mathbb{F}_1$ is also a symmetric cone on its span and $\mathbb{F}_2$ is a face of $\mathbb{F}_1$, $\{0\} \oplus \mathcal{COP}(\mathbb{F}_2)$ is a face of $\mathcal{COP}(\mathbb{F}_1)$.

In general, the facial structure of the copositive cone over a symmetric cone is much more complicated than the underlying symmetric cone. For example, while the nonnegative orthant is polyhedral, the standard copositive cone is not even facially exposed. The same is true for general symmetric cones. Although symmetric cones satisfy good properties including facial exposedness[2], copositive cones over symmetric cones are not facially exposed, as shown in the following theorem.

Theorem 6.5. *Let $\mathbb{K}$ be a symmetric cone of dimension greater than or equal to 2 in a finite-dimensional real inner product space. Then, for any c generating an extreme ray of $\mathbb{K}$, $\mathbb{R}_+ c^{\otimes 2}$ is a non-exposed face of $\mathcal{COP}(\mathbb{K})$. In particular, $\mathcal{COP}(\mathbb{K})$ is not facially exposed.*

Proof. It follows from Corollary 6.4 that $\{0\} \oplus \mathcal{COP}(\mathbb{R}_+ c)$ is a face of $\mathcal{COP}(\mathbb{K})$. In addition, as a linear transformation, since $c^{\otimes 2}$ is a basis for $\mathcal{S}(\mathbb{R}c)$, $\{0\} \oplus \mathcal{COP}(\mathbb{R}_+ c)$ is equal to $\mathbb{R}_+ c^{\otimes 2}$. In what follows, we show that the face $\mathbb{R}_+ c^{\otimes 2}$ of $\mathcal{COP}(\mathbb{K})$ is not exposed.

We assume that the face $\mathbb{R}_+ c^{\otimes 2}$ is exposed. Then, there exists $\mathcal{H} \in \mathcal{CP}(\mathbb{K})$ such that

$$\mathbb{R}_+ c^{\otimes 2} = \mathcal{COP}(\mathbb{K}) \cap \{\mathcal{H}\}^\perp. \tag{6.16}$$

[2] In fact, symmetric cones satisfy a stronger form of facial exposedness called *orthogonal projectional exposedness*, see [72, Proposition 33] and also Propositions 9 and 13 therein.

As $\mathcal{H} \in \mathcal{CP}(\mathbb{K})$, there exist $h_1, \ldots, h_k \in \mathbb{K}$ such that $\mathcal{H}$ decomposes into $\sum_{i=1}^k h_i^{\otimes 2}$. We see from (6.16) that $c^{\otimes 2}$ is orthogonal to $\mathcal{H}$, i.e.,

$$0 = \langle c^{\otimes 2}, \mathcal{H} \rangle = \sum_{i=1}^k (c \bullet h_i)^2,$$

where the second equality is a consequence of (2.3). This implies that $c \bullet h_i = 0$ for all $i = 1, \ldots, k$.

Since the dimension of $\mathbb{K}$ is greater than or equal to 2 and that of $\mathbb{R}_+ c$ is 1, $\mathbb{R}_+ c$ is a face strictly contained in $\mathbb{K}$. Recalling that symmetric cones are facially exposed, there exists a supporting hyperplane that exposes the face $\mathbb{R}_+ c$, i.e., there exists $d \in \mathbb{K}^*$ such that $\mathbb{R}_+ c = \mathbb{K} \cap \{d\}^\perp$. Since $\mathbb{R}_+ c$ is strictly contained in $\mathbb{K}$ and $\mathbb{K}$ is self-dual, we have $d \neq 0$ and $d \in \mathbb{K} \cap \{c\}^\perp$.

Let $\mathcal{A} := c \otimes d + d \otimes c \in \mathcal{S}(\operatorname{span} \mathbb{K})$. Note that

$$\|\mathcal{A}\|_{\mathrm{F}}^2 = 2\|c\|^2 \|d\|^2 > 0. \tag{6.17}$$

For any $x \in \mathbb{K}$, we have

$$x \bullet \mathcal{A}(x) = 2(c \bullet x)(d \bullet x) \geq 0,$$

where we use $c, d \in \mathbb{K}$ and the self-duality of $\mathbb{K}$ to derive the inequality. Therefore, we obtain $\mathcal{A} \in \mathcal{COP}(\mathbb{K})$. In addition, since

$$\langle \mathcal{A}, \mathcal{H} \rangle = 2 \sum_{i=1}^k (c \bullet h_i)(d \bullet h_i) = 0,$$

we see that $\mathcal{A} \in \mathcal{COP}(\mathbb{K}) \cap \{\mathcal{H}\}^\perp$. Combining it with (6.16) implies that there exists $\alpha \geq 0$ such that $\mathcal{A} = \alpha c^{\otimes 2}$. Therefore, we have

$$\|\mathcal{A}\|_{\mathrm{F}}^2 = \langle c \otimes d + d \otimes c, \alpha c^{\otimes 2} \rangle = 2\alpha \|c\|^2 (c \bullet d) = 0,$$

which contradicts (6.17). Thus, $\mathbb{R}_+ c^{\otimes 2}$ is a non-exposed face of $\mathcal{COP}(\mathbb{K})$. $\square$

Part IV

Epilogue

Chapter 7

Conclusion

7.1 Summary

This thesis concerns approximations and the facial structure of completely positive and copositive cones over symmetric cones in Parts II and III, respectively.

In Chapter 3 of Part II, we provided approximation hierarchies for copositive cones over symmetric cones and compared them with existing hierarchies. We provided the three approximation hierarchies: the NN-type inner-, dP-type inner-, and Yıldırım-type outer-approximation hierarchies. The NN-type inner-approximation hierarchy approaches copositive tensor cones over general symmetric cones and permits a sum-of-squares representation. Although the dP- and Yıldırım-type approximation hierarchies are applicable only to copositive matrix cones over a direct product of a nonnegative orthant and a single second-order cone, they are not affected much by the increase in the size of the second-order cone, unlike the NN-type and existing approximation hierarchies. Combining the proposed approximation hierarchies especially with the ZVP-type inner-approximation hierarchy, we obtained nearly optimal values of copositive programming problems when the size of the nonnegative orthant is small. At the same time, the results of the numerical experiments posed us a question of whether the zeroth levels of the ZVP- and NN-type inner-approximation hierarchies are identical.

In Chapter 4 of Part II, we tackled the question arising in the previous chapter. We defined the dual cones of the zeroth levels of the ZVP- and NN-type inner-approximation hierarchies as the ZVP- and NN-type generalized doubly nonnegative cones, respectively. We compared the strength of the relaxation of them and the BD-type generalized doubly nonnegative cone. As an underlying cone, we considered two types of cones: a direct product of a nonnegative orthant and second-order cones and a direct product of a nonnegative orthant and positive semidefinite cones. When the underlying cone is a direct product of a nonnegative orthant and second-order cones, no theoretical inclusion relationship is obtained between ZVP- and BD-type generalized doubly nonnegative cones, but the ZVP-type cone provides better relaxation numerically than BD-type cone. In contrast, when the underlying cone is a direct product of a nonnegative orthant and positive semidefinite cones, the ZVP-type cone numerically provides worse relaxation than the BD-type cone. In both cases, the NN-type generalized doubly nonnegative cone is theoretically strictly included in the ZVP-type cone. The results of our numerical experiments showed that the three generalized doubly nonnegative cones yielded much tighter bounds for generalized completely positive programming than positive semidefinite cones.

In Chapter 5 of Part III, we constructed a chain of faces of length $\ell_{\mathcal{K}} = T_n + 1$ for any closed convex cone $\mathcal{K}$ satisfying either $\mathcal{DD}_+^n \subseteq \mathcal{K} \subseteq \mathcal{N}^n$ or $\mathcal{N}^n \subseteq \mathcal{K} \subseteq (\mathcal{SDD}_+^n)^*$. Furthermore, for any closed convex cone $\mathcal{K}$ satisfying $\mathcal{CP}^n \subseteq \mathcal{K} \subseteq \mathcal{DNN}^n$ or $\mathcal{SPN}^n \subseteq \mathcal{K} \subseteq \mathcal{COP}^n$ with $n \geq 2$, we established that $\ell_{\mathrm{poly}}(\mathcal{K}) = T_n - 2$ as well as $\ell_{\mathcal{K}} = T_n + 1$, which are the maximum possible for any closed convex cone in $\mathbb{S}^n$. Notably, such cones $\mathcal{K}$ include not only the standard completely positive, doubly nonnegative, SPN, and standard copositive cones but also the closure of the completely positive semidefinite cone and some approximation hierarchies.

In Chapter 6 of Part III, we showed that the facial structure of a symmetric cone induces that of the copositive cone over the symmetric cone injectively. However, the exposedness of the faces of the symmetric cone is not passed on to the induced faces of the copositive cone. Specifically, we proved that if the symmetric cone has dimension at least 2, then for any c generating an extreme ray of the symmetric cone, $c^{\otimes 2}$ generates a non-exposed extreme ray of the copositive cone.

7.2 Future Directions

We believe that the results of this thesis help us solve completely positive and copositive programming problems and understand the geometry of completely positive and copositive cones. At the same time, however, they also bring us interesting questions to study in the future.

7.2.1 Connection between Parts II and III

In Corollary 6.4, we provided a class of faces of copositive cones over symmetric cones. As stated in the following proposition, by using the characterization of copositive cones over symmetric cones shown in Theorem 3.17, we can provide another class of their faces.

Proposition 7.1. *Under the same notation as in Theorem 3.17, if $\mathcal{F}$ is a face of $\mathcal{COP}^{r,m}$, then*

$$\mathcal{COP}^{n,m}(\mathbb{E}_+;\mathcal{F}) \coloneqq \bigcap_{(c_1,\dots,c_r)\in\mathfrak{F}} \{\mathcal{A} \in \mathcal{S}^{n,m}(\mathbb{E}) \mid \mathcal{A}(c_1,\dots,c_r) \in \mathcal{F}\}$$

is a face of $\mathcal{COP}^{n,m}(\mathbb{E}_+)$.

Proof. Since $\mathcal{A}(c_1,\dots,c_r)$ is linear with respect to $\mathcal{A} \in \mathcal{S}^{n,m}(\mathbb{E})$ for each $(c_1,\dots,c_r) \in \mathfrak{F}$, we see that $\mathcal{COP}^{n,m}(\mathbb{E}_+;\mathcal{F})$ is a convex subcone of $\mathcal{COP}^{n,m}(\mathbb{E}_+)$. For $\mathcal{A},\mathcal{B} \in \mathcal{COP}^{n,m}(\mathbb{E}_+)$, we assume that

$$\mathcal{A} + \mathcal{B} \in \mathcal{COP}^{n,m}(\mathbb{E}_+;\mathcal{F}). \tag{7.1}$$

Let $(c_1, \ldots, c_r) \in \mathfrak{F}$ be arbitrary. Then, it follows from (7.1) that

$$\mathcal{A}(c_1, \ldots, c_r) + \mathcal{B}(c_1, \ldots, c_r) = (\mathcal{A} + \mathcal{B})(c_1, \ldots, c_r) \in \mathcal{F}.$$

The two tensors $\mathcal{A}(c_1, \ldots, c_r)$ and $\mathcal{B}(c_1, \ldots, c_r)$ belong to $\mathcal{COP}^{r,m}$ since $\mathcal{A}, \mathcal{B} \in \mathcal{COP}^{n,m}(\mathbb{E}_+)$. Given that $\mathcal{F}$ is a face of $\mathcal{COP}^{r,m}$, we see that $\mathcal{A}(c_1, \ldots, c_r)$ and $\mathcal{B}(c_1, \ldots, c_r)$ belong to $\mathcal{F}$. As $(c_1, \ldots, c_r) \in \mathfrak{F}$ is arbitrary, we obtain $\mathcal{A}, \mathcal{B} \in \mathcal{COP}^{n,m}(\mathbb{E}_+; \mathcal{F})$. □

This proposition suggests the relationship between the facial structure of copositive cones over symmetric cones and that of the standard copositive cone. We are wondering to what extent the facial structure of the standard copositive cone is inherited by that of copositive cones over symmetric cones. For example, is the mapping that maps a face $\mathcal{F}$ of $\mathcal{COP}^{r,m}$ to the face $\mathcal{COP}^{n,m}(\mathbb{E}_+; \mathcal{F})$ of $\mathcal{COP}^{n,m}(\mathbb{E}_+)$ surjective or injective? If a face $\mathcal{F}$ of $\mathcal{COP}^{r,m}$ is exposed, is the face $\mathcal{COP}^{n,m}(\mathbb{E}_+; \mathcal{F})$ of $\mathcal{COP}^{n,m}(\mathbb{E}_+)$ also exposed?

Furthermore, using the extreme rays of completely positive and copositive cones, we can verify how exact an inner approximation for them is. Let $\mathcal{I}$ be a convex subcone of a copositive cone, for example. Then, it follows from Minkowski's theorem for cones [106, Corollary 18.5.2] that if $\mathcal{I}$ contains every extreme ray of the copositive cone, $\mathcal{I}$ agrees with it. We are curious about what types of extreme rays copositive cones over symmetric cones have in addition to those provided in Theorem 6.5 and whether they belong to inner-approximation hierarchies for the copositive cones or not. For example, as shown in the following proposition, the extreme rays provided in Theorem 6.5 are included in the NN-type inner-approximation hierarchy.

Proposition 7.2. *Under the same notation as in Section 3.2.2, for any primitive idempotent c in the Euclidean Jordan algebra $(\mathbb{E}, \circ, \bullet)$, the tensor $c^{\otimes 2}$ generating an extreme ray of $\mathcal{COP}(\mathbb{E}_+)$ belongs to $\mathcal{I}_{\mathrm{NN},0}^{n,2}(\mathbb{E}_+)$. Consequently, it belongs to $\mathcal{I}_{\mathrm{NN},l}^{n,2}(\mathbb{E}_+)$ for all $l \in \mathbb{N}$.*

Proof. It is sufficient to see that $\langle c^{\otimes 2}, (x \circ x)^{\otimes 2} \rangle \in \Sigma^{n,4}(\mathbb{E})$. This is true because $\langle c^{\otimes 2}, (x \circ x)^{\otimes 2} \rangle = (c \bullet (x \circ x))^2$ and $c \bullet (x \circ x)$ belongs to $H^{n,2}(\mathbb{E})$. □

7.2.2 Other topics

Under an appropriate condition, the sequence of approximate optimal values obtained by replacing completely positive and copositive cones with approximation hierarchies converges to true optimal values of completely positive and copositive programming problems [125, Theorem 2]. However, this result does not tell us how large we should increase an approximation order l to obtain a solution that induces an ϵ-optimal value. To put approximation hierarchies into practical use, it is necessary to investigate how fast the gap between a true optimal value and its approximated value converges to zero as the approximation order l increases. When the underlying cones are nonnegative orthants, some studies discuss the relationship between the approximation order and the residual [3, 7, 21, 22, 109, 121].

The theoretical inclusion relationship between the BD-type and other generalized doubly nonnegative cones is of interest when the underlying cone $\mathbb{K}$ involves positive semidefinite cones. In addition, the case of $\mathbb{K}$ being neither a direct product of a nonnegative orthant and second-order cones nor a direct product of a nonnegative orthant and positive semidefinite cones is also intriguing. Seemingly, these are more challenging tasks than the cases we treated in Section 4.3 for several reasons. First, as exemplified in Example 4.4, the zeroth level of the ZVP-type inner-approximation hierarchy is not necessarily closed in general. This would preclude us from investigating the strict inclusion relationship between ZVP- and NN-type generalized doubly nonnegative cones in the same way as in Section 4.3. Second, although we can characterize NN-type generalized doubly nonnegative cones over general symmetric cones in the same way as (4.7), the characterization needs a closure operator in general. Third, when the underlying cone involves a matrix cone such as a positive semidefinite cone, the vectorization of a matrix and matrization of a vector are required to deal with the BD-type generalized doubly nonnegative cone. Consider the case where the underlying cone is the positive semidefinite cone $\mathrm{svec}(\mathbb{S}^n_+)$. In this scenario, a symmetric matrix $\boldsymbol{X} \in \mathbb{S}^{T_n}$ belongs to $\mathcal{DNN}_{\mathrm{BD}}(\mathrm{svec}(\mathbb{S}^n_+))$ only if $\mathrm{smat}(\boldsymbol{X}\,\mathrm{svec}(\boldsymbol{v}\boldsymbol{v}^\top)) \in \mathbb{S}^n_+$ for all $\boldsymbol{v} \in S^{n-1}$.[1] This condition appears to be much more difficult to verify theoretically than the discussion conducted in Section 4.3.1.

As introduced in Section 1.1.2, using the length of the longest chain of faces and the distance to polyhedrality, we can provide upper bounds for the Carathéodory number and singularity degree. However, the results of this thesis imply that the upper bounds are not exact in general. It is known that the Carathéodory number p_n of the standard completely positive cone $\mathcal{CP}^n$ satisfies $p_n = n$ if $n \le 4$ [6, Section 3.1] and $p_n \le T_n - 4$ if $n \ge 5$ [112, Corollary 5.2]. When $n \ge 2$, it is strictly smaller than its upper bound $\ell_{\mathcal{CP}^n} - 1 = T_n$. Furthermore, the singularity degree of conic linear feasibility problems over $\mathcal{DNN}^n$ does not exceed n [73, Corollary 20]. When $n \ge 3$, it is strictly smaller than its upper bound $\ell_{\mathrm{poly}}(\mathcal{DNN}^n) + 1 = T_n - 1$. On the other hand, the upper bounds may also be (nearly) exact. In the case of positive semidefinite cones, the Carathéodory number of the positive semidefinite cone $\mathbb{S}^n_+$ agrees with its upper bound $\ell_{\mathbb{S}^n_+} - 1 = n$ [59, Section 5.3], and the maximum singularity degree of conic linear feasibility problems over $\mathbb{S}^n_+$ is smaller than its upper bound $\ell_{\mathrm{poly}}(\mathbb{S}^n_+) + 1 = n$ only by 1 [73, page 2315]. It is an open problem to determine how tight the upper bounds for these quantities are for other cones.

In Chapter 6, we proved that $\mathcal{COP}(\mathbb{K})$ is not facially exposed for any symmetric cone $\mathbb{K}$ of dimension at least 2. However, the situation for $\mathcal{CP}(\mathbb{K})$, the dual of $\mathcal{COP}(\mathbb{K})$, is significantly less clear. Zhang [123, Theorem 3.4] showed that the standard completely positive cone $\mathcal{CP}^n$ is not facially exposed for $n \ge 5$. Surprisingly, if $\mathbb{K}$ is a single second-order cone, then $\mathcal{CP}(\mathbb{K})$ is facially exposed. This is because it can be expressed as the intersection of a positive semidefinite cone and a half-space, see (4.10), and the intersection of facially exposed cones is facially exposed. However, if $\mathbb{K}$ is a positive semidefinite cone, whether $\mathcal{CP}(\mathbb{K})$ is facially exposed or not seems to be unknown and

[1] Any extreme ray of $\mathbb{S}^n_+$ is generated by the matrix $\boldsymbol{v}\boldsymbol{v}^\top$ for some $\boldsymbol{v} \in S^{n-1}$. Conversely, for every $\boldsymbol{v} \in S^{n-1}$, the matrix $\boldsymbol{v}\boldsymbol{v}^\top$ generates an extreme ray of $\mathbb{S}^n_+$.

it would be an interesting question to explore.

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Appendix A

Explicit Exchange Method for BD-type Generalized Doubly Nonnegative Programming

For a symmetric cone $\mathbb{K}$, the algorithm used to solve a BD-type generalized doubly nonnegative programming problem obtained by replacing $\mathcal{CP}(\mathbb{K})$ in problem (4.17) with $\mathcal{DNN}_{\mathrm{BD}}(\mathbb{K})$ is presented in Algorithm 1. Although we preliminarily set γ_k to 0.5^k, in the branch of Step 1-2, if case (b) with $r = 0$ occurred five times in a row, γ_{k+1} was set to τ at the end of Step 2 and returned to Step 1 to accelerate convergence. The threshold value τ in Algorithm 1 was set to 10^{-5} and subset $\mathfrak{J}^{(0,0)}$ was set to $\emptyset$. We tried to identify $\boldsymbol{s}_{\mathrm{new}}^{(k,r)}$ using the following method. Before starting the algorithm, we fixed a finite subset $\mathfrak{J}_{\mathrm{fix}}$ of a compact set $\mathfrak{J}$ satisfying [1]

$$\mathrm{Ext}\,\mathbb{K} = \{\alpha \boldsymbol{s} \mid \alpha > 0,\ \boldsymbol{s} \in \mathfrak{J}\}. \tag{A.1}$$

In Section 4.4.1, because

$$\mathfrak{J} = \{(\boldsymbol{e}_i; \boldsymbol{0}_n) \mid i = 1, \dots, 2n+1\} \cup \left\{ \left(\boldsymbol{0}_{2n+1}; \frac{1}{2}; \frac{\boldsymbol{v}}{2} \right) \,\middle|\, \boldsymbol{v} \in S^{n-2} \right\}$$

satisfies (A.1), we took

$$\mathfrak{J}_{\mathrm{fix}} = \{(\boldsymbol{e}_i; \boldsymbol{0}_n) \mid i = 1, \dots, 2n+1\} \cup \left\{ \left(\boldsymbol{0}_{2n+1}; \frac{1}{2}; \frac{\boldsymbol{v}_i}{2} \right) \,\middle|\, i = 1, \dots, 1000 \right\},$$

where each $\boldsymbol{v}_i$ was generated from S^{n-2} randomly. In Section 4.4.2, because

$$\begin{aligned}\mathfrak{J} = \{(1; \boldsymbol{0}_{3T_n})\} &\cup \{(0; \mathrm{svec}(\boldsymbol{v}\boldsymbol{v}^\top); \boldsymbol{0}_{2T_n}) \mid \boldsymbol{v} \in S^{n-1}\} \\ &\cup \{(0; \boldsymbol{0}_{T_n}; \mathrm{svec}(\boldsymbol{v}\boldsymbol{v}^\top); \boldsymbol{0}_{T_n}) \mid \boldsymbol{v} \in S^{n-1}\} \cup \{(0; \boldsymbol{0}_{2T_n}; \mathrm{svec}(\boldsymbol{v}\boldsymbol{v}^\top)) \mid \boldsymbol{v} \in S^{n-1}\}\end{aligned}$$

satisfies (A.1), we took

$$\begin{aligned}\mathfrak{J}_{\mathrm{fix}} = \{(1; \boldsymbol{0}_{3T_n})\} &\cup \{(0; \mathrm{svec}(\boldsymbol{v}_i\boldsymbol{v}_i^\top); \boldsymbol{0}_{2T_n}) \mid i = 1, \dots, 1000\} \\ &\cup \{(0; \boldsymbol{0}_{T_n}; \mathrm{svec}(\boldsymbol{v}_i\boldsymbol{v}_i^\top); \boldsymbol{0}_{T_n}) \mid i = 1001, \dots, 2000\} \\ &\cup \{(0; \boldsymbol{0}_{2T_n}; \mathrm{svec}(\boldsymbol{v}_i\boldsymbol{v}_i^\top)) \mid i = 2001, \dots, 3000\},\end{aligned}$$

[1] For example, the set of primitive idempotents in a Euclidean Jordan algebra associated with $\mathbb{K}$ is a compact set satisfying (A.1) (see [34, Exercise IV.5] and [59, Corollary 12]).

Algorithm 1 Explicit exchange method for BD-type generalized doubly nonnegative programming.

Step 0. Choose a positive sequence $\{\gamma_k\}_k$ such that $\gamma_k \downarrow 0$ $(k \to \infty)$, a positive value τ, a compact subset $\mathfrak{J}$ of $\mathrm{Ext}\,\mathbb{K}$ such that $\mathrm{Ext}\,\mathbb{K} = \{\alpha \boldsymbol{s} \mid \alpha > 0,\ \boldsymbol{s} \in \mathfrak{J}\}$, and a finite subset $\mathfrak{J}^{(0,0)}$ of $\mathfrak{J}$. Solve $\mathrm{P}(\mathfrak{J}^{(0,0)})$ to obtain an optimal solution $\boldsymbol{X}^{(0,0)}$, where $\mathrm{P}(\mathfrak{J}')$ is defined as

$$
\mathrm{P}(\mathfrak{J}') : \quad \begin{array}{ll} \underset{\boldsymbol{X}}{\text{minimize}} & \langle \boldsymbol{C}, \boldsymbol{X} \rangle \\ \text{subject to} & \langle \boldsymbol{A}_i, \boldsymbol{X} \rangle = b_i \ (i = 1, \ldots, m), \\ & \boldsymbol{X} \in \mathbb{S}^n_+, \\ & \boldsymbol{X}\boldsymbol{s} \in \mathbb{K} \ (\forall \boldsymbol{s} \in \mathfrak{J}') \end{array}
$$

for a finite subset $\mathfrak{J}'$ of $\mathfrak{J}$. Set $k := 0$.

Step 1. Obtain $\boldsymbol{X}^{(k+1,0)}$ and $\mathfrak{J}^{(k+1,0)}$ by the following procedure.

Step 1-1. Set $r := 0$.

Step 1-2. Try to find $\boldsymbol{s}^{(k,r)}_{\mathrm{new}} \in \mathfrak{J}$ such that $\lambda_{\min}(\boldsymbol{X}^{(k,r)}\boldsymbol{s}^{(k,r)}_{\mathrm{new}} + \gamma_k \boldsymbol{e}) < 0$.

(a) If there exists such $\boldsymbol{s}^{(k,r)}_{\mathrm{new}}$, let $\bar{\mathfrak{J}}^{(k,r+1)} := \mathfrak{J}^{(k,r)} \cup \{\boldsymbol{s}^{(k,r)}_{\mathrm{new}}\}$ and solve $\mathrm{P}(\bar{\mathfrak{J}}^{(k,r+1)})$ to find an optimal solution $\boldsymbol{X}^{(k,r+1)}$ and optimal dual variables $\boldsymbol{v}^{(k,r+1)}_{\boldsymbol{s}}$ $(\boldsymbol{s} \in \bar{\mathfrak{J}}^{(k,r+1)})$. Let $\mathfrak{J}^{(k,r+1)} := \{\boldsymbol{s} \in \bar{\mathfrak{J}}^{(k,r+1)} \mid \boldsymbol{v}^{(k,r+1)}_{\boldsymbol{s}} \neq \boldsymbol{0}\}$. Set $r := r + 1$ and go to the beginning of Step 1-2.

(b) Otherwise, let $\boldsymbol{X}^{(k+1,0)} := \boldsymbol{X}^{(k,r)}$ and $\mathfrak{J}^{(k+1,0)} := \mathfrak{J}^{(k,r)}$.

Step 2. If $\gamma_k \leq \tau$, stop the algorithm. Otherwise, set $k := k + 1$ and go to Step 1.

---

where each $\boldsymbol{v}_i$ was generated from S^{n-1} randomly. In Step 1-2, we first checked whether there exists $\boldsymbol{s}_{\mathrm{new}} \in \mathfrak{J}_{\mathrm{fix}}$ such that $\lambda_{\min}(\boldsymbol{X}^{(k,r)}\boldsymbol{s}_{\mathrm{new}} + \gamma_k \boldsymbol{e}) < 0$ where $\lambda_{\min}(\cdot)$ is the minimum eigenvalue in the sense of Euclidean Jordan algebras [34, Theorem III.1.1]. If such $\boldsymbol{s}_{\mathrm{new}}$ exists, we let $\boldsymbol{s}^{(k,r)}_{\mathrm{new}} = \boldsymbol{s}_{\mathrm{new}}$. Otherwise, we next checked whether the optimal value of the optimization problem $\min_{\boldsymbol{s} \in \mathfrak{J}} \lambda_{\min}(\boldsymbol{X}^{(k,r)}\boldsymbol{s} + \gamma_k \boldsymbol{e})$ is less than 0. In addition, $\boldsymbol{v}^{(k,r+1)}_{\boldsymbol{s}}$ was regarded as $\boldsymbol{0}$ if $\|\boldsymbol{v}^{(k,r+1)}_{\boldsymbol{s}}\|_2 \leq 10^{-12}$.

Appendix B

On the Eleven Cases in the Proof of Lemma 6.3

In this appendix, we explain the reason why a quadruple (i, j, k, l) satisfying (6.10) must fall into exactly one of the eleven cases listed in the proof of Lemma 6.3. The case separation we describe next is based on checking how many among the indices (i, j, k, l) are equal. First we recall that (i, j, k, l) satisfies (6.10) if and only if one of the following conditions is satisfied

$$i \leq j, \ k \leq l, \ i = k, \ j \leq l, \tag{B.1}$$

or

$$i \leq j, \ k \leq l, \ i < k. \tag{B.2}$$

The first case is when all the indices i, j, k, l are equal, which corresponds to Case 1. We also note that (i, j, k, l) satisfying (6.10) falls into Case 1 if and only if $i = l$. In order to see that, we assume that $i = l$. From (B.1) and (B.2), we have $i \leq k$ which implies $l \leq k$, by assumption. Together with $k \leq l$, we obtain $i = k = l$. In addition, by $ij \preceq kl = ii$, we obtain $j \leq i$. Combining this with $i \leq j$ implies that $i = j$. Therefore, i, j, k, l are identical. In other words, unless i, j, k, l are identical, i is not equal to l.

Next, we examine what happens when the quadruple (i, j, k, l) is such that exactly three of i, j, k, l are identical. They must fall into exactly one of the following four cases:

(i-1) $j = k = l \neq i$ (Case 5);

(i-2) $i = k = l \neq j$;

(i-3) $i = j = l \neq k$;

(i-4) $i = j = k \neq l$ (Case 2).

If the quadruple (i, j, k, l) satisfies (i-2) or (i-3), as shown in the previous paragraph, i, j, k, l must be identical, which contradicts the assumption that exactly three indices are equal. Therefore, (i-2) and (i-3) do not occur.

Our next task is to consider the case where two among the i, j, k, l are identical, and the remaining indices are also identical, but the two pairs are different from each other. The quadruple (i, j, k, l) satisfying this condition must fall into exactly one of the following three cases:

(ii-1) $i = j \neq k = l$ (Case 3);

(ii-2) $i = k \neq j = l$ (Case 4);

(ii-3) $i = l \neq j = k$.

If the quadruple (i, j, k, l) satisfies (ii-3), i, j, k, l must be identical, which contradicts the assumption.

Next, the quadruple (i, j, k, l) such that exactly two among the i, j, k, l are identical must fall into exactly one of the following six cases: Cases 6 through 10, and the case where $i = l$ and i, j, k are different from each other. However, the last case of $i = l$ cannot occur since this implies that all the i, j, k, l are equal.

Finally, the quadruple (i, j, k, l) such that all the indices are different from each other corresponds to Case 11.