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INSTITUTE OF SCIENCE TOKYO

DOCTORAL THESIS

**Kinetic Approach to the Schwinger
Effect on the Expanding Universe
towards the Analysis of the
Inflationary Magnetogenesis
Scenario**

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*A thesis submitted in fulfillment of the requirements
for the degree of Doctor of Philosophy*

in the

Suyama Group

February 17, 2025

Declaration of Authorship

I, So OKANO, declare that this thesis titled, “Kinetic Approach to the Schwinger Effect on the Expanding Universe towards the Analysis of the Inflationary Magnetogenesis Scenario” and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.

Signed: SO OKANO

Date: 2025 / 02 / 17

"Thanks to my solid academic training, today I can write hundreds of words on virtually any topic without possessing a shred of information, which is how I got a good job in journalism."

Dave Barry

INSTITUTE OF SCIENCE TOKYO

*Abstract***Kinetic Approach to the Schwinger Effect on the Expanding Universe
towards the Analysis of the Inflationary Magnetogenesis Scenario**

In this thesis, we introduce the kinetic equation of the Schwinger effect, which is the non-perturbative particle production from the strong electric field background, in inflation with the charged particles and the subsequent inflaton oscillation era, considering the backreaction to the electric field and the inflaton dynamics. The inflation model with the electromagnetic field and the charged particles is broadly analyzed in terms of the inflationary magnetogenesis scenario, where the strong electromagnetic field can appear, and the Schwinger effect can occur. Applying the kinetic approach to that model, we obtain the time dependence of the particle production in the uniformly varying background, which in general cannot be known by other approaches. We show the time evolution of the system by the kinetic approach, which can be useful for the detailed analysis of the Inflationary Magnetogenesis.

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Dedicated to my late grandfather, Akira...

Chapter 1

Introduction

Magnetogenesis lies inside the domain of modern cosmology. Magnetic fields are ubiquitous in our universe. The magnetic field localized around the clusters of galaxies are the weakest and largest¹ in the directly observed magnetic field. The origin of this magnetic field is often discussed and still not cleared so far. One of the plausible hypotheses is the inflationary magnetogenesis scenario. In this scenario, the inflation mechanism amplified the quantum fluctuations of the electromagnetic field to the large-scale magnetic field, which became the “seed” of the galactic magnetic field. Recent multi-frequency blazar observations imply the existence and the lower bound of the larger and weaker magnetic field than the galactic magnetic field in the intergalactic void region [1] and this result supports the inflationary magnetogenesis since the other hypotheses do not generate the large scale and strong magnetic field.

However, inflationary magnetogenesis has some difficulties in producing a magnetic field above the lower bound implied by the observation. One of the difficulties originates from the fact that the electromagnetic field in the standard model has conformal symmetry. Inflation is often described in the de Sitter spacetime, which is obtained from the Minkowski spacetime by the conformal transformation. In the context of a standard photon field, because of the conformal symmetry, the de Sitter spacetime is equivalent to the flat spacetime and cannot be amplified by inflation. Therefore we should assume the beyond standard model (BSM) coupling to the electromagnetic field violates the conformal symmetry. For example, we can construct a simple magnetogenesis scenario using a BSM coupling as $I^2 F_{\mu\nu} F^{\mu\nu}$ [2], where I and $F_{\mu\nu}$ are the kinetic coupling function and electromagnetic field strength, respectively. In this scenario, by assuming the backreaction on spacetime and the inflaton to be negligible, a magnetic field satisfying the observational constraints is generated. Nonetheless, since the energy density of the electromagnetic field generated in this way becomes comparable to that of the inflaton, it contradicts the assumption of neglecting the backreaction. For this reason, even assuming the BSM coupling, it is not sufficient to rely on a simple scenario setup. A magnetic field generation scenario that considers the backreaction must be constructed. This property applies not only to the coupling presented in the example but also to all models. Moreover, when the electromagnetic field is generated with an energy density comparable to

¹The size of the magnetic field is determined by the correlation length.

that of the inflaton, the Schwinger effect will also need to be considered [3, 4], which is a non-perturbative pair production of the charged particle from the strong electromagnetic field. The contribution of the Schwinger effect significantly influences the generation of the magnetic field, and until the analysis that incorporates the Schwinger effect is conducted, it remains unclear whether the inflationary magnetogenesis scenario can adequately explain the origin of the “seed” magnetic field. The production of the charged particles affects the evolution of the electromagnetic field, and the Schwinger effect including the backreaction to the electromagnetic field should be taken into account in the analysis. Furthermore, both the backreaction and the Schwinger effect should be considered dynamically. According to refs. [5–7], when the amplification of the electric field (without magnetic field) during inflation is considered, dynamically taking into account both the backreaction and the Schwinger effect, it was found that the charged particles generated by the Schwinger effect could immediately dominate the universe after the end of inflation. In the analysis presented in ref. [6, 7] and chapter 4 of this thesis, it has been shown that this phenomenon arises from the dynamical interaction. Since this phenomenon implies that the reheating of the universe also depends on the charged particles generated by the Schwinger effect, it must be taken into account when constructing a scenario that explains the primordial magnetic field. Therefore, the ultimate goal of this research is to construct the inflationary magnetogenesis scenario, with the dynamical analysis of spacetime, and electromagnetic field, incorporating the backreaction and Schwinger effect. However, it is a complicated task and various approaches have been discussed [3, 8–20]. The difficulties of the analysis arise from the time dependence of the electric field and the curved spacetime.

In fact, even in flat spacetime, the Schwinger effect in a general dynamical electric field is difficult to analyze. For the dynamical analysis, the particle production rate depending on time is given by performing the spatial integration including the mode functions of the charged particles which are obtained by solving the Dirac equation with the dynamical electromagnetic field. At the same time, we should consider the backreaction of the charged particles to the electromagnetic field. Performing the numerical or analytical calculation along the above discussion is highly costed. The difficulty in evaluating the particle production is attributed to the process of finding the mode functions. If there exists an equation that directly relates the particle production rate with the electric field, we can greatly reduce the computational cost. In the kinetic approach, where we solve the time evolution equation for the one-particle distribution functions or corresponding other statistical functions instead of the equation for the elementary field, the direct relation between the electromagnetic field and the particle production rate appears. Since the kinetic approach makes the numerical analysis for the time evolution of the charged particles low-cost, this approach provides the most convenient route to our final goal. Wigner formalism [21, 22], where we solve the time evolution of the Wigner function, also gives us the relation between the electric field and the particle production rate. This formalism is applied

for a more general background than the kinetic approach, but it is more difficult to numerically solve equations. Therefore in this thesis, we will take the kinetic approach. Quantum kinetic approach [5, 23–29], where the time evolution of the momentum distribution function of the charged particles is described by the quantum Boltzmann equation, is equivalent to the Wigner formalism for the uniform and dynamical electric field background. As we see in the following chapter, the source term derived by the field-theoretic method, obtained from the first principle, in the time evolution equation is non-Markovian, where we cannot extract the information of the state at some time slice only by the one before. This feature becomes an obstacle to performing the long-term analysis of the quantum kinetic approach. In the case of the low-density and the higher energy of the particle than an electric field, the source term becomes Markovian [25].

As the first step to the application of the kinetic approach to the inflationary magnetogenesis model, we consider the phenomenologically obtained Markovian source term which is equivalent to the source term derived by the field theoretical method at the constant electric field, and we focus on the inflation model with the kinetic coupling discussed in the above sentences, $I(\phi)^2 F^{\mu\nu} F_{\mu\nu}/4$, where ϕ is the inflaton. This inflation model can generate a homogeneous electric field. By the numerical calculation, we have the time evolution of the system including produced charged particles, which in general cannot be obtained by other approaches. This model, which does not consider the magnetic field, is not a scenario for explaining magnetogenesis, but it does reveal several interesting phenomena. First, the produced particles and the electric field begin to oscillate after inflation. Second, the produced charged particles temporarily dominate the universe after inflation, which is a phenomenon we refer to as Schwinger preheating. These two phenomena are not observed in other approaches which focus on the current. Refs. [6, 7] and chapter 4 of this thesis provide the numerical analysis of the kinetic coupling model in the inflationary background, incorporating both the Schwinger effect and the backreaction. In our research, we additionally consider anisotropy as a backreaction of the uniform electric field in spacetime. Although the effect of anisotropy is considered to be small from the equations of motion, it has the effect of accelerating the end of inflation. Since a large number of charged particles are produced near the end of inflation, it may influence the Schwinger preheating. We also take into account the influence of the thermal mass of the charged particles. Then we find the necessary condition for the Schwinger preheating to happen and numerically analyze the effect of the thermal mass and anisotropy.

The following contents will be discussed in each of the succeeding chapters.

In chapter 2, we briefly review inflation and some representative models of inflation. Then we introduce inflationary magnetogenesis. In this section, we explain the observational result for the implication of a magnetic field in a void, which supports inflationary magnetogenesis. We also show two types of inflationary magnetogenesis models. One is the kinetic coupling model, which is used in the following chapter, and the other is the hybrid model,

which seems to be a successful model but this model does not include the Schwinger effect.

In chapter 3, we provide a short review of the proper time method which is a useful method to analyze the Schwinger effect and derive the reaction rate in the static background. Then we explain another method for the Schwinger effect, where we use the Bogoliubov transformation. After that, we explain the kinetic approach with the Schwinger source term derived from the theoretical/phenomenological method.

In chapter 4, we analyze the dynamical Schwinger effect on the inflationary background. Although this will not be developed in this thesis, it is essential for the ultimate goal of analyzing the inflationary magnetogenesis with consideration of the dynamical Schwinger effect. We briefly review the Schwinger effect in the de-Sitter spacetime. Subsequently, we explore the kinetic approach to the inflation model in the uniform electric field with the Schwinger effect. Building upon refs. [6, 7], this thesis presents new findings, including: 1) The potential does not influence the occurrence of Schwinger preheating in the Ratra coupling model, 2) Necessary condition for the Schwinger preheating, 3) The impact of thermal mass, and 4) The effects of anisotropy.

Finally, in chapter 5, we summarize the thesis and discuss the extension of the kinetic approach to the more general background case including the magnetic field. Then in the context of inflationary magnetogenesis, we address the unresolved issues related to the Schwinger effect. Appendices A give us the analytical details.

Chapter 2

Inflationary Cosmology

2.1 Introduction to the Inflationary Cosmology

The standard cosmology model can well explain a lot of phenomena of the universe. In particular inflation [30–32], namely, the period of the accelerating expansion of spacetime at the early stage of the universe, has been one of the main topics of cosmology and it has two major advantages to consider. One is that inflation solves the serious problems of the standard cosmology known as the horizon problem, the flatness problem, and the monopole problem. The other one is that inflation can make seeds of the structure of the universe as cosmological primordial perturbations.

2.1.1 Motivation of inflation

Homogeneous and isotropic universe

Let us introduce the Friedmann–Lemaître–Robertson–Walker (FLRW) universe based on the cosmological principle, which is the assumption that the universe seems to be homogeneous and isotropic on a sufficiently large scale. The FLRW metric is given as,

$$ds^2 = dt^2 - a^2(t) \left[dx^2 + K \frac{(x \cdot dx)^2}{1 - Kx^2} \right] = a^2(t) \left[d\eta^2 - dx^2 + K \frac{(x \cdot dx)^2}{1 - Kx^2} \right], \quad (2.1)$$

where a and K represent the scale factor of the spacetime and the spatial curvature, respectively. $\eta = \int \frac{dt}{a(t)}$ is the comoving time. The spacetime becomes flat FLRW when $K = 0$ ¹.

There exists the matter field in the universe. The distribution of the matter field and the spacetime metric are associated with the Einstein equation. We consider the matter field as the perfect fluids substituting the energy-momentum tensor of the matter into the Einstein equation, and we obtain

¹The sign of K determines the sign of the curvature. The 3-dimensional space is spherical for $K = 1$, and hyperbolic for $K = -1$.

the equations of the spacetime as,

$$H^2 = \frac{\rho}{3M_{\text{Pl}}^2} - \frac{K}{a^2} + \frac{\Lambda}{3}, \quad (2.2)$$

$$\frac{\ddot{a}}{a} = -\frac{\rho + 3p}{6M_{\text{Pl}}^2} + \frac{\Lambda}{3}. \quad (2.3)$$

Here ρ , p , and Λ represent the energy density, pressure of the perfect fluids, and the cosmological constant, respectively. $H = \frac{\dot{a}}{a}$, and M_{Pl} are the Hubble parameter and the reduced Planck mass, respectively. Eq. (2.2) is known as the Friedmann equation, which denotes energy conservation. We can derive the fluid equation by removing $\ddot{a}$ from Eqs. (2.2) and (2.3) as,

$$\dot{\rho} = -3H(\rho + p). \quad (2.4)$$

The above equation is transformed as follows,

$$\frac{d}{dt}(\rho a^3) = -p \frac{d}{dt}(a^3). \quad (2.5)$$

This equation denotes the time evolution of the energy per unit comoving volume. From this equation, we regard the spacetime expansion as adiabatic. To solve Eqs. (2.2) and (2.3), we need the equation of state (EOS) for the matter field, as

$$p = w\rho, \quad (2.6)$$

where w is the EOS parameter which determines the behavior of the matter field. The matter field behave as radiation for $w = \frac{1}{3}$, non-relativistic for 0, and the cosmological constant for $w = -1$. Thanks to the EOS, we can solve the Equations² as,

$$\rho \propto a^{-3(1+w)}. \quad (2.7)$$

When we assume the flat FLRW spacetime, we solve the spacetime equations about $a(t)$ as,

$$a \propto \begin{cases} t^{\frac{2}{3(1+w)}} & (w \neq -1), \\ \exp(Ht) & (w = -1). \end{cases} \quad (2.8)$$

²Here we assume that a simple matter field dominates the universe.

Definition of inflation

Cosmic inflation is the period of the accelerated expansion of the universe before the Big bang nucleosynthesis as,

$$\ddot{a} > 0 \leftrightarrow \frac{d}{dt} \left(\frac{1}{aH} \right) < 0 \quad (t < t_{\text{BBN}}), \quad (2.9)$$

where (R.H.S) of the above equation denotes the comoving Hubble horizon decrease during inflation. We rewrite the condition of inflation by the Hubble parameter,

$$\epsilon_H \equiv -\frac{\dot{H}}{H^2} < 1 \quad (2.10)$$

where we introduce the slow-roll parameter, ϵ_H . When we consider the extreme situation, in other words $\epsilon_H \rightarrow 0$, the scale factor is solved as $a \propto \exp(Ht)$. In this case, the spacetime becomes the $K = 0$ flat chart of de-Sitter spacetime. From Eq (2.3), we derive the condition of inflation for the matter field as,

$$w = \frac{p}{\rho} < -\frac{1}{3}, \quad (2.11)$$

where we restrict the sign of an energy density is always plus.

Horizon problem

The horizon problem claims the fine-tuning of the initial condition for the standard cosmology. The temperature of the whole sky universe measured by the CMB observation takes the same value as $T = 2.725 \pm 0.002K$ even if the areas depart from the particle horizon estimated in the standard cosmology. The particle horizon which denotes the length of the light trajectory from $t=0$ to $t, l_{ph}(t)$, is defined as,

$$l_{\text{PH}}(t) \equiv a(t) \int_0^t \frac{dt'}{a(t')}. \quad (2.12)$$

In addition, we define the physical length to the earth at the given time t as,

$$d(t) \equiv a(t) \int_t^{t_0} \frac{dt'}{a(t')}, \quad (2.13)$$

where the index 0 means the present time. Taking the ratio of the $a(t)l_{\text{PH}}(t)$ and $d(t)$ at $t = t_L$, when the CMB photon emitted to the Earth³, we obtain the maximum angle between the causally connected two points in the sky as $\Delta\theta \sim 1.7^\circ$. Here we assume that the matter field dominates the universe. Thus the number of uncorrelated regions is $\frac{4\pi}{\pi(1.7^\circ)^2} \sim 4.5 \times 10^3$, and these

³ $z(t_L) = 1100$, which is the redshift at $t = t_L$.

take the same initial condition with an accuracy of 10^{-4} . This fine-tuning is removed by considering inflation. When we assume the exponential expansion occurs during inflation, $a \propto e^{-Ht}$, we estimate the particle horizon at the last scattering as,

$$l_{\text{PH}}(t_L) \simeq \frac{a(t_L)}{a_e H_e} [e^N - 1], \quad (2.14)$$

where the index e denotes the inflation end. $N \equiv \ln(a_e/a_i)$ represents the e-folding number, where the index i denotes the initial point of inflation. Since the fine-tuning is removed when $a(t_L)l_{\text{PH}}(t_L) > d(t_L)$, the e-folding number should satisfies the following inequality as,

$$N > \ln \left(\frac{a_e H_e}{a_0 H_0} \right). \quad (2.15)$$

The typical value of the e-folding number satisfying the above inequality is 60.

Flatness problem

We introduce the energy fraction $\Omega \equiv \frac{\rho}{3M_{\text{Pl}}^2 H^2}$ in the Friedmann equation (2.2), we have the fraction equation as follows,

$$\Omega - 1 = \frac{K}{a^2 H^2} \equiv \Omega_K, \quad (2.16)$$

where ρ includes the radiation, non-relativistic matter, and the cosmological constant. From the CMB observation by the Planck satellite and the galaxy survey, Ω_K is constrained as [33],

$$\Omega_K = 0.0007 \pm 0.0037 (95\% \text{CL}), \quad (2.17)$$

where the above result includes CMB lensing and BAO. Ω_K is small but the standard cosmological model does not explain why the contribution of the curvature to the total energy is small. Since the scale factor dependence of curvature fraction is $\Omega_K \propto a^{1+3w}$, the flatness problem is solved when inflation occurs for a long time.

Monopole problem

In the grand unified theory, some simple local symmetry breaks into the gauge symmetry of the standard model belongs to $SU(3) \times SU(2) \times U(1)$ at the energy scale around $M \sim 10^{16} \text{GeV}$. From the symmetry breaking, the twisted phase of the scalar field generates the magnetic monopole as the topological defect. If the phase transition occurred in the early universe, the many produced magnetic monopoles may survive today, but such magnetic monopoles are not observed so far. This is the statement of the monopole problem. This problem can be also solved by inflation. Since the phase of the

scalar is equal when the distance of the two points is smaller than l_{PH} , the number of the monopole is one in the particle horizon. When the inflation occurs after the symmetry breaking, the number density of the monopole is diluted by the exponential spacetime expansion. On the other hand, when the inflation occurs before the symmetry breaking, the particle horizon is exponentially expanded and the number of the monopole becomes small. Therefore the monopole problem can be resolved by inflation.

2.1.2 Dynamics of inflation

In the previous section, we confirm that inflation can solve the three problems in the standard cosmology. In this section, we consider how to cause the acceleration of the spacetime and how to decide the end of inflation. For this purpose, we introduce the action of the inflaton field and slow-roll parameters. In this section, we assume the FLRW spacetime.

Minimally coupled scalar action

The action of the scalar field, ϕ , is given by,

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right], \quad (2.18)$$

where $V(\phi)$ is the potential, and the scalar field which causes inflation is called inflaton. From the above action, we obtain the energy-momentum tensor of the scalar as,

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_\phi}{\delta g^{\mu\nu}} = \partial_\mu \phi \partial_\nu \phi + g_{\mu\nu} \left(-\frac{1}{2} \partial_\alpha \phi \partial^\alpha \phi - V(\phi) \right). \quad (2.19)$$

Since we assume the FLRW metric, we can associate the scalar field and the perfect fluids as $T_{00} = \rho$ and $T_{ii} = a^2 p$. Then the EOS parameter of the scalar field, w_ϕ is given by,

$$w_\phi = \frac{p_\phi}{\rho_\phi} = \frac{\frac{1}{2} [\dot{\phi}^2 + a^{-2} (\partial_i \phi)^2] - V(\phi)}{\frac{1}{2} [\dot{\phi}^2 + a^{-2} (\partial_i \phi)^2] + V(\phi)} = \frac{K_\phi - V}{K_\phi + V}, \quad (2.20)$$

where K_ϕ is the kinetic energy of the scalar field. From the condition of inflation for the matter field, Eq. (2.11), we obtain $K_\phi < \frac{1}{2} V(\phi)$ as a condition for inflation. Since we assume the global homogeneity of the universe, the spatial dependence of the scalar field is ignored. Thus we rewrite the inflation condition as $\dot{\phi}^2 < V(\phi)$. The exponential expansion of the spacetime occurs when the scalar field satisfies,

$$\dot{\phi}^2 \ll V(\phi). \quad (2.21)$$

This is called the slow-roll approximation.

Slow-roll parameters

Let us introduce the slow-roll parameters from the discussion so far. In Eq.(2.10), we define ϵ_ϕ as the slow-roll parameter based on the spacetime. Here we define the slow-roll parameters of the scalar field. The equation of motion of the scalar field and the Friedmann equation are given as,

$$\ddot{\phi} + 3H\dot{\phi} = -V'(\phi), \quad (2.22)$$

$$3M_{\text{Pl}}^2 H^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi). \quad (2.23)$$

We can remove the potential from the above two equations and obtain,

$$\dot{\phi}^2 = -2M_{\text{Pl}}^2 \dot{H}. \quad (2.24)$$

By using the slow-roll condition for the scalar field in Eq. (2.21), we can ignore the acceleration of the scalar field, and therefore we have the approximated equations from Eq. (2.23) as,

$$3M_{\text{Pl}}^2 H^2 \simeq V(\phi), \quad 3H\dot{\phi} \simeq -V'(\phi). \quad (2.25)$$

From these equations, we can rewrite the slow-roll condition by the potential of the scalar field as,

$$\epsilon_V \equiv \frac{M_{\text{Pl}}^2}{2} \left(\frac{V'}{V} \right)^2 \ll 1, \quad \eta_V \equiv M_{\text{Pl}}^2 \frac{V''}{V} \ll 1, \quad (2.26)$$

where we take the time derivative of Eq. (2.22) to derive the second condition. In case the slow-roll condition is satisfied, we can represent the equations of the system by slow-roll parameters as,

$$\dot{\phi}^2 \simeq \frac{2V}{3}\epsilon, \quad \ddot{\phi} \simeq H\dot{\phi}(\epsilon - \eta), \quad \dot{H} \simeq -H^2\epsilon. \quad (2.27)$$

2.1.3 Perturbation of the field

Here we review the perturbations of the scalar and vector field in the model-independent way on the flat spacetime. We confirmed that inflation could resolve the problems of Big Bang cosmology, but we did not claim how we tested the inflation models. Actually, we can observe the remnants of inflation as the fluctuation of the CMB spectra. The fluctuations are generated from the perturbations of the scalar, vector, and tensor fields amplified by the inflation mechanism. These perturbations give the initial conditions for the structure formation of the universe. In this section, we review the basics of the amplification of the quantum fluctuation in the inflationary background. Then we only consider the free scalar perturbation here. The other type of perturbation is considered in sec.2.4.1.

Scalar perturbation

We briefly review the amplification of the scalar fluctuation during inflation. The inflaton lagrangian is given in Eq. (2.18). Here we consider the first order perturbation of the scalar field, and we do not ignore the spatial derivative for the scalar field⁴. Let us consider the simplest case, where the classical action of the free scalar field in the spatially flat FLRW spacetime is written as

$$S = \frac{1}{2} \int d\eta d^3x a^2 \left[\phi'^2 - (\partial_i \phi)^2 - a\mu^2 \phi^2 \right]. \quad (2.28)$$

where $'$ denotes the derivative with respect to the conformal time, η . μ is the mass of the scalar field. By redefining the fluctuation as $\varphi \equiv a\phi$, and taking the Fourier transformation for the scalar field, we obtain the following action,

$$S = \int \frac{d\eta d^3k}{(2\pi)^3} \left[\varphi'_k \varphi'_{-k} - \left(k^2 - \frac{a''}{a} + \mu^2 a^2 \right) \varphi_k \varphi_{-k} \right], \quad (2.29)$$

where φ_k is the k-mode Fourier component of φ defined as,

$$\varphi(\eta, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k} \cdot \mathbf{x}} \varphi_k(\eta). \quad (2.30)$$

We obtain the canonical EoM for the scalar field in the case of the de-Sitter spacetime, $a = -(\eta H)^{-1}$, as

$$\varphi''_k + \left[k^2 - \left(2 - \frac{\mu^2}{H^2} \right) \frac{1}{\eta^2} \right] \varphi_k = 0. \quad (2.31)$$

The general solution of the above EoM is written as,

$$\varphi_k(\eta) = a_k u_k(\eta) + a_{-k}^* u_k^*(\eta), \quad (2.32)$$

where u_k , a_k , and a_{-k}^* are the mode function, two coefficients for the mode function, respectively. Let us quantize φ by imposing the canonical commutation relation for the coefficients,

$$[a_k, a_{-k'}^\dagger] = (2\pi)^3 \delta(\mathbf{k} + \mathbf{k}'), \quad [a_k, a_{k'}] = [a_k^\dagger, a_{k'}^\dagger] = 0, \quad (2.33)$$

where $a_k^\dagger$ and a_k are the creation and annihilation operators, respectively. By using the above commutation relation, we confirm the following commutation relations for the φ and conjugate momentum $\Pi = \varphi'$,

$$[\hat{\varphi}(\eta, \mathbf{x}), \hat{\Pi}(\eta, \mathbf{y})] = i\delta(\mathbf{x} - \mathbf{y}), \quad [\hat{\varphi}(\eta, \mathbf{x}), \hat{\varphi}(\eta, \mathbf{y})] = [\hat{\Pi}(\eta, \mathbf{x}), \hat{\Pi}(\eta, \mathbf{y})] = 0, \quad (2.34)$$

⁴Since the scalar perturbation has the spatial dependence, we should also consider the metric perturbation for the consistency, but here we impose that the inflaton fluctuation does not affect the background spacetime for the simplicity and ignore the metric perturbation.

where we used the normalization condition for the mode function as $u_k u_k^{*'} - u_k^* u_k' = i$, which is the Wronskian. We should determine the initial condition to solve the EoM for the mode function, eq. (2.31). At the past infinity, the scale factor goes to 0 and the EoM can be written as,

$$u_k'' + k^2 u_k = 0, \quad (2.35)$$

where the scalar field does not feel the gravity at the past infinity. Here we assume that the scalar field is in vacuum ($a_k |0\rangle = 0$ for all k). Then we can specify the initial condition for the mode function,

$$u_k^{\text{BD}}(\eta) \equiv \lim_{\eta \rightarrow \infty} u_k(\eta) = \frac{1}{\sqrt{2k}} e^{ik\eta}. \quad (2.36)$$

The above vacuum solution is called the Bunchi-Davies vacuum. By using the initial condition, we can solve the EoM for the mode function,

$$u_k^*(\eta) = \frac{\sqrt{-\pi\eta}}{2} [J_n(-k\eta) - iY_n(-k\eta)], \quad (2.37)$$

where $n \equiv \sqrt{\frac{9}{4} - \frac{\mu^2}{H^2}}$. J_n and Y_n are the Bessel functions. It is meaningful to see the behaviors of the solution in the extreme case. In the case of $k\eta \rightarrow 0$, which is called in the super-horizon, the Bessel functions become $J_n(-k\eta) \rightarrow \Gamma(n+1)^{-1}(-k\eta/2)^n \rightarrow 0$ and $Y_n(-k\eta) \rightarrow -\frac{\Gamma(n)}{\pi}(-2/k\eta)^n$, and therefore the solution asymptotically goes to

$$u_k^*(\eta) \rightarrow \frac{i\Gamma(n)}{2} \sqrt{\frac{-\eta}{\pi}} \left(\frac{2}{-k\eta} \right)^n (k\eta \rightarrow 0). \quad (2.38)$$

The dimensionless power spectrum of the scalar fluctuation is,

$$\mathcal{P}_\phi \equiv \frac{k^3}{(2\pi)^2} \langle |\hat{\phi}|^2 \rangle = \frac{H^2}{8\pi} (k\eta)^3 [J_n(-k\eta)^2 + Y_n(-k\eta)^2], \quad (2.39)$$

$$\rightarrow \left(\frac{H}{2\pi} \right)^2 \left(\frac{k\eta}{2} \right)^{\frac{2\mu^2}{3H^2}} (k\eta \rightarrow 0). \quad (2.40)$$

If the mass of the scalar field is much smaller than the Hubble parameter, the power spectrum of the scalar perturbation becomes scale invariant. In the sub-horizon, the scalar operator behaves as,

$$\lim_{k\eta \rightarrow -\infty} \hat{\phi}_k = \frac{1}{\sqrt{2k}} [a_k e^{-ik\eta} - a_{-k}^\dagger e^{ik\eta}], \quad (2.41)$$

where the eigenstates change from time to time. On the other hand in the super-horizon, the scalar operator behaves as,

$$\lim_{k\eta \rightarrow 0} \hat{\phi}_k = \frac{i\Gamma(n)}{2} \sqrt{\frac{-\eta}{\pi}} \left(\frac{2}{-k\eta} \right)^n [a_k - a_{-k}^\dagger], \quad (2.42)$$

where the eigenstates of the scalar operator remain in the same eigenstates even after time evolution. The super-horizon modes behave like classical numbers and they are considered as the seed of the structure of our universe. During inflation, the physical horizon scale, aH , becomes large and the short wavelength (in other words, large k) modes cross the horizon one after another. This is why the quantum fluctuation can be amplified by inflation. This is the simplest case of the generation of the perturbation by inflation, but the method is almost the same as the other type of perturbation. Firstly we find the EoM for the perturbation and impose the commutation relations on the operators for the quantization. Then we set the initial condition and solve the EoM for the perturbation. In sec.2.4.1, we consider the amplification of the fluctuation of the electromagnetic field in the spatially flat FLRW spacetime.

2.2 Model Analysis of Inflation

In the last section, we considered the mechanism of inflation and the slow-roll conditions for the minimally coupled scalar field in a model-independent way. Here we apply the previous analysis for the three concrete models. One is called the chaotic inflation [34] model, which has the quadratic potential. The second one is called the Starobinsky inflation [30], which has the curvature-squared corrections to the gravity action as a special case of $f(R)$ modified gravity, and this model. The Starobinsky inflation is equivalently transformed to the inflation with the minimally coupled scalar field by the conformal transformation [35], which has the special form of the potential. The latter model is consistent with the CMB observation. On the other hand, the former model is almost rejected, confirmed in Ref. [33] but the potential is quite simple and this model tells us the basics of the inflaton dynamics. The last one is the Watanabe-Kanno-Soda (WKS) inflation model [36], where inflaton field coupled to the uniform electric field. Since the electric field determines the special direction in the spacetime, the isotropy of the spacetime is broken. Then the FLRW spacetime is no longer a solution for this setup. Instead of the FLRW spacetime, we apply the Bianchi type I spacetime [37] which is the homogeneous but anisotropic solution of the spacetime. The anisotropy contributes to the CMB observation. The WKS model is almost excluded by the CMB constraint when we consider the quantum effect in the attractor solution [38].

2.2.1 Chaotic model

The potential of the chaotic inflation is given as,

$$V(\phi) = \frac{1}{2}\mu^2\phi^2, \quad (2.43)$$

where μ is the mass of the inflaton. We can consider the general potential as $V_n = \frac{\lambda}{n} M_{\text{Pl}}^{4-\alpha} \phi^\alpha$. Since the potential has no local minimum when the α takes

odd numbers, we only consider α as even numbers. Let us estimate the slow-roll parameters for this potential. The derivatives of the potential are given as,

$$V' = \mu^2 \phi, \quad V'' = \mu^2, \quad (2.44)$$

and we obtain the slow-roll parameters as,

$$\epsilon_V = \eta_V = \frac{2M_{\text{Pl}}^2}{\phi^2}. \quad (2.45)$$

Since inflation finishes when the slow-roll parameters become unity, the end-point of inflation is given as $\phi_{\text{end}} = \sqrt{2}M_{\text{Pl}}$. The e-folding number, N , is estimated by,

$$N \equiv \ln \left(\frac{a_{\text{end}}}{a} \right) = \int_t^{t_{\text{end}}} H dt \sim \frac{1}{M_{\text{Pl}}^2} \int_{\phi_{\text{end}}}^{\phi} \frac{V'}{V} d\phi, \quad (2.46)$$

where we used the slow-roll approximations, Eqs. (2.25). Then by introducing the potential, Eq. (2.43) into the above equation, we obtain the e-folding number in this model as,

$$N(\phi) \simeq \frac{\phi^2}{4M_{\text{Pl}}^2} - \frac{1}{2}. \quad (2.47)$$

Since the typical value of N is 60, the inflaton takes $\phi(t_i) \sim 16M_{\text{Pl}}^2$ at the beginning of the inflation, $t = t_i$. The beginning of the chaotic inflation is explained as follows. Before inflation, $t \sim M_{\text{Pl}}^{-1}$, there exist quantum fluctuations of the inflaton field, which have the high energy density, $\rho_\phi \sim M_{\text{Pl}}^4$, because of the uncertainty relation of energy and time. The energy density of the inflaton consists of the kinetic energy and the potential energy, whose values are the same order with M_{Pl}^4 . Since the mass parameter of the inflaton is much smaller than the Planck mass, the field value of the inflaton is estimated as $\phi \sim (M_{\text{Pl}}/\mu)M_{\text{Pl}} \gg M_{\text{Pl}}$. Therefore at the beginning of inflation, the slow-roll condition is satisfied and inflation occurs regardless of the tuning of the initial conditions.

2.2.2 Starobinsky model

In this section, we show the R^2 -correction gives rise to the exact de Sitter solution, and by using the conformal transformation, we derive the minimally coupled scalar field with the special form of potential. The $f(R)$ -modified theory given as,

$$S = -\frac{M_{\text{Pl}}^2}{2} \int d^4x \sqrt{-g} f(R) + \int d^4x \mathcal{L}_M(g_{\mu\nu}, \Psi_M) \quad (2.48)$$

where $\mathcal{L}_M(g_{\mu\nu}, \Psi_M)$ denotes the lagrangian for the matter field. The function $f(R)$ is the arbitrary function of the Ricci scalar. Taking the variation with respects to $\delta g^{\mu\nu}$, we have the geometric equation as,

$$\varphi R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu \nabla_\nu \varphi + g_{\mu\nu} \square \varphi = \frac{1}{M_{\text{Pl}}^2} T_{\mu\nu} \quad (2.49)$$

Here we introduced $\varphi \equiv f'(R)$. By taking the trace of the subscripts for the spacetime, we obtain the equation for the φ as,

$$\square \varphi + \frac{R}{3} \varphi - \frac{2}{3} f(R) = \frac{1}{3M_{\text{Pl}}^2} T. \quad (2.50)$$

Let us consider the vacuum solution, $T = 0$. We assume the de Sitter solution where the Ricci scalar is constant, and the above equation becomes,

$$\frac{R}{3} \varphi - \frac{2}{3} f(R) = 0 \rightarrow f(R) = \alpha R^2. \quad (2.51)$$

Then, the R^2 - correction term to the Einstein-Hilbert action can cause inflation. If we set $f(R) = R + \alpha R^2$ and we can rewrite Eq. (2.50) as,

$$\square \varphi - \frac{\varphi}{6\alpha} = \frac{1}{6\alpha M_{\text{Pl}}^2} T. \quad (2.52)$$

Then Eq. (2.52) is the EoM for the φ with $\mu^2 = 1/6\alpha$ as the mass parameter of φ . The Starobinsky inflation model is given by the special case of the $f(R)$ -modified theory as,

$$f_s(R) = R + \frac{R^2}{6\mu^2}. \quad (2.53)$$

The $f(R)$ - modified theory is equivalent to the Einstein-Hilbert theory with the minimally coupled scalar field to the gravity given by the conformal transformation on the $f(R)$ theory. This equivalence is shown in terms of one-loop correction [39] and the curvature perturbation [40]. First, we rewrite the action by the Legendre transformation as,

$$S_A = -\frac{M_{\text{Pl}}^2}{2} \int d^4x \sqrt{-g} [RA - Z(A)], \quad (2.54)$$

where $Z(A) = RA - f(R)$, $R = Z'(A)$, and $A = f'(R)$. We consider the conformal transformation as,

$$g_{\mu\nu} \mapsto \Omega^{-2} g_{\mu\nu}^{(E)},$$

$$\Omega(\varphi)^2 \equiv \exp \left(\sqrt{\frac{2}{3}} \frac{\varphi}{M_{\text{Pl}}} \right), \quad (2.55)$$

where the superscript E denotes the transformed frame. Then Eq. (2.54) is written as,

$$\begin{aligned} S_A &= \frac{1}{2\kappa} \int d^4x \sqrt{-g^{(E)}} \left[\Omega^2 R - Z(A) \right], \\ &= \int d^4x \sqrt{-g^{(E)}} \left[-\frac{M_{\text{Pl}}^2}{2} R^{(E)} + \frac{1}{2} g^{(E)\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + \frac{M_{\text{Pl}}^2}{2} Z(A(\varphi)) \exp \left(-\frac{4}{\sqrt{6}} \frac{\varphi}{M_{\text{Pl}}} \right) \right], \end{aligned} \quad (2.56)$$

where we used the following relation as,

$$\Omega^2 R = R^E - 6\Box \ln \Omega - 6(\nabla \ln \Omega)^2 \simeq R^E - g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi. \quad (2.57)$$

Here we cut the surface term from the above equation, $\Box \ln \Omega$. Let us substitute the Starobinsky model, Eq. (2.53), into Eq. (2.58) as follows,

$$\begin{aligned} S &= \int d^4x \sqrt{-g^{(E)}} \left[-\frac{M_{\text{Pl}}^2}{2} R^{(E)} + \frac{1}{2} g^{(E)\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + V_S(\varphi) \right], \\ V_S(\varphi) &= \frac{3}{4} \mu^2 M_{\text{Pl}}^2 \left(1 - \exp \left(-\sqrt{\frac{2}{3}} \frac{\varphi}{M_{\text{Pl}}} \right) \right)^2. \end{aligned} \quad (2.58)$$

Here we call $V_S(\varphi)$ as Starobinsky-Maeda potential. We regard φ as inflaton in this frame. Finally, we derived the minimally coupled scalar theory to gravity. We also estimate the slow-roll parameters from the Starobinsky-Maeda potential as,

$$\epsilon_V = \frac{4}{3} \left(\exp \left(\sqrt{\frac{2}{3}} \frac{\varphi}{M_{\text{Pl}}} \right) - 1 \right)^{-2}, \quad \eta_V = \epsilon_V \left(\exp \left(\sqrt{\frac{2}{3}} \frac{\varphi}{M_{\text{Pl}}} \right) - 2 \right). \quad (2.59)$$

When φ takes the much larger value than M_{Pl} , the slow-roll parameters behave as $\epsilon_V \propto \exp \left(-2\sqrt{\frac{2}{3}} \frac{\varphi}{M_{\text{Pl}}} \right)$ and $\eta_V \propto \sqrt{\epsilon_V}$. The endpoint of inflation is $\phi_{\text{end}} = \sqrt{\frac{3}{2}} \ln \left(1 + \frac{2}{\sqrt{3}} \right) M_{\text{Pl}} \sim 0.94 M_{\text{Pl}}$, obtained by solving $\epsilon_V = 1$ with respect to ϕ .

2.2.3 Inflation in the anisotropic background

Here we briefly review the inflation dynamics in the anisotropic background and derive the background equations for the spacetime, inflaton, and electric field. Our Lagrangian for the gravity, inflaton, electromagnetic field, and charged particles are given by

$$\mathcal{L} = -\frac{M_{\text{Pl}}^2}{2} R + \frac{1}{2} (\partial_\mu \phi)^2 - V(\phi) - \frac{1}{4} f(\phi)^2 F_{\mu\nu} F^{\mu\nu} + \mathcal{L}_\psi. \quad (2.60)$$

Here R , ϕ , $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ are the Ricci scalar, the inflaton, the field strength of the $U(1)$ gauge field A_μ , respectively. $V(\phi)$ is the potential of the inflaton, and $f(\phi)$ is the kinetic coupling function of the inflaton to the $U(1)$ gauge field, which will be specified later. $\mathcal{L}_\psi$ is the lagrangian of charged particles interacting with the electromagnetic field. To be concrete, we consider the Dirac fermions $\mathcal{L}_\psi = \bar{\psi}(i\mathcal{D} - m)\psi$, while the extension to charged scalars is straightforward. Without loss of generality, we can choose the x -axis as the direction of the electric field. For simplicity, we assume that the direction of the electric field does not change in time. In the homogeneous universe, we consider $\phi = \phi(t)$ and $A_\mu = (0, A(t), 0, 0)$, where we used the gauge fixing condition $A_0 = 0$ due to the gauge invariance. Note that the magnetic field is neglected in this treatment. The consistent background spacetime metric for this field configuration is known as the Bianchi-I spacetime,

$$ds^2 = dt^2 - e^{2\alpha(t)} \left(e^{-4\sigma(t)} dx^2 + e^{2\sigma(t)} (dy^2 + dz^2) \right). \quad (2.61)$$

Here $\alpha(t)$ and $\sigma(t)$ represent the e-foldings of the isotropic scale factor and deviation from the isotropy respectively. We solve the following evolution equations of the spacetime $\alpha(t), \sigma(t)$, and the electric energy density $\rho_E(t)$:

$$\dot{\alpha}^2 - \dot{\sigma}^2 = \frac{1}{3M_{\text{Pl}}^2} \left(\rho_E + \rho_\psi + V(\phi) + \frac{1}{2}\dot{\phi}^2 \right), \quad (2.62)$$

$$\ddot{\sigma} + 3\dot{\alpha}\dot{\sigma} = \frac{2\rho_E}{3M_{\text{Pl}}^2}, \quad (2.63)$$

$$\ddot{\phi} + 3\dot{\alpha}\dot{\phi} + V'(\phi) = 2\frac{f'}{f}\rho_E, \quad (2.64)$$

$$\dot{\rho}_E + 4(\dot{\alpha} + \dot{\sigma})\rho_E = -2\frac{f'}{f}\dot{\phi}\rho_E - e^\alpha \tilde{j}_{\text{tot}} \mathcal{E} + \frac{\dot{\alpha}^3}{4\pi^2} \left[\dot{\alpha}^2 + \left(\frac{\dot{f}}{f} \right)^2 \right], \quad (2.65)$$

where dot denotes the cosmic time derivative and prime denotes the inflaton derivative, e.g. $f' \equiv \partial_\phi f(\phi)$. The physical electric field is defined as $E(t) = -e^{-\alpha+2\sigma}\dot{A}$ and its energy density is $\rho_E = \frac{1}{2}f^2 E^2$. ρ_ψ represents the energy density of the charged particles. $\tilde{j}_{\text{tot}}$ is the total electric current of the charged particles, which flows along the electric field E . We used $e\tilde{j}_{\text{tot}} \cdot E = \tilde{j}_{\text{tot}} \mathcal{E}$ by introducing $\mathcal{E} \equiv eE(t)$, where e is the coupling constant of the fermion with the electric field. The last term in the right-hand side of Eq. (2.65) is phenomenologically added. With the varying kinetic function f , the fluctuation of the gauge field is produced in the inflationary background and its contribution to the background electric energy is roughly estimated as [4]

$$(\dot{\rho}_E)_H = \left. \frac{d\rho_E}{dk} \right|_{k=k_H} \cdot \frac{dk_H}{dt} \simeq \frac{\dot{\alpha}^3}{4\pi^2} \left[\dot{\alpha}^2 + \left(\frac{\dot{f}}{f} \right)^2 \right]. \quad (2.66)$$

Here the index “H” of $(\dot{\rho}_E)$ denotes the contribution from the horizon crossing mode during inflation and k_H represents the wavenumber of the mode

crossing the horizon. This contribution gives a finite initial amplitude for the electric field which is subsequently amplified by the inflaton. Since the electric field is eventually stabilized at its attractor value, the initial amplitude is not very important and we adopt the same expression for $(\dot{\rho}_E)_H$ as Ref. [4]. The drawback of Eq. (2.65) is that we cannot track the direction of the electric field. Therefore, when we numerically solve the set of the equations, we use not Eq. (2.65) but the EoM for electric field $\mathcal{E} = ef^{-1}\sqrt{2\rho_E}$,⁵

$$\dot{\mathcal{E}} + 2(\dot{\alpha} + \dot{\sigma})\mathcal{E} + 2\frac{\dot{f}}{f}\mathcal{E} = -\frac{e^2}{f^2}j_{\text{tot}} + \frac{e^2}{f^2}\frac{\dot{\alpha}^3}{4\pi^2\mathcal{E}}\left[\dot{\alpha}^2 + \left(\frac{\dot{f}}{f}\right)^2\right], \quad (2.67)$$

where we introduce $j_{\text{tot}} \equiv e^\alpha \tilde{j}_{\text{tot}}$, which is the electric current density for the comoving observer. For simplicity, we ignore the contribution from the charged particles in this section, and we will discuss the charged particles in Chapter 4. Then Eqs. (??) and (2.65) equation are given as,

$$\dot{\alpha}^2 - \dot{\sigma}^2 = \frac{1}{3M_{\text{Pl}}^2}\left(\rho_E + V(\phi) + \frac{1}{2}\dot{\phi}^2\right), \quad (2.68)$$

$$\dot{\rho}_E + 4(\dot{\alpha} + \dot{\sigma})\rho_E = -2\frac{f'}{f}\dot{\phi}\rho_E + \frac{\dot{\alpha}^3}{4\pi^2}\left[\dot{\alpha}^2 + \left(\frac{\dot{f}}{f}\right)^2\right], \quad (2.69)$$

This system is expected to have an attractor solution of the electric field for the following reasons. The electric field is amplified by the inflaton, because the inflaton's kinetic energy is transferred to the electric field through the kinetic function. The amplification might be weaker than the dilution effect of the spacetime expansion in the early stage of the inflationary phase. Towards the end of inflation, however, the inflaton velocity $\dot{\phi}$ is accelerated by the potential force, the amplification effect is enhanced and eventually overcomes the dilution effect, and the electric field begins to grow. When the electric field is sufficiently amplified and its backreaction on the inflaton becomes significant, the growth stops and ρ_E is stabilized. This is because, if the electric field is further amplified, the inflaton is slowed down by the backreaction, its kinetic energy pumped to the electric field is reduced, and they decay. If the electric field is smaller than the stabilized value, the inflaton is accelerated and gives more energy to the electric field. Therefore, the inflaton and the electric field constitute a negative feedback system and they should have an attractor solution.

In what follows, we derive the analytic expression for ρ_E in the attractor phase as a function of V and f .

⁵In our numerical calculation, we multiply the last term by the Heaviside step function $\theta(\mathcal{E} - 10^{-100}\mu^2)$ to avoid the spurious singularity at $\mathcal{E} = 0$. Since no new mode crosses the horizon and contributes to the background electric field, the last terms in Eqs. (2.65) and (2.67) disappear after inflation. In addition, the electric field becomes 0 after inflation. Although we keep the boundary term even after inflation for a technical reason in our numerical analysis, we expect that it does not significantly affect the results.

The attractor solution for ρ_E

Here we find the general attractor solution for the energy density of the electric field, ρ_E^{att} . As we discussed above, when the electric field reaches the attractor, its energy density is stabilized. Solving Eq. (2.69) without the perturbations, we obtain

$$e^{4\alpha} f^2 \rho_E = \text{const.} \quad (2.70)$$

Thus, in the attractor phase, the kinetic function is expected to satisfy $f \propto e^{-2\alpha}$. We rewrite this relation as

$$\frac{df}{dt} = -2\dot{\alpha}f \implies \dot{\phi} = -2 \frac{\dot{\alpha}}{(\ln f)'}. \quad (2.71)$$

Substituting it into Eq. (2.64) and ignoring $\ddot{\phi}$ as a slow-roll correction term, we obtain

$$\rho_E^{\text{att}} = -\frac{V}{M_{\text{Pl}}^2 (\ln f)'^2} + \frac{V'}{2(\ln f)'}, \quad (2.72)$$

where we used the slow-roll equation, $3M_{\text{Pl}}^2 \dot{\alpha}^2 \simeq V$.

In the WKS model, the inflaton potential and the kinetic coupling function are given by

$$V_W(\phi) = \frac{1}{2}\mu^2\phi^2, \quad f_W(\phi) = \exp\left(\frac{c\phi^2}{2M_{\text{Pl}}^2}\right), \quad (2.73)$$

where μ is the mass of inflaton, and c is a dimensionless coupling constant.

$$\rho_E^{\text{att}} = \frac{3}{2} \frac{c-1}{c^2} \epsilon_V M_{\text{Pl}}^2 \dot{\alpha}^2, \quad (2.74)$$

with $\epsilon_V = \frac{M_{\text{Pl}}^2}{2} \left(\frac{V'}{V}\right)^2$. Note that we should set $c > 1$ to have a non-vanishing value of the electric field in the attractor solution.

Let us confirm the above discussion by performing the numerical calculation. We found two different slow-roll phases in the left panel in Fig. 2.1. At first, the inflaton goes down the potential slope without resistant force. After a while, the inflaton feels the friction force given by $3\dot{\alpha}\dot{\phi}$ in the EoM for the inflaton. This phase also satisfies the slow-roll conditions. Since the electric field grows up by the energy injection by the kinetic energy of the inflaton, the backreaction from the electric field balances with the potential force. Then we found the second slow-roll phase. In the inflation model without the background electromagnetic field such as the chaotic and Starobinsky models, we do not find the second slow-roll phase. This phase is a special feature of the anisotropic inflation. In the right panel of Fig. 2.1, we can confirm that the energy density of the electric field behaves as the attractor solution given in Eq. (2.74) during the second slow-roll phase. We can derive the relation

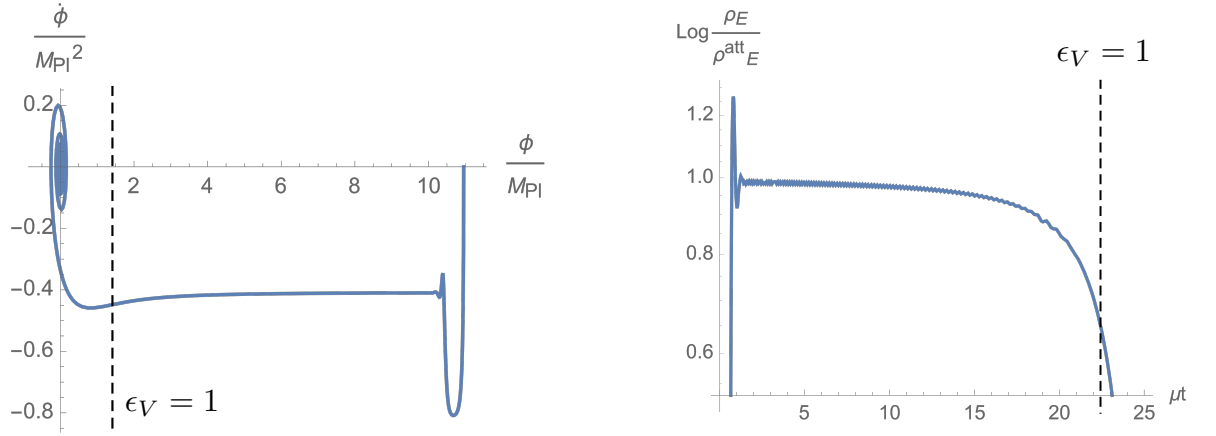


FIGURE 2.1: Left: Phase flow for ϕ in case of $(c, \mu/M_{\text{Pl}}) = (2, 10^{-5})$. The initial value of $\phi = \sqrt{2N} \sim 11$. The black dashed line denotes the inflation end where $\epsilon_V = 1$. At the beginning of inflation, the inflaton goes down the potential and the kinetic energy becomes larger. Then the inflaton gives the energy to the electric field and the backreaction pushes back the inflaton. After a short time, the potential force and the backreaction force are balanced and the slow-roll condition is satisfied. After inflation, the inflaton begins to oscillate around the local minimum of the potential, which is related to the reheating era. right: A time evolution of the ratio of electric energy densities and the attractor solution. The setup and the normalization are equal to the left panel. The attractor solution is a good approximation to the energy density of the electric field while the slow-rolling condition is satisfied.

between the anisotropy and the Hubble parameter as,

$$\frac{\dot{\sigma}}{\dot{a}} = \frac{c-1}{3c^2} \epsilon_V = \frac{c-1}{3c} \epsilon_H, \quad (2.75)$$

where we used the slow-roll approximation for the anisotropy and the attractor solution of the electric field, Eq. (2.72). We estimated the slow-roll parameter, ϵ_H , given in Eq. (2.10) by using the slow-roll conditions and found the relation as $\epsilon_V = c \epsilon_H$. Though the end time of inflation is originally given by $\epsilon_H = 1$, we idiomatically used $\epsilon_V = 1$ to determine the inflation end in this section. The anisotropy becomes larger when the c takes a larger value. The anisotropy is tested by the observation of the quadrupole anisotropy in the CMB angular power spectrum. We should remark that Ref. [38] showed $c-1 < 10^{-7}$ by using the stochastic formalism for the electric field, in which the sub-horizon modes can affect the background as the stochastic noise. Then the WKS model is not favored in the observation.

2.3 Inflationary Magnetogenesis

Here we introduce Inflationary Magnetogenesis, which is a kind of scenario explaining the origin of the cosmic magnetic field. In this section, we explain the cosmic magnetic field, the observational implication of the magnetic field in the Voids, and the lower bound of the magnetic field. Then, we review the concrete inflationary magnetogenesis models. In this section, we ignore the anisotropy of the spacetime, since we focus on the situation that the energy density of the electromagnetic field is much smaller than the energy density of inflaton.

2.3.1 Galactic and Void Magnetic field

It is well known that galaxies and their clusters have magnetic fields with the typical strength $\mathcal{O}(10^{-6}\text{G})$ [41–43]. The strength and the direction of these magnetic fields are by observation of radiation phenomena, such as the synchrotron radiation or the Faraday rotation. The charged particle twists around the magnetic field because of the Lorentz force and emits the radiation along the direction of the velocity of the charged particle. This type of radiation is called synchrotron radiation, whose intensity and polarization depend on the strength and the direction of the magnetic field, respectively. When the linearly polarized light passes through the ionized magnetized region, the polarization is rotated by the magnetic field, which is called the Faraday rotation. We can estimate the strength of the magnetic field directed along the line of sight by the multi-frequency observation of the radiation.

We still do not know where the galactic magnetic field comes from. The origin of such a magnetic field has been discussed for a long time. The primordial origin, the dynamo-generated origin, or the combination of these two are mainly considered. The primordial scenario claimed that there exists the large-scale “seed” magnetic field in the early universe and they are amplified with the growth of the galaxy. The dynamo-generated scenario is to generate the magnetic field from the hydrodynamics of the ionized objects in the galaxy. In the latter case, the initial condition of the magnetic field is given as e.g. the battery mechanism. When the density and the temperature gradient in the ionized plasma are misaligned, the electrons go down the pressure gradient. Then the electric current turns on and the small-scale magnetic field is generated. This is called the Biermann battery [44], and the time evolution of the vector potential is given as $\dot{\mathbf{A}} \propto \nabla p_e$, where p_e is the pressure of electrons. One of the other types of battery mechanisms is called the Durrive battery [45], where the time derivative of the pressure of electrons also affects the generation of the initial magnetic field.

The 3-D numerical simulation claims that both primordial and dynamo-generated scenarios can explain the present magnetic field in the galaxy [46]. The former scenario does not assume the mechanism of the ionized plasma to produce the “seed” magnetic field. Then, if we find the magnetic field in the intergalactic region, where few objects exist, the primordial scenario is favored.

Implication of the void magnetic field

Recent multi-frequency blazar observations imply the existence of the magnetic field in the void region and the lower bound of such intergalactic magnetic field (IGMFs) [1, 47–55]. We briefly explain the experience as follows.

The spectrum of the gamma-ray from the blazar is represented as the power-law with the cut-off energy. When the cut-off energy scales in TeV, the blazar is called TeV blazar⁶. We do not observe all the TeV scale gamma rays from the TeV blazar. The parts of the gamma rays interact with the Extragalactic background light (EBL) and they cannot propagate the cosmological distance between the blazar and the earth. D_γ , the mean free path of the TeV scale gamma ray is represented as,

$$D_\gamma \sim 800\kappa \left(\frac{\text{TeV}}{E_\gamma^i} \right) \text{Mpc}, \quad (2.76)$$

where E_γ^i and κ denote the energy of the gamma-ray from the blazar and the $\mathcal{O}(1)$ numerical factor from the model uncertainty [57]. The pair creation of the electron and positron occurs when the TeV scale gamma ray interacts with the EBL. Since the mass of the electron is about 0.5MeV, the pair creation needs at least MeV energy scale. When we represent the E_{EBL} as the energy of the EBL, the Mandelstum variable s of two photons is written as $s = 4E_\gamma^i E_{\text{EBL}}$. Then we have the inequality for the pair creation as

$$\sqrt{4E_\gamma^i E_{\text{EBL}}} \gtrsim 1\text{MeV}, \rightarrow E_{\text{EBL}} \gtrsim 0.1 \left(\frac{\text{TeV}}{E_\gamma^i} \right) \text{eV}. \quad (2.77)$$

The spectrum of the EBL is observed in the broad wavelengths, and the intensity of EBL is peaked around 0.1eV region [58]. Since the energy scale of the CMB photon, E_{CMB} , is 10^{-4}eV , the CMB photon does not interact with the TeV scale gamma ray during the propagation from the blazar to the earth.

Here the TeV scale electron-positron pair is generated. The pair gives the kinetic energy to the CMB photons by the inverse Compton scattering. For simplicity, the angles of incidence and reflection are 0. In the rest frame of the electron, the energy of the scattered photon is written as $\gamma_e E_{\text{CMB}}$, where $\gamma_e = E_\gamma^i/m_e$ is the gamma factor multiplied by the Lorentz boost. We need one more gamma factor when we come back to the rest frame for the observer. Then the CMB photon obtains the energy by the inverse Compton scattering as,

$$E_\gamma^f = \gamma_e^2 E_{\text{CMB}} \sim 0.1\text{GeV}, \quad (2.78)$$

where E_γ^f is the energy of the scattered CMB photons observed in the earth. The gamma factor is estimated as 10^6 in case of $E_{\text{CMB}} \ll E_\gamma^i = \text{TeV}$. The parts of the emitted TeV scale gamma rays from the blazar are observed as

⁶More than 70 TeV blazars are observed so far, see [56].

the GeV scale gamma rays by the above mechanism. Then, when we observe the gamma rays from the blazar, we may obtain the TeV and GeV scale gamma rays. However, the intensity of the observed GeV scale gamma ray is smaller than the simulated GeV scale gamma ray estimated by the above discussion. These observational results imply the existence of the magnetic field in the void region. When the magnetic field is in the intergalactic region, the Lorentz force bends the trajectories of the paired charged particles and reduces the intensity of the gamma-ray on the GeV scale.

In the rest of this section, we introduce the lower bound of the implied void magnetic field. In ref. [59], they use four blazars, 1ES 0229+200 ($z=0.14$, comoving radial distance = 580Mpc), 1E0347-121 ($z=0.188$, 780Mpc), 1E1101-232 ($z=0.186$, 770Mpc), and H2356-309 ($z=0.165$, 680Mpc). The distance between these blazars and the earth is enough large to cause the pair creation of the charged particles, eq. (2.76). If the void magnetic field exists, we observed GeV scale gamma ray as not the point light source, but the spread light source because of the bending orbits of the pair created charged particles. We define the spread angle of the light source as θ_{PSF} , which is determined by the detector and the energy of photons, and we also define the estimated spread angle of the light caused by the IGMFs as θ_{ext} . When $\theta_{\text{ext}} > \theta_{\text{PSF}}$, the intensity of the observed GeV gamma-ray seems to become small. Since the θ_{ext} is proportional to the IGMFs, we obtain the lower bound for the magnetic field from the above inequality as,

$$B_{\text{Void}} \gtrsim 3 \times 10^{-16} \text{G} \times \begin{cases} 1 & (\lambda_B \geq 1\text{Mpc}) \\ \sqrt{\lambda/1\text{Mpc}} & (\lambda_B < 1\text{Mpc}) \end{cases}, \quad (2.79)$$

where λ_B is the correlation length of the magnetic field. The above result is the lower bound for the IGMFs. There is the upper bound for the magnetic field given by the CMB observation as $B_{\text{Void}} 10^{-9} \text{G}$ [60].

2.4 Inflationary Magnetogenesis Model

The cosmological scenario hypothesizes that primordial magnetic fields (PMFs) are formed in the early universe before the recombination, and they are amplified to the order of 10^{-6}G via the galactic dynamo which is driven by the interaction between the magnetic field and plasma [61]. This scenario can explain the existence of IGMFs. However, we do not know the origin of such PMFs. One of the candidates is the cosmological phase transition [62–65] in which bubbles from the first order phase transition of the universe serve the kinetic energy to the electromagnetic field via the bubble collisions. Another candidate is the Harrison mechanism [66, 67], in which the second-order perturbations of the electron, proton, and photon induce the electric current and it becomes the source of the magnetic field. Inflationary magnetogenesis, in which the quantum fluctuation is the origin of the PMFs, is one of the most studied scenarios. Since the standard $U(1)$ gauge field on the flat-FLRW universe can not be amplified by inflation due to its conformal symmetry [68],

several models are devised to break the conformal symmetry during inflation.

In this section, we review the two inflationary magnetogenesis models. One is the kinetic coupling model [2, 69] was first proposed by Ratra [2], where a rolling scalar field Φ coupled to the electromagnetic field as $I^2(\Phi)F_{\mu\nu}F^{\mu\nu}$ and the electromagnetic field are generated while $I(\Phi)$ evolves. Unfortunately, the original model inevitably violates one of the following conditions [70–73]; (i) the effective coupling constant between the canonical electromagnetic field and charged particles should be small enough to validate the perturbative treatment, (ii) the energy density of the electromagnetic field should not overwhelm the inflaton energy density during inflation, and (iii) the curvature perturbation induced by the generated magnetic field should be consistent with the CMB observations. Since these three conditions are often broken in the models aiming to explain Eq. (2.79) in the literature, they are known as the serious problems to achieve viable inflationary magnetogenesis and called (i) the strong coupling problem, (ii) the back reaction problem, and (iii) the curvature perturbation problem, respectively. Recent studies have proposed to introduce the IR cut-off into the spectrum of the produced magnetic field [74, 75] and the post-inflationary phase of magnetogenesis [76–78] to satisfy these conditions.

The other one is the kinetic and axial hybrid model. Before going to the hybrid model, we introduce the axial coupling model [79–83], which is first proposed by Turner and Widrow [79]. Its detailed analytic study was provided by Anber and Sorbo [82]. A rolling pseudo scalar serves its kinetic energy to the electromagnetic field via the axial coupling $\phi F_{\mu\nu}\tilde{F}^{\mu\nu}$. Ref. [83] numerically found that the significant amplification of the magnetic field occurs around the end of the inflation and Ref. [84] showed that the generated magnetic field reaches 10^{-16}G by the lattice simulation.

Caprini and Sorbo [85, 86] proposed the hybrid model which contains both the kinetic and axial couplings, $I^2(\chi)(FF - \gamma F\tilde{F})$, where χ represents a rolling pseudo scalar. Since this model violates the parity symmetry, the generated magnetic field has non-zero helicity. Then the correlation length of the helical magnetic field in the plasma grows faster than the cosmic expansion via the inverse cascade [87–89], which is a well-known mechanism in magnetohydrodynamics. By using the inverse cascade in the instantaneous reheating case, which is an ideal situation, they show that this model can produce the helical magnetic field with the strength $B_{\text{Void}} = \mathcal{O}(10^{-17}\text{G})$. This model was extended by including magnetogenesis during the reheating era, and the produced magnetic field in Ref. [90–92] can be even larger.

2.4.1 Kinetic coupling Model

Here we review the kinetic coupling model first suggested by Ratra [2], where the lagrangian on a spatially flat FLRW metric $ds^2 = a^2(\eta)[-d\eta^2 + d\mathbf{x}^2]$ as

$$\mathcal{L} = \frac{M_{\text{Pl}}^2}{2}R - \frac{1}{2}(\partial_\mu\phi)^2 - V(\phi) - \frac{1}{4}I^2(\phi)F_{\mu\nu}F^{\mu\nu} \quad (2.80)$$

where the inflaton potential and the kinetic coupling function are written as,

$$V(\phi) = V_0 \exp\left(-\frac{\kappa_1 \phi}{M_{\text{Pl}}}\right), \quad (2.81)$$

$$I(\phi)^2 = \exp\left(\frac{\kappa_2 \phi}{M_{\text{Pl}}}\right), \quad (2.82)$$

where κ_1 and κ_2 are the dimensionless factors. This scenario assumes that the electromagnetic field does not affect the dynamics of the inflaton field. In other words, the energy density of the inflaton is much larger than the electromagnetic field. In this assumption, we have the exact solution for the system. The slow-roll parameters are given as,

$$\eta_V = \epsilon_V = \kappa_1^2. \quad (2.83)$$

Therefore the inflation occurs when $\kappa_1 \ll 1$. By imposing $a(t) \propto t^\alpha$, the Friedmann equation (2.2) and the EoM for the inflaton (2.22) are solved as follows,

$$a \propto t^{\frac{2}{\kappa_1^2}}, \quad \phi = \frac{2M_{\text{Pl}}^2}{\kappa_1} \ln\left(\frac{\kappa_1 t}{M_{\text{Pl}}} \sqrt{\frac{V_0}{2(\frac{6}{\kappa_1^2} - 1)}}\right). \quad (2.84)$$

Thus the kinetic function is represented as,

$$I(\phi)^2 = \left[\frac{t}{M_{\text{Pl}}} \sqrt{\frac{V_0}{\frac{2}{\kappa_1^2}(\frac{6}{\kappa_1^2} - 1)}} \right]^{\frac{2\kappa_2}{\kappa_1}} \propto a^{\kappa_1 \kappa_2}. \quad (2.85)$$

Then we rewrite the kinetic function by the power law for the scale factor, which breaks the conformal symmetry of the electromagnetic field and their fluctuations can be amplified by the spacetime expansion.

From now on, we explain the amplification of the electromagnetic fluctuations with the kinetic function written by the power law of a scale factor, in the same way as sec. 2.1.3. The action of the electromagnetic field with the kinetic coupling is written as,

$$S_{\text{EM}} = \int d\eta d^3x \sqrt{-g} \left[-\frac{1}{4} I^2(\eta) F_{\mu\nu} F^{\mu\nu} \right], \quad (2.86)$$

and we decompose the perturbation of the electromagnetic field into the scalar and the vector parts given as follows,

$$A_i = A_i^{(v)} + \partial_i \sigma, \quad \partial_i A_i^{(v)} = 0 \quad (2.87)$$

, where the superscript (v) means the vector component of the perturbation. By using the above decomposition, we obtain the following action,

$$S_{\text{EM}} = \frac{1}{2} \int d\eta d^3x I(\eta)^2 \left[(A'^{(v)})^2 + (\partial_i \sigma')^2 + (\partial_i A_0)^2 - (\partial_i A_j^{(v)})^2 + 2A_0 \partial_i^2 \sigma' \right]. \quad (2.88)$$

Since $\dot{A}_0$ term is not included in the action, A_0 is a non-dynamical degree of freedom. By using the EoM for the A_0 from the above action, we obtain the condition for the scalar perturbation as,

$$A_0 = \sigma' + c, \quad (2.89)$$

where c is a constant. Substituting the above condition with $c = 0$ into eq. (2.88), we can remove the scalar parts and A_0 from the action as,

$$S_{\text{gauge}} = \frac{1}{2} \int \frac{d\eta d^3k}{(2\pi)^3} \left[(I\mathcal{A}(\eta, \mathbf{k}))' (I\mathcal{A}(\eta, -\mathbf{k}))' - \left(k^2 - \frac{I''}{I} \right) (I\mathcal{A}(\eta, \mathbf{k})) (I\mathcal{A}(\eta, -\mathbf{k})) \right], \quad (2.90)$$

where $\mathcal{A}(\eta, \mathbf{k})$ is the mode function of the electromagnetic fluctuation defined as,

$$\hat{A}_i^{(v)}(\eta, \mathbf{x}) = \sum_{\lambda=\pm} \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} e_i^{(\lambda)}(\hat{\mathbf{k}}) \left[a_{\mathbf{k}}^{(\lambda)} \mathcal{A}_\lambda(\eta, k) + a_{-\mathbf{k}}^{(\lambda)\dagger} \mathcal{A}_\lambda^*(\eta, k) \right]. \quad (2.91)$$

Here we introduced the polarization vector, e_i^λ , which determines the polarization of the fluctuation, and we imposed the canonical commutation relations as,

$$[a_{\mathbf{k}}^{(\lambda)}, a_{-\mathbf{k}'}^{(\lambda')\dagger}] = (2\pi)^3 \delta(\mathbf{k} + \mathbf{k}') \delta^{\lambda\lambda'}, \quad [a_{\mathbf{k}}^{(\lambda)}, a_{\mathbf{k}'}^{(\lambda')}] = [a_{\mathbf{k}}^{(\lambda)\dagger}, a_{\mathbf{k}'}^{(\lambda')\dagger}] = 0. \quad (2.92)$$

Then we obtain the EoM for the electromagnetic mode functions as as,

$$\left[\partial_\eta^2 + k^2 - \frac{\partial_\eta^2 I}{I} \right] (I(\eta) \mathcal{A}_\lambda(\eta, k)) = 0. \quad (2.93)$$

Before solving the above equation, we give the power spectra of the electric field and the magnetic field. The vacuum expectation values of the electric and magnetic field are written as,

$$\langle |E|^2 \rangle = \langle (\dot{A}_i(\eta, \mathbf{x}))^2 \rangle = \int \frac{dk}{k} \mathcal{P}_E(\eta, k), \quad (2.94)$$

$$\langle |B|^2 \rangle = \langle (\epsilon_{ijk} \partial_j A_k(\eta, \mathbf{x}))^2 \rangle = \int \frac{dk}{k} \mathcal{P}'_B(\eta, k), \quad (2.95)$$

where $\mathcal{P}_E$, and $\mathcal{P}_B$ are the power spectra of the electric and magnetic field represented as,

$$\mathcal{P}_E^\pm(\eta, k) \equiv \frac{k^3 I^2}{2\pi^2 a^4} |\partial_\eta \mathcal{A}_\pm|^2, \quad \mathcal{P}_B^\pm(\eta, k) \equiv \frac{k^5 I^2}{2\pi^2 a^4} |\mathcal{A}_\pm|^2. \quad (2.96)$$

Taking the square root of the power spectra, we obtain the strength of the electromagnetic field at the given time.

Let us go back to solve eq. (2.93). Since we obtained the power-law type kinetic function in the Ratra model, we assume the following kinetic function as,

$$I(\eta) = \left(\frac{a(\eta)}{a_e} \right)^{-n} \quad (n > 0), \quad (2.97)$$

where $n = -\kappa_1 \kappa_2$ in the Ratra model. Here we set $n > 0$ to avoid the strong coupling of the electromagnetic field with the charged particles. When we consider the interaction between the electromagnetic field and electrons, the interaction term is written as $\mathcal{L}_{\text{int}} = -e\bar{\psi}\gamma^\mu\psi A_\mu$, where ψ and e are the electrons and the coupling constant, respectively. Here we find the canonical electromagnetic field is IA_μ , and hence the effective coupling constant is given as e/I . If $n < 0$, the effective coupling constant becomes larger than the standard model and we cannot use the perturbation theory to analyze the interaction theory at the early stage of inflation in case the effective coupling approaches the unity. Then we do not consider the $n < 0$ case here. By using $a = -(\eta H)^{-1}$ in the de-Sitter spacetime, the EoM for the electromagnetic mode functions is rewritten as,

$$\left[\partial_\eta^2 + k^2 - \frac{n(n+1)}{\eta^2} \right] (I\mathcal{A}_\lambda(\eta, k)) = 0. \quad (2.98)$$

This is the same differential equation as the scalar perturbation case by setting $2 - \mu^2/H^2 = n(n+1)$ in eq. (2.31). Then the solution can be written by the Bessel functions. Here we roughly estimate the strength of the magnetic field by using the super-horizon power spectrum. The electromagnetic mode function in the super-horizon is approximately given as,

$$\lim_{k\eta \rightarrow 0} I\mathcal{A} \rightarrow \frac{1}{\sqrt{2k}} (-k\eta)^{-n}, \quad (2.99)$$

Thus we can estimate the magnetic field at the end of inflation as,

$$B(k, \eta_e) = \sqrt{\langle |\mathbf{B}|^2 \rangle} \approx \frac{H^2}{\pi\sqrt{2}} \left(\frac{a_e H}{k} \right)^{n-3} \quad (2.100)$$

where the scale-invariant magnetic field appears in the case of $n = 3$. When we substitute the typical values of each parameter into the above relation, we can obtain the present void magnetic field satisfying the observational lower bound given in eq. (2.79). However, when we consider the backreaction from

the electromagnetic field to the spacetime expansion, the production of the magnetic field is highly suppressed and cannot explain the void magnetic field by the original kinetic coupling model.

“Here we introduce the extended kinetic coupling model, where we consider the post-inflationary growth of the electromagnetic field. I have to write the details”

2.4.2 Kinetic and Axial Hybrid Model

In this section, we show the kinetic and axial hybrid model, which can produce the magnetic field satisfying the observational lower bound given in eq. (2.31) with a consideration of the backreaction from the electromagnetic field to the background.

Model setup

Let us consider the following Lagrangian proposed in Ref. [85] as

$$\mathcal{L} = \frac{M_{\text{Pl}}^2}{2} R - \frac{1}{2} (\partial_\mu \phi)^2 - V(\phi) - \frac{1}{2} (\partial_\mu \chi)^2 - U(\chi) - \frac{1}{4} I^2(\chi) (F_{\mu\nu} F^{\mu\nu} - \gamma F_{\mu\nu} \tilde{F}^{\mu\nu}). \quad (2.101)$$

Here M_{Pl} is the reduced Planck mass, R is the Ricci scalar, ϕ is the inflaton, χ is a spectator scalar field, $F_{\mu\nu}$ is the field strength of U(1) gauge field, $\tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} / (2\sqrt{-g}) F_{\alpha\beta}$ is the dual field strength, γ is a constant, and the term including γ explicitly violates the parity symmetry. $V(\phi)$ and $U(\chi)$ are the potentials of ϕ, χ respectively. We assume that χ has a non-zero vacuum expectation value and is coupled to the U(1) gauge field via the kinetic coupling function $I(\chi)$ but its energy density is always much smaller than the total energy density. Thus, we call χ a “spectator”. One can derive the equation of motion (EoM) for χ and see that the kinetic energy of χ is transferred to the electromagnetic field through the time derivative of $I(\chi)$. Since electromagnetic fields are produced while $I(\chi)$ evolves in time, for simplicity, we treat I as a function of time by ignoring the perturbation of χ . We also assume that $I(\eta)$ varies during inflation and the subsequent reheating era (i.e. the inflaton oscillation era). In order to study the generation of the gauge field, we promote the classical vector potential to the quantum operator as

$$A_i(\eta, \mathbf{x}) = \sum_{\lambda=\pm} \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} e_i^{(\lambda)}(\hat{\mathbf{k}}) \left[a_{\mathbf{k}}^{(\lambda)} \mathcal{A}_\lambda(\eta, k) + a_{-\mathbf{k}}^{(\lambda)\dagger} \mathcal{A}_\lambda^*(\eta, k) \right], \quad (2.102)$$

where we work in Coulomb gauge, $A_0 = \partial_i A_i = 0$, $e_i^{(\pm)}(\hat{\mathbf{k}})$ are the right/left-handed polarization vectors which satisfy $i\epsilon_{ilm} k_l e_m^{(\pm)}(\hat{\mathbf{k}}) = \pm k e_i^{(\pm)}(\hat{\mathbf{k}})$, $\mathcal{A}_\lambda$ is the mode function of the gauge field, and we impose the usual canonical commutation relation, $[a_{\mathbf{k}}^{(\alpha)}, a_{-\mathbf{k}'}^{(\beta)\dagger}] = (2\pi)^3 \delta(\mathbf{k} + \mathbf{k}') \delta^{\alpha\beta}$. Here $\hat{\mathbf{n}}$ represents

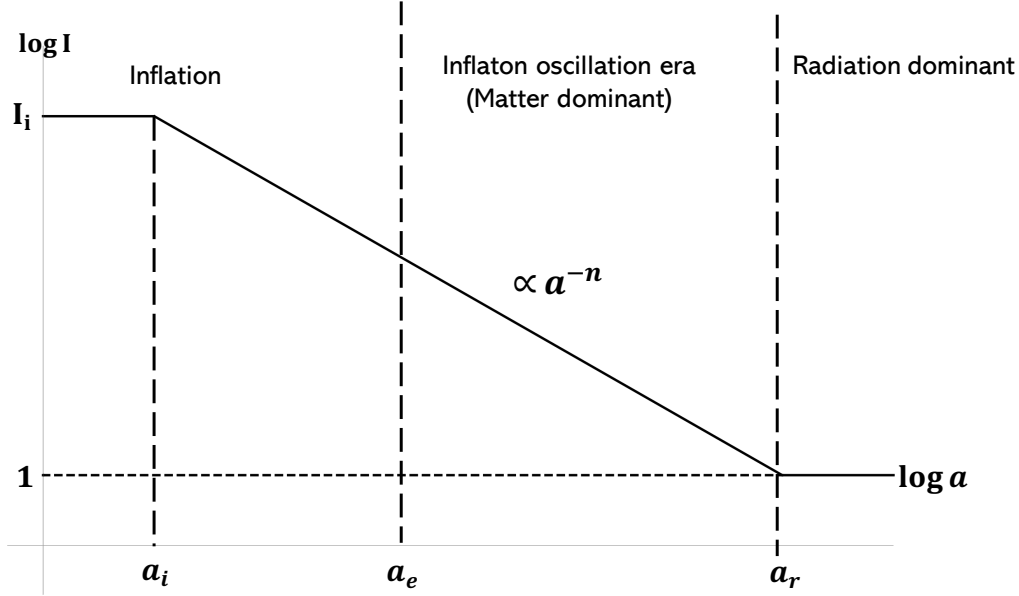


FIGURE 2.2: The behavior of $I(a)$ given in Eq. (2.104). $I(a)$ decreases in proportion to a^{-n} for $a_i < a < a_r$. In this paper, we assume that $I(a)$ stops at the reheating completion, a_r . The power-law behavior is achieved when the spectator field satisfies the slow-roll condition. We assume that the spectator field violates the slow-roll condition before $a = a_i$. Related behaviors are studied in a different context, where the curvature perturbation on the super horizon scale is amplified by the sudden violation of the slow-roll condition [93].

the unit vector parallel to $\mathbf{n}$. The EoM for the mode function is written as

$$\left[\partial_\eta^2 + k^2 \pm 2\gamma k \frac{\partial_\eta I}{I} - \frac{\partial_\eta^2 I}{I} \right] \left(I(\eta) \mathcal{A}_\pm(\eta, k) \right) = 0. \quad (2.103)$$

Note that we restore the usual EoM in normal electromagnetism when the function I is constant, and the gauge modes are not amplified. To solve Eq. (2.103), we need to specify I as the function of the conformal time. In a similar way to Ref. [91], we assume that the kinetic function I is given by

$$I(\eta) = \begin{cases} (a_i/a_r)^{-n} \equiv I_i & (a < a_i) \\ (a(\eta)/a_r)^{-n} & (a_i < a < a_r) \\ 1 & (a_r < a) \end{cases}, \quad (2.104)$$

where $I(\eta)$ begins to vary at a_i during inflation, and $I(\eta)$ becomes unity at the reheating completion a_r . The shape of the function $I(a)$ is illustrated in

Fig. 2.2. The conformal time in each era behaves as

$$\eta = \begin{cases} -1/aH_{\text{inf}} \propto a^{-1} & (a < a_e) \\ 2/aH \propto a^{1/2} & (a_e < a < a_r) \\ 1/aH \propto a & (a_r < a) \end{cases}, \quad (2.105)$$

where inflation ends at a_e and the reheating is completed at a_r . By using Eq. (2.103) and Eq. (2.104), we obtain the EoM in each era as

$$\left[\partial_\eta^2 + k^2 \right] \left(I\mathcal{A}_\pm^{\text{BD}} \right) = 0, \quad (a < a_i) \quad (2.106)$$

$$\left[\partial_\eta^2 + k^2 \pm 2\tilde{\zeta} \frac{k}{\eta} - \frac{n(n+1)}{\eta^2} \right] \left(I\mathcal{A}_\pm^{\text{inf}} \right) = 0, \quad (a_i < a < a_e) \quad (2.107)$$

$$\left[\partial_\eta^2 + k^2 \mp 4\tilde{\zeta} \frac{k}{\eta} - \frac{2n(2n+1)}{\eta^2} \right] \left(I\mathcal{A}_\pm^{\text{osc}} \right) = 0, \quad (a_e < a < a_r) \quad (2.108)$$

$$\left[\partial_\eta^2 + k^2 \right] \left(I\mathcal{A}_\pm^{\text{fin}} \right) = 0, \quad (a_r < a), \quad (2.109)$$

where $\tilde{\zeta} \equiv n\gamma$. The term including $\tilde{\zeta}$ generates the polarization of the gauge field through the tachyonic instability which is effective around the horizon crossing of a fluctuation mode, $k\eta \sim \tilde{\zeta}$. Since we are interested in the efficient production of the polarized gauge field, we assume that $\tilde{\zeta} > n$ (i.e. $\text{fl} > 1$). In addition, the super-horizon mode is amplified by the term proportional to η^{-2} . These amplifications occur while the function $I(a)$ varies.

Solving the dynamics of the electromagnetic field

Now we can solve the EoM, Eqs. (2.106)-(2.109) by using Bunch-Davies initial conditions and the junction conditions at the boundary between different eras. From Bunch-Davies initial conditions, the solutions for $a < a_i$ is written as $I_i\mathcal{A}_\pm^{\text{BD}} = e^{-ik(\eta-\eta_i)}/\sqrt{2k}$. The gauge field and their time derivatives are continuous at any time and hence we impose the junction conditions, $\mathcal{A}_\pm(\eta_* - \delta) = \mathcal{A}_\pm(\eta_* + \delta)$ and $\partial_\eta \mathcal{A}_\pm(\eta_* - \delta) = \partial_\eta \mathcal{A}_\pm(\eta_* + \delta)$ in the limit $\delta \rightarrow 0$ at the boundary times $\eta_* = \eta_i$ and η_e . These junction conditions ensure that the energy density of the electromagnetic field is continuous. The solution during inflation is already given in Ref. [91]. In this paper, we are interested in the solution during reheating, and we show only the solution for Eq. (2.108).

The solution for Eq. (2.108) can be written as

$$I\mathcal{A}_\pm^{\text{reh}} = \frac{1}{\sqrt{2k}} \left[D_1^\pm(|k\eta_e|) M_{\pm 2i\tilde{\zeta}, 2n+\frac{1}{2}}(2ik\eta) + D_2^\pm(|k\eta_e|) W_{\pm 2i\tilde{\zeta}, 2n+\frac{1}{2}}(2ik\eta) \right], \quad (2.110)$$

where $W_{\alpha,\beta}(z)$ and $M_{\alpha,\beta}(z)$ are Whittaker functions, and the coefficients $D_1^\pm, D_2^\pm$ which depend on the wave number of the mode through $|k\eta_e|$ are given by

$$D_1^\pm(y) = \frac{1}{4iy} \frac{\Gamma(2n+1 \mp 2i\tilde{\zeta})}{\Gamma(4n+2)} C_2^\pm(k\eta_i) \left[W_{1 \pm 2i\tilde{\zeta}, 2n+1/2}(4iy) W_{\mp i\tilde{\zeta}, n-1/2}(-2iy) \right. \\ \left. - 2W_{1 \mp i\tilde{\zeta}, n-1/2}(-2iy) W_{\pm 2i\tilde{\zeta}, 2n+1/2}(4iy) \right], \quad (2.111)$$

$$D_2^\pm(y) = \frac{1}{4iy} \frac{\Gamma(2n+1 \mp 2i\tilde{\zeta})}{\Gamma(4n+2)} C_2^\pm(k\eta_i) \left[2M_{\pm 2i\tilde{\zeta}, 2n+1/2}(4iy) W_{1 \mp i\tilde{\zeta}, n-1/2}(-2iy) \right. \\ \left. - (1+2n \pm 2i\tilde{\zeta}) 2W_{1 \pm i\tilde{\zeta}, 2n+1/2}(4iy) W_{1 \mp i\tilde{\zeta}, n-1/2}(-2iy) \right]. \quad (2.112)$$

Here, we introduced $y \equiv |k\eta_e|$ and the function $C_2^\pm$ is

$$C_2^\pm(k\eta_i) = \frac{\Gamma(n \pm i\tilde{\zeta})}{2ik\eta_i \Gamma(2n)} \left[(n \mp i\tilde{\zeta}) M_{1 \mp i\tilde{\zeta}, n-\frac{1}{2}}(2ik\eta_i) - (n \mp i\tilde{\zeta} - 2ik\eta_i) M_{\mp i\tilde{\zeta}, n-\frac{1}{2}}(2ik\eta_i) \right]. \quad (2.113)$$

In the sub-horizon limit at the onset of the $I(\eta)$ evolution, $-k\eta_i \gg 1$, one can use $C_2^\pm \simeq e^{\pm \frac{1}{2}\pi\tilde{\zeta}}$. On the other hand, in the super-horizon limit, $-k\eta_i \ll 1$, one obtains $C_2^\pm \simeq \Gamma(n \pm i\tilde{\zeta}) |2ik\eta_i|^n / \Gamma(2n)$. One finds that in Eq. (2.110), the term including $D_1^\pm$ corresponds to the growing mode and the other is the decaying mode, and the growing mode is always the dominant component of the mode function⁷. For the super-horizon modes at the end of inflation, $|k\eta_e| \ll 1$, D_1 is approximated as,

$$D_1^\pm(y) \approx \frac{2^{-5n} \Gamma(2n)}{(4n+1) \Gamma(n \pm i\tilde{\zeta})} y^{-3n} C_2^\pm \quad (|k\eta_e| \ll 1). \quad (2.114)$$

The mode function at the super-horizon scale is written as,

$$I\mathcal{A}_\pm^{\text{reh}} \simeq \frac{1}{\sqrt{2k}} D_1^\pm(|k\eta_e|) (2k\eta)^{2n+\frac{3}{2}} \quad (k\eta \ll 1), \quad (2.115)$$

where we used the super-horizon approximation of the Whittaker function, $M_{a,b}(x) \simeq x^{b+1/2}$ for $x \ll 1$. In this case, one can safely use Eq. (2.114) since $|k\eta_e| < k\eta \ll 1$. On the other hand, the mode function at the sub-horizon

⁷This statement is valid for $n = 3$. At the beginning of the reheating era ($k\eta \simeq 2|k\eta_e|$), the ratio of the growing and decaying modes is approximated as

$$\left| \frac{D_2^\pm W_{\pm 2i\tilde{\zeta}, \frac{1}{2}(4n+1)}(2ik\eta)}{D_1^\pm M_{\pm 2i\tilde{\zeta}, \frac{1}{2}(4n+1)}(2ik\eta)} \right| \simeq \frac{3\Gamma(2n-1)\Gamma(4n+1)}{2\Gamma(2n)\Gamma(4n+2)} \leq 1 \quad \left(n \geq \frac{5}{4} \right).$$

Thus, the growing modes are the dominant component during the reheating era.

scale is obtained as

$$IA_{\pm}^{\text{reh}} \simeq \frac{1}{\sqrt{2k}} D_1^{\pm}(|k\eta_e|) \frac{e^{\pi\tilde{\zeta}} \Gamma(4n+3)}{\Gamma(2n + \frac{3}{2} \mp 2i\tilde{\zeta})} e^{ik\eta} (k\eta)^{-2i\tilde{\zeta}} \quad (k\eta \gg 1). \quad (2.116)$$

Here we used the sub-horizon approximation of the Whittaker function, $M_{a,b}(x) \simeq x^{-a} e^{x/2} \Gamma(2b+1)/\Gamma(b-a+1/2)$ for $x \gg 1$. In this case, one cannot always use the super-horizon approximation for D_1 .

Electromagnetic Power spectra

By using the analytic solutions of $\mathcal{A}_{\pm}^{\text{reh}}(\eta)$, we can calculate the power spectra of the electromagnetic field. We define the power spectra of the electric and magnetic fields as

$$\mathcal{P}_E^{\pm}(\eta, k) \equiv \frac{k^3 I^2}{2\pi^2 a^4} |\partial_{\eta} \mathcal{A}_{\pm}|^2, \quad \mathcal{P}_B^{\pm}(\eta, k) \equiv \frac{k^5 I^2}{2\pi^2 a^4} |\mathcal{A}_{\pm}|^2. \quad (2.117)$$

In Figs. 2.3 and 2.4, we show the magnetic and electric power spectra for both circular polarizations and compare them. One can see that the electric and magnetic fields are maximally helical and they have peaks. Both the magnetic and electric power spectra have sharp peaks on the same scale which is relatively small, and only the electric power spectrum has a peak on a larger scale at $k \sim |\eta_i|^{-1}$. Let us scrutinize the origin of these small-scale peaks. The tachyonic instability during inflation occurs slightly before the modes exit the horizon $k \sim -2\tilde{\zeta}/\eta \propto a$ and only the right-handed modes are amplified. On super-horizon scales, all the modes are increased at the same rate due to the last term in Eq. (2.107), and the smooth spectra are formed for $-1/\eta_i \lesssim k \lesssim -2\tilde{\zeta}/\eta$ as shown as the dotted lines in Fig. 2.3. On the other hand, the second tachyonic amplification during the subsequent reheating era takes place slightly after the modes re-enter the horizon $k \sim 4\tilde{\zeta}/\eta \propto a^{-1/2}$. Thus the modes in $4\tilde{\zeta}/\eta < k < 2\tilde{\zeta}/|\eta_e|$ acquire the double tachyonic amplifications. Since the modes quickly decay well inside the horizon, the mode that has just gone through the second tachyonic amplification has the largest amplitude. As a result, the power spectra obtain such sharp peaks at $k = 4\tilde{\zeta}/\eta$. While the electric field is stronger than the magnetic field at the large scale, the magnetic field is stronger than the electric field at the peak scale as shown in Fig. 2.4. One analytically confirms these behaviors by using super/sub horizon approximation for the mode function. Note that the small scale peaks are located at $k\eta = 4\tilde{\zeta}$ which is well inside the horizon for $4\tilde{\zeta} \gg 1$. The ratio of the power spectra is estimated as,

$$\frac{\mathcal{P}_B}{\mathcal{P}_E}(\eta_r) = k^2 \frac{|\mathcal{A}_{\pm}|^2}{|\partial_{\eta} \mathcal{A}_{\pm}|^2} \sim \begin{cases} 4 & (k\eta_r = 4\tilde{\zeta}) \\ \frac{4k\eta^2}{(3+4n)^2} \ll 1 & (k\eta_r \ll 1) \end{cases}. \quad (2.118)$$

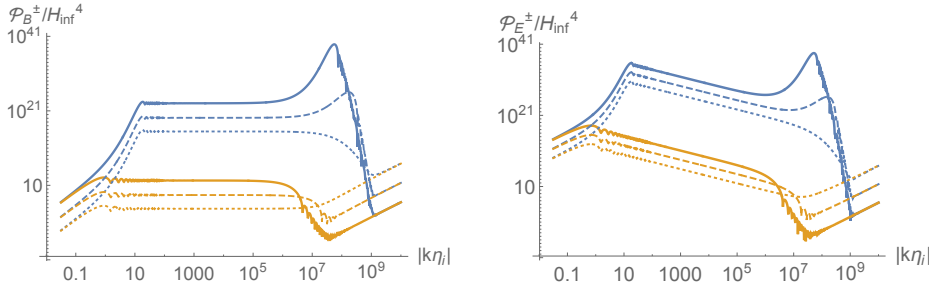


FIGURE 2.3: The magnetic and the electric power spectra are shown in the left and right panels, respectively. The dotted, dashed, and solid lines denote the power spectra at $a = a_e$, $20a_e$ and $400a_e$ respectively. We fixed the parameters as $n = 3$, $\xi = 7.6$ and $a_e = 7.7 \times 10^7 a_i$. The right-handed modes (blue) are amplified by the tachyonic instabilities, while the left-handed modes (orange) are not, because we chose $\xi > 0$.

Here we used Eqs. (2.115) and (2.116). From now on, following the previous paper [91], we use these fiducial values of the model parameters,

$$n = 3, \quad \xi = 7.6. \quad (2.119)$$

Consistency conditions

For consistent magnetogenesis, there should not be a significant back reaction from the electromagnetic field to the background evolution of the universe which was assumed in Eq. (2.105). Then the maximum value of the energy fraction of the electromagnetic field is much less than unity,

$$\Omega_{\text{EM}} \equiv \frac{1}{2\rho_{\text{tot}}} \int \frac{dk}{k} (\mathcal{P}_E(k) + \mathcal{P}_B(k)) \ll 1, \quad (2.120)$$

where $\rho_{\text{tot}} = 3M_{\text{pl}}^2 H^2$ is the total energy density of the universe. Since the power spectra of the electromagnetic field have prominent peaks at two different scales, we separate the main contributions to Ω_{EM} as,

$$\Omega_{\text{EM}}(\eta_e < \eta \leq \eta_r) \simeq \Omega_{\text{EM}}|_{k\eta \sim 4\xi} + \Omega_{\text{EM}}|_{|k\eta_i| \sim 2\xi}. \quad (2.121)$$

Here $\Omega_{\text{EM}}|_{k\eta \sim 4\xi}$ denotes the energy fraction contributed from the small scale peaks, and $\Omega_{\text{EM}}|_{|k\eta_i| \sim 2\xi}$ comes from the large scale peak. $\Omega_{\text{EM}}(\eta)$ takes its maximum value when the electromagnetic field stops growing. Thus we impose the back reaction condition on the energy fraction at $\eta = \eta_r$ as

$$\Omega_{\text{EM}}(\eta_r)|_{k\eta \sim 4\xi} = 10^{-2}, \quad \Omega_{\text{EM}}(\eta_r)|_{|k\eta_i| \sim 2\xi} \ll 10^{-2}, \quad (2.122)$$

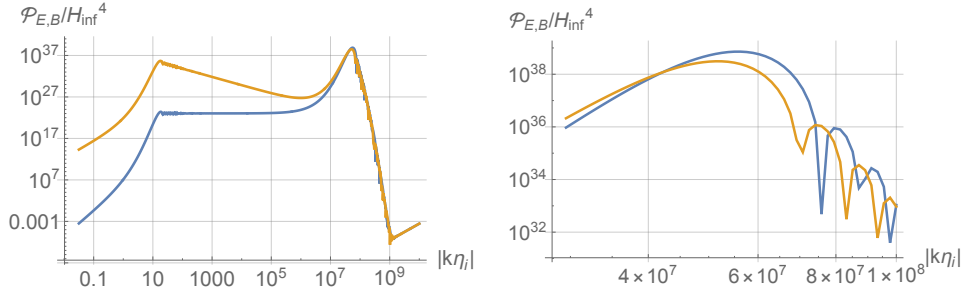


FIGURE 2.4: The magnetic (blue) and electric (orange) power spectra are compared at $a = 10^2 a_e$. The right panel is the enlarged figure of the left panel around the small-scale peak, $k = 4\xi/\eta$. The magnetic field is stronger than the electric field around the small-scale peak, while the electric field is stronger than the magnetic field on larger scales. The parameters are the same as Fig. 2.3.

where we require that the large-scale peak has a negligible contribution compared to the small-scale peaks. In the following, we evaluate these contributions to determine the model parameters which satisfy the above conditions.

Let us evaluate the energy fraction from the small-scale peaks at $k\eta = 4\xi$. Since the magnetic field is four times stronger than the electric field at the peak scale (see Eq. (2.118)), the right-handed magnetic part is the main component of $\Omega_{\text{EM}}(\eta_r)|_{k\eta \sim 4\xi}$. Hence we evaluate the following integral,

$$\begin{aligned} \Omega_{\text{EM}}(\eta_c)|_{k\eta \sim 4\xi} &\simeq \frac{1}{2\rho_{\text{tot}}} \int \frac{dk}{k} \mathcal{P}_B(k) \\ &\simeq \frac{1}{96\pi^2 M_{\text{Pl}}^2 a_r^2 \eta_r^2} \int_1^{4\xi} \frac{dy}{y} y^4 \left| D_1^+ \left(\frac{1}{2} e^{-\frac{N_r}{2}} y \right) M_{2i\xi, 2n+\frac{1}{2}}(2iy) \right|^2, \end{aligned} \quad (2.123)$$

where we used the $H = 2/a\eta$ at the matter dominant era and we introduced the dummy variable $y \equiv k\eta_r$. Since the significant contributions to $\Omega_{\text{EM}}(\eta_c)|_{k\eta \sim 4\xi}$ are in sub-horizon scale, we ignore the super-horizon modes, $0 < k\eta_r < 1$. Additionally, the modes on smaller scales than the peak scale, $k > 4\xi/\eta_r$, have negligible contributions, because they have smaller amplitudes and are highly oscillating. Therefore the interval of the integration of $k\eta_r$ can be limited to $1 < k\eta_r < 4\xi$. N_r represents the e-folding number between the end of inflation and that of the reheating phase,

$$\begin{aligned} N_r &= \ln \left(\frac{\eta_r}{2|\eta_e|} \right)^2 = \ln \left(\frac{a_r}{a_e} \right) = \frac{1}{3} \ln \left(\frac{\rho_{\text{inf}}}{\frac{\pi^2}{30} g_* T_r^4} \right) \\ &\approx 8.05 + \frac{4}{3} \ln \left(\frac{\rho_{\text{inf}}^{1/4}}{10^7 \text{GeV}} \right) - \frac{4}{3} \ln \left(\frac{T_r}{10^4 \text{GeV}} \right) - \frac{1}{3} \ln \left(\frac{g_*}{100} \right), \end{aligned} \quad (2.124)$$

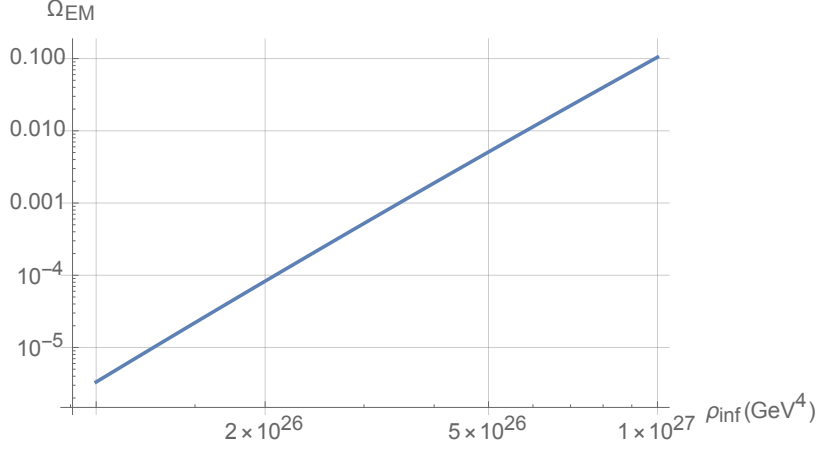


FIGURE 2.5: The relation between Ω_{EM} and ρ_{inf} for $T_r = 2.2 \times 10^4 \text{ GeV}$. To satisfy the consistency condition, $\Omega_{\text{EM}} \ll 1$, we choose $\rho_{\text{inf}} < 5.85 \times 10^{26} \text{ GeV}^4$. There is the lower bound on ρ_{inf} since the energy density at reheating completion is smaller than ρ_{inf} , $\rho_{\text{inf}} > \pi^2 g_* T_r^4 / 30 \approx 10^{18} \text{ GeV}^4$.

where T_r and g_* denote the temperature and the number of degrees of freedom at the reheating completion, respectively. By using the equation of the entropy conservation we obtain the condition for a_r as

$$a_r \approx 8.0 \times 10^{-18} \left(\frac{T_r}{10^4 \text{ GeV}} \right)^{-1} \left(\frac{g_{*s}}{100} \right)^{-\frac{1}{3}}, \quad (2.125)$$

where we introduce the number of degrees of freedom for entropy, g_{*s} , and we assume that $g_{*s} = g_* = 100$ from now on. η_r is also represented by T_r as,

$$\eta_r = 2e^{\frac{1}{2}N_r - N_i} |\eta_i| \approx 1.18 \times 10^{-11} \left(\frac{T_r}{10^4 \text{ GeV}} \right)^{-1} \text{ Mpc}, \quad (2.126)$$

where $N_i \equiv \ln(a_e/a_i) \approx 29.9 + 2/3 \ln(\rho_{\text{inf}}/10^7 \text{ GeV}) + 1/3 \ln(T_r/10^4 \text{ GeV}) - \ln(k_i/1 \text{ Mpc}^{-1})$. One can numerically evaluate Eq. (2.123) by substituting particular values into T_r and ρ_{inf} . We show the relation between ρ_{inf} and Ω_{EM} in Fig. 2.5 for a fixed reheating temperature, $T_r = 2.2 \times 10^4 \text{ GeV}$. In this case, one finds that $\rho_{\text{inf}} \approx 6 \times 10^{26} \text{ GeV}^4$ satisfies our consistency condition, Eq. (2.122).

Let us consider the large-scale contribution to Ω_{EM} . The large-scale peak can be larger than the small-scale peak since the electric power spectrum depends on k^{-2} at super-horizon scales for $n = 3$. We evaluate the contribution

of the large-scale electric field

$$\begin{aligned}
\Omega_{\text{EM}}(\eta_r)|_{k\eta_i \sim 2\tilde{\zeta}} &\approx \frac{1}{2\rho_{\text{tot}}} \int \frac{dk}{k} \mathcal{P}_E(k\eta \ll 1), \\
&= \frac{25}{2\pi^3} \frac{\rho_{\text{inf}}}{M_{\text{Pl}}^4} e^{2N_i+5N_r} \frac{\mathcal{I}(\tilde{\zeta}) \sinh(\pi\tilde{\zeta})}{\tilde{\zeta}^5 + 5\tilde{\zeta}^3 + 4\tilde{\zeta}}, \\
&\approx 10^{-3} \left(\frac{T_r}{10^4 \text{GeV}} \right)^{-6} \left(\frac{\rho_{\text{inf}}^{1/4}}{10^7 \text{GeV}} \right)^3 \left(\frac{k_i}{10^7 \text{Mpc}^{-1}} \right)^{-2}.
\end{aligned} \tag{2.127}$$

Here we introduce the numerical fits $\mathcal{I}(\tilde{\zeta}) \equiv \int_0^\infty \frac{d(-k\eta_i)}{(-k\eta_i)^3} |C_2^+(-k\eta_i)| \approx \frac{e^{\pi\tilde{\zeta}}}{4.7\tilde{\zeta}^2 + 2.7\tilde{\zeta} + 10}$ valid for $1 \leq \tilde{\zeta} \leq 50$ [91]. Then, one can suppress the contribution from the super-horizon mode by sending k_i to a sufficiently small scale.

Inverse cascade and the present magnetic field strength

Here we evaluate the present magnetic field strength by considering the inverse cascade process in magnetohydrodynamics. It is known that when helical magnetic field and plasma tightly interact with each other in the turbulent and high conductivity regime, the correlation length of the magnetic field increases, because of the magnetic helicity conservation. This process is called the inverse cascade. Our magnetogenesis scenario generates a maximally helical, small-scale, and strong magnetic field. Thus the subsequent inverse cascade automatically works and enables us to obtain a magnetic field whose correlation length is much larger than the case only with the adiabatic expansion. This property is advantageous to explain the observational lower bound, Eq. (2.79).

The magnetic helicity density $\mathcal{H}(\eta)$ is defined as the volume average of the local helicity $\mathbf{A} \cdot \mathbf{B}(\eta, \mathbf{x})$,

$$\mathcal{H} \equiv \frac{1}{V} \int_V d^3x \epsilon_{ijk} A_i \partial_j A_k \tag{2.128}$$

$$\simeq a^3 \lambda_{\text{phys}} B_{\text{phys}}^2 \quad (\text{maximally helical}) \tag{2.129}$$

where V is the comoving volume, $\mathbf{B}_{\text{phys}} \equiv -a^{-2} \nabla \times \mathbf{A} \simeq a^{-1} \mathbf{A} / \lambda_{\text{phys}}$ is the physical magnetic field and λ_{phys} is its physical correlation length. The magnetic field on the boundary of the volume V is assumed to have no normal component for the gauge invariance. It is known that the helicity is conserved in the magnetohydrodynamics limit in which the electric conductivity becomes infinite. In the context of cosmology, the helicity approximately conserves after the reheating completion. It can be also shown that the helicity represents the difference between the right-handed and left-handed polarization contributions to Eq. (2.128). In our case, the right-handed component is much stronger than the left-handed one and the generated magnetic field is maximally helical. Then the simplified evaluation, Eq. (2.129), is available.

Now let us consider the inverse cascade process for our maximally helical magnetic field. Provided that the helicity density $\mathcal{H}$ in Eq. (2.128) is conserved, the helicity at the end of the magnetogenesis and the present time are the same. B_{phys} and λ_{phys} can be evaluated by the maximum value of the magnetic power spectrum $\mathcal{P}_B(k_{\text{peak}})$ and $2\pi a/k_{\text{peak}}$, respectively. Thus the helicity evaluated at $\eta = \eta_r$ is given by

$$\mathcal{H} \simeq a_s^3 \lambda_{\text{phys}}(\eta_r) B_{\text{phys}}^2(\eta_r) \simeq a_r^3 \frac{2\pi a_r \eta_r}{4\zeta} \mathcal{P}_B(\eta_r, k_{\text{peak}}), \quad (2.130)$$

where we used $k_{\text{peak}} \equiv 4\zeta/\eta_r$, $I^2 B_{\text{phys}}^2 \simeq \mathcal{P}_B(k_{\text{peak}})$ and $I(\eta_r) = 1$. On the other hand, the blazar observations are sensitive to B_{eff} defined in Eq. (2.79) which is directly related to the helicity at the present time as $B_{\text{eff}} \simeq \sqrt{\mathcal{H}/1\text{Mpc}}$ for $\lambda < 1\text{Mpc}$. Therefore, we obtain B_{eff} as,

$$B_{\text{eff}} \approx 8 \times 10^{-14} \text{G} \left(\frac{\Omega_{\text{EM}}}{10^{-2}} \right)^{1/2} \left(\frac{T_r}{10^4 \text{GeV}} \right)^{-1/2} \left(\frac{\zeta}{10} \right)^{-1/2}. \quad (2.131)$$

Here we used $M_{\text{Pl}} = 2.43 \times 10^{18} \text{GeV}$ and $G = 6.8 \times 10^{-20} \text{GeV}^2$, $\text{Mpc} = 1.56 \times 10^{38} \text{GeV}^{-1}$, and $\mathcal{P}_B(\eta_r, k_{\text{peak}}) \approx 2\rho_{\text{tot}}\Omega_{\text{EM}}$. Hence, the observational constraint Eq. (2.79) can be explained in our model.

2.5 Important phenomena on Inflationary Magnetogenesis

Finally in this chapter, we introduce the ignored phenomena when we estimate the present magnetic field in the previous sections. We did not consider the interaction between the electromagnetic field and the charged particles in detail. We assumed instant dissipation and ignored the non-perturbative phenomena such as the Schwinger effect. The Schwinger effect is the non-perturbative phenomenon in QED, in which charged particles and antiparticles are created by the strong electric field. When the variation time scale of the electric field, $\tau_E = E/\dot{E}$, is larger than the Hubble time, $\tau_H = 1/H$ ($\tau_H < \tau_E$), the current of the charged particles is proportional to the electric field and the coefficient is electric conductivity. The Schwinger effect suppresses the evolution of the electric field since the current represented by Ohm's law behaves as the friction term in the EoM for the electric field and the growth of the magnetic field are also suppressed [3, 4, 10, 12–14, 16, 17, 94–96]. The estimations given in the above references are based on the assumption of $\tau_H < \tau_E$. However, there exists an abrupt change of the electric field in the inflationary magnetogenesis, in other words, $\tau_E < \tau_H$. As a concrete example, we can confirm the sudden variation of the electric field caused by the tachyonic instability in the previous section. To deal with such a sudden variation of the electric field precisely, we need to consider other prescriptions for the Schwinger effect instead of the induced current calculation. Then, in this

thesis, we will focus on the estimation of the Schwinger effect in the context of the Inflationary Magnetogenesis.

We did not consider other potentially important phenomena related to primordial magnetogenesis, such as chiral anomaly and baryogenesis in the model. The chiral anomaly associates the helicity of the electromagnetic field with the chiral asymmetry of the charged fermions which we neglect in this paper. The chiral asymmetry modifies the inverse cascade through the chiral magnetic effect and the final magnetic field would be significantly different [17, 97–104]. It has been pointed out that the helical primordial magnetic field can be responsible for the generation of the baryon asymmetry in our universe [105–108]. This mechanism may also constrain magnetogenesis models when the over-production of baryons is predicted. However, the chiral plasma instability may cancel the helicity of the magnetic field and the over-production of the baryons may be suppressed [109]. Careful calculations are needed to determine the amount of the cancellation, and it is beyond the scope of this thesis. The above phenomena require dedicated investigations to evaluate their implication and it is worth studying them in broader contexts than primordial magnetogenesis.

Chapter 3

Schwinger Effect

The production and the dynamics of the charged particles in the strong electromagnetic field are intensively investigated in several areas of physics. One of the interesting phenomena in the strong electromagnetic field is the Schwinger effect, which is the non-perturbative pair production from the strong electromagnetic field [110–112]. The original setup is in the static electric field, E , and the production rate is proportional to $\exp\left(-\pi\frac{m^2}{eE}\right)$, where m and e represent the mass and the coupling constant of the produced charged particles, respectively. This production rate is exponentially suppressed until the electric field reaches the critical value, $E_{\text{cr}} = \pi m^2/e \sim 10^{18} \text{ V/m}$ ¹ for the electrons and the positrons, and which has not yet been achieved experimentally. The LUXE experiment (Laser Und XFEL Experiment), which is a new laser experiment using the European X-ray free electron laser (XFEL), is expected to achieve the field strength close to the Schwinger critical values [113], and therefore the Schwinger effect would be experimentally verified soon. In the perturbation theory, we consider the coefficients of the Taylor expansion around $e = 0$. The contribution to the observables by the higher order correction is generally so smaller than one by the lower order correction that we can cut off the higher order contribution depending on use. For the Schwinger effect, the coefficients of the Taylor expansion around $e = 0$ are zero for all orders and we cannot use the same analysis as the perturbation theory. This is because this effect is non-perturbative. We can intuitively understand this non-analytical behavior of the transition amplitude as the tunneling effect of the particles from the Dirac Sea to the positive energy states. Let us consider the simple case of tunneling, where the massive charged particle is in the constant electric field. m , e , and $V(x) = -Ex$ are the particle mass, the charge, and the potential of the external constant electric field. For the particles in the boundary of the Dirac sea, the potential barrier is $2m - eEx$ ($0 < x < 2m/eE$). Then by using the WKB approximation, we can roughly estimate the transition probability as $\exp\left(\frac{16m^2}{3eE}\right)$. This result is similar to the production rate of the Schwinger effect. Several physical situations which may cause the Schwinger effect are discussed, such as relativistic heavy ion collisions [114], neutron stars [115, 116], charged black holes [117,

¹The critical value of the electric field is converted as, $E_{\text{cr}} \sim 10^{-6} \text{ GeV}^2 \sim 10^{14} \text{ G}$. When the magnetic field is switched on, the effective magnetic mass takes the place of mass in the production rate, which is called the quantum hole effect in the context of the condensed matter physics, see ref. [PhysRevLett.45.494].

118], and inflation with the amplification of the electromagnetic field [2, 69, 76–83, 85, 86, 91, 92]. In particular, such inflation models are intensively studied in the context of inflationary magnetogenesis, which explain the origin of the magnetic field in our universe by inflation and subsequent high energy phenomena, such as preheating or reheating. We will focus on the Schwinger effect in the inflationary background in the next chapter.

3.1 Proper Time Method

Here we briefly review the proper time method to derive the Euler-Heisenberg Lagrangian, which is the effective Lagrangian of the QED. Schwinger's original paper applies the proper time method to preserve the gauge invariance of the analysis. Before considering the proper time method to derive the effective Lagrangian, we focus on the calculation of the Schwinger pair production from the given Lagrangian.

3.1.1 Reaction rate in case of the constant electric field.

The Euler-Heisenberg Lagrangian contains the 1-loop correction to the electromagnetic dynamics with the electron propagators, which is written as,

$$\mathcal{L}^{(1)} = -\frac{1}{8\pi^2} \int_0^\infty \frac{ds}{s^3} e^{-m^2 s} \left\{ \left[(es)^2 \mathcal{G} \frac{\text{Re} \cosh(esX)}{\text{Im} \cosh(esX)} \right] - \frac{2}{3} (es)^2 \mathcal{F} - 1 \right\}, \quad (3.1)$$

where we introduced the gauge invariant variables of the electromagnetic field as,

$$\begin{aligned} X^2 &= 2(\mathcal{F} + i\mathcal{G}) = (\mathbf{B} + i\mathbf{E})^2, \\ \mathcal{F} &= \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{\mathbf{B}^2 - \mathbf{E}^2}{2}, \quad \mathcal{G} = \frac{1}{4} F_{\mu\nu} \tilde{F}^{\mu\nu} = \mathbf{E} \cdot \mathbf{B}. \end{aligned} \quad (3.2)$$

Here we denote $\tilde{F}^{\mu\nu} \equiv \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$. Let us consider the constant electric field in the above Lagrangian. The Lagrangian is rewritten as,

$$\mathcal{L}^{(1)} = -\frac{1}{8\pi^2} \int_0^\infty \frac{ds}{s^3} e^{-m^2 s} \left\{ esE \cot[esE] + \frac{1}{3} (es) E^2 - 1 \right\}, \quad (3.3)$$

where we used $\lim_{x \rightarrow 0} \frac{x}{e^{b(\sqrt{a^2+x^2}-a)^{1/2}} - e^{-b(\sqrt{a^2+x^2}-a)^{1/2}}} = \frac{\sqrt{a}}{\sqrt{2b}}$. Since the above integration have the poles as $r_n = \frac{n\pi}{eE}$ for $n \in \mathbb{N}$, we can perform the integration by adding a small imaginary part $i\epsilon$ to the denominator and applying the Sokhotski-Plemelj formula,

$$\lim_{\epsilon \rightarrow 0} \frac{1}{r - r_n \mp i\epsilon} = \text{P} \frac{1}{r - r_n} \pm i\pi\delta(r - r_n), \quad (3.4)$$

where P means the principal value of the integration. Then we have the following equation,

$$\mathcal{L}^{(1)} = \frac{1}{8\pi^2} \left(i\pi \sum_{n=1}^{\infty} \frac{(eE)^2}{(n\pi)^2} e^{-m^2 \frac{n\pi}{eE}} + \frac{1}{3} \int_0^{\infty} \frac{ds}{s} e^{-m^2 s} \right). \quad (3.5)$$

The probability of the transition from vacuum to vacuum is represented as,

$$\langle 0_{\text{out}} | 0_{\text{in}} \rangle^2 = \left| \exp \left(i \int d^4x \mathcal{L} \right) \right|^2 = \exp \left(- \int d^4x 2\text{Im}\mathcal{L} \right) \quad (3.6)$$

where the subscripts, out and in, denote the out and in vacuum, respectively. Since the QED original Lagrangian is a real number, the imaginary parts of the Lagrangian are extracted from eq. (3.5). The production rate of the particles, p , is estimated as,

$$p = 1 - \langle 0_{\text{out}} | 0_{\text{in}} \rangle^2 \approx \int d^4x 2\text{Im}\mathcal{L}, \quad (3.7)$$

where we used the assumption that the exponent of the transition amplitude is much smaller than unity. Then the reaction rate of the pair production per unit volume and unit time, Γ is written as,

$$\Gamma = \frac{2(eE)^2}{(2\pi)^3} \sum_{n=1}^{\infty} \frac{1}{n^2} \exp \left(-\frac{n\pi m^2}{eE} \right), \quad (3.8)$$

This is the original representation of the production rate for the Schwinger effect.

3.1.2 Derivation of the Euler-Heisenberg Lagrangian

Then, let us derive the Euler-Heisenberg Lagrangian by using the proper time method based on ref. [112]. The effective action with the external electromagnetic field, $W[A]$, decomposed into the classical part and quantum correction part as,

$$W = W^{(0)} + W^{(1)}, \quad (3.9)$$

and the effective action satisfies the following relation,

$$\begin{aligned} \frac{\delta W^{(1)}[A]}{\delta A_\mu} &= \langle j^\mu \rangle = \langle 0 | \frac{e}{2} [\bar{\psi}, \gamma^\mu \psi] | 0 \rangle, \\ &= ie \text{tr}[\gamma^\mu G_A(x, x)], \end{aligned} \quad (3.10)$$

where ψ is the fermion field coupled to the electromagnetic field, and tr indicates the diagonal sum with spinor indices. $G_A(x, x)$ is the Green function of the Dirac equation, defined as $(\gamma^\mu D_\mu - m) G_A(x, y) = \delta(x - y)$, where we defined $D_\mu \equiv i\partial_\mu - eA_\mu$. We introduce the operator, $\hat{G}$, corresponds to the

green function, and the operator satisfies the following operator equation as,

$$\hat{G} = \frac{1}{\gamma^\mu \hat{D}_\mu - m}, \quad (3.11)$$

where the 1 is the unit operator. The above equation can be formally solved by introducing the proper time as,

$$\hat{G}[A] = (m + \gamma^\mu \hat{D}_\mu) i \int_0^\infty ds e^{-is(-m^2 + i\epsilon + (\gamma \hat{D})^2)}, \quad (3.12)$$

where we introduced the $i\epsilon$ prescription for the convergence of the integration. We can easily confirm that the solution of eq. (3.10) is represented as,

$$W^{(1)}[A] = \frac{i}{2} \int_0^\infty \frac{ds}{s} e^{ism^2} \text{Tr} e^{-is(\gamma \hat{D})^2} \quad (3.13)$$

where Tr means taking trace for x and spinor indices. Thus we obtain the formal solution of the effective Lagrangian, $\mathcal{L}^{(1)}$, as,

$$\mathcal{L}^{(1)} = \frac{i}{2} \text{tr} \int_0^\infty \frac{ds}{s} e^{ism^2} \langle x | e^{-is(\gamma \hat{D})^2} | x \rangle. \quad (3.14)$$

Thus we need to evaluate the x-representation of $e^{is(\gamma \hat{D})^2}$. Regarding this operator as the evolution operator, the problem can be interpreted into the quantum mechanics for the “particle” depending upon the proper time and the evolution operator. We introduce the “Hamiltonian” from the effective Lagrangian as,

$$\langle x(s) | y(0) \rangle = \langle x | e^{-isH} | y \rangle \equiv \langle x | e^{-is(\gamma \hat{D})^2} | y \rangle. \quad (3.15)$$

Here $\langle x(s) | y(0) \rangle$ represents the transition amplitude from a state where $x(s=0)$ has the value y to a state where $x(s)$ has the value x . H is estimated as,

$$H = \Pi_\mu^2 - \frac{1}{2} e \sigma^{\mu\nu} F_{\mu\nu}, \quad \Pi_\mu \equiv p_\mu - e A_\mu \quad (3.16)$$

Here we used the commutation relation as, $[\Pi^\mu, \Pi^\nu] = -ieF^{\mu\nu}$. We also introduce, $\sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu]$. We also obtain the evolution operator of the system, $U(s) = e^{-isH}$, which satisfies the Schrödinger equation as,

$$i \frac{\partial U(s)}{\partial s} = H U(s). \quad (3.17)$$

To find the solution of the transition amplitude, we should solve the equations for the particles. We have the Heisenberg equations for the particle as,

$$\frac{dx_\mu(s)}{ds} = -i[x_\mu, H] = 2\Pi_\mu, \quad (3.18)$$

$$\frac{d\Pi_\mu(s)}{ds} = -i[\Pi_\mu, H] = 2eF_{\mu\nu}\Pi^\nu - ie\partial^\nu F_{\mu\nu} + \frac{e}{2}\sigma_{\lambda\nu}\partial_\mu F^{\lambda\nu}. \quad (3.19)$$

Let us consider the constant electromagnetic field, and the second line of the above equations can be simplified. Then we focus on the following equations as,

$$\frac{dx_\mu(s)}{ds} = 2\Pi_\mu, \quad \frac{d\Pi_\mu(s)}{ds} = 2eF_{\mu\nu}\Pi^\nu. \quad (3.20)$$

We can formally obtain the solutions from the above equations in the matrix form,

$$\Pi(s) = e^{2esF}\Pi(0), \quad (3.21)$$

$$x(s) - x(0) = (eF)^{-1} \left(e^{2esF} - 1 \right) \Pi(0). \quad (3.22)$$

We can rewrite the solutions about Π as,

$$\Pi(0) = \frac{1}{2}eFe^{-esF}\sinh^{-1}(esF)(x(s) - x(0)), \quad (3.23)$$

$$\Pi(s) = \frac{1}{2}eFe^{esF}\sinh^{-1}(esF)(x(s) - x(0)) = \frac{1}{2}(x(s) - x(0))eFe^{-esF}\sinh^{-1}(esF), \quad (3.24)$$

where we consider the transposition of the matrix in the last equality on the second line. By the transposition, the field strength of the electromagnetic field changes the sign as $F_{\mu\nu}^{(t)} = -F_{\mu\nu}$, which is used to confirm the commutation relation between $x(s)$ and $x(0)$ as,

$$[x(s), x(0)] = i(eF)^{-1}(e^{2esF} - 1) \quad (3.25)$$

Then we can evaluate the Hamiltonian by the above expression,

$$\begin{aligned} H + \frac{1}{2}e\sigma^{\mu\nu}F_{\mu\nu} &= (x(s) - x(0))K(x(s) - x(0)) \\ &= x(s) K x(s) - 2x(s) K x(0) + x(0) K x(0) - \frac{i}{2}\text{tr}eF\coth(esF) \end{aligned} \quad (3.26)$$

where we define $K \equiv \frac{1}{4}(eF)^2 (\sinh(esF))^{-2}$. Thus we obtain the differential equation for the transition amplitude, $\langle x(s)|y(0) \rangle$ as,

$$i \frac{\partial}{\partial s} \langle x(s)|y(0) \rangle = [(x-y)K(x-y) - \frac{i}{2} \text{tr } eF \coth(esF) - \frac{1}{2} e\sigma^{\mu\nu} F_{\mu\nu}] \langle x(s)|y(0) \rangle. \quad (3.27)$$

We have the solution of the above equation as,

$$\langle x(s)|y(0) \rangle = Z(x, y) e^{L(s)} s^{-2} \exp \left[\frac{i}{4} (x-y) eF \coth(esF) (x-y) + \frac{i}{2} \sigma esF \right], \quad (3.28)$$

where $L(s) \equiv \frac{1}{2} \text{tr} \ln[(esF)^{-1} \sinh(esF)]$. We can easily confirm the solutions by substituting the eq. (3.28) into eq. (3.27). $Z(x, y)$ is the time independent function. Considering $\langle x(s)|\Pi_\mu|y(0) \rangle$ same as eq. (3.17), we obtain the equation for $Z(x, y)$. Then we obtain the condition for $Z(x, y)$ as,

$$Z(x, y) = \frac{-i}{(4\pi)^2} \Phi(x, y) \\ \Phi(x, y) \equiv \exp \left(ie \int_x^y dx \cdot A(x) \right), \quad (3.29)$$

where the coefficient of $Z(x, y)$ is determined by the absence of the external field at $s \rightarrow 0$. Finally, we can determine the transition amplitude and we can calculate the effective Lagrangian as,

$$\begin{aligned} \mathcal{L}^{(1)} &= \frac{i}{2} \int_0^\infty \frac{ds}{s} e^{im^2 s} \text{tr} \langle x(s)|y(0) \rangle_{y \rightarrow x} \\ &= \frac{1}{32\pi^2} \int \frac{ds}{s^3} e^{-m^2 s} e^{-\frac{1}{2} \text{tr} \ln[(esF)^{-1} \sinh(esF)]} \text{tr} e^{\frac{1}{2} e\sigma s F} \quad (s \rightarrow is), \end{aligned} \quad (3.30)$$

where we changed the integration path as $s \rightarrow is$ to exhibit the Lagrangian more explicitly as a real quantity. Then, we evaluate the Dirac matrix trace in the above equation, and by using the definition of the gauge invariant quantity given in eq. (3.2), we obtain the Euler-Heisenberg Lagrangian shown at the beginning of this section. Here we apply the proper time method for the constant electromagnetic field. Of course, we can investigate other electromagnetic configurations, such as plane waves. This method was originally suggested by Schwinger, and other methods to evaluate the Schwinger effect are developed one after another. In the next section, we introduce a more simple (and easy to perform the numerical calculation) approach to the Schwinger effect.

3.2 Equivalent Method for the Schwinger Effect

In the previous section, we consider the original formulation of the Schwinger effect, called the proper time method. The strategy of the proper time formulation is as follows: first, by assuming the configuration of the electromagnetic field, we solve the operator EoM, eqs. (3.18) and (3.19), for the moment, $\Pi(s)$ and $\Pi(0)$; secondary, we substitute the solution into the Hamiltonian and solve the Schrödinger equation for the time evolution operator to obtain the transition amplitude in the coordinate representation; and finally, by considering the amplitude given by the momentum, we find the coefficient of the transition amplitude and substitute it into the Green function represented by the proper time integration.

In this section, we consider another type of formulation for the Schwinger effect, where the particle production is described by the Bogoliubov transformation [119]. We have already confirmed that the reaction rate of the Schwinger effect is related to the probability of the vacuum collapse or pair production of the particle and has the no information about the number of the particles. The number of particles sometimes affects the dynamics of the system. For example, the generated particles move along the electromagnetic field and make an electric current which reduces the intensity of the external electric field. In addition, if the number of the charged fermions is large, the reaction of the Schwinger effect can be suppressed by the Pauli blocking. Considering the above discussion, it is wiser to focus on the number of particles rather than the reaction rate itself. The difficulty of this formulation is the ambiguity of the definition of the particle. If we have the global timelike Killing vector, we have the criterion of the negative and positive frequency. On the other hand, if the spacetime loses the timelike Killing vector because of the external field, we do not define the particle according to the standard. Thus we note that an ambiguity about the definition of the particle remains in this formulation, even though we instantaneously define the particles here.

In this formulation, we obtain the time evolution equation for the number density as the momentum spectrum of the particles, which can be solved by using the solution of the mode functions. In the case of the uniform external field, we can formulate the time evolution equation for the distribution function related to the momentum spectrum as the Boltzmann equation. Such an approach, where we see the time evolution of the distribution function of the particles, is called the kinetic approach. We can obtain the explicit form of the source term by using field-theoretic methods or phenomenological construction where the asymptotic behavior is correct.

From now on, we first introduce the formulation of the Bogoliubov transformation method. Second, we explain the kinetic approach based on the field-theoretic method and the phenomenological construction method.

3.2.1 Bogoliubov Transformation

Here we introduce the other idea to calculate the Schwinger effect, where we apply the Bogoliubov transformation. This approach is equivalent to the

canonical quantization of the field where the particle production is denoted by the Bogoliubov transformation.

The following discussion is based on Refs. [17, 120, 121]. We consider the fermion field coupled to the electric field as

$$\mathcal{L} = \bar{\psi} i \gamma^\mu (\partial_\mu + ie A_\mu) \psi - m \bar{\psi} \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \quad (3.31)$$

Here ψ represents the fermion field, $\bar{\psi}$ is defined as $\psi^\dagger \gamma^0$ and m is the mass of the fermions. A_μ , e , and $F_{\mu\nu}$ are the electromagnetic field, the coupling constant, and the field strength of A_μ . Here we introduce the method to find the solution of the field equation from the above lagrangian in case the electromagnetic field is classical and spatially homogeneous, which means $F_{\mu\nu} = F_{\mu\nu}(t)$. The field equations derived from Eq. (3.31) are written as,

$$\begin{aligned} (i \gamma^\mu \partial_\mu - e \gamma^\mu A_\mu - m) \psi &= 0, \\ \partial_\mu F^{\mu\nu} &= j^\nu = e \bar{\psi} \gamma^\nu \psi, \end{aligned} \quad (3.32)$$

where j^ν is the current whose components are defined in the (RHS) in the above equation. Since we assume the spatial homogeneity, $j^0=0$. The first equation is the Dirac equation with the classical homogeneous electric field.

First, we consider the past infinity case ($t \rightarrow -\infty$), where the electric interaction vanishes as

$$\lim_{t \rightarrow -\infty} A_\mu = 0. \quad (3.33)$$

Moreover, we choose the Weyl gauge where the time component of the electromagnetic field is fixed to 0. Then we can solve the Dirac equation there and we can also linearly expand the fermion field as,

$$\hat{\psi} = \sum_{\lambda=\pm} \int d^3x \left[\psi_{p,\lambda}^{\text{in},+}(x) a_{p,\lambda}^{\text{in}} + \psi_{p,\lambda}^{\text{in},-}(x) b_{p,\lambda}^{\text{in},\dagger} \right], \quad (3.34)$$

where $\psi_{p,\lambda}^{\text{in},\pm}$ is the classical solution of the Dirac equation and $b_{p,\lambda}^{\text{in}}$ are the annihilation operator for the particles and antiparticles with the momentum p and spin s , respectively. The superscript $\pm$ in each operator denotes the positive/negative energy modes, and the superscript represents the solution satisfying the condition where the mode function becomes free at the past infinity, $t \rightarrow -\infty$. Thus we obtain the asymptotic condition for the in-mode functions as,

$$\begin{aligned} \lim_{t \rightarrow -\infty} \psi_{p,\lambda}^{\text{in},+}(x) &= u(p, \lambda) e^{-ip \cdot x}, \\ \lim_{t \rightarrow -\infty} \psi_{p,\lambda}^{\text{in},-}(x) &= v(p, \lambda) e^{ip \cdot x}, \end{aligned} \quad (3.35)$$

where $u(\mathbf{p}, \lambda)$ and $v(\mathbf{p}, \lambda)$ denote the free spinors normalized as,

$$\bar{u}(\mathbf{p}, \lambda) \gamma^\nu u(\mathbf{p}, \lambda') = \bar{v}(\mathbf{p}, \lambda) \gamma^\nu v(\mathbf{p}, \lambda') = 2p^\nu \delta_{\lambda, \lambda'}. \quad (3.36)$$

Here we define the inner product at the time slice t by

$$(\psi | \chi) \equiv \int d^3 \mathbf{x} \psi^\dagger(t, \mathbf{x}) \chi(t, \mathbf{x}). \quad (3.37)$$

On the above definition, the mode functions normalized as,

$$\begin{aligned} (\psi_{\mathbf{p}, \lambda}^{\text{in}, \pm} | \psi_{\mathbf{p}', \lambda'}^{\text{in}, \pm}) &= 2\omega_{\mathbf{p}} (2\pi)^3 \delta(\mathbf{p} - \mathbf{p}') \delta_{\lambda, \lambda'}, \\ (\psi_{\mathbf{p}, \lambda}^{\text{in}, \pm} | \psi_{\mathbf{p}', \lambda'}^{\text{in}, \mp}) &= 0, \end{aligned} \quad (3.38)$$

where $\omega_{\mathbf{p}}$ is the energy of the particle with momentum $\mathbf{p}$. By using the above conditions, we can represent $a_{\mathbf{p}, \lambda}^{\text{in}}$ and $b_{\mathbf{p}, \lambda}^{\text{in} \dagger}$ by the inner products $(\psi_{\mathbf{p}, \lambda}^{\text{in}, \pm} | \hat{\psi})$ as follows,

$$a_{\mathbf{p}, \lambda}^{\text{in}} = \frac{1}{2\omega_{\mathbf{p}}} (\psi_{\mathbf{p}, \lambda}^{\text{in}, +} | \hat{\psi}), \quad b_{\mathbf{p}, \lambda}^{\text{in} \dagger} = \frac{1}{2\omega_{\mathbf{p}}} (\psi_{\mathbf{p}, \lambda}^{\text{in}, -} | \hat{\psi}). \quad (3.39)$$

When we assume the canonical anti-commutation relation for the quantization of the fermion,

$$\{\hat{\psi}(t, \mathbf{x}), \hat{\psi}^\dagger(t, \mathbf{y})\} = \delta(\mathbf{x} - \mathbf{y}), \quad (3.40)$$

we have the following anti-commutation relations for creation and annihilation operators,

$$\{a_{\mathbf{p}, \lambda}^{\text{in}}, a_{\mathbf{p}', \lambda'}^{\text{in} \dagger}\} = \{b_{\mathbf{p}, \lambda}^{\text{in}}, b_{\mathbf{p}', \lambda'}^{\text{in} \dagger}\} = \frac{1}{2\omega_{\mathbf{p}}} (2\pi)^3 \delta(\mathbf{p} - \mathbf{p}') \delta_{\lambda, \lambda'}. \quad (3.41)$$

We can also have the same discussion for the out-solutions, which satisfy the free boundary condition at the future infinity, $t \rightarrow \infty$ same as Eqs. (3.35). Though we assume the Dirac equation becomes free at the asymptotic time, $t \rightarrow \pm\infty$, the spatial parts of the gauge field, A^i , can be non-zero at $t \rightarrow \infty$. These non-zero parts of the gauge field behave as a pure gauge because the field strength vanishes². Thus we can have the out-solution free at future infinity as,

$$\begin{aligned} \lim_{t \rightarrow \infty} \psi_{\mathbf{p}, \lambda}^{\text{out}, +}(x) &= U^\dagger(x) u(\mathbf{p}, \lambda) e^{-ip \cdot x}, \\ \lim_{t \rightarrow \infty} \psi_{\mathbf{p}, \lambda}^{\text{out}, -}(x) &= U^\dagger(x) v(\mathbf{p}, \lambda) e^{ip \cdot x}, \end{aligned} \quad (3.42)$$

²The U(1) gauge transformation represented for the gauge field is $A'_\mu = A_\mu - \partial_\mu \theta$, where θ is the arbitrary functions which do not affect the field strength. Thus in general, the gauge field is non-zero even if the field strength is zero.

where $U(x)$ is the gauge factor represented as,

$$U(x) \equiv \exp \left[-ie \int^x dx' \cdot A(t, x') \right]. \quad (3.43)$$

Since the gauge field is a pure gauge at $t \rightarrow \infty$, a in the above equation is a pure gauge, and in the case of the global gauge transformation, the gauge factor becomes $U = \exp[-ie\theta]$, where θ is constant. The integration path at the above definition should be on the constant t -plane, and since the gauge field is pure gauge, $A = \partial\theta$, the integration is not affected by the shape of the path but only the endpoints of the path. Then the field operator can also be expressed by the out-mode functions as follows,

$$\hat{\psi} = \sum_{\lambda=\pm} \int d^3x \left[\psi_{p,\lambda}^{\text{out},+}(x) a_{p,\lambda}^{\text{out}} + \psi_{p,\lambda}^{\text{out},-}(x) b_{p,\lambda}^{\text{out},\dagger} \right]. \quad (3.44)$$

In the same way as in-solutions, we can also have the normalization for the mode functions and anti-commutation relations for the operators as,

$$a_{p,\lambda}^{\text{out}} = \frac{1}{2\omega_p} (\psi_{p,\lambda}^{\text{out},+} | \hat{\psi}), \quad b_{p,\lambda}^{\text{out}\dagger} = \frac{1}{2\omega_p} (\psi_{p,\lambda}^{\text{out},-} | \hat{\psi}), \quad (3.45)$$

$$\{a_{p,\lambda}^{\text{in}}, a_{p',\lambda'}^{\text{in}\dagger}\} = \{b_{p,\lambda}^{\text{in}}, b_{p',\lambda'}^{\text{in}\dagger}\} = \frac{1}{2\omega_p} (2\pi)^3 \delta(\mathbf{p} - \mathbf{p}') \delta_{\lambda,\lambda'}. \quad (3.46)$$

Thus we have the two different definitions of the particle in Eqs. (3.34) and (3.44), based on the in-basis and out-basis, respectively. In general, these two definitions differ when the non-zero electromagnetic field appears at some points in the time evolution of the system. This difference is the consequence of the particle production from the vacuum by the background electromagnetic field. Here we consider the initial state to be vacuum, and the number density of the particles observed at the future infinity, $t \rightarrow \infty$, is obtained by the in-vacuum expectation value, defined by $a_{p,s}^{\text{in}} |\text{in}\rangle = b_{p,s}^{\text{in}} |\text{in}\rangle = 0$, of out-particle number operator as follows,

$$\frac{dN_\lambda}{d^3\mathbf{p}} = \frac{2\omega_p}{(2\pi)^3} \langle \text{in} | a_{p,\lambda}^{\text{out}\dagger} a_{p,\lambda}^{\text{out}} | \text{in} \rangle. \quad (3.47)$$

To calculate the number density, we need to rewrite the out-particle operators by the in-particle operators. By substituting Eq. (3.34) into Eq. (3.45), we have the following relationship between two different bases,

$$\begin{aligned} a_{p,\lambda}^{\text{out}} &= \frac{1}{2\omega_p} \sum_{\lambda'=\pm} \int \frac{d^3\mathbf{p}'}{(2\pi)^3} \left[(\psi_{p,\lambda}^{\text{out},+} | \psi_{p',\lambda'}^{\text{in},+}) a_{p',\lambda'}^{\text{in}} + (\psi_{p,\lambda}^{\text{out},+} | \psi_{p',\lambda'}^{\text{in},-}) b_{p',\lambda'}^{\text{in}\dagger} \right], \\ b_{p,\lambda}^{\text{out}\dagger} &= \frac{1}{2\omega_p} \sum_{\lambda'=\pm} \int \frac{d^3\mathbf{p}'}{(2\pi)^3} \left[(\psi_{p,\lambda}^{\text{out},-} | \psi_{p',\lambda'}^{\text{in},+}) a_{p',\lambda'}^{\text{in}} + (\psi_{p,\lambda}^{\text{out},-} | \psi_{p',\lambda'}^{\text{in},-}) b_{p',\lambda'}^{\text{in}\dagger} \right]. \end{aligned} \quad (3.48)$$

Without the loss of generality, we can constantly rotate the phase of the

spinor in order to satisfy the following conditions for the inner product of the mode functions,

$$\begin{aligned} (\psi_{\mathbf{p},\lambda}^{\text{out}+} | \psi_{\mathbf{p}',\lambda'}^{\text{in}+})^* &= (\psi_{-\mathbf{p},\lambda}^{\text{out}-} | \psi_{-\mathbf{p}',\lambda'}^{\text{in}-}), \\ (\psi_{\mathbf{p},\lambda}^{\text{out}+} | \psi_{-\mathbf{p}',\lambda'}^{\text{in}-})^* &= -(\psi_{-\mathbf{p},\lambda}^{\text{out}-} | \psi_{\mathbf{p}',\lambda'}^{\text{in}+}). \end{aligned} \quad (3.49)$$

By substituting the above expression, we can rewrite Eq. (3.48) as,

$$\sqrt{2\omega_{\mathbf{p}}} a_{\mathbf{p},\lambda}^{\text{out}} = \sum_{\lambda'} \int \frac{d^3 \mathbf{p}'}{(2\pi)^3} \sqrt{2\omega_{\mathbf{p}'}} \left[\alpha(\mathbf{p}, \lambda; \mathbf{p}', \lambda') a_{\mathbf{p}',\lambda'}^{\text{in}} + \beta(\mathbf{p}, \lambda; \mathbf{p}', \lambda') b_{-\mathbf{p}',-\lambda'}^{\text{in}} \right], \quad (3.50)$$

$$\sqrt{2\omega_{\mathbf{p}}} b_{-\mathbf{p},\lambda}^{\text{out}+} = \sum_{\lambda'} \int \frac{d^3 \mathbf{p}'}{(2\pi)^3} \sqrt{2\omega_{\mathbf{p}'}} \left[\alpha^*(\mathbf{p}, \lambda; \mathbf{p}', \lambda') b_{-\mathbf{p}',-\lambda'}^{\text{in}} - \beta^*(\mathbf{p}, \lambda; \mathbf{p}', \lambda') a_{\mathbf{p}',\lambda'}^{\text{in}} \right], \quad (3.51)$$

where we introduced the coefficients α and β as,

$$\alpha(\mathbf{p}, \lambda; \mathbf{p}', \lambda') \equiv \frac{1}{2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}} (\psi_{\mathbf{p},\lambda}^{\text{out}+} | \psi_{\mathbf{p}',\lambda'}^{\text{in}+}) \quad (3.52)$$

$$\beta(\mathbf{p}, \lambda; \mathbf{p}', \lambda') \equiv \frac{1}{2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}} (\psi_{\mathbf{p},\lambda}^{\text{out}+} | \psi_{-\mathbf{p}',\lambda'}^{\text{in}-}). \quad (3.53)$$

Then Eqs. (3.50) and (3.51) take the form of a Bogoliubov transformation, where α and β are called as Bogoliubov coefficient. By using this expression, we can find the number of anti-particles with momentum $-\mathbf{p}$ is equal to particles with momentum $\mathbf{p}$ as,

$$\langle \text{in} | a_{\mathbf{p},\lambda}^{\text{out}+} a_{\mathbf{p},\lambda}^{\text{out}} | \text{in} \rangle = \langle \text{in} | b_{-\mathbf{p},\lambda}^{\text{out}+} b_{-\mathbf{p},\lambda}^{\text{out}} | \text{in} \rangle. \quad (3.54)$$

Finally, we have the expression of the number density of the produced particles by the difference of the definition of the particles as,

$$\frac{dN_{\lambda}}{d^3 \mathbf{p}} = \frac{1}{(2\pi)^3} \sum_{\lambda'} \int \frac{d^3 \mathbf{p}'}{(2\pi)^3} |\beta(\mathbf{p}, \lambda; \mathbf{p}', \lambda')|^2. \quad (3.55)$$

Then we can evaluate the production of the particles by calculating somehow the Bogoliubov coefficients. If we evaluate β in Eq. (3.55) at the future infinity, $t \rightarrow \infty$, by the out mode functions given in Eq. (3.42), the number density of the particle is rewritten as,

$$\frac{dN_s}{d^3 \mathbf{p}} = \frac{1}{(2\pi)^3 2\omega_{\mathbf{p}}} \sum_{\lambda'} \int \frac{d^3 \mathbf{p}'}{(2\pi)^3 2\omega_{\mathbf{p}'}} \lim_{t \rightarrow \infty} \left| u^{\dagger}(\mathbf{p}, \lambda) \int d^3 \mathbf{x} e^{-i\mathbf{p} \cdot \mathbf{x}} U(t, \mathbf{x}) \psi_{\mathbf{p}',\lambda'}^{\text{in}-}(t, \mathbf{x}) \right|^2. \quad (3.56)$$

Since this expression for the particle production is only for the asymptotic case, where the field strength of the background gauge field vanishes, we

cannot obtain the time evolution of the production by this formula. However, if we define the instantaneous state of the quasiparticle at some time when the electromagnetic field turning on, we can heuristically obtain the time dependence of the production. The time-dependent momentum spectrum of the produced particles is interpolated by removing the limit of $t \rightarrow \infty$ in Eq. (3.56). This procedure is equivalent that we define the instantaneous state of the quasiparticle with the following mode function,

$$\begin{aligned}\psi_{\mathbf{p},\lambda}^{(t)+}(x) &= U^\dagger(x)u(\mathbf{p},\lambda)e^{-ip \cdot x}, \\ \psi_{\mathbf{p},\lambda}^{(t)-}(x) &= U^\dagger(x)v(\mathbf{p},\lambda)e^{ip \cdot x}.\end{aligned}\tag{3.57}$$

The field operator is expanded by the above mode functions and the corresponding creation and annihilation operators. Since the gauge field is not pure gauge at the middle of the evolution of the system, the integration of the gauge factor, $U(x)$, in Eq. (3.57) depends on the path of the line integral defined in Eq. (3.43). Although we only consider the simple case that the path is uniquely determined, the choice of the path is the origin of the ambiguity of the particle definition in the case of the general gauge field background.

3.2.2 Uniform electric field case

We derived the momentum spectrum of the produced particles by the Bogoliubov coefficients, which can be obtained by solving the Dirac equation. The analytical solution of the Dirac equation is given in some special configuration of the electromagnetic field in Refs.[119]. When the electric field is uniform, we can directly obtain the time evolution equation for the Bogoliubov coefficients. Then, in this section, we consider the uniform background electric field and derive the time evolution of the coefficients. Here we consider the uniform electric field background case. The configuration of the gauge field is as follows,

$$A_\mu = (0, \mathbf{A}(t)).\tag{3.58}$$

Since the gauge field is spatially homogeneous, the line integral in the gauge factor of the instantaneous quasiparticle is uniquely obtained as,

$$U(x) = e^{-ie\mathbf{A}(t) \cdot \mathbf{x}},\tag{3.59}$$

and the spatial dependence of each mode function becomes trivial as,

$$\psi_{\mathbf{p},\lambda}^{\text{in},\pm}(x) = \Psi_{\mathbf{p},\lambda}^\pm(t)e^{\pm ip \cdot x}.\tag{3.60}$$

In this set up we can estimate the Bogoliubov coefficients of the instantaneous state by using Eqs. (3.52) and (3.53) as,

$$\begin{aligned}
 \alpha(t; \mathbf{p}, \lambda, \mathbf{p}', \lambda') &= \frac{1}{2\sqrt{\omega_p \omega_{p'}}} (\psi_{\mathbf{p}, \lambda}^{(t)+} | \psi_{\mathbf{p}', \lambda'}^{\text{in}+}), \\
 &= \frac{e^{i\omega_p t}}{2\sqrt{\omega_p \omega_{p'}}} u^\dagger(\mathbf{p}, \lambda) \Psi_{\mathbf{p}', \lambda'}^+(t) \int d^3x e^{-i(\mathbf{p}+e\mathbf{A}-\mathbf{p}') \cdot \mathbf{x}}, \\
 &= \frac{e^{i\omega_p t}}{2\sqrt{\omega_p \omega_{\mathbf{p}+e\mathbf{A}}}} u^\dagger(\mathbf{p}, \lambda) \Psi_{\mathbf{p}+e\mathbf{A}, \lambda'}^+(t) (2\pi)^3 \delta(\mathbf{p} + e\mathbf{A} - \mathbf{p}'),
 \end{aligned} \tag{3.61}$$

$$\beta(t; \mathbf{p}, \lambda, \mathbf{p}', \lambda') = \frac{e^{i\omega_p t}}{2\sqrt{\omega_p \omega_{\mathbf{p}+e\mathbf{A}}}} u^\dagger(\mathbf{p}, \lambda) \Psi_{-\mathbf{p}-e\mathbf{A}, \lambda'}^-(t) (2\pi)^3 \delta(\mathbf{p} + e\mathbf{A} - \mathbf{p}'). \tag{3.62}$$

When we substitute the above expressions into the Bogoliubov transformation in Eqs. (3.50) and (3.51), the momentum integrations can be performed because of the delta function in the above equations. Thus we redefine the Bogoliubov transformation as

$$\begin{aligned}
 a_{\mathbf{p}, \lambda} &= \sqrt{\frac{\omega_{\mathbf{p}+e\mathbf{A}}}{\omega_p}} \sum_{\lambda'} \left[\alpha(t; \mathbf{p}, \lambda, \lambda') a_{\mathbf{p}+e\mathbf{A}, \lambda'}^{\text{in}} + \beta(t; \mathbf{p}, \lambda, \lambda') b_{-\mathbf{p}-e\mathbf{A}, \lambda'}^{\text{int}} \right], \tag{3.63} \\
 b_{-\mathbf{p}, \lambda}^\dagger &= \sqrt{\frac{\omega_{\mathbf{p}+e\mathbf{A}}}{\omega_p}} \sum_{\lambda'} \left[\alpha^*(t; \mathbf{p}, \lambda, \lambda') b_{-\mathbf{p}-e\mathbf{A}, -\lambda'}^{\text{int}} - \beta^*(t; \mathbf{p}, \lambda, \lambda') a_{\mathbf{p}+e\mathbf{A}, \lambda'}^{\text{in}} \right],
 \end{aligned} \tag{3.64}$$

where the regularized time-dependent Bogoliubov coefficients are given by,

$$\alpha(t; \mathbf{p}, \lambda, \lambda') = \frac{e^{i\omega_p t}}{2\sqrt{\omega_p \omega_{\mathbf{p}+e\mathbf{A}}}} u^\dagger(\mathbf{p}, \lambda) \Psi_{\mathbf{p}+e\mathbf{A}, \lambda'}^+(t), \tag{3.65}$$

$$\beta(t; \mathbf{p}, \lambda, \lambda') = \frac{e^{i\omega_p t}}{2\sqrt{\omega_p \omega_{\mathbf{p}+e\mathbf{A}}}} u^\dagger(\mathbf{p}, \lambda) \Psi_{-\mathbf{p}-e\mathbf{A}, \lambda'}^-(t). \tag{3.66}$$

Here we confirm that the momentum spectrum of produced particles estimated in this formula does not violate the Pauli exclusion principle. Since definition of the anti-commutation relation is independent of the time slice, the we can find the relation between the Bogoliubov coefficients as,

$$\begin{aligned}
 &\frac{1}{2\omega_p} (2\pi)^3 \delta(\mathbf{p} - \mathbf{p}') \delta_{\lambda, \lambda'} \\
 &= \left\{ a_{\mathbf{p}, \lambda}, a_{\mathbf{p}', \lambda'}^\dagger \right\}, \\
 &= \frac{1}{2\omega_p} (2\pi)^3 \delta(\mathbf{p} - \mathbf{p}') \sum_{\sigma} \left[\alpha(t; \mathbf{p}, \lambda, \sigma) \alpha^*(t; \mathbf{p}, \lambda', \sigma) + \beta(t; \mathbf{p}, \lambda, \sigma) \beta^*(t; \mathbf{p}, \lambda', \sigma) \right], \\
 \delta_{\lambda, \lambda'} &= \left[\alpha(t; \mathbf{p}, \lambda, \sigma) \alpha^*(t; \mathbf{p}, \lambda', \sigma) + \beta(t; \mathbf{p}, \lambda, \sigma) \beta^*(t; \mathbf{p}, \lambda', \sigma) \right].
 \end{aligned} \tag{3.67}$$

The distribution function, f_p , of the produced particles is obtained by the momentum spectrum of the particle number per unit volume as

$$f_p = \frac{(2\pi)^3}{V} \frac{dN_\lambda}{d^3p} = \sum_{\lambda'} |\beta(t; \mathbf{p}, \lambda, \lambda')|^2 \leq 1, \quad (3.68)$$

where $V = (2\pi)^3 \delta(\mathbf{0})$ is the volume of the system. Since the distribution function represents the occupation number, then the Bogoliubov transformation method does not violate the Pauli exclusion principle.

Let us consider the time derivative of Eqs. (3.65) and (3.66). The direction of the uniform electric field can vary in time, though we choose the spin basis to diagonalize the interaction term along the z-directed electric field. We also remark that the produced particles can be accelerated by the background electric field, $\mathbf{E} = -\dot{\mathbf{A}}$. Therefore the time derivative of the momentum of the particles becomes $\dot{\mathbf{p}} = e\mathbf{E}$. Taking the time derivative, we obtain the equations for the coefficients as,

$$\frac{d\alpha(t; \mathbf{p})}{dt} = i[\epsilon(t) + S(t)]\alpha(t; \mathbf{p}) - e^{2i\omega_p t} P(t) \beta^*(t; \mathbf{p}), \quad (3.69)$$

$$\frac{d\beta(t; \mathbf{p})}{dt} = i[\epsilon(t) + S(t)]\beta(t; \mathbf{p}) + e^{2i\omega_p t} P(t) \alpha^*(t; \mathbf{p}). \quad (3.70)$$

For the simple representation, we used the following definitions,

$$\epsilon \equiv \frac{e\mathbf{E} \cdot \mathbf{p}}{\omega_p} t \quad (3.71)$$

$$P(t) \equiv \frac{eE_z E_p^2 - (e\mathbf{E} \cdot \mathbf{p})p_z}{2\omega_p^2 m_\perp} \mathbf{1} + i \frac{eE_x p_y - eE_y p_x}{2\omega_p m_\perp} \boldsymbol{\sigma}^3 + i \frac{m}{2\omega_p m_\perp} (eE_x \boldsymbol{\sigma}^3 + eE_y \boldsymbol{\sigma}^1), \quad (3.72)$$

$$S(t) \equiv \frac{m(eE_x \boldsymbol{\sigma}^2 + eE_y \boldsymbol{\sigma}^1)}{2\omega_p (\omega_p - p_z)} + \frac{(eE_x p_y - eE_y p_x) \boldsymbol{\sigma}^3}{2\omega_p (\omega_p - p_z)}, \quad (3.73)$$

where $m_\perp \equiv \sqrt{m^2 + p_\perp^2}$ and $p_\perp$ is the transverse momentum to the electric field. We used the spin basis which is the eigenstate of $\boldsymbol{\sigma}^3$ when we solved the Dirac equation for the mode functions.

3.2.3 Kinetic Approach

In the previous section, we derive the time derivatives of the Bogoliubov coefficients, which directly give us information on the time evolution of the produced particles. Here we take the kinetic approach, where we mainly focus on the time evolution of the distribution function. As discussed before, the source term of the kinetic equation of the distribution is given by the field-theoretic method and phenomenological construction method. First, we derive the former as the continuation of the previous section. Second, we show the latter.

Field theoretic source term

In addition to the uniform electric field, we impose that the electric field is parallel to the z -axis. When the direction of the electric field is time-independent, we can simplify the time derivative equations, eqs. (3.69) and (3.70), and obtain the time evolution equation for the particle distribution function as,

$$\frac{df_p(t)}{dt} = \frac{eE_z m_\perp}{\omega_p^2} \text{Re}(g_p(t)), \quad (3.74)$$

where we introduce the function $g_p(t)$ as,

$$g_p(t) \equiv e^{-2i\omega_p t} \alpha(t, \mathbf{p}) \beta(t, \mathbf{p}) \quad (3.75)$$

By using eqs. (3.69) and (3.70), the time derivative of $g_p(t)$ is represented as,

$$\frac{dg_p(t)}{dt} = -2i\omega_p g_p(t) + \frac{eE_z m_\perp}{2\omega_p} [1 - 2f_p(t)]. \quad (3.76)$$

Here the factor $1 - 2f_p$ represents Pauli-blocking, with the coefficient accounting for the effects of both particle and antiparticle creation. When f_p exceeds $1/2$, the effects of pair annihilation become significant³. Then we can solve the above equation by just performing the time integration as,

$$g_p(t) = \int^t d\tau \frac{eE_z(\tau) m_\perp}{2\omega_p^2(\tau)} [1 - 2f_p(\tau)] e^{-i(\theta(t) - \theta(\tau))}, \quad (3.77)$$

where θ is known as the dynamical phase as

$$\theta(t) \equiv 2 \int^t dt' \omega_p. \quad (3.78)$$

Thus we find the source term of the kinetic equation for the distribution function as,

$$\begin{aligned} \frac{df_p}{dt} &= \frac{eE_z(t) m_\perp}{2\omega_p(t)^2} \int^t d\tau \frac{eE_z(\tau) m_\perp}{\omega_p(\tau)^2} [1 - 2f_p(\tau)] \cos(\theta(t) - \theta(\tau)), \\ &\equiv \mathcal{S}[f_p] \end{aligned} \quad (3.79)$$

Because of the time integration in the (R.H.S) of the above equation, the time evolution of the distribution function is non-Markovian, where the information of the time step is not determined by only the previous time step, but determined by all the past information. In other words, the source term is non-local in time. This non-locality stems from the statistical factor, $1 - 2f_p$,

³For numerical calculations that incorporate the impact of pair annihilation, please refer to ref. [122]

which represents the Pauli blocking. In the case of the low density and constant electric field limit, the momentum integration of the source term becomes the reaction rate of the Schwinger effect in the constant electric field as,

$$\lim_{t \rightarrow \infty} \int d^3 p \mathcal{S}[f_p] = \frac{e^2 E^2}{4\pi^3} e^{-\frac{\pi m^2}{eE}}. \quad (3.80)$$

Phenomenological construction of the source term

Here we show the Schwinger source term derived from the phenomenological construction, where the behavior of the source term in the case of the constant electric background is consistent with the reaction rate given in eq. (3.8). The phenomenological construction of the Schwinger source term for the massive fermion field is suggested by ref. [123] as,

$$\mathcal{S}[f] = eE \exp\left(-\pi \frac{m^2 + \mathbf{p}_\perp^2}{eE}\right) \delta(\mathbf{p}_\parallel), \quad (3.81)$$

where $\mathbf{p}_\perp$ and $\mathbf{p}_\parallel$ are the transverse and longitudinal momenta to the electric field, respectively. This source term is derived from the WKB calculation of the tunneling probability from the Dirac Sea. They set 0 longitudinal momentum, but the typical momentum of the produced particle is of order $p_\parallel \sim \sqrt{eE}$. Then we replace the delta function with the normal distribution function as,

$$\delta(\mathbf{p}_\parallel) \rightarrow \frac{1}{\sqrt{eE}} \exp\left(-\pi \frac{\mathbf{p}_\parallel^2}{eE}\right). \quad (3.82)$$

We obtain the source term with the non-zero longitudinal momentum as,

$$\mathcal{S}[f] = (1 \pm 2f_p) \sqrt{eE} \exp\left(-\pi \frac{m^2 + \mathbf{p}^2}{eE}\right), \quad (3.83)$$

where we multiply the Pauli blocking (Bose enhancement for boson field) factor. The equation above has a Markovian source term because it assumes a low-density approximation. The disappearance of non-Markovianity in the low-density limit is discussed in ref. [120]. Then we can confirm that this source term can reproduce the same reaction rate with eq. (3.8) in the low-density limit as,

$$\Gamma = \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \mathcal{S}[f=0] = \frac{(eE)^2}{(2\pi)^3} \exp\left(\frac{\pi m^2}{|eE|}\right). \quad (3.84)$$

Thus eq. (3.83) is the consistent source term.

Chapter 4

Application of the Schwinger Effect to Inflation

In this chapter, we will apply the Schwinger effect to inflation models and investigate the dynamics of inflaton, electric field, and charged particles. In the Schwinger effect in curved spacetime, the background field generating the charged particles is not only the electric field but also the gravitational field. We have already confirmed in previous chapters that the spacetime expansion can produce particles from the vacuum state. The combination of these different generation sources makes the Schwinger effect more complicated to apply for the inflationary magnetogenesis scenario. Once the charged particles are produced, they move along the line of the electric field and induce the electric current which suppresses the amplification of the electromagnetic field. This induced current can be quantitatively estimated by taking the vacuum expectation value of the current operator, but the current operator contains the ultraviolet divergences, which can be removed by the subtraction scheme, such as the adiabatic subtraction and the point splitting renormalization. By evaluating the time evolution of the induced current, we can estimate the suppression of the amplification of the electromagnetic field without calculating the time evolution of the production rate. Several attempts investigate the Schwinger current in the curved spacetime, for example in $(1 + 1)$ -dimensional [10], $(2 + 1)$ -dimensional [13], and $(3 + 1)$ -dimensional [3, 14, 17, 20, 124] de-Sitter spacetime for the scalar [3, 124] and spinor [14, 17, 124] charged particles. The interesting phenomena related to the Schwinger effect appear in the de-Sitter spacetime, which is absent in the flat spacetime.

One such phenomenon, which happens when the charged particles are bosons, is the IR hyperconductivity, in which the induced current is significantly enhanced in the small mass and weak electric field regime. We have already confirmed that the light scalar field can have large mean squared values. The effective mass of the scalar field coupled to the electric field depends on the strength of the electric field, and hence the light scalar field can have large field values in the weak electric field case. However, if the scalar field is near the origin in our Hubble patch, the effective mass of the electromagnetic field is still small and the hyperconductivity is to be ignored.

Another interesting phenomenon is the negative conductivity, where the induced current flows in the direction opposite to the electric field. Such a situation is not prohibited in curved spacetime. When the pairs of the charged

particle move slower than the expansion rate of the spacetime, H , the current flows opposite to the electric field in the expanding coordinate. The negative current can amplify the electric field and relate the instability of the de-Sitter spacetime to cause spontaneous electrification. However in ref. [94], the negative conductivity does not cause the instability in the weak coupling regime considered in this thesis.

The Schwinger effect in the cosmological background has been intensively investigated, but still there exist open issues to address. The evaluation of the induced current by the constant electric field is justified only in case the electric field adiabatically evolves. When the time scale of the variation of the electric field is smaller than the Hubble time, the delayed reaction of the charged particles occurs and the formula of the induced current derived by the constant electric field breaks down. In addition, in that case, the process of the thermalization of the charged particles should be taken into account. Then there is the possibility that the dynamics of the charged particles can affect the evolution of the electric field and inflaton field. This background makes us apply the kinetic approach to investigate the Schwinger effect in curved spacetime. At first, we review the scalar QED in the curved spacetime and show the induced current with the interesting property discussed above. After that, we consider the kinetic approach to the Schwinger effect on inflation.

4.1 Schwinger effect in de Sitter spacetime

Here we briefly review the dynamics of the scalar QED in the (3+1)-dimensional de-Sitter spacetime based on ref. [3].

The solution of the mode function

The Lagrangian for the charged scalar minimally coupled to the electromagnetic field in de Sitter spacetime is written as,

$$\int d^4\sqrt{-g}\mathcal{L} = \int d^4\sqrt{-g} \left[(D_\mu\phi)^* D^\mu\phi - m^2\phi^*\phi \right], \quad (4.1)$$

where $D_\mu = \partial_\mu - ieA_\mu$, ϕ is the complex scalar field, and m is the mass parameter of the scalar. Note that ϕ is not the inflaton and the spacetime is fixed to the de Sitter spacetime as,

$$ds^2 = a(\eta)^2 (d\eta^2 - dx^2 - dy^2 - dz^2), \quad (4.2)$$

where $\eta = -\frac{1}{aH}$ is the conformal time. We also deal with the electric field as the external background field, and we set the uniform and constant electric field into the z direction as,

$$A_\mu = \frac{E}{H^2\eta} \delta_\mu^z, \quad (4.3)$$

where $E = \text{const}$. The EoM for the canonical scalar field, $q = a\phi$, is written as,

$$q'' - (\partial_i + ieA_i)^2 q + a^2 \left((m^2 - 2H^2) \right) q = 0, \quad (4.4)$$

where $'$ denote the derivative by η . When we impose the canonical commutation relation for the q and its conjugate momentum Π in the same way as sec. 2.1.3, the mode function satisfies the following equation as,

$$q_k'' + \omega_k^2 q_k = 0, \quad (4.5)$$

where ω_k is the effective frequency, which is defined as,

$$\omega_k^2 = \left(k_z + \frac{eE}{H^2\eta} \right)^2 + k_x^2 + k_y^2 + \frac{m^2}{H^2\eta^2} - \frac{2}{\eta^2} \quad (4.6)$$

$$= k^2 - \frac{2}{\eta} \lambda k r + \frac{1}{\eta^2} \left(\frac{1}{4} - \mu^2 \right). \quad (4.7)$$

where we introduce the following notation as,

$$\lambda = \frac{eE}{H^2}, \quad r = \frac{k_z}{k} \quad \text{and} \quad \mu^2 = \frac{9}{4} - \frac{m^2}{H^2} - \lambda^2. \quad (4.8)$$

We can confirm that the EoM for the mode function has the same form with eq. (2.103), and hence we know the solution is written by the Whittaker functions, $W_{i\lambda r, \mu}(2ik\eta)$, and $M_{i\lambda r, \mu}(2ik\eta)$. The former one represents the positive frequency solution, and then we choose the mode function as,

$$q_k = \frac{e^{-\pi\lambda r/2}}{\sqrt{2k}} W_{i\lambda r, \mu}(2ik\eta). \quad (4.9)$$

Induced current and conductivity

The charged particles produced via the Schwinger effect move along the electric field, and the induced current appears. The current is estimated by the vacuum expectation value of the current operator, but the operator itself is not well defined, which has the ultraviolet divergence. Here we just introduce the definition of the induced current, the expectation value of the current with the divergence, and the regularized expectation value. By using the solution derived in eq. (4.9) and the definition of the current operator as,

$$J_\mu = -\frac{ie}{2} \left\{ \phi^\dagger D_\mu \phi - \phi (D_\mu \phi)^\dagger \right\} + \text{h.c.}, \quad (4.10)$$

we can represent $\langle J_\mu \rangle$ by the mode function as,

$$\langle J_z \rangle = \frac{2e}{a^2} \int \frac{d^3k}{(2\pi)^3} (k_z - eA_z) |q_k|^2. \quad (4.11)$$

When we try to perform the integration at the past infinity, $-k\eta \rightarrow \infty$, we find the quadratic and logarithmic divergence. We can perform the integration with the momentum cut-off, ζ as,

$$\langle J_z \rangle = \frac{eaH^3}{(2\pi)^2} \lim_{\zeta \rightarrow \infty} \left[\frac{2\lambda}{3} \left(\frac{\zeta}{aH} \right)^2 + \frac{\lambda}{3} \ln \left(\frac{2\zeta}{aH} \right) - \frac{25\lambda}{36} + \frac{\mu^2\lambda}{3} + \frac{\lambda^3}{15} + K(\lambda, \mu, r) \right], \quad (4.12)$$

where for convenience, we define $K(\lambda, \mu, r)$ as

$$\begin{aligned} K(\lambda, \mu, r) \equiv & \frac{45 + 4\pi^2(-2 + 3\lambda^2 + 2\mu^2)}{12\pi^3} \frac{\mu \cosh(2\pi\lambda)}{\lambda \sin(2\pi\mu)} - \frac{45 + 8\pi^2(-1 + 9\lambda^2 + \mu^2)}{24\pi^4} \frac{\mu \sinh(2\pi\lambda)}{\lambda^2 \sin(2\pi\mu)} \\ & + \text{Re} \left\{ \int_{-1}^1 dr \frac{i\lambda}{16 \sin(2\pi\mu)} \left(-1 + 4\mu^2 + (7 + 12\lambda^2 - 12\mu^2)r^2 - 20\lambda^2 r^4 \right) \right. \\ & \left. \times \left(\left(e^{2\pi r\lambda} + e^{2\pi i\mu} \right) \psi \left(\frac{1}{2} + \mu + ir\lambda \right) - \left(e^{2\pi r\lambda} + e^{-2\pi i\mu} \right) \psi \left(\frac{1}{2} - \mu + ir\lambda \right) \right) \right\}, \end{aligned} \quad (4.13)$$

where $\psi(x) \equiv \frac{\Gamma'(x)}{\Gamma(x)}$ is the digamma function. The detailed calculation is shown in the appendix of ref. [3]. To remove the ultraviolet divergence from the above current expression, we use a subtraction scheme. The original paper employs the adiabatic subtraction scheme, where we subtract the quantity computed in the slow variation of the background. Ref. [94] uses the Pauli-Villars regularization, where the auxiliary ghost field in order to cancel the divergence and we set the mass of the auxiliary field infinity to make them non-dynamic. Both schemes give us the same result,

$$\langle J_z \rangle_{\text{reg}} = \frac{eaH^3}{(2\pi)^2} \left[-\frac{2\lambda^3}{15} + \frac{\lambda}{3} \ln \left(\frac{m}{H} \right) + K(\lambda, \mu, r) \right]. \quad (4.14)$$

We consider the extreme case for the regularized current, eq. (4.14). In the strong electric field limit, $E \gg H$, we obtain the same result with the flat space case as,

$$\langle J_z \rangle_{\text{reg}} \sim \text{sign}(E) \frac{a}{12\pi^2} \frac{e^3 E^2}{H^2} \exp \left(\frac{-\pi m^2}{|eE|} \right), \quad (4.15)$$

On the other hand, when we consider the light mass and weak field limit, $\frac{m^2}{H^2} + \lambda^2 \ll 1$, we can find the implication of the infrared hyperconductivity as,

$$\langle J_z \rangle_{\text{reg}} \sim \frac{3e^2 EaH}{4\pi^2} \frac{1}{\frac{m^2}{H^2} + \lambda^2}. \quad (4.16)$$

If $m^2/H^2 \ll \lambda^2$, the electric current depends on the inverse of the electric field, $J \propto E^{-1}$. The infrared hyperconductivity is one of the interesting features.

We have field-theoretically considered the Schwinger effect on the curved spacetime in this section, but this method has not yet been investigated for the dynamical background case.

4.2 Kinetic approach to the Schwinger effect on inflation

In this section, which is the main part of this thesis, we consider the kinetic approach with the phenomenological source term. Here, this section is mainly based on ref. [6].

4.2.1 Boltzmann equation in the anisotropic background

We employ the Boltzmann equation in the curved spacetime with the electromagnetic field to describe the dynamics of the charged particles,

$$\left[P^\mu \partial_\mu - \Gamma_{\nu\lambda}^\mu P^\nu P^\lambda \frac{\partial}{\partial P^\mu} - e P^\mu F_{\mu j} \frac{\partial}{\partial P_j} \right] \mathcal{F}(x, P) = u_\mu P^\mu (\mathcal{S}[\mathcal{F}] + \mathcal{C}[\mathcal{F}]), \quad (4.17)$$

where $u, P, \mathcal{S}, \mathcal{C}$ and $\mathcal{F}(x, P)$ represent the four-velocity of the local flow, the four-momentum of the particle, production rate of charged particles due to the Schwinger effect, the collision term, and the distribution function, respectively. $\Gamma_{\nu\lambda}^\mu$ denotes the Levi-Civita connection. When we consider the anisotropic background, Eq. (2.61), and the electric field in the x -direction, the Boltzmann equation can be rewritten as

$$\frac{\partial \mathcal{F}_p}{\partial t} + (\mathcal{E}(t) + 3\dot{\sigma} p_\parallel) \frac{\partial \mathcal{F}_p}{\partial p_\parallel} - (\dot{\alpha} + \dot{\sigma}) \mathbf{p} \cdot \frac{\partial \mathcal{F}_p}{\partial \mathbf{p}} = \mathcal{S}[\mathcal{F}_p] + \mathcal{C}[\mathcal{F}_p], \quad (4.18)$$

where we introduced the physical momentum $\mathbf{p}$ with $P^\mu = (\epsilon_p, e^{-\alpha+2\sigma} p_\parallel, e^{-\alpha-\sigma} \mathbf{p}_\perp)$ and $\mathbf{p}^2 = p_\parallel^2 + \mathbf{p}_\perp^2$. One can reproduce the equation in the isotropic background by taking the limit $\dot{\sigma} \rightarrow 0$.

To solve the Boltzmann equation, we need to specify the collision term and the Schwinger source term. In this part, we follow the procedure used in the previous work [5]. We assume that the collision term can be approximated by a restoring force towards the thermal equilibrium with a relaxation time τ ,

$$\mathcal{C}[\mathcal{F}] = -\frac{\mathcal{F}_p - \mathcal{F}_p^{\text{eq}}}{\tau} \quad (4.19)$$

where $\mathcal{F}_p^{\text{eq}}$ represents the Fermi-Dirac distribution function without chemical potential $\mathcal{F}_p^{\text{eq}} = (\exp[(u^\nu P_\nu/T)] + 1)^{-1}$ with $T = [60\rho_c/(7\pi^2)]^{1/4}$. It is

known that the relaxation time of the electron in the flat spacetime is $\tau = \frac{c_0(4\pi)^2}{e^4 T \ln e^{-1}}$, where c_0 is an order 1 parameter [125]. We adjust this expression to our setup. e should be replaced by the effective coupling $e_{\text{eff}} = e/f$. The temperature T in τ is approximated by the average kinetic energy of the electron, ρ_c/n . Finally, we chose $c_0 = 1$. Then we obtain the relaxation time as

$$\tau = \frac{(4\pi)^2}{\left(\frac{\rho_c(t)}{n(t)}\right) e_{\text{eff}}^4 \ln e_{\text{eff}}^{-1}}. \quad (4.20)$$

The Schwinger source term in the kinetic approach in the inflationary universe was estimated in eq. (3.83),

$$\mathcal{S}[\mathcal{F}] = (1 - 2\mathcal{F}(t, \mathbf{p})) \sqrt{|\mathcal{E}(t)|} \exp\left(-\pi \frac{m^2 + \mathbf{p}^2}{|\mathcal{E}(t)|}\right), \quad (4.21)$$

where $(1 - 2\mathcal{F})$ represents the Pauli blocking factor.¹

One can show that Eq. (4.21) reproduces the pair production rate for the constant electric field,

$$\Gamma = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \mathcal{S}[\mathcal{F} = 0] = \frac{|\mathcal{E}|^2}{(2\pi)^3} \exp\left(-\frac{\pi m^2}{|\mathcal{E}|}\right), \quad (4.22)$$

where we ignored the Pauli blocking factor. We also find the polarization current density, j_{pol} , of the produced charged particles which is not yet accelerated by the electric field. In the strong field limit, $|\mathcal{E}| \gg H^2, m^2$, which is applicable to our case, j_{pol} is computed as

$$j_{\text{pol}} = \frac{2g}{\mathcal{E}} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \sqrt{m^2 + \mathbf{p}^2} \mathcal{S}[\mathcal{F} = 0] \simeq g \text{sign}(\mathcal{E}) \frac{|\mathcal{E}|^{3/2}}{(2\pi)^3}, \quad (4.23)$$

where $g = 2s + 1$ is the number of spin projection and $s = 1/2$ in our case. Here we ignored $\mathcal{F}$ again. We distinguish j_{pol} from the conductive current, j_{cond} , which is the current density of the charged particles accelerated by the electric field after their production. When we consider the interaction between the electric field and charged particles, we should include both current densities, $j_{\text{tot}} = j_{\text{pol}} + j_{\text{cond}}$.

Our numerical calculations use Eqs. (4.22) and (4.23) which assume that the number of the charged particles per unit volume in the phase space is negligible, $\mathcal{F} \approx 0$. The approximation of the vanishing $\mathcal{F}$ may be justified since the rapid expansion of the spacetime and the acceleration by the electric field change the momenta of the charged particles and empty the phase space that was filled up by the previous particle production. Note that Ref. [5] directly solved the Boltzmann equation without using this approximation

¹For the stimulated pair creation for bosons, we replace the factor to $(1 + 2\mathcal{F})$. The explanations for these Schwinger source terms in the kinetic approach are written in Ref. [24] for fermions and Refs. [18, 25] for bosons.

$\mathcal{F} \approx 0$ and confirmed that the hydrodynamic approach reproduced the qualitatively equivalent results with this approximation. Thus we adopt the same approximation.

Let us introduce a (k, r) moment to apply the hydrodynamic approach for Eq. (4.18) as,

$$\mathcal{J}_{i_1, \dots, i_k}^{(k, r)}(t) = 2g \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{p_{i_1} \cdots p_{i_k}}{\epsilon(\mathbf{p})^{r+2k-1}} \mathcal{F}(t, \mathbf{p}), \quad (4.24)$$

where the factor 2 appears from the contribution of the electron and positron. Multiplying the Boltzmann equation by $\frac{p_{i_1} \cdots p_{i_k}}{\epsilon(\mathbf{p})^{r+2k-1}}$ and integrating over the momentum space, we obtain the following chain of ordinary differential equations:

$$\begin{aligned} & \left[\frac{d}{dt} + \frac{1}{\tau} - (r+k-4)(\dot{\alpha} + \dot{\sigma}) - 3\dot{\sigma} \right] \mathcal{J}_{i_1 \dots i_k}^{(k, r)} - 3\dot{\sigma} \sum_{\ell=1}^k \mathcal{J}_{i_1 \dots \hat{i}_\ell \dots i_k x}^{(k, r)} + 3\dot{\sigma}(r+2k-1) \mathcal{J}_{i_1 \dots i_k xx}^{(k+2, r-2)} \\ & + \mathcal{E} \left[(r+2k-1) \mathcal{J}_{i_1 \dots i_k x}^{(k+1, r)} - \sum_{\ell=1}^k \delta_{\ell, x} \mathcal{J}_{i_1 \dots \hat{i}_\ell \dots i_k}^{(k-1, r+2)} \right] + (r+2k-1)(\dot{\alpha} + \dot{\sigma}) m^2 \mathcal{J}_{i_1 \dots i_k}^{(k, r+2)} \\ & = 2g \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{p_{i_1} p_{i_2} \cdots p_{i_k}}{\epsilon(\mathbf{p})^{r+2k-1}} \mathcal{S}(\mathbf{p}) + 2g \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{p_{i_1} p_{i_2} \cdots p_{i_k}}{\epsilon(\mathbf{p})^{r+2k-1}} \mathcal{F}_{\text{eq}}(\mathbf{p}, T). \end{aligned} \quad (4.25)$$

The detailed derivation is described in Appendix. A. Here $\hat{i}_\ell$ means that the index i_ℓ is absent in the sequence. Since we consider that the charged particles are relativistic and $\mathcal{E} \gg m^2$, we ignore the contribution from the mass term in the above equation. We cannot solve the infinite chains of equations and hence we need closure conditions to cut off the chains. Now we extract the three pieces of the chains of equations for the $(0, 1)$, $(0, 0)$, $(1, 0)$ moments, which are important to describe the Schwinger pair production. We specify each moment as the energy density of the charged particles, $\mathcal{J}^{(0, 0)} \equiv \rho_c$, their number density, $\mathcal{J}^{(0, 1)} \equiv n$, and their conductive current density along electric field, $\mathcal{J}_x^{(1, 0)} = j_{\text{cond}}$. The evolution equations for them are written as,

$$\frac{dn}{dt} + \left(\frac{1}{\tau} + 3\dot{\alpha} \right) n = 2g\Gamma + \frac{n_{\text{eq}}}{\tau}, \quad (4.26)$$

$$\frac{d\rho_c}{dt} + (4\dot{\alpha} + \dot{\sigma}) \rho_c - 3\dot{\sigma} \mathcal{J}_{xx}^{(2, -2)} = \mathcal{E} j_{\text{pol}} + \mathcal{E} j_{\text{cond}}, \quad (4.27)$$

$$\frac{dj_{\text{cond}}}{dt} + \left(\frac{1}{\tau} + 3\dot{\alpha} \right) j_{\text{cond}} - 3\dot{\sigma} j_{\text{cond}} + 3\dot{\sigma} \mathcal{J}_{xxx}^{(3, -2)} = \mathcal{E} \mathcal{J}_{\perp\perp}^{(2, 0)}, \quad (4.28)$$

where $n_{\text{eq}} = 2g \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \mathcal{F}_{\text{eq}}$. The reaction rate Γ and the polarization current j_{pol} are given by Eqs. (4.22) and (4.23), respectively. The variables whose expressions have not yet been specified are $\mathcal{J}_{xx}^{(2, -2)}$, $\mathcal{J}_{xxx}^{(3, -2)}$ and $\mathcal{J}_{\perp\perp}^{(2, 0)}$.

Here, for simplicity, we ignore the anisotropy for a while. Taking the isotropic limit, $\dot{\sigma} \rightarrow 0$, one finds that $\mathcal{J}_{xx}^{(2, -2)}$ and $\mathcal{J}_{xxx}^{(3, -2)}$ become irrelevant to

the evolution of the system. $\mathcal{J}_{\perp\perp}^{(2,0)}$ is the only variable remaining relevant but unspecified. Note that we will restore the anisotropy and discuss its influence in Sec. 4.4.

Let us identify $\mathcal{J}_{\perp\perp}^{(2,0)}$ through a microscopic arguments. Assuming the conductive current is approximated by the bulk flow of the charged particles in the x -direction, $j_{\text{cond}} = nv_x$, we obtain the EoM for the charged particle by combining Eq. (4.26) and Eq. (4.28) with $\sigma \rightarrow 0$ as,

$$\frac{dv_x}{dt} = -\frac{v_x}{n} \left(2g\Gamma + \frac{n_{\text{eq}}}{\tau} \right) + \frac{\mathcal{E}}{n} \mathcal{J}_{\perp\perp}^{(2,0)}. \quad (4.29)$$

The first term in the R.H.S. of the above equation describes the slowing down of the particle by creating new particles whose averaged velocity is zero and scattering with the other particles. The second term represents the Lorentz force from the electric field. The EoM for the relativistic charged particle in the isotropic background, $ds^2 = dt^2 - e^{2\alpha} (dx^2 + dy^2 + dz^2)$ under the electric field in the x -direction is written as,

$$\frac{dv_x}{dt} = -Cv_x + \mathcal{E} \frac{n}{\rho_c} (1 - v_x^2), \quad (4.30)$$

where C is the decay constant and we replaced the typical momentum of the particle p by ρ_c/n in the second term [91]. Comparing Eq. (4.29) with Eq. (4.30), we obtain the closure condition as,

$$\mathcal{J}_{\perp\perp}^{(2,0)} = \frac{n^2 - j_{\text{cond}}^2}{\rho_c}. \quad (4.31)$$

4.2.2 Time evolution of the system

In this section, we will present the numerical results of the time evolution of the system. We solve the following EoMs for the inflaton, the spacetime, and the electric field in the isotropic background:

$$\dot{\alpha}^2 = \frac{1}{3M_{\text{Pl}}^2} \left(\rho_E + \rho_c + V(\phi) + \frac{1}{2}\dot{\phi}^2 \right), \quad (4.32)$$

$$\ddot{\phi} = -3\dot{\alpha}\dot{\phi} - V'(\phi) + 2\frac{f'}{f}\rho_E,$$

$$\dot{\mathcal{E}} + 2\dot{\alpha}\mathcal{E} + 2\frac{\dot{f}}{f}\mathcal{E} = -\frac{e^2}{f^2}j_{\text{tot}} + \frac{e^2}{f^2} \frac{\dot{\alpha}^3}{4\pi^2\mathcal{E}} \left[\dot{\alpha}^2 + \left(\frac{\dot{f}}{f} \right)^2 \right]. \quad (4.33)$$

We simultaneously solve the evolution of the hydrodynamic variables introduced in the previous section. The hydrodynamic equations are given by

$$\frac{dn}{dt} + \left(\frac{1}{\tau} + 3\dot{\alpha} \right) n = 2g\Gamma + \frac{n_{\text{eq}}}{\tau}, \quad (4.34)$$

$$\frac{d\rho_c}{dt} + 4\dot{\alpha}\rho_c = \mathcal{E}j_{\text{pol}} + \mathcal{E}j_{\text{cond}}, \quad (4.35)$$

$$\frac{dj_{\text{cond}}}{dt} + \left(\frac{1}{\tau} + 3\dot{\alpha} \right) j_{\text{cond}} = \mathcal{E} \frac{n^2 - j_{\text{cond}}^2}{\rho_c}. \quad (4.36)$$

To fix its mass, we identify the fermions as electrons and positrons. Their mass is given by the Yukawa coupling and the Higgs field VEV, $h(t)$, which obtains a large root mean square value due to the quantum fluctuations during inflation. The mass of the electron is estimated as

$$m(t) = \frac{y_e}{\sqrt{2}} h(t) \simeq \frac{y_e}{\sqrt{2}} \lambda^{-1/4} \dot{\alpha}(t), \quad (4.37)$$

where $y_e = 3 \times 10^{-6}$ and $\lambda = 0.26$. Note that when the particle production via the Schwinger effect is the most efficient, this mass is negligibly smaller than the electric field, $m \ll \mathcal{E}$ and thus its precise value does not affect our final results.

Model specification

From now on, we show the numerical analysis. For the analysis, the forms of the coupling function and the inflaton potential should be specified. For the coupling function, we will analyze the Ratra function, $\sim \exp(\phi)$, and the coupling that is used in the representative anisotropic inflation model, $\sim \exp(\phi^2)$. The occurrence of the phenomenon of Schwinger preheating, as discussed later, is determined by the coupling function. In the case of the Ratra coupling, this is independent of the potential. In contrast, for the Gaussian coupling, the magnitude of the inflaton field at the end of inflation also plays a significant role, and the potential must be considered. Furthermore, for the Gaussian coupling, Schwinger preheating did not occur with the potential used in our analysis. These points will be discussed in detail in the subsequent analysis. In this thesis, we adopt two types of inflaton potentials: the Starobinsky potential, which is consistent with observations, and the chaotic potential used in the WKS model. Initially, we perform numerical calculations for two combinations: the Ratra coupling with the Starobinsky potential and the Gaussian coupling with the chaotic potential, focusing on the characteristics of each case. Subsequently, we demonstrate that the coupling function alone plays an important role in the dynamics of the Ratra coupling case. Then we explain that in the case of the Gaussian coupling, the inflaton value at the end of inflation also plays a significant role in the dynamics. In summary, the models analyzed in this thesis are as follows,

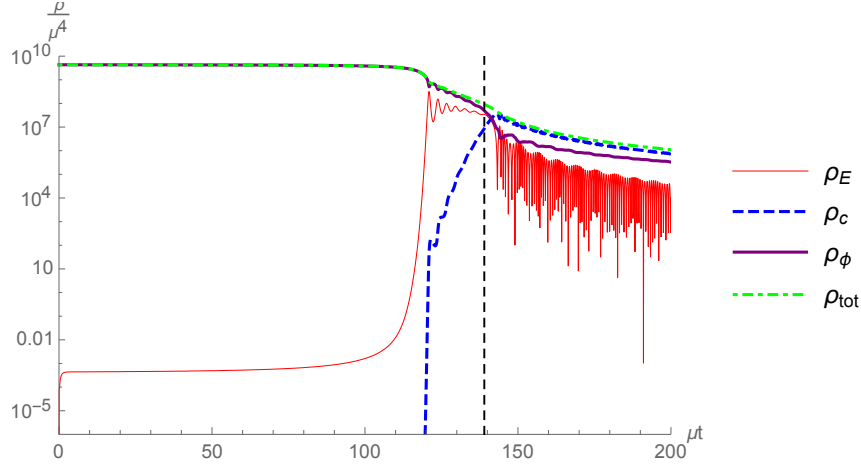


FIGURE 4.1: The numerical result in the case of the Starobinsky model V_S and the Ratra coupling f_R with the coupling constant $\beta = 10$. We show the time evolution of the energy densities of the electric field ρ_E (red solid), the charged fermions ρ_c (blue dashed), the inflaton ρ_ϕ (purple solid) and their total ρ_{tot} (green dot-dashed). The calculation starts at $\mu t = 0$ corresponding to $N \simeq 60$ and inflation ends at $\mu t = 139$ (vertical black dashed line) where $\epsilon_H \equiv -\ddot{\alpha}/\dot{\alpha}^2 = 1$. The produced fermions dominate the universe after inflation, $\rho_c > \rho_\phi, \rho_E$, which we call the Schwinger preheating. The electric field begins to oscillate at the end of the inflation.

1. Starobinsky potential with Ratra coupling

$$V_S(\phi) = \frac{3\mu^2 M_{\text{Pl}}^2}{4} \left[1 - \exp \left(-\sqrt{\frac{2}{3}} \frac{\phi}{M_{\text{Pl}}} \right) \right]^2, \quad f_R(\phi) = \exp \left(\beta \frac{\phi}{M_{\text{Pl}}} \right), \quad (4.38)$$

where $\mu \simeq 1.3 \times 10^{-5} M_{\text{Pl}}$ is the mass of the inflaton around $\phi = 0$.

2. WKS model(chaotic potential with Gaussian coupling)

$$V_W(\phi) = \frac{1}{2} \mu^2 \phi^2, \quad f_W(\phi) = \exp \left(\frac{c \phi^2}{2 M_{\text{Pl}}^2} \right), \quad (4.39)$$

where μ represents the mass of the inflaton, and we adopt the same value as in the Starobinsky model. c is a dimensionless coupling constant.

Starobinsky model with Ratra coupling

The first model that we study is the Starobinsky inflation model [30] with the Ratra coupling function [2] given in Eq. (4.38). To perform the numerical

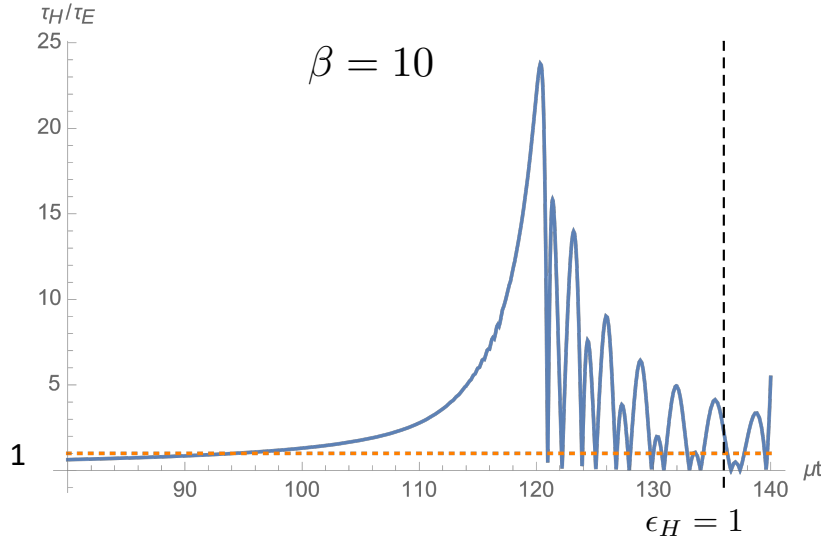


FIGURE 4.2: Time evolution of the time scale ratio $\tau_H/\tau_E = \dot{\mathcal{E}}/\ddot{\mathcal{E}}$. The estimation of the induced current is based on the assumption of $\tau_H/\tau_E < 1$. Then we cannot apply the induced current method for the suppression of the electric field around the inflation end in this case.

calculation, we set the initial condition at $N = 60$ based on

$$\phi(0) = \sqrt{\frac{3}{2}} M_{\text{Pl}} \ln \left(\frac{4}{3} N \right), \quad \phi'(0) = -\frac{\mu M_{\text{Pl}}}{2N} \sqrt{\frac{3}{2}}, \quad (4.40)$$

where N is the e-folding number.

The time evolution of the energy densities obtained by the numerical calculation is presented in Fig. 4.1. For most of the inflationary phase, the electric field sustained by the quantum fluctuation is not amplified and the fermion production is negligible. A few e-folds before the inflation ends, however, the electric field rapidly grows and enters the attractor (see also Fig. 4.5), and the Schwinger effect becomes active. After the inflation end, ρ_c overwhelms ρ_ϕ and ρ_E . We refer to the phenomenon in which the particles generated by the Schwinger effect during inflation rapidly dominate after inflation and satisfy $\rho_c/\rho_{\text{tot}} > 1/2$ as “Schwinger preheating.” The time scale ratio between the variation of the electric field, $\tau_E = \mathcal{E}/\dot{\mathcal{E}}$, and Hubble time scale, $\tau_H = 1/\dot{\alpha}$, is shown in Fig. 4.2.

The Schwinger preheating is very fascinating since it is a novel mechanism of preheating, it may provide some inflation models with a new channel to reheat the universe, and it might change the thermal history. Nevertheless, we do not know the generality of this phenomenon. We investigate if the Schwinger preheating occurs in another model in the next section.

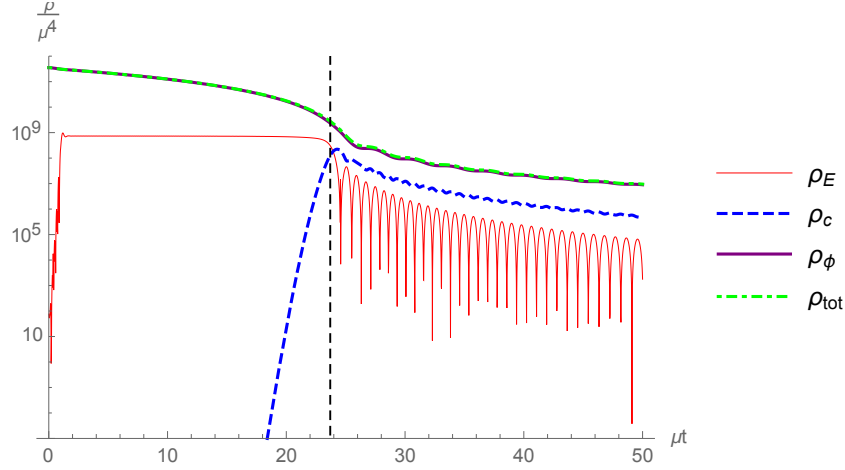


FIGURE 4.3: The numerical result in the case of the WKS model V_W and f_W with the coupling constant $c = 2$. The plot scheme is the same as Fig. 4.1. $\mu t = 0$ corresponds to $N \simeq 56$ and inflation ends at $\mu t = 23.5$. The Schwinger preheating does not occur in this case.

WKS model

As the second model, we investigate the WKS model which was originally proposed in Ref. [36]. Its potential and kinetic coupling have been given in Eq. (4.39). The advantage of this model is that its dynamics have been well studied and some analytic solutions are available (e.g. Eq. (2.74)). Setting the initial condition as

$$\phi(0) = M_{\text{Pl}}\sqrt{2N}, \quad \phi'(0) = 0, \quad (4.41)$$

we run the numerical calculation in the same way as the previous model. The result is shown in Fig. 4.3. Although the electric field enters the attractor much earlier than the previous model and the Schwinger effect lets ρ_c exceed ρ_E at around the inflation end, the produced fermions never dominate the universe. Thus, we find a counter-example of the Schwinger preheating. At least, the WKS inflation model does not exhibit it for the coupling constant $c = 2$.

For the charged particles to satisfy the condition for Schwinger preheating, $\rho_c/\rho_{\text{tot}} > \frac{1}{2}$, the energy of the electric field must be sufficiently high. From the Maxwell equations, eq. (4.33), it can be seen that the energy transfer from the inflaton to the electric field occurs through the coupling function. Therefore, we can expect that the occurrence of the Schwinger preheating depends on the coupling constant. We show the dependence in both models in Fig. 4.4. When the maximum value of ρ_c/ρ_{tot} exceeds $1/2$, we observe the Schwinger preheating in the numerical calculations and vice versa. We numerically find that the WKS model does not exhibit the Schwinger preheating, at least for $1 < c \leq 10$. In addition, we numerically find that the behavior of the maximum value of the energy fraction of the charged particles is well described by the orange dashed curve, $(c - 1)/(2c)$, in Fig. 4.4,

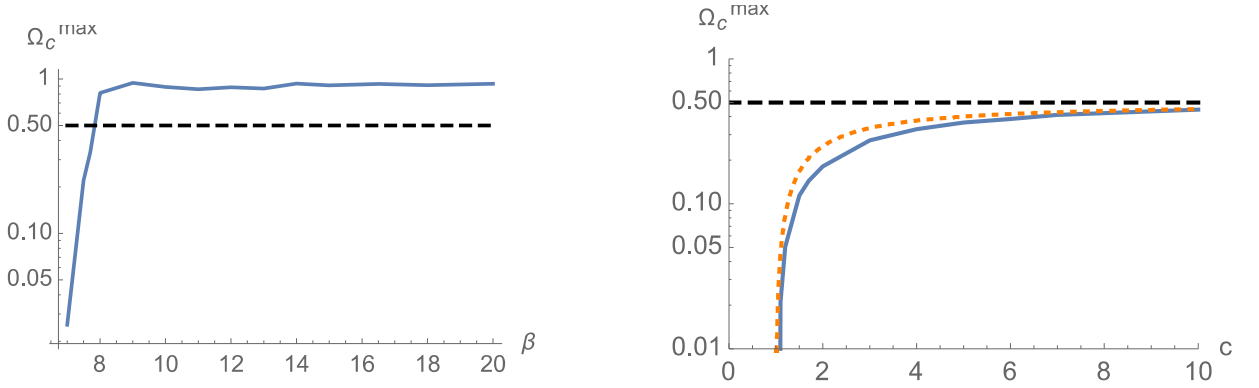


FIGURE 4.4: The maximum value of the energy fraction of the charged particles against the coupling constant. The left and right panels adopt the Starobinsky model with the Ratra coupling and the WKS model, respectively. The black dashed line denotes the energy fraction equal to $1/2$, and the Schwinger preheating is observed above this line. The orange dotted line in the right panel represents $(c - 1)/2c$ which asymptotes to $1/2$. The Schwinger preheating occurs only in the former model with the sufficiently strong coupling constant, $\beta \gtrsim 8$.

but we have no analytical basis that the fraction is estimated by such a function. Since we cannot apply the slow-roll condition to the behavior of the electric field at the end of inflation, it is difficult to analytically describe the behaviors of the charged particles. Here, although one may think that even $\rho_c/\rho_{\text{tot}} \approx 0.4$ is practically eligible for the Schwinger preheating, we adopt $1/2$ as the criteria. On the other hand, the Schwinger preheating occurs in the first model, if the coupling constant is large enough, $\beta \gtrsim 8$. Therefore, we find that the occurrence of the Schwinger preheating depends on models and coupling constants. We will further explore the dependence on the coupling constant in the next section.

4.2.3 Analytic Conditions for the Occurrence of the Schwinger Preheating

In this section, we analytically derive the conditions for the Schwinger preheating. The necessary condition for the Schwinger preheating to occur is $\rho_E \sim \rho_{\text{tot}}$. Then we can obtain a model-independent requirement for the Schwinger preheating by using the analytic expression for ρ_E , which is a function of $V(\phi)$ and $f(\phi)$.

Now we confirm that the solution, Eq. (2.72), is the attractor solution by studying the behavior of ρ_E around ρ_E^{att} . Considering a small deviation from the attractor solution, $\rho_E = \rho_E^{\text{att}} + \delta\rho_E$, we rewrite the EoM for the inflaton,

Eq. (2.64), as

$$\delta\rho_E = \frac{3\dot{\alpha}}{2(\ln f)^2} [2\dot{\alpha} + (\ln f)'\dot{\phi}]. \quad (4.42)$$

The EoM for ρ_E , Eq. (2.65), reads $\dot{\rho}_E + 2[2\dot{\alpha} + (\ln f)'\dot{\phi}]\rho_E = 0$ in the isotropic background without the fermions. Substituting $\rho_E = \rho_E^{\text{att}} + \delta\rho_E$ and ignoring $\dot{\rho}_E^{\text{att}}$ as a slow-roll correction as well as the second order perturbation terms with $\delta\rho_E^2$, we obtain the evolution equation for $\delta\rho_E$ as

$$\delta\dot{\rho}_E = -\frac{4}{3} \frac{(\ln f)^2 \rho_E^{\text{att}}}{\dot{\alpha}} \delta\rho_E. \quad (4.43)$$

This equation implies that $\delta\rho_E$ exponentially decays with a time scale, $\tau^{\text{att}} = \frac{3\dot{\alpha}}{4(\ln f)^2 \rho_E^{\text{att}}}$. Therefore, ρ_E^{att} is an attractor solution, when τ^{att} is sufficiently short. The comparison between τ^{att} and the Hubble time $\dot{\alpha}^{-1}$ is given by

$$\dot{\alpha}\tau^{\text{att}} = \frac{\rho_{\text{tot}}/\rho_E^{\text{att}}}{4M_{\text{Pl}}^2(\ln f)^2} = 2.5 \times 10^{-2} \times \begin{cases} \left(\frac{\beta}{10}\right)^{-2} \left(\frac{\rho_E^{\text{att}}/\rho_{\text{tot}}}{10^{-1}}\right)^{-1} & (f = e^{\beta \frac{\phi}{M_{\text{Pl}}}}) \\ c^{-2} \left(\frac{\phi}{10M_{\text{Pl}}}\right)^{-2} \left(\frac{\rho_E^{\text{att}}/\rho_{\text{tot}}}{10^{-1}}\right)^{-1} & (f = e^{\frac{c}{2} \left(\frac{\phi}{M_{\text{Pl}}}\right)^2}) \end{cases}. \quad (4.44)$$

Since $\dot{\alpha}\tau^{\text{att}}$ is proportional to $\rho_{\text{tot}}/\rho_E^{\text{att}}$, this attractor is very weak for the negligibly small electric field, $\rho_E \ll \rho_{\text{tot}}$. However, the attractor becomes stronger, as the electric field is amplified and significantly backreacts on the inflaton, as expected.

In the case of the WKS model (i.e. $V = \mu^2\phi^2/2$ in the second line of Eq. (4.44)), we can further reduce the time scale as $\dot{\alpha}\tau^{\text{att}} = \frac{1}{4(c-1)}$, which is consistent with the known solution of ρ_E . Thus, in the WKS model, ρ_E^{att} is always the attractor solution irrespective of ϕ unless $c - 1$ is negative or tiny.

We find that our attractor solution, Eq. (2.72), matches the numerical results very well. Fig. 4.5 presents the comparisons between the numerically computed ρ_E and our analytic expression ρ_E^{att} . Note that these numerical calculations do not include the fermion for illustrative purposes. The right panel shows that the electric field stays at the attractor for most of the inflationary phase in the second model. However, the left panel shows that the electric field enters the attractor only a few e-folds before the inflation ends in the first model for $\beta = 10$. Since $\dot{\alpha}\tau^{\text{att}} \propto \beta^{-2}$ in the Ratra coupling case, for a smaller coupling constant, the attractor becomes weaker and the electric field may not enter the attractor phase during inflation.

In Fig. 4.6, we show the evolution of the energy densities in the first model for the coupling constant $\beta = 7.7$, which is slightly smaller than the threshold value for the Schwinger preheating to occur, $\beta \approx 8$. In this case, the electric field never enters the attractor and the Schwinger preheating does not occur. From this observation, we infer that whether the electric field reaches the attractor solution is crucial to the Schwinger preheating. We will study this issue in more detail in Sec. 4.2.3. Before that, let us first investigate the cases

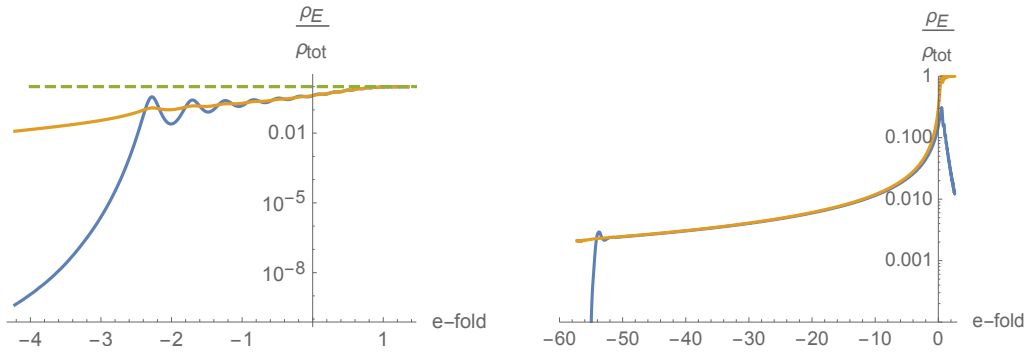


FIGURE 4.5: The energy density of the electric field ρ_E (blue) and that in the attractor phase ρ_E^{att} given in Eq. (2.72) (orange), which are normalized by the total energy density ρ_{tot} . These numerical calculations do not include the fermions. The left and right panels adopt the Starobinsky model with the Ratra coupling for $\beta = 10$ and the WKS for $c = 2$, respectively. The horizontal axis denotes the e-folding number which is zero at the end of inflation ($\epsilon_H = 1$) and takes negative values during inflation. In the left panel, ρ_E does not reach the attractor until the last few e-folds of inflation, while it is well approximated by ρ_E^{att} even after inflation ends. In the right panel, on the other hand, ρ_E stays at the attractor solution during inflation and deviates from it at around the end of inflation. When we include the fermions in the numerical calculation, the right panel does not change much, but in the left panel, ρ_E abruptly drops after the inflation ends.

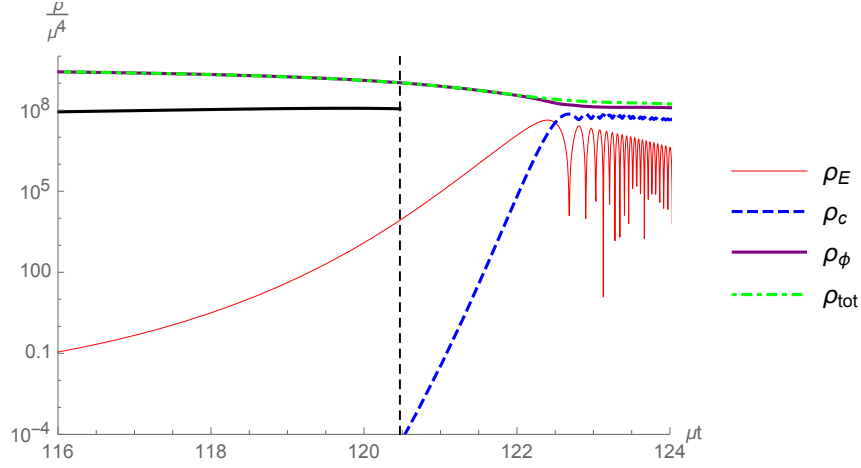


FIGURE 4.6: The same plot as Fig. 4.1 for a smaller coupling constant, $\beta = 7.7$. The calculation starts at $\mu t = 0$ corresponding to $N \simeq 60$ and inflation ends at $\mu t = 120$ (vertical black dashed line). The attractor solution ρ_E^{att} given in Eq.(2.72) is shown as a black solid line until inflation ends. The electric field does not enter the attractor phase during inflation. The maximum value of $(\rho_E + \rho_c)/\rho_\phi$ is roughly equal to 1, and the Schwinger preheating is marginally absent.

where the electric field enters the attractor.

The inflation end with the attractor

In this section, we first examine whether Schwinger preheating occurs independently of the inflaton potential when employing the Ratra coupling function. Conversely, we demonstrate that with a Gaussian coupling function, the occurrence of Schwinger preheating depends on the specific potential used. Subsequently, we consider the possibility of preheating for the relevant coupling function.

We have found that the Schwinger preheating occurs in the Starobinsky model with the Ratra coupling if the coupling constant is large enough, $\beta \gtrsim 8$. In the same model, the electric field enters the attractor during inflation for the same range of the coupling constant, $\beta \gtrsim 8$. Now we investigate the inflation end in the case that the electric field is in the attractor phase. One finds that the slow-roll parameter is rewritten as

$$\epsilon_H \equiv -\frac{\ddot{\alpha}}{\dot{\alpha}^2} = \frac{\dot{\phi}^2/2 + 2\rho_E/3}{\dot{\alpha}^2 M_{\text{Pl}}^2} = \frac{2}{(\ln f)^2 M_{\text{Pl}}^2} + 2\Omega_E, \quad (\text{attractor phase}) \quad (4.45)$$

where $\dot{\phi}$ was evaluated with Eq. (2.71) which holds in the attractor phase, and $\Omega_E \equiv \rho_E/\rho_{\text{tot}}$. The first and the second terms in the R.H.S. represent the contributions from the inflaton kinetic energy and the electric energy density, respectively. As they grow, the slow-roll parameter ϵ_H increases and inflation eventually ends at $\epsilon_H = 1$.

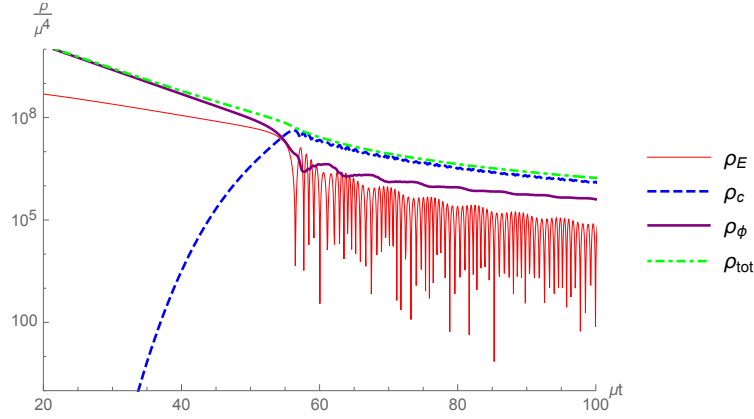


FIGURE 4.7: The numerical result in case of the chaotic model V_W and the Ratra coupling f_R with the coupling constant $\beta = 10$. We show the time evolution of the energy densities of the electric field ρ_E (red solid), the charged fermions ρ_c (blue dashed), the inflaton ρ_ϕ (purple solid) and their total ρ_{tot} (green dot-dashed). Indeed, it is found that charged particles with the Ratra coupling can dominate regardless of the difference in potential.

In the case of the Ratra coupling f_R , we have $(\ln f)' = \beta/M_{\text{Pl}}$ and obtain

$$\epsilon_H = \frac{2}{\beta^2} + 2\Omega_E. \quad (\text{attractor phase \& Ratra coupling}) \quad (4.46)$$

When we assume $\beta \gg 1$, the first term becomes negligible, and thus, at the end of inflation ($\epsilon_H = 1$), Ω_E will always be approximately $1/2$. Ω_E does not have to stop increasing at $1/2$ and may become even larger after inflation until the electric field is significantly consumed to produce and accelerate the charged particles. This is a general property of the inflationary models with the Ratra coupling in the attractor phase of the electric field. It can be confirmed in Fig. 4.7 that Schwinger preheating also occurs in the case of the Ratra coupling with the chaotic inflaton potential.

On the other hand, in the case of the Gaussian coupling, $f_W = \exp(\frac{c\phi^2}{2M_{\text{Pl}}^2})$, we have $(\ln f)' = c\phi/M_{\text{Pl}}^2$, and obtain,

$$\epsilon_H = \frac{2M_{\text{Pl}}^2}{c^2\phi^2} + 2\Omega_E. \quad (\text{attractor phase \& Gaussian coupling}). \quad (4.47)$$

We have discovered that at the end of inflation, the first term becomes dominant, preventing the electric field from becoming sufficiently large. This first term depends on the value of the inflaton at the end of inflation², which is determined not only by the coupling function but also by the inflaton potential. In this study, we do not conduct a thorough analysis of how the occurrence

²In the case of $c = 2$, $\phi = 0.76M_{\text{Pl}}$ at the end of inflation.

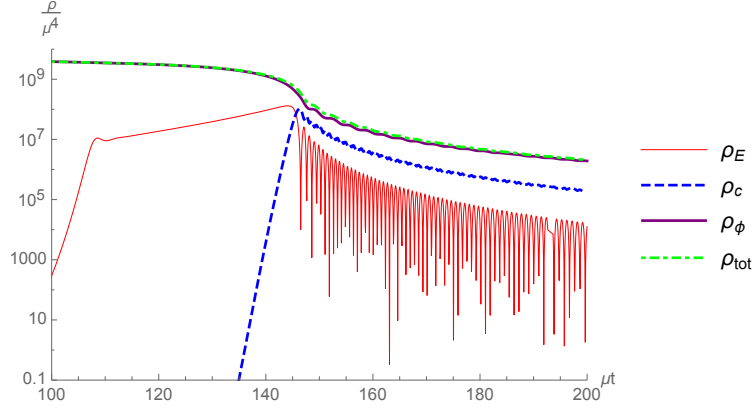


FIGURE 4.8: The numerical result in case of the Starobinsky model V_S and the Gaussian coupling f_W with the coupling constant $c = 8$. We show the time evolution of the energy densities of the electric field ρ_E (red solid), the charged fermions ρ_c (blue dashed), the inflaton ρ_ϕ (purple solid) and their total ρ_{tot} (green dot-dashed).

of Schwinger preheating depends on the different inflation models. This aspect will be pursued in future research. It can be confirmed in Fig. 4.8 that Schwinger preheating also occurs in the case of the Ratra coupling with the chaotic inflaton potential. It has been found that for the Ratra coupling, a large value of β induces Schwinger preheating. Within the context of the Schwinger preheating, what is the specific range of β within which this phenomenon is observed? As demonstrated in Fig. 4.4, the Starobinsky model with the Ratra coupling does not cause the Schwinger preheating for $\beta \lesssim 8$. This is because, for a smaller coupling constant, the electric field does not reach the attractor solution, as seen in Fig. 4.6.

In summary, under the current setup, for charged particles to dominate, the solution must enter the attractor phase, and the electric field needs to become sufficiently large. In the attractor phase, regardless of the inflaton potential, there exists a relationship with the slow-roll parameter, electric field, and a coupling function such as Eq. (4.45). Based on this relationship, Ratra coupling can result in charged particle dominance if the coupling coefficient is sufficiently large (see Eq. (4.46)). We have also found that in the case of Gaussian coupling with the chaotic and Starobinsky inflaton potential, the dominance of charged particles does not occur. This outcome depends not only on the coupling function but also on the value of the inflaton at the end of inflation. Therefore, even using the slow-roll parameter relations (see Eq. (4.47)), it is not possible to determine whether Schwinger preheating occurs. However, in this study, we did not identify any cases where it occurs.

Condition to enter the attractor

Entering the attractor phase of the electric field is a requirement to cause the Schwinger preheating in the Starobinsky model with the Ratra coupling. In this section, we will derive the condition to reach the attractor solution.

Let us introduce the inflaton value ϕ^* at which the electric field begins to grow by overcoming the cosmic dilution. At this moment, the kinetic function transiently satisfies $\dot{\phi}^* = -2\dot{\alpha}/(\ln f)' = -2\dot{\alpha}M_{\text{Pl}}/\beta$ for the same reason as Eq. (2.71). Using the slow-roll approximation, we obtain ϕ^* as

$$\phi^* = \sqrt{\frac{3}{2}} \log \left(1 + \frac{\sqrt{6}}{3} \beta \right) M_{\text{Pl}}. \quad (4.48)$$

At $\phi = \phi^*$, the energy density of the electric field sourced by the perturbation is also obtained as

$$\dot{\rho}_E + 8\dot{\alpha}\rho_E = \frac{5\dot{\alpha}^5}{4\pi^2}, \quad \Rightarrow \quad \rho_E(\phi^*) = \frac{5\dot{\alpha}_*^4}{32\pi^2} = \frac{5V^2(\phi^*)}{288\pi^2 M_{\text{Pl}}^4}. \quad (4.49)$$

Here we ignored the anisotropy and the fermions and used $\dot{f}/f = 2\dot{\alpha}$ in Eq. (2.65). After $\phi = \phi^*$, from Eq. (2.70), the electric energy density is given by

$$\rho_E(\phi) = \frac{5V^2(\phi^*)}{288\pi^2 M_{\text{Pl}}^4} \exp[2\beta(\phi^* - \phi) + 4N], \quad (4.50)$$

where we introduced the e-folding number which is computed as,

$$N = \int_{t^*}^t H dt = \int_{\phi}^{\phi^*} \frac{V}{V'} d\phi = \left[\frac{3}{4} e^{\sqrt{\frac{2}{3}}\phi} - \sqrt{6}\phi \right]_{\phi}^{\phi^*}. \quad (4.51)$$

Now we can calculate when $\rho_E(\phi)$ reaches ρ_E^{att} . Numerically solving

$$\rho_E(\phi^{\text{att}}) = \rho_E^{\text{att}}(\phi^{\text{att}}), \quad (4.52)$$

we obtain the inflaton value $\phi^{\text{att}}(\beta)$, at which the electric field enters the attractor phase, as a function of β . Finally, we impose a condition that the electric field enters the attractor before the end of inflation at $\phi = \phi_{\text{end}}$,

$$\phi^{\text{att}}(\beta) > \phi_{\text{end}}. \quad (4.53)$$

In Fig. 4.9, we plot $\phi^{\text{att}}(\beta)$ and compare it with the inflaton values satisfying $\epsilon_V(\phi) = 0.1, 0.3$ and 1 . If one adopts $\epsilon_V(\phi_{\text{end}}) = 1$ to characterize the inflation end, one obtains $\beta = 6.5$ as the threshold value, which is a slightly milder condition than the numerical result shown in Fig. 4.4. Note that since we extrapolated the dynamics around the end of inflation by using the expressions derived under the slow-roll approximation, we may well have a residual error. In fact, when one uses $\epsilon_V(\phi_{\text{end}}) = 0.3$ instead, the threshold becomes $\beta = 7.6$ which is closer to the numerical result.

Consequently, we found that the occurrence of the Schwinger preheating in the Starobinsky model with the Ratra coupling is determined by whether the electric field enters the attractor solution during inflation or not. For a weaker coupling than $\beta = 8$, the amplification of the electric field is too slow

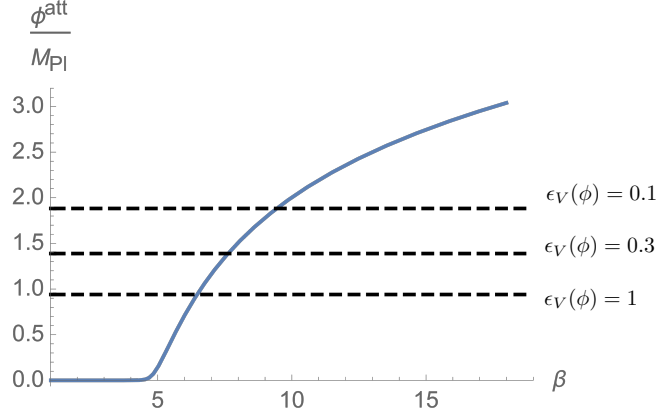


FIGURE 4.9: The inflaton value $\phi^{\text{att}}(\beta)$ at which the electric field enters the attractor phase and which solves Eq. (4.52). For the electric field to enter the attractor during inflation, $\phi^{\text{att}}(\beta)$ must be larger than ϕ_{end} , which sets the lower bound on β . The dashed lines represent the values of ϕ satisfying $\epsilon_V(\phi) = 1, 0.3$ and 0.1 from bottom to top, and they cross $\phi^{\text{att}}(\beta)$ at $\beta = 6.6, 7.6$ and 9.4

to reach the attractor by the end of inflation, and the charged fermions cannot receive sufficiently large energy to dominate the universe.

4.3 Effects of thermal mass of the charged particle

In the previous discussion, the effect of the thermal mass of charged particles was not considered. When particles exist in a high-temperature plasma, their interactions with particles in the thermal background (such as scattering or exchange processes) contribute to their kinetic energy and rest mass. This contribution appears as a self-energy correction in the propagator of the particle, derived from finite-temperature field theory. In other words, a correction is applied to the mass of charged particles.

The corrected effective mass is called the thermal mass, and it depends on the temperature T of the thermal background. It is generally expressed in refs. [126, 127]

$$m_{\text{eff}}^2 = m^2 + \Delta m(T)^2. \quad (4.54)$$

Since the Schwinger effect is suppressed when the mass of the charged particles is larger than the electric field, the inclusion of the thermal mass may suppress the Schwinger effect, potentially hindering the occurrence of preheating. For simplicity, in this thesis, we convert only the mass term in the reaction rate defined in Eq. (4.22) as $m^2 \rightarrow m_0^2 + m_{\text{Th}}^2$ while temporarily ignoring the thermal mass terms in other evolution equations, focusing solely on capturing the changes. Note that m_0 is the mass of the charged particles in the absence of thermal mass, as given by Eq. (4.37). The thermal mass, m_{Th} ,

is defined as follows³, using the temperature of the charged particles:

$$m_{\text{Th}}^2 = e_{\text{eff}}^2 T_c^2 = e_{\text{eff}}^2 \left(\frac{\rho_c}{n} \right)^2, \quad (4.55)$$

where we define the temperature of the charged particles as the average energy, as well as in Eq. (4.20). We note that the T_c is always larger than the Fermi-Dirac distribution, $T = [60\rho_c/(7\pi^2)]^{1/4}$ ⁴. Since this analysis captures the effect of the thermal mass more strongly than it should, it is important to note that the results are conservative. To confirm that the Schwinger effect is suppressed due to the effect of thermal mass, we define the thermal mass using the larger of the two temperatures in this analysis. In the following, we evaluate the impact of the thermal mass in the Ratra model. This section is mainly based on Ref. [7]

4.3.1 Numerical analysis of the effect of thermal mass

As demonstrated in Fig. 4.11, when thermal mass is taken into account, the charged particles become dominant, similar to the case without thermal mass (see Fig. 4.12). However, when considering the thermal mass, the time at which the charged particles become dominant is delayed. Based on these numerical calculation results, there are two questions to answer,

1. Has the Schwinger effect been properly suppressed due to the effect of thermal mass given by Eq. (4.55)?
2. If the Schwinger effect has been suppressed, why were the charged particles still able to become dominant?

The reason for this will be discussed in the next section.

4.3.2 Suppression of the Schwinger effect by the thermal mass

Here we will verify whether the Schwinger effect is properly suppressed based on the numerical analysis. First, we examine the time evolution of each term in Eq. (4.34). As shown in Fig. 4.13, it can be observed that the Schwinger production term becomes ineffective at some point before inflation ends. Subsequently, the contributions from terms dependent on space-time expansion and relaxation time become dominant⁵. On the other hand, in the case of non-thermal mass, as shown in fig. 4.14, the Schwinger particle production term remains dominant until just before the end of inflation.

³According to refs. [126, 127], the expression for the thermal mass includes a coefficient, but since the coefficient is at most of order 1, we proceed with the calculation by assuming it to be 1.

⁴This can be numerically verified in Fig. 4.10

⁵ The term dependent on the relaxation time represents the process of transitioning from the non-equilibrium state to the equilibrium state, and this effect increases as the relaxation time decreases. Indeed, as shown in Fig. 4.15, the order of the relaxation time is approximately 10 relative to μ , which is comparable to the timescale of inflation.

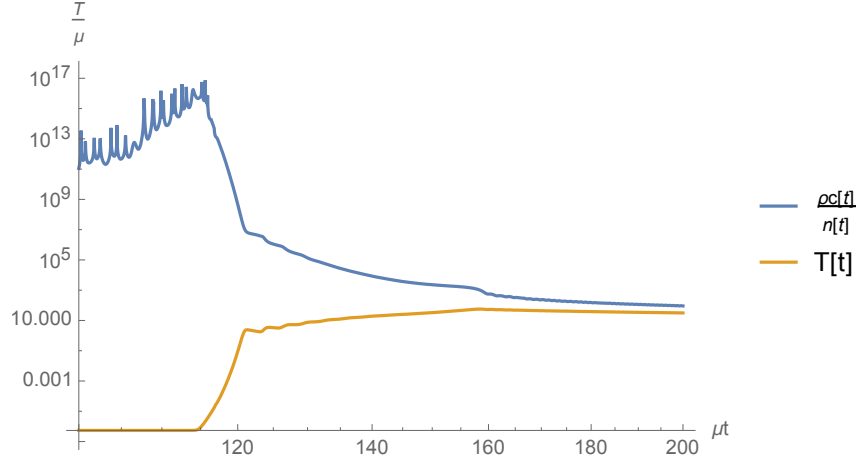


FIGURE 4.10: A comparison of the temperature of the thermal bath in the Fermi-Dirac distribution, $T = [60\rho_c / (7\pi^2)]^{\frac{1}{4}}$, and the average energy of the charged particles, ρ_c/n using the numerical calculation results for the Ratra coupling function with $\beta = 10$.

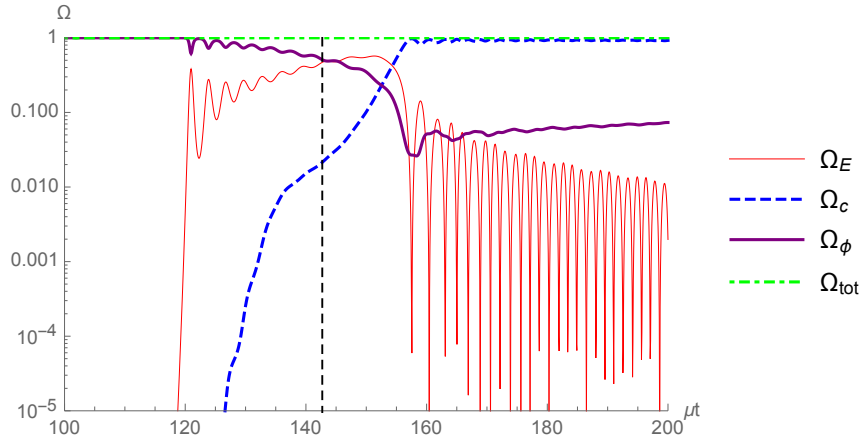


FIGURE 4.11: The time evolution of the energy fraction when thermal mass is considered. The subscripts represent E : electric field, c : charged particles, ϕ : inflaton, and tot : total energy fraction. The black dashed line represents the end of inflation, $\mu t = 142$. It can be seen that around the inflation end, where charged particles are expected to dominate, the behavior of Ω_E changes. However, after that, the energy of the charged particles increases again, and they become dominant in the same way as ignoring thermal mass.

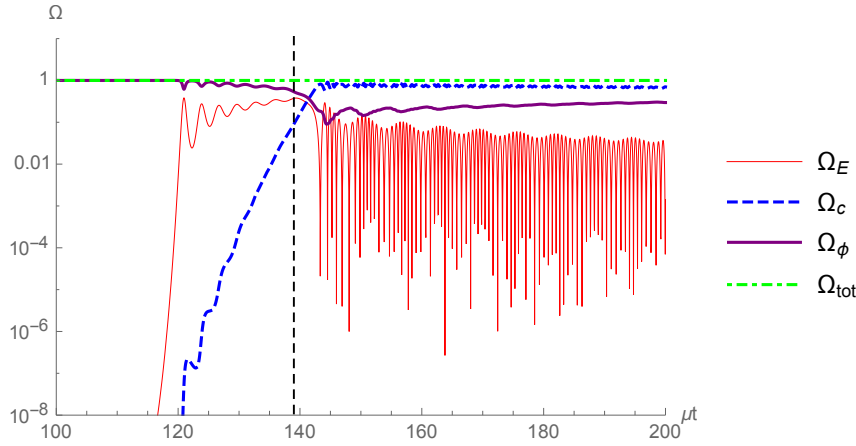


FIGURE 4.12: The time evolution of the energy fraction when thermal mass is ignored. The subscripts are defined in the same way as in Fig. 4.11. The end of inflation occurs at $\mu t = 139$. The charged particles become dominant immediately after the end of inflation.

It has been confirmed that Schwinger particle production is indeed suppressed by the thermal mass. Next, we will examine why charged particles become dominant even when Schwinger particle production is suppressed, by numerical analysis of the evolution equations for ρ_c .

4.3.3 The reason why the charged particles dominate the universe

As with the case of the number density, we also consider which terms contribute to the time evolution of ρ_c .

Fig. 4.16 and fig. 4.17 show the breakdown of the terms contributing to the time derivative of ρ_c , with and without considering the thermal mass, respectively. In fig. 4.16, the term from the vacuum polarization current initially increases and after that the conductive current becomes dominant. This situation indicates that the generated charged particles are being accelerated by the background electric field. Unlike in the case where thermal mass is ignored, the contribution from the conductive current temporarily decreases near the end of inflation and then increases again. As seen in Fig. 4.13, this is because the particle number density temporarily decreases due to the expansion of spacetime, and then increases again due to thermalization.

To summarize the discussion, by considering the thermal mass, the particle production due to the Schwinger effect is suppressed near the end of inflation (see fig. 4.13). As a result, the particle number density decreases by the expansion of spacetime, and the current becomes smaller, but at the same time, the decay of the electric field is also suppressed (see the first term of the right-hand side of eq. (4.33)). On the other hand, the particles already generated undergo thermalization with the relaxation time, τ , and they will be accelerated by the background electric field (see fig. 4.16). In other words, the increase in particle number density due to thermalization, and the increase in

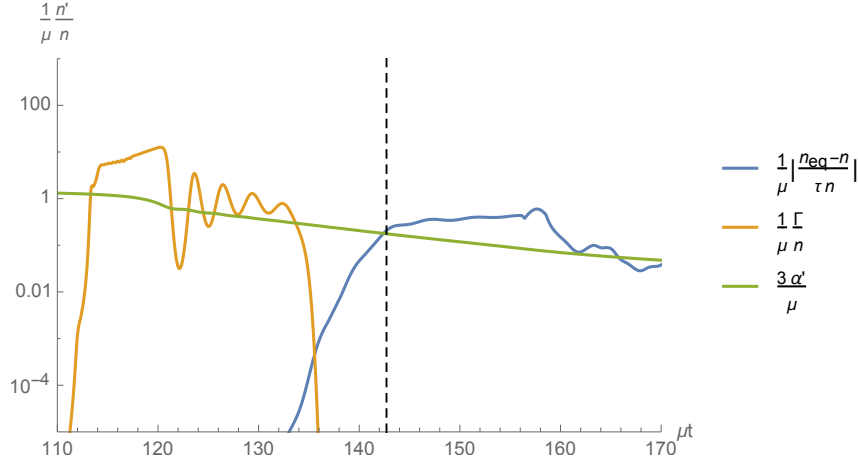


FIGURE 4.13: The relative difference in the number density per unit time, considering thermal mass for the Ratra coupling model with $\beta = 10$. The blue line represents the component dependent on the relaxation term, the orange line corresponds to the Schwinger production term, and the green line represents the contribution from spacetime expansion. The black dashed line indicates the end of inflation. It can be observed that the Schwinger production is suppressed before the end of inflation, and after inflation, the component dependent on the relaxation term becomes dominant.

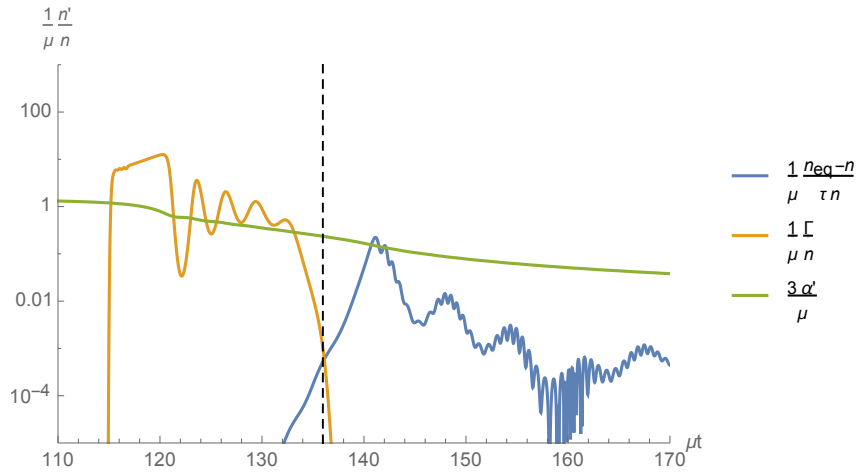


FIGURE 4.14: The relative difference in the number density per unit time, ignoring thermal mass for the Ratra coupling model with $\beta = 10$. Each line corresponds to those in 4.13. Here, the particle production term remains dominant until just before the end of inflation, after which the particle number evolves due to dilution caused by spacetime expansion.

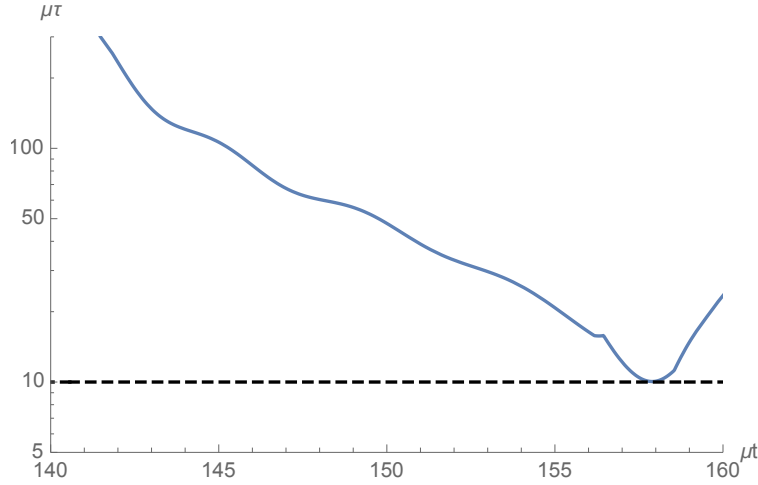


FIGURE 4.15: This figure illustrates the time evolution of the relaxation time from the non-equilibrium state to the equilibrium state around the end of inflation, based on numerical analysis with the thermal mass. When the order of the relaxation time approaches $\mu t \sim \mathcal{O}(10)$, it is expected that equilibration can significantly influence the overall dynamics.

the conductive current due to the electric field, cause the energy transfer from the electric field into the charged particles. Therefore, even when considering the thermal mass, the dominance of charged particles is still realized.

However, in the present analysis, the effect of thermal mass has been considered only in the particle production process via the Schwinger effect, while its influence on the subsequent acceleration of particles by the electric field has not been taken into account. This omission is expected to suppress the growth of the current density and reduce the energy transferred from the electric field to the charged particles. A more precise analysis would require a reconsideration of the fluid equations and the vacuum polarization term. However, such an investigation lies beyond the scope of this paper and is left for future work.

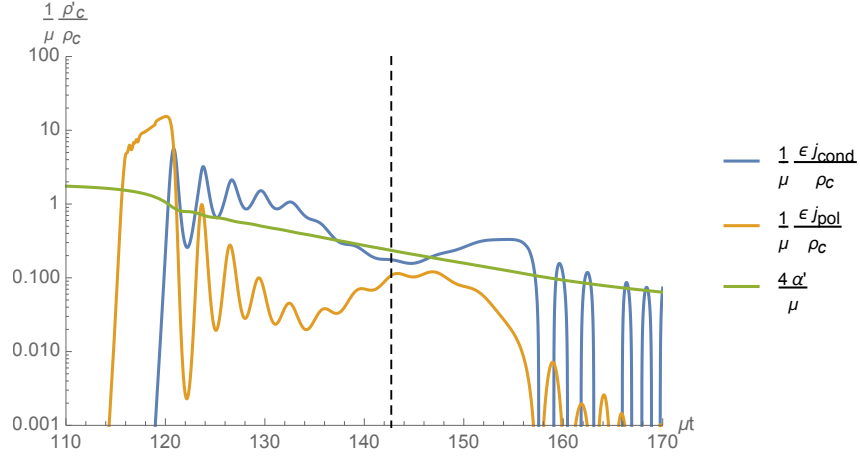


FIGURE 4.16: The relative difference per unit time for the energy density of charged particles, separated by each term. The blue line represents the contribution from the conductive current j_{cond} , the orange line represents the contribution from vacuum polarization current, j_{pol} , the green line represents the contribution from spacetime dilution, and the black dashed line indicates the end of inflation. The orange line increasing near the end of inflation is due to the assumption of massless particles in the j_{pol} equation, in Eq. (4.23). However, since the contribution from this term is not dominant, it does not affect the overall result.

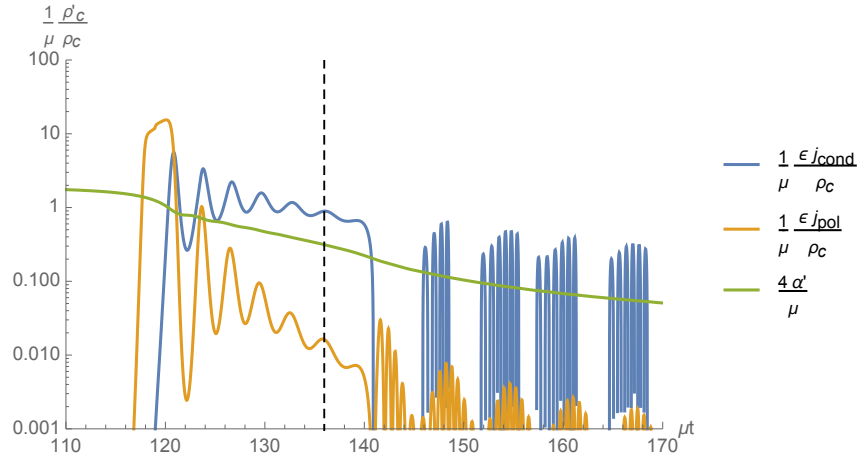


FIGURE 4.17: The relative difference per unit time for the energy density of charged particles, separated by each term. Each line corresponds to those in 4.16. In contrast to the case with thermal mass, right after inflation ends, energy from the electric field is immediately transferred to the charged particles through the conductive current.

4.4 Influence of the Anisotropy

In the previous analysis, we have neglected the back reaction of the electric field on spacetime and considered the time evolution under isotropic spacetime. However, when the electric field becomes sufficiently large, the back reaction on spacetime becomes non-negligible, and it potentially can disturb the evolution of the electric field and current density. The reaction rate of the Schwinger production depends on the strength of the electric field, and the energy from the electric field is transferred to the charged particle through the current density (see Sec. 4.3). Therefore, we would like to understand the growth of the electric field and current density with the anisotropy. In this section, we first derive the evolution equations considering anisotropy and then evaluate its impact through numerical calculations.

Derivation of the anisotropic EoM

First, let us express the terms dependent on $\dot{\sigma}$ in terms of known quantities. In Eqs. (4.26)-(4.28), we wrote the hydrodynamic equations in the anisotropic background, while we left $\mathcal{J}_{xx}^{(2,-2)}$ and $\mathcal{J}_{xxx}^{(3,-2)}$ unspecified, which became irrelevant in the isotropic limit $\dot{\sigma} \rightarrow 0$. Here let us determine their expressions through phenomenological arguments in similar ways to $\mathcal{J}_{xx}^{(2,0)}$. Assuming $j_{\text{cond}} = nv_x$ and combining Eq. (4.26) and (4.28), we obtain the EoM for the charged particles as

$$\frac{dv_x}{dt} = -\frac{v_x}{n} \left(2g\Gamma + \frac{n_{\text{eq}}}{\tau} \right) + \frac{\mathcal{E}}{n} \mathcal{J}_{\perp\perp}^{(2,0)} + 3\dot{\sigma} \left(v_x - \frac{\mathcal{J}_{xxx}^{(3,-2)}}{n} \right). \quad (4.56)$$

Compared to the isotropic case, Eq. (4.29), the anisotropy induces the last term in the R.H.S.⁶ The EoM for the relativistic charged particles in the anisotropic background (2.61) under the electric field in the x -direction is written as

$$\frac{dv_x}{dt} = -Cv_x + \mathcal{E} \frac{n}{\rho_c} (1 - v_x^2) + 3\dot{\sigma} (v_x - v_x^3). \quad (4.57)$$

Comparing Eq. (4.56) with Eq. (4.57), we obtain an additional closure condition as

$$\mathcal{J}_{xxx}^{(3,-2)} = j_{\text{cond}} \left(\frac{j_{\text{cond}}}{n} \right)^2. \quad (4.58)$$

⁶Since we took the massless limit, the contribution from the isotropic expansion vanishes.

In the meanwhile, the same expressions for $\mathcal{J}_{xxx}^{(3,-2)}$ as well as $\mathcal{J}_{\perp\perp}^{(2,0)}$ can be obtained by approximately manipulating their definitions,

$$\begin{aligned}\mathcal{J}_{\perp\perp}^{(2,0)} &\equiv n \left\langle \frac{p_{\perp}^2}{\epsilon_p^3} \right\rangle \simeq n \frac{1 - \langle \frac{p_{\parallel}}{\epsilon_p} \rangle^2}{\langle \epsilon_p \rangle} = \frac{n^2 - j_{\text{cond}}^2}{\rho_c}, \\ \mathcal{J}_{xxx}^{(3,-2)} &\equiv n \left\langle \left(\frac{p_{\parallel}}{\epsilon(\mathbf{p})} \right)^3 \right\rangle \simeq n \left\langle \frac{p_{\parallel}}{\epsilon(\mathbf{p})} \right\rangle^3 = j_{\text{cond}} \left(\frac{j_{\text{cond}}}{n} \right)^2,\end{aligned}\quad (4.59)$$

where we used $v_x = \langle \frac{p_{\parallel}}{\epsilon_p} \rangle$, $\rho_c = n \langle \epsilon_p \rangle$, and the bracket $\langle \cdots \rangle$ denotes the average over the phase space weighted by $\mathcal{F}(t, \mathbf{p})$. Although the phenomenological argument to identify $\mathcal{J}_{xx}^{(2,-2)}$ is unknown, we can apply a similar approximation to find its expression as

$$\mathcal{J}_{xx}^{(2,-2)} \equiv n \left\langle \frac{p_{\parallel}^2}{\epsilon} \right\rangle \simeq n \langle \epsilon \rangle \left\langle \frac{p_{\parallel}}{\epsilon} \right\rangle^2 = \rho_c \left(\frac{j_{\text{cond}}}{n} \right)^2. \quad (4.60)$$

Without the verification from the microscopic EoM, one may be concerned with the accuracy of the closure condition for $\mathcal{J}_{xx}^{(2,-2)}$.

Summarizing these results, the full equations for this system including anisotropy can be described as follows,

$$\dot{\alpha}^2 - \dot{\sigma}^2 = \frac{1}{3M_{\text{Pl}}^2} \left(\rho_E + \rho_c + V(\phi) + \frac{1}{2}\dot{\phi}^2 \right), \quad (4.61)$$

$$\ddot{\phi} = -3\dot{\alpha}\dot{\phi} - V'(\phi) + 2\frac{f'}{f}\rho_E, \quad (4.62)$$

$$\ddot{\sigma} = -3\dot{\alpha}\dot{\sigma} + \frac{2}{3M_{\text{Pl}}^2}\rho_E, \quad (4.63)$$

$$\dot{\mathcal{E}} + 2(\dot{\alpha} + \dot{\sigma})\mathcal{E} + 2\frac{\dot{f}}{f}\mathcal{E} = -\frac{e^2}{f^2}j_{\text{tot}} + \frac{e^2}{f^2}\frac{\dot{\alpha}^3}{4\pi^2\mathcal{E}} \left[\dot{\alpha}^2 + \left(\frac{\dot{f}}{f} \right)^2 \right], \quad (4.64)$$

$$\frac{dn}{dt} + \left(\frac{1}{\tau} + 3\dot{\alpha} \right) n = 2g\Gamma + \frac{n_{\text{eq}}}{\tau}, \quad (4.65)$$

$$\frac{d\rho_c}{dt} + 4\dot{\alpha}\rho_c + \dot{\sigma}\rho_c \left\{ 1 + 3 \left(\frac{j_{\text{cond}}}{n} \right)^2 \right\} = \mathcal{E}j_{\text{pol}} + \mathcal{E}j_{\text{cond}}, \quad (4.66)$$

$$\frac{dj_{\text{cond}}}{dt} + \left(\frac{1}{\tau} + 3\dot{\alpha} \right) j_{\text{cond}} - 3\dot{\sigma}j_{\text{cond}} \left\{ 1 - \left(\frac{j_{\text{cond}}}{n} \right)^2 \right\} = \mathcal{E} \frac{n^2 - j_{\text{cond}}^2}{\rho_c}. \quad (4.67)$$

Since j_{cond} represents the current density in the x - direction, an increase in the anisotropic expansion in the x - direction would promote the acceleration of charged particles. However, if j_{cond} is equal to the particle number density, $\frac{j_{\text{cond}}}{n} = 1$, the contribution from the anisotropic components vanish in the

EoM for j_{cond} ⁷. This is because the charged particles will move isotropically in that situation.

Discussion

We obtained the full hydrodynamic equations for this system, and we can perform numerical calculations in the anisotropic background. Since we showed that differences in the potential do not affect the outcome of whether charged particles dominate (see 4.2.3.), we will use only the Starobinsky potential in the numerical analysis and employ the Ratra and Gaussian coupling functions as the coupling functions. The numerical results for the time evolution of the energy fraction in each model with anisotropy are shown in Fig. 4.18. Even with the inclusion of anisotropy, it can be confirmed that there is no significant global impact. In fact, when we examine the evolution of the energy fraction over a longer time scale in the Ratra model, the dynamics remain unchanged: after the charged particles dominate, the inflaton becomes dominant again. Therefore, we focus on the regions where anisotropy becomes more pronounced. During the charged particle dominance, the energy fraction associated with anisotropy decreases. Therefore, we will analyze the end of inflation when the anisotropy has a significant energy fraction.

Fig. 4.19 shows $\frac{\rho_E + \rho_c}{\rho_\phi}$ near the end of inflation for the Ratra and Gaussian coupling. From this figure, we can see that in both cases, the position of the peak shifts to an earlier time. Specifically, for the Ratra coupling case, the peak becomes larger, while for the Gaussian coupling case, the peak becomes smaller. This phenomenon is attributed to the fact that the timing of the end of attractor inflation is shifted earlier. During the attractor phase, the energy density of the electric field decreases over time in the case of the Ratra coupling, while it increases in the Gaussian coupling⁸. Since the charged particles receive energy from the electric field, in the Ratra case, a greater number of charged particles are produced, whereas in the Gaussian case, fewer charged particles are generated.

In the following analysis, we show that the anisotropy leads to an earlier termination of inflation. For the purpose of this analysis, we neglect the charged particles. The slow-roll parameter including anisotropy is given as follows,

$$\epsilon_H^{\text{ani}} = \epsilon_H^{\text{iso}} + 3 \left(\frac{\dot{\sigma}}{\dot{\alpha}} \right)^2, \quad (4.68)$$

$$\epsilon_H^{\text{iso}} = \frac{1}{M_{\text{Pl}}^2 \dot{\alpha}^2} \left(\frac{1}{2} \dot{\phi}^2 + \frac{2}{3} \rho_E \right) \quad (4.69)$$

⁷By using the condition, $\dot{\sigma} j_{\text{cond}} \left\{ 1 - \left(\frac{j_{\text{cond}}}{n} \right)^2 \right\} \leq \dot{\sigma}$, and $\left(\frac{\dot{\sigma}}{\dot{\alpha}} \right)^2 \simeq \frac{2}{3} \Omega_E^2$, we can conclude that the anisotropic term in eq. (4.67) cannot dominate the equation.

⁸We can confirm this argument by analyzing the derivatives of the attractor solution ρ_E^{att} , given in eq. (2.72), with respect to ϕ .

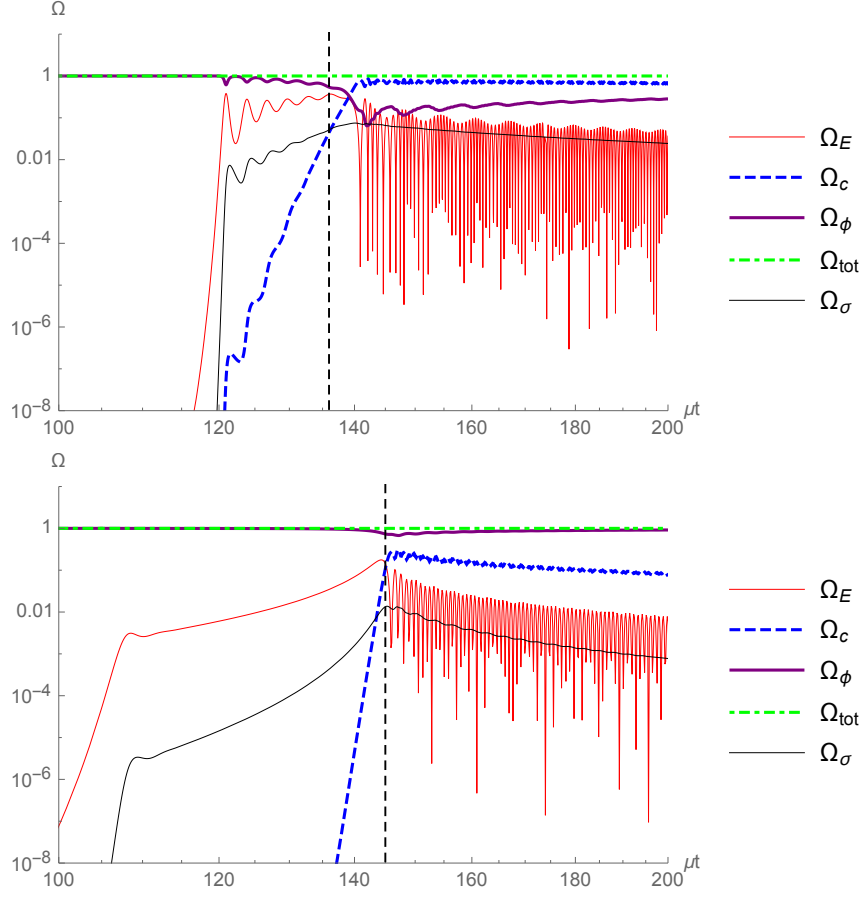


FIGURE 4.18: (Top figure) This figure shows the time evolution of the energy fraction for the Ratra coupling with $\beta = 10$. The subscripts represent E : electric field, c : charged particles, ϕ : inflaton, tot : total energy fraction, and σ : anisotropy, respectively. The black dashed line represents the end of inflation, $\mu t = 136$. Without considering anisotropy, the inflation end is at $\mu t = 139$. Even when considering anisotropy, the charged particles remained dominant. (Bottom figure) The time evolution of the energy fraction for the case of Gaussian coupling with $c=8$ is shown. The notation for the subscripts is the same as in the top figure. The inflation end is $\mu t = 145$. Without considering anisotropy, the inflation end is at $\mu t = 145.5$.

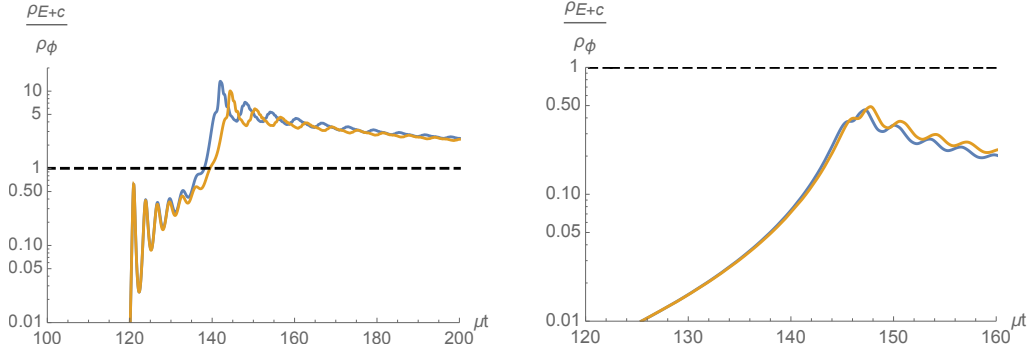


FIGURE 4.19: The time evolution of the energy fraction $(\rho_E + \rho_c)/\rho_\phi$ computed in the anisotropic (blue) and the isotropic background (orange). The left and right panels adopt the Starobinsky potential with the Ratra coupling for $\beta = 10$ and the Starobinsky potential with the Gaussian coupling for $c = 8$, respectively. The background anisotropy causes the peak height of the energy fraction to increase slightly, and its timing to occur slightly earlier in the left panel. On the other hand, in the right panel, while the peak timing also becomes slightly earlier, the peak height decreases slightly.

where the above equations are derived by using the Friedmann equation, eq. (4.61). Fig. ?? illustrates the time evolution of slow-roll parameters, $\epsilon_H^{\text{ani}}, \epsilon_H^{\text{ani}} - 3(\frac{\dot{\sigma}}{\dot{\alpha}})^2, \epsilon_H^{\text{iso}}$. As shown, the inclusion of anisotropy causes an increase in the slow-roll parameter, leading to an earlier termination of inflation.

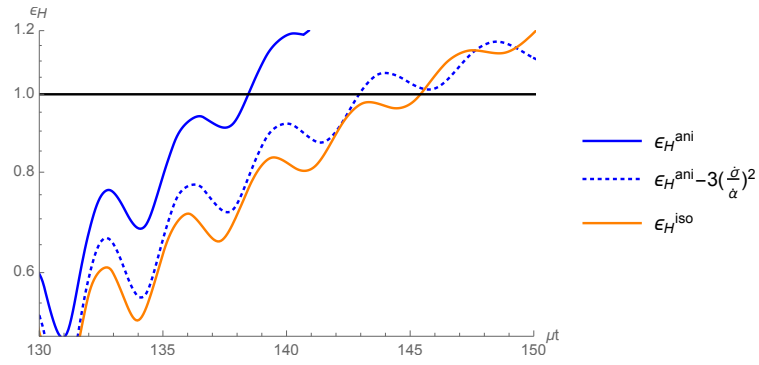


FIGURE 4.20: This figure represents the time evolution of slow-roll parameters for the Starobinsky-Ratra inflation model without charged particles in the isotropic and anisotropic background. Considering slow-roll condition for the anisotropy, we can estimate $\left(\frac{\dot{\sigma}}{\bar{\alpha}}\right)^2 = \frac{2}{3}(\Omega_E)^2$

Chapter 5

Conclusion

5.1 Summary

5.1.1 Chapter 2

In chapter 2, we first reviewed the basics of inflationary cosmology. Second, we introduced the representative inflation models, chaotic, Starobinsky, and anisotropic inflation models. These three types of inflation models are related to the analysis of the main parts. In the anisotropic inflation section, we showed the formula to give us the attractor solutions independent from the kinetic function and the potential of the inflaton. The latter half of the chapter 2 is dedicated to the introduction of the inflationary magnetogenesis. The main motivation for using inflation for magnetogenesis is that long-wavelength fluctuations are generated during inflation. We explain the observational implication of the existence of the large-scale magnetic field in the void region in our universe and the lower bound of such magnetic field. The other mechanism for magnetogenesis is suffering from generating a large-scale magnetic field. In this thesis, we introduce two types of inflationary magnetogenesis models. One is the simplest model, the Ratra model, where the kinetic coupling between the inflation and electromagnetic field is considered. However, this model has a serious problem in that the electric field becomes too strong, close to the inflaton energy when the magnetic field satisfying the observational lower bound appears. The other type of model is the hybrid model, where the kinetic and axial couplings between the electromagnetic and spectator field, not the inflaton, are considered. We confirm that in some parameter regions, the magnetic field consistent with the observation can be generated in this model without too strong an electric field like the Ratra coupling model. However, the Schwinger effect is not considered in this model.

5.1.2 Chapter 3

In chapter 3, we focus on the Schwinger effect in the flat spacetime. First, we estimate the reaction rate of the Schwinger effect from the Euler-Heisenberg Lagrangian, which is the 1-loop effective QED Lagrangian. Then we consider the proper time method to derive the Euler-Heisenberg Lagrangian from QED in the same way as Schwinger's original work. In this method,

by solving the dynamics of one particle, we evaluate the trace of the operator. After that, we explain another approach (but equivalent to the proper time method) to the Schwinger effect, where we used the Bogoliubov transformation to obtain the number density of the produced particles. In the case of the uniform electric field, we can derive the time evolution equation for the Bogoliubov coefficients directly related to the electric field. Since the square of the Bogoliubov coefficient β is the distribution function itself, in this case, we can directly (without solving the EoM for the mode function one by one) estimate the distribution function. Then we obtained the kinetic equation for the distribution function with the field theoretically derived Schwinger source term. On the other hand, we also introduce the kinetic equation with the phenomenologically derived Schwinger source term. The former is the exact form but difficult to perform the long time analysis, because of the non-locality in time. Since our aim, application of the dynamical Schwinger effect to the inflation model needs long-term analysis, we did not use the field theoretical source in the numerical calculation in chapter 3. We note that the field-theoretic source term is not derived in the curved spacetime so far.

5.1.3 Chapter 4

In chapter 4, we first consider the Schwinger effect in the (3+1) dimensional de Sitter spacetime. In curved spacetime, the electric current induced by the produced charged particles is often discussed because of its interesting features. We show the representation of the electric current in the scalar QED regularized by the adiabatic subtraction. We confirmed that the electric current shows the hyperconductivity in case of light mass and weak electric field, which is observed in the fermionic theory. Since the electric current disturbs the development of the electromagnetic field, we can take in the Schwinger effect in the analysis when we estimate the electric current. However, the current analysis is justified only when the electric field evolves adiabatically, where the time scale of the electric field evolution is smaller than the Hubble time. On the other hand, the kinetic approach can be applied to the non-adiabatically varying electric field. This is the clear advantage of the kinetic approach.

Then, in the latter half of this chapter, we numerically performed the kinetic approach with the phenomenologically derived source term in the inflation model with the Ratra coupling and the Gaussian coupling functions. The former coupling functions with the Starobinsky inflaton potential were analyzed in the previous work [5], and they find the "Schwinger preheating", which is the phenomenon that the charged particles produced via the Schwinger effect during inflation dominate the universe without directly decaying from the inflaton. They conclude that the coefficient of the Ratra coupling satisfies $\beta \geq 8$ without any analytical discussion. While we confirmed the results of the previous work, we have shown the analytical necessary condition of the Schwinger preheating for the Ratra coupling model. We have demonstrated that the analytic condition for the Gaussian coupling model depends on both the inflaton potential and the coupling function. However,

we have not identified any cases where Schwinger preheating occurs in this model. Schwinger preheating occurs when the electric field strong enough to dominate the universe is generated during the attractor transfer of its energy to the charged particles. Subsequently, we conducted a more detailed analysis of the capability of the Schwinger preheating for the Ratra coupling model. In the analysis, we assumed the Starobinsky inflaton potential. Using the attractor solution, we evaluated the electric field energy density at the inflation end. In the Ratra coupling, the analytic evaluation apparently indicates that ρ_E always dominates the universe at the end of inflation, independent of β . However, we found that ρ_E does not reach the attractor, and hence it remains much smaller than ρ_ϕ in the case of a small β . The threshold to enter the attractor is estimated as $\beta = 7.5$ when the inflation end is evaluated as $\epsilon_H = 0.3$, while the numerical results indicate the threshold is $\beta \approx 8$.

Next, we analyzed the case where charged particles have thermal mass. Since it is difficult to fully consideration for the thermal effects, we assumed the thermal mass to be $m_0^2 + e_{\text{eff}}^2 T^2$ and the average energy of the charged particles to be a temperature. The effect of thermal mass becomes significant near the end of inflation and the particle production is suppressed at that point, but it was found that the charged particles already generated up to that time are accelerated by the electric field energy, and as a result, the Schwinger preheating is still realized.

We have also addressed the impact of spacetime anisotropy. Two more closure conditions to cut the chain equation are needed, and we specified them. These anisotropic terms contribute to increasing the energy of the charged particles, but we found that their effect becomes very small. We revealed that the effect of anisotropy becomes most prominent around the inflation end when the electric field becomes large. Therefore, we investigate the region around that point in detail. We analyze the time evolution of $(\rho_E + \rho_\psi) / \rho_\phi$, energy ratio of the inflaton, electric field, and charged particles, and discovered that the peak value of the quantity changes due to anisotropy. This phenomenon occurs due to the advancement of the inflation end caused by anisotropy. The slow-roll parameter becomes larger than the isotropic case because of the correction from the anisotropy proportional to $(\dot{\sigma}/\dot{\chi})^2$ and inflation will terminate earlier. Whether the peak increases or decreases depends on whether the derivative of the electric field attractor solution with respect to the inflaton is positive or negative.

5.2 Conclusion

The following works are our new findings. We analyzed the kinetic approach for the Schwinger effect in the inflationary models with consideration of the backreaction to the spacetime. We found the analytic condition for the Schwinger preheating. We also found that even with the consideration of the thermal mass, the charged particles generated by the Schwinger effect dominate the universe. Finally, we numerically showed that the influence of the

anisotropy of the spacetime is smaller than the other components but it shifts the timing of inflation end and changes the peak value of the energy fraction of the charged particles.

Our ultimate goal is to dynamically analyze the Schwinger effect in the realistic magnetogenesis model. In the realistic model, the electromagnetic field is dealt with as the quantum fluctuations, and the modes after the horizon crossing can be regarded as the background electromagnetic field. We still do not know how the fluctuations affect the dynamics and the production of the charged particles. This is the main difficulty in applying our method to the realistic magnetogenesis model. In the magnetogenesis model, we often use the assumption that the electric components are diluted during the reheating era because of the interaction with the Ohmic current induced by the charged particles. However, based on our dynamic analysis, we have found the electric field does not dissipate but oscillates even though the electric field interacts with the current induced by the charged particles. This result implies that the dynamical analysis for the charged particles is indispensable in inflationary magnetogenesis.

We suggest the other outlook for this research. The field theoretical derivation of the Schwinger source term is only given in the flat spacetime. Then, it is a challenging but very interesting attempt to do it in the curved spacetime. If we could derive it, we can dynamically obtain the exact current. We pointed out that the current in the curved spacetime shows strange phenomena such as negative conductivity and infrared hyperconductivity. Since the current in our kinetic approach is not estimated in the field theoretically, we did not find such phenomena. Although ref. [94] showed that those phenomena do not cause a serious impact on the magnetogenesis, they may intensely affect the dynamics of the magnetogenesis.

Appendix A

Derivation of the hydrodynamic Boltzmann equation

Let us obtain the Eq. (4.25) from the Boltzmann equation in the Bianchi-I spacetime, Eq. (4.18).

The first term in Eq. (4.18)

Since the moment of the distribution function is independent of the momentum, the partial time derivative for $\mathcal{F}_p$ becomes a total derivative as,

$$2g \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{p_{i_1} \cdots p_{i_k}}{\epsilon(\mathbf{p})^{r+2k-1}} \frac{\partial \mathcal{F}_p}{\partial t} = \frac{d}{dt} \mathcal{J}_{i_1, \dots, i_k}^{(k,r)}. \quad (\text{A.1})$$

The second term in Eq. (4.18)

By using the integration by part, the term proportional to $\mathcal{E}(t)$ is written as

$$2g \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{p_{i_1} \cdots p_{i_k}}{\epsilon(\mathbf{p})^{r+2k-1}} \frac{\partial \mathcal{F}_p}{\partial p_{\parallel}} = (r+2k-1) \mathcal{J}_{i_1, \dots, i_k, x}^{(k+1,r)} - \sum_{\ell=1}^k \delta_{i_{\ell}, x} \mathcal{J}_{i_1, \dots, \hat{i}_{i_{\ell}}, \dots, x'}^{(k+1, r+2)}, \quad (\text{A.2})$$

where we used the relation,

$$\epsilon(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}, \quad (\text{A.3})$$

$$\frac{\partial}{\partial p_{\parallel}} \left(\epsilon(\mathbf{p})^{-r-2k+1} \right) = -(r+2k-1) \epsilon(\mathbf{p})^{-r-2k-1} p_{\parallel}. \quad (\text{A.4})$$

In the same way as $\mathcal{E}(t)$ term, the term proportional to $3\dot{\sigma}$ is written as,

$$\begin{aligned} 2g \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{p_{i_1} \cdots p_{i_k}}{\epsilon(\mathbf{p})^{r+2k-1}} p_{\parallel} \frac{\partial \mathcal{F}_p}{\partial p_{\parallel}} &= -2g \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{\partial}{\partial p_{\parallel}} \left(\frac{p_{i_1} \cdots p_{i_k} p_{\parallel}}{\epsilon(\mathbf{p})^{r+2k-1}} \right) \mathcal{F}_p, \\ &= (r+2k-1) \mathcal{J}_{i_1 \cdots i_k x x}^{(k+2, r-2)} - \mathcal{J}_{i_1 \cdots i_k}^{(k,r)} - \sum_{\ell=1}^k \mathcal{J}_{i_1 \cdots \hat{i}_{i_{\ell}} \cdots i_k x'}^{(k,r)}, \end{aligned} \quad (\text{A.5})$$

where we used the following relation,

$$\frac{\partial}{\partial p_{\parallel}} \left(\frac{p_{i_1} \cdots p_{i_k} p_{\parallel}}{\epsilon(\mathbf{p})^{r+2k-1}} \right) = -(r+2k-1) \frac{p_{i_1} \cdots p_{i_k} p_{\parallel}^2}{\epsilon(\mathbf{p})^{r+2k+1}} + \sum_{\ell=1}^k \delta_{\ell,x} \frac{p_{i_1} \cdots \hat{p}_{i_{\ell}} \cdots p_{i_k} p_{\parallel}}{\epsilon(\mathbf{p})^{r+2k-1}} + \frac{p_{i_1} \cdots p_{i_k}}{\epsilon(\mathbf{p})^{r+2k-1}}. \quad (\text{A.6})$$

The third term in Eq. (4.18)

The term proportional to the $(\dot{\alpha} + \dot{\sigma})$ is written as,

$$2g(\dot{\alpha} + \dot{\sigma}) \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{p_{i_1}, \dots, p_{i_k}}{\epsilon(\mathbf{p})^{r+2k-1}} \mathbf{p} \cdot \frac{\partial \mathcal{F}_p}{\partial \mathbf{p}} = -(r+2k-1)(\dot{\alpha} + \dot{\sigma}) m^2 \mathcal{J}_{i_1, \dots, i_k}^{(k, r+2)} - (4-k-r)(\dot{\alpha} + \dot{\sigma}) \quad (\text{A.7})$$

where we used the relations,

$$2g(\dot{\alpha} + \dot{\sigma}) \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{p_{i_1}, \dots, p_{i_k}}{\epsilon(\mathbf{p})^{r+2k-1}} \mathbf{p} \cdot \frac{\partial \mathcal{F}_p}{\partial \mathbf{p}} = -2g(\dot{\alpha} + \dot{\sigma}) \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{\partial}{\partial \mathbf{p}} \left(\frac{p_{i_1} \cdots p_{i_k} \mathbf{p}}{\epsilon(\mathbf{p})^{r+2k-1}} \right) \mathcal{F}_p, \quad (\text{A.8})$$

and

$$\begin{aligned} \frac{\partial}{\partial \mathbf{p}} \left(\frac{p_{i_1} \cdots p_{i_k} \mathbf{p}}{\epsilon(\mathbf{p})^{r+2k-1}} \right) &= (k+3) \frac{p_{i_1} \cdots p_{i_k}}{\epsilon(\mathbf{p})^{r+2k-1}} - (r+2k-1) |\mathbf{p}|^2 \frac{p_{i_1} \cdots p_{i_k}}{\epsilon(\mathbf{p})^{r+2k+1}}, \\ &= (k+3) \frac{p_{i_1} \cdots p_{i_k}}{\epsilon(\mathbf{p})^{r+2k-1}} - (r+2k-1)(\epsilon^2 - m^2) \frac{p_{i_1} \cdots p_{i_k}}{\epsilon(\mathbf{p})^{r+2k+1}}, \\ &= -(r+k-4) \frac{p_{i_1} \cdots p_{i_k}}{\epsilon(\mathbf{p})^{r+2k-1}} + (r+2k-1) m^2 \frac{p_{i_1} \cdots p_{i_k}}{\epsilon(\mathbf{p})^{r+2k+1}}. \end{aligned} \quad (\text{A.9})$$

Thus we obtain Eq. (4.25) by using these discussions.

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