LETTER

Linear Phase Condition for Complex FIR Digital Filters

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SUMMARY A novel condition for complex FIR digital filters to have linear phase is presented. It naturally includes the linear phase conditions for real FIR filters as special cases.

Digital filters having complex coefficients, which were proposed 20 years ago(1) receive much interest recently(2). In the case of real coefficient filters, FIR filters are widely used since they can realize complete phase linearity by making their impulse responses even or odd symmetric about the center(3). It is known that odd-length complex FIR filters with conjugate symmetric impulse response have linear phase response(4). This letter shows a more general condition for complex FIR filters to have linear phase.

[Theorem] An N-tap complex FIR filter of the form

$$H(z)=\sum_{n=0}^{N-1} h(n)z^{-n} \tag{1}$$

has linear phase if

$$\{h(n)e^{-j\theta}\}^*=h(N-1-n)e^{-j\theta}, \quad \text{for all } n \tag{2}$$

holds, where asterisk denotes complex conjugate and θ is the phase at DC. ■

(Proof) For odd N, M is defined as $(N-1)/2$. Then the frequency response of Eq. (1) is written by

$$\begin{aligned}H(e^{j\omega})&=e^{-jM\omega}\Big[\sum_{n=0}^{M-1} h(n)e^{j(M-n)\omega}+h(M)\\&\quad+\sum_{n=M+1}^{N-1} h(n)e^{j(M-n)\omega}\Big]\\&=e^{-jM\omega}\Big[\sum_{m=1}^{M} h(M-m)e^{jm\omega}+h(M)\\&\quad+\sum_{m=1}^{M} h(M+m)e^{-jm\omega}\Big]\\&=e^{-j(M\omega-\theta)}\Big[\sum_{m=1}^{M} h(M-m)e^{j(m\omega-\theta)}\\&\quad+h(M)e^{-j\theta}+\sum_{m=1}^{M} h(M+m)e^{-j(m\omega+\theta)}\Big]\end{aligned} \tag{3}$$

where ω is the normalized angular frequency. If Eq. (2) holds, it can easily be shown that $h(M-m)e^{j(m\omega-\theta)}$ and $h(M+m)e^{-j(m\omega+\theta)}$ are mutually conjugate for all m and $h(M)e^{-j\theta}$ is real. Then the quantity inside the brackets in the last expression of Eq. (3) is real. Thus the phase response of this filter is $-(M\omega-\theta)$, which is linear with respect to ω and is equal to θ at DC. The case of even N is quite similar. ■

This general condition naturally includes all four cases (even or odd N, and even or odd symmetry)(3) for real FIR filters as special cases.

For the design of a linear-phase complex FIR filter, θ can be assumed to be zero without any loss of generality. Then Remez exchange algorithm can efficiently solve the design problem(4). The complete solution is obtained by multiplying $e^{j\theta}$ to all the coefficients.

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References

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