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LETTER Special Section on the 1992 IEICE Fall Conference

Efficient Design of N-D Hyperspherically Symmetric FIR Filters

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SUMMARY The design of N-dimensional (N-D) FIR filters requires in general an enormous computational effort. One of the most successful methods for design and implementation is the McClellan transformation. In this paper a numerically simple technique for determining the coefficients of the transformation is suggested. This appears to be the simplest available method for the design of N-D hyperspherically symmetric FIR filters with excellent symmetry.

key words: *digital filters, digital signal processing, multidimensional digital filters*

1. Introduction

In recent years an entire multidimensional approach to the processing of video signals has emerged and as a result the design of N-dimensional (N-D) filters is an active research topic. It is well known that in image processing the phase response must be linear [1]. FIR filters are preferred in practice, because they can have exactly linear phase and avoid the stability problems. Several approaches exist for the design of 2-D and 3-D filters[1]. Window functions and frequency sampling may be used, but the results are not satisfactory. Optimal filters have very good performance, but the design is a difficult task. Linear programming may also be used. All these techniques, however, have important disadvantages: they require extremely large computational effort and the computer time increases exponentially with respect to the number of dimensions. Also the developed computer programs are not readily available. Engineers need a simple design method that can be widely used. In this communication our aim is to suggest a very simple technique for the design of N-D linear-phase FIR filters with hyperspherical symmetry. Since multivariate polynomials are not factorable, somehow the design and implementation must be considered simultaneously. As a result a frequency transformation, known as McClellan transformation, has emerged as the most popular design technique for linear-phase 2-D FIR digital filters, owing to its simplicity and ease of implementation[1]. For the design of N-D hyperspherically symmetric filters the McClellan transform has the form of

$$\sin^2(\omega/2) = f(\omega_1, \omega_2, \cdots, \omega_N), \tag{1}$$

or

$$\begin{aligned}\sin^2\frac{\omega}{2} &= a_0 + a_1\left(\sin^2\frac{\omega_1}{2} + \sin^2\frac{\omega_2}{2} + \cdots + \sin^2\frac{\omega_N}{2}\right) \\ &\quad + a_2\left(\sin^2\frac{\omega_1}{2}\sin^2\frac{\omega_2}{2} + \sin^2\frac{\omega_1}{2}\sin^2\frac{\omega_3}{2} \right. \\ &\quad \left. + \cdots + \sin^2\frac{\omega_{N-1}}{2}\sin^2\frac{\omega_N}{2}\right) \\ &\quad + \cdots + a_N \sin^2\frac{\omega_1}{2}\sin^2\frac{\omega_2}{2}\cdots\sin^2\frac{\omega_N}{2},\end{aligned} \tag{2}$$

where ω is the frequency variable of the 1-D prototype and $\omega_1, \omega_2, \cdots, \omega_N$ are the frequency variables of the N-D filter. The problem is to find the coefficients a_i so that the iso-contours in the N-D frequency plane are as close as possible to hyperspheres. Real frequencies on the 1-D frequency axis must correspond to real frequencies in the N-D frequency space, or in other words, because of Eq. (1) $f(\omega_1, \omega_2, \cdots, \omega_N)$ must not be less than 0 and greater than 1:

$$f_{\min}(\omega_1, \omega_2, \cdots, \omega_N) = 0, \tag{3}$$

$$f_{\max}(\omega_1, \omega_2, \cdots, \omega_N) = 1. \tag{4}$$

If Eqs. (3) and (4) are inherent characteristics of the function, it is called scaling-free. Otherwise after determining the coefficients of the transformation scaling must be performed[1]. The original idea of J. McClellan for 2-D filters is $a_0=0$; $a_1=-a_2=1$. Charalambous[3] extended straightforwardly the original idea to the 3-D case to include $a_3=1$. For the N-D case, quite straightforwardly, we have $a_{2k+1}=1$ and $a_{2k}=-1$. The properties of the transformation (2) can be best understood by thinking of it as a mapping between the 1-D frequency axis and the N-D frequency hyperplane. The transformation corresponding to $a_0=0$, $a_1=-a_2=a_3=1$ meets Eqs. (3) and (4) (i.e. it is a scaling-free transformation), but maps all points $(0, 0, \pi)$, $(0, \pi, 0)$, $(\pi, 0, 0)$, $(\pi, \pi, 0)$, $(\pi, 0, \pi)$, $(0, \pi, \pi)$, (π, π, π) onto the point $\omega=\pi$ on the 1-D frequency axis. The result is a poor spherical symmetry. Hazra and Reddy in Ref. (2) suggested a simple, analytic

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technique for determining the coefficients for 2-D filters. In this communication this approach is extended to 3-D filters, which have increasing practical importance. A further extension is made, which leads to a scaling-free transformation in the general N-D case. To the best of our knowledge, this is the simplest technique for the design of arbitrary-dimensional linear-phase FIR digital filters with excellent hyper-spherical symmetry.

2. Outline of the Method

For the sake of clarity first the 3-D case will be discussed.

2.1 Design of 3-D Spherically Symmetric FIR Filters

If the transformation[(3)] is used, the deviation from the spherical shape is maximum at a point $\omega_1=\omega_2=\omega_3=R/\sqrt{3}$, where R is the radius of the sphere. Improvement of the shape approximation can be obtained by forcing this point to lie on the transform contour. The coefficients of the transformation can be found as follows.

1) The point $\omega=0$ should map onto $(0, 0, 0)$ or $\alpha_0=0$.

2) The points $(\omega_{1c}, 0, 0)$, $(0, \omega_{2c}, 0)$, $(0, 0, \omega_{3c})$ should map onto the cutoff frequency of the 1-D filter $\omega=\omega_c$. For a spherically symmetric digital filter the cutoff frequencies are equal $\omega_{1c}=\omega_{2c}=\omega_{3c}$. This leads to

$$\alpha_1=A/b_1, \tag{5}$$

where $A=\sin^2(\omega_c/2)$, $b_1=\sin^2(\omega_{1c}/2)$.

3) The points $(\omega_{1c}/\sqrt{2}, \omega_{1c}/\sqrt{2}, 0)$, $(0, \omega_{1c}/\sqrt{2}, \omega_{1c}/\sqrt{2})$ and $(\omega_{1c}/\sqrt{2}, 0, \omega_{1c}/\sqrt{2})$ should map onto $\omega=\omega_c$. Therefore

$$A=2\alpha_1 b_2+\alpha_2 b_2^2, \tag{6}$$

or $\alpha_2=[A-2\alpha_1 b_2]/b_2^2$, $b_2=\sin^2(\omega_{1c}/2\sqrt{2})$.

4) $(\omega_{1c}/\sqrt{3}, \omega_{1c}/\sqrt{3}, \omega_{1c}/\sqrt{3})$ should map onto $\omega=\omega_c$. In turn, this leads to

$$A=\alpha_1 3b_3+3\alpha_2 b_3^2+\alpha_3 b_3^3, \tag{7}$$

or

$$\alpha_3=[A-3\alpha_1 b_3-3\alpha_2 b_3^2]/b_3^3,\ b_3=\sin^2(\omega_{1c}/2\sqrt{3}).$$

Two important problems remain to be solved. Since the 3-D filter design specifications include the passband cutoff frequency ω_{1c} and stopband cutoff frequency ω_{1s} the corresponding cutoff frequencies of the 1-D prototype must be found. From Eqs. (5), (6) and (7) we have $A=b_1/G$,

$$G=3+3(2b_2-b_1)/b_2^2+b_1[b_1 b_2^2-3b_3 b_2^2 +3b_3^2(2b_2-b_1)]/b_3^3.$$

The passband cutoff frequency of the prototype is $\omega_c=2\arcsin(\sqrt{A})$. For the stopband cutoff frequency, the points $(\omega_{1s}, 0, 0)$, $(0, \omega_{1s}, 0)$, $(0, 0, \omega_{1s})$ are mapped onto the point $\omega=\omega_s$. From this mapping we have $V=W/G$, $W=\sin^2(\omega_{1s}/2)$, and finally $\omega_s=2\arcsin(\sqrt{V})$

2.2 Design of N-D FIR Filters

Now the extension to the N-D case is easier. The procedure is as follows:

$$\alpha_0=0,\ \alpha_1=A/b_1,\ \alpha_2=[A-2\alpha_1 b_2]/b_2^2,$$
$$\alpha_i=[A-C_i^1\alpha_1 b_i-C_i^2\alpha_2 b_i^2-\cdots -C_i^{i-1}\alpha_{i-1}b_i^{i-1}]/b_i^i,$$

where C_n^m are the binomial coefficients $n!/((n-m)!m!)$ and

$$b_i=[\sin(\omega_{1c}/(2\sqrt{i}))]^2,\ i=1,\cdots,N. \tag{8}$$

The cutoff frequencies are calculated using the same relationships. The expression for G in the N-D case is

$$G=C_N^1\beta_1+C_N^2\beta_2+\cdots+C_N^N\beta_N. \tag{9}$$

where $\beta_i=(\alpha_i b_1/A)$, $i=1,\cdots,N$. Let us summarize the complete design procedure.

Step 1: Specify the number of dimensions N, the passband cutoff ω_{1c} and the stopband cutoff ω_{1s} frequencies;

Step 2: Find b_i, $i=1, 2,\cdots, N$ from Eq. (8);

Step 3: Find β_i, $i=1, 2,\cdots, N$: $\beta_1=1$, $\beta_2=[b_1-2b_2]/b_2^2$,

$$\beta_i=[b_1-C_i^1\beta_1 b_i-\cdots-C_i^{i-1}\beta_{i-1}b_i^{i-1}]/b_i^i;$$

Step 4: Find G from Eq. (9);

Step 5: Find $A=b_1/G$, $V=W/G$ and the cutoff frequencies for the 1-D filter

$$\omega_c=2\arcsin(\sqrt{A}),\ \omega_s=2\arcsin(\sqrt{V});$$

Step 6: Find the coefficients of the transformation $\alpha_i=A\beta_i/b_1$, $i=1,\cdots,N$;

Step 7: Finally, design a suitable 1-D FIR zero-phase prototype with the corresponding cutoff frequencies.

3. Examples and Comments

Several examples were computed and the results are summarized in Table 1. In Fig. 1 the passband cutoff contour of the filter with an ideal passband frequency $\omega_{1c}=0.5\pi$ and stopband cutoff frequency $\omega_{1s}=0.6\pi$ is drawn. It is important to measure the quality of the approximation, achieved by the new technique. The ideal sphere is described by the equation $\omega_3=f_1(\omega_1, \omega_2)$, or

Table 1 Requirements, coefficients of the transformation and resulting cutoff frequencies of the 1-D prototype.

ω_{1c}	ω_{1s}	α_1	α_2	α_3	ω_c	ω_s
0.1π	0.2π	0.65374551	−0.43726712	0.35056482	0.080739π	0.160765π
0.2π	0.3π	0.65853004	−0.44484513	0.35894527	0.161366π	0.240197π
0.3π	0.4π	0.66674391	−0.45790414	0.37348069	0.241767π	0.318692π
0.4π	0.5π	0.67877322	−0.47713967	0.39509935	0.321824π	0.395905π
0.5π	0.6π	0.69521705	−0.50364060	0.42527065	0.401416π	0.471330π
0.6π	0.7π	0.71696015	−0.53903400	0.46622157	0.480416π	0.544189π
0.7π	0.8π	0.74529189	−0.58572529	0.52130019	0.558699π	0.613222π
0.8π	0.9π	0.78210248	−0.64730053	0.59559414	0.636161π	0.676281π
0.9π	π	0.83021542	−0.72922027	0.69701455	0.712785π	0.729626π

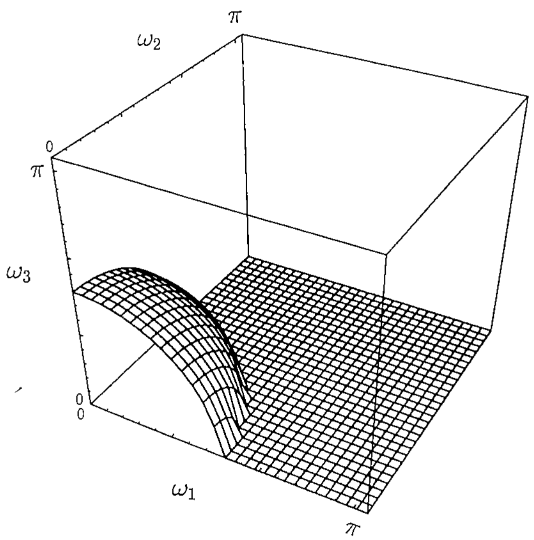

Fig. 1 Passband of the designed 3-D digital filter.

$$f_1(\omega_1, \omega_2)=\begin{cases}\sqrt{r^2-\omega_1^2-\omega_2^2} & \text{if } r^2>\omega_1^2+\omega_2^2\\ 0 & \text{otherwise}\end{cases}$$

From Eq. (2) we can express ω_3 in terms of ω_1 and ω_2, $\omega_3=f_2(\omega_1, \omega_2)$:

$$f_2(\omega_1, \omega_2)=\begin{cases}2\arcsin\sqrt{D} & \text{if } D>0\\ 0 & \text{otherwise}\end{cases}$$

where

$$D=\frac{\begin{array}{c}-\alpha_1(\sin^2\omega_1/2+\sin^2\omega_2/2)\\ -\alpha_2\sin^2\omega_1/2\,\sin^2\omega_2/2+\sin^2\omega/2\end{array}}{\begin{array}{c}\alpha_1+\alpha_2(\sin^2\omega_1/2+\sin^2\omega_2/2)\\ +\alpha_3\sin^2\omega_1/2\,\sin^2\omega_2/2\end{array}} \qquad (10)$$

The error function can be defined as

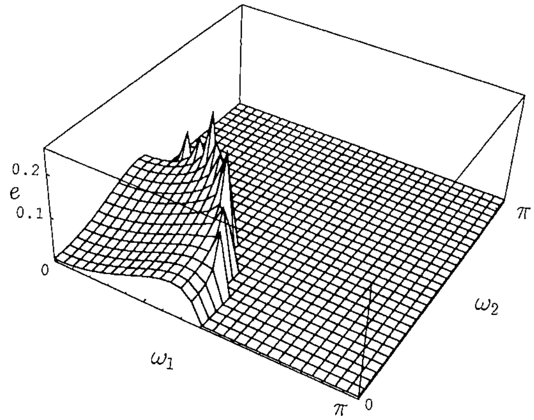

Fig. 2 The approximation error using the original McClellan transformation.

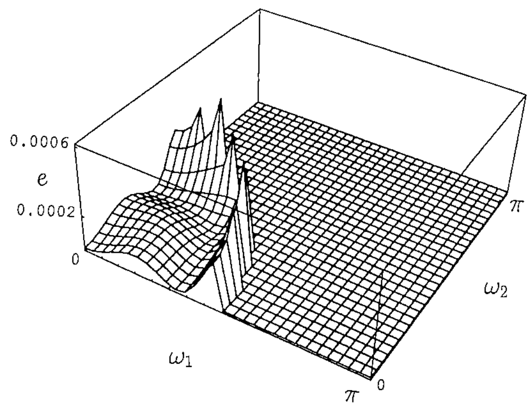

Fig. 3 The approximation error using the new approach.

$$e(\omega_1, \omega_2)=f_1(\omega_1, \omega_2)-f_2(\omega_1, \omega_2). \qquad (11)$$

One might argue that other error expressions are more suitable. The functions f_1 and f_2 are zero over different regions and the relative error cannot be used. Another possibility is to use the L_p norm, but the error is a two-variable function and there are problems with the numerical accuracy in calculating the double integral. From a qualitative viewpoint the absolute error Eq. (11) does give an idea of the spherical symmetry. In order to appreciate the improvement in the shape approximation, offered by the new technique, the error functions for the original transformation (Fig. 2) and for the new technique (Fig. 3) are plotted. Note that the error functions are not usually smooth, and the maxima are tiny peaks, omitted in these figures. Nevertheless, the improvement in the approximation is considerable and obvious. For other cutoff frequencies the results are essentially the same. It should be pointed that the cross-section for $\omega_3=0$ does not coincide with the 2-D transformation in Ref. (2). The circular symmetry of the cross-section is not so good in order to make the overall spherical symmetry better. Since this is a 3-D filter the spherical symmetry is important, rather than the circular symmetry of the cross-sections.

4. Conclusions

A simple approach for the design of N-D zero-phase FIR digital filters with hyperspherical symmetry has been described. The suggested method is a further development of the techniques in Refs. (2),(3) and (4). This approach involves two steps: a transformation from 1-D to N-D and the design of a 1-D filter. These steps are independent, but it is important the coefficients of the transformation to be found first and then an 1-D prototype to be designed. The resulting N-D filters can be implemented efficiently using the Chebyshev structure[(1),(3)]. The technique for determining the coefficients is extremely numerically simple and results in filters with excellent hyperspherical symmetry. In the general N-D case no other technique seems competitive, if at all feasible.

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