論文 / 著書情報
Article / Book Information

題目(和文)	
Title(English)	Mathematical Foundations for Algebraic Decomposition Theory
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出典(和文)	学位:工学博士, 学位授与機関:東京工業大学, 報告番号:甲第1437号, 授与年月日:1983年1月31日, 学位の種別:課程博士, 審査員:
Citation(English)	Degree:Doctor of Engineering, Conferring organization: , Report number:甲第1437号, Conferred date:1983/1/31, Degree Type:Course doctor, Examiner:
学位種別(和文)	博士論文
Type(English)	Doctoral Thesis

MATHEMATICAL FOUNDATIONS FOR ALGEBRAIC DECOMPOSITION THEORY

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Preface :

This dissertation reports on the unified treatment of decomposition theory from the general systems theoretical point of view. The theory [1] was introduced in 1960's by M.D.Mesarovic, Y. Takahara and others. Since it is based on formalization, it has the following pay-offs [2] ;

1) formalizing a connected family of concepts is one way to bring out their meaning in an explicit fashion ;

2) formalization results in the standardization of terminology and the methods of conceptual analysis for various branches of science ;

3) the generality provided by formalization enables us to determine the essential feautures of theories ;

4) formalization provides a degree of objectivity which is impossible without formalization ;

5) formalization makes clear exactly what is being assumed, and thus is a safeguard against ad hoc and post hoc verbalization ;

6) formalization enables one to determine what the minimal assumption are which a theory requires.

The interest of this dissertation is strictly in the algebraic aspects of decomposition theory. Applications and philosophical implications are not included.

It will be seen that the essential feauture of decomposition is how to find a class of congruence relations with an appropriate property. Universal Algebra gives this insight and it seems to be natural to consider a system as an algebra for the development of systems theory. Initially, the problem was introduced by Professor Y.Takahara, who is my supervisor, and I have concentrated on this problem over the years in

the doctor course of Tokyo Institute of Technology. I express my heartfelt thanks to Professor Y.Takahara. In my academic years in Europe, I could proceed the study in a good situation. I appreciate for much discussions my polish collegues, Dr.I.Sierocki, Dr.W.Jacak and Dr.J.Hajdul, and austrian collegues,Dr.H.Kellermayr and Dr.G.Straka. Especially, I greatly appreciate Professor J.Jaron of Technical University of Wroclaw and Professor F.Pichler of Linz University who were helpful and gave me much insights. Finally, I thank to Professor B.Nakano, Dr.H.Ikeshoji, Dr.K.Kijima and all my collegues of our group.

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1. Introduction

In systems engineering, we often have to treat a complicated system. One of the most useful and popular method for investigating a complicated system is decomposition method. That is, we decompose a complicated system into simple systems and synthesize a system by those simple systems so that it is connected to the original system in an appropriate relation. Since the synthesized system is simpler, we can easily investigate the original system by the synthesized system according to the relation.

Before considering what problems are important in decomposition theory, we introduce the following concepts [3]. A <u>global system</u> is an object of decomposition and we denote it by $\hat{S} \subset \hat{X} \times \hat{Y}$. A <u>complex system</u> over a class of <u>component systems</u> $\underline{S} = \{ S_i | i \varepsilon I \}$ is a subset of $\Pi \underline{S}$ and we denote it by S, that is, $S \subset \Pi \underline{S}$. Those concepts are precisely defined in section 2.2. We often identify a complex system S as an input-output system $S \subset X \times Y$,

where $X=\underline{D}(S)= \{ [x_i|i \varepsilon I] | (\exists[y_i|i \varepsilon I])(([x_i|i \varepsilon I],[y_i|i \varepsilon I]) \varepsilon S) \}$

and $Y=\underline{R}(S)= \{ [y_i|i \varepsilon I] | (\exists[x_i|i \varepsilon I])(([x_i|i \varepsilon I],[y \;|i \varepsilon I]) \varepsilon S) \}$,

respectively. The class of focused systems, that is, the class of systems which have some common property, is denoted by $\overline{S}$.

The central problem in decomposition theory is how to find properties of a global system $\hat{S}$ by investigating those of component systems $\underline{S}= \{ S_i | i \varepsilon I \}$ when there is a relation between $\hat{S}$ and a complex system $S \subset \Pi \underline{S}$. Among properties, we must consider systemic ones such as stability and controllability. A problem how to find a property of a function defined on a global system or a complex system when the function is induced by those of component systems must be also included as a special case of this problem. Furthermore, as a converse of this problem, we must also investigate what kind of global systems or complex systems have a given property.

To solve these problems, the following problems are fundamental in decomposition theory.

1) Characterization of Interactions

A complex system $S \subset X \times Y$ over $\underline{S} = \{ S_i | i \in I \}$ is defined as a subset of $\Pi \underline{S}$. If $S = \Pi \underline{S}$, we can say that there is no interaction among component systems in S, that is, S is a non-interacted system. Generally S is a proper subset of $\Pi \underline{S}$ because of the existence of interactions among component systems in S. The first step of characterization of interactions has been made by Takahara and Nakano[4]. However, in order to classify complex systems according to their structures, we need more detail characterization. Furthermore, it is necessary to clarify the difference of hierarchical systems from non-hierarchical ones.

2) Decomposition Problem

In this problem, we pay attention to a necessary and sufficient condition under which a global system can be decomposed into a certain complex system.

Suppose that a global system $\hat{S} \subset \hat{X} \times \hat{Y}$ and a type of a complex system are given. Let R be a binary relation on $\overline{S}$, where $\hat{S}$ is an element of $\overline{S}$. Find a necessary and sufficient condition for constructing a complex system S of the given type such that $(\hat{S}, S) \in R$.

By investigating a relationship between complex systems, we can also solve a problem how to find a condition for decomposition of a global system into a given complex system $S \subset X \times Y$ over $\underline{S}= \{ S_i | i \in I \}$.

3) Property Preserving or Reflecting Decomposition

In this problem, we investigate preservation and reflection of properties under some relations.

Suppose that a global system $\hat{S} \subset \hat{X} \times \hat{Y}$ and a complex system $S \subset X \times Y$ over $\underline{S}= \{ S_i | i \in I \}$ are given. Let R be a binary relation on $\overline{S}$ such that $(\hat{S},S) \in R$ and P a property on $\overline{S}$.

a) Property Preservation Problem

(i) What kind of properties can be preserved under R.

(ii) What kind of relations preserves a property P.

b) Property Reflection Problem

(i) What kind of properties can be reflected under R.

(ii) What kind of relations reflects a property P.

Furthermore, the following problem is also important in Property Reflection Problem ;

(iii) Investigate a property of a function $\hat{f}$ defined on a global system $\hat{S}$ or a function f defined on a complex system S over $\underline{S}= \{ S_i | i \in I \}$ when $(\hat{S},S) \in R$ by using functions f_i on S_i for $i \in I$, where $\hat{f}$ and f are induced by $\underline{f}= \{ f_i | i \in I \}$ in an appropriate way.

The main goal of this dissertation is to construct the mathematical foundations for algebraic decomposition theory. We introduce some concepts and provide the mathematical basis in order to solve the problems mentioned above.

Mainly we pay attention to decomposition problem of systems. Since non-interacted system, parallel connected system, cascade connected system and feedback connected system are basic, we solve decomposition problem into these connections. In order to make concepts transparent some examples from Automata Theory and Linear Systems Theory are also checked. Other two problems are also investigated for the further study.

The dissertation is organized as follows ;

In Part 0, we first define the object of decomposition such as input-output systems, functional systems and transition systems and consider the meaning of decomposition in Chapter 2. Since the approach is mainly based on Universal Algebra, we introduce some universal algebraic concepts in that Chapter. Furthermore, we investigate some properties of congruence relations, which will play a central role in decomposition.

In Part I, we define primary connections such as non-interacted connection, parallel connection, cascade connection and feedback connection in Chapter 3. And in Chapter 4, we construct the general characterization theory of intercactions.

In Part II, we investigate a condition for decomposition. First, we construct the basic scheme of decomposition in Chapter 5. Decomposition problem of input-output systems are investigated in Chapter 6. Other types of decomposition are also taken into account in Chapter 7. While in Chapter 8 and Chapter 9, we investigate decomposition problem of functional systems and transition systems, respectively.

In Part III, we move our attention to property reflecting decomposition in Chapter 10. A well-known example from Artificial Intelligence is investigated to illustrate the usefulness of decomposition theory.

Finally, we make conclusions of this dissertation.

2. Preliminaries

In this chapter, we provide the mathematical preliminaries for the dissertation. Since we develop the decomposition theory in the framework of the mathematical general systems theory introduced by M.D.Mesarovic and Y.Takahara[1], we first review the concept of "system". Second we define systems in decomposition. Because our approach is mainly algebraic (universal algebraic), some basic concepts in universal algebra are introduced in section 2.3. It is an algebra with a group operation (Ω -group) that is one of the basic units for decomposition. Therefore we next introduce the concept of Ω -group and investigate its property. Since we recognize the basic idea of decomposition is to realize a system by using a class of congruence relations, we investigate some properties of a class of congruence relations in section 2.5.

2.1 Basic Concepts[1]

As this dissertation is based on the mathematical general systems theory. Let us first review the concept of "system".

<u>Definition 2.1.1</u> System

Let $\underline{V} = \{ V_i | i \varepsilon I \}$ be a class of non-void sets. Then we call a subset S of $\Pi \underline{V}$, a system over $\underline{V}$ and V_i is called the i-th object of S. That is, we consider a system as a relation among objects.

When we give an input-output recognition to a system, we obtain an input-output system.

Definition 2.1.2 Input-Output System

A system $S \subset \Pi \underline{V}$ over $\underline{V} = \{ V_i | i \in I \}$ is called an input-output system if there is a partition of I, $I = I_x \cup I_y$ such that $S \subset X \times Y$, where $X = \underline{D}(S) \subset \Pi (V_i | i \in I_x)$ and $Y = \underline{R}(S) \subset \Pi (V_i | i \in I_y)$. In this case, X is refered to as the input set of S and Y the output set of S, respectively.

From now on, we assume $\underline{D}(S) = \{ x | (\exists y)((x,y) \in S) \} = X$ and $\underline{R}(S) = \{ y | (\exists x)((x,y) \in S) \} = Y$ in every input-output system $S \subset X \times Y$.

The concept of an input-output system is very broad so that every object of decomposition can be written as an input-output system.

An input-output system $S \subset X \times Y$ is generally a relation, however, it is easy to be handled and gives us much insight when S is a function from X to Y.

Definition 2.1.3 Functional System

An input-output system $S \subset X \times Y$ is called to be functional if $(\forall (x,y) \in S)(\forall (x',y') \in S)(x=x' \rightarrow y=y')$.

One of the most important system is a transition system. It is known that every causal time system has a canonical representation $(\overline{\phi} , \overline{\mu})$[1], where $\overline{\phi} = \{ \phi_{tt'} : C \times X_{tt'} \rightarrow C | t,t' \in T \text{ and } t \leqslant t' \}$ is the family of state transition functions and $\overline{\mu} = \{ \mu_t : C \times A \rightarrow B | t \in T \}$ is the family of output functions. A transition system is the dynamical part of a canonical representation and defined as follows ;

Definition 2.1.4 Transition System

An input-output system $S \subset X \times Y$ is called a transition system if $C = X = Y$, and there is a nonvoid set U and a mapping $\phi : C \times U \to C$ such that

$$(x,y) \in S \leftrightarrow (\exists u)(\phi(x,u)=y).$$

In this case, we refer to C as the state set and U as the input set, respectively and the transition system is denoted by $T = [U, C, \phi]$.

We often consider U as $(T \times T) \times X$ and $\phi(c,t,t',x) = \phi_{tt'}(c, x_{tt'})$. The latter expression is more familiar in the general systems theory[1]. So we denote a transition system as $T=[\overline{X}, C, \overline{\phi}]$, where $\overline{X} = \{ x_{tt'} \in X_{tt'} | t,t' \in T \text{ and } t \leqslant t' \}$ and $\phi = \{ \phi_{tt'} : C \times X_{tt'} \to C \, | x_{tt'} \in X_{tt'} \}$.

2.2 Systems in Decomposition[3]

Decomposition theory is, in other words, a theory of the relationship between a real system and a model when a model consists of some component systems. From now on, we refer to a global system as a real system and a complex system as a model which consists of some component systems. That is,

Definition 2.2.1 Global System

An input-output system is called a global system if it is an object of the decomposition and we often denote it by $\hat{S} \subset \hat{X} \times \hat{Y}$.

Definition 2.2.2 Complex System

A complex system S over a class of input-output systems $\underline{S}$ = $\{ S_i | i \in I \}$ is a subset of the direct product of $\underline{S}$ satisfying $p_i(S) = S_i$, where $p_i : S \to S_i$ is the i-th projection defined by $p_i([s_i | i \in I]) = s_i$ and S_i is called the i-th component system of S. We often identify S as an input-output system $S \subset X \times Y$, where $X = \underline{D}(S) = \{ [x_i | i \in I] | (\exists [y_i | i \in I])([(x_i, y_i) | i \in I] \in S) \} \subset \Pi (X_i | i \in I)$ and $Y = \underline{R}(S) = \{ [y_i | i \in I] | (\exists [x_i | i \in I])([(x_i, y_i) | i \in I] \in S) \} \subset \Pi (Y_i | i \in I)$.

Definition 2.2.3 Modelling Morphism (Fig. 2.2.1)

Let $S \subset X \times Y$ and $S' \subset X' \times Y'$ be input-output systems. Suppose that $h_x : X \to X'$ and $h_y : Y \to Y'$ are mappings. If $(x,y) \in S$ implies $(h_x(x), h_y(y)) \in S'$, the pair $h = (h_x, h_y)$ is called a modelling morphism from S to S'. If there is a modelling morphism from S to S', S' is called a model of S. And when there is a modelling morphism from a global system to a complex system (or from a complex system to a global system), we refer the situation as inductive modelling (or deductive modelling, respectively).

Among modelling morphisms, the following two types are important.

(a) h_x and h_y are injective;

(b) h is surjective as a mapping, that is,

$$h(S) = \{ (h_x(x), h_y(y)) | (x,y) \in S \} = S';$$

, where (a) is refered to as an injective modelling morphism and (b) a surjective modelling morphism, respectively. If h_x and h_y are injective, the image h(S) of h is generally a proper subsystem of S', that is, $h(S) \subsetneq S'$. Hence the model S' contains much information than S. While if h(S) = S', the model can be considered as a simplified form of S.

$$\begin{tikzcd} X \arrow[r, "S"] \arrow[d, "h_X"'] & Y \arrow[d, "h_Y"] \\ X' \arrow[r, "S'"'] & Y' \end{tikzcd}$$

Fig. 2.2.1 Modelling Morphism

Proposition 2.2.1

Let $S \subset X \times Y$ be a complex system over $\underline{S} = \{ S_i | i \varepsilon I \}$. Then each component system $S_i \varepsilon \underline{S}$ is a surjective model of S.

We can construct the category[5),6)] of input-output systems with a modelling morphism[7),8)].

Proposition 2.2.2

Let $\overline{S}$ be the class of all input-output systems. And $\mathrm{Hom}_{\underline{MOD}}(S,S') = \{ h = (h_x,h_y):S \rightarrow S' | \text{ h is a modelling morphism from S to S}' \}$. Let the composition operation be defined by $h' \circ h = (h_x',h_y') \circ (h_x,h_y) = (h_x' \cdot h_x, h_y' \cdot h_y)$, where " $\cdot$ " is the juxtaposition of mappings. Then $\underline{MOD} = [\overline{S}, \{ \mathrm{Hom}_{\underline{MOD}}(S,S') | S,S' \varepsilon \overline{S} \}, \circ]$ is a category and we call it the category of systems in modelling.

The next propositions state the relationship among some important types of morphisms in $\underline{MOD}$.

Proposition 2.2.3[3),7)]

Let $h = (h_x,h_y)$ be a modelling morphism from S to S'. Then

(1) h is injective as a mapping if and only if h is a monomorphism.

(2) h is a monomorphism if h is an injective modelling morphism. The converse does not hold.

(3) h is an injective modelling morphism if h is a section. The converse does not hold.

(4) h is a section if h is an isomorphism. The converse does not hold.

Proposition 2.2.4[3),7)]

Let $h = (h_x, h_y)$ be a modelling morphism. Then

(1) h is an epimorphism if and only if both h_1 and h_2 are surjective.

(2) h is an epimorphism if h is a surjective modelling morphism. The converse does not hold.

(3) h is a surjective modelling morphism if h is a retraction. The converse does not hold.

(4) h is a retraction if h is an isomorphism. The converse does not hold.

Proposition 2.2.5[7),8)]

$h=(h_1,h_2)$ is an isomorphic modelling morphism (isomorphism) if and only if it is a surjective and injective modelling morphism.

When there is an isomorphism from S to S', we denote it by $S \cong S'$

In most cases of decomposition, we pay attention to modelling morphisms. However, the following morphism is also of interest.

Definition 2.2.4 Simulation Morphism (Fig. 2.2.2)

Let $S \subset X \times Y$ and $S' \subset X' \times Y'$ be input-output systems. Suppose that $h_x: X \to X'$ and $h_y: Y' \to Y$ are mappings. If $(x,y) \in S$ implies that there exists $y' \in Y'$ such that $(h_x(x), y') \in S'$ and $h_y(y')=y$, the pair $h^s = (h_x, h_y)$ is called a simulation morphism from S to S'. If there is a simulation morphism h^s from S to S', S' is called a simulation model of S.

In order to distinguish a simulation morphism from a modelling morphism, we use the notation h^s for a simulation morphism.

$$\begin{tikzcd}
X \arrow[r, "S"] \arrow[d, "h_X"'] & Y \\
X' \arrow[r, "S'"'] & Y' \arrow[u, "h_Y"']
\end{tikzcd}$$

Fig. 2.2.2   Simulation Morphism

A relation between modelling morphisms and simulation morphisms will be given in Section 7.2.

When we take a simulation morphism as a morphism in MOD, we also obtain a category.

Proposition 2.2.6

Let $\overline{S}$ be the class of all input-output systems and $\mathrm{Hom}_{\mathrm{SIM}}(S,S')$ $= \{ h^S = (h_x, h_y) \mid h^S$ is a simulation morphism from S to S' $\}$. Let the composition operation be defined by

$h^{S'} \circ h^S = (h_x', h_y') \circ (h_x, h_y) = (h_x' \cdot h_x, h_y \cdot h_y')$, where " $\cdot$ " is the juxtaposition of mappings. Then $\mathrm{SIM} = [\overline{S}, \{ \mathrm{Hom}_{\mathrm{SIM}}(S,S') \mid S,S' \in \overline{S} \}, \circ]$ is a category and we call it the category of input-output systems in simulation.

2.3 Universal Algebra

In this dissertation, we often use some concepts of universal algebra. Let us first review the basic concepts in universal algebra[9),10)].

Definition 2.3.1 (Universal) Algebra

A universal algebra or, briefly algebra $\underline{A}$ is a pair [A;F], where A is a nonvoid set and F is a family of finitary operations on A.

Definition 2.3.2 Type

A type of algebras τ is a sequence $[n_o, n_1, \ldots, n_\gamma, \ldots]$ of non-negative integers, $\gamma < o(\tau)$, where $o(\tau)$ is an ordinal, called the order of τ. For every $\gamma < o(\tau)$ we have a symbol $\overline{f}_\gamma$ of n_γ-ary operation.

Definition 2.3.3 Algebra of Type τ

An algebra of type τ is a pair $\underline{A}=[A;F]$, where A is a nonvoid set called the base set of $\underline{A}$ and for every $\gamma < o(\tau)$, we realize $\overline{f}_\gamma$ as an n_γ-ary operation on A : $(\overline{f}_\gamma)_A$ and $F=[(\overline{f}_o)_A, \ldots, (\overline{f}_\gamma)_A, \ldots]$.
For notational convenience, we write f_γ instead of $(\overline{f}_\gamma)_A$ without any confusion.

Example : The set N of non-negative integers is a monoid $\underline{N}=[N; \{0,+\}]$, where $0: \{\phi\} \to N$ is a nullary operation defined by $0(\phi)=0 \in N$ and $+ : N \times N \to N$ is a binary operation defined by $+(m,n)= m + n$.

The most important algebraic concepts are those of subalgebras, homomorphisms and congruence relations. Let us next review such concepts.

Definition 2.3.4 Subalgebra

Let $\underline{A}=[A;F]$ be an algebra of type τ and B a nonvoid subset of A. $\underline{B}=[B;F]$ is called a subalgebra of $\underline{A}$ if and only if

$(\forall b_o, \ldots, \forall b_{n\gamma -1} \in B)(f_\gamma(b_o, \ldots, b_{n\gamma -1}) \in B)$ for all $\gamma < o(\tau)$.

The class of all subalgebras of $\underline{A}$ is denoted by Sub($\underline{A}$).

Definition 2.3.5 Homomorphism

Let $\underline{A}$=[A;F] and $\underline{B}$=[B;F] be algebras with type τ . A mapping ϕ :A → B such that

$$\phi\,(f_\gamma\,(a_o,\ldots,a_{n\gamma}\,-1))=f_\gamma\,(\,\phi\,(a_o),\ldots,\,\phi\,(a_{n\gamma}\,-1))$$

for all $\gamma < o(\,\tau\,)$ is called a homomorphism from $\underline{A}$ to $\underline{B}$.

If $\underline{A} = \underline{B}$, ϕ is called an endomorphism on $\underline{A}$ and the class of all endomorphisms on $\underline{A}$ is denoted by End($\underline{A}$).

And if a homomorphism is bijective, it is called an isomorphism. If there is an isomorphism from $\underline{A}$ to $\underline{B}$, we denote it by $A \cong B$.

Definition 2.3.6 Congruence Relation

Let $\underline{A}$=[A;F] be an algebra and Θ a binary relation on A. Θ is called a congruence relation on $\underline{A}$ if it is an equivalence relation on A and satisfies the substitution property (SP) ;

$$\text{(SP)}\quad (\forall\,\gamma < o(\,\tau\,))(\forall a_o,\ldots,\forall a_{n\gamma}\,-1\ \varepsilon\ A)(\forall b_o,\ldots,\forall b_{n\gamma}\,-1\ \varepsilon\ A)$$
$$(a_o\ \Theta\ b_o,\ldots,a_{n\gamma}\,-1\ \Theta\ b_{n\gamma}\,-1$$
$$\rightarrow\ f_\gamma\,(a_o,\ldots,a_{n\gamma}\,-1)\ \Theta\ f_\gamma\,(b_o,\ldots,b_{n\gamma}\,-1)).$$

The class of all congruence relations on $\underline{A}$ is denoted by Con($\underline{A}$).

An important property of a congruence relation is that we can construct a new algebra called the quotient algebra.

Let $\underline{A}$=[A;F] be an algebra and Θ a congruence relation on $\underline{A}$. And let $A\ /\ \Theta = \{\,[a]\ \Theta\ |a\ \varepsilon\ A\,\}$ be the quotient set of A modulo Θ . For any $f_\gamma\ \varepsilon$ F, define the corresponding operation by $f_\gamma\,([a_0]\ \Theta\,,\ldots,[a_{n\gamma}\,-1]\ \Theta\,)$ $=[f_\gamma\,(a_0,\ldots,a_{n\gamma}\,-1)]\ \Theta$. Since Θ is a congruence relation on $\underline{A}$, a new operation f_γ defined above is well-defined. Then $\underline{A}\ /\ \Theta$ =[A/ Θ ;F] is an algebra of the same type as $\underline{A}$ and called the quotient algebra of $\underline{A}$

modulo Θ .

2.4 Ω -group

In the previous section, we defined three classes ; Sub($\underline{A}$), End($\underline{A}$) and Con($\underline{A}$) of an algebra $\underline{A}$. Let us next investigate a relation among them.

Theorem 2.4.1

Let $\underline{A}=[A;F]$ be an algebra and $Ker \subset End(\underline{A}) \times Con(\underline{A})$ defined by $(\phi, \Theta) \in Ker \leftrightarrow \Theta = \{ (a,b) \in A^2 | \phi(a) = \phi(b) \}$. Then Ker is a mapping.

When we restrict our attention to the following algebra, we obtain a remarkable relation among End($\underline{A}$), Con($\underline{A}$) and Sub($\underline{A}$).

Definition 2.4.1 Ω -group

Let $\underline{A}_G=[A;F_G]$ be a group and $F_U = \{ f_\omega | \omega \in \Omega \} \subset End(\underline{A}_G)$. Then $\underline{A}=[A;F_G \cup F_U]$ is called an Ω -group.

For notational convenience, we express the group structure F_G as $\{ 0,-,+ \}$, where + is the binary operation, - the inverse operation and 0 the unit.

Example 2.4.1

1) A group $\underline{G}=[G; \{ 1, ^{-1}, \cdot \}]$ is of course an Ω -group with $F_U = \phi$.
2) A ring $\underline{R}=[R; \{ 0,-,+ \} \cup \{ 1, \cdot \}]$ can be considered as an Ω - group $\underline{R}=[R; \{ 0,-,+ \} \cup F_U]$, where $R_G=[R; \{ 0,-,+ \}]$ is an abelian group, $F_U = \{ f_r:R \rightarrow R | r \in R \} \subset End(\underline{R}_G)$, $f_r(r')=r \cdot r'$ and F_U is a monoid.
3) A vector space $\underline{V}=[V; \{ 0,-,+ \} \cup F_U]$ can be considered as an Ω -group , where $F_U = \{ f_\lambda :V \rightarrow V | \lambda \in \Lambda \}$, $f_\lambda(v) = \lambda v$ (scalar product) and

F_U is a field.

The following subalgebras of an Ω -group are important.

Definition 2.4.2 Ω -Normal Subgroup

Let $\underline{A}=[A;F_G \cup F_U]$ be an Ω -group. A subalgebra $\underline{B}=[B;F_G \cup F_U]$ of $\underline{A}$ is called an Ω -normal subgroup if for any $b \in B$, any $a \in A$, $a + b + (-a) \in B$ And the class of all Ω -normal subgroups of an Ω -group $\underline{A}$ is denoted by $NSub(\underline{A})$.

Proposition 2.4.1

Let $\underline{A}=[A;F_G \cup F_U]$ be an Ω -group. Then $NSub(\underline{A}) \subset Sub(\underline{A})$. Moreover the equality holds if the binary operation in F_G is commutative.

The reason why we consider Ω -groups is that we can treat the concept of normal subgroups, ideals and linear subspaces in the unified way. That is ;

Example 2.4.2

1) An Ω -normal subgroup of a group is equal to a normal subgroup.

2) An Ω -normal subgroup of a ring is equal to an ideal.

3) An Ω -normal subgroup of a vector space is equal to a subspace.

Remark : It is noted that an ideal I of a ring R is not generally a subring of R. In our formulation, however, it is an Ω -normal subgroup of R.

Theorem 2.4.2

Let $\underline{A}=[A;F_G \cup F_U]$ be a commutative Ω -group and Ψ :Con($\underline{A}$) $\to$ Sub($\underline{A}$) be defined by

$$\Psi (\Theta)= \{ a + (-b)|(a,b) \varepsilon \Theta \} .$$

Then Ψ is bijective.

Let us next investigate a property of the bijection Ψ .

Definition 2.4.3 Complementarity

Let $\underline{B_1}$ and $\underline{B_2}$ be Ω -subgroup of an Ω -group $\underline{A}=[A;F_G \cup F_U]$. If $\underline{B_1}$ and $\underline{B_2}$ satisfy the following conditions, $\underline{B_1}$ and $\underline{B_2}$ are called to be complementary.

1) $B_1 \cap B_2= \{ 0 \}$, where 0 is the identity of F_G,

2) $B_1 + B_2= \{ b_1+b_2|b_1 \varepsilon B_1$ and $b_2 \varepsilon B_2 \} =A$.

In this case, we denote it by $(\underline{B_1}, \underline{B_2}) \varepsilon \perp (\underline{A})$.

Proposition 2.4.2

Let $\underline{B_1}$ and $\underline{B_2}$ be Ω subgroups of a commutative Ω -group $\underline{A}=[A;F_G \cup F_U]$, where the binary operation + in F_G is commutative. Suppose that $(B_1,B_2) \varepsilon \perp (\underline{A})$. Then for any a ε A, there is exactly one decomposition of a such that $a=b_1+b_2$, where $b_1 \varepsilon B_1$ and $b_2 \varepsilon B_2$.

The following theorem states the bijection Ψ from Con($\underline{A}$) to Sub($\underline{A}$) preserves some structure when $\underline{A}$ is a commutative Ω -group.

Theorem 2.4.3

Let $\underline{B}_1$ and $\underline{B}_2$ be Ω -subgroups of a commutative Ω -group $\underline{A}=[A;F_G \cup F_U]$, Then $(\underline{B}_1, \underline{B}_2) \in \perp (\underline{A})$ if and only if $\underline{\Theta} = \{ \Psi^{-1}(\underline{B}_1), \Psi^{-1}(\underline{B}_2) \}$ is full and separating.

Theorem 2.4.4

Let $\underline{A}=[A;F_G \cup F_U]$ be an Ω -group and $\mathrm{Ker}^G \subset \mathrm{End}(\underline{A}) \times \mathrm{NSub}(\underline{A})$ be defined by

$$(\phi ,\underline{B}) \in \mathrm{Ker}^G \leftrightarrow B= \{ a \in A | \ \phi (a)=0 \} .$$

Then Ker^G is a mapping.

Theorem 2.4.5

Let $\underline{A}=[A;F_G \cup F_U]$ be a commutative Ω -group. Then the following diagram commutes ;

$$\begin{tikzcd} \mathrm{End}(\underline{A}) \arrow[d, "\mathrm{Ker}"'] \arrow[dr, "\mathrm{Ker}^G"] & \\ \mathrm{Con}(\underline{A}) \arrow[r, "\Psi"'] & \mathrm{Sub}(\underline{A}) \end{tikzcd}$$

2.5 Congruence Relation

A congruence relation on an algebra $\underline{A}=[A;F]$ is an equivalence relation on A with the substitution property. It will be seen that the essential point of decomposition is the existence of a class of congruence relations satisfying some conditions. The following properties[10)] are important in decomposition.

Definition 2.5.1 Separating

Let $\underline{A}$=[A;F] be an algebra and $\underline{\Theta} = \{ \Theta_1, \ldots, \Theta_n \}$ a class of congruence relations on $\underline{A}$. If $\cap \underline{\Theta} = Id_A$, $\underline{\Theta}$ is called to be separating.

Definition 2.5.2 Full[11)]

Let $\underline{A}$=[A;F] be an algebra and $\underline{\Theta} = \{ \Theta_1, \ldots, \Theta_n \}$ a class of congruence relations on $\underline{A}$. If $(\Theta_1 \cap \ldots \cap \Theta_{i-1}) \circ \Theta_i = A^2$ (i=2,...,n), $\underline{\Theta}$ is called to be full.

Example : Let $\underline{V}=[V^n;F]$ be a vector space over V. Suppose that $\underline{W_1}$ and $\underline{W_2}$ are subspaces of $\underline{V}$ and $\underline{W_1} \oplus \underline{W_2} = \underline{V}$. Then $(\underline{W_1}, \underline{W_2}) \in \perp (\underline{V})$. In this case the induced congruence relations $\equiv_{W_i}$ is equal to $\Psi^{-1}(\underline{W_i})$ and $\underline{\Theta} = \{ \equiv_{W_1}, \equiv_{W_2} \}$ is separating and full. This fact will be used in decomposition of a transition system in Chapter 9.

The following theorems show the importance of the above properties in decomposition.

Theorem 2.5.1

Let $\underline{A}$=[A;F] be an algebra and $\underline{\Theta} = \{ \Theta_1, \ldots, \Theta_n \}$ a class of congruence relations on $\underline{A}$. Let a subalgebra $\underline{B}$=[B;F] of the direct product $\Pi (A/ \Theta_i | i=1,\ldots,n)$ be defined by $B = \{ ([a]\Theta_1, \ldots, [a]\Theta_n) | a \in A \}$. Then there is an epimorphism from $\underline{A}$ to $\underline{B}$. The subalgebra $\underline{B}$=[B;F] of $(\Pi \underline{A}/ \Theta_i | i=1,\ldots,n)$ is called the natural subdirect product of $\{ \Pi \underline{A}/ \Theta_i | i=1,\ldots,n \}$ and denoted by $\underline{A} / \underline{\Theta}$.

Theorem 2.5.2[11)]

Let $\underline{A}=[A;F]$ be an algebra and $\underline{\Theta} = \{ \Theta_1,\ldots, \Theta_n \}$ a class of congruence relations on $\underline{A}$. Then

if $\underline{\Theta}$ is separating, $\underline{A} \cong \underline{A}/\underline{\Theta}$.

Moreover,

if $\underline{\Theta}$ is separating and full,

$\underline{A}/\underline{\Theta} \cong \Pi(\underline{A}/\Theta_i | i=1,\ldots,n)$, that is $\underline{A} \cong \Pi(\underline{A}/\Theta_i | i=1,\ldots,n)$.

Conversely, suppose that there is an isomorphism h from $\underline{A}$ to $\Pi(\underline{A}_i | i=1,\ldots,n)$. Let $\Theta_i = \equiv_{p_i}$ for $i=1,\ldots,n$, where $p_i:A \to A_i$ is defined by $p_i=p_i' \circ h$ with the i-th projection $p_i': \Pi(A_i | i=1,\ldots,n) \to A_i$. then $\underline{\Theta} = \{ \Theta_i | i=1,\ldots,n \}$ is separating and full.

Example : Let $\underline{V}=[V^n;F]$ be a vector space over V. Suppose that $\underline{W_1}$ and $\underline{W_2}$ are subspaces of $\underline{V}$ such that $\underline{W_1} \oplus \underline{W_2} = \underline{V}$. Then the class $\underline{\Theta} = \{ \Theta_1, \Theta_2 \}$ of induced congruence relations is separating and full. Therefore by Theorem 2.5.2, $\underline{V} \cong \underline{V/W_1} \times \underline{V/W_2}$ and since $\underline{V/W_1} \cong \underline{W_2}$ and $\underline{V/W_2} \cong \underline{W_1}$, $\underline{V} \cong \underline{W_1} \times \underline{W_2}$.

Part I : Connections and Interactions

3. Connections

3.1 Connections of Input-Output Systems

In this section, connections of input-output systems such as parallel connection, cascade connection and feedback connection[1)] are represented as complex systems. At the beginning, we define a complex system that has no interaction among component systems.

Definition 3.1.1 Non-Interacted System (Fig. 3.1.1)

A complex system S over $\underline{S} = \{ S_1, S_2 \}$ is called the non-interacted system over $\underline{S}$ if $S = S_1 \times S_2$. In this case, S is denoted by $NI(\underline{S})$. When $S = \Pi \underline{S}$, each S_i can behave in S without interacting the other component system. It is why S is called a non-interacted system.

Remark : NI can be considered as a binary operation on $\overline{S}$ such that $NI : \overline{S} \times \overline{S} \to \overline{S}$. It is easily seen that NI can be extended to a finitary operation such that $NI : \overline{S}^n \to \overline{S}$; $NI(S_1, \ldots, S_n) = \Pi\ (S_i | i=1,2,\ldots,n)$

Mostly, complex systems are proper subset of $\Pi\ \underline{S}$. Because there are some interactions among component systems. Let us now define three connections of input-output systems as complex systems.

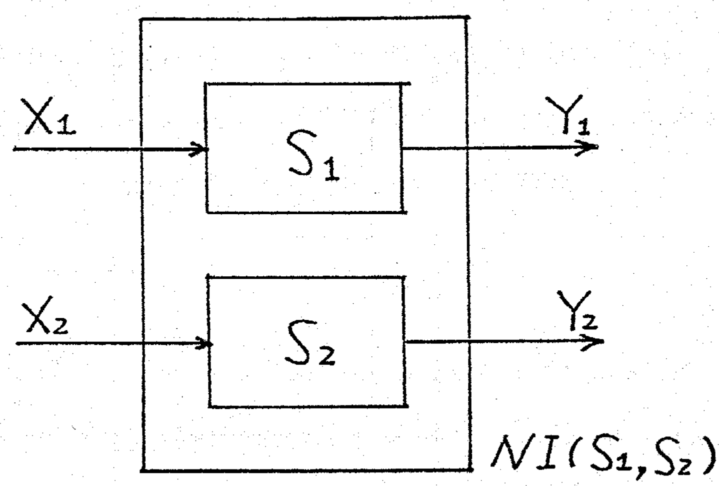


Fig. 3.1.1 Non-Interacted System

Definition 3.1.2 Parallel Complex System (Fig. 3.1.2)

Let $S_1 \subset (X^*_1 \times Z) \times Y_1$ and $S_2 \subset (X^*_2 \times Z) \times Y_2$ be component systems. Let $S \subset S_1 \times S_2$ be defined by

$$(((x_1,z_1),y_1),((x_2,z_2),y_2)) \in S \leftrightarrow z_1 = z_2.$$

Then S satisfies $p_i(S) = S_i$ for i=1,2. Hence S is a complex system. We call S the parallel complex system over $\underline{S} = \{ S_1,S_2 \}$ and denote it by $P(\underline{S})$. It is noted that the parallel complex system $P(\underline{S})$ is uniquely determined when input sets and output sets of component systems are specified.

Remark : P can be considered as a partial binary operation on $\overline{S}$ such that $P : (\overline{S} \times \overline{S}) \to \overline{S}$, where $\underline{D}(P) = \{ (S_1,S_2) | (\exists Z)(\exists X^*_1)(\exists X^*_2)(\underline{D}(S_1)=X^*_1 \times Z$ & $\underline{D}(S_2)=X^*_2 \times Z) \}$.

When we pay attention to decomposition, the input-output behaviour of a parallel complex system is of interest.

Definition 3.1.3 External Representation of Parallel Complex System (Fig. 3.1.2)

Let $S \subset S_1 \times S_2$ be a parallel complex system, where $S_1 \subset (X^*_1 \times Z) \times Y_1$ and $S_2 \subset (X^*_2 \times Z) \times Y_2$. Let $S' \subset (X^*_1 \times Z \times X^*_2) \times (Y_1 \times Y_2)$ be defined by

$$((x_1,z,x_2),(y_1,y_2)) \in S' \leftrightarrow ((x_1,z),y_1),((x_2,z),y_2)) \in S.$$

Then S' is called the external representation of S and denoted by $EX(P(\underline{S}))$.

The following propositions are useful in decomposition of an input-output system into a parallel connected system.

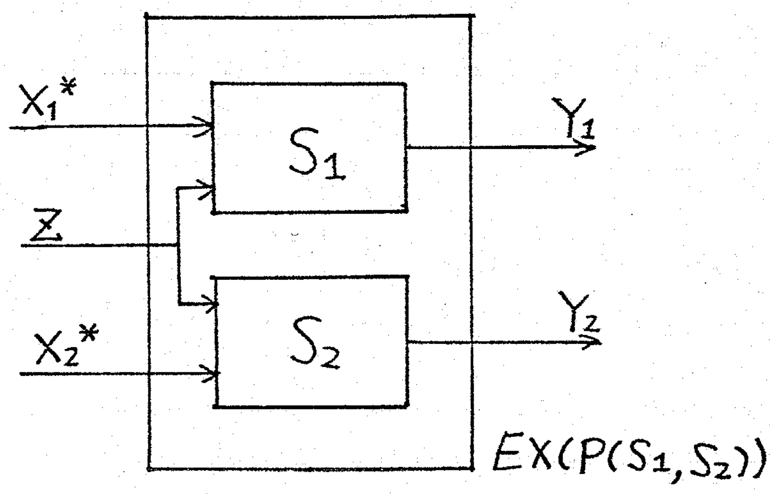


Fig. 3.1.2 Parallel Complex System

ned

Proposition 3.1.1

EX(P($\underline{S}$)) is an isomorphic model of P($\underline{S}$).

If there is a modelling (surjective, injective or isomorphic modelling) morphism from a global system to EX(P($\underline{S}$)), S is called to be parallel decomposable with respect to $\underline{S}$. In this case, if $\underline{S}$ is not specified, S is simply called to be parallel decomposable.

While in simulation, the following proposition holds.

Proposition 3.1.2

EX(P($\underline{S}$)) simulates P($\underline{S}$).

A cascade complex system is defined as follows.

Definition 3.1.4 Cascade Complex System (Fig. 3.1.3)

Let $S_1 \subset X_1 \times (Y^*_1 \times Z)$ and $S_2 \subset (X^*_2 \times Z) \times Y_2$ be component systems. Let $S \subset S_1 \times S_2$ be defined by

$$((x_1,(y_1,z_1)),((x_2,z_2),y_2)) \in S \leftrightarrow z_1 = z_2 .$$

Then S satisfies $p_i(S)=S_i$ for i=1,2. Hence S is a complex system. We call it the cascade complex system over $\underline{S} = \{ S_1,S_2 \}$ and denote it by C($\underline{S}$). It is also noted that the cascade complex system C($\underline{S}$) is uniquely determined when input sets and output sets of component systems are specified.

Remark : C can be considered as a partial binary operation on $\overline{S}$ such that $C : (\overline{S} \times \overline{S}) \to \overline{S}$, where $\underline{D}(C)= \{ (S_1,S_2) \mid (\exists Z)(\exists X^*_2)(\exists Y^*_1)(\underline{R}(S_1)=Y^*_1 \times Z$ & $\underline{D}(S_2)=X^*_2 \times Z) \}$

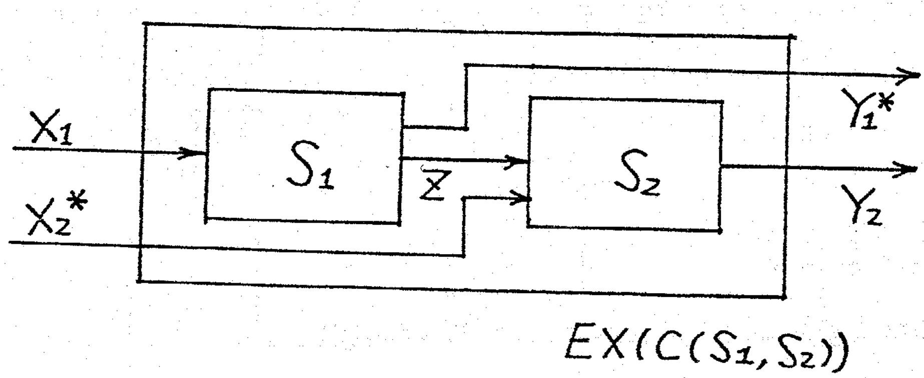


Fig. 3.1.3 Cascade Complex System

Similar to a parallel complex system, the input-output behaviour of a cascade complex system is noteworthy in decomposition.

Definition 3.1.5 External Representation of Cascade Complex System (Fig. 3.1.3)

Let $S \subset S_1 \times S_2$ be a cascade complex system, where $S_1 \subset X_1 \times (Y^*_1 \times Z)$ and $S_2 \subset (X^*_2 \times Z) \times Y_2$. Let $S' \subset (X_1 \times X^*_2) \times (Y^*_1 \times Y_2)$ be defined by

$$((x_1,x_2),(y_1,y_2)) \in S' \leftrightarrow (\exists z \in Z)(((x_1,(z,y_1)),((x_2,z),y_2)) \in S).$$

Then S' is called the external representation of S and denoted by $EX(C(\underline{S}))$.

Proposition 3.1.3

$EX(C(\underline{S}))$ is a surjective model of $C(\underline{S})$.

A feedback complex system is defined as follows.

Definition 3.1.6 Feedback Complex System (Fig. 3.1.4)

Let $S_1 \subset (X^* \times Z_x) \times (Y^* \times Z_y)$ and $S_2 \subset Z_y \times Z_x$ be input-output systems. Let $S \subset S_1 \times S_2$ be defined by

$$(((x,z_x),(y,z_y)),(z'_y,z'_x)) \in S$$
$$\leftrightarrow \quad z_x=z'_x \text{ and } z_y=z'_y.$$

Then S satisfies $p_i(S)=S_i$ for $i=1,2$. Hence S is a complex system. We call S the feedback complex system over $\underline{S} = \{ S_1,S_2 \}$ and denote it by $F(\underline{S})$. It is also noted that the feedback complex system $F(\underline{S})$ is uniquely determined when input sets and output sets of component systems are specified.

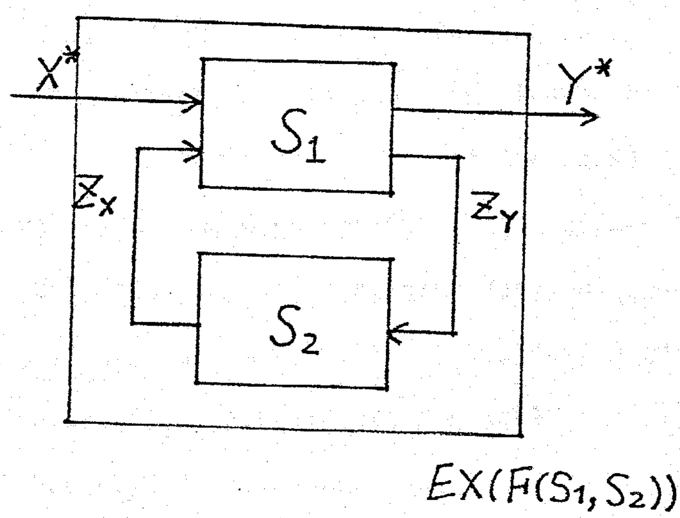


Fig. 3.1.4 Feedback Complex System

Remark : F can be considered as a partial binary operation on $\underline{S}$ such that $F:(\overline{S} \times \overline{S}) \to \overline{S}$, where $\underline{D}(F)= \{ (S_1,S_2) | (\exists X^*)(\exists Y^*)(\underline{D}(S_1)=X^* \times \underline{R}(S_2)$ and $\underline{R}(S_1)=Y^* \times \underline{D}(S_2) \}$.

Similar to a parallel and a cascade complex system, it is the input-output behaviour that is noteworthy in decomposition.

Definition 3.1.7 External Representation of Feedback Complex System (Fig. 3.1.4)

Let $S_1 \subset (X^* \times Z_x) \times (Y^* \times Z_y)$ and $S_2 \subset Z_y \times Z_x$ be input-output systems and $F(\underline{S})$ the feedback complex system over $\underline{S} = \{ S_1,S_2 \}$.
Let $S' \subset X^* \times Y^*$ be defined by

$$(x,y) \in S' \leftrightarrow (\exists z_x \in Z_x)(\exists z_y \in Z_y)(((x,z_x),(y,z_y)),(z_y,z_x)) \in S).$$

Then S' is called the external representation of $F(\underline{S})$ and denoted by $EX(F(\underline{S}))$.

Proposition 3.1.4

$EX(F(\underline{S}))$ is a surjective model of $F(\underline{S})$.

When we identify $A \times \{\phi\}$ with A, We can consider EX as a unary operation on $\overline{S}$ such that

$$EX(S)=S' \leftrightarrow S'= \{ (x,y) | (\exists z_1)(\exists z_2)(((x,z_1),(y,z_2)) \in S \}$$

, where $S \subset (X^* \times Z_1) \times (Y^* \times Z_2)$. Therefore the class of focused systems is a partial algebra [S; { P,C,F,EX }].

In most cases of interest, a complex system can be regarded as a finite combination of those operations such as P, C and F.

3.2 Connections of Functional Systems

An input-output system $S \subset X \times Y$ is called to be functional if (x,y) and $(x,y') \in S$ imply $y = y'$. In this section, we pay attention to basic connections of functional systems. Especially, it is the case where a functional system $S:X \to Y$ satisfies $X = Y$,and X and Y are an Ω -group $\underline{A}$ $=[A;F_G \cup F_U]$ and S is an endomorphism on $\underline{A}$ that must be investigated.

Let us first define a modelling morphism of functional systems.

<u>Definition 3.2.1</u> Modelling Morphism of Functional System

Let $S:X \to Y$ and $S':X' \to Y'$ be functional systems. Suppose that $h_x:X \to X'$ and $h_y:Y \to Y'$ are mappings. If the following diagram commutes, $h=(h_x,h_y)$ is called a modelling morphism from S to S'.

$$\begin{tikzcd} X \arrow[r, "S"] \arrow[d, "h_x"'] & Y \arrow[d, "h_y"] \\ X' \arrow[r, "S'"'] & Y' \end{tikzcd}$$

This definition is essentially same as that of input-output systems.

<u>Proposition 3.2.1</u>

Let $S:X \to Y$ and $S':X' \to Y'$ be input-output systems. Then $h=(h_x,h_y)$ is a modelling morphism from S to S' in the above sense if and only if so is as between input-output systems.

Definition 3.2.2 Category of Functional Systems

Let $\overline{S}^F = \{ S:X \to Y \mid S \text{ is a functional system} \}$ and $\mathrm{Hom}_{\underline{MODF}}(S,S') = \{ h=(h_x,h_y):S \to S' \mid h \text{ is a modelling morphism from } S \text{ to } S' \}$. The composition operation " o " is defined by the componentwise juxtaposition. Then $\underline{MODF} = [S^F, \{ \mathrm{Hom}_{\underline{MODF}}(S,S') \mid S,S' \varepsilon S^F \}, o]$ is a category and called the category of functional systems.

Definition 3.2.3 Non-Interacted Connection of Functional Systems (Fig. 3.2.1)

Let $S_1:X_1 \to Y_1$ and $S_2:X_2 \to Y_2$ be functional systems. And let $S:X_1 \times X_2 \to Y_1 \times Y_2$ be defined by $S(x_1,x_2)=(S_1(x_1),S_2(x_2))$. Then S is called the non-interacted connection of $\underline{S} = \{ S_1,S_2 \}$.

Definition 3.2.4 Parallel Connection of Functional Systems (Fig. 3.2.2)

Let $S_1:X \to Y_1$ and $S_2:X \to Y_2$ be functional systems. And let $S:X \to Y_1 \times Y_2$ be defined by $S(x)=(S_1(x),S_2(x))$. Then S is called the parallel connection of $\underline{S} = \{ S_1,S_2 \}$ and denoted by $P_F(\underline{S})$.

Proposition 3.2.2

$$P_F(\underline{S}) \cong EX(P(\underline{S}))$$

Definition 3.2.5 Serial Connection of Functional Systems[12) (Fig. 3.2.3)

Let $S_1:X \to Z$ and $S_2:Z \to Y$ be functional systems. And let $S:X \to Y$ be defined by $S(x)=S_2(S_1(x))$. Then S is called the serial connection of $\underline{S} = \{ S_1,S_2 \}$ and denoted by $S = S_1 \bullet S_2$.

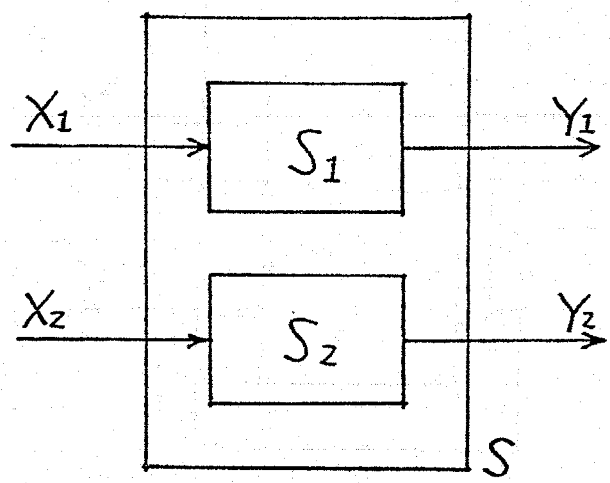


Fig. 3.2.1 Non-Interacted Connection of Functional Systems

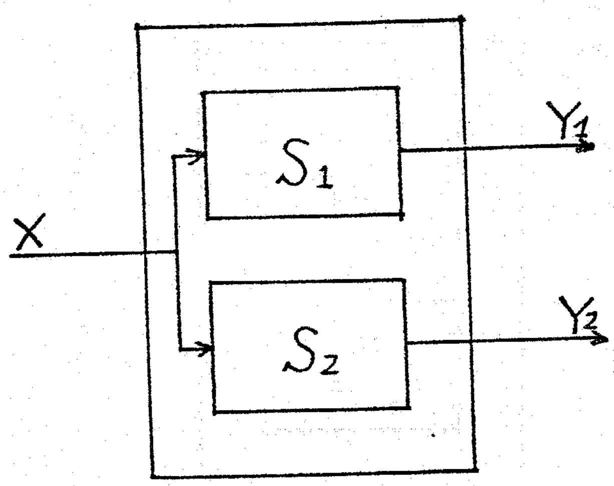


Fig. 3.2.2 Parallel Connection of Functional Systems

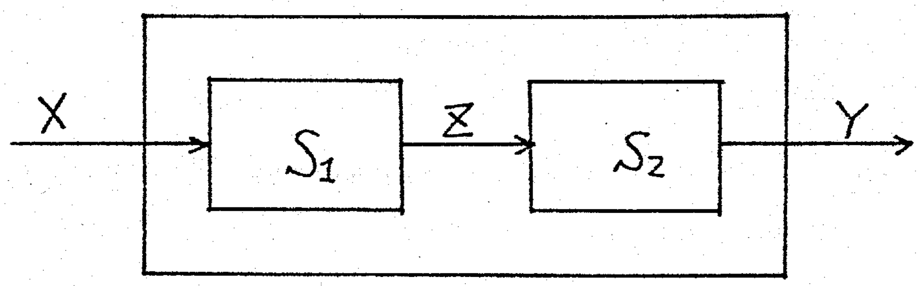


Fig. 3.2.3 Serial Connection of Functional Systems

Remark : When X=Z=Y is an Ω -group $\underline{A}=[A;F_G \cup F_U]$, the operation " $\cdot$ " is exactly a binary operation on End($\underline{A}$). That is, $\cdot$:End($\underline{A}$) $\times$ End($\underline{A}$) $\to$ End($\underline{A}$).

Proposition 3.2.3

$$S_1 \cdot S_2 \cong EX(C(\underline{S})).$$

If both of X and Y are a commutative Ω -group $\underline{A} = [A;F_G \cup F_U]$, parallel connection of endomorphisms on $\underline{A}$ is given by

Definition 3.2.6 Parallel Connection of Endomorphisms[12)] (Fig. 3.2.4)

Let $S_1:\underline{A} \to \underline{A}$ and $S_2:\underline{A} \to \underline{A}$ be endomorphisms of a commutative Ω -group $\underline{A} = [A;F_G \cup F_U]$. And let $S:\underline{A} \to \underline{A}$ be defined by $S(a) = S_1(a) + S_2(a)$, where + is the binary operation in F_G. Then S is also an endomorphism of $\underline{A}$ and called the parallel connection of $\underline{S} = \{ S_1,S_2 \}$ and denoted by $S = S_1 + S_2$.

Remark : "+" is a binary operation on End($\underline{A}$), that is, +:End($\underline{A}$) $\times$ End($\underline{A}$) $\to$ End($\underline{A}$).

Proposition 3.2.4

Let $\underline{S}= \{S_1,S_2 \} \subset$ End($\underline{A}$). Then $P_F(\underline{S})$ is a surjective model of $S_1 + S_2$.

Definition 3.2.7 Feedback Connection of Endomorphisms[12)]

(Fig.3.2.5)

Let $S_1:\underline{A} \to \underline{A}$ and $S_2:\underline{A} \to \underline{A}$ be endomorphisms of a commutative Ω-group $\underline{A}=[A;F_G \cup F_U]$ such that $(I-S_1S_2)$ is an isomorphism, where $I:\underline{A} \to \underline{A}$ is the identity morphism. Let $S:\underline{A} \to \underline{A}$ be defined by $S(a)=(I-S_1S_2)^{-1}S_1(a)$. Then S is called the feedback connection of S_1 and S_2, and denoted by $F(S_1,S_2)$.

Remark : F is a partial binary operation on End($\underline{A}$), where $\underline{D}(F) = \{ (S_1,S_2) | (I-S_1S_2) \text{ is invertible} \}$.

3.3 Connections of Transition Systems

A transition system is the dynamical part of a canonical representation of a causal system. One of the reasons why we investigate a transition system is that it can be naturally described as an algebra $\underline{C}(\phi)= [C; \{ \phi_{tt'}(-,x_{tt'}) | x_{tt'} \in \overline{X} \}]$, where $\phi_{tt'}(-,x_{tt'}):C \to C$ is defined by $\phi_{tt'}(-,x_{tt'})(c)= \phi_{tt'}(c,x_{tt'})$.

A modelling morphism of transition systems needs rather strict condition than that of input-output systems.

Definition 3.3.2 Morphism of Transition System

Let $T = [\overline{X}, C, \overline{\phi}]$ and $T' = [\overline{X}, C', \overline{\phi}']$ be transition systems with same input set $\overline{X}$. If there is a mapping h from C to C' such that the following diagram commutes for all $x_{tt'} \in \overline{X}, t \leqslant t'$, h is called a morphism from T to T'. If h is bijective, it is called an iso-

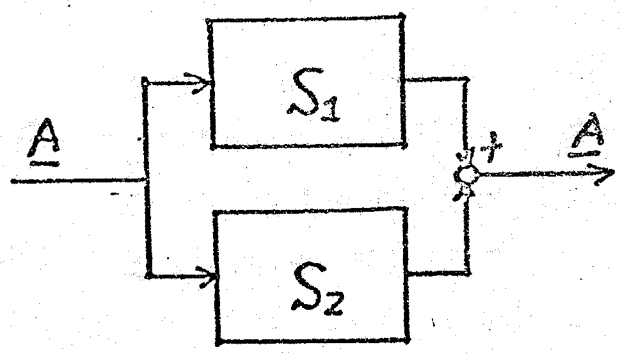


Fig. 3.2.4 Parallel Connection of Endomorphisms.

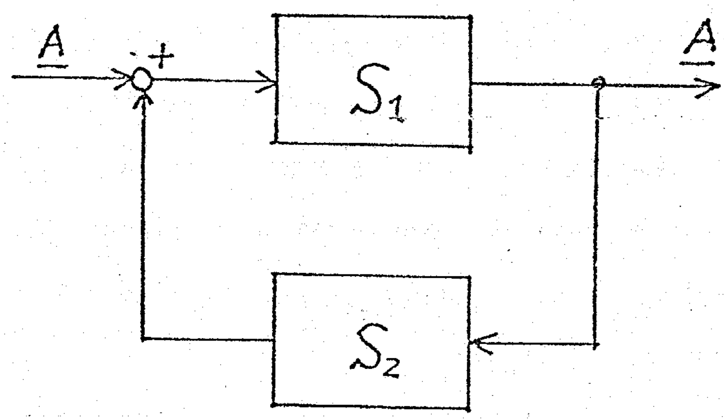


Fig. 3.2.5 Feedback Connection of Endomorphisms

morphism from T to T'.

$$\begin{array}{ccc} C \times X_{tt'} & \xrightarrow{\phi_{tt'}} & C \\ h \downarrow \ \downarrow Id & & \downarrow h \\ C' \times X_{tt'} & \xrightarrow[\phi'_{tt'}]{} & C' \end{array}$$

Proposition 3.3.1

Let $T = [\overline{X}, C, \overline{\phi}]$ and $T' = [\overline{X}, C', \overline{\phi}']$ be transition systems whose input-output system representation are given by $S \subset C \times C$ and $S' \subset C' \times C'$. And let $h: C \rightarrow C'$ be a morphism from T to T'. Then $\underline{h} = (h,h)$ is a modelling morphism from S to S'.

Proposition 3.3.2

Let $T=[\overline{X}, C, \overline{\phi}]$ and $T'=[\overline{X}, C', \overline{\phi}']$ be transition systems. Then $h: C \rightarrow C'$ is a morphism from T to T' if and only if h is a homomorphism from $\underline{C}(\overline{\phi})$ to $\underline{C}(\overline{\phi}')$, where $\underline{C}(\overline{\phi})=[C; \{ \phi_{tt'}(-,x_{tt'}) | x_{tt'} \in \overline{X} \}]$ and $\underline{C}(\overline{\phi}')=[C'; \phi'_{tt'}(-,x_{tt'}) | x_{tt'} \in X_{tt'} \}]$.

Let us next define connections of transition systems.

Definition 3.3.3 Parallel Connection of Transition Systems (Fig.3.3.1)

Let $T_1 = [\overline{X}, C_1, \overline{\phi}^1]$ and $T_2 = [\overline{X}, C_2, \overline{\phi}^2]$ be transition systems. Let $C = C_1 \times C_2$ and $\overline{\phi} = \{ \phi_{tt'}: C \times X_{tt'} \rightarrow C | x_{tt'} \in X_{tt'} \}$ be defined by

$$\phi_{tt'}((c_1,c_2),x_{tt'}) = (\phi^1_{tt'}(c_1,x_{tt'}), \phi^2_{tt'}(c_2,x_{tt'})).$$

Then the resultant transition system $T = [\overline{X}, C, \overline{\phi}]$ is called the parallel connection of $\underline{T} = \{ T_1,T_2 \}$ and denoted by $P_T(\underline{T})$.

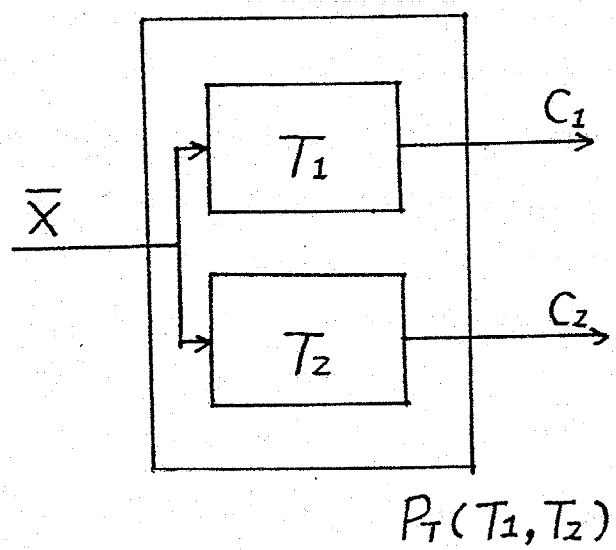


Fig. 3.3.1 Parallel Connection of Transition Systems

Proposition 3.3.3

Let $T=[\overline{X}, C, \overline{\phi}]$ be the parallel connection of $T_1=[\overline{X}, C_1, \overline{\phi}^1]$ and $T_2=[\overline{X}, C_2, \overline{\phi}^2]$. Then $\underline{C}(\overline{\phi})=\underline{C}(\overline{\phi}^1) \times \underline{C}(\overline{\phi}^2)$, where $\underline{C}(\overline{\phi})$, $\underline{C}(\overline{\phi}^1)$ and $\underline{C}(\overline{\phi}^2)$ are algebraic expressions of T, T_1 and T_2, respectively.

In this dissertation, we investigate serial connection of transition systems in the case where a transition system is a state automaton.

Definition 3.3.4 Serial Connection of State Automata (Fig.3.3.2)

Let $T_1 = [U, C_1, \delta_1]$ and $T_2 = [C_1 \times U, C_2, \delta_2]$ be state automata. And let $C=C_1 \times C_2$ and $\delta : C \times U \to C$ be defined by

$$\delta((c_1,c_2),u)=(\delta_1(c_1,u), \delta_2(c_2,(c_1,u))).$$

Then the resultant state automaton $T= [U, C, \delta]$ is called the serial connection of $\underline{T} = \{T_1,T_2\}$ and denoted by $S_T(\underline{T})$.

In contrast to parallel connection, unfortunately, we cannot express serial connection of transition systems in universal algebraic way because component transition systems cannot be recognized as algebras with the same type.

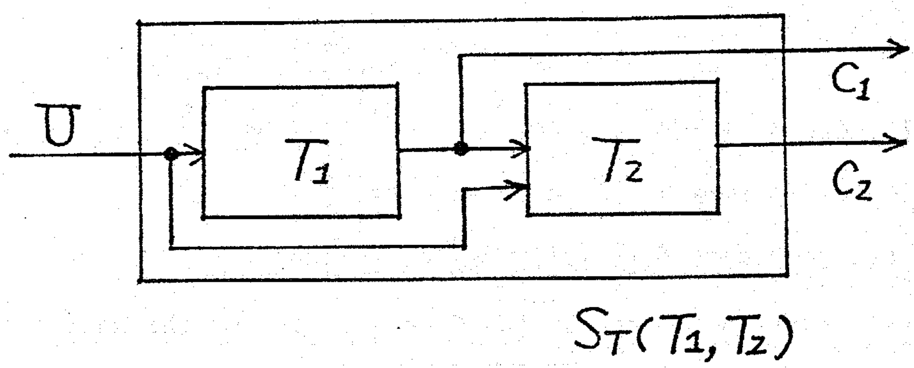


Fig. 3.3.2 Serial Connection of Transition Systems

4. General Theory of Interactions[4)]

One of the main problems in decomposition theory is to characterize interaction among component systems of a complex system. In the reference [4], two sorts of interaction concepts, which are called the process interaction and the system interaction, have been proposed and it has been shown that it is enough to consider them in order to characterize interaction of a complex system. The interaction concepts that have been proposed in the reference [4], however, concern only interactions between one component subsystem and others. In other words, though it is useful to clarify what kind of interactions a component system has, we need another concept when we pay attention to interaction between two component systems.

Our aim of this chapter is to define the interaction structure of a complex system with which we can clarify what kind of interactions occurs between two component systems of a complex system.

At the beginning, we define three kinds of interaction relation between two component systems of a complex system which naturally correspond to the previous concepts of interactions, and show that they characterize interaction between two component systems.

Furhthermore we define the interaction structure of a complex system as a triple of the interaction relations and show that it is invariant under a certain isomorphism.

4.1 Basic Concepts

As notations, we use the following ones ;

X_i : $X_i=\underline{D}(S_i)=\{ x_i|(\exists y_i)((x_i,y_i) \varepsilon S_i) \}$

$\underline{S}$: $\underline{S}=\{ S_i|i \varepsilon I \}$, where $I=\{ 1,2,\ldots,n \}$, a class of n component systems

$\underline{S}^F$: $S^F=\{ S_i:X_i \rightarrow Y_i|i \varepsilon I \}$, a class of n functional component systems

$\overline{S}$: the class of all complex systems over n component systems

$\overline{S}_F$: the class of all functional complex systems over n component systems

$p_i=(p_{ix},p_{iy})$: $p_i:S \rightarrow S_i$, the i-th projection defined by $p_i([s_i|i \varepsilon I])=s_i$.

$p_J=(p_{Jx},p_{Jy})$: $p_J:S \rightarrow \Pi (S_i|i \varepsilon J)$, the J-projection defined by

$$p_J([s_i|i \varepsilon I])=[s_i|i \varepsilon J]$$

$\overline{S}_i$: $\overline{S}_i=p_{I/\{ i \}}(S)$

$\overline{S}_{ij}$: $\overline{S}_{ij}=p_{I/\{ i,j \}}(S)$

S^*_i : $S^*_i:\overline{S}_i \rightarrow P(S_i)$ defined by $S^*_i(\overline{s}_i)=\{ s_i|(s_i,\overline{s}_i) \varepsilon S \}$, where $P(S_i)$ denotes the power set of S_i and $S^*_i(\overline{s}_i):X_i \rightarrow P(Y_i)$ is defined by $S^*_i(\overline{s}_i)(x_i)=\{ y_i|(x_i,y_i) \varepsilon S^*_i(s_i) \}$

S^*_{ij} : $S^*_{ij}:S_j \rightarrow P(S_i)$ defined by $S^*_{ij}(s_j)=\{ s_i|(s_i,s_j) \varepsilon p_{(i,j)}(S) \}$

$S^*_{ij}(-)$: $S^*_{ij}(s_j):X_i \rightarrow P(Y_i)$ defined by $S^*_{ij}(s_j)(x_i)=\{ y_i|(x_i,y_i) \varepsilon S^*_{ij}(s_j) \}$

$\overline{S}^*_i$: $\overline{S}^*_i:S_i \rightarrow P(\overline{S}_i)$ defined by $\overline{S}^*_i(s_i)=\{ \overline{s}_i|(s_i,\overline{s}_i) \varepsilon S \}$

DS^*_i : $DS^*_i:\overline{S}_i \rightarrow P(X_i)$ defined by $DS^*_i(\overline{s}_i)=\{ x_i|(\exists y_i)(((x_i,y_i),\overline{s}_i) \varepsilon S) \}$

DS^*_{ij} : $DS^*_{ij}:S_j \rightarrow P(X_i)$ defined by $DS^*_{ij}(s_j)=\{ x_i|(\exists y_i)(((x_i,y_i),s_j) \varepsilon p_{(i,j)}(S)$

Id_I : $Id_I \subset I \times I$ defined by $(i,j) \varepsilon Id_I \leftrightarrow i=j$.

Using the above notations, let us review the previous concepts of interactions.

Definition 4.1.1 Non-Interaction (NI)

Let $S \subset \Pi \underline{S}$ be a complex system. If $S = S_i \times \overline{S}_i$, S_i is called to have non-interaction (briefly, NI).

Proposition 4.1.1

$S = \Pi \underline{S}$ if and only if S_i has NI for all $i \in I$.

If $S = \Pi \underline{S}$, S is called a non-interacted system.

Definition 4.1.2 Non-Process Interaction (NPI)

Let $S \subset \Pi \underline{S}$ be a complex system. Let $\overline{s}_i, \overline{s}_i' \in \overline{S}_i$ and $x_i \in X_i$ be arbitrary. If $x_i \in DS_i^*(\overline{s}_i) \cap DS^*_i(\overline{s}_i')$ implies $S^*_i(\overline{s}_i)(x_i) = S^*_i(\overline{s}_i')(x_i)$, then S_i is called to have non-process interaction (briefly, NPI).
If S_i has NPI for all $i \in I$, S is called a non-process interacted system.

Definition 4.1.3 Strong Non-System Interaction (SNSI)

Let $S \subset \Pi \underline{S}$ be a complex system. If $DS^*_i(\overline{s}_i) = X_i$ holds for all $\overline{s}_i \in \overline{S}_i$, S_i is called to have strong non-system interaction (briefly, SNSI).
If S_i has SNSI for all $i \in I$, S is called a strongly non-system interacted system.

Definition 4.1.4 Weak Non-System Interaction (WNSI)

Let $S \subset \Pi \underline{S}$ be a complex system. If $X = X_i \times \overline{X}_i$, S_i is called to have weak non-system interaction (briefly, WNSI). If S_i has WNSI for all $i \in I$, S is called a weakly non-system interacted system.

4.2 Interaction Relations

In this section, we define four types of interaction relation induced by a complex system, which naturally correspond to the previous concepts of interaction.

At the beginning, let us define the relation which represents that two component systems are not interacted each other.

Definition 4.2.1 Non-Interaction Relation Induced by S (NI(S))

Let $S \subset \Pi \underline{S}$ be a complex system. If for $i \neq j$ $p_{(i,j)}(S) = S_i \times S_j$, S_i is called to be non-interacted with S_j in S and we express it by $(i,j) \in NI(S)$. NI(S) is refered to as the non-interaction relation induced by S. If $(i,j) \in NI(S)$ & $i \neq j$, S_i is called to be interacted with S_j in S.

Proposition 4.2.1

$NI(S) = NI(S)^{-1}$ for all $S \in \overline{S}$.

Remark : The transitivity of NI(S), however, does not hold in general. That is, it happens that S_i is non-interacted with S_j and so is S_j with S_k, but S_i is interacted with S_k.

According to the previous definitioin, what S_i is non-interacted means that $S = S_i \times \overline{S}_i$.

Proposition 4.2.2

If $S = S_i \times \overline{S}_i$, $(i,j) \varepsilon NI(S)$ for all $j \varepsilon I- \{ i \}$.

Remark : The converse, however, does not hold in general.

The process interaction relation is defined as follows.

Definition 4.2.2 Process Interaction Relation Induced by S

Let $S \subset \Pi \underline{S}$ be a complex system. Let $i \neq j$. If for any $s_j, s_j' \varepsilon S_j$ and any $x_i \varepsilon X_i$, $x_i \varepsilon DS_{ij}*(s_j) \cap DS_{ij}*(s_j')$ implies $S_{ij}*(s_j)(x_j) = S_{ij}*(s_j')(x_i)$, then S_i is called to be non-process interacted with S_j in S and we express it by $(i,j) \varepsilon NPI(S)$. NPI(S) is refered to as the non-process interaction relation induced by S.

If $(i,j) \varepsilon PI(S)=(NPI(S) \cup Id_I)^C$, S_i is called to be process interacted with S_j in S and PI(S) is refered to as the process interaction relation induced by S.

Proposition 4.2.3

If S_i has NPI in S, $(i,j) \varepsilon NPI(S)$ for all $j \varepsilon I- \{ i \}$.

Remark : The converse, however, does not hold in general.

According to the previous concepts of the system interaction, we define the system interaction relation in two ways.

Definition 4.2.3 Strong System Interaction Relation Induced by S

Let $S \subset \Pi \underline{S}$ be a complex system. If for $i \neq j$, $p_{(i,j)}(X) = X_i \times X_j$, S_i is called to be weakly non-system interacted with S_j in S and we express it by $(i,j) \in$ WNSI(S). WNSI(S) is refered to as the weak non-system interaction relation induced by S. If $(i,j) \in$ SSI(S) $=(\mathrm{WNSI}(S) \cup Id_I)^C$, S_i is called to be strongly system interacted with S_j in S and SSI(S) is refered to as the strong system interaction relation induced by S.

If S_i is weakly non-system interacted with S_j in S, every pair of inputs of S_i and S_j is acceptable.

Proposition 4.2.4

$\mathrm{WNSI}(S) = \mathrm{WNSI}(S)^{-1}$ for all $S \in \overline{S}$.

Remark : The transitivity of WNSI(S), however, does not hold in general. That is, it happens that S_i is weakly non-system interacted with S_j and so is S_j with S_k, but S_i is strongly system interacted with S_k.

Proposition 4.2.5

If S_i has WNSI in S, $(i,j) \in$ WNSI(S) for all $j \in I- \{ i \}$. That is, if S_i is strongly system interacted with some S_j, S_i has SSI.

Remark : The converse, however, does not hold in general.

Definition 4.2.4 Weak System Interaction Relation Induced by S

Let $S \subset \Pi \underline{S}$ be a complex system. Let $i \neq j$. If $DS_{ij}{}^*(s_j)=X_i$ holds for any $s_j \in S_j$, S_i is called to be strongly non-system interacted with S_j in S and we express it by $(i,j) \in SNSI(S)$. SNSI(S) is refered to as the strong non-system interaction relation induced by S. If $(i,j) \in WSI(S)=(SNSI(S) \cup Id_I)^C$, S_i is called to be weakly system interacted with S_j in S and WSI(S) is refered to as the weak system interaction relation induced by S.

What SNSI(S) means is that if $(i,j) \in SNSI(S)$, every input of S_i is acceptable whatever S_j behaves.

Proposition 4.2.6

If S_i has SNSI in S, $(i,j) \in SNSI(S)$ for all $j \in I-\{i\}$. That is, if S_i is weakly system interacted with some S_j in S, S_i has WSI.

Remark : The converse, however, does not hold in general.

As shown in the reference[4)], if S_i has SNSI, then it also has WNSI. Similarly, SNSI(S) is a subset of WNSI(S).

Proposition 4.2.7

$SNSI(S) \subset WNSI(S)$, or equivalently $SSI(S) \subset WSI(S)$ for all $S \in \overline{S}$.

We have now four kinds of interaction relations, that is, NI(S), PI(S)(NPI(S),SSI(S)(WNSI(S)) and WSI(S)(SNSI(S)). They are relations on I induced by S, however, in other words, NI, PI, SSI and WSI can be considered as mappings from $\overline{S}$ into $P(I^2)$.

Let us next investigate the properties of interaction relations.

Theorem 4.2.1

The following statements hold for all complex system over $\underline{S}$.

1) $WNSI(S) \cap NPI(S)^{-1} \subset SNSI(S)$

2) $NI(S)=WNSI(S) \cap NPI(S)^{-1} \cap NPI(S)=SNSI(S) \cap NPI(S)$.

The above theorem states that it is enough to consider the three interaction relations, that is, NPI(S),WNSI(S),SNSI(S), when we pay attention to interaction between component systems of a complex system. Therefore the triple [PI(S), SSI(S), WSI(S)] characterizes the structure of interactions of a complex system.

Definition 4.2.5 Interaction Structure of Complex System

IS(S) = [PI(S), SSI(S), WSI(S)] is called the interaction structure of a complex system S.

Remark : IS can be considered as a mapping from $\overline{S}$ to $P(I^2)^3$.

In Section 4.4, we will consider the meaning of the interaction structure of a complex system.

4.3 Property of Interaction Relations of Functional Complex Systems

In systems theory, we often treat a functional complex system. An autonomous discrete time system whose state set is n-dimensional is a typical example of such a system.

In this section, we investigate some properties of interaction relations in the case of a functional complex system. It is obvious that if every $S_i \varepsilon \underline{S}$ is functional, so is any $S \varepsilon \overline{S}$. the following proposition states a weaker condition for $S \varepsilon \overline{S}$ to be functional.

Proposition 4.3.1

Let $S \subset \Pi \underline{S}$ be a complex system and F a subset of I^2 that satisfies $\underline{D}(F) \cup \underline{R}(F)=I$. If $p_{(i,j)}(S)$ is functional for all $(i,j) \varepsilon F$, S is functional.

Conversely, if $S = \Pi \underline{S}$ and S is functional, $p_{(i,j)}(S)$ is functional if $i \neq j$.

When we restrict our attention to $FC(S)= \{ (i,j) \varepsilon I \times I | p_{(i,j)}(S)$ is functional $\}$, the interaction relations have some disirable properties.

Lemma 4.3.1

Let $S \subset \Pi \underline{S}$ be a complex system. If $(i,j) \varepsilon SNSI(S) \cap FC(S)$, S_j is functional.

Lemma 4.3.2

Let $S \subset \Pi \underline{S}$ be a complex system. If S_i is functional, $(i,j) \varepsilon NPI(S)$ for all $j \varepsilon I- \{ i \}$.

Therefore,

Proposition 4.3.1

$SNSI(S) \cap FC(S) \subset NPI(S)^{-1}$ for all $S \in \overline{S}$.

Theorem 4.3.1

$WNSI(S) \cap NPI(S)^{-1} \cap FC(S) = SNSI(S) \cap FC(S)$ for all $S \in \overline{S}$.

Since $p_{(i,j)}(S)$ is functional if both of S_i and S_j are functional, $(i,j) \in FC(S)$ if $i \neq j$ when we restrict our attention to a complex system over $\underline{S}^F{}_n$. In this case, the interaction relations have some disirable properties.

Proposition 4.3.2

$NPI(S)=Id_I{}^C$ for all $S \in \overline{S^F}$.

Proposition 4.3.3

$WNSI(S) \subset SNSI(S)$ for all $S \in S^F$.

Hence,

Theorem 4.3.2

$NI(S)=SNSI(S)=WNSI(S)$ for all $S \in \overline{S^F}$.

Therefore if all component systems are functional, the weak system interaction characterizes the interaction between two component systems of a complex system.

4.4 Interaction Structure

In section 4.2, we have defined the interaction structure of a complex system S as a triple IS(S)=[PI(S),SSI(S),WSI(S)]. Let us now consider a relation between the interaction structure of S and a modelling morphism from S.

At the beginning, let us define the following category.

<u>Proposition 4.4.1</u>

Let $S \subset \Pi\, \underline{S}$ and $S' \subset \Pi\, \underline{S}'$ be complex systems. Let us define $\mathrm{Hom}_{\underline{CS}}(S,S')$ be $\mathrm{Hom}_{\underline{CS}}(S,S') = \{\, \Pi\,(h_i \mid i \in I)|S : S \to S' \mid (\forall i \in I)(h_i : S_i \to S_i'$ is an isomorphism in $\underline{MOD}$) and $\Pi\,(h_i \mid i \in I)|S$ is an isomorphism in $\underline{MOD}\,\}$.
And let us define the composition operation as componentwise. Then $\underline{CS} = [\, \overline{S},\ \{\, \mathrm{Hom}(S,S') \mid S,S' \in \overline{S}\,\}\ ,\ \circ\ \}\ ]$ is a subcategory of $\underline{MOD}$ and we call it the category of complex systems with order n.

Let us now consider a relationship between interaction and isomorphisms.

<u>Theorem 4.4.1</u>

Let $S \subset \Pi\, \underline{S}$ and $S' \subset \Pi\, \underline{S}'$ be complex systems and $\Pi\,(h_i \mid i \in I)|S : S \to S' \in \mathrm{Hom}_{\underline{CS}}(S,S')$. Then

1) S_i has NI in S if and only if S_i' has NI in S'
2) S_i has NPI in S if and only if S_i' has NPI in S'
3) S_i has SNSI in S if and only if S_i' has SNSI in S'
4) S_i has WNSI in S if and only if S_i' has WNSI in S'

hold for all $i \in I$.

Sinse S is a non-interacted system (non-process interacted system, strongly non-system interacted system or weakly non-system interacted system) if and only if S_i has NI (NPI,SNSI or WNSI, respectively) for all $i \in I$, the following corollary is a direct consequence of Theorem 4.4.1.

Corollary 4.4.1

Let $S \subset \Pi \underline{S}$ and $S' \subset \Pi \underline{S}'$ be complex systems, and $\Pi (h_i | i \in I)|S : S \to S' \in \mathrm{Hom}_{\underline{CS}}(S,S')$. Then

1) S is a non-interacted system if and only if so is S'.

2) S is non-process interacted system if and only if so is S'.

3) S is a strongly non-system interacted system if and only if so is S'.

4) S is a weakly non-system interacted system if and only if so is S'.

Let us now consider a relation between interaction structures and isomorphisms.

Theorem 4.4.2

Let $S \subset \Pi \underline{S}$ and $S' \subset \Pi \underline{S}'$ be complex systems and $\Pi (h_i | i \in I)|S : S \to S' \in \mathrm{Hom}_{\underline{CS}}(S,S')$. Then $IS(S) = IS(S')$.

Therefore the interaction structure is invariant under a certain isomorphism.

Part III : Decomposition

5. Basic Scheme of Decomposition

5.1 Quotient System

In decomposition, it is very natural to stress component systems simpler than a given global system. It is is, however, fairly difficult to define the concept of "simplicity". One of naturally accepted definitions is by using a surjective modelling morphism when it is the case of general input-output systems.

As known in universal algebra, an algebra can be simplified by a non-trivial congruence relation. When we adopt it to the mathematical systems theory, at first we must impose a congruence relation to preserve input-output behaviour of a system. In order to simplify our discussion, from now on, we only consider the case when an input-output system is simply a set, that is a general input-output system.

Definition 5.1.1 Input-Output Compatibility[3)]

Let $S \subset X \times Y$ be an input-output system. An equivalence relation R on S is called to be input-output compatible if there are equivalence relations R_x and R_y such that

$$(x,y)\ R\ (x',y') \leftrightarrow x\ R_x\ x' \text{ and } y\ R_y\ y'.$$

In this case, a pair (R_x,R_y) is called to be an associated pair of equivalence relations (briefly, an associated pair) with R.

It is easily seen that all equivalence relations on S are not input-output compatible.

Since an input-output system is a set itself, we can define a quotient set S/R of S modulo R. However, if we impose R to be input-output compatible, the resultant system can also be considered as an input-output system.

Definition 5.1.2 Quotient System (Fig.5.1.1)

Let $S \subset X \times Y$ be an input-output system and R an input-output compatible equivalence relation on S. Suppose that (R_x,R_y) is an associated pair with R. Let $S' \subset X/R_x \times Y/R_y$ be defined by

$$([x]R_x,\ [y]R_y) \in S' \leftrightarrow ([x]R_x \times [y]R_y) \cap S \neq \phi .$$

Then S' is called a quotient system of S modulo R and denoted by S/R.

Proposition 5.1.1

Let $S \subset X \times Y$ be a general input-output system and S/R its quotient system modulo R. Then S/R is a surjective model of S.

Remark : If $R \neq Id$, S/R can be considered to be simpler than S.

It is important to note that a quotient system can be constructed by a modelling morphism.

Proposition 5.1.2[3)]

Let $S \subset X \times Y$ and $S' \subset X' \times Y'$ be input-output systems, and $h=(h_x,h_y):S \rightarrow S'$ a modelling morphism. Let $\equiv_h$ on S be defined by

$$(x,y) \equiv_h (x',y') \leftrightarrow h_x(x)=h_x(x') \text{ and } h_y(y)=h_y(y').$$

Then $\equiv_h$ is an input-output compatible equivalence relation on S.

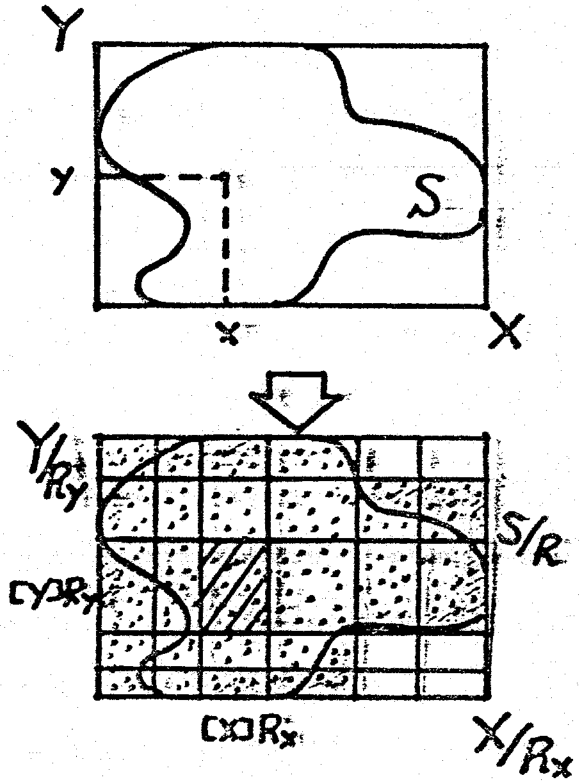


Fig. 5.1.1 Quotient System

The quotient system of S modulo $\equiv_h$ is denoted by S/h. Actually,
$S/h = \{ ([x]h_x, [y]h_y) \mid (x,y) \in S \}$.

Then we obtain an epi-mono factorization of S by a modelling morphism.

Proposition 5.1.3[3)]

Let $S \subset X \times Y$ and $S' \subset X' \times Y'$ be general input-output systems, and $h=(h_x,h_y):S \to S'$ a modelling morphism. Then the following diagram commutes ;

$$\begin{tikzcd} S \arrow[rr, "h"] \arrow[dr, "\pi"'] & & S' \\ & S/h \arrow[ur, "\sigma"'] & \end{tikzcd}$$

, where $\pi = (\pi_x, \pi_y)$ is a canonical projection and $\sigma = (\sigma_x, \sigma_y)$ is an embedding. Moreover, π is a surjective modelling morphism and σ is an injective modelling morphism.

The next proposition states the relationship between a quotient system of a global system and a component system of a complex system when there is a modelling morphism from the global system to the complex system.

Proposition 5.1.4

Let $\hat{S} \subset \hat{X} \times \hat{Y}$ be an input-output system and $S \subset \Pi \underline{S}$ a complex system over $\underline{S} = \{ S_i \mid i \in I \}$. Suppose that $h=(h_x,h_y)$ is a modelling morphism from $\hat{S}$ to S. Then S_i is an injective model of $\hat{S}/p_ih$,where $p_i=(p_{ix},p_{iy}):S \to S_i$ is the i-th projection of S. Moreover, if h is a surjective modelling morphism, $\hat{S}/p_ih \cong S_i$.

$$\begin{tikzcd} \hat{S} \arrow[r, "h"] \arrow[dr, "\pi_i"'] & S \arrow[r, "p_i"] & S_i \\ & \hat{S}/p_ih \arrow[ur, "\sigma"'] & \end{tikzcd}$$

Let $\underline{S}_q = \{ S/R^i | i \varepsilon I \}$ be a class of quotient systems of S. The next question is how to construct a complex system over $\underline{S}_q$ which has some disirable relationship with S.

Definition 5.1.2 Canonical Complex System over $\underline{S}_q$

Let $S \subset X \times Y$ be an input-output system and $\underline{R} = \{ R^i | i \varepsilon I \}$ a class of input-output compatible equivalence relations on S. Suppose that $(R_x{}^i, R_y{}^i)$ is an associated pair with R^i for $i \varepsilon I$. Let $S' \subset \Pi\, S/R^i$ be defined by

$([[x_i]R_x{}^i | i \varepsilon I], [[y_i]R_y{}^i | i \varepsilon I]) \varepsilon S'$

$$\leftrightarrow \quad (\cap ([x_i]R_x{}^i | i \varepsilon I) \times \cap ([y_i]R_y{}^i | i \varepsilon I)) \cap S \neq \phi$$

,where $(R_x{}^i, R_y{}^i)$ is an associated pair with R^i for all $i \varepsilon I$. Then S' is a complex system over $\underline{S}_q = \{ S/R^i | i \varepsilon I \}$ and called the canonical complex system over $\underline{S}_q$ denoted by $S/\underline{R}$.

Then,

Theorem 5.1.1

Let $S \subset X \times Y$ be an input-output system and $\underline{R} = \{ R^1, R^2 \}$ a class of input-output compatible equivalence relations on S. Let $\underline{R}_x = \{ R_x{}^1, R_x{}^2 \}$ and $\underline{R}_y = \{ R_y{}^1, R_y{}^2 \}$, where $(R_x{}^i, R_y{}^i)$ is an associated pair with R^i for $i=1,2$. Then,

(1) $S/\underline{R}$ is a surjective model of S ;

(2) if $\underline{R}_x$ and $\underline{R}_y$ are separating, $S \cong S/\underline{R}$;

(3) if $\underline{R}$ is full,and $\underline{R}_x$ and $\underline{R}_y$ are separating, $S/\underline{R} \cong S/R^1 \times S/R^2$, that is, $S \cong S/R_1 \times S/R_2$.

As we will see, a quotient system is a candidate of a component system when a complex system has no interaction between input of a component system and output of another. However, in cascade decomposition, another kind of concepts is required. It will be stated later.

5.2 General Theory of Decomposition

We first consider a general condition for a global system to have a complex system representation in inductive modelling. An input-output system is called to be decomposable if it has a complex system representation as its model. For notational convenience, we only discuss the case where a complex system consists of two component systems. It is, however, easy to extend it to the case of finite component systems.

Proposition 5.2.1[3)]

Every input-output system is decomposable.

If we impose a modelling morphism to be injective,

Proposition 5.2.2[3)]

A global system $\hat{S} \subset \hat{X} \times \hat{Y}$ has a complex system representation as its injective model if and only if there is a class $\underline{R} = \{ R^1, R^2 \}$ of input-output compatible equivalence relations on $\hat{S}$ such that $\underline{R}_x = \{ R_x{}^1, R_x{}^2 \}$ and $\underline{R}_y = \{ R_y{}^1, R_y{}^2 \}$ are separating, where $(R_x{}^i, R_y{}^i)$ is an associated pair with R^i for i=1,2.

While in surjective decomposition,

Proposition 5.2.3[8)]

A global system $\hat{S} \subset \hat{X} \times \hat{Y}$ has a complex system representation as its surjective model if and only if there is a class $\underline{R} = \{ R^1, R^2 \}$ of input-output compatible equivalence relations on $\hat{S}$.

Finally, for isomorphic decomposition we obtain the following proposition.

Propositio 5.2.4

A global system $\hat{S} \subset \hat{X} \times \hat{Y}$ has a complex system representation as its isomorphic model if and only if there is a class $\underline{R} = \{ R^1, R^2 \}$ of input-output compatible equivalnce relations on $\hat{S}$ such that $\underline{R}_x = \{ R_x{}^1, R_x{}^2 \}$ and $\underline{R}_y = \{ R_y{}^1, R_y{}^2 \}$ are separating, where $(R_x{}^i, R_y{}^i)$ is an associated pair with R^i for i=1,2.

5.3 Modelling Morphisms between Complex Systems

In section 5.2, a complex system into which a given global system was decomposed was not a pri ori given. When we have a class $\underline{S} = \{ S_1, S_2 \}$ of component systems and are required to represent the global system by a complex system over $\underline{S}$, we can easily find a decomposable condition by theorems obtained before if we develope the relationship between complex systems and modelling morphisms. So, we investigate the following problem in this section ; Let $S \subset S_1 \times S_2$ and $S' \subset S_1' \times S_2'$ be complex systems. If modelling morphisms $k_1=(k_{1x},k_{1y}):S_1 \to S_1'$ and $k_2=(k_{2x}, k_{2y}):S_2 \to S_2'$ are given, how naturally can we construct a modelling morphism from S to S' and what property does such a modelling morphism have ?

<u>Definition 5.3.1</u> Generated Modelling Morphism of Complex Systems

Let $S \subset S_1 \times S_2$ and $S' \subset S_1' \times S_2'$ be complex systems. Suppose that $k_1=(k_{1x},k_{1y}):S_1 \to S_1'$ and $k_2=(k_{2x},k_{2y}):S_2 \to S_2'$ are modelling morphisms. Let $k_1 \times k_2|S=(k_{1x} \times k_{2x}|\underline{D}(S),k_{1y} \times k_{2y}|\underline{R}(S))$, where $k_{1x} \times k_{2x}$ and $k_{1y} \times k_{2y}$ are usual products of mappings. If $k_1 \times k_2|S$ is a modelling morphism from S to S', it is called a modelling morphism generated by (k_1,k_2). From now on, whenever we write $k_1 \times k_2|S$ with modelling morphisms k_1 and k_2, it denotes a modelling morphism from S.

<u>Proposition 5.3.1</u>

If $S'=S_1' \times S_2'$, $k_1 \times k_2|S$ is a modelling morphism for all $S \subset S_1 \times S_2$.

<u>Proposition 5.3.2</u>

The following diagram commutes for all $S \subset S_1 \times S_2$ and $S' \subset S_1' \times S_2'$;

$$\begin{array}{ccc} S & \xrightarrow{k_1 \times k_2|S} & S' \\ p_i \downarrow & & \downarrow p_i' \\ S_i & \xrightarrow[k_i]{} & S_i' \end{array}$$

, where $k_1 \times k_2|S$, k_1 and k_2 are modelling morphisms, and p_i and p_i' are i-th projections on S and on S', respectively.

Most important modelling morphisms between complex systems are injective , surjecticive and isomorphic ones.

<u>Proposition 5.3.3</u>

If $k_1 \times k_2|S$ is a surjective modelling morphism, so is k_i for i=1,2. Conversely, if $S = S_1 \times S_2$ and k_1 and k_2 are surjective modelling morphisms, then so is $k_1 \times k_2|S$.

<u>Proposition 5.3.4</u>

If k_1 and k_2 are injective modelling morphism, so is $k_1 \times k_2|S$ for all $S \subset S_1 \times S_2$.

<u>Proposition 5.3.5</u>

If k_1 and k_2 are isomorphisms and $S=S_1 \times S_2$, then $k_1 \times k_2|S$ is also an isomorphism.

6. Decomposition of Input-Output System

6.1 Non-Interacted Decomposition

In this section, we pay attention to a condition for non-interacted decomposition, that is, a condition under which a global system is decomposable into a non-interacted system.

Proposition 6.1.1 (General Case)

A global system $\hat{S} \subset \hat{X} \times \hat{Y}$ has a non-interacted system as its model if and only if there is a class of input-output compatible equivalence relations $\underline{R} = \{ R^1, R^2 \}$ on $\hat{S}$.

Proposition 6.1.2 (Surjective Case)[8]

A global system $\hat{S} \subset \hat{X} \times \hat{Y}$ has a non-interacted system as its surjective model if and only if there is a full class $\underline{R} = \{ R_1, R_2 \}$ of equivalence relations on $\hat{S}$.

Proposition 6.1.3 (Injective Case)

A global system $\hat{S} \subset \hat{X} \times \hat{Y}$ has a non-interacted system as its injective model if and only if there is a class $\underline{R} = \{ R_1, R_2 \}$ of equivalence relations on $\hat{S}$ such that $\underline{R}_x = \{ R_x{}^1, R_x{}^2 \}$ and $\underline{R}_y = \{ R_y{}^1, R_y{}^2 \}$ are separating, where $(R_x{}^i, R_y{}^i)$ is an associated pair of equivalence relations with R^i for $i=1,2$.

Proposition 6.1.4 (Isomorphic Case)

A global system $\hat{S} \subset \hat{X} \times \hat{Y}$ has a non-interacted system as its isomorphic model if and only if there is a full class $\underline{R} = \{ R^1, R^2 \}$ of equivalnce relations on $\hat{S}$ such that $\underline{R}_x = \{ R_x^{\ 1}, R_x^{\ 2} \}$ and $\underline{R}_y = \{ R_y^{\ 1}, R_y^{\ 2} \}$ are separating, where $(R_x^{\ i}, R_y^{\ i})$ is an associated pair with R^i for i=1,2.

6.2 Parallel Decomposition[15)]

In this section, we investigate a condition for decomposition of a global system into a parallel complex system.

At first, let us consider how to construct a parallel complex system by quotient systems.

Definition 6.2.1 Decomposable Equivalence Relations

Let R_o, R_1 and R_2 be equivalence relations on a non-void set A. If there are equivalence relations R_1' and R_2' such that

$$R_1 = R_1' \cap R_o \text{ and } R_2 = R_2' \cap R_o,$$

R_1 and R_2 are called to be decomposable with R_o.

Lemma 6.2.1

Let R_o be an equivalence relation on a non-void set A and $\underline{R} = \{ R_1, R_2 \}$ a class of equivalence relations on A satisfying $R_o = R_1 \cap R_2$. Then $A/R_o \cong A/\underline{R}$, where $A/\underline{R} = \{ ([a]R_1, [a]R_2) | a \in A \}$.

Definition 6.2.2 Parallel Connection of Quotient Systems

Let $\underline{R}= \{ R^1,R^2 \}$ be a class of input-output compatible equivalence relations on an input-output system $S \subset X \times Y$. Suppose that $(R_x{}^i,R_y{}^i)$ is an associated pair with R_i for i=1,2 such that $R_x{}^1$ and $R_x{}^2$ are decomposable with some R_z, that is there are equivalence relations $R'_x{}^1$ and $R'_x{}^2$ such that $R_x{}^i = R'_x{}^i \cap R_z$ for i=1,2. Then we can identify S/R^i with $S'_i \subset (X/R'_x{}^i \times X/R_z) \times Y/R_y{}^i$ be Lemma 6.2.1, where $S'_i= \{ (([x]R'_x{}^i,[x]R_z),[y]R_y{}^i) | (x,y) \varepsilon S \}$. Therefore we can define the parallel connection of S/R^1 and S/R^2. The resultant parallel complex system is denoted by $P(S/R^1,S/R^2)$.

Lemma 6.2.2

Let $\underline{R}= \{ R^1,R^2 \}$ be a class of input-output compatible equivalence relations on an input-output system $S \subset X \times Y$. Suppose that $R_x{}^1$ and $R_x{}^2$ are decomposable with R_z, where $(R_x{}^i,R_y{}^i)$ is an associated pair with R^i for i=1,2. Then $S/\underline{R}$ can be embedded in $P(S/R^1,S/R^2)$.

Let us now consider a condition for parallel decomposition. If a global system $\hat{S} \subset \hat{X} \times \hat{Y}$ has the external representation of a parallel complex system as its model (surjective, injective or isomorphic model), we say that $\hat{S}$ is parallel (surjectively, injectively or isomorphicly parallel, respectively) decomposable.

Lemma 6.2.3

Let $\underline{R}$= { R^1, R^2 } be a class of input-output compatible equivalence relations on an input-output systems $S \subset X \times Y$. Suppose that $R_x{}^1$ and $R_x{}^2$ are decomposable with R_z, where $(R_x{}^i, R_y{}^i)$ is an associated pair with R^i for i=1,2. If $\underline{R}$ satisfies the following condition, $P(S/R^1, S/R^2) \cong S/\underline{R}$.

$(\forall(x_1,y_1),\forall(x_2,y_2) \in S)([x_1]R_z=[x_2]R_z \rightarrow (\exists(x,y) \in S)$

$(x,y) \in ([x_1]R_x{}^1 \cap [x_2]R_x{}^2) \times ([y_1]R_y{}^1] \cap [y_2]R_y{}^2) \cap S.$

Remark : If $\underline{R}$ is full, $\underline{R}$ satisfies the condiiton.

Proposition 6.2.1 (General Case)

A global system $\hat{S} \subset \hat{X} \times \hat{Y}$ is parallel decomposable if and only if there is a class $\underline{R}$= { R^1,R^2 } of input-output compatible equivalence relations on $\hat{S}$ such that $R_x{}^1$ and $R_x{}^2$ are decomposable with some R_z, where $(R_x{}^i, R_y{}^i)$ is an associated pair with R^i for i=1,2.

Proposition 6.2.2 (Surjective Case)

A global system $\hat{S} \subset \hat{X} \times \hat{Y}$ is surjectively parallel decomposable if and only if there is a class $\underline{R}$= { R^1,R^2 } of equivalence relations on $\hat{S}$ such that $R_x{}^1$ and $R_x{}^2$ are decomposable with some R_z and $\underline{R}$ satisfies the condition of Lemma 6.2.3, where $(R_x{}^i, R_y{}^i)$ is an associated pair with R^i for i=1,2.

Proposition 6.2.3 (Injective Case)

A global system $\hat{S} \subset \hat{X} \times \hat{Y}$ is injectively parallel decomposable if and only if there is a class $\underline{R}= \{ R^1,R^2 \}$ of equivalence relations on $\hat{S}$ such that $R_x^{\ 1}$ and $R_x^{\ 2}$ are decomposable with some R_z, where $(R_x^{\ i},R_y^{\ i})$ is an associated pair with R^i for i=1,2, and $\underline{R_x}= \{ R_x^{\ 1},R_x^{\ 2} \}$ and $\underline{R_y}= \{ R_y^{\ 1}, R_y^{\ 2} \}$ are separating.

Theorem 6.2.1 (Isomorphic Case)

A global system $\hat{S} \subset \hat{X} \times \hat{Y}$ is isomorphicly parallel decomposable if and only if there is a class $\underline{R}= \{ R1,R^2 \}$ of equivalence relations on $\hat{S}$ such that

(1) $R_x^{\ 1}$ and $R_x^{\ 2}$ are decomposable with some R_z ;

(2) $\underline{R_x}= \{ R_x^{\ 1},R_x^{\ 2} \}$ and $\underline{R_y}= \{ R_y^{\ 1},R_y^{\ 2} \}$ are separating ;

(3) $\underline{R}$ satisfies the condition of Lemma 6.2.3.

, where $(R_x^{\ i}, R_y^{\ i})$ is an associated pair with R^i for i=1,2.

Example 6.2.1 : Let $T=[A,C, \delta]$ be a state automaton and $\hat{S} \subset C \times C$ be defined by $$(c,c') \in \hat{S} \leftrightarrow (\exists a \in A)(\delta (c,a)=c').$$
Let us now consider to decompose $\hat{S}$ into a parallel connections of input-output systems. Suppose that there are equivalence relations π_1 and π_2 on C such that $\pi_1 \cap \pi_2=Id$ and

$(\forall(c,c_1'))(\forall(c,c_2'))(\exists\hat{c})((c,\hat{c}) \in S$ and $(c_1',\hat{c}) \in \pi_1$ and $(c_2',\hat{c}) \in \pi_2)$.

Let $R^i \subset S \times S$ be defined by $(c_1,c_1')R^i(c_2,c_2') \leftrightarrow c_1=c_2$ and $(c_1',c_2') \in \pi_i)$.

Then $R_x^{\ 1}=R_x^{\ 2}=Id$ and obviously they are decomposable with Id. And $\underline{R_x}= \{ Id \}$ and $\underline{R_y}= \{ \pi_1, \pi_2 \}$ are separating. If $(c_1,c_1'),(c_2,c_2') \in S$ and $c_1=c_2$, there is c_2'' such that $c_2'' \in [c_1']\pi_1 \cap [c_2']\pi_2$ and $(c_1,c_2'') \in S$.

Therefore the condition 3) of Theorem 6.2.4 are satisfied. Consequently, we can decompose S into a parallel connection of input-output systems. Actually, if we define $\underline{S} = \{ S_1 \subset C \times C/\pi_1, S_2 \subset C \times C/\pi_2 \}$ by

$$(c,[c']\pi_i) \in S_i$$
$$\leftrightarrow \quad (\exists c'' \in [c']\pi_i)((c,c'') \in S),$$

then (Id,h_y) is an isomorphism from $\hat{S}$ to $P(\underline{S})$, where $h_y : C \rightarrow C/\pi_1 \times C/\pi_2$ is defined by $h_y(c)=([c]\pi_1,[c]\pi_2)$.

It is noted that there is no assurance that S_1 and S_2 can be realized by state automata because we do not take into account of the input.

6.3 Cascade Decomposition[16)]

In this section, we investigate a condition for decomposition of a global system into a cascade complex system. In parallel decomposition, a parallel complex system and the external representation of it are isomorphic. However, it is not the case in cascade decomposition. Therefore we cannot use the concept of quotient systems directly to cascade decomposition. In cascade decomposition, we use the following concepts.

<u>Definition 6.3.1</u> Semi-Quotient Systems

Let $\underline{R} = \{ R^1, R^2 \}$ be a class of input-output compatible equivalence relations on an input-output system $S \subset X \times Y$. Suppose that $(R_x{}^i, R_y{}^i)$ is an associated pair with R^i for i=1,2. Let a subset $S'_1 \subset X/R_x{}^1 \times (Y/R_y{}^1 \times S)$ and $S'_2 \subset (X/R_x{}^2 \times S) \times Y/R_y{}^2$ be defined by

$$([x]R_x{}^1,([y]R_y{}^1,(x',y'))) \in S'_1 \leftrightarrow (x',y') \in [x]R_x{}^1 \times [y]R_y{}^1 \cap S$$
$$(([x]R_x{}^2,(x',y')),[y]R_y{}^2) \in S'_2 \leftrightarrow (x',y') \in [x]R_x{}^2 \times [y]R_y{}^2 \cap S.$$

Then S'_1 and S'_2 are called the first and second semi-quotient systems with modulo R^1 and R^2, respectively and denoted by $S'_1=FSQ(S;R^1)$ and $S'_2=SSQ(S;R^2)$.

<u>Definition 6.3.2</u> Cascade Connection of Semi-Quotient Systems

Let $\underline{R}=\{R^1,R^2\}$ be a class of input-output compatible equivalence relations on an input-output system $S \subset X \times Y$. And let $S' \subset FSQ(S;R^1) \times SSQ(S;R^2)$ be defined by

$$(([x_1]R_x^{\,1},([y_1]R_y^{\,1},(x_1,y_1))),(([x_2]R_x^{\,2},(x_2,y_2)),[y_2]R_y^{\,2})) \in S'$$
$$\leftrightarrow (x_1,y_1)=(x_2,y_2).$$

Then S' is called the cascade complex system over $(FSQ(S;R^1),SSQ(S;R^2))$ and denoted by $C(FSQ(S;R^1),SSQ(S;R^2))$. And the external representation $EX(C(FSQ(S;R^1),SSQ(S;R^2)))$ is defined by

$$((([x_1]R_x^{\,1},[x_2]R_x^{\,2}),([y_1]R_y^{\,1},[y_2]R_y^{\,2})) \in EX(C(FSQ(S;R^1),SSQ(S;R^2)))$$
$$\leftrightarrow (\exists(x,y) \in ([x_1]R_x^{\,1} \cap [x_2]R_x^{\,2}) \times ([y_1]R_y^{\,1} \cap [y_2]R_y^{\,2}) \cap S.$$

<u>Proposition 6.3.1</u>

$EX(C(FSQ(S;R^1),SSQ(S;R^2))) = S/\underline{R}$, where $\underline{R}=\{R^1,R^2\}$.

Let us now consider a condition for cascade decomposition. If a global system $\hat{S} \subset \hat{X} \times \hat{Y}$ has the external representation of a cascade complex system as its model (surjective, injective or isomorphic model), we say that S is cascade (surjectively, injectively or isomorphicly cascade, respectively) decomposable.

Proposition 6.3.2 (General and Surjective Case)

A global system $\hat{S} \subset \hat{X} \times \hat{Y}$ is (surjectively) cascade decomposable if and only if there is a class $\underline{R}= \{ R^1,R^2 \}$ of input-output compatible equivalence relations on $\hat{S}$.

Proposition 6.3.3 (Injective and Isomorphic Case)

A global system $\hat{S} \subset \hat{X} \times \hat{Y}$ is injectively (isomorphicly) cascade decomposable if and only if there is a class $\underline{R}= \{ R^1,R^2 \}$ of input-output compatible equivcalence relations on $\hat{S}$ such that $\underline{R}_x= \{ R_x{}^1,R_x{}^2 \}$ and $\underline{R}_y= \{ R_y{}^1,R_y{}^2 \}$ are separating, where $(R_x{}^i,R_y{}^i)$ is an associated pair with R^i for i=1,2.

Example : Let $T=[A,C, \delta]$ be a state automaton and $\hat{S} \subset C \times C$ be defined as in the previous example. Let us consier to decompose $\hat{S}$ into a cascade connection of input-output systems. Suppose that there are equivalence relations π_1 and π_2 on C such that $\pi_1 \cap \pi_2=Id$. Let R^i be defined as in the previous example. Then all conditions of Proposition 6.3.3 are satisfied. Therefore we can decompose $\hat{S}$ into a serial connection of input-output systems. Actually if we define $\underline{S}= \{ S_1 \subset C \times (C/ \pi_1 \times S),S_2 \subset (C \times S) \times C/ \pi$ by

$$(c,([c'] \pi_1,(c_1,c_1'))\ \varepsilon\ S_1$$

$$\leftrightarrow c=c_1 \text{ and } [c'] \pi_1=[c_1'] \pi_1$$

and
$$((c,(c_2,c_2')),[c'] \pi_2)\ \varepsilon\ S_2$$

$$\leftrightarrow c=c_2 \text{ and } [c'] \pi_2=[c_2'] \pi_2,$$

then (Id,h_y) is an isomorphism, where $h_y:C \rightarrow C/ \pi_1 \times C/ \pi_2$ is defined as in the previous example. If we define $S_1' \subset C \times C/ \pi_1$ by

$$(c,[c'] \pi_1)\ \varepsilon\ S_1' \leftrightarrow S(c) \cap [c'] \pi_1 \neq \phi$$

and $S_2' \subset (C \times C/\pi_1) \times C/\pi_2$ by

$$((c,[c_1]\pi_1),[c_2]\pi_2) \in S_2' \leftrightarrow S(c) \cap [c_1]\pi_1 \cap [c_2]\pi_2 \neq \phi ,$$

this type of decomposition is more familiar in serial decomposition of state automata. It is also noted that there is no assurance that S_1 and S_2 can be realized by state automata because we do not take into account of the input.

6.4 Feedback Decomposition

In this section, we investigate a condition for decomposition of a global system into a feedback connection of input-output systems. Let us first consider a feedback transformable type.

Definition 6.4.1 Feedback Transformable Type

Let $S \subset X \times Y$ be an input-output system. Let $S' \subset (X \times S) \times (Y \times S)$ be defined by

$$((x,(x',y')),(y,(x'',y''))) \in S'$$

$$\leftrightarrow \quad (x,y)=(x',y')=(x'',y'').$$

Then S' is called the feedback transformable type of S and denoted by FT(S) and the external representation of $F_T(S)$ is defined by

$$EX(F_T(S))= \{ (x,y) | (\exists(x',y'))(\exists(x'',y''))(((x,(x',y')),(y,(x'',y''))) \in F_T(S) \} .$$

Proposition 6.4.1

Every input-output system $S \subset X \times Y$ is feedback decomposable. Actually, $S = EX(FT(S)))$.

7. Different Forms of Decomposition

7.1 Deductive Modelling

In section 5.1, we introduced the concept of quotient systems. However, more broad concept is required in deductive modelling. An input-output system $S \subset X \times Y$ is a set itself and so we can define a covering C of S, that is $\phi \,\underline{\varepsilon}\, C$, $C \subset P(S)$ and $\cup C = S$. Let us now extend the concept of quotient systems by using a covering.

Most important covering of an input-output system $S \subset X \times Y$ is what reflects the input-output behaviour of S.

<u>Definition 7.1.1</u> Input-Output Compatible System Covering[3)]

Let $S \subset X \times Y$ be an input-output system and $C = \{ C_\lambda \mid \lambda \varepsilon \Lambda \}$ a covering of S. C is called to be input-output compatible if there are coverings $C_x = \{ C_x{}^\alpha \mid \alpha \varepsilon \Lambda_x \}$ and $C_y = \{ C_y{}^\beta \mid \beta \varepsilon \Lambda_y \}$ of X and Y, respectively such that

$$C_\lambda \varepsilon C \leftrightarrow (\exists C_x{}^\alpha \varepsilon C_x)(\exists C_y{}^\beta \varepsilon C_y)$$
$$((C_x{}^\alpha \times C_x{}^\beta) \cap S = C_\lambda \neq \phi).$$

In this case, the pair (C_x, C_y) is called an associated pair of coverings (briefly, an associated pair) with C.

Similar to Definition 2.5.1, we define the property of separatingness of a class of coverings.

Definition 7.1.2 Separating

A class $\underline{C} = \{ C^1,..,C^n \}$ of coverings of a non-void set A is called to be separating if $(\forall (C^i{}_\alpha \mid i=1,\ldots,n) \in \Pi\, \underline{C})(\forall a \in A)(\forall a' \in A)$ $((a,a') \in \cap (C^i{}_\alpha \mid i=1,\ldots,n) \rightarrow a=a')$, where $C^i = \{ C^i{}_\alpha \mid \alpha \in \Lambda_i \}$.

Let us next extend the concept of quotient systems to the case of coverings.

Definition 7.1.3 Quasi-Quotient System

Let $S \subset X \times Y$ be an input-output system. Suppose that C is an input-output compatible covering of S, where (C_x, C_y) is an associated pair. Let $S' \subset C_x \times C_y$ be defined by

$$(C_x{}^\alpha, C_y{}^\beta) \in S' \leftrightarrow C_x{}^\alpha \times C_y{}^\beta \cap S \in C.$$

Then S' is called a quasi-quotient system of S modulo C and denoted by S/C.

It is noted that a quotient system is also a quasi-quotient system. The next proposition tells us how to construct an input-output compatible covering.

Proposition 7.1.1

Let $S \subset \Pi\, \underline{S}$ be a complex system over $\underline{S} = \{ S_i \mid i \in I \}$. Then $C^i = \{ (p_{ix}{}^{-1}(x_i) \times p_{iy}{}^{-1}(y_i)) \cap S \mid (x_i, y_i) \in S_i \}$ is an input-output compatible system covering of S for all $i \in I$, where $p_i = (p_{ix}, p_{iy})$ is the i-th projection on S.

Let $\underline{S_c} = \{ S/C^i | i \in I \}$ be a class of quasi-quotient systems. Then we can naturally extend the concept of a canonical complex system over $\underline{S_c}$.

Definition 7.1.4   Canonical Complex System over $\underline{S_c}$

Let $S \subset X \times Y$ be a general input-output system and $\underline{C} = \{ C^i | i \in I \}$ a class of input-output compatible coverings of S. Let $S' \subset \Pi \underline{S_c}$ be defined by

$$([C_x{}^i{}_\alpha | i \in I], [C_y{}^i{}_\beta | i \in I]) \in S'$$
$$\leftrightarrow \quad (\cap C_x{}^i{}_\alpha \times \cap C_y{}^i{}_\beta) \cap S \neq \phi .$$

Then S' is called the canonical complex system over $\underline{S_c}$ and denoted by $S/\underline{C}$.

In the deductive modelling, we use quasi-quotient systems instead of quotient systems in decomposition. It is, however, difficult to find a general condition in this case. We just mention about surjective decomposition.

Theorem 7.1.1[3)]

If a global system $\hat{S} \subset \hat{X} \times \hat{Y}$ is a surjective model of a complex system $S \subset X \times Y$, there is a class $\underline{C} = \{ C^1, C^2 \}$ of input-output compatible coverings of $\hat{S}$.

Conversely, if there is a class $\underline{C} = \{ C^1, C^2 \}$ of input-output compatible coverings of $\hat{S}$ and $\underline{C_x} = \{ C_x{}^1, C_x{}^2 \}$ and $\underline{C_y} = \{ C_y{}^1, C_y{}^2 \}$ are separating, where $(C_x{}^i, C_y{}^i)$ is an associated pair with C^i for i=1,2, then $\hat{S} \subset \hat{X} \times \hat{Y}$ is a surjecive model of a complex system.

Example 7.1.1 : Let $T=[A,C,\delta]$ be a state automaton and $\hat{S} \subset C \times C$ be defined as in Example 6.2.1. It is known that if there is a covering $\underline{C}=\{C_\alpha \mid \alpha \in I\}$ of C such that $(\forall u)(\forall \alpha)(\exists \beta)(a \in A \to (C_\alpha, a) \subset C_\beta)$, then there are state automata T_1 and T_2 such that T can be realized by the serial connection of T_1 and T_2. In this case, in order to construct state automata, we use another covering $\underline{P}$ of C such that $\{\underline{C}, \underline{P}\}$ is separating. It is noted that the condition for $\underline{C}$ ensures that the resultant input-output system can be realized by a state automaton.

7.2 Simulation

Since a simulation morphism is obtained by reversing the arrow of output mapping h_y of a modelling morphism $h=(h_x,h_y)$, we naturally obtain the following proposition.

Proposition 7.2.1

If a global system $\hat{S} \subset \hat{X} \times \hat{Y}$ has a simulation model $S \subset S_1 \times S_2$, then there is a class $\underline{C} = \{C^1, C^2\}$ of input-output compatible coverings. Conversely, if there is a class $\underline{R} = \{R^1, R^2\}$ of input-output compatible equivalence relations satisfying that $\underline{R}_y = \{R_y^{\ 1}, R_y^{\ 2}\}$ is separating, where $(R_x^{\ i}, R_y^{\ i})$ is an associated pair with R^i, $\hat{S}$ is decomposed into a complex system in simulation.

The category SIM of input-output systems with simulation morphisms was introduced in Proposition 2.2.6. Let us next consider functors from MOD to SIM and from SIM to MOD.

Proposition 7.2.2

Let MOD and SIM be categories of input-output systems with modelling morphisms and simulation morphisms, respectively. Let $\underline{MOD}_{1b}$ and $\underline{SIM}_{1b}$ be subcategories of MOD and SIM defined by

$$\mathrm{Obj}(\underline{MOD}_{1b})=\mathrm{Obj}(\underline{SIM}_{1b})=\mathrm{Obj}(\underline{MOD})$$

$$\underset{\underline{MOD}_{1b}}{\mathrm{Hom}}(S,S')=\{\ h=(h_x,h_y):S\to S'\mid h_y \text{ is bijective }\}\subset \underset{\underline{MOD}}{\mathrm{Hom}}(S,S')$$

and
$$\underset{\underline{SIM}_{1b}}{\mathrm{Hom}}(S,S')=\{\ h^s=(h_x,h_y):S\to S'\mid h_y \text{ is bijective }\}\subset \underset{\underline{SIM}}{\mathrm{Hom}}(S,S').$$

Let $F_{MS}:\underline{MOD}_{1b}\to\underline{SIM}_{1b}$ be defined by

$$F_{MS}(S)=S \qquad \text{for } S\in \mathrm{Obj}(\underline{MOD}_{1b})$$

$$F_{MS}(h=(h_x,h_y))=(h_x,{h_y}^{-1}),\text{ where } {h_y}^{-1} \text{ is the inverse of } h_y,$$

and $F_{SM}:\underline{SIM}_{1b}\to\underline{MOD}_{1b}$ be defined by

$$F_{SM}(S)=S \qquad \text{for } S\in \mathrm{Obj}(\underline{SIM}_{1b})$$

$$F_{SM}(h^s=(h_x,h_y))=(h_x,{h_y}^{-1}),\text{ where } {h_y}^{-1} \text{ is the inverse of } h_y.$$

Then F_{MS} and F_{SM} are functors. Morover, $F_{MS}\circ F_{SM}$ and $F_{SM}\circ F_{MS}$ are identity functors.

Therefore, we can transfer a decomposable condition in inductive modelling into simulation by the functor F_{MS} in this case. Especially, in isomorphic decomposition a decomposable condition that we investigated before is also valid for simulation.

Chapter 8 Decomposition of Functional Systems

In this chapter, we investigate a condition for decomposition of a functional system. Since a functional system $S:X \to Y$ is an input-output system $S \subset X \times Y$, the decomposability condition is directly derived by theorems obtained in Chapter 6. It is, however, natural to require a component system to be also functional, we impose an equivalence relation to satisfy such a condition.

One of the important systems is an endomorphism on an Ω -group $\underline{A}=[A;F_G \cup F_U]$. The connection of endomorphisms on $\underline{A}$ was defined in Section 3.2. It will be seen that the decomposability condition for an endomorphism is a natural consequence of that of functional systems.

8.1 Functional Component System

Let $\hat{S} \subset \hat{X} \times \hat{Y}$ be a functional system. If there are functional systems $S_1:X_1 \to Y_1$ and $S_2:X_2 \to Y_2$ such that a functional complex system over $\underline{S}= \{ S_1,S_2 \}$ is a model of $\hat{S}$, we say $\hat{S}$ to be functionally decomposable. In other words, $\hat{S}$ is called to be functionally decomposable if there are functional systems $S_1:X_1 \to Y_1$ and $S_2:X_2 \to Y_2$ such that the following diagram commutes for some functional complex system $S:X \to Y$, where $S \subset S_1 \times S_2$, and $h_x:\hat{X} \to X$ and $h_y:\hat{Y} \to Y$ are mappings.

$$\begin{array}{ccc} \hat{X} & \xrightarrow{\hat{S}} & \hat{Y} \\ h_x \downarrow & & \downarrow h_y \\ X & \xrightarrow[S]{} & Y \end{array}$$

The above condition is exactly same as that of input-output system.

Proposition 8.1.1

Let $S:X \to Y$ and $S':X' \to Y'$ be functional systems. Suppose that $h_x:X \to X'$ and $h_y:Y \to Y'$ are mappings. Then $h=(h_x,h_y)$ is a modelling morphism from S to S' if and only if the following diagram commutes ;

$$\begin{array}{ccc} X & \xrightarrow{S} & Y \\ h_x \downarrow & & \downarrow h_y \\ X' & \xrightarrow[S']{} & Y' \end{array}$$

Moreover, $h=(h_x,h_y)$ is an isomorphism if and only if h_x and h_y are bijective, and the diagram commutes.

If a global system $\hat{S} \subset \hat{X} \times \hat{Y}$ is functional, it is natural to impose component systems to be functional. Therefore we need equivalence relations which satisfy such a condition.

Proposition 8.1.2

Let $S:X \to Y$ be a functional system and R an input-output compatible equivalence relation on S. Suppose that (R_x,R_y) is an associated pair with R. Then the quotient system S/R is functional if and only if

$$(x,x') \in R_x \to (S(x),S(x')) \in R_y \text{ for all } x,x' \in X.$$

If an input-output compatible equivalence relation satisfies the above condition, it is called to be quotient functional.

In order to clarify our idea, we just investigate the case where h_x and h_y are bijective, that is, isomorphic decomposition in the following section.

8.2 Non-Interacted Decomposition

The condition for non-intercated decomposition of a functional system is directly derived by Proposition 6.1.4.

<u>Theorem 8.2.1</u>

A functional global system $\hat{S}:\hat{X} \rightarrow \hat{Y}$ has a non-interacted functional system as its isomorphic model if and only if there is a full class $\underline{R} = \{ R^1,R^2 \}$ of input-output compatible and quotient functional equivalence relations on $\hat{S}$ such that $\underline{R}_x = \{ R_x{}^1,R_x{}^2 \}$ and $\underline{R}_y = \{ R_y{}^1,R_y{}^2 \}$ are separating, where $(R_x{}^i,R_y{}^i)$ is an associated pair with R^i for i=1,2.

8.3 Parallel Decomposition

Parallel connection of functional systems was defined in Chapter 3.2. We say a functional system $S:X \rightarrow Y$ is parallel decomposable if there are a class of functional systems $\underline{S} = \{ S_i:X_i \rightarrow Y_i | i=1,2 \}$ and a bijection $h:Y \rightarrow Y'$ such that the following diagram commutes, where $Y' = \{ (S_1(x),S_2(x)) | x \in X \}$.

$$\begin{tikzcd} X \arrow[r, "S"] \arrow[d, "Id"'] & Y \arrow[d, "h"] \\ X \arrow[r, "P_F(\underline{S})"'] & Y' \end{tikzcd}$$

Since $P_F(\underline{S}) \cong EX(P(\underline{S}))$ by Proposition 3.2.2, the term of parallel decomposition of functional systems is exactly consistent with that of input-output systems.

The following theorem is a version of Proposition 6.2.4 for functional case.

Theorem 8.3.1

A functional global system $\hat{S}:\hat{X} \to \hat{Y}$ is parallel decomposable if and only if there is a separating class of $\underline{R}_y = \{ R_y{}^1, R_y{}^2 \}$ of equivalence relations on $\hat{Y}$.

If functional systems are endomorphisms of an Ω -group $\underline{A}=[A;F_G \cup F_U]$, a parallel connection of them was defined as in Definition 3.2.6. In this case parallel connection of $S_1:\underline{A} \to \underline{A}$ and $S_2:\underline{A} \to \underline{A}$ is the sum S_1+S_2 induced by the binary operation + in F_G. We say an endomorphism $S:\underline{A} \to \underline{A}$ to be parallel decomposable if there are endomorphisms $S_1:\underline{A} \to \underline{A}$ and $S_2:\underline{A} \to \underline{A}$ such that $S = S_1 + S_2$.

We obtain the following theorem for parallel decomposition of an endomorphism.

Theorem 8.3.2

Let $S:\underline{A} \to \underline{A}$ be an endomorphism on a commutative Ω -group $\underline{A}=[A;F_G \cup F_U]$. Then S is parallel decomposable if there is a full and separating class of congruence relations on $S(\underline{A})$. Conversely, if S is injective, there is a full and separating class of congruence relations on $S(\underline{A})$.

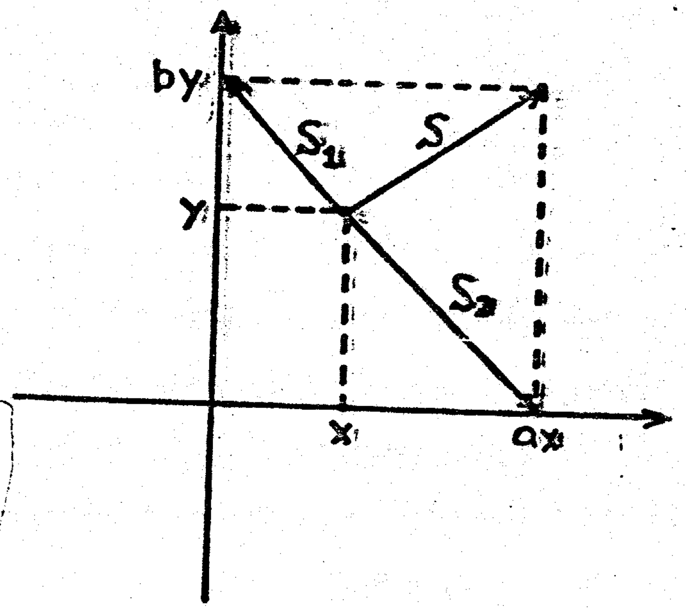


Fig. 8.3.1 Example of Parallel Decomposition of Endomorphism

Example : (Fig.8.3.1)

Let $\underline{A} = R^2$ and $S:\underline{A} \to \underline{A}$ be defined by

$$S((x,y))=(ax,by)$$

, where a,b ε A are non-zero elements. Let Θ_1 and Θ_2 be defined by

$$((x,y),(x',y')) \varepsilon \Theta_1 \leftrightarrow x=x'$$

$$((x,y),(x',y')) \varepsilon \Theta_2 \leftrightarrow y=y'.$$

Then $\underline{\Theta} = \{ \Theta_1, \Theta_2 \}$ is a full and separating class of congruence relations on $S(\underline{A})=\underline{A}$. Let $S_1:\underline{A} \to \underline{A}$ and $S_2:\underline{A} \to \underline{A}$ be defined by $S_1((x,y)) \varepsilon [S((x,y))] \Theta_1 \cap [0] \Theta_2$ and $S_2((x,y)) \varepsilon [S((x,y))] \Theta_2 \cap [0] \Theta_1$,that is $S_1((x,y))=(ax,0)$ and $S_2((x,y))=(0,by)$. Then $S_1+S_2((x,y))=(ax,0)+(0,by)=(ax,by)=S((x,y))$ and hence S is parallel decomposable.

8.4 Serial Decomposition

Serial connection of functional systems was defined in Section 3.2. However, it is difficult to find a general condition of a serial decomposition of a functional system. So, in this section we restrict our attention to that of an endomorphism of an Ω -group. We say an endomorphism $S:\underline{A} \to \underline{A}$ to be serial decomposable if there are endomorphisms $S_1:\underline{A} \to \underline{A}$ and $S_2:\underline{A} \to \underline{A}$ such that $S = S_1 \cdot S_2$. A sufficient condition for serial decomposition of an endomorphism is given by the following theorem.

Theorem 8.4.1

Let $S:\underline{A} \to \underline{A}$ be an endomorphism on an Ω -group $\underline{A}=[A;F_G \cup F_U]$. If there is a full and separating class of congruence relations $\underline{\Theta} = \{ \Theta_1, \Theta_2 \}$ and one of them is also a congruence relation on $\underline{A}'=[A;F_G \cup F_U \cup \{ S \}]$, then

S is serial decomposable.

Example : (Fig.8.4.1)

Let $\underline{A}=R^2$ and $S:\underline{A} \to \underline{A}$ be defined by $S((x,y))=(ax,by)$.

Let Θ_1 and Θ_2 be defined by

$$((x,y),(x',y')) \in \Theta_1 \leftrightarrow x=x'$$

$$((x,y),(x',y')) \in \Theta_2 \leftrightarrow y=y'.$$

Then $\underline{\Theta} = \{ \Theta_1, \Theta_2 \}$ is a full and separating class of congruence relations on $\underline{A}$ and moreover, $((x,y),(x',y')) \in \Theta_2$ implies $((ax,by),(ax',by')) \in \Theta_2$. Therefore Θ_2 satisfies the condition.

Let $S_1:\underline{A} \to \underline{A}$ and $S_2:\underline{A} \to \underline{A}$ be defined by $S_1((x,y)) \in [S((x,y))]\Theta_1 \cap [(x,y)]\Theta_2$ and $S_2((x,y)) \in [S((x,y))]\Theta_2 \cap [(x,y)]\Theta_1$, that is $S_1((x,y))=(ax,y)$ and $S_2((x,y))=(x,by)$. Then $S_1 \bullet S_2((x,y))=S_2(S_1(x,y)))=S_2((ax,y))$ $=(ax,by)=S((x,y))$. Hence S is serial decomposable.

8.5 Feedback Decomposition of Endomorphisms

Feedback connection of endomorphisms of a commutative Ω-group $\underline{A}=[A;F_G \cup F_U]$ was defined in Section 3.2. We say an endomorphism $S:\underline{A} \to \underline{A}$ to be feedback decomposable if there are endomorphisms $S_1:\underline{A} \to \underline{A}$ and $S_2:\underline{A} \to \underline{A}$ such that $S=S_1 \bullet (I - S_1S_2)^{-1}$. A condition for feedback decomposition of an endomorphism is given by the following theorem.

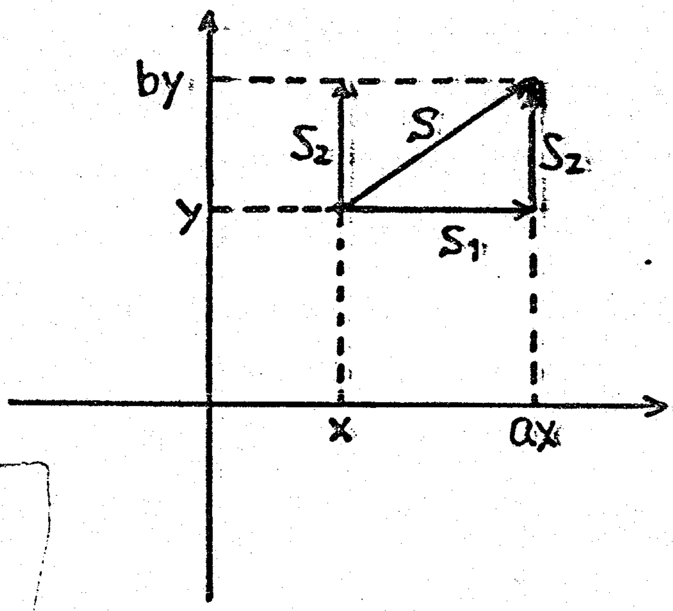


Fig. 8.4.1 Example of Serial Decomposition of Endomorphism

Theorem 8.5.1

Let $S:\underline{A} \rightarrow \underline{A}$ be an endomorphism on a commutative Ω -group $\underline{A}=[A;F_G \cup F_U]$. Then S is feedback decomposable if there is a full and separating class $\Theta = \{ \Theta_1, \Theta_2 \}$ of congruence relations on $\underline{A}$ such that

1) one of Θ_i is a congruence relation on $A'=[A;F_G \cup F_U \cup \{ S \}]$;

2) $(\forall b)(\exists! a)((a,b) \in \Theta_1$ and $(S(a),b) \in \Theta_2)$;

3) $(\forall b)(\exists! a)((a,b) \in \Theta_2$ and $(S(a),b) \in \Theta_1)$.

9. Decomposition of Transition Systems

In this chapter, we investigate decomposition problem of transition systems. A transition system, which was defined in chapter 3, is the dynamical part of a dynamical system, however, it can be recognized as an Ω -group when the state space is also an Ω -group. That is, if $\phi = \{ \phi_{tt'}:C \times X_{tt'} \to C|t,t' \in T \text{ and } t \leqslant t' \}$ is a transition system and C is an Ω -group $\underline{C}=[C;F_G \cup F_U]$, then $\overline{\phi}$ can be understood as an Ω -group $\underline{C}(\phi)=[C;F_G \cup F_U \cup F_T]$, where $F_T=\{ \phi_{tt'}(-,x_{tt'}):C \to C|x_{tt'} \in X_{tt'} \}$. Since a transition system is an Ω -group in the sense above, decomposition of a transition system just becomes that of an Ω -group in contrast to Chapter 8, where we studied decomposition of endomorphisms of an Ω -group.

9.1 Parallel Decomposition

Parallel connection of transition systems was defined in Section 3.3. Since a transition system $\overline{\phi}$ can be recognized as an input-output system $S \subset C \times C$;

$$(c,c') \in S \leftrightarrow (\exists t)(\exists t')(\exists x_{tt'})(\phi_{tt'}(c,x_{tt'})=c')$$

, the decomposability condition can be derived from theorems developed in Chapter 6. It is, however, natural to require a component system to be also realizable by a transition system. In order to do this, our universal algebraic approach is appropriate.

Definition 9.1.1 Parallel Decomposability

Let $\hat{T}=[X, \hat{C}, \hat{\phi}]$ be a transition system, where $\overline{X}=\{ x_{tt'} \in X_{tt'} | t,t' \in T \text{ and } t \leq t' \}$ and $\hat{\phi} = \{ \phi_{tt'}:\hat{C} \times X_{tt'} \rightarrow \hat{C} | t,t' \in T, t \leq t' \}$ If there exist transition systems $T_1=[X, C_1, \phi^1]$ and $T_2=[X, C_2, \phi^2]$ such that there is an isomorphism h from T to the parallel connection of T_1 and T_2, $\hat{T}$ is called to be parallel decomposable.

A necessary and sufficient condition for parallel decomposition of a transition system is given by the following theorem.

Theorem 9.1.1

A transition system $\overline{\phi} = \{ \phi_{tt'}:\hat{C} \times X_{tt'} \rightarrow \hat{C} \}$ is parallel decomposable if and only if there is a full and separating class $\underline{\Theta} = \{ \Theta_1, \Theta_2 \}$ of congruence relations on $\underline{C}(\overline{\phi})$, where $\underline{C}(\overline{\phi})$ is defined above.

Let us next illustrate the meaning of Theorem 9.1.1 by some well-known examples from Automata Theory and Linear System Theory.

Example 9.1.1 : Parallel Decomposition of Automaton[13)]

"Let $M=[A,B,C, \delta, \lambda]$ be a Mealy type automaton. Then M is non-trivial parallel decomposable if and only if there exist two non-trivial S.P. partitions π_1 and π_2 on C such that $\pi_1 \cap \pi_2=\text{Id}$."

The above statement is a well-known theorem by Hartmanis-Stearns, where S.P. partition means a partition π satisfying that

$$(\forall B \in \pi)(\exists B' \in \pi)(\forall a \in A)(\delta(B,a) \subset B').$$

Since a partitioin on C corresponds to an equivalence ralation on C, it is easily seen that the above condition for partition is exactly same as the following condition ;

there are two non-trivial equivalence relations π_1, π_2 on C such that $\pi_1 \cap \pi_2 = \mathrm{Id}$ and

$$(c,c') \in \pi_i \rightarrow (\forall a \in A)((\delta(c,a), \delta(c',a)) \in \pi_i) \qquad i=1,2.$$

It is noted that the condition of an equivalence relation to have subsitution property is exactly same as to require an equivalence relation to be a congruence relation on $[C; \{\delta(-,a):C \rightarrow C | a \in A\}$, where $\delta(-,a)$ $:C \rightarrow C$ is defnied by $\delta(-,a)(c)= \delta(c,a)$. Therefore it is easily seen that Theorem 9.1.1 is a generalization of Hartmanis-Stearns's one.

The essencial part of decomposition of an automaton M is how to decompose the dynamical part of M. Then the focused dynamical system is a transition system $T=[A,C,\delta]$. Since C is just a set, the algebraic expression of T is an Ω-group $\underline{C}(\delta)=[C; \{\delta(-,a)|a \in A\}]$. Then by Theorem 9.1.1, the necessary and sufficient conditiion for parallel decomposition of T is the existence of full and separating class $\underline{\Theta} = \{\Theta_1, \Theta_2\}$ of congruence relations on $\underline{C}(\delta)$. Since in Hartmanis-Stearns's parallel decomposition, they considered a non-trivial decomposition and require a morphism to be only injective, let us omit the condition of fullness and require Θ_1 and Θ_2 to be non-trivial. Then we obtain the necessity part of Hartmanis-Stearns's Theorem.

Conversely, if an automaton is parallel decomposable, we can easily construct a separating class $\underline{\Theta}$ of congruence relations on $\underline{C}(\delta)$.

Example 9.1.2 : Modal Decomposition[17] (Fig. 9.1.2)

Let a system be described by the following differential equation ;

$$z' = A\,z + b\,x$$

, where $z \in C \subset R^n$, $A \in R^{n \times n}$, $b \in R^n$ and $x \in R$. Then if there are subspaces W_1 and W_2 of C such that both of them are A-invariant and $W_1 \oplus W_2 = C$, then the system can be described by

$$\left\{ \begin{array}{l} z_1 = A_1\,z_1 + b_1\,x \\ z_2 = A_2\,z_2 + b_2\,x \,. \end{array} \right.$$

And the following diagram commutes ;

$$\begin{tikzcd} & C \arrow[r, "A"] \arrow[d, "e"'] & C \arrow[d, "e"] \\ X \arrow[ur, "b"] \arrow[r, "b_1 \times b_2"'] & C/W_1 \times C/W_2 \arrow[r, "A_1 \times A_2"'] & C/W_1 \times C/W_2 \end{tikzcd}$$

, where $e: C \to C/W_1 \times C/W_2$ is defined by $e(c)=([c]\equiv_{W_1}, [c]\equiv_{W_2})$.
The above is called the modal decomposition in Linear Systems Theory.
This fact can also be obtained as a corollary of Theorem 9.1.1.

Let $\phi_{tt'}: C \times X_{tt'} \to C$ be defined by

$$\phi_{tt'}(c_t, x_{tt'}) = \exp(A(t'-t))c_t + \int_t^{t'} \exp(A(t'-\tau)) b x_{tt'}\, d\tau\,,$$

then the above system is a transition system $T=[\overline{X}, C, \overline{\phi}\,]$ and its algebraic expression is $\underline{C}(\phi)=[C;F_U \cup F_T]$, where $\underline{C}=[C;F_U]$ is a vector space called state space and $F_T = \{\phi_{tt'}(-,x_{tt'}): C \to C \mid x_{tt'} \in X_{tt'} \text{ and } t \leqslant t'\}$.
Then by the theorem, if there is a full and separating class $\underline{\Theta} = \{\Theta_1, \Theta_2\}$ of congruence relations on $\underline{C}(\phi)$, $\underline{C}(\phi)$ is parallel decomposable.
Suppose that there are A-invariant subspaces W_1 and W_2 of C such that $W_1 \oplus W_2 = C$. Since W_i is A-invariant, $c \in W_i$ implies $\phi_{tt'}(c, x_{tt'}) \in W_i$ for all $x_{tt'} \in X_{tt'}, t \leqslant t'$. Because $\phi_{tt'}(c, x_{tt'})$ $= \exp(A(t'-t))c + \int_t^{t'} \exp(A(t'-\tau)) b x_{tt'}(\tau)\, d\tau$

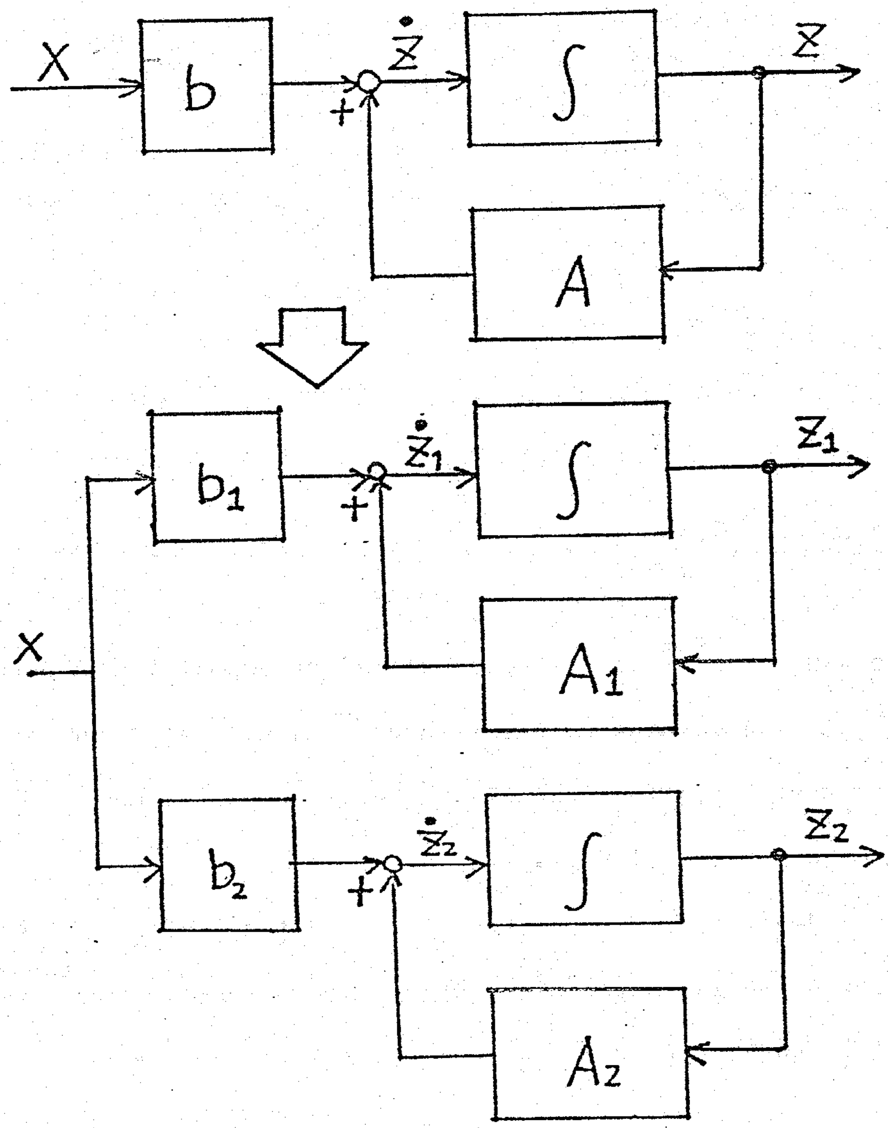


Fig. 9.1.2 Modal Decomposition

$= \sum_{i=1}^{n} A^i \phi_i(t'-t)c + \sum_{i=1}^{n} A^i b \int_t^{t'} \phi_i'(t'-\tau)x_{tt'}(\tau)d\tau \in W_i$[18].

Therefore W_i is also a subspace of $\underline{C}(\overline{\phi})$. Hence if we define $\Theta_i = \Psi^{-1}(W_i)$, Θ_i is a congruence relation on $\underline{C}(\overline{\phi})$. Since $W_1 \oplus W_2 = C$, $\underline{\Theta}$ is full and separating by Proposition 2.4.2 and Theorem 2.4.3. Therefore all conditions of Theorem 9.1.1 is satisfied. Hence $\underline{C}(\phi)$ is parallel decomposable into $\underline{C}(\overline{\phi}^1) \times \underline{C}(\overline{\phi}^2)$, where $\overline{\phi}^i = \{\phi^i_{tt'}: C/\Theta_i \times X_{tt'} \to C/\Theta_i\}$ and $\phi^i_{tt'}([c]\Theta_i, x_{tt'}) = [\phi_{tt'}(c, x_{tt'})]\Theta_i$ for $i=1,2$.

9.2 Serial Decomposition

Hartmanis-Stearns[13] also studied serial decomposition of an automaton $M=[A,B,C,\delta,\lambda]$. According to their theorem, a necessary and sufficient condition for serial decomposition is the existence of a nontrivial partition π with substitution property on C.

In serial decomposition of a transition system, unfortunately, we cannot use the universal algebraic approach. Because component systems $T_1=[A, C_1, \delta_1]$ and $T_2=[C_1 \times A, C_2, \delta_2]$ cannot be representaed as algebras of same type. It is, however, expected to be treated by the heterogeneous algebraic approach[19],[20],[21].

A necessary and sufficient condition for serial decomposition of a dynamical system was given by Pichler and Ottendoerfer[22]. In their paper, they extended Hartmanis-Stearns's Theorem by using the concept of partition pair, which was originally introduced by Hartmanis-Stearns[13].

In this chapter, we modify the concept of a partition pair as follows.

Definition 9.2.1 Congruence Pair of Transition System

Let $T=[\overline{X}, C, \overline{\phi}]$ be a transition system, where C is an Ω -group. Suppose that Θ_1 and Θ_2 are congruence relations on C. Then the ordered pair (Θ_1, Θ_2) is called a congruence pair of T if

$$(c,c') \varepsilon \Theta_1 \rightarrow (\phi_{tt'}(c,x_{tt'}), \phi_{tt'}(c',x_{tt'})) \varepsilon \Theta_2$$

for all $x_{tt'} \varepsilon X_{tt'}$, $t \leqslant t'$.

It is noted that this condition is exactly similar to a condition that (Θ_1, Θ_2) is quotient functional congruence relation of $\phi_{tt'}(-,x_{tt'})$ $:C \rightarrow C$.

Proposiotion 9.2.1

(Θ, Θ) is a congruence pair of $T=[\overline{X}, C, \overline{\phi}]$ if and only if Θ is a congruence relation on $\underline{C}(\overline{\phi})=[C;F_T]$, where

$F_T= \{\phi_{tt'}(-,x_{tt'}):C \rightarrow C | x_{tt'} \varepsilon X_{tt'}\}$.

Proposition 9.2.2

(Id, Θ) is a congruence pair for any congruence relation Θ on C.

What a state automaton is serial decomposable means as follows.

Definition 9.2.3 Serial Decomposability

Let $\hat{T}=[A, \hat{C}, \hat{\delta}]$ be a state automaton. If there are state automata $T_1=[A, C_1, \delta_1]$ and $T_2=[C_1 \times A, C_2, \delta_2]$ such that there is an isomorphism from $\hat{C}$ to C and the diagram commutes, where $T=[A, C, \delta]$

's th serial connection of T_1 and T_2, then $\hat{T}$ is called to be serial decomposable to the serial connection of T_1 and T_2.

$$\begin{array}{ccc} \hat{C} \times A & \xrightarrow{\hat{\delta}} & \hat{C} \\ h \downarrow \ \downarrow Id & & \downarrow h \\ C \times A & \xrightarrow[\delta]{} & C \end{array}$$

The following theorem is similar to that by Pichler and Ottendoerfer[22].

Theorem 9.2.1

A state automaton $\hat{T}=[A,\hat{C}, \hat{\delta}]$ is serial decomposable if and only if there is a full and separating class $\underline{\Theta} = \{ \Theta_1, \Theta_2 \}$ of congruence relations on $\hat{C}$ such that Θ_1 is a congruence relation on $\underline{C}(\delta)=[C; \{ \delta_a : C \to C | a \in A \}]$, where $\delta_a : C \to C$ is defined by $\delta_a(c) = \delta(c,a)$.

Example 9.2.1 : Let us consider the following serial decomposition ; "Let $T=[A,C, \delta]$ be a state automaton and C,G groups, $\phi : C \to G$ a homomorphism satisfying

$$\phi(c) = \phi(c') \to \phi(\delta(c,a)) = \phi(\delta(c',a)) \qquad \text{for } \forall a \in A.$$

Then T can be decomposed into the serial connection of state automata $T_1=[A, C/E_\phi, \delta_1]$ and $T_2=[C/E_\phi \times A, H_\phi, \delta_2]$, where $E_\phi = \{ (a,b) | \phi(a) = \phi(b) \}$ and $H_\phi = \{ a | \phi(a)=0 \}$."

In this decomposition, the resultant mapping $R: C/E_\phi \times H_\phi \to C$ is not an isomorphism but a surjective mapping. By Theorem 9.2.1, if $\underline{\Theta} = \{ \Theta_1, \Theta_2 \}$ is full and separating and Θ_1 is a congruence relation on $\underline{C}(\delta)$, T can be decomposed into the serial connection of $T_1=[A, C/\Theta_1, \delta_1]$

and $T_2=[C/\Theta_1 \times A, C/\Theta_2, \delta_2]$. It is noted that E_ϕ is a congruence relation on $\underline{C}(\delta)$. Since if $\underline{\Theta}$ satisfies $C/\Theta_1 \cong \Psi(\Theta 2)$,the condition is satisfied, we require that the normal subgroup H_ϕ has a congruence relation Θ_2 such that $C/\Theta_2 \cong H_\phi$, we can construct an isomorphism $h: C/E_\phi \times H_\phi \to C$.

Part III : Property Reflecting Decomposition

10. Property Reflecting Decomposition[23)]

In property reflecting decomposition, we investigate what properties can be reflected under a certain morphism. In this chapter, we pay attention to one of property reflecting decomposition problems and illustrate it with a simple example from the Artificial Intelligence area. First, we show a necessary and sufficient condition for decomposition of a global system into a disjunctive complex system form, which is essentially equal to a product of graphs. Next, we show that the value of Sprague-Grundy Function of a global system can be calculated by those of component systems of disjunctive complex system form when there is a strong modelling morphism from the global system into the disjunctive complex system form. Finally, we illustrate the meaning of our theorem by using a typical example from the Artificial Intelligence area[24)].

10.1 Disjunctive Complex System

Let us first define a disjunctive complex system.

<u>Definition 10.1.1</u> Disjunctive Complex System

Let $\underline{S}=\{ S_i \subset X_i \times Y_i | i \in I, X_i=Y_i \text{ and } Id_{Xi} \subsetneq S_i \}$ be a class of component systems. Let a subset $S' \subset \Pi \underline{S}$ be defined by

$$([x_i | i \in I],[y_i | i \in I]) \in S'$$
$$\leftrightarrow (\exists! i \in I)(x_i \neq y_i \text{ and } (\forall j \in I)(j \neq i \rightarrow x_j=y_j)).$$

Then S' is called the disjunctive complex system over $\underline{S}$ and denoted by DC($\underline{S}$).

In this chapter, we use a strong modelling morphism instead of modelling morphism. It is defined as follows.

Definition 10.1.2 Strong Modelling Morphism

Let $S \subset X \times Y$ and $S' \subset X' \times Y'$ be input-output systems. Suppose that there are mappings $h_x : X \to X'$ and $h_y : Y \to Y'$. If the following diagram commutes ;

$$\begin{array}{ccc} X & \xrightarrow{S} & P(Y) \\ h_x \downarrow & & \downarrow h_y^* \\ X' & \xrightarrow[S']{} & P(Y') \end{array}$$

, $h=(h_x,h_y)$ is called a strong modelling morphism from S to S', where P(Y) and P(Y') denote the power set of Y and Y', respectively, and

$S:X \to P(Y)$; $S(x)= \{ y|(x,y) \in S \}$

$S':X' \to P(Y')$; $S'(x')= \{ y'|(x',y') \in S' \}$

$h_y^*:P(Y) \to P(Y')$ is defined by $h_y^*(Y'')= \{ h_y(y)|y \in Y'' \}$ for $Y'' \in P(Y)$. If h_x and h_y are injective, h is called an injective strong modelling morphism. And if h is also a surjective modelling morphism, it is called a surjective strong modelling morphism.

Proposition 10.1.1

$h=(h_x,h_y)$ is a strong modelling morphism from $S \subset X \times Y$ to $S' \subset X' \times Y'$ if and only if h is a modellig morphism from S to S' and satisfies the following condition ;

$$(h_x(x),y') \in S' \to (\exists y)((x,y) \in S \text{ and } h_y(y)=y')$$

That is, a strong modelling morphism is a modelling morphism.

Lemma 10.1.1

Let $h=(h_x,h_y)$ be a strong modelling morphism from $S \subset X \times Y$ to $S' \subset X' \times Y'$ and let $\tilde{S} \subset \tilde{X} \times \tilde{Y}$ be an arbitrary subsystem of S, that is $\tilde{S} \subset S$. If $S|\tilde{X}=\tilde{S}$, then $h|\tilde{S}$ is also a strong modelling morphism, where $S|\tilde{X} = \{ (x,y) \in S | x \in \tilde{X} \}$ and $h|\tilde{S}=(h_x|\tilde{X},h_y|\tilde{Y})$.

Lemma 10.1.2

Let $S \subset X \times Y$ be the disjunctive complex system over $\underline{S}= \{ S_i \subset X_i \times Y_i | i \in I \}$. Then $X= \Pi (X_i | i \in I)= \Pi (Y_i | i \in I)=Y$. Moreover, the i-th projection $p_i=(p_{ix},p_{iy}):S \to S_i$ is a surjective strong modelling morphism for all $i \in I$.

Let us now consider a necessary and sufficient condition for decomposition of a global system into a disjunctive complex system under a strong modelling morphism.

Lemma 10.1.3

Let $S \subset X \times Y$ and $S' \subset X' \times Y'$ be the disjunctive complex systems over $\underline{S}= \{ S_i \subset X_i \times Y_i | i \in I \}$ and $\underline{S}'= \{ S_i' \subset X_i' \times Y_i' | i \in I \}$, respectively. Suppose that $h_i=(h_{ix},h_{iy}):S_i \to S_i'$ is a modelling morphism satisfying $h_{ix}=h_{iy}$ for all $i \in I$. Then $\Pi (h_i | i \in I)|S$ is an injective strong modelling morphism if and only if so is h_i for all $i \in I$, where $\Pi (h_i | i \in I)|S([x_i | i \in I],[y_i | i \in I])=([h_{ix}(x_i) | i \in I],[h_{iy}(y_i) | i \in I])$.

Lemma 10.1.4

Let $\underline{S/R}$= { $S/R^i | i \in I$ } be a class of quotient systems of $S \subset X \times Y$, where X=Y. Suppose that $S/\underline{R} = DC(\underline{S/R})|\underline{D}(S/\underline{R})$. Then the natural modelling morphism $1=(1_x,1_y):S \to S/\underline{R}$ is a strong modelling morphism if and only if the following condition holds ;

(P1) if $(x,y) \in S$ and $[x]R_x{}^i \neq [y]R_y{}^i$, then

$$(\forall x' \in X)([x]R_x{}^i=[x']R_x{}^i \to (\exists y')((x',y') \in S \text{ and } [y]R_y{}^i=[y']R_y{}^i))$$

for all $i \in I$.

Lemma 10.1.5

Let $\underline{S/R}$ = { $S/R^i | i \in I$ } be a class of quotient systems of $S \subset X \times Y$, where X=Y. Suppose that $S/\underline{R} = DC(\underline{S/R})|\underline{D}(S/\underline{R})$. Then the inclusion from $S/\underline{R}$ to $DC(\underline{S/R})$ is a strong modelling morphism.

Then,

Theorem 10.1.1

Let $\hat{S} \subset \hat{X} \times \hat{Y}$ be a global system satisfying $\hat{X}=\hat{Y}$. And let $\underline{S}$ = { $S_i \subset X_i \times Y_i | i \in I$ } be a class of component systems. Then there exists a strong modelling morphism $h=(h_x,h_y)$ from $\hat{S}$ to the disjunctive complex system $DC(\underline{S})$ over $\underline{S}$ satisfying $h_x=h_y$ if and only if there is a class of quotient systems $\hat{S}/\underline{R}$= { $\hat{S}/R^i | i \in I$ } such that the following conditions hold ;

1) The canonical complex system $\hat{S}/\underline{R}$ is equal to the restriction of the disjunctive complex system over $\underline{\hat{S}/R}$ to the domain of the disjunctive complex system over $\underline{\hat{S}/R}$ to the domain of it, that is, $\hat{S}/\underline{R}$ $= DC(\hat{S}/\underline{R})|\underline{D}(\hat{S}/\underline{R})$.

2) (P1)

3) There is an injective strong modelling morphism $k_i=(k_{ix},k_{iy})$ from $\hat{S}/R^i$ to S_i for all $i \in I$, where $k_{ix}=k_{iy}$.

Let us consider the sufficiency of the conditions. From the condition 2), the natural modelling morphism $l=(l_x,l_y):\hat{S} \to \hat{S}/\underline{R}$ is a strong modelling morphism. And the inclusion $i=(i_x,i_y):\hat{S}/\underline{R} \to DC(\underline{\hat{S}/R})$ is also a strong modelling morphism from the condition 1). From the condition 3) and Lemma 10.1.3, $\Pi\ (k_i|i \in I)|DC(\underline{\hat{S}/R}):DC(\underline{\hat{S}/R}) \to DC(\underline{S})$ is also a strong modelling morphism. Hence ($\Pi\ (k_i|i \in I) \cdot i \cdot l$ is a strong modelling morphism. These conditions are also necessary for the decomposition.

10.2 Relation between Decomposition to Disjunctive Complex System and Sprague-Grundy Function

In Section 10.1, we have shown a necessary and sufficient condition for decomposition of a global system into a disjunctive complex system by a strong modelling morphism. The rreason why the decomposition is important stems from the fact that when we recognize a global system $\hat{S}$ as a graph, we can calculate the value of Sprague-Grundy Function of $\hat{S}$, which determines the kernel of $\hat{S}$, by those of component systems of the disjunctive complex system.

In the reference[24], the similar fact has been proved in the algebraic way. However, we cannot directly apply it to our case because our formulation is different from the reference. We will modify the Sprague-Grundy Function so as to make it compatible with our formulation and will show that the same fact is also true in our formulation.

<u>Definition 10.2.1</u> Sprague-Grundy Function (SGF)

Let $S \subset X \times Y$ be an input-output system satisfying that $X \cap Y \neq \phi$.
If a mapping $g:X \cup Y \to N$ satisfies the following condition, we call g a Sprague-Grundy Function (briefly, SGF) of S, where $N = \{ 0,1,\ldots,n,\ldots \}$.

1) $(x,y) \in S$ and $x \neq y \to g(x) \neq g(y)$

2) $m < g(x) \to (\exists y)((x,y) \in S$ and $g(y)=m)$

The above definition is essentially same as that of the reference [24].

<u>Proposition 10.2.1</u>

SGF g of S is uniquely determined if S is progressively bounded.

The relation between SGF and a strong modelling morphism is given by the following Lemma.

<u>Lemma 10.2.1</u>

Let $S \subset X \times Y$ and $S' \subset X' \times Y'$ be input-output systems satisfying that $X \cap Y \neq \phi$ and $X' \cap Y' \neq \phi$. And let $h=(h_x,h_y):S \to S'$ a strong modelling morphism satisfying that

1) $h_x|X \cap Y=h_y|X \cap Y$

2) $(x,y) \in S$ and $x \neq y \to h_x(x) \neq h_y(y)$.

Suppose that g' is the SGF of S'. Then the SGF g of S is given by

$$g:X \cup Y \to N \; ; \; x \to \begin{cases} g'h_x(x) & x \in X \\ g'h_y(y) & y \in Y \; . \end{cases}$$

In the reference [24], the similar fact is proved by using a "D-morphism". A strong modelling morphism can be considered as a systems theoretical version of a D-morphism.

Let us next consider a relation between the value of SGF of a disjunctive complex system and those of component systems. The following is a modification of the similar result of the reference [24].

Lemma 10.2.2

Let $S=DC(\underline{S})$ be the disjunctive complex system over $\underline{S}= \{ S_i | i \in I \}$. Suppose that SGF g_i of S_i is given for all $i \in I$. Then the SGF g of $S \subset X \times Y$ is given by

$$g : X \cup Y \rightarrow N ;$$

$$[x_i | i \in I] \rightarrow \bigoplus_{i \in I} g_i(x_i) = g_1(x_1) \oplus g_2(x_2) \oplus \cdots \oplus g_n(x_n) \oplus \cdots$$

, where $\oplus$ is a binary operation on N defined by

$$c = a \oplus b \leftrightarrow c = \sum_1^m 2^i c_i, \text{ where } a = \sum_1^m 2^i a_i \text{ and } b = \sum_1^m 2^i b_i$$

$$c_i = 0 \leftrightarrow a_i = b_i,\ a_i, b_i \text{ and } c_i \in \{ 0,1 \} .$$

Let us now prove that the value of SGF of a global system $\hat{S}$ can be calculated by using those of component systems S_i if there is a strong modelling from $\hat{S}$ to the disjunctive complex system $DC(\underline{S})$, where $\underline{S}= \{ S_i | i \in I \}$.

Theorem 10.2.1

Suppose that a global system $\hat{S} \subset \hat{X} \times \hat{Y}$ which satisfies $\hat{X}=\hat{Y}$ is decomposed to a disjunctive complex system $S=DC(\underline{S})$ over $\underline{S}= \{ S_i | i \varepsilon I \}$ by a strong modelling morphism. Then if a subsystem $\tilde{S} \subset \tilde{X} \times \tilde{Y}$ of S satisfies that $\hat{S}|\tilde{X}=\tilde{S}$, the SGF $\tilde{g}$ of $\tilde{S}$ is given by

$$\tilde{g}:\tilde{X} \cup \tilde{Y} \to N$$

$$x \to \begin{cases} \bigoplus_{i \varepsilon I} g_i p_{ix} h_x(x) & x \varepsilon \tilde{X} \\ \bigoplus_{i \varepsilon I} g_i p_{iy} h_y(x) & x \varepsilon \tilde{Y} \end{cases}$$

, where $h=(h_x,h_y):\hat{S} \to S$ is a strong modelling morphism and $h_x=h_y$, and $p_i=(p_{ix},p_{iy}):S \to S_i$ is the i-th projection of S.

An outline of the proof of this theorem is as follows ; By Lemma 10.2.2, SGF g of $DC(\underline{S})$ can be defined by those of S_i as the binary sum of g_i and SGF $\hat{g}$ of $\hat{S}$ can be defined by the compositioin of h and g by Lemma 10.2.1.. Since $h|\tilde{S}$ is also a strong modelling morphism by Lemma 10.1.1, SGF $\tilde{g}$ of $\tilde{S}$ can be defined as the above form.

10.3 Example

Let us now illustrate the meaning of Theorem 10.2.1 by using the "Stair Case" game which is also used in the reference [24].

This game is a two-person game. Initially some sticks are placed on stairs. A player who finally brings down the sticks onto the lowest stair is the winner. Following the rule, one can bring any number of sticks from exactly one stair down to the next lower stair.

Let us consider the state transition of this game as a global system $\hat{S}$. Then $\underline{D}(\hat{S})=\underline{R}(\hat{S})$ is the set of state and $\hat{S} \subset X \times Y$ is defined by

$$([x_k|k \in N],[y_k|k \in N]) \in \hat{S}$$

$$\leftrightarrow \quad (\exists! k_o \in N)(x_{ko} + x_{ko-1} = y_{ko} + y_{ko-1} \text{ and } x_{ko} \geq y_{ko}$$

$$\text{and } (\forall k \in N)(k \neq k_o \text{ and } k \neq k_{o-1} \rightarrow x_k=y_k)).$$

This definition states that a move of this game is to bring some sticks on the k_o-stair down the (k_o-1)-stair for some $k_0 \in N$.

Let us now pay attention to the even stairs and represent its move by $S_{2k} = N \times N$. That is, any pair of non-negative integers is assumed to be possible for the move on each even stair. Let us now construct the disjunctive complex system over $\underline{S}= \{ S_{2k}|k \in N \}$. Let us define an equivalence relation E_i on $\hat{X} =\hat{Y}$ by

$$[x_k|k \in N] \; E_i \; [x_k'|k \in N] \leftrightarrow x_i=x_i'$$

and $\hat{E}_i$ on $\hat{S}$ by

$$([x_k|k \in N],[y_k|k \in N]) \; \hat{E}_i \; ([x_k'|k \in N],[y_k'|k \in N])$$

$$\leftrightarrow \; [x_k|k \in N] \; E_i \; [x_k'|k \in N] \text{ and } [y_k|k \in N] \; E_i \; [y_k'|k \in N].$$

Then $\hat{E}_i$ is an input-output compatible equivalence relation for $i \in N$ and the quotient system is given by

$$\hat{S}/E_{2k}= \{ ([[x_k|k \in N]]E_{2k},[[y_k|k \in N]]E_{2k})|$$

$$([[x_k|k \in N]]E_{2k} \times [[y_k|k \in N]]E_{2k}) \cap \hat{S} \neq \phi \}.$$

Then it can be easily shown that the conditions of Theorem 10.1.1 are satisfied and hence $\hat{S}$ can be decomposed into the disjunctive complex system $S=DC(\underline{S})$ by a strong modelling morphism.

Let us now calculate the value of SGF of a simple example. Suppose that there are four stairs and initially one stick is on each even stair. Let us represent the initial state by (0101). Then the tree of this game is as shown in Fig.10.3.1 and $\tilde{S} \subset \tilde{X} \times \tilde{Y}$ satisfies $\tilde{S}=\hat{S}|X$. Let $h=(h_x,h_y)$ $:\hat{S} \to S$ be a strong modelling morphism whose existence is ensured by the above discussion and $p_i=(p_{ix},p_{iy}):S \to S_i$ is the i-th projection. Suppose that SGF g_{2k} of S_{2k} is given for i=1,2 (Fig.10.3.2). Then the value of SGF $\tilde{g}$ of $\tilde{S}$ can be calculated by

$$\tilde{g}(x) = \begin{cases} g_2p_{2x}h_x(x) \oplus g_4p_{4x}h_x(x) & x \in \tilde{X} \\ g_2p_{2y}h_y(y) \oplus g_4p_{4y}h_y(y) & y \in \tilde{Y} \end{cases}$$

from Theorem 10.2.1. k in Fig.10.3.1 represents the value of SGF. Concerning with the practical meaning of SGF, the reference [24] should be consulted.

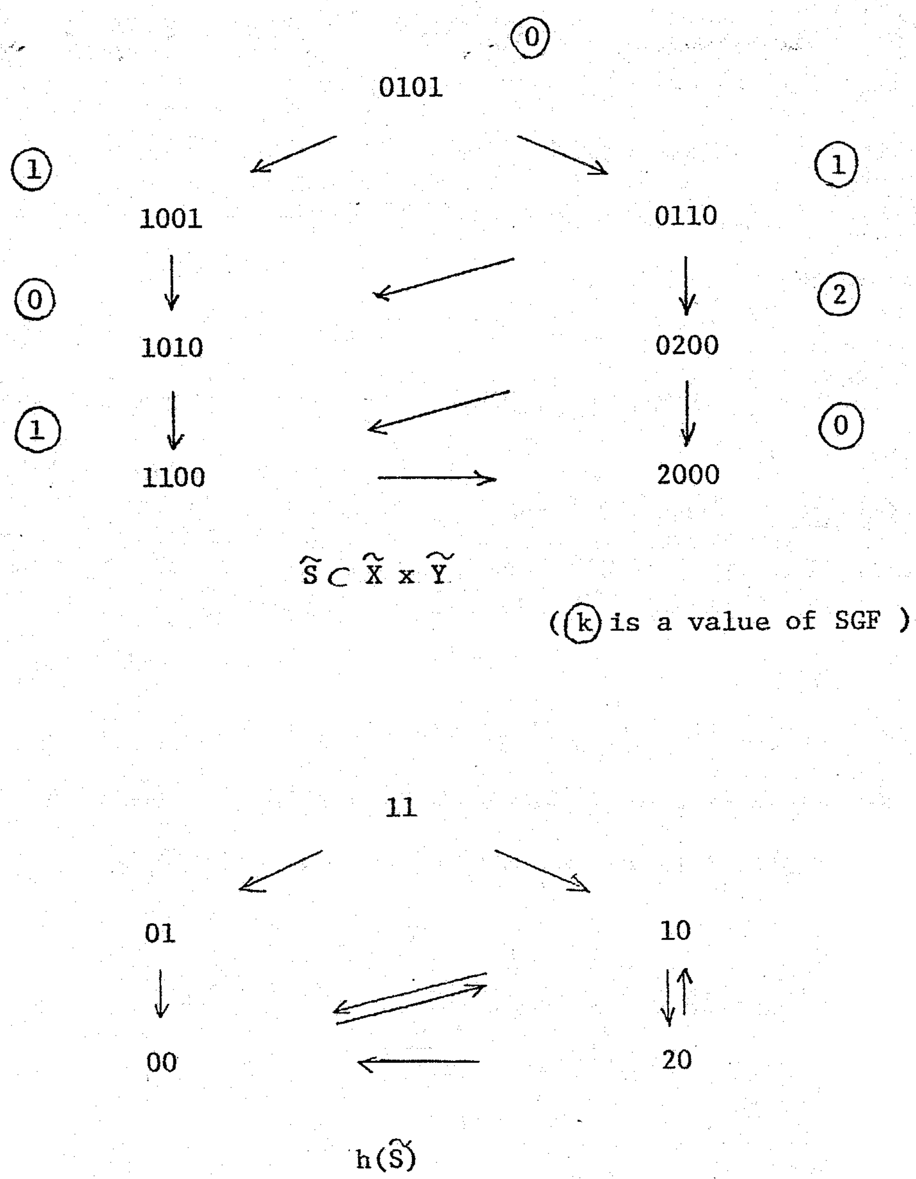
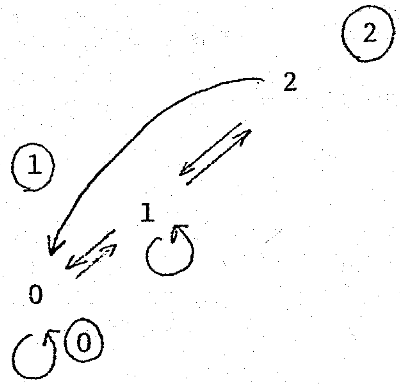
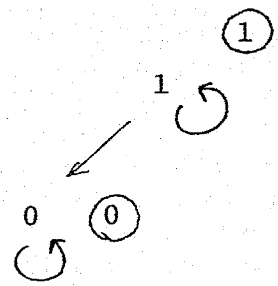


Fig. 10.3.1 $\tilde{S}$ and $h(\tilde{S})$



$p_2 h(\widetilde{S})$



$p_4 h(\widetilde{S})$

(k is a value of each SGF g_i)

Fig. 10.3.2 $p_{2k} h(\widetilde{S})$ (k=1,2) and Its SGF

Part IV : Conclusions

11. Conclusions

Decomposition is a useful tool for investigating a complicated system. After decomposition, we can get some information on a property of a global system by those of component systems.

In this dissertation, we have constructed the mathematical basis for algebraic decomposition theory. In decomposition theory, the following problems are fundamental.

1) Characterization of Interactions ;

2) Decomposition Problem ;

3) Property Preserving or Reflecting Decomposition.

We have provided the foundations to solve the above problems.

Systems of our concern in decomposition were general input-output systems, functional systems and transition systems. Since the essential point of decomposition is whether or not we can find a class of congruence relations with a disirable property. It was clarified that universal algebraic approach is useful for this purpose.

In Part 0, we have provided preliminaries for this dissertation. An input-output system, a functional system and a transition system were precisely defined and some universal algebraic notions were introduced.

The first problem was investigated in Part I. It was clarified that the concepts of interactions such as the system interaction and the process interaction are enough for characterization of interactions. It seems, however, that other concepts must be introduced in order to distinguish hierarchical systems from non-hierarchical one.

Primary types of connections which are well-known in systems theory were also introduced in Part I. It seems that these connections are enough when we consider connections of component systems of a non-hierarchical system. Intuitively, a complex system expressed by a signal flow diagram can be decomposed into these connections of component systems.

The second problem was investigated in Part II. We mainly consider the case where a global system has a complex system in modelling (inductive modelling). By using universal algebraic approach, we introduced the concept of quotient systems. The essencial point of decomposition is, in other words, whether or not it is possible to find a class of quotient systems of a global system by which we can construct a complex system with a given type. Though cascade and feedback decomposition of an input-output system was also investigated, it must be further developed in future.

Other forms of decomposition of an input-output system such as deductive modelling and simulation were also considered in Part I.

For decomposition of a functional system, we mainly considered the case where a functional system is an endomorphism of an Ω -group. In decomposition of a transition system, the main idea is that a transition system can be expressed as an Ω -group. In contrast to decomposition of an endomorphism, decomposition of a transition system is considered as that of a Ω -group. It is, however, difficult to solve the serial decomposition problems because component systems are algebras with different types in this case. It seems that heterogeneous algebraic approach[19),20),21)] is useful for this type of decomposition.

The third problem was discussed in Part III. We only investigated one of property reflecting decomposition. The type of a complex system considered in this part is not so special and the method how to calculate SGF of a global system by those of component systems can be extended to other property reflecting decomposition.

Whether or not a systemic property such as stability, controllability can be preserved or reflected in decomposition must be investigated in future. It seems that model theoretic approach is useful for this problem.

Consequently, the following problems remain to be solved.

1) Formalize a hierarchical system and a non-hierarchical one in the framework developed here.

2) According to 1), show that primary types of connections such as parallel, cascade and feedback connections are enough to realize a non-hierarchical complex system.

3) Develope a method for cascade and feedback decomposition of a transition system by using heterogeneous algebraic approach.

4) Investigate property preserving and reflecting problems for systemic properties using model theoretic approach.

Adding that other types of decomposition problems in systems theory must be of course investigated and formalized in the framework developed here.

APPENDIX

1. Proofs for Chapter 2

Proof of Proposition 2.2.1 : Since $X \subset \Pi (X_i | i \in I)$ and $Y \subset \Pi (Y_i | i \in I)$, let $p_{ix}: X \to X_i$ and $p_{iy}: Y \to Y_i$ be the i-th projection on X and on Y, respectively. Then $p_i' = (p_{ix}, p_{iy}): S \to S_i$ is a modelling morphism for all $i \in I$ and moreover p_i' is a surjective modelling morphism. Because $p_i'(([x_i | i \in I], [y_i | i \in I])) = (p_{ix}([x_i | i \in I]), p_{iy}([y_i | i \in I])) = (x_i, y_i) = p_i([s_i | i \in I])$ and p_i is a surjective map from S to S_i by Definition 2.2.2, where $s_i = (x_i, y_i)$ for all $i \in I$.

Q.E.D.

Proof of Proposition 2.2.2 : See the reference [1].

Proof of Proposition 2.2.3, 2.2.4 : See the reference [3], [7].

Proof of Proposition 2.2.5 : See the reference [7], [8].

Proof of Proposition 2.2.6 : It is enough to check that the composition is well-defined. Let $h^S = (h_x, h_y) \in \mathrm{Hom}(S, S')$ and $h^{S'} = (h_x', h_y') \in \mathrm{Hom}(S', S'')$. Then $h^{S'} \circ h^S = (h_x' \cdot h_x, h_y \cdot h_y')$ by the definition. For any $(x,y) \in S$, there is $y' \in R(S') = Y'$ such that $(h_x(x), y') \in S'$ and $h_y(y') = y$. And hence there is $y'' \in R(S'') = S''$ such that $(h_x'(h_x(x)), y'') = (h_x' \cdot h_x(x), y'') \in S''$ and $h_y'(y'') = y'$. Because $h^{S'}$ is also a simulation morphism. Since $h_y(y') = y$, $h_y \cdot h_y'(y'') = h_y(y') = y$. Therefore $h^{S'} \circ h^S$ is a simulation morphism.

Hence $\underline{SIM}$ is a category.

Q.E.D.

Proof of Theorem 2.4.1 : For any $\phi \in End(\underline{A})$, Θ is uniquely determined by Ker. Therefore it is enough to show that $\Theta \in Con(\underline{A})$. It is easily seen that Θ is an equivalence relation. For any $f_\gamma \in F$, any $(a_o,b_o),\ldots,(a_{n\gamma-1},b_{n\gamma-1}) \in \Theta$, $\phi(f_\gamma(a_o,\ldots,a_{n\gamma-1}))=f_\gamma(\phi(a_o),\ldots,\phi(a_{n\gamma-1}))$ $=f_\gamma(\phi(b_0),\ldots,\phi(b_{n\gamma-1}))=\phi(f_\gamma(b_o,\ldots,b_{n\gamma-1}))$.
Therefore $(f_\gamma(a_o,\ldots,a_{n\gamma-1}),f_\gamma(b_o,\ldots,b_{n\gamma-1})) \in \Theta$.
Hence Θ is a congruence relation and so ϕ is a mapping.

Q.E.D.

Proof of Proposition 2.4.1 : It is obvious that $NSub(\underline{A}) \subset Sub(\underline{A})$ because every Ω-normal subgroup is an Ω-subgroup. Let $\underline{B}$ be a. Ω-subgroup of $\underline{A}$. For any $b \in B$, any $a \in A$, if the binary operation is commutative, $a+b+(-a)=a+(-a)+b=b \in B$. Therefore $\underline{B}$ is an Ω-normal subgroup. Hence $NSub(\underline{A}) = Sub(\underline{A})$ if it is the case.

Q.E.D.

Proof of Theorem 2.4.2 : It is easily seen that $\Psi(\Theta)$ is closed under F_G for all $\Theta \in Con(\underline{A})$. For any $f_\lambda \in F_U$, any $c=a-b \in \Psi(\Theta)$, $f_\lambda(c)=f_\lambda(a-b)=f_\lambda(a)-f_\lambda(b)$. Since $f_\lambda \in End(\underline{A_G})$, $(f_\lambda(a),f_\lambda(b)) \in \Theta$. Then $f_\lambda(c) \in \Psi(\Theta)$. For any $c=a-b$ $\Psi(\Theta)$, any $\hat{a} \in A$, $\hat{a}+c+(-\hat{a})=\hat{a}+a-b-\hat{a}=\hat{a}+a-(\hat{a}+b)$. Since $(\hat{a},\hat{a}) \in \Theta$ and $(a,b) \in \Theta$, $(\hat{a}+a,\hat{a}+b) \in \Theta$. Then $\hat{a}+c+(-\hat{a}) \in \Psi(\Theta)$. Therefore $\Psi(\Theta) \in Sub(\underline{A})$ and Ψ is well-defined.

Let Θ , $\Theta' \varepsilon \text{Con}(\underline{A})$ be $\Theta \neq \Theta'$. Then we can suppose that there exists $(a,b) \varepsilon A \times A$ such that $(a,b) \,\bar{\varepsilon}\, \Theta$ and $(a,b) \varepsilon \Theta'$ without loss of generality. If $a-b \varepsilon \Psi(\Theta)$, there is $(a',b') \varepsilon \Theta$ such that $a'-b'= a-b$. Since $(a,b)=(a+(-a'+a'),b+(-b'+b'))=((a-a')+a',(b-b')+b')$ and $(a-a',b-b') \varepsilon \Theta$, $(a,b) \varepsilon \Theta$, because Θ is a congruence relation. It contradicts the assumption. Therefore $a-b \varepsilon \Psi(\Theta)$ and Ψ is injective.

For any $\underline{B} \varepsilon \text{Sub}(\underline{A})$, let Θ be defined by $(a,b) \varepsilon \Theta \leftrightarrow a-b \varepsilon B$. Then it is easily seen that $\Psi(\Theta) = \underline{B}$. Hence Ψ is bijective.

Q.E.D.

<u>Proof of Proposition 2.4.2</u> : Suppose that there are two different decomposition of $a \varepsilon A$ such that $a=b_1+b_2=b_1'+b_2'$, where $b_1, b_1' \varepsilon B_1$ and $b_2,b_2' \varepsilon B_2$. Then $0=a-a=(b_1+b_2)-(b_1'+b_2')=(b_1-b_1')+(b_2-b_2') \varepsilon B_1+B_2$. Therefore b_2-b_2' is the inverse of (b_1-b_1') and hence $b_2-b_2'=-(b_1-b_1') \varepsilon B_1$. Then $b_2-b_2' \varepsilon B_1 \cap B_2$ and therefore $b_2=b_2'$. Similarly, $b_1=b_1'$. Hence a is uniquely decomposed. Q.E.D.

<u>Proof of Theorem 2.4.3</u> : Let that $\Psi^{-1}(\underline{B}_1)= \Theta_1$, $\Psi^{-1}(\underline{B}_2)= \Theta_2$ and $(\underline{B}_1 ,\underline{B}_2) \varepsilon \perp (\underline{A})$. For any $a,a' \varepsilon A$, there are $b_1,b_1' \varepsilon B_1$ and $b_2,b_2' \varepsilon B_2$ such that $a=b_1+b_2$ and $a=b_1'+b_2'$. Let $c=b_1'+b_2$. Then $c-a=(b_1'+b_2)-(b_1+b_2) =b_1'-b_1 \varepsilon B_1$. Therefore $(a,c) \varepsilon \Theta_1$. Similarly, $(c,a') \varepsilon \Theta_2$. Hence $(a,a') \varepsilon \Theta_1 \circ \Theta_2$, then $\underline{\Theta} = \{ \Theta_1, \Theta_2 \}$ is full. Let $(a,b) \varepsilon \Theta 1 \cap \Theta 2$. Then $b-a \varepsilon B_1 \cap B_2$. Hence $a=b$ and therefore $\underline{\Theta}$ is separating.

Conversely, suppose that $\underline{\Theta} = \{ \Theta_1, \Theta_2 \}$ is full and separating. Since $\underline{B_1}= \Psi (\Theta_1)$ and $\underline{B_2}= \Psi (\Theta_2)$ are normal subalgebras of $\underline{A}$, $0 \varepsilon B_1 \cap B_2$. If there is $a \varepsilon B_1 \cap B_2$, $(a,0) \varepsilon \Theta_1 \cap \Theta_2$. Since $\underline{\Theta}$ is separating, $a=0$. Hence $B_1 \cap B_2= \{ 0 \}$. For any $a \varepsilon A$, let $b_1 \varepsilon [a] \Theta_2$

$\cap$ $[0]\,\Theta_1$ and $b_2 \in [a]\,\Theta_1 \cap [0]\,\Theta_2$. Since $\underline{\Theta}$ is full and separating, such b_1 and b_2 exist. Then $b_1 \in B_1$, $b_2 \in B_2$, $a-(b_1+b_2)=(a-b_1)+b_2 \in B_2$ and $a-(b_1+b_2)=(a-b_2)+b_1 \in B_1$. Therefore $a-(b_1+b_2) \in B_1 \cap B_2$ and hence $a=b_1+b_2$.

Q.E.D.

<u>Proof of Theorem 2.4.4</u> : Let $\phi \in End(\underline{A})$. Then it is easily seen that $Ker^G(\phi)$ is closed under F_G. For any $f_\lambda \in F_U$, any $a \in Ker^G(\phi)$, $\phi(f_\lambda(a))=f_\lambda(\phi(a))=f_\lambda(0)=0$, because $f_\lambda \in End(\underline{A_G})$. Therefore $Ker^G(\phi)$ is a subalgebra of $\underline{A}$. For any $b \in Ker^G(\phi)$, any $a \in A$, $\phi(a+b+(-a)) = \phi(a)+\phi(b)-\phi(a)=\phi(a)-\phi(a)=0$. Therefore $Ker^G(\phi)$ is a normal subalgebra of $\underline{A}$ and Ker^G is a mapping of $End(\underline{A})$ to $NSub(\underline{A})$.

Q.E.D.

<u>Proof of Theorem 2.4.5</u> : For any $\phi \in End(\underline{A})$, $\Psi\,Ker(\phi)=\{a-b \mid (a,b) \in Ker\,\phi\} = \{a-b \mid \phi(a)=\phi(b)\}$, while $Ker^G(\phi)=\{a \mid \phi(a)=0\}$. Since for any $a-b \in \Psi\,Ker(\phi)$, $\phi(a-b)=\phi(a)-\phi(b)=0$, therefore $a-b \in Ker^G(\phi)$ and $\Psi\,Ker(\phi) \subset Ker^G(\phi)$. And for any $a \in Ker^G\,\phi$, $\phi(a)=0=\phi(0)$. Then $a=a-0 \in \Psi\,Ker(\phi)$. Then $Ker^G\,\phi \subset \Psi\,Ker\,\phi$. Hence $Ker^G\,\phi = \Psi\,Ker\,\phi$.

Q.E.D.

<u>Proof of Theorem 2.5.1</u> : See the reference [9].

<u>Proof of Theorem 2.5.2</u> : See the reference [10].

2. Proofs for Chapter 3

Proof of Proposition 3.1.1 : Let $\underline{S} = \{ S_1, S_2 \}$, where $S_1 \subset (X_1{*} \times Z) \times Y_1$ and $S_2 \subset (X_2{*} \times Z) \times Y_2$. Let $h=(h_x,h_y)$ be defined by

$$h_x : \underline{D}(P(\underline{S})) \to \underline{D}(EX(P(\underline{S})))$$

$$h_x((x_1,z),(x_2,z))=(x_1,z,x_2)$$

$$h_y : \underline{R}(P(\underline{S})) \to \underline{R}(EX(P(\underline{S}))) \text{ ; identity.}$$

Then obviously, $h=(h_x,h_y)$ is a modelling morphism. Moreover, it is an isomorphism in $\underline{MOD}$. Q.E.D.

Proof of Proposition 3.1.2 : Let $\underline{S} = \{ S_1, S_2 \}$, where $S_1 \subset (X_1{*} \times Z) \times Y_1$ and $S_2 \subset (X_2{*} \times Z) \times Y_2$. Let h_x be defined as in the proof of Proposition 3.1.1 and $h_y : \underline{R}(EX(P(\underline{S}))) \to \underline{R}(P(\underline{S}))$ the identity. Then for any $(((x_1,z),(x_2,z)),(y_1,y_2)) \in P(\underline{S})$, $(y_1,y_2) \in \underline{R}(EX(P(\underline{S})))$ and $(h_x((x_1,z),(x_2,z)),(y_1,y_2)) \in EX(P(\underline{S}))$. Therefore h^s is a simulation morphism from $P(\underline{S})$ to $EX(P(\underline{S}))$.

Q.E.D.

Proof of Proposition 3.1.3 : Let $\underline{S} = \{ S_1, S_2 \}$, where $S_1 \subset X_1 \times (Y_1{*} \times Z)$ and $S_2 \subset (X_2{*} \times Z) \times Y_2$. Let $h=(h_x,h_y)$ be defined by

$$h_x : X_1 \times (X_2{*} \times Z) \to X_1 \times X_2{*} \text{ ; } h_x(x_1,(x_2,z))=(x_1,x_2)$$

and $$h_y : (Y_1{*} \times Z) \times Y_2 \to Y_1{*} \times Y_2 \text{ ; } h_y((y_1,z),y_2)=(y_1,y_2).$$

For any $(((x_1,(y_1,z)),((x_2,z),y_2)) \in C(\underline{S})$, $h((x_1,(x_2,z)),((y_1,z),y_2)) =((x_1,x_2),(y_1,y_2)) \in EX(C(\underline{S}))$.Therefore $h=(h_x,h_y)$ is a modelling morphism from $C(\underline{S})$ to $EX(C(\underline{S}))$. It is easily seen that $h(C(\underline{S}))=EX(C(\underline{S}))$. Hence $EX(C(\underline{S}))$ is a surjective model of $C(\underline{S})$).

Q.E.D.

Proof of Proposition 3.1.4 : Let $\underline{S} = \{ S_1, S_2 \}$, where $S_1 \subset (X^* \times Z_x) \times (Y^* \times Z_y)$ and $S_2 \subset Z_y \times Z_x$. Let $h=(h_x,h_y)$ be defined by

$$h_x:(X^* \times Z_x) \times Z_y \to X^*$$

$$h_x((x,z_x),z_y)=x$$

$$h_y:(Y^* \times Z_y) \times Z_x \to Y^*$$

$$h_y((y,z_y),z_x)=y.$$

For any $(((x,z_x),(y,z_y),(z_y,z_x)) \in F(\underline{S})$, $h(((x,z_x),(y,z_y)),(z_y,z_x)) =(x,y) \in EX(F(\underline{S}))$. Therefore $h=(h_x,h_y)$ is a modelling morphism from $F(\underline{S})$ to $EX(F(\underline{S}))$. It is easily seen that $h(F(\underline{S}))=EX(F(\underline{S}))$. Hence $EX(F(\underline{S}))$ is a surjective model of $F(\underline{S})$.

Q.E.D.

Proof of Proposition 3.2.1 : Let $h=(h_x,h_y):S \to S'$ be a modelling morphism in the sense of Definition 3.2.1. Then for any $(x,y) \in S$, $h_yS(x) =S'h_x(x)$. Therefore $(h_x(x),h_y(S(x)))=(h_x(x),h_y(y)) \in S'$. Hence it is also a modelling morphism from S to S' as between input-output systems.

Conversely, suppose that $h=(h_x,h_y)$ is a modelling morphism from $S \subset X \times Y$ to $S' \subset X' \times Y'$. For any $x \in X$, there is $y \in Y$ such that $(x,y) \in S$. Since $h=(h_x,h_y)$ is a modelling morphism, $(h_x(x),h_y(y)) \in S'$. And since S' is functional, $S'h_x(x)=h_y(y)=h_yS(x)$. Therefore h is also a modelling morphism in the sense of Definition 3.2.1.

Q.E.D.

Proof of Proposition 3.2.2 : Let $\underline{S}=\{S_1,S_2\}$, where $S_1:X \to Y_1$ and $S_2:X \to Y_2$ are functional systems. Then $P_F(\underline{S})=\{(x,(S_1(x),S_2(x)))|x \in X\}$ and $EX(P(\underline{S})) \cong \{(\phi,x,\phi),(S_1(x),S_2(x))|x \in X\}$, where we identify X with $\{\phi\} \times X$. Then $P_F(\underline{S}) \cong EX(P(\underline{S}))$.

Q.E.D.

Proof of Proposition 3.2.3 : Let $\underline{S}$ $\{S_1,S_2\}$, where $S_1:X \to Z$ and $S_2:Z \to Y$ are functional systems. Then $S_1 \cdot S_2=\{(x,S_2S_1(x))|x \in X\}$ and $EX(C(\underline{S}))$ $\cong \{(x,\phi),(\phi,S_2S_1(x))|x \in X\}$, where we identify $X_2{*} \times Z$ and $Y_1{*} \times Z$ with $\{\phi\} \times Z$. Then $S_1 \cdot S_2 \cong EX(C(\underline{S}))$.

Q.E.D.

Proof of Proposition 3.2.4 : Let $\underline{S}=\{S_1,S_2\}$, where $S_1,S_2 \in End(\underline{A})$. Then $S_1+S_2=\{(x,S_1(x)+S_2(x)))|x \in A\}$ and $P_F(\underline{S})=\{(x,(S_1(x),S_2(x)))|x \in A\}$.

Let $h=(h_x,h_y)$ be defined by

$h_x:A \to A$; identity

$h_y:\underline{R}(P_F(\underline{S})) \to \underline{R}(S_1+S_2)$

$h_y(S_1(x),S_2(x))=S_1(x)+S_2(x)$.

Then $h(P_F(\underline{S}))=S_1+S_2$. Hence h is a surjective modelling morphism .

Q.E.D.

Proof of Proposition 3.3.1 : For any $(c,c') \in S$, there is $u \in U$ such that $\phi(c,u)=c'$. Since $h:C \to C'$ is a morphism from T to T', $\phi'(h(c),u)$ $=h\,\phi(c,u)=h(c')$. Therefore $(h(c),h(c')) \in S'$. Hence $\underline{h}=(h,h)$ is a modelling morphism from S to S'.

Q.E.D.

Proof of Proposition 3.3.2 : Suppose that $h:C \to C'$ is a morphism from T to T'. For any $c \in C$, any $x_{tt'} \in X_{tt'}$, $h\ \phi_{tt'}(-,x_{tt'})(c)=h\ \phi_{tt'}(c,x_{tt'})$ $= \phi'_{tt'}(h(c),x_{tt'})= \phi'_{tt'}(-,x_{tt'})(h(c))$. Therefore h is a homomorphism from $\underline{C}(\bar{\phi})$ to $\underline{C}(\bar{\phi}')$.

Conversely, if h is a homomorphism from $\underline{C}(\bar{\phi})$ to $\underline{C}(\bar{\phi}')$, $h\ \phi_{tt'}(c,x_{tt'})=h\ \phi_{tt'}(-,x_{tt'})(c)= \phi'_{tt'}(-,x_{tt'})(h(c))= \phi'_{tt'}(h(c),x_{tt'})$. Therefore h is a morphism from T to T'. Q.E.D.

Proof of Proposition 3.3.3 : By definition, it is directly shown that $\underline{C}(\bar{\phi})=[C_1 \times C_2;F_T]$, where for any $\phi_{tt'}(-,x_{tt'}) \in F_T$, $\phi_{tt'}(-,x_{tt'}):C_1 \times C_2 \to C_1 \times C_2$ is defined by $\phi_{tt'}(-,x_{tt'})(c_1,c_2)$ $=(\phi_{tt'}{}^1(-,x_{tt'})(c_1), \phi_{tt'}{}^2(-,x_{tt'})(c_2))$. Therefore $\underline{C}(\phi)=\underline{C}(\phi^1) \times \underline{C}(\phi^2)$. Q.E.D.

3. Proofs for Chapter 4

Proof of Proposition 4.1.1 : Suppose that $S= \Pi \underline{S}$. Let $(s_i,\overline{s_i}) \varepsilon S_i \times \overline{S_i}$ be arbitrary. Then $s_i \varepsilon S_i$ and $\overline{s_i} \varepsilon \Pi (S_j | j \varepsilon I- \{ i \})$. Therefore $(s_i,\overline{s_i}) \varepsilon \Pi \underline{S} = S$. Hence S_i has NI.

Conversely, suppose that S_i has NI, that is, $S=S_i \times \overline{S_i}$, for all $i \varepsilon I$. Let $[s_i | i \varepsilon I] \varepsilon \Pi \underline{S}$ be arbitrary. Then there exists $\overline{s_1} \varepsilon \overline{S_1}$ such that $(s_1,\overline{s_1}) \varepsilon S$. Since $(s_1,p_{I-\{1,2\}}(\overline{s_1})) \varepsilon \overline{S_2}$, $(s_1,s_2,p_{I-\{1,2\}}(\overline{s_1})) \varepsilon S_2 \times \overline{S_2}=S$. Continueing this process, we finally get $[s_i | i \varepsilon I] \varepsilon S$ because I is finite. Q.E.D.

Proof of Proposition 4.2.1 : Suppose that $(i,j) \varepsilon NI(S)$. Then $p_{(i,j)}(S) =S_i \times S_j$ by the definition and hence $p_{(j,i)}(S)=S_j \times S_i$. Therefore $(i,j) \varepsilon NI(S)^{-1}$ and so $NI(S) \subset NI(S)^{-1}$. Similarly $NI(S)^{-1} \subset NI(S)$. Hence $NI(S)=NI(S)^{-1}$. Q.E.D.

Proof of Proposition 4.2.2 : Suppose that $S=S_i \times \overline{S_i}$. Then it is obvious that $p_{(i,j)}(S) \subset S_i \times S_j$ for any $j \varepsilon I- \{ i \}$. Therefore it is enough to show that $S_i \times S_j \subset p_{(i,j)}(S)$. For any $(s_i,s_j) \varepsilon S_i \times S_j$, there exists $\overline{s_i} \varepsilon \overline{S_i}$ such that $p_j(\overline{s_i})=s_j$. Therefore there exists $s=(s_i,\overline{s_i}) \varepsilon S$ such that $p_{(i,j)}(s)=(s_i,s_j)$. Hence $S_i \times S_j \subset p_{(i,j)}(S)$ and $S_i \times S_j=p_{(i,j)}(S)$ for all $j \varepsilon I- \{ i \}$. Q.E.D.

Proof of Proposition 4.2.3 : Suppose that S_i has NPI. Let $s_j, s_j' \in S_j$ be arbitrary and $x_i \in DS_{ij}^*(s_j) \cap DS_{ij}^*(s_j')$. Then there exists y_i, $y_i' \in Y_i$ and $\overline{s}_{ij}, \overline{s}_{ij}' \in \overline{S}_{ij}$ such that $((x_i,y_i),s_j,\overline{s}_{ij}) \in S$ and $((x_i,y_i'), s_j',\overline{s}_{ij}') \in S$. Let $\overline{s}_i=(s_j,\overline{s}_{ij})$ and $\overline{s}_i'=(s_j',\overline{s}_{ij}')$. Then $x_i \in DS_i^*(\overline{s}_i) \cap DS_i^*(\overline{s}_i')$. Since S_i has NPI, $S_i^*(\overline{s}_i)(x_i)=S_i^*(\overline{s}_i')(x_i)$. Therefore $y_i \in S_i^*(\overline{s}_i')(x_i)$ and $y_i' \in S_i^*(\overline{s}_i)(x_i)$. Hence $y_i \in S_{ij}^*(s_j')(x_i)$ and $y_i' \in S_{ij}^*(s_j)(x_i)$. Therefore $S_{ij}^*(s_j)(x_i)=S_{ij}^*(s_j')(x_i)$. Hence $(i,j) \in NPI(S)$.

Q.E.D.

Proof of Proposition 4.2.4 : Let $(i,j) \in WNSI(S)$. Then $p(i,j)(X)=X_i \times X_j$. Therefore $p_{(j,i)}(X)=X_j \times X_i$ and hence $(i,j) \in WNSI(S)^{-1}$. Similarly, $WNSI(S)^{-1} \subset WNSI(S)$. Hence $WNSI(S)^{-1}=WNSI(S)$.

Q.E.D.

Proof of Proposition 4.2.5 : Suppose that S_i has WNSI in S. Then $X=X_i \times \overline{X}_i$. Let $(x_i,x_j) \in X_i \times X_j$ be arbitrary. Since $x_j \in X_j$, there exists $\overline{x}_i \in \overline{X}_i$ such that $p_j(\overline{x}_i)=x_j$. Therefore $(x_i,x_j)=p(i,j)(x_i,\overline{x}_i) \in p(i,j)(X)$. Hence $X_i \times X_j \subset p(i,j)(X)$. Since it is obvious that $p(i,j)(X) \subset X_i \times X_j$, $p(i,j)(X)=X_i \times X_j$. It follows that $(i,j) \in WNSI(S)$ for all $j \in I-\{i\}$.

Q.E.D.

Proof of Proposition 4.2.6 : Suppose that S_i has SNSI in S. Then $DS_i^*(\overline{s}_i)=X_i$ for any $\overline{s}_i \in \overline{S}_i$. Let $s_j \in S_j$ and $x_i \in X_i$ be arbitrary. Then there exists $s \in S$ such that $p_j(s)=s_j$. Since S_i has SNSI, $x_i \in DS_i^*(p_{I-\{i\}}(s))=DS_{ij}^*(s_j)$. Hence $DS_{ij}^*(s_j)=X_i$ and $(i,j) \in SNSI(S)$.

Q.E.D.

Proof of Proposition 4.2.7 : Let $(i,j) \in SNSI(S)$. Then $DS_{ij}^*(s_j)=X_i$ for any $s_j \in S_j$. Let $(x_i,x_j) \in X_i \times X_j$ be arbitrary. Then there exists $y_j \in Y_j$ such that $s_j=(x_j,y_j) \in S_j$. Since $(i,j) \in SNSI(S)$, $x_i \in DS_{ij}^*(s_j)$. Then there exists $y_i \in Y_i$ and $\bar{s}_{ij}$ such that $((x_i,y_i),s_j,\bar{s}_{ij}) \in S$. Therefore $(x_i,x_j) \in P(i,j)(X)$ and hence $(i,j) \in WNSI(S)$. Q.E.D.

Proof of Theorem 4.2.1 : 1) Let $(i,j) \in WNSI(S) \cap NPI(S)^{-1}$. Let $s_j=(x_j,y_j)$, $s_j'=(x_j',y_j') \in S_j'$ be arbitrary. It is enough to show that $DS_{ij}^*(s_j)$ $=DS_{ij}^*(s_j')$ because any $x_i \in X_i$ should be an element of $DS_{ij}^*(s_j)$ for some $s_j \in S_j$. Let $x_i \in DS_{ij}^*(s_j)$ and $x_i' \in DS_{ij}^*(s_j')$. Then there exists $y_i,y_i' \in Y_i$ such that $((x_i,y_i),s_j) \in P(i,j)(S)$ and $((x_i',y_i'),s_j') \in P(i,j)(S)$. Since $P(i,j)(X)=X_i \times X_j$, there exists $y_i",y_j"$ such that $((x_i,y_i"),(x_j',y_j")) \in P(i,j)(S)$. Then $x_j' \in DS_{ji}^*((x_i,y_i")) \cap DS_{ji}^*((x_i',y_i'))$. Since $(i,j) \in NPI(S)^{-1}$, $((x_i,y_i"),(x_j',y_j')) \in P(i,j)(S)$. Therefore $x_i \in DS_{ij}^*((x_j'y_j'))=DS_{ij}^*(s_j')$. Hence $DS_{ij}^*(s_j) \subset DS_{ij}^*(s_j')$. Similarly, $DS_{ij}^*(s_j') \subset DS_{ij}^*(s_j)$. Hence $(i,j) \in SNSI(S)$.

2)-(i): $NI(S) \subset WNSI(S) \cap NPI(S)^{-1} \cap NPI(S)$

Let $(i,j) \in NI(S)$ and let $(x_i,x_j) \in X_i \times X_j$ be arbitrary. Then there exists $y_i \in Y_i$ and $y_j \in Y_j$ such that $(x_i,y_i) \in S_i$ and $(x_j,y_j) \in S_j$. Since $P(i,j)(S)=S_i \times S_j$, $((x_i,y_i),(x_j,y_j)) \in P(i,j)(S)$. Therefore (x_i,x_j) $\in P(i,j)(X)$ and hence $(i,j) \in WNSI(S)$.

Let us next show that $(i,j) \in NPI(S)$. Suppose that $x_i \in DS_{ij}^*(s_j)$ $\cap DS_{ij}^*(s_j')$ for arbitrary elements $s_j,s_j' \in S_j$. Let $y_i \in S_{ij}^*(s_j)(x_i)$ be arbitrary. Then $((x_i,y_i),s_j') \in S_i \times S_j=P(i,j)(S)$. Therefore $y_i \in$ $S_{ij}^*(s_j')(x_i)$ and hence $S_{ij}^*(s_j)(x_i) \subset S_{ij}^*(s_j')(x_i)$. Similarly, $S_{ij}^*(s_j')(x_i)$ $\subset S_{ij}^*(s_j)(x_i)$ and hence $S_{ij}^*(s_j')(x_i)=S_{ij}^*(s_j)(x_i)$. Therefore $NI(S)$

$\subset$ NPI(S). Since NI(S)=NI(S)$^{-1}$ and NI(S) $\subset$ NPI(S), NI(S) $\subset$ NPI(S)$^{-1}$.

2-(ii) : WNSI(S) $\cap$ NPI(S)$^{-1}$ $\cap$ NPI(S) $\subset$ SNSI(S) $\cap$ NPI(S)

It is obvious from 1) in this theorem.

2-(iii) : SNSI(S) $\cap$ NPI(S) $\subset$ NI(S)

Suppose that (i,j) ε SNSI(S) $\cap$ NPI(S). It is enough to show that $S_i \times S_j \subset p_{(i,j)}(S)$ because $p_{(i,j)}(S) \subset S_i \times S_j$ holds by the definition. Let$((x_i,y_i),(x_j,y_j)) \in S_i \times S_j$ be arbitrary. Then there exists (x_j',y_j') ε S_j such that $((x_i,y_i),(x_j',y_j')) \in p_{(i,j)}(S)$. And there exists y_i'' ε Y_i such that $((x_i,y_i''),(x_j,y_j)) \in p_{(i,j)}(S)$ because (i,j) ε SNSI(S). Then $x_i \in DS_{ij}*((x_j',y_j')) \cap DS_{ij}*((x_j,y_j))$. Since (i,j) ε NPI(S), $y_i \in S_{ij}*$ $((x_j,y_j))(x_i)$. Therefore $((x_i,y_i),(x_j,y_j)) \in S_i \times S_j$. Q.E.D.

Proof of Proposition 4.3.1 : Let F be a relation on I satisfying $\underline{D}$(F) $\cup$ $\underline{R}$(F)=I. Let $([x_i|i \in I],[y_i|i \in I])$, $([x_i|i \in I],[y_i'|i \in I]) \in S$. let k_1=1. Since $k_1 \in I=\underline{D}(F) \cup \underline{R}(F)$, there exists $j_1 \in I$ such that $(k_1,j_1) \in F \cup F^{-1}$. Since $((x_{k1},x_{j1}),(y_{k1},y_{j1})) \in p_{(k1,j1)}(S)$ and $((x_{k1},x_{j1}),$ $(y_{k1}',y_{j1}')) \in p_{(k1,j1)}(S)$, and $p_{(k1,j1)}(S)$ or $p_{(j1,k1)}(S)$ is functional, $y_{k1}=y_{k1}'$ and $y_{j1}=y_{j1}'$. Let $k_2 = \min[I- \{ k_1,j_1 \}]$. Similarly, we can find $j_2 \in I$ such that $y_{k2}=y_{k2}'$ and $y_{j2}=y_{j2}'$. Continueing this process, we can show that S is functional because $\underline{D}$(F) $\cup$ $\underline{R}$(F)=I.

Conversely, if S= Π $\underline{S}$ and S is functional, S_i is functional for all i ε I. Since $p_{(i,j)}(S)$ is functional if S_i and S_j are functional, $p_{(i,j)}$ (S) is functional for i $\neq$ j. Q.E.D.

Proof of Lemma 4.3.1 : Let $(i,j) \in SNSI(S) \cap FC(S)$. Suppose that (x_j,y_j), $(x_j,y_j') \in S_j$. Sinse $(i,j) \in SNSI(S)$, $DS_{ij}^*((x_j,y_j))=DS_{ij}^*((x_j,y_j'))=X_i$. Therefore there exists $(x_i,y_i),(x_i,y_i') \in S_i$ such that $((x_i,y_i),(x_j,y_j)) \in p_{(i,j)}(S)$ and $((x_i,y_i'),(x_j,y_j')) \in p_{(i,j)}(S)$. Since $(i,j) \in FC(S)$, $(y_i,y_j)=(y_i',y_j')$. It follows that S_j is functional.

Q.E.D.

Proof of Lemma 4.3.2 : Let S_i be functional. Suppose that $x_i \in DS_{ij}^*(s_j) \cap DS_{ij}^*(s_j')$ for any $s_j,s_j' \in S_j$. Since S_i is functional, $S_i(x_i)= S_{ij}^*(s_j)(x_i)=S_{ij}^*(s_j')(x_i)$. Hence $(i,j) \in NPI(S)$.

Q.E.D.

Proof of Proposition 4.3.1 : Let $(i,j) \in SNSI(S) \cap FC(S)$. Then S_j is functional by Lemma 4.3.1. Therefore $(i,j) \in NPI(S)^{-1}$ by Lemma 4.3.2.

Q.E.D.

Proof of Theorem 4.3.1 : Since $WNSI(S) \cap NPI(S)^{-1} \subset SNSI(S)$ by Theorem 4.2.1, $WNSI(S) \cap NPI(S)^{-1} \cap FC(S) \subset SNSI(S) \cap FC(S)$. And by Proposition 4.2.7 and Proposition 4.3.1, $SNSI(S) \cap FC(S) \subset WNSI(S) \cap NPI(S)^{-1} \cap FC(S)$. Hence $WNSI(S) \cap NPI(S)^{-1} \cap FC(S)=SNSI(S) \cap FC(S)$.

Q.E.D.

Proof of Proposition 4.3.2 : It is directly from Lemma 4.3.2.

Proof of Proposition 4.3.3 : Since $NPI(S)^{-1}=(Id_I{}^C)^{-1}=Id_I{}^C$ by Proposition 4.3.2, $WNSI(S)=WNSI(S) \cap Id_I{}^C=WNSI(S) \cap NPI(S)^{-1} \subset SNSI(S)$ by Theorem 4.2.1.

Q.E.D.

Proof of Theorem 4.3.2 : NI(S)=SNSI(S) ∩ NPI(S)=SNSI(S) by Theorem 4.2.1 and Proposition 4.3.2. And SNSI(S)=WNSI(S) by Proposition 4.2.7 and Proposition 4.3.3.

Q.E.D.

Proof of Proposition 4.4.1 : At first, let us show that $\underline{CS_n}$ is a category. By the definition of the composition, the composition operation is well-defined and admits associativity. Let $Id_S = \Pi\ (Id_{Si} | i \in I)|S$, where $S \subset \Pi\ (S_i | i \in I)$ and Id_{Si} is the identity morphism of S_i in MOD. Then Id_S is an identity morphism. Hence CS is a category and it is obvious that it is a subcategory of MOD.

Q.E.D.

Proof of Theorem 4.4.1 :

1) S_i has NI in S if and only if S_i has SNSI and NPI[4]. Therefore it is directly from 2) and 3).

2) Suppose that S_i has NPI in S. Let $\bar{s}_i'^1$, $\bar{s}_i'^2 \in \bar{S}_i'$ and $x_i' \in X_i'$ be arbitrary. Suppose that $x_i' \in DS_i'^*(\bar{s}_i'^1) \cap DS_i'^*(\bar{s}_i'^2)$. Then there exist $y_i'^1$, $y_i'^2 \in Y_i'$ such that $((x_i', y_i'^1), \bar{s}_i'^1) \in S'$ and $((x_i', y_i'^2), \bar{s}_i'^2) \in S'$. Since $\Pi\ (h_i | i \in I)|S$ is an isomorphism, there exist $((x_i^1, y_i^1), \bar{s}_i^1) \in S$ and $((x_i^2, y_i^2), \bar{s}_i^2) \in S$ such that $\Pi\ (h_i | i \in I)((x_i^1, y_i^1), \bar{s}_i^1)$ $=((x_i', y_i'^1), s_i'^1)$ and $\Pi\ (h_i | i \in I)((x_i^1, y_i^2), \bar{s}_i^2) = ((x_i', y_i'^2), \bar{s}_i'^2)$. Since $h_i = (h_{ix}, h_{iy})$ is an isomorphism, h_{ix} is injective[3] Therefore $x_i^1 = x_i^2$. Let $x_i = x_i^1$. Then $x_i \in DS_i^*(s_i^1) \cap DS_i^*(s_i^2)$ and hence $S_i^*(s_i^1)(x_i) = S_i^*(s_i^2)(x_i)$ because S_i has NPI.
Therefore $y_i^1 \in S_i^*(\bar{s}_i^2)(x_i)$ and hence $y'^1_i \in S_i'^*(\bar{s}'^2_i)(x'_i)$. Similarly, $S'_i{}^*(\bar{s}'^2_i)(x'_i) \subset S'_i{}^*(\bar{s}'^1_i)(x'_i)$. Hence S'_i has NPI.

Conversely, suppose that S'_i has NPI in S'. Let $\bar{s}_i^1$, $\bar{s}_i^2 \in \bar{S}_i$ and $x_i \in X_i$ be arbitrary. Suppose that $x_i \in DS_i^*(\bar{s}_i^1) \cap DS_i^*(\bar{s}_i^2)$. Then there exist y_i^1, $y_i^2 \in Y_i$ such that $((x_i,y_i),\bar{s}_i^1) \in S$ and $((x_i,y_i^2),\bar{s}_i^2) \in S$. Since $\Pi\,(h_i|i \in I)|S$ is a modelling morphism, $(h_i(x_i,y_i^1),\bar{h}_i(\bar{s}_i^1)) \in S'$ and $(h_i(x_i,y_i^2),\bar{h}_i(\bar{s}_i^2)) \in S'$, where $\bar{h}_i = \Pi\,(h_j|j \in I-\{\,i\,\})|\bar{S}_i$. Therefore $h_{ix}(x_i) \in DS'_i{}^*(\bar{h}_i(\bar{s}_i^1)) \cap DS'_i{}^*(\bar{h}_i(\bar{s}_i^2))$ and hence $h_{iy}(y_i^1) \in S'_i{}^*(h_i(\bar{s}_i^2))(h_{ix}(x_i))$ because S'_i has NPI. Since $\Pi\,(h_i|i \in I)|S$ is surjective, there exists $((\hat{x}_i,\hat{y}_i),\bar{s}_i) \in S$ such that $(h_i(\hat{x}_i,\hat{y}_i), \bar{h}_i(\bar{s}_i))=(h_i(x_i,y_i), \bar{h}_i(\bar{s}_i^2))$. Then $(\hat{x}_i,\hat{y}_i)=(x_i,y_i^1)$ and $\bar{s}_i=\bar{s}_i^2$ because $\Pi\,(h_i|i \in I)|S$ is an isomorphism. Therefore $y_i^1 \in S_i^*(\bar{s}_i^2)(x_i)$ and hence $S_i^*(\bar{s}_i^1)(x_i) \subset S_i^*(\bar{s}_i^2)(x_i)$. Similarly, $S_i^*(\bar{s}_i^2)(x_i) \subset S_i^*(\bar{s}_i^1)(x_i)$. Hence S_i has NPI in S.

3) Suppose that S_i has SNSI in S. Let $\bar{s}_i'^1$, $\bar{s}_i'^2 \in \bar{S}_i'$ be arbitrary. Then there exists $s_i'^1, s_i'^2 \in S_i'$ such that $(s_i'^1,\bar{s}_i'^1) \in S'$ and $(s_i'^2,\bar{s}_i'^2) \in S'$. Since $\Pi\,(h_i|i \in I)|S$ is surjective, there exist $(s_i^1,\bar{s}_i^1) \in S$ and $(s_i^2, \bar{s}_i^2) \in S$ such that $\Pi\,(h_i|i \in I)(s_i^1,\bar{s}_i^1)=(s_i'^1,\bar{s}_i'^1)$ and $\Pi\,(h_i|i \in I)(s_i^2,\bar{s}_i^2)=(s_i'^2, \bar{s}_i'^2)$. Since S_i has SNSI, $DS_i^*(\bar{s}_i^1)=DS_i^*(\bar{s}_i^2)$. Suppose that $x_i' \in DS_i'^*(\bar{s}'^1_i)$. Then there exists $x_i \in X_i$ such that $x_i \in DS_i^*(\bar{s}_i^1)$ and $h_{ix}(x_i)=x_i'$ because $\Pi\,(h_i|i \in I)|S$ is an isomorphism. Then $x_i' \in DS_i'^*(\bar{s}_i'^2)$ and hence $DS_i'^*(\bar{s}_i'^1) \subset DS_i'^*(\bar{s}_i'^2)$. Similarly, $DS_i'^*(\bar{s}_i'^2) \subset DS_i'^*(\bar{s}_i'^1)$. Therefore $DS_i'^*(\bar{s}_i'^1)=DS_i'^*(\bar{s}_i'^2)$ and hence S_i' has SNSI in S'.

Conversely, suppose that S_i' has SNSI in S'. Let $\bar{s}_i^1,\bar{s}_i^2 \in \bar{S}_i$ be arbitrary. Let $x_i \in DS_i^*(\bar{s}_i^1)$. Then $h_{ix}(x_i) \in DS_i'^*(\bar{h}_i(\bar{s}_i^1))$. Since S_i' has SNSI, $h_{ix}(x_i) \in DS_i'^*(\bar{h}_i(\bar{s}_i^2))$. Then there exists $((\hat{x}_i,\hat{y}_i),\bar{s}_i) \in S$ such that $h_{ix}(\hat{x}_i)=h_{ix}(x_i)$ and $\bar{h}_i(\bar{s}_i)=\bar{h}_i(\bar{s}_i^2)$. Since $\Pi\,(h_i|i \in I)|S$

is an isomorphism, $\hat{x}_i=x_i$ and $\bar{s}_i=\bar{s}_i^2$. Then $x_i \in DS_i*(\bar{s}_i^2)$ and hence $DS_i*(\bar{s}_i^1) \subset DS_i*(\bar{s}_i^2)$. Similarly, $DS_i*(\bar{s}_i^2) \subset DS_i*(\bar{s}_i^1)$. Therefore $DS_i*(\bar{s}_i^1)=DS_i*(\bar{s}_i^2)$ and hence S_i has SNSI in S.

4) Suppose that S_i has WNSI in S. Let $(x_i',\bar{x}_i') \in X_i' \times \bar{X}_i'$ be arbitrary. Then there exists $(x_i,\bar{x}_i) \in X_i \times \bar{X}_i$ such that $h_{ix}(x_i)=x_i'$ and $\bar{h}_{ix}(\bar{x}_i)=\bar{x}_i'$. Since S_i has WNSI, $(x_i,\bar{x}_i) \in X$. Therefore $(\bar{x}_i',x_i')=(h_{ix}(\bar{x}_i),\bar{h}_{ix}(x_i))$ $= \Pi\,(h_{ix}|i \in I)((x_i,\bar{x}_i)) \in X'$. Hence S_i' has WNSI in S'.

Conversely, suppose that S_i' has WNSI. Let $(x_i,\bar{x}_i) \in X_i \times \bar{X}_i$ be arbitrary. Then $(h_{ix}(x_i),\bar{h}_{ix}(\bar{x}_i)) \in X_i' \times \bar{X}_i'=X'$. Since $\Pi\,(h_i|i \in I)|S$ is surjective and h_{ix} and h_{iy} are injective, $(x_i,\bar{x}_i) \in X$. Hence S_i has WNSI in S. Q.E.D.

<u>Proof of Corollary 4.4.2</u> :

1) Suppose that S is a non-interacted system. Then S_i has NI for all $i \in I$. By Theorem 4.4.1, S_i' has NI for all $i \in I$. It follows that S' is also a non-interacted system. Similarly, if S' is a non-interacted system, so is S.

2) ~ 4) are similarly proved. Q.E.D.

<u>Proof of Theorem 4.4.2</u> : Since IS(S) is defined by IS(S)=[PI(S),SSI(S), WSI(S)], it is enough to prove that 1) NPI(S)=NPI(S'), 2) WNSI(S)=WNSI(S') and 3) SNSI(S)=SNSI(S').

1) Suppose that $(i,j) \in NPI(S)$. Let $s_j'^1,\ s_j'^2 \in S_j'$ and $x_i' \in X_i'$ be arbitrary. Suppose that $x_i' \in DS_{ij}'*(s_j'^1) \cap DS_{ij}'*(s_j'^2)$. Then there exists $y_i'^1,\ y_i'^2 \in Y_i'$ such that $((x_i',y_i'^1),s_j'^1) \in p_{(i,j)}(S')$

and $((x_i', y_i'^2), s_j'^2) \varepsilon p_{(i,j)}(S')$. Since $\Pi (h_i | i \varepsilon I)|S$ is an isomorphism, there exist $((x_i, y_i^1), s_j^1) \varepsilon p_{(i,j)}(S)$ and $((x_i, y_i^2), s_j^2) \varepsilon p_{(i,j)}(S)$ such that $h_i \times h_j((x_i, y_i^1), s_j^1) = ((x_i', y_i'^1), s_j'^1)$ and $h_i \times h_j((x_i, y_i^2), s_j^2) = ((x_i', y_i'^2), s_j'^2)$. Therefore $x_i \varepsilon DS_{ij}*(s_j^1) \cap DS_{ij}*(s_j^2)$ and hence $S_{ij}*(s_j^1)(x_i) = S_{ij}*(s_j^2)(x_i)$ because $(i,j) \varepsilon NPI(S)$. Therefore $S_{ij}'*(s_j'^1)(x_i') = S_{ij}'*(s_j'^2)(x_i')$. Hence $(i,j) \varepsilon NPI(S')$.

Conversely, suppose that $(i,j) \varepsilon NPI(S')$. Let $s_j^1, s_j^2 \varepsilon S_j$ and $x_i \varepsilon X_i$ be arbitrary. Suppose that $x_i \varepsilon DS_{ij}*(s_j^1) \cap DS_{ij}*(s_j^2)$. Then there exist $y_i^1, y_i^2 \varepsilon Y_i$ such that $((x_i, y_i^1), s_j^1) \varepsilon p_{(i,j)}(S)$ and $((x_i, y_i^2), s_j^2) \varepsilon p_{(i,j)}(S)$. Since $\Pi (h_i | i \varepsilon I)|S$ is a modelling morphism, $(h_i(x_i, y_i^1), h_j(s_j^1)) \varepsilon p_{(i,j)}(S')$ and $(h_i(x_i, y_i^2), h_j(s_j^2)) \varepsilon p_{(i,j)}(S')$. Therefore $h_{ix}(x_i) \varepsilon DS_{ij}'*(h_j(s_j^1)) \cap DS_{ij}'*(h_j(s_j^2))$ and hence $S_{ij}'*(h_j(s_j^1))(h_{ix}(x_i)) = S_{ij}'*(h_j(s_j^2))(h_{ix}(x_i))$ because $(i,j) \varepsilon NPI(S')$. Then $h_{iy}(y_i^1) \varepsilon S_{ij}'*(h_j(s_j^2))(h_{ix}(x_i))$. Since $\Pi (h_i | i \varepsilon I)$ is a surjective modelling morphism, there exists $((\hat{x}_i, \hat{y}_i), \hat{s}_j) \varepsilon p_{(i,j)}(S)$ such that $(h_i(\hat{x}_i, \hat{y}_i), h_j(\hat{s}_j)) = (h_i(x_i, y_i^1), h_j(s_j^2))$.
Since h_i and h_j are isomorphisms, $(\hat{x}_i, \hat{y}_i) = (x_i, y_i^1)$ and $\hat{s}_j = s_j^2$. Therefore $y_i^1 \varepsilon S_{ij}*(s_j^2)(x_i)$ and hence $S_{ij}*(s_j^1)(x_i) \subset S_{ij}*(s_j^2)(x_i)$. Similarly, $S_{ij}*(s_j^2)(x_i) \subset S_{ij}*(s_j^1)(x_i)$. Therefore $S_{ij}*(s_j^1)(x_i) = S_{ij}*(s_j^2)(x_i)$ and hence $(i,j) \varepsilon NPI(S)$.

2) Suppose that $(i,j) \varepsilon WNSI(S)$. Let $(x_i', x_j') \varepsilon X_i' \times X_j'$ be arbitrary. Then there exists $(x_i, x_j) \varepsilon X_i \times X_j$ such that $h_{ix} \times h_{jx}(x_i, x_j) = (x_i', x_j')$. Since $(i,j) \varepsilon WNSI(S)$, $(x_i, x_j) \varepsilon p_{(i,j)}(X)$. Therefore $(x_i', x_j') = (h_{ix}(x_i), h_{jx}(x_j)) \varepsilon p_{(i,j)}(X')$. Hence $(i,j) \varepsilon WNSI(S')$.

Conversely, suppose that $(i,j) \varepsilon WNSI(S')$. Let $(x_i, x_j) \varepsilon X_i \times X_j$ be arbitrary. Then $(h_{ix}(x_i), h_{jx}(x_j)) \varepsilon X_i' \times X_j'$. Since $(i,j) \varepsilon WNSI(S')$, $(h_{ix}(x_i), h_{jx}(x_j)) \varepsilon p_{(i,j)}(X')$. Then $(x_i, x_j) \varepsilon p_{(i,j)}(X)$ because $\Pi (h_i | i \varepsilon I)$

is an isomorphism and h_{ix} and h_{jx} are injective. Hence $(i,j) \in WNSI(S)$.

3) Suppose that $(i,j) \in SNSI(S)$. Let $s_j'^1, s_j'^2 \in S_j'$ be arbitrary. Let $x_i' \in DS_{ij}*(s_j'^1)$. Then there exists $y_i'^1 \in Y_i'$ such that $((x_i', y_i'^1), s_j'^1) \in P(i,j)(S')$. Since $\Pi\ (h_i | i \in I)|S$ is surjective, there exists $((x_i, y_i^1), s_j^1) \in P_{(i,j)}(S)$ such that $h_i \times h_j\ ((x_i, y_i^1), s_j^1) = ((x_i', y_i'^1), s_j'^1)$. Then $x_i \in DS_{ij}*(s_j^1)$. Since $(i,j) \in SNSI(S)$, $x_i \in DS_{ij*}(s_j^2)$, where $h_j(s_j^2) = s_j'^2$. Then $x_i' = h_{ix}(x_i) \in DS_{ij}'*(h_j(s_j^2)) = DS_{ij}*(s_j'^2)$. Therefore $DS_{ij}'*(s_j'^1) \subset DS_{ij}'*(s_j'^2)$. Similarly, $DS_{ij}'*(s_j'^2) \subset DS_{ij}'*(s_j'^1)$. Hence $(i,j) \in SNSI(S')$.

Conversely, suppose that $(i,j) \in SNSI(S')$. Let $s_j^1, s_j^2 \in S_j$ be arbitrary. Let $x_j \in DS_{ij}*(s_j^1)$. Then $h_{ix}(x_i) \in DS_{ij}'*(h_j(s_j^1))$. Since $(i,j) \in SNSI(S')$, $h_{ix}(x_i) \in DS_{ij}'*(h_j(s_j^2))$. Therefore $x_i \in DS_{ij}*(s_j^2)$ because $\Pi\ (h_i | i \in I)|S$ is an isomorphism. Hence $DS_{ij}*(s_j^1) \subset DS_{ij}*(s_j^2)$. Similarly, $DS_{ij}*(s_j^2) \subset DS_{ij}*(s_j^1)$. Hence $(i,j) \in SNSI(S)$.

Q.E.D.

4. Proofs for Chapter 5

Proof of Proposition 5.1.1 : Let (R_X,R_y) be an associated pair with R.

Let $p=(p_X,p_y)$ be defined by

$$p_X:X \to X/R_X \; ; \; x \mapsto [x]R_X$$

$$p_y:Y \to Y/R_y \; ; \; y \mapsto [y]R_y \; .$$

For any $(x,y) \varepsilon S$, $(x,y) \varepsilon [x]R_X \times [y]R_y \cap S$. Therefore $([x]R_X,[y]R_y) \varepsilon$ S/R and $p=(p_X,p_y)$ is a modelling morphism. And for any $([x]R_X,[y]R_y) \varepsilon S/R$, there is $(x',y') \varepsilon S$ such that $(x',y') \varepsilon [x]R_X \times [y]R_y \cap S$. Therefore $p(x',y')=(p_X(x'),p_y(y')) =([x']R_X,[y']R_Y) =([x]R_X,[y]R_y)$. Hence $p=(p_X,p_y)$ is a surjective modelling morphism, that is S/R is a surjective model of S.

Q.E.D.

Proof of Proposition 5.1.2, 5.1.3 : See the Reference [3].

Proof of Proposition 5.1.4 : Since $\sigma =(\sigma_X, \sigma_y) :S/p_ih \to S_i$ is an injective modelling morphism from Proposition 5.1.3. It is enough to show that σ is a surjective modelling morphism if so is h.

Since $p_i \circ h$ is a surjective modelling morphism, for any $(x_i,y_i) \varepsilon S_i$, there is $(x,y) \varepsilon S$ such that $p_i \circ h(x,y)=(x_i,y_i)$. Therefore $([x]p_{iX}h_X,[y]p_{iy}h_y) \varepsilon S/p_ih$ and $\sigma([x],[y]) =(x_i,y_i)$. Hence σ is a surjective modelling morphism and $S/p_ih \cong S_i$.

Q.E.D.

Proof of Theorem 5.1.1 :

(1) Let $l=(l_x,l_y)$ be defined by

$$l_x : X \to \underline{D}(S/\underline{R}) \ ; \ l_x(x)=([x]R_x{}^1,[x]R_x{}^2)$$

$$l_y : Y \to \underline{R}(S/\underline{R}) \ ; \ l_y(y)=([y]R_y{}^1,[y]R_y{}^2).$$

For any $(x,y) \in S$, $(x,y) \in ([x]R_x{}^1 \cap [x]R_x{}^2) \times ([y]R_y{}^1 \cap [y]R_y{}^2) \cap S$. Therefore $(([x]R_x{}^1,[x]R_x{}^2),([y]R_y{}^1,[y]R_y{}^2)) \in S/\underline{R}$ and $l=(l_x,l_y)$ is a modelling morphism. For any $(([x_1]R_x{}^1,[x_2]R_x{}^2),([y_1]R_y{}^1,[y_2]R_y{}^2)) \in S/\underline{R}$, there is $(x,y) \in ([x_1]R_x{}^1 \cap [x_2]R_x{}^2) \times ([y_1]R_y{}^1 \cap [y_2]R_y{}^2) \cap S$. Then

$$\begin{aligned} l(x,y)=(l_x(x),l_y(y))&=(([x]R_x{}^1,[x]R_x{}^2),([y]R_y{}^1,[y]R_y{}^2)) \\ &=(([x_1]R_x{}^1,[x_2]R_x{}^2),([y_1]R_y{}^1,[y_2]R_y{}^2)). \end{aligned}$$

Hence l is a surjective modelling morphism and $S/\underline{R}$ is a surjective model of S.

(2) Suppose that $\underline{R}_x$ and $\underline{R}_y$ are separating. If $l_x(x)=l_x(x')$, $(x,x') \in R_x{}^1 \cap R_x{}^2$. Then $x=x'$ and l_x is injective. Similarly, so is l_y. Therefore $l=(l_x,l_y)$ is an injective modelling morphism. Since l is a surjective modelling morphism as well by (1), l is an isomorphism.

(3) Suppose that $\underline{R}$ is full, and $\underline{R}_x$ and $\underline{R}_y$ are separating. Since $S \cong S/\underline{R}$ by (1), it is enough to prove $S/\underline{R} \cong S/R^1 \times S/R^2$. Let $S' \subset X' \times Y'$ be defined by

$$\begin{aligned} &(([x]R_x{}^1,[x_2]R_x{}^2),([y_1]R_y{}^1,[y_2]R_y{}^2)) \in S' \\ &\qquad \leftrightarrow ([x_1]R_x{}^1,[y_1]R_y{}^1),([x_2]R_x{}^2,[y_2]R_y{}^2)) \in S/R^1 \times S/R^2 \end{aligned}$$

, where $X' \subset X/R_x{}^1 \times X/R_x{}^2$ and $Y' \subset Y/R_y{}^1 \times Y/R_y{}^2$. Since obviously $S' \cong S/R^1 \times S/R^2$ in the sense of set theory, it is enough to show that $S/\underline{R} \cong S'$. For any $(([x_1]R_x{}^1,[x_2]R_x{}^2),([y_1]R_y{}^1,[y_2]R_y{}^2)) \in S'$, there are $(x_1',y_1') \in [x_1]R_x{}^1 \times [y_1]R_y{}^1 \cap S$ and $(x_2',y_2') \in [x_2]R_x{}^2 \times [y_2]R_y{}^2 \cap S$. Since $\underline{R}$ is full, there is $(x,y) \in S$ such that $(x_1,y_1)R^1(x,y)$ and $(x,y)R^2(x_2,y_2)$. Then $(([x_1]R_x{}^1,[y_1]R_y{}^1),([x_2]R_x{}^2,[y_2]R_y{}^2))=(([x]R_x{}^1,[y]R_y{}^1),([x]R_x{}^2,[y]R_y{}^2))$

ε $S/\underline{R}$. Since it is obvious that $S/\underline{R} \subset S'$, it follows that $S/\underline{R}=S'$.

Q.E.D.

<u>Proof of Proposition 5.2.1[3)]</u> see the reference [3].

<u>Proof of Proposition 5.2.2</u>

Let us first prove the if part. Let $\hat{S}/\underline{R}$ be the canonical complex system over $\underline{S}\hat{}/\underline{R}$. Then by Theorem 5.1.1, $\hat{S}/\underline{R} \cong \hat{S}$. Therefore $\hat{S}/\underline{R}$ is an injective (moreover, isomorphic) model of $\hat{S}$.

Conversely, suppose that $S \subset S_1 \times S_2$ is an injective model of $\hat{S}$ and $h=(h_x,h_y):\hat{S} \to S$ is an injective modelling morphism. Let $\equiv_{pi \circ h}$ denote a relation on $\hat{S}$ defined by

$$(x,y) \equiv_{pi \circ h} (x',y') \leftrightarrow p_i \circ h(x,y)=p_i \circ h(x'y')$$

for i=1,2. Then it is easy to see that $\equiv_{pi \circ h}$ is an input-output compatible equivalence relation on $\hat{S}$ with $(\equiv_{pix \cdot hx}, \equiv_{piy \cdot hy})$ as its associated pair by Proposition 5.1.2, where

$$(x,x') \varepsilon \equiv_{pix \cdot hx} \leftrightarrow p_{ix} \cdot h_x(x)=p_{ix} \cdot h_x(x')$$

$$(y,y') \varepsilon \equiv_{piy \cdot hy} \leftrightarrow p_{iy} \cdot h_y(y)=p_{iy} \cdot h_y(y').$$

For any $x,x' \varepsilon \hat{X}$, let $(x,x') \varepsilon \equiv p_{ix} \cdot h_x$ for i=1,2. Then $h_x(x)=h_x(x')$. Since h_x is injective, $x=x'$. Therefore $\underline{R}_x = \{ \equiv_{p1x \cdot hx}, \equiv_{p2x \cdot hx} \}$ is separating. Similarly, so is $\underline{R}_y = \{ \equiv_{p1y \cdot hy}, \equiv_{p2y \cdot hy} \}$. This is what to be proved.

Q.E.D.

Proof of Proposition 5.2.3[8] : By Theorem 5.1.1, the if part is obvious. And if we construct relations R^1 and R^2 on $\hat{S}$ as in the proof of Proposition 5.2.2, R^1 and R^2 are input-output compatible. Q.E.D.

Proof of Proposition 5.2.4 : This is directly derived from Proposition 5.2.2 and Proposition 5.2.3. Q.E.D.

Proof of Proposition 5.3.1 : Let $S \subset S_1 \times S_2$ be a complex system. For any $((x_1,x_2),(y_1,y_2)) \in S$, since k_1 and k_2 are modelling morphisms, $k_1(x_1,y_1) \in S_1'$ and $k_2(x_2,y_2) \in S_2'$. Therefore $k_1 \times k_2((x_1,y_1),(x_2,y_2)) =(k_1(x_1,y_1),k_2(x_2,y_2)) \in S_1' \times S_2'=S'$. Hence $k_1 \times k_2|S$ is a modelling morphism. Q.E.D.

Proof of Proposition 5.3.2 : For any $((x_1,y_1),(x_2,y_2)) \in S$, $k_i \circ p_i((x_1,y_1),(x_2,y_2))=k_i(x_i,y_i) \in S_i'$, because $k_i:S_i \to S_i'$ is a modelling morphism for $i=1,2$. While, $p_i' \circ (k_1 \times k_2)((x_1,y_1),(x_2,y_2))=p_i'(k_1(x_1,y_1),k_2(x_2,y_2))= k_i(x_i,y_i)$. Hence the diagram is commutative. Q.E.D.

Proof of Proposition 5.3.3 : Suppose that $k_1 \times k_2|S$ is a surjective modelling morphism from $S \subset S_1 \times S_2$ to $S' \subset S_1' \times S_2'$. For any $(x_1',y_1') \in S_1'$, there is $(x_2',y_2') \in S_2'$ such that $((x_1',y_1'),(x_2',y_2')) \in S'$ because S' is a complex system. Since $k_1 \times k_2|S$ is a surjective modelling morphism, there is $((x_1,y_1),(x_2,y_2)) \in S$ such that $k_1 \times k_2|S((x_1,y_1),(x_2,y_2)) =((x_1',y_1'),(x_2',y_2'))$. Therefore there is $(x_1,y_1) \in S_1$ such that $k_1(x_1,y_1)=(x_1',y_1')$. Hence k_1 is a surjective modellig morphism. Similarly, so is k_2.

Conversely, suppose that $S=S_1 \times S_2$,and $k_1:S_1 \to S_1'$ and $k_2:S_2 \to S_2'$ are surjective modelling morphisms. For any $((x_1',y_1'),(x_2',y_2')) \varepsilon S'$, since k_1 and k_2 are surjective modelling morphisms, there are $(x_1,y_1) \varepsilon S_1$ and $(x_2,y_2) \varepsilon S_2$ such that $k_1(x_1,y_1)=(x_1',y_1')$ and $k_2(x_2,y_2)=(x_2',y_2')$. Since $S=S_1 \times S_2$, $((x_1,y_1),(x_2,y_2)) \varepsilon S$. Therefore $k_1 \times k_2|S$ is a surjecive modelling morphism. Q.E.D.

<u>Proof of Proposition 5.3.4</u> : It is enough to show that $k_{1x} \times k_{2x}|X$ and $k_{1y} \times k_{2y}|Y$ are injective as mappings. For any $(x_1,x_2),(x_1'x_2') \varepsilon X$, if $k_{1x} \times k_{2x}(x_1,x_2)=k_1x \times k_2x(x_1',x_2')$, $k_{1x}(x_1)=k_{1x}(x_1')$ and $k_{2x}(x_2)=k_{2x}(x_2')$. Since k_{1x} and k_{2x} are injective, $x_1=x_1'$ and $x_2=x_2'$. Therefore $k_{1x} \times k_{2x}|X$ is injective. Similarly, so is $k_{1y} \times k_{2y}|Y$. Hence $k_1 \times k_2|S$ is an injective modelling morphism. Q.E.D.

<u>Proof of Proposition 5.3.5</u> : By Proposition 5.3.3 and 5.3.4, it is obvious that $k_1 \times k_2|S$ is an isomorphism. Q.E.D.

5. Proofs for Chapter 6

Proof of Proposition 6.1.1 : The if part is obvious by Theorem 5.1.1.

Let us next prove the only if part. Suppose that $\hat{S}$ has a non-interacted system $S_1 \times S_2$ as its model, where $S_i \subset X_i \times Y_i$ for i=1,2. Let $h=(h_x,h_y):\hat{S} \to S_1 \times S_2$ be a modelling morphism from $\hat{S}$ to $S_1 \times S_2$ and $p_i=(p_{ix},p_{iy})$ the i-th projection on $S_1 \times S_2$ for i=1,2. Since $p_i \circ h$ is also a modelling morphism from $\hat{S}$ to S_i for i=1,2, the induced equivalence relation $\equiv_{pi \circ h}$ is an input-output compatible equivalence relation for i=1,2 by Proposition 5.1.2. Q.E.D.

Proof of Proposition 6.1.2 : Let us prove the if part first. Suppose that $(R_x{}^i,R_y{}^i)$ is an associated pair with R^i for i=1,2. Let $\hat{S}/R^i$ be the quotient system of $\hat{S}$ modulo R^i for i=1,2. Let $k_i=(k_{ix},k_{iy})$ be defined by

$$k_{ix}:\hat{X} \to \hat{X}/R_x{}^i \;;\; k_{ix}(\hat{x})=[\hat{x}]R_x{}^i$$

$$k_{iy}:\hat{Y} \to \hat{Y}/R_y{}^i \;;\; k_{iy}(\hat{y})=[\hat{y}]R_y{}^i \quad .$$

for i=1,2. For any $(([x_1]R_x{}^1,[x_2]R_x{}^2),[y_1]R_y{}^1, [y_2]R_y{}^2)) \in \hat{S}/R^1 \times \hat{S}/R^2$, there is $(\hat{x},\hat{y}) \in \hat{S}$ such that $(\hat{x},\hat{y}) \in ([x_1]R_x{}^1 \times [y_1]R_y{}^1 \cap \hat{S}) \cap ([x_2]R_x{}^2 \times [y_2]R_y{}^2 \cap \hat{S})$ because $\underline{R}$ is full. Therefore $k_1 \times k_2|\hat{S}(\hat{x},\hat{y})$
$=(([\hat{x}]R_x{}^1,[\hat{y}]R_y{}^1),([\hat{x}]R_x{}^2,[\hat{y}]R_y{}^2))$
$=(([x_1]R_x{}^1,[y_1]R_y{}^1),([x_2]R_x{}^2,[y_2]R_y{}^2))$. Hence $k_1 \times k_2|\hat{S}$ is surjective modelling morphism.

Let us next prove the only if part. Suppose that $\hat{S}$ has a non-interacted system $S_1 \times S_2$ as its surjective model. Let $\equiv_{pi \circ h}$ be the induced equivalence relation for i=1,2. For any $(\hat{x}_1,\hat{y}_1),(\hat{x}_2,\hat{y}_2) \in \hat{S}$, let $(x_1,y_1)=p_1 \circ h(\hat{x}_1,\hat{y}_1)$ and $(x_2,y_2)=p_2 \circ h(\hat{x}_2,\hat{y}_2)$.
Since $((x_1,\cdot y_1),(x_2,y_2)) \in S_1 \times S_2$ and h is a surjective modelling morphism,

there is $(\hat{x},\hat{y}) \varepsilon \hat{S}$ such that $h(\hat{x},\hat{y})=((x_1,x_2),(y_1,y_2))$. Therefore $((\hat{x}_1,\hat{y}_1),(\hat{x},\hat{y})) \varepsilon \equiv_{p_1 \circ h}$ and $((\hat{x},\hat{y}),(\hat{x}_2,\hat{y}_2)) \varepsilon \equiv_{p_2 \circ h}$. Hence $\underline{R}= \{ \equiv_{p1 \circ h}, \equiv_{p2 \circ h} \}$ is full.

Q.E.D.

Proof of Proposition 6.1.3 : Let us first prove the if part. Suppose that $(R_x{}^i,R_y{}^i)$ is an associated pair with R^i for i=1,2. Let $\hat{S}/R^i$ be the quotient system of $\hat{S}$ modulo R^i for i=1,2. Let $k_i=(k_{ix},k_{iy})$ be defined as in the proof of Proposition 6.1.2. Let $k_1 \times k_2|\hat{S}=(k_x,k_y)$. Obviously, $k_1 \times k_2|\hat{S}$ is a modelling morphism. For any $\hat{x}_1,\hat{x}_2 \varepsilon \hat{X}$ such that $\hat{x}_1 \neq \hat{x}_2$, there are $\hat{y}_1,\hat{y}_2$ such that $(\hat{x}_1,\hat{y}_1),(\hat{x}_2,\hat{y}_2) \varepsilon \hat{S}$. Since $(\hat{x}_1,\hat{x}_2) \underline{\varepsilon} R_x{}^1 \cap R_x{}^2$, $([\hat{x}_1]R_x{}^1,[\hat{x}_1]R_x{}^2) \neq ([\hat{x}_2]R_x{}^1,[\hat{x}_2]R_x{}^2)$. Hence k_x is injective. Similarly, so is k_y. Therefore $k_1 \times k_2|S$ is an inejctive modelling morphism.

Let us next prove the only if part. Suppose that $\hat{S}$ has a non-interacted system $S_1 \times S_2$ as its injective model, where $S_i \subset X_i \times Y_i$ for i=1,2. Let $p_i \circ h$ be defined as in the proof of Proposition 6.1.1. Let $(\hat{x}_1,\hat{x}_2) \varepsilon \equiv_{p_{1x} \circ h_x} \cap \equiv_{p_{2x} \circ h_x}$. Then $p_{ix} \circ h_x(\hat{x}_1)=p_{ix} \circ h_x(\hat{x}_2)$ for i=1,2. Since h_x is injective, $\hat{x}_1=\hat{x}_2$. Therefore $\underline{R}_x= \{ \equiv_{p_{1x} \circ h_x}, \equiv_{p_{2x} \circ h_x} \}$ is separating. Similarly, so is $\underline{R}_y= \{ \equiv_{p_{1y} \circ h_y}, \equiv_{p_{2y} \circ h_y} \}$.

Q.E.D.

Proof of Proposition 6.1.4 : It is a direct consequence by Proposition 6.1.2 and 6.1.3. See also Theorem 5.1.1. Q.E.D.

Proof of Lemma 6.2.1 : Let $f:A/R_o \to A/\underline{R}$ be defined by

$$f([a]R_o)=([a]R_1,[a]R_2).$$

For any $a' \in [a]R_o$, $([a]R_1,[a]R_2)=([a']R_1,[a']R_2)$ because $R_o=R_1 \cap R_2$. Therefore f is well defined. It is easy to see that f is surjective. Let $([a]R_1,[a]R_2)=([a']R_1,[a']R_2)$. Then $(a,a') \in R_1 \cap R_2=R_o$. Therefore $[a]R_o=[a']R_o$ and hence f is injective. Consequently, f is bijective. Hence $A/R_o \cong A/\underline{R}$. Q.E.D.

Proof of Lemma 6.2.2 : Let $f_i:X/R_x{}^i \to X/\underline{R}_x{}^i$ be defined as in the proof of Lemma 6.2.1, where $\underline{R}_x{}^i= \{ R'_x{}^i,R_z \}$ such that $R_x{}^i=R'_x{}^i \cap R_z$. For any $(([\hat{x}]R_x{}^1,[\hat{y}]R_y{}^1),([\hat{x}]R_x{}^2,[\hat{y}]R_y{}^2)) \in S/\underline{R}$, since $f_i([\hat{x}]R_x{}^i)$ $=([\hat{x}]R'_x{}^i,[\hat{x}]R_z)$ for i=1,2, $(f_1 \times f_2) \times (Id \times Id)((([\hat{x}]R_x{}^1,[\hat{x}]R_x{}^2)$ $([\hat{y}]R_y{}^1,[\hat{y}]R_y{}^2)=((([\hat{x}]R'_x{}^1,[\hat{x}]R_z),([\hat{x}]R'_x{}^2,[\hat{x}]R_z)),$ $([\hat{y}]R_y{}^1,[\hat{y}]R_y{}^2)) \in P(\hat{S}/R^1,\hat{S}/R^2)$. Since $(f_1 \times f_2) \times (Id \times Id)|S/\underline{R}$ is injective modelling morphism, by Proposiotion 5.3.4, $S/\underline{R}$ is embedded in $P(S/R^1,S/R^2)$. Q.E.D.

Proof of Lemma 6.2.3 : Suppose that $\underline{R}$ satisfies the condition. Let $f_i:X/R_x{}^i \to X/\underline{R}_x{}^i$ be defined as in Lemma 6.2.2. Let us show that $(f_1 \times f_2) \times (Id \times Id)|S/\underline{R}$ is an isomorphism from $S/\underline{R}$ to $P(S/R^1,S/R^2)$. Since $(f_1 \times f_2) \times(Id \times Id)|S/\underline{R}$ is an injective modelling morphism by Lemma 6.2.2, it is enough to show that it is also a surjective modelling morphism by Proposition 2.2.5. For any $s'=((([x_1]R'_x{}^1,[z]R_z),([x_2]R'_x{}^2,[z]R_z)),$ $([y_1]R_y{}^1,[y_2]R_y{}^2)) \in P(S/R^1,S/R^2)$, since $[x_1]R'_x{}^1 \cap [z]R_z \neq \phi$ and $[x_2]R'_x{}^2$

$\cap\ [z]R_z \neq \phi$, s' can be written as $s'=((([x_1]R'^{\,1}_x,[x_1]R_z),([x_2]R'^{\,2}_x,[x_2]R_z)),([y_1]R_y^1,[y_2]R_y^2))$. By the condition of $\underline{R}$, there is $(x,y) \in S$ such that $(x,y) \in (([x_1]R'^{\,1}_x \cap [x_1]R_z) \times [y_1]R_y^1) \times (([x_2]R'^{\,2}_x \cap [x_2]R_z) \times [y_2]R_y^2) \cap S)$. Therefore $s'=((([x]R'^{\,1}_x,[x]R_z),([x]R'^{\,2}_x,[x]R_z),([y]R_y^1,[y]R_y^2))$. Since $s=(([x]R_x^1,[x]R_x^2),([y]R_y^1,[y]R_y^2)) \in S/\underline{R}$, $(f_1 \times f_2) \times (Id \times Id)(s)=s'$. Therefore $(f_1 \times f_2) \times (Id \times Id)|S/\underline{R}$ is a surjective modelling morphism. From now on, we use $f=(f_x,f_y)$ as $f_x=f_1 \times f_2|\underline{D}(S/\underline{R})$ and $f_y=Id \times Id|\underline{R}(S/\underline{R})$.

Q.E.D.

Proof of Proposition 6.2.1 : Let us first prove the if part. Suppose that $\underline{R}$ satisfies the conditions and $R_x^i=R'^{\,i}_x \cap R_z$ for i=1,2. Since $\hat{S}/\underline{R}$ can be embedded in $P(\hat{S}/R^1,\hat{S}/R^2)$ by Lemma 6.2.2, we can easily construct a modelling morphism from $\hat{S}$ to $P(\hat{S}/R^1,\hat{S}/R^2)$. Hence $EX(P(\hat{S}/R^1,\hat{S}/R^2))$ is a model of $\hat{S}$. Because $P(\hat{S}/R^1,\hat{S}/R^2) \cong EX(P(\hat{S}/R^1,\hat{S}/R^2))$ by Proposition 3.1.1.

Let us next prove the only if part. Suppose that there are component systems S_1 and S_2 such that $EX(P(S_1,S_2))$ is a model of $\hat{S}$, that is, $P(S_1,S_2)$ is a model of $\hat{S}$. Let $h=(h_x,h_y):\hat{S} \to P(S_1,S_2)$ be a modelling morphism. Then $\underline{R}= \{ \equiv p_1 \circ h, \equiv p_2 \circ h \}$ is a class of input-output compatible equivalence relations. Let $R_z \subset \hat{X} \times \hat{X}$ be defined by

$$(z,z') \in R_z \leftrightarrow (z,z') \in \equiv p_{1z} \bullet h_x \cap \equiv p_{2z} \bullet h_x$$

, where $p_{iz}:X_i^* \times Z \to Z$ is the projection. And $R'^{\,i}_x \subset \hat{X} \times \hat{X}$ be defined by $R'^{\,i}_x = \equiv p_{ix_i^*} \bullet h_x$, where $p_{ix_i^*}:X_i^* \times Z \to X_i^*$ is the projection. Then $R_x^i = \equiv p_{ix} \bullet h_x = \equiv p_{ix_i^*} h_x \cap \equiv p_{iz} \bullet h_x$ and hence R_x^1 and R_x^2 are decomposable with $R_z = \equiv p_{iz} \bullet h_x$.

Q.E.D.

Proof of Proposition 6.2.2 : Suppose that there is a class $\underline{R}=\{R^1,R^2\}$ of input-output compatible equivalence realtions on $\hat{S}$ such that R_x^1 and R_x^2 are decomposable with R_z and $\underline{R}$ satisfies the condition. Let R'^1_x and R'^2_x be equivalence relations on $\hat{X}$ such that $R_x^i=R'^i_x \cap R_z$ for $i=1,2$. Let $f=(f_x,f_y):\hat{S}/\underline{R} \to P(\hat{S}/R^1,\hat{S}/R^2)$ be defined as in the proof of Lemma 6.2.3. Then $f \circ 1:\hat{S} \to P(\hat{S}/R^1,\hat{S}/R^2)$ is a surjective modelling morphism. Because the natural modelling morphism $1:\hat{S} \to \hat{S}/\underline{R}$ is a surjective modelling morphism. Since $EX(P(\hat{S}/R^1,\hat{S}/R^2)) \cong P(\hat{S}/R^1,\hat{S}/R^2)$ by Proposition 3.1.1, $\hat{S}$ is surjectively parallel decomposable.

Conversely, suppose that there are input-output systems $S_1 \subset (X_1{}^* \times Z) \times Y_1$ and $S_2 \subset (X_2{}^* \times Z) \times Y_2$ such that $EX(P(S_1,S_2))$ is a surjective model of $\hat{S}$. Let $h=(h_x,h_y):\hat{S} \to EX(P(S_1,S_2))$ be a surjective modelling morphism. Let us define equivalence relations as follows.

$$R'^i_x \subset \hat{X} \times \hat{X} : (x,x') \in R'^i_x \leftrightarrow p_{ix^*}h_x(x)=p_{ix^*}h_x(x')$$
$$R_z \subset \hat{X} \times \hat{X} : (x,x') \in R_z \leftrightarrow p_z h_x(x)=p_z h_x(x')$$
$$R_y^i \subset \hat{Y} \times \hat{Y} : (y,y') \in R_y^i \leftrightarrow p_{iy}h_y(y)=p_{iy}h_y(y')$$

, where $p_{ix^*}:X_1{}^* \times Z \times X_2{}^* \to X_i{}^*$, $p_z:X_1{}^* \times Z \times X_2{}^* \to Z$ and $p_{iy}:Y_1 \times Y_2$ are projections. Let $R_x^i=R'^i_x \cap R_z$ and let $R^i \subset \hat{S} \times \hat{S}$ be defined by $((x,y),(x',y')) \in R^i \leftrightarrow (x,x') \in R_x^i$ and $(y,y') \in R_y^i$ for $i=1,2$. Then R^i is an input-output compatible equivalence relations on $\hat{S}$ for $i=1,2$. Let $(x_1,y_1),(x_2,y_2) \in \hat{S}$ be arbitrary such that $[x_1]R_x=[x_2]R_z$. Let $s=((p_{1x^*}h_x(x_1),p_z h_x(x_1),p_{2x^*}h_x(x_2)),(p_{1y}h_y(y_1),p_{2y}h_y(y_2)) \in EX(P(S_1,S_2))$. Since h is a surjective modelling morphism, there is $(x,y) \in \hat{S}$ such that $h(x,y)=s$. Therefore $(x,y) \in ([x_1]R_x^1 \times [y_1]R_y^1 \cap S) \cap ([x_2]R_x^2 \times [y_2]R_y^2 \cap S)$. Hence $\underline{R}$ satisfies the condition.

Q.E.D.

Proof of Proposition 6.2.3 : Suppose that there is a class $\underline{R}=\{R^1,R^2\}$ of equivalence relations on $\hat{S}$ satisfying the conditions. Since the natural modelling morphism $1=(1_x,1_y):\hat{S}\to\hat{S}/\underline{R}$ is an isomorphism by Theorem 5.1.1 and $f=(f_x,f_y):\hat{S}/\underline{R}\to P(\hat{S}/R^1,\hat{S}/R^2)$ defined in the proof of Lemma 6.2.3 is also an injective modelling morphism, it is easily seen that $\hat{S}$ is injectively parallel decomposable.

Conversely, suppose that there are input-output systems $S_1\subset(X_1^*\times Z)\times Y_1$ and $S_2\subset(X_2^*\times Z)\times Y_2$ such that $\hat{S}$ is injectively parallel decomposable into $EX(P(S_1,S_2))$. Let $h=(h_x,h_y):\hat{S}\to EX(P(S_1,S_2))$ be an injective modelling morphism. Let $R'^{\,i}_x,R_z,R_y^{\,i},R_x^{\,i}$ and R^i be defined as in the proof of Proposition 6.2.2. Then R^i is an input-output compatible equivalence relation on $\hat{S}$ for i=1,2. Let $(x,x')\in R_x^{\,1}\cap R_x^{\,2}$. Then $(x,x')\in R'^{\,1}_x\cap R_z\cap R'^{\,2}_x$. Since h_x is injective, $x=x'$. Therefore $\underline{R}_x=\{R_x^{\,1},R_x^{\,2}\}$ is saparating. Similarly, so is $\underline{R}_y=\{R_y^{\,1},R_y^{\,2}\}$.

Q.E.D.

Proof of Proposition 6.2.4 : It is a direct consequence from Proposition 6.2.2 and 6.2.3.

Q.E.D.

Proof of Proposition 6.3.1 : For any $s=(([x_1]R_x^{\,1},[y_1]R_y^{\,1}),([x_2]R_x^{\,2},[y_2]R_y^{\,2}))$ $\in EX(C(FSQ(\hat{S};R^1),SSQ(\hat{S};R^2)))$, there is $(x,y)\in\hat{S}$ such that $x\in[x_1]R_x^{\,1}\cap[x_2]R_x^{\,2}$ and $y\in[y_1]R_y^{\,1}\cap[y_2]R_y^{\,2}$. Then $s=(([x]R_x^{\,1},[y]R_y^{\,1}),([x]R_x^{\,2},[y]R_y^{\,2}))\in\hat{S}/\underline{R}$. Therefore $EX(FSQ(\hat{S};R^1),SSQ(\hat{S};R^2)))\subset\hat{S}/\underline{R}$. Similarly, $\hat{S}/\underline{R}\subset EX(C(FSQ(\hat{S};R^1),SSQ(\hat{S};R^2)))$. Hence $\hat{S}/\underline{R}=EX(C(FSQ(\hat{S};R^1),SSQ(\hat{S},R^2)))$.

Q.E.D.

Proof of Proposition 6.3.2 : Suppose that there is a class $\underline{R}=\{R^1,R^2\}$ of input-output compatible equivalence relations on $\hat{S}$. Since $EX(C(FSQ(\hat{S};R^1), SSQ(\hat{S};R^2)))=\hat{S}/\underline{R}$ by Proposition 6.3.1, it is obvious that $EX(C(FSQ(\hat{S};R^1), SSQ(\hat{S};R^2)))$ is a surjective model of $\hat{S}$ by Theorem 5.1.1.

Conversely, suppose that there are input-output systems $S_1 \subset X_1 \times (Z \times Y_1{}^*)$ asnd $S_2 \subset (X_2{}^* \times Z) \times Y_2$ such that $EX(C(S_1,S_2))$ is a model of $\hat{S}$. Let $h=(h_x,h_y):\hat{S} \rightarrow EX(C(S_1,,S_2))$ be a modelling morphism. Let $R_x{}^i$ and $R_y{}^i$ be defined by

$$(x,x') \varepsilon R_x{}^i \leftrightarrow p_{ix}h_x(x)=p_{ix}h_x(x')$$

$$(y,y') \varepsilon R_y{}^i \leftrightarrow p_{iy}h_y(y)=p_{iy}h_y(y')$$

for i=1,2, where $p_{ix}:X_1 \times X_2{}^* \rightarrow X_1(\text{or } X_2{}^*)$ and $p_{iy}:Y_1{}^* \times Y_2 \rightarrow Y_1{}^*(\text{or } Y_2)$ are projections. Let R^i be defined by

$$((x,y),(x',y')) \varepsilon R^i \leftrightarrow (x,x') \varepsilon R_x{}^i \text{ and } (y,y') \varepsilon R_y{}^i \text{ for } i=1,2.$$

Then it is easily seen that R^i is an input-output compatible equivalence relations on $\hat{S}$ for i=1,2. Q.E.D.

Proof of Proposition 6.3.3 : By Theorem 5.1.1, it is similarly proved as Proposition 6.3.2.

Proof of Proposition 6.4.1 : Let $S \subset X \times Y$ be an input-output system. By Definition 6.4.1, $FT(S) \subset (X \times S) \times (Y \times S)$ is defined by

$$FT(S)=\{(x,(x,y)),(y,(x,y))|(x,y) \varepsilon S\}.$$

And therefore $EX(FT(S))=\{(x,y) \varepsilon S\}=S$. Hence every input-output system is feedback decomposable into $F(S,Id)$. Q.E.D.

6. Proofs for Chapter 7

Proof of Proposition 7.1.1 : Since $p_i=(p_{ix},p_{iy})$ is a surjective modelling morphism, $p_{ix}^{-1}(x_i) \times p_{iy}^{-1}(y_i) \cap S \neq \phi$ for all $(x_i,y_i) \in S_i$. $\cup C_i \subset S$
1*) is obvious. For any $(x,y) \in S$, $p_i(x,y)=(p_{ix}(x),p_{iy}(y)) \in S_i$ and $(x,y) \in p_{ix}^{-1}(p_{ix}(x)) \times p_{iy}^{-1}(p_{iy}(y)) \cap S$. Therefore $\cup C_i=S$. Hence C_i is a covering of S. Let $C_x^i=\{ p_{ix}^{-1}(x_i) | x_i \in X_i \}$ and $C_y^i=\{ p_{iy}^{-1}(y_i) | y_i \in Y_i \}$. Then obviously, C_x^i and C_y^i are coverings of X and Y, respectively, and (C_x^i,C_y^i) is an associated pair with C_i. Hence C_i is input-output compatible. Q.E.D.

Proof of Proposition 7.1.2 : See the Reference [3].

Proof of Proposition 7.2.1 : Suppose that $\hat{S} \subset \hat{X} \times \hat{Y}$ has a simulation model $S \subset S_1 \times S_2$. Let $h^S=(h_x,h_y):\hat{S} \to S$ be a simulation morphism. Let $C_x^i=\hat{X}/p_{ix}h_x$ and $C_y^i=\{ h_y p_{iy}^{-1}(y_i) | y_i \in Y_i \}$. By assumption, for any $\hat{y}$, there is $\hat{x} \in \hat{X}$ such that $(\hat{x},\hat{y}) \in \hat{S}$ and h^S is a simulation morphism, there is $y \in Y$ such that $(h_x(\hat{x}),y) \in S$ and $h_y(y)=\hat{y}$. Since $\hat{y} \in h_y p_{iy}^{-1}(p_{iy}(y))$, C_y^i is a covering of $\hat{Y}$. And obviously, C_x^i is a covering of $\hat{X}$. Let $C^i=\{ [x] \equiv p_{ix}h_x \times h_y p_{iy}^{-1}(y_i) \cap \hat{S} | x \in \hat{X}$ and $y_i \in Y_i \}$. For any $(\hat{x},\hat{y}) \in \hat{S}$, there is $y \in Y$ such that $(h_x(\hat{x}),y) \in S$ and $h_y(y)=\hat{y}$. Therefore $(\hat{x},\hat{y}) \in [\hat{x}] \equiv p_{ix}h_x \times h_y p_{iy}^{-1}(p_{iy}(y)) \cap \hat{S}$. Hence C^i is an input-output compatible covering of $\hat{S}$ for i=1,2.

Conversely, suppose that there is a class $\underline{R}=\{ R^1,R^2 \}$ of input-output compatible equivalence relations satisfying that $\underline{R}_y=\{ R_y^1,R_y^2 \}$ is separating. Let $\hat{S}/\underline{R}$ be the canonical complex system over $\underline{\hat{S}}_q=\{ \hat{S}/R^1,\hat{S}/R^2 \}$. Let $h_x:\hat{X} \to \underline{D}(\hat{S}/\underline{R})$ and $h_y:\underline{R}(\hat{S}/\underline{R}) \to \hat{Y}$ be defined by

$$h_x(\hat{x})=([\hat{x}]R_x{}^1,[\hat{x}]R_x{}^2)$$

and
$$h_y([y_1]R_y{}^1,[y_2]R_y{}^2)=\begin{cases} \hat{y} & \varepsilon\ [y_1]R_y{}^1 \cap [y_2]R_y{}^2 \neq \phi \\ y_o\ \varepsilon\ \hat{Y} & \text{otherwise} \end{cases}$$

Since $\underline{R}_y$ is separating, h_y is well-defined. For any $(\hat{x},\hat{y})\ \varepsilon\ \hat{S}$, $(h_x(\hat{x}),([\hat{y}]R_y{}^1,[\hat{y}]R_y{}^2))\ \varepsilon\ \hat{S}/\underline{R}$ and $h_y([\hat{y}]R_y{}^1,[\hat{y}]R_y{}^2)=\hat{y}$. Hence $\hat{S}/\underline{R}$ is a simulation model of $\hat{S}$. Q.E.D.

<u>Proof of Proposition 7.2.2</u> : Let $h=(h_x,h_y)\ \varepsilon\ \mathrm{Hom}_{\underline{MOD}_{1b}}(S,S')$. Then for any $(x,y)\ \varepsilon\ S$, $(h_x(x),h_y(y))\ \varepsilon\ S'$. Since h_y is bijective, the inverse $h_y{}^{-1}$ of h_y is well defined. Therefore for any $(x,y)\ \varepsilon\ S$, $(h_x(x),h_y(y))\ \varepsilon\ S'$ and $h_y{}^{-1}h_y(y)=y$. Hence $F_{MS}(h)\ \varepsilon\ \mathrm{Hom}_{\underline{SIM}_{1b}}(S,S')$. Similarly, $F_{SM}(h^s)\ \varepsilon\ \mathrm{Hom}_{\underline{MOD}_{1b}}(S,S')$ for any $h^s\ \varepsilon\ \mathrm{Hom}_{\underline{SIM}_{1b}}(S,S')$. Since $F_{MS}(h' \circ h)=F_{MS}((h_x' \cdot h_x,h_y' \cdot h_y))=(h_x' \cdot h_x,h_y{}^{-1} \cdot h_y{}'^{-1})=F_{MS}(h') \circ F_{MS}(h)$ and $F_{MS}(Id)=Id$, F_{MS} is a functor. Similarly, so is F_{SM}. It is obvious that $F_{MS} \circ F_{SM}$ and $F_{SM} \circ F_{MS}$ are identity functors.

Q.E.D.

7. Proofs for Chapter 8

Proof of Proposition 8.1.1 : Let $h=(h_x,h_y)$ be a modelling morphism. For any $x \in X$, there is $y \in Y$ such that $S(x)=y$. Since h is a modelling morphism, $(h_x(x),h_y(y)) \in S'$, that is $S'h_y(x)=h_yS(x)$. Hence the diagram is commutative.

Conversely, suppose that the diagram is commutative. Then for any $(x,y) \in S$, $h_y(y)=h_yS(x)=S'h_x(x)$. Therefore $(h_x(x),h_y(y)) \in S'$. Hence h is a modelling morphism.

Let us next prove the second part. Let $h=(h_x,h_y)$ be an isomorphism. Then it is obvious that both of h_x and h_y are bijective by Proposition 2.2.4 and 2.2.5.

Conversely, suppose that h_x and h_y are bijective and the diagram is commutative. Then $h=(h_x,h_y)$ is a modelling morphism as we proved above. For any $(x',y') \in S'$, there is $x \in X$ and $y \in Y$ such that $h_x(x)=x'$ and $h_y(y)=y'$ because h_x and h_y are surjective. Since S is functional, $h_yS(x)$ $=S'h_x(x)=S'(x')=y'=h_y(y)$. Since h_y is injective, $S(x)=y$. Therefore $(x,y) \in S$ and h is a surjective modelling morphism.
Hence h is an isomorphism because both of h_x and h_y are injective.

Q.E.D.

Proof of Proposition 8.1.2 : Suppose that R_x and R_y satisfy the condition. Let $([x]R_x,[y]R_y),([x]R_x,[y']R_y) \in S/R$. Then $S(x) \in [y]R_y \cap [y']R_y$. Therefore $[y]R_y=[y']R_y$ and hence S/R is functional.

Conversely, suppose that S/R is functional. Let $x,x' \in X$ be arbitrary elements such that $(x,x') \in R_x$. Since S/R is functional, $[S(x)]R_y$ $=S/R([x]R_x)=S/R([x']R_x)=[S(x')]R_y$. Therefore $(S(x),S(x')) \in R_y$. Q.E.D.

Proof of Theorem 8.2.1 : It is directly shown by Proposition 6.1.4 and Proposition 8.1.2.

Q.E.D.

Proof of Theorem 8.3.1 : Suppose that $\underline{R}_y = \{ R_y^1, R_y^2 \}$ is separating. Let $S_1 \subset \hat{X} \times \hat{Y}/R_y^1$ and $S_2 \subset \hat{X} \times \hat{Y}/R_y^2$ be defined by

$$(x,[y]R_y^1) \in S_1 \leftrightarrow S(x) \in [y]R_y^1$$
$$(x,[y]R_y^2) \in S_2 \leftrightarrow S(x) \in [y]R_y^2.$$

Then S_1 and S_2 are functional. Let $Y' = \{ ([y]R_y^1,[y]R_y^2) | y \in \hat{Y} \}$ and $h:Y \to Y'$ be defined by $h(y)=([y]R_y^1,[y]R_y^2)$. Then $h(\hat{S}(x)) =([\hat{S}(x)]R_y^1,[\hat{S}(x)]R_y^2)=(S_1(x),S_2(x))$. Since $\underline{R}_y$ is separating, h is bijective. Therefore $\hat{S}$ is parallel decomposable.

Conversely, suppose that $\underline{S} = \{ S_1,S_2 \}$ is a class of functional component systems and $\hat{S}$ is parallel decomposable into $P_F(\underline{S})$. Then there is a bijection $h:\hat{Y} \to Y$ such that the diagram commutes. Let R_y^1 and R_y^2 be defined by

$$(y,y') \in R_y^1 \leftrightarrow p_{1y}h(y)=p_{1y}h(y')$$
$$(y,y') \in R_y^2 \leftrightarrow p_{2y}h(y)=p_{2y}h(y') .$$

Then for any $(y,y') \in R_y^1 \cap R_y^2$, $h(y)=h(y')$. Since h is injective, $y=y'$. Hence $\underline{R}_y = \{ R_y^1, R_y^2 \}$ is separating.

Q.E.D.

Proof of Theorem 8.3.2 : Let $\underline{\Theta} = \{ \Theta_1, \Theta_2 \}$ be a full and separating class of congruence relations on $S(\underline{A})$. Then $\Psi(\Theta_1)$ and $\Psi(\Theta_2)$ are Ω-normal subgroups of $S(\underline{A})$. By Theorem 2.4.2, $(\Psi(\Theta_1), \Psi(\Theta_2)) \in \perp(S(\underline{A}))$. Therefore any $S(a) \in S(A)$ can be uniquely decomposed into $S(a)=b_1+b_2$, where $b_1 \in \Psi(\Theta_1)$ and $b_2 \in \Psi(\Theta_2)$ by Proposition 2.4.2. Let $S_1:\underline{A} \to \underline{A}$ and $S_2:\underline{A} \to \underline{A}$ be defined by

$$S_1(a)=b_1 \in [S(a)]\,\Theta_2 \cap [0]\,\Theta_1$$

and $$S_2(a)=b_2 \in [S(a)]\,\Theta_1 \cap [0]\,\Theta_2 .$$

Then by the proof of Theorem 2.4.2, S_1 and S_2 are well defined and $S(a)=b_1+b_2=S_1(a)+S_2(a)=(S_1+S_2)(a)$. Hence S is parallel decomposable.

Conversely, suppose that S is injective and $S=S_1+S_2$. Let Θ_1 and Θ_2 be relations on S(A) defined by

$$(b,b') \in \Theta_1 \leftrightarrow S_1S^{-1}(b)=S_1S^{-1}(b')$$

$$(b,b') \in \Theta_2 \leftrightarrow S_2S^{-1}(b)=S_2S^{-1}(b'),$$

where $S^{-1}:S(\underline{A}) \to \underline{A}$ is the inverse of S. Then obviously, Θ_1 and Θ_2 are congruence relations on $S(\underline{A})$. Let $(b,b') \in \Theta_1 \cap \Theta_2$. Then $S_1S^{-1}(b)=S_1S^{-1}(b')$ and $S_2S^{-1}(b)=S_2S^{-1}(b')$. Therefore $b=SS^{-1}(b)=S_1S^{-1}(b)+S_2S^{-1}(b)=S_1S^{-1}(b')$ $+S_2S^{-1}(b')=SS^{-1}(b')=b'$. Hence $\Theta = \{\Theta_1, \Theta_2\}$ is separating. For any $a,b \in S(A)$, let $c \in S(A)$ be defined by $c=S_1S^{-1}(a)+S_2S^{-1}(b)$. Then $S_1S^{-1}(c)$ $=S_1S^{-1}S_1S^{-1}(a)+S_1S^{-1}S_2S^{-1}(b)=S_1S^{-1}(a)$ and $S_2S^{-1}(c)=S_2S^{-1}(b)$. Therefore $(a,b) \in \Theta_1 \circ \Theta_2$. Hence $\underline{\Theta}$ is full.

Q.E.D.

<u>Proof of Theorem 8.4.1</u> : Let $\underline{\Theta} = \{\Theta_1, \Theta_2\}$ be full and separating class of congruence relations on $\underline{A}$ and Θ_2 a congruence relation on $\underline{A}'=[A;F_G \cup F_U \cup \{S\}]$. Let $S_1:\underline{A} \to \underline{A}$ and $S_2:\underline{A} \to \underline{A}$ be defined by

$$S_1(a) \in [S(a)]\,\Theta_1 \cap [a]\,\Theta_2$$

and $$S_2(a) \in [S(a)]\,\Theta_2 \cap [a]\,\Theta_1.$$

Since $\underline{\Theta}$ is full and separating, S_1 and S_2 are well defined. Let $S_1(a)=b$ and $S_2(b)=c$. Then $c=S_2(b) \in [S(b)]\,\Theta_2 \cap [b]\,\Theta_1$ and $b=S_1(a) \in [S(a)]\,\Theta_1 \cap [a]\,\Theta_2$. Therefore $(c,S(b)) \in \Theta_2, (b,c) \in \Theta_1$, $(S(a),b) \in \Theta_1$ and $(a,b) \in \Theta_2$. Since Θ_2 is a congruence relation on A', $(S(a),S(b)) \in \Theta_2$. Then $(S(a),c) \in \Theta_2$. Therefore $(S(a),c) \in \Theta_1 \cap$

Θ_2 and therefore $S(a)=c$. Because $\underline{\Theta}$ is separating. Hence $S(a)=S_2(S_1(a))$ $=(S_1 \cdot S_2)(a)$.

Q.E.D.

<u>Proof of Theorem 8.5.1</u> : Suppose that $\Theta_2 \varepsilon \underline{\Theta}$ is a congruence relation on $\underline{A}'$. Since $\underline{\Theta}$ satisfies the conditions of Theorem 8.4.1, S is serial decomposable

such that $S=S_1 \cdot S_2'$, where $S_1:\underline{A} \to \underline{A}$ and $S_2':\underline{A} \to \underline{A}$ are defined by

$$S_1(a) \ \varepsilon \ [S(a)] \ \Theta_2 \cap [a] \ \Theta_1$$

$$S_2'(a) \ \varepsilon \ [S(a)] \ \Theta_1 \cap [a] \ \Theta_2 \ .$$

Then by the condition 3) and 4), S_1 and S_2' are bijective. Therefore there exist the inverse of S_1 and S_2', call it S_1^{-1} and $S_2'^{-1}$. Let $S_2=S_1^{-1}(I-S_2'^{-1})$, where $I:\underline{A} \to \underline{A}$ is the identity morphism. Then $S_1S_2=I-S_2'^{-1}$ and $S_2'=(I-S_1S_2)^{-1}$. Therefore $S=S_1 \cdot S_2'=S_1 \cdot (I-S_1S_2)^{-1}$. Hence S is feedback decomposable.

Q.E.D.

8. Proofs for Chapter 9

Proof of Theorem 9.1.1 : Suppose that there is a full and separating class $\underline{\Theta} = \{ \Theta_1, \Theta_2 \}$ of congruece relations on $\underline{C}(\hat{\phi})$. Let $\underline{C}^1(\overline{\phi}^1) = \underline{C}(\hat{\phi})/\Theta_1$ and $\underline{C}^2(\overline{\phi}^2) = \underline{C}(\hat{\phi})/\Theta_2$. Since Θ_1 and Θ_2 are congruence relations on $\underline{C}(\hat{\phi})$, quotient algebras are well-defined. Then $\underline{C}(\overline{\phi}^1) \times \underline{C}(\overline{\phi}^2) = \underline{C}(\hat{\phi})/\Theta_1 \times \underline{C}(\hat{\phi})/\Theta_2 \cong \underline{C}(\hat{\phi})$ by Theorem 2.5.2. Since $\underline{C}(\overline{\phi}^1) \times \underline{C}(\overline{\phi}^2)$ is the parallel connection of $\underline{C}(\overline{\phi}^1)$ and $\underline{C}(\overline{\phi}^2)$ by Proposition 3.3.3, $\underline{C}(\hat{\phi})$ is parallel decomposable, that is, $\hat{T}$ is parallel decomposable.

Conversely, suppose that $\underline{C}(\hat{\phi})$ is parallel decomposable. That is, there are transition systems $T_1=[C_1,\overline{X}, \overline{\phi}^1]$ and $T_2=[C_2,\overline{X}, \overline{\phi}^2]$, and an isomorphism $h:\hat{C} \to C_1 \times C_2$ such that

$$\hat{\phi}_{tt'}(\hat{c},x_{tt'}) = \phi_{tt'}(h(\hat{c}),x_{tt'})$$
$$=(\phi^1_{tt'}(p_1h(\hat{c}),x_{tt'})),\ \phi^2_{tt'}(p_2h(\hat{c}),x_{tt'}))$$

, where $T=[\overline{X}, C_1 \times C_2, \overline{\phi}]$ is the parallel connection of T_1 and T_2.
Let Θ_1 and Θ_2 be relations on $\underline{C}(\hat{\phi})$ defined by

$$(c,c') \in \Theta_1 \leftrightarrow p_1h(c) = p_1h(c')$$

and $$(c,c') \in \Theta_2 \leftrightarrow p_2h(c) = p_2h(c').$$

Then Θ_i is an equivalence relations on $\hat{C}$ for i=1,2. Moreover, since p_ih is a homomorphism from $\underline{C}(\hat{\phi})$ to $\underline{C}(\overline{\phi}^i)$, Θ_i satisfies the substitution property. Hence $\underline{\Theta} = \{ \Theta_1, \Theta_2 \}$ is a class of congruence relations on $\underline{C}(\hat{\phi})$. If $(c,c') \in \Theta_1 \cap \Theta_2$, $h(c)=h(c')$. Since h is injective, c=c'. And hence $\underline{\Theta}$ is separating. For any $\hat{c}_1,\hat{c}_2 \in \hat{C}$, let $c=(p_1h(\hat{c}_1),p_2h(\hat{c}_2)) \in C_1 \times C_2$. Since h is surjective, there is $\hat{c} \in \hat{C}$ such that $h(\hat{c})=(p_1h(\hat{c}),p_2h(\hat{c}))=c=(p_1h(\hat{c}_1),p_2h(\hat{c}_2))$.
Therefore $(c,c') \in \Theta_1 \circ \Theta_2$ and hence $\underline{\Theta}$ is full. Q.E.D.

Proof of Proposition 9.2.1 : Suppose that Θ is a congruence relation on $\underline{C}(\phi)$. Then if $(c,c') \varepsilon \Theta$, $(\phi_{tt'}(c,x_{tt'}), \phi_{tt'}(c',x_{tt'})) \varepsilon \Theta$ for any $x_{tt'} \varepsilon X_{tt'}$. Therefore (Θ, Θ) is a congruence pair of T.

Conversely, if (Θ, Θ) is a congruence pair of T, it is obvious that Θ is a congruence relation on $\underline{C}(\phi)$. Q.E.D.

Proof of Proposition 9.2.2 : Since Id $\subset \Theta$ for every congruence relation Θ on C, it is obvious from definition. Q.E.D.

Proof of Theorem 9.2.1 : Suppose that there is a full and separating class $\underline{\Theta} = \{\Theta_1, \Theta_2\}$ of congruence relations on $\hat{C}$ satisfying the conditions. Let $C_1=\hat{C}/\Theta_1$ and $C_2=\hat{C}/\Theta_2$ and construct state automata $T_1=[A,C_1,\delta_1]$ and $T_2=[C_1 \times A,C_2,\delta_2]$ by

$$\delta_1 : C_1 \times A \to C_1$$

$$([c]\Theta_1,a) \to [\hat{\delta}(c,a)]\Theta_1$$

$$\delta_2 : C_2 \times (C_1 \times A) \to C_2$$

$$([c_2]\Theta_2,([c_1]\Theta_1,a)) \to [\hat{\delta}(c,a)]\Theta_2$$

$$\text{, where } c \ \varepsilon\ [c_1]\Theta_1 \cap [c_2]\Theta_2$$

Since Θ_1 is a congruence relation on $\underline{C}(\delta)$, δ_1 is well defined. And since $\underline{\Theta}$ is full and separating and so $(\Theta_1 \cap \Theta_2, \Theta_2)$ is a congruence pair of T by Proposition 9.2.2, δ_2 is also well defined.

Let $T=[A,C,\delta]$ be the serial connection of T_1 and T_2, where $C=C_1 \times C_2$ and δ is defined as follows.

$$\delta : C \times A \to C ;$$

$$\delta(([c_1]\Theta_1,[c_2]\Theta_2),a)$$

$$=(\delta_1([c_1]\Theta_1,a), \delta_2([c_2]\Theta_2,([c_1]\Theta_1,a))).$$

Let $h:\hat{C} \to C$ be defined by

$$h(c)=([c]\,\Theta_1,[c]\,\Theta_2).$$

Then h is an isomorphism by Theorem 2.5.2. For any $(c,a)\ \varepsilon\ \hat{C}\times A$,

$$h\,\hat{\delta}\,(c,a)=([\,\hat{\delta}\,(c,a)]\,\Theta_1,[\,\hat{\delta}\,(c,a)]\,\Theta_2)$$

and $$\delta\,(h(c),a)=(\,\delta_1([c]\,\Theta_1,a),\ \delta_2([c]\,\Theta_2,([c]\,\Theta_1,a)))$$
$$=([\,\hat{\delta}\,(c,a)]\,\Theta_1,[\,\hat{\delta}\,(c,a)]\,\Theta_2).$$

Therefore the following diagram is commutative.

$$\begin{array}{ccccc}
\hat{C} & \times & A & \xrightarrow{\hat{\delta}} & \hat{C} \\
h\downarrow & & \downarrow Id & & \downarrow h \\
C & \times & A & \xrightarrow[\delta]{} & C
\end{array}$$

Hence $\hat{T}$ is serial decomposable.

Conversely, suppose that there are state automata $T_1=[A,C_1,\ \delta_1]$ and $T_2=[C_1\times A,C_2,\ \delta_2]$ such that $\hat{T}$ is decomposable to the serial connection of T_1 and T_2. Let $h:\hat{C}\to C_1\times C_2$ be an isomorphism and the diagram commutes. Let Θ_1 and Θ_2 be relations on $\hat{C}$ defined by

$$(c,c')\ \varepsilon\ \Theta_1 \leftrightarrow p_1h(c)=p_1h(c')$$

and $$(c,c')\ \varepsilon\ \Theta_2 \leftrightarrow p_2h(c)=p_2h(c'),$$

where $p_i:C_1\times C_2\to C_i$ is the i-th projection. Then it is easily seen that Θ_1 and Θ_2 are congruence relations on $\hat{C}$ and moreover $\underline{\Theta}=\{\Theta_1,\Theta_2\}$ is full and separating because h is bijective. If $(c,c')\ \varepsilon\ \Theta_1$,

$$p_1h(\,\hat{\delta}\,(c,a))=\delta_1(p_1h(c),a)=\delta_1(p_1h(c'),a)=p_1h(\,\hat{\delta}\,(c'a))$$

for all $a\ \varepsilon\ A$. Then Θ_1 is also a congruence relation on $\underline{C}(\,\delta\,)$. Hence we obtain the required results. Q.E.D.

9. Proofs for Chapter 10

<u>Proof of Proposition 10.1.1</u> : Suppose that $h=(h_x,h_y)$ is a strong modelling morphism. Let $(x,y) \varepsilon S$. Then $y \varepsilon S(x)$ and hence $h_y(y) \varepsilon h_y^*(S(x))=S(h_x(x))$. Therefore $(h_x(x),h_y(y)) \varepsilon S'$ and h is a modelling morphism.

Let $(h_x(x),y') \varepsilon S'$. Then $y' \varepsilon S'(h_x(x))=h_y^*(S(x))$. Therefore there exists $y \varepsilon Y$ such that $(x,y) \varepsilon S$ and $h_y(y)=y'$ and hence the condition is satisfied.

Conversely, suppose that h is a modellig morphism and satisfies the condition. Let $x \varepsilon X$ be fixed. For any $y' \varepsilon h_y^*(S(x))$, there exists $y \varepsilon S(x)$ such that $h_y(y)=y'$. Since h is a modelling morphism, $y'=h_y(y)$ $\varepsilon S'(h_x(x))$. Therefore $h_y^*(S(x)) \subset S'(h_x(x))$. And for any $y' \varepsilon S'(h_x(x))$, there exists $y \varepsilon S(x)$ such that $h_y(y)=y'$ by the condition. Therefore $y' \varepsilon h_y(y) \varepsilon h_y^*(S(x))$ and hence $S'(h_x(x)) \subset h_y^*(S(x))$. Consequently, $h=(h_x,h_y)$ is a strong modelling morphism.

Q.E.D.

<u>Proof of Lemma 10.1.1</u> : It is easy to show that $h|\tilde{S}:\tilde{S} \rightarrow S'$ is a modelling morphism because $\tilde{S} \subset S$. Let us show that $h|\tilde{S}$ satisfies the condition mentioned in Proposition 10.1.1. Let $(h_x(x),y') \varepsilon S'$ for any $x \varepsilon \tilde{X}$. Since $h:S \rightarrow S'$ is a strong modelling morphism, there is $y \varepsilon Y$ such that $(x,y) \varepsilon S$ and $h_y(y)=y'$. Then there is $y \varepsilon \tilde{Y}$ such that $(x,y) \varepsilon \tilde{S}$ and $h_y(y)=y'$ because $S|\tilde{X}=\tilde{S}$. Therefore $h|\tilde{S}$ is a strong modelling morphism.

Q.E.D.

Proof of Lemma 10.1.2 : For any $[x_i|i \in I] \in \Pi(X_i|i \in I)$, let $i_o \in I$ be fixed. Then there is $y_{io} \in Y_{io}$ such that $(x_{io},y_{io}) \in S_{io}$, $x_{io} \neq y_{io}$ and $([x_i|i \in I],[[x_i|i \in I-\{i_o\}],y_{io}]) \in S$. Therefore $X = \Pi(X_i|i \in I)$. Similarly, $Y = \Pi(Y_i|i \in I)$ and $X = Y$ because $X_i = Y_i$ for all $i \in I$.

Let $(p_{ix}(x),y_i) \in S_i$. If $p_{ix}(x) \neq y_i$, let $y=([p_{jx}(x)|j \in I-\{i\}], y_i)$. Then $(x,y) \in S$ and $p_{iy}(y)=y_i$. If $p_{ix}(x)=y_i$, let $y = [[p_{jx}(x)|j \in I-\{i_o\}],y_{io}]$, where $i \neq i_o$, $(p_{io}(x),y_{io}) \in S_{io}$ and $p_{io}(x) \neq y_{io}$. Then $(x,y) \in S$ and $p_{iy}(y)=y_i$. Therefore $p_i=(p_{ix},p_{iy})$ is a strong modelling morphism in both cases. Since $p_i=(p_{ix},p_{iy})$ is a surjective modelling morphism, p_i is a surjective strong modelling morphism.

Q.E.D.

Proof of Lemma 10.1.3 : Suppose that $\Pi(h_i|i \in I)|S$ is an injective strong modelling morphism. Let x_i and x_i' be arbitrary elements of X_i satisfying that $x_i \neq x_i'$. Let $x_j \in X_j$ be arbitrary for all $j \in I-\{i\}$. Then $([[x_j|j \in I-\{i\}],x_i],]]x_j|j \in I-\{i\}],x_i'])\in S$ from the definition of a disjunctive complex system. Since $\Pi(h_i|i \in I)([[x_j|j \in I-\{i\}],x_i], [[x_j|j \in I-\{i\}],x_i']) \in S'$, $h_{ix}(x_i) \neq h_{iy}(x_i')$. Therefore $h_{ix}(=h_{iy})$ is injective and hence $h_i=(h_{ix},h_{iy})$ is an injective modelling morphism.

Let $(h_{ix}(x_i),y_i') \in S_i'$. Suppose that $h_{ix}(x_i) \neq y_i'$. Let $x_j \in X_j$ be arbitrary for all $j \in I-\{i\}$. Then $[x_i|i \in I] \in X$ and $([h_{ix}(x_i)|i \in I], [[h_{jy}(x_j)|j \in I-\{i\}],y_i']) \in S'$. Since $\Pi(h_i|i \in I)|S$ is a strong modelling morphism, there is $[y_i|i \in I] \in Y$ such that $([x_i|i \in I],[y_i|i \in I]) \in S$ and $\Pi(h_{iy}|i \in I)(y_i|i \in I)=[[h_{jy}(x_j)|j \in I-\{i\}],y_i']$. Therefore there is $y_i \in Y_i$ such that $(x_i,y_i) \in S_i$ and $h_{iy}(y_i)=y_i'$. Let us next suppose that $h_{ix}(x_i)=y_i'$. Since $(x_i,x_i) \in S_i$, it is obvious that h_i is a strong modelling morphism. Hence h_i is an injective strong modelling morphism in both cases.

Conversely, let us suppose that $h_i=(h_{ix},h_{iy})$ is an injective strong modelling morphism for all $i \varepsilon I$. For any $([x_i|i \varepsilon I],[y_i|i \varepsilon I]) \varepsilon S$, there is $i \varepsilon I$ such that $(x_i,y_i) \varepsilon S_i$, $x_i \neq y_i$ and $x_j=y_j$ for all $j \varepsilon I- \{ i \}$. Then $\Pi (h_i|i \varepsilon I)(([x_i|i \varepsilon I],[y_i|i \varepsilon I])) \varepsilon S'$. Because h_i is an injective modelling morphism. Therefore $\Pi (h_i|i \varepsilon I)|S$ is a modelling morphism. Let $[x_i|i \varepsilon I] \neq [x_i'|i \varepsilon I]$. Then there is $i \varepsilon I$ such $x_i \neq x_i'$. Since h_i is an injective modelling morphism, $\Pi (h_{ix}|i \varepsilon I)([x_i|i \varepsilon I]) \neq \Pi (h_{ix}|i \varepsilon I)([x_i'|i \varepsilon I])$. Therefore $\Pi (h_i|i \varepsilon)|S=(\Pi (h_{ix}|i \varepsilon I), \Pi (h_{iy}|i \varepsilon I))$ is also an injective modelling morphism because $\Pi (h_{ix}|i \varepsilon I)= \Pi (h_{iy}|i \varepsilon I)$.

Let $(\Pi (h_{ix}|i \varepsilon I)([x_i|i \varepsilon]),[y_i'|i \varepsilon I]) \varepsilon S'$. Then there is $i \varepsilon I$ such that $(h_{ix}(x_i),y_i') \varepsilon S_i'$ and $h_{ix}(x_i) \neq y_i'$, and $h_{jx}(x_j)=y_j'$ for $j \varepsilon I- \{ i \}$. Since h_i is an injective modelling morphism, there is y_i such that $(x_i,y_i) \varepsilon S_i$, $x_i \neq y_i$ and $h_{iy}(y_i)=y_i'$. Therefore $([x_i|i \varepsilon I],[[x_j|j \varepsilon I- \{ i \}],y_i]) \varepsilon S$ and $\Pi (h_{iy}|i \varepsilon I)([[x_j|j \varepsilon I- \{ i \}],y_i])=[y_i'|i \varepsilon I]$. Hence $\Pi (h_i|i \varepsilon I)|S =(\Pi (h_{ix}|i \varepsilon I)|X, \Pi (h_{iy}|i \varepsilon I)|Y)$ is an injective strong morphism.

Q.E.D.

Proof of Lemma 10.1.4 : Suppose that $1=(1_x,1_y)$ is a strong modelling morphism. Let $(x,y) \varepsilon S$ and $[x]R_x{}^i \neq [y]R_y{}^i$. Suppose that $[x]R_x{}^i=[x']R_x{}^i$ for arbitrary $x' \varepsilon X$. Let $\overline{y}=[[[x']R_x{}^j|j \varepsilon I- \{ i \}],[y]R_x{}^i]$. Then $(1_x(x'),\overline{y}) \varepsilon DC(\underline{S})|\underline{D}(S/\underline{R})=S/\underline{R}$. Therefore there is $y' \varepsilon Y$ such that $(x',y') \varepsilon S$ and $1_y(y')=\overline{y}$ because 1 is a strong modelling morphism. Hence there is $y' \varepsilon Y$ such that $(x',y') \varepsilon S$ and $[y']R_y{}^i=[y]R_y{}^i$.

Conversely, suppose that the condition is satisfied. Let $(1_x(x), [[y_i]R_y{}^i|i \varepsilon I]) \varepsilon S/\underline{R} \subset DC(\underline{S})$. Then there is $i \varepsilon I$ such that $[x]R_x{}^i \neq [y_i]R_y{}^i$ and $[x]R_x{}^j=[y_j]R_y{}^j$ for all $j \varepsilon I- \{ i \}$. By the definition of $S/\underline{R}$, there is $(\hat{x},\hat{y}) \varepsilon S$ such that $\hat{x} \varepsilon \cap 1_x(x)$ and $\hat{y} \varepsilon \cap ([y_i]R_y^i | i \varepsilon I])$. Since $[x]R_x^i=[\hat{x}]R_x^i$, there is $y \varepsilon Y$ such that $(x,y) \varepsilon S$ and $[y]R_y^i=[\hat{y}]R_y^i=[y_i]R_y^i$. And since $1(x,y) \varepsilon DC(\underline{S})$, $[y]R_y^j =[x_j]R_x^j$ for all $j \varepsilon I-\{ i \}$. Therefore $1_y(y)=[[y]R_y^i|i \varepsilon I]=[[[y]R_y^j |j \varepsilon I-\{ i \}],[y]R_y^i]=[[[x]R_x^j|j \varepsilon I-\{ i \}],[y_i]R_y^i]=[[y_i]R_y^i|i \varepsilon]$ and hence $1=(1_x,1_y)$ is a strong modelling morphism. Q.E.D.

<u>Proof of Lemma 10.1.5</u> : It is easy to see that $i=(i_x,i_y)$ is a modelling morphism. For any $(i_x([[x]R_x^i|i \varepsilon I]),[[y_i]R_y^i|i \varepsilon I]) \varepsilon DC(\underline{S})$, there is $(x,y) \varepsilon S$ such that $1_x(x)=[[x]R_x^i|i \varepsilon I]=i_x([[x]R_x^i|i \varepsilon I])$ and $y \varepsilon \cap 1_y(y)= \cap [[y]R_y^i|i \varepsilon I]$. Therefore $(1_x(x),1_y(y)) \varepsilon S/\underline{R}$ and $i_y \cdot 1_y (y)=i_y([[y]R_y^i|i \varepsilon I])=[[y]R_y^i|i \varepsilon I]=[[y_i]R_y^i|i \varepsilon I]$. Hence $i=(i_x,i_y)$ is a strong modelling morphism. Q.E.D.

<u>Proof of Theorem 10.1.1</u> :

<u>Only if part</u> : Suppose that $h=(h_x,h_y):\hat{S} \rightarrow S$ is a strong modelling morphism and $h_x=h_y$. Let $R_x^i=R_y^i$ be an equivalence relation on $\hat{X}(=\hat{Y})$ defined by

$$x \; R_x^i \; x' \leftrightarrow p_{ix}h_x(x) = p_{ix}h_x(x').$$

And let R^i be an equivalence relation on $\hat{S}$ induced by (R_x^i,R_y^i), that is,

$$(x,y) \; R^i \; (x',y') \leftrightarrow x \; R_x^i \; x' \text{ and } y \; R_y^i \; y'.$$

Then R^i is input-output compatible. And let $k_i=(k_{ix},k_{iy}):\hat{S}_i \rightarrow S_i$ be defined by

$$k_{ix}:X/R_x^i \rightarrow X_i \; ; \; [x]R_x^i \rightarrow p_{ix}h_x(x)$$
$$k_{iy}:Y/R_y^i \rightarrow Y_i \; ; \; [y]R_y^i \rightarrow p_{iy}h_y(y).$$

Then $k_{ix}=k_{iy}$ and k_i is an embedding strong modelling morphism because h is a strong modelling morphism.

For any $([[x_i]R_x^i|i \in I]],[[y_i]R_y^i|i \in I]]) \in DC(\underline{S})|\underline{D}(S/\underline{R})$, there is $x \in \cap [[x_i]R_x^i|i \in I]$ such that $(h_x(x), [p_{iy}h_y(y_i)|i \in I]) \in S$. Since h is a strong modelling morphism, there is $y \in \hat{Y}$ such that $(x,y) \in \hat{S}$ and $p_{iy}h_y(y)=p_{iy}h_y(y_i)$ for any $i \in I$. Therefore $DC(\underline{S})|\underline{D}(S/\underline{R}) \subset S/\underline{R}$.

For any $([[x]R_x^i|i \in I],[[y]R_y^i|i \in I]]) \in S/\underline{R}$, $(h_x(x),h_y(y)) \in S$. Since S is a disjunctive complex system, $([[x]R_x^i|i \in I],[[y]R_y^i|i \in I])$ $\in DC(\underline{S})$ by the definition of R^i. Hence $S/\underline{R}=DC(\underline{S})|\underline{D}(S/\underline{R})$.

Let $(x,y) \in S$ and $[x]R_x^i \neq [y]R_y^i$. Suppose that $[x]R_x^i=[\hat{x}]R_x^i$ for arbitrary $\hat{x} \in X$. Let $y'=[[p_{jx}h_x(\hat{x})|j \in I- \{ i \}],p_{iy}h_y(y)]$. Then $(h_x(\hat{x}),y') \in S$. Since h is a strong modelling morphism, there is $\hat{y} \in \hat{Y}$ such that $(\hat{x},\hat{y}) \in S$ and $h_y(\hat{y})=y'$. Hence $[\hat{y}]R_y^i=[y]R_y^i$.

<u>If part</u> : Suppose that all the conditions are satisfied. Let us define $h_x= \Pi (k_{ix}|i \in I) \cdot 1_x$ and $h_y= \Pi (k_{iy}|i \in I) \cdot 1_y$. Then $h=(h_x,h_y)$ is a strong modelling morphism. Because $1=(1_x,1_y)$ is a strong modelling morphism by Lemma 10.1.4, so is the inclusion operation from $S/\underline{R}$ to $DC(\underline{S})$ by Lemma 10.1.5 and $\Pi (k_i|i \in I)=(\Pi (k_{ix}|i \in I)$, $\Pi (k_{iy}|i \in I)):DC(\underline{S}) \to DC(\underline{S})$ is also a strong modelling morphism by Lemma 10.1.3. Q.E.D.

<u>Proof of Proposition 10.2.1</u> : At first, let us show that SGF g of S satisfies the following condition ;

$$g(x)=\min(N- \{ g(y)|y \in S(x) \text{ and } x \neq y \}).$$

Suppose that $g(x) < \min(N- \{ g(y)|y \in S(x) \text{ and } x \neq y \})$. Then $g(x)$ $\in \{ g(y)|y \in S(x) \text{ and } x \neq y \}$ and hence there is $y \in S(x)$ such that $x \neq y$ and $g(x)=g(y)$. It contradicts the condition 1) of Definition 10.2.1. Let us now suppose that $\min(N- \{ g(y)|y \in S(x) \text{ and } x \neq y \}) < g(x)$.

Then there is $y \varepsilon S(x)$ such that $g(y)=\min(N- \{ g(y)|y \varepsilon S(x)$ and $x \neq y \})$ from the condition 2) of Definition 10.2.1. Therefore $g(y) \underline{\varepsilon} \{ g(y)|y \varepsilon S(x)$ and $x \neq y \}$ and it is also a contradiciton. Hence $g(x)=\min(N- \{ g(y)|y \varepsilon S(x)$ and $x \neq y \})$.

Let us now show that SGF is unique. Suppose that g and g' are SGFs of S. Let $W_o=Y-X$. Then $g(W_o)=g'(W_o)=0$. Because $x \varepsilon W_o$ implies that $g(x)$ $=g'(x)=\min(N- \phi)=0$. Suppose that $g(x) \neq g'(x)$ for some $x \varepsilon X-Y$. Then there is $y \varepsilon S(x)$ such that $x \neq y$ and $g(y) \neq g'(y)$ because $\{ g(y)|y \varepsilon S(x)$ and $x \neq y \} \neq \{ g'(y)|y \varepsilon S(x)$ and $x \neq y \}$. Continueing this process, we get $g(W) \neq g(W')$ for some $w \varepsilon W_o$ because S is progressively bounded. Then this is a contradiction and hence SGF is uniquely determined.

Q.E.D.

Proof of Lemma 10.2.1 : Let us show that g satisfies the conditions of Definition 10.2.1. Let $(x,y) \varepsilon S$ and $x \neq y$. Then $(h_x(x),h_y(y)) \varepsilon S'$ and $h_x(x) \neq h_y(y)$ and hence $g(x)=g'h_x(x) \neq g'h_y(y)=g(y)$ because g' is SGF. Let $m < g(x)=g'h_x(x)$ for arbitrary $x \varepsilon X$. Then there is $y' \varepsilon Y'$ such that $(h_x(x),y') \varepsilon S'$ and $g'(y')=m$. Since h is a strong modelling morphism, there is $y \varepsilon Y$ such that $(x,y) \varepsilon S$ and $h_y(y)=y'$, hence $g(y)$ $=g'h_y(y)=g'(y')=m$. Let $m < g(x)$ for arbitrary $x \varepsilon Y-X$. If $h_y(y) \varepsilon X'$, there is $y' \varepsilon Y'$ such that $(h_x(x),y') \varepsilon S$ and $g'(y')=m$. Since h is a strong modelling morphism, there is $y \varepsilon Y$ such that $(x,y) \varepsilon S$ and $h_y(y)=y'$. This is a contradiction. Therefore g is SGF of S.

Q.E.D.

Proof of Lemma 10.2.2 : See the reference [24].

<u>Proof of Theorem 10.2.1</u> : Since S is a disjunctive complex system, SGF g of S can be defined by

$$g([x_i | i \in I]) = \bigoplus (g_i(x_i) | i \in I)$$

by Lemma 10.2.2. And $h=(h_x,h_y)$ is a strong modelling morphism, SGF $\hat{g}$ of $\hat{S}$ can be defined by

$$\hat{g}(x) = \begin{cases} gh_x(x) = \bigoplus (g_i p_{ix} h_x(x) | i \in I) & x \in \hat{X} \\ gh_y(y) = \bigoplus (g_i p_{iy} h_y(y) | i \in I) & x \in \hat{Y} \end{cases}$$

by Lemma 10.2.1. Since $h|\tilde{S}$ is also a strong modelling morphism by Lemma 10.1.1, SGF $\tilde{g}$ of $\tilde{S}$ can be defined by

$$\tilde{g}(x) = \begin{cases} gh_x(x) = \bigoplus (g_i p_{ix} h_x(x) | i \in I) & x \in \tilde{X} \\ gh_y(x) = \bigoplus (g_i p_{iy} h_y(x) | i \in I) & y \in \tilde{Y} \end{cases}$$

Q.E.D.

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