T2R2 東京科学大学 リサーチリポジトリ
Science Tokyo Research Repository

論文 / 著書情報
Article / Book Information

題目(和文)	
Title(English)	Theory of Fuzzy Integrals and Its Applications
著者(和文)	菅野道夫
Author(English)	MICHIO SUGENO
出典(和文)	学位:工学博士, 学位授与機関:東京工業大学, 報告番号:乙第586号, 授与年月日:1975年1月22日, 学位の種別:論文博士, 審査員:
Citation(English)	Degree:Doctor of Engineering, Conferring organization: Tokyo Institute of Technology, Report number:乙第586号, Conferred date:1975/1/22, Degree Type:Thesis doctor, Examiner:
学位種別(和文)	博士論文
Type(English)	Doctoral Thesis

Powered by T2R2 (Science Tokyo Research Repository)

THEORY OF FUZZY INTEGRALS AND ITS APPLICATIONS

by

Michio Sugeno

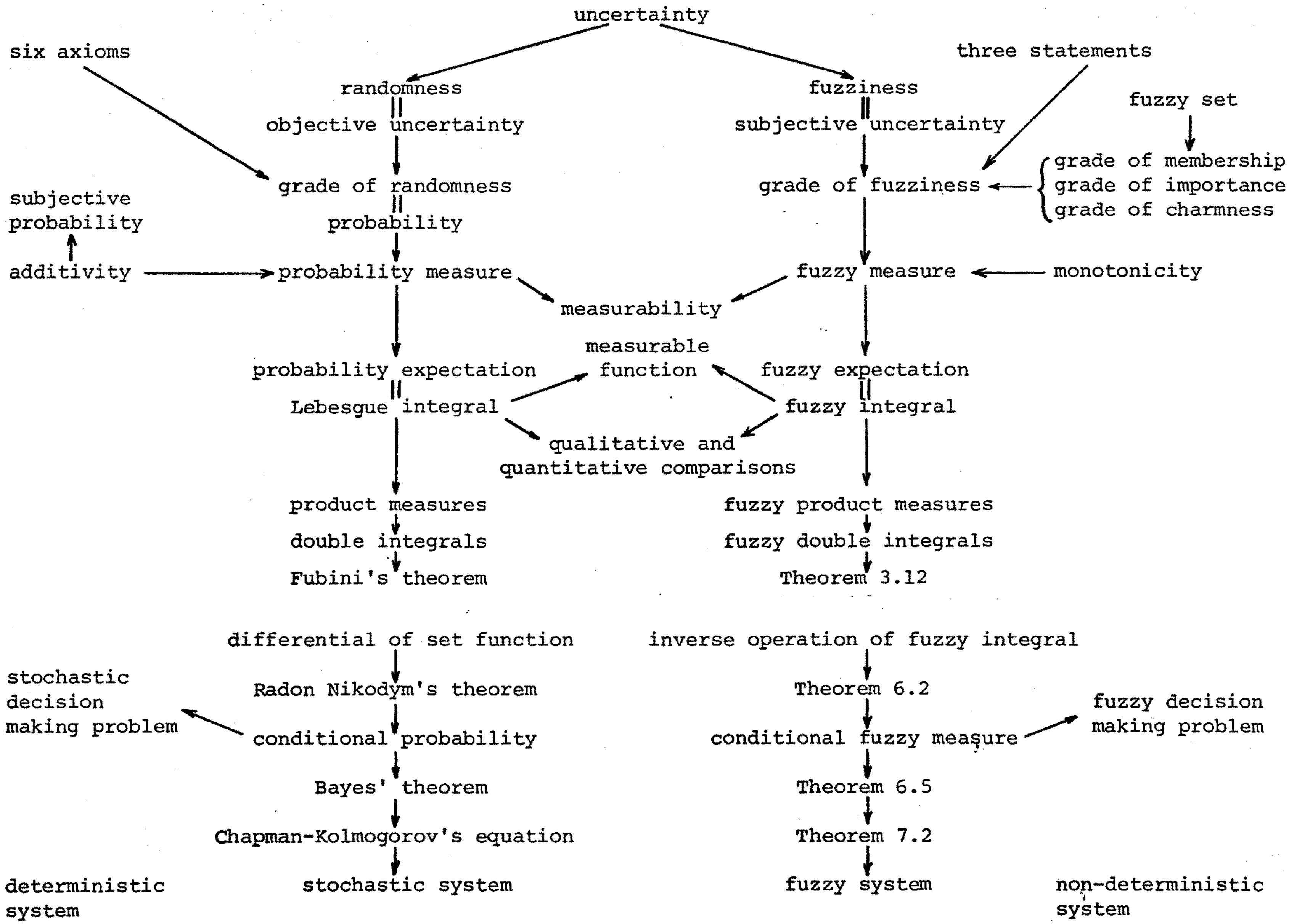
A Dissertation
Submitted in Partial Fulfillment of the
Requirements for the Degree of
Doctor of Engineering

at

Tokyo Institute of Technology

1974

東京工業大学 353373


uncertainty
six axioms
randomness
objective uncertainty
grade of randomness
probability
subjective
probability
additivity
probability measure
three statements
fuzziness
subjective uncertainty
fuzzy set
grade of membership
grade of importance
grade of charmness
grade of fuzziness
fuzzy measure
monotonicity
measurability
probability expectation
Lebesgue integral
measurable
function
fuzzy expectation
fuzzy integral
qualitative and
quantitative comparisons
product measures
double integrals
Fubini's theorem
fuzzy product measures
fuzzy double integrals
Theorem 3.12
differential of set function
Radon Nikodym's theorem
stochastic
decision
making problem
conditional probability
Bayes' theorem
Chapman-Kolmogorov's equation
stochastic system
deterministic
system
inverse operation of fuzzy integral
Theorem 6.2
conditional fuzzy measure
fuzzy decision
making problem
Theorem 6.5
Theorem 7.2
fuzzy system
non-deterministic
system

THEORY OF FUZZY INTEGRALS AND ITS APPLICATIONS

by

Michio Sugeno

A Dissertation

Submitted in Partial Fulfillment of the

Requirements for the Degree of

Doctor of Engineering

at

Tokyo Institute of Technology

1974

ACKNOWLEDGEMENTS

The author wishes to express his gratitude to his supervisor Professor T. Terano for directing his interest toward the fuzzy integrals theory. He further wishes to thank Professor Y. Morita and Mr. Y. Tsukamoto for their continuous encouragement and fruitful discussions. Also, he is grateful to Professor K. Kunisawa, Professor A. Ichikawa, Professor M. Mori and Professor K. Furuta for their comments and suggestions during the preparation of the dissertation. Special appreciation goes to his wife, Hikaru, for her patience and understanding throughout the time the author spent for the dissertation.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

LIST OF SYMBOLS

Symbol	Description
$a \vee b$	: max(a,b)
$a \wedge b$	: min(a,b)
$\bigvee_{i=1}^{n} a_i$	: $\max_{1 \leq i \leq n} \{a_i\}$
X	: arbitrary and non-fuzzy set
x	: element of X
χ_E	: characteristic function of non-fuzzy set E
h_A	: membership function of fuzzy set A
$\mathcal{F}$	: monotone family of subsets of X
$\mathcal{B}$	: Borel field of X
$\tilde{\mathcal{B}}$	: fuzzy extension of $\mathcal{B}$
2^X	: power set of X
$\tilde{2}^X$	: family of all fuzzy subsets of X
E^c	: complement of set E
A^{c_λ}	: λ-complement of set A
K	: finite set
s	: element of K
R^1	: one dimensional Euclidean space
[0,1]	: unit closed interval of R^1
$g(\cdot)$	: fuzzy measure
g^i	: fuzzy density
H	: fuzzy distribution function
$p(\cdot)$	: probability measure

p^i	: probability density
λ	: $-1 < \lambda < \infty$
$⨍$	: symbol of fuzzy integral
h	: mapping from X to [0,1]
$\sup_{\alpha}$	: $\sup_{\alpha \in [0,1]}$
$g^{(\phi)}$	: fuzzy measure induced by mapping ϕ
$\rho(\cdot \mid \phi=y)$	: fuzzy measure under condition of $\phi = y$
$\rho_Y(\cdot \mid x)$	: conditional fuzzy measure from X to Y

Chapter 1

INTRODUCTION

In recent years, artificial intelligence, behavioural science, and human engineering, etc. which originated in cybernetics have found many applications in all fields of engineering. Together with this tendency, a variety of problems on human subjectivity which was studied first mainly in psychology have become problems in engineering. Here, a fundamental doubt is directed toward the fact that engineering has been inquiring objectivity by eliminating subjectivity.

Problems on human beings have caught a general interest also in the field of systems engineering, where it is often pointed out that control systems should be regarded as essentially man-machine systems.

One of the reasons for these facts would be that human abilities of judgement, analogy based on experience, and adaptation to any unfamiliar environment, etc. have become again considered important compared with computers.

From the viewpoint of attaching importance to human subjectivity, we may question, when we talk about an optimal control system and its criterion. "For whom is the system optimal?" or "To whom does the criterion belong?"

Concerning subjectivity among the characteristics of men which are superior to those of machines, L. A. Zadeh presented in 1965 the concept of fuzzy sets [1], which has given us a powerful means to deal with subjectivity by methods of mathematics as well as engineering.

Since his proposal, fuzzy sets theory have been widely applied in the fields of automata [4,5,6], linguistics [8,16], algorithm [7], pattern recognition [17], and so on.

The concept of "fuzziness" corresponding to randomness in probability theory is introduced in the fuzzy sets theory. Here, fuzziness is defined as a kind of uncertainty which is caused by subjectivity and belongs to the side of subject. On the other hand, randomness can be considered as one caused by random phenomena, i.e., objective and physical phenomena.

This dissertation divides uncertainties into two classes: randomness and fuzziness, and it discusses fuzziness in comparison with randomness. The purposes of this dissertation are to propose the concept of fuzzy measures and fuzzy integrals [11, 12] as a way for expressing human subjectivity and to discuss their applications.

Fuzzy measures are defined as subjective scales for fuzziness. Fuzzy integrals are the functionals with monotonicity defined by using fuzzy measures. Those correspond to probability expectations and are discussed in comparison with Lebesgue integrals.

Algebraic methods have been mainly used to approach fuzziness so far, while analytical methods have been seldom explored. Fuzzy measures and fuzzy integrals belong to analytical methods which enable us to deal with fuzziness qualitatively and quantitatively.

Fuzzy measures are set functions with monotonicity which have not necessarily additivity, while the set functions which have been investigated in mathematics are mostly endowed with additivity such as Lebesgue measures. With this point of view, the feature of this dissertation will be seen where monotone set functions are studied and their applications are discussed.

We start with the concept of "grade of fuzziness", which is the main argument of this dissertation. Fuzzy measures are interpreted as "measures" expressing grade of fuzziness and are compared with probability measures expressing grade of randomness. This comparison will also serve to make clear the difference between fuzziness and randomness.

The fuzzy integrals theory is constructed and developed almost independently of the fuzzy sets theory. However, the concept of fuzzy sets are referred often in this dissertation, since the operations used in fuzzy integrals match with those in fuzzy sets. Therefore, let us mention briefly the definition of fuzzy sets.

Let X be an arbitrary set which is treated in the ordinary sets theory. Denote an element of X by x. As is widely known, an arbitrary subset E of X can be defined by its characteristic function such that

$$\chi_E : X \to \{0,1\} .$$

That is, $x \in E$ if $\chi_E(x) = 1$ and $x \notin E$ if $\chi_E(x) = 0$. (Note: in this dissertation, the characteristic function of a non-fuzzy set is always denoted by χ.) The set, $\{0,1\}^X$, of all mappings from X to $\{0,1\}$ is isomorphic to the power set 2^X of X. Now let us consider $[0,1]^X$, i.e., the set of all mappings such that

$$h : X \to [0,1].$$

We attach a label to an element of $[0,1]^X$ and write it, for instance, as h_A. Then A is called a fuzzy subset of X and h_A the membership function of A. Here, $h_A(x)$ expresses the grade of membership of x in A. By $\tilde{2}^X$, denote the family of all fuzzy subsets of X. Then $[0,1]^X$ and $\tilde{2}^X$ are isomorphic. From $\{0,1\}^X \subset [0,1]^X$ follows $2^X \subset \tilde{2}^X$. Therefore a fuzzy set can be regarded as an extension of an ordinary set.

We omit to explain the meaning of fuzzy sets, regarding it as already known. The operations of union, intersection, complement and the relations containment and equivalence are defined for all members of $\tilde{2}^X$ as follows:

$$h_{A \cup B}(x) = h_A(x) \vee h_B(x)$$

$$h_{A \cap B}(x) = h_A(x) \wedge h_B(x)$$

$$h_{A^c}(x) = 1 - h_A(x)$$

$$A \subset B \text{ iff } h_A(x) \leq h_B(x) \text{ for any } x \in X$$

$$A = B \text{ iff } h_A(x) = h_B(x) \text{ for any } x \in X$$

where $a \vee b$ means $\max(a,b)$ and $a \wedge b$ $\min(a,b)$.

Further in this dissertation the next notation is used:

$$\bigvee_{i=1}^{n} a_i = \max_{1 \leq i \leq n} \{a_i\}.$$

A fuzzy subset R of X x Y is called a fuzzy relation. Let R_1 be a fuzzy relation in X x Y and R_2 in Y x Z. Then the composition of R_1 and R_2 is defined as follows:

$$h_{R_1 \circ R_2}(x,z) = \sup_{y \in Y} [h_{R_1}(x,y) \wedge h_{R_2}(y,z)].$$

This dissertation is organized as follows. Chapter 2 defines fuzzy measures of a set X based on the statements concerned with grade of fuzziness and discusses the method of their construction. The set X is an arbitrary set. There is no assumption on its topological aspects before Chapter 6. Fuzzy measures constructed in Section 2.2 are very useful ones which are used in the applications of Chapter 5 and Section 7.2.

Chapter 3 defines fuzzy integrals by using fuzzy measures and states their properties. All Chapters following this Chapter are concerned with the matters which are obtained half inevitably and half derivatively.

In Section 3.2, fuzzy integrals are compared with Lebesgue integrals qualitatively as well as quantitatively. In Section 3.3, fuzzy double integrals are defined and a theorem corresponding to Fubini's theorem in the theory of Lebesgue integrals is proved. In Section 3.4, the domain of fuzzy measures is extended onto a subset of $\tilde{2}^X$. Here, the relation between grade of fuzziness and that of membership is discussed.

Chapter 4 rewrites, restricting X to a finite set, some results of Chapters 2 and 3 for convenience in calculating fuzzy integrals, giving actual examples.

Chapter 5 deals with two problems of subjective evaluation as applications of fuzzy integrals. It is attempted to express subjectivity through fuzzy measures. The subjectivity of human being in general is discussed in Section 5.2 and the subjectivity of individuals in Section 5.3.

Chapter 6 finds an inverse operation of fuzzy integrals by which conditional fuzzy measures are defined. After Chapter 6, a set X is assumed to be a locally compact Hausdorff space with the second countability axiom. Conditional fuzzy measures correspond to conditional probabilities.

Chapter 7 discusses two applications of conditional fuzzy measures. One of them is a concrete application concerned with fuzzy decision-making problems. The other is an application to the representation of fuzzy systems. Here, the concept of fuzzy disturbances corresponding to that of stochastic disturbances is presented, and an expression of fuzzy systems is derived as in the case of stochastic systems.

Chapter 2

FUZZY MEASURES

2.1 Definition of fuzzy measures

The measures discussed so far in the theory of Lebesgue integrals or in probability theory are the set functions with additivity. Here, extending the concept of the measures, we consider "measures" as monotone set functions which are not necessarily additive. The concept of "measures" discussed in this chapter can be summarized in three statements concerned with grade of fuzziness. These statements are similar to the axioms of probability theory. It should be noticed, however, that there are essential differences between them.

Now, let X be an arbitrary set and ϕ an empty set. Let x denote an element of X and let A, B, etc. denote subsets of X. $\{F_n | 1 \leq n < \infty\}$, a family of subsets of X, is written merely $\{F_n\}$. If $F_1 \subset F_2 \subset \ldots \subset F_n \subset \ldots$ or $F_1 \supset F_2 \supset \ldots \supset F_n \supset \ldots$, then $\{F_n\}$ is called a monotone sequence.

We now enter the main argument. First, suppose that a person picks up an element x out of X, but does not know which one he has picked up. Next, suppose that he guesses if x belongs to a given subset A. It is uncertain and fuzzy for him whether $x \in A$ or not. His guess would become subjective when there are few clues for guessing.

Assume in general that a human being has a subjective quantity called the grade of fuzziness measuring fuzziness such as stated above.

Then our statements are described as follows:

(1) Grade of $x \in \phi = 0$ and grade of $x \in X = 1$.

(2) If $A \subset B$, then grade of $x \in A \leq$ grade of $x \in B$.

The third statement concerned with continuity will be seen in the definition of fuzzy measures.

By the term, "the grade of fuzziness", we imply the quantity which depends heavily on human subjectivity. When one says that an object is uncertain, we can consider two kinds of uncertainties. One is uncertainty due to the lack of information and knowledge. This uncertainty is an objective one which is characterized by the nature of objects and the circumstance surrounding them. For instance, the probability of the result of throwing a die is independent of a subjectivity and dependent only on the nature of the die and its circumstance. The other is the subjective uncertainty due to human subjectivity: the niceness of a woman's face is affected by a man's subjectivity besides her looks. The objective uncertainty is called randomness and the subjective one fuzziness.

We reenter the main argument. The grade of $x \in A$ is merely an abstract example of the grade of fuzziness. As more concrete examples we can consider "the grade of importance" or "the grade of charmness" which is stated later in the applications. Though it may be adequate for understandings of our statements that the grade of importance is picked up, we do not mention it in this chapter.

Now, let us introduce fuzzy measures for expressing the grade of fuzziness.

[Definition 2.1] $\mathcal{F}$, a monotone family of subsets of X, has the next properties.

1) Φ, $X \in \mathcal{F}$.

2) If $F_n \in \mathcal{F}$ and $\{F_n\}$ is monotone, then $\lim_{n\to\infty} F_n \in \mathcal{F}$.

[Definition 2.2] A set function g defined on $\mathcal{F}$ which has the following properties is called a fuzzy measure.

1) $g(\phi) = 0$ and $g(X) = 1$.

2) If A, $B \in \mathcal{F}$ and $A \subset B$, then $g(A) \leq g(B)$.

3) If $F_n \in \mathcal{F}$ and $\{F_n\}$ is monotone, then $\lim_{n\to\infty} g(F_n) = g(\lim_{n\to\infty} F_n)$.

Here, (1) means boundedness and non-negativity, (2) monotonicity, and (3) continuity. The property (2) is the most important one and (3) is essential only when X is an infinite set.

In the above definition g(A) is the expression of grade of $x \in A$. In general, let us interpret g(A) as a subjective measure expressing the grade of fuzziness of a set A. Of course, this does not necessarily mean that A is a fuzzy set.

Though A exists objectively for any one, we regard it imaginarily fuzzy since it is associated with subjectivity when a person guesses, for instance, grade of $x \in A$. In probability theory, a set A is called an event. But we do not use the terminology "event" because we wish to distinguish grade of fuzziness from probability (grade of randomness). We shall discuss in Section 3.4, the relation between the grade of fuzziness of A and the grade of membership of x in a fuzzy set.

Now, an order can be introduced into a family of subsets of X by $g(\cdot)$. It is, however, doubtful that human subjectivity is fine and reasonable enough to arrange the all subsets of X in an order. Therefore we restrict the application of grade of fuzziness to the subsets which can be compared. The definition of $\mathfrak{F}$ implies this limitation. For instance, it is necessary that ϕ and X can be compared by g and $g(\phi) \leq g(X)$ is satisfied.

[Definition 2.3] $(X, \mathfrak{F})$ is called an F-measurable space or simply a measurable space.

[Definition 2.4] $(X, \mathfrak{F}, g)$ is called a fuzzy measure space.

Here g is called a fuzzy measure of $(X, \mathfrak{F})$. When the domain of g is evident, g is simply called a fuzzy measure of X.

In this paper, a Borel field $\mathfrak{B}$ is mainly adopted as the domain of g except in Section 3.4, because this enables us to broaden the region of measurable functions defined in the next chapter. The Borel field $\mathfrak{B}$ has the following properties.

1) $\phi \in \mathfrak{B}$.
2) If $E \in \mathfrak{B}$, then $E^c \in \mathfrak{B}$.
3) If $E_n \in \mathfrak{B}$ for $1 \leq n < \infty$, then $\bigcup_{n=1}^{\infty} E_n \in \mathfrak{B}$.

$\mathfrak{B}$ satisfies the all properties of $\mathfrak{F}$, since it follows from these that $X \in \mathfrak{B}$ and if $F_n \in \mathfrak{B}$ and $\{F_n\}$ is monotone, then $\lim_{n\to\infty} F_n \in \mathfrak{B}$.

Now, additivity is the most important property among the properties of ordinary measures. It is, however, doubtful that an individual uses a "measure" with additivity when he subjectively measures fuzziness. Though a reasonable man is imagined in the theory of subjective probabilities, it would be more realistic to assume that an actual man has no additive measure, because his behaviours are often contradictory to the assumption that he uses an additive measure in evaluating things. As will be seen in the example of applications in Chapter 7.1, the identified fuzzy measure, which is interpreted as the grade of importance subject to a person, does not satisfy the condition of additivity.

Monotonicity is a very natural assumption on the subjective judgements of an actual man, while additivity is a restrictive one. In many applications, we can easily accept that if $A \subset B$, then grade of $x \in A \leq$ grade of $x \in B$. However in some applications, it is not surprising that there exists a man who regards grade of $x \in A \geq$ grade of $x \in B$. Note that if there holds grade of $x \in A \geq$ grade of $x \in B$ for all A and B such as $A \subset B$, then it brings no difficulty to our discussions. Because, by defining $g(A) = 1 -$ grade of $x \in A$, we obtain $g(A) \leq g(B)$.

Further if our statements are adopted for the conditions satisfied by a man's subjective measure, it would be pointed out that the interpretation of subjective measures becomes rather free in comparison with probabilities. It is very difficult to explain, for instance, the grade of charmness in terms of probabilities, which will be discussed in Section 7.2.

[Proposition 2.1] Let g be a fuzzy measure of $(X, \mathcal{B})$. It follows that if $E, E' \in \mathcal{B}$, then

1) $g(E \cup E') \geq g(E) \vee g(E')$ (2.1)

2) $g(E \cap E') \geq g(E) \wedge g(E')$. (2.2)

Here 1) or 2) is also a sufficient condition for monotonicity of a set function g on $\mathcal{B}$.

[Definition 2.5] If $g(E \cup E') = g(E) \vee g(E')$ for all $E, E' \in \mathcal{B}$, then g is said to be F-additive.

Let us next show a few examples of fuzzy measures.

[Example 1] We have shown the concept of grade of $x \in A$ at the beginning of this chapter. Here, let us assume that a person actually knows which x he has picked up. Let this x be x_0. Then it follows that the grade of $x_0 \in A$ is equal to 1 if $x_0 \in A$ and equal to 0 if $x_0 \notin A$. Define $g(x_0, A) = \chi_A(x_0)$ where χ_A is the characteristic function of A. It is clear that $g(x_0, \cdot)$ has the properties of a fuzzy measure.

[Example 2] Let $h : X \to [0,1]$ and define $\Psi(A) = \sup_{x \in A} h(x)$. Then it follows that if $A \subset B$, then $\Psi(A) \leq \Psi(B)$. Here Ψ satisfies the conditions of fuzzy measures under the proper assumptions on h. (This type of monotone set functions is seen in [13].) Note: $\Psi(\cdot)$ is F-additive.

[Example 3] Let $(X, \mathcal{B}, P)$ be a probability space. The probability measure P has the following properties.

1) $0 \leq P(E) \leq 1$ for all $E \in \mathcal{B}$, particularly, $P(X) = 1$.

2) If $E_n \in \mathcal{B}$ for $1 \leq n < \infty$ and $E_i \cap E_j = \phi$ for $i \neq j$, then $P(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} P(E_n)$.

It follows from these that if $A \subset B$, then $P(A) \leq P(B)$. Here P is one of fuzzy measures since $P(\phi) = 0$ and continuity are derived from 2).

As is seen in Example 3, we obtain a fuzzy measure by loosing the properties of a probability measure. Therefore a fuzzy measure may be regarded as the extended Lebesgue measure.

2.2 Construction of fuzzy measures

In order to execute fuzzy integrals later defined in Section 3.1 on $(X, \mathcal{B}, g)$, the values of g must be given over the whole domain of g. By the way, since g has monotonicity, it is necessary that this property is satisfied for all the members of $\mathcal{B}$ without contradictions. Let us assume that a finite set K with n members is taken as X and the power set 2^K as $\mathcal{B}$. The number of monotone sequences of subsets in 2^K is n!. So it is almost impossible to determine g for each of them.

In this chapter let us determine g for one monotone sequence and construct a rule by which g is defined for all other members of $\mathcal{B}$. This leads us to the problem of how $g(E \cup E')$ should be constructed by $g(E)$ and $g(E')$ where $E, E' \in \mathcal{B}$ and $E \cap E' = \emptyset$.
That is, we wish to determine f when $g(E \cup E')$ is written as follows.

$$g(E \cup E') = f(g(E), g(E')) \text{ where } E \cap E' = \phi. \qquad (2.3)$$

Now let us define the function f by

$$f(x, y) = x + y + \lambda xy \text{ where } -1 < \lambda < \infty. \qquad (2.4)$$

Using the function f, the second condition of fuzzy measures in Definition 2.2 can be replaced by Eq. (2.3).

[Definition 2.6] A set function $\psi(\cdot)$ having the next properties is written $g_\lambda(\cdot)$.

1) $0 \leq \psi(E) \leq 1$ for $E \in \mathcal{B}$ and particularly $\psi(X) = 1$.

2) If $E, E' \in \mathcal{B}$ and $E \cap E' = \phi$, then

$$\psi(E \cup E') = \psi(E) + \psi(E') + \lambda\psi(E)\psi(E'), \ -1 < \lambda < \infty. \quad (2.5)$$

3) If $F_n \in \mathcal{B}$ for $1 \leq n < \infty$ and $\{F_n\}$ is monotone, then

$$\lim_{n\to\infty} \psi(F_n) = \psi(\lim_{n\to\infty} F_n).$$

(This type of set functions is also seen in [15], in which λ is set to -1.)

[Proposition 2.2] $g_\lambda(\cdot)$ satisfies the conditions of the fuzzy measures.

(Proof) Let $E = X$ and $E' = \phi$. Then we obtain $\psi(\phi) = 0$ from the second condition. Monotonicity of $g_\lambda(\cdot)$ is obtained from 1) and 2).

(Q.E.D.)

Now we have the problem of finding f by which a fuzzy measure is constructed. The answer to this problem will be seen in Appendix A.

[Proposition 2.3]

$$g_\lambda(F' - F) = \frac{g_\lambda(F') - g_\lambda(F)}{1 + \lambda g_\lambda(F)} \quad \text{where } F \subset F'. \quad (2.6)$$

$$g_\lambda(F^c) = \frac{1 - g_\lambda(F)}{1 + \lambda g_\lambda(F)} \quad (2.7)$$

$$g_\lambda(\bigcup_{i=1}^{\infty} E_i) = \frac{1}{\lambda}[\prod_{i=1}^{\infty}(1 + \lambda g_\lambda(E_i)) - 1]$$

$$\text{where } E_i \cap E_j = \phi \text{ for } i \neq j. \quad (2.8)$$

(Proof) Eqs. (2.6) and (2.7) are easy to prove. So let us prove Eq. (2.8).

From the second condition of g_λ, it follows that

$$g_\lambda(\bigcup_{i=1}^{n} E_i) = \frac{1}{\lambda}[\prod_{i=1}^{n}(1 + \lambda g_\lambda(E_i)) - 1]. \tag{2.9}$$

Define $F_n = \bigcup_{i=1}^{n} E_i$. Then $\{F_n\}$ becomes a monotone sequence. From continuity of g_λ, Eq. (2.8) is obtained. (Q.E.D.)

If we allow $\lambda = -1$ in Definition 2.6, then we have $g_\lambda(F^c) = 1$ for any $F \in \mathcal{B}$ and $g_\lambda((F^c)^c) \neq g_\lambda(F)$. Therefore it is necessary to assume $\lambda > -1$ if we wish to define $g_\lambda(F^c)$.

If $\lambda = 0$, then g_λ becomes equal to a probability measure. In practice, we can derive $g_0(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} g_0(E_i)$ from Eq. (2.8). Expanding the right-hand side of Eq. (2.9) into power series of λ, we have

$$\prod_{i=1}^{n}(1 + \lambda g(E_i)) = 1 + \lambda \sum_{i=1}^{n} g(E_i) + O(\lambda^2).$$

From this we obtain $\lim_{\lambda \to 0} g_\lambda(\sum_{i=1}^{n} E_i) = \sum_{i=1}^{n} g_0(E_i)$. Putting $n \to \infty$, complete additivity is obtained, which can be, of course, derived directly from Definition 2.6.

Assume $E \cap E' = \phi$ in general. Then we have

$$g_\lambda(E \cup E') \geq g_\lambda(E) + g_\lambda(E') \text{ for } \lambda \geq 0, \tag{2.10}$$

$$g_\lambda(E \cup E') < g_\lambda(E) + g_\lambda(E') \text{ for } \lambda < 0. \tag{2.11}$$

Therefore, in addition to the simplicity of the rule of construction, g_λ varies its characteristics by the parameter λ. For this reason, g_λ may be used for an approximate expression of other arbitrary fuzzy measures. Now, a probability measure P separates E and E^c at 0.5

if $P(E) \geq 0.5$, then $P(E^C) \leq 0.5$; on the other hand, g_λ separates E and E^C at $(-1 + \sqrt{1 + \lambda})/\lambda$, which can be easily derived from Eq. (2.7).

[Proposition 2.4]

1) If $g_\lambda(E) = 0$, then $g_\lambda(A) = g_\lambda(A \cup E)$ for any $A \subset \mathcal{B}$.

2) If $A \subset B$ and $g_\lambda(A) = g_\lambda(B)$, then $g_\lambda(B - A) = 0$.

3) If $g_\lambda(E_n) = 0$ for $1 \leq n < \infty$, then $g_\lambda(\bigcup_{n=1}^{\infty} E_n) = 0$.

The above properties will be used in Chapter 6. Note that these properties are not always satisfied by an arbitrary fuzzy measure; a fuzzy measure which is F-additive does not satisfy the property 2). Next, restricting our discussions to R^1, we will show a concrete method for constructing g_λ on the Borel field $\mathcal{B}$ of R^1: minimum σ-field consisted of closed sets in R^1.

[Definition 2.7] A function with the following properties on R^1 is called an F-distribution function.

1) $0 \leq H(x) \leq 1$ for $x \in R^1$.

2) If $x \leq y$, then $H(x) \leq H(y)$.

3) $\lim_{x \to a+0} H(x) = H(a)$.

4) $\lim_{x \to -\infty} H(x) = 0$.

5) $\lim_{x \to \infty} H(x) = 1$.

H(x) has the same properties as the distribution function of a probability measure. Using this H, for a half open interval $(a,b] \in \mathcal{B}$, let us construct g_λ.

[Definition 2.8] A set function $\psi(\cdot)$ is defined for $(a,b]$ as follows.

$$\psi((a,b]) = \frac{H(b) - H(a)}{1 + \lambda H(a)} \quad \text{where } -1 < \lambda < \infty. \tag{2.12}$$

[Proposition 2.5] $\psi(\cdot)$ satisfies the properties of g_λ on $\mathcal{B}$.

(Proof) It suffices to prove that the second property is satisfied. We can easily obtain that for $a < b < c$,

$$\frac{H(c) - H(a)}{1 + \lambda H(a)} = \frac{H(b) - H(a)}{1 + \lambda H(a)} + \frac{H(c) - H(b)}{1 + \lambda H(b)} + \lambda \cdot \frac{H(b) - H(a)}{1 + \lambda H(a)} \cdot \frac{H(c) - H(b)}{1 + \lambda H(b)}$$

(Q.E.D.)

[Proposition 2.6] Let g_λ be a fuzzy measure of $(R^1, \mathcal{B})$. Then there exists an open set O such that $A \subset O$ and $g_\lambda(O - A) < \varepsilon$ for any $\varepsilon > 0$ and for any $A \in \mathcal{B}$.

(Proof) It suffices to prove the case in which $A = (a,b]$.
Let $O_n = (a, b(1 + \frac{1}{n}))$. Then it follows that $O_n \supset O_{n-1} \supset \dots \supset A$ and $\lim_{n\to\infty} O_n = A$. From continuity of g_λ, we obtain $\lim_{n\to\infty} g_\lambda(O_n - A) = 0$.

(Q.E.D.)

Proposition 2.6 will be also used in Chapter 6.

Thus, starting from H, we can define a fuzzy measure g_λ over all members of $\mathcal{B}$ without contradictions. To give $H(x)$ implies to give g_λ for a monotone sequence of intervals $(-\infty,x]$ as $g_\lambda((-\infty,x]) = H(x)$. Our purpose stated at the beginning of this section has been now accomplished.

Chapter 3

FUZZY INTEGRALS

3.1 Definition of fuzzy integrals

In this section, fuzzy integrals are defined by using fuzzy measures shown in Definition 2.2. First, let us consider the fuzzy measures of the F-measurable space $(X, 2^X)$ first and the fuzzy measures of $(X, \mathfrak{F})$ and $(X, \mathfrak{B})$ later.

[Definition 3.1] Given a function $h : X \to [0,1]$, a fuzzy integral over A in $(X, 2^X)$ is defined as follows.

$$⨍_A h(x) \circ g(\cdot) = \sup_{F \in 2^X} [\inf_{x \in F} h(x) \wedge g(A \cap F)] \qquad (3.1)$$

where g is a fuzzy measure of $(X, 2^X)$.

In the above definition, the symbol $⨍$ is an integral with a small bar and also shows a symbol of the letter f. The small circle is the symbol of the composition used in the fuzzy sets theory.

[Theorem 3.1] A fuzzy integral can be expressed in the following form.

$$⨍_A h(x) \circ g(\cdot) = \sup_{\alpha \in [0,1]} [\alpha \wedge g(A \cap F_\alpha)], \qquad (3.2)$$

where $F_\alpha = \{x | h(x) \geq \alpha\}$.

(Proof) Let $\alpha_F = \inf_{x \in F} h(x)$. Then there holds

$$\sup_{F \in 2^X} [\inf_{x \in F} h(x) \wedge g(A \cap F)] \leq \sup_{F \in 2^X} [\alpha_F \wedge g(A \cap \{x | h \geq \alpha_F\})]$$

since $\{x | h \geq \alpha_F\} \supset F$. Further we obtain

$$\sup_{F \in 2^X} [\alpha_F \wedge g(A \cap \{x | h \geq \alpha_F\})] \leq \sup_{\alpha \in [0,1]} [\alpha \wedge g(A \cap F_\alpha)]$$

since $\{\alpha_F | F \in 2^X\} \subset [0,1]$. From $\alpha \leq \inf_{x \in F_\alpha} h(x)$ follows the inequality

$$\sup_{\alpha \in [0,1]} [\alpha \wedge g(A \cap F_\alpha)] \leq \sup_{\alpha \in [0,1]} [\inf_{x \in F_\alpha} h(x) \wedge g(A \cap F_\alpha)].$$

Now, it is valid that $2^X \supset \{F_\alpha | \alpha \in [0,1]\}$. From this, it follows that

$$\sup_{F \in 2^X} [\inf_{x \in F} h(x) \wedge g(A \cap F)] \geq \sup_{\alpha \in [0,1]} [\inf_{x \in F_\alpha} h(x) \wedge g(A \cap F_\alpha)].$$

Thus Eq. (3.2) is derived. (Q.E.D.)

According to Theorem 3.1, it is not necessary to take 2^X for the domain of g. We can merely take a family of sets including a monotone sequence $\{F_\alpha | \alpha \in [0,1]\}$ of subsets. Hence, we define fuzzy integrals in the fuzzy measure space $(X, \mathfrak{F}, g)$ by Eq. (3.2). Now let us discuss the concept of measurability.

[Definition 3.2] A subset E of X is called an $\mathfrak{F}$-measurable set, if $E \in \mathfrak{F}$.

[Definition 3.3] Let $h : X \to [0,1]$. A function h is said to be $\mathfrak{F}$-measurable or merely measurable, if $\{x | h(x) \geq \alpha\} \in \mathfrak{F}$ for any $\alpha \in [0,1]$.

The fuzzy integral of h over X can be defined according to Theorem 3.1, if $F_\alpha = \{x | h(x) \geq \alpha\} \in \mathfrak{F}$. If we consider a monotone decreasing sequence $F_n = \{x | h > \alpha(1 - \frac{1}{n})\} \in \mathfrak{F}$, then we have $F_\alpha = \lim_{n \to \infty} F_n \in \mathfrak{F}$. And so $\{x | h > \alpha\} \in \mathfrak{F}$ can be used as the definition of measurability instead of $\{x | h \geq \alpha\} \in \mathfrak{F}$. Furthermore, it is sufficient that $\{x | h > r\} \in \mathfrak{F}$ holds for any rational number $r \in [0,1)$.

Corresponding to the monotonicity of a fuzzy measure g, monotone sequences will be discussed mainly in this paper. Generally, the condition $A \cap \{x|h > \alpha\} \in \mathcal{F}$ is necessary in order that the fuzzy integral of a function h over $A \subset X$ may be defined. To avoid this difficulty and to broaden the region of measurable functions, hereafter, we adopt a Borel field $\mathcal{B}$ for the domain of fuzzy measures.

The definition of measurability of functions is the same as in the theory of Lebesgue integrals, in which it is necessary for the definition of integrals that $\{x|h = \alpha\}$ is measurable for $\alpha \in (-\infty, \infty)$. It is easily shown that $\{x|h \geq \alpha\} \in \mathcal{B}$ is equivalent to $\{x|h = \alpha\} \in \mathcal{B}$.

Hereafter, we assume that all functions discussed in this paper, including constants, have the range [0,1]. For simplification, we write a fuzzy integral as $\oint_A h \circ g(\cdot)$ or $\oint_A h \circ g$. In the case of A = X, we write it briefly as $\oint h \circ g$. Further we will denote $\sup\limits_{\alpha \in [0,1]}$ by $\sup\limits_{\alpha}$.

[Proposition 3.1] If h and h' are $\mathcal{B}$-measurable, then $h \vee h'$ and $h \wedge h'$ are $\mathcal{B}$-measurable.

[Proposition 3.2] Denote $1 - h$ by h^c. If h is $\mathcal{B}$-measurable, then h^c is also $\mathcal{B}$-measurable.

[Proposition 3.3] If $\{h_n\}$ is a monotone sequence of $\mathcal{B}$-measurable functions, then $\lim\limits_{n\to\infty} h_n$ is $\mathcal{B}$-measurable.

[Proposition 3.4] Let $a \in [0,1]$, then

$$\oint a \circ g(\cdot) = a. \tag{3.3}$$

[Theorem 3.2] If $h \leq h'$, then there holds

$$\oint h \circ g(\cdot) \leq \oint h' \circ g(\cdot). \tag{3.4}$$

(Proof) Define $F_\alpha = \{x | h \geq \alpha\}$ and $F_\alpha' = \{x | h' \geq \alpha\}$. Then there follows $F_\alpha \subset F_\alpha'$ for all $\alpha \in [0,1]$ since $h \leq h'$. The inequality is obtained by using Theorem 3.1. (Q.E.D.)

[Lemma]

$$\oint (h_1 \vee h_2) \circ g \geq \oint h_1 \circ g \vee \oint h_2 \circ g \tag{3.5}$$

$$\oint (h_1 \wedge h_2) \circ g \leq \oint h_1 \circ g \wedge \oint h_2 \circ g \tag{3.6}$$

In Eq. (3.5), equality holds when g is F-additive.

[Proposition 3.5]

$$g(F) = \oint \chi_F(x) \circ g(\cdot) \tag{3.7}$$

[Theorem 3.3]

$$\oint_A h(x) \circ g(\cdot) = \oint [\chi_A(x) \wedge h(x)] \circ g(\cdot) \tag{3.8}$$

(Proof) Define $F_\alpha = \{x | h(x) \geq \alpha\}$ and $E_\alpha = \{x | \chi_A(x) \wedge h(x) \geq \alpha\}$. It suffices to prove $A \cap F_\alpha = E_\alpha$ for any $\alpha \in [0,1]$.

Assume $\alpha \neq 0$, If $x \in A \cap F_\alpha$, then $x \in E_\alpha$. From this follows $A \cap F_\alpha \subset E_\alpha$. The reverse $A \cap F_\alpha \supset E_\alpha$ is also true. Therefore we obtain $A \cap F_\alpha = E_\alpha$ for any $\alpha \in (0,1]$. If $\alpha = 0$, then $\alpha \wedge g(A \cap F_\alpha) = \alpha \wedge g(E_\alpha)$.

(Q.E.D.)

[Theorem 3.4] If $A \subset B$, then there holds

$$\int_A h \circ g \leq \int_B h \circ g. \tag{3.9}$$

(Proof) This inequality can be easily obtained from Theorems 3.2 and 3.3. (Q.E.D.)

[Lemma]

$$\int_{A \cup B} h \circ g \geq \int_A h \circ g \vee \int_B h \circ g \tag{3.10}$$

$$\int_{A \cap B} h \circ g \leq \int_A h \circ g \wedge \int_B h \circ g \tag{3.11}$$

In Eq. (3.10), equality holds when g is F-additive.

[Corollary 3.1]

If

$$\sup_{\alpha \in [0,a]} [\alpha \wedge g(F_\alpha)] < \sup_{\alpha \in (a,1]} [\alpha \wedge g(F_\alpha)],$$

then

$$\sup_{\alpha \in [0,a]} [\alpha \wedge g(F_\alpha)] = a.$$

(Proof) There holds $\sup_{\alpha \in [0,a]} [\alpha \wedge g(F_\alpha)] \leq a$ for an arbitrary a. Now let us assume $\sup_{\alpha \in [0,a]} [\alpha \wedge g(F_\alpha)] \neq a$ and state that this leads to a contradiction. The inequality $g(F_a) < a$ is valid from the above assumption, since $g(F_\alpha)$ decreases as α increases. We obtain

$$\sup_{\alpha \in [0,a]} [\alpha \wedge g(F_\alpha)] \geq \sup_{\alpha \in (a,1]} g(F_\alpha)$$

from

$$\sup_{\alpha \in [0,a]} [\alpha \wedge g(F_\alpha)] \geq a \wedge g(F_a) \text{ and } g(F_a) \geq \sup_{\alpha \in (a,1]} g(F_\alpha).$$

On the other hand, we obtain

$$\sup_{\alpha \in (a,1]} g(F_\alpha) = \sup_{\alpha \in (a,1]} [\alpha \wedge g(F_\alpha)]$$

from

$$a > g(F_a) \geq \sup_{\alpha \in (a,1]} g(F_\alpha).$$

Thus

$$\sup_{\alpha \in [0,a]} [\alpha \wedge g(F_\alpha)] \geq \sup_{\alpha \in (a,1]} [\alpha \wedge g(F_\alpha)]$$

is derived and this contradicts our assumption in the statement of the corollary. (Q.E.D.)

[Theorem 3.5] Let $a \in [0,1]$. Then there hold

1) $$\int (a \vee h) \circ g(\cdot) = a \vee \int h \circ g(\cdot), \tag{3.12}$$

2) $$\int (a \wedge h) \circ g(\cdot) = a \wedge \int h \circ g(\cdot). \tag{3.13}$$

(Proof) Let $F_\alpha = \{x | h \geq \alpha\}$, $G_\alpha = \{x | a \quad h \geq \alpha\}$ and $H_\alpha = \{x | a \quad h \geq \alpha\}$.

(i) We can write

$$\int (a \vee h) \circ g(\cdot) = \sup_{\alpha \in [0,a]} [\alpha \wedge g(G_\alpha)] \vee \sup_{\alpha \in (a,1]} [\alpha \wedge g(G_\alpha)].$$

If

$$\sup_{\alpha \in [0,a]} [\alpha \wedge g(G_\alpha)] \geq \sup_{\alpha \in (a,1]} [\alpha \wedge g(G_\alpha)],$$

then

$$\int (a \vee h) \circ g(\cdot) = \sup_{\alpha \in [0,a]} [\alpha \wedge g(G_\alpha)].$$

This equals to $\int a \circ g$ since $G_\alpha = X$ for $\alpha \in [0,a]$.

We obtain $\int (a \vee h) \circ g = \int a \circ g \vee \int h \circ g$ from Lemma of Theorem 3.2.

Next assume

$$\sup_{\alpha \in [0,a]} [\alpha \wedge g(G_\alpha)] < \sup_{\alpha \in (a,1]} [\alpha \wedge g(G_\alpha)],$$

then we obtain

$$\int (a \vee h) \circ g = \sup_{\alpha \in (a,1]} [\alpha \wedge g(G_\alpha)],$$

since $\sup_{\alpha \in [0,a]} [\alpha \wedge g(G_\alpha)] = a$ follows from Corollary 3.1.

Now consider

$$\int h \circ g = \sup_{\alpha \in [0,a]} [\alpha \wedge g(F_\alpha)] \vee \sup_{\alpha \in (a,1]} [\alpha \wedge g(F_\alpha)].$$

Since $\sup_{\alpha \in [0,a]} [\alpha \wedge g(F_\alpha)] \leq a$ and $F_\alpha = G_\alpha$ for $\alpha \in (a,1]$,

we obtain

$$\int h \circ g = \int (a \vee h) \circ g.$$

Thus Eq. (3.12) holds in this case.

(ii) We have, analogously, $H_\alpha =$ for $\alpha > a$ and $H_\alpha = F_\alpha$ for $\alpha \leq a$.

From this we obtain

$$\int (a \wedge h) \circ g = \sup_{\alpha \in [0,a]} [\alpha \wedge g(F_\alpha)] \leq a.$$

Now we can write

$$\int h \circ g = \sup_{\alpha \in [0,a]} [\alpha \wedge g(F_\alpha)] \vee \sup_{\alpha \in (a,1]} [\alpha \wedge g(F_\alpha)].$$

There follows

$$\oint (a \wedge h) \circ g = a \wedge \oint h \circ g,$$

since

$$\oint (a \wedge h) \circ g = \oint h \circ g$$

holds, if

$$\sup_{\alpha \in [0,a]} [\alpha \wedge g(F_\alpha)] \geq \sup_{\alpha \in (a,1]} [\alpha \wedge g(F_\alpha)].$$

Next, if

$$\sup_{\alpha \in [0,a]} [\alpha \wedge g(F_\alpha)] < \sup_{\alpha \in (a,1]} [\alpha \wedge g(F_\alpha)],$$

then $\sup_{\alpha \in [0,a]} [\alpha \wedge g(F_\alpha)] = a$ is obtained from Corollary 3.1.

So we have

$$\oint h \circ g > a \text{ and } \oint (a \wedge h) \circ g = a.$$

Thus $\oint (a \wedge h) \circ g = a \wedge \oint h \circ g$ is concluded. (Q.E.D.)

[Theorem 3.6] If $\{h_n\}$ is a monotone sequence of $\mathcal{B}$-measurable functions, then

$$\lim_{n \to \infty} \oint h_n \circ g = \oint \lim_{n \to \infty} h_n \circ g. \tag{3.14}$$

(Proof) Let us define $F_\alpha^n = \{x | h_n \geq \alpha\}$, which is monotone with n. There holds $g(\{x | \lim_{n \to \infty} h_n \geq \alpha\}) = \lim_{n \to \infty} g(F_\alpha^n)$ from the continuity of g. Hence holds Eq. (3.14). (Q.E.D.)

[Theorem 3.7] If $\{h_i\}$ is a monotone decreasing (increasing) sequence of $\mathcal{B}$-measurable functions and $\{a_i\}$ is a monotone increasing (decreasing) sequence of real numbers, then $\bigvee_{i=1}^{\infty} [a_i \wedge h_i]$ is also $\mathcal{B}$-measurable and there holds

$$\int [\bigvee_{i=1}^{\infty} (a_i \wedge h_i)] \circ g = \bigvee_{i=1}^{\infty} [a_i \wedge \int h_i \circ g]. \tag{3.15}$$

(Proof) First, let us prove Eq. (3.15) by replacing ∞ by n. The equality is valid from 2) of Theorem 3.5 when n = 1, so let n = 2.

Define $E_\alpha = \{x | a_1 \wedge h_1 \geq \alpha\}$, $F_\alpha = \{x | a_2 \wedge h_2 \geq \alpha\}$ and $G_\alpha = \{x | \bigvee_{i=1}^{2} (a_i \wedge h_i) \geq \alpha\}$. It is valid that $a_1 \wedge h_1$ and $a_2 \wedge h_2$ are $\mathcal{B}$-measurable. We have $E_\alpha = F_\alpha = \phi$ if $\alpha > a_2$, $E_\alpha = \phi$ if $a_2 \geq \alpha > a_1$, and $E_\alpha \supset F_\alpha$ from $h_1 \geq h_2$ if $a_1 \geq \alpha$. So $G_\alpha (= E_\alpha \cup F_\alpha)$ equals ϕ, F_α, and E_α, respectively. Therefore $\bigvee_{i=1}^{2} [a_i \wedge h_i]$ is $\mathcal{B}$-measurable.

Now we have

$$\int [\bigvee_{i=1}^{2} (a_i \wedge h_i)] \circ g = \sup_{\alpha \in [0,a_1]} [\alpha \wedge g(E_\alpha)] \vee \sup_{\alpha \in (a_1,a_2]} [\alpha \wedge g(F_\alpha)]$$

and

$$\int (a_1 \wedge h_1) \circ g \vee \int (a_2 \wedge h_2) \circ g =$$

$$\sup_{\alpha \in [0,a_1]} [\alpha \wedge g(E_\alpha)] \vee \sup_{\alpha \in [0,a_1]} [\alpha \wedge g(F_\alpha)] \vee \sup_{\alpha \in (a_1,a_2]} [\alpha \wedge g(F_\alpha)].$$

Since $E_\alpha \supset F_\alpha$ for $\alpha \in [0,a_1]$, there holds

$$\int [\bigvee_{i=1}^{2} (a_i \wedge h_i)] \circ g = \int (a_1 \wedge h_1) \circ g \vee \int (a_2 \wedge h_2) \circ g.$$

The proof is omitted for $n \geq 3$. Thus Eq. (3.15) holds for a finite n.

Now, let $k_n = \bigvee_{n=1}^{n} (a_i \wedge h_i)$. Then k_n becomes a monotone sequence. We obtain Eq. (3.15) by using Theorem 3.6. (Q.E.D.)

[Lemma] If $\{h_\alpha(x)\}$ is monotone with respect to $\alpha \in [0,1]$, then

$$\int \sup_\alpha [\alpha \wedge h_\alpha(x)] \circ g = \sup_\alpha [\alpha \wedge \int h_\alpha(x) \circ g]. \tag{3.16}$$

[Proposition 3.6] (Zadeh [9]) Define $F_\alpha = \{x | h(x) \geq \alpha\}$. Then there holds

$$h(x) = \sup_\alpha [\alpha \wedge \chi_{F_\alpha}(x)]. \tag{3.17}$$

Here $\chi_{F_\alpha}(x)$ is a specific form of $h_\alpha(x)$ in the above Lemma.

From Lemma of Theorem 3.7 and Proposition 3.6, we have

$$\begin{aligned} \int h(x) \circ g &= \int \sup_\alpha [\alpha \wedge \chi_{F_\alpha}(x)] \circ g \\ &= \sup_\alpha [\alpha \wedge \int \chi_{F_\alpha}(x) \circ g] \\ &= \sup_\alpha [\alpha \wedge g(F_\alpha)]. \end{aligned}$$

The last term is obtained directly from Theorem 3.1. As can be seen in Lemma of Theorem 3.2, sup and $\int$ are not always commutative. In general, there holds the next theorem.

[Theorem 3.8] Let $h : X \times A \to [0,1]$ and let $h(x, a)$ be a $\mathcal{B}$-measurable function of x for an arbitrary parameter $a \in A$. Then there hold

1) $\sup_{a \in A} \int h(x, a) \circ g \leq \int \sup_{a \in A} h(x, a) \circ g,$ (3.18)

2) $\inf_{a \in A} \int h(x, a) \circ g \geq \int \inf_{a \in A} h(x, a) \circ g.$ (3.19)

(Proof) Define $F_\alpha(a) = \{x | h(x, a) \geq \alpha\}$ and $\hat{F}_\alpha = \{x | \sup_{a \in A} h(x, a) \geq \alpha\}$.

Since $\hat{F}_\alpha \supset F_\alpha(a)$ for any $a \in A$, we obtain $\sup_\alpha [\alpha \wedge g(F_\alpha(a)] \leq \sup_\alpha [\alpha \wedge g(\hat{F}_\alpha)]$.

Therefore the first inequality holds. The next one is similarly obtained.

(Q.E.D.)

Theorem 3.8 implies the generalization of Lemma of Theorem 3.2.

[Corollary 3.2] Let $F_\alpha = \{x | h \geq \alpha\}$. Then $\lim_{\alpha' \to \alpha - 0} g(F_{\alpha'}) = g(F_\alpha)$

for $0 < \alpha \leq 1$.

(Proof) Let $F_n = \{x | h \geq \alpha(1 - \frac{1}{n})\}$, then it holds that $\{F_n\}$ is monotonously decreasing and $\bigcap_{n=1}^{\infty} F_n = F_\alpha$. From continuity of g follows

$\lim_{n \to \infty} g(F_n) = g(F_\alpha)$. This implies that $\lim_{\alpha' \to \alpha - 0} g(F_{\alpha'}) = g(F_\alpha)$.

(Q.E.D.)

Now, let us denote $\lim_{\alpha' \to \alpha - 0} F_{\alpha'}$ by $F_{\alpha - 0}$ and $\lim_{\alpha' \to \alpha + 0} F_{\alpha'}$ by $F_{\alpha + 0}$. It is valid that $F_{\alpha - 0} = F_\alpha$ and $F_{\alpha + 0} = \{x | h > \alpha\}$.

[Theorem 3.9] There holds $\int_A h \circ g = M$ if and only if $g(A \cap F_M) \geq M \geq g(A \cap F_{M+0})$.

(Proof) Sufficiency is obvious. To prove necessity, assume $\int_A h \circ g = M$.

We can write

$$\oint_A h \circ g = \sup_{\alpha \in [0,M]} [\alpha \wedge g(A \cap F_\alpha)] \vee \sup_{\alpha \in (M,1]} [\alpha \wedge g(A \cap F_\alpha)]$$

where

$$F_\alpha = \{x | h(x) \geq \alpha\}.$$

From Corollary 3.1 follows

$$M = \sup_{\alpha \in [0,M]} [\alpha \wedge g(A \cap F_\alpha)],$$

if

$$\sup_{\alpha \in [0,M]} [\alpha \wedge g(A \cap F_\alpha)] < \sup_{\alpha \in (M,1]} [\alpha \wedge g(A \cap F_\alpha)].$$

This contradicts the assumption since we have $\oint_A h \circ g > M$. Therefore, it is necessary that

$$\sup_{\alpha \in [0,M]} [\alpha \wedge g(A \cap F_\alpha)] \geq \sup_{\alpha \in (M,1]} [\alpha \wedge g(A \cap F_\alpha)].$$

From the assumption, there must holds

$$M = \sup_{\alpha \in [0,M]} [\alpha \wedge g(A \cap F_\alpha)].$$

Thus we obtain $g(A \cap F_M) \geq M$. It is valid that $M \geq g(A \cap F_{M+0})$.

(Q.E.D.)

[Lemma] Let $E_\alpha = \{x | h > \alpha\}$, then there holds

$$\oint_A h \circ g(\cdot) = \sup_{\alpha} [\alpha \wedge g(A \cap E_\alpha)]. \tag{3.20}$$

(Proof) In the same way as in the proof of Theorem 3.9, it can be easily proved that if $M = \sup_{\alpha} [\alpha \wedge g(A \cap E_\alpha)]$, then there holds

$g(E_{M-0}) \geq M \geq g(E_M)$. There follows clearly $F_M = E_{M-0}$ and $F_{M+0} = E_M$. From this we obtain $g(F_M) \geq M \geq g(F_{M+0})$. Hence holds $\int_A h \circ g = M$.

(Q.E.D.)

3.2 Comparison of fuzzy and Lebesgue integrals

The fuzzy integrals defined in the preceding section are very similar to the Lebesgue integrals at their definition. We shall make it clear in this section.

Now, let $h(x)$ be a simple function such that

$$h(x) = \sum_{i=1}^{n} \alpha_i \chi_{E_i}(x) \tag{3.21}$$

where $X = \sum_{i=1}^{n} E_i$, $E_i \in \mathcal{B}$, and $E_i \cap E_j = \phi (i \neq j)$.

In the measure space $(X, \mathcal{B}, \mu)$, the Lebesgue integral of h over A is defined as

$$\int_A h \, d\mu = \sum_{i=1}^{n} \alpha_i \mu(A \cap E_i) . \tag{3.22}$$

The value of integration is independent of the expression at the right-hand side of Eq. (3.21). Here let us assume $0 \leq \alpha_i \leq 1$ $(1 \leq i \leq n)$ and $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$. Then we have the next corollary.

[Corollary 3.3] Let $F_i = E_i + E_{i+1} + \ldots + E_n$ $(1 \leq i \leq n)$. Then a simple function can be also written as

$$h(x) = \bigvee_{i=1}^{n} [\alpha_i \wedge \chi_{F_i}(x)], \tag{3.23}$$

and two expressions are identical.

(Proof) Let us define h_1 by Eq. (3.21) and h_2 by Eq. (3.23). If $x \in E_j$, then $h_1(x) = \alpha_j$ and $x \in F_j$, $x \notin F_{j+1}$ since $E_j = F_j - F_{j+1}$. Further we have $x \in F_i$ for $i \leq j$ and $x \notin F_i$ for $i > j$ since $\{F_j\}$ monotonously decreases as j increases. From this follows $h_2(x) = \bigvee_{i=1}^{j} \alpha_i = \alpha_j$. Thus $h_1 = h_2$. (Q.E.D.)

The generalization of Eq. (3.23) is seen in Proposition 3.6.

Hereafter, assuming

$$0 \leq \alpha_1 \leq \alpha_2 \leq \cdots\cdot \leq \alpha_n \leq 1$$

and

$$F_1 \supset F_2 \supset \cdots\cdot \supset F_n,$$

we will use Eq. (3.23) as the expression of simple functions.

[Theorem 3.10] With respect to a simple function h on X, there holds

$$\int_A h \circ g(\cdot) = \bigvee_{i=1}^{n} [\alpha_i \wedge g(A \cap F_i)], \; F_1 = X \tag{3.24}$$

and the value of the right side is uniquely determined independent of the expressions of h.

(Proof) We have $F_i \subset \{x | h \geq \alpha_i\}$, since $\{\alpha_i\}$ is monotonously increasing and $\{F_i\}$ decreasing. Now let us define i' by the smallest j satisfying $\alpha_j = \alpha_i$. There follows clearly $F_{i'} = \{x | h \geq \alpha_i\}$, and $F_\alpha = \{x | h \geq \alpha\}$ $= F_{i'}$ when $\alpha_{i'} \geq \alpha \geq \alpha_{i'-1}$. Regarding i' as a function of i, we obtain

$$\sup_{\alpha \in [0,1]} [\alpha \wedge g(A \cap F_\alpha)] = \bigvee_{i=1}^{n} [\alpha_{i'} \wedge g(A \cap F_{i'})].$$

Since

$$\alpha_{i'} = \alpha_i \text{ and } g(A \cap F_{i'}) \geq g(A \cap F_i),$$

there follows

$$\bigvee_{i=1}^{n} [\alpha_{i'} \wedge g(A \cap F_{i'})] \geq \bigvee_{i=1}^{n} [\alpha_i \wedge g(A \cap F_i)].$$

On the contrary, the reverse inequality holds always. Let us turn now to the proof of the latter half. To do this, let us assume that there are two different expressions of h such as

$$\bigvee_{i=1}^{n} [\alpha_i \wedge \chi_{F_i}(x)]$$

and

$$\bigvee_{i=1}^{m} [\alpha'_i \wedge \chi_{F'_i}(x)].$$

Let

$$\alpha_\ell \wedge g(A \cap F_\ell) = \bigvee_{i=1}^{n} [\alpha_i \wedge g(A \cap F_i)].$$

Then there must be at least one α'_i such that $\alpha'_i = \alpha_\ell$ from the assumption on α_i and α'_i. So let k be the smallest i. From $\alpha_\ell = \alpha'_k$ and $F_\ell \subset F'_k$ which is valid by the properties of h, we have

$$\alpha_\ell \wedge g(A \cap F_\ell) \leq \alpha'_k \wedge g(A \cap F'_k).$$

Thus

$$\bigvee_{i=1}^{n} [\alpha_i \wedge g(A \cap F_i)] \leq \bigvee_{i=1}^{m} [\alpha'_i \wedge g(A \cap F'_i)].$$

The reverse inequality is also valid. (Q.E.D.)

In Theorem 3.10 we can assume that $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n$ and $F_1 \subset F_2 \subset \dots \subset F_n$. As concerns an arbitrary function h, taking a monotone sequence $\{h_n\}$ of simple functions such that $\lim_{n\to\infty} h_n = h$, we obtain $⨍ h \circ g = \lim_{n\to\infty} ⨍ h_n \circ g$ from Theorem 3.6. We can prove, as in the Lebesgue integral, that the value of $\lim_{n\to\infty} ⨍ h_n \circ g$ is unique for any sequence $\{h_n\}$ which converges to the same function, but we omit this proof. This fact enables us to define the fuzzy integrals by Eq. (3.24). The similarity of Lebesgue and fuzzy integrals is clarified by comparing Eq. (3.21) with Eq. (3.23) and Eq. (3.22) with Eq. (3.24), respectively.

Let us next try a quantitative comparison. Let h be a $\mathcal{B}$-measurable function. Then we can define both integrals, fuzzy and Lebesgue, with respect to a probability measure P, and obtain the following inequality.

[Theorem 3.11] Let $(X, \mathcal{B}, P)$ be a probability space and $h : X \to [0,1]$ be a $\mathcal{B}$-measurable function. Then there holds

$$\left| \int_X h(x)\, dP - ⨍_X h \circ P(\cdot) \right| \leq \frac{1}{4}. \tag{3.25}$$

(Proof) Let $\text{suph} = \sup_{x \in X} h(x)$, $\text{infh} = \inf_{x \in X} h(x)$, and $F_\alpha = \{x \mid h \geq \alpha\}$.

There holds

$$\int_X h\,dP = \int_{X-F_\alpha} h\,dP + \int_{F_\alpha} h\,dP$$
$$\leq \int_{X-F_\alpha} \alpha\,dP + \int_{F_\alpha} \sup h\,dP$$
$$= \alpha P(X - F_\alpha) + \sup h\,P(F_\alpha).$$

Taking into account that

$$P(X - F_\alpha) = 1 - P(F_\alpha),$$

we obtain

$$\int_X h\,dp \leq \alpha(1 - P(F_\alpha)) + \sup h\,P(F_\alpha) \text{ for all } \alpha \in [0,1]$$

Assume

$$⨍_X h \circ P = M$$

and let

$$\alpha \to M + 0.$$

Then, using Theorem 3.9, we obtain

$$\int_X h\,dP - ⨍_X h \circ P \leq (\sup h - M)P(F_{M+0})$$
$$\leq (\sup h - P(F_{M+0}))P(F_{M+0}).$$

Analogously we have

$$\int_X h\,dP \geq \inf h(1 - P(F_\alpha)) + \alpha P(F_\alpha).$$

Let

$$\alpha \to M - 0,$$

then there follows

$$\int_X h\, dP - \oint_X h \circ P \geq -(P(F_M) - \inf h)(1 - P(F_M)).$$

From these, we obtain

$$-\left(\frac{1 - \inf h}{2}\right)^2 \leq \int_X h\, dP - \oint_X h \circ P \leq \left(\frac{\sup h}{2}\right)^2. \quad \text{(Q.E.D.)}$$

Since the operations of fuzzy integrals include only comparisons of grades, Theorem 3.11 implies that using only $\vee$ and $\wedge$, we can obtain a value different by at most 1/4 from a probabilistic expectation obtained from an additive measure.

In this paper, a fuzzy integral is also called a fuzzy expectation in the sense of comparing it with a probabilistic expectation.

As can be clarified from the preceding discussions, the essential difference between a probabilistic quantity and a fuzzy one is that the former has additivity while the latter has only monotonicity. Therefore we can grasp indirectly the meaning of difference between "randomness" and "fuzziness" through the difference between a probability measure P and a fuzzy measure g.

Now again, let us consider the difference between ordinary integrals and fuzzy ones. As is well known, the essential property of ordinary integrals is additivity stating that the area of a figure consisting of a triangle and a square equals the area of a triangle added by that of the square. Apart from visual figures such as a triangle or a square, let us consider what an "area" is. An area in a mathematically abstracted world is something with additivity hidden behind our objects. This "area" can be measured by means of integrals which are constructed by measures with additivity. Thus we can state that measures with additivity are used to measure quantities with additivity and also suitable for this purpose.

Now let us assume that our objects have not additivity but at least monotonicity. By what means can such objects be measured? Of course it may be possible that we measure those by the ordinary measures. But their additivity does not seem to suit the objects which have no additivity. The fuzzy measures introduced in this chapter have monotonicity but not always additivity. Is it not expected that fuzzy measures are more suitable than ordinary ones to measure the objects with only monotonicity?

Our remaining problem is how we can reach the concept of integrals by using fuzzy measures. The fuzzy integral defined here is one which meets our purpose. However, it should be noted that a fuzzy integral is merely one of the functionals defined by using a fuzzy measure. Of course, we will be able to define a functional in another way.

Additivity is generally a restrictive condition, particularly, in dealing with fuzzy objects or events. From the viewpoint of mathematics, fuzzy integrals have some fuzziness because additivity is removed from the properties of measures.

3.3 Fuzzy product measures and fuzzy double integrals

In this section, we discuss fuzzy integrals over the product space X x Y. We adopt $\mathfrak{F}$ shown at Definition 2.1 as the domain of fuzzy measures. Given fuzzy measure spaces $(X, \mathfrak{F}_X, g_X)$ and $(Y, \mathfrak{F}_Y, g_Y)$, let us compose a fuzzy measure on X x Y from g_X and g_Y in the following way.

[Definition 3.4] Let $H = \bigcup_{i=1}^{\infty} [E_i \times F_i]$ be a subset of $Z = X \times Y$ such that $\{E_i | E_i \in \mathfrak{F}_X\}$ is a monotone increasing sequence and $\{F_i | F_i \in \mathfrak{F}_Y\}$ a monotone decreasing one. Define a monotone family $\mathcal{F}_Z$ by subsets as written H.

In the above definition, we can take a monotone decreasing sequence as $\{E_i\}$ and a monotone increasing one as $\{F_i\}$.

[Proposition 3.7] We have

1) $\phi \in \mathcal{F}_Z$ and $Z \in \mathcal{F}_Z$,

2) If $H_n \in \mathcal{F}_Z$ and $\{H_n\}$ is monotone, then $\lim_{n\to\infty} H_n \in \mathcal{F}_Z$.

[Definition 3.5] In $(Z, \mathcal{F}_Z)$, we call $g_Z = g_X \times g_Y$ a fuzzy product measure, which is defined in the following way.

$$\text{For } H = \bigcup_{i=1}^{\infty} [E_i \times F_i],\ g_Z(H) = \bigvee_{i=1}^{\infty} [g_X(E_i) \wedge g_Y(F_i)]. \qquad (3.26)$$

Here it must be proved that $g_Z(H)$ is independent of such expressions of H. But we omit it because we can prove it in an analogous way as in Theorem 3.10. Now given $h(z) = h(x, y)$ defined on Z, we have, as fuzzy couble integrals,

$$\oint_Y [\oint_X h(x, y)] \circ g_X] \circ g_Y$$

and

$$\oint_X [\oint_Y h(x, y) \circ g_Y] \circ g_X.$$

Generally both values are not equal. We will now find the conditions which enable us to change the order of integration.

First let us define a subset of X as a function of y by $E(y) = \{x | (x, y) \in H\}$ where $H \subset X \times Y$. Similarly define $F(x) = \{y | (x, y) \in H\}$.

[Corollary 3.4] If $H \in \mathfrak{F}_Z$, then there hold

1) $E(y) \in \mathfrak{F}_X$, $F(x) \in \mathfrak{F}_Y$,

2) $g_X(E(y))$ is $\mathfrak{F}_Y$-measurable as a function of y and $g_Y(F(x))$ is $\mathfrak{F}_X$-measurable as a function of x,

3)
$$g_Z(H) = \int_Y g_X(E(y)) \circ g_Y$$
$$= \int_X g_Y(F(x)) \circ g_X. \tag{3.27}$$

(Proof) An arbitrary $H \in \mathfrak{F}_Z$ can be expressed, using $E_i \in \mathfrak{F}_X$ and $F_i \in \mathfrak{F}_Y$ $(1 \leq i < \infty)$, as in Eq. (3.26) or as a limit of such a monotone sequence.

First let

$$H = \bigcup_{i=1}^{\infty} [E_i \times F_i].$$

Since

$$(F_i - F_{i+1}) \cap (F_j - F_{j+1}) = \phi \text{ for } i \neq j,$$

H can be rewritten as

$$H = \sum_{i=1}^{\infty} E_i x (F_i - F_{i+1}).$$

(i) If $y \in F_n - F_{n+1}$, then $E(y) = E_n \in \mathfrak{F}_X$, which holds for all $y \in Y$ since $\lim_{n\to\infty} E_n \in \mathfrak{F}_X$. Analogously we will have $F(x) \in \mathfrak{F}_Y$ for all $x \in X$.

(ii) From the properties of H follows $F_n \subset \{y | E(y) \supset E_n\}$.

From (i), E(y) is $\mathfrak{F}_X$-measurable and, hence, $g_1(E(y))$ can be defined. Using the monotonicity of g_X, we have

$$F_n \subset \{y \mid g_X(E(y)) \geq g_X(E_n)\}.$$

Taking into account that

$$g_X(E_1) \leq g_X(E_2) \leq \dots \leq g_X(E_n)$$

and

$$F_1 \supset F_2 \supset \dots \supset F_n,$$

we obtain

$$F_{n'} = \{y \mid g_X(E(y)) \geq g_X(E_n)\}$$

where n' denotes the smallest m such that $g_X(E_m) = g_X(E_n)$. Thus $g_X(E(y))$ is $\mathcal{F}_Y$-measurable as a function of y since $F_{n'} \in \mathcal{F}_Y$ for all n. It is true also for $g_Y(F(x))$ but in this case

$$g_Y(F_1) \geq g_Y(F_2) \geq \dots \geq g_Y(F_n)$$

and

$$E_1 \subset E_2 \subset \dots \subset E_n.$$

iii) Using the results of Theorem 3.10, we have

$$\int_Y g_X(E(y)) \circ g_Y = \bigvee_{i=1}^{\infty} [g_X(E_i) \wedge g_Y(F_i)],$$

which is equal to $g_Z(H)$. We can prove in an analogous way that $g_Z(H) = \int_X g_Y(F(x)) \circ g_X$. If there further exists a monotone sequence $\{H_n\}$ as assumed at the beginning and $\lim_{n\to\infty} H_n = H$, then $E_n(y)$ and $F_n(x)$ are also monotone for n. Hence 1) and 2) are valid.

Concerning 3), taking into account that

$$g_Z(H_n) = \int_Y g_X(E_n(y)) \circ g_Y,$$

we obtain that

$$\lim_{n\to\infty} g_Z(H_n) = \int_Y \lim_{n\to\infty} g_X(E_n(y)) \circ g_Y = \int_Y g_X(E(y)) \circ g_Y.$$

(Q.E.D.)

[Proposition 3.8] $g_Z = g_X \times g_Y$ is a fuzzy measure on $(Z, \mathfrak{F}_Z)$.

In accordance with Proposition 3.8, we can consider the fuzzy integral of h(z) given on Z. The next theorem is proved by using Corollary 3.4 and Lemma of Theorem 3.7.

[Theorem 3.12] If $h(z) = h(x, y)$ is an $\mathfrak{F}_Z$-measurable function, then

$$\int_Z h(z) \circ g_Z(\cdot) = \int_X [\int_Y h(x, y) \circ g_Y] \circ g_X$$

$$= \int_Y [\int_X h(x, y) \circ g_X] \circ g_Y. \quad (3.28)$$

(Proof) We will conduct the proof in two stages.

(i) If $h(z) = \chi_H(z)$ is $\mathfrak{F}_Z$-measurable, then $H \in \mathfrak{F}_Z$. From Corollary 3.4 follows $F(x) \in \mathfrak{F}_Y$. Hence

$$\int_Y \chi_H(x, y) \circ g_Y = \int_Y \chi_{F(x)}(y) \circ g_Y$$

$$= g_Y(F(x)).$$

Since $g_Y(F(x))$ is $\mathfrak{F}_X$-measurable as a function of x, we can define a fuzzy integral with respect to g_X. Consequently we obtain

$$g_Z(H) = \int_X g_Y(F(x)) \circ g_X.$$

The equality for the latter half of Eq. (3.28) may be proved similarly.

(ii) According to proposition 3.6, let us write h(z) as $h(z) = \sup_\alpha [\alpha \wedge \chi_{H_\alpha}(z)$

where $H_\alpha = \{z | h(z) \geq \alpha\}$. Define $F_\alpha(x) = \{y | (x, y) \in H_\alpha\}$. If h(z) is $\mathfrak{F}_Z$-measurable, then $H_\alpha \in \mathfrak{F}_Z$ for all $\alpha \in [0,1]$. From (i) above it follows that

$$g_Z(H_\alpha) = \int_X g_Y(F_\alpha(x)) \circ g_X.$$

We obtain, using Lemma of Theorem 3.7,

$$\sup_\alpha [\alpha \wedge g_Z(H_\alpha)] = \sup_\alpha [\alpha \wedge \int_X g_Y(F_\alpha(x)) \circ g_X]$$

$$= \int_X \sup_\alpha [\alpha \wedge g_Y(F_\alpha(x))] \circ g_X.$$

Now it is clear that

$$\sup_\alpha [\alpha \wedge g_Y(F_\alpha(x))] = \int_Y h(x, y) \circ g_Y.$$

Therefore we obtain

$$\int_Z h(z) \circ g_Z(\cdot) = \int_X [\int_Y h(x, y) \circ g_Y] \circ g_X.$$

The proof of the latter half can be carried in the same way. Proof is now complete. (Q.E.D.)

Corollary 3.4 and Theorem 3.12 state the same contents as Fubini's theorem in the theory of Lebesgue integrals. As can be seen from the definition of $\mathfrak{F}_Z$, even if H, H' $\in \mathfrak{F}_Z$, $H \cup H'$ or $H \cap H'$ has not always such an expression as $\bigcup_{i=1}^{\infty} [E_i \times F_i]$, and therefore it may not be included in $\mathfrak{F}_Z$. This implies that $\mathfrak{F}_Z$ is not a Borel field, even if $\mathfrak{F}_X$ and $\mathfrak{F}_Y$ are Borel fields. If $\mathfrak{B}_Z$ is adopted instead of $\mathfrak{F}_Z$, it will not always be possible to construct a fuzzy product measure on $(Z, \mathfrak{B}_Z)$ so that Theorem 3.12 holds.

3.4 Extension of fuzzy measures

The fuzzy measures defined on a family of subsets of X can be extended in a natural manner onto that including fuzzy sets. (This extension for probability measures was already performed by Zadeh [3])

It is not necessary to distinguish ordinary sets from fuzzy ones in the theory of fuzzy integrals. That is, we can consider the fuzzy measures for fuzzy sets and the fuzzy integrals over them.

Hereafter, we will write a fuzzy set A as $\tilde{A}$ particularly when we wish to distinguish it from an ordinary set. As for the membership function, we will write not $h_{\tilde{A}}$ but merely h_A.

[Definition 3.6] Let $\tilde{\mathcal{B}}$ denote a Borel-field $\mathcal{B}$ which includes all fuzzy sets with $\mathcal{B}$-measurable membership functions. $\tilde{\mathcal{B}}$ is called a fuzzy extension of $\mathcal{B}$. (A member of $\tilde{\mathcal{B}}$ is called a Borel fuzzy set in [3]).

[Proposition 3.9] $\tilde{\mathcal{B}}$ is closed under the operations, such as taking countable unions, countable intersections and complements of its members.

[Definition 3.7] A set function $\tilde{g}$ defined as

$$\tilde{g}(\tilde{A}) = \int_X h_A(x) \circ g(\cdot)$$

for $\tilde{A} \in \tilde{\mathcal{B}}$ is called an extension of g onto $\tilde{\mathcal{B}}$.

It is due to the following facts that $\tilde{g}$ is said to be an extension of g.

[Proposition 3.10] There holds $\tilde{g} = g$ on $\mathcal{B}$. Further $\tilde{g}$ has the properties:

1) $0 \leq \tilde{g}(\tilde{A}) \leq 1$ for $\tilde{A} \in \tilde{\mathcal{B}}$. (3.29)

2) If $\tilde{A}$, $\tilde{B} \in \tilde{\mathcal{B}}$ and $\tilde{A} \subset \tilde{B}$, then $\tilde{g}(\tilde{A}) \leq \tilde{g}(\tilde{B})$. (3.30)

3) If $\tilde{A}_n \in \tilde{\mathcal{B}}$ and $\{\tilde{A}_n\}$ is monotone, then

$$\lim_{n\to\infty} \tilde{g}(\tilde{A}_n) = \tilde{g}(\lim_{n\to\infty} \tilde{A}_n). \tag{3.31}$$

(Proof) From Proposition 3.5, it follows that $g(F) = \int_X \chi_F(x) \circ g(\cdot)$ for $F \in \mathcal{B}$. Hence $\tilde{g} = g$ on $\mathcal{B}$. 1) is valid. We can prove 2) by using Eq. (3.9) since $\int h_A \circ g \leq \int h_B \circ g$ for $h_A \leq h_B$. The property 3) is valid from Eq. (3.14). (Q.E.D.)

[Proposition 3.11] If $\tilde{A}$, $\tilde{B} \in \tilde{\mathcal{B}}$, then there hold

1) $\tilde{g}(\tilde{A} \cup \tilde{B}) \geq \tilde{g}(\tilde{A}) \vee \tilde{g}(\tilde{B})$, (3.32)

2) $\tilde{g}(\tilde{A} \cap \tilde{B}) \leq \tilde{g}(\tilde{A}) \wedge \tilde{g}(\tilde{B})$. (3.33)

It is shown by Proposition 3.10 that $\tilde{g}$ keeps on $\tilde{\mathcal{B}}$ the properties of g. Therefore let us write $\tilde{g}(\cdot)$ merely as $g(\cdot)$. We can further write $g(A)$ instead of $g(\tilde{A})$ without distinguishing a fuzzy set from an ordinary one. Thus it is concluded that we can measure both fuzzy and ordinary sets by a common measure g.

Here, let us make clear the relation between "grade of fuzziness" of a set and "grade of membership" of an element in a fuzzy set. We use the fuzzy measure $g(x_0, \cdot)$ which is shown in Example 1 in Chapter 2. In Example 1 the element x_0 ($\in X$) has been assumed to be known. From the definition of fuzzy sets, the grade of membership of x_0 in $\tilde{A}$ is $h_A(x_0)$.

Now let us compare $h_A(x_0)$ and $g(x_0, \tilde{A})$. We have, according to the definition, $g(x_0, E) = \chi_E(x_0)$ which implies the grade of $x_0 \in E$. Therefore it should hold that the grade of $x_0 \in \tilde{A} = h_A(x_0)$.

[Theorem 3.13]

$$g(x_0, \tilde{A}) = h_A(x_0) \qquad (3.34)$$

(Proof) Define $F_\alpha = \{x | h_A(x) \geq \alpha\}$. Then we obtain

$$g(x_0, \tilde{A}) = \int h_A(x) \circ g(x_0, \cdot)$$

$$= \sup_\alpha [\alpha \wedge g(x_0, F_\alpha)].$$

If $\alpha > h_A(x_0)$, then $x_0 \notin F_\alpha$ and $g(x_0, F_\alpha) = 0$. On the contrary, if $\alpha \leq h_A(x_0)$, then $x_0 \in F_\alpha$ and $g(x_0, F_\alpha) = 1$. Therefore we obtain

$$g(x_0, \tilde{A}) = \sup_{\alpha \in [0, h_A(x_0)]} [\alpha \wedge g(x_0, F_\alpha)]$$

$$= h_A(x_0). \qquad \text{(Q.E.D.)}$$

[Definition 3.8] A fuzzy integral over $\tilde{A}$ is defined in the following way.

$$\int_{\tilde{A}} h(x) \circ g(\cdot) = \int_X [h_A(x) \wedge h(x)] \circ g(\cdot) \qquad (3.35)$$

[Proposition 3.12] There hold

1) $\int_{\tilde{A} \cup \tilde{B}} h \circ g \geq \int_{\tilde{A}} h \circ g \vee \int_{\tilde{B}} h \circ g,$ (3.36)

2) $\int_{\tilde{A} \cap \tilde{B}} h \circ g \leq \int_{\tilde{A}} h \circ g \wedge \int_{\tilde{B}} h \circ g.$ (3.37)

(Proof) Using Eq. (3.35), we have that $\int_{\tilde{A} \cup \tilde{B}} h \circ g = \int_X [(h_A \vee h_B) \wedge h] \circ g$ $= \int_X [(h_A \wedge h) \vee (h_B \wedge h)] \circ g \geq \int_X (h_A \wedge h) \circ g \vee \int_X (h_B \wedge h) \circ g.$

We can prove 2) in an analogous way. (Q.E.D.)

As previously stated, the fuzzy integral over F is expressed as

$$\oint_F h(x) \circ g(\cdot) = \oint_X [\chi_F(x) \wedge h(x)] \circ g(\cdot).$$

Thus Definition 3.8 shows a natural extension of the domain of integration to a fuzzy set. Proposition 3.12 states that the properties of Eqs. (3.10) and (3.11) are kept in this extension.

We have discussed an extension of g onto $\tilde{\mathcal{B}}$. In particular, $\tilde{g}_\lambda$ keeps the property shown in Eq. 2.7. To show this, let us define λ-complements of fuzzy sets. The complement of a fuzzy set A is defined by the membership function $1-h_A(x)$, which corresponds to $\chi_{E^c}(x) = 1 - \chi_E(x)$. However, comparing it with the definitions of unions and intersections, it seems to be unsatisfactory from the viewpoint of fuzziness that the concept of complements is defined uniquely as stated above. For instance, it is doubtful that the complement of a fuzzy set of "charming ladies", i.e., a fuzzy set of "not charming ladies", should be defined through the operation $1-h_A$. The definition by $1-h_A$ is so restrictive that a flexible definition seems to be more suitable for our ordinary senses. But the comparative relation with non-fuzzy sets should not be thrown away even at the new definition of complements.

[Definition 3.9] A fuzzy set with the membership function h defined by Eq. (3.38) is called a λ-complement of $\tilde{A}$ and written $\tilde{A}^{c_\lambda}$.

$$h(x) = \frac{1 - h_A(x)}{1 + \lambda h_A(x)}, \quad -1 < \lambda < \infty \tag{3.38}$$

In the above definition, putting $\lambda = 0$, $\tilde{A}^{c_\lambda}$ agrees with that defined by Zadeh. The λ-complement has flexibility for a parameter and there holds

$$E^{c_\lambda} = E^c$$

for non-fuzzy sets since

$$\frac{1 - \chi_E(x)}{1 + \lambda\chi_E(x)} = 1 - \chi_E(x) \text{ for } -1 < \lambda < \infty.$$

[Theorem 3.14] Let h be a $\mathcal{B}$-measurable function. Then $\frac{1 - h}{1 + \lambda h}$ is also $\mathcal{B}$-measurable and there holds

$$\oint \frac{1 - h}{1 + \lambda h} \circ g_\lambda(\cdot) = \frac{1 - \oint h \circ g_\lambda(\cdot)}{1 + \lambda \oint h \circ g_\lambda(\cdot)}. \qquad (3.39)$$

(Proof) Define $F_\alpha = \{x | h \geq \alpha\}$ and $G_\beta = \{x | \frac{1 - h}{1 + \lambda h} > \beta\}$. If $\beta = \frac{1 - \alpha}{1 + \lambda\alpha}$, then $G_\beta = F_\alpha^c$ holds clearly. Since h is $\mathcal{B}$-measurable, we obtain $F_\alpha^c \in \mathcal{B}$ for all $\alpha \in [0,1]$, from which $G_\beta \in \mathcal{B}$ for all $\beta \in [0,1]$. Therefore $\frac{1 - h}{1 + \lambda h}$ is also $\mathcal{B}$-measurable. From Lemma of Theorem 3.9, the left-hand side of Eq. (3.39) is expressed as $\sup_\beta [\beta \wedge g_\lambda(G_\beta)]$.

Now let

$$M' = \oint \frac{1 - h}{1 + \lambda h} \circ g_\lambda.$$

Then, as is shown in the proof of Lemma of Theorem 3.9, there holds

$$g_\lambda(G_{M'-0}) \geq M' \geq g_\lambda(G_{M'}).$$

Let

$$M' = \frac{1 - M}{1 + \lambda M}.$$

We obtain

$$G_{M'-0} = \{x | M \geq h\}$$

since

$$G_{M'-0} = \{x | \frac{1 - h}{1 + \lambda h} \geq M'\},$$

from which

$$G_{M'-0} = F^c_{M+0}.$$

Hence holds :

$$g_\lambda(F^c_{M+0}) \geq \frac{1 - M}{1 + \lambda M} \geq g_\lambda(F^c_M).$$

Using Eq. (2.7), it is derived that

$$g_\lambda(F_M) \geq M \geq g_\lambda(F_{M+0}).$$

Thus we obtain $M = \int h \circ g_\lambda$ from Theorem 3.9. Proof is now complete.

(Q.E.D.)

For a non-fuzzy set E, there follows from Eq. (2.7)

$$g_\lambda(E^{c_\lambda}) = \frac{1 - g_\lambda(E)}{1 + \lambda g_\lambda(E)}$$

since

$$E^{c_\lambda} = E^c.$$

Corresponding to this relation, we have the next lemma for a fuzzy set $\tilde{A}$ which shows another expression of Theorem 3.14.

[Lemma]

$$g_\lambda(\tilde{A}^{c_\lambda}) = \frac{1 - g_\lambda(\tilde{A})}{1 + \lambda g_\lambda(\tilde{A})} \tag{3.40}$$

As is shown in Section 2.2, g_λ converges to a probability measure as $\lambda \to 0$. The fuzzy integral $⨍ h \circ g_\lambda$, however, does not converge to a Lebesgue integral as $\lambda \to 0$. In accordance with Theorem 3.11, there remains the difference such that $|⨍ h \circ g_0 - \int h dg_0| \leq 1/4$.

The reason for this is that a fuzzy integral is defined for an arbitrary g but not specially for g_λ. However, we can find an operation by using g_λ which actually converges to $\int h\, dg_0$ as $\lambda \to 0$. It will be shown in Appendix B.

Chapter 4

FUZZY INTEGRALS ON FINITE SETS

In this chapter, let us discuss the fuzzy integrals on finite sets. If X of a fuzzy measure space $(X, \mathcal{B}, g)$ is a finite set, it is not necessary to assume continuity of fuzzy measures.

Denote a finite set with n members by $K = \{s_1, s_2, \ldots, s_n\}$. We consider a fuzzy measure space $(K, 2^K, g)$. According to Definition 3.1, a fuzzy integral over A of a function $h : K \to [0,1]$ can be written as follows.

$$\int_A h(s) \circ g(\cdot) = \max_{K' \in 2^K} [\min_{s \in K'} h(s) \wedge g(A \cap K')] \qquad (4.1)$$

Here if it is assumed that $h(s_i) \leq h(s_{i+1})$ for $1 \leq i \leq n-1$ (if not so, rearrange $h(s_i)$), then the next theorem holds.

[Theorem 4.1] A fuzzy integral in $(K, 2^K, g)$ can be written as

$$\int_A h(s) \circ g = \bigvee_{i=1}^{n} [h(s_i) \wedge g(A \cap K_i)], \qquad (4.2)$$

where $K_i = \{s_i, s_{i+1}, \ldots, s_n\}$.

(Proof) Let $h(s_i) = \min_{s \in K'} h(s)$, then we obtain

$$\max_{K' \in 2^K} [\min_{s \in K'} h(s) \wedge g(A \cap K')] \leq \bigvee_{i=1}^{n} [h(s_i) \wedge g(A \cap K_i)].$$

On the other hand, the reverse inequality holds since $2^K \supset \{K_i | 1 \leq i \leq n\}$.

(Q.E.D.)

Though the power set 2^K has 2^n members, according to Theorem 4.1, it is necessary for fuzzy integration that g is defined merely for a monotone sequence of subsets of K such as $A\cap K_1 \supset A\cap K_2 \supset \cdots \supset A\cap K_n$. This fact has been already mentioned in Section 3.1. We can integrate a given function by calculating $h(s_i)\wedge g(A\cap K_i)$ at n points at most.

Let us here pick up again Theorem 3.9 and Theorem 3.11.

[Theorem 3.9'] There holds

$$\bigvee_{i=1}^{n} [h(s_i)\wedge g(A\cap K_i)] = h(s_\ell)\wedge g(A\cap K_\ell),$$

if and only if

$$h(s_{\ell-1}) \leq g(A\cap K_\ell) \leq h(s_\ell)$$

or

$$g(A\cap K_\ell) > h(s_\ell) \geq g(A\cap K_{\ell+1}).$$

(Proof) First, let us prove necessity. Since $h(s_i)$ is monotonously increasing and $g(A\cap K_i)$ monotonously decreasing, $h(s_i)\wedge g(A\cap K_i)$ is a function with a single peak for i. Therefore, it is necessary that

$$h(s_{\ell-1})\wedge g(A\cap K_{\ell-1}) \leq h(s_\ell)\wedge g(A\cap K_\ell) \geq h(s_{\ell+1})\wedge g(A\cap K_{\ell+1}).$$

Assume $h(s_\ell) \geq g(A\cap K_\ell)$. Then we obtain $h(s_{\ell-1}) \leq g(A\cap K_\ell) \leq h(s_\ell)$ from the above inequality. On the contrary, assume $h(s_\ell) < g(A\cap K_\ell)$. Then $g(A\cap K_\ell) > h(s_\ell) \geq g(A\cap K_{\ell+1})$ is derived. Sufficiency is valid.

(Q.E.D.)

Now, a probability measure P on a finite set K is defined in the following way. Let $0 \leq p^i \leq 1$ for $1 \leq i \leq n$ and $\sum_{i=1}^{n} p^i = 1$.

1) $P(\phi) = 0$

2) $P(K') = \sum_{s_i \in K'} p^i$, where $K' \subset K$.

[Theorem 3.11']

Let

$$h(s_i) \leq h(s_{i+1}) \text{ for } 1 \leq i \leq n-1$$

and

$$K_i = \{s_i, s_{i+1}, \ldots, s_n\}.$$

Then there holds

$$\left| \sum_{i=1}^{n} h(s_i)p^i - \bigvee_{i=1}^{n} [h(s_i) \wedge p(K_i)] \right| \leq 1/4. \tag{4.3}$$

(Proof) Let $E_p = \sum_{i=1}^{n} h(s_i)p^i$ and $E_f = \bigvee_{i=1}^{n} [h(s_i) \wedge P(K_i)]$. Let further for simplicity $h_i = h(s_i)$ and $P_i = P(K_i)$. Assume $E_f = h_\ell \wedge P_\ell$.

(i) If $h_\ell \geq P_\ell$, then, taking into account that $\{h_i\}$ increases monotonously and $P_i = \sum_{j=i}^{n} p^j$, we obtain from Theorem 3.9'

$$\begin{aligned} E_p &= \sum_{i=1}^{\ell-1} h_i p^i + \sum_{i=\ell}^{n} h_i p^i \\ &\leq \sum_{i=1}^{\ell-1} h_{\ell-1} p^i + \sum_{i=\ell}^{n} h_n p^i \\ &= h_{\ell-1}(1 - P_\ell) + h_n P_\ell \\ &\leq P_\ell (1 - P_\ell) + h_n P_\ell. \end{aligned}$$

On the other hand, there holds similarly

$$E_f \geq h_1(1 - P_\ell) + h_\ell P_\ell$$
$$\geq h_1(1 - P_\ell) + P_\ell^2.$$

From the above inequalities it follows that

$$-(\frac{1-h_1}{2})^2 \leq E_p - E_f \leq (\frac{h_n}{2})^2.$$

(ii) Conversely if $h_\ell < P_\ell$, then we have the same inequality.
Hence holds $|E_p - E_f| \leq 1/4$. (Q.E.D.)

In Theorem 3.9', it is better for understanding that we rewrite the probability expectation as follows.

$$E_p = \sum_{i-1}^{n} h(s_i)(P(K_i) - P(K_{i+1})), \text{ where } K_{n+1} = \phi. \quad (4.4)$$

The fuzzy integrals on finite sets can be also used to execute the continuous integrals numerically because an analytical calculation is almost impossible.

Now let us show how to calculate fuzzy integrals. Here, we use the fuzzy measure g_λ defined in Section 2.2. First let us define a fuzzy distribution function H(s) for a monotone sequence $K = K_1 \supset K_2 \supset \ldots . \supset K_n$ in the following way.

$$1 = H(s_1) \geq H(s_2) \geq \ldots . \geq H(s_n) \geq 0 \quad (4.5)$$

We next define g^i for $1 \leq i \leq n$

$$g^i = \frac{H(s_i) - H(s_{i+1})}{1 + \lambda H(s_{i+1})}, \; 1 \leq i \leq n-1, \quad (4.6)$$

$$g^n = H(s_n).$$

By using g^i, $g_\lambda(\cdot)$ is defined by the next equation.

$$g_\lambda(K') = \frac{1}{\lambda}\left[\prod_{s_i \in K'} (1 + \lambda g^i) - 1\right], \quad -1 < \lambda < \infty \tag{4.7}$$

where $K' \subset K$.

From Eqs. (4.6) and (4.7) we obtain

$$H(s_i) = g_\lambda(K_i) \text{ where } K_i = \{s_i, s_{i+1}, \ldots, s_n\}, \tag{4.8}$$

$$g^i = g_\lambda(\{s_i\}). \tag{4.9}$$

In particular, we obtain

$$g_\lambda(\{s_i, s_j\}) = g^i + g^j + \lambda g^i g^j, \text{ where } i \neq j. \tag{4.10}$$

As is clear from the method of constructing $g_\lambda(\cdot)$, it is possible to give g^i at first instead of $H(s_i)$ such that

$$0 \leq g^i \leq 1, \quad 1 \leq i \leq n. \tag{4.11}$$

In this case we have

$$H(s_i) = \frac{1}{\lambda}\left[\prod_{k=i}^{n} (1 + \lambda g^k) - 1\right] = 1, \quad 1 \leq i \leq n. \tag{4.12}$$

Since $g_\lambda(K) = 1$ is required, we must choose λ so that

$$\frac{1}{\lambda}\left[\prod_{k=1}^{n} (1 + \lambda g^k) - 1\right] = 1. \tag{4.13}$$

By the way, the degree of freedom of $H(s)$ and λ is just n. The degree of freedom of g^i and λ is also n. The parameter λ can be arbitrarily chosen in the first method but must be adjusted to normalize $g_\lambda(K)$ in the second method. This normalization can be easily done since the equation (4.13) is merely an algebraic equation for λ.

[Proposition 4.1] Within $(-1, \infty)$, Eq. (4.13) has only one solution.

Differentiate $f(\lambda)$ with respect to λ.

$$f'(\lambda) = \frac{1}{\lambda} [(1 + \lambda f(\lambda)) \sum_{i=1}^{n} \frac{g^i}{1 + \lambda g^i} - f(\lambda)]. \tag{4.14}$$

Using Newton-Raphson's method, the k-th approximation of the solution of (4.13) is obtained.

$$\lambda_k = \lambda_{k-1} - \frac{f(\lambda_{k-1}) - 1}{f'(\lambda_{k-1})} \tag{4.15}$$

If the value of λ is necessary to be free in the second method, we can multiply each g^i by a scalar and choose it as (4.13) is satisfied. Let us next show, given $h_i = h(s_i)$ for $1 \le i \le n$, a procedure of fuzzy integration. We assume that g^i is given for $1 \le i \le n$.

Step 1 Rearrange h_i such that $h_{r_1} \le h_{r_2} \le \dots \le h_{r_n}$.

Step 2 Calculate

$$H(r_{i-1}) = g^{i-1} + H(r_i) + \lambda g^{i-1}(r_i) \text{ for } 1 \le i \le n-1$$

where $H(r_n) = g^n$.

Step 3 Find for $1 \le i \le n$ the maximum of $h_{r_i} \wedge H(r_i)$.

It can be easily shown that the fuzzy integral can be written as

$$\oint h(s) \circ g_\lambda = \bigvee_{i=1}^{n} [h_{r_i} \wedge H(r_i)]. \tag{4.16}$$

So we can obtain the integrated value of h by Step 3. In Step 2, it is not necessary to calculate $H(r_i)$ for all i because $h_{r_i} \wedge H(r_i)$ is a function with a single peak for i. That is, we can stop calculating when its peak is found. More precisely, it is necessary for the calculation of a fuzzy integral to evaluate $H(r_i)$ at least for only three different i's, even if $n \to \infty$. This fact is a very excellent convenience in comparison with the ordinary integral calculus.

Tsukamoto [18] has proposed another construction of fuzzy measures on finte sets.

Define

$$g_\mu^*(K') = (1 - \mu) \bigvee_{s_i \in K'} g^i + \mu \sum_{s_i \in K'} g^i, \quad 0 \le \mu \le 1. \qquad (4.17)$$

Here, μ is determined from $g_\mu^*(K) = 1$. Let $K' \cap K'' = \phi$. If $\mu = 0$, then

$$g_\mu^*(K' \cup K'') = g_\mu^*(K') \vee g_\mu^*(K'').$$

If $\mu = 1$, then

$$g_\mu^*(K' \cup K'') = g_\mu^*(K') + g_\mu^*(K'').$$

Chapter 5

APPLICATIONS OF FUZZY INTEGRALS

5.1 Introduction

In this chapter, let us apply fuzzy integrals to the problems of subjective evaluation of fuzzy objects [12, 14, 18]. Fuzzy measures, as has been discussed in Chapter 2, are considered as subjective measures for grade of fuziness. When application problems are discussed, however, it is convenient to interpret fuzzy measures more concretely. This will be discussed in the examples of applications in this Chapter.

Now when a human being tries to measure and evaluate the objects which seems fuzzy, his evaluation is related to both, the nature of the objects and his own subjectivity. In general, there appears in the process of subjective evaluation the complicated interplay between the objects and the evaluater's subjectivity. In this sense, fuzzy measures should be considered to change their properties affected by the both of the objects and his subjectivity.

The evaluation problems treated so far in systems engineering are mostly those which are based on objective standards, e.g., the performance indices of optimal control systems. However, let us discuss here the evaluation problems which lead to different results according to the subjectivity of the individuals evaluating the objects. The concept of fuzzy measures is powerful particularly for dealing with these problems.

First, for preparation, the problem of grading the similarity of one dimensional patterns is discussed in Section 5.2 where a fuzzy measure is not considered as one belonging to an individual but as a common measure of several individuals. There is an explanation of relation between the nature of the objects and a fuzzy measure.

In Section 5.3 the evaluation problem of female faces by male is picked up. It is attempted to distinguish the subjectivities of individuals through the identified fuzzy measures.

Both problems are concerned with the evaluation of the objects which are decomposed into several elements. The overall evaluation and the partial evaluation of the elements are connected by using fuzzy integrals.

5.2 Grading similarity of patterns

Let us discuss, through grading the similarity of fuzzy patterns, how fuzzy integrals can be used for evaluating fuzzy objects. Assuming that there are some fuzzy patterns similar to each other, we consider the problem of how a human being grades their similarity. If we let a person compare a test pattern with standard patterns, then he, finding the characteristics of the pattern by some means, can answer the grade of similarity in the numerical values according to his standard of evaluation. Here we assume that he evaluates the patterns by his subjective measure. Let us now adopt an expression with a fuzzy integral as a mathematical model of the subjective evaluation of patterns as stated above.

First let us suppose a pattern with some black and white points on a plane. Dividing the plane into suitable meshes and giving the numerical values to the parts, such as 1 to the black and 0 to the white, we can obtain a characteristic function of the pattern. (The problem of distinguishing patterns by their characteristic functions with binary values is known as the design problem of perceptrons.) When the boundaries between black and white parts of the patterns are unclear and look gray, the characteristic functions of the patterns will take the values in the interval [0,1]. The fuzzy integrals of these functions can be used as the discriminant functions of the patterns by constructing a suitable g_λ.

Let $P = \{A, B\} \cup \{C, D, \ldots.\}$ be a set of patterns. Here A and B are standard patterns. C and D, etc. are general patterns which should be discriminated. Let us define a function p_C on a finite set K such that $p_C : K \to [0,1]$. Here p_C is considered as the characteristic function of a pattern C, and K is a set of points showing the characteristics of the patterns. Comparing the given $C \in P$ with A and B, we express the similarity "which of them C is similar to" in the numerical value of [0,1] Define

$$\Psi(C) = \oint_K p_C \circ g_\lambda(\cdot). \tag{5.1}$$

Assuming that A and B have been selected so that $\Psi(A) \geq \Psi(C) \geq \Psi(B)$, let us next define

$$w(C) = \frac{\Psi(C) - \Psi(B)}{\Psi(A) - \Psi(B)}. \tag{5.2}$$

Then we have $0 \leq w(C) \leq 1$, $w(A) = 1$, and $w(B) = 0$. Therefore, the function

$$w : P \to [0,1] \tag{5.3}$$

can be considered as a membership function of the fuzzy set of the patterns which are similar to A but not to B.

If we regard p_C as a membership function of a certain fuzzy set, then $\Psi(C)$ in Eq. (5.1) becomes equal to $\tilde{g}_\lambda(C)$ in Section 3.4 and keeps the properties of the extended fuzzy measure $\tilde{g}_\lambda$. In Eq. (5.2), w is the normalized $\tilde{g}_\lambda$.

On the other hand, if we show a person these patterns and let him determine a function

$$d : P \to [0,1] \tag{5.4}$$

so that there hold $0 \leq d(C) \leq 1$, $d(A) = 1$ and $d(B) = 0$, then another membership function d is obtained.

Here let us analyze the process in which a human being evaluates the similarity of the patterns. First he grasps the characteristics of the standard patterns, from which he constructs his subjective measure. Next he evaluates the similarity by using this measure. The characteristics of the standard patterns are, for instance, such that the pattern A is convex and the pattern B concave. His measure would be one which shows to what extent he attaches importance to a part of a pattern, which can be expressed as a subset of K. We consider the grade of "importance" of K' for $K' \subset K$. If the similarity of the patterns is fuzzy for a person, the importance of a part of a pattern is also fuzzy. Therefore we can consider that the grade of importance is an example of the grade of fuzziness discussed in Section 2.1.

It is clear that the grade of importance of a subset satisfies monotonicity; if $K' \subset K'' \subset K$, then the grade of importance of K" should be larger than that of K'. Now we can interpret the fuzzy measure g_λ in Eq. (5.1) as a subjective measure expressing the grade of importance.

Adopting Eqs. (5.1) and (5.2) as a macro model in which a person determines d, and choosing g_λ under a suitable criterion so that w becomes nearly equal, we can obtain his subjectivity through the fuzzy measure g_λ. That is, if we assume that a person discriminates these fuzzy patterns by a certain fuzzy measure, then his fuzzy measure can be identified in the above procedure.

A simple experiment was performed using one dimensional patterns shown in Fig. 5.1. These patterns were formed by plotting eight points which take values of 0.8, 0.7, 0.5 and 0.2, just two points taking each of them. Here the function p_C itself was regarded as a pattern. Selecting twenty patterns besides A and B, we asked the subjects to answer the values of d. We adopted finally, as d of man, the average value of d which was given by the ten subjects. Next, picking up fifteen points at equal intervals from the pattern, and forming K and the function p_C, we tried to identify g_λ so that the following J could become small.

$$J = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (d_i - w_i)^2}, \text{ where N is the number of patterns.} \tag{5.5}$$

The results are shown in Figs. 5.2 and 5.3. Fig. 5.3 shows the fuzzy density with $\lambda = -0.5$ which gives the optimal g_λ. Fig. 5.2 shows the comparison of d_i and w_i. If a small circle with lies on the straight line, it implies that there holds $d_i = w_i$. As can be seen in Fig. 5.2, the similarity of the patterns obtained from the model approximately agrees with that evaluated by several persons. As has been already mentioned, the value of g^i at the i-th point shows to what extent one attaches importance to that point when he discriminates the patterns.

From this consideration, the shape of g^i can be imagined; it is expected that a function with a peak at the center, which is similar to the pattern A, is suitable for g^i $(1 \leq i \leq n)$. Actually we can obtain a fairly good value of J by the method of choosing an adequate g^i for $1 \leq i \leq n$ and changing the value of λ in steps. The value of λ was about -0.5 in the both methods. From these results, it seems that the value of λ is partially reflected by the property of the objects. Let us next consider this.

The patterns A and B in Fig. 5.1 are in a complementary relation with each other. So let us examine the relation of p_A and $(1 - p_B)/(1 + \lambda p_B)$ by using the concept of the λ-complement shown in Definition 3.9 in Section 3.4. The function $(1 - p_B)/(1 + \lambda p_B)$ is shown in Fig. 5.4 where the parameter λ takes three different values. The dotted line shows p_A, i.e., the pattern A. In Fig. 5.4, it can be seen that there holds

$$p_A \simeq (1 - p_B)/(1 - 0.5p_B) \tag{5.6}$$

Now let us write, in accordance with the notation of λ-complements, the pattern expressed by $(1 - p_B)/(1 + \ p_B)$ as B^{c_λ}. Then we obtain $A \simeq B^{c_\lambda}$ when $\lambda \simeq -0.5$. That is, the patterns A and B are in a complementary relation corresponding to $\lambda \simeq -0.5$ and this value of λ is just equal to that obtained by the identification of g_λ. However, it cannot be suggested from this simple example of applications that the relation clarified above holds generally with respect to λ of the fuzzy measure g_λ.

As has been already mentioned, the identified λ is negative. This implies, assuming that the grades in which a human being attaches importance to the subsets K' and K" of K are $g_\lambda(K')$ and $g_\lambda(K'')$, respectively where $K' \cap K'' = \phi$, that the grade in the case of $K' \cup K''$ is $g_\lambda(K' \cup K'') < g_\lambda(K') + g_\lambda(K'')$; the grade of union is smaller than the addition of grades.

In this application, g_λ was adopted as a fuzzy measure. However, denoting the set of all set functions which satisfy the fuzzy measures by G, the set $\{g_\lambda | -1 < \lambda < \infty\}$ is merely a subset of G. (Recall a probability measure P is merely an element of $\{g_\lambda | -1 < \lambda < \infty\}$.) Therefore, our identification problem should be that we find the optimal set function in G, i.e., $\min_{g \in G} J$. Since g is a set function, but not an ordinary function, it is a new problem different from the problems previously encountered from the viewpoint of engineering.

5.3 Subjective evaluation of female faces

Here, let us discuss the problem of the evaluation of female faces. Pictures of about 100 young ladies were taken. The boundary conditions of these pictures are kept constant carefully. Thirty pictures are chosen at random and enlarged to actual size. Each of these pictures is cut into five pieces; those are eyes, nose, mouth, chin and all the remains, as are shown in Figs. 5.5 and 5.6. Those pieces are shown to student (male) separately and according to his preference they are scored with a numerical value between zero and one. The ideal picture is scored one and the worst is zero. Now we obtain five values for each face. Next the complete picture is shown to the student, who is asked score it by the same scoring rule. The problem is how to connect the score of a whole face with those of pieces.

Generally, when a system is perfectly decomposed into mutually independent factors, a linear model is usually used to relate overall and the partial evaluations. However, if the boundaries among the factors are not sharp and the factors influence each other, a fuzzy integral model is one of the powerful means to evaluate such fuzzy objects. A fuzzy integral model is derived analogously as in Section 5.2.

The symbol s_1, s_2, ..., s_5 are used for eyes, nose, mouth, chin and the remains.

Define

$$K = \{s_1, s_2, s_3, s_4, s_5\}.$$

From the above experiments, we obtain the function

$$h_j : K \rightarrow [0, 1] \tag{5.7}$$

where j is the number of pictures.

If a linear model is used, the preference w_j of j-th face is expressed as follows.

$$w_j = \sum_{i=1}^{5} a_i \, h_j(s_i) \tag{5.8}$$

Using the fuzzy measure g_λ which is a student's subjective scale concerned with grade of importance, a fuzzy integral model is introduced as follows.

Define

$$e_j = \int_K h_j(s) \circ g(\cdot). \tag{5.9}$$

Let $\bar{e} = \max_{1\le j\le N} \{e_j\}$ and $\underline{e} = \min_{1\le j\le N} \{e_j\}$, where N is the total number of faces. Let d_j denote the score of the whole face which is obtained from the experiment. Similarly, $\bar{d}$ and $\underline{d}$ are defined. Now, e_j is normalized so that $\bar{e} = \bar{d}$ and $\underline{e} = \underline{d}$. The preference w_j is obtained as follows.

$$w_j = \frac{\bar{d} - \underline{d}}{\bar{e} - \underline{e}} e_j + \frac{\underline{d}\bar{e} - \bar{d}\underline{e}}{\bar{e} - \underline{e}} \quad (5.10)$$

The fuzzy measure g_λ is identified so as to minimize the criterion function J in Eq. (5.5). When "complex method" was used for hill-climing, the minimum value of J was about 0.1.

The comparison of the calculated value w with the experimental one d is shown in Fig. 5.7. Fig. 5.8 shows the fuzzy measures of two students. In this figure, if g^i for a specific i is larger than the others, it means that the student thinks the i-th piece very important. So we can know from Fig. 5.8 the characteristics of an individual who evaluates ladies' faces. As is shown in Fig. 5.7, the experimental results show a good agreement with the calculation by the model. As a matter of course, the difference between w and d decreases, if the faces are decomposed into fewer elements.

The process of subjective evaluation can be explained qualitatively by using the concept of fuzzy measures. When a linear model is adopted, it is difficult to interpret the weighting coefficients. In Eq. (5.8), if a coefficient a_i is large, then a partial evaluation $h(s_i)$ is enlarged. This implies that the value of the overall evaluation increases in the linear model even if a partial evaluation is small. However, a man will give actually a relatively small value to the overall evaluation when $h(s_i)$ is small.

On the contrary, in the fuzzy integral model, if a partial evaluation $h(s_i)$ is smaller than a fuzzy density g^i, then $h(s_i)$ contributes directly the overall evaluation. If $h(s_i)$ is larger than g^i, then the value of $h(s_i)$ is cut at that of g^i. This implies that a large value of a partial evaluation is cut when the grade of importance of i-th element is small. Therefore, we could say that a fuzzy integral model can explain a human evaluation process more qualitatively than a linear model. Further it should be pointed out that the concept of the grade of importance is convenient in representing a subjective evaluation process.

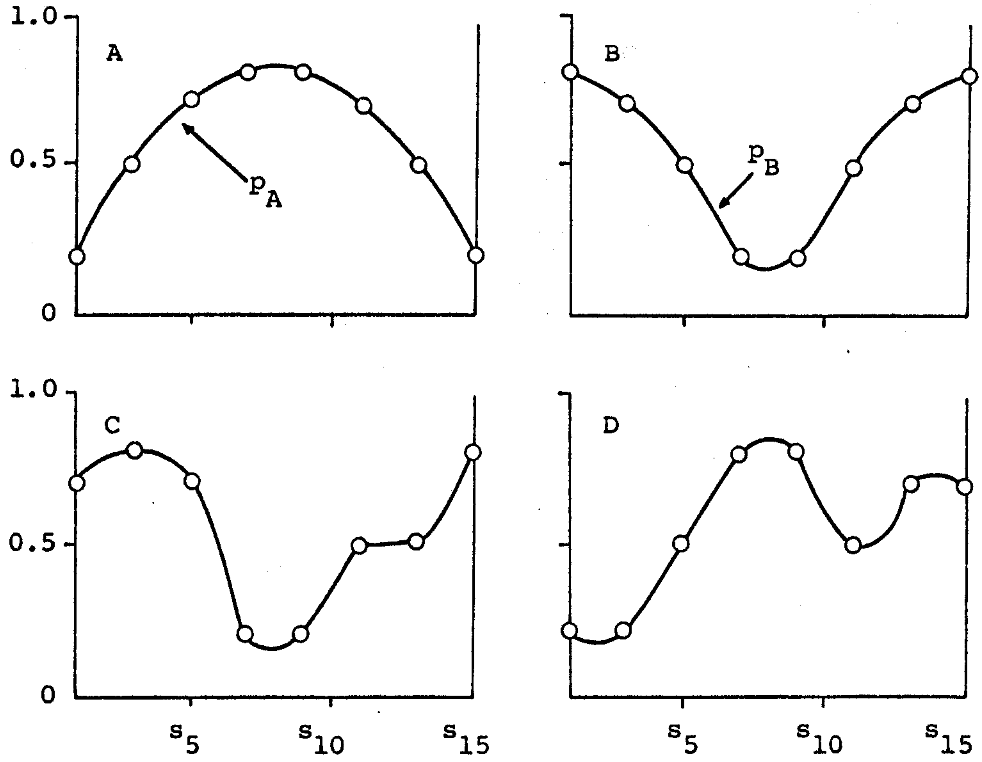


Fig. 5.1 Examples of Patterns

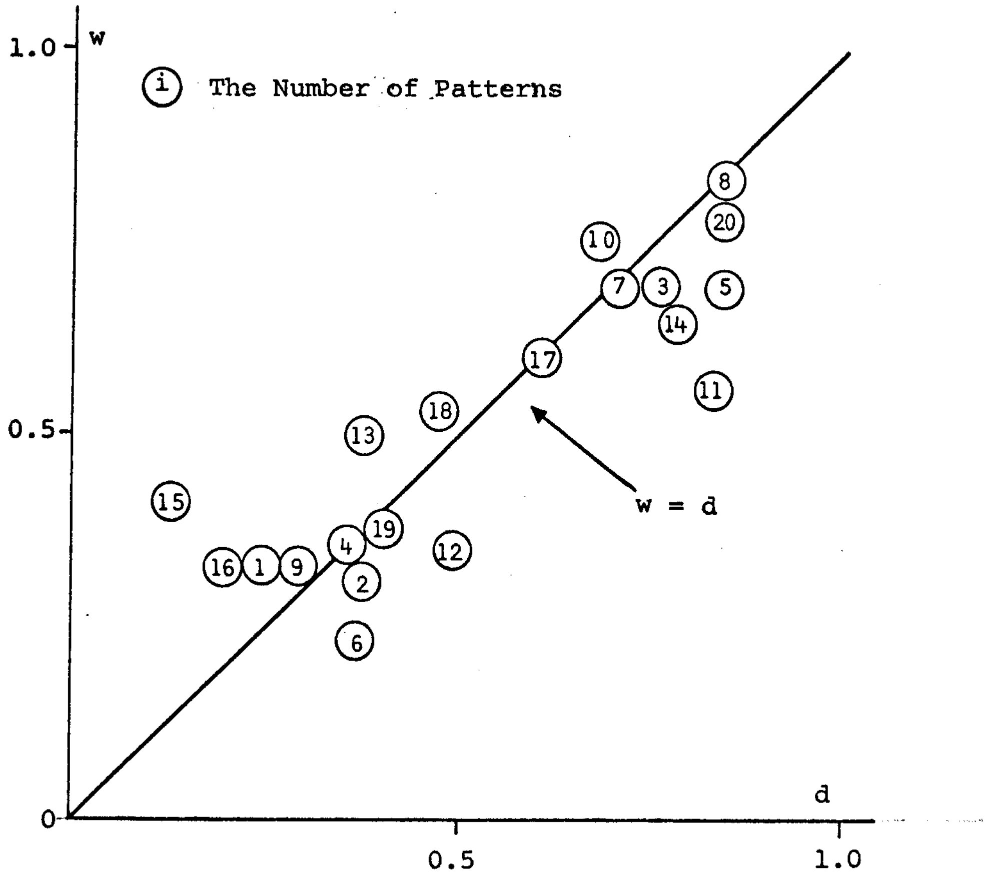


Fig. 5.2 Comparison of d and w

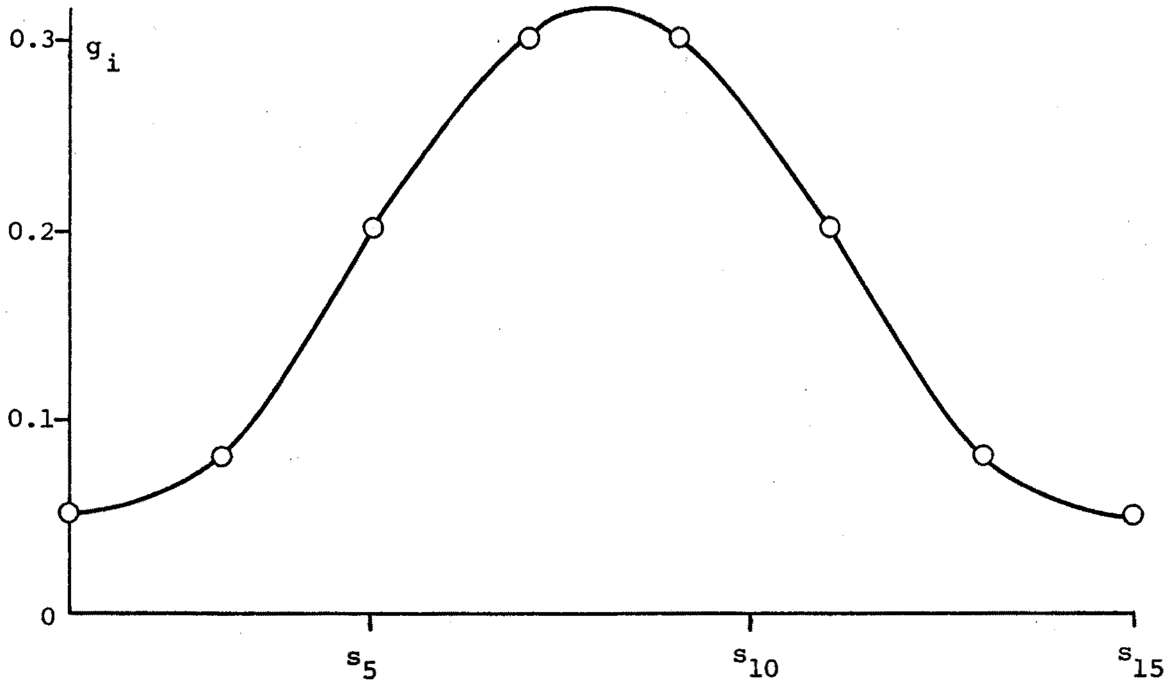


Fig. 5.3 Fuzzy Density of Identified g_λ, $\lambda = -0.5$

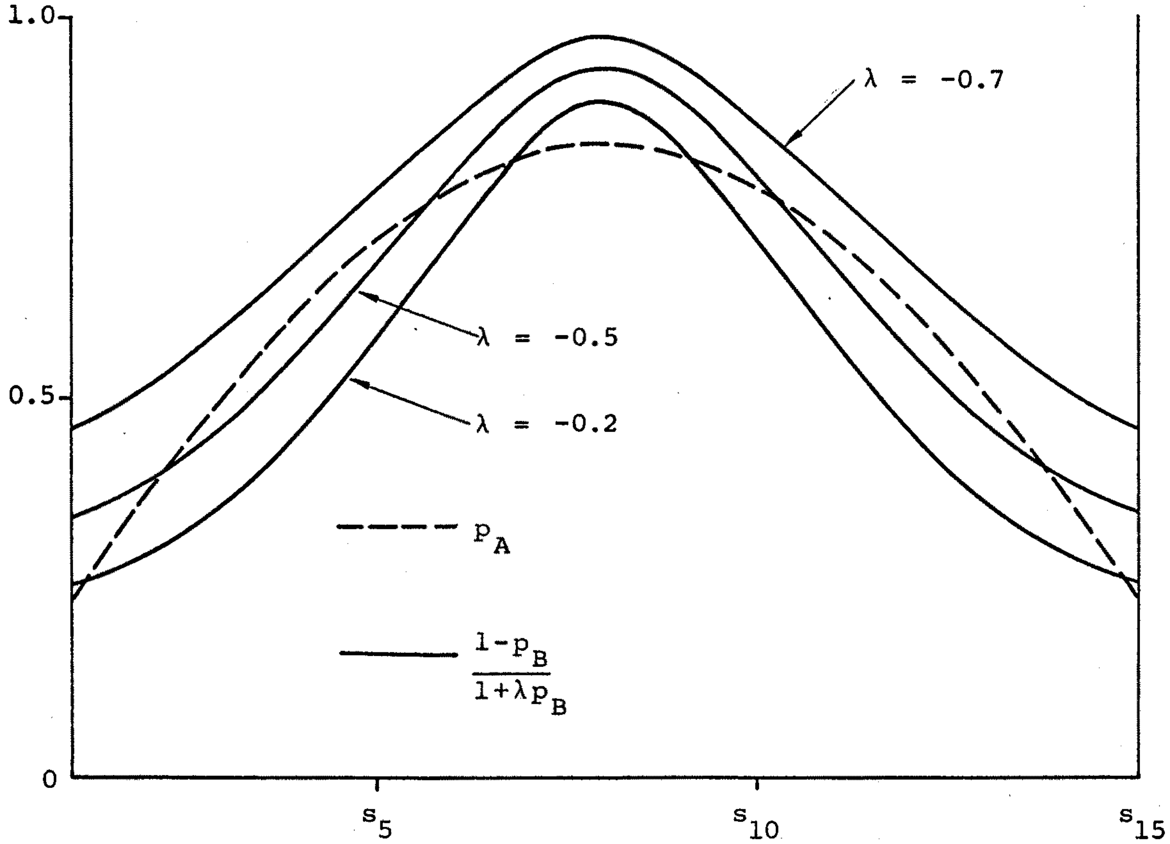


Fig. 5.4 Comparison of A and B^{c_λ}

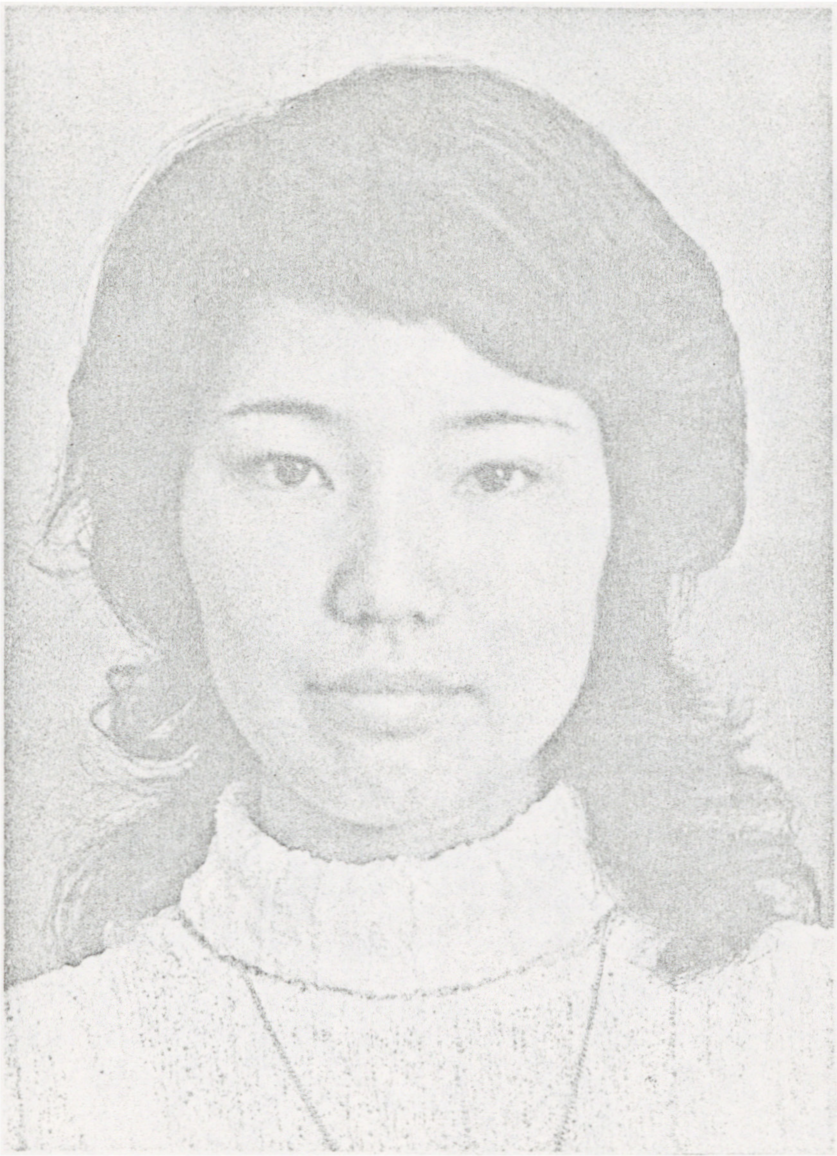


Fig. 5.5 Example of Female Face

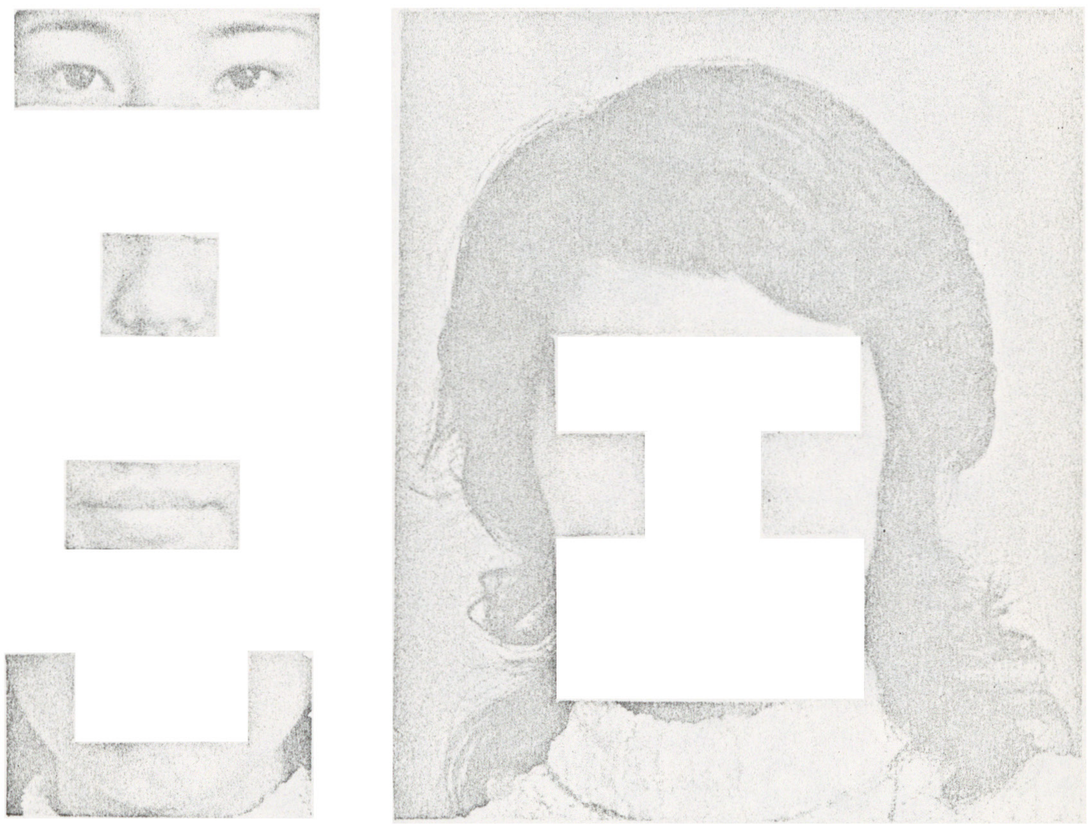


Fig. 5.6 Elements of Face

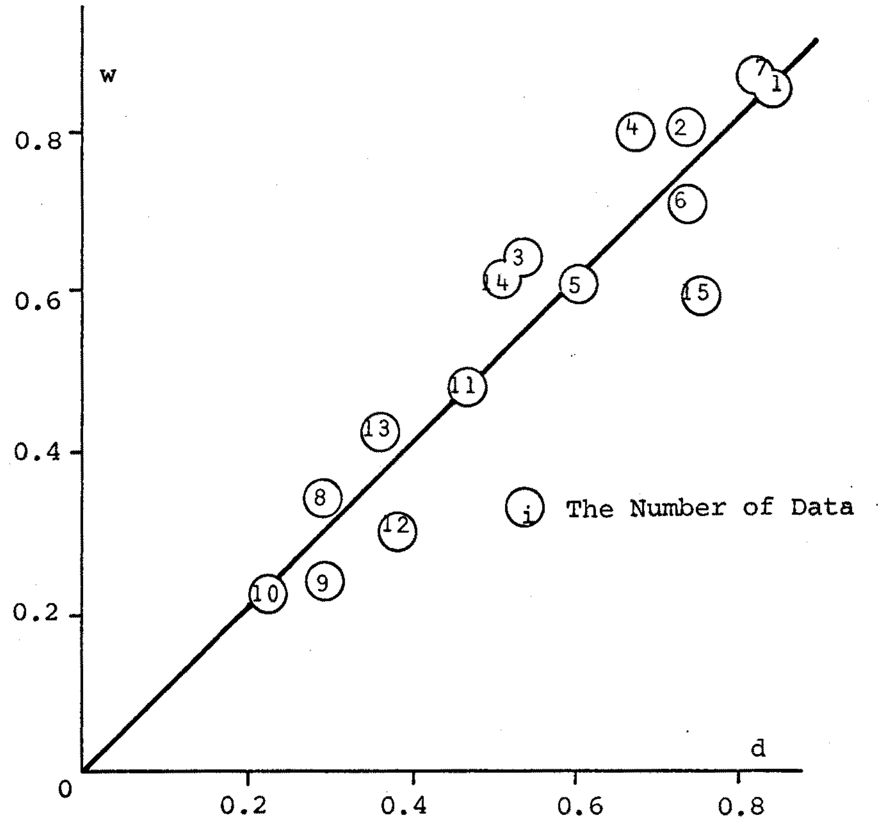


Fig. 5.7 Comparison of w(model) and d(experiment)

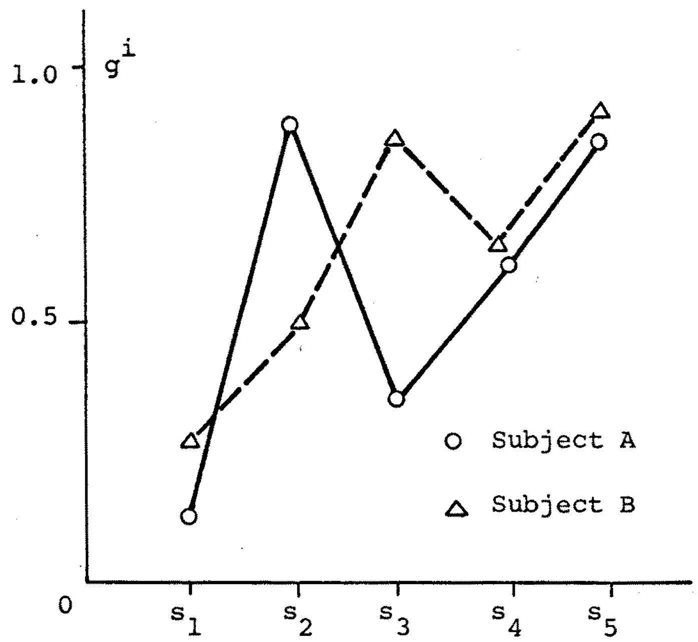


Fig. 5.8 Identified Fuzzy Densities of Subjects A and B

Chapter 6

CONDITIONAL FUZZY MEASURES

6.1 Inverse operation of fuzzy integrals

In this section, let us consider an inverse operation of fuzzy integrals which corresponds formally to differentiation. In the next section, conditional fuzzy measures are defined and their existence theorem is proved by using this operation.

Define a set function $\Phi(\cdot)$ by

$$\Phi(A) = \int_A h(x) \circ g(\cdot) \text{ where } A \in \mathcal{B}. \tag{6.1}$$

It is easily shown from the property of fuzzy integrals in Theorem 3.4 that $\Phi(\cdot)$ is a monotone set function. The purpose of this section is to find $h(x)$ assuming that $\Phi(\cdot)$ and $g(\cdot)$ are given. The operation for finding h corresponds to the differentiation of set functions in the theory of Lebesgue integrals. First, we make the necessary assumptions.

[Assumptions]

1. In the fuzzy measure space $(X, \mathcal{B}, g)$, X is a locally compact Hausdorff space with the second countability axiom.
2. The domain of g, $\mathcal{B}$, is the minimum σ-field including all open sets of X.
3. If $g(E) = 0$ then $g(A) = g(A \cup E)$ for any $A \in \mathcal{B}$. The reverse is also true. E is called a null-set. Further if $g(E_n) = 0$ for $i \leq n < \infty$, then there holds $g(\bigcup_{n=1}^{\infty} E_n) = 0$; the union of countable numbers of null-sets is also a null-set.
4. There exists an open set O for any $A \in \mathcal{B}$ such that $A \subset O$ and $g(O - A) < \varepsilon$ for any $\varepsilon > 0$.

For instance, the space R^1 and a fuzzy measure g_λ satisfy all the assumptions. Out of the properties of X derived from Assumption 1, we recapitulate what are necessary to prove Corollaries and Theorems in this section.

1) If x is a point of X, O is an open set of X and $x \in O$, then there exists an open set O_1 such that $x \in O_1$ and $\bar{O}_1 \subset O$.

2) If A is a closed set of X, O is an open set of X and $A \subset O$, then there exists an open set O_1 such that $A \subset O_1$ and $\bar{O}_1 \subset O$.

3) The intersection of all closed neighbourhoods of x is $\{x\}$ for any $x \in X$.

4) Let $A = \bigcup_{\lambda \in \Lambda} O_\lambda$ where Λ is an arbitrary set and O_λ is an open set. Then A can be expressed as $A = \bigcup_{n=1}^{\infty} O_n$ by choosing at most countable numbers of O_n out of $\{O_\lambda | \lambda \in \Lambda\}$.

Now, for a set function $\psi(\cdot)$ which is defined on $\mathcal{B}$ and satisfies

$$0 \leq \psi(\cdot) \leq g(\cdot), \tag{6.2}$$

let us define an operation D_g with respect to g such that $D_g \psi : X \to [0,1]$.

[Definition 6.1] Let I_x be a closed neighbourhood of x such that

$$\psi(I_x) = g(I_x).$$

Define

$$(D_g \psi)(x) = \sup_{I_x} \psi(I_x). \tag{6.3}$$

If there exists no I_x satisfying $\psi(I_x) = g(I_x)$, define for an arbitrary I_x

$$(D_g\psi)(x) = \inf_{I_x} \psi(I_x). \tag{6.4}$$

Hereafter we write $(D_g\psi)(x)$ merely as $D_g\psi(x)$ without confusing a set function $\psi(\cdot)$ with an ordinary function $(D_g\psi)(x)$.

[Definition 6.2] When a proposition holds except on a null-set, it is said to hold almost everywhere with respect to g. We write it as g-a.e.

[Corollary 6.1] Let $\{\psi_n(\cdot)\}$ is a monotone sequence for n, Then there holds

$$\lim_{n\to\infty} D_g\psi_n(x) = D_g \lim_{n\to\infty} \psi_n(x). \tag{6.5}$$

Proof is omitted.

[Corollary 6.2] Let $\Phi_G(\cdot) = g(G\cap\cdot)$ for an open set G. Then there holds

$$D_g\Phi_G(x) = g(G) \wedge \chi_G(x), \text{ (g-a.e.)}. \tag{6.6}$$

(Proof) (i) Assume $x \in G$. From Assumption 1, there exists at least one closed neighbourhood of x such as $I_x \subset G$. There holds $\Phi_G(I_x) = g(I_x)$. From Assumption 1, it is derived that there exists an open set O such as $I_x \subset O$ and $\bar{O} \subset G$, since I_x is a closed set and G is an open set. This implies that there exists a closed neighbourhood arbitrarily close to G since $\bar{O}$ is a closed neighbourhood of x. Therefore we obtain that $D_g\Phi_G(x) = \sup_{I_x} \Phi_G(I_x) = g(G)$.

(ii) Assume $x \notin G$ on the contrary. If there exists I_x such that $\Phi_G(I_x) =$

$g(G \cap I_x) = g(I_x)$, then it is obtained from Assumption 3 that $x \in I_x - G$ and $g(I_x - G) = 0$. On the other hand, if it is assumed that there holds always $g(G \cap I_x) < g(I_x)$, then follows $D_g \Phi_G(x) = \inf_{I_x} g(G \cap I_x) = g(G \cap \bigcap\{I_x\})$. From Assumption 1, it is shown that the intersection of all I_x, i.e., $\bigcap\{I_x\}$, is a set with only one element x. There follows $D_g \Phi_G(x) = 0$ since $x \notin G$.

When $x \in I_x - G$ and $g(I_x - G) = 0$, there exists an open set O such that $x \in O$ and $O \subset I_x - G$. Here O is clearly a null-set. Hence holds $D_g \Phi_G(x) = 0$ except the union of all such O as mentioned above, i.e., $\bigcup O$. Denote $\bigcup O$ by A, then, from Assumption 1, A can be expressed as $A = \bigcup_{n=1}^{\infty} O_n$ by choosing countable numbers of O_n. From Assumption 3, it follows that A is also a null-set. Thus we obtain $D_g \Phi_G(x) = 0$, (g-a.e.)

(Q.E.D.)

[Theorem 6.1] Let

$$\Phi(A) = \int_A h(x) \circ g(\cdot). \tag{6.7}$$

Then $D_g \Phi(x)$ is written as

$$D_g \Phi(x) = \sup_{\alpha} [\alpha \wedge g(F_\alpha) \wedge \chi_{F_\alpha}(x)], \quad \text{(g-a.e.)} \tag{6.8}$$

where $F_\alpha = \{x | h \geq \alpha\}$

and there holds

$$\Phi(A) = \int_A D_g \Phi(x) \circ g(\cdot). \tag{6.9}$$

(Proof) For an arbitrary $G \in \mathcal{B}$, choose a sequence of open sets O_n such as $O_1 \supset O_2 \supset \dots \supset G$ and define $H = \bigcap_{n=1}^{\infty} O_n$. Further, define $\Phi_n(\cdot) = g(O_n \cap \cdot)$ and $\Phi_H(\cdot) = g(H \cap \cdot)$. Then we obtain $\lim_{n\to\infty} \Phi_n(\cdot) = \Phi_H(\cdot)$ from continuity of g.

From Corollaries 6.1 and 6.2, it follows that

$$D_g\Phi_H(x) = \lim_{n\to\infty} D_g\Phi_n(x)$$

$$= \lim_{n\to\infty} [g(O_n) \wedge \chi_{O_n}(x)]$$

$$= g(H) \wedge \chi_H(x), \quad (g\text{-a.e.}).$$

Since it is possible to choose a sequence of open sets so that $g(G) = g(H)$, we obtain $\Phi_G(\cdot) = \Phi_H(\cdot)$, hence, $D_g\Phi_G(x) = D_g\Phi_H(x)$. Further it is derived that $g(H) \wedge \chi_H(x) = g(G) \wedge \chi_G(x)$, (g-a.e.), since $g(H - G) = 0$ from Assumption 3. Therefore, for an arbitrary $G \in \mathcal{B}$, we obtain $D_g\Phi_G(x) = g(G) \wedge \chi_G(x)$, (g-a.e.).

Next, for an arbitrary h(x), we obtain

$$\Phi(\cdot) = \sup_\alpha [\alpha \wedge g(F_\alpha \cap \cdot)]$$

where $F_\alpha = \{x | h \geq \alpha\}$.

From this, it is shown that there holds

$$D_g\Phi(x) = \sup_\alpha [\alpha \wedge D_g\Phi_{F_\alpha}(x)].$$

Hence

$$D_g\Phi(x) = \sup_\alpha [\alpha \wedge g(F_\alpha) \wedge \chi_{F_\alpha}(x)], \quad (g\text{-a.e.}).$$

(Q.E.D.)

[Lemma] Let $M = \int_X h(x) \circ g(\cdot)$. Then there holds

$$D_g\Phi(x) = \begin{cases} M & \text{for } x \in F_M \\ h(x) & \text{for } x \notin F_M \end{cases} \quad (g\text{-a.e.}) \tag{6.10}$$

(Proof) Let $M = \int_X h(x) \circ g(\cdot)$. From Theorem 3.9, we obtain $g(F_M) \geq M \geq g(F_{M+0})$. By monotonicity of g, it is derived that $g(F_\alpha) \geq \alpha$ for $\alpha \leq M$ and $g(F_\alpha) < \alpha$ for $\alpha > M$. From this we obtain that

$$D_g\Phi(x) = \sup_{\alpha \in [0,M]} [\alpha \wedge g(F_\alpha) \wedge \chi_{F_\alpha}(x)] \vee \sup_{\alpha \in (M,1]} [\alpha \wedge g(F_\alpha) \wedge \chi_{F_\alpha}(x)]$$

$$= \sup_{\alpha \in [0,M]} [\alpha \wedge \chi_{F_\alpha}(x)] \vee \sup_{\alpha \in (M,1]} [g(F_\alpha) \wedge \chi_{F_\alpha}(x)].$$

Now assume $x \in F_M$. It follows that $D_g\Phi(x) = M$ since $\chi_{F_\alpha}(x) = 1$ for $\alpha \leq M$. On the contrary, assume $x \notin F_M$. Since $\chi_{F_\alpha}(x) = 0$ for $\alpha > M$, we have

$$D_g\Phi(x) = \sup_{\alpha \in [0,M]} [\alpha \wedge \chi_{F_\alpha}(x)]$$

$$= \sup_{\alpha \in [0,1]} [\alpha \wedge \chi_{F_\alpha}(x)].$$

From Proposition 3.6 follows $D_g\Phi(x) = h(x)$. (Q.E.D.)

Now let μ be a Lebesgue measure. The differential quotient of a set function with respect to μ is, if we omit the details, written as follows:

$$D_\mu\psi(x) = \lim_{I_x \to \{x\}} \frac{\psi(I_x)}{\mu(I_x)}. \tag{6.11}$$

For a specific set function such that $\Phi_G(\cdot) = \mu(G \cap \cdot)$, there holds

$$D_\mu \Phi(x) = \chi_G(x), \quad (\mu\text{-a.e.}). \tag{6.12}$$

Similarity between the inverse operation of fuzzy integrals and the ordinary differentiation is clarified by comparing Eqs. (6.3) and (6.6) with Eqs. (6.11) and (6.12). The operation D_g in Definition 6.1, however, does not include such concept as differentiation.

By Theorem 6.1, it has been shown that if $\Phi(\cdot)$ is expressed by fuzzy integrals, then $\Phi(A)$ can be written $\int_A D_g\Phi(x) \circ g$. Our next problem is under what conditions a general monotone set function $\Phi(\cdot)$ can be written as $\psi(A) = \int_A h(x) \circ g$. There exists, in the theory of Lebesgue integrals, Radon-Nikodym's theorem which states that a μ-absolutely continuous and additive set function can be expressed by a integral of a certain function with respect to the measure μ. In probability theory this is a very important theory because, using this theorem, Kolmogorov proved the existence of conditional probabilities. The next Theorem 6.2 corresponds to Radon-Nikodym's theorem.

Assume that a set function ψ such as $\psi : \mathcal{B} \to [0,1]$ has the following properties:

(1) If $B \subset A$, then $\psi(B) \leq \psi(A)$.

(2) There exists an open set O such that $g(O) = \psi(O) = \psi(X)$.

(3) Define $H_{\psi(A)} = \bigcup \{O \mid g(O) = \psi(O) \geq \psi(A),\ O \text{ is an open set}\}$. Then for an arbitrary A, there holds $\psi(A) = g(A \cap H_{\psi(A)})$.

It is easily derived from these properties that $\psi(\cdot) \leq g(\cdot)$ and, in particular, $\psi(\phi) = 0$. Further it is shown that if $\psi(B) \leq \psi(A)$, then $H_{\psi(B)} \supset H_{\psi(A)}$.

[Corollary 6.3] If $\psi(A) = g(A)$, then $\psi(B) = g(B)$ for any $B \subset A$ where $B \in \mathcal{B}$.

(Proof) Taking into account that $\psi(A)$ is expressed as $\psi(A) = g(A \cap H_{\psi(A)})$ by the properties of $\psi(\cdot)$, we obtain $g(A - A \cap H_{\psi(A)}) = 0$. On the other hand, we obtain also $g(B - B \cap H_{\psi(B)}) = 0$ since $\psi(B)$ is written as $\psi(B) = g(B - B \cap H_{\psi(B)})$ and there holds $B - B \cap H_{\psi(B)} \subset A - A \cap H_{\psi(A)}$. From this follows $\psi(B) = g(B)$. (Q.E.D.)

[Corollary 6.4] If $D_g\psi(x) > 0$, then there exists an open neighbourhood of x such as $g(O_x^*) = \psi(O_x^*)$ and holds $D_g\psi(x) = g(O_x^*)$ almost everywhere with respect to g.

(Proof) (i) Assume that there exists a closed neighbourhood of x such as $g(I_x) = \psi(I_x)$ for a certain x. From the properties of ψ, $\psi(I_x)$ is written $g(I_x \cap H_{\psi(I_x)})$. Here assume $x \notin H_{\psi(I_x)}$, then we obtain $x \in I_x - I_x \cap H_{\psi(I_x)}$ and $g(I_x - I_x \cap H_{\psi(I_x)}) = 0$. On the contrary, assume $x \in H_{\psi(I_x)}$, then there exists an open neighbourhood of x such that $g(O_x) = \psi(O_x) \geq \psi(I_x)$. From this, it is derived that there exists O_x^* such that $g(O_x^*) = \psi(O_x^*)$ and holds $D_g\psi(x) = \sup_{I_x} \psi(I_x) \leq g(O_x^*)$. From Corollary 6.3, for an arbitrary I_x' such as $I_x' \subset O_x^*$, it follows that $g(I_x') = \psi(I_x')$ and $\sup_{I_x'} \psi(I_x') = g(O_x^*)$. Hence holds $D_g\psi(x) = g(O_x^*)$. In an analogous way as in Proof of Corollary 6.2, it is easily proved that $D_g\psi(x) = g(O_x^*)$, (g-a.e.).

(ii) Assume that $g(I_x) > \psi(I_x)$ for an arbitrary closed neighbourhood of x. Now, if we assume that there holds $g(O_x) = \psi(O_x)$ for a certain open neighbourhood of x, then, by using Corollary 6.3, it is derived that

$g(I'_x) = \psi(I'_x)$ for $I'_x \subset O_x$. This contradicts the above assumption. Therefore, there must hold always $g(O_x) > \psi(O_x)$.

Next if $x \in H_{\psi(I_x)}$ is assumed, then it is derived that there exists O_x such that $g(O_x) = \psi(O_x) \geq \psi(I_x)$. Therefore $x \notin H_{\psi(I_x)}$ must hold. We obtain that $\inf_{I_x} \psi(I_x) = \inf_{I_x} g(I_x \cap H_{\psi(I_x)}) = 0$, since $\bigcap I_x = \{x\}$ and $\{x\} \cap H_{\psi(\{x\})} = \phi$. Hence holds $D_g\psi(x) = 0$. By summing up the above results, Proof is now complete. (Q.E.D.)

[Theorem 6.2] Let $\psi(\cdot)$ have Properties (1)-(3). Then there exists a function $h(x)$ such that

$$\psi(A) = \int_A h(x) \circ g. \tag{6.13}$$

(Proof) It suffices to prove that there exists at least one $h(x)$. Here, let us prove that it is possible to adopt $D_g\psi(x)$ as $h(x)$.

(i) Assume $\psi(A) > 0$.

Define

$$F_{\psi(A)} = \{x \mid D_g\psi(x) \geq \psi(A)\}.$$

From Corollary 6.4, it is shown that there exists an open neighbourhood of x almost everywhere with respect to g for an arbitrary $x \in F_{\psi(A)}$ such that $g(O^*_x) = \psi(O^*_x) \geq \psi(A)$. We obtain $x \in H_{\psi(A)}$ since $O^*_x \subset H_{\psi(A)}$. Therefore if we subtract a null-set from $F_{\psi(A)}$, then the remainder is included in $H_{\psi(A)}$. This implies $g(F_{\psi(A)} - H_{\psi(A)}) = 0$. More precisely, the set $F_{\psi(A)} - H_{\psi(A)}$ can be covered by countable numbers of null-sets. On the other hand, if $x \in H_{\psi(A)}$ is assumed, then it follows that $D_g\psi(x) \geq \psi(A)$, hence, $H_{\psi(A)} \subset F_{\psi(A)}$. Thus we obtain $\psi(A) = g(A \cap F_{\psi(A)})$ since $g(A \cap H_{\psi(A)})$

$= g(A \cap F_{\psi(A)})$. From Theorem 3.7, it follows that $\psi(A) = \int_A D_g\psi(x) \circ g(\cdot)$.

(ii) Assume $\psi(A) = 0$. Then $g(A \cap H_{\psi(A)}) = 0$.

For $x \in A - A \cap H_{\psi(A)}$, there exists no open neighbourhood of x such as $g(O_x) = \psi(O_x)$, hence, exists neither a closed neighbourhood of x such as $g(I_x) = \psi(I_x)$. This implies $D_g\psi(x) = 0$ for $x \in A - A \cap H_{\psi(A)}$.

Therefore,

$$\int_A D_g\psi(x) \circ g = \int_{A \cap H_{\psi(A)}} D_g\psi(x) \circ g$$

$$\leq g(A \cap H_{\psi(A)}),$$

from which it follows that

$$\int_A D_g\psi(x) \circ g = 0.$$

Thus we obtain also in this case $\psi(A) = \int_A D_g\psi(x) \circ g$. Proof is now complete. (Q.E.D.)

As is shown above, Theorem 6.2 is proved by using Corollaries 6.3 and 6.4. It seems difficult to examine that given a certain set function, it has Property (3). However, for instance, it is easily proved that Φ_G in Corollary 6.1 has this property. In the next section, the existence theorem of conditional fuzzy measures is proved by using the result of Theorem 6.2.

6.2 Definition of conditional fuzzy measures

Let ϕ be a mapping from X onto a set Y, where X is a fuzzy measure space $(X, \mathcal{B}, g)$. Then, in the same manner as in a probability space, Y becomes a fuzzy measure space by the mapping ϕ. This construction can be performed as follows.

For an arbitrary $F \subset Y$, define that F is an open set of Y if and only if $\phi^{-1}(F)$ is that of X. (In this section, we use the notation $\phi^{-1}(F) = \{x | \phi(x) \in F, F \subset Y\}$, $\phi(E) = \{y | y = \phi(x), x \in E, E \subset X\}$.) Then a topology is induced into Y and a Borel field $\mathcal{B}^{(\phi)}$ corresponding to $\mathcal{B}$ is also induced. Further a fuzzy measure $g^{(\phi)}$ is defined on $\mathcal{B}^{(\phi)}$.

[Definition 6.3] $(Y, \mathcal{B}^{(\phi)}, g^{(\phi)})$ is defined such that

1) $F \in \mathcal{B}^{(\phi)}$ if and only if $\phi^{-1}(F) \in \mathcal{B}$,

2) $g^{(\phi)}(F) = g(\phi^{-1}(F))$. (6.14)

It is valid that $\mathcal{B}^{(\phi)}$ is a Borel field of Y.

[Proposition 6.1] $g^{(\phi)}$ is a fuzzy measure of F-measurable space $(Y, \mathcal{B}^{(\phi)})$.

Thus the fuzzy measure space $(Y, \mathcal{B}^{(\phi)}, g^{(\phi)})$ is induced from $(X, \mathcal{B}, g)$ by the mapping $\phi : X \to Y$. We can interpret $(Y, \mathcal{B}^{(\phi)}, g^{(\phi)}$ in the following way. If Y is related to X by a mapping ϕ, then a fuzzy measure of Y by which an individual measures fuzziness in Y should be also related to that of X. This relation is shown in the definitions of $\mathcal{B}^{(\phi)}$ and $g^{(\phi)}$.

[Definition 6.4] If

$$\oint_A h \circ g = \oint_A h' \circ g \text{ for any } A \in \mathcal{B},$$

then h and h' are said to be equivalent with respect to g. This is written as g-eqv.

[Definition 6.5] Let $E \in \mathcal{B}$ and $F \in \mathcal{B}^{(\phi)}$. By $\rho(E|\phi = y)$, denote the representative of all functions equivalent to $h(y)$ with respect to g such that

$$g(E \cap \phi^{-1}(F)) = \int_F h(y) \circ g^{(\phi)}(\cdot). \tag{6.15}$$

Here $\rho(\cdot|\phi = y)$ is called a conditional fuzzy measure under the condition of $\phi = y$.

In accordance with this definition, $\rho(\cdot|\phi = y)$ is determined uniquely in the sense of $g^{(\phi)}$-eqv., if $h(y)$ exists. The next theorem is the existence theorem of conditional fuzzy measures.

[Theorem 6.3] There exists $\rho(\cdot|\phi = y)$ uniquely in the sense of $g^{(\phi)}$-eqv.

(Proof) Define a set function $\Phi_E(\cdot)$ on $\mathcal{B}^{(\phi)}$ such that

$$\Phi_E(F) = g(E \cap \phi^{-1}(F)),\ F \in \mathcal{B}^{(\phi)}.$$

If it is shown that $\Phi_E(\cdot)$ has the properties of $\psi(\cdot)$, then the existence of $h(y)$ can be proved by Theorem 6.2. Now it is valid that $\Phi_E(\cdot)$ satisfies Property (1).

(i) Let E be an open set of X. Choose F so that $\phi^{-1}(F) = E$. Then F is an open set of Y and $g^{(\phi)}(F) = \Phi_E(F) = g(E)$. Since $\Phi_E(Y) = g(E)$, there exists an open set O' such that $g^{(\phi)}(O') = \Phi_E(O') = \Phi_E(Y)$. Therefore Property (2) is satisfied. Next define for an arbitrary $A \in \mathcal{B}^{(\phi)}$

$$H = \bigcup \{O' | g^{(\phi)}(O') = \Phi_E(O') \geq \Phi_E(A)\}.$$

If $g^{(\phi)}(O') = \Phi_E(O')$, then it is derived from the definitions of $g^{(\phi)}$ and Φ_E that $g(\phi^{-1}(O')) = g(E \cap \phi^{-1}(O'))$, from which it follows that $g(\phi^{-1}(O') - E \cap \phi^{-1}(O')) = 0$. From the assumption of the space X

(Assumption 1 in Section 6.1), H can be expressed by countable numbers of O'_n such that $g^{(\phi)}(O'_n) = \Phi_E(O'_n) \geq \Phi_E(A)$. From this it follows that

$H = \bigcup_{n=1}^{\infty} O'_n$ and $\phi^{-1}(H) = \bigcup_{n=1}^{\infty} \phi^{-1}(O'_n)$.

Now since

$$\phi^{-1}(O'_n) = (E \cap \phi^{-1}(O'_n)) \cup (\phi^{-1}(O'_n) - E \cap \phi^{-1}(O'_n)),$$

it is derived that

$$g(\bigcup_{n=1}^{\infty} (\phi^{-1}(O'_n) - E \cap \phi^{-1}(O'_n))) = 0.$$

Hence

$$g(\phi^{-1}(H)) = g(\bigcup_{n=1}^{\infty} (E \cap \phi^{-1}(O'_n))).$$

We obtain $E \subset \phi^{-1}(H)$ since $O' \subset H$ for O' such that $\phi^{-1}(O') = E$. Therefore there holds $g(\phi^{-1}(H)) = g(E)$. From this, it follows that

$$g(\phi^{-1}(A) \cap \phi^{-1}(H)) = g(\phi^{-1}(A) \cap E),$$

which equals $\Phi_E(A)$. On the other hand, it follows from the definition of $g^{(\phi)}$ that

$$g^{(\phi)}(H \cap A) = g(\phi^{-1}(H \cap A)) = g(\phi^{-1}(H) \cap \phi^{-1}(A)).$$

This implies that $\Phi_E(\cdot)$ satisfies Property (3).

(ii) Let $E \in \mathcal{B}$ be an arbitrary set. Choosing an open set $O \supset E$ and taking into account that $g(E)$ can be arbitrarily approximated by $g(O)$, Proof is obtained in the same manner. (Q.E.D.)

In the specific case in which ϕ is a mapping from X onto X and $\phi(x) = x$, we obtain that

$$\rho(E|\phi = x) = g(E) \wedge \chi_E(x), \text{ (g-eqv.)}. \tag{6.16}$$

Clearly from the definition, we have

$$g(E \cap \phi^{-1}(F)) = \int_F \rho(E|\phi = y) \circ g^{(\phi)}(\cdot). \tag{6.17}$$

In particular let F = Y. Then we obtain, since $\phi^{-1}(Y) = X$,

$$g(E) = \int_Y \rho(E|\phi = y) \circ g^{(\phi)}(\cdot). \tag{6.18}$$

Now, conditional fuzzy measures have the following properties.

1) For a fixed $E \in \mathcal{B}$, $\rho(E|\phi = y)$ is, as a function of y, a $\mathcal{B}^{(\phi)}$-measurable function.

2) For a fixed y, $\rho(\cdot|\phi = y)$ is a fuzzy measure of $(X, \mathcal{B})$ in the sense of $g^{(\phi)}$-a.e.

It is clear by the definition that the property 1) holds. We omit to prove 2) since it can be easily proved.

[Theorem 6.4]

$$\int_A h(x) \circ g = \int_Y [\int_A h(x) \circ \rho(\cdot|\phi = y) \circ g^{(\phi)} \tag{6.19}$$

(Proof) Let $E_\alpha = \{x|h \geq \alpha\}$. Then the left-hand side of Eq. (6.19) is written $\sup_\alpha [\alpha \wedge g(E_\alpha)]$. Now, since $g(E_\alpha) = \int_Y \rho(E_\alpha|\phi = y) \circ g^{(\phi)}$,

$$\int_A h(x) \circ g = \int_Y \sup_\alpha [\alpha \wedge \rho(E_\alpha|\phi = y)] \circ g^{(\phi)}.$$

On the other hand, we obtain

$$\int_A h(x) \circ \rho(\cdot|\phi = y) = \sup_{\alpha} [\alpha \wedge \rho(E_\alpha|\phi = y)].$$

The proof is now complete. (Q.E.D.)

Theorem 6.4 shows the formula in the change of variable of integration. As is clear in the process of the proof, Eq. (6.19) is derived by using only Eq. (6.18). If ϕ is one to one, then a conditional fuzzy measure $\rho^{-1}(\cdot|\phi^{-1} = x)$ under the condition of $\phi^{-1} = x$ can be considered by $\phi^{-1} : Y \to X$. We have the next theorem.

[Theorem 6.5] Let $E \in \mathcal{B}$ and $F \in \mathcal{B}^{(\phi)}$. Then

$$\int_F \rho(E|\phi = y) \circ g^{(\phi)} = \int_E \rho^{-1}(F|\phi^{-1} = x) \circ g. \qquad (6.20)$$

(Proof) Since ϕ is one to one, $\phi^{-1}(\phi(E) \cap F) = E \cap \phi^{-1}(F)$. Consequently, we obtain that $g(E \cap \phi^{-1}(F)) = g^{(\phi)}(\phi(E) \cap F)$. Eq. (6.20) follows from the definition of conditional fuzzy measures. (Q.E.D.)

The definition of conditional fuzzy measures and their properties have been now verified. In words, $\rho(\cdot|\phi = y)$ relates $(Y, \mathcal{B}^{(\phi)}, g^{(\phi)})$ to $(X, \mathcal{B}, g)$ instead of ϕ. Even if two fuzzy measure spaces, $(X, \mathcal{B}_X, g_X)$ and $(Y, \mathcal{B}_Y, g_Y)$, are related to each other, ϕ may be not explicit in general. We then write $\rho(\cdot|\phi = y)$ as $\rho_X(\cdot|y)$ which is called a conditional fuzzy measure from Y to X. In this case, there must hold similarly

$$g_X(\cdot) = \int_Y \rho_X(\cdot|y) \circ g_Y. \qquad (6.21)$$

If $g_Y(\cdot)$ and $\rho_X(\cdot|y)$ are given at first, then $g_X(\cdot)$ is obtained by Eq. (6.21). In accordance with Eq. (6.20), we have

$$\int_F \rho_X(E|y) \circ g_Y = \int_E \rho_Y(F|x) \circ g_X. \qquad (6.22)$$

[Theorem 6.6] Define $H_\alpha = \{(x,y) | h(x,y) \geq \alpha\}$. If $H_\alpha = \bigcup_{i=1}^{\infty} (E_\alpha^i \times F_\alpha^i)$ where $\{E_\alpha^i\}$ is monotonously increasing (decreasing) and $\{F_\alpha^i\}$ monotonously decreasing (increasing), then

$$\int_X [\int_Y h(x,y) \circ \rho_Y(\cdot|x)] \circ g_X$$

$$= \int_Y [\int_X h(x,y) \circ \rho_X(\cdot|y)] \circ g_Y. \tag{6.23}$$

(Proof) From the assumption, it follows that

$$\chi_{H_\alpha}(x,y) = \bigvee_{i=1}^{\infty} [\chi_{E_\alpha^i}(x) \wedge \chi_{F_\alpha^i}(y)].$$

Taking into account that h(x,y) can be written as

$$h(x,y) = \sup_\alpha [\alpha \wedge \chi_{H_\alpha}(x,y)],$$

we obtain

$$\int_X [\int_Y h(x,y) \circ \rho_Y(\cdot|x)] \circ g_X$$

$$= \int_X \sup_\alpha [\alpha \wedge \int_Y \chi_{H_\alpha}(x,y) \circ \rho_Y(\cdot|x)] \circ g_X$$

using Theorem 3.7 and its Lemma,

$$= \sup_\alpha [\alpha \wedge \int_X [\int_Y \chi_{H_\alpha}(x,y) \circ \rho_Y(\cdot|x)] \circ g_X]$$

$$= \sup_\alpha [\alpha \wedge \int_X \bigvee_{i=1}^{\infty} [\chi_{E_\alpha^i}(x) \wedge \rho_Y(F_\alpha^i|x)] \circ g_X]$$

$$= \sup_\alpha [\alpha \wedge \bigvee_{i=1}^{\infty} \int_X [\chi_{E_\alpha^i}(x) \wedge \rho_Y(F_\alpha^i|x)] \circ g_X]$$

from Theorem 3.3,

$$= \sup_\alpha [\alpha \wedge \bigvee_{i=1}^{\infty} \int_{E_\alpha^i} \rho_Y(F_\alpha^i| y) \circ g_X]$$

using Eq. (6.22),

$$= \sup_{\alpha} [\alpha \wedge \bigvee_{i=1}^{\infty} \int_{F_\alpha^i} \rho_X(E_\alpha^i|y) \circ g_Y].$$

(Q.E.D.)

[Lemma] If $\{h_i(x)\}$ is monotonously increasing (decreasing) with i and $\{h_i(y)\}$ monotonously decreasing (increasing), then

$$\int_X [\int_Y \bigvee_{i=1}^{\infty} (h_i(x) \wedge h_i(y)) \circ \rho_Y(\cdot|x) \circ g_X$$

$$= \int_Y [\int_X \bigvee_{i=1}^{\infty} (h_i(x) \wedge h_i(y)) \circ \rho_X(\cdot|y)] \circ g_Y. \qquad (6.24)$$

Now, as is shown in the definition, conditional fuzzy measures correspond to conditional probabilities. Here $\rho^{-1}(\cdot|\phi^{-1} = x)$ corresponds to a posteriori probability and Theorem 6.5 stands for Bayes' theorem by which a posteriori probabilities are found.

Chapter 7

APPLICATIONS OF CONDITIONAL FUZZY MEASURES

7.1 Introduction

In this chapter, we consider two applications of conditional fuzzy measures defined in the previous chapter. One is concerned with fuzzy decision-making problems and the other the representation of fuzzy systems.

Since conditional fuzzy measures, as has been mentioned in Section 6.2, are similar to conditional probabilities, a broad field of their applications could be expected.

We have dealt with static evaluation problems in Chapter 5. By introducing the concept of conditional fuzzy measures, we can consider dynamic evaluation problems in which human subjectivity varies as time goes; it is natural to suppose that subjectivity varies due to outer or inner causes.

In general, the concept "conditional" is very important in systems theory as well as in probability theory. For example, systems are often described such that the output of a system is y under the condition of the input = x at time t. That is, by using the concept "conditional", we can deal with causality or dynamics.

7.2 Fuzzy decision-making problems

In this section, we consider fuzzy decision-making problems corresponding to stochastic ones. The problem of buying residences is discussed as an example in which conditional fuzzy measures are concretely applied.

First let us define fuzzy decision-making problems.

[Definition 7.1] A fuzzy decision-making problem, FDP, is the following 6 tuples.

$$FDP = < \Theta, X, A, g_\Theta(\cdot), \sigma_X(\cdot|\theta), \ell >$$

where

Θ	:	a finite set of states of nature
X	:	a finite set of samples
A	:	a finite set of actions
$g_\Theta(\cdot)$	:	a fuzzy measure for states of nature
$\sigma_X(\cdot\|\theta)$	:	a conditional fuzzy measure for samples under the condition that a state of nature is θ
ℓ	:	the membership function of a fuzzy relation in $\Theta \times A$ which implies fuzzy loss when an action a is adopted for θ.

In the above definition, a state of nature does not always imply that of real nature. Since there is not an adequate word, we use the same terminology as in stochastic decision-making problems. All sets are restricted to finite ones for simplifying our discussions. A fuzzy measure $g_\Theta(\cdot)$ is one for the grade of fuzziness of states of nature and $\sigma_X(\cdot|\theta)$ for that of samples.

Let us first consider the case in which a sample x is not available, i.e., no-data problem. Given g_Θ and ℓ, the fuzzy expectation of loss with respect to an action a is given as follows:

$$<\ell>_a = \oint_\Theta \ell(\theta,a) \circ g_\Theta. \tag{7.1}$$

When $\langle \ell \rangle_a$ is adopted as criterion of an action a, it is usual to choose the optimal a^o which minimizes $\langle \ell \rangle_a$. However, since loss is assumed to be fuzzy in FDP, it is doubtful that the optimal action in the sense of the above description is really excellent in comparing with another a. Therefore let us choose a fuzzy set of actions called a likely-good action for the decision-maker; we stop picking up one optimal a^o.

[Definition 7.2] A fuzzy action $\tilde{a}$ is a fuzzy subset of A associated with the membership function $h_{\tilde{a}} : A \to [0,1]$.

Loss with respect to a fuzzy action is obtained by extending ℓ in the following way.

[Definition 7.3]

$$\ell(\theta,\tilde{a}) = 1 - \max_{a \in A} [h_{\tilde{a}}(a) \wedge (1 - \ell(\theta,a))] \tag{7.2}$$

By Eq. (7.2), the loss function is extended to a set function. This extension is, of course, not unique. If a is a non-fuzzy set E, then

$$\ell(\theta,E) = \min_{a \in E} \ell(\theta,a) \text{ where } E \subset A. \tag{7.3}$$

In particular, we obtain

$$\ell(\theta,\{a\}) = \ell(\theta,a). \tag{7.4}$$

[Definition 7.4] The likely-good action $\tilde{a}^o$ is a fuzzy action defined as follows:

$$h_{\tilde{a}^o}(a) = \int (1 - \ell(\theta,a)) \circ g_{\Theta}. \tag{7.5}$$

Here the term "likely-good" implies that if $h_{\tilde{a}^o}(a)$ is large for a, then the action a becomes good since the fuzzy expectation of loss becomes small. Actually, the decision-maker can choose, according to his preference, one action out of those for which $h_{\tilde{a}^o}$ is large.

The next Theorem holds for the fuzzy expectation of loss with respect to likely-good action $\tilde{a}^o$. In the sense of this Theorem, $\tilde{a}^o$ is actually a good action.

[Theorem 7.1] Let $\ell^c(\theta,a) = 1 - \ell(\theta,a)$, then

$$\int_\Theta \ell^c(\theta,\tilde{a}^o) \circ g_\Theta = \max_a \int_\Theta \ell^c(\theta,a) \circ g_\Theta. \tag{7.6}$$

(Proof) Taking into account that

$$h_{\tilde{a}^o}(a) = \int_\Theta \ell^c(\theta,a) \circ g,$$

it is clear from Theorem 3.5 that

$$\int_\Theta \ell^c(\theta,a) \circ g_\Theta = \int_\Theta [h_{\tilde{a}^o}(a) \wedge \ell^c(\theta,a)] \circ g_\Theta.$$

From Theorem 3.8 follows

$$\max_a \int_\Theta \ell^c(\theta,a) \circ g_\Theta \leq \int_\Theta \max_a [h_{\tilde{a}^o}(a) \wedge \ell^c(\theta,a)] \circ g_\Theta.$$

Since

$$\max_a [h_{\tilde{a}^o}(a) \wedge \ell^c(\theta,a)] \leq \max_a h_{\tilde{a}^o}(a),$$

we obtain using Proposition 3.4

$$\int_\Theta \max_a [h_{\tilde{a}^o}(a) \wedge \ell^c(\theta,a)] \circ g_\Theta \leq \int_\Theta \max_a h_{\tilde{a}^o}(a) \circ g_\Theta = \max_a h_{\tilde{a}^o}(a).$$

Proof is now complete. (Q.E.D.)

Next let us consider the case in which X and a conditional fuzzy measure from Θ to X are given. In this case, our problem is to find a strategy $t : X \to A$. For a strategy t, the fuzzy expectation of loss is written as follows:

$$\langle \ell \rangle_t = \int_\Theta \left[\int_X \ell(\theta, t(x)) \circ \sigma_X(\cdot|\theta) \right] \circ g_\Theta. \tag{7.7}$$

We shall again find a likely-good strategy instead of the optimal strategy minimizing $\langle \ell \rangle_t$.

[Definition 7.5] A fuzzy strategy is a fuzzy relation in X x A associated with the membership function $h_{\tilde{t}} : X \times A \to [0,1]$.

[Definition 7.6] A fuzzy action $\tilde{t}(x)$ based on a fuzzy strategy $\tilde{t}$ is defined by

$$h_{\tilde{t}(x)}(a) = h_{\tilde{t}}(x,a). \tag{7.8}$$

According to Definition 7.3, loss with respect to $\tilde{t}(x)$ can be written

$$\ell(\theta, \tilde{t}(x)) = 1 - \max_{a \in A} [h_{\tilde{t}}(x,a) \wedge (1 - \ell(\theta,a))]. \tag{7.9}$$

and a fuzzy expectation of loss can be expressed as follows:

$$\langle \ell \rangle_{\tilde{t}} = \int_\Theta \left[\int_X \ell(\theta, \tilde{t}(x)) \circ \sigma_X(\cdot|\theta) \right] \circ g_\Theta. \tag{7.10}$$

[Definition 7.7] A fuzzy strategy $\tilde{t}^o$ in Eq. (7.11) is called a likely-good strategy.

$$h_{\tilde{t}^o}(x,a) = \int_\Theta (1 - \ell(\theta,a)) \circ \sigma_\Theta(\cdot|x) \tag{7.11}$$

In Eq. (7.11), $\sigma_\Theta(\cdot|x)$ is a posteriori fuzzy measure defined in Section 6.2 The right-hand side of Eq. (7.12) is used for finding the Bayes' solution in a stochastic decision theory, if a posteriori fuzzy measure is replaced by a posteriori probability and the fuzzy integral by Lebesgue one.

From Theorem 7.1, it follows that

$$\oint_X [\oint_\Theta (1 - \ell(\theta,\tilde{t}^o(x)) \circ \sigma_\Theta(\cdot|x)] \circ g_X$$
$$= \oint_X [\max_a \oint_\Theta (1 - \ell(\theta,a)) \circ \sigma_\Theta(\cdot|x)] \circ g_X. \quad (7.12)$$

As has been mentioned in Section 6.2, $\sigma_\theta(\cdot|x)$ can be obtained from Eqs. (6.21) and (6.22). Therefore, from Eq. (7.11) a solution of FDP can be obtained.

Now let us solve, as an example of FDP, the problem of buying residences. The reason of taking this problem is because it is considered that this problem could not be treated in the stochastic decision theory.

We consider the problem of what action one should take when he wishes to buy a residence by reading several advertisements. The actions which he can take are such as a_1 : to buy in cash, a_2 : to buy by an instalment plan, a_3 : to think for a while, a_4 : not to buy. It seems reasonable that one takes a_2 rather than a_1 when he wants to buy a residence profitably. From the dealer's point of view, one offering cash is a better customer than one proposing an instalment plan. Therefore it is assumed that if there is someone taking a_1, he has the right to buy the residence. By the assumption stated above, the actions a_1-a_4 are put under the same condition.

We consider the advertisements for such four objects as x_1 : in Yokohama, x_2 : in Tokyo, x_3 : in Saitama, x_4 : in Yokosuka. These advertisement which appeared in a newspaper have been fuzzificated by modifying the parts written clearly. Those describe the time which one takes from the residence to go to his office located at the center of Tokyo, the surrounding conditions of the house, and its facilities and furniture. Further each advertisement is associated with a picture from which one can estimate the area.

In this application, two subjects have been selected and the likely-good strategies for them have been found according to Eq. (7.11). Now, the procedure for finding a likely-good strategy is shown below.

Step 0 Translation of the problem into FDP

First we must translate the given problem into FDP. Clearly we have

$$A = \{a_1, a_2, a_3, a_4\},$$

$$X = \{x_1, x_2, x_3, x_4\}.$$

As for Θ, we can consider as follows. A decison-maker will evaluate the fineness of a residence by his standard of evaluation in order that given $x \in X$, he chooses $a \in A$.

When one evaluates the fineness of a residence, he will decompose it into its elements and evaluate it by superposing the partial evaluations. The evaluation problem like this has been already discussed in Chapter 5.

Under the above consideration, let Θ be a set of elements constructing residences.

$$\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4, \theta_5\}$$

where θ_1 : facilities and furniture, θ_2 : area, θ_3 : natural environment, θ_4 : time from residence to office, θ_5 : circumstance of life (shops, hospitals, etc.).

Next we define $g_\Theta(\cdot)$ and $\sigma_X(\cdot|\theta)$.

$g_\Theta(\cdot)$: the grade of importance for elements of Θ

$\sigma_X(\cdot|\theta)$: the grade of charmness for advertised residences when one evaluates them from the point of view of one element θ

For instance, $\sigma_X(\{x_1\}|\theta_1)$ implies to what extent the residence in Yokohama attracts a subject when he evaluates it with respect to its facilities. By $\sigma_X(\{x_1,x_2\}|\theta)$, we mean the grade of charmness of the imaginary residence constructed by superposing x_1 and x_2 in the same space. (Superposition is possible in an imaginary world.) One might imagine a new residence in which only excellent parts are selected from two residences. Of course another one might imagine the contrary. Anyway we can consider, for any subset E of X, the grade of charmness of E. Here monotonicity must be assumed.* Step 0 is now complete by defining loss.

$\ell(\theta,a)$: loss expected when one decides to take an action a only from the point of view of θ

As can be seen in the above discussions, it is assumed that

(1) a decision-maker has a subjective measure of importance for the elements of general residences,

(2) he has also, in the sense of "conditional", a subjective measure of charmness for actual residences, and

(3) he takes an action according to the evaluation performed by using these fuzzy measures.

These assumptions can be regarded as an expression of a human decision-making process.

Step 1 Indentification of g

In the applications in Chapter 5, a fuzzy measure of a subject has been identified indirectly. Here let us identify it directly.

For all subsets of Θ, a subject is asked the grade of importance, which is denoted by $d_\Theta(\cdot)$. We adopt again $g_\lambda(\cdot)$ as the model of $d_\Theta(\cdot)$.

$$g_\Theta(E) = \frac{1}{\lambda}\left[\prod_{\theta_i \in E}(1 + \lambda g^i) - 1\right] \text{ where } -1 < \lambda < \infty \tag{7.13}$$

Define

$$J = \sqrt{\sum_{E \subset \Theta} (d_\Theta(E) - g_\Theta(E))^2}\,. \tag{7.14}$$

Here, g^i $(1 \leq i \leq 5)$ and λ are determined so as to minimize J.

The results are shown in Tables 7.1 and 7.2. As is shown in these Tables, λ is positive for subject A and negative for subject B, which implies that their fuzzy measures are very different. The difference between d_Θ and g_Θ is shown in Figs. 7.1 and 7.2 in which a point shows $(d_\Theta(E), g_\Theta(E))$. In particular, Fig. 7.2 shows a very good agreement between d_Θ (obtained experimentally) and g_Θ(a model).

Step 2 Identification of $\sigma_X(\cdot|\theta)$

A subject is also asked the grade of charmness for residences with respect to each θ. Here $\sigma_X(\cdot|\theta)$ is constructed in the following manner.

Let

$$\eta^{ij} = \sigma_X(\{x_i\}|\theta_j), \tag{7.15}$$

$$\frac{1}{\mu_j}\left[\prod_{i=1}^{4}(1 + \mu_j\eta^{ij}) - 1\right] = 1 \text{ where } -1 < \mu_j < \infty. \tag{7.16}$$

For simplicity, only $\sigma_X(\{x_i\}|\theta_j)$ has been asked in this case. Now, μ_j is calculated so that Eq. (7.16) is satisfied.

Next assuming that μ_j is independent of j, η^{ij} of a subject is corrected so that μ_j is identical with respect to j. That is, letting μ be a mean of μ_j $(1 \le j \le 5)$, find α such that

$$\frac{1}{\mu}\left[\prod_{i=1}^{4}(1 + \mu\alpha\eta^{ij}) - 1\right] = 1 \tag{7.17}$$

and define

$$\sigma^{ij} = \alpha\eta^{ij}. \tag{7.18}$$

Finally we obtain

$$\sigma_X(F|\theta_j) = \frac{1}{\mu}\left[\prod_{x_i \in F}(1 + \mu\sigma^{ij}) - 1\right]. \tag{7.19}$$

This correction is not always necessary since our method does not require the identical μ. The results are shown in Tables 7.3 and 7.4.

Step 3 Determination of $\ell(\theta,a)$

A subject is asked $\ell(\theta,a)$ for all $\theta \in \Theta$ and $a \in A$. The results is shown in Table 7.5.

Step 4 Calculation of $\sigma_\Theta(\cdot|x)$

We rewrite Eqs. (6.21) and (6.22).

$$g_X(F) = \int_\Theta \sigma_X(F|\theta) \circ g_\Theta \tag{7.20}$$

$$\int_F \sigma_\Theta(E|x) \circ g_X = \int_E \sigma_X(F|\theta) \circ g_\Theta \tag{7.21}$$

Let $F = \{x_j\}$ in Eq. (7.21). Then we obtain

$$\sigma_\Theta(E|x_j) \wedge g_X(\{x_j\}) = \int_E \sigma_X(\{x_j\}|\theta) \circ g_\Theta. \tag{7.22}$$

From Eq. (7.22) it follows that

(1) if $g_X(\{x_j\}) > \int_E \sigma_X(\{x_j\}|\theta) \circ g_\theta$, then

$$\sigma_\Theta(E|x_j) = \int_E \sigma_X(\{x_j\}|\theta) \circ g_\Theta, \tag{7.23}$$

(2) if $g_X(\{x_j\}) = \int_E \sigma_X(\{x_j\}|\theta) \circ g_\Theta$, then

$$\sigma_\Theta(E|x_j) \geq \int_E \sigma_X(\{x_j\}|\theta) \circ g_\Theta. \tag{7.24}$$

Note that there holds $g_X(\{x_j\}) \geq \int_E \sigma_X(\{x_j\}|\theta) \circ g_\Theta$. In the case (2), $\sigma_\Theta(E|x_j)$ is not uniquely determined. We can, for instance, let $\sigma_\Theta(E|x_j)$ be unity.

Thus we can calculate $\sigma_\Theta(E|x_j)$ by obtaining $g_X(\cdot)$ from Eq. (7.20) since $g_\Theta(\cdot)$ has been obtained in Step 1 and $\sigma_X(\cdot|\theta)$ in Step 2.

Table 7.6 shows the fuzzy densities : $\sigma_\Theta(\{\theta_i\}|x)$, $1 \leq i \leq 5$. Note that we cannot calculate $\sigma_\Theta(E|x)$ for an arbitrary E by using the fuzzy densities, since Eq. (7.19) does not hold in this case.

Step 5 Calculation of the likely-good strategy

The likely-good strategy, $\tilde{t}^o$, for subject A is shown in Table 7.7. As can be seen in Table 7.7, it seems better to buy the residence x_1, but not to buy the residences x_3 and x_4. As for x_2, it is a question which is better for subject A to buy it or not since the values of $h_{\tilde{t}^o}(x_2,a)$ are almost equal for all a.

After the experiments, subject A was interviewed on this results. He answered that $\tilde{t}^o$ seemed to be really a likely-good strategy.

Now, we can regard FDP as a model of a man's decision-making for fuzzy objects. In particular, the concept of conditional fuzzy measures plays an important role in the model. We could say that this concept can represent the structure of subjective evaluation in decision-making.

The differences between stochastic decision theory and fuzzy decision theory may be characterized from two points of view. One is that between probability measures and fuzzy ones, and the other between a Bayes' solution and a fuzzy strategy.

The concept of subjective probabilities is necessary when stochastic desision theory is used as a theory of a man's decision-making under uncertainties. However subjective probabilities have been seldom obtained experimentally for a specific individual. For instance, as we have seen in this application, it cannot be expected that the subjective measure of an individual satisfies additivity, unless we force him to answer so that his subjective measure becomes additive.

As can be seen Tables 7.1 and 7.2, $\lambda \neq 0$ implies that $d_\Theta(\cdot)$ does not satisfy additivity. Figs. 7.3 and 7.4 show the results when $d_\Theta(\cdot)$ is fitted by the probability measure P_Θ. The differences are evident by comparing those with Figs. 7.1 and 7.2.

Here we could conclude that a fuzzy measure can represent subjectivity better than a probability measure since a severe condition, i.e., additivity need not be always assumed.

Further, a fuzzy strategy does not require a person to take only one action, but a Bayes' solution does. Therefore one can reach more freely at the final decision. We could say that a fuzzy strategy is one matching for the fuzziness of the problem. Note that in the stochastic decision theory, a mixed strategy is available when a priori probability is unknown. In this case, a fuzzy strategy can be comparable with a mixed strategy at the same level.

Finally it is pointed out that we can deal with more variety of problems by using the fuzzy decision theory than the stochastic one, because we are not constrained by probability theory. It is very difficult to interpret the example treated in this section in terms of probabilities.

7.3 Representation of fuzzy systems

In general, systems which have been dealt with in systems theory can be divided into three classes: non-deterministic systems, stochastic ones, and deterministic ones. In the above, non-deterministic systems have the largest amount of uncertainty. It is well-known that we have the probabilistic representation of systems when uncertainty is caused by random phenomena. Deterministic systems have, of course, no uncertainty.

The fuzzy systems discussed in this section are those in which uncertainty is caused by fuzzy phenomena. That is, our fuzzy systems are placed between non-deterministic ones and deterministic ones, at the same level with stochastic systems.

Fuzzy systems studied so far [10] are those derived by replacing merely a deterministic input-output relation or a transition of states by a fuzzy relation. These fuzzy systems are, therefore, algebraic ones; a fuzzy automaton is a typical example. If we look at this derivation, however, from the point of view of describing actual systems, it is a question whether algebraic systems can have "reality" or not.

We shall attempt, in this section, to derive the representation of fuzzy systems, assuming that "fuzzy disturbance" inputs are put into deterministic systems. This method is the same as used for the derivation of stochastic systems when probabilistic disturbance inputs are put into the systems. There is another way to derive our fuzzy systems. That is the way to obtain their representation by putting some constraints on non-deterministic systems. Here we show simply this since it will be useful for making clear the difference between stochastic systems and fuzzy ones.

Let us consider a non-deterministic automaton. Let K be a finite set of states and Σ a finite set of inputs. State transitions are described by δ such that $\delta : K \times \Sigma \to 2^K$. Define $K' = \delta(s,u)$ where $s \in K$ and $u \in \Sigma$. The state s changes to a certain state s' in K' ($\subset K$).

The statement "the next state s' exists in K'" shows uncertainty since s' is not specified. This uncertainty may be said to be that of $s' \in K'$, which can be decreased as follows. Let us interpret the uncertainty of $s' \in K'$ as the probability of $s' \in K'$ by introducing a probability measure.

Let

$$0 \le p(s,u,s') \le 1 \text{ for all } s' \in K, \tag{7.25}$$

$$\sum_{s' \in K} p(s,u,s') = 1. \tag{7.26}$$

Then the probability of $s' \in K'$ is expressed by

$$P(s,u,K') = \sum_{s' \in K'} p(s,u,s'). \tag{7.27}$$

On the other hand, if a fuzzy measure $g(s,u,\cdot)$ is introduced, then the grade of $s' \in K'$ is expressed similarly by $g(s,u,K')$. It is clear that the uncertainty is decreased by the both methods. Because $P(s,u,K'')$ and $g(s,u,K'')$ can be defined for any $K'' \subset K'$.

Further we can assume arbitrary properties as for the uncertainty of $s' \in K'$. These properties are nothing but the properties of a certain set function defined on 2^K. However, if we derive fuzzy systems, so to speak, by an artificial method, we must assume a priori without any proof, the property of state transitions corresponding to Chapman-Kolmogoroff's equation as shown later. Thus this method becomes unrealistic.

Now, we return to the main argument. When we consider on fuzzy systems, it is necessary to find where fuzziness is in the system and where it is not. Throughout this paper we have discussed up to now fuzziness caused by subjectivity. Therefore it is particularly important for us to make clear in what manner subjectivity is involved in an actual and objective system.

The matter described above have been seldom discussed in dealing with fuzzy systems. Hereafter we say that a system is fuzzy, when objectivity is associated with subjectivity in the system. It does not imply that the system has merely uncertainty. Subjectivity is a kind of uncertainty, but this uncertainty would not disappear even if we study more on it. That is, it seems that as for subjectivity, there exists something like the "principle of uncertainty" as in quantum mechanics.

Now let U and Ω be sets of inputs of a system, X a set of state variable, and Y a set of outputs. Assume that the elements of U can be controlled and those of Ω are disturbance inputs which are uncontrollable. For taking notice of $\omega \in \Omega$, we regard $u \in U$ as a parameter. Let $\phi_{u_{tt'}}$ be a state transition function and ψ_{u_t} an output function.

Then a dynamical system can be expressed as follows:

$$x_{t'} = \phi_{u_{tt'}} (x_t, \omega_{tt'}) \text{ for } t' \geq t, \tag{7.28}$$

$$y_t = \psi_{u_t} (x_t, \omega_t) \tag{7.29}$$

where it is assumed that

$$\phi_{u_{tt''}}(x_t,\omega_{tt''}) = \phi_{u_{t't''}}(\phi_{u_{tt'}}(x_t,\omega_{tt'}),\omega_{t't''})$$

$$\text{for } t'' \geq t' \geq t. \tag{7.30}$$

Now we make the necessary assumptions:

(1) Ω is a fuzzy measure space $(\Omega, \mathcal{B}, g)$.

(2) The fuzzy measure g is independent of time, i.e., stationary.

Let us call ω a fuzzy disturbance input. If Ω is a probability space, then ω becomes a stochastic disturbance input. Our problem is to show in what manner fuzziness in Ω is transmitted to the states and outputs of the system. For simplification of the following discussions, time is regarded discrete and the matters concerned with outputs are omitted.

For discrete times t_1 and t_2 which are denoted simply by 1 and 2, we rewrite Eq. (7.28).

$$x_1 = \phi_{u_1}(x_0,\omega_1) \tag{7.31}$$

$$x_2 = \phi_{u_2}(x_1,\omega_2)$$

$$= \phi_{u_1 u_2}(x_0,\omega_1\omega_2) \tag{7.32}$$

where x_0 is an initial state of the system.

Eq. (7.30) is also rewritten as

$$\phi_{u_1 u_2}(x_0,\omega_1\omega_2) = \phi_{u_2}(\phi_{u_1}(x_0,\omega_1),\ \omega_2). \tag{7.33}$$

Here ϕ_{u_1}, ϕ_{u_2}, and $\phi_{u_1 u_2}$ can be considered as the following mappings.

$$\phi_{u_1} : X_0 \times \Omega_1 \to X_1$$

$$\phi_{u_2} : X_1 \times \Omega_2 \to X_2$$

$$\phi_{u_1 u_2} : X_0 \times \Omega_1 \times \Omega_2 \to X_2$$

From the assumptions, Ω_1 and Ω_2 are spaces with the common fuzzy measure g.

We attempt to induce fuzzy measures into X_1 and X_2 in an analogous way as in Section 6.2. For this purpose, it is necessary to define a fuzzy measure of X_0, i.e., the set of initial states. Assuming that an initial state x_0 is known, a fuzzy measure is naturally defined as has been showed in Example 1 in Section 2.1. Let

$$X_0 = (X, \mathcal{B}^{(\phi_e)}, g^{(\phi_e)})$$

$$\text{where } g^{(\phi_e)}(A) = \chi_A(x_0) \text{ for } A \in \mathcal{B}^{(\phi_e)}.$$

Here, e implies the empty input and ϕ_e is a one to one mapping such that $\phi_e : X \to X$.

What is necessary next is to consider the fuzzy product measure space:

$$X_0 \times \Omega_1 = (X \times \Omega, \mathcal{B}^{(\phi_e)} \times \mathcal{B}, g^{(\phi_e)} \times g).$$

If $g^{(\phi_e)} \times g$ can be defined, a fuzzy measure is induced into X_1. Consequently we have

$$X_1 = (X, \mathcal{B}^{(\phi_{u_1})}, g^{(\phi_{u_1})})$$

Here $\mathcal{B}^{(\phi_e)} \times \mathcal{B}$ can be defined in an ordinary way. However, a fuzzy product measure can not be defined in such manner as enabling us to change the order of fuzzy integrations: it has been mentioned in Section 3.3. Here, we define $g^{(\phi_e)} \times g$ on $\mathcal{B}^{(\phi_e)} \times \mathcal{B}$ as follows:

$$(g^{(\phi_e)} \times g)(F) = \int_{X_0} [\int_{\Omega_1} \chi_F(x_0, \omega_1) \circ g] \circ g^{(\phi_e)} \qquad (7.34)$$

where $F \subset X_0 \times \Omega_1$ and $F \in \mathcal{B}^{(\phi_e)} \times \mathcal{B}$.

Now let us construct the fuzzy measure space $(X_1, \mathcal{B}^{(\phi_{u_1})}, g^{(\phi_{u_1})})$

(1) $E \in \mathcal{B}^{(\phi_{u_1})}$ if and only if $\phi_{u_1}^{-1}(E) \in \mathcal{B}^{(\phi_e)} \times \mathcal{B}$,

(2) $g^{(\phi_{u_1})}(E) = (g^{(\phi_e)} \times g)(F)$

where $F = \{(x_0, \omega_1) | \phi_{u_1}(x_0, \omega_1) \in E\}$.

Define for $E \in \mathcal{B}^{(\phi_{u_1})}$

$$\rho^{(\phi_{u_1})}(E|x_0) = \int_{\Omega_1} \chi_F(x_0, \omega_1) \circ g. \qquad (7.35)$$

Then we obtain

$$g^{(\phi_{u_1})}(E) = \int_{X_0} \rho^{(\phi_{u_1})}(E|x_0) \circ g^{(\phi_e)}. \qquad (7.36)$$

It is clear that $\rho^{(\phi_{u_1})}(\cdot|x_0)$ satisfies the properties of a conditional fuzzy measure. We omit the proof.

In an analogous way, we obtain

$$X_2 = (X, \mathcal{B}^{(\phi_{u_2})}, g^{(\phi_{u_2})}).$$

For $H \in \mathcal{B}^{(\phi_{u_2})}$

$$g^{(\phi_{u_2})}(H) = \int_{X_1} [\int_{\Omega_2} \chi_G(x_1,\omega_2) \circ g] \circ g^{(\phi_{u_1})} \tag{7.37}$$

where

$$G = \{(x_1,\omega_2) | \phi_{u_2}(x_1,\omega_2) \in H\}.$$

Further define

$$\rho^{(\phi_{u_2})}(H|x_1) = \int_{\Omega_2} \chi_G(x_1,\omega_2) \circ g. \tag{7.38}$$

From Eq. (7.37) follows

$$g^{(\phi_{u_2})}(H) = \int_{X_1} \rho^{(\phi_{u_2})}(H|x_1) \circ g^{(\phi_{u_1})}. \tag{7.39}$$

Now, another fuzzy measure is induced into X_2 also by the mapping

$$\phi'_{u_1u_2} : X_0 \times \Omega_1 \times \Omega_2 \rightarrow X_2.$$

Define

$$X'_2 = (X, \mathcal{B}^{(\phi_{u_1u_2})}, g^{(\phi_{u_1u_2})}).$$

For $H \in \mathcal{B}^{(\phi_{u_1u_2})}$, $g^{(\phi_{u_1u_2})}(H)$ can be defined as follows:

$$g^{(\phi_{u_1u_2})}(H) = \oint_{X_0 \times \Omega_1} [\oint_{\Omega_2} \chi_K(x_0,\omega_1,\omega_2) \circ g] \circ g^{(\phi_e)} \times g$$

$$= \oint_{X_0} [\oint_{\Omega_1} [\oint_{\Omega_2} \chi_K(x_0,\omega_1,\omega_2) \circ g] \circ g] \circ g^{(\phi_e)}. \tag{7.40}$$

where

$$K = \{(x_0,\omega_1,\omega_2) | \phi_{u_1u_2}(x_0,\omega_1,\omega_2) \in H\}.$$

Define

$$\rho^{(\phi_{u_1u_2})}(H|x_0) = \oint_{\Omega_1} [\oint_{\Omega_2} \chi_K(x_0,\omega_1,\omega_2) \circ g] \circ g. \tag{7.41}$$

Then we obtain

$$g^{(\phi_{u_1u_2})}(H) = \oint_{X_0} \rho^{(\phi_{u_1u_2})}(H|x_0) \circ g^{(\phi_e)}. \tag{7.42}$$

From the property of the mapping ϕ_u in Eq. (7.33), X_2 and X_2' must be the same measure space. We have the next Theorem.

[Theorem 7.2]

(1) $\mathcal{B}^{(\phi_{u_1u_2})} = \mathcal{B}^{(\phi_{u_2})}$

(2) $$\rho^{(\phi_{u_1u_2})}(H|x_0) = \oint_{X_1} \rho^{(\phi_{u_2})}(H|x_1) \circ \rho^{(\phi_{u_1})}(\cdot|x_0) \tag{7.43}$$

(Proof) (i) From the definition of $\mathcal{B}^{(\phi_{u_1u_2})}$,

we obtain that

$$H \in \mathcal{B}^{(\phi_{u_1u_2})},$$

if and only if

$$\{(x_0,\omega_1,\omega_2)\,|\,\phi_{u_1u_2}(x_0,\omega_1,\omega_2)\in H\}\in \mathcal{B}^{(\phi_e)} \times \mathcal{B} \times \mathcal{B}.$$

On the other hand, let $H' \in \mathcal{B}^{(\phi_{u_2})}$ and define

$$G = \{(x_1,\omega_2)\,|\,\phi_{u_2}(x_1,\omega_2)\in H'\}.$$

Then we have $G \in \mathcal{B}^{(\phi_{u_1})} \times \mathcal{B}$ which implies that

$$\{(x_0,\omega_1,\omega_2)\,|\,(\phi_{u_1}(x_0,\omega_1),\omega_2)\in G\}\in \mathcal{B}^{(\phi_e)} \times \mathcal{B} \times \mathcal{B}.$$

Now, from the definition of G and the property of ϕ_u, it follows that

$$\{(x_0,\omega_1,\omega_2)\,|\,(\phi_{u_1}(x_0,\omega_1),\omega_2)\in G\}$$

$$= \{(x_0,\omega_1,\omega_2)\,|\,\phi_{u_2}(\phi_{u_1}(x_0,\omega_1),\omega_2)\in H'\}$$

$$= \{(x_0,\omega_1,\omega_2)\,|\,\phi_{u_1u_2}(x_0,\omega_1,\omega_2)\in H'\}.$$

Consequently if $H \in \mathcal{B}^{(\phi_{u_1u_2})}$, then $H \in \mathcal{B}^{(\phi_{u_2})}$. The reverse is also true.

(ii) Define $E_\alpha = \{x_1\,|\,\rho^{(\phi_{u_2})}(H|x_1) \geq \alpha\}$ and $F_\alpha = \{(x_0,\omega_1)\,|\,\phi_{u_1}\in E_\alpha\}$.

Then the right-hand side of Eq. (7.43) can be written

$$\sup_\alpha\,[\alpha \wedge \rho^{(\phi_{u_1})}(E_\alpha|x_0)]$$

$$= \sup_\alpha\,[\alpha \wedge \int_{\Omega_1} \chi_{F_\alpha}(x_0,\omega_1) \circ g]$$

$$= \int_{\Omega_1} \sup_\alpha\,[\alpha \wedge \chi_{F_\alpha}(x_0,\omega_1)] \circ g.$$

From the definition of E_α, F_α can be rewritten as

$$F_\alpha = \{(x_0,\omega_1) | \rho^{(\phi_{u_2})} (H|\phi_{u_1}(x_0,\omega_1)) \geq \alpha \}.$$

From Proposition 3.6 it follows that

$$\sup_\alpha [\alpha \wedge \chi_{F_\alpha}(x_0,\omega_1)] = \rho^{(\phi_{u_2})} (H|\phi_{u_1}(x_0,\omega_1)).$$

Therefore the right-hand side of Eq. (7.43) is written as

$$\int_{\Omega_1} \rho^{(\phi_{u_2})} (H|\phi_{u_1}(x_0,\omega_1)) \circ g$$

$$= \int_{\Omega_1} [\int_{\Omega_2} \chi_G(\phi_{u_1}(x_0,\omega_1),\omega_2) \circ g] \circ g$$

where

$$G = \{(\phi_{u_1}(x_0,\omega_1),\omega_2) | \phi_{u_2}(\phi_{u_1}(x_0,\omega_1),\omega_2) \in H\}$$

$$= \{(\phi_{u_1}(x_0,\omega_1),\omega_2) | \phi_{u_1u_2}(x_0,\omega_1,\omega_2) \in H\}.$$

Now we have

$$\rho^{(\phi_{u_1u_2})} (H|x_0) = \int_{\Omega_1} [\int_{\Omega_2} \chi_K(x_0,\omega_1,\omega_2) \circ g] \circ g$$

where

$$K = \{(x_0,\omega_1,\omega_2) | \phi_{u_1u_2}(x_0,\omega_1,\omega_2) \in H\}.$$

There holds clearly

$$\chi_G(\phi_{u_1}(x_0,\omega_1),\omega_2) = \chi_K(x_0,\omega_1,\omega_2).$$

Proof is now complete. (Q.E.D.)

[Lemma]

$$g^{(\phi_{u_1u_2})}(\cdot) = g^{(\phi_{u_2})}(\cdot) \tag{7.44}$$

Our purpose has been now accomplished. From the above Lemma follows

$$g^{(\phi_{u_1u_2})}(H) = \int_{X_0} [\int_{X_1} \rho^{(\phi_{u_2})}(H|x_1) \circ \rho^{(\phi_{u_1})}(\cdot|x_0)] \circ g^{(\phi_e)}. \tag{7.45}$$

Eq. (7.43) corresponds to Chapman-Kolmogoroff's equation.

Now, let us use the following notations:

$$\sigma_X(\cdot|u_1u_2\ldots\ldots u_n,x_0) = \rho^{(\phi_{u_1u_2\ldots\ldots u_n})}(\cdot|x_0), \tag{7.46}$$

$$g_X^n(\cdot) = g^{(\phi_{u_1u_2\ldots\ldots u_n})}(\cdot). \tag{7.47}$$

By generalizing Eq. (7.45), we obtain that

$$\begin{aligned} g_X^n(H) &= \int_{X_{n-1}} \sigma_X(H|u_n, x_{n-1}) \circ g_X^{n-1}(\cdot) \\ &= \int_{X_0} \sigma_X(H|u_1u_2\ldots\ldots u_n, x_0) \circ g_X^0(\cdot), \end{aligned} \tag{7.48}$$

where

$$\sigma_X(H|u_1u_2\ldots\ldots u_n, x_0) = \int_{X_k} \sigma_X(H|u_{k+1}\ldots\ldots u_n, x_k) \circ \sigma_X(\cdot|u_1u_2\ldots\ldots u_k, x_0). \tag{7.49}$$

In general, we can assume that an initial state x_0 is unknown, though it was assumed at first to be known. In this case, $g_X^0(\cdot)$ can be considered as a fuzzy measure for fuzziness of initial states.

It corresponds to an initial distribution of a stochastic system.

Further, the conditional fuzzy measure $\sigma_Y(K|u_n, x_n)$ can be derived as for outputs of the system. The fuzzy measure with respect to the output y_n can be obtained as

$$g_Y^n(K) = \int_{X_n} \sigma_Y(K|u_n, x_n) \circ g_X^n(\cdot). \qquad (7.50)$$

Now, $g_X^n(H)$ expresses the grade of fuzziness of $x_n \in H$. By using the extension of fuzzy measures defined in Section 3.4, the grade of fuzziness of the statement "x_n at time t_n is within a fuzzy set A" can be expressed as

$$g_X^n(A) = \int_{X_n} h_A(x_n) \circ g_X^n(\cdot). \qquad (7.51)$$

On the other hand, $\sigma_X(\cdot|u_1u_2.....u_n, x_0)$ can be interpreted as a fuzzy measure for the fuzzy transition of states as a result that the input sequence $u_1u_2.....u_n$ is put into the system under the condition that the initial state is x_0.

We have now obtained the representation of a fuzzy system under the condition that fuzzy disturbance inputs are put into a deterministic system. Note: if a time sequence $t_1t_2......t_n$ is substituted for $u_1u_2......u_n$ in Eq. (7.48), then we obtain a system equation corresponding to a Markoff chain.

Table 7.1 A Priori Fuzzy Measure (experimentally obtained) of Subject A

Subset* of Θ	g_Θ	Subset of Θ	g_Θ
1	0.2	123	0.6
2	0.15	124	0.7
3	0.1	125	0.7
4	0.1	134	0.6
5	0.05	135	0.6
		145	0.6
12	0.5	234	0.7
13	0.4	235	0.6
14	0.3	245	0.6
15	0.15	345	0.5
23	0.3		
24	0.5	1234	0.95
25	0.3	1235	0.9
34	0.2	1245	0.85
35	0.15	1345	0.8
45	0.2	2345	0.9

*) $123 = \{\theta_1, \theta_2, \theta_3\}$

Table 7.2 Identified Fuzzy Densities of Subjects A and B

	A (λ = 0.25)	B (λ = -0.51)
θ_1	0.170	0.143
θ_2	0.257	0.400
θ_3	0.216	0.261
θ_4	0.212	0.350
θ_5	0.061	0.131

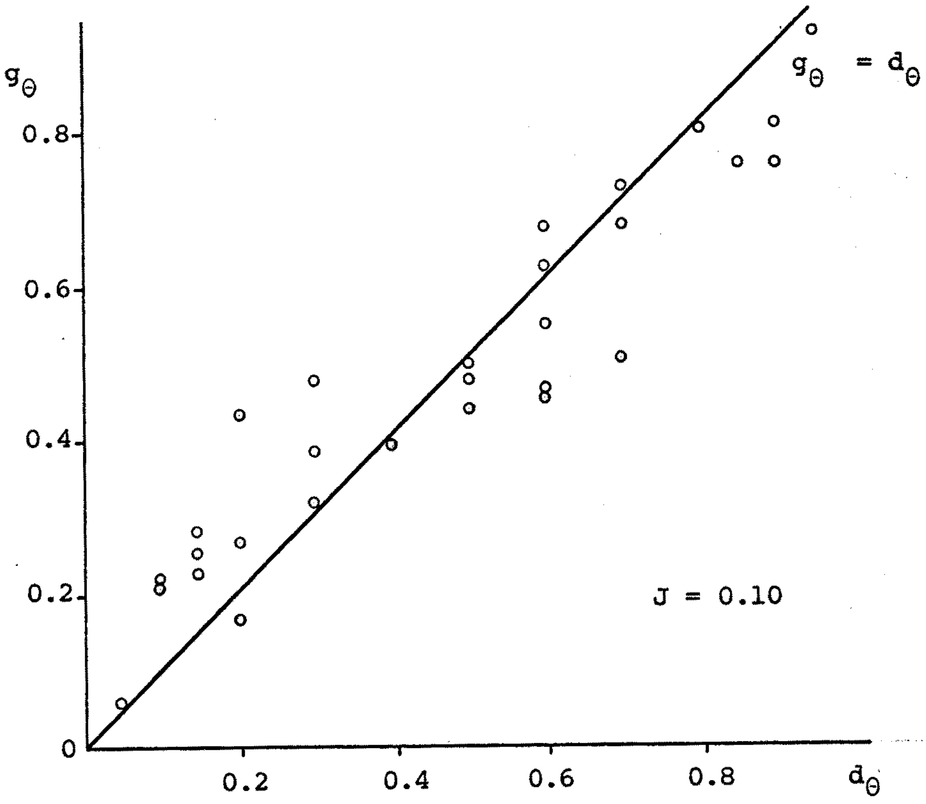


Fig. 7.1 Comparison of g_Θ and d_Θ - Subject A

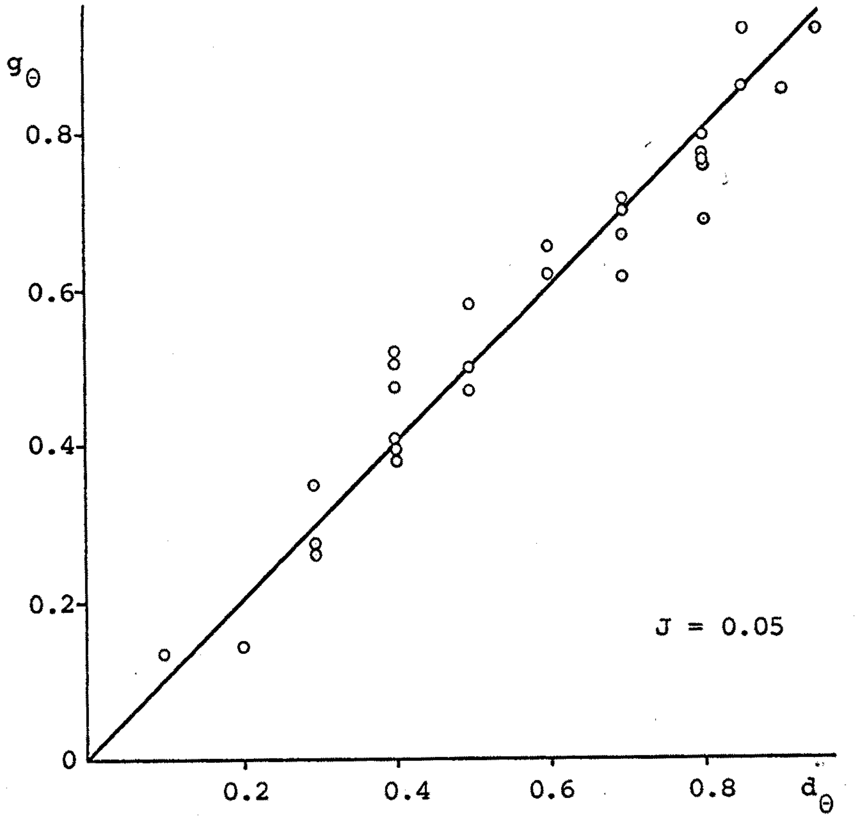


Fig. 7.2 Comparison of g_Θ and d_Θ - Subject B

Table 7.3 Conditional Fuzzy Density η^{ij} (experimentally obtained) of Subject A

	x_1	x_2	x_3	x_4
θ_1	0.4	0.3	0.2	0.1
θ_2	0.5	0.4	0.2	0.1
θ_3	0.4	0.3	0.2	0.2
θ_4	0.2	0.1	0.4	0.5
θ_5	0.5	0.2	0.2	0.4

Table 7.4 Conditional Fuzzy Density σ^{ij} (corrected) of Subject A

	x_1	x_2	x_3	x_4
θ_1	0.46	0.35	0.23	0.12
θ_2	0.48	0.38	0.19	0.10
θ_3	0.42	0.32	0.21	0.21
θ_4	0.19	0.10	0.38	0.48
θ_5	0.44	0.18	0.18	0.36

Table 7.5 Fuzzy Loss $\ell(\theta,a)$ of Subject A

	a_1	a_2	a_3	a_4
θ_1	0.7	0.3	0.2	0.5
θ_2	0.3	0.5	0.6	0.2
θ_3	0.2	0.3	0.5	0.7
θ_4	0.7	0.5	0.4	0.0
θ_5	0.6	0.4	0.3	0.2

Table 7.6 Fuzzy Densities of A Posteriori Fuzzy Measure $\sigma_\Theta(\cdot|x)$ of Subject A

	θ_1	θ_2	θ_3	θ_4	θ_5
x_1	0.17	0.26	0.22	0.19	0.06
x_2	0.17	0.26	0.22	0.10	0.06
x_3	0.17	0.19	0.21	0.21	0.06
x_4	0.12	0.10	0.21	0.21	0.06

Table 7.7 Likely-good Strategy $h_{\tilde{t}}(x,a)$ of Subject A

	a_1	a_2	a_3	a_4
x_1	0.70	0.68	0.68	0.60
x_2	0.70	0.70	0.68	0.68
x_3	0.30	0.70	0.79	0.79
x_4	0.40	0.60	0.70	0.79

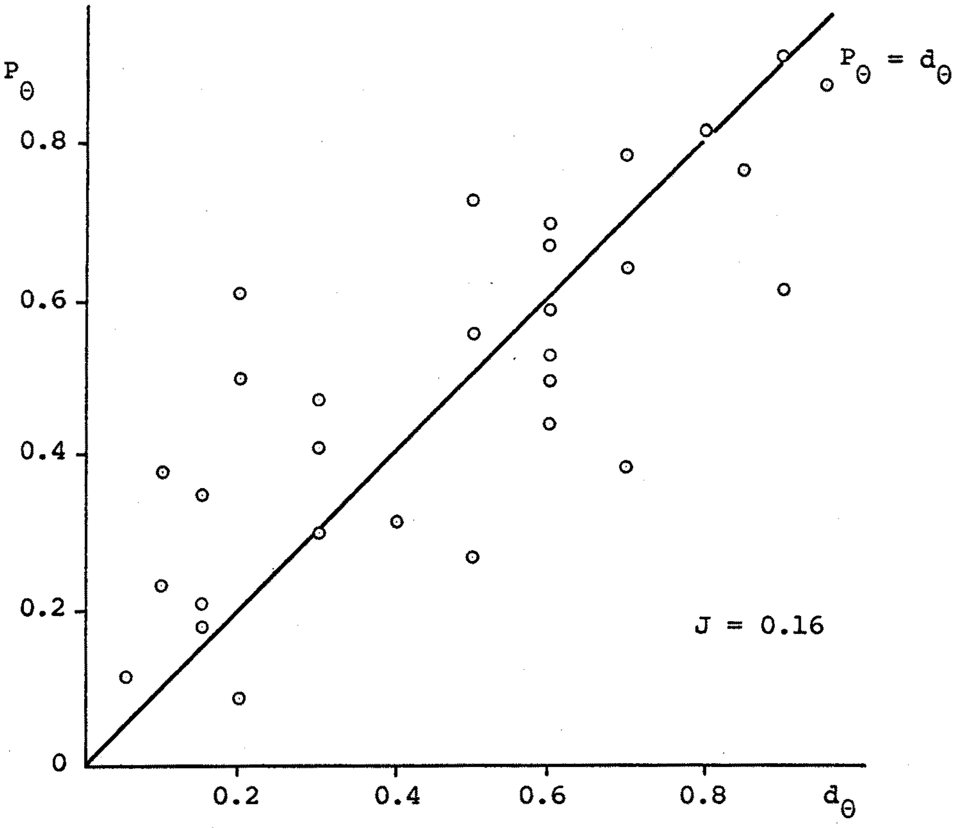


Fig. 7.3 Comparison of P_Θ and d_Θ - Subject A

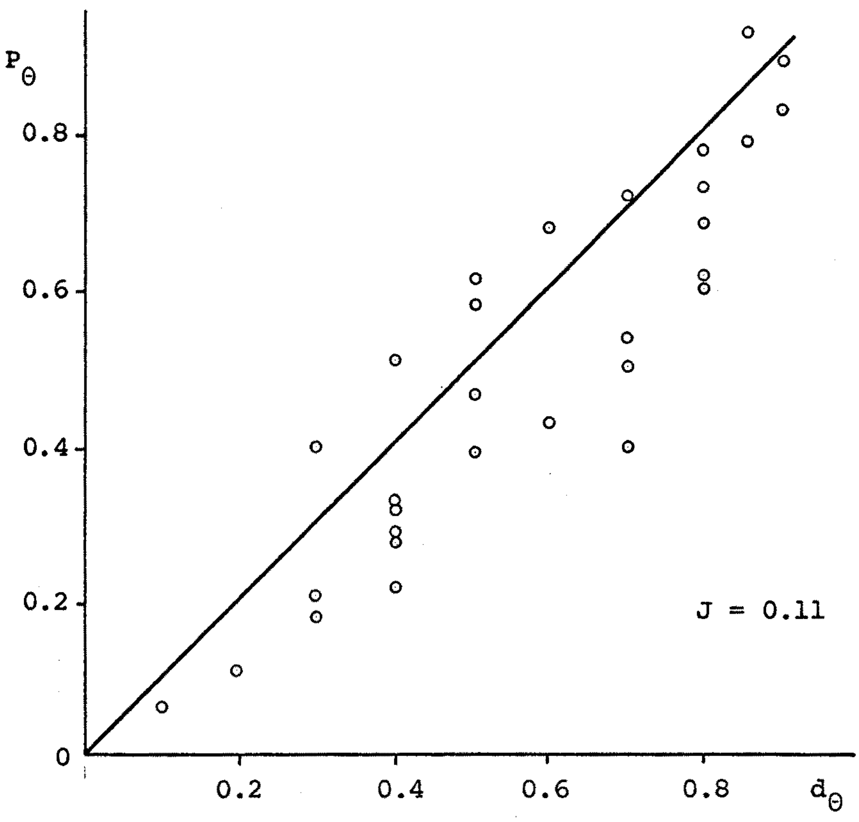


Fig. 7.4 Comparison of P_Θ and d_Θ - Subject B

Chapter 8

CONCLUSIONS

The research reported here has dealt with human subjectivity from a stand point of engineering and presented a method of measuring fuzziness which is caused by subjectivity. It develops an analytical method for approaching fuzziness by introducing a new' concept on "measures" and "integrals", while fuzziness has been mainly approached by an algebraic method, i.e., fuzzy sets theory.

The ordinary concepts of measures and integrals are based on "additivity". However, this research starts from a question: "Is additivity really necessary to deal with fuzziness?" From this point of view, the concept of fuzzy measures and fuzzy integrals based on monotonicity are proposed.

Uncertainties are classified into fuzziness and randomness, and fuzzy measures are interpreted as subjective measures of individuals expressing "grade of fuzziness". Three statements are assumed a priori on grade of fuzziness. The concept of grade of fuzziness is the most fundamental theme in this research as that of grade of membership in fuzzy sets theory.

On the other hand, fuzzy integrals are taken instead of linear models as new models which express the subjective evaluation of fuzzy objects. Those are also called fuzzy expectations corresponding to probability expectations. As for fuzzy integrals, it is assumed tacitly that there exists "quantity" having only monotonicty in a mathematical world.

The author has tried, through the whole, not to lose "reality". For this purpose, there are some applications in which actual problems are treated.

An inverse operation of fuzzy integrals defined in Section 6.1 would have its own development. However, only the properties necessary for the definition of conditional fuzzy measures are clarified. The concept of conditional fuzzy measures is a very useful one. It is expected that this concept as well as the representation of fuzzy systems will be applied in a variety of fields.

It is particularly hoped that this research will serve in future for the studies of artificial intelligence.

REFERENCES

[1] L. A. Zadeh : "Fuzzy Sets", Information and Control, Vol. 8, 338/353, 1965.

[2] L. A. Zadeh : "Fuzzy Algorithms", ibid., Vol. 12, 94/102, 1968.

[3] L. A. Zadeh : "Probability Measures of Fuzzy Events", Journal of Mathematical Analysis and Applications, Vol. 23, 421/427, 1968.

[4] E. S. Santos : "General Formulation of Sequential Machines", Information and Control, Vol. 12, 5/10, 1968.

[5] W. G. Fee and K. S. Fu : "A Formulation of Fuzzy Automata and its Application as a Model of Learning Systems", IEEE transactions on SSC, Vol. SSC-5, No. 3, 215/223, 1969.

[6] M. Mizumoto, J. Toyoda, and K. Tanaka : "Some Considerations on Fuzzy Automata", Journal of Computers and Systems Science, Vol. 3, No. 4, 409/422, 1969.

[7] E. S. Santos : "Fuzzy Algorithms", Information and Control, Vol. 17, 236/339, 1970.

[8] L. A. Zadeh : "Quantitative Fuzzy Semantics", Information Sciences, Vol. 3, 159/176, 1971.

[9] L. A. Zadeh : "Similarity Relations and Fuzzy Orderings", ibid., Vol. 3, 177/200, 1971.

[10] S. S. L. Chang and L. A. Zadeh : "On Fuzzy Mapping and Control", IEEE transactions on SMC, Vol. SMC-2, 30/34, 1972.

[11] M. Sugeno : "Fuzzy Measures and Fuzzy Integrals", Transactions of Society of Instrument and Control Engineers (in Japanese), Vol. 8, No. 2, 1972.

[12] M. Sugeno : "Constructing Fuzzy Measures and Grading Similarity of Patterns by Fuzzy Integrals", ibid., Vol. 9, No. 3, 1973.

[13] H. Tanaka, T. Okuda, and K. Asai : "Fuzzy Mathematical Programming", ibid., Vol. 9, No. 5, 1973.

[14] M. Sugeno and T. Terano : "An Approach to the Identification of Human Characteristics by Applying Fuzzy Integrals", Proceedings of the 3rd IFAC Symposium on Identification and System Parameter Estimation, 1973.

[15] L. Fung and K. S. Fu : "The k-th Optimal Policy Algorithm for Decision Making in Fuzzy Environments", ibid.

[16] S. Tamura and K. Tanaka : "Learning of Fuzzy Languages", IEEE transactions on SMC, Vol. SMC-3, 98/102, 1973.

[17] P. Siy : "Fuzzy Logic for Handwritten Character Recognition", Ph. D Thesis, the University of Alkon, 1973.

[18] M. Sugeno, Y. Tsukamoto, and T. Terano : "Subjective Evaluation of Fuzzy Objects", Preprint of IFAC Symposium on Stochastic Control, 1974.

Appendix A

It is necessary that a function f with two variables has the following properties.

(1) $f(x, 0) = x$

(2) $f(x, y) = f(y, x)$

(3) $f(f(x, y), z) = f(x, f(y, z))$

(4) $f(x, y) \geq f(x, 0) \vee f(0, y)$

(5) For $z = f(x, y)$, there exists uniquely f^* such that $x = f^*(z, y)$.

The last property is necessary to define $g(E^c)$. It can be omitted if $g(E^c)$ is not necessary.

Appendix B

Assuming $E_i \cap E_j = \emptyset$ for $i \neq j$, let us consider a simple function such that $h(x) = \sum_{i=1}^{n} \alpha_i \chi_{E_i}(x)$, where $0 \leq \alpha_i \leq 1$ for $1 \leq i \leq n$. Define an operation with respect to a fuzzy measure g as follows:

$$(h, g_\lambda) = \frac{1}{\lambda} [\prod_{i=1}^{n} (1 + \lambda\alpha_i g_\lambda(E_i)) - 1], \; -1 < \lambda < \infty.$$

Then we have $0 \leq (h, g_\lambda) \leq 1$ and further

$$\lim_{\lambda \to 0} (h, g_\lambda) = \sum_{i=1}^{n} \alpha_i g_0(E_i).$$

The right-hand side of the above equation is just the definition of the Lebesgue integral $\int h \, dg$.

If

$$\lambda \geq 0,$$

then

$$(h, g_\lambda) \geq \int h \, dg.$$

If

$$0 > \lambda > -1,$$

then

$$(h, g_\lambda) < \int h \, dg.$$

The functional (h, g_λ) is continuous with respect to λ and converges to a Lebesgue integral as $\lambda \to 0$, while as previously stated, g_λ converges to a probability measure.