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Authors	Yu Kurokawa, Masaharu Taniguchi
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Multi-dimensional pyramidal travelling fronts in the Allen–Cahn equations

Yu Kurokawa and Masaharu Taniguchi

Department of Mathematical and Computing Sciences,
Tokyo Institute of Technology, 2-12-1-W8-38 Ookayama,
Meguro-ku, Tokyo 152-8552, Japan
(kurokaw3@is.titech.ac.jp; masaharu.taniguchi@is.titech.ac.jp)

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We study travelling-front solutions of pyramidal shapes in the Allen–Cahn equation in $\mathbb{R}^N$ with $N \geq 3$. It is well known that two-dimensional V-form travelling fronts and three-dimensional pyramidal travelling fronts exist and are stable. The aim of this paper is to show that for $N \geq 4$ there exist N -dimensional pyramidal travelling fronts. We construct a supersolution and a subsolution, and find a pyramidal travelling-front solution between them. For the construction of a supersolution we use a multi-scale method.

1. Introduction

Multi-dimensional travelling fronts in the Allen–Cahn equation or the Nagumo equation have been studied recently. For example, two-dimensional V-form front solutions have been studied by Ninomiya and Taniguchi [8, 9], Hamel *et al.* [5, 6] and Haragus and Scheel [7]; cylindrically symmetric travelling fronts have been studied by Hamel *et al.* [5, 6] and Chen *et al.* [1]; three-dimensional travelling fronts with pyramidal shapes were studied in [12, 13]. The aim of this paper is to study the existence of travelling fronts with pyramidal shapes in the four-or-higher-dimensional space. A new stationary wave was found by del Pino *et al.* [2, 3] for dimension $N \geq 9$ (see also [11] for the non-existence in lower dimensions). Thus, the spatial dimension is crucial for the existence and non-existence of multi-dimensional travelling fronts. It is an interesting but extremely difficult problem to classify all possible travelling-front solutions in $\mathbb{R}^N$. This problem becomes more interesting and difficult as the spatial dimension N becomes higher. As a first step to approaching this problem, we consider the existence of pyramidal travelling fronts in $\mathbb{R}^N$.

In this paper we consider the parabolic equation of the form

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u + f(u), & \mathbf{x} \in \mathbb{R}^N, \quad t > 0, \\ u|_{t=0} &= u_0, & \mathbf{x} \in \mathbb{R}^N, \end{aligned}$$

where $N \geq 3$ and a given function u_0 is of class $\text{BU}(\mathbb{R}^N)$, and Δ stands for the Laplacian $\sum_{i=1}^N \partial^2 / \partial x_i^2$. Here $\text{BU}(\mathbb{R}^N)$ is the space of bounded uniformly continuous functions with the supremum norm.

In one-dimensional space, let $\Phi(x - kt)$ be a travelling wave that connects two stable equilibrium states ± 1 with speed k . By putting $\mu = x - kt$, Φ satisfies

$$\left. \begin{aligned} -\Phi''(\mu) - k\Phi'(\mu) - f(\Phi(\mu)) &= 0, & -\infty < \mu < \infty, \\ \Phi(-\infty) &= 1, & \Phi(\infty) = -1. \end{aligned} \right\} \quad (1.1)$$

To fix the phase we set $\Phi(0) = 0$.

The following are the assumptions on f throughout this paper:

(A1) f is of class $C^1[-1, 1]$ with $f(1) = 0$, $f(-1) = 0$, $f'(1) < 0$ and $f'(-1) < 0$;

(A2) f satisfies $\int_{-1}^1 f > 0$;

(A3) there exists $\Phi(\mu)$ that satisfies (1.1) for some $k \in \mathbb{R}$.

In order to study the travelling-wave solutions in one direction, we adopt the moving coordinates of speed c towards the x_N -axis without loss of generality. We write $\mathbf{x} = (x_1, \dots, x_N) \in \mathbb{R}^N$ and $\mathbf{x}' = (x_1, \dots, x_{N-1}) \in \mathbb{R}^{N-1}$, and put $s = x_N - ct$ and $u(\mathbf{x}, t) = w(\mathbf{x}', s, t)$. We denote $w(\mathbf{x}', s, t)$ by $w(\mathbf{x}, t)$ for simplicity. Then we have

$$\left. \begin{aligned} \frac{\partial w}{\partial t} - \Delta w - c \frac{\partial w}{\partial x_N} - f(w) &= 0 & \text{in } \mathbb{R}^N, \quad t > 0, \\ w|_{t=0} &= u_0 & \text{in } \mathbb{R}^N. \end{aligned} \right\} \quad (1.2)$$

We denote the solution of this equation by $w(\mathbf{x}, t; u_0)$. If v is a travelling wave with speed c , it must satisfy

$$-\Delta v - c \frac{\partial v}{\partial x_N} - f(v) = 0 \quad \text{in } \mathbb{R}^N. \quad (1.3)$$

To study this equation, we introduce a nonlinear operator

$$\mathfrak{L}[v] := -\Delta v - c \frac{\partial v}{\partial x_N} - f(v) \quad (1.4)$$

for a function $v \in C^2(\mathbb{R}^N)$. We assume $c > k$ throughout this paper, because the curvature effect is expected to accelerate the speed.

Let $n \geq 3$ be a given integer, and

$$m := \frac{\sqrt{c^2 - k^2}}{k} > 0. \quad (1.5)$$

Let $\{\mathbf{A}_j\}_{j=1}^n$ be a set of unit vectors in $\mathbb{R}^{N-1}$ such that $\mathbf{A}_i \neq \mathbf{A}_j$ for $i \neq j$. Then $\mathbf{A}_j = (A_{1,j}, \dots, A_{N-1,j}) \in \mathbb{R}^{N-1}$ satisfies

$$|\mathbf{A}_j|^2 = \sum_{i=1}^{N-1} (A_{i,j})^2 = 1 \quad \text{for } j = 1, \dots, n. \quad (1.6)$$

Now $(-m\mathbf{A}_j, 1) \in \mathbb{R}^N$ is the normal vector of $\{\mathbf{x} \in \mathbb{R}^N \mid x_N = m(\mathbf{A}_j, \mathbf{x}')\}$, where $(\mathbf{A}_j, \mathbf{x}')$ is the inner product of $\mathbf{A}_j$ and $\mathbf{x}'$ given by

$$(\mathbf{A}_j, \mathbf{x}') := \sum_{i=1}^{N-1} A_{i,j} x_i.$$

We set

$$\left. \begin{aligned} h_j(\mathbf{x}') &:= m(\mathbf{A}_j, \mathbf{x}'), \\ h(\mathbf{x}') &:= \max_{1 \leq j \leq n} h_j(\mathbf{x}') = m \max_{1 \leq j \leq n} (\mathbf{A}_j, \mathbf{x}') \end{aligned} \right\} \quad (1.7)$$

in $\mathbf{x}' \in \mathbb{R}^{N-1}$. We call $\{\mathbf{x} = (\mathbf{x}', x_N) \in \mathbb{R}^N \mid x_N = h(\mathbf{x}')\}$ an N -dimensional pyramid in $\mathbb{R}^N$. Setting

$$\Omega_j := \{\mathbf{x}' \in \mathbb{R}^{N-1} \mid h(\mathbf{x}') = h_j(\mathbf{x}')\}$$

for $j = 1, \dots, n$, we have

$$\mathbb{R}^{N-1} = \bigcup_{j=1}^n \Omega_j. \quad (1.8)$$

We denote the boundary of Ω_j by $\partial\Omega_j$. Now we put

$$S_j := \{\mathbf{x} \in \mathbb{R}^N \mid x_N = h_j(\mathbf{x}') \text{ for } \mathbf{x}' \in \Omega_j\}$$

for each j , and call $\bigcup_{j=1}^n S_j \subset \mathbb{R}^N$ the lateral faces of a pyramid. We put

$$\Gamma_j := \{\mathbf{x} \in \mathbb{R}^N \mid x_N = h_j(\mathbf{x}') \text{ for } \mathbf{x}' \in \partial\Omega_j\}$$

for $j = 1, \dots, n$. Then $\bigcup_{j=1}^n \Gamma_j$ represents the set of all edges of a pyramid.

For every $\mathbf{A}_j$ with (1.6), (1.3) has a planar front solution $\Phi(k(x_N - h_j(\mathbf{x}'))/c)$. We define

$$v(\mathbf{x}) := \Phi\left(\frac{k}{c}(x_N - h(\mathbf{x}'))\right) = \max_{1 \leq j \leq n} \Phi\left(\frac{k}{c}(x_N - h_j(\mathbf{x}'))\right). \quad (1.9)$$

Then it is easy to see that v becomes a subsolution to (1.3). We define

$$D(\gamma) := \left\{ \mathbf{x} \in \mathbb{R}^N \mid \text{dist}\left(\mathbf{x}, \bigcup_{j=1}^n \Gamma_j\right) > \gamma \right\} \quad (1.10)$$

for $\gamma > 0$. We will show that the function

$$\bar{v}(\mathbf{x}) := \Phi\left(\frac{\alpha x_N - \varphi(\alpha \mathbf{x}')}{\alpha \sqrt{1 + |\nabla \varphi(\alpha \mathbf{x}')|^2}}\right) + \varepsilon S(\alpha \mathbf{x}')$$

becomes a suitable supersolution with $v < \bar{v}$ to obtain a solution of (1.3) between them by the comparison principle. Then this solution has a contour surface of a pyramidal front shape. See (4.7) below for the precise definition of $\bar{v}$.

The following theorem is the main assertion in this paper.

THEOREM 1.1. *Let $c > k$, and let $v(\mathbf{x})$ be given by (1.9). Under assumptions (A1)–(A3), there exists a solution $V(\mathbf{x})$ to (1.3) such that*

$$\lim_{\gamma \rightarrow \infty} \sup_{\mathbf{x} \in D(\gamma)} |V(\mathbf{x}) - v(\mathbf{x})| = 0, \quad v(\mathbf{x}) < V(\mathbf{x}) < 1 \text{ for all } \mathbf{x} \in \mathbb{R}^N, \quad (1.11)$$

and

$$\frac{\partial V}{\partial x_N}(\mathbf{x}) < 0 \quad \text{for all } \mathbf{x} \in \mathbb{R}^N. \quad (1.12)$$

Most of this paper is devoted to constructing a supersolution $\bar{v}$. We introduce a small positive parameter α and rescale the spatial variable as $\boldsymbol{\xi} = \alpha \mathbf{x}$. In §§ 2 and 3, we take a suitable positive function $S(\boldsymbol{\xi})$ that is used to construct the supersolution $\bar{v}$. This function takes positive values near edges $\bigcup_{j=1}^n \Gamma_j$ and decays to 0 as $\boldsymbol{\xi}$ moves away from the edges. On the other hand, the first term of $\bar{v}$ converges to a planar front travelling with a slower speed $k \in (c, \infty)$ as $\alpha \rightarrow +0$. If we use the moving coordinate of a speed c , we expect that $w(\mathbf{x}, t; \bar{v})$ is monotone decreasing in $t > 0$. This suggests that $\bar{v}$ is a supersolution of (1.3). We carry out this argument and prove the main theorem in § 4.

2. Preliminaries

In this section we make preparations. We state known results for travelling fronts, and prepare to construct supersolutions for pyramidal travelling-front solutions.

LEMMA 2.1 (Fife and McLeod [4]). *Under assumptions (A1) and (A3), $\Phi(\mu)$ as in (1.1) satisfies*

$$\begin{aligned} \Phi'(\mu) &< 0 \quad \text{for all } \mu \in \mathbb{R}, \\ \max\{|\Phi'(\mu)|, |\Phi''(\mu)|, |\mu\Phi'(\mu)|\} &\leq K_0 \exp(-\kappa_0|\mu|). \end{aligned}$$

Here K_0, κ_0 are positive constants.

There exists a positive constant δ_* with $0 < \delta_* < \frac{1}{4}$ and

$$-f'(s) > \beta \quad \text{if } |s+1| < 2\delta_* \text{ or } |s-1| < 2\delta_*,$$

where

$$\beta = \frac{1}{2} \min\{-f'(-1), -f'(1)\} > 0.$$

Since the unit vectors $\{\mathbf{A}_j\}_{j=1}^n$ satisfy $\mathbf{A}_i \neq \mathbf{A}_j$ for $i \neq j$, we have

$$-1 \leq (\mathbf{A}_i, \mathbf{A}_j) < 1 \quad \text{for } i \neq j.$$

For $j, i, i' \in \{1, \dots, n\}$ with $j \neq i$, $j \neq i'$ and $i \neq i'$, let $\theta_{i,i'}(j)$ be the angle between $\mathbf{A}_j - \mathbf{A}_i$ and $\mathbf{A}_j - \mathbf{A}_{i'}$. Then we have $0 < \theta_{i,i'}(j) < \pi$ and

$$(\mathbf{A}_j - \mathbf{A}_i, \mathbf{A}_j - \mathbf{A}_{i'}) = |\mathbf{A}_j - \mathbf{A}_i| |\mathbf{A}_j - \mathbf{A}_{i'}| \cos \theta_{i,i'}(j).$$

We define

$$\begin{aligned} \theta_{\min} &:= \min\{\theta_{i,i'}(j) \mid j \neq i, j \neq i', i \neq i'\}, \\ \theta_{\max} &:= \max\{\theta_{i,i'}(j) \mid j \neq i, j \neq i', i \neq i'\} \end{aligned}$$

and have $0 < \theta_{\min} \leq \theta_{\max} < \pi$.

Let $\tilde{\rho}(r) \in C^\infty[0, \infty)$ be a function with the following properties:

$$\begin{aligned} \tilde{\rho}(r) &> 0, \quad \tilde{\rho}_r(r) \leq 0 \quad \text{for } r \geq 0, \\ \tilde{\rho}(r) &= \begin{cases} 1 & \text{if } r > 0 \text{ is small enough,} \\ e^{-r} & \text{if } r > 0 \text{ is large enough, say } r > R_0, \end{cases} \\ \int_{\mathbb{R}^{N-1}} \tilde{\rho}(|\mathbf{x}'|) d\mathbf{x}' &= 1. \end{aligned}$$

We assume $R_0 > 1$ without loss of generality. We have

$$\int_{\mathbb{R}^{N-1}} \tilde{\rho}(|\mathbf{x}'|) d\mathbf{x}' = \frac{(N-1)\pi^{(N-1)/2}}{\Gamma((N+1)/2)} \int_0^\infty r^{N-2} \tilde{\rho}(r) dr,$$

where Γ is the Gamma function.

We put $\rho(\mathbf{x}') := \tilde{\rho}(|\mathbf{x}'|)$. Then $\rho : \mathbb{R}^{N-1} \rightarrow \mathbb{R}$ belongs to $C^\infty(\mathbb{R}^{N-1})$ and satisfies

$$\int_{\mathbb{R}^{N-1}} \rho(\mathbf{x}') d\mathbf{x}' = 1, \quad (\rho * h_j)(\mathbf{x}') = h_j(\mathbf{x}')$$

for all $\mathbf{x}' \in \mathbb{R}^{N-1}$ and $j = 1, 2, \dots, n$. Here $\rho * h_j$ implies the convolution of ρ and h_j given by

$$\rho * h_j(\mathbf{x}') := \int_{\mathbb{R}^{N-1}} \rho(\mathbf{y}') h_j(\mathbf{x}' - \mathbf{y}') d\mathbf{y}'.$$

For all non-negative integers $j_1, j_2, \dots, j_{N-1}$ with $0 \leq \sum_{p=1}^{N-1} j_p \leq 3$, we have

$$|D_1^{j_1} D_2^{j_2} \cdots D_{N-1}^{j_{N-1}} \rho(\mathbf{x}')| \leq M_1 \rho(\mathbf{x}') \quad \text{for all } \mathbf{x}' \in \mathbb{R}^{N-1}, \quad (2.1)$$

where $M_1 > 0$ is a constant.

We put $\varphi := \rho * h$, that is,

$$\varphi(\mathbf{x}') := \int_{\mathbb{R}^{N-1}} \rho(\mathbf{x}' - \mathbf{y}') h(\mathbf{y}') d\mathbf{y}' = \int_{\mathbb{R}^{N-1}} \rho(\mathbf{y}') h(\mathbf{x}' - \mathbf{y}') d\mathbf{y}' \quad (2.2)$$

for $\mathbf{x}' \in \mathbb{R}^{N-1}$. We call $x_N = \varphi(\mathbf{x}')$ a mollified pyramid for a pyramid $x_N = h(\mathbf{x}')$. We put

$$S(\mathbf{x}') := \frac{c}{\sqrt{1 + |\nabla \varphi(\mathbf{x}')|^2}} - k, \quad (2.3)$$

where $\nabla \varphi(\mathbf{x}') = (\partial \varphi / \partial x_1, \dots, \partial \varphi / \partial x_{N-1})$. We denote $(\partial / \partial x_p)^i$ by D_p^i . We have the following lemma.

LEMMA 2.2. *Let φ and S be as in (2.2) and (2.3), respectively. For any fixed $(i_1, \dots, i_{N-1}) \neq (0, \dots, 0)$ with $i_p \geq 0$ ($p = 1, \dots, N-1$), one has*

$$\sup_{\mathbf{x}' \in \mathbb{R}^{N-1}} |D_1^{i_1} D_2^{i_2} \cdots D_{N-1}^{i_{N-1}} \varphi(\mathbf{x}')| < \infty.$$

One has

$$h(\mathbf{x}') < \varphi(\mathbf{x}') \leq h(\mathbf{x}') + m \int_{\mathbb{R}^{N-1}} |\mathbf{y}'| \rho(\mathbf{y}') d\mathbf{y}',$$

$$|\nabla \varphi(\mathbf{x}')| < m, \quad 0 < S(\mathbf{x}') \leq c - k$$

for all $\mathbf{x}' \in \mathbb{R}^{N-1}$.

Proof. We have

$$\int_{\mathbb{R}^{N-1}} |\mathbf{x}'| \rho(\mathbf{x}') d\mathbf{x}' = \frac{(N-1)\pi^{(N-1)/2}}{\Gamma((N+1)/2)} \int_0^\infty r^{N-1} \tilde{\rho}(r) dr.$$

Without loss of generality we assume $i_1 \geq 1$ and have

$$\begin{aligned} D_1^{i_1-1} D_2^{i_2} \cdots D_{N-1}^{i_{N-1}} \varphi(\mathbf{x}') &= \int_{\mathbb{R}^{N-1}} D_1^{i_1-1} D_2^{i_2} \cdots D_{N-1}^{i_{N-1}} \rho(\mathbf{x}' - \mathbf{y}') h(\mathbf{y}') d\mathbf{y}' \\ &= \int_{\mathbb{R}^{N-1}} D_1^{i_1-1} D_2^{i_2} \cdots D_{N-1}^{i_{N-1}} \rho(\mathbf{y}') h(\mathbf{x}' - \mathbf{y}') d\mathbf{y}'. \end{aligned}$$

Then we get

$$D_1^{i_1} D_2^{i_2} \cdots D_{N-1}^{i_{N-1}} \varphi(\mathbf{x}') = \int_{\mathbb{R}^{N-1}} D_1^{i_1-1} D_2^{i_2} \cdots D_{N-1}^{i_{N-1}} \rho(\mathbf{y}') D_1 h(\mathbf{x}' - \mathbf{y}') d\mathbf{y}',$$

and thus

$$|D_1^{i_1} D_2^{i_2} \cdots D_{N-1}^{i_{N-1}} \varphi(\mathbf{x}')| \leq m \int_{\mathbb{R}^{N-1}} |D_1^{i_1-1} D_2^{i_2} \cdots D_{N-1}^{i_{N-1}} \rho(\mathbf{y}')| d\mathbf{y}' < \infty.$$

Using $\rho > 0$, $h_j(\mathbf{x}') \leq h(\mathbf{x}')$, $\rho * h_j(\mathbf{x}') = h_j(\mathbf{x}')$ and $h_j(\mathbf{x}') \neq h(\mathbf{x}')$, we have

$$h_j(\mathbf{x}') = \rho * h_j(\mathbf{x}') < \rho * h(\mathbf{x}') = \varphi(\mathbf{x}').$$

Thus, we get $h(\mathbf{x}') = \max_{1 \leq j \leq n} h_j(\mathbf{x}') < \varphi(\mathbf{x}')$ for all $\mathbf{x}' \in \mathbb{R}^{N-1}$. Since we have

$$|h_j(\mathbf{x}' - \mathbf{y}') - h_j(\mathbf{x}')| \leq m|\mathbf{y}'|$$

for all $j = 1, 2, \dots, n$, we get

$$|h(\mathbf{x}' - \mathbf{y}') - h(\mathbf{x}')| \leq m|\mathbf{y}'|.$$

Using this inequality and

$$\int_{\mathbb{R}^{N-1}} \rho(\mathbf{y}') d\mathbf{y}' = 1,$$

we obtain

$$\begin{aligned} \varphi(\mathbf{x}') - h(\mathbf{x}') &= \int_{\mathbb{R}^{N-1}} h(\mathbf{x}' - \mathbf{y}') \rho(\mathbf{y}') d\mathbf{y}' - h(\mathbf{x}') \int_{\mathbb{R}^{N-1}} \rho(\mathbf{y}') d\mathbf{y}' \\ &\leq \int_{\mathbb{R}^{N-1}} |h(\mathbf{x}' - \mathbf{y}') - h(\mathbf{x}')| \rho(\mathbf{y}') d\mathbf{y}' \\ &\leq m \int_{\mathbb{R}^{N-1}} |\mathbf{y}'| \rho(\mathbf{y}') d\mathbf{y}'. \end{aligned}$$

Next we have

$$(\nabla \varphi)(\mathbf{x}') = \int_{\mathbb{R}^{N-1}} \rho(\mathbf{y}') (\nabla h)(\mathbf{x}' - \mathbf{y}') d\mathbf{y}' = m \sum_{j=1}^n \int_{\Omega_j} \rho(\mathbf{x}' - \mathbf{y}') d\mathbf{y}' \mathbf{A}_j.$$

Here we used $\nabla h(\mathbf{x}') = m\mathbf{A}_j$ if $\mathbf{x}' \in \Omega_j$ for $j = 1, 2, \dots, n$. Thus, we have an inequality

$$|(\nabla \varphi)(\mathbf{x}')| < m \sum_{j=1}^n \int_{\Omega_j} \rho(\mathbf{x}' - \mathbf{y}') d\mathbf{y}' = m.$$

Since $S < c$ is valid, we obtain $0 < S \leq c - k$. This completes the proof. $\square$

The following proposition plays a key role in this paper.

PROPOSITION 2.3. *One has*

$$0 < \inf_{\mathbf{x}' \in \mathbb{R}^{N-1}} \frac{\varphi(\mathbf{x}') - h(\mathbf{x}')}{S(\mathbf{x}')} \leq \sup_{\mathbf{x}' \in \mathbb{R}^{N-1}} \frac{\varphi(\mathbf{x}') - h(\mathbf{x}')}{S(\mathbf{x}')} < \infty.$$

For every integer $j_p \geq 0$ for $p = 1, \dots, N-1$ with $2 \leq \sum_{p=1}^{N-1} j_p \leq 3$,

$$\sup_{\mathbf{x}' \in \mathbb{R}^{N-1}} \left| \frac{D_1^{j_1} D_2^{j_2} \cdots D_{N-1}^{j_{N-1}} \varphi(\mathbf{x}')}{S(\mathbf{x}')} \right| < \infty$$

holds true.

We give the proof of this proposition in § 3. In preparation we study $\varphi(\mathbf{x}') - h(\mathbf{x}')$. We set

$$\tilde{\varphi}(\mathbf{x}') := \varphi(\mathbf{x}') - h(\mathbf{x}') \quad \text{for all } \mathbf{x}' \in \mathbb{R}^{N-1}. \quad (2.4)$$

For each $1 \leq j \leq n$ we have

$$\tilde{\varphi}(\mathbf{x}') = \varphi(\mathbf{x}') - h_j(\mathbf{x}') = \rho * (h - h_j) \quad \text{for all } \mathbf{x}' \in \Omega_j. \quad (2.5)$$

We put $q(\mathbf{x}') := \max\{-x_1, 0\}$ for $\mathbf{x}' = (x_1, \dots, x_{N-1}) \in \mathbb{R}^{N-1}$. We set $\mathbf{e}_1 = (1, 0, \dots, 0) \in \mathbb{R}^{N-1}$ and introduce P as

$$\begin{aligned} P(x_1) &:= (\rho * q)(x_1 \mathbf{e}_1) \\ &= \int_{\mathbb{R}^{N-1}} \rho(\mathbf{y}') \max\{(\mathbf{y}', \mathbf{e}_1) - x_1, 0\} d\mathbf{y}' \\ &= - \int_{\mathbb{R}^{N-2}} \left(\int_{x_1}^{\infty} \rho(y_1, \mathbf{y}'') (x_1 - y_1) dy_1 \right) d\mathbf{y}'' \\ &> 0 \end{aligned} \quad (2.6)$$

for $x_1 \in \mathbb{R}$. Since

$$P(x_1) = - \int_{\mathbb{R}^{N-2}} \left(x_1 \int_{x_1}^{\infty} \rho(y_1, \mathbf{y}'') dy_1 - \int_{x_1}^{\infty} y_1 \rho(y_1, \mathbf{y}'') dy_1 \right) d\mathbf{y}'',$$

we have

$$\begin{aligned} P'(x_1) &= - \int_{\mathbb{R}^{N-2}} \left(\int_{x_1}^{\infty} \rho(y_1, \mathbf{y}'') dy_1 \right) d\mathbf{y}'' < 0, \\ P''(x_1) &= \int_{\mathbb{R}^{N-2}} \rho(x_1, \mathbf{y}'') d\mathbf{y}'' \\ &= \int_{\mathbb{R}^{N-2}} \tilde{\rho} \left(\sqrt{x_1^2 + |\mathbf{y}''|^2} \right) d\mathbf{y}'' > 0, \\ P'''(x_1) &= \int_{\mathbb{R}^{N-2}} \frac{x_1}{\sqrt{x_1^2 + |\mathbf{y}''|^2}} \tilde{\rho}_r \left(\sqrt{x_1^2 + |\mathbf{y}''|^2} \right) d\mathbf{y}'' \leq 0. \end{aligned} \quad (2.7)$$

In particular, we have

$$P''(x_1) = \int_{\mathbb{R}^{N-2}} \exp\left(-\sqrt{x_1^2 + |\mathbf{y}''|^2}\right) d\mathbf{y}'', \quad (2.8)$$

$$P'''(x_1) = - \int_{\mathbb{R}^{N-2}} \frac{x_1}{\sqrt{x_1^2 + |\mathbf{y}''|^2}} \exp\left(-\sqrt{x_1^2 + |\mathbf{y}''|^2}\right) d\mathbf{y}'', \quad (2.9)$$

if $x_1 > 0$ is large enough. We have

$$P'(x) > -1 \quad \text{for } x \geq 0 \quad (2.10)$$

from (2.7).

Now we have the following lemma.

LEMMA 2.4. *Let P be as in (2.6). Then one has*

$$\lim_{x \rightarrow \infty} \frac{P(x)}{\frac{1}{2} \Gamma(\frac{1}{2}N - 1)(2x)^{(N/2)-1} e^{-x}} = 1. \quad (2.11)$$

Moreover,

$$\lim_{x \rightarrow \infty} \frac{|P^{(i)}(x)|}{P(x)} = 1, \quad 0 < \inf_{x \geq 1} \frac{|P^{(i)}(x)|}{P(x)} \leq \sup_{x \geq 1} \frac{|P^{(i)}(x)|}{P(x)} < +\infty$$

hold true for all i with $1 \leq i \leq 3$.

Proof. We use the polar coordinates in $\mathbb{R}^{N-1}$ given by

$$\begin{aligned} \Psi(r, \theta_1, \dots, \theta_{N-2}) \\ := \left(r \cos \theta_1, r \sin \theta_1 \cos \theta_1, \dots, r \left(\prod_{i=1}^{N-3} \sin \theta_i \right) \cos \theta_{N-2}, r \left(\prod_{i=1}^{N-3} \sin \theta_i \right) \sin \theta_{N-2} \right) \end{aligned}$$

for $(\theta_1, \dots, \theta_{N-2}) \in [0, \pi]^{N-3} \times [0, 2\pi]$. We put $I_0 := [0, \pi]^{N-3} \times [0, 2\pi]$. Next we have

$$P''(x) = \frac{(N-2)\pi^{(N-2)/2}}{\Gamma(\frac{1}{2}N)} \int_0^\infty \exp(-\sqrt{x^2 + r^2}) r^{N-3} dr \quad (2.12)$$

if $x > R_0$. Putting $r = \sqrt{s^2 + 2sx}$, we have

$$\begin{aligned} \int_0^\infty \exp(-\sqrt{x^2 + r^2}) r^{N-3} dr \\ = \int_0^\infty e^{-x-s} (s^2 + 2sx)^{(N-4)/2} (s+x) ds \\ = \frac{1}{2} (2x)^{(N/2)-1} e^{-x} \int_0^\infty e^{-s} \left(s + \frac{s^2}{2x} \right)^{(N-4)/2} \left(1 + \frac{s}{x} \right) ds. \end{aligned}$$

By the Lebesgue convergence theorem we have

$$\lim_{x \rightarrow \infty} \int_0^\infty e^{-s} \left(s + \frac{s^2}{2x} \right)^{(N-4)/2} \left(1 + \frac{s}{x} \right) ds = \int_0^\infty s^{(N/2)-2} e^{-s} ds = \Gamma(\tfrac{1}{2}N - 1).$$

Thus, we get

$$\lim_{x \rightarrow \infty} \frac{P''(x)}{\frac{1}{2} \Gamma(\frac{1}{2}N - 1)(2x)^{(N/2)-1} e^{-x}} = 1.$$

Similarly, we calculate

$$\begin{aligned} P'''(x) &= - \int_{\mathbb{R}^{N-2}} \frac{x}{\sqrt{x^2 + |\mathbf{y}''|^2}} \exp(-\sqrt{x^2 + |\mathbf{y}''|^2}) d\mathbf{y}'' \\ &= - \frac{(N-2)\pi^{(N-2)/2}}{\Gamma(\frac{1}{2}N)} \int_0^\infty \frac{x}{\sqrt{x^2 + r^2}} \exp(-\sqrt{x^2 + r^2}) r^{N-3} dr \end{aligned}$$

if $x > R_0$.

Putting $r = \sqrt{s^2 + 2sx}$ again, we have

$$P'''(x) = - \int_0^\infty x e^{-x-s} (s^2 + 2sx)^{(N-4)/2} ds$$

if $x > R_0$. Thus, we get

$$\lim_{x \rightarrow \infty} \frac{-P'''(x)}{\frac{1}{2} \Gamma(\frac{1}{2}N - 1)(2x)^{(N/2)-1} e^{-x}} = 1.$$

Thus, we obtain

$$\lim_{x \rightarrow \infty} \frac{-P'''(x)}{P''(x)} = 1.$$

Now the Cauchy mean-value theorem gives

$$\frac{P''(x)}{P'(x)} = \frac{P'''(x')}{P''(x')}$$

for some $x' > x$. This yields

$$\lim_{x \rightarrow \infty} \frac{P''(x)}{-P'(x)} = 1.$$

Similarly, we have

$$\lim_{x \rightarrow \infty} \frac{-P'(x)}{P(x)} = \lim_{x \rightarrow \infty} \frac{P''(x)}{-P'(x)} = 1.$$

Thus, we obtain

$$\lim_{x \rightarrow \infty} \frac{-P'(x)}{P(x)} = \lim_{x \rightarrow \infty} \frac{P''(x)}{-P'(x)} = \lim_{x \rightarrow \infty} \frac{-P'''(x)}{P''(x)} = 1.$$

Using these equalities, we obtain

$$0 < \inf_{x \geq 1} \frac{|P^{(i)}(x)|}{P(x)} \leq \sup_{x \geq 1} \frac{|P^{(i)}(x)|}{P(x)} < +\infty.$$

□

We choose $\tilde{\eta} \in C^\infty[0, \infty)$ with

$$\begin{aligned}\tilde{\eta}(r) &= 1 & \text{if } 0 \leq r \leq 1, \\ 0 < \tilde{\eta}(r) &< 1 & \text{if } 1 < r < 2, \\ \tilde{\eta}(r) &= 0 & \text{if } r \geq 2.\end{aligned}$$

We put $\eta(\mathbf{x}') := \tilde{\eta}(|\mathbf{x}'|)$ for $\mathbf{x}' \in \mathbb{R}^{N-1}$.

Let $a > 0$ be any given constant. For any $\zeta \in L^\infty(\mathbb{R}^{N-1})$ with

$$\left. \begin{aligned}\zeta(0) &= 0, \\ \text{supp } \zeta &\subset \{(x_1, \dots, x_{N-1}) \mid x_1 \leq 0\}, \\ |\zeta(\mathbf{x}')| &\leq |\mathbf{x}'| \quad \text{for all } \mathbf{x}' \in \mathbb{R}^{N-1},\end{aligned} \right\} \quad (2.13)$$

we set

$$\begin{aligned}Q(x; a, \zeta) &:= \int_{\mathbb{R}^{N-1}} \rho(x\mathbf{e}_1 - \mathbf{y}') \eta\left(\frac{\mathbf{y}'}{ax}\right) \zeta(\mathbf{y}') \, d\mathbf{y}' & \text{for } x > 0, \\ R(x; a, \zeta) &:= \int_{\mathbb{R}^{N-1}} \rho(x\mathbf{e}_1 - \mathbf{y}') \left(1 - \eta\left(\frac{\mathbf{y}'}{ax}\right)\right) \zeta(\mathbf{y}') \, d\mathbf{y}' & \text{for } x > 0.\end{aligned}$$

Then we have

$$P(x) = Q(x; a, q) + R(x; a, q) \quad \text{for all } x > 0.$$

LEMMA 2.5. *Let $a > 0$ be any given constant. Let $\zeta \in L^\infty(\mathbb{R}^{N-1})$ be a given function with (2.13). Then there exists $K(a) > 0$ depending on a such that one has*

$$\max_{i=0,1,2,3} |R^{(i)}(x; a, \zeta)| \leq K(a) \exp\left(-\frac{1 + \sqrt{1 + a^2}}{2}x\right) \quad \text{for all } x > 0.$$

Hence, lemma 2.4 holds true by replacing $P(x)$ by $Q(x; a, q)$.

Proof. The latter statement follows from the former one and lemma 2.4. It suffices to prove the former statement.

We use the polar coordinate in $\mathbb{R}^{N-1}$ given by

$$\mathbf{x}' = \Psi(r, \theta_1, \dots, \theta_{N-2}) \quad \text{for } (\theta_1, \dots, \theta_{N-2}) \in [0, \pi]^{N-3} \times [0, 2\pi].$$

Here $I_0 = [0, \pi]^{N-3} \times [0, 2\pi]$. We have

$$0 \leq \zeta(\mathbf{y}') \leq ax \quad \text{if } \mathbf{y}' \in B(0; ax)$$

for $x > 0$. Here $B(0; ax)$ is a ball in $\mathbb{R}^N$ whose centre is the origin and whose radius is ax . Then we get

$$0 < R(x; a, \zeta) \leq 2\pi^{N-2} \int_{ax}^{\infty} \exp(-\sqrt{x^2 + r^2}) r^{N-1} \, dr \quad \text{if } r > R_0.$$

We obtain

$$\begin{aligned}
0 &< \int_{ax}^{\infty} \exp(-\sqrt{x^2 + r^2}) r^{N-1} dr \\
&= \int_{(\sqrt{1+a^2}-1)x}^{\infty} e^{-x-s} (s^2 + 2sx)^{(N-2)/2} (s+x) ds \\
&\leq \left(\frac{a^2 + 2a + 4}{a^2} \right)^{N/2} e^{-x} \int_{(\sqrt{1+a^2}-1)x}^{\infty} s^{N-1} e^{-s} ds.
\end{aligned}$$

Here we used $(\sqrt{1+a^2}-1)x < s$, which gives $x \leq a^{-2}(2+a)s$. We also get

$$\begin{aligned}
0 &< \int_{(\sqrt{1+a^2}-1)x}^{\infty} s^{N-1} e^{-s} ds \\
&\leq \exp\left(-\frac{\sqrt{1+a^2}-1}{2}x\right) \int_0^{\infty} s^{N-1} e^{-s/2} ds \\
&\leq 2^N (N-1)! \exp\left(-\frac{\sqrt{1+a^2}-1}{2}x\right).
\end{aligned}$$

Thus, we obtain

$$\lim_{x \rightarrow \infty} R(x; a, \zeta) \exp\left(\frac{1 + \sqrt{1+a^2}}{2}x\right) < \infty.$$

Next we estimate derivatives of $R(x; a, \zeta)$. Let $i = 1, 2, 3$. We have

$$R^{(i)}(x; a, \zeta) = \int_{\mathbb{R}^{N-1}} D_x^i \left(\rho(xe_1 - \mathbf{y}') \left(1 - \eta\left(\frac{\mathbf{y}'}{ax}\right) \right) \right) \zeta(\mathbf{y}') d\mathbf{y}' \quad \text{for } x > 0.$$

By direct calculation, we obtain

$$\sup_{x \geq 1, \mathbf{y}' \in \mathbb{R}^{N-1}} \left| D_x^i \left(1 - \tilde{\eta}\left(\frac{|\mathbf{y}'|}{ax}\right) \right) \right| \leq M_2 \quad (2.14)$$

for all $i = 0, 1, 2, 3$. Here $M_2 > 0$ is a constant depending only on $\tilde{\eta}$ and is independent of a . We have

$$D_x^i \left(1 - \tilde{\eta}\left(\frac{|\mathbf{y}'|}{ax}\right) \right) = 0 \quad \text{for all } \mathbf{y}' \in B(0; ax).$$

Using (2.1) yields

$$|R^{(i)}(x, a, \zeta)| \leq 2^i M_1 M_2 \int_{\mathbb{R}^{N-1} \setminus B(0; ax)} \rho(xe_1 - \mathbf{y}') |\zeta(\mathbf{y}')| d\mathbf{y}'.$$

We obtain

$$\int_{\mathbb{R}^{N-1} \setminus B(0; ax)} \rho(xe_1 - \mathbf{y}') |\zeta(\mathbf{y}')| d\mathbf{y}' \leq 2\pi^{N-2} \int_{ax}^{\infty} \exp(-\sqrt{x^2 + r^2}) r^{N-1} dr$$

if $r > R_0$. From the argument stated above we obtain the estimates of the derivatives of $R^{(i)}(x; a, \zeta)$. This completes the proof. $\square$

3. Proof of proposition 2.3

We study Ω_j for each $1 \leq j \leq n$. Without loss of generality we can assume that $\mathbf{A}_j = \mathbf{e}_{N-1}$, where $\mathbf{e}_{N-1} := (0, \dots, 0, 1) \in \mathbb{R}^{N-1}$. For any $i \neq j$, we have $A_{N-1,i} < 1$ from $\mathbf{A}_i \neq \mathbf{e}_{N-1}$. Now $h_j(\mathbf{x}') \geq h_i(\mathbf{x}')$ is equivalent to

$$x_{N-1} \geq (1 - A_{N-1,i})^{-1} \sum_{p=1}^{N-2} A_{p,i} x_p$$

for every $i \neq j$. Then from the definition of Ω_j , we have

$$\Omega_j = \left\{ \mathbf{x}' \in \mathbb{R}^{N-1} \mid x_{N-1} \geq \max_{i \neq j} \frac{\sum_{p=1}^{N-2} A_{p,i} x_p}{1 - A_{N-1,i}} \right\} \quad (3.1)$$

if $\mathbf{A}_j = \mathbf{e}_{N-1}$. This fact implies that Ω_j is an $(N-1)$ -dimensional pyramid in $\mathbb{R}^{N-1}$ for every $j \in \{1, \dots, n\}$. In particular, Ω_j is a convex set in $\mathbb{R}^{N-1}$. The lateral faces of Ω_j are given by

$$\partial\Omega_j = \bigcup_{i \in A'(j)} (\Omega_j \cap \Omega_i).$$

Here $A'(j)$ is a subset of $\{1, \dots, n\} \setminus \{j\}$. We determine $A(j)$ as the minimum of such an $A'(j)$, that is,

$$A(j) := \bigcap \left\{ A'(j) \mid \partial\Omega_j = \bigcup_{i \in A'(j)} (\Omega_j \cap \Omega_i) \right\}.$$

We see that $i \in A(j)$ is equivalent to $j \in A(i)$. We call Ω_i and Ω_j are adjacent if and only if $i \in A(j)$.

LEMMA 3.1. *If $\mathbf{A}_j = \mathbf{e}_{N-1}$, one has*

$$\Omega_j = \left\{ \mathbf{x}' \in \mathbb{R}^{N-1} \mid x_{N-1} \geq \max_{i \in A(j)} \frac{\sum_{p=1}^{N-2} A_{p,i} x_p}{1 - A_{N-1,i}} \right\}.$$

Proof. We obtain this lemma from the definition of $A(j)$ and (3.1). $\square$

A set $A(j)$ has at least two elements. For any $i \in A(j)$, $\mathcal{H}_i$ is given by

$$x_{N-1} = (1 - A_{N-1,i})^{-1} \sum_{p=1}^{N-2} A_{p,i} x_p.$$

The normal vector of $\mathcal{H}_i$ is given by $\mathbf{e}_{N-1} - \mathbf{A}_i$. For each $i, i' \in A(j)$ with $i \neq i'$, let $\theta_{i,i'}$ be the angle between $\mathcal{H}_i$ and $\mathcal{H}_{i'}$ with $0 < \theta_{i,i'} < \pi$. We have

$$0 < \theta_{\min} \leq \min_{i, i' \in A(j), i \neq i'} \theta_{i,i'} \leq \max_{i, i' \in A(j), i \neq i'} \theta_{i,i'} \leq \theta_{\max} < \pi.$$

Let $\mathbf{x}' \in \Omega_j$; the length of a perpendicular onto $\mathcal{H}_i$ is given by

$$\frac{(\mathbf{A}_j - \mathbf{A}_i, \mathbf{x}')}{|\mathbf{A}_j - \mathbf{A}_i|} = (\mathbf{b}_{i,j}, \mathbf{x}') \geq 0 \quad (3.2)$$

for each $i \in A(j)$. Here we set

$$\mathbf{b}_{i,j} := \frac{\mathbf{A}_j - \mathbf{A}_i}{|\mathbf{A}_j - \mathbf{A}_i|} \in \mathbb{R}^{N-1}$$

for all $i \neq j$. The foot of this perpendicular is given by

$$\mathbf{g}_i(\mathbf{x}') := \mathbf{x}' - (\mathbf{b}_{i,j}, \mathbf{x}') \mathbf{b}_{i,j} \in \mathbb{R}^{N-1}$$

for all $i \neq j$.

Now we return to studying $\tilde{\varphi}_j(\mathbf{x}')$ and $S(\mathbf{x}')$. By the definition of S we have

$$S(\mathbf{x}') = \frac{k^2(m^2 - |\nabla\varphi(\mathbf{x}')|^2)}{\sqrt{1 + |\nabla\varphi(\mathbf{x}')|^2}(c + k\sqrt{1 + |\nabla\varphi(\mathbf{x}')|^2})}. \quad (3.3)$$

By lemma 2.2, we have

$$\begin{aligned} 0 &< m^2 - |\nabla\varphi|^2 \\ &= m^2 - |\nabla(\tilde{\varphi} + h_j)|^2 \\ &= m^2 - |\nabla\tilde{\varphi}|^2 - |\nabla h_j|^2 - 2(\nabla\tilde{\varphi}, \nabla h_j) \\ &= m^2 - |\nabla\tilde{\varphi}|^2 - m^2 - 2m(\mathbf{A}_j, \nabla\tilde{\varphi}) \\ &= -|\nabla\tilde{\varphi}|^2 - 2m(\mathbf{A}_j, \nabla\tilde{\varphi}) \end{aligned} \quad (3.4)$$

for all $\mathbf{x}' \in \Omega_j$. Then using lemma 2.2, we have

$$\frac{k^3}{2c^2}(-2m(\mathbf{A}_j, \nabla\tilde{\varphi}) - |\nabla\tilde{\varphi}|^2) \leq S(\mathbf{x}') \leq \frac{k^2}{c+k}(-2m(\mathbf{A}_j, \nabla\tilde{\varphi}) - |\nabla\tilde{\varphi}|^2)$$

for all $\mathbf{x}' \in \Omega_j$.

We study $S(\mathbf{x}')$ when $\mathbf{x}'$ lies near $\bigcup_{j=1}^n \partial\Omega_j$.

LEMMA 3.2. *For any given $b > 0$ one has*

$$0 < \inf \left\{ S(\mathbf{x}') \mid \text{dist} \left(\mathbf{x}', \bigcup_{j=1}^n \partial\Omega_j \right) \leq b \right\}.$$

Proof. Let $b > 0$ be given arbitrarily. Using lemma 3.1, we have

$$\inf_{\mathbf{x}' \in \Omega_j} \int_{\Omega_j} \rho(\mathbf{x}' - \mathbf{y}') \, d\mathbf{y}' > 0, \quad \inf_{\text{dist}(\mathbf{x}', \Omega_i) \leq b} \int_{\Omega_i} \rho(\mathbf{x}' - \mathbf{y}') \, d\mathbf{y}' > 0$$

for all $i, j \in \{1, \dots, n\}$. Without loss of generality we can assume $\mathbf{x}' \in \Omega_j$ for some j and $\text{dist}(\mathbf{x}', \Omega_i) \leq b$ for some $i \in A(j)$.

By lemma 2.2 and its proof we have

$$\nabla\varphi(\mathbf{x}') = m \sum_{j=1}^n \int_{\Omega_j} \rho(\mathbf{x}' - \mathbf{y}') \, d\mathbf{y}' \mathbf{A}_j.$$

Using the inequalities stated above, we obtain

$$0 < \inf \left\{ m - |\nabla \varphi| \mid \text{dist} \left(\mathbf{x}', \bigcup_{j=1}^n \Gamma_j \leq b \right) \right\}.$$

Using this inequality and (3.3), we complete the proof. $\square$

We study $\tilde{\varphi}$ in Ω_j . Without loss of generality we assume $\mathbf{A}_j = \mathbf{e}_{N-1}$. Let H be the Heaviside function given by

$$H(x) := \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

For each $i \in A(j)$ we set

$$\chi_i(\mathbf{x}') := H \left((1 - A_{N-1,i})^{-1} \sum_{p=1}^{N-2} A_{p,i} x_p - x_{N-1} \right)$$

for all $\mathbf{x}' = (x_1, \dots, x_{N-1}) \in \mathbb{R}^{N-1}$. We set $A(j) = \{i_1, \dots, i_\ell\}$, where $1 \leq \ell \leq n$. We have

$$0 = (h - h_j) \prod_{\kappa=1}^{\ell} (1 - \chi_\kappa) \quad \text{in } \mathbb{R}^{N-1}.$$

We set

$$1 - \prod_{\kappa=1}^{\ell} (1 - \chi_\kappa) = \sum_{\kappa=1}^{\ell} \chi_\kappa p_\kappa(\chi_1, \dots, \chi_\ell),$$

where p_κ is a polynomial. The degree of p_κ is no greater than ℓ .

Then we have

$$0 = (h - h_j) \left(1 - \sum_{\kappa=1}^{\ell} \chi_\kappa p_\kappa(\chi_1, \dots, \chi_\ell) \right)$$

and thus

$$h - h_j = \sum_{\kappa=1}^{\ell} \chi_\kappa p_\kappa(\chi_1, \dots, \chi_\ell) (h - h_j). \quad (3.5)$$

For any $\kappa \in \{1, \dots, \ell\}$ we set

$$c_\kappa := \max\{|p_\kappa(\chi_1, \dots, \chi_\ell)| \mid \chi_i = 0 \text{ or } 1 \text{ for all } 1 \leq i \leq \ell\}.$$

We put $b_* := \max\{1, \max_{1 \leq \kappa \leq \ell} c_\kappa\}$ and obtain $1 \leq b_* < \infty$.

LEMMA 3.3. *Let $\mathbf{A}_j = \mathbf{e}_{N-1}$. One has*

$$|h(\mathbf{x}') - h_j(\mathbf{x}')| \leq b_* \sum_{i \in A(j)} (h(\mathbf{x}') - h_j(\mathbf{x}')) \chi_i(\mathbf{x}') \quad \text{for all } \mathbf{x}' \in \mathbb{R}^{N-1}.$$

Proof. We obtain $0 \leq h - h_j \leq h$. Then we get this lemma immediately by (3.5). $\square$

We set

$$a_* := \frac{1}{4} \sin \frac{1}{2} \theta_{\min} \cot \frac{1}{2} \theta_{\max},$$

and obtain $0 < a_* < \infty$.

LEMMA 3.4. *For any $\mathbf{x}'$ that lies in the interior of Ω_j , one has*

$$\overline{B(\mathbf{g}_i(\mathbf{x}'); 2a_*(\mathbf{b}_{i,j}, \mathbf{x}'))} \subset \Omega_i \cup \Omega_j$$

with respect to any $i \in A(j)$ with $\text{dist}(\mathbf{x}', \partial\Omega) = \text{dist}(\mathbf{x}', \mathcal{H}_i)$.

Proof. For any $s \in A(j)$ with $s \neq i$ we have

$$\mathcal{H}_s \cap \overline{B(\mathbf{g}_i(\mathbf{x}'); \cot(\frac{1}{2}\theta_{\max})(\mathbf{b}_{i,j}, \mathbf{x}'))} = \emptyset.$$

Thus, we obtain

$$\overline{\mathcal{H}_i \cap B(\mathbf{g}_i(\mathbf{x}'); \cot(\frac{1}{2}\theta_{\max})(\mathbf{b}_{i,j}, \mathbf{x}'))} \subset \Omega_i \cap \Omega_j.$$

Then we get

$$\overline{B(\mathbf{g}_i(\mathbf{x}'); \sin(\frac{1}{2}\theta_{\min}) \cot(\frac{1}{2}\theta_{\max})(\mathbf{b}_{i,j}, \mathbf{x}'))} \subset \Omega_i \cup \Omega_j.$$

This completes the proof. $\square$

Proof of proposition 2.3. It suffices to prove this proposition by assuming $\mathbf{x}' \in \Omega_j$ with $\text{dist}(\mathbf{x}', \partial\Omega_j) \geq 1$ due to lemma 2.2 and lemma 3.2. For $\mathbf{x}' \in \Omega_j$ let

$$C(j) := \{i \in A(j) \mid \text{dist}(\mathbf{x}', \partial\Omega_j) = \text{dist}(\mathbf{x}', \mathcal{H}_i)\}.$$

There exists a constant $\nu > 1$ such that we have

$$\nu \text{dist}(\mathbf{x}', \mathcal{H}_i) \leq \text{dist}(\mathbf{x}', \mathcal{H}_s)$$

for all $\mathbf{x}' \in \Omega_j$ and all $s \in A(j) \setminus C(j)$. We set

$$\lambda := (\mathbf{b}_{i,j}, \mathbf{x}') = \text{dist}(\mathbf{x}', \mathcal{H}_i) \quad \text{for } i \in C(j).$$

Note that λ is independent of $i \in C(j)$.

It suffices to prove this proposition by assuming that $C(j)$ remains unchanged, say C_0 , as $|\mathbf{x}'| \rightarrow +\infty$. Here C_0 is a subset of $\{1, \dots, n\}$. We set

$$\Delta_j := \{\mathbf{x}' \in \Omega_j \mid \text{dist}(\mathbf{x}', \partial\Omega_j) \geq 1, \{i \in A(j) \mid \text{dist}(\mathbf{x}', \partial\Omega_j) = \text{dist}(\mathbf{x}', \mathcal{H}_i)\} = C_0\}.$$

We assume $\mathbf{x}' \in \Delta_j$. We have

$$B(\mathbf{g}_i(\mathbf{x}'); 2a_*\lambda) \subset \Omega_i \cap \Omega_j$$

for all $i \in C(j)$ and $\mathbf{x}' \in \Delta_j$.

We have

$$1 - \sum_{i \in C(j)} \eta\left(\frac{\mathbf{y}' - \mathbf{g}_i(\mathbf{x}')}{a_*\lambda}\right) = 0 \quad \text{if } \mathbf{y}' \in \bigcup_{i \in C(j)} B(\mathbf{g}_i(\mathbf{x}'); a_*\lambda).$$

For any $\mathbf{x}' \in \Omega_j$ with $\text{dist}(\mathbf{x}', \partial\Omega_j) \geq 1$ we have $\tilde{\varphi}(\mathbf{x}') = I(\mathbf{x}') + J(\mathbf{x}')$, where

$$I(\mathbf{x}') := \int_{\mathbb{R}^{N-1}} \rho(\mathbf{x}' - \mathbf{y}') \sum_{i \in C(j)} \eta\left(\frac{\mathbf{y}' - \mathbf{g}_i(\mathbf{x}')}{a_* \lambda}\right) (h(\mathbf{y}') - h_j(\mathbf{y}')) \, d\mathbf{y}',$$

$$J(\mathbf{x}') := \int_{\mathbb{R}^{N-1}} \rho(\mathbf{x}' - \mathbf{y}') \left(1 - \sum_{i \in C(j)} \eta\left(\frac{\mathbf{y}' - \mathbf{g}_i(\mathbf{x}')}{a_* \lambda}\right)\right) (h(\mathbf{y}') - h_j(\mathbf{y}')) \, d\mathbf{y}'.$$

We have

$$I(\mathbf{x}') = m \sum_{i \in C(j)} |\mathbf{A}_i - \mathbf{A}_j| Q(\lambda; a_*, q). \quad (3.6)$$

Using lemma 3.3, we obtain

$$\begin{aligned} |J(\mathbf{x}')| &\leq 2mb_* \sum_{i \in C(j)} \int_{\mathbb{R}^{N-1}} \rho(\mathbf{x}' - \mathbf{y}') \left(1 - \sum_{i \in C(j)} \eta\left(\frac{\mathbf{y}' - \mathbf{g}_i(\mathbf{x}')}{a_* \lambda}\right)\right) \\ &\quad \times \chi_i(\mathbf{y}') \frac{h(\mathbf{y}') - h_j(\mathbf{y}')}{2m} \, d\mathbf{y}' \\ &\quad + 2mb_* \sum_{i \in A(j) \setminus C(j)} \int_{\mathbb{R}^{N-1}} \rho(\mathbf{x}' - \mathbf{y}') \chi_i(\mathbf{y}') \frac{h(\mathbf{y}') - h_j(\mathbf{y}')}{2m} \, d\mathbf{y}' \\ &\leq 2mb_* \sum_{i \in C(j)} R(\lambda; a_*, q) + 2mb_* \sum_{i \in A(j) \setminus C(j)} P(\lambda; a_*, q) \\ &\leq 2mb_* n K(a_*) \exp\left(-\frac{1 + \sqrt{1 + (a_*)^2}}{2} \lambda\right) + 2mb_* n P(\nu \lambda; a_*, q). \end{aligned}$$

Putting

$$a_0 := \min\left\{\nu, \frac{1 + \sqrt{1 + (a_*)^2}}{2}\right\} \in (1, \infty),$$

we have

$$|J(\mathbf{x}')| \leq M_3 e^{-a_0 \lambda},$$

where $M_3 > 0$ is a constant.

By (2.14) we have

$$\left| \left(\frac{\partial}{\partial x_1}\right)^{j_1} \cdots \left(\frac{\partial}{\partial x_{N-1}}\right)^{j_{N-1}} \eta\left(\frac{\mathbf{y}' - \mathbf{g}_i(\mathbf{x}')}{a_* \lambda}\right) \right| \leq M_2 \quad (3.7)$$

for $j_1 \geq 0, \dots, j_{N-1} \geq 0$ with $\sum_{p=1}^{N-1} j_p \leq 3$.

Using (2.1) and (3.7), we have

$$\begin{aligned} |D_1^{j_1} \cdots D_{N-1}^{j_{N-1}} J(\mathbf{x}')| \\ \leq 8M_1 M_2 \int_{\mathbb{R}^{N-1} \setminus \bigcup_{i \in C(j)} B(\mathbf{g}_i(\mathbf{x}'); a_* \lambda)} \rho(\mathbf{x}' - \mathbf{y}') (h(\mathbf{y}') - h_j(\mathbf{y}')) \, d\mathbf{y}'. \end{aligned}$$

Now we have

$$\begin{aligned}
& \int_{\mathbb{R}^{N-1} \setminus \bigcup_{i \in C(j)} B(\mathbf{g}_i(\mathbf{x}'); a_* \lambda)} \rho(\mathbf{x}' - \mathbf{y}')(h(\mathbf{y}') - h_j(\mathbf{y}')) d\mathbf{y}' \\
& \leq 2mb_* \sum_{i \in C(j)} \int_{\mathbb{R}^{N-1} \setminus B(\mathbf{g}_i(\mathbf{x}'); a_* \lambda)} \rho(\mathbf{x}' - \mathbf{y}') \chi_i(\mathbf{y}') \frac{h(\mathbf{y}') - h_j(\mathbf{y}')}{2m} d\mathbf{y}' \\
& \quad + 2mb_* \sum_{i \in A(j) \setminus C(j)} \int_{\mathbb{R}^{N-1}} \rho(\mathbf{x}' - \mathbf{y}') \chi_i(\mathbf{y}') \frac{h(\mathbf{y}') - h_j(\mathbf{y}')}{2m} d\mathbf{y}' \\
& \leq 2mb_* nR(\lambda; a_*, q) + 2mb_* nP(\nu\lambda; a_*, q).
\end{aligned}$$

Thus, we obtain

$$|D_1^{j_1} \cdots D_{N-1}^{j_{N-1}} J(\mathbf{x}')| \leq M_4 e^{-a_0 \lambda} \quad (3.8)$$

for all $\lambda > 0$, $j_1 \geq 0, \dots, j_{N-1} \geq 0$ with $\sum_{p=1}^{N-1} j_p \leq 3$. From lemma 2.5, and (3.3), (3.6) and (3.8), we obtain

$$\lim_{\lambda \rightarrow \infty} \sup_{\mathbf{x}' \in \Delta_j, (\mathbf{b}_{i,j}, \mathbf{x}') = \lambda} \frac{S(\mathbf{x}')}{\sum_{i \in C(j)} (1 - (\mathbf{A}_i, \mathbf{A}_j)) Q(\lambda; a_*, q)} = \frac{2m^2 k^2}{c + k} \quad (3.9)$$

and

$$|D_1^{j_1} \cdots D_{N-1}^{j_{N-1}} \tilde{\varphi}(\mathbf{x}')| \leq M_5 Q(\lambda; a_*, q) \quad \text{for } \mathbf{x}' \in \Delta_j, (\mathbf{b}_{i,j}, \mathbf{x}') = \lambda > 0. \quad (3.10)$$

Here $M_5 > 0$ is a constant, and $j_1 \geq 0, \dots, j_{N-1} \geq 0$ satisfy $\sum_{p=1}^{N-1} j_p \leq 3$. This completes the proof of proposition 2.3. $\square$

4. Proof of theorem 1.1

In this section we construct a pyramidal travelling-front solution and prove theorem 1.1 by constructing a supersolution and a subsolution and by using the methods of [8, 12].

We put $\boldsymbol{\xi} = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N$ and $\boldsymbol{\xi}' = (\xi_1, \dots, \xi_{N-1}) \in \mathbb{R}^{N-1}$. We study $\xi_N = \varphi(\boldsymbol{\xi}')$.

For $\alpha \in (0, 1)$ we consider the graph of

$$\left\{ \mathbf{x} \in \mathbb{R}^N \mid x_N = \frac{1}{\alpha} \varphi(\alpha \mathbf{x}') \right\}. \quad (4.1)$$

Later we will choose $\alpha > 0$ to be small enough. We note that

$$\frac{1}{\alpha} h(\alpha \mathbf{x}') = h(\mathbf{x}').$$

Putting $\boldsymbol{\xi} = \alpha \mathbf{x}$, we have $\xi_N = \varphi(\boldsymbol{\xi}')$. For a given constant $\mathbf{b}' \in \mathbb{R}^{N-1}$, the tangent plane of (4.1) at $(\mathbf{b}', (1/\alpha)\varphi(\alpha \mathbf{b}'))$ is expressed by

$$-(\nabla \varphi(\alpha \mathbf{b}'), (\mathbf{x}' - \mathbf{b}')) + x_N - \frac{1}{\alpha} \varphi(\mathbf{b}') = 0. \quad (4.2)$$

Here we denote $(\partial/\partial\xi_1, \partial/\partial\xi_2, \dots, \partial/\partial\xi_{N-1})$ by ∇ . The length of the perpendicular from $\mathbf{b} = (\mathbf{b}', b_N) \in \mathbb{R}^N$ onto the tangent plane is given by

$$\frac{|b_N - \varphi(\alpha\mathbf{b}')/\alpha|}{\sqrt{1 + |\nabla\varphi(\alpha\mathbf{b}')|^2}}.$$

We set

$$\hat{\mu} := \frac{x_N - \varphi(\alpha\mathbf{x}')/\alpha}{\sqrt{1 + |\nabla\varphi(\alpha\mathbf{x}')|^2}} = \frac{1}{\alpha} \frac{\xi_N - \varphi(\boldsymbol{\xi}')}{\sqrt{1 + |\nabla\varphi(\boldsymbol{\xi}')|^2}}, \quad (4.3)$$

and obtain

$$\frac{\partial\hat{\mu}}{\partial x_N} = \frac{1}{\sqrt{1 + |\nabla\varphi(\boldsymbol{\xi}')|^2}}, \quad \frac{\partial^2\hat{\mu}}{\partial x_N^2} = 0.$$

We have

$$\frac{\partial\hat{\mu}}{\partial x_i} = -\frac{1}{\sqrt{1 + |\nabla\varphi|^2}} \frac{\partial\varphi}{\partial\xi_i} + \alpha\hat{\mu}F_i, \quad \frac{\partial^2\hat{\mu}}{\partial x_i^2} = \alpha G_i + \alpha^2\hat{\mu}H_i,$$

where

$$F_i(\boldsymbol{\xi}') := \sqrt{1 + |\nabla\varphi|^2} \frac{\partial}{\partial\xi_i} \left(\frac{1}{\sqrt{1 + |\nabla\varphi|^2}} \right), \quad (4.4)$$

$$G_i(\boldsymbol{\xi}') := -\frac{\partial}{\partial\xi_i} \left(\frac{1}{\sqrt{1 + |\nabla\varphi|^2}} \frac{\partial\varphi}{\partial\xi_i} \right) - \frac{F_i(\boldsymbol{\xi}')}{\sqrt{1 + |\nabla\varphi|^2}} \frac{\partial\varphi}{\partial\xi_i}, \quad (4.5)$$

$$H_i(\boldsymbol{\xi}') := \frac{\partial F_i}{\partial\xi_i} + (F_i(\boldsymbol{\xi}'))^2 \quad (4.6)$$

for $i = 1, \dots, N-1$. We define

$$\bar{v}(\mathbf{x}) = \Phi(\hat{\mu}) + \sigma(\mathbf{x}'), \quad (4.7)$$

where $\hat{\mu}$ is as in (4.3) and $\sigma(\mathbf{x}') := \varepsilon S(\alpha\mathbf{x}')$. Here we will fix $\varepsilon > 0$ in (4.9). We have

$$\frac{\partial\bar{v}}{\partial x_N} = \frac{1}{\sqrt{1 + |\nabla\varphi|^2}} \Phi'(\hat{\mu}), \quad \frac{\partial^2\bar{v}}{\partial x_N^2} = \frac{1}{1 + |\nabla\varphi|^2} \Phi''(\hat{\mu}).$$

Since

$$\frac{\partial\bar{v}}{\partial x_i} = \Phi'(\hat{\mu}) \frac{\partial\hat{\mu}}{\partial x_i} + \frac{\partial\sigma}{\partial x_i}, \quad \frac{\partial^2\bar{v}}{\partial x_i^2} = \Phi''(\hat{\mu}) \left(\frac{\partial\hat{\mu}}{\partial x_i} \right)^2 + \Phi'(\hat{\mu}) \frac{\partial^2\hat{\mu}}{\partial x_i^2} + \frac{\partial^2\sigma}{\partial x_i^2}$$

and

$$\begin{aligned} \sum_{i=1}^{N-1} \left(\frac{\partial^2\hat{\mu}}{\partial x_i^2} \right)^2 &= \sum_{i=1}^{N-1} \left(-\frac{1}{\sqrt{1 + |\nabla\varphi|^2}} \frac{\partial\varphi}{\partial\xi_i} + \alpha\hat{\mu}F_i \right)^2 \\ &= \frac{|\nabla\varphi|^2}{1 + |\nabla\varphi|^2} + \sum_{i=1}^{N-1} \left(\frac{-2\alpha\hat{\mu}F_i}{\sqrt{1 + |\nabla\varphi|^2}} \frac{\partial\varphi}{\partial\xi_i} + \alpha^2\hat{\mu}^2(F_i)^2 \right) \end{aligned}$$

for $i = 1, \dots, N-1$, we have

$$\begin{aligned} \sum_{i=1}^{N-1} \frac{\partial^2 \bar{v}}{\partial x_i^2} &= \Phi'(\hat{\mu}) \sum_{i=1}^{N-1} \frac{\partial^2 \hat{\mu}}{\partial x_i^2} + \Phi''(\hat{\mu}) \sum_{i=1}^{N-1} \left(\frac{\partial^2 \hat{\mu}}{\partial x_i^2} \right)^2 + \sum_{i=1}^{N-1} \frac{\partial^2 \sigma}{\partial x_i^2} \\ &= \Phi'(\hat{\mu}) \sum_{i=1}^{N-1} (\alpha G_i + \alpha^2 \hat{\mu} H_i) + \Phi''(\hat{\mu}) \left(\frac{|\nabla \varphi|^2}{1 + |\nabla \varphi|^2} \right) \\ &\quad + \Phi''(\hat{\mu}) \sum_{i=1}^{N-1} \left(\frac{-2\alpha \hat{\mu} F_i}{\sqrt{1 + |\nabla \varphi|^2}} \frac{\partial \varphi}{\partial \xi_i} + \alpha^2 \hat{\mu}^2 (F_i)^2 \right) + \sum_{i=1}^{N-1} \frac{\partial^2 \sigma}{\partial x_i^2}. \end{aligned}$$

We calculate $\mathfrak{L}[\bar{v}]$ as

$$\begin{aligned} \mathfrak{L}[\bar{v}] &= - \sum_{i=1}^N \frac{\partial^2 \bar{v}}{\partial x_i^2} - c \frac{\partial \bar{v}}{\partial x_N} - f(\bar{v}) \\ &= -\Phi'(\hat{\mu}) \sum_{i=1}^{N-1} (\alpha G_i + \alpha^2 \hat{\mu} H_i) - \Phi''(\hat{\mu}) \left(\frac{|\nabla \varphi|^2}{1 + |\nabla \varphi|^2} \right) \\ &\quad + \Phi''(\hat{\mu}) \sum_{i=1}^{N-1} \left(\frac{2\alpha \hat{\mu} F_i}{\sqrt{1 + |\nabla \varphi|^2}} \frac{\partial \varphi}{\partial \xi_i} - \alpha^2 \hat{\mu}^2 (F_i)^2 \right) - \sum_{i=1}^{N-1} \frac{\partial^2 \sigma}{\partial x_i^2} \\ &\quad - \frac{1}{1 + |\nabla \varphi|^2} \Phi''(\hat{\mu}) - \frac{c}{\sqrt{1 + |\nabla \varphi|^2}} \Phi'(\hat{\mu}) - f(\bar{v}) \\ &= -\Phi''(\hat{\mu}) - \frac{c}{\sqrt{1 + |\nabla \varphi|^2}} \Phi'(\hat{\mu}) - f(\Phi(\hat{\mu}) + \sigma) - \Phi'(\hat{\mu}) \sum_{i=1}^{N-1} (\alpha G_i + \alpha^2 \hat{\mu} H_i) \\ &\quad + \Phi''(\hat{\mu}) \sum_{i=1}^{N-1} \left(\frac{2\alpha \hat{\mu} F_i}{\sqrt{1 + |\nabla \varphi|^2}} \frac{\partial \varphi}{\partial \xi_i} - \alpha^2 \hat{\mu}^2 (F_i)^2 \right) - \varepsilon \alpha^2 \sum_{i=1}^{N-1} \frac{\partial^2 S}{\partial \xi_i^2}. \end{aligned}$$

We define

$$\begin{aligned} Y(\xi', \mu; \varepsilon, \alpha) &:= -\Phi'(\mu) \sum_{i=1}^{N-1} (G_i + \alpha \mu H_i) \\ &\quad + \Phi''(\mu) \sum_{i=1}^{N-1} \left(\frac{2\mu F_i}{\sqrt{1 + |\nabla \varphi|^2}} \frac{\partial \varphi}{\partial \xi_i} - \alpha \mu^2 (F_i)^2 \right) - \varepsilon \alpha \sum_{i=1}^{N-1} \frac{\partial^2 S}{\partial \xi_i^2}. \end{aligned}$$

Thus, we have

$$\mathfrak{L}[\bar{v}] = -\Phi''(\hat{\mu}) - \frac{c}{\sqrt{1 + |\nabla \varphi|^2}} \Phi'(\hat{\mu}) - f(\Phi(\hat{\mu}) + \sigma) + \alpha Y(\xi', \hat{\mu}; \varepsilon, \alpha).$$

Using the first equality of (1.1),

$$\sigma \int_0^1 f'(\Phi(\hat{\mu}) + s\sigma) ds = f(\Phi(\hat{\mu}) + \sigma) - f(\Phi(\hat{\mu}))$$

and (2.3), we then obtain

$$\mathfrak{L}[\bar{v}] = -\Phi'(\hat{\mu})S(\xi') - \sigma \int_0^1 f'(\Phi(\hat{\mu} + s\sigma)) \, ds + \alpha Y(\xi', \hat{\mu}; \varepsilon, \alpha).$$

We have

$$\begin{aligned} |Y(\xi', \mu; \varepsilon, \alpha)| &\leq \max\{|\Phi'(\mu)|, |\mu\Phi'(\mu)|, |\mu\Phi''(\mu)|, |\mu^2\Phi''(\mu)|\} \\ &\quad \times \left(\sum_{i=1}^{N-1} \left(|G_i| + |H_i| + 2|F_i| + |F_i|^2 + \left| \frac{\partial^2 S}{\partial \xi_i^2} \right| \right) \right) \end{aligned}$$

for $0 < \alpha < 1$. Using lemmas 2.1 and 2.2, we have

$$\sum_{i=1}^{N-1} \left(|G_i| + |H_i| + 2|F_i| + |F_i|^2 + \left| \frac{\partial^2 S}{\partial \xi_i^2} \right| \right) \leq \nu_0 \sum_{2 \leq \sum_{p=1}^{N-1} i_p \leq 3} |D_1^{i_1} D_2^{i_2} \cdots D_{N-1}^{i_{N-1}} \varphi(\xi')|$$

with a constant $\nu_0 > 0$ for all $\xi' \in \mathbb{R}^{N-1}$. Using proposition 2.3, there exists $\nu_* > 0$ satisfying

$$\frac{|Y(\xi', \mu; \varepsilon, \alpha)|}{S(\xi')} < \nu_*$$

for all $\xi' \in \mathbb{R}^{N-1}$, $\mu \in \mathbb{R}$, $\varepsilon \in (0, 1)$ and $\alpha \in (0, 1)$. Constants ν_0 and ν_* are independent of ξ' , $\alpha \in (0, 1)$ and $\varepsilon \in (0, 1)$. We continue to calculate $\mathfrak{L}[\bar{v}]$ as

$$\mathfrak{L}[\bar{v}] = S(\xi') \left(-\Phi'(\hat{\mu}) - \varepsilon \int_0^1 f'(\Phi(\hat{\mu}) + s\sigma) \, ds + \alpha \frac{Y(\xi', \hat{\mu}; \varepsilon, \alpha)}{S(\xi')} \right).$$

Thus, we have

$$\mathfrak{L}[\bar{v}] \geq S(\xi') \left(-\Phi'(\hat{\mu}) + \varepsilon \int_0^1 (-f'(\Phi(\hat{\mu}) + s\sigma)) \, ds - \alpha \nu_* \right). \quad (4.8)$$

We set

$$\omega := \inf_{\mathbf{x}' \in \mathbb{R}^{N-1}} \frac{\varphi(\mathbf{x}') - h(\mathbf{x}')}{S(\mathbf{x}')} \in (0, \infty)$$

by using proposition 2.3. Now we choose ε small enough to satisfy

$$0 < \varepsilon < \min \left\{ \frac{1}{2}, \frac{\delta_*}{c}, \frac{4K_0}{ek\kappa_0}, \frac{\min_{-1+\delta_* \leq \Phi(p) \leq 1-\delta_*} (-\Phi'(p))}{4 \max_{|s| \leq 1+\delta_*} |f'(s)|} \right\}. \quad (4.9)$$

Then we choose α small enough to satisfy

$$0 < \alpha < \min \left\{ \frac{1}{2}, \frac{\varepsilon\beta}{\nu_*}, \frac{\min_{-1+\delta_* \leq \Phi(p) \leq 1-\delta_*} (-\Phi'(p))}{4\nu_*}, \frac{k^2\omega\kappa_0}{2c \log(4K_0/ek\kappa_0\varepsilon)} \right\}. \quad (4.10)$$

Now we show that $\bar{v}$ is a supersolution and is larger than our subsolution.

LEMMA 4.1. *Assume ε and α satisfy (4.9) and (4.10), respectively. Let $\underline{v}$ and $\bar{v}$ be as in (1.9) and (4.7), respectively. Then*

$$\mathfrak{L}[\bar{v}] > 0 \quad \text{in } \mathbb{R}^N$$

holds true. Moreover, one has $\underline{v}(\mathbf{x}) < \bar{v}(\mathbf{x})$ in $\mathbb{R}^N$.

Proof. By lemma 2.2 and (4.9) we have

$$|s\varepsilon S| \leq s\varepsilon c \leq \delta_* \quad \text{for all } s \in [0, 1].$$

We consider the case of $\Phi(\hat{\mu}) < -1 + \delta_*$ or $1 - \delta_* < \Phi(\hat{\mu})$. We have $\Phi(\hat{\mu}) + s\varepsilon S \leq -1 + 2\delta_*$ or $1 - 2\delta_* \leq \Phi(\hat{\mu}) + s\varepsilon S$. Since $\sigma = \varepsilon S$, we have

$$\Phi(\hat{\mu}) + s\sigma \leq -1 + 2\delta_* \text{ or } 1 - 2\delta_* \leq \Phi(\hat{\mu}) + s\sigma.$$

From the definition of δ_* we have

$$\int_0^1 -f'(\Phi(\hat{\mu}) + s\sigma) ds > \beta.$$

Combining this inequality, $-\Phi'(\hat{\mu}) > 0$ and (4.8), we have

$$\mathfrak{L}[\bar{v}] \geq S(\xi')(\varepsilon\beta - \alpha\nu_*) > 0.$$

Next, we consider the case of $-1 + \delta_* \leq \Phi(\hat{\mu}) \leq 1 - \delta_*$. We have

$$-\Phi'(\hat{\mu}) \geq \min_{-1+\delta_* \leq \Phi(p) \leq 1-\delta_*} (-\Phi'(p))$$

and

$$-\varepsilon \int_0^1 f'(\Phi + \sigma s) ds \geq -\varepsilon \max_{|s| \leq 1+\delta_*} |f'(s)|.$$

Combining these inequalities and (4.8), we have

$$\mathfrak{L}[\bar{v}] \geq S(\xi') \left(\min_{-1+\delta_* \leq \Phi(p) \leq 1-\delta_*} (-\Phi'(p)) - \varepsilon \max_{|s| \leq 1+\delta_*} |f'(s)| - \alpha\nu_* \right) > 0.$$

Thus, $\bar{v}$ is a supersolution.

Now we prove the latter statement. It suffices to prove that

$$\Phi\left(\frac{k}{c}(x_N - h_j(\mathbf{x}'))\right) < \bar{v}(\mathbf{x}) \quad (4.11)$$

for any fixed j . If we have

$$\hat{\mu} \leq \frac{k}{c}(x_N - h_j(\mathbf{x}')),$$

we obtain

$$\Phi\left(\frac{k}{c}(x_N - h_j(\mathbf{x}'))\right) \leq \Phi(\hat{\mu}) < \bar{v}(\mathbf{x})$$

using $\Phi' < 0$. Thus, it suffices to consider the case when

$$\frac{k}{c}(x_N - h_j(\mathbf{x}')) < \hat{\mu}.$$

Substituting the definition of $\hat{\mu}$ into this inequality, we obtain

$$\frac{k}{c}(x_N - h_j(\mathbf{x}')) < \frac{x_N - \varphi(\xi')/\alpha}{\sqrt{1 + |\nabla\varphi|^2}} = \frac{x_N - h_j(\mathbf{x}') + (h_j(\mathbf{x}') - \varphi(\xi')/\alpha)}{\sqrt{1 + |\nabla\varphi|^2}}$$

and thus

$$\frac{c}{\alpha} \left(\frac{\varphi(\xi') - h_j(\xi')}{\sqrt{1 + |\nabla \varphi|^2}} \right) < \left(\frac{c}{\sqrt{1 + |\nabla \varphi|^2}} - k \right) (x_N - h_j(\mathbf{x}')).$$

By (2.3), we have

$$\frac{c}{\alpha} \left(\frac{1}{\sqrt{1 + |\nabla \varphi|^2}} \right) \left(\frac{\varphi(\xi') - h_j(\xi')}{S(\xi')} \right) < x_N - h_j(\mathbf{x}').$$

By the definition of ω and lemma 2.2 we have

$$\frac{k\omega}{\alpha} < \frac{c}{\alpha} \frac{\omega}{\sqrt{1 + |\nabla \varphi|^2}} < x_N - h_j(\mathbf{x}'). \quad (4.12)$$

Since

$$h_j(\mathbf{x}') = \frac{1}{\alpha} h_j(\xi') \leq \frac{1}{\alpha} h(\xi') \leq \frac{1}{\alpha} \varphi(\xi'),$$

we have

$$\begin{aligned} \bar{v}(\mathbf{x}) - \Phi \left(\frac{k}{c} (x_N - h_j(\mathbf{x}')) \right) &= \Phi \left(\frac{x_N - \frac{1}{\alpha} \varphi(\xi')}{\sqrt{1 + |\nabla \varphi|^2}} \right) + \varepsilon S(\xi') - \Phi \left(\frac{k}{c} (x_N - h_j(\mathbf{x}')) \right) \\ &\geq \Phi \left(\frac{x_N - h_j(\mathbf{x}')}{\sqrt{1 + |\nabla \varphi|^2}} \right) - \Phi \left(\frac{k}{c} (x_N - h_j(\mathbf{x}')) \right) + \varepsilon S(\xi'). \end{aligned}$$

Since

$$\begin{aligned} &\Phi \left(\frac{x_N - h_j(\mathbf{x}')}{\sqrt{1 + |\nabla \varphi|^2}} \right) - \Phi \left(\frac{k}{c} (x_N - h_j(\mathbf{x}')) \right) \\ &= \frac{(x_N - h_j(\mathbf{x}')) S(\xi')}{c} \int_0^1 \Phi' \left(\left(\frac{\theta}{\sqrt{1 + |\nabla \varphi|^2}} + \frac{k}{c} (1 - \theta) \right) (x_N - h_j(\mathbf{x}')) \right) d\theta, \end{aligned}$$

we have

$$\begin{aligned} \bar{v}(\mathbf{x}) - \Phi \left(\frac{k}{c} (x_N - h_j(\mathbf{x}')) \right) &\geq \frac{(x_N - h_j(\mathbf{x}')) S(\xi')}{c} \\ &\quad \times \int_0^1 \Phi' \left(\left(\frac{\theta}{\sqrt{1 + |\nabla \varphi|^2}} + \frac{k}{c} (1 - \theta) \right) (x_N - h_j(\mathbf{x}')) \right) d\theta + \varepsilon S(\xi'). \end{aligned}$$

By lemma 2.2 and (4.12), we have

$$\frac{k}{c} \leq \frac{\theta}{\sqrt{1 + |\nabla \varphi|^2}} + \frac{k}{c} (1 - \theta) \leq 1.$$

Then we have

$$\begin{aligned} \bar{v}(\mathbf{x}) - \Phi \left(\frac{k}{c} (x_N - h_j(\mathbf{x}')) \right) &\geq -\frac{S(\xi')}{c} \sup_{|\mu| \geq k^2 \omega / c\alpha} \left| \frac{c}{k} \mu \Phi'(\mu) \right| + \varepsilon S(\xi') \\ &= S(\xi') \left(\varepsilon - \frac{1}{k} \sup_{|\mu| \geq k^2 \omega / c\alpha} |\mu \Phi'(\mu)| \right). \end{aligned}$$

By virtue of lemma 2.1 and (4.10) we have

$$\frac{1}{k} \sup_{|\mu| \geq k^2 \omega / c\alpha} |\mu \Phi'(\mu)| < \frac{1}{2} \varepsilon.$$

Then we get

$$\bar{v}(\mathbf{x}) - \Phi\left(\frac{k}{c}(x_N - h_j(\mathbf{x}'))\right) > \frac{1}{2} \varepsilon S(\boldsymbol{\xi}') \geq 0$$

and obtain (4.11). This completes the proof. $\square$

Now we prove the main theorem in this paper.

Proof of theorem 1.1. Recall (1.9) and (4.7), and consider solutions of (1.2) given by $w(\mathbf{x}, t; \underline{v})$ and $w(\mathbf{x}, t; \bar{v})$, respectively. Since $\underline{v}$ is a subsolution and $\bar{v}$ is a supersolution, we have

$$\underline{v} \leq w(\mathbf{x}, t; \underline{v}) \leq w(\mathbf{x}, t; \bar{v}) \leq \bar{v}$$

for $\mathbf{x} \in \mathbb{R}^N$ and $t \geq 0$ by using [10, theorem 3.4]. Then

$$V(\mathbf{x}) := \lim_{t \rightarrow \infty} w(\mathbf{x}, t; \underline{v}) \quad (4.13)$$

exists in $L^\infty(\mathbb{R}^N)$ with

$$\underline{v}(\mathbf{x}) < V(\mathbf{x}) < \bar{v}(\mathbf{x}) \quad \text{for } \mathbf{x} \in \mathbb{R}^N.$$

This $V(\mathbf{x})$ is a solution of (1.3). See [10, theorem 3.6] for detailed arguments.

Now we prove (1.11). For any given $\varepsilon > 0$ we prove

$$\sup_{\mathbf{x} \in D(\gamma)} (\bar{v}(\mathbf{x}) - \underline{v}(\mathbf{x})) < 2\varepsilon \quad (4.14)$$

if $\gamma > 0$ is large enough. We use the same ε as in (4.7). Assume the contrary. Then there exist sequences $(\gamma_i)_{i=1}^\infty \subset \mathbb{R}$ and $(\mathbf{x}_i)_{i=1}^\infty \subset \mathbb{R}^N$ such that we have

$$\lim_{i \rightarrow \infty} \gamma_i = \infty, \quad \mathbf{x}_i \in D(\gamma_i), \quad (4.15)$$

and

$$\left| \Phi(\hat{\mu}_i) - \Phi\left(\frac{k}{c}(x_{N,i} - h(\mathbf{x}'_i))\right) \right| \geq \varepsilon, \quad (4.16)$$

where $\mathbf{x}_i = (x_{1,i}, \dots, x_{N,i})$ and $\mathbf{x}'_i = (x_{1,i}, \dots, x_{N-1,i})$ for $i \in \mathbb{N}$. Here we put $\boldsymbol{\xi}'_i = \alpha \mathbf{x}'_i$ and

$$\begin{aligned} \hat{\mu}_i &:= \frac{1}{\alpha} \frac{\xi_{N,i} - \varphi(\boldsymbol{\xi}'_i)}{\sqrt{1 + |\nabla \varphi(\boldsymbol{\xi}'_i)|^2}} \\ &= \frac{x_{N,i} - h(\mathbf{x}'_i) - (\varphi(\boldsymbol{\xi}'_i) - h(\boldsymbol{\xi}_i))/\alpha}{\sqrt{1 + |\nabla \varphi(\boldsymbol{\xi}'_i)|^2}} \\ &= \frac{x_{N,i} - h(\mathbf{x}'_i) - \tilde{\varphi}(\boldsymbol{\xi}')/\alpha}{\sqrt{1 + |\nabla \varphi(\boldsymbol{\xi}'_i)|^2}}. \end{aligned}$$

From (3.9) and (3.10) we have

$$\lim_{i \rightarrow \infty} |\tilde{\varphi}(\xi_i)| = 0, \quad |\nabla \varphi(\xi'_i)| = m, \quad \lim_{i \rightarrow \infty} S(\xi'_i) = 0.$$

Then we obtain

$$\lim_{i \rightarrow \infty} \left| \hat{\mu}_i - \frac{k}{c}(x_{N,i} - h(\mathbf{x}'_i)) \right| = 0.$$

This implies

$$\lim_{i \rightarrow \infty} \left| \Phi(\hat{\mu}_i) - \Phi\left(\frac{k}{c}(x_{N,i} - h(\mathbf{x}'_i))\right) \right| = 0.$$

This contradicts (4.16). This completes the proof of theorem 1.1. $\square$

For the uniqueness and stability of pyramidal travelling fronts, see [13] in $\mathbb{R}^3$. In $\mathbb{R}^N$ ($N \geq 4$) those properties are left to be an interesting open problem.

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