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On Group-Delay Sensitivity Properties of Complex Allpass Lattice Filters

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SUMMARY The group-delay sensitivity is studied theoretically for Gray and Markel allpass lattice filter realizing complex transfer function. Recursive expressions which were derived for phase and group-delay in real lattice are rederived for the complex case. These expressions are used to obtain upper bounds on group-delay sensitivity. The minimum number of frequencies where group-delay sensitivity becomes zero is discussed. Results corresponding real allpass lattice are also shown. Phase sensitivity properties of these filters are analyzed and compared with existing results. A new bound on phase sensitivity is also obtained.

1. Introduction

The digital allpass filter plays an important role in Digital Signal Processing (DSP) systems. It has several applications such as, equalizing (linearizing) group-delay (phase) characteristics of an arbitrary filter, notch filtering, multirate filtering, low sensitivity filter synthesis and so on[1]. On the other hand the existence of structures which enable us to realize the filter in an efficient and economical way makes it an attractive building block in DSP. Among these structures, Gray and Markel lattice filter[2] (allpass lattice filter) has a lot of excellent properties.

The minimum multiplier structure for allpass lattice filter gives rise to a zero gain sensitivity structure. Regalia et al. analyzed phase sensitivity properties of the lattice allpass filter in a recent paper[4]. In this paper we try to analyze the group-delay sensitivity properties of the filter, which is unknown and is important in most applications of it. In order to make a general analysis, the complex allpass lattice is assumed.

2. Recursive Expressions for Phase and Group-Delay

Takebe et al. have derived recursive expressions for phase and group-delay of the real allpass lattice[3]. In this section we extend their method to the complex allpass lattice and derive the generalized form of their results. The real allpass lattice will become a special case then.

Consider an $N+1$th order Gray-Markel allpass lattice network implemented using cascaded lattice sections C_i $(i=0, 1, \cdots, N)$. Each section is a complex digital two-pair characterized by multiplier k_i $(=\alpha_i+\sqrt{-1}\beta_i)$, as shown in Fig. 1. The transfer function of the cascaded sections from C_0 to C_i is an $i+1$th order complex allpass function designated as

$$A_{i+1}(z)=\frac{z^{-N-1}\sum_{i=0}^{N+1} a_i^* z^i}{\sum_{i=0}^{N+1} a_i z^{-i}}$$

(* indicates complex conjugation) whose frequency response is ††

$$A_{i+1}(z)\big|_{z=e^{j\omega}}=e^{-j\Theta_{i+1}(\omega)} \qquad i=0, 1, \cdots, N.$$

Thus we have $N+1$ phase functions, $\Theta_{i+1}(\omega)$, and the same number of group-delay functions defined as

$$\mathcal{T}_{i+1}(\omega)=\frac{\partial\Theta_{i+1}(\omega)}{\partial\omega}. \tag{1}$$

From Fig. 1, $A_{i+1}(z)$ can be implemented by transforming the delay element of the 1st order complex lattice realizing $(k_i^*+z^{-1})/(1+k_i z^{-1})$ with $z^{-1}A_i(z)$. In frequency domain this will be interpreted as †††

$$\omega \to \omega+\Theta_i.$$

So we can calculate Θ_{i+1} if Θ_i and k_i are known.

$$\Theta_{i+1}=\omega+\Theta_i-2\arctan$$

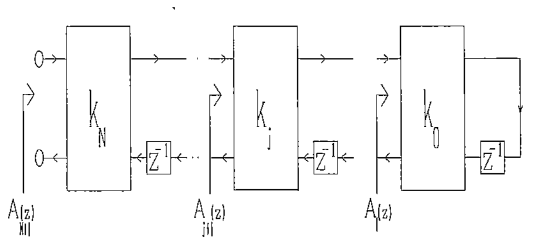

Fig. 1 Block diagram of $N+1$th order Gray and Markel IIR allpass lattice.

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†† Sampling period is assumed unity in this analysis.
††† For notational convenience (ω) will be abbreviated in all functions.

$$\left[\frac{\alpha_i \sin(\omega+\Theta_i)-\beta_i \cos(\omega+\Theta_i)}{1+\alpha_i \cos(\omega+\Theta_i)+\beta_i \sin(\omega+\Theta_i)}\right]$$

$$\text{for} \quad i=0, 1, \cdots, N \tag{2}$$

$$\Theta_0=0.$$

The entire lattice phase corresponds to $i=N$. From Eq. (2), we conclude that

$$\Theta_{i+1}|_{\omega=2\pi}-\Theta_{i+1}|_{\omega=0}=2(i+1)\pi. \tag{3}$$

The above relation is important in determining the number of zero-sensitivity frequencies. Differentiating Eq. (2) with respect to ω, group-delay will become Ref. (3)

$$\mathcal{T}_{i+1}=\tau_{i+1,0}(1+\mathcal{T}_i)$$

$$\tau_{i+1,0}=\frac{1-\alpha_i^2-\beta_i^2}{1+\alpha_i^2+\beta_i^2+2\alpha_i\cos(\omega+\Theta_i)+2\beta_i\sin(\omega+\Theta_i)}$$

$$\text{for} \quad i=0, 1, \cdots, N \tag{4}$$

$$\mathcal{T}_0=0.$$

The entire lattice group-delay can also be shown using $\tau_{i+1,0}$.

$$\mathcal{T}_{N+1}=\sum_{l=0}^{N}\prod_{i=l}^{N}\tau_{i+1,0} \tag{5}$$

In a stable filter $\alpha_i^2+\beta_i^2<1$ and $\tau_{i+1,0}$ is positive at all frequencies. Equations (2) and (4) are consistent with the results of Ref. (3) for $\beta_i=0$.

3. Phase Sensitivity Properties

In this section we use the recursive expression of phase function and derive the phase sensitivity. The sensitivity function used here is the entire network's phase derivative with respect to an arbitrary parameter, defined by

$$S_m^{\Theta_{N+1}}=\frac{\partial\Theta_{N+1}}{\partial m}. \tag{6}$$

This definition is used here because of the convenience of comparing the results with those of Ref. (4). Parameter m may be the real or imaginary part of k_i. Let m be the real part of the jth section's parameter α_j, then from Eq. (2), we have

$$\frac{\partial\Theta_{N+1}}{\partial\alpha_j}=\begin{cases}\dfrac{\partial\Theta_{N+1}}{\partial(\omega+\Theta_N)}\dfrac{\partial(\omega+\Theta_N)}{\partial\alpha_j}=\tau_{N+1,0}\dfrac{\partial\Theta_N}{\partial\alpha_j} \\ \text{for} \quad j<N \\ -2\dfrac{\beta_j+\sin(\omega+\Theta_j)}{1+\alpha_j^2+\beta_j^2+2\alpha_j\cos(\omega+\Theta_j)+2\beta_j\sin(\omega+\Theta_j)} \\ \text{for} \quad j=N.\end{cases} \tag{7}$$

This can be summarized as

$$\frac{\partial\Theta_{N+1}}{\partial\alpha_j}=\left(\prod_{l=j+2}^{N+1}\tau_{l,0}\right)\frac{\partial\Theta_{j+1}}{\partial\alpha_j}. \tag{8}$$

Results regarding the imaginary part β_j can be found in Appendix A.

3.1 Number of Zero-Sensitivity Frequencies

From Eq. (8), the zero-sensitivity frequencies of $S_{\alpha_j}^{\Theta_{N+1}}$ are the frequencies where $\partial\Theta_{j+1}/\partial\alpha_j$ becomes zero or the roots of

$$\sin(\omega+\Theta_j)=-\beta_j.$$

Stability condition of the filter causes $|\beta_j|<1$ and the above equation must have a root in stable filters. Considering Eq. (3), sensitivity function becomes zero at $2(j+1)$ frequencies when $\omega\in[0, 2\pi]$. $S_{\beta_i}^{\Theta_{N+1}}$ does also possess the same number of zero-sensitivity frequencies.

Real lattice case: In this case each phase function is determined at $\omega=0$ and $\omega=\pi$ as,

$$\theta_i|_{\omega=0}=0$$

$$\theta_i|_{\omega=\pi}=i\pi.$$

Thus there are $j+2$ zero-sensitivity frequencies in $[0, \pi]$ where $\sin(\theta_j+\omega)=0$. This property of real lattice has been already shown in Ref. (4).

3.2 Bounds on Phase Sensitivity

Next, we derive upper bounds on the phase sensitivity with respect to each lattice parameter. From Eq. (8), we can write

$$\max_{\omega}\left|\frac{\partial\Theta_{N+1}}{\partial\alpha_j}\right|\leqq\max_{\omega}\left(\prod_{l=j+2}^{N+1}\tau_{l,0}\right)\max_{\omega}\left|\frac{\partial\Theta_{j+1}}{\partial\alpha_j}\right|$$

$$\max_{\omega}\left|\frac{\partial\Theta_{N+1}}{\partial\alpha_j}\right|\leqq\left(\prod_{l=j+1}^{N}\frac{1+\sqrt{\alpha_l^2+\beta_l^2}}{1-\sqrt{\alpha_l^2+\beta_l^2}}\right)\frac{2(1+\beta_j)}{1-\alpha_j^2-\beta_j^2}. \tag{9}$$

Phase sensitivity has larger bounds when parameters' magnitude approach unity. Note that for $j=N$ (the leftmost section in Fig. 1), the first term of Eq. (9)'s right-hand side disappears and $\leqq$ becomes equal sign.

It seems that the above bound may become too large for a lattice whose parameters absolute values are near unity when j is too small, i.e., $j\ll N$. A useful bound on phase sensitivity for this case is derived here. From Eqs. (5) and (8),

$$\frac{\partial\Theta_{N+1}}{\partial\alpha_j}<\mathcal{T}_{N+1}\frac{\partial\Theta_{j+1}}{\partial\alpha_j} \tag{10}$$

and consequently,

$$\max_{\omega}\left|\frac{\partial\Theta_{N+1}}{\partial\alpha_j}\right|<\max_{\omega}(\mathcal{T}_{N+1})\frac{2(1+\beta_j)}{1-\alpha_j^2-\beta_j^2}. \tag{11}$$

Real lattice case: In a real lattice, $k_i=\alpha_i$ $(i=0, 1, \cdots, N)$ and we obtain

$$\max_{\omega}\left|\frac{\partial\theta_{N+1}}{\partial k_j}\right| \leq \left(\prod_{l=j+1}^{N}\frac{1+|k_l|}{1-|k_l|}\right)\frac{2}{1-k_j^2} \qquad (12)$$

which is consistent with the bound given in Ref.(4). The bound shown in Eq.(11) can also be used in this case. To see how effective Eq.(11) is, consider the 4th order real lattice characterized by $k_0=0.43$, $k_1=0.71$, $k_2=0.67$, $k_3=0.25$, and let $j=0$. Using Eq.(12), we obtain an upper bound of about 122 for $|\partial\theta_4/\partial k_0|$, while Eq.(11) decreases it to 62.

4. Group-Delay Sensitivity Properties

For group-delay, we define the first order sensitivity by

$$\mathcal{S}_m^{\mathcal{T}_{N+1}}=\frac{m}{\mathcal{T}_{N+1}}\frac{\partial\mathcal{T}_{N+1}}{\partial m} \qquad (13)$$

with respect to parameter m. Note that the same definition as phase sensitivity can be used here too. By substituting Eq.(4) for $\mathcal{T}_{N+1}$, group-delay sensitivity with respect to the real part of an arbitrary lattice parameter α_j is shown by the following recursive expression:

$$\mathcal{S}_{\alpha_j}^{\mathcal{T}_{N+1}}=\alpha_j\underbrace{\frac{2\alpha_N\sin(\omega+\Theta_N)-2\beta_N\cos(\omega+\Theta_N)}{1+\alpha_N^2+\beta_N^2+2\alpha_N\cos(\omega+\Theta_N)+2\beta_N\sin(\omega+\Theta_N)}}_{g_N}\frac{\partial\Theta_N}{\partial\alpha_j}+\frac{\mathcal{T}_N}{1+\mathcal{T}_N}\mathcal{S}_{\alpha_j}^{\mathcal{T}_N}$$

$$\vdots$$

$$\mathcal{S}_{\alpha_j}^{\mathcal{T}_{j+1}}=\underbrace{\frac{-2\alpha_j}{1-\alpha_j^2-\beta_j^2}\,\frac{2\alpha_j+(1+\alpha_j^2-\beta_j^2)\cos(\omega+\Theta_j)+2\alpha_j\beta_j\sin(\omega+\Theta_j)}{1+\alpha_j^2+\beta_j^2+2\alpha_j\cos(\omega+\Theta_j)+2\beta_j\sin(\omega+\Theta_j)}}_{g_j}. \qquad (14)$$

4.1 Number of Zero-Sensitivity Frequencies

$\partial\mathcal{T}_{N+1}/\partial\alpha_j$ is the phase sensitivity's derivative with respect to ω. Using the results of Sect. 3.1, phase sensitivity with respect to α_j is a continuous periodic function and has $2(j+1)$ roots in $[0, 2\pi]$. As a result, group-delay sensitivity becomes zero *at least* $2(j+1)$ times while ω rotates around the unit circle. For $j=N$, we have *exactly* $2(N+1)$ zero-sensitivity frequencies which are the roots of the following equation:

$$2\alpha_N+(1+\alpha_N^2-\beta_N^2)\cos(\omega+\Theta_N)+2\alpha_N\beta_N\sin(\omega+\Theta_N)=0. \qquad (15)$$

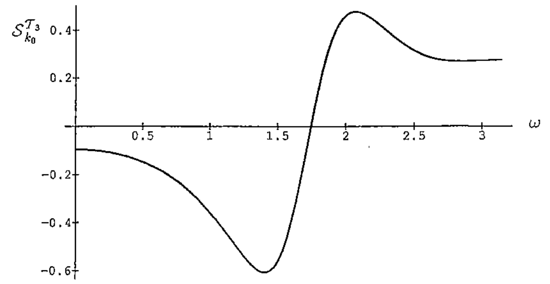

Fig. 2 Group-delay sensitivity of the 3rd order real lattice characterized by: $k_0=0.33$, $k_1=0.41$, $k_2=0.17$.

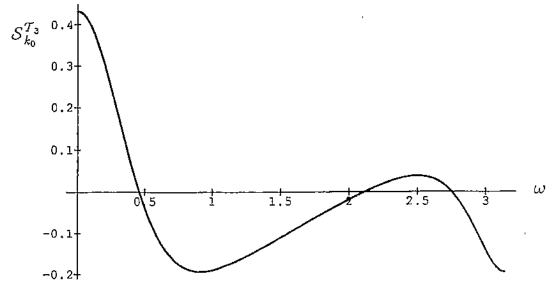

Fig. 3 Group-delay sensitivity of the 3rd order real lattice characterized by: $k_0=-0.33$, $k_1=-0.41$, $k_2=0.17$.

The same number of zero-sensitivity points exist for the imaginary part β_j.

Real lattice case: From Sect. 3.1, there are *at least* $j+1$ of these points when $j<N$, and *exactly* $N+1$ when $j=N$ in $[0, \pi]$. As an example we show the position of these ptoints in a 3rd order lattice in Figs. 2 and 3. Though the absolute value of lattice parameters are equal, the number of zero-sensitivity points with respect to k_0 differs.

4.2 Bounds on Group-Delay Sensitivity

In a stable allpass filter group-delay is always positive and inequality

$$0<\frac{\mathcal{T}_N}{1+\mathcal{T}_N}<1$$

holds for all frequencies. Applying triangle inequality to the absolute value of $\mathcal{S}_{\alpha_j}^{\mathcal{T}_{N+1}}$ yields

$$\max_{\omega}|\mathcal{S}_{\alpha_j}^{\mathcal{T}_{N+1}}|\leq\max_{\omega}|g_N|\max_{\omega}\left|\frac{\partial\Theta_N}{\partial\alpha_j}\right|+\cdots$$
$$+\max_{\omega}|g_{j+1}|\max_{\omega}\left|\frac{\partial\Theta_{j+1}}{\partial\alpha_j}\right|+\max_{\omega}|g_j|. \qquad (16)$$

Now, we can obtain each $\max_{\omega}$ in terms of lattice parameter and derive upper bounds on phase sensitivity.

$$\max_{\omega}|\mathcal{S}^{\mathcal{T}_{N+1}}_{\alpha_j}|=\frac{2|\alpha_N|}{1-\alpha_N^2-\beta_N^2}$$

$$\max_{\omega}|\mathcal{S}^{\mathcal{T}_{N+1}}_{\alpha_j}|\leqq|\alpha_{N-1}|\left(\frac{2}{1-\alpha_{N-1}^2-\beta_{N-1}^2}+\frac{2(1+\beta_{N-1})}{1-\alpha_{N-1}^2-\beta_{N-1}^2}\,\frac{2\sqrt{\alpha_N^2+\beta_N^2}}{1-\alpha_N^2-\beta_N^2}\right)$$

$$\vdots$$

$$\max_{\omega}|\mathcal{S}^{\mathcal{T}_{N+1}}_{\alpha_j}|\leqq|\alpha_j|\left(\frac{2}{1-\alpha_j^2-\beta_j^2}+\frac{2(1+\beta_j)}{1-\alpha_j^2-\beta_j^2}\,\frac{2\sqrt{\alpha_{j+1}^2+\beta_{j+1}^2}}{1-\alpha_{j+1}^2-\beta_{j+1}^2}+\frac{2(1+\beta_j)}{1-\alpha_j^2-\beta_j^2}\cdot\sum_{i=j+2}^{N}\left[\left\{\prod_{l=j+1}^{i-1}\frac{1+\sqrt{\alpha_l^2+\beta_l^2}}{1-\sqrt{\alpha_l^2+\beta_l^2}}\right\}\frac{2\sqrt{\alpha_i^2+\beta_i^2}}{1-\alpha_i^2-\beta_i^2}\right]\right) \quad (17)$$

Results regarding the imaginary part can be found in Appendix B.

Real lattice case: Upper bounds on group-delay sensitivities are shown here, for the real allpass lattice.

$$\max_{\omega}|\mathcal{S}^{\mathcal{T}_{N+1}}_{k_N}|=\frac{2|k_N|}{1-k_N^2}$$

$$\max_{\omega}|\mathcal{S}^{\mathcal{T}_{N+1}}_{k_N}|\leqq\frac{2|k_{N-1}|}{1-k_{N-1}^2}\left[1+\frac{2|k_N|}{1-k_N^2}\right]$$

$$\vdots$$

$$\max_{\omega}|\mathcal{S}^{\mathcal{T}_{N+1}}_{k_N}|\leqq\frac{2|k_j|}{1-k_j^2}\left(1+\frac{2|k_{j+1}|}{1-{k_{j+1}}^2}+\sum_{i=j+2}^{N}\left[\left\{\prod_{l=j+1}^{i-1}\frac{1+|k_l|}{1-|k_l|}\right\}\frac{2|k_i|}{1-k_i^2}\right]\right) \quad (18)$$

5. Conclusion

The group-delay sensitivity properties of complex allpass lattice filters have been studied. The minimum number of zero sensitivity frequencies were shown and upper bounds on sensitivity functions were derived in terms of lattice parameters. Since each parameter may consist of real and/or imaginary parts, group-delay sensitivity with respect to both of them were considered.

Upper bounds on phase sensitivity of these filters, which are the generalized form of the existing results corresponding the real lattice, were also shown. A new bound on phase sensitivity was derived which may have smaller values for parameters with small subscript numbers and large absolute values in long lattices.

Both group-delay and phase sensitivity possess larger bounds when lattice parameters approach unity in magnitude.

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Appendix A: Phase Sensitivity

Recursive formula for phase sensitivity:

$$\frac{\partial\Theta_{N+1}}{\partial\beta_j}=\begin{cases}\tau_{N+1,0}\dfrac{\partial\Theta_N}{\partial\beta_j} & \text{for } j<N\\ -2\dfrac{\alpha_j+\cos(\omega+\Theta_j)}{1+\alpha_j^2+\beta_j^2+2\alpha_j\cos(\omega+\Theta_j)+2\beta_j\sin(\omega+\Theta_j)} & \text{for } j=N.\end{cases} \quad (19)$$

Bounds on phase sensitivity:

$$\max_{\omega}\left|\frac{\partial\Theta_{N+1}}{\beta_j}\right|\leqq\left(\prod_{l=j+1}^{N}\frac{1+\sqrt{\alpha_l^2+\beta_l^2}}{1-\sqrt{\alpha_l^2+\beta_l^2}}\right)\frac{2(1-\alpha_j)}{1-\alpha_j^2-\beta_j^2}. \quad (20)$$

Appendix B: Group-Delay Sensitivity

Recursive formula for group-delay sensitivity:

$$\mathcal{S}^{\mathcal{T}_{N+1}}_{\beta_j}=\beta_j\frac{2\alpha_N\sin(\omega+\Theta_N)-2\beta_N\cos(\omega+\Theta_N)}{1+\alpha_N^2+\beta_N^2+2\alpha_N\cos(\omega+\Theta_N)+2\beta_N\sin(\omega+\Theta_N)}\cdot\frac{\partial\Theta_N}{\partial\beta_j}+\frac{\mathcal{T}_N}{1+\mathcal{T}_N}\mathcal{S}^{\mathcal{T}_N}_{\beta_j}$$

$$\vdots$$

$$\mathcal{S}^{\mathcal{T}_{j+1}}_{\beta_j}=\frac{-2\beta_j}{1-\alpha_j^2-\beta_j^2}\,\frac{2\beta_j+(1-\alpha_j^2+\beta_j^2)\sin(\omega+\Theta_j)+2\alpha_j\beta_j\cos(\omega+\Theta_j)}{1+\alpha_j^2+\beta_j^2+2\alpha_j\cos(\omega+\Theta_j)+2\beta_j\sin(\omega+\Theta_j)}. \quad (21)$$

Bounds on group-delay sensitivity:

$$\max|\mathcal{S}^{\mathcal{T}_{N+1}}_{\beta_N}|=\frac{2|\beta_N|}{1-\alpha_N^2-\beta_N^2}$$

$$\max|\mathcal{S}_{\beta_{N-1}}^{\mathcal{T}_{N+1}}| \leqq |\beta_{N-1}|\left(\frac{2}{1-\alpha_{N-1}^2-\beta_{N-1}^2}\right.$$
$$\left.+\frac{2(1-\alpha_{N-1})}{1-\alpha_{N-1}^2-\beta_{N-1}^2}\frac{2\sqrt{\alpha_N^2+\beta_N^2}}{1-\alpha_N^2-\beta_N^2}\right)$$
$$\vdots$$
$$\max|\mathcal{S}_{\beta_j}^{\mathcal{T}_{N+1}}| \leqq |\beta_j|\left(\frac{2}{1-\alpha_j^2-\beta_j^2}\right.$$
$$+\frac{2(1-\alpha_j)}{1-\alpha_j^2-\beta_j^2}\frac{2\sqrt{\alpha_{j+1}^2+\beta_{j+1}^2}}{1-\alpha_{j+1}^2-\beta_{j+1}^2}+\frac{2(1-\alpha_j)}{1-\alpha_j^2-\beta_j^2}$$
$$\left.\cdot\sum_{i=j+2}^{N}\left[\left\{\prod_{l=j+1}^{i-1}\frac{1+\sqrt{\alpha_l^2+\beta_l^2}}{1-\sqrt{\alpha_l^2+\beta_l^2}}\right\}\frac{2\sqrt{\alpha_i^2+\beta_i^2}}{1-\alpha_i^2-\beta_i^2}\right]\right). \quad (22)$$

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